PHENOMENOLOGY WITH MAGNETIZED D-BRANES

By

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Binata Panda

To My Parents and Prof. Alok Kumar ......

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Contents

Synopsis

Superstring theory is currently considered to be one of the most promising candidates for unifying the different particles and their interactions in nature. This is due to the fact that it provides a description of gauge and gravitational interactions in a unified framework consistently at the quantum level. If String theory is indeed realized in nature, it's certain low energy limit should reproduce the Standard Model (SM), a unified model of strong and electroweak interactions, which has been so successful in describing the particle world. As it is well known, the superstring theories are consistent in ten dimensional spacetime, and usually have a high degree of supersymmetry. In the process of describing models reducing at low energies to four dimensions with less or no supersymmetry, there is an enormous arbitrariness in the choice of the background configuration. To reproduce four-dimensional physics at low energies, one needs to compactify the theory on a six-dimensional manifold. This leads to the existence of large number of unobserved neutral massless scalar particles (moduli fields). Geometrically the vacuum expectation values of these moduli fields parametrize, among other things, the size and shape of the compactification manifolds. These values are also related to the parameters like gauge coupling constants or masses of the effective four dimensional theory. By not being able to provide these expectation values via minimization of some effective potential, string models generally lose the predictive power. One of the main focus of present day research is to generate, in various ways, potentials for these moduli fields, minima of which could give masses to these fields. This goes by the name of 'moduli stabilization'.

The search for realistic string vacua is one of the most ambitious tasks in Superstring theory, and thus essentially covers the branch known as String Phenomenology. A phenomenologically viable string compactification should contain three chiral fermion generations, the Standard Model gauge group or some extension of it e.g. GUT models and broken space-time supersymmetry. In addition to this basic structure, it should reproduce the exact gauge and Yukawa couplings. Moreover, it should satisfy a set of conditions in order to produce a consistent anomaly free theory. Further, all the modulis are needed to be stabilized. There have been a lot of effort devoted along this direction in past years. Consequently there exists a good number of string constructions like heterotic string compactification on Calabi-Yau threefolds, M-theory compactifications on G2-holonomy spaces, intersecting D-brane models, compactification with non-trivial fluxes etc. aiming to reproduce the physics of the Standard Model at low energies.

In the present thesis, we discuss a simple framework of toroidal compactification of type I string theory with magnetized D-branes (D-branes with worldvolume fluxes along compactified tori), that offers an interesting self-consistent set up for string phenomenology. In such models, the gauge bosons and the chiral fermions come from the open string sector. In particular, the gauge bosons appear due to strings attached to stacks of D-branes and chiral matter arises from the strings stretching between different stacks of D-branes. Gravity, as usual, originates from the closed string sector. The fluxes that are turned on, can be used to build phenomenological models with an exact chiral fermion spectrum and gauge group, where some/all the moduli are stabilized and spacetime supersymmetry is broken.

We begin with a discussion of compactification of type I strings on a torus with additional background gauge flux on the D9-branes and review the necessary constraints needed for constructing semi-realistic models in such a framework. Switching on constant internal magnetic fields has important consequences in type I string compactifications to four-dimensions [1,2]. Such magnetic fluxes are described by exact conformal field theories and they give a spin dependent shift (for states which are charged under the corresponding gauge transformation) in the masses leading to a spectrum described by various Landau energy levels. This leads to chiral massless spectra in four space-time dimensions. Moreover, when the magnetic field is turned on along the compact directions, it has to satisfy Dirac quantization conditions. Fluxes, in general, break supersymmetry. However, in some special cases, a part of the supersymmetry can be preserved provided fluxes satisfy certain constraints. These constraints, in turn, can be used for stabilizing the closed string moduli because they correspond to stable minima of the scalar potential. However, in order to stabilize all 36 closed string geometric moduli of the torus T^6, one needs to include both 'diagonal' and 'oblique' fluxes [5,6]. These methods can also be employed for the open string moduli stabilization in any specific model. We also study the tadpole cancellation conditions which are required for consistency of type I string vacua. Since a crucial step in a three generation model building is the introduction of a Neveu Schwarz - Neveu Schwarz B-field background, the effect of non-zero B on the chirality and tadpoles is summarized following [3,4].

We then carry out the computations of Yukawa couplings in such magnetized brane constructions and find the close form expressions for them. In such a framework, the computation of the Yukawa couplings amounts to evaluating overlap integrals of three wavefunctions (contributing to the interaction) along internal directions. To perform the task, knowledge of the fermion (scalar) wavefunctions on toroidally compactified spaces (in the presence of fluxes) is required. However, technical difficulties arise in dealing with the explicit form of the fermion wavefunctions on tori in the presence of magnetic fluxes. Particularly, the presence of 'oblique' fluxes adds extra complexity to the problem.

We summarize the results for the fermion (scalar) wave functions and the Yukawa interaction for factorized tori and 'diagonal' fluxes [7]. In this case, the fermion wavefunctions are given by Jacobi Theta functions. The Yukawas are obtained by performing the overlap integrals of these wavefunctions and using certain identity [8] satisfied by Jacobi theta functions. We present a proof of the identity. We then generalize the results to write down the expression for the Yukawa interaction when oblique fluxes are present [10]. In order to perform this task, fermion (scalar) wavefunctions on toroidally compactified spaces are presented for general fluxes. These are parametrized by Hermitian matrices with eigenvalues of arbitrary signatures. The wavefunctions, so obtained, are given by general Riemann Theta functions with matrix valued modular parameter. We also give explicit mappings among fermion wavefunctions, of different internal chiralities on the tori, which interchange the role of the flux components with the complex structure of the torus. By evaluating the overlap integral of the wave functions, the expressions for Yukawa couplings among chiral multiplets, arising from an arbitrary set of branes are obtained. This essentially leads us to construct certain mathematical identities for general Riemann theta functions. We generalize the theta identity for Riemann theta functions and present a proof of this. We then use this new mathematical relation for writing down the expression for the Yukawa interaction when oblique fluxes consistent with supersymmetry and 'Riemann condition' requirements are present. In order to relax the later, the results are further generalized to include the wavefunctions of the other internal chiralities, in order to accommodate general fluxes consistent with supersymmetry restrictions.

Finally, we present a minimal example of a supersymmetric grand unified model in a toroidal compactification of type I string theory with magnetized $D9$-branes [9]. We obtain general solutions for fluxes along magnetized $D9$-branes yielding the chiral spectrum and gauge group of a three generation SU(5) GUT model, with no extra chiral matter nor U(1) factors. The gauge symmetry is just $SU(5)$ and the gauge non-singlet chiral spectrum contains only three families of quarks and leptons transforming in the $\mathbf{10} + \bar{\mathbf{5}}$ representations. Brane stacks with oblique fluxes play a central role in this construction, in order to stabilize all close string moduli, in a manner restricting the chiral matter content to precisely that of $SU(5)$ GUT. Another interesting feature of this model is that it is free from any chiral exotics that often appear in such brane constructions. The flux solutions also satisfy the RR tadpole cancellation conditions resulting the model to be consistent. However, the model contains extra non-chiral matter that is expected to become massive at a high scale, close to that of $SU(5)$ breaking. Finally, we present a brief analysis of the superpotential and D-terms for the model in order to show the mass generation for several non-chiral fermion multiplets in a supersymmetric ground state [10]. Using the results for Yukawa couplings, we show that a ground state allowing masses for the above matter multiplets is possible. This exercise further fine tunes our $SU(5)$ GUT model to the ones used in conventional grand unification.

References:

[1] C. Bachas, *"A Way to break supersymmetry,"* hep-th/9503030.

[2] R. Blumenhagen, L. Görlich, B. Körs and D. Lüst, *"Magnetic flux in toroidal type I compactification,"* Fortsch. Phys. **49**, 591 (2001), hep-th/0010198.

[3] C. Angelantonj, *"Comments on open string orbifolds with a nonvanishing B(ab),"* Nucl. Phys. B **566**, 126 (2000) [arXiv:hep-th/9908064].

[4] R. Blumenhagen, B. Körs and D. Lüst, *"Type I strings with F- and B-flux,"* JHEP **0102**, 030 (2001), hep-th/0012156.

[5] I. Antoniadis and T. Maillard,*"Moduli stabilization from magnetic fluxes in type I string theory,"* Nucl. Phys. B **716** (2005) 3, [arXiv:hep-th/0412008].

[8] I. Antoniadis, A. Kumar, T. Maillard, *"Magnetic fluxes and moduli stabilization"* Nucl. Phys. B **767** (2007) 139, [arXiv:hep-th/0610246].

[9] D. Cremades, L. E. Ibanez and F. Marchesano, *"Computing Yukawa Couplings from Magnetized Extra Dimensions,"* JHEP **0405** 079 (2004),. hep-th/0404229.

[10] D. Mumford, *Tata lectures on Theta I*, Berkhauser, Boston, 1983.

[11] I. Antoniadis, A. Kumar and B. Panda, *"Supersymmetric SU(5) GUT with stabilized moduli,"* Nucl. Phys. **B** (2008), arXiv: 0709.2799.

[12] I. Antoniadis, A. Kumar and B. Panda, *"Fermion Wavefunctions in Magnetized branes: Theta identities and Yukawa couplings,"* Nucl. Phys. B **823**, 116 (2009) [arXiv:0904.0910 [hep-th]].

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B. Chandrasekhar, Binata Panda,
Int. J. Mod. Phys. A **26** (2011) 2377 [arXiv:0909.3061[hep-th]].

[4] *"Black hole phase transitions via Bragg-Williams"*,
Souvik Banerjee, Sayan K. Chakrabarti, Sudipta Mukherji, Binata Panda,
Int. J. Mod. Phys. **A26**, (2011) 3469 [arXiv:1012.3256 [hep-th]].

A (*) indicates papers on which this thesis is based.

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Introduction

1.1 An Overview

General theory of relativity and the Standard Model (SM) of particle physics perhaps are the two greatest discoveries in physics during the last century. However, their domains of applicability remained largely disjoint. While general theory of relativity was found to dominate at large distances (for instance, describing the motion of a planet), SM described interactions at small length scales. But, surely, there are situations where these two forces become equally crucial. Universe at a very early time, behaviour near the horizon of a not-so-large black hole provide such situations. In these cases, gravitational force becomes strong even at small distances and, therefore, there is a need to consider gravitational interaction along with the other three interactions of the Standard Model.

Preceding discussion, therefore, suggests that the Standard Model alone is not enough in describing our universe at high energies. There are other reasons to believe that this model indeed is not complete. SM contains nineteen free parameters which are fixed, a posteriori, by experimental data. Furthermore, it suffers from well known hierarchy and naturalness problems. A correct description of the observed masses and mixing of quarks and leptons require very different values for the Yukawa coupling constants for different generations. Although many approaches have been put forward to describe the hierarchical structure of Yukawa couplings between the Higgs field and the SM fermions, it is perhaps fair to say that we do not have, at the moment, a compelling theory for quark and lepton masses. On the other hand, naturalness technically refers to the necessity of fine-tuning the tree level parameters to accommodate for experimentally acceptable values given the size of the perturbative quantum corrections.

These lead us to believe that there is a more fundamental theory which incorporates gravity along with SM in a unified framework and, in turn, fixes all the arbitrariness of the

SM. Among all the possibilities that have so far been put forward, supersymmetric string theory or superstring theory, surely, is the most promising one. Instead of point particles, here, fundamental objects are the strings and particles appear as different vibrational modes of the string. Strings with open and closed ends constitute two different sectors known as open and closed string sectors respectively. While closed strings have, in it's spectrum, a mass-less spin two particle known as graviton [1], the open strings carry gauge charges at it's end points. Therefore, strings provide a possibility to unify gauge and gravitational interactions in a natural way. The scale of this unification is dictated by the inverse of the size of a string. This scale is, however, much higher than the accessible energies in present day accelerators and hence, strings remained un-observable. At the same time, since via accelerators, the correctness of the SM have been tested to a very high accuracy, string theory must reproduce just the SM at low energies. In spite of several attempts, getting just the SM from string theory has so far remained an illusive task.

Consistency requires superstrings to live in ten space-time dimensions with space-time supersymmetry. As we will discuss in the later sections, consistency also requires five different kinds of superstrings in ten dimensions. The connection to our four dimensional observational world is made via compactification of six space dimensions. Unfortunately, it turns out that, there are several consistent compactification schemes which produce different effective field theories in four dimensions at low energy scale. Even if strings at ten dimensions do not have any free parameter, arbitrariness in compactification introduces many undetermined parameters in the low energy theory. Among them, for example, are the sizes and shapes of the compact manifold. In four dimensional theories, these parameters appear as the vacuum expectation values (vev) of the scalars. These are commonly known as the moduli fields. Continuous deformations in size and shape of the compact manifold show up as continuous changes in the vevs of these moduli. This, in turn, means that these scalars are not accompanied by any potentials. One of the main focus of present day research is to find ways to remove these flat directions of the moduli by generating their masses. Unless these moduli fields are de-coupled at a scale higher than the presently accessible scale, relating string theory to SM remains a difficult task. In spite of this vexing problem, exciting progress, however, has been made in achieving partial stabilization of these moduli. This will be discussed in details in the later sections.

The present thesis serves as an attempt to construct low energy string models by partially stabilizing the moduli and constructing an extension of the SM. A crucial ingredient in our model building will be the Dirichlet branes or the D-branes in short. D-branes, discovered in [2, 3], are the solitonic configurations in sting theory on which open string can end. As we will see, magnetized D-branes contain several phenomenologically appealing general features suggesting that they may offer an interesting self-consistent set up to construct semi-realistic models.

Any phenomenologically viable string compactification should contain three chiral generations, the SM gauge group or some extension of it and broken space-time supersymmetry. In addition it should reproduce the exact gauge and Yukawa couplings. It must satisfy a set of conditions in order to produce a consistent anomaly free theory. Moreover, all the moduli fields are needed to be stabilized. In this thesis, we start with type I string theory (one out of five consistent string theories in 10 dimensions) compactified on a six dimensional torus T^6. In type I string theory, there exists two known ways of achieving chirality in the effective lower dimensional theory. Either, one can compactify on curved spaces, in particular on orbifolds, leading to supersymmetric and non-supersymmetric chiral models in four dimensions. Or, one can obtain chiral spectra by introducing D-branes with magnetic flux [4]. We follow the later approach and discuss the toroidal compactification of type I string theory with additional background gauge flux on the D9-branes. A D9-brane is a soliton in type I theory with $9 + 1$ world-volume directions filling up the whole space time. We review the necessary constraints required for constructing phenomenological models in such a framework. For arbitrary magnetic fields, supersymmetry is spontaneously broken. However, a part of the supersymmetry can be preserved provided fluxes satisfy certain constraints. These constraints, in turn, can be used for stabilizing the closed string moduli. However, in order to stabilize all closed string geometric moduli of the torus T^6, one needs to include both diagonal and oblique fluxes.

The main aim of the thesis is to build phenomenological models, with an exact chiral fermion spectrum and gauge group, where some/all the moduli are stabilized and space-time supersymmetry is broken, in the framework described above. Moreover, we carry out the computations of Yukawa couplings in such magnetized brane constructions and find the close form expressions for them. In such a framework, the computation of the Yukawa couplings amounts to evaluating overlap integrals of three wavefunctions (contributing to the interaction) along internal directions. In the course of this work, we explicitly solve for the fermion (scalar) wavefunctions on toroidally compactified spaces in the presence of general fluxes. The wavefunctions, so obtained, are given by general Riemann Theta functions with matrix valued modular parameter. By evaluating the overlap integrals of these wave functions, the expressions for Yukawa couplings among chiral multiplets are obtained [5]. This essentially leads us to construct certain mathematical identities for general Riemann theta functions. We generalize the existing theta identity, satisfied by Jacobi theta functions, for Riemann theta functions and present a proof of this. We then use these new mathematical relations to write down the expressions for the Yukawa interactions. In special cases, our results reproduce the results obtained in [6] for factorized tori and 'diagonal' fluxes.

Finally, we present an example of a three generation $SU(5)$ supersymmetric grand unified (GUT) model in simple toroidal compactifications of type I string theory with magnetized $D9$ branes in [7]. The gauge group is just $SU(5)$ and the chiral gauge non-singlet spectrum consists of three families with the quantum numbers of quarks and leptons, transforming in the $\mathbf{10} + \bar{\mathbf{5}}$ representations of $SU(5)$. The fluxes also satisfy the RR tadpole cancellation conditions yielding a consistent model. Brane stacks with oblique fluxes play a central role in this construction, in order to stabilize all close string moduli, in a manner restricting the chiral matter content to precisely that of $SU(5)$ GUT. Another interesting feature of this model is that it is free from any chiral exotics that often appear in such brane constructions. However, the model contains extra non-chiral matter that is expected to become massive at a high scale, close to that of $SU(5)$ breaking. Using the results for Yukawa couplings, we show the mass generation for several non-chiral fermion multiplets in a supersymmetric ground state which further fine tunes the $SU(5)$ GUT model[5].

Before we go on to present our results in the later chapters, in the next section of this chapter, we give a brief historical survey on the search of the SM or Grand Unified Theory (GUT) models in the context of string theory. The aim of this survey is to motivate our work, as well as giving an account of all the efforts that have made in the branch of Superstring Phenomenology. We will use elements and notations that are already standard in string theory literature, and are common in the basic texts [8, 9, 10, 11, 12, 13]. We refer the reader to these texts for the details and completeness on the basic aspects of the theory.

1.2 The Search for the Standard Model

The First String Revolution took place around 1984, when Green and Schwarz discovered a new mechanism to formulate consistent superstring theories in ten dimensions [14]. Until then, two such consistent theories had been constructed, namely type IIA and type IIB superstring theories. Both involved closed strings only, and the effective field theories derived from the low energy spectrum amounted to the two different $\mathcal{N} = 2$ Supergravity (SUGRA) theories in ten dimensions, named in the same manner. Both of these effective theories are free of inconsistencies such as chiral, mixed and gravitational anomalies. On the contrary, the superstring theory known as type I, which involved both open and closed strings, seemed to have such quantum anomalies. With the discovery of the Green-Schwarz mechanism, however, it was possible to show that if type I theory was endowed with a Yang-Mills theory with gauge group $SO(32)$, then the anomalies could factorize and be canceled, finally obtaining a consistent theory. This was followed by the subsequent construction of the heterotic superstring theory in ten dimensions, [15, 16, 17]. These two theories involve a 10D SUGRA $\mathcal{N} = 1$ effective theory and are endowed with gauge groups which are, respectively, $SO(32)$ and $E_8 \times E_8$.

The first attempts to build realistic string models were based on $E_8 \times E_8$ heterotic string compactifications. A phenomenologically viable compactification requires obtaining an effective theory in four dimensions with a chiral spectrum and a gauge group containing $SU(3) \times SU(2) \times U(1)$. Since gravity was also to be a part of the low energy spectrum, the string scale was fixed at the order of the Planck scale, and the hierarchy problem was avoided by imposing local $\mathcal{N} = 1$ supersymmetry (SUSY). As it was shown in [18, 19], such conditions required the six *extra* dimensions to fulfill some constraints, namely it should be a compact Riemannian manifold with $SU(3)$ holonomy group. Such manifolds are known as Calabi-Yau threefolds, or $\mathbf{CY_3}$ [20, 21, 22]. An explicit model with three generations based on heterotic superstring compactification is presented in [23, 24]. Although $E_8 \times E_8$ heterotic compactifications on Calabi-Yau manifolds have provided rather realistic models, it is difficult to perform computations in such manifolds where, in most cases, not even the metric is known. An interesting class of spaces where computations are much more tractable is given by the toroidal orbifolds [25, 26]. Since the geometry is simpler than that of a $\mathbf{CY}$ and the metric is flat outside the singularities, computations can be easily carried out, and quantities of physical interest are thus more easily computable. Subsequently, exact heterotic string solutions on six dimensional orbifold spaces were constructed [27, 28, 29]. This was followed by a series of constructions, such as the Gepner models [30], the free-fermion models [31, 32] or heterotic string-derived flipped $SU(5)$ models [33].

The Second Superstring Revolution took place around 1995, and it mainly concerned the non-perturbative aspects of string theory. Until then, string theory was understood as five different superstring theories, apparently independent, known as type I, type II (A and B) and the two heterotic theories. However, in the context of this second revolution, it was learnt that they were all related to each other by a web of string dualities. The duality establishes a one-to-one correspondence between parameters and fields defining one theory (compactification radii, coupling constants, etc.) and the same set of quantities defining its dual. Duality involves strong-weak coupling exchange either in sigma- model or in space-time. The string duality web revealed that these five string theories were not isolated independent theories, but actually limiting cases of a deeper, more fundamental theory, named M-theory, whose precise nature has not yet been unraveled [34, 35]. Such theory would be formulated in eleven space time dimensions, and its basic dynamical objects would be membranes rather than strings. These membranes naturally appear as solitonic objects of $D = 11$ SUGRA, which would be another limiting case of M-theory. The other five limiting cases, i.e., the five superstring theories, would then be obtained from compactifying the eleventh dimension in a very small length. Figure 1.1 shows an schematic representation of the situation.

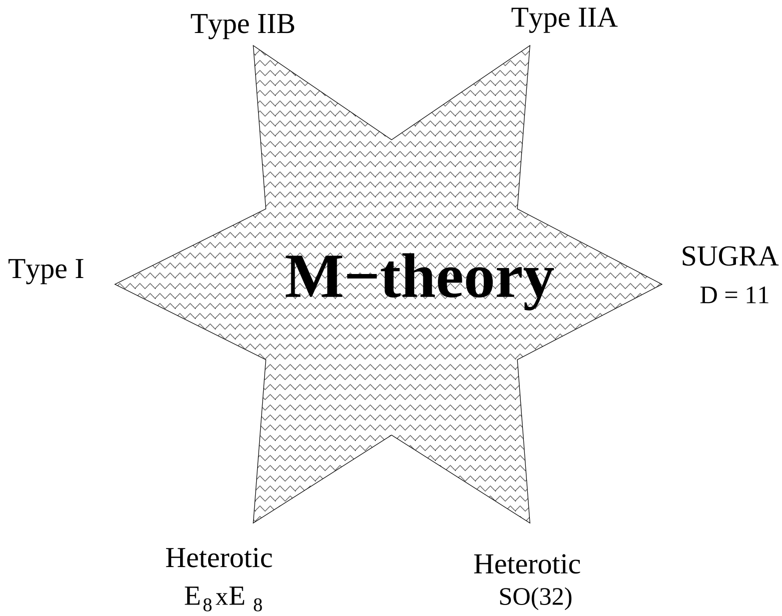

Figure 1.1: Situation of string theory after the second superstring revolution. The previously disconnected five superstring theories are nothing but specific (limiting) points in the parameter space of a more fundamental theory: M-theory.

In the formulation of these new string dualities, non-perturbative objects of the theory such as the so-called D-branes played a prime role. D-branes naturally emerge while considering a toroidal compactification of type I theory and performing a T-duality on one of the compact dimensions [2, 3]. Generically, a Dp-brane is a BPS solitonic object of spatial dimension p where the open strings localize their ends. Type IIA theory contains Dp-branes with p even, whereas, for type IIB, p must be odd. The lowest excitation modes of the open strings gives rise to massless gauge fields and their fermionic superpartners. The supersymmetric effective theory arising from the worldvolume of a D-brane is endowed with a $U(1)$ gauge group. And a stack of N D-branes of the same kind on top of each other have $U(N)$ gauge symmetry. Moreover, the effective field theory is defined on the $p+1$ dimensional D-brane worldwolume, and the fields on it are confined to propagate on such worldvolume. D-branes being solitonic in nature, are massive within the perturbative string theory. Their mass scales as $\frac{1}{g_s}$ where g_s is the string coupling constant.

The properties of these D-branes make them promising candidates for string model building and indeed, new semi-realistic models based on type I and type II theories started appearing. Actually, the first consistent compactifications of type I theory on orbifold spaces were realized long time ago in [36, 37], obtaining $D = 6$ supersymmetric effective theories. In addition, such type I orbifolds were related to type II *orientifold* compactifications [38, 39, 40, 41]. Roughly, an orientifold is a generalization of the orbifold, where an element Ω for changing string orientation is included. Such $D = 6$ constructions were rediscovered in the modern language of D-branes in [42, 43, 44]. Such class of compactifications was then generalized to orbifolds and orientifolds of type I and type II theories on six compact dimensions, yielding $\mathcal{N} = 1$ chiral theories in four dimensions [45, 46, 47, 48, 49, 50, 51, 52]. At the same time, such compactifications were related with their heterotic duals. Some semi-realistic models were achieved in this particular context [53, 54, 55]. A review of the phenomenology associated to these constructions can be found in [56].

D-brane constructions not only allowed to re-derive the previous achievements of heterotic compactification, but its properties as extended objects gave new possibilities into semi-realistic model-building, allowing to consider non-supersymmetric models. In heterotic models, both gauge and gravitational interactions have the same origin, as massless modes of the closed heterotic string. So they correspond to fields that propagate through the whole target space and they are unified at the string scale M_s. In order to reproduce two energy scales which differ by several orders of magnitude, such as the Planck and the Electroweak scale, one needs to introduce in general new parameters or a new scale and the predictive power is essentially lost. On the otherhand, in D-brane constructions the gauge and gravitational interactions have different origin. The latter are described by closed strings, while the former emerge as excitations of open strings with end points confined on Dp-branes with ($p < 9$). The gauge theory is confined to the $p + 1$ dimensions of the D-brane worldvolume, whereas gravitation, arising from the closed string sector, will propagate on the full ten-dimensional target space or *bulk* of the theory. As it was shown in [57, 58, 59, 60, 61], from this simple observation, we can obtain a difference of scales between gauge and gravitational interactions. In particular, we can obtain realistic compactifications where the string scale M_s should not necessarily be of the order of the Planck scale, but as low at the TeV region or at some intermediate scale [62, 63, 64]. In this way, we can consider non-supersymmetric models free from the scale hierarchy problem. Non-supersymmetric orientifold compactifications were first constructed in [65, 66], whereas the semi-realistic models and the phenomenology associated to them were provided in [67, 68].

The theoretical development in these new class of constructions, where D-branes played a central role, allowed to take one step further in semirealistic model building. So far, the quest for the SM had been based on considering a family of consistent compactifications in a certain superstring theory (as e.g., $\mathbf{CY_3}$ heterotic compactifications) and exploring the parameter or moduli space of such family (Euler characteristic, Wilson lines, etc.) looking for a low energy theory which resembled as much as possible to the SM. In [69], a new strategy for finding the SM in a string-based model was proposed. Since the gauge group and chiral matter content of the SM may arise as an effective theory from a set of Dp-branes, and the physics of this effective theory is not very sensitive to the rest of the details of the compactification, one may conceive the construction of a realistic model in two steps. First, we consider a consistent D-brane configuration with the low-energy spectrum of the SM. Second, we complete the construction by adding all the extra elements necessary to yield a fully-fledged compactification, including four-dimensional gravity. This, so-called *bottom-up* philosophy, enables us to find the simplest semi-realistic models. In such models, the SM was obtained from a bunch of D3-branes filling four-dimensional Minkowski space time and localized at an orbifold singularity in the compact space. Consistency conditions known as *tadpole* conditions imposed the presence of additional D-branes, namely D7-branes.

The bottom-up philosophy has indeed produced a whole set of D-brane models whose semi-realistic effective theories contain either the SM gauge group, or some extension of it. After it was realized that chiral fermions appear on the intersection of two D-branes [70], model building involving configurations of D-branes at angles or *intersecting D-branes* were intensively studied. Generically, these configurations yield a non-supersymmetric chiral low-energy spectrum. Each stack of N D-branes will be endowed with a $U(N)$ gauge theory, so that the construction of the SM gauge group or some extension of it basically reduces to consider the appropriate set of D-brane stacks. The chiral matter fields appear at their intersection, transforming in the bi fundamental representations. The number of zero modes i.e. the generation number is given by the intersection number in the compact six-dimensional space. These class of models, baptized as *Intersecting Brane Worlds* presents an interesting hierarchy on the different sectors of the effective theory. The gravity sector propagates on the whole target space i.e. on the four non-compact dimensions and on the six compact dimensions. The gauge sector, on the other hand, remains confined to the D-brane worldvolume, which fills the four non-compact dimensions and a submanifold of the compact space. Chiral matter is localized at D-branes intersections so, generically, they fill the non-compact dimensions and stuck at a point in the compact space. This natural hierarchy allows one to implement the low string scale scenario discussed above, as well as to consider non-supersymmetric models. Intersecting brane worlds provide a scenario to address some well known phenomenological problems and features of SM physics, by translating them to a more geometrical language. The specific examples for this type of models are discussed in the context of type IIA strings, see for example [71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83].

The intersecting D-brane models are related by T-duality to the magnetized D-brane models. In the language of T-duality, the intersecting angle of two D-branes in the type IIA side is interpreted as the magnetic flux inside two internal dimensions in the type IIB picture. Roughly, two T-dual theories yield the same physics: same interactions, same operators, same Hilbert space. It implies a one-to-one correspondence between two theories. On one side of the T-dual picture we have D-branes with magnetic fluxes, whereas on the other side we have D-branes at angles. The T-dual models i.e. the magnetized D-brane models also have been investigated. In particular, it was observed in [4] that turning on a non-vanishing magnetic field in a simple toroidal compactification of type I string theory implies both chiral spectra and supersymmetry breaking. Some theoretical aspects, as well as semi-realistic constructions have been analyzed in this framework in [6, 84, 85, 86, 87, 88, 89, 90, 92, 93, 94]. Non-supersymmetric toroidal compactifications of type I string theory with both constant background Neveu Schwarz - Neveu Schwarz (NSNS) two-form flux and non-trivial magnetic flux on the various D9-branes are discussed in [90, 94]. The solutions to the cancellation of the RR tadpoles display various phenomenologically attractive features: supersymmetry breaking, chiral fermions and the opportunity to reduce the rank of the gauge group. The non-vanishing B-flux admits four-dimensional models with three generations of chiral fermions in standard model like gauge groups. We refer to [95, 96] for more details on type I constructions.

In recent years a renewal of the local model building has been developing, for instance, F-theory model buildings [97, 98]. F-theory models naturally include exceptional gauge groups beyond the type IIB D-branes. Since the flavor structures are different from that of D-branes models, there are a lot of developments in phenomenological studies.

Although the general features get us quite close to obtain a realistic D-brane construction, in any string model one always find large number of unobserved light neutral scalar particles (moduli fields), extra chiral fermions and U(1) gauge groups in the low energy spectrum. Geometrically, the vacuum expectation values of the so called moduli fields parametrize the size and shape of the compactification manifold or positions of D-branes. These values are also related to the parameters like gauge coupling constants or masses of the effective four dimensional theory. Without uniquely determining these expectation values by means of minimizing an effective potential, which could then also induce mass terms for the moduli, string models are not predictive. This led to an intensive study on the problem of moduli stabilization to discover a controllable mechanism which generates a potential for the moduli fields. Such stabilizations employ various supergravity [99, 100], non-perturbative [101] and string theory [102, 103, 104] techniques to generate potentials for the moduli fields.

The superstring spectrum in ten dimensions contains various anti-symmetric tensor fields, the so called p-form fields C_p. It has been realized that by allowing the corresponding field strengths, schematically $F_{p+1} = dC_p$ to take non-trivial expectation values along the internal space, one can fix the vevs of the moduli fields and therefore provide the possibility for choosing a ground state as a local isolated minimum of the scalar potential of the theory. Moreover, when the fluxes are turned on along the compact directions, they have to satisfy Dirac quantization conditions and hence take discrete values. By a suitable choice of NS-NS and Ramond - Ramond (R-R) 3-form fluxes, one can find $\mathcal{N} = 1$ supersymmetric vacua where all complex structure moduli, as well as the dilaton, are fixed [100]. A disadvantage of this method is that there is no exact string description of such fluxes and thus the analysis is restricted to the lowest order in α' expansion, described by the effective field theory. Moreover, generalization of the stabilization mechanism to Kähler class moduli requires introduction of non-perturbative effects which are again treated in the low-energy supergravity approximation [101].

An alternative mechanism of moduli stabilization based on open string constant magnetic backgrounds that have an exact description in string theory [4, 105] is presented in [102, 103]. In fact, magnetic fluxes can be turned on in any 2-cycle of the internal compactification manifold. In the simplest case, magnetic backgrounds on (1,1)-cycles fix the Kähler class moduli [106, 107], while backgrounds on holomorphic (2,0)-cycles fix the complex structure moduli. In the generic Calabi-Yau case, this method can stabilize mainly the Kähler moduli [102, 106, 107] and is thus complementary to 3-form closed string fluxes that stabilize the complex structure and the dilaton [100]. On the other hand, it can also be used in simple toroidal compactifications, stabilizing all the geometric moduli in a supersymmetric vacuum using only magnetized $D9$-branes. This has an exact perturbative string description [4, 105]. RR tadpole cancellation requires some charged scalar fields from the branes to acquire non-vanishing vacuum expectation values, breaking partly the gauge symmetry in order to preserve supersymmetry [103]. Alternatively, one can break supersymmetry by D-terms and fix the dilaton at weak string coupling, by going “slightly" off-criticality and thus generating a tree-level bulk dilaton potential [108]. One of the main ingredients for this approach of moduli stabilization is the inclusion of ‘oblique’ fluxes given by mutually non-commuting matrices, in order to fix all off-diagonal components of the metric. This mechanism can be combined with the presence of closed type IIB string 3-form fluxes, allowing to fix the dilaton and the complex structure of more general compactification manifolds.

However, despite enormous efforts, very few examples are known so far of a complete stabilization of closed string moduli in any specific model, while the known ones are too constrained to accommodate interesting models from physical perspective. Hence, there have been very few attempts to construct a concrete model of particle physics even with partially stabilized moduli. Nevertheless, in view of the importance of the task at hand, we revisit the type I string constructions with moduli stabilizations [102, 103, 104], to explore the possibility of incorporating particle physics models, such as the SM or GUT models based on grand unified groups, in such a framework.

In the quest for obtaining a realistic string-based model, generic properties of the low-energy effective Lagrangian such as $D = 4$ chirality and unitary gauge groups are of fundamental importance. Once these have been found in a particular setup of string theory, there are still many other issues to face in order to reproduce some realistic physics at low energies. In particular, even if one manages to obtain a massless spectrum quite close to the SM (or some extension of it), one is eventually faced with the problem of computing some finer data defining a Quantum Field Theory. These data may tell us how close are we of reproducing the SM which, as we know, is not a group of chiral fermions with appropriate quantum numbers, but an intricate theory with lots of well-measured parameters. One should know the Yukawa couplings in any string model.

Close form expressions for Yukawa couplings have been written down for Type IIA models with intersecting branes [81, 109]. In this case, one has to perform a sum over string worldsheet instanton contributions to obtain the final expression of Yukawa couplings, a pure stringy (non-field theoretical) computation. These results have been further generalized to include Euclidean $D2$ brane instanton contributions to the Yukawa couplings [110, 111, 112, 113, 114, 115, 116, 117, 118], generating up quark and right handed neutrino masses through a Higgs mechanism, in a particular class of models. On the other hand, in the T-dual picture, the calculations of the Yukawa couplings are purely field theoretical. Yukawa interactions can be calculated by overlap integrals over internal spaces with three wavefunctions as the following forms

$$Y = \int dy^6 \psi_i(y)\psi_j(y)\phi(y) \tag{1.1}$$

where $\psi_{i,j}(y)$ correspond to the internal wavefunctions of chiral matter fields and $\phi(y)$ is the internal wavefunctions of Higgs scalar fields. The explicit calculations of the overlap integrals can tell us the form of the Yukawa couplings. It is found that two different approaches of stringy and field theory calculations lead to the consistent results of the Yukawa couplings after proper transformation of moduli parameters [6]. A limitation on the exercise performed in these papers comes from the factorized structure of the tori, which arises from the orientations of the brane wrappings that are classified by angles in three different T^2 planes or fluxes that are diagonal along three T^2's. These results require generalizations further to obtain the interactions involving branes with oblique fluxes, in view of the importance of such fluxes for obtaining phenomenologically viable models.

In this thesis, we discuss a simple framework of toroidal string models with magnetized branes, that offers an interesting self-consistent set up for string phenomenology. We will see, in the following chapters, how one can address the issues of moduli stabilization (fixing the geometric parameters of the compactification), building calculable particle physics models (gauge group, chiral fermions, family triplication, anomaly cancellation etc.) and computations of the Yukawa couplings in such a framework.

With this brief introduction, in the next section, we discuss the structure of the thesis.

1.3 Plan of the thesis

In *chapter 2*, we briefly review the string construction using magnetized branes. We discuss the compactification of type I strings on a torus with additional background gauge flux on the D9-branes and summarize the necessary constraints needed for constructing semi-realistic models in such a framework. We recall the main properties of the six-dimensional toroidal compactification and its moduli space. We consider the open string propagation in the presence of constant internal magnetic fields [4] and summarize the conditions for unbroken supersymmetry. We analyze the conditions for the unbroken supersymmetry in the presence of a stack of magnetized D9-branes and discuss the closed string moduli stabilization. We also study the tadpole cancellation conditions which are required for consistency of type I string vacua. Then we discuss the low-energy spectrum of the effective theory within this compactification scheme. Here we pay special attention to the massless open string of the theory, where unitary gauge groups and chiral fermions charged under them arise. Since a crucial step in a three generation model building is the introduction of a NS-NS B-field background (without which only even generation models can be built), the effects of non-zero B on the chirality and the tadpoles is summarized.

The next chapter is dedicated to obtain close form expressions for Yukawa couplings in such magnetized brane constructions. We first review the known results on the Jacobi theta identity given in [119] and present a proof of its validity. We also give an expression for the

Yukawa interaction for factorized tori and 'diagonal' fluxes using the theta identity [6]. We then generalize the results to writing down expressions for the Yukawa interactions when oblique fluxes are present. In order to perform this task, fermion (scalar) wavefunctions on toroidally compactified spaces are presented for general fluxes, parametrized by Hermitian matrices with eigenvalues of arbitrary signatures. We also give explicit mappings among fermion wavefunctions, of different internal chiralities on the tori, which interchange the role of the flux components with the complex structure of the torus. By evaluating the overlap integral of the wavefunctions, we give the expressions for Yukawa couplings among chiral multiplets arising from an arbitrary set of branes (or their orientifold images). The method is based on constructing certain mathematical identities for general Riemann theta functions with matrix valued modular parameter.

After developing this theoretical framework, we present a specific model in the *chapter 4*. We construct a minimal example of a supersymmetric grand unified model in a toroidal compactification of type I string theory with magnetized $D9$-branes. We obtain general solutions for fluxes along magnetized $D9$-branes yielding the chiral spectrum and gauge group of a three generation SU(5) GUT model, with no extra chiral matter nor U(1) factors. The gauge symmetry is just $SU(5)$ and the gauge non-singlet chiral spectrum contains only three families of quarks and leptons transforming in the $\mathbf{10}+\bar{\mathbf{5}}$ representations. Moreover, all geometric moduli are stabilized in terms of the background internal magnetic fluxes which are of "oblique" type (mutually non-commuting). The flux solutions also satisfy the RR tadpole cancellation conditions yielding a consistent model. Finally, we present a brief analysis of the superpotential and D-terms for the model in order to show the mass generation for several non-chiral fermion multiplets in a supersymmetric ground state.

We end this thesis with a conclusion. In the appendix, we collect all the technical details required for the main text.

2

Magnetic Flux in Toroidal Type I Compactification

2.1 Introduction

In this chapter we introduce the basic class of objects upon the whole thesis is based: D-branes with magnetic fluxes or magnetized branes. We study some of their salient features, which motivate their role as building blocks of semirealistic string-based constructions.

As it is discussed in the previous chapter, string theory is known to possess a large number of vacua which contain the basic structure of grand unified theories and in particular of the Standard Model. However, the presence of moduli fields with flat directions has remained one of the major stumbling blocks in making further progress. Consequently, closed string moduli stabilization has been intensively studied during the last years for its implication towards a comprehensive understanding of the superstring vacua[99, 101], as well as due to its significance in deriving definite low energy predictions for particle models from string theory. Such stabilizations employ various supergravity[99, 100], non-perturbative[101] and string theory[102, 103, 104] techniques to generate potentials for the moduli fields. However, very few examples are known so far of a complete stabilization of all closed string moduli in any specific model. The known models with stabilized moduli are too constrained to accommodate interesting models from physical point of view. Hence, there have been very few attempts to construct a concrete model of particle physics even with partially stabilized moduli. With the above motivation, we revisit the type I string constructions[95, 96] and moduli stabilizations[102, 103, 104], to explore the possibility of incorporating particle physics models, such as the Standard Model or GUT models based on grand unified groups.

A new calculable method of moduli stabilization was recently proposed, using constant internal magnetic fields in four-dimensional (4d) type I string compactifications[102, 103]. In the generic Calabi-Yau case, this method can stabilize mainly the Kähler moduli [102, 106] and is thus complementary to 3-form closed string fluxes that stabilize the complex structure and the dilaton [100]. On the other hand, it can also be used in simple toroidal compactifications, stabilizing all geometric moduli in a supersymmetric vacuum using only magnetized $D9$-branes that have an exact perturbative string description [4, 105]. RR tadpole cancellation requires then some charged scalar fields from the branes to acquire non-vanishing vacuum expectation values (VEVs), breaking partly the gauge symmetry in order to preserve supersymmetry [103]. Alternatively, one can break supersymmetry by D-terms and fix the dilaton at weak string coupling, by going "slightly" off-criticality and thus generating a tree-level bulk dilaton potential [108].

There are two main ingredients for this approach of moduli stabilization [102, 103]: (1) A set of nine magnetized $D9$-branes is needed to stabilize all 36 moduli of the torus T^6 by the supersymmetry conditions [89, 120]. Moreover, at least six of them must have oblique fluxes given by mutually non-commuting matrices, in order to fix all off-diagonal components of the metric. On the other hand, all nine $U(1)$ brane factors become massive by absorbing the RR partners of the Kähler class moduli [89]. (2) Some extra branes are needed to satisfy the RR tadpole cancellation conditions, with non-trivial charged scalar VEVs turned on, in order to maintain supersymmetry.

However, as already pointed out in [103], our moduli stabilization scheme is restricted to closed string moduli space that may be enlarged if one takes into account open string fields[1]. Unfortunately, their effects cannot be taken into account exactly at the string level, as the geometric toroidal closed string moduli. Moreover, they have $N = 1$ superpotential leading to non-trivial F-flatness conditions, besides the D-terms arising from the magnetic fields. A recent analysis shows that a generalization of the stabilization mechanism may be possible in the quadratic approximation and, for reasonable conditions on the spectrum, open string 'recombination' fields can also be fixed [121]. In the present work, we apply the following algorithm for moduli stabilization in toroidal type I compactifications: (1) All geometric moduli are first fixed using a minimal set of (nine in the present case) magnetized branes, in the absence of charged scalar VEVs. This has the advantage of being exact in α' (world-sheet) perturbation theory, but does not satisfy tadpole cancellation. (2) The latter is achieved by adding extra magnetized branes on which some charged scalars are forced to acquire non-vanishing VEVs in order to cancel the induced Fayet-Iliopoulos

[1]Many open string moduli are charged and their VEVs break local and global symmetries. For instance they play the role of ordinary higgses either for GUT or Standard Model breaking. These VEVs could be driven from soft supersymmetry breaking terms. The issue is related to supersymmetry breaking, however in the present thesis we are interested in $\mathcal{N} = 1$ supersymmetric vacuum.

terms. Since the inclusion of charged fields in the D-terms is not known exactly, their VEVs can be determined only perturbatively in α', when their values are small compared to the string scale. As a result, any 'back-reaction' of the charged scalar VEVs, coming from this perturbative brane action, is expected to be small on the closed string moduli, and therefore not of any significant phenomenological consequence.

We apply the above method to construct phenomenologically interesting models. In this chapter, we briefly describe the construction based on D-branes with magnetic fluxes in type I string theory, or equivalently type IIB with orientifold $O9$-planes and magnetized $D9$-branes, in a T^6 compactification. The rest of the chapter is structured as follows. We start with summarizing the main properties of the six dimensional toroidal compactification and its moduli space in Section 2.2. In Section 2.3, we consider open string propagation in the presence of constant internal magnetic fields. Further, we discuss the general setup with the magnetized branes, including the gauge fluxes that can be turned on, in a consistent manner. In Section 2.4, we write down the conditions that guarantee the existence of one unbroken supersymmetry preserved by stacks magnetized D9-branes. We then discuss the stabilization of complex structure and kähler class moduli using such conditions. We study the tadpole cancellation conditions which are required for consistency of type I string vacua in the presence of internal magnetic fields in Section 2.5. Further, in Section 2.6, we discuss the low energy spectrum, in particular fermion degeneracies, of the effective theory in this compactification. Since a crucial step in a three generation model building is the introduction of a NS-NS B-field background without which only even generation models can be built, the effect of non-zero B on the chirality and tadpoles is summarized in Section 2.7.

2.2 Torus compactification : Parametrization of T^6 and Moduli space

Consider a six-dimensional torus T^6 having six coordinates x^i, y_i with $i = 1, 2, 3$ and periodicity normalized to unity $x^i = x^i + 1$, $y_i = y_i + 1$ [102]. We choose the orientation

$$\int_{T^6} dx^1 \wedge dy_1 \wedge dx^2 \wedge dy_2 \wedge dx^3 \wedge dy_3 = 1 \tag{2.1}$$

and define the basis of the cohomology $H^3(T^6, \mathbb{Z})$

$$
\begin{aligned}
\alpha_0 &= dx^1 \wedge dx^2 \wedge dx^3 \\
\alpha_{ij} &= \frac{1}{2}\epsilon_{ilm} dx^l \wedge dx^m \wedge dy_j \\
\beta^{ij} &= -\frac{1}{2}\epsilon^{ilm} dy_l \wedge dy_m \wedge dx^j \\
\beta^0 &= dy_1 \wedge dy_2 \wedge dy_3,
\end{aligned}
\tag{2.2}
$$

forming a symplectic structure on T^6:

$$
\int_{T^6} \alpha_a \wedge \beta^b = -\delta_a^b \ , \quad \text{for } a, b = 1, \cdots, h_3/2 \,, \tag{2.3}
$$

with $h_3 = 20$, the dimension of the cohomology $H^3(T^6, \mathbb{Z})$.

The 36 moduli of T^6 correspond to 21 independent deformations of the internal metric and 15 deformations of the two-index antisymmetric tensor C_2 from the RR closed string sector. They form nine complex parameters of Kähler class and nine of complex structure. Indeed, the geometric moduli of T^6 decompose in a complex structure variation which is parametrized by the matrix Ω^{ij} entering in the definition of the complex coordinates

$$
z^i = x^i + \Omega^{ij} y_j \,, \tag{2.4}
$$

and in the Kähler variation of the mixed part of the metric described by the real $(1,1)$-form

$$
J = i\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^j . \tag{2.5}
$$

Choosing the basis $e^{i\bar{j}}$ of the cohomology $H^{1,1}$ to be of the form

$$
e^{i\bar{j}} = i dz^i \wedge d\bar{z}^j, \tag{2.6}
$$

the Kähler form can be parametrized as

$$
J = J_{i\bar{j}} e^{i\bar{j}} \,. \tag{2.7}
$$

The dimension of the space of complex structure moduli is given by the dimension of the cohomology $H^{2,1}$ on the torus T^6, $h_{2,1} = 9$. The elements $J_{i\bar{j}}$ satisfy the reality condition $J^\dagger_{i\bar{j}} = J_{j\bar{i}}$, implying that J depends on nine real parameters. They can be used to parametrize the space of Kähler deformations whose dimension is given by the dimension of the cohomology $H^{1,1}$ on the torus T^6, $h_{1,1} = 9$. The Kähler form is complexified with the corresponding RR two-form deformation.

2.3 Magnetized $D9$-branes: Fluxes and Windings

Let's consider a stack of N coincident $D9$-branes, giving rise to a $U(N)$ $\mathcal{N}=4$ supersymmetric gauge theory. We pick up a $U(1)$ subgroup in the Cartan subalgebra of $U(N)$ with gauge potential A, and turn on a constant magnetic field. Thus, the corresponding field strength $F_{\alpha\beta}$ is constant and $A_\alpha = \frac{1}{2}F_{\alpha\beta}u^\beta$, where u^β stands for all six coordinates of T^6, x^i and y^i. This constant magnetic background couples to the boundary of the open string on the brane by quadratic terms in the world-sheet action S_{ws} [105]. The corresponding conformal field theory can therefore be solved exactly:

$$\begin{aligned} S_{ws} \;&=\; -\frac{1}{4\pi\alpha'}\int_\Sigma dtd\sigma\left(\partial_\lambda X^\mu\partial^\lambda X_\mu - i\bar{\psi}^\mu\rho^\lambda\partial_\lambda\psi_\mu\right) \\ &\quad - \int dtq_L F_{\alpha\beta}\left(X^\alpha\partial_t X^\beta - \frac{i}{2}\bar{\psi}^\alpha\rho^0\psi^\beta\right)_{\sigma=0} \\ &\quad - \int dtq_R F_{\alpha\beta}\left(X^\alpha\partial_t X^\beta - \frac{i}{2}\bar{\psi}^\alpha\rho^0\psi^\beta\right)_{\sigma=\pi} \end{aligned} \tag{2.8}$$

where α' is the Regge slope, ψ^μ are the real Majorana fermionic superpartners of the coordinates X^μ and ρ^λ with $\lambda = 0,1$ are the two-dimensional gamma-matrices. The indices α,β run over the magnetized dimensions $\alpha,\beta = 4,\cdots,9$, whereas the indices μ,ν run over all ten-dimensional spacetime coordinates $\mu,\nu = 0,\cdots,9$. The couplings of the left and right endpoints of the open string to the background are given by the corresponding charges q_L and q_R.

The field $F_{\alpha\beta}$ corresponds to a non trivial $U(1)$ gauge bundle over the torus T^6 with transition function around the cycles u_α:

$$A_\alpha\big|_{u^\beta+1} = \left(A_\alpha - ie^{-iq\theta}\partial_\alpha e^{iq\theta}\right)\Big|_{u^\beta}, \quad \theta = F_{\alpha\beta}u^\beta \tag{2.9}$$

with $q = q_L + q_R$. Imposing the phase over each cycle u^α to be single-valued leads to the usual Dirac quantization condition

$$q\cdot F_{\alpha\beta} = 2\pi m_{\alpha\beta}, \quad \forall\,\alpha,\beta = 4,\ldots,9\,, \tag{2.10}$$

where $m_{\alpha\beta}$ are integers corresponding to the first Chern class of the $U(1)$ gauge bundle.

Let us now be more specific and assume the presence of K stacks of N_a magnetized $D9$-branes, $a = 1,\ldots,K$. Each stack is associated with a corresponding $U(N_a)$ gauge symmetry. We choose K linear combinations of the generators of $U(N_a)$ which lie in the Cartan subalgebra and denote their abelian gauge potentials by A^a; for simplicity, we identify them with $U(1)_a$. Their field strengths are assumed to take constant values on the torus T^6. Thus there is a set of K $U(1)$ gauge potentials A^a with constant background field strengths

$$A^a_\alpha = \frac{1}{2} F^a_{\alpha\beta} X^\beta \quad \text{where} \quad a = 1, \ldots, K\,. \tag{2.11}$$

The stacks of $D9$-branes are characterized by three independent sets of data: (a) their multiplicities N_a, (b) winding matrices $W_I^{\hat{I},a}$ and (c) 1st Chern numbers $m^a_{\hat{I}\hat{J}}$ of the $U(1)$ background on their world-volume Σ^a, $a = 1, \ldots, K$. And $I, \hat{I}$ run over the target space and world-volume indices, respectively. These parameters are described below:

(a) Multiplicities: The first quantity N_a describes the rank of the the unitary gauge group $U(N_a)$ on each $D9$ stack.

(b) Winding Matrices: The second set of parameters $W_I^{\hat{I},a}$ is the covering of the world-volume of each stack of $D9$-branes on the ambient space. In other words, they give the winding of the branes around the different cycles of the internal space. They are characterized by the wrapping numbers of the branes around the different 1-cycles of the torus[2], which are encoded in the covering matrices $W_I^{\hat{I},a}$ defined as

$$W_J^{\hat{I}} = \frac{\partial \xi^{\hat{I}}}{\partial X^J} \qquad \text{for} \;\; \hat{I}, J = 0, \ldots, 9\,, \tag{2.12}$$

where the coordinates on the world-volume are denoted by $\xi^{\hat{I}}$, while the coordinates of the space-time $\mathcal{M}_{10}$ are X^I. Since space-time is assumed to be a direct product of a four-dimensional Minkowski manifold with a six-dimensional torus, the covering matrix is of the form:

$$W_J^{\hat{I},a} = \begin{pmatrix} \delta_\mu^{\hat{\mu}} & 0 \\ 0 & W_\alpha^{\hat{\alpha},a} \end{pmatrix} \qquad \text{for} \;\; \mu, \hat{\mu} = 0, \ldots, 3 \;\; \text{and} \;\; \alpha, \hat{\alpha} = 1, \ldots, 6\,, \tag{2.13}$$

with the upper block corresponding to the covering of Σ_4^a on the four-dimensional space-time $\mathcal{M}_4$. Since these are assumed to be identical, the associated covering map $W_\mu^{\hat{\mu}}$ is the identity, $W_\mu^{\hat{\mu}} = \delta_\mu^{\hat{\mu}}$. The entries of the lower block, on the other hand, describe the wrapping numbers of the $D9$-branes around the different 1-cycles of the torus T^6 which are therefore restricted to be integers $W_\alpha^{\hat{\alpha}} \in \mathbb{Z}$, $\forall\, \alpha, \hat{\alpha} = 1, \ldots, 6$ [104]. The K $D9$ stacks are then ten-dimensional objects which fill the four-dimensional space-time and cover the

[2]There is no wrap factor here because the fluxes are magnetic (at the disk level) with an exact CFT description, in contrast to the closed string fluxes.

internal torus T^6. Thus there are K different coverings $\mathcal{T}_6^a$ of the torus T^6 described by the K covering maps $W_\alpha^{\hat{\alpha},\,a}$, for $a = 1,\ldots,K$.

For simplicity, in the examples we consider in this thesis, the winding matrix $W_\alpha^{\hat{\alpha}}$ in the internal directions is also chosen to be a six-dimensional diagonal matrix, implying an embedding such that the six compact $D9$ world-volume coordinates are identified with those of the internal target space T^6, up to a winding multiplicity factor n_α^a, for a brane stack-a:

$$n_\alpha^a \equiv W_\alpha^{\hat{\alpha},a}. \tag{2.14}$$

We will also use the notation

$$\hat{n}_1^a \equiv n_1^a n_2^a, \quad \hat{n}_2^a \equiv n_3^a n_4^a, \quad \hat{n}_3^a \equiv n_5^a n_6^a, \quad \text{(no sum on a)} \tag{2.15}$$

to define the diagonal wrapping of the $D9$'s on the three orthogonal T^2's inside T^6, given by:

$$x^i \equiv X^\alpha, \;\; \alpha = 1,3,5\,; \quad y^i \equiv X^\alpha, \;\; \alpha = 2,4,6\,, \tag{2.16}$$

with periodicities: $x^i = x^i + 1$, $y^i \equiv y^i + 1$:

$$\mathbb{T}^6 = \bigotimes\nolimits_{i=1}^{3} \mathbb{T}_i^2\,, \tag{2.17}$$

and coordinates of the orthogonal 2-tori (T_i^2) being (x^i, y^i) for $i = 1,2,3$.

For further simplification, in our example, we will choose for all stacks trivial windings:

$$n_\alpha^a \equiv W_\alpha^{\hat{\alpha},a} = 1, \;\text{ for } \alpha = 1,..,6. \tag{2.18}$$

However in this section, in order to describe the formalism, we keep still general winding matrices $W_\alpha^{\hat{\alpha},a}$.

(c) First Chern numbers: The parameters $m_{\hat{I}\hat{J}}^a$ of the brane data given above are the 1st Chern numbers of the $U(1) \subset U(N_a)$ background on the world-volume of the $D9$-branes. For each stack $U(N_a) = U(1)_a \times SU(N_a)$, the $U(1)_a$ has a constant field strength on the covering of the internal space. These are subject to the Dirac quantization condition which implies that all internal magnetic fluxes $F_{\hat{\alpha}\hat{\beta}}^a$, on the world-volume of each stack of $D9$-branes, are integrally quantized. The Dirac quantization condition applies independently to the K fluxes $F_{\hat{\alpha}\hat{\beta}}^a$.

Explicitly, the world-volume fluxes $F_{\hat{\alpha}\hat{\beta}}^a$ and the corresponding target space induced fluxes $p^a_{\alpha\beta}$ are quantized as

$$\left\{\begin{array}{ll} F^a_{\hat\alpha\hat\beta} = m^a_{\hat\alpha\hat\beta} \in \mathbb{Z} & \forall\,\hat\alpha,\hat\beta = 1,\ldots,6 \\ p^a_{\alpha\beta} = (W^{-1})^{\hat\alpha,\,a}_{\alpha}(W^{-1})^{\hat\beta,\,a}_{\beta}\, m^a_{\hat\alpha\hat\beta} \in \mathbb{Q}, & \forall\,\alpha,\beta = 1,\ldots,6 \end{array}\right. \qquad \forall a = 1,\ldots,K\,. \tag{2.19}$$

When fluxes are turned on only along three factorized T^2's of eq. (2.17), as will be the case for some of our brane stacks, we make use of the following convenient notation:

$$\hat m^a_1 \equiv m^a_{12} \equiv m^a_{x^1y^1},\quad \hat m^a_2 \equiv m^a_{34} \equiv m^a_{x^2y^2},\quad \hat m^a_3 \equiv m^a_{56} \equiv m^a_{x^3y^3}. \tag{2.20}$$

The magnetized $D9$-branes couple only to the $U(1)$ flux associated with the gauge fields located on their own world-volume. In other words, the charges of the endpoints q_R and q_L of the open strings stretched between the i-th and the j-th $D9$-brane can be written as $q_L \equiv q_i$ and $q_R \equiv -q_j$, while the Cartan generator h is given by $h = \mathrm{diag}(h_1\,\mathbb{1}_{N_1},\ldots,h_N\,\mathbb{1}_{N_K})$, with $\mathbb{1}_{N_a}$ being the $N_a \times N_a$ identity matrix. In addition, in type I string theory, the number of magnetized $D9$-branes must be doubled. Since the orientifold projection $\mathcal{O}$ is defined by the world-sheet parity, it maps the field strength $F_a = dA_a$ of the $U(1)_a$ gauge potential A_a to its opposite, $\mathcal{O}: F_a \to -F_a$. Therefore, the magnetized $D9$-branes are not an invariant configuration and for each stack a mirror stack must be added with opposite flux on its world-volume [3].

A general gauge flux, on T^6 with coordinates $X^I \equiv (x^i, y^i)$, $i = 1,2,3$, has the form:

$$\begin{aligned} \mathcal{F} &\equiv p_{IJ} dX^I \wedge dX^J \\ &= p_{x^ix^j} dx^i \wedge dx^j + p_{y^iy^j} dy^i \wedge dy^j + p_{x^iy^j} dx^i \wedge dy^j + p_{y^ix^j} dy^i \wedge dx^j\ . \end{aligned} \tag{2.21}$$

Then using the definition of a general complex structure matrix Ω as defined in eq.(2.4) :

$$dz^i = dx^i + \Omega^i_j dy^j,\ \ d\bar z^i = dx^i + \bar\Omega^i_j dy^j,$$

we obtain:

$$\mathcal{F} = F_{z^iz^j} dz^i \wedge dz^j + F_{z^i\bar z^j}(i dz^i \wedge d\bar z^j) + F_{\bar z^i\bar z^j} d\bar z^i \wedge d\bar z^j. \tag{2.22}$$

Choosing the basis $e^{i\bar j}$ of the cohomology $H^{1,1}$ to be of the form $e^{i\bar j} = i dz^i \wedge d\bar z^j$, we get:

$$F_{(2,0)} = F_{z^iz^j} = (\bar\Omega - \Omega)^{-1^T}\left(\bar\Omega^T p_{xx}\bar\Omega - \bar\Omega^T p_{xy} + p^T_{xy}\bar\Omega + p_{yy}\right)(\bar\Omega - \Omega)^{-1} \tag{2.23}$$

[3] There are no O_5 planes in our model. However every magnetic flux creates also 5-brane charges that are cancelled among various stacks of magnetized $D9$-branes.

and

$$F_{(1,1)} = F_{z^i\bar{z}^j} = (-i)(\bar{\Omega} - \Omega)^{-1^T} \left(\bar{\Omega}^T p_{xx}\Omega - \bar{\Omega}^T p_{xy} + p_{xy}^T \Omega + p_{yy}\right)(\bar{\Omega} - \Omega)^{-1}. \tag{2.24}$$

where the matrices $(p^a_{x^ix^j})$, $(p^a_{x^iy^j})$ and $(p^a_{y^iy^j})$ are the quantized field strengths in target space, given in eq. (2.19). For our choice (2.18), they coincide with the Chern numbers m^a along the corresponding cycles. The field strengths $F^a_{(2,0)}$ and $F^a_{(1,1)}$ are 3×3 matrices that correspond to the upper half of the matrix $\mathcal{F}^a$:

$$\mathcal{F}^a \equiv -(2\pi)^2 i\alpha' \begin{pmatrix} F^a_{(2,0)} & F^a_{(1,1)} \\ -F^{a\dagger}_{(1,1)} & F^{a*}_{(2,0)} \end{pmatrix}, \tag{2.25}$$

which is the total field strength in the cohomology basis $e_{i\bar{j}} = i dz^i \wedge d\bar{z}^j$. In addition, $F_{\bar{z}^i\bar{z}^j}$ is complex conjugate to $F_{z^iz^j}$ and $F_{\bar{z}^iz^j} = -F_{z^j\bar{z}^i}$.

In this thesis, we consider the fluxes for which a four dimensional supersymmetric theory can be recovered. As it will be discussed in the following sections, supersymmetry demands all fluxes to be of $(1,1)$ form which gives us the condition:

$$\left(\bar{\Omega}^T p_{xx}\bar{\Omega} - \bar{\Omega}^T p_{xy} + p_{xy}^T \bar{\Omega} + p_{yy}\right) = 0, \tag{2.26}$$

or equivalently:

$$\left(\Omega^T p_{xx}\Omega - \Omega^T p_{xy} + p_{xy}^T \Omega + p_{yy}\right) = 0. \tag{2.27}$$

Eqs. (2.26) and (2.27) together give two real matrix equations. These equations can then be used to eliminate some of the variables and write the final $(1,1)$ form in terms of certain independent variables only.

Using eq. (2.27), eq. (2.24) reduces to the following form,

$$F_{z^i\bar{z}^j} = -i\left(p_{xx}\Omega - p_{xy}\right)(\bar{\Omega} - \Omega)^{-1}. \tag{2.28}$$

On the other hand, use of eq. (2.26) in eq. (2.24) gives,

$$F_{z^i\bar{z}^j} = -i(\bar{\Omega} - \Omega)^{-1^T}\left(-\bar{\Omega}^T p_{xx} - p_{xy}^T\right). \tag{2.29}$$

We also notice that the $(1,1)$ form $F_{z^i\bar{z}^j}$ given in eq. (2.24) satisfies the hermiticity property: $F_{z^i\bar{z}^j} = F^\dagger_{z^i\bar{z}^j}$. To explicitly see that, we use eqs. (2.28), (2.29).

$$\begin{aligned} F^\dagger_{z^i\bar{z}^j} &= \left[\left(-i\left(p_{xx}\Omega - p_{xy}\right)(\bar{\Omega} - \Omega)^{-1}\right)^*\right]^T \\ &= -i(\bar{\Omega} - \Omega)^{-1^T}\left(-\bar{\Omega}^T p_{xx} - p_{xy}^T\right) = F_{z^i\bar{z}^j} \end{aligned} \tag{2.30}$$

There are some special cases, however, in which eqs. (2.26) and (2.27) simplify further and the resulting $F_{z^i\bar{z}^j}$ can be written more compactly. One such case arises when p_{xx} and p_{yy} components are turned off. In such a situation $F_{(2,0)} = 0$ condition (2.27), reduces to:

$$\Omega^T p_{xy} = p_{xy}^T \Omega. \tag{2.31}$$

Thus far, we have concentrated on the spatial components of the gauge fluxes, but ignored the gauge indices. In the magnetized D-brane construction, gauge quantum numbers arise from the Chan-Paton factors associated with the end points of the open strings for a given stack of branes. The simplest possibility is to consider fluxes with gauge indices given by an $n \times n$ identity matrix for a stack of D-branes:

$$F = mI_n, \tag{2.32}$$

with m an arbitrary integer giving the 1st Chern number. All spatial indices of the gauge flux above have been suppressed, which are given as in eq. (2.21) by the components : $p_{x^iy^j}$, $p_{x^ix^j}$, $p_{y^iy^j}$. Actually, eq. (2.32) corresponds to the situation when all the wrapping numbers are trivial: $n^{x^i} = n^{y^i} = 1$ as discussed in eq.(2.18). F, then represents a stack of n magnetized D-branes with a $U(1)^n$ gauge flux. The first Chern number for each of the $U(1)$ fluxes is equal to m. Moreover, D-brane wrapping numbers on the internal directions, are all unity, given by a diagonal embedding of the brane in target space and winding around each 1-cycle once. In most of the thesis, we will consider fluxes of the above type.

For multiple stacks of n_i branes with respective 1st Chern numbers m_i, the flux matrix is of block diagonal form:

$$F = \begin{pmatrix} m_1 I_{n_1} & & & & \\ & m_2 I_{n_2} & & & \\ & & . & & \\ & & & . & \\ & & & & m_{n_p} I_{n_p} \end{pmatrix} \tag{2.33}$$

and corresponds to gauge fluxes in the diagonal $U(1)$'s of $U(n_1) \times U(n_2) \times \cdots$ gauge group.

Gauge fluxes on branes with higher wrapping numbers can also be given a gauge theoretic interpretation. The method is based on a representation of the magnetized brane constructions [6] in terms of fluxes along internal directions in a compactified gauge theory. In this picture, the effect of windings of branes around T^6 is simulated by the rank of the gauge group. In particular, due to the Dirac quantization condition on fluxes, a $U(n)$ flux on, say T^2:

$$F = \frac{m}{n} I_n, \tag{2.34}$$

with I_n being the n-dimensional identity matrix, and (n, m) relatively prime, represents a single brane wound n times around T^2 with flux quantum m and resulting gauge symmetry being only $U(1)$. On the other hand, if m is an integer multiple of n such that $m = pn$, then each of the entries in the identity matrix represents a well defined $U(1)$ flux of quantum p and the gauge symmetry is $U(n)$, given by a stack of n such magnetized branes, as described in the last paragraph. It turns out that explicit realization of fluxes with (n, m) relatively prime, needs gauge configurations with non-abelian Wilson lines.

In the next section, we write down the supersymmetry conditions for magnetized $D9$-branes in the context of type I toroidal compactifications and discuss the stabilization of complex structure and Kähler class moduli using such conditions.

2.4 Supersymmetry Conditions and Moduli Stabilization

The presence of constant internal magnetic fields breaks supersymmetry by shifting the masses of the four dimensional scalars and fermions [4]. A single magnetized $D9$-brane in type I string theory is not generically supersymmetric. Indeed, the orientifold projection implies the presence of mirror branes. Twisted scalars from the Neveu-Schwarz sector of open string stretched between a brane and its image are generically massive, while some chiral spinors from the Ramond sector remain massless. In other words, the $D9$-brane does not preserve the same supersymmetry as the orientifold projection. However, for suitable choice of the fluxes and moduli, a four-dimensional supersymmetric theory can be recovered [89]. In this section, we summarize the conditions under which a supersymmetric vacuum can exist.

Written in the complex basis (eq. (2.4)) where the field strength $\mathcal{F}$ splits in purely (anti-) holomorphic ($F_{(0,2)}$), $F_{(2,0)}$ and mixed $F_{(1,1)}$ parts, the condition for $\mathcal{N} = 1$ supersymmetry in four dimensions can be written as [102, 103]:

$$(iJ + \mathcal{F})^3 = e^{i\theta} \sqrt{|g_6 + \mathcal{F}|} \frac{V_6}{\sqrt{|g_6|}} \tag{2.35}$$

$$F_{(2,0)} = 0\,, \tag{2.36}$$

where V_6 is the volume form of T^6 and g_6 is its metric. Eq. (2.35) can be rewritten in the form:

$$\tan\theta\,(J\wedge J\wedge\mathcal{F}-\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F})=J\wedge J\wedge J-J\wedge\mathcal{F}\wedge\mathcal{F}\,, \tag{2.37}$$

where the wedge product A^N is defined with an implicit normalization factor $1/N!$. Note that only the $(1,1)$-part of $\mathcal{F}$ contributes in this formula. Formally, (2.37) can be also written as

$$\mathrm{Im}\left(\mathrm{e}^{-\mathrm{i}\theta}\Phi\right)=0\,, \tag{2.38}$$

with

$$\Phi=(iJ+\mathcal{F})\wedge(iJ+\mathcal{F})\wedge(iJ+\mathcal{F})\,. \tag{2.39}$$

The constant phase θ selects which supersymmetry the magnetized D9-brane preserves. In the case of type I string theory, the supercharges preserved by the magnetic background field is consistent with the presence of the orientifold plane O9 for the choice of $\theta=-\frac{\pi}{2}$.

Similarly, for a given configuration of K stacks of magnetized branes, one may ask whether the different stacks forming the brane configuration preserve some common supersymmetries. All θ_a's, for $a=1,\ldots,K$, have to be the same in order to preserve the same supersymmetry. We then have $\theta_a=-\frac{\pi}{2}\ \forall\, a$. The supersymmetry conditions for each stack then read[102, 103]:

$$\begin{aligned}
&F^a_{(2,0)}=0;\\
&\mathcal{F}_a\wedge\mathcal{F}_a\wedge\mathcal{F}_a=\mathcal{F}_a\wedge J\wedge J;\\
&\det W_a\,(J\wedge J\wedge J-\mathcal{F}_a\wedge\mathcal{F}_a\wedge J)>0\,,
\end{aligned} \tag{2.40}$$

for each $a=1,\ldots,K$.

The first set of conditions of eq. (2.40) states that the purely holomorphic flux vanishes. For given flux quanta and winding numbers, this matrix equation restricts the complex structure Ω. Using eq. (2.23), the supersymmetry conditions for each stack can first be seen as a restriction on the parameters of the complex structure matrix elements Ω:

$$F^a_{(2,0)}=0 \qquad \rightarrow \qquad \left(\Omega^T p_{xx}\Omega-\Omega^T p_{xy}+p_{xy}^T\Omega+p_{yy}\right)=0\,, \tag{2.41}$$

giving rise to at most six complex equations for each brane stack a.

The second set of conditions of eq. (2.40) gives rise to a real equation and restricts the Kähler moduli. This can be understood as a D-flatness condition. In the four-dimensional effective action, the magnetic fluxes give rise to topological couplings for the different axions of the compactified field theory. These arise from the dimensional reduction of the Wess Zumino action. In addition to the topological coupling, the $\mathcal{N}=1$ supersymmetric action yields a Fayet-Iliopoulos (FI) term of the form:

$$\frac{\xi_a}{g_a^2} = \frac{1}{(4\pi^2\alpha')^3}\int_{T^6}\big(\mathcal{F}_a\wedge\mathcal{F}_a\wedge\mathcal{F}_a - \mathcal{F}_a\wedge J\wedge J\big)\,. \tag{2.42}$$

The D-flatness condition in the absence of charged scalars requires then that $<\mathrm{D}_a>=\xi_a=0$, which is equivalent to the second equation of eq. (2.40). Finally, the last inequality in eq. (2.40) may also be understood from a four-dimensional viewpoint as the positivity of the $U(1)_a$ gauge coupling g_a^2, since its expression in terms of the fluxes and moduli reads

$$\frac{1}{g_a^2} = \frac{1}{(4\pi^2\alpha')^3}\int_{T^6}\big(J\wedge J\wedge J - \mathcal{F}_a\wedge\mathcal{F}_a\wedge J\big)\,. \tag{2.43}$$

The above supersymmetry conditions, get modified in the presence of VEVs for scalars charged under the $U(1)$ gauge groups of the branes. The D-flatness condition, in the low energy field theory approximation, then reads:

$$\mathrm{D}_a = -\left(\sum_\phi q_a^\phi|\phi|^2 G_\phi + M_s^2\xi_a\right) = 0\,, \tag{2.44}$$

where $M_s=\alpha'^{-1/2}$ is the string scale[4], and the sum is extended over all scalars ϕ charged under the a-th $U(1)_a$ with charge q_a^ϕ and metric G_ϕ. Such scalars arise in the compactification of magnetized $D9$-branes in type I string theory, for instance from the NS sector of open strings stretched between the a-th brane and its image $a^\star$, or between the stack-a and another stack-b or its image b^*. When one of these scalars acquire a non-vanishing VEV $\langle|\phi|\rangle^2 = v_\phi^2$, the calibration condition of eq. (2.40) is modified to:

$$q_a v_a^2\int_{T^6}\big(J\wedge J\wedge J-\mathcal{F}_a\wedge\mathcal{F}_a\wedge J\big) \;=\; -\frac{M_s^2}{G_\phi}\int_{T^6}\big(\mathcal{F}_a\wedge\mathcal{F}_a\wedge\mathcal{F}_a-\mathcal{F}_a\wedge J\wedge J\big) \tag{2.45}$$

$$\det W_a\,(J\wedge J\wedge J-\mathcal{F}_a\wedge\mathcal{F}_a\wedge J)\;>\;0\quad,\qquad\forall a=1,\ldots,K\,. \tag{2.46}$$

Note that our computation is valid for small values of v_a (in string units), since the inclusion of the charged scalars in the D-term is in principle valid perturbatively.

Actually, the fields appearing in (2.44) are not canonically normalized since the metric G_ϕ appears explicitly also in their kinetic terms. Thus, the physical VEV is $v_\phi\sqrt{G_\phi}$. However, to estimate the validity of the perturbative approach, it is more appropriate to keep v_ϕ instead of $v_\phi\sqrt{G_\phi}$. The reason is that the next to leading correction to the D-term involves a quartic term of the type $|\phi|^4$, proportional to a new coefficient $\mathcal{K}$, and

[4] When mass scales are absent, string units are implicit throughout the thesis.

the condition of validity of perturbation theory is $\mathcal{K}v_\phi^2/G_\phi << 1$. A rough estimate is then obtained by approximating $\mathcal{K} \sim G_\phi$, which gives our condition.

The metric G_ϕ of the scalars living on the brane has been computed explicitly for the case of diagonal fluxes[122]. In this special case, the fluxes are denoted by three angles θ_i^a, $(i = 1, 2, 3)$.[5] Then suppressing index-a, we have:

$$\tan \pi\theta_i = \frac{p_{x^i y^i}}{J_i} \equiv \frac{(F_{(1,1)})_{z^i \bar{z}^i}}{J_i}, \tag{2.47}$$

and

$$G = e^{\gamma_E(\theta_1+\theta_2+\theta_3)} \times \sqrt{\frac{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}{\Gamma(1-\theta_1)\Gamma(1-\theta_2)\Gamma(1-\theta_3)}}, \tag{2.48}$$

with γ_E being the Euler constant[6].

The above results will be applied in Section 4.4 to find out the FI parameters and charged scalar VEVs along three of the twelve brane stacks: U_1, A and B. The other nine stacks, U_5, $O_1, \ldots, O_8$, stabilizing all the geometric moduli, will satisfy the calibration condition $\xi^a = 0$ in the absence of open string scalar VEVs. Moreover, the RR moduli that appear in the same chiral multiplets as the geometric Kähler moduli, become Goldstone modes which get absorbed by the $U(1)$ gauge bosons [102] corresponding to each of the D-terms that stabilize the relevant geometric moduli.

2.5 Tadpoles

In this section, we discuss the the consistency conditions that a magnetized D-brane configuration must satisfy. Such restrictions play a crucial role when constructing a consistent effective field theory. Necessary conditions for a consistent construction involving K stacks of N_a magnetic $D9$-branes on a compact orientifold compactification follow from the RR tadpole cancellations. These account for the absence of UV divergences in the one loop amplitude and ensure, via a generalized Green-Schwarz mechanism, the cancellation of gauge anomalies in the associated four dimensional field theories.

In toroidal compactifications of type I string theory, the magnetized $D9$-branes induce 5-brane charges as well, while the 3-brane and 7-brane charges automatically vanish due to the presence of mirror branes with opposite flux. For general magnetic fluxes, RR tadpole

[5] See examples in Appendix A for the precise map between $p_{x^i y^i}$ and $(F_{(1,1)})_{z^i \bar{z}^i}$.

[6] The T^6 metric is diagonal in our case a posteriori, since the moduli are fixed in this way. To leading order in α' (corresponding to keep the matter scalar VEVs small) the matter metric is diagonal but its elements have a non-trivial (torus) moduli dependence due to the magnetic fluxes, that we calculated explicitly using the relations given in equation (2.47) and (2.48).

conditions can be written in terms of the Chern numbers and winding matrix [103, 104] as:

$$16 = \sum_{a=1}^{K} N_a \,\mathrm{det}W_a \equiv \sum_{a=1}^{K} Q^{9,\,a}, \tag{2.49}$$

$$0 = \sum_{a=1}^{K} N_a \,\mathrm{det}W_a \,\mathcal{Q}^{a,\,\alpha\beta} \equiv \sum_{a=1}^{K} Q^{5,\,a}_{\alpha\beta}, \qquad \forall\alpha,\beta = 1,\ldots,6 \tag{2.50}$$

where

$$\mathcal{Q}^{a,\,\alpha\beta} = \epsilon^{\alpha\beta\delta\gamma\sigma\tau} p^a_{\delta\gamma} p^a_{\sigma\tau}\,.$$

The l.h.s. of eq. (2.49) arises from the contribution of the $O9$-plane. On the other hand, in toroidal compactifications there are no $O5$-planes and thus the l.h.s. of eq. (2.50) vanishes.

For our choice of windings (2.18), $W_i^{\hat{i}} = 1$, the $D9$ tadpole contribution from a given stack-a of branes is simply equal to the number of branes, N_a. The $D5$ tadpole expression also takes a simple form for the fluxes satisfying the $F^a_{(2,0)} = 0$ condition (2.40). The fluxes are then represented by three-dimensional Hermitian matrices ($F^a_{(1,1)}$) which appeared in eq. (2.25) and the $D5$ tadpoles $\mathcal{Q}^{5,\,a}_{i\bar{j}}$ are the Cofactors of the $i\bar{j}$ matrix elements $(F^a_{(1,1)})_{i\bar{j}}$. Fluxes and tadpoles in such a form are given in Appendix A.

2.6 Spectrum

Analyzing the low energy spectrum of a string based model is the first step towards building a semirealistic $D = 4$ compactification from a superstring theory. In particular, in order to build a semirealistic model important issues as chirality, family triplication and realistic gauge group must be possible to achieve. In this section, we will study the four dimensional low energy spectrum that we get in a magnetized D-brane constructions involving K stacks of N_a magnetic D9-branes.

As a D9-brane with $\mathcal{F} \neq 0$ is not invariant under orientifold projection, but maps to the brane of opposite flux, there is no orientifold projection in its open string spectrum. The resulting gauge group on a stack of N such branes is therefore $U(N)$ instead of $SO(N)$ or $Sp(N)$. For the configuration involving K stacks of N_a magnetized D9-branes, the gauge sector of the spectrum follows from the open string states corresponding to strings starting and ending on the same brane stack. The gauge symmetry group is given by a product of unitary groups $\otimes_a U(N_a)$, upon identification of the associated open strings attached on a given stack with the ones attached on the mirror (under the orientifold transformation) stack. In addition to these vector bosons, the massless spectrum contains adjoint scalars and fermions forming $\mathcal{N} = 4,\ D = 4$ supermultiplets.

In the matter sector, the massless spectrum is obtained from the following open string states[75, 89, 93]:

1. Open strings stretched between the a-th and b-th stack give rise to chiral spinors in the bi fundamental representation $(N_a, \bar{N}_b)$ of $U(N_a) \times U(N_b)$. Their multiplicity I_{ab} is given by [104]:

$$I_{ab} = \frac{\det W_a \det W_b}{(2\pi)^3} \int_{T^6} \left(q_a F^a_{(1,1)} + q_b F^b_{(1,1)}\right)^3 , \tag{2.51}$$

   where $F^a_{(1,1)}$ (given in eqs. (2.24) and (2.25)) is the pullback of the integrally quantized world-volume flux $m^a_{\hat{\alpha}\hat{\beta}}$ on the target torus in the complex basis (eq. 2.4), and q_a is the corresponding $U(1)_a$ charge; in our case $q_a = +1$ (-1) for the fundamental (anti-fundamental representation). The transformation under the gauge group and their multiplicities are thus determined in terms of the data $(N_a, W_I^{\hat{I},a}, m_{\hat{I}\hat{J}})$.

   For factorized toroidal compactifications $(T^2)^3$ (eq. 2.17) with only diagonal fluxes $p_{x^i y^i}$ $(i = 1, 2, 3)$, the multiplicities of chiral fermions, arising from strings starting from stack a and ending at b or vice verse, take the simple form (using notations of eqs. (2.15) and (2.20)):

$$\begin{aligned} (N_a, \overline{N}_b) : I_{ab} &= \prod_i (\hat{m}^a_i \hat{n}^b_i - \hat{n}^a_i \hat{m}^b_i), \\ (N_a, N_b) : I_{ab^*} &= \prod_i (\hat{m}^a_i \hat{n}^b_i + \hat{n}^a_i \hat{m}^b_i) . \end{aligned} \tag{2.52}$$

   where i is the label of the i-th two-tori T^2_i, and the integers $\hat{m}^a_i, \hat{n}^a_i$ enter in the multiplicity expressions through the magnetic field as in eq. (2.19).

   In the model that we construct, however, we need stacks with fluxes which contain both diagonal and oblique flux components, for the purpose of complete Kähler and complex structure moduli stabilization.

2. Open strings stretched between the a-th brane and its mirror $a^\star$ give rise to massless modes associated to $I_{aa^\star}$ chiral fermions. These transform either in the antisymmetric or symmetric representation of $U(N_a)$. For factorized toroidal compactifications $(T^2)^3$, the multiplicities of chiral fermions are given by;

$$\text{Antisymmetric}: \quad \frac{1}{2}\left(\prod_i 2\hat{m}^a_i\right)\left(\prod_j \hat{n}^a_j + 1\right),$$

$$\text{Symmetric}: \quad \frac{1}{2}\left(\prod_i 2\hat{m}_i^a\right)\left(\prod_j \hat{n}_j^a - 1\right). \tag{2.53}$$

In generic configurations, where supersymmetry is broken by the magnetic fluxes, the scalar partners of the massless chiral spinors in twisted open string sectors (*i.e.* from non-trivial brane intersections) are massive (or tachyonic). Moreover, when a chiral index I_{ab} vanishes, the corresponding intersection of stacks a and b is non-chiral. The multiplicity of the non-chiral spectrum is then determined by extracting the vanishing factor and calculating the corresponding chiral index in higher dimensions. This analysis is done explicitly in section 4.2.7, once explicit semi-realistic examples are constructed.

2.7 Constant NS-NS B-field background

In toroidal models with vanishing B-field, the net generation number of chiral fermions is in general even[94]. Thus, it is necessary to turn on a constant B-field background in order to obtain a Standard Model like spectrum with three generations. Due to the world-sheet parity projection $\mathcal{O}$, the NS-NS two-index field $B_{\alpha\beta}$ is projected out from the physical spectrum and constrained to take the discrete values 0 or 1/2 (in string units) along a 2-cycle $(\alpha\beta)$ of T^6 [91, 92].

For branes at angles, $B_{\alpha\beta} = 1/2$ changes the number of intersection points of the two branes. For the case of magnetized $D9$-branes, if B is turned on only along the three diagonal 2-tori:

$$B_{x^i y^i} \equiv b_i = \frac{1}{2}, \quad i = 1, 2, 3, \tag{2.54}$$

the effect is accounted for by introducing an effective world-volume magnetic flux quantum, defined by $\tilde{\hat{m}}_j^a = \hat{m}_j^a + \frac{1}{2}\hat{n}_j^a$, while the first Chern numbers along all other 2-cycles remain unchanged (and integral). Thus, the modification can be summarized by

$$(\hat{m}_j^a, \hat{n}_j^a) \ \text{ for } b_j = 0 \quad \rightarrow (\hat{m}_j^a + \frac{1}{2}\hat{n}_j^a, \hat{n}_j^a) \equiv (\tilde{\hat{m}}_j^a, \hat{n}_j^a), \ \text{ for } b_j = \frac{1}{2}, \tag{2.55}$$

along the particular 2-cycles where the NS-NS B-field is turned on. This transformation also takes into account the changes in the fermion degeneracies given in eqs. (2.52) and (2.53) (as well as in (2.59), (2.60) below), due to the presence of a non-zero B:

$$(N_a, \overline{N}_b) : I_{ab} = \prod_i (\tilde{\hat{m}}_i^a \hat{n}_i^b - \hat{n}_i^a \tilde{\hat{m}}_i^b),$$

$$(N_a, N_b) : I_{ab^*} = \prod_i (\tilde{\hat{m}}_i^a \hat{n}_i^b + \hat{n}_i^a \tilde{\hat{m}}_i^b)\,, \tag{2.56}$$

$$\text{Antisymmetric} : I_{aa^*}^A = \quad \frac{1}{2}\left(\prod_i 2\tilde{\hat{m}}_i^a\right)\left(\prod_j \hat{n}_j^a + 1\right), \tag{2.57}$$

$$\text{Symmetric} : I_{aa^*}^S = \quad \frac{1}{2}\left(\prod_i 2\tilde{\hat{m}}_i^a\right)\left(\prod_j \hat{n}_j^a - 1\right). \tag{2.58}$$

In addition, similar modifications take place in the tadpole cancellation conditions, as well. Note that for non trivial B, if $\hat{n}_i^a$ is odd $\tilde{\hat{m}}_i^a$ is half-integer, while if $\hat{n}_i^a$ is even $\tilde{\hat{m}}_i^a$ must be integer.

When restricting to the trivial windings of eq. (2.18) that we use in constructing explicit semirealistic examples, $\hat{n}_i^a = 1$, the degeneracy formula (2.51) simplifies to:

$$(N_a, \overline{N}_b) : I_{ab} = \det\left(\tilde{F}^a_{(1,1)} - \tilde{F}^b_{(1,1)}\right)\,, \tag{2.59}$$

$$(N_a, N_b) : I_{ab^*} = \det\left(\tilde{F}^a_{(1,1)} + \tilde{F}^b_{(1,1)}\right), \tag{2.60}$$

where $\tilde{F} = F + B$ and we have assumed the canonical volume normalization (2.1) on T^6. Similarly, the multiplicity of chiral antisymmetric representations is given by:

$$\text{Antisymmetric} : \quad I_{aa^*}^A = \prod_i \left(2\tilde{\hat{m}}_i^a\right)\,, \tag{2.61}$$

while there are no states in symmetric representations. Finally, the tadpole cancellation conditions (2.49) and (2.50) become:

$$\sum_{a=1}^K N_a = 16 \quad ; \quad \sum_{a=1}^K N_a \,\mathrm{Co}(\tilde{F}^a_{(1,1)})_{i\bar{j}} = 0 \qquad \forall\, i,j = 1,\ldots,3\,. \tag{2.62}$$

3

Fermion Wavefunctions in Magnetized branes: Theta identities and Yukawa couplings

3.1 Introduction

One of the most outstanding puzzles of the Standard Model (SM) of particle physics is the structure of the Yukawa couplings between the Higgs field and the SM fermions. A correct description of the observed masses and mixing of quarks and leptons require very different values for the Yukawa coupling constants for the different generations. In the context of semirealistic model building from string theory, one should look for the possibility of computing Yukawa couplings in terms of the extra-dimensional geography. Starting from a (D+4)-dimensional field theory and compactifying D dimensions one may get massless modes with factorized wavefunctions $\chi(x) \times \psi(y)$, with x, y denoting Minkowski and extra dimensions respectively. Gauge boson components A^i in extra dimensions give rise to scalars at low energies and Yukawa couplings are thus expected to appear upon compactification from the higher dimensional gauge vertex interaction $A^M \Psi \Gamma_M \Psi$. The Yukawa coupling constants are then computed from overlap integrals over the extra dimensions.

The aim of this chapter is to address the issue of computating Yukawa couplings, in the context of magnetized D-brane models. We consider, as our starting point, ten dimensional super-Yang-Mills (SYM) theory as the best motivated extra dimensional field theory, since it appears in the low-energy limit of Type I, Type IIB and heterotic string theories. We compactify D=10 $\mathcal{N} = 1$ SYM on a 6-torus T^6 and, in order to obtain chiral fermions, we add constant magnetic flux through the torus. We solve Dirac and Laplace equations to find out the explicit form of wavefunctions in extra dimensions. The Yukawa couplings are obtained by performing the overlap integrals of these wavefunctions.

Close form expressions for Yukawa couplings have been written down for string constructions involving branes at angles [81, 109] or those with magnetized branes [4, 6, 89, 90, 93, 96, 102, 103, 104, 107, 123, 124, 125, 126, 127]. In the IIA picture, the interaction is described by the worldsheet instanton contributions from the sum of areas of various triangles that are formed by three $D6$ branes intersecting at three vertices, forming a triangle. This is due to the fact that the intersection of branes relevant for Yukawa interactions are those which are point-like giving chiral multiplets. Line or surface like intersections, on the other hand, would give rise to interactions of non-chiral matter. In these discussions, the orientation of the branes themselves are parameterized by three angles in the three orthogonal 2-planes, inside T^6. These results have been further generalized to include Euclidean $D2$ brane instanton contributions to the Yukawa couplings [110, 111, 112, 113, 114, 115, 116, 117, 118], generating up quark and right handed neutrino masses through a Higgs mechanism, in a particular class of models. A limitation on the exercise performed in these papers comes from the factorized structure of the tori, which arises from the orientations of the brane wrappings that are classified by angles in three different T^2 planes, rather than their general orientations in the internal six dimensional space parameterized for instance by the $SU(3)$ angles in supersymmetric situations.

Similar results for perturbative Yukawa couplings have also been obtained in the magnetized brane picture, based on their gauge theoretic representation [6]. In this case, the interactions are given by the overlap integral of three wavefunctions (contributing to the interaction) along internal directions. The wavefunctions correspond, in the ordinary field theory context, to those belonging to two fermions and a scalar, and are given by Jacobi theta functions, when fluxes are turned on along three diagonal 2-tori. The relationship between the Yukawa interactions in the magnetized brane constructions and those involving $D6$ branes, have also been established using T-duality rules. However, these exercises have once again been of limited scope due to the fact that explicit expressions are written down only for magnetized branes with fluxes that are diagonal along three T^2's.

Technically, the wavefunctions of chiral fields participating in Yukawa interactions are defined in terms of Jacobi theta functions, with a modular parameter identified as a product of the complex structure of the T^2, with the flux that is turned on along it. The Yukawa interactions are therefore computed for the case when the six dimensional internal space is of a factorized form:

$$T^2 \times T^2 \times T^2 \in T^6. \tag{3.1}$$

As advocated in [7, 102, 103, 104], one, in general, needs to include both 'diagonal' and 'oblique' fluxes for applications to model building with moduli stabilization. Therefore it is imperative that we generalize previous results further and obtain interactions involving branes with oblique fluxes. As stated, in the language of $D6$ branes such generalizations would amount to intersections of branes with orientations given by $SU(3)$ rotation angles, resulting to $\mathcal{N} = 1$ supersymmetry in $D = 4$ with chiral matter. In view of the importance of such fluxes in obtaining realistic particle physics models with stabilized moduli, and to describe the interactions among the chiral fields, we shall study the explicit construction of fermion (and scalar) wavefunctions on compact toroidal spaces with arbitrary constant fluxes.

Scattered results on fermion wavefunctions in presence of constant gauge fluxes, on tori of arbitrary dimensions, exist already in the literature [6, 128]. However, they are of limited use for our purpose. First, any wavefunction obtained through a diagonalization process of the gauge fluxes [128], is not in general suitable for obtaining an overlap integral of wavefunctions. This is because the flux matrices need not commute along different stacks of branes that participate in the interaction through the chiral multiplets, arising from the strings that join these branes and therefore they are not simultaneously diagonalizable.

In [6], a set of wavefunctions was given for constant gauge fluxes. However, once again, explicit results are valid only for those fluxes which satisfy a set of 'Riemann conditions', including a positivity criterion on the flux matrices. As the analysis in this chapter will clarify, the positivity restrictions on the fluxes is due to the fact that the given wavefunction in [6] corresponds to a specific component of the 2^n dimensional Dirac spinor for a $2n$-dimensional torus T^{2n}. We will show that this restriction is relaxed, if one considers wavefunctions of various chiralities, such that all possible flux matrices are allowed, though in our case we restrict to only those fluxes that are consistent with the requirements of space-time supersymmetry .

In fact, we give explicit solutions for the wavefunctions for arbitrary fluxes, that are well defined globally on the toroidal space. We also give explicit mappings among the wave-functions of different chiralities, satisfying different consistency criterion. These mappings are shown to relate wavefunctions corresponding to different fluxes and complex structures of the tori. We further reconfirm that our wavefunctions, as well as mappings are indeed correct, by showing that equations of motion also map into each other for the fermion wavefunctions just described, corresponding to different internal chiralities.

Apart from the lack of enough knowledge about the fermion wavefunctions, the limitations on available information about the Yukawa couplings for general gauge fluxes also arose from the technicalities in dealing with general Riemann theta functions that are used for defining the wavefunctions on toroidal spaces. Internal wavefunctions of chiral fermions participating in the interaction are given by a general Riemann theta function whose modular parameter argument is determined in terms of the complex structure of T^6 as well as the 'oblique' fluxes that we turn on. Hence, the limitations on available results for Yukawa interactions in the literature, arise due to the intricacies involved in evaluating the overlap integrals of the trilinear product of general Riemann theta functions over the six dimensional internal space. In particular, even for positive chirality wavefunctions along the internal T^6 given in [6], one finds that theta identities [119] need to be further generalized, in order to compute the Yukawa interactions with oblique fluxes. The task goes beyond the identity given in [119], since one needs to evaluate the overlap integral of three wavefunctions, all having different modular parameter matrices as arguments, due to the presence of different fluxes along the three brane stacks involved in generating the Yukawa coupling.

In this chapter, first, we generalize the identities used in [6] (available from mathematical literature [119]) for the known positive chirality wavefunctions to those with general Riemann theta functions representing the fermion wavefunctions. This gives an explicit answer for the Yukawa interaction in a close form and generalizes the results of [6, 81]. In particular, we generalize the result further for the positive chirality wavefunction, when general (hermitian) fluxes with all nine parameters rather than the six components, considered before, are turned on.

Furthermore, as already stated earlier, we give explicit constructions of the other T^6 spinor wavefunctions, as well. In these cases too, we obtain the selection rules among chiral multiplets giving nonzero Yukawa couplings. Now, however, the final answer is left as a real finite integration of a theta function, over three toroidal coordinate variables. This integration can be evaluated numerically for any given example.

The chapter is organized as follows. In the next section we briefly discuss the origin of Yukawa couplings in extra dimensional theories. We motivate the study of magnetized compactification in order to achieve $D = 4$ chiral models from extra dimensions. We describe the general strategy that we follow to compute three-point functions in such models [6]. In Section 3.3, we give the chiral fermion wavefunctions in the presence of constant fluxes. In Section 3.4, we review the known results on the Jacobi theta identity given in [119] and present a proof of its validity. We also give an expression for the Yukawa interaction for factorized tori and 'diagonal' fluxes using the theta identity. In Section 3.5, we construct a similar identity, but now for the general Riemann theta function. We then use this new mathematical relation for writing down the expression for the Yukawa interaction when oblique fluxes are present and satisfy the 'Riemann conditions' of [6]. Results are further generalized to include the most general flux matrices consistent with supersymmetry and 'Riemann condition' requirements. In order to relax the later, in Section 3.6, we present the generalizations to include the wavefunctions of the other internal chiralities, in order to accommodate general fluxes consistent with supersymmetry restrictions. Conclusions are presented in Section 3.7.

3.2 Ten Dimensional $\mathcal{N} = 1$ Super Yang-Mills compactification with magnetic fluxes

Let us consider $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in ten dimensions. Its Lagrangian density is given by

$$\mathcal{L}_{SYM} = -\frac{1}{4g^2} \operatorname{Tr}\left\{F^{MN}F_{MN}\right\} + \frac{i}{2g^2} \operatorname{Tr}\left\{\bar{\lambda}\Gamma^M D_M \lambda\right\} \tag{3.2}$$

where $M, N = 0, \dots, 9$. Here, λ denotes gaugino field, g is the Yang-Mills coupling constant in $D = 10$, and Γ^M is the gamma matrix for ten dimensions. The gauge group field strength F_{MN} and covariant derivative D_M are given by

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N] \tag{3.3}$$

$$D_M \lambda = \partial_M \lambda - i[A_M, \lambda] \tag{3.4}$$

where A_M is the ten-dimensional vector field.

In order to obtain a $D = 4$ theory at low energies, we should consider the above theory compactified on a six-dimensional compact manifold $\mathcal{M}_6$, so that we recover four-dimensional physics at energies below the compactification scale M_c. Here we consider the torus T^6 as the extra dimensional compact space. The ten-dimensional fields A_M and λ are decomposed as

$$\lambda(X^\mu, x^m) = \sum_n \chi_n(X^\mu) \otimes \psi_n(x^m) \tag{3.5}$$

$$A_M(X^\mu, x^m) = \sum_n \varphi_{n,M}(X^\mu) \otimes \phi_{n,M}(x^m) \tag{3.6}$$

where X^μ, $\mu = 0, \dots, 3$ and x^m, $m = 4, \dots, 9$ stand for the non-compact and internal dimensions, respectively. The internal wavefunctions ψ_n, $\phi_{n,M}$ can be chosen to be eigenstates of the corresponding internal wave operator

$$i\not{D}_6 \psi_n = 0 \tag{3.7}$$

$$\Delta_6 \phi_{n,M} = M^2_{n,M} \phi_{n,M} \tag{3.8}$$

By introducing non-trivial expectation values for the gauge field A_M, one can obtain chiral fermions in four dimension. Indeed, since we are only interested in preserving Poincaré invariance in the four non-compact dimensions, we are entitled to consider non-vanishing v.e.v.'s $\langle A_m(x) \rangle$, $m = 4, \ldots, 9$. A non-trivial gauge field modifies the Dirac operator and hence the computation of the Dirac index, and may introduce a chiral asymmetry that allows for a chiral massless spectrum. We hence find that compactifications with non-trivial gauge fields $\langle A_m(x) \rangle$, or equivalently, magnetized $\mathcal{M}_6$ compactifications with $\langle F_{mn} \rangle \neq 0$, provide a natural way of achieving $D = 4$ chiral theories with reduced gauge group.

In addition, the introduction of a magnetic field in the compactification may not only lead to chiral matter but also to replication of chiral fermions, since the Dirac equation for the internal fermionic wavefunction $\not{D}_6 \psi = 0$ may yield several independent degenerate solutions, labeled by $\psi_j(x)$. In order to get canonical kinetic terms, these internal wavefunctions must satisfy

$$\int_{\mathcal{M}_6} d^6 y \ \psi_j(x)^\dagger \psi_k(x) = \delta_{jk} \tag{3.9}$$

the same condition applying to bosonic wavefunctions.

Finally, given the internal wavefunctions ψ_j, ϕ_k corresponding to the $D = 4$ chiral fermions and lightest scalars, it is possible to compute the Yukawa couplings between them, as an overlap between three wavefunctions. Indeed, the fermionic part of the $D = 10$ SYM action (3.2) contains a term of the form $A \cdot \lambda \cdot \lambda$, which upon dimensional reduction yields the Yukawa coupling

$$Y_{ijk} = \int_{\mathcal{M}} \psi_i^{a\dagger} \ \Gamma^m \ \psi_j^b \ \phi_{k,m}^c \ f_{abc} \tag{3.10}$$

where f_{abc} are the structure constants of the higher dimensional gauge group.

3.3 Toroidal Wavefunctions

We first present the construction of chiral fermion wavefunctions on tori and give their representation in terms of theta functions. For definiteness we first discuss the case of 4-tori, though T^6 chiral multiplet structure can be analyzed in a similar manner. To be explicit, for the moment we restrict ourselves to the canonical complex structure: $\Omega = iI_2$ and $\Omega = iI_3$ for T^4 and T^6 respectively, where I_d represents a d-dimensional identity matrix. The general complex structure is restored while writing the wavefunctions as well as interaction vertices.

To obtain the Dirac wavefunctions in T^4, we start by writing four Dirac Gamma matrices (in a complex basis) :

$$\Gamma^{z_1} = \sigma^z \times \sigma^3 = \begin{pmatrix} 0 & 2 & & \\ 0 & 0 & & \\ & & 0 & -2 \\ & & 0 & 0 \end{pmatrix}, \quad \Gamma^{z_2} = I \times \sigma^z = \begin{pmatrix} & & 2 & 0 \\ & & 0 & 2 \\ 0 & 0 & & \\ 0 & 0 & & \end{pmatrix}, \tag{3.11}$$

where the information about the complex structure in the above expression is hidden in the fact that we have used the definitions: $z_i = x_i + iy_i$ in writing these Dirac matrices. Similarly,

$$\Gamma^{\bar{z}_1} = \sigma^{\bar{z}} \times \sigma^3 = \begin{pmatrix} 0 & 0 & & \\ 2 & 0 & & \\ & & 0 & 0 \\ & & -2 & 0 \end{pmatrix}, \quad \Gamma^{\bar{z}_2} = I \times \sigma^{\bar{z}} = \begin{pmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 2 & 0 & & \\ 0 & 2 & & \end{pmatrix}. \tag{3.12}$$

They satisfy the anti-commutation relations:

$$\{\Gamma^{z_i}, \Gamma^{z_j}\} = 0, \quad \{\Gamma^{\bar{z}_i}, \Gamma^{\bar{z}_j}\} = 0, \quad \{\Gamma^{z_i}, \Gamma^{\bar{z}_j}\} = 4\delta_{ij} \tag{3.13}$$

with $i, j = 1, 2$. In the above basis Γ^5 takes the form:

$$\Gamma^5 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \tag{3.14}$$

with 4-component Dirac wavefunctions having the form:

$$\Psi = \begin{pmatrix} \Psi^1_+ \\ \Psi^2_- \\ \Psi^1_- \\ \Psi^2_+ \end{pmatrix}. \tag{3.15}$$

In such a decomposition of Ψ, Dirac equations for fermions in the adjoint representation are of the form:

$$\bar{\partial}_1 \Psi^1_+ + \partial_2 \Psi^2_+ + [A_{\bar{z}_1}, \Psi^1_+] + [A_{z_2}, \psi^2_+] = 0,$$

$$\begin{aligned}
&\bar{\partial}_2\Psi^1_+ - \partial_1\Psi^2_+ + [A_{\bar{z}_2}, \Psi^1_+] - [A_{z_1}, \Psi^2_+] = 0,\\
&\partial_1\Psi^2_- + \partial_2\Psi^1_- + [A_{z_1}, \Psi^2_-] + [A_{z_2}, \Psi^1_-] = 0,\\
&\bar{\partial}_2\Psi^2_- - \bar{\partial}_1\Psi^1_- + [A_{\bar{z}_2}, \Psi^2_-] - [A_{\bar{z}_1}, \Psi^1_-] = 0.
\end{aligned} \tag{3.16}$$

In a generic model, chiral fermions arise either from the string starting at a brane stack-a and ending at another brane stack-b (or its image b^*) or from strings starting at a brane stack a and ending at its image a^*. We already showed the correspondence between a stack of magnetized branes and flux quanta in supersymmetric gauge theory, in eq. (2.34). The correspondence is easily generalized when several stacks of branes are present. Explicitly, in a construction with P number of stacks of branes, with number of branes being n_i for the i'th stack, the flux (for a given target space component $(i\bar{j})$) takes a form:

$$F_{i\bar{j}} = \begin{pmatrix} F^1 I_{n_1} & & & & \\ & F^2 I_{n_2} & & & \\ & & . & & \\ & & & . & \\ & & & & F^{n_p} I_{n_p} , \end{pmatrix} \tag{3.17}$$

with I_{n_i} being the n_i-dimensional identity matrix and we have hidden the $i\bar{j}$ indices in the RHS of eq. (3.17) in constants F^i that are all integrally quantized, as given earlier explicitly in eqs. (2.32) and (2.33). The corresponding gauge potentials will also then have a block diagonal structure:

$$A_i = \begin{pmatrix} A^1_i I_{n_1} & & & \\ & A^2_i I_{n_2} & & \\ & & . & \\ & & . & A^{n_p}_i I_{n_p} \end{pmatrix} . \tag{3.18}$$

Now, in order to understand the wavefunctions associated with chiral fermion bifundamentals, in such a representation of the brane stacks, we consider the flux matrix $F_{i\bar{j}}$ in eq. (3.17) and gauge potential in eq. (3.18) with only two blocks ($P = 2$). The chiral fermion bilinears between stack-a and stack-b are then represented by:

$$\Psi_{ab} = \begin{pmatrix} C_{n_a} & \chi_{ab} \\ & C_{n_b} \end{pmatrix} , \tag{3.19}$$

with C_{n_a}, C_{n_b} being constant matrices of dimensions n_a and n_b respectively [7]. We can easily derive the equation satisfied by the various Dirac components, as given in eq. (3.15), for χ_{ab} such that Ψ_{ab} satisfies the Dirac equation (3.16). We obtain:

$$\begin{aligned}
&\bar{\partial}_1\chi^1_+ + \partial_2\chi^2_+ + (A^1 - A^2)_{\bar{z}_1}\chi^1_+ + (A^1 - A^2)_{z_2}\chi^2_+ = 0,\\
&\bar{\partial}_2\chi^1_+ - \partial_1\chi^2_+ + (A^1 - A^2)_{\bar{z}_2}\chi^1_+ - (A^1 - A^2)_{z_1}\chi^2_+ = 0,\\
&\partial_1\chi^2_- + \partial_2\chi^1_- + (A^1 - A^2)_{z_1}\chi^2_- + (A^1 - A^2)_{z_2}\chi^1_- = 0,\\
&\bar{\partial}_2\chi^2_- - \bar{\partial}_1\chi^1_- + (A^1 - A^2)_{\bar{z}_2}\chi^2_- - (A^1 - A^2)_{\bar{z}_1}\chi^1_- = 0,
\end{aligned} \tag{3.20}$$

with subscript a, b being dropped from χ_{ab} to make the expressions simpler. We will, however, restore the indices at a later stage while evaluating the overlap of three such wave functions from different intersections. In particular, for the chiral components, χ^1_+ equations reduce to:

$$\begin{aligned}
&\bar{\partial}_1\chi^1_+ + (A^1 - A^2)_{\bar{z}_1}\chi^1_+ = 0,\\
&\bar{\partial}_2\chi^1_+ + (A^1 - A^2)_{\bar{z}_2}\chi^1_+ = 0.
\end{aligned} \tag{3.21}$$

The generalization of eq. (3.21) to the T^6 case is straightforward and can be written as:

$$\bar{D}_i\chi^{ab}_+ \equiv \bar{\partial}_i\chi^{ab}_+ + (A^1 - A^2)_{\bar{z}_i}\chi^{ab}_+ = 0, \quad (i = 1, 2, 3). \tag{3.22}$$

Eq. (3.22) matches with the results obtained in [6] for $\Omega = iI_3$, with the identification:

$$(A^1 - A^2)_{\bar{z}_i} \equiv \frac{\pi}{2}\left([\mathbf{N}.(\tilde{\mathbf{z}} + \tilde{\zeta})].(\mathbf{Im\Omega})^{-1}\right)_i, \tag{3.23}$$

with $\vec{\zeta}$ being the complex constants representing the Wilson lines and N is the difference of fluxes between the two stacks a and b, having constant fluxes F^1 and F^2, giving the fermion bilinears in the representation $(n_1, \bar{n}_2)$.

Such a solution for eq. (3.22) and (3.23) is given in [6] for arbitrary complex structure Ω by the basis elements:

$$\psi^{\vec{j},\mathbf{N}}(\vec{z}, \mathbf{\Omega}) = \mathcal{N} \cdot e^{\{i\pi[\mathbf{N}.\vec{z}].(\mathbf{N}.Im\mathbf{\Omega})^{-1}Im[\mathbf{N}.\vec{z}]\}} \cdot \vartheta\begin{bmatrix}\vec{j}\\ 0\end{bmatrix}(\mathbf{N}.\vec{z}, \mathbf{N}.\mathbf{\Omega}), \tag{3.24}$$

[7]The constant matrices correspond to gaugino wavefunction. The $\mathcal{N} = 1$ gauginos are massless as long as supersymmetry remains unbroken. The other gauginos coming from the $\mathcal{N} = 4$, that acquire high scale masses, and decouples from the massless spectrum, which we are interested in.

with general definition of Riemann theta function:

$$\vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\vec{\nu}|\mathbf{\Omega}) = \sum_{\vec{m}\in\mathbf{Z}^n} e^{\pi(\vec{m}+\vec{a}).\mathbf{\Omega}.(\vec{m}+\vec{a})} e^{2\pi i(\vec{m}+\vec{a}).(\vec{\nu}+\vec{b})}. \tag{3.25}$$

Moreover, the matrix $\mathbf{N}$ should satisfy the following conditions in order to have well defined bifundamental wavefunctions. These are the so-called Riemann conditions [6] and are written as:

$$\begin{aligned} \mathbf{N}_{\bar{i}j} &\in \mathbf{Z}\,, \\ (\mathbf{N}.Im\Omega)^T &= \mathbf{N}.Im\Omega\,, \\ \mathbf{N}.Im\Omega &> 0. \end{aligned} \tag{3.26}$$

The first condition in eq. (3.26) is the integrality of the elements of $\mathbf{N}$, that we discuss later on, in the absence of any non-abelian Wilson lines [6], following from the Dirac quantization of fluxes. To understand the last condition of eq. (3.26), one rewrites the $(1,1)$ form $F_{z^i\bar{z}^j}$, for the case when $p_{xx} = p_{yy} = 0$. Indeed using eq. (2.31), one obtains:

$$F_{z^i\bar{z}^j} = -ip_{xy}(\Omega - \bar{\Omega})^{-1}, \tag{3.27}$$

which matches with the expression for H in eq. (4.73) of [6] upon the identification $\mathbf{N^T} = \mathbf{p_{xy}}$ and $H = \frac{1}{2}\mathbf{N}^T.Im\Omega^{-1}$. Also using (2.31), it follows that:

$$(\mathbf{N}\Omega)^T = (\mathbf{N}\Omega). \tag{3.28}$$

The positivity requirement on H then arises from the condition that the solutions of the Dirac equation, corresponding to chiral wavefuntions, be normalizable.

Again, $\mathbf{N}$ satisfies the constraints given in eqs. (3.26) as well as:

$$\vec{j}.\mathbf{N} \in \mathbf{Z}^n, \tag{3.29}$$

implying that $\vec{j}.\mathbf{N}$ is an n-dimensional vector with integer entries. Also, the normalization factor $\mathcal{N}$ in eq. (3.24) is given by:

$$\mathcal{N} = (2^n|\det \mathbf{N}|.\det(Im\Omega))^{\frac{1}{4}} \left(Vol(T^{2n})\right)^{-\frac{1}{2}}. \tag{3.30}$$

Then wavefunctions satisfy the orthonormality relations:

$$\int_{T^{2n}} (\psi^{\vec{j},\mathbf{N}})^* \psi^{\vec{k},\mathbf{N}} = \delta_{\vec{j},\vec{k}}. \tag{3.31}$$

These results are useful in determining the interaction terms in Section 3.5. However, to have well-defined wavefunctions, $\mathbf{N}$'s must satisfy the Riemann conditions given in eq. (3.26).

The wavefunctions of the chiral fermion bifundamentals, with both abelian and non-abelian Wilson lines, involved in Yukawa computations, are given in [6] for the case of the factorized tori, eq. (3.1), and diagonal fluxes. For oblique fluxes, we postpone the discussion of non-abelian Wilson lines and rational fluxes to the last section of the chapter and for the moment we consider the case of integral fluxes only. This restriction, nevertheless, allows for a rich structure of phenomenological value, since semi-realistic models with three generations of chiral fermions and stabilized moduli can be built even in the context of such integral fluxes, by turning on NS-NS antisymmetric tensor background. For example, a three generation $SU(5)$ GUT with stabilized moduli given in [7] was constructed with all winding numbers, $n = 1$, for different stacks of branes. Also, the presence of a half-integral NS-NS antisymmetric tensor does not modify any of our results, since all the relevant chiral fermion wavefunctions depend on the difference of fluxes along pairs of brane stacks which is always integral.

3.4 Yukawa computation on factorized tori

3.4.1 Wavefunction

A detail discussion of the chiral fermion wavefunctions in the presence of constant gauge fluxes is presented in the previous section for general tori and fluxes. In the case of factorized tori, eq. (3.1), the six dimensional chiral/anti-chiral wavefunctions are written as a product of wavefunctions on T^2. To show this explicitly, we present the case of T^4 as an example, with T^6 case working out in a similar fashion. More precisely, considering that on two T^2's, fermion wavefunctions

$$\psi^{(1)} = \begin{pmatrix} \psi_+^{(1)} \\ \psi_-^{(1)} \end{pmatrix}, \quad \psi^{(2)} = \begin{pmatrix} \psi_+^{(2)} \\ \psi_-^{(2)} \end{pmatrix}, \tag{3.32}$$

with their internal $U(n_1) \times U(n_2)$ structure being represented in a manner as in eq. (3.19), satisfy the equations:

$$\begin{aligned} \bar{\partial}_1 \psi_+^{(1)} + (A^1 - A^2)_{\bar{z}_1} \psi_+^{(1)} &= 0, \\ \partial_1 \psi_-^{(1)} + (A^1 - A^2)_{z_1} \psi_-^{(1)} &= 0, \\ \bar{\partial}_2 \psi_+^{(2)} + (A^1 - A^2)_{\bar{z}_2} \psi_+^{(2)} &= 0, \\ \partial_2 \psi_-^{(2)} + (A^1 - A^2)_{z_2} \psi_-^{(2)} &= 0. \end{aligned} \tag{3.33}$$

T^4 fermion wavefunctions are then constructed through a direct product of ψ^1 and ψ^2 (in the notations of Section 3.3):

$$\begin{pmatrix} \Psi_+^1 \\ \Psi_-^2 \\ \Psi_-^1 \\ \Psi_+^2 \end{pmatrix} \equiv \begin{pmatrix} \psi_+^{(1)} \\ \psi_-^{(1)} \end{pmatrix} \otimes \begin{pmatrix} \psi_+^{(2)} \\ \psi_-^{(2)} \end{pmatrix}. \tag{3.34}$$

In particular,

$$\Psi_+^1 \equiv \psi_+^{(1)} \otimes \psi_+^{(2)} \tag{3.35}$$

satisfies precisely the equations (3.21) for chiral fermions on T^4. We can further extend these results to show that T^6 chiral wavefunctions can also be written as a product of the chiral wavefunctions on three T^2's in the decomposition (3.1).

Yukawa interaction on T^6 is then also given by an expression which is a direct product of the interaction terms for the three T^2's. Wavefunctions for the chiral fermions on a T^2 (with coordinates x, y) are expressed in terms of the basis wavefunctions $\psi^{j,N}$ [6]:

$$\psi^{j,N}(\tau, z) = \mathcal{N} \cdot e^{i\pi N z \mathrm{Im}\, z / \mathrm{Im}\, \tau} \cdot \vartheta \begin{bmatrix} \frac{j}{N} \\ 0 \end{bmatrix} (Nz, N\tau), \qquad j = 0, \ldots, N-1\ , \tag{3.36}$$

with N denoting the difference of the $U(n_a)$ and $U(n_b)$ magnetic gauge fluxes, turned on along the Cartan generators, representing stacks of n_a and n_b branes respectively and gives the degeneracy of the chiral fermions:

$$N = m_a - m_b \equiv I_{ab}, \tag{3.37}$$

with m_a and m_b being the 1st Chern number of fluxes along stacks a and b, with unit windings.

Using such a basis, the chiral and anti-chiral (left and right handed fermions) basis wavefunctions:

$$\psi^j = \begin{pmatrix} \psi^j_+ \\ \psi^j_- \end{pmatrix}, \tag{3.38}$$

are given by:

$$\begin{aligned} \psi^j_+ &= \psi^{j,N}(\tau, z+\zeta),\ (\psi^j_+)^* = \psi^{-j,-N}(\bar\tau, \bar z+\bar\zeta), \\ \psi^j_- &= \psi^{j,N}(\bar\tau, \bar z+\bar\zeta),\ (\psi^j_-)^* = \psi^{-j,-N}(\tau, z+\zeta), \end{aligned} \tag{3.39}$$

and satisfy the equations:

$$\begin{aligned} D\psi^j_+ = 0,\ D^\dagger(\psi^j_+)^* = 0, \\ D^\dagger\psi^j_- = 0,\ D(\psi^j_-)^* = 0 \end{aligned}$$

Expanding as in 3.33 and by substituting the corresponding gauge potentials,

$$\begin{aligned} (A^1)_{\bar z_1} = \frac{\pi m_a}{2Im\tau}(z+\zeta),\ (A^2)_{\bar z_1} = \frac{\pi m_b}{2Im\tau}(z+\zeta), \\ (A^1)_{z_1} = \frac{-\pi m_a}{2Im\tau}(\bar z+\bar\zeta),\ (A^2)_{z_1} = \frac{-\pi m_b}{2Im\tau}(\bar z+\bar\zeta). \end{aligned}$$

one gets,

$$\begin{aligned} \left(\bar\partial + \frac{\pi N}{2Im\tau}(z+\zeta)\right)\psi^j_+ = 0, \\ \left(\partial - \frac{\pi N}{2Im\tau}(\bar z+\bar\zeta)\right)(\psi^j_+)^* = 0, \\ \left(\partial - \frac{\pi N}{2Im\tau}(\bar z+\bar\zeta)\right)\psi^j_- = 0, \\ \left(\bar\partial + \frac{\pi N}{2Im\tau}(z+\zeta)\right)(\psi^j_-)^* = 0, \end{aligned} \tag{3.40}$$

with ζ representing the Wilson lines. In the following we set the Wilson lines $\zeta = 0$. Furthermore, expressions of the chiral and anti-chiral solutions, as given in eqs. (3.39) and (3.36), are well defined provided $N > 0$ for the wavefunctions ψ^j_+ and $N < 0$ for the wavefunctions ψ^j_-. In these cases, for ψ^j_+ and ψ^j_- to be properly normalized:

$$\int_{T^2} dz d\bar z\, \psi^j_\pm (\psi^k_\pm)^* = \delta_{jk}, \tag{3.41}$$

an additional factor

$$\mathcal{N}_j = \left(\frac{2Im\tau|N|}{\mathcal{A}^2}\right)^{\frac{1}{4}} \tag{3.42}$$

needs to be introduced, with $\mathcal{A}$ being the area of the T^2.

In fact, the basis functions (3.36) are also eigenfunctions of the Laplacian. We elaborate on this point more in Section 3.6.4 and now proceed to make use of these fermion and boson basis functions to determine the Yukawa interaction in the case of factorized tori and 'diagonal' fluxes.

3.4.2 Interaction for factorized tori

We now summarize the basic results of [6] regarding the computations of Yukawa interactions. As discussed in Section 3.2, such four dimensional interaction terms were obtained through a dimensional reduction of the $D = 10$, $N = 1$ super-Yang-Mills theory to four dimensions in the presence of constant magnetic fluxes. The Yukawa coupling is given by

$$Y_{ijk} = \int_{\mathcal{M}} \psi_i^{a\dagger}\Gamma^m \psi_j^b \phi_{k,m}^c f_{abc}, \tag{3.43}$$

where $\mathcal{M}$ is the internal space on which the gauge theory has been compactified and ψ and ϕ are the internal zero mode fluctuations of the gaugino and Yang-Mills fields with f_{abc} being the structure constants of the higher dimensional gauge group. For the torus compactification that we are discussing, the internal wavefunctions are factorized into those depending on the coordinates of three T^2's. In turn, these involve the evaluation of terms of the type:

$$\int_{T^2} dzd\bar{z}Tr\{\psi_+.[\phi_-,\psi_+]\} \quad \text{and} \quad \int_{T^2} dzd\bar{z}Tr\{\psi_-.[\phi_+,\psi_-]\}, \tag{3.44}$$

with $\phi_\pm$ being the wavefunctions of the bosonic fluctuations of the ten dimensional gauge fields with helicity ± 1 along the particular T^2 direction. Similarly $\psi_\pm$ denotes the spinor fluctuations with helicities $\pm\frac{1}{2}$. Therefore, In the factorized case of eq. (3.1), the full interaction term is computed as a product of three such integrals. To evaluate these integrals, one uses the wavefunctions (3.32) and basis functions as given in eq. (3.36).

In the language of string construction with magnetized branes, $N \equiv I_{ab}$ corresponds to the intersection number for the string starting at a stack a and ending on another one b. The Yukawa interaction then reads:

$$Y_{ijk} = g\sigma_{abc}\int_{T^2} dzd\bar{z}\, \psi^{i,I_{ab}}(\tau,z).\psi^{j,I_{ca}}(\tau,z).(\psi^{k,I_{cb}}(\tau,z))^* \tag{3.45}$$

with $I_{bc} < 0$, corresponding to the fact that when the intersection numbers I_{ab} and I_{ca} are positive, then I_{bc} has to be negative, since $I_{ab} + I_{bc} + I_{ca} = 0$. A similar expression exists for $I_{bc} > 0$ as well. To evaluate this integral, one uses an identity, satisfied by the theta functions appearing in the definition of the basis functions (3.36). The aim of this relation is to establish a connection between the wavefunctions with intersection numbers N_1 and N_2 for bifundamental states in brane intersections ab and ca with the one in the intersection bc with $N_3 = N_1 + N_2$. However, in view of the further generalization to the oblique flux case, we establish this identity explicitly in the next subsection and generalize it further in Section 3.5.

3.4.3 Jacobi theta function identities

We now explicitly prove the following theta function identity[119] used in [6] for computing the Yukawa couplings:

$$\vartheta\begin{bmatrix}\frac{r}{N_1}\\0\end{bmatrix}(z_1,\tau N_1)\cdot\vartheta\begin{bmatrix}\frac{s}{N_2}\\0\end{bmatrix}(z_2,\tau N_2) = \sum_{m\in\mathbf{Z}_{N_1+N_2}}\vartheta\begin{bmatrix}\frac{r+s+N_1m}{N_1+N_2}\\0\end{bmatrix}(z_1+z_2,\tau(N_1+N_2))$$
$$\times\vartheta\begin{bmatrix}\frac{N_2r-N_1s+N_1N_2m}{N_1N_2(N_1+N_2)}\\0\end{bmatrix}(z_1N_2-z_2N_1,\tau N_1N_2(N_1+N_2)), \tag{3.46}$$

where ϑ is the Jacobi theta-function:

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\nu,\tau) = \sum_{l\in\mathbf{Z}} e^{\pi i(a+l)^2\tau}\ e^{2\pi i(a+l)(\nu+b)}. \tag{3.47}$$

To proceed with the proof of the above identity, we write its LHS explicitly as:

$$\vartheta\begin{bmatrix}\frac{r}{N_1}\\0\end{bmatrix}(z_1,\tau N_1)\cdot\vartheta\begin{bmatrix}\frac{s}{N_2}\\0\end{bmatrix}(z_2,\tau N_2) = \sum_{l_1\in\mathbf{Z}}\sum_{l_2\in\mathbf{Z}} e^{\pi i(\frac{r}{N_1}+l_1)^2\tau N_1}\ e^{2\pi i(\frac{r}{N_1}+l_1)z_1}$$
$$\cdot\ e^{\pi i(\frac{s}{N_2}+l_2)^2\tau N_2}\ e^{2\pi i(\frac{s}{N_2}+l_2)z_2}. \tag{3.48}$$

Similarly the RHS of the identity (3.46) can be written as:

$$\sum_{m\in\mathbf{Z}_{N_1+N_2}}\vartheta\begin{bmatrix}\frac{r+s+N_1m}{N_1+N_2}\\0\end{bmatrix}(z_1+z_2,\tau(N_1+N_2))$$
$$\times\vartheta\begin{bmatrix}\frac{N_2r-N_1s+N_1N_2m}{N_1N_2(N_1+N_2)}\\0\end{bmatrix}(z_1N_2-z_2N_1,\tau N_1N_2(N_1+N_2))$$

$$= \sum_{m \in \mathbf{Z}_{N_1+N_2}} \sum_{l_3 \in \mathbf{Z}} \sum_{l_4 \in \mathbf{Z}} e^{\pi i (\frac{r+s+N_1 m}{N_1+N_2}+l_3)^2 \tau (N_1+N_2)} \; e^{2\pi i (\frac{r+s+N_1 m}{N_1+N_2}+l_3)(z_1+z_2)}$$
$$\times e^{\pi i (\frac{N_2 r - N_1 s + N_1 N_2 m}{N_1 N_2 (N_1+N_2)}+l_4)^2 \tau N_1 N_2 (N_1+N_2)} \; e^{2\pi i (\frac{N_2 r - N_1 s + N_1 N_2 m}{N_1 N_2 (N_1+N_2)}+l_4)(z_1 N_2 - z_2 N_1)}. \quad (3.49)$$

Now, to match the z_1, z_2 terms in both sides of eq. (3.46), we first note the identity:

$$\left(\frac{r+s}{N_1+N_2}\right)(z_1+z_2)+\left(\frac{N_2 r - N_1 s}{N_1 N_2 (N_1+N_2)}\right)(z_1 N_2 - z_2 N_1) = \left(\frac{r}{N_1} z_1 + \frac{s}{N_2} z_2\right), \quad (3.50)$$

and find coefficients p_1, p_2, q_1, q_2 such that,

$$(p_1 l_1 + p_2 l_2)(z_1 + z_2) + (q_1 l_1 + q_2 l_2)(z_1 N_2 - z_2 N_1) = (l_1 z_1 + l_2 z_2). \quad (3.51)$$

Eq. (3.51) leads to the following values for p_1, p_2, q_1, q_2 :

$$p_1 = \frac{N_1}{N_1+N_2}, \;\; p_2 = \frac{N_2}{N_1+N_2},$$
$$q_1 = \frac{1}{N_1+N_2}, \;\; q_2 = \frac{-1}{N_1+N_2}. \quad (3.52)$$

Then the two terms, containing z_1, z_2, in the RHS of eq. (3.48) can be rewritten as:

$$e^{2\pi i (\frac{r}{N_1}+l_1) z_1} \; e^{2\pi i (\frac{s}{N_2}+l_2) z_2} = e^{2\pi i (\frac{r+s}{N_1+N_2}+\frac{N_1 l_1}{N_1+N_2}+\frac{N_2 l_2}{N_1+N_2})(z_1+z_2)} \; e^{2\pi i (\frac{N_2 r - N_1 s}{N_1 N_2 (N_1+N_2)}+\frac{l_1 - l_2}{N_1+N_2})(z_1 N_2 - z_2 N_1)}. \quad (3.53)$$

Similarly, coefficients p, q satisfying identity:

$$p\left[\frac{r+s}{N_1+N_2}+\frac{N_1 l_1}{N_1+N_2}+\frac{N_2 l_2}{N_1+N_2}\right]^2 + q\left[\frac{N_2 r - N_1 s}{N_1 N_2 (N_1+N_2)}+\frac{l_1-l_2}{N_1+N_2}\right]^2 = \quad (3.54)$$
$$\left[\frac{r}{N_1}+l_1\right]^2 N_1 + \left[\frac{s}{N_2}+l_2\right]^2 N_2,$$

are given as:

$$p = N_1 + N_2, \;\; q = N_1 N_2 (N_1 + N_2). \quad (3.55)$$

Using eqs. (3.50), (3.51), (3.53) and (3.55), the RHS of eq. (3.48) (appearing in the LHS of eq. (3.46)) can be re-written :

$$\sum_{l_1 \in \mathbf{Z}} \sum_{l_2 \in \mathbf{Z}} e^{\pi i (\frac{r}{N_1}+l_1)^2 \tau N_1} \; e^{2\pi i (\frac{r}{N_1}+l_1) z_1} \cdot e^{\pi i (\frac{s}{N_2}+l_2)^2 \tau N_2} \; e^{2\pi i (\frac{s}{N_2}+l_2) z_2} =$$

$$\sum_{l_1\in\mathbf{Z}}\sum_{l_2\in\mathbf{Z}} e^{\pi i(\frac{r+s}{N_1+N_2}+\frac{N_1 l_1}{N_1+N_2}+\frac{N_2 l_2}{N_1+N_2})^2\tau(N_1+N_2)}\; e^{2\pi i(\frac{r+s}{N_1+N_2}+\frac{N_1 l_1}{N_1+N_2}+\frac{N_2 l_2}{N_1+N_2})(z_1+z_2)}.$$
$$e^{\pi i(\frac{N_2 r-N_1 s}{N_1N_2(N_1+N_2)}+\frac{l_1-l_2}{N_1+N_2})^2\tau N_1N_2(N_1+N_2)}\; e^{2\pi i(\frac{N_2 r-N_1 s}{N_1N_2(N_1+N_2)}+\frac{l_1-l_2}{N_1+N_2})(z_1N_2-z_2N_1)}. \tag{3.56}$$

Proving the identity, eq. (3.46), now amounts to showing that the RHS of eq. (3.49) matches precisely with that of eq. (3.56) with m in eq. (3.49) taking value as $m = 0, 1, ..., (N_1 + N_2 - 1)$. We note:

1. When $l_1 = l_2$ in eq. (3.56), the terms in the RHS are identical to those in the RHS of eq. (3.49), with $m = 0, l_4 = 0$, if we identify l_2 with l_3.
   When $l_1 = l_2 + 1$, the terms in eq. (3.56) exactly match with those in eq. (3.49) obtained for the values $m = 1, l_4 = 0$ with the identification of l_2 with l_3.
   This goes on up to $l_1 = l_2 + (N_1 + N_2 - 1)$ which corresponds to the case for $l_3(= l_2), m = (N_1 + N_2 - 1)$ and $l_4 = 0$.

2. The terms obtained in eq. (3.56) for $l_1 = l_2+(N_1+N_2)$ corresponds to $m = 0, l_4 = 1$ and $l_2 + N_1$ identified with l_3 in eq. (3.49).
   When $l_1 = l_2 + (N_1 + N_2) + 1$ the terms correspond to the case $m = 1, l_4 = 1$ and $l_2 + N_1$ identified with l_3 in eq. (3.49).
   This goes on till $l_1 = l_2 + (N_1 + N_2) + (N_1 + N_2 - 1)$ when they correspond to $m = (N_1 + N_2 - 1), l_4 = 1$ and $l_2 + N_1$ identified with l_3 in eq. (3.49).

3. Similarly the terms for $l_1 = l_2+2(N_1+N_2)$ correspond to the terms for $m = 0, l_4 = 2$ and $l_3 = (l_2 + 2N_1)$. And so on....

We have therefore shown a one-to-one correspondence between the terms in the RHS of eqs. (3.49) and (3.56). The identity eq. (3.46) has thus been proved explicitly.

3.4.4 Application to Yukawa computation for factorized tori

We now make use of the above Jacobi theta identity as well as of the explicit forms of the fermion and scalar wavefunctions, defined in terms of the basis functions in eq. (3.36) to write the expression for the Yukawa interaction term. More precisely, in order to evaluate the Yukawa coupling given in eq. (3.45), one uses the theta identity of eq. (3.46) and the basis function in eq. (3.36) and proceeds by writing down:

$$\psi^{i,I_{ab}}(\tau,z).\psi^{j,I_{ca}}(\tau,z) = \left(\frac{2Im\tau}{\mathcal{A}^2}\right)^{\frac{1}{2}} (I_{ab}I_{ca})^{\frac{1}{4}} e^{i\pi(N_1+N_2)z\mathrm{Im}\, z/\mathrm{Im}\,\tau}\times$$
$$\times\vartheta\begin{bmatrix}\frac{i}{N_1}\\ 0\end{bmatrix}(N_1z, N_1\tau)\cdot\vartheta\begin{bmatrix}\frac{j}{N_2}\\ 0\end{bmatrix}(N_2z, N_2\tau),\ i = 0,\ldots,N_1-1,\ j = 0,\ldots,N_2-1.$$

$$(3.57)$$

where we have also made use of the normalization factor, $\mathcal{N}$ given in eq. (3.42), and identified for a T^2 compactification:

$$N_1 = I_{ab}, \quad N_2 = I_{ca}, \tag{3.58}$$

with

$$I_{ab} = m_a - m_b, \quad etc. \tag{3.59}$$

giving

$$N_3 = (N_1 + N_2) = I_{cb}. \tag{3.60}$$

Now, using the theta identity (3.46), eq. (3.57) can be rewritten in the form:

$$\begin{aligned} \psi^{i,I_{ab}}(\tau,z).\psi^{j,I_{ca}}(\tau,z) = \left(\frac{2Im\tau}{\mathcal{A}^2}\right)^{\frac{1}{4}} \left(\frac{I_{ab}I_{ca}}{I_{cb}}\right)^{\frac{1}{4}} \sum_{m\in\mathbf{Z}_{I_{cb}}} \psi^{i+j+I_{ab}m,I_{cb}}(\tau,z)\times \\ \times\vartheta\begin{bmatrix} \frac{I_{ca}i-I_{ab}j+I_{ab}I_{ca}m}{I_{ab}I_{ca}I_{cb}} \\ 0 \end{bmatrix}(0,\tau I_{ab}I_{ca}I_{cb}). \end{aligned} \tag{3.61}$$

The Yukawa interaction (3.45), is then evaluated using the orthogonality property of the wavefunctions given in eq. (3.41) and reads [8]:

$$Y_{ijk} = \sigma_{abc}g\left(\frac{2Im\tau}{\mathcal{A}^2}\right)^{\frac{1}{4}} \left(\frac{I_{ab}I_{ca}}{I_{cb}}\right)^{\frac{1}{4}} \sum_{m\in\mathbf{Z}_{I_{cb}}} \delta_{k,i+j+I_{ab}m}\cdot\vartheta\begin{bmatrix} \frac{I_{ca}i-I_{ab}j+I_{ab}I_{ca}m}{I_{ab}I_{ca}I_{cb}} \\ 0 \end{bmatrix}(0,\tau I_{ab}I_{ca}I_{cb}). \tag{3.62}$$

After imposing the Kronecker delta constraint, we obtain:

$$Y_{ijk} = \sigma_{abc}g\left(\frac{2Im\tau}{\mathcal{A}^2}\right)^{\frac{1}{4}} \left(\frac{I_{ab}I_{ca}}{I_{cb}}\right)^{\frac{1}{4}} \vartheta\begin{bmatrix} -\left(\frac{j}{I_{ca}}+\frac{k}{I_{bc}}\right)/I_{ab} \\ 0 \end{bmatrix}(0,\tau I_{ab}I_{ca}I_{cb}). \tag{3.63}$$

The final answer can be expressed as :

$$Y_{ijk} = \sigma_{abc}g\left(\frac{2Im\tau}{\mathcal{A}^2}\right)^{\frac{1}{4}} \left(\frac{I_{ab}I_{ca}}{I_{cb}}\right)^{\frac{1}{4}} \vartheta\begin{bmatrix} \delta_{ijk} \\ 0 \end{bmatrix}(0,\tau|I_{ab}I_{bc}I_{ca}|), \tag{3.64}$$

[8]In eq. (3.62), the computed Yukawa coupling is evaluated from the expression of Yukawa interaction given in eq. (3.45) which is a triple overlap of basis wavefunctions given in eq. (3.36). These basis functions not only represent zero modes of the Dirac operator but also eigenfunctions of the Laplacian. This is explicitly shown in Ref. [6] for the positive chirality wavefunction and in Section (3.6.4) for the negative chirality wavefunction.

with

$$\delta_{ijk} = \frac{i}{I_{ab}} + \frac{j}{I_{ca}} + \frac{k}{I_{bc}}. \tag{3.65}$$

The result can be easily extended to the case of factorized T^6 (3.1) and the interaction is then written in terms of the products of theta functions of the type appearing in eq. (3.64). We refer the reader to [6] for the details and now go on to the generalization when fluxes of both oblique and diagonal forms are present. Such magnetic fluxes do not respect the factorization and hence involve the wavefunctions written in terms of the general Riemann theta functions.

3.5 General tori and 'oblique' fluxes

Let us now consider the more general case where the $2n$-dimensional torus is not necessarily factorizable. A generic flat $2n$-dimensional torus, $T^{2n} \simeq \mathbf{C}^n/\Lambda$, inherits a complex structure from the covering space $\mathbf{C}^n$. Its geometry can hence be described in terms of a Kähler metric and complex structure as

$$\begin{aligned} ds^2 &= h_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}} \\ dz^\mu &= dx^\mu + \Omega^\mu_\nu dy^\nu \end{aligned} \tag{3.66}$$

where $x^\mu, y^\mu \in (0,1)$, $\mu = 1, \ldots, n$, parametrize the $2n$ vectors of the lattice Λ. The natural generalization of the Jacobi theta function (3.47) to this higher-dimensional tori is known as Riemann ϑ-functions, as defined in eq. (3.25):

$$\vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\vec{\nu}|\Omega) = \sum_{\vec{l} \in \mathbf{Z}^n} e^{i\pi(\vec{l}+\vec{a}).\Omega.(\vec{l}+\vec{a})} e^{2\pi i(\vec{l}+\vec{a}).(\vec{\nu}+\vec{b})}. \tag{3.67}$$

As already elaborated upon earlier, in our case, although the geometry itself may be such that T^6 is factorizable as in eq. (3.1), the fluxes turned on, may violate in general the factorizable structure of the tori. Indeed, the general wavefunctions for bifundamentals given in terms of basis functions (3.24):

$$\begin{aligned} \psi^{\vec{j},\mathbf{N}}(\vec{z},\Omega) &= \mathcal{N} \cdot e^{\{i\pi[\mathbf{N}.\vec{z}].(\mathbf{N}.Im\Omega)^{-1}Im[\mathbf{N}.\vec{z}]\}} \cdot \vartheta \begin{bmatrix} \vec{j} \\ 0 \end{bmatrix} (\mathbf{N}.\vec{z}|\mathbf{N}.\Omega), \\ &= \mathcal{N} \cdot e^{i\pi[\mathbf{N}\cdot\vec{z}]\cdot(\mathrm{Im}\,\Omega)^{-1}\cdot\mathrm{Im}\,\vec{z}} \cdot \vartheta \begin{bmatrix} \vec{j} \\ 0 \end{bmatrix} (\mathbf{N}\cdot\vec{z}\,|\mathbf{N}\cdot\Omega)\,, \end{aligned} \tag{3.68}$$

with $\mathbf{N}$'s being the intersection matrices, depend on such fluxes explicitly in terms of its modular parameter argument: $\mathbf{N}\Omega$; this breaks in general the factorized structure, even if the complex structure Ω is diagonal. The explicit form of the normalization factor $\mathcal{N}$ appearing in eq. (3.68) is given eq. (3.30). One needs to obtain an overlap integral of three basis functions of the type (3.68), in order to generalize the results of Yukawa computations given in eqs. (3.45), (3.61) - (3.64).

3.5.1 Riemann theta function identity

We now generalize eq. (3.46) to the case of general Riemann theta functions given in eq. (3.67). Explicitly, we consider the LHS of our identity to be given by an expression:

$$\vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1|\mathbf{N_1}\cdot\Omega)\cdot\vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2|\mathbf{N_2}\cdot\Omega) \tag{3.69}$$

where Ω is an $n{\times}n$ complex matrix and $\mathbf{N_1}$, $\mathbf{N_2}$ are $n{\times}n$ integer-valued symmetric matrices satisfying the constraints (3.26). These constraints, in turn, follow from the convergence of theta series expansion, as well as from the holomorphicity of fluxes: for instance, eq. (2.31) when p_{xx} and p_{yy} components of fluxes are zero, with x^i, y^i, $(i = 1, 2, 3)$ denoting the coordinates of three T^2's in the decomposition (3.1) and (3.66). Generalization to the case when p_{xx} and p_{yy} flux components are also present is discussed later on in subsection 3.5.7, and is relevant for evaluating the Yukawa couplings in models with moduli stabilization, such as the one of [7].

Initially, we also restrict ourselves to the case when $\Omega = \tau I_n$ with I_n being a $n \times n$ identity matrix, implying that the geometric structure is factorized as in eq. (3.1). However, in Section 3.5.6, we generalize the results further to the case when Ω is an arbitrary matrix satisfying the $F_{(2,0)} = 0$ supersymmetry condition, as given in eqs. (2.26) and (2.27). Then, using the definition of Riemann ϑ-functions (3.67), the expression in eq. (3.69) can be expanded as:

$$\vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1|\mathbf{N_1}\tau)\cdot\vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2|\mathbf{N_2}\tau) = \sum_{\vec{l}_1,\vec{l}_2\in\mathbf{Z}^n} e^{\pi i(\vec{j}_1+\vec{l}_1)\cdot\mathbf{N_1}\tau\cdot(\vec{j}_1+\vec{l}_1)}e^{2\pi i(\vec{j}_1+\vec{l}_1)\cdot\vec{z}_1}\cdot e^{\pi i(\vec{j}_2+\vec{l}_2)\cdot\mathbf{N_2}\tau\cdot(\vec{j}_2+\vec{l}_2)}e^{2\pi i(\vec{j}_2+\vec{l}_2)\cdot\vec{z}_2}. \tag{3.70}$$

Now, by defining $2n$-dimensional vectors:

$$(\vec{\mathbf{j}}+\vec{\mathbf{l}}) = \begin{pmatrix} \vec{j}_1+\vec{l}_1 \\ \vec{j}_2+\vec{l}_2 \end{pmatrix},\ \ \vec{\mathbf{z}} = \begin{pmatrix} \vec{z}_1 \\ \vec{z}_2 \end{pmatrix}, \tag{3.71}$$

and the $2n \times 2n$ dimensional matrix:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{N_1}\tau & 0 \\ 0 & \mathbf{N_2}\tau \end{pmatrix}, \tag{3.72}$$

eq. (3.70) can be re-written as:

$$\vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z_1}|\mathbf{N_1}\tau) \cdot \vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z_2}|\mathbf{N_2}\tau) = \sum_{\vec{l} \in Z^{2n}} e^{\pi i (\vec{\mathbf{j}}+\vec{\mathbf{l}})^T \cdot \mathbf{Q} \cdot (\vec{\mathbf{j}}+\vec{\mathbf{l}})} e^{2\pi i (\vec{\mathbf{j}}+\vec{\mathbf{l}})^T \cdot \vec{\mathbf{z}}}. \tag{3.73}$$

Our aim in combining the terms into $2n$ dimensional vectors and matrices is to generalize the procedure outlined in [119] to our situation, namely when two theta functions appearing in the LHS of the identity (that we propose below) carry independent modular parameter matrices $\mathbf{N_1}\tau$ and $\mathbf{N_2}\tau$, which generally may not commute. Note that the results of [119] are insufficient to give such an identity as they involve theta functions whose modular parameter matrices are proportional to each other and therefore commute. In order to proceed, we note that using a transformation matrix:

$$T = \begin{pmatrix} 1 & 1 \\ \alpha\mathbf{N_1}^{-1} & -\alpha\mathbf{N_2}^{-1} \end{pmatrix}, \tag{3.74}$$

$$T^T = \begin{pmatrix} 1 & \mathbf{N_1}^{-1}\alpha^T \\ 1 & -\mathbf{N_2}^{-1}\alpha^T \end{pmatrix}, \tag{3.75}$$

and

$$T^{-1} = (\mathbf{N_1}^{-1} + \mathbf{N_2}^{-1})^{-1} \begin{pmatrix} \mathbf{N_2}^{-1} & \alpha^{-1} \\ \mathbf{N_1}^{-1} & -\alpha^{-1} \end{pmatrix}, \tag{3.76}$$

with α being an arbitrary matrix (to be determined below) and $\mathbf{N_1}, \mathbf{N_2}$ being real symmetric matrices, due to the condition (3.26) (for $\Omega = \tau I_n$), one obtains:

$$\mathbf{Q}' \equiv T \cdot \mathbf{Q} \cdot T^T = \begin{pmatrix} (\mathbf{N_1} + \mathbf{N_2})\tau & 0 \\ 0 & \alpha(\mathbf{N_1}^{-1} + \mathbf{N_2}^{-1})\tau\alpha^T \end{pmatrix}. \tag{3.77}$$

In the following we also make use of the identities:

$$(\mathbf{N_1}^{-1} + \mathbf{N_2}^{-1}) = \mathbf{N_1}^{-1}(\mathbf{N_1} + \mathbf{N_2})\mathbf{N_2}^{-1} = \mathbf{N_2}^{-1}(\mathbf{N_1} + \mathbf{N_2})\mathbf{N_1}^{-1} \tag{3.78}$$

and

$$(\mathbf{N_1}^{-1} + \mathbf{N_2}^{-1})^{-1} = \mathbf{N_1}(\mathbf{N_1} + \mathbf{N_2})^{-1}\mathbf{N_2} = \mathbf{N_2}(\mathbf{N_1} + \mathbf{N_2})^{-1}\mathbf{N_1} \tag{3.79}$$

in simplifying certain expressions.

The transformation matrix T defined above is used to transform the product of theta functions in the LHS of eq. (3.73), in terms of a finite sum over another product of theta's, now with modular parameter matrices: $(\mathbf{N_1}+\mathbf{N_2})\tau$ and $\alpha({\mathbf{N_1}}^{-1}+{\mathbf{N_2}}^{-1})\tau\alpha^T$. Explicitly, we can write the terms appearing in the exponents in the RHS of eq. (3.73) as:

$$(\vec{\mathbf{j}}+\vec{\mathbf{l}})^T\cdot\mathbf{Q}\cdot(\vec{\mathbf{j}}+\vec{\mathbf{l}}) = (\vec{\mathbf{j}}+\vec{\mathbf{l}})^T\cdot(T^{-1}T)\cdot\mathbf{Q}\cdot(T^T(T^{-1})^T)\cdot(\vec{\mathbf{j}}+\vec{\mathbf{l}}) \tag{3.80}$$

$$(\vec{\mathbf{j}}+\vec{\mathbf{l}})^T\cdot\vec{\mathbf{z}} = (\vec{\mathbf{j}}+\vec{\mathbf{l}})^T(T^{-1}T)\cdot\vec{\mathbf{z}}. \tag{3.81}$$

Then using:

$$T\cdot\vec{\mathbf{z}} = \begin{pmatrix} \vec{z_1}+\vec{z_2} \\ \alpha{\mathbf{N_1}}^{-1}\vec{z_1}-\alpha{\mathbf{N_2}}^{-1}\vec{z_2} \end{pmatrix}, \tag{3.82}$$

$$(\vec{\mathbf{j}}+\vec{\mathbf{l}})^T T^{-1} = \begin{pmatrix} (\vec{j_1}+\vec{l_1})({\mathbf{N_1}}^{-1}+\mathbf{N}_2^{-1})^{-1}\mathbf{N}_2^{-1}+(\vec{j_2}+\vec{l_2})(\mathbf{N}_1^{-1}+\mathbf{N}_2^{-1})^{-1}\mathbf{N}_1^{-1} \\ ((\vec{j_1}+\vec{l_1})-(\vec{j_2}+\vec{l_2}))(\mathbf{N}_1^{-1}+\mathbf{N}_2^{-1})^{-1}\alpha^{-1} \end{pmatrix}^T, \tag{3.83}$$

and

$$(T^{-1})^T(\vec{\mathbf{j}}+\vec{\mathbf{l}}) = \begin{pmatrix} \mathbf{N}_2^{-1}(\mathbf{N}_1^{-1}+\mathbf{N}_2^{-1})^{-1}(\vec{j_1}+\vec{l_1})+\mathbf{N}_1^{-1}(\mathbf{N}_1^{-1}+\mathbf{N}_2^{-1})^{-1}(\vec{j_2}+\vec{l_2}) \\ (\alpha^{-1})^T(\mathbf{N}_1^{-1}+\mathbf{N}_2^{-1})^{-1}[(\vec{j_1}+\vec{l_1})-(\vec{j_2}+\vec{l_2})] \end{pmatrix} \tag{3.84}$$

we can re-write eq. (3.70) as,

$$\begin{aligned}
&\vartheta\begin{bmatrix}\vec{j_1}\\0\end{bmatrix}(\vec{z_1}|N_1\tau)\cdot\vartheta\begin{bmatrix}\vec{j_2}\\0\end{bmatrix}(\vec{z_2}|N_2\tau) = \\
&\sum_{\vec{l_1},\vec{l_2}\in\mathbf{Z}^n} e^{\pi i\left[\{((\vec{j_1}+\vec{l_1})\mathbf{N_1}+(\vec{j_2}+\vec{l_2})\mathbf{N_2})(\mathbf{N}_1+\mathbf{N}_2)^{-1}\}\cdot(\mathbf{N}_1+\mathbf{N}_2)\tau\cdot\{(\mathbf{N}_1+\mathbf{N}_2)^{-1}(\mathbf{N}_1(\vec{j_1}+\vec{l_1})+\mathbf{N}_2(\vec{j_2}+\vec{l_2}))\}\right]} \\
&\quad\times e^{2\pi i\{[((\vec{j_1}+\vec{l_1})\mathbf{N}_1+(\vec{j_2}+\vec{l_2})\mathbf{N}_2)(\mathbf{N_1}+\mathbf{N_2})^{-\mathbf{1}}]\cdot[\tilde{\mathbf{z_1}}+\tilde{\mathbf{z_2}}]\}}\times \\
&e^{\pi i\{[((\vec{j_1}-\vec{j_2})+(\vec{l_1}-\vec{l_2}))\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1}]\cdot[\alpha({\mathbf{N}_1}^{-1}(\mathbf{N}_1+\mathbf{N}_2){\mathbf{N}_2}^{-1})\tau\alpha^T]\cdot[(\alpha^{-1})^T\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1((\vec{j_1}-\vec{j_2})+(\vec{l_1}-\vec{l_2}))]\}} \\
&\quad\times e^{2\pi i\{[((\vec{j_1}-\vec{j_2})+(\vec{l_1}-\vec{l_2}))\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1}]\cdot[\alpha N_1^{-1}\vec{z_1}-\alpha N_2^{-1}\vec{z_2}]\}}.
\end{aligned} \tag{3.85}$$

Now, to reexpress the above series expansion in terms of a sum over theta functions with modular parameter matrices: $\mathbf{N_1}+\mathbf{N_2}$ and $\alpha({\mathbf{N_1}}^{-1}+{\mathbf{N_2}}^{-1})\alpha^T$, we rearrange the series in eq. (3.85) in terms of new summation variables $\vec{l_3},\vec{l_4},\vec{m}$, whose values and ranges will be assigned later. In the course of going from eq. (3.85) to (3.87) below, however, one needs to make sure that such redefined variables are integers. This requirement constrains the matrix α whose 'minimal' solution will be taken to be

$$\alpha = (\det \mathbf{N_1} \det \mathbf{N_2}) I. \tag{3.86}$$

We will later on discuss also the possibility of choosing other forms of α and show that such choices lead to the cyclicity of the superpotential coefficients, as in eqs. (3.64), (3.65). Using eq. (3.86), the RHS of eq. (3.85) takes the form:

$$\begin{aligned}
&\sum_{\vec{l_3},\vec{l_4}\in \mathbf{Z}^n} \sum_{\vec{m}} e^{\pi i[(\vec{j_1}\mathbf{N}_1+\vec{j_2}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot(\mathbf{N}_1+\mathbf{N}_2)\tau\cdot[(\mathbf{N}_1+\mathbf{N}_2)^{-1}(\mathbf{N}_1\vec{j_1}+\mathbf{N}_2\vec{j_2}+\mathbf{N}_1\vec{m})+\vec{l_3}]}\\
&\cdot e^{2\pi i[(\vec{j_1}\mathbf{N}_1+\vec{j_2}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot[\vec{z_1}+\vec{z_2}]}\times\\
&e^{\pi i[(\vec{j_1}-\vec{j_2}+\vec{m})\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det \mathbf{N_1}\det \mathbf{N_2}}+\vec{l_4}]\cdot[(\det \mathbf{N_1}\det \mathbf{N_2})^2\mathbf{N}_1^{-1}(\mathbf{N}_1+\mathbf{N}_2)\mathbf{N}_2^{-1}]\tau\cdot[\frac{\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1}{\det \mathbf{N_1}\det \mathbf{N_2}}(\vec{j_1}-\vec{j_2}+\vec{m})+\vec{l_4}]}\\
&\cdot e^{2\pi i[(\vec{j_1}-\vec{j_2}+\vec{m})\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det \mathbf{N_1}\det \mathbf{N_2}}+\vec{l_4}]\cdot\det \mathbf{N_1}\det \mathbf{N_2}[\mathbf{N}_1^{-1}\vec{z_1}-\mathbf{N}_2^{-1}\vec{z_2}]}.
\end{aligned} \tag{3.87}$$

This series can now be reexpressed in terms of a finite sum over product of generalized theta functions given in eq. (3.67), leading to a generalization of the identity (3.46) to:

$$\begin{aligned}
&\vartheta\begin{bmatrix}\vec{j_1}\\ 0\end{bmatrix}(\vec{z_1}|\mathbf{N_1}\tau)\cdot\vartheta\begin{bmatrix}\vec{j_2}\\ 0\end{bmatrix}(\vec{z_2}|\mathbf{N_2}\tau)=\\
&\sum_{\vec{m}}\vartheta\begin{bmatrix}(\vec{j_1}\mathbf{N_1}+\vec{j_2}\mathbf{N_2}+\vec{m}.\mathbf{N_1})(\mathbf{N_1}+\mathbf{N_2})^{-1}\\ 0\end{bmatrix}(\vec{z_1}+\vec{z_2}|(\mathbf{N_1}+\mathbf{N_2})\tau)\times\\
&\vartheta\begin{bmatrix}[(\vec{j_1}-\vec{j_2})+\vec{m}]\frac{\mathbf{N_1}(\mathbf{N_1}+\mathbf{N_2})^{-1}\mathbf{N_2}}{\det\mathbf{N_1}\det\mathbf{N_2}}\\ 0\end{bmatrix}\\
&\quad((\det\mathbf{N_1}\det\mathbf{N_2})(\mathbf{N_1}^{-1}\vec{z_1}-\mathbf{N_2}^{-1}\vec{z_2})|(\det\mathbf{N_1}\det\mathbf{N_2})^2(\mathbf{N_1}^{-1}(\mathbf{N_1}+\mathbf{N_2})\mathbf{N_2}^{-1})\tau),
\end{aligned} \tag{3.88}$$

where $\vec{m}=\sum_i m_i\vec{e_i}$ are all vectors generated by the basis vectors $\vec{e_i}$:

$$\begin{pmatrix}1\\0\\.\\.\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\.\\.\\0\end{pmatrix}\quad etc., \tag{3.89}$$

and lied within the unit-cell defined by the new basis vectors:

$$\vec{e'} = \vec{e}(\det \mathbf{N_1} \det \mathbf{N_2})(\mathbf{N_1}^{-1}(\mathbf{N_1} + \mathbf{N_2})\mathbf{N_2}^{-1}). \tag{3.90}$$

The above identity already assumes the form $\Omega = \tau I_n$ for the complex structure of T^{2n}. As mentioned already, in subsection 3.5.6 below, we make further generalization to include arbitrary complex structure Ω as well. Also, note that, due to the identities (3.78) and (3.79), the theta functions appearing in the RHS of eq. (3.88) satisfy the constraint (3.29) with respect to their own arguments.

3.5.2 Proof of the identity

We now show the equality of the series expansions (3.85) and (3.87) to establish the identity eq. (3.88). We also show that matrix α needs to be chosen as in eq. (3.86) for showing the equality of eqs. (3.85) and (3.87) for the case when $\det \mathbf{N_1}$ and $\det \mathbf{N_2}$ are relatively prime. In other cases α can be chosen as the least common multiple of $\det \mathbf{N_1}$ and $\det \mathbf{N_2}$. Here we assume them to be relatively prime, while the remaining cases can be worked out in a similar fashion.

We now follow an exercise similar to the one in Section 3.4.3, to show that series in eqs. (3.85) and (3.87) precisely match with $\vec{m}$ restricted to be an integer, provided α is given by eq. (3.86).

1. When $\vec{l_1} = \vec{l_2}$ in eq. (3.85), we have:

$$(\vec{l_1}\mathbf{N_1} + \vec{l_2}\mathbf{N_2})(\mathbf{N_1} + \mathbf{N_2})^{-1} = \vec{l_2} \tag{3.91}$$

   and

$$(\vec{l_1} - \vec{l_2})\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1} = 0 \tag{3.92}$$

   These terms are exactly same if we consider the series given in eq. (3.87) for the values $\vec{l_3}(\equiv \vec{l_2}), \vec{l_4} = 0$ and $\vec{m} = 0$, irrespective of the choice for the matrix α.

2. In order to see the restriction on the matrix α, one needs to understand how the nonzero integers $\vec{l_4} \neq 0$ in eq. (3.87) are generated from the terms in eq. (3.85). In other words, one needs to make sure that

$$(\vec{l_1} - \vec{l_2})\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1} \equiv \vec{l_4} \tag{3.93}$$

is an integer. This in turn is possible only if $\vec{l_1}$ is of the form:

$$\vec{l_1} = \vec{l_2} + \vec{l_4}\alpha \mathbf{N}_2^{-1}(\mathbf{N}_1 + \mathbf{N}_2)\mathbf{N}_1^{-1}. \tag{3.94}$$

However, since $\vec{l_4}$, $\mathbf{N}_1$, $\mathbf{N}_2$, take integral values, the RHS in eq. (3.94) is an integer only if $\alpha(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})$ is an integer. In other words, for $\det \mathbf{N_1}$ and $\det \mathbf{N_2}$ relatively prime, α needs to be of the form:

$$\alpha = (\det \mathbf{N_1} \det \mathbf{N_2})P. \tag{3.95}$$

with P being an arbitrary invertible integer matrix. 'Minimal' choice also demands $\det P = 1$, otherwise $\vec{l_4}$ will not span over *all* integers. Then, since P is invertible, it is fixed to be the identity matrix. We have therefore established the restriction on α as in eq. (3.86). At the same time, we have also proved that the series in eqs. (3.85) and (3.87) precisely match for $\vec{m} = 0$ provided $\vec{l_2} + \det \mathbf{N_1} \det \mathbf{N_2}\vec{l_4}\mathbf{N}_2^{-1}$ is identified with $\vec{l_3}$ in eq.(3.87). Note that $(\det \mathbf{N_1} \det \mathbf{N_2})\mathbf{N}_2^{-1}$ is also integer valued and ensures that such an identification with $\vec{l_3}$ holds.

3. On the other hand, When $\vec{l_1} = \vec{l_2} + \vec{m}$ in eq. (3.85), we end up with terms like:

$$(\vec{l_1}\mathbf{N_1} + \vec{l_2}\mathbf{N_2})(\mathbf{N}_1 + \mathbf{N}_2)^{-1} = \vec{l_2} + \vec{m}.\mathbf{N_1}(\mathbf{N_1} + \mathbf{N_2})^{-1} \tag{3.96}$$

and

$$(\vec{l_1} - \vec{l_2})\frac{\mathbf{N_1}(\mathbf{N_1} + \mathbf{N_2})^{-1}\mathbf{N_2}}{\det \mathbf{N_1} \det \mathbf{N_2}} = \vec{m}\frac{\mathbf{N_1}(\mathbf{N_1} + \mathbf{N_2})^{-1}\mathbf{N_2}}{\det \mathbf{N_1} \det \mathbf{N_2}} \tag{3.97}$$

These terms can also be obtained in the series (3.87), for the following values of the variables: $\vec{l_3}(\equiv \vec{l_2})$, $\vec{l_4} = 0$, $\vec{m}$ arbitrary. However, as seen above in eqs. (3.93), (3.94), the sum over $\vec{m}$ is finite due to the fact that

$$\vec{l_1} - \vec{l_2} = \vec{m} = \vec{L} \det \mathbf{N_1} \det \mathbf{N_2}\mathbf{N}_2^{-1}(\mathbf{N}_1 + \mathbf{N}_2)\mathbf{N}_1^{-1}, \tag{3.98}$$

for $\vec{L}$ arbitrary integers, contributes to the values of $\vec{l_4}$ in the RHS of eq. (3.87) by an amount $\vec{L}$, while setting $\vec{m}$ to zero, $\vec{l_3}$ is identified with $\vec{l_2} + \det \mathbf{N_1} \det \mathbf{N_2}\vec{L}\mathbf{N}_2^{-1}$. In other words, we have shown that the sum over $\vec{m}$ in (3.87) is over all integrally defined vectors in the unit cell generated by the basis elements:

$$\vec{e'} = \vec{e} \det \mathbf{N_1} \det \mathbf{N_2}\mathbf{N}_2^{-1}(\mathbf{N}_1 + \mathbf{N}_2)\mathbf{N}_1^{-1} \tag{3.99}$$

with $\vec{e}$ being the elements of the canonical basis (3.89).

We have therefore proved that identity eq. (3.88) holds by explicitly showing a one to one correspondence between the series in eqs. (3.85) and (3.87).

3.5.3 Yukawa expressions for oblique fluxes

We now use the wavefunctions given in eqs. (3.68) and (3.67), to obtain the expression of Yukawa interactions when oblique fluxes, specified by intersection matrices

$$\mathbf{N_1} = F_a - F_b,\ \mathbf{N_2} = F_c - F_a,\ \mathbf{N_3} = F_c - F_b. \tag{3.100}$$

are turned on along branes a, b and c. $\mathbf{N_1}$, $\mathbf{N_2}$ and $\mathbf{N_3}$ are all real symmetric matrices (in the absence of components p_{xx}, p_{yy}) and in addition the complex structure matrix is chosen to be proportional to the identity: τI_n, with τ complex. We then have:

$$\begin{aligned}\psi^{\vec{i},\mathbf{N_1}}(\vec{z},\mathbf{\Omega}=\tau I_n)\cdot\psi^{\vec{j},\mathbf{N_2}}(\vec{z},\mathbf{\Omega}=\tau I_n) &= \left(2^{\frac{n}{2}}\right)\left(Vol(T^{2n})\right)^{-1}\left(|\det\mathbf{N_1}|.|\det\mathbf{N_2}|(Im\tau)^6\right)^{\frac{1}{4}}\\ &\times e^{i\pi\mathbf{N_3}.\vec{z}\mathrm{Im}\,\vec{z}/\mathrm{Im}\,\tau}\vartheta\begin{bmatrix}\vec{i}\\0\end{bmatrix}(\mathbf{N_1}\cdot\vec{z}|\mathbf{N_1}\cdot\tau)\cdot\vartheta\begin{bmatrix}\vec{j}\\0\end{bmatrix}(\mathbf{N_2}\cdot\vec{z}|\mathbf{N_2}\cdot\tau).\end{aligned} \tag{3.101}$$

Using the Riemann theta identity derived earlier in eq. (3.88), eq. (3.101) can be rewritten as:

$$\begin{aligned}&\psi^{\vec{i},\mathbf{N_1}}(\vec{z})\cdot\psi^{\vec{j},\mathbf{N_2}}(\vec{z}) = \sum_{\vec{m}}\left(2^{\frac{n}{2}}\right)^{\frac{1}{2}}\left(Vol(T^{2n})\right)^{-\frac{1}{2}}\left[\frac{(|\det\mathbf{N_1}|.|\det\mathbf{N_2}|(Im\tau)^3)}{|\det\mathbf{N_3}|}\right]^{\frac{1}{4}}\times\\ &\psi^{(\vec{i}\mathbf{N_1}+\vec{j}\mathbf{N_2}+\vec{m}\mathbf{N_1}).\mathbf{N_3}^{-1},\mathbf{N_3}}(\vec{z})\cdot\vartheta\begin{bmatrix}[(\vec{i}-\vec{j})+\vec{m}]\frac{\mathbf{N_1}\mathbf{N_3}^{-1}\mathbf{N_2}}{\det\mathbf{N_1}\det\mathbf{N_2}}\\0\end{bmatrix}\\ &\qquad(0|(\det\mathbf{N_1}\det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1})\tau).\end{aligned} \tag{3.102}$$

Note that the integrality condition (3.29) is maintained by $\psi^{(\vec{i}\mathbf{N_1}+\vec{j}\mathbf{N_2}+\vec{m}\mathbf{N_1})\mathbf{N_3}^{-1},\mathbf{N_3}}(\vec{z})$ appearing in the RHS of the above equation, since the expression

$$\left[(\vec{i}\mathbf{N_1}+\vec{j}\mathbf{N_2}+\vec{m}\mathbf{N_1})\mathbf{N_3}^{-1}\right]\cdot\mathbf{N_3} \tag{3.103}$$

is always an integer. On the other hand, the sum $\vec{m}$ in eq. (3.102) is over the integers inside the cell generated by the lattice vectors in eq. (3.99) and total number of them is given by the volume of this compact space. The size of the cell, i.e., its volume matches with those in eq. (3.60) and (3.61) for the T^2 case which is just the number, $N_3 = I_{cb}$ in eq. (3.60), of chiral states for brane intersection bc. However, the situation is different for T^{2n}, $n > 1$. This becomes clear by observing that the size of the cell given in eq. (3.99) is bigger than the number of states ($\vec{k}$) in the intersection $\mathbf{N_3}$ between the branes b and c by a factor $det(\det \mathbf{N_1} \det \mathbf{N_2} \mathbf{N}_2^{-1} \mathbf{N}_1^{-1})$. This factor, on the other hand, for T^2 is unity. We therefore notice that the sum $\vec{m}$ is over many more terms than the actual number of states ($\vec{k}$) in the intersection $\mathbf{N_3}$ between the branes b and c.

The extra factor of terms appearing in eq. (3.102) can be explained by noticing that the sum over terms in eqs. (3.102) and (3.104) is over the states $\psi^{(\vec{i}\mathbf{N_1}+\vec{j}\mathbf{N_2}+\vec{m}\mathbf{N_1}).\mathbf{N_3}^{-1},\mathbf{N_3}}(\vec{z})$ that are inside the cell in eq. (3.99) and contribute to the Yukawa coupling by the orthogonality relation eq. (3.31). As any state (with more details given in the subsection-3.5.4) $\vec{k}$, satisfying integrality conditions such as (3.29) is defined only upto the integer lattice shifts, one therefore has appearance of the same states inside the volume of lattice (3.99), multiple times. In other words, for any given state, in the RHS of eqs. (3.102), all those integer vector ($\vec{m}$) shifts also contribute to the sum which satisfy the integrality condition for $\vec{m}\mathbf{N_1}\mathbf{N_3}^{-1}$ inside the cell (3.99). Explicit solution of this condition is presented later on in section 3.5.4 in eq. (3.110).

Then, as in the T^2 case, orthonormality of wavefunctions (3.31), implies that the Yukawa coupling, whose explicit form is given in section 3.5.4, can be 'formally' written in a form :

$$Y_{ijk} = g\sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(Vol(T^{2n})\right)^{-\frac{1}{2}} \left[\frac{(|\det \mathbf{N_1}|.|\det \mathbf{N_2}|(Im\tau)^3)}{|\det \mathbf{N_3}|}\right]^{\frac{1}{4}} \times \sum_{\vec{m}\in\{\vec{e'}\}} \delta_{\vec{k},\mathbf{N_3}^{-1}(\mathbf{N_1}\vec{i}+\mathbf{N_2}\vec{j}+\mathbf{N_1}\vec{m})}$$
$$\times \vartheta \begin{bmatrix} [(\vec{i}-\vec{j})+\vec{m}]\frac{\mathbf{N_1}\mathbf{N_3}^{-1}\mathbf{N_2}}{\det \mathbf{N_1} \det \mathbf{N_2}} \\ 0 \end{bmatrix} (0|(\det \mathbf{N_1} \det \mathbf{N_2})^2 (\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1})\tau), \qquad (3.104)$$

where by the summation index $\vec{m} \in \{\vec{e'}\}$, one means to sum over all integer points inside the lattice generated by $\vec{e'_1}, \vec{e'_2} \cdots \vec{e'_n}$ in eq. (3.99) and the Kronecker delta is to identify all the states $\vec{k}$ upto integer shifts.

The above expression reduces in the case of T^2 flux compactification to eq. (3.63), since the Kronecker delta constraint has a unique solution in such a situation. To compare the two expressions, note that the indices i, j, k in the factorized case are scaled with respect to the one of general tori, by the factors $\frac{1}{N_1}$, $\frac{1}{N_2}$ and $\frac{1}{N_3}$, respectively. Then, the Kronecker delta constraint in eq. (3.104) precisely matches with the one in eq. (3.62). In the case of general tori, however, the constraint implies that the interaction terms involve the states which satisfy the equation

$$\mathbf{N_3}\vec{k} = (\mathbf{N_1}\vec{i} + \mathbf{N_2}\vec{j} + \mathbf{N_1}\vec{m}) \qquad (3.105)$$

among the vectors $\mathbf{N_1}\vec{i}$, $\mathbf{N_2}\vec{j}$, $\mathbf{N_3}\vec{k}$ for $\vec{m}$ integers inside the unit cell given in eq. (3.90) and corresponding states $\vec{k}$ are only defined upto integer lattice shifts. We now find all such solutions of the lattice shifts in the next subsection and present the explicit answer for the Yukawa coupling for general tori.

3.5.4 Explict Yukawa coupling expressions

In this subsection we now present the set of terms that contribute to eqs. (3.102) and (3.104). In order to clarify the situation we analyze the correspondence between the chiral multiplet families of states such as the ones appearing in eq. (3.103) and the fluxes along the branes. Our discussion is restricted to $\mathbf{N}$ being real symmetric matrices, due to the imposition of the Riemann conditions (3.26) for the special complex structure $\Omega = \tau I_n$ under discussion.

For a given pair of brane-stacks with intersection matrix $\mathbf{N}$, the condition eq. (3.29) that a state $\hat{i}$ needs to satisfy is $N.\hat{i} = integer$. The solution of this condition is: $\hat{i} = \mathbf{N}^{-1}\vec{e}$, with $\vec{e}$ being the integer basis vectors in an n-dimensional space as given in eq. (3.89). The states are therefore generated by the set of n vectors: $\hat{i}_i = \vec{e_i}\mathbf{N}^{-1}$, with subscript $i = 1, 2 \cdots n$ and are $det(\mathbf{N})$ in number, namely those which are inside the cell generated by $\vec{e_i}$'s. Here and in following we also keep in mind that all the chiral multiplet states that we are discussing, are defined only upto the shift by integer lattice vectors $\vec{e_i}$'s.

To give an example: for $n = 2$ (corresponding to T^4), with

$$\mathbf{N} = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}, \tag{3.106}$$

we have the basis vectors for generating the states:

$$\hat{i}_1 = \frac{1}{(\alpha\beta - \gamma^2)} \begin{pmatrix} \beta \\ -\gamma \end{pmatrix}, \quad \hat{i}_2 = \frac{1}{(\alpha\beta - \gamma^2)} \begin{pmatrix} -\gamma \\ \alpha \end{pmatrix}. \tag{3.107}$$

To obtain the degeneracy count, we note that for the above example we have:

$$\begin{aligned} \vec{e_1} &= \alpha\vec{i_1} + \gamma\vec{i_2}, \\ \vec{e_2} &= \gamma\vec{i_1} + \beta\vec{i_2}. \end{aligned} \tag{3.108}$$

The number of independent states inside the cell with lattice vectors $\vec{e_1}$ and $\vec{e_2}$ is then the determinant of the above transformation which is $det\mathbf{N}$. A generic state appearing in eq.

(3.105) then has a form:

$$\vec{i} = m_1\vec{i_1} + m_2\vec{i_2}, \;\; \vec{j} = n_1\vec{j_1} + n_2\vec{j_2}, \;\; \vec{k} = p_1\vec{k_1} + p_2\vec{k_2}. \tag{3.109}$$

with $\vec{j_i}$, $\vec{k_i}$ defined in a similar way as in eq. (3.107) with respect to the corresponding intersection matrices. Also, integers m_i, n_i, p_i label the states of a chiral family in a given brane stack.

We now go on to give explicit solution for the vector $\vec{m}$ that contribute to the sum of terms in Yukawa coupling expressions (3.102) and (3.104), namely those inside the cell defined in eq. (3.99). The size of the cell, namely the number of states that it contains is equal to $det(\det \mathbf{N_1} \det \mathbf{N_2} \mathrm{N}_2^{-1}(\mathrm{N}_1 + \mathrm{N}_2)\mathrm{N}_1^{-1})$, as stated earlier. In a situation with 2×2 matrices, for example, it is $det\mathbf{N_1}det\mathbf{N_2}det\mathbf{N_3}$. For illustration purposes we restrict ourselves to the discussion with 2×2 matrices. However, all the results we write below are valid for other situations as well.

Now, restricting to this 2×2 case for the simplicity of discussion, we write all possible solutions for $\vec{m}$ that provide integer solutions for $\vec{m}\mathbf{N_1}\mathbf{N_3}^{-1}$, as appearing in the definition of states in eqs. (3.102), (3.103), and show that they are $det\mathbf{N_1}det\mathbf{N_2}$ in number. So that the degeneracy of the state matches with $det\mathbf{N_1}det\mathbf{N_2}det\mathbf{N_3}$ given in the last paragraph. To compare, note that for a diagonal flux situation, as in section-3.4, we have $m = n_3$ as a single solution of an analogous condition $mn_1n_3^{-1} = integer$, corresponding to the state degeneracy which is n_3.

The integer solutions for $\vec{m}\mathbf{N_1}\mathbf{N_3}^{-1}$ are:

$$\vec{m} = \vec{p}det\mathbf{N_1}\mathbf{N_3}\mathbf{N_1}^{-1} + \vec{\tilde{p}}det\mathbf{N_2}\mathbf{N_3}\mathbf{N_2}^{-1}, \tag{3.110}$$

where $\vec{p}$ is all integer vectors within a cell generated by $\vec{e}det\mathbf{N_2}\mathbf{N_2}^{-1}$ and $\vec{\tilde{p}}$ is all integer vectors within a cell generated by $\vec{e}det\mathbf{N_1}\mathbf{N_1}^{-1}$. It is easy to see that $\vec{m}$ satisfies $\vec{m}\mathbf{N_1}\mathbf{N_3}^{-1} = integer$ (by making use of $\mathbf{N_1} = \mathbf{N_3} - \mathbf{N_2}$). Together, for every solution of the first term in $\vec{m}$ we have $det\mathbf{N_1}$ solution for the second term and this goes on for $det\mathbf{N_2}$ number of terms from the first term. So that total degeneracy of such $\vec{m}$ is $det\mathbf{N_1}det\mathbf{N_2}$, as stated earlier.

About the states: $\vec{m}$ given in eq. (3.110) defines a periodic set, in the same way as for the T^2 case $m = n_3$ defines the periodic set of states in the RHS of eqs. (3.61) and (3.62). There the states are explicitly given as $k = (0), (n_1/n_3), (2n_1/n_3), \cdots [(n_3 - 1)n_1/n_3]$ with a periodicity n_3 for this series. Various states inside the cell (3.99) can also be found using eq. (3.105) and making use of the condition: $\mathbf{N_1} = \mathbf{N_3} - \mathbf{N_2}$ as: (also the fact that any state is defined upto integer vectors). The states are:

$$\vec{k} = \vec{p} det \mathbf{N_1} \mathbf{N_3}^{-1} + \vec{\tilde{p}} det \mathbf{N_2} \mathbf{N_3}^{-1} \ \ etc. \tag{3.111}$$

and the state degeneracy is $det\mathbf{N_1} det\mathbf{N_2} det\mathbf{N_3}$.

The Yukawa coupling can now be written in an explicit form given by a sum of $det\mathbf{N_1} det\mathbf{N_2}$ number of terms, which can be read off from eq. (3.102) directly, with $\vec{m}$ replaced by

$$\vec{\tilde{m}} + \vec{p} det \mathbf{N_1} \mathbf{N_3} \mathbf{N_1}^{-1} + \vec{\tilde{p}} det \mathbf{N_2} \mathbf{N_3} \mathbf{N_2}^{-1} \tag{3.112}$$

and now such $\vec{\tilde{m}}$ are the unique solutions of eq. (3.105) where all other solutions defined upto the shifts in $\vec{\tilde{m}}$ by $\vec{p} det N_1 N_3 N_1^{-1} + \vec{\tilde{p}} det N_2 N_3 N_2^{-1}$ have been identified.

Eq. (3.104) now reads as:

$$\begin{aligned} Y_{ijk} &= g\sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(Vol(T^{2n})\right)^{-\frac{1}{2}} \left[\frac{(|\det \mathbf{N_1}|.|\det \mathbf{N_2}|(Im\tau)^3)}{|\det \mathbf{N_3}|}\right]^{\frac{1}{4}} \times \sum_{\vec{p},\vec{\tilde{p}}} \\ &\times \vartheta \left[\begin{array}{c} [\{(\vec{i} - \vec{j}) + (\vec{k}\mathbf{N_3} - \vec{i}\mathbf{N_1} - \vec{j}\mathbf{N_2})\mathbf{N_1}^{-1}\} \frac{\mathbf{N_1}\mathbf{N_3}^{-1}\mathbf{N_2}}{\det \mathbf{N_1} \det \mathbf{N_2}} + (\vec{p}\frac{\mathbf{N_2}}{det\mathbf{N_2}} + \vec{\tilde{p}}\frac{\mathbf{N_1}}{det\mathbf{N_1}})] \\ 0 \end{array} \right] \\ &\qquad (0|(\det \mathbf{N_1} \det \mathbf{N_2})^2 (\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau), \end{aligned} \tag{3.113}$$

or equivalently:

$$\begin{aligned} Y_{ijk} &= g\sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(Vol(T^{2n})\right)^{-\frac{1}{2}} \left[\frac{(|\det \mathbf{N_1}|.|\det \mathbf{N_2}|(Im\tau)^3)}{|\det \mathbf{N_3}|}\right]^{\frac{1}{4}} \times \sum_{\vec{p},\vec{\tilde{p}}} \\ &\times \vartheta \left[\begin{array}{c} [(-\vec{j} + \vec{k}) \frac{\mathbf{N_2}}{\det \mathbf{N_1} \det \mathbf{N_2}} + (\vec{p}\frac{\mathbf{N_2}}{det\mathbf{N_2}} + \vec{\tilde{p}}\frac{\mathbf{N_1}}{det\mathbf{N_1}})] \\ 0 \end{array} \right] \\ &\qquad (0|(\det \mathbf{N_1} \det \mathbf{N_2})^2 (\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau). \end{aligned} \tag{3.114}$$

Note that the sum over $\vec{m}$ is now broken into sum over $\vec{p}$ and $\vec{\tilde{p}}$. We end this discussion by reminding ourselves once again that $\vec{p}$ runs over all the states inside the cell generated by $\vec{e_1} det\mathbf{N_2}\mathbf{N_2}^{-1}$ and $\vec{e_2} det\mathbf{N_2}\mathbf{N_2}^{-1}$. Similarly $\vec{\tilde{p}}$ runs over all the states inside the cell generated by $\vec{e_1} det\mathbf{N_1}\mathbf{N_1}^{-1}$ and $\vec{e_2} det\mathbf{N_1}\mathbf{N_1}^{-1}$.

We now present two explicit examples, one for the oblique situation and the other for the commuting diagonal fluxes. We show that our answer for the diagonal flux is identical to the one for the diagonal yukawa coupling expression given in [6] for T^{2n}. In fact this holds for any set of fluxes with $\mathbf{N_1}$, $\mathbf{N_2}$, $\mathbf{N_3}$ diagonal. On the other hand, we also show that the set of terms given above in eqs. (3.113) and (3.114) can also be summed up in a number of cases, for the oblique cases as well.

Example : Oblique flux

For the oblique case, by taking two noncommuting matrices:

$$\mathbf{N_1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{N_2} = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}, \tag{3.115}$$

we have:

$$(det\mathbf{N_1})\mathbf{N_1}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (det\mathbf{N_2})\mathbf{N_2}^{-1} = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}. \tag{3.116}$$

The set of integer points inside the cell generated by $\vec{e_1} det\mathbf{N_2}\mathbf{N_2}^{-1} = (2,0)$ and $\vec{e_2} det\mathbf{N_2}\mathbf{N_2}^{-1} = (0,1)$, are: $(0,0)$ and $(1,0)$, as $det(det\mathbf{N_2}\mathbf{N_2}^{-1}) = 2$. The set of integer points inside the cell generated by $\vec{e_1} det\mathbf{N_1}\mathbf{N_1}^{-1} = (2,-1)$ and $\vec{e_2} det\mathbf{N_1}\mathbf{N_1}^{-1} = (-1,2)$, are : $(0,0)$, $(1,0)$ and $(0,1)$, as $det(det\mathbf{N_1}\mathbf{N_1}^{-1}) = 3$.[9]

Now, to illustrate our method, we concentrate on finding a particular Yukawa interaction among states: $\vec{i} = \vec{j} = \vec{k} = (0,0)$. This particular Yukawa now has the form, making use of Eq. (3.113) as:

$$Y_{000} = g\sigma_{000}\left(2^{\frac{n}{2}}\right)^{\frac{1}{2}}\left(Vol(T^{2n})\right)^{-\frac{1}{2}}\left[\frac{(|\det\mathbf{N_1}|.|\det\mathbf{N_2}|(Im\tau)^3)}{|\det\mathbf{N_3}|}\right]^{\frac{1}{4}}\times\sum_{\vec{p},\vec{\tilde{p}}}$$
$$\vartheta\begin{bmatrix} [(\vec{p}\frac{\mathbf{N_2}}{det\mathbf{N_2}} + \vec{\tilde{p}}\frac{\mathbf{N_1}}{det\mathbf{N_1}})] \\ 0 \end{bmatrix}(0|(\det\mathbf{N_1}\det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau)),$$

To see what terms in $\vec{p}$ and $\vec{\tilde{p}}$ dependent arguements appear in the sum, we write down all the possibilities that arise from the combinations:

$$(\vec{p}\frac{\mathbf{N_2}}{det\mathbf{N_2}} + \vec{\tilde{p}}\frac{\mathbf{N_1}}{det\mathbf{N_1}}) = \vec{p}\begin{pmatrix} \frac{1}{2} & \\ & 1 \end{pmatrix} + \vec{\tilde{p}}\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{3.117}$$

with $\vec{p} = (0,0), (1,0)$ and $\vec{\tilde{p}} = (0,0), (0,1), (1,0)$. All the six possibilities then imply that

[9] Another example with mixed eigenvalues for the matrix $\mathbf{N_1}$ can be constructed by exchanging the off-diagonal and diagonal entries in eq. (3.115) for $\mathbf{N_1}$. Such an example will be relevant for the situtation discussed in later sections where intersection matrices with both positive and negative eigenvalues are discussed.

in Theta function we get the following explicit sum:

$$\Bigg(\vartheta\begin{bmatrix}[(0,0)]\\0\end{bmatrix}+\vartheta\begin{bmatrix}[(\frac{1}{2},0)]\\0\end{bmatrix}+\vartheta\begin{bmatrix}[(\frac{2}{3},\frac{1}{3})]\\0\end{bmatrix}+\vartheta\begin{bmatrix}[(\frac{1}{3},\frac{2}{3})]\\0\end{bmatrix}+\vartheta\begin{bmatrix}[(\frac{1}{6},\frac{1}{3})]\\0\end{bmatrix}+ \vartheta\begin{bmatrix}[(\frac{5}{6},\frac{2}{3})]\\0\end{bmatrix}\Bigg)(0|(\det\mathbf{N_1}\det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau)) \tag{3.118}$$

where a common modular parameter arguement of the all the six Theta terms have been written outside of the bracket for saving space. The integer sums of the six terms over integer $\vec{l}$ are of the forms:

$$\sum_{\vec{l}} e^{[\vec{l}+(q_1,q_2)](\det\mathbf{N_1}\det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau)[\vec{l}+(q_1,q_2)]} \tag{3.119}$$

with $\vec{l}+(q_1,q_2)$ given explicitly as:

$$\vec{l}+(0,0),\ \vec{l}+(\frac{1}{2},0),\ \vec{l}+(\frac{2}{3},\frac{1}{3}),\ \vec{l}+(\frac{1}{3},\frac{2}{3}),\ \vec{l}+(\frac{1}{6},\frac{1}{3}),\ \vec{l}+(\frac{5}{6},\frac{2}{3}), \tag{3.120}$$

for the six terms in eq. (3.118). It can also be seen that we can write them as:

$$\begin{pmatrix}l_1\\l_2\end{pmatrix}+\begin{pmatrix}\frac{m}{2}+\frac{2n}{3}\\\frac{n}{3}\end{pmatrix}\equiv\begin{pmatrix}l_1\\l_2\end{pmatrix}+\frac{1}{6}\begin{pmatrix}3&4\\0&2\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix} \tag{3.121}$$

with $m=0,1$ and $n=0,1,2$. Now, using the inverse of the matrix

$$P=\frac{1}{6}\begin{pmatrix}3&4\\0&2\end{pmatrix}, \tag{3.122}$$

appearing in eq. (3.121):

$$P^{-1}=\begin{pmatrix}2&-4\\0&3\end{pmatrix}, \tag{3.123}$$

we can write eq. (3.121) as:

$$\frac{1}{6}\begin{pmatrix}3&4\\0&2\end{pmatrix}\left[\begin{pmatrix}2l_1-4l_2\\3l_2\end{pmatrix}+\begin{pmatrix}m\\n\end{pmatrix}\right] \tag{3.124}$$

with $m=0,1$ and $n=0,1,2$.

It can now be seen that as l_1,l_2 vary over all integers, and $m=0,1$ and $n=0,1,2$, then the combination of terms in the big square bracket in eq. (3.124) also span over ALL integers. As a result we are able to take the factor of matrix P out by summing over all the six terms, while reducing the six terms in eq.(3.118) to one. The net result is then the arguement of theta function modifies by the factor:

$$(\det \mathbf{N_1} \det \mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau) \to P^T(\det \mathbf{N_1} \det \mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau)P \qquad (3.125)$$

and final answer for Yukawa coupling is:

$$\begin{aligned} Y_{000} = g\sigma_{000} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(Vol(T^{2n})\right)^{-\frac{1}{2}} \left[\frac{(|\det \mathbf{N_1}|.|\det \mathbf{N_2}|(Im\tau)^3)}{|\det \mathbf{N_3}|}\right]^{\frac{1}{4}} \times \\ \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0|P^T(\det \mathbf{N_1} \det \mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau)P). \end{aligned}$$

We can similarly take care of other nonzero values $\vec{i}, \vec{j}, \vec{k}$ etc. as well, but details are being left.

Example : Diagonal Flux

We take another example, now with diagonal fluxes :

$$\mathbf{N_1} = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}, \quad \mathbf{N_2} = \begin{pmatrix} 5 & \\ & 2 \end{pmatrix}. \qquad (3.126)$$

Then:

$$(det\mathbf{N_1})\mathbf{N_1}^{-1} = \begin{pmatrix} 3 & \\ & 2 \end{pmatrix}, \quad (det\mathbf{N_2})\mathbf{N_2}^{-1} = \begin{pmatrix} 2 & \\ & 5 \end{pmatrix}. \qquad (3.127)$$

Set of integer points inside the cell generated by $\vec{e_1} det\mathbf{N_2}\mathbf{N_2}^{-1} = (2,0)$ and $\vec{e_2} det\mathbf{N_2}\mathbf{N_2}^{-1} = (0,5)$, are: $(0,0)$, $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(1,0)$, $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, as $det(det\mathbf{N_2}\mathbf{N_2}^{-1}) = 10$. On the other hand, set of integer points inside the cell generated by $\vec{e_1} det\mathbf{N_1}\mathbf{N_1}^{-1} = (3,0)$ and $\vec{e_2} det\mathbf{N_1}\mathbf{N_1}^{-1} = (0,2)$, are: $(0,0)$, $(1,0)$ $(0,1)$, $(1,1)$, $(2,0)$, $(2,1)$, as $det(det\mathbf{N_1}\mathbf{N_1}^{-1}) = 6$.

We now have:

$$\vec{l} + (\vec{p}\frac{\mathbf{N_2}}{det\mathbf{N_2}} + \vec{\tilde{p}}\frac{\mathbf{N_1}}{det\mathbf{N_1}}) = \vec{l} + \vec{p}\begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{5} \end{pmatrix} + \vec{\tilde{p}}\begin{pmatrix} \frac{1}{3} & \\ & \frac{1}{2} \end{pmatrix}, \qquad (3.128)$$

which can also be written as:

$$\vec{l} + (\vec{p}\frac{\mathbf{N_2}}{det\mathbf{N_2}} + \vec{\tilde{p}}\frac{\mathbf{N_1}}{det\mathbf{N_1}}) = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{5} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & \\ & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix}, \tag{3.129}$$

with $p_1 = 0, 1$, $p_2 = 0, 1, 2, 3, 4$, $\tilde{p}_1 = 0, 1, 2$, $\tilde{p}_2 = 0, 1$.

By taking a factor of $\frac{\mathbf{N_1 N_2}}{det\mathbf{N_1} det\mathbf{N_2}}$ out, the above equation can also be rewritten as:

$$\frac{\mathbf{N_1 N_2}}{det\mathbf{N_1} det\mathbf{N_2}}[\vec{l} + (\vec{p}\frac{\mathbf{N_2}}{det\mathbf{N_2}} + \vec{\tilde{p}}\frac{\mathbf{N_1}}{det\mathbf{N_1}})] = \begin{pmatrix} \frac{1}{6} & \\ & \frac{1}{10} \end{pmatrix} \left[\begin{pmatrix} 6l_1 \\ 10l_2 \end{pmatrix} + \begin{pmatrix} 3p_1 \\ 2p_2 \end{pmatrix} + \begin{pmatrix} 2\tilde{p}_1 \\ 5\tilde{p}_2 \end{pmatrix} \right] \tag{3.130}$$

with $p_1 = 0, 1$, $p_2 = 0, 1, 2, 3, 4$, $\tilde{p}_1 = 0, 1, 2$, $\tilde{p}_2 = 0, 1$. It can again be checked explicitly that it leads to ALL integer variables inside the square bracket. The net result of summing over different terms in the diagonal case therefore is the appearance of the matrix outside the square bracket: $\frac{\mathbf{N_1 N_2}}{det\mathbf{N_1} det\mathbf{N_2}}$. When multiplying the modular parameter arguement as appearing in eq. (3.113), from both left and the right, this precisely reproduces a modified modular parameter which matches with the known diagonal flux solultion for Yukawa coupling in [6]. This holds for the diagonal flux in general, not restricted to the example above.

3.5.5 arbitrary-α

The results, obtained so far in this section, are derived for a particular choice of α given in the eq. (3.86). However, all the results can be re-derived for arbitrary α, appearing in eq. (3.74) etc.. For the factorized case, we saw in that the Yukawa coupling expression (3.63) can be recast into a symmetric form in eq. (3.64) (apart from the prefactor), where the arguments of the Jacobi theta functions are invariant under a cyclic change: $a \to b \to c$. This is due to the cyclic property of the superpotential coefficients obtained by a third derivative of the superpotential W_{ijk}. The prefactor does not obey in general this symmetry, since it depends on the wave function normalizations (Kähler metric). Here, we show a similar cyclic property in the non-factorized case, given above in the Yukawa coupling expression (3.114), by making different choices of the matrix α in eq. (3.86). Note that different choices of this matrix provide equivalent expressions for the wavefunctions, and in turn Yukawa couplings, since they are related though a change of variables inside the theta sum. The α matrix can be chosen appropriately so that the redefined variables in eqs. (3.94) and (3.98) are well defined integers. Below we present a few examples with different choices of α, to demonstrate the cyclicity mentioned above.

Eq. (3.87), for arbitrary α, can be written as:

$$\sum_{\vec{l_3},\vec{l_4}\in\mathbf{Z}^n}\sum_{\vec{m}}\Big(e^{\pi i[(\vec{j_1}\mathbf{N}_1+\vec{j_2}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot(\mathbf{N}_1+\mathbf{N}_2)\tau\cdot[(\mathbf{N}_1+\mathbf{N}_2)^{-1}(\mathbf{N}_1\vec{j_1}+\mathbf{N}_2\vec{j_2}+\mathbf{N}_1\vec{m})+\vec{l_3}]}$$
$$\times e^{2\pi i[(\vec{j_1}\mathbf{N}_1+\vec{j_2}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot[\vec{z_1}+\vec{z_2}]}\Big)$$
$$\times\Big(e^{\pi i[(\vec{j_1}-\vec{j_2}+\vec{m})\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1}+\vec{l_4}]\cdot[\alpha\mathbf{N}_1^{-1}(\mathbf{N}_1+\mathbf{N}_2)\mathbf{N}_2^{-1}\tau]\alpha^T\cdot[(\alpha^{-1})^T\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1(\vec{j_1}-\vec{j_2}+\vec{m})+\vec{l_4}]}$$
$$\times e^{2\pi i[(\vec{j_1}-\vec{j_2}+\vec{m})\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1}+\vec{l_4}]\cdot[\alpha\mathbf{N}_1^{-1}\vec{z_1}-\mathbf{N}_2^{-1}\vec{z_2}]}\Big), \tag{3.131}$$

provided $\vec{l_4}$, defined in eq. (3.93), is an integer vector, and so is $\vec{m}$ given in eq. (3.98). In addition the unit-cell, within which $\vec{m}$ lie, is now defined by the basis vectors :

$$\vec{e'} = \vec{e}\alpha(\mathbf{N_1}^{-1}(\mathbf{N_1}+\mathbf{N_2})\mathbf{N_2}^{-1}). \tag{3.132}$$

Moreover, eq. (3.88) takes the form:

$$\vartheta\begin{bmatrix}\vec{j_1}\\0\end{bmatrix}(\vec{z_1}|\mathbf{N_1}\tau)\cdot\vartheta\begin{bmatrix}\vec{j_2}\\0\end{bmatrix}(\vec{z_2}|\mathbf{N_2}\tau) =$$
$$\sum_{\vec{m}}\vartheta\begin{bmatrix}(\vec{j_1}\mathbf{N_1}+\vec{j_2}\mathbf{N_2}+\vec{m}.\mathbf{N_1})(\mathbf{N_1}+\mathbf{N_2})^{-1}\\0\end{bmatrix}(\vec{z_1}+\vec{z_2}|(\mathbf{N_1}+\mathbf{N_2})\tau)$$
$$\times\vartheta\begin{bmatrix}[(\vec{j_1}-\vec{j_2})+\vec{m}]\mathbf{N_1}(\mathbf{N_1}+\mathbf{N_2})^{-1}\mathbf{N_2}\alpha^{-1}\\0\end{bmatrix}$$
$$\big(\alpha(\mathbf{N_1}^{-1}\vec{z_1}-\mathbf{N_2}^{-1}\vec{z_2})|\alpha(\mathbf{N_1}^{-1}(\mathbf{N_1}+\mathbf{N_2})\mathbf{N_2}^{-1}\tau)\alpha^T\big). \tag{3.133}$$

It is then easy to see, all equations from (3.101) to (3.104) go through for arbitrary α, giving the following expression for the Yukawa couplings:

$$Y_{ijk} = g\sigma_{abc}\left(2^{\frac{n}{2}}\right)^{\frac{1}{2}}\left(Vol(T^{2n})\right)^{-\frac{1}{2}}\left[\frac{(|\det\mathbf{N_1}|.|\det\mathbf{N_2}|(Im\tau)^3)}{|\det\mathbf{N_3}|}\right]^{\frac{1}{4}}\times\sum_{\vec{m}}$$
$$\vartheta\begin{bmatrix}(-\vec{j}+\vec{k})\mathbf{N_2}\alpha^{-1}+\vec{m}\mathbf{N_1}\mathbf{N_3}^{-1}\mathbf{N_2}\alpha^{-1}\\0\end{bmatrix}(0|\alpha(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau)\alpha^T). \tag{3.134}$$

where the sum $\vec{m}$ is now over all the integer solutions of $\vec{m}\mathbf{N_1}\mathbf{N_3}^{-1}$ in the cell given in eq. (3.132). Explicit contributions to this sum, of course, will depend on the exact form of α. In subsection 3.5.4, we have presented the case of $\alpha = det\mathbf{N_1}det\mathbf{N_2}$.

We now study how the above expression (3.134) reduces for another choice of α, such as:

$$\alpha = \mathbf{N_3}^{-1}\mathbf{N_1}(\det\mathbf{N_2}.\det\mathbf{N_3}). \tag{3.135}$$

Note, for this choice of α, that the degeneracy of states in the cell given in eq. (3.132) is $det(det\mathbf{N_3}det\mathbf{N_2}\mathbf{N_2}^{-1})$. As a result, for the case of 2×2 matrices for example, one now expects the sum over $\vec{m}$ to run over $det\mathbf{N_2}det\mathbf{N_3}$ values. Explicit solutions are now given as:

$$\vec{m} = \vec{p}det\mathbf{N_2}\mathbf{N_3}\mathbf{N_2}^{-1} + \vec{\tilde{p}}det\mathbf{N_3}, \tag{3.136}$$

where $\vec{p}$ is all integer vectors within a cell generated by $\vec{e}det\mathbf{N_3}\mathbf{N_3}^{-1}$ and $\vec{\tilde{p}}$ is all integer vectors within a cell generated by $\vec{e}det\mathbf{N_2}\mathbf{N_2}^{-1}$. It is again easy to see that $\vec{m}$ satisfies $\vec{m}\mathbf{N_1}\mathbf{N_3}^{-1} = integer$ (by making use of $\mathbf{N_1} = \mathbf{N_3} - \mathbf{N_2}$).

The characteristic of the ϑ-function in eq. (3.134), becomes:

$$\begin{aligned}(-\vec{j}+\vec{k})\mathbf{N_2}\alpha^{-1} &= (-\vec{k}\mathbf{N_1} + \vec{i}\mathbf{N_1} + \vec{m}\mathbf{N_1})\frac{\mathbf{N_1}^{-1}\mathbf{N_3}}{(\det\mathbf{N_2}.\det\mathbf{N_3})}\\ &= \frac{(-\vec{k}+\vec{i})\mathbf{N_3}}{(\det\mathbf{N_2}.\det\mathbf{N_3})},\end{aligned} \tag{3.137}$$

where in the first equality we have made use of eq. (3.105). Also we have,

$$\begin{aligned}\alpha(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\tau)\alpha^T &= (\mathbf{N_3}^{-1}\mathbf{N_1})(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1})(\mathbf{N_1}\mathbf{N_3}^{-1})\tau(\det\mathbf{N_2}.\det\mathbf{N_3})^2\\ &= (\mathbf{N_2}^{-1}\mathbf{N_1}\mathbf{N_3}^{-1}\tau)(\det\mathbf{N_2}.\det\mathbf{N_3})^2.\end{aligned} \tag{3.138}$$

The Yukawa couplings then read (following the exercise performed in subsection 3.5.4):

$$\begin{aligned}Y_{ijk} = g\sigma_{abc}\left(2^{\frac{n}{2}}\right)^{\frac{1}{2}}\left(Vol(T^{2n})\right)^{-\frac{1}{2}}\left[\frac{(|\det\mathbf{N_1}|.|\det\mathbf{N_2}|(Im\tau)^3)}{|\det\mathbf{N_3}|}\right]^{\frac{1}{4}}\times\\ \sum_{\vec{p},\vec{\tilde{p}}}\vartheta\begin{bmatrix}(-\vec{k}+\vec{i})\frac{\mathbf{N_3}}{\det\mathbf{N_2}\det\mathbf{N_3}} + (\vec{p}\frac{\mathbf{N_3}}{det\mathbf{N_3}} + \vec{\tilde{p}}\frac{\mathbf{N_2}}{det\mathbf{N_2}})\\ 0\end{bmatrix}(0|(\det\mathbf{N_2}\det\mathbf{N_3})^2(\mathbf{N_2}^{-1}\mathbf{N_1}\mathbf{N_3}^{-1})\tau),\end{aligned} \tag{3.139}$$

where the summation over indices $\vec{p}$ and $\vec{\tilde{p}}$ is explained earlier after eq. (3.136). We can also explicitly obtain the sums, as done for various examples in the last subsection.

Now, a comparison of eqs. (3.114) and (3.139) shows a symmetry between the ϑ-function characteristics in these cases, including the summation variables $\vec{p}$ and $\vec{\tilde{p}}$. It is obvious that the replacement $\vec{i}\to\vec{j}, \vec{j}\to\vec{k}, \vec{k}\to\vec{i}$ and $\mathbf{N_1}\to\mathbf{N_2}, \mathbf{N_2}\to\mathbf{N_3}, \mathbf{N_3}\to\mathbf{N_1}$ in eq. (3.114) results eq. (3.139). We have thus established that just as in the factorized case, for oblique fluxes too, one can show the cyclicity property of the Yukawa superpotential coefficients, as naively expected.

3.5.6 General complex structure

In the previous subsections 3.5.1 - 3.5.3, we have confined ourselves to the complex structure matrix $\Omega = \tau I_n$ for a $2n$ dimensional torus. This implies the restriction to orthogonal tori, a solution which is already used in many phenomenologically interesting models. However, the results are easily generalized to complex structure with arbitrary Ω. More precisely, to write down an identity generalizing eq. (3.88) one starts with the product expression given in eq. (3.69) and rescales $\mathbf{N_1}$, $\mathbf{N_2}$ in eqs. (3.72) - (3.90) to $\mathbf{N_1}\Omega/\tau$, $\mathbf{N_2}\Omega/\tau$. At the same time, the matrix α in eq. (3.86) is also rescaled :

$$\alpha \to \tilde{\alpha} = \det \mathbf{N_1} \det \mathbf{N_2} \Omega/\tau = \frac{\alpha\Omega}{\tau}. \tag{3.140}$$

Moreover, one needs to take into account that in relations such as (3.75) earlier, we have made use of the property $\mathbf{N}^T = \mathbf{N}$, which is true for the complex structure of the form: τI_n. Replacements: $\mathbf{N}\tau \to \mathbf{N}\Omega$ are, however, to be done in the original expression.

Explicitly, under the changes mentioned, the transformation matrix T in eq. (3.74) remains unchanged, while its transposition in eq. (3.75) is now written as:

$$T^T = \begin{pmatrix} 1 & {\mathbf{N_1}^{-1}}^T \alpha^T \\ 1 & -{\mathbf{N_2}^{-1}}^T \alpha^T \end{pmatrix}. \tag{3.141}$$

Also, (3.76) is unchanged, whereas $\mathbf{Q'}$ in eq. (3.77) goes over to

$$\mathbf{Q'} \equiv T \cdot \mathbf{Q} \cdot T^T = \begin{pmatrix} (\mathbf{N_1} + \mathbf{N_2})\Omega & 0 \\ 0 & \alpha({\mathbf{N_1}}^{-1} + {\mathbf{N_2}}^{-1})\Omega^T \alpha^T \end{pmatrix}, \tag{3.142}$$

where we have made use of the fact that both $(\mathbf{N_1} + \mathbf{N_2})\Omega$ and $({\mathbf{N_1}}^{-1} + {\mathbf{N_2}}^{-1})\Omega^T$ are symmetric matrices, due to the condition (2.31), with $\mathbf{N}$ defined as $\mathbf{N}^T = p^a_{xy} - p^b_{xy}$. Then expressions (3.82) and (3.83) remain unchanged, while (3.84) is modified to:

$$(T^{-1})^T(\vec{\mathbf{j}} + \vec{\mathbf{l}}) = \begin{pmatrix} {\mathbf{N}_2^{-1}}^T ({\mathbf{N}_1^{-1}}^T + {\mathbf{N}_2^{-1}}^T)^{-1}(\vec{j_1} + \vec{l_1}) + {\mathbf{N}_1^{-1}}^T ({\mathbf{N}_1^{-1}}^T + {\mathbf{N}_2^{-1}}^T)^{-1}(\vec{j_2} + \vec{l_2}) \\ (\alpha^{-1})^T ({\mathbf{N}_1^{-1}}^T + {\mathbf{N}_2^{-1}}^T)^{-1}[(\vec{j_1} + \vec{l_1}) - (\vec{j_2} + \vec{l_2})] \end{pmatrix} \tag{3.143}$$

The identity (3.88) then takes the form:

$$\vartheta\begin{bmatrix}\vec{j_1}\\0\end{bmatrix}(\vec{z_1}|\mathbf{N_1}\Omega)\cdot\vartheta\begin{bmatrix}\vec{j_2}\\0\end{bmatrix}(\vec{z_2}|\mathbf{N_2}\Omega) = \tag{3.144}$$
$$\sum_{\vec{m}}\vartheta\begin{bmatrix}(\vec{j_1}\mathbf{N_1}+\vec{j_2}\mathbf{N_2}+\vec{m}.\mathbf{N_1})(\mathbf{N_1}+\mathbf{N_2})^{-1}\\0\end{bmatrix}(\vec{z_1}+\vec{z_2}|(\mathbf{N_1}+\mathbf{N_2})\Omega)\times$$
$$\vartheta\begin{bmatrix}[(\vec{j_1}-\vec{j_2})+\vec{m}]\frac{\mathbf{N_1}(\mathbf{N_1}+\mathbf{N_2})^{-1}\mathbf{N_2}}{\det\mathbf{N_1}\det\mathbf{N_2}}\\0\end{bmatrix}$$
$$((\det\mathbf{N_1}\det\mathbf{N_2})(\mathbf{N_1}^{-1}\vec{z_1}-\mathbf{N_2}^{-1}\vec{z_2})|(\det\mathbf{N_1}\det\mathbf{N_2})^2(\mathbf{N_1}^{-1}(\mathbf{N_1}+\mathbf{N_2})\mathbf{N_2}^{-1}\Omega^T)),$$

leading to the expression for the Yukawa interaction:

$$Y_{ijk} = \sigma_{abc}g\left(2^{\frac{n}{2}}\right)^{\frac{1}{2}}\left(Vol(T^{2n})\right)^{-\frac{1}{2}}\left[\frac{(|\det\mathbf{N_1}|.|\det\mathbf{N_2}||\det\Omega|)}{|\det\mathbf{N_3}|}\right]^{\frac{1}{4}}\times\sum_{\vec{p},\vec{\tilde{p}}}$$
$$\vartheta\begin{bmatrix}(-\vec{j}+\vec{k})\frac{\mathbf{N_2}}{\det\mathbf{N_1}\det\mathbf{N_2}}+(\vec{p}\frac{\mathbf{N_2}}{det\mathbf{N_2}}+\vec{\tilde{p}}\frac{\mathbf{N_1}}{det\mathbf{N_1}})\\0\end{bmatrix}(0|(\det\mathbf{N_1}\det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{N_3}\mathbf{N_2}^{-1}\Omega^T)). \tag{3.145}$$

We leave the rest of the details, which can be worked out easily.

3.5.7 Hermitian intersection matrices

In subsections 3.5.1, 3.5.2, 3.5.3, we have assumed that intersection matrices $\mathbf{N_1}, \mathbf{N_2}$ etc. are real symmetric. As explained, this restriction originates from the case when fluxes p_{xx}, p_{yy} are zero and the intersection matrix $\mathbf{N}$ is represented by the real matrix p_{xy}, which is symmetric whenever the complex structure is of the canonical form: $\Omega = iI_d$. Moreover, the Yukawa coupling expression was generalized nicely in the last subsection to the case of arbitrary complex structure, as well.

In this subsection we discuss the case when fluxes p_{xx} and p_{yy} are also present, in addition to those of the type p_{xy} and p_{yx}. Furthermore, all these fluxes are constrained by the conditions (2.26) and (2.27) giving a resulting $(1,1)$ - form flux which can be represented by the Hermitian matrix (2.28), (2.29). We explicitly present the case of $\Omega = iI_d$ solution (I_d : d-dimensional Identity matrix), which is particularly simple, since in this case due to constraints (D.1), the Hermitian flux has the simple final form of eq. (D.2). The generalization to arbitrary complex structure Ω can also be done, but is left as an exercise.

Wavefunctions on T^6, as given in eq. (3.68), satisfy the following field equations (3.22) and (3.23):

$$\bar{\partial}_i \chi_+^{ab} + (A^1 - A^2)_{\bar{z}_i} \chi_+^{ab} = 0, \quad (i = 1, 2, 3). \tag{3.146}$$

We now show that the solution for the above equation, together with proper periodicity requirements on T^6, is given by the basis elements:

$$\begin{aligned} \psi^{\vec{j},\mathbf{N}}(\vec{z}) &= \mathcal{N}_{\vec{j}} \cdot f(z, \bar{z}) \cdot \hat{\Theta}(z, \bar{z}) \\ &= \mathcal{N}_{\vec{j}} \cdot e^{i\pi[(\mathbf{N_R} - i\mathbf{N_I}) \cdot \vec{z}] \cdot \operatorname{Im} \vec{z}} \cdot \vartheta \begin{bmatrix} \vec{j} \\ 0 \end{bmatrix} (\mathbf{N_R} \cdot \vec{z} \,|\mathbf{N_R} \cdot iI_3) \end{aligned} \tag{3.147}$$

where $\mathbf{N_R}$ is a real, symmetric matrix.

The wavefunction given in eq. (3.147) satisfies the Dirac equations (3.146) for the following gauge potentials.

$$(A^1 - A^2)_{\bar{z}_j} = \left(\frac{\pi}{2}\right) z_i (\mathbf{N_R} - i\mathbf{N_I})_{i\bar{j}}, \tag{3.148}$$

which exactly matches with eq. (3.23) for the complex structure $\Omega = iI_3$. The intersection matrix is therefore given by :

$$\mathbf{N} = \mathbf{N_R} - i\mathbf{N_I}, \tag{3.149}$$

where we identify,

$$\mathbf{N_R} = p^a_{xy} - p^b_{xy}, \qquad \mathbf{N_I} = p^a_{xx} - p^b_{xx}. \tag{3.150}$$

The wavefunction described in eq. (3.147) can be re-written in terms of the real coordinates $\vec{x}$ and $\vec{y}$ as well as matrices $\mathbf{N_R}$, $\mathbf{N_I}$. By a slight abuse of notation, below, only for this subsection, we use $\mathbf{N_R} = p_{xy}$, $\mathbf{N_I} = p_{xx}$, by setting p^b's to zero in eq. (3.150) and suppressing the superscript a in p^a. Such a notational change, helps to make comparison of the transformation rules we derive for the wavefunction written above in eq. (3.147) with general transition functions, consistent with the gauge transformations along the $2n$ non-contractible cycles of T^n, given in [6]. These transition functions are written in equations (4.40), (4.41) of [6] for the fields that transform in fundamental representation rather than as bifundamentals. Hence, the notation changes above are meant to make the expressions consistent with the ones of [6].

The wavefunction (3.147), in the real coordinates $\vec{x}$ and $\vec{y}$, then reads:

$$\begin{aligned} \psi^{\vec{j},\mathbf{N}}(\vec{z}) &= \mathcal{N}_{\vec{j}} \cdot e^{i\pi\left[(x^i \cdot p_{x^i y^j} \cdot y^j) + i(-x^i \cdot p_{x^i x^j} \cdot y^j + y^i \cdot p_{x^i y^j} \cdot y^j)\right]} \\ &\cdot \sum_{l_i \in \mathbf{Z}^n} e^{i\pi(i)\left[(l_i + j_i) \cdot p_{x^i y^j} \cdot (l_j + j_j)\right]} e^{2i\pi\left[(l_i + j_i) \cdot p_{x^i y^j} \cdot (x^j + iy^j)\right]}. \end{aligned} \tag{3.151}$$

This expression in terms of real coordinates is useful in comparing the transformation properties of the wavefunction over T^6 with the one in [6]. The transformation properties, as derived from eq. (3.147), are given by,

$$\begin{aligned} \psi^{\vec{j},\mathbf{N}}(\vec{z} + \vec{n}) &= e^{i\pi([\mathbf{N}\cdot\vec{n}]\cdot \mathrm{Im}\,\vec{z})} \cdot \psi^{\vec{j},\mathbf{N}}(\vec{z}), \\ \psi^{\vec{j},\mathbf{N}}(\vec{z} + i\vec{n}) &= e^{-i\pi([\mathbf{N}^t\cdot\vec{n}]\cdot Re\vec{z})} \cdot \psi^{\vec{j},\mathbf{N}}(\vec{z}), \end{aligned} \tag{3.152}$$

provided that

- $(\mathbf{N_R})_{i\bar{j}} \equiv p_{x^i y^j} \in \mathbf{Z}$, i.e $\mathbf{N_R}$ is integrally quantized,
- $\vec{j}$ satisfies $\vec{j} \cdot \mathbf{N_R} \in \mathbf{Z}^n$.

We therefore notice that the integer quantization is imposed only on the symmetric part $\mathbf{N_R}$ of the intersection matrix from the periodicity of the wavefunction as well. However, Dirac quantization already imposes both p_{xy} and p_{xx} to be integral for unit windings, as discussed in Section 2.3.

Using eq. (3.151), the expressions (3.152) can be re-written in terms of real coordinates as:

$$\psi^{\vec{j},\mathbf{N}}(\vec{x} + \vec{n} + i\vec{y}) = e^{i\pi[n_i(p_{x^i y^j} - ip_{x^i x^j})y^j]} \cdot \psi^{\vec{j},\mathbf{N}}(\vec{x} + i\vec{y}), \tag{3.153}$$

$$\psi^{\vec{j},\mathbf{N}}(\vec{x} + i[\vec{y} + \vec{n}]) = e^{-i\pi[n_i(p_{x^j y^i} - ip_{x^j x^i})x^j]} \cdot \psi^{\vec{j},\mathbf{N}}(\vec{x} + i\vec{y}). \tag{3.154}$$

In order to see that eqs. (3.153) and (3.154) are the proper transformation properties of the fermion wavefunction over T^6, let us compare them with the the transition functions eq. (4.41) of [6] given for a fundamental representation in six real coordinates X_I, $I = 1, \cdots, 6$, as used in our eq. (2.21) as well. After changing variables first to the coordinates x^i, y^i, $i = 1, 2, 3$ and then making coordinate transformation to $z^i, i\bar{z^i}$, as described in Section 2.3, the general transition function is given by,

$$\chi(x_i, y_i) = e^{i\pi[(m_i + in_i).F_{i\bar{j}}(y^j + ix^j) + (im_i + n_i).F_{\bar{i}j}(x^j + iy^j)]}. \tag{3.155}$$

In correspondence to the transformation along the 1-cycles, the integer parameters on x_i and y_i are denoted as m_i and n_i respectively. One then has two cases:
Case -I : When $n_i = 0$, i.e $\vec{x} \longrightarrow (\vec{x} + \vec{m})$, eq. (3.155) reduces to

$$\begin{aligned} \chi(x_i, y_i) &= e^{i\pi\{[m_i.F_{i\bar{j}}.y^j - m_i.F_{\bar{i}j}.y^j] + i[m_i.F_{i\bar{j}}.x^j + m_i.F_{\bar{i}j}.x^j]\}}, \\ &= e^{2i\pi(m_i.F_{i\bar{j}}.y^j)}, \end{aligned} \tag{3.156}$$

where we used the hermiticity property of F. Using the expression (D.2) in eq. (3.156), we recover the transformation given in eq. (3.153).
Case -II : When $m_i = 0$ i.e $\vec{y} \longrightarrow (\vec{y} + \vec{n})$, eq. (3.155) takes the form,

$$\begin{aligned} \chi(x_i, y_i) &= e^{i\pi\{[-n_i F_{i\bar{j}} x^j + n_i F_{\bar{i}j} x^j] + i[n_i F_{i\bar{j}} y^j + n_i F_{\bar{i}j} y^j]\}}, \\ &= e^{-2i\pi[n_i.F_{i\bar{j}}.x^j]}. \end{aligned} \tag{3.157}$$

Again, using eq. (D.2) in eq. (3.157), we reproduce the transformation (3.154).

It can also be easily seen that the basis wavefunctions given in eqs. (3.147) and (3.151) satisfy the orthonormality condition

$$\int_{T^{2n}} (\psi^{\vec{k},\mathbf{N}})^\dagger \psi^{\vec{j},\mathbf{N}} = \delta_{\vec{j},\vec{k}}\,, \tag{3.158}$$

by fixing the normalization constant to

$$\mathcal{N}_{\vec{j}} = (2^n |\det \mathbf{N_R}|)^{1/4} \cdot \mathrm{Vol}(T^{2n})^{-1/2}, \quad \forall j\ . \tag{3.159}$$

We have therefore confirmed that the wavefunction written in (3.147) is not only a solution of the field equation, but also has the correct periodicity properties on the torus under the gauge transformation. Now, regarding the Yukawa interaction, since only $\mathbf{N_R}$, which is real symmetric matrix, appears in the $\hat{\Theta}(z, \bar{z})$ part of the wavefunction (3.147), all the theta function identities described in Sections 3.5.1, 3.5.2 hold for this new $\hat{\Theta}(z, \bar{z})$. Similarly, as in the expression (3.114), the Yukawa coupling Y_{ijk} now has the following form,

$$\begin{aligned} Y_{ijk} = g\sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(Vol(T^{2n})\right)^{-\frac{1}{2}} \left[\frac{(|\det \mathbf{N^1_R}|.|\det \mathbf{N^2_R}|)}{|\det \mathbf{N^3_R}|}\right]^{\frac{1}{4}} \times \sum_{\vec{p},\vec{\tilde{p}}} \\ \vartheta \begin{bmatrix} (-\vec{j} + \vec{k})\frac{\mathbf{N^2_R}}{\det \mathbf{N^1_R} \det \mathbf{N^2_R}} + (\vec{p}\frac{\mathbf{N^2_R}}{det \mathbf{N^2_R}} + \vec{\tilde{p}}\frac{\mathbf{N^1_R}}{det \mathbf{N^1_R}}) \\ 0 \end{bmatrix} (0|(\det \mathbf{N^1_R} \det \mathbf{N^2_R})^2 (\mathbf{N^1_R}^{-1} \mathbf{N^3_R} \mathbf{N^2_R}^{-1})\tau) \end{aligned} \tag{3.160}$$

with $\vec{p}$ running over all the states inside the cell generated by $\vec{e_1} det \mathbf{N^2_R} \mathbf{N^2_R}^{-1}$ and $\vec{e_2} det \mathbf{N^2_R} \mathbf{N^2_R}^{-1}$. Similarly $\vec{\tilde{p}}$ runs over all the states inside the cell generated by $\vec{e_1} det \mathbf{N^1_R} \mathbf{N^1_R}^{-1}$ and $\vec{e_2} det \mathbf{N^1_R} \mathbf{N^1_R}^{-1}$.

3.5.8 Constraints on the results in section-3.5 and further generalization

To summarize, in this section we have given a close form expression for the Yukawa couplings in the magnetized brane constructions, when in general both oblique and diagonal fluxes are present along the branes. However, the results of this section are somewhat restrictive, since the basis wavefunctions used for the computations are well defined only when the intersection matrices satisfy a positivity condition given in eq. (3.26) for arbitrary complex structure Ω. A similar positivity criterion, for the case when $p_{x^ix^j}$ and $p_{y^iy^j}$ are nonzero, can be written using the wavefunction (3.147), as well; it implies simply the positivity of $\mathbf{N_R}$.

On the other hand, in realistic string model building, one may need intersection matrices that are not necessarily positive definite. The simplest examples correspond simply to diagonal intersection matrices, having some positive and some negative elements along the diagonal. In such a factorized torus case, there is a unique prescription, to define the basis functions corresponding to the negative elements in the intersection matrix, as given in [6], consisting of taking complex conjugates of the wavefunctions for the positive elements. Such a prescription also works, in the case of oblique + diagonal fluxes, when some intersection matrices are 'negative-definite' rather than being positive definite. One can then take a complete complex conjugation over all the coordinates, in order to obtain a well defined wavefunction.

Such a process, however, does not work when oblique fluxes are present and intersection matrices have mixed eigenvalues. Note that a diagonal flux of the type $F_{z^i\bar{z}^i}$ preserves its $(1,1)$-form structure, under the interchange : $z^i \to \bar{z}^i$, required by supersymmetry. This is, however, no longer true when oblique fluxes are present, since off diagonal elements of a $(1,1)$-form flux, say $F_{z^1\bar{z}^2}$, does not remain of the $(1,1)$ form when complex conjugation is taken only along z^1 or z^2.

In order to cure the problem, one needs to construct new basis functions. We present the results of our investigation in the next section, where we first restrict to the case of a T^4 compactification, for simplicity. The complications arising from the oblique nature of the fluxes are manifest in the T^4 example as well, though it is possible to generalize the result to the full T^6, which is discussed in Section 3.6.8.

3.6 Negative-chirality fermion wavefunction

As already mentioned, the basis wavefunctions given in eq. (3.68), used for deriving the Yukawa coupling expression in eq. (3.145), are constrained by the Riemann conditions (3.26), which imply in particular the positive-definiteness of the matrix $\mathbf{N}Im\Omega$.

Now, first restricting to T^4, we will show that the basis function (3.68) corresponds to the positive chirality spinor on T^4. On the other hand, to accommodate intersection matrices, having two eigenvalues of opposite signature, one needs to find out the basis function corresponding to negative chirality spinor. The need to use such basis functions, for intersection matrices with mixed eigenvalues, can be easily seen in the case when the T^4 factorizes into $T^2 \times T^2$ and one turns on only non-oblique (diagonal) fluxes. In this case, the intersection matrix has one positive diagonal element along the first T^2 and one negative diagonal element along the second one. Good basis functions are then products of two T^2 wavefunctions of opposite chiralities[6], and the total wavefunction on T^4 is of negative chirality.

Our task therefore amounts to searching for the basis functions corresponding to negative chirality spinors on T^4 with oblique fluxes. Search for fermion wavefunctions in the presence of arbitrary fluxes (in general oblique) has been pursued in [128]. However, the resulting wavefunctions are presented in terms of diagonalized coordinates and eigenvalues of fluxes. Any such solution is however unsuitable for the Yukawa computation, both for the purpose of extracting the selection rules of the type given in eq. (3.105), as well as in actual evaluation, since the diagonalized coordinates become 'stack dependent' and inherent nonlinearities involved in the diagonalization process appear in the wavefunctions, prohibiting the derivation of Yukawa couplings in a concrete form.

In this section, we are able to write both the positive and negative chirality basis functions in a 'unified' fashion, by showing that all basis functions have a form similar to the one given in eq. (3.68). However, the complex structure Ω appearing in eq. (3.68) for a positive chirality wavefunction needs to be replaced by an 'effective' modular parameter matrix $\tilde{\Omega} = \hat{\Omega}\Omega$, in order to accommodate the negative chirality wavefunctions, where $\hat{\Omega}$ is given in terms of the elements of the intersection matrices (as explicitly obtained later). We also show that our results reduce to the ones in [6] for the case of diagonal fluxes.

First, in the next subsection we present new basis functions, relevant for the situation when the intersection matrices are neither positive nor negative definite. In a later subsection, we show how the negative chirality spinor basis functions can be identified with the positive chirality ones given in eq. (3.68), with an effective modular parameter, defined in terms of the fluxes. We verify this fact by mapping the wavefunctions into each other, as well as, by showing explicitly that the relevant field equations transform into each other through such a mapping. As a result, we are able to absorb the complications associated in the diagonalization process of the modular parameter matrix, and the final wavefunction once again has an identical form as given in eq. (3.68), however, with a flux dependent modular parameter argument.

3.6.1 Construction of the wavefunction

In this subsection, as mentioned earlier, we discuss the case of 4-tori, though T^6 generalization can be analyzed in a similar manner. We first also restrict ourselves to the situation with canonical complex structure: $\Omega = iI_2$ and $\Omega = iI_3$ for T^4 and T^6 respectively, where I_d represents the d-dimensional identity matrix. The generalization to arbitrary Ω is given in subsections 3.6.6 - 3.6.8. Now, in oder to avoid the restriction to the positivity condition (3.26), we present an explicit solution of a wavefunction of negative chirality satisfying both the equations of motion, as well as the periodicity requirements on T^4.

Going back to the positive chirality wavefunctions, note that the two equations for the component χ^1_+ in eq. (3.21) (derived from the original Dirac equation (3.16)) can be simultaneously solved, since when acting on χ^1_+ with two covariant derivatives, we have: $[D_{\bar{1}}, D_{\bar{2}}] \sim F^{ab}_{\bar{1}\bar{2}}$ and the RHS is zero, since all the $(0, 2)$ components of the gauge fluxes are zero in order to maintain supersymmetry. The superscript ab in this relation implies that we need to take the difference of fluxes in brane stacks a and b due to the combination $A^a - A^b$ that appears in eq. (3.21) for the bifundamental wavefunction. Same is true for the two χ^2_+ equations, since $(2, 0)$ components of the fluxes are zero as well. On the other hand, the relevant equations for the negative chirality spinors are:

$$D_1\chi^2_- + D_2\chi^1_- = 0, \tag{3.161}$$

and

$$\bar{D}_2\chi^2_- - \bar{D}_1\chi^1_- = 0. \tag{3.162}$$

When only one of the two components $\chi^{1,2}_-$ is excited at a time, $\chi^{1,2}_-$ satisfy: $\bar{D}_1\chi^1_- = D_2\chi^1_- = 0$ or $D_1\chi^2_- = \bar{D}_2\chi^2_- = 0$. But none of these sets of equations can be consistently solved when oblique fluxes are present, since $[D_1, \bar{D}_2] \sim F_{1\bar{2}} \neq 0$.

The two negative chirality components $\chi^{1,2}_-$ therefore need to be mixed up in order to obtain a solution of the relevant Dirac equations, when oblique fluxes are present. In other words, we need to simultaneously excite both $\chi^{1,2}_-$. Then, taking

$$\chi^1_- = \alpha\psi, \;\; \chi^2_- = \beta\psi, \tag{3.163}$$

equations (3.6.1) and (3.162) become:

$$(\beta\bar{D}_2 - \alpha\bar{D}_1)\psi = 0, \tag{3.164}$$

and

$$(\beta D_1 + \alpha D_2)\psi = 0. \tag{3.165}$$

In order for these two equations to have simultaneous solution, one obtains the condition:

$$-\alpha\beta F^{ab}_{1\bar{1}} - \alpha^2 F^{ab}_{2\bar{1}} + \beta^2 F^{ab}_{1\bar{2}} + \alpha\beta F^{ab}_{2\bar{2}} = 0, \tag{3.166}$$

where $F^{ab}_{i\bar{j}} \equiv \mathbf{N}_{i\bar{j}}$ is again the difference of fluxes in brane stacks a and b and $\mathbf{N}_{i\bar{j}}$ is the same hermitian intersection matrix, eq. (3.149), used in writing the positive chirality wavefunction and Yukawa couplings in eq. (3.68), and other parts of Section 3.5. When $p_{x^i x^j} = 0$, and $\Omega = iI_3$, $\mathbf{N}$ reduces to the real symmetric matrix.

Fortunately, equation (3.166) has arbitrary solutions of the type:

$$F^{ab} \equiv \mathbf{N} \equiv \hat{N}_{1\bar{1}} \begin{pmatrix} 1 & -q \\ -q & q^2 \end{pmatrix} + \tilde{N}_{2\bar{2}} \begin{pmatrix} q^2 & q \\ q & 1 \end{pmatrix}, \tag{3.167}$$

with $q = \frac{\beta}{\alpha}$ and $\hat{N}_{1\bar{1}}$, $\tilde{N}_{2\bar{2}}$ being arbitrary integers whose notation will become clear later (see eq. (3.186) below). The RHS of the above relation is a general parameterization of a 2×2 symmetric matrix, since the two terms can be written as

$$F^{ab} \equiv \mathbf{N} \equiv \hat{N}_{1\bar{1}} \begin{pmatrix} 1 \\ -q \end{pmatrix} \begin{pmatrix} 1 & -q \end{pmatrix} + \tilde{N}_{2\bar{2}} \begin{pmatrix} q \\ 1 \end{pmatrix} \begin{pmatrix} q & 1 \end{pmatrix}. \tag{3.168}$$

After having shown the possible existence of the solution of the type (3.163), we proceed to find the explicit form of the wavefunction ψ by applying the allowed orthogonal transformations on the wavefunction of the negative chirality fermion on a T^4 which is factorized into $T^2 \times T^2$. To obtain the explicit form of this orthogonal transformation, we start by writing the coordinate T^4 coordinate, $X^M = z^i, \bar{z}^i$ $(i = 1, 2)$, in the spinor basis. We note, for the choice of Dirac Gamma matrices (in a real basis) given in eqs. (3.11), (3.12) that

$$\Gamma^M X_M = \begin{pmatrix} & \bar{z}_1 & \bar{z}_2 & \\ z_1 & & & \bar{z}_2 \\ z_2 & & & -\bar{z}_1 \\ & z_2 & -\bar{z}_1 & \end{pmatrix}, \tag{3.169}$$

with $z_i = x_i + iy_i$ and $\bar{z}_i = x_i - iy_i$, $(i = 1, 2)$, which factorizes into 2×2 blocks providing the basis on which $SU(2)$'s in the Lorentz group : $SU(2)_L \times SU(2)_R \sim SO(1,3)$ act. We get x^i in the spinor basis in the form of a 2×2 matrix:

$$X_{\alpha\dot{\alpha}} = \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ -z_2 & z_1 \end{pmatrix}. \tag{3.170}$$

Now to understand the transformation properties of the fermions on T^4, we consider the following transformations on $X_{\alpha\dot{\alpha}}$:

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ -z_2 & z_1 \end{pmatrix} \begin{pmatrix} e^{-i\theta_2} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} = \begin{pmatrix} e^{i(\theta_1-\theta_2)}\bar{z}_1 & e^{i(\theta_1+\theta_2)}\bar{z}_2 \\ -e^{-i(\theta_1+\theta_2)}z_2 & e^{-i(\theta_1-\theta_2)}z_1 \end{pmatrix} \tag{3.171}$$

We learn from eq. (3.171) that when T^4 factorizes into $T^2 \times T^2$, the transformations of the positive and negative chirality fermions on the two T^2's can be read off from the transformation rules of z_1 and z_2 given above[10]. Indeed, the transformation rules for the fermions $\psi_\pm^{(i)}$ on the two T^2's, denoted by indices $i = 1, 2$ are:

$$\begin{aligned} &\psi_+^{(1)} \longrightarrow e^{-i\frac{(\theta_1-\theta_2)}{2}}\psi_+^{(1)}; \quad \psi_-^{(1)} \longrightarrow e^{i\frac{(\theta_1-\theta_2)}{2}}\psi_-^{(1)}, \\ &\psi_+^{(2)} \longrightarrow e^{-i\frac{(\theta_1+\theta_2)}{2}}\psi_+^{(2)}; \quad \psi_-^{(2)} \longrightarrow e^{i\frac{(\theta_1+\theta_2)}{2}}\psi_-^{(2)}. \end{aligned} \tag{3.172}$$

In this case, as described in the section 3.4.1, the T^4 fermion wavefunctions can be written as a direct product of the ones on two T^2's as in eq. (3.34). We obtain the transformation of T^4 wavefunctions (eq. (3.34)):

$$\begin{aligned} &\Psi_+^1 \longrightarrow e^{-i\theta_1}\Psi_+^1, \ \ \Psi_+^2 \longrightarrow e^{i\theta_1}\Psi_+^2, \\ &\Psi_-^1 \longrightarrow e^{i\theta_2}\Psi_-^1, \ \ \Psi_-^2 \longrightarrow e^{-i\theta_2}\Psi_-^2. \end{aligned} \tag{3.173}$$

It follows that a left transformation $(\theta_1 \neq 0, \theta_2 = 0)$ acts independently on (left handed) positive chirality wavefunctions, and a right transformation $(\theta_1 = 0, \theta_2 \neq 0)$ acts on the negative-chirality (right handed) wavefunctions. Now, consider the following complex transformation on vectors in spinor basis:

[10] The equation number (3.171) is matrix multiplication defining the transformations on $X_{\alpha\dot{\alpha}}$. The equation number (3.172) gives the transformation rules for the fermions $\psi_\pm^{(i)}$ on the two T^2's, which are read off from the transformation rules of z_1 and z_2 given in equation (3.171).

$$\begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ z_2 & -z_1 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ z_2 & -z_1 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \tag{3.174}$$

Case-I: For $e = h = 1$, $f = g = 0$, $c = -b$, $a = d$, i.e a left transformation results in the following orthogonal coordinate transformation,

$$z_1 \longrightarrow a z_1 + b\bar{z}_2; \quad z_2 \longrightarrow a z_2 - b\bar{z}_1. \tag{3.175}$$

Case-II: Similarly, for $a = d = 1$, $c = b = 0$, $h = e$, $f = -g$, i.e a right transformation leads to

$$z_1 \longrightarrow e z_1 - f z_2; \quad z_2 \longrightarrow e z_2 + f z_1. \tag{3.176}$$

In order to maintain the holomorphicity of the gauge fluxes, one therefore needs to make use of the later transformation, in order to generate a general wavefunction, starting with the one which corresponds to the diagonal (non-oblique) flux. In addition, we need to maintain the integrality of the fluxes, as we make such orthogonal transformations. However, in our case, we do not make use of any specific form of the transformation and rather use the above analysis as a guide for writing down a general solution. We then verify the equations of motion directly, in order to confirm that the solution we propose is indeed the correct one.

3.6.2 New wavefunction

We now use the transformation (3.176) to obtain the wavefunction associated with the negative chirality fermion bifundamentals, starting with a wavefunction associated with a negative chirality spinor for a diagonal flux. In the notations of eq. (3.15), it corresponds to exciting only the negative chirality component

$$\begin{pmatrix} \Psi_-^2 \\ \Psi_-^1 \end{pmatrix} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}. \tag{3.177}$$

We ignore the explicit form of ψ, except to note that after the transformation (3.176), one generates

$$\begin{pmatrix}\psi\\0\end{pmatrix} \longrightarrow \begin{pmatrix}\Psi^2_-\\\Psi^1_-\end{pmatrix} = \begin{pmatrix}\beta\psi\\\alpha\psi\end{pmatrix}, \tag{3.178}$$

while $(\Psi^1_+,\ \Psi^2_+)$ remain zero. In the gauge sector, such wavefunctions are parameterized in the bifundamental representations by:

$$\Psi_{ab} = \begin{pmatrix} C_{n_a} & \chi_{ab} \\ & C_{n_b}\end{pmatrix}, \tag{3.179}$$

as also given in eq. (3.19). For negative chirality components, the equations to be satisfied by the various components are: (see eq. (3.20))

$$\begin{aligned}
&\partial_1\chi^2_- + \partial_2\chi^1_- + (A^1 - A^2)_{z_1}\chi^2_- + (A^1 - A^2)_{z_2}\chi^1_- = 0,\\
&\bar\partial_2\chi^2_- - \bar\partial_1\chi^1_- + (A^1 - A^2)_{\bar z_2}\chi^2_- - (A^1 - A^2)_{\bar z_1}\chi^1_- = 0.
\end{aligned} \tag{3.180}$$

We now show that the solution to eqs. (3.180), together with proper periodicity requirements on T^4, is given by the basis elements:

$$\psi^{\vec j,\hat{\mathbf{N}},\tilde{\mathbf{N}}} = \mathcal{N}\cdot f(z,\bar z)\cdot \hat\Theta(z,\bar z) \tag{3.181}$$

where,

$$f(z,\bar z) = e^{i\pi[(\hat{\mathbf{N}}_{i\bar j} z_i Im z_j) - (\tilde{\mathbf{N}}_{\mathbf{i}\bar{\mathbf{j}}}\bar z_i Im\bar z_j)]}\ , \tag{3.182}$$

$$\hat\Theta(z,\bar z) = \sum_{m_1,m_2\in\mathbf{Z}^n} e^{\pi i(i)[(m_i+j_i)\mathbf{M}_{i\bar j}(m_j+j_j)]} e^{2\pi i[(m_i+j_i)\hat{\mathbf{N}}_{i\bar j} z_j} e^{2\pi i (m_i+j_i)\tilde{\mathbf{N}}_{\mathbf{i}\bar{\mathbf{j}}}\bar z_j}\ , \tag{3.183}$$

with

$$\mathbf{M}_{i\bar j} = \hat{\mathbf{N}}_{i\bar j} - \tilde{\mathbf{N}}_{\mathbf{i}\bar{\mathbf{j}}} \tag{3.184}$$

where both $\hat{\mathbf{N}}$, $\tilde{\mathbf{N}}$ are real, symmetric matrices, given earlier in eq. (3.167), and so also is $\mathbf{M}$ ($\mathbf{M}_{i\bar j} = \mathbf{M}_{j\bar i}$). We retain, however, both types of indices: i and $\bar j$ to incorporate real as well as complex components of the $(1,1)$-form fluxes $F_{i\bar j}$. Also, an extra factor of i in the exponent of $\hat\Theta(z,\bar z)$ corresponds to the fact that we are working with the canonical complex structure : $\Omega = iI_2$ for the present example of the fermion wavefuncton on T^4.

The wavefunction (3.181) satisfies the Dirac equations (3.180) for the following gauge potentials:

$$\begin{aligned}(A^1 - A^2)_{\bar{z}_1} &= (\hat{\mathbf{N}}_{1\bar{1}} + \tilde{\mathbf{N}}_{\mathbf{1\bar{1}}})z_1 + (\hat{\mathbf{N}}_{1\bar{2}} + \tilde{\mathbf{N}}_{\mathbf{1\bar{2}}})z_2 \\ (A^1 - A^2)_{\bar{z}_2} &= (\hat{\mathbf{N}}_{1\bar{2}} + \tilde{\mathbf{N}}_{\mathbf{1\bar{2}}})z_1 + (\hat{\mathbf{N}}_{2\bar{2}} + \tilde{\mathbf{N}}_{\mathbf{2\bar{2}}})z_2.\end{aligned} \tag{3.185}$$

The intersection matrix $\mathbf{N}$ is therefore given by:

$$\mathbf{N} = \hat{\mathbf{N}} + \tilde{\mathbf{N}}, \tag{3.186}$$

as appearing previously in eqs. (3.166), (3.167). Also, we have imposed the following constraints, in order to retain the holomorphicity of gauge potentials:

$$\frac{\alpha}{\beta} = \frac{-\hat{\mathbf{N}}_{1\bar{1}}}{\hat{\mathbf{N}}_{1\bar{2}}} = \frac{-\hat{\mathbf{N}}_{1\bar{2}}}{\hat{\mathbf{N}}_{2\bar{2}}} = \frac{\tilde{\mathbf{N}}_{1\bar{2}}}{\tilde{\mathbf{N}}_{1\bar{1}}} = \frac{\tilde{\mathbf{N}}_{2\bar{2}}}{\tilde{\mathbf{N}}_{1\bar{2}}} = \frac{1}{q}. \tag{3.187}$$

Note that the ratios of the matrix elements of $\hat{\mathbf{N}}$ and $\tilde{\mathbf{N}}$ are identical to those given in eq. (3.167). We have therefore explicitly shown that the solution given in eqs. (3.181) - (3.183) satisfies the equations of motion. The transformation properties of this wavefunction (3.181) along the four 1-cycles of T^4, are given by:

$$\begin{aligned}\psi^{\vec{j},\hat{\mathbf{N}},\tilde{\mathbf{N}}}(\vec{z}+\vec{n}) &= e^{i\pi([\hat{\mathbf{N}}\cdot\vec{n}]\cdot\mathrm{Im}\,\vec{z} - [\tilde{\mathbf{N}}\cdot\vec{n}]\cdot\mathrm{Im}\,\bar{\vec{z}})} \cdot \psi^{\vec{j},\hat{\mathbf{N}},\tilde{\mathbf{N}}}(\vec{z}), \\ \psi^{\vec{j},\hat{\mathbf{N}},\tilde{\mathbf{N}}}(\vec{z}+i\vec{n}) &= e^{-i\pi([\hat{\mathbf{N}}\cdot\vec{n}]\cdot Re\vec{z} + [\tilde{\mathbf{N}}\cdot\vec{n}]\cdot Re\bar{\vec{z}})} \cdot \psi^{\vec{j},\hat{\mathbf{N}},\tilde{\mathbf{N}}}(\vec{z}),\end{aligned} \tag{3.188}$$

provided that

- $\mathbf{N}_{\mathbf{i\bar{j}}} \equiv (\hat{\mathbf{N}} + \tilde{\mathbf{N}})_{i\bar{j}} \in \mathbf{Z}$, i.e $(\hat{\mathbf{N}} + \tilde{\mathbf{N}})$ is integrally quantized,
- $\vec{j}$ satisfies: $\vec{j}\cdot(\hat{\mathbf{N}} + \tilde{\mathbf{N}}) \in \mathbf{Z}^n$.

We therefore notice that the integer quantization is imposed only on the intersection matrix $\mathbf{N}$ given in eq. (3.186) and does not necessarily hold for the matrix $\mathbf{M}$ in eq. (3.184). Explicitly, we have:

$$\begin{aligned}\mathbf{N} = \hat{\mathbf{N}} + \tilde{\mathbf{N}} = \hat{\mathbf{N}}_{1\bar{1}}\begin{pmatrix}1 & -q \\ -q & q^2\end{pmatrix} + \tilde{\mathbf{N}}_{2\bar{2}}\begin{pmatrix}q^2 & q \\ q & 1\end{pmatrix}, \\ \mathbf{M} = \hat{\mathbf{N}} - \tilde{\mathbf{N}} = \hat{\mathbf{N}}_{1\bar{1}}\begin{pmatrix}1 & -q \\ -q & q^2\end{pmatrix} - \tilde{\mathbf{N}}_{2\bar{2}}\begin{pmatrix}q^2 & q \\ q & 1\end{pmatrix},\end{aligned} \tag{3.189}$$

where the first eq. in (3.189) is identical to the solutions in eq. (3.167).

Note that the wavefunction given in eqs. (3.181), (3.182) and (3.183) is now well defined, as the series expansion in eq. (3.183) is now convergent. To show this, we note the following relation:

$$\det \mathbf{N} = -\det \mathbf{M} = \hat{\mathbf{N}}_{1\bar{1}} \tilde{\mathbf{N}}_{2\bar{2}} (1+q^2)^2. \tag{3.190}$$

As a result, in the case when $\det \mathbf{N}$ is negative (when $\mathbf{N}$ has two eigenvalues of opposite signatures), $\det \mathbf{M} > 0$. So, the series (3.183) is now convergent when the two eigenvalues are of positive signature, since it is the quadratic part, in the summation index in theta series, that dominates in the exponent of this expansion. An overall complex conjugation will be required, for the case when two eigenvalues are negative rather than positive.

3.6.3 Normalization

Now that we have found a basis of wavefunctions, classified by the index j_i in the exponent in (3.183), we proceed to show its orthonormality. The wavefunctions described in eqs. (3.181), (3.182), (3.183) can be re-written in terms of the real coordinates $\vec{x}$ and $\vec{y}$ as follows:

$$\psi^{\vec{j},\mathbf{N},\mathbf{M}} = \mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x}\cdot\mathbf{N}\cdot\vec{y}+i\vec{y}\cdot\mathbf{M}\cdot\vec{y}]} \sum_{\vec{m}\in\mathbf{Z}^n} e^{\pi i(i)[(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot(\vec{m}+\vec{j})]} e^{2\pi i[(\vec{m}+\vec{j})\cdot\mathbf{N}\cdot\vec{x}+i(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot\vec{y}]}. \tag{3.191}$$

Then the following orthonormality conditions are satisfied:

$$\int_{T^4} (\psi^{\vec{k},\mathbf{N},\mathbf{M}})^* \psi^{\vec{j},\mathbf{N},\mathbf{M}} = \delta_{\vec{j},\vec{k}}. \tag{3.192}$$

To verify the orthogonality relation and obtain the normalization factor, we note that, in terms of the wavefunctions (3.191) we have:

$$\begin{aligned}
(\psi^{\vec{k},\mathbf{N},\mathbf{M}})^* \psi^{\vec{j},\mathbf{N},\mathbf{M}} \quad &= \mathcal{N}_{\vec{k}} \cdot e^{-i\pi[\vec{x}\cdot\mathbf{N}\cdot\vec{y}-i\vec{y}\cdot\mathbf{M}\cdot\vec{y}]} \sum_{\vec{l}\in\mathbf{Z}^n} e^{\pi i(i)[(\vec{l}+\vec{k})\cdot\mathbf{M}\cdot(\vec{l}+\vec{k})]} \cdot e^{-2\pi i[(\vec{l}+\vec{k})\cdot\mathbf{N}\cdot\vec{x}-i(\vec{l}+\vec{k})\cdot\mathbf{M}\cdot\vec{y}]} \\
&\mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x}\cdot\mathbf{N}\cdot\vec{y}+i\vec{y}\cdot\mathbf{M}\cdot\vec{y}]} \sum_{\vec{m}\in\mathbf{Z}^n} e^{\pi i(i)[(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot(\vec{m}+\vec{j})]} \cdot e^{2\pi i[(\vec{m}+\vec{j})\cdot\mathbf{N}\cdot\vec{x}+i(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot\vec{y}]} \\
&= \mathcal{N}_{\vec{j}}\mathcal{N}_{\vec{k}} \cdot e^{-2\pi(\vec{y}\cdot\mathbf{M}\cdot\vec{y})} \sum_{\vec{m},\vec{l}\in\mathbf{Z}^n} e^{\pi i(i)[(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot(\vec{m}+\vec{j})]} \cdot e^{\pi i(i)[(\vec{l}+\vec{k})\cdot\mathbf{M}\cdot(\vec{l}+\vec{k})]} \\
&e^{2\pi i[(\vec{m}+\vec{j})-(\vec{l}+\vec{k})]\cdot\mathbf{N}\cdot\vec{x}} \cdot e^{2\pi i(i)[(\vec{m}+\vec{j})+(\vec{l}+\vec{k})]\cdot\mathbf{M}\cdot\vec{y}}.
\end{aligned} \tag{3.193}$$

The integration over $\vec{x}$ in eq. (3.192) imposes the condition $\vec{j} = \vec{k}$ and equality on the summation indices $\vec{m} = \vec{l}$. In particular, the condition $\vec{j} = \vec{k}$ gives our orthogonality condition

(3.192). One can now obtain the normalization factor by performing the integration:

$$\int_0^1 d^2\vec{y} \qquad \left[e^{-2\pi\vec{y}\cdot\mathbf{M}\cdot\vec{y}} \sum_{\vec{m}\in\mathbf{Z}^n} e^{-2\pi(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot(\vec{m}+\vec{j})} \cdot e^{-4\pi(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot\vec{y}} \right]$$
$$= \int_0^1 d^2\left(\vec{y}\right) \left[\sum_{\vec{m}\in\mathbf{Z}^n} e^{-2\pi\left((\vec{m}+\vec{j})+\vec{y}\right)\cdot\mathbf{M}\cdot\left((\vec{m}+\vec{j})+\vec{y}\right)} \right]. \tag{3.194}$$

One can integrate over $\vec{y}$, using

$$\int_0^1 d^2\vec{y} \left[\sum_{\vec{m}\in\mathbf{Z}^n} e^{-2\pi\left((\vec{m}+\vec{j})+\vec{y}\right)\cdot\mathbf{M}\cdot\left((\vec{m}+\vec{j})+\vec{y}\right)} \right] = \sum_{\vec{m}\in\mathbf{Z}^n} \int_0^1 d^2\vec{y} \left[e^{-2\pi\left[(\vec{m}+\vec{j})+\vec{y}\right]\cdot\mathbf{M}\cdot\left((\vec{m}+\vec{j})+\vec{y}\right)} \right]$$
$$= \int_{-\infty}^{\infty} d^2\vec{y'} \left[e^{-2\pi\vec{y'}\cdot\mathbf{M}\cdot\vec{y'}} \right] \tag{3.195}$$

The integration (3.195) fixes then the normalization constant to

$$\mathcal{N}_{\vec{j}} = (2|\mathrm{det}\mathbf{M}|)^{1/4} \cdot \mathrm{Vol}(T^4)^{-1/2}, \quad \forall j. \tag{3.196}$$

3.6.4 Eigenfunctions of the Laplace equation

The wavefunctions (3.181) not only represent zero modes of the Dirac operator, but are also eigenfunctions of the Laplacian. In order to see this, we start with computing the Dirac operator in four dimensions. In our notations:

$$\Gamma^\mu\partial_\mu = \begin{pmatrix} & \bar{\partial}_1 & \bar{\partial}_2 & \\ \partial_1 & & & \bar{\partial}_2 \\ \partial_2 & & & -\bar{\partial}_1 \\ & \partial_2 & -\partial_1 & \end{pmatrix}, \tag{3.197}$$

which leads to

$$
\begin{aligned}
(\not{D})^2 &= \begin{pmatrix} \bar{D}_1 D_1 + \bar{D}_2 D_2 & & & \\ & D_1\bar{D}_1 + \bar{D}_2 D_2 & & \\ & & D_2\bar{D}_2 + \bar{D}_1 D_1 & \\ & & & D_1\bar{D}_1 + D_2\bar{D}_2 \end{pmatrix} \\
&= \Delta + \begin{pmatrix} F_{1\bar{1}} + F_{2\bar{2}} & & & \\ & -F_{1\bar{1}} + F_{2\bar{2}} & & \\ & & F_{1\bar{1}} - F_{2\bar{2}} & \\ & & & -(F_{1\bar{1}} + F_{2\bar{2}}) \end{pmatrix} .
\end{aligned} \tag{3.198}
$$

The Dirac equation $\not{D}\Psi = 0$, with Ψ given in eq. (3.34), implies that such basis functions are also eigenfunctions of the Laplacian Δ. The question whether massless scalars exist, depends on whether some combination of fluxes appearing in eq. (3.198) vanish[11]. Of course, their existence is guaranteed in the supersymmetric case.

3.6.5 Mapping of basis functions from positive to negative chirality

We now show that the basis for the negative chirality wavefunction, given in eqs. (3.181), (3.182), (3.183) can in fact be obtained by a mapping from the basis of the positive chirality wavefunction given in eq. (3.68). We also present the mapping between the corresponding field equations. Our mapping reduces to the ones in [6] for the case of factorized tori.

More precisely, we show that our negative chirality wavefunction, given in eqs. (3.181), (3.182), (3.183), as well as (3.191) (for a trivial modular parameter matrix : $\Omega = iI_2$) is identical to the positive chirality wavefunction (3.68) for a 'nontrivial' (flux dependent) modular parameter matrix $\Omega = i\hat{\Omega}$. Explicitly, $\hat{\Omega}$ is given in terms of the ratios (q) of flux components. This result gives a 'unified' picture of all the relevant basis functions. Later on, in Section 3.6.7, we show that a similar mapping holds for nontrivial complex structure on T^4, by examining the equations of motion.

Let us write down explicitly the wavefunction (3.68) for complex structure with arbi-

[11] The condition $F_{1\bar{1}} = -F_{2\bar{2}}$ implies massless scalar and supersymmetry in T^4. The other two scalars become tachyonic.

trary Ω $(= i\hat{\Omega})$.

$$\begin{aligned}
\psi^{\vec{j},\mathbf{N}'}(\vec{z},\Omega) &= \mathcal{N}\cdot e^{i\pi[(\vec{x}+i\hat{\Omega}\vec{y}).\mathbf{N}'\hat{\Omega}^{-1}.\hat{\Omega}\vec{y}]}\cdot\sum_{\vec{m}\in\mathbf{Z}^n} e^{i\pi[(\vec{m}+\vec{j}).i\mathbf{N}'\hat{\Omega}.(\vec{m}+\vec{j})]}e^{2i\pi[(\vec{m}+\vec{j})(\mathbf{N}'\vec{x}+i\mathbf{N}'\hat{\Omega}.\vec{y})]}\\
&\sim e^{i\pi[\vec{x}.\mathbf{N}'.\vec{y}+i\hat{\Omega}\vec{y}.\mathbf{N}'.\vec{y}]}\cdot\sum_{\vec{m}\in\mathbf{Z}^n} e^{i\pi[(\vec{m}+\vec{j}).i\mathbf{N}'\hat{\Omega}.(\vec{m}+\vec{j})]}e^{2i\pi[(\vec{m}+\vec{j})(\mathbf{N}'\vec{x}+i\mathbf{N}'\hat{\Omega}.\vec{y})]}, \qquad (3.199)
\end{aligned}$$

where $\mathbf{N}$ is changed to $\mathbf{N}'$ to show a distinction between the two wavefunctions for the purpose of defining the mapping as given below. Next consider the negative chirality wavefunction (3.191), written in terms of real coordinates $\vec{x}$ and $\vec{y}$,

$$\psi^{\vec{j},\mathbf{N},\mathbf{M}} \sim e^{i\pi[\vec{x}\cdot\mathbf{N}\cdot\vec{y}+i\vec{y}\cdot\mathbf{M}\cdot\vec{y}]}\sum_{\vec{m}\in\mathbf{Z}^n} e^{\pi i(i)[(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot(\vec{m}+\vec{j})]}e^{2\pi i[(\vec{m}+\vec{j})\cdot\mathbf{N}\cdot\vec{x}+i(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot\vec{y}]}. \qquad (3.200)$$

It is now easy to check that the above equations (3.199) and (3.200) precisely match with the following identification :

$$\begin{aligned}
&\mathbf{N} = \hat{\mathbf{N}} + \tilde{\mathbf{N}} = \mathbf{N}',\\
&\mathbf{M} = \hat{\mathbf{N}} - \tilde{\mathbf{N}} = \mathbf{N}'\hat{\Omega} \Rightarrow \hat{\Omega} = \mathbf{N}^{-1}\mathbf{M}, \qquad (3.201)
\end{aligned}$$

with $\Omega = i\hat{\Omega}$, and $\hat{\Omega}$ is a real matrix. For the $\mathbf{N}$ and $\mathbf{M}$, defined in eq. (3.189), $\mathbf{N}^{-1}$ and $\hat{\Omega}$ are given by;

$$\mathbf{N}^{-1} = \frac{1}{(1+q^2)^2}\left[\frac{1}{\tilde{\mathbf{N}}_{1\bar{1}}}\begin{pmatrix}1 & -q\\ -q & q^2\end{pmatrix} + \frac{1}{\tilde{\mathbf{N}}_{2\bar{2}}}\begin{pmatrix}q^2 & q\\ q & 1\end{pmatrix}\right], \qquad (3.202)$$

$$\hat{\Omega} = \frac{1}{(1+q^2)}\begin{pmatrix}1-q^2 & -2q\\ -2q & q^2-1\end{pmatrix} = (\hat{\Omega})^{-1}. \qquad (3.203)$$

We have therefore shown explicitly that the positive chirality basis wavefunction (3.68), known earlier in the literature, can be mapped to the negative chirality wavefunctions that we have constructed in eqs. (3.181)-(3.183), (3.191). Such a map also confirms the validity of our construction for the negative chirality basis functions, presented using basic principles, such as equations of motion as well as periodicity requirement. In fact, in the next subsection, the same mapping is also obtained through comparison of the relevant equations of motion, which further confirms our results for the construction of the basis functions. Note that for $q = 0$ or $q \to \infty$, corresponding to the case when both matrices

$\mathbf{N}$ and $\mathbf{M}$ in eq. (3.189) are diagonal, we have:

$$\hat{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{or} \quad \hat{\Omega} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.204}$$

respectively. As a result, one reproduces the known mapping of the wavefunctions between positive and negative chirality spinors in the case when T^4 is factorized into $T^2 \times T^2$ [6].

3.6.6 Mapping the equations of motion

In order to derive a similar mapping of the equations of motion, we show below that the covariant derivative operators appearing in eqs. (3.22) for the positive chirality wavefunction, with a nontrivial complex structure $(i\hat{\Omega})$, are equivalent to the derivative operators appearing in eqs. (3.164), (3.165) for the negative chirality wavefunction (with complex structure $\Omega = iI_2$). The mapping of corresponding gauge potentials can also be shown in the same manner, since they have similar dependence on the complex structure as the derivative operator. Note that the complex structure appears in the wavefunctions as modular parameter matrices. We therefore reconfirm the mapping between the two wavefunctions by comparing the equations of motion as well.

We now examine the Dirac equations for both cases. For the first one, with arbitrary $\Omega(= i\hat{\Omega})$, we have

$$\vec{z} = \vec{x} + i\hat{\Omega}\vec{y}; \quad \vec{\bar{z}} = \vec{x} - i\hat{\Omega}\vec{y} \quad \Rightarrow \quad \vec{x} = \frac{\vec{z} + \vec{\bar{z}}}{2}; \quad \vec{y} = (\hat{\Omega})^{-1}\left(\frac{\vec{z} - \vec{\bar{z}}}{2i}\right),$$

which implies

$$\begin{aligned} \frac{\partial}{\partial z_i} &= \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i(\hat{\Omega})^{-1}_{ji}\frac{\partial}{\partial y^j}\right), \\ \frac{\partial}{\partial \bar{z}_i} &= \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i(\hat{\Omega})^{-1}_{ji}\frac{\partial}{\partial y^j}\right). \end{aligned} \tag{3.205}$$

Then, the Dirac equation for the positive chirality wavefunction is:

$$\bar{D}_{\bar{z}_i}\psi^{\vec{j},\mathbf{N}'}(\vec{z},\Omega) \equiv \frac{1}{2}\left(D_{x^i} + i(\hat{\Omega})^{-1}_{ji}D_{y^j}\right)\psi^{\vec{j},\mathbf{N}'}(\vec{z},\Omega) = 0, \quad i,j = 1,2. \tag{3.206}$$

On the other hand, for the negative chirality solution (3.180), with complex structure $\Omega = iI_2$, the relevant derivative operators are:

$$(\beta D_1 + \alpha D_2)\,\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0; \quad \left(\beta\bar{D}_2 - \alpha\bar{D}_1\right)\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0. \tag{3.207}$$

These equations, using the definitions $z^i = x^i + iy^i$, $\bar{z}_i = x^i - iy^i$, i.e. substituting

$$D_i = \frac{1}{2}\left(D_{x^i} - iD_{y^i}\right),$$
$$\bar{D}_i = \frac{1}{2}\left(D_{x^i} + iD_{y^i}\right). \quad i = 1, 2$$

can be rewritten as:

$$\left\{\frac{\beta}{2}\left(D_{x^1} - iD_{y^1}\right) + \frac{\alpha}{2}\left(D_{x^2} - iD_{y^2}\right)\right\}\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0,$$
$$\left\{\frac{\beta}{2}\left(D_{x^2} + iD_{y^2}\right) - \frac{\alpha}{2}\left(D_{x^1} + iD_{y^1}\right)\right\}\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0.$$

These two equations upon simplification leads to,

$$\frac{1}{2}\left\{D_{x^1} + i\left(\frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}D_{y^1} - \frac{2\alpha\beta}{\alpha^2 + \beta^2}D_{y^2}\right)\right\}\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0,$$
$$\frac{1}{2}\left\{D_{x^2} + i\left(\frac{-2\alpha\beta}{\alpha^2 + \beta^2}D_{y^1} + \frac{\beta^2 - \alpha^2}{\alpha^2 + \beta^2}D_{y^2}\right)\right\}\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0. \tag{3.208}$$

Now using $\frac{\beta}{\alpha} = q$ from eq. (3.187) and comparing the equations (3.206) and (3.208), one finds that they precisely match for the following complex structure:

$$(\hat{\Omega})^{-1} = \frac{1}{(1+q^2)}\begin{pmatrix} 1-q^2 & -2q \\ -2q & q^2-1 \end{pmatrix}, \tag{3.209}$$

which is exactly the same as eq. (3.203). Thus, the wavefunctions as well as the Dirac equations for both cases match exactly. This mapping can be generalized further, as given in subsection 3.6.8 below.

3.6.7 Mapping for arbitrary complex structure Ω

In this subsection, we generalize the mapping between the equations of motion associated with the positive and negative chirality wavefunction to the case of T^4 compactification with arbitrary complex structure Ω. Now, the negative chirality basis functions satisfy:

$$\frac{1}{2}\left\{D_{x^1} + i(\Omega)^{-1}_{i1}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}D_{y^i}\right) - i(\Omega)^{-1}_{i2}\left(\frac{2\alpha\beta}{\alpha^2 + \beta^2}D_{y^i}\right)\right\}\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0$$
$$\frac{1}{2}\left\{D_{x^2} + i(\Omega)^{-1}_{i1}\left(\frac{-2\alpha\beta}{\alpha^2 + \beta^2}D_{y^i}\right) + i(\Omega)^{-1}_{i2}\left(\frac{\beta^2 - \alpha^2}{\alpha^2 + \beta^2}D_{y^i}\right)\right\}\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0, \tag{3.210}$$

which can be identified with the equations satisfied by the positive chirality wavefunction with $\tilde{\Omega} = \hat{\Omega}\Omega$, as can be seen through the decomposition:

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i(\tilde{\Omega})^{-1}_{ji}\frac{\partial}{\partial y^j}\right),$$
$$\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i(\tilde{\Omega})^{-1}_{\bar{j}i}\frac{\partial}{\partial y^j}\right). \tag{3.211}$$

Thus, eq. (3.68) with $\tilde{\Omega} = \hat{\Omega}\Omega$, with $\hat{\Omega}$ given in eq. (3.209), provides the negative chirality solution for arbitrary complex structure Ω, where both 'oblique' and diagonal fluxes are turned on.

3.6.8 Generalization for the T^6- case

In this subsection, we generalize the results obtained so far for negative chirality fermions on T^4 to the more general T^6 case. We only consider the wavefunctions that are well defined with two positive and one negative eigenvalues of the 3×3 Hermitian intersection matrices, since these will complete the list of well defined wavefunctions, once complex conjugations are taken into account. For the case of T^6, the relevant equations, obtained by generalization of eqs. (3.164) and (3.165) to be examined, are:

$$(\alpha\bar{D}_1 - \beta_i\bar{D}_i)\psi = 0, \tag{3.212}$$

and

$$(\alpha D_i + \beta_i D_1)\psi = 0. \tag{3.213}$$

Note that in these equations and below, the indices $i, j = 1, 2$ (used for the T^4 with wavefunctions of positive chirality). In order for the above two equations to have simultaneous solution, one obtains the condition :

$$\alpha^2 F^{ab}_{i\bar{1}} + \alpha\beta_i F^{ab}_{1\bar{1}} - \alpha\beta_j F^{ab}_{i\bar{j}} - \beta_i\beta_j F^{ab}_{1\bar{j}} = 0, \tag{3.214}$$

where $F^{ab} \equiv \mathbf{N}$ is the difference of fluxes in brane stacks a and b. The general solution of this equation is of the following type:

$$F^{ab} \equiv \mathbf{N} \equiv \hat{N}\begin{pmatrix} 1 & -(\vec{q})^T \\ -\vec{q} & \vec{q}(\vec{q})^T \end{pmatrix} + \begin{pmatrix} (\vec{q})^T\tilde{N}\vec{q} & \vec{q}^T\tilde{N} \\ \tilde{N}\vec{q} & \tilde{N} \end{pmatrix}, \tag{3.215}$$

where $\tilde{N}$ is a 2×2 matrix and $\hat{N}$ is a number. Also, $\vec{q}$ is the two-dimensional ($2d$) vector defined as:

$$\vec{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{3.216}$$

with $q_i = \frac{\beta_i}{\alpha}$.

Now, after showing the possible existence of the solution by defining F^{ab} in (3.215), for the negative chirality wavefunction on T^6, we proceed to present a mapping between the equations of motion for negative chirality and positive chirality wavefunctions on T^6. As described before in section 3.6.6. Here also we show that the covariant derivative operators appearing in eqs. (3.22), for the positive chirality wavefunction, with a nontrivial complex structure are equivalent to the derivative operators appearing in eqs. (3.212), (3.213) for the negative chirality wavefunction (with complex structure $\Omega = iI_3$) and the corresponding gauge potentials map in the same manner.

For the positive chirality case, with arbitrary $\Omega(= i\hat{\Omega})$ and eqs. (3.205), (3.205), the Dirac equation reads:

$$\bar{D}_{\bar{z}_\mu} \psi^{\vec{j},\mathbf{N}'}(\vec{z}, \Omega) \equiv \frac{1}{2} \left(D_{x^\mu} + i(\hat{\Omega})^{-1}_{\nu\mu} D_{y^\nu} \right) \psi^{\vec{j},\mathbf{N}'}(\vec{z}, \Omega) = 0, \quad \mu, \nu = 1, 2, 3 \; . \tag{3.217}$$

On the other hand, for the negative chirality solution, with complex structure $\Omega = iI_3$, the relevant derivative operators, given in eqs. (3.212), (3.213), take the form:

$$\begin{aligned} &\frac{1}{2} \left\{ \left(\alpha^2 \delta_{ij} + \beta_i \beta_j \right) D_{x^j} - i \left(2\alpha\beta_i \right) D_{y^1} + i \left(\beta_i \beta_j - \alpha^2 \delta_{ij} \right) D_{y^j} \right\} \psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0, \\ &\frac{1}{2} \left\{ \left(\beta_i^2 + \alpha^2 \right) D_{x^1} + i \left(\alpha^2 - \beta_i^2 \right) D_{y^1} - i \left(2\beta_i \alpha \right) D_{y^i} \right\} \psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0. \end{aligned} \tag{3.218}$$

Now, defining new 2×2 matrices,

$$A_{ij} = \left(\alpha^2 \delta_{ij} + \beta_i \beta_j \right), \quad B_{ij} = \left(\beta_i \beta_j - \alpha^2 \delta_{ij} \right),$$

and

$$P_i = (2\alpha\beta_i), \tag{3.219}$$

eqs. (3.218) can be re-written as:

$$\begin{aligned}
&\frac{1}{2}\left\{D_{x^i} - i\left(A^{-1}P\right)_i D_{y^1} + i\left(A^{-1}B\right)_{ij} D_{y^j}\right\}\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0 \\
&\frac{1}{2}\left\{D_{x^1} + i\left(\frac{\alpha^2-\beta_i^2}{\beta_i^2+\alpha^2}\right) D_{y^1} - i\left(\frac{2\alpha\beta_i}{\beta_i^2+\alpha^2}\right) D_{y^i}\right\}\psi^{\vec{j},\mathbf{N},\mathbf{M}} = 0\ .
\end{aligned} \tag{3.220}$$

A comparison of equations (3.217) and (3.220) implies that they precisely match for the following complex structure:

$$\begin{aligned}
(\hat{\Omega})^{-1}_{11} &= \left(\frac{\alpha^2-\beta_i^2}{\beta_i^2+\alpha^2}\right), \quad (\hat{\Omega})^{-1}_{1i} = \left(-A^{-1}P\right)_i\,, \\
(\hat{\Omega})^{-1}_{i1} &= -\left(\frac{2\alpha\beta_i}{\beta_i^2+\alpha^2}\right), \quad (\hat{\Omega})^{-1}_{ij} = \left(A^{-1}B\right)_{ij}\,.
\end{aligned} \tag{3.221}$$

This expression for the complex structure generalizes the one derived earlier in eq. (3.203) for the T^4 case. The results are also easily generalizable to arbitrary complex structure Ω following the discussions in subsection 3.6.7 for the special case of T^4 (see eq. (3.211)).

3.6.9 Computation of Yukawa couplings

Now that we have derived both the fermionic and bosonic internal wavefunctions and expressed them as an orthonormal basis, we compute the Yukawa couplings using the basis wavefunctions (3.191). We also point out how the results derived below reduce to the ones in section 3.5.

Starting with basis functions described in eq. (3.191), for the case of the canonical complex structure $\Omega = iI_2$ (in the T^4 case), we have:

$$\begin{aligned}
\psi^{\vec{i},\mathbf{N_1},\mathbf{M_1}}(\vec{z})\cdot\psi^{\vec{j},\mathbf{N_2},\mathbf{M_2},}(\vec{z}) \;=\;& \mathcal{N}_{\vec{i}}\cdot\mathcal{N}_{\vec{j}}\cdot e^{i\pi[\vec{x}\cdot(\mathbf{N_1}+\mathbf{N_2})\cdot\vec{y}+i\vec{y}\cdot(\mathbf{M_1}+\mathbf{M_2})\cdot\vec{y}]} \\
&\cdot \sum_{\vec{l_1},\vec{l_2}\in\mathbf{Z}^n} e^{\pi i(i)[(\vec{l_1}+\vec{i})\cdot\mathbf{M_1}\cdot(\vec{l_1}+\vec{i})+(\vec{l_2}+\vec{j})\cdot\mathbf{M_2}\cdot(\vec{l_2}+\vec{j})]} \\
&\cdot\; e^{2\pi i[(\vec{l_1}+\vec{i})\cdot\mathbf{N_1}+(\vec{l_2}+\vec{j})\cdot\mathbf{N_2}]\cdot\vec{x}}e^{2\pi i(i)[(\vec{l_1}+\vec{i})\cdot\mathbf{M_1}+(\vec{l_2}+\vec{j})\cdot\mathbf{M_2}]\cdot\vec{y}}
\end{aligned} \tag{3.222}$$

This expression can be re-written as:

$$\begin{aligned}
\psi^{\vec{i},\mathbf{N_1},\mathbf{M_1}}(\vec{z})\cdot\psi^{\vec{j},\mathbf{N_2}\mathbf{M_2},}(\vec{z}) \;=\;& \mathcal{N}_{\vec{i}}\cdot\mathcal{N}_{\vec{j}}\cdot e^{i\pi[\vec{x}\cdot(\mathbf{N_1}+\mathbf{N_2})\cdot\vec{y}+i\vec{y}\cdot(\mathbf{M_1}+\mathbf{M_2})\cdot\vec{y}]} \\
&\cdot \sum_{\vec{l_1},\vec{l_2}\in\mathbf{Z}^n} e^{\pi i(i)(\vec{\mathbf{I}}^T\cdot\hat{\mathbf{Q}}\cdot\vec{\mathbf{I}})}e^{2\pi i(\vec{\mathbf{I}}^T\cdot\mathbf{Q}\cdot\vec{\mathbf{X}})}.e^{2\pi i(i)(\vec{\mathbf{I}}^T\cdot\hat{\mathbf{Q}}\cdot\vec{\mathbf{Y}})}\,,
\end{aligned} \tag{3.223}$$

where we defined the $4d$-vectors:

$$\vec{\mathbf{I}} = \begin{pmatrix} \vec{i} + \vec{l_1} \\ \vec{j} + \vec{l_2} \end{pmatrix}, \quad \vec{\mathbf{X}} = \begin{pmatrix} \vec{x} \\ \vec{x} \end{pmatrix}, \quad \vec{\mathbf{Y}} = \begin{pmatrix} \vec{y} \\ \vec{y} \end{pmatrix}, \tag{3.224}$$

and the $4d$-matrices:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{N_1} & 0 \\ 0 & \mathbf{N_2} \end{pmatrix}, \quad \hat{\mathbf{Q}} = \begin{pmatrix} \mathbf{M_1} & 0 \\ 0 & \mathbf{M_2} \end{pmatrix}. \tag{3.225}$$

Using the transformation matrix T, defined in eq. (3.74), and eqs. (3.75)-(3.79), we explicitly write the terms appearing in the exponents in the RHS of eq. (3.223) as:

$$\begin{aligned} (\vec{\mathbf{I}})^T \cdot \hat{\mathbf{Q}} \cdot (\vec{\mathbf{I}}) &= (\vec{\mathbf{I}})^T \cdot (T^{-1}T) \cdot \hat{\mathbf{Q}} \cdot (T^T(T^{-1})^T) \cdot (\vec{\mathbf{I}}), \\ (\vec{\mathbf{I}}^T \cdot \mathbf{Q} \cdot \vec{\mathbf{X}}) &= \vec{\mathbf{I}}^T \cdot (T^{-1}T) \cdot \mathbf{Q} \cdot (T^T(T^{-1})^T) \cdot \vec{\mathbf{X}}, \\ (\vec{\mathbf{I}}^T \cdot \hat{\mathbf{Q}} \cdot \vec{\mathbf{Y}}) &= \vec{\mathbf{I}}^T \cdot (T^{-1}T) \cdot \hat{\mathbf{Q}} \cdot (T^T(T^{-1})^T) \cdot \vec{\mathbf{Y}}. \end{aligned} \tag{3.226}$$

Then using:

$$\mathbf{Q}' \equiv T \cdot \mathbf{Q} \cdot T^T = \begin{pmatrix} (\mathbf{N_1} + \mathbf{N_2}) & 0 \\ 0 & \alpha(\mathbf{N_1}^{-1} + \mathbf{N_2}^{-1})\alpha^T \end{pmatrix}, \tag{3.227}$$

$$\hat{\mathbf{Q}}' \equiv T \cdot \hat{\mathbf{Q}} \cdot T^T = \begin{pmatrix} (\mathbf{M_1} + \mathbf{M_2}) & (\mathbf{M_1}\mathbf{N_1}^{-1} - \mathbf{M_2}\mathbf{N_2}^{-1})\alpha^T \\ \alpha(\mathbf{N_1}^{-1}\mathbf{M_1} - \mathbf{N_2}^{-1}\mathbf{M_2}) & \alpha(\mathbf{N_1}^{-1}\mathbf{M_1}\mathbf{N_1}^{-1} + \mathbf{N_2}^{-1}\mathbf{M_2}\mathbf{N_2}^{-1})\alpha^T \end{pmatrix},$$

$$(\vec{\mathbf{I}})^T T^{-1} = \begin{pmatrix} (\vec{i} + \vec{l_1})(\mathbf{N_1}^{-1} + \mathbf{N}_2^{-1})^{-1}\mathbf{N}_2^{-1} + (\vec{j} + \vec{l_2})(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}\mathbf{N}_1^{-1} \\ \left[(\vec{i} + \vec{l_1}) - (\vec{j} + \vec{l_2})\right](\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}\alpha^{-1} \end{pmatrix}^T, \tag{3.228}$$

and

$$(T^{-1})^T(\vec{\mathbf{I}}) = \begin{pmatrix} \mathbf{N}_2^{-1}(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}(\vec{i} + \vec{l_1}) + \mathbf{N}_1^{-1}(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}(\vec{j} + \vec{l_2}) \\ (\alpha^{-1})^T(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}[(\vec{i} + \vec{l_1}) - (\vec{j} + \vec{l_2})] \end{pmatrix}, \tag{3.229}$$

$$(T^{-1})^T(\vec{\mathbf{X}}) = \begin{pmatrix} \vec{x} \\ 0 \end{pmatrix}; \quad (T^{-1})^T(\vec{\mathbf{Y}}) = \begin{pmatrix} \vec{y} \\ 0 \end{pmatrix}, \tag{3.230}$$

we can re-write eq. (3.223) as

$$
\begin{aligned}
&\psi^{\vec{i},\mathbf{N_1},\mathbf{M_1}}(\vec{z}) \cdot \psi^{\vec{j},\mathbf{N_2}\mathbf{M_2},}(\vec{z}) = \mathcal{N}_{\vec{i}} \cdot \mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x}\cdot(\mathbf{N_1}+\mathbf{N_2})\cdot\vec{y}+i\vec{y}\cdot(\mathbf{M_1}+\mathbf{M_2})\cdot\vec{y}]} \times \\
&\sum_{\vec{l_1},\vec{l_2}\in\mathbf{Z}^n} e^{\pi i(i)\left(\{[(\vec{l_1}+\vec{i})\mathbf{N_1}+(\vec{l_2}+\vec{j})\mathbf{N_2}](\mathbf{N}_1+\mathbf{N}_2)^{-1}\}\cdot(\mathbf{M}_1+\mathbf{M}_2)\cdot\{(\mathbf{N}_1+\mathbf{N}_2)^{-1}(\mathbf{N}_1(\vec{i}+\vec{l_1})+\mathbf{N}_2(\vec{j}+\vec{l_2}))\}\right)} \times \\
&e^{2\pi i\{[(\vec{l_1}+\vec{i})\cdot\mathbf{N_1}+(\vec{l_2}+\vec{j})\cdot\mathbf{N_2}](\mathbf{N}_1+\mathbf{N}_2)^{-1}\}\cdot(\mathbf{N}_1+\mathbf{N}_2)\vec{x}} \cdot e^{2\pi i(i)\{[(\vec{l_1}+\vec{i})\cdot\mathbf{N_1}+(\vec{l_2}+\vec{j})\cdot\mathbf{N_2}](\mathbf{N}_1+\mathbf{N}_2)^{-1}\}\cdot(\mathbf{M}_1+\mathbf{M}_2)\vec{y}} \times \\
&e^{2\pi i(i)\{[(\vec{i}+\vec{l_1})-(\vec{j}+\vec{l_2})](\mathbf{N}_1^{-1}+\mathbf{N}_2^{-1})^{-1}\alpha^{-1}\}\cdot\alpha(\mathbf{N_1}^{-1}\mathbf{M_1}-\mathbf{N_2}^{-1}\mathbf{M_2})\cdot\vec{y}} \times \\
&e^{\pi i(i)\left(\{[(\vec{l_1}+\vec{i})\mathbf{N_1}+(\vec{l_2}+\vec{j})\mathbf{N_2}](\mathbf{N}_1+\mathbf{N}_2)^{-1}\}\cdot(\mathbf{M_1}\mathbf{N_1}^{-1}-\mathbf{M_2}\mathbf{N_2}^{-1})\alpha^T\{(\alpha^{-1})^T\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1[(\vec{i}-\vec{j})+(\vec{l_1}-\vec{l_2})\}\right)} \times \\
&e^{\pi i(i)\{[((\vec{i}-\vec{j})+(\vec{l_1}-\vec{l_2}))\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1}]\cdot[\alpha(\mathbf{N_1}^{-1}\mathbf{M_1}-\mathbf{N_2}^{-1}\mathbf{M_2})](\mathbf{N}_1+\mathbf{N}_2)^{-1}(\mathbf{N}_1(\vec{i}+\vec{l_1})+\mathbf{N}_2(\vec{j}+\vec{l_2}))\}} \times \\
&e^{\pi i(i)\{[((\vec{i}-\vec{j})+(\vec{l_1}-\vec{l_2}))\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1}][\alpha(\mathbf{N_1}^{-1}\mathbf{M_1}\mathbf{N_1}^{-1}+\mathbf{N_2}^{-1}\mathbf{M_2}\mathbf{N_2}^{-1})\alpha^T][(\alpha^{-1})^T\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1[(\vec{i}-\vec{j})+(\vec{l_1}-\vec{l_2})]\}}
\end{aligned}
\tag{3.231}
$$

Now, in a similar exercise as the one performed earlier in sections 3.5.2, 3.5.3, 3.5.4, we rearrange the series in eq. (3.231) in terms of new summation variables $\vec{l_3}, \vec{l_4}, \vec{m}$, whose values and ranges are assigned as in these sections.[12] With the value of $\alpha = (\det \mathbf{N_1} \det \mathbf{N_2})I$, defined in eq. (3.86), eq. (3.231) takes the form:

$$
\begin{aligned}
&\psi^{\vec{i},\mathbf{N_1},\mathbf{M_1}}(\vec{z}) \cdot \psi^{\vec{j},\mathbf{N_2}\mathbf{M_2},}(\vec{z}) = \mathcal{N}_{\vec{i}} \cdot \mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x}\cdot(\mathbf{N_1}+\mathbf{N_2})\cdot\vec{y}+i\vec{y}\cdot(\mathbf{M_1}+\mathbf{M_2})\cdot\vec{y}]} \\
&\sum_{\vec{l_3},\vec{l_4}\in\mathbf{Z}^n}\sum_{\vec{m}} e^{\pi i(i)[(\vec{i}\mathbf{N}_1+\vec{j}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot(\mathbf{M}_1+\mathbf{M}_2)\cdot[(\mathbf{N}_1+\mathbf{N}_2)^{-1}(\mathbf{N}_1\vec{i}+\mathbf{N}_2\vec{j}+\mathbf{N}_1\vec{m})+\vec{l_3}]} \times \\
&e^{2\pi i[(\vec{i}\mathbf{N}_1+\vec{j}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot(\mathbf{N}_1+\mathbf{N}_2)\vec{x}} \cdot e^{2\pi i(i)[(\vec{i}\mathbf{N}_1+\vec{j}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot(\mathbf{M}_1+\mathbf{M}_2)\vec{y}} \times \\
&e^{2\pi i(i)[(\vec{i}-\vec{j}+\vec{m})\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det\mathbf{N_1}\det\mathbf{N_2}}+\vec{l_4}]\cdot[(\det\mathbf{N_1}\det\mathbf{N_2})(\mathbf{N_1}^{-1}\mathbf{M_1}-\mathbf{N_2}^{-1}\mathbf{M_2})]\cdot\vec{y}} \times \\
&e^{\pi i(i)[(\vec{i}\mathbf{N}_1+\vec{j}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot[(\det\mathbf{N_1}\det\mathbf{N_2})(\mathbf{M_1}\mathbf{N_1}^{-1}-\mathbf{M_2}\mathbf{N_2}^{-1})]\cdot[\frac{\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1}{\det\mathbf{N_1}\det\mathbf{N_2}}(\vec{i}-\vec{j}+\vec{m})+\vec{l_4}]} \times \\
&e^{\pi i(i)[(\vec{i}-\vec{j}+\vec{m})\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det\mathbf{N_1}\det\mathbf{N_2}}+\vec{l_4}]\cdot[(\det\mathbf{N_1}\det\mathbf{N_2})(\mathbf{N_1}^{-1}\mathbf{M_1}-\mathbf{N_2}^{-1}\mathbf{M_2})]\cdot[(\mathbf{N}_1+\mathbf{N}_2)^{-1}(\mathbf{N}_1\vec{i}+\mathbf{N}_2\vec{j}+\mathbf{N}_1\vec{m})+\vec{l_3}]} \times \\
&e^{\pi i(i)[(\vec{i}-\vec{j}+\vec{m})\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det\mathbf{N_1}\det\mathbf{N_2}}+\vec{l_4}][(\det\mathbf{N_1}\det\mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{M_1}\mathbf{N_1}^{-1}+\mathbf{N_2}^{-1}\mathbf{M_2}\mathbf{N_2}^{-1})][\frac{\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1}{\det\mathbf{N_1}\det\mathbf{N_2}}(\vec{i}-\vec{j}+\vec{m})+\vec{l_4}]}
\end{aligned}
\tag{3.232}
$$

Using from eq.(3.191):

$$
\begin{aligned}
(\psi^{\vec{k},\mathbf{N_3},\mathbf{M_3}})^* = \mathcal{N}_{\vec{k}} \cdot e^{-i\pi[\vec{x}\cdot\mathbf{N_3}\cdot\vec{y}-i\vec{y}\cdot\mathbf{M_3}\cdot\vec{y}]} \\
\times \sum_{\vec{l_3'}\in\mathbf{Z}^n} e^{\pi i(i)[(\vec{l_3'}+\vec{k})\cdot\mathbf{M_3}\cdot(\vec{l_3'}+\vec{k})]} \cdot e^{-2\pi i[(\vec{l_3'}+\vec{k})\cdot\mathbf{N_3}\cdot\vec{x}-i(\vec{l_3'}+\vec{k})\cdot\mathbf{M_3}\cdot\vec{y}]},
\end{aligned}
\tag{3.233}
$$

we can then proceed to calculate the Yukawa coupling:

$$
Y_{ijk} = \sigma_{abc} g \int_{T^4} dz_i d\bar{z}_i \cdot \psi^{\vec{i},\mathbf{N_1},\mathbf{M_1}} \cdot \psi^{\vec{j},\mathbf{N_2}\mathbf{M_2}} \cdot (\psi^{\vec{k},\mathbf{N_3},\mathbf{M_3}})^* \quad (i = 1, 2) \; . \tag{3.234}
$$

[12] For details see sections 3.5.1, 3.5.2, 3.5.3, 3.5.4.

Consider first the integration over $\vec{x}$:

$$\int d^2\vec{x}\, e^{i\pi\{\vec{x}\cdot[(\mathbf{N_1}+\mathbf{N_2})-\mathbf{N_3}]\cdot\vec{y}\}} \sum_{\vec{l_3},\vec{l_4},\vec{l'_3}\in\mathbf{Z}^n} \sum_{\vec{m}} e^{2\pi i[(\vec{i}\mathbf{N}_1+\vec{j}\mathbf{N}_2+\vec{m}\mathbf{N}_1)(\mathbf{N}_1+\mathbf{N}_2)^{-1}+\vec{l_3}]\cdot(\mathbf{N}_1+\mathbf{N}_2)\vec{x}} e^{-2\pi i(\vec{l'_3}+\vec{k})\cdot\mathbf{N_3}\cdot\vec{x}} \tag{3.235}$$

which implies, using $(\mathbf{N_1}+\mathbf{N_2}) = \mathbf{N_3}$, the following conditions:

- equality of the summation indices $\vec{l_3} = \vec{l'_3}$,
- the relation $(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1)(\mathbf{N}_3)^{-1} = \vec{k}$.

Note that $(\mathbf{N_1}+\mathbf{N_2}) = \mathbf{N_3}$ is a valid condition in a triple intersection since $I_{ab}+I_{bc} = I_{ac}$, with complex conjugation taking care of the fact that $I_{ac} = -I_{ca}$, which changes the signs of $\mathbf{N}_3$ and $\mathbf{M}_3$. Also, as in section 3.5.3, 3.5.4, for any given solution of the above constraint equation for $\vec{i}, \vec{j}, \vec{k}, \vec{m}$, other solutions inside the cell of eq. (3.99) that are shifted by $\vec{m}$'s satisfying $\vec{m}\mathbf{N_1}{\mathbf{N_3}}^{-1}$: integer are also allowed. In view of this, as in eq. (3.112), we break the sum over $\vec{m}$ into two parts, one corresponding to $\vec{\tilde{m}}$, which is a given specific solution of eq. (3.105) and the other ones as given by sum over integer variables $\vec{p}$ and $\vec{\tilde{p}}$ whose ranges are as defined in eq. (3.110).

Imposing the constraints from the $\vec{x}$ integration, we obtain:

$$\begin{aligned}
Y_{ijk} = {} & \sigma_{abc} g \cdot \mathcal{N}_{\vec{i}} \cdot \mathcal{N}_{\vec{j}} \cdot \mathcal{N}_{\vec{k}} \\
& \int d^2\vec{y}\{e^{-\pi[\vec{y}\cdot(\mathbf{M_1}+\mathbf{M_2}+\mathbf{M_3})\cdot\vec{y}]} \sum_{\vec{l_3},\vec{l_4}\in\mathbf{Z}^n} \sum_{\vec{p},\vec{\tilde{p}}} e^{\pi i(i)[\vec{k}+\vec{l_3}]\cdot(\mathbf{M_1}+\mathbf{M}_2)\cdot[\vec{k}+\vec{l_3}]} \times \\
& e^{\pi i(i)[\vec{k}+\vec{l_3}]\cdot[(\det\mathbf{N_1}\det\mathbf{N_2})(\mathbf{M_1}{\mathbf{N_1}}^{-1}-\mathbf{M_2}{\mathbf{N_2}}^{-1})]\cdot[\frac{\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1}{\det\mathbf{N_1}\det\mathbf{N_2}}(\vec{i}-\vec{j}+\vec{\tilde{m}})+\vec{l_4}]} \times \\
& e^{\pi i(i)[(\vec{i}-\vec{j}+\vec{\tilde{m}})\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det\mathbf{N_1}\det\mathbf{N_2}}+\vec{l_4}]\cdot[(\det\mathbf{N_1}\det\mathbf{N_2})({\mathbf{N_1}}^{-1}\mathbf{M_1}-{\mathbf{N_2}}^{-1}\mathbf{M_2})]\cdot[\vec{k}+\vec{l_3}]} \times \\
& e^{\pi i(i)[(\vec{i}-\vec{j}+\vec{\tilde{m}})\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det\mathbf{N_1}\det\mathbf{N_2}}+\vec{l_4}][(\det\mathbf{N_1}\det\mathbf{N_2})^2({\mathbf{N_1}}^{-1}\mathbf{M_1}{\mathbf{N_1}}^{-1}+{\mathbf{N_2}}^{-1}\mathbf{M_2}{\mathbf{N_2}}^{-1})][\frac{\mathbf{N}_2(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_1}{\det\mathbf{N_1}\det\mathbf{N_2}}(\vec{i}-\vec{j}+\vec{m})+\vec{l_4}]} \\
& \times e^{\pi i(i)[\vec{k}+\vec{l_3}]\cdot(\mathbf{M}_1+\mathbf{M}_2)\cdot\vec{y}} \cdot e^{\pi i(i)[(\vec{i}-\vec{j}+\vec{m})\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det\mathbf{N_1}\det\mathbf{N_2}}+\vec{l_4}]\cdot[(\det\mathbf{N_1}\det\mathbf{N_2})(\mathbf{M_1}{\mathbf{N_1}}^{-1}-\mathbf{M_2}{\mathbf{N_2}}^{-1})]\cdot\vec{y}}\},
\end{aligned} \tag{3.236}$$

where the range of the sum over $\vec{p}, \vec{\tilde{p}}$ is as used in eq. (3.110) in section 3.5.3.

The above expression for the Yukawa interaction can be written as following:

$$
\begin{aligned}
Y_{ijk} &= \sigma_{abc} g \cdot (2^3)^{\frac{1}{4}} (|\det \mathbf{M_1}|.|\det \mathbf{M_2}|.|\det \mathbf{M_3}|)^{\frac{1}{4}} \left(Vol(T^4)\right)^{-\frac{3}{2}} \\
&\int d^2\vec{y}\{e^{-\pi[\vec{y}\cdot(\mathbf{M_1}+\mathbf{M_2}+\mathbf{M_3})\cdot\vec{y}]} \sum_{\vec{l_3},\vec{l_4}\in\mathbf{Z}^n} \sum_{\vec{p},\vec{\tilde{p}}} e^{\pi i(i)[\vec{\mathbf{K}}+\vec{\mathbf{L}}]\cdot\hat{\mathbf{Q}}'\cdot[\vec{\mathbf{K}}+\vec{\mathbf{L}}]} e^{2\pi i(i)[\vec{\mathbf{K}}+\vec{\mathbf{L}}]\cdot\vec{\mathbf{Y}}'} \\
&= \sigma_{abc} g \cdot (2^3)^{\frac{1}{4}} (|det\mathbf{M_1}|.|det\mathbf{M_2}|.|det\mathbf{M_3}|)^{\frac{1}{4}} \left(Vol(T^4)\right)^{-\frac{3}{2}} \times \\
&\sum_{\vec{p},\vec{\tilde{p}}} \int d^2\vec{y}\{e^{-\pi[\vec{y}\cdot(\mathbf{M_1}+\mathbf{M_2}+\mathbf{M_3})\cdot\vec{y}]} \cdot \vartheta \begin{bmatrix} \vec{\mathbf{K}} \\ 0 \end{bmatrix} (\vec{\mathbf{Y}}'|i\hat{\mathbf{Q}}')\}
\end{aligned} \tag{3.237}
$$

where we defined new $4d$-vectors:

$$
\vec{\mathbf{L}} = \begin{pmatrix} \vec{l_3} \\ \vec{l_4} \end{pmatrix}, \quad \vec{\mathbf{K}} = \begin{pmatrix} \vec{k} \\ [(\vec{i}-\vec{j}+\vec{m})][\frac{\mathbf{N}_1(\mathbf{N}_1+\mathbf{N}_2)^{-1}\mathbf{N}_2}{\det \mathbf{N_1} \det \mathbf{N_2}}] \end{pmatrix}, \tag{3.238}
$$

$$
\vec{\mathbf{Y}}' = \begin{pmatrix} (\mathbf{M_1}+\mathbf{M_2})\vec{y} \\ [(\det \mathbf{N_1} \det \mathbf{N_2})(\mathbf{M_1}\mathbf{N_1}^{-1} - \mathbf{M_2}\mathbf{N_2}^{-1})] \cdot \vec{y} \end{pmatrix} \tag{3.239}
$$

and the $4d$-matrix:

$$
\hat{\mathbf{Q}}' = \begin{pmatrix} (\mathbf{M_1}+\mathbf{M_2}) & (\det \mathbf{N_1} \det \mathbf{N_2})(\mathbf{M_1}\mathbf{N_1}^{-1} - \mathbf{M_2}\mathbf{N_2}^{-1}) \\ (\det \mathbf{N_1} \det \mathbf{N_2})(\mathbf{N_1}^{-1}\mathbf{M_1} - \mathbf{N_2}^{-1}\mathbf{M_2}) & (\det \mathbf{N_1} \det \mathbf{N_2})^2(\mathbf{N_1}^{-1}\mathbf{M_1}\mathbf{N_1}^{-1} + \mathbf{N_2}^{-1}\mathbf{M_2}\mathbf{N_2}^{-1}) \end{pmatrix} \tag{3.240}
$$

with $\vec{k}$ appearing in eq. (3.238) restricted by the Kronecker delta relation written above, as following from the x integration, in eq. (3.235) and the range of the sum over $\vec{p}, \vec{\tilde{p}}$ is as used in eq. (3.110) in section 3.5.3, we skip the details regarding them.

In fact, the form of the result (3.237) is valid for all basis functions, whether corresponding to positive or negative chirality wavefunctions, since the negative chirality wavefunction (3.191), written for the complex structure $\Omega = iI_2$ and used in obtaining the final answer for Yukawa coupling in eq. (3.237), reduces to the one for positive chirality wavefunction for the same complex structure when $\mathbf{M}$ is set to $\mathbf{N}$ (see eq. (3.68) for the general form of the positive chirality wavefunction). For such a choice: $\mathbf{M_i} = \mathbf{N_i}$, $\hat{\mathbf{Q}}'$ has a factorized block form and the vector $\vec{\mathbf{Y}}'$ in eq. (3.239) now has a form:

$$
\vec{\mathbf{Y}}' = \begin{pmatrix} (\mathbf{N_1}+\mathbf{N_2})\vec{y} \\ 0 \end{pmatrix}. \tag{3.241}
$$

The theta function in eq. (3.237) then factorizes and the final answer reduces to the form given in eqs. (3.104), (3.114) for the choice $\tau = i$ corresponding to the complex structure of our choice in the negative chirality wavefunction (3.181).

The Yukawa coupling expression (3.237) can be further generalized to other situations. First, although the above analysis was very specific to the case of T^4 due to our choice of wavefunction in eq. (3.191), the generlization to the T^6 is staightforward. Mapping between matrices $\mathbf{N}$ and $\mathbf{M}$ is identical and follows from the definition of $\hat{\Omega}$ in subsection 3.6.8. The final answer is identical to the one given in eq. (3.237).

Further generalization to the situation of arbitrary complex structure should also be possible, using the wavefunctions that emerge due to the mappings obtained in subsection (3.6.7) and scaling procedure presented in section (3.5.6) for the positive chirality wavefunctions. One, however, also needs to examine the symmetry property of the matrices $\mathbf{N}\hat{\Omega}\Omega$ etc., appearing in the definition of the wavefunction. We leave further details for future work.

3.7 Discussions and Conclusions

In this concluding section, we first comment on the case of magnetized branes with higher winding numbers. The form of the wrapping matrices [104] for $D9$ branes on T^6 was discussed in [7, 103]. They are real 6×6 matrices giving the embedding of the brane along spatial internal directions. The situation where worldvolume coordinates are identified with the spatial coordinates corresponds to W being diagonal. Then, for example, for a canonical complex structure $\Omega = iI_3$, the spatial components of the flux matrices are of the form given in eqs. (D.3), (D.4), (D.5). Taking into account the gauge indices, one obtains a block diagonal matrix structure for the fluxes, that reduces in the case of factorized tori to the form:

$$F = \begin{pmatrix} \frac{m_i^a}{n_i^a} I_{N^a} & \\ & \frac{m_i^b}{n_i^b} I_{N^b} \end{pmatrix}, \tag{3.242}$$

with a and b representing the brane-stacks and i denotes the i'th T^2. Also $m_i^{a,b}$ are the first Chern numbers, as given in eqs. (D.3) and (D.4), whereas $n_i^{a,b}$ are the product of the winding numbers along various 1-cycles of $(T^2)^3 \in T^6$. Also, N^a and N^b are the number of branes in stacks a and b respectively and the above expression has a straightforward generalization when many such brane stacks are involved.

In [6], a gauge theoretic picture of the magnetic fluxes along brane stacks with higher winding numbers (> 1) was given. For instance, consider the simplest choice $N^a = N^b =$

1. In this case, the configuration of the brane stacks a and b with one D-brane each, having wrapping numbers n^a, n^b and 1st Chern numbers m^a, m^b, is given by a flux matrix associated with a $U(n^a+n^b)$ gauge group with flux having the internal (gauge) components:

$$F = \begin{pmatrix} \frac{m_i^a}{n_i^a} I_{n_i^a} & \\ & \frac{m_i^b}{n_i^b} I_{n_i^b} \end{pmatrix}, \tag{3.243}$$

along the i'th T^2 and m_i^a, n_i^a etc. are relatively prime.

Given the $U(n^a + n^b)$ flux in eq. (3.243), the fermion wavefunctions associated with bifundamentals were constructed in [6]. The new feature is that, to have proper periodicity property for these fermion wavefunctions, non-abelian Wilson lines need to be turned on. In turn, these non-abelian Wilson lines mix up $n_i^a \times n_i^b$ components and the set of periodicity constraints only allows the bifundamentals belonging to the representations of the gauge group: $U(P_i^a) \times U(P_i^b)$, with $P_i^a = g.c.d.(m_i^a, n_i^a)$. In our example above we have $P_i^a = P_i^b = 1$.

The case of oblique fluxes brings in extra complexities in the analysis due to the presence of six independent 1-cycles along which non-abelian Wilson line actions need to be fixed. Given the action of these Wilson lines, one can then proceed to obtain the wavefunctions as well as the Yukawa couplings. However, unlike the factorized situation in [6], one finds that the action of non-abelian Wilson lines on the wavefunction, is dependent on the particular model, or more precisely, on the details of the oblique fluxes that are turned on. Further analysis along this line is, though cumbersome, possible.

To summarize, in this work, we have been able to explicitly generalize the Yukawa coupling expressions to the situation when the worldvolume fluxes, that are responsible for moduli stabilization, chiral mass generation, supersymmetry breaking to $N = 1$ etc., do not respect the factorization of T^6 into $(T^2)^3$. For the factorized tori, the mappings of the Yukawa couplings, superpotentials and Kähler potential between the type IIB and IIA expressions was discussed in [6]. In the IIA case, the results are obtained through a 'diagonal' wrapping of the $D6$ branes in three T^2's.

It will also be interesting to map our IIB expressions, given in this chapter to the IIA side and find the corresponding intersecting brane picture. Due to the presence of magnetic fluxes, obtaining the Type IIA picture by simply applying T-duality is not trivial. When fluxes are turned on along the three diagonal 2-tori, the corresponding T-dual picture is given by intersecting $D6$-branes, the angle of intersection being related to the magnetic flux turned along that tori. However, when there are 'oblique' fluxes present, the corresponding intersecting brane picture is not very illustrative. As stated earlier, such a

IIA construction will require putting the branes along general $SU(3)$ rotation angles and then obtain the area of the triangles corresponding to the intersections of three branes giving chiral multiplets.

Finally, it will be interesting to explore the generalization of our results to higher-point functions (computing couplings of higher dimensional effective operators) [129] and make explicit comparisons of our results with those in [124, 125], where the situation with diagonal intersection matrices $\mathbf{N_i}$, but non-factorized complex structure, is addressed through a computation of twist field correlations. However, one then needs to examine the effect of supersymmetry conditions (2.26) and (2.27) to see if the interaction indeed remains nontrivial in a supersymmetric set up.

4

Supersymmetric $SU(5)$ GUT model with Stabilized Moduli:

4.1 Introduction

In this chapter, we apply the framework described in the previous chapters, as well as the theoretical results derived in them, to construct semi-realstic models. In particular, we discuss the construction of a three generation $SU(5)$ supersymmetric grand unified (GUT) model in simple toroidal compactifications of type I string theory with magnetized $D9$ branes. The final gauge group is just $SU(5)$ and the chiral gauge non-singlet spectrum consists of three families with the quantum numbers of quarks and leptons, transforming in the $\mathbf{10} + \bar{\mathbf{5}}$ representations of $SU(5)$. Brane stacks with oblique fluxes play a central role in this construction, in order to stabilize all close string moduli. Moreover, the model is free from any chiral exotics that often appear in such brane constructions.

In the minimal case, three stacks of branes are needed to embed locally the Standard Model (SM) gauge group and the quantum numbers of quarks and leptons in their intersections [123]. They give rise to the gauge group $U(3)\times U(2)\times U(1)$, with the hypercharge being a linear combination of the three $U(1)$'s. Three different models can then be obtained, one of which corresponds to an $SU(5)$ Grand Unified Theory (GUT) when $U(3)$ and $U(2)$ are coincident. Here, we focus precisely on this $U(5)\times U(1)$ model employing two magnetized $D9$-brane stacks. Open strings stretched in the intersection of $U(5)$ with its orientifold image give rise to 3 chiral generations in the antisymmetric representation $\mathbf{10}$ of $SU(5)$, while the intersection of $U(5)$ with the orientifold image of $U(1)$ gives 3 chiral states transforming as $\bar{\mathbf{5}}$. Finally, the intersection of $U(5)$ with the $U(1)$ is non chiral, giving rise to Higgs pairs $\mathbf{5} + \bar{\mathbf{5}}$.

In order to obtain an odd number (3) of fermion generations, a NS-NS 2-form B-

field background[91, 92] must be turned on [94]. This requires the generalization of the minimal set of branes with oblique magnetic fluxes that generate only diagonal 5-brane tadpoles on the three orthogonal tori of $T^6 = \prod_{i=1}^{3} T_i^2$. We find indeed a set of eight such "oblique" branes which combined with $U(5)$ can fix all geometric moduli by the supersymmetry conditions. The metric is fixed in a diagonal form, depending on six radii given in terms of the magnetic fluxes. At the same time, all nine corresponding $U(1)$'s become massive yielding an $SU(5) \times U(1)$ gauge symmetry. This $U(1)$ factor cannot be made supersymmetric without the presence of charged scalar VEVs. Moreover, two extra branes are needed for RR tadpole cancellation, which also require non-vanishing VEVs to be made supersymmetric. As a result, all extra $U(1)$'s are broken and the only leftover gauge symmetry is an $SU(5)$ GUT. Furthermore, the intersections of the $U(5)$ stack with any additional brane used for moduli stabilization are non-chiral, yielding the three families of quarks and leptons in the $\mathbf{10}+\bar{\mathbf{5}}$ representations as the only chiral spectrum of the model (gauge non-singlet).

To elaborate further, the model is described by twelve stacks of branes, namely U_5, U_1, $O_1 \ldots, O_8$, A, and B. The $SU(5)$ gauge group arises from the open string states of stack-U_5 containing five magnetized branes. The remaining eleven stacks contain only a single magnetized brane. Also, the stack-U_5 containing the GUT gauge sector, contributes to the GUT particle spectrum through open string states which either start and end on itself[13] or on the stack-U_1, having only a single brane and therefore contributing an extra $U(1)$. For this reason we will also refer to these stacks as U_5 and U_1 stacks.

The matter sector of the $SU(5)$ GUT is specified by 3 generations of fermions in the group representations $\bar{\mathbf{5}}$ and $\mathbf{10}$ of $SU(5)$, both of left-handed helicity. In the magnetized branes construction, the $\mathbf{10}$ dimensional (antisymmetric) representation of left-handed fermions:

$$\mathbf{10} \equiv \begin{pmatrix} 0 & u_3^c & u_2^c & u_1 & d_1 \\ & 0 & u_1^c & u_2 & d_2 \\ & & 0 & u_3 & d_3 \\ & & & 0 & e^+ \\ & & & & 0 \end{pmatrix}_{\mathbf{L}} \tag{4.1}$$

arises from the doubly charged open string states starting on the stack-U_5 and ending at its orientifold image: U_5^* and vice verse. They transform as $\mathbf{10}_{(\mathbf{2},\mathbf{0})}$ of $SU(5)\times U(1)\times U(1)$, where the first $U(1)$ refers to stack-U_5 and the second one to stack-U_1, while the subscript denotes the corresponding $U(1)$ charges. The $\bar{\mathbf{5}}$ of $SU(5)$ containing left-handed chiral

[13] For simplicity, we do not distinguish a brane stack with its orientifold image, unless is explicitly stated.

fermions, or alternatively the **5** with right-handed fermions:

$$\mathbf{5} \equiv \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ \nu^c \end{pmatrix}_{\mathbf{R}} \tag{4.2}$$

are identified as states of open strings starting from stack-U_5 (with five magnetized branes) and ending on stack-U_1^* (i.e. the orientifold image of stack-U_1) and vice verse. The magnetic fluxes along the various branes are constrained by the fact that the chiral fermion spectrum, mentioned above, of the $SU(5)$ GUT should arise from these two sectors only. The appearance of this form is discussed in later Subsection (4.2.1).

Our aim, in this chapter, is to give a supersymmetric construction which incorporates the above features of $SU(5)$ GUT while stabilizing all the Kähler and complex structure moduli. More precisely, for fluxes to be supersymmetric, one demands that their holomorphic $(2,0)$ part vanishes. This condition then leads to complex structure moduli stabilization[102]. In our case we show that, for the fluxes we turn on, the complex structure Ω of T^6 is fixed to

$$\Omega = i\, \mathbb{1}_3, \tag{4.3}$$

with $\mathbb{1}_3$ being the 3×3 identity matrix.

In this chapter, we make use of the conventions given in *chapter 2*, for the parametrization of the torus T^6, as well as for the general definitions of the Kähler and complex structure moduli. In particular, the coordinates of three factorized tori: $(T^2)^3 \in T^6$ are given by x_i, y_i $i = 1, 2, 3$ with a volume normalization:

$$\int dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 = 1. \tag{4.4}$$

For Kähler moduli stabilization, we make use of the mechanism based on the magnetized D-branes supersymmetry conditions as discussed in [102, 103, 120]. Physically this corresponds to the requirement of vanishing of the potential which is generated for the moduli fields from the Fayet-Iliopoulos (FI) D-terms associated with the various branes. Even in this simplified scenario, the mammothness of the exercise is realized by noting that every magnetic flux that is introduced along any brane also induces charges corresponding to lower dimensional branes, giving rise to new tadpoles that need to be canceled. In particular, for the type I string that we are discussing, there are induced $D5$ tadpoles from fluxes along the magnetized $D9$ branes. These fluxes, in turn, are forced to be non-zero not only in order to satisfy the condition of zero net chirality among the U_5 and the extra brane stacks (except with the U_1), but in order to implement the mechanism of complex structure and Kähler moduli stabilization, as well. Specifically, for stabilizing the non-diagonal components of the metric, one is forced to introduce 'oblique' fluxes along the $D9$-branes, thus generating 'oblique' $D5$-brane tadpoles, and all these need to be canceled.

However, as mentioned earlier, we are able to find eight brane stacks $O_1, \ldots, O_8$, with different oblique fluxes, such that the combined net induced $D5$-brane charge lies only along the three diagonal directions $[x_i, y_i]$. The holomorphicity conditions of fluxes, namely the vanishing of field strengths with purely holomorphic indices, for these brane stacks stabilizes the complex structure moduli to the value (4.3). These fluxes also introduce D-term potential for the Kähler moduli. Once the complex structure is fixed as in (4.3), the fluxes in the nine stacks $U_5, O_1, \ldots, O_8$ generate potential in such a a way that all the nine Kähler moduli, $J_{i\bar{j}}$, $(i, j = 1, 2, 3)$ are completely fixed by the D-flatness conditions, imposing the vanishing of the FI terms. The residual diagonal tadpoles of the branes in the stacks U_5, U_1, $O_1, \ldots, O_8$ are then canceled by introducing the last two brane stacks A and B. D-flatness conditions for the brane stacks U_1, A and B are also satisfied, provided some VEVs of charged scalars living on these branes are turned on to cancel the corresponding FI parameters. Magnetized D-branes provide exact CFT (conformal field theory) construction of the GUT model. However, in the presence of the these non-vanishing scalar VEVs, exact CFT description is lost. The validity of the approximation then requires these VEVs to be smaller than unity in string units, a condition which is met in our case. We explicitly determine the charged scalar VEVs and verify that they all take values $v^a << 1$. Our model therefore corresponds to the Higgsing of a magnetized $D9$-brane model to be made supersymmetric through the VEVs of certain charged scalar fields on the intersections of the branes U_1, A and B.

At this point we would like to point out that, our strategy is to start with a suitable ansatz for both the complex structure (4.3) and Kähler moduli leading to diagonal internal metric. Using this ansatz, we then determine fluxes along the branes satisfying all the constraints we elaborated upon earlier. We then use the flux solutions, to show explicitly that the moduli are indeed completely fixed, consistent with our ansatz.

The chapter is organized as follows. In Section 4.2, we obtain general solutions for fluxes along magnetized $D9$-branes satisfying the necessary constraints, as described in *chapter 2*, for building the model. Moduli stabilization is discussed in Section 4.3. In Section 4.4, the VEVs of charged scalars on the stacks U_1, A and B are determined. In Section 4.5, we briefly present an analysis of the superpotential and D-terms for the model in order to show how masses for several non-chiral fermion multiplets can be generated, without evaluating explicitly the superpotential coefficients. This chapter ends with a discussion, Section 4.6, of our results. In Appendix A, the fluxes along branes are written explicitly for the stacks $O_1, \ldots, O_8$ and the associated $D5$-brane tadpoles are given. The absence of chiral fermions is also shown from these sectors. In Appendix B, complex structure stabilization is shown explicitly using the fluxes given in Appendix A. Finally, the Kähler moduli stabilization is shown in Appendix C.

4.2 Constructing a three generation $SU(5)$ GUT model

In this section, we first present in subsection 4.2.1 the brane stacks U_5 and U_1, on which the $SU(5)$ GUT, with three generations of chiral fermions, lives. Then, in subsection 4.2.2, we write down the conditions which any extra stacks, called O_a have to satisfy, so that there are no net $SU(5)$ non-singlet chiral fermions corresponding to open strings of the type: $U_5 - O_a$ and $U_5 - O_a^*$. In other words:

$$I_{U_5 O_a} + I_{U_5 O_a^*} = 0. \tag{4.5}$$

In addition, we also write down, in subsection 4.2.3, the condition that such stacks are mutually supersymmetric with the stack U_5, without turning on any charged scalar VEVs on these branes. The solution of these conditions giving eight branes $O_1, ..., O_8$ is presented in subsections 4.2.4 and 4.2.5. They are all supersymmetric, stabilize all Kähler moduli (together with stack-U_5) and cancel all tadpoles along the oblique directions, $x_i x_j$, $x_i y_j$, $y_i y_j$ for $i \neq j$. Finally in subsection 4.2.6, two more stacks A and B are found which cancel the overall $D9$ and $D5$-brane tadpoles (together with the U_1 stack).

As stated earlier, our strategy to find solutions for branes and fluxes is to first assume a canonical complex structure and Kähler moduli which have non-zero components only along the three factorized orthogonal 2-tori. In other words, we look for solutions where Kähler moduli are eventually stabilized such that

$$J_{i\bar{j}} = 0, \;\; i \neq j, \;\; (i, j = 1, 2, 3). \tag{4.6}$$

By assuming the complex structure and Kähler moduli as in eqs. (4.3) and (4.6), we then find fluxes needed to be turned on in order to cancel tadpoles. These fluxes are also used in the stabilization equations, in section 4.3 and Appendices B and C, to show that moduli are indeed completely fixed in a way that the six-torus metric becomes diagonal.

4.2.1 SU(5) GUT brane stacks

We now present the two brane stacks U_5 and U_1 which give the particle spectrum of $SU(5)$ GUT. For this purpose, we consider diagonally magnetized $D9$-branes on a factorized six-dimensional internal torus (2.17), in the presence of a NS-NS B-field turned on according to eq. (2.54). The stacks of $D9$-branes have multiplicities $N_{U_5} = 5$ and $N_{U_1} = 1$, so that an $SU(5)$ gauge group can be accommodated on the first one. Next, we impose a constraint on the windings $\hat{n}_i^{U_5}$ (defined in eq.(2.15)) of this stack by demanding that chiral fermion multiplicities in the symmetric representation of $SU(5)$ is zero. Then from eqs. (2.58), we obtain the constraint:

$$\prod_j \hat{n}_j^{U_5} = 1. \tag{4.7}$$

We solve eq. (4.7) by making the choice (2.18): $n_\alpha^{U_5} \equiv W_\alpha^{\hat{\alpha},U_5} = 1$ for the stack U_5. This also implies $\hat{n}_i^{U_5} = 1$ for $i = 1, 2, 3$. Moreover, since from (2.49) the total $D9$-brane charge has to be sixteen and higher winding numbers give larger contributions to the $D9$ tadpole, the windings in all stacks will be restricted[14] to $n_i^a = 1$ so that a maximum number of brane stacks can be accommodated (with $Q^9 = 16$), in view of fulfilling the task of stabilization.

Indeed, the stack U_5 already saturates five units of $D9$ charge while stabilizing only a single Kähler modulus. One more unit of $D9$ charge is saturated by the U_1 stack, responsible for producing the chiral fermions in the representation $\bar{\mathbf{5}}$ of $SU(5)$ at its intersection with U_5. Moreover, it cannot be made supersymmetric in the absence of charged scalar VEVs, as we will see below. Thus, stabilization of the eight remaining Kähler moduli, apart from the one stabilized by the U_5 stack, needs eight additional branes $O_1, \ldots, O_8$, contributing at least that many units of $D9$ charge (when windings are all one). These leave only two units of $D9$ charge yet to be saturated, which are also required to cancel any $D5$-brane tadpoles generated by the ten stacks, U_5, U_1 and $O_1, \ldots, O_8$. We find that this is achieved by two stacks A and B, also of windings one, so that the total $D9$ charge is $Q^9 = 16$ and all $D5$ tadpoles vanish $Q^5_{\alpha\beta} = 0$.

Now, after having imposed the condition that symmetric doubly charged representation of $SU(5)$ is absent, we find solutions for the first Chern numbers and fluxes, so that the degeneracy of chiral fermions in the antisymmetric representation ($\mathbf{10}$) of $SU(5)$ is equal to three. These multiplicities are given in eqs. (2.57), (2.61), and when applied to the stack U_5 give the constraint:

$$(2\hat{m}_1^{U_5} + 1)(2\hat{m}_2^{U_5} + 1)(2\hat{m}_3^{U_5} + 1) = 3, \tag{4.8}$$

[14] detW is restricted to be positive definite in order to avoid the presence of anti-branes.

with a solution:

$$\hat{m}_1^{U_5} = -2,\ \ \hat{m}_2^{U_5} = -1,\ \ \hat{m}_3^{U_5} = 0. \tag{4.9}$$

The corresponding flux components are:

$$p^{U_5}_{x^1y^1} = -\frac{3}{2},\ \ p^{U_5}_{x^2y^2} = -\frac{1}{2},\ \ p^{U_5}_{x^3y^3} = \frac{1}{2}, \tag{4.10}$$

associated to the total (target space) flux matrix

$$\tilde{F}^{U_5}_{(1,1)} = \begin{pmatrix} -\frac{3}{2} & & \\ & -\frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix}. \tag{4.11}$$

At this level, the choice of signs is arbitrary and is taken for convenience.

Next, we solve the condition for the presence of three generations of chiral fermions transforming in $\bar{\mathbf{5}}$ of $SU(5)$. These come from singly charged open string states starting from the U_5 stack and ending on the U_1 stack or its image. In other words, we use the condition:

$$I_{U_5U_1} + I_{U_5U_1^*} = -3. \tag{4.12}$$

To solve this condition for diagonal fluxes, one can use the formulae (2.56), or alternatively eqs. (2.59) and (2.60). In the presence of the NS-NS $B_{\alpha\beta}$-field of our choice (2.54), and using the fluxes along the U_5 stack (4.10) or (4.11), the formulae take a form:

$$(N_{U_5}, \overline{N}_{U_1}):\quad I_{U_5U_1} = (-\frac{3}{2} - F_1^{U_1})(-\frac{1}{2} - F_2^{U_1})(\frac{1}{2} - F_3^{U_1})\,, \tag{4.13}$$

$$(N_{U_5}, N_{U_1}):\quad I_{U_5U_1^*} = (-\frac{3}{2} + F_{U_1}^{1})(-\frac{1}{2} + F_2^{U_1})(\frac{1}{2} + F_3^{U_1})\,, \tag{4.14}$$

where we have used the notation $F_i^a \equiv (\tilde{F}^a_{(1,1)})_{i\bar{i}}$ for a given stack-a. We will also demand that all components $F_1^{U_1}, F_2^{U_1}, F_3^{U_1}$ are half-integers, due to the shift in 1st Chern numbers $\hat{m}_i^{U_1}$ by half a unit, in the presence of a non-zero NS-NS B-field along the three T^2's (2.17). We then get a solution of eq. (4.12):

$$I_{U_5U_1} = 0,\ \ I_{U_5U_1^*} = -3, \tag{4.15}$$

for flux components on the stack U_1:

$$F_1^{U_1} = -\frac{3}{2},\ \ F_2^{U_1} = \frac{3}{2},\ \ F_3^{U_1} = \frac{1}{2}. \tag{4.16}$$

Stack no. a	No. of branes: N_a	Windings $(\hat{n}_1^a, \hat{n}_2^a, \hat{n}_3^a)$	Chern no. $(\hat{m}_1^a, \hat{m}_2^a, \hat{m}_3^a)$	Fluxes $[\frac{(\hat{m}_1^a+\hat{n}_1^a/2)}{\hat{n}_1^a}, \frac{(\hat{m}_2^a+\hat{n}_2^a/2)}{\hat{n}_2^a}, \frac{(\hat{m}_3^a+\hat{n}_3^a/2)}{\hat{n}_3^a}]$
Stack-U_5	5	$(1,1,1)$	$(-2,-1,0)$	$[-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}]$
Stack-U_1	1	$(1,1,1)$	$(-2,1,0)$	$[-\frac{3}{2}, \frac{3}{2}, \frac{1}{2}]$

Table 4.1: Basic branes for the $SU(5)$ model

One can ask whether solutions other than (4.16) are possible for the U_1 stack. For instance, instead of the choice $(0,-3)$ of eq. (4.15) for the intersections U_5-U_1 and $U_5-U_1^*$ subject to the condition (4.12), one could try $(-3,0)$ or in general $(n,-n-3)$, for n any integer. Note that n (for $n>0$) or $-n-3$ (for $n<-3$) is the number of electroweak Higgs pairs contained in $\mathbf{5}+\bar{\mathbf{5}}$ of $SU(5)$. Thus, the cases $(-1,-2)$ and $(-2,-1)$ were excluded because of the absence of higgses, but other cases such as $n=1$ or $n=-4$ (containing one Higgs pair) are worth to explore. We leave these as exercises for the future.

The present results, including the quanta $(\hat{m}_i, \hat{n}_i)$ for both U_5 and U_1 stacks, are summarized in Table 4.1.

Moreover, the (chiral) massless spectrum under the resulting gauge group $U(5)\times U(1)$ is summarized in Table 4.2. The intersection of U_5 with U_1 is non-chiral since $I_{U_5U_1}$ vanishes. The corresponding non-chiral massless spectrum shown in the table consists of four pairs of $\mathbf{5}+\bar{\mathbf{5}}$ and will be discussed in section 4.2.7.

4.2.2 Non-chiral stacks

So far, we have obtained the gauge and matter chiral spectrum of the $SU(5)$ GUT using two stacks of magnetized branes [15]. However, in order to complete the model and stabilize all moduli, one needs to add additional stacks of magnetized branes. This has to be done in a manner such that the supersymmetries of all the brane stacks are mutually compatible. To this end, we first examine whether the first two stacks U_5 and U_1 can have mutually

[15] The gauge sector of the SU(5) arises from the open string states starting and ending on the stack-U_5.

$SU(5)\times U(1)^2$	number
$(\mathbf{10};2,0)$	3
$(\mathbf{5};1,1)$	-3
$(\overline{\mathbf{5}};-1,1)$	$4-4$

Table 4.2: Massless spectrum

compatible supersymmetry in a way suitable for moduli stabilization. The Kähler moduli stabilization conditions are written in eqs. (2.40) and (2.45), corresponding to the cases where charged scalar VEVs are respectively zero or non-zero.

Since the VEV of any charged scalar on the U_5 stack is required to be zero, in order to preserve the gauge symmetry, the supersymmetry conditions for the U_5 stack read:

$$\frac{3}{8}-\frac{1}{2}(J_1J_2-3J_2J_3-J_1J_3)=0, \tag{4.17}$$

$$J_1J_2J_3-\frac{1}{4}(-J_1-3J_2+3J_3)>0, \tag{4.18}$$

where we have used the fact that all windings are equal to unity and that eventually the Kähler moduli are stabilized according to our ansatz (4.6), such that $J_{i\bar{j}}=0$ for $i\neq j$, and we have also defined

$$J_{i\bar{i}}\equiv J_i. \tag{4.19}$$

For the U_1 stack on the other hand, one has the option of turning on a charged scalar VEV without breaking $SU(5)$ gauge invariance. However, since all windings are equal to unity, there are no charged states under $U(1)$ which are $SU(5)$ singlets. Indeed, there is no antisymmetric representation for $U(1)$, while symmetric representations are absent because of our winding choice. The only charged states then come from the intersection of U_1 with U_5 (or its image). Thus, the supersymmetry condition for the U_1 stack follows from eq. (2.40), with the fluxes given in eq. (4.16) and Table 4.1:

$$-\frac{9}{8}-\frac{1}{2}(J_1J_2-3J_2J_3+3J_1J_3)=0, \tag{4.20}$$

$$J_1J_2J_3 - \frac{1}{4}(3J_1 - 3J_2 - 9J_3) > 0. \tag{4.21}$$

Subtracting eq. (4.20) from eq. (4.17) one obtains: $J_1J_3 = -\frac{3}{4}$ which is clearly not allowed. We then conclude that the U_1 stack is not suitable for closed string moduli stabilization without charged scalar VEVs from its intersection with other brane stacks (besides U_5). We therefore need eight new $U(1)$ stacks for stabilizing all the nine geometric Kähler moduli, in the absence of open string VEVs.

In order to find such new stacks, one needs to impose the condition that any chiral fermions, other than those discussed in section 4.2.1, are $SU(5)$ singlets and thus belong to the 'hidden sector', satisfying:

$$I_{U_5a} + I_{U_5a^*} = 0, \quad \text{for } a = 1,..,8\,. \tag{4.22}$$

We then introduce eight new stacks $O_1, \ldots, O_8$, which carry in general both oblique and diagonal fluxes in order to stabilize eight of the geometric Kähler moduli, using the supersymmetry constraints (2.40). The remaining one is stabilized by the stack U_5. More precisely, to determine the brane stacks $O_1, \ldots, O_8$, we start with our ansatz for both Kähler and complex structure moduli, and use them to find out the allowed fluxes, consistent with zero net chirality and supersymmetry. Later on, we use the resulting fluxes to show the complete stabilization of moduli, and thus prove the validity of our ansatz.

In general, along a stack-a, the fluxes can be denoted by 3×3 Hermitian matrices,

$$F^a_{(1,1)} = \begin{pmatrix} f_1 & a & b \\ a^* & f_2 & c \\ b^* & c^* & f_3 \end{pmatrix}, \tag{4.23}$$

with f_i's being real numbers, and we have suppressed the superscript 'a' on the matrix components in the rhs of eq. (4.23). The relationships between the matrix elements $(F^a_{(1,1)})_{i\bar{j}}$ and the flux components $p^a_{x^ix^j}$, $p^a_{x^iy^j}$, $p^a_{y^iy^j}$ are:

$$f_i = p_{x^iy^i}\,, \quad a = p_{x^1y^2} + ip_{x^1x^2}\,, \quad b = p_{x^1y^3} + ip_{x^1x^3}\,, \quad c = p_{x^2y^3} + ip_{x^2x^3}\,. \tag{4.24}$$

The subscript $(1,1)$ will also sometimes be suppressed for notational simplicity. We now solve the non-chirality condition (4.22) that a general flux of the type (4.23) must satisfy:

$$I_{U_5a} + I_{U_5a^*} = \det(F^{U_5} - F^a) + \det(F^{U_5} + F^a) = 0\,. \tag{4.25}$$

The general solution for the flux (4.23) is:

$$\frac{3}{4} + (f_1 f_2 - 3f_2 f_3 - f_1 f_3) + (3cc^* - aa^* + bb^*) = 0. \tag{4.26}$$

All additional stacks, including $O_1, \ldots, O_8$, are required to satisfy this condition.

4.2.3 Supersymmetry constraint

We now impose an additional requirement on the fluxes along the stacks $O_1, \ldots, O_8$, that together with the stack U_5 they should satisfy the supersymmetry conditions (2.40), in the absence of charged scalar VEVs. Using F^a of eq. (4.23), the supersymmetry equations analogous to (4.17) and (4.18) for a stack O_a read:

$$\begin{aligned}
&(f_1 f_2 f_3 - cc^* f_1 - bb^* f_2 - aa^* f_3 + a^* bc^* + ab^* c) \\
&-(J_1 J_2 f_3 + J_2 J_3 f_1 + J_1 J_3 f_2) = 0, \qquad (4.27)\\
&J_1 J_2 J_3 - [J_1(f_2 f_3 - cc^*) + J_2(f_3 f_1 - bb^*) + J_3(f_1 f_2 - aa^*)] > 0. \qquad (4.28)
\end{aligned}$$

Next, we obtain two sets of fluxes of the form (4.23) which satisfy eqs. (4.26) and (4.27). The two sets, $O_1, \ldots, O_4$ and $O_5, \ldots, O_8$, are characterized by the diagonal entries in the matrix F^a (4.23), which will be the same for the branes of each set. The motivation behind such choices is dictated by the fact that once the off diagonal components of $J_{i\bar{j}}$ are fixed to zero, these two sets of fluxes along the diagonal, together with the flux of U_5 stack, determine the three diagonal elements J_i (4.19), completely.

4.2.4 Solution for the stacks $O_1, \ldots, O_4$

In order to find a constraint on the flux components f_1, f_2, f_3 and a, b, c arising out of the requirement that equations (4.17) and (4.27) should be satisfied simultaneously, we start with a particular one-parameter solution of eq. (4.17):

$$J_1 = \frac{3}{4\epsilon^2}, \quad J_2 = \frac{1}{2\epsilon} + \frac{1}{2}, \quad J_3 = \frac{1}{2\epsilon} - \frac{1}{2} \tag{4.29}$$

for arbitrary parameter $\epsilon \in (0,1)$.[16] Then, by inserting (4.29) into eq. (4.27), one obtains the relation:

$$\frac{3}{4\epsilon^3}(\frac{f_2+f_3}{2}) + \frac{1}{4\epsilon^2}[\frac{3}{2}(f_3-f_2)+f_1]$$
$$= (f_1f_2f_3 - cc^*f_1 - bb^*f_2 - aa^*f_3 + a^*bc^* + ab^*c) + \frac{f_1}{4}\,. \qquad (4.30)$$

In solving eqs. (4.26) and (4.30), satisfying also the positivity condition (4.28), we have to keep in mind that f_i's take half-integer values due to the NS-NS B-field background (2.54). On the other hand the parameters a, b, c must be integers, since the windings are all one and there is no B-field turned on along any off-diagonal 2-cycle. Our approach is then to first look for a solution of eq. (4.26) and then examine whether such a solution gives an ϵ from eq. (4.30) such that all the J_i's in eq. (4.29) are positive. In addition, both positivity conditions (4.18) and (4.28) have to be satisfied.

To solve eq. (4.26), we impose the relation $f_2 = -f_3$. The two equations (4.26) and (4.30) are then reduced to

$$\frac{3}{4} + 2f_1f_2 + 3f_2^2 + 3cc^* + bb^* - aa^* = 0, \qquad (4.31)$$

and

$$\frac{1}{4\epsilon^2}(-3f_2+f_1) = -f_1f_2^2 - cc^*f_1 - bb^*f_2 + aa^*f_2 + a^*bc^* + ab^*c + \frac{f_1}{4}. \qquad (4.32)$$

A solution of eq. (4.31) with purely real flux components is found to be:

$$f_1 = \frac{5}{2}\,, \quad f_2 = \frac{1}{2}\,, \quad f_3 = -\frac{1}{2}\,, \quad a = 4\,, \quad b = 3\,, \quad c = 1\,. \qquad (4.33)$$

Moreover, we notice from eqs. (4.31), (4.32) and the identity:

$$a^*bc^* + ab^*c = 2a_1(b_1c_1 + b_2c_2) + 2a_2(b_2c_1 - b_1c_2)\,, \qquad (4.34)$$

with $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, that other solutions can be found simply by replacing some of the real components of a, b, c by imaginary ones modulo signs, as long as the values of the products aa^*, bb^*, cc^*, as well as that of $(a^*bc^* + ab^*c)$ remain unchanged. We make use of such choices for canceling off-diagonal $D5$-brane tadpoles

[16] One can also write down a full two-parameter solution of eq. (4.17), however we prefer to use two different one-parameter families with appropriate parametrization for convenience in model building. The second one-parameter solution will be used in section 4.2.5. Equation (4.40) represents the second one-parameter solution.

which for a general flux matrix (4.23) read (using eq. (2.50)):

$$\begin{aligned} Q^{5,a}_{1\bar{1}} &= (f_2 f_3 - cc^*)\,, \quad Q^{5,a}_{2\bar{2}} = (f_3 f_1 - bb^*)\,, \quad Q^{5,a}_{3\bar{3}} = (f_1 f_2 - aa^*)\,, \\ Q^{5,a}_{1\bar{2}} &= (b^* c - a^* f_3)\,, \quad Q^{5,a}_{2\bar{3}} = (b^* a - c^* f_1)\,, \quad Q^{5,a}_{3\bar{1}} = (ac - bf_2)\,. \end{aligned} \tag{4.35}$$

Here we have used the complex coordinates $z^i, \bar{z}^i$ and the assumption that complex structure is eventually stabilized as in eq. (4.3).

The result of our analysis above, giving fluxes for the brane stacks $O_1, \ldots, O_4$, (including the solution (4.33)) is presented in Appendix A, in eqs. (A.2), (A.7), (A.12), (A.17). In this Appendix, we also show that the net chiral fermion contribution from the intersection of each of the four stacks $O_1, \ldots, O_4$ with U_5 (and its image) is zero, as shown in eqs. (A.3), (A.8), (A.13), (A.18). Oblique tadpoles $Q^5_{1\bar{2}}$, $Q^5_{2\bar{3}}$, $Q^5_{3\bar{1}}$ are given in eqs. (A.4), (A.9), (A.14), (A.19) and their cancellations among these branes is also apparent. This leaves only diagonal $D5$ tadpoles, given in eqs. (A.5), (A.10), (A.15), (A.20). The fluxes in real basis are given in eqs. (A.6), (A.11), (A.16), (A.21). In Table 4.3, we summarize all Chern numbers and windings for the stacks $O_1, \ldots, O_4$, as well as those for the stacks $O_5, \ldots, O_8$ appearing in the next subsection.

From eqs. (4.27) and (4.32), the stacks $O_1, \ldots, O_4$ satisfy the supersymmetry condition:

$$\frac{195}{8} - \frac{1}{2}[-J_1 J_2 + 5 J_2 J_3 + J_1 J_3] = 0, \tag{4.36}$$

for $\epsilon = \frac{1}{10}$ in eq. (4.29). The positivity condition (4.28) for all of them has the following final form:

$$J_1 J_2 J_3 + \frac{5}{4} J_1 + \frac{41}{4} J_2 + \frac{59}{4} J_3 > 0, \tag{4.37}$$

which is obviously satisfied for the solution (4.29) with $\epsilon = \frac{1}{10}$. Also, the chiral fermion degeneracies on the intersections $U_5 - O_a$ and $U_5 - O_a^*$ are equal to

$$I_{U_5 O_a} = -23\,, \quad I_{U_5 O_a^*} = 23\,, \quad a = 1, \ldots, 4\,, \tag{4.38}$$

giving vanishing net chirality for all of them individually. The non-trivial tadpole contributions from the four stacks are:

$$Q^9 = 4\,, \quad Q^5_{x^1 y^1} = -5\,, \quad Q^5_{x^2 y^2} = -41\,, \quad Q^5_{x^3 y^3} = -59\,. \tag{4.39}$$

Stack	No. of branes: N_{O_a}	Windings $(n^{O_a}_{x^1}, n^{O_a}_{x^2}, n^{O_a}_{x^3})$ $(n^{O_a}_{y^1}, n^{O_a}_{y^2}, n^{O_a}_{y^3})$	Diag. Chern no. $(m^{O_a}_{x^1y^1}, m^{O_a}_{x^2y^2}, m^{O_a}_{x^3y^3})$	Diagonal fluxes $[f^a_1, f^a_2, f^a_3]$	Oblique Chern no.
O_1	1	$(1,1,1)$ $(1,1,1)$	(2,0,-1)	$[\frac{5}{2},\frac{1}{2},-\frac{1}{2}]$	$m^{O_1}_{x^1y^2} = m^{O_1}_{x^2y^1} = 4$ $m^{O_1}_{x^1y^3} = m^{O_1}_{x^3y^1} = 3$ $m^{O_1}_{x^2y^3} = m^{O_1}_{x^3y^2} = 1$
O_2	1	$(1,1,1)$ $(1,1,1)$	(2,0,-1)	$[\frac{5}{2},\frac{1}{2},-\frac{1}{2}]$	$m^{O_2}_{x^1y^2} = m^{O_2}_{x^2y^1} = 4$ $m^{O_2}_{x^1y^3} = m^{O_2}_{x^3y^1} = -3$ $m^{O_2}_{x^2y^3} = m^{O_2}_{x^3y^2} = -1$
O_3	1	$(1,1,1)$ $(1,1,1)$	(2,0,-1)	$[\frac{5}{2},\frac{1}{2},-\frac{1}{2}]$	$m^{O_3}_{x^1y^2} = m^{O_3}_{x^2y^1} = -4$ $m^{O_3}_{x^3x^1} = m^{O_3}_{y^3y^1} = 3$ $m^{O_3}_{x^2x^3} = m^{O_3}_{y^2y^3} = 1$
O_4	1	$(1,1,1)$ $(1,1,1)$	(2,0,-1)	$[\frac{5}{2},\frac{1}{2},-\frac{1}{2}]$	$m^{O_4}_{x^1y^2} = m^{O_4}_{x^2y^1} = -4$ $m^{O_4}_{x^3x^1} = m^{O_4}_{y^3y^1} = -3$ $m^{O_4}_{x^2x^3} = m^{O_4}_{y^2y^3} = -1$
O_5	1	$(1,1,1)$ $(1,1,1)$	(-13,0,0)	$[\frac{-25}{2},\frac{1}{2},\frac{1}{2}]$	$m^{O_5}_{x^1x^2} = m^{O_5}_{y^1y^2} = -2$ $m^{O_5}_{x^3x^1} = m^{O_5}_{y^3y^1} = 1$ $m^{O_5}_{x^2y^3} = m^{O_5}_{x^3y^2} = 1$
O_6	1	$(1,1,1)$ $(1,1,1)$	(-13,0,0)	$[\frac{-25}{2},\frac{1}{2},\frac{1}{2}]$	$m^{O_6}_{x^1x^2} = m^{O_6}_{y^1y^2} = -2$ $m^{O_6}_{x^3x^1} = m^{O_6}_{y^3y^1} = -1$ $m^{O_6}_{x^2y^3} = m^{O_6}_{x^3y^2} = -1$
O_7	1	$(1,1,1)$ $(1,1,1)$	(-13,0,0)	$[\frac{-25}{2},\frac{1}{2},\frac{1}{2}]$	$m^{O_7}_{x^1x^2} = m^{O_7}_{y^1y^2} = 2$ $m^{O_7}_{x^1y^3} = m^{O_7}_{x^3y^1} = -1$ $m^{O_7}_{x^2x^3} = m^{O_7}_{y^2y^3} = 1$
O_8	1	$(1,1,1)$ $(1,1,1)$	(-13,0,0)	$[\frac{-25}{2},\frac{1}{2},\frac{1}{2}]$	$m^{O_8}_{x^1x^2} = m^{O_8}_{y^1y^2} = 2$ $m^{O_8}_{x^1y^3} = m^{O_8}_{x^3y^1} = 1$ $m^{O_8}_{x^2x^3} = m^{O_8}_{y^2y^3} = -1$

Table 4.3: Chern numbers and windings of the oblique stacks $O_1, \ldots, O_8$

4.2.5 Additional stacks: $O_5, \ldots, O_8$

In the last subsection we found four stacks $O_1, \ldots, O_4$ with oblique fluxes but diagonal 5-brane charges. Clearly, in order to stabilize all the Kähler moduli, we need at least four additional stacks with oblique fluxes. The search for such branes is simplified by observing that the supersymmetry condition (4.17) for the stack U_5 has another one parameter family of solutions, independent of (4.29), which solves also the condition (4.36) for the stacks $O_1, \ldots, O_4$:

$$J_1 = \frac{300\alpha}{4\alpha^2 - 99}\,, \quad J_2 = \alpha\,, \quad J_3 = \frac{99}{4\alpha}\,, \quad \text{with} \quad \alpha^2 > \frac{99}{4}\,. \tag{4.40}$$

By inserting expressions (4.40) into the general supersymmetry condition (4.27), and following steps similar to those of the last subsection, we find the set of stacks $O_5, \ldots, O_8$ given in Appendix A, with fluxes as in eqs. (A.22), (A.27), (A.32), (A.37). One of these solutions has flux components:

$$f_1 = -\frac{25}{2}\,, \quad f_2 = \frac{1}{2}\,, \quad f_3 = \frac{1}{2}\,, \quad a = -2i\,, \quad b = -i\,, \quad c = 1\,, \tag{4.41}$$

while the others can be obtained by trivial changes of the off-diagonal elements, as for the stacks $O_1, \ldots, O_4$ discussed in the previous subsection. Oblique $D5$ tadpoles are written in eqs. (A.24), (A.29), (A.34), (A.39) and the diagonal ones in eqs. (A.25), (A.30), (A.35), (A.40). The net $SU(5)$ non-singlet fermion chirality for these stacks is also zero, as shown in eqs. (A.23), (A.28), (A.33), (A.38). Once again, all off-diagonal $D5$ tadpoles of the type $Q^5_{1\bar{2}}$, $Q^5_{2\bar{3}}$ and $Q^5_{3\bar{1}}$ cancel among the contributions of the four brane stacks. In Table 4.3, we summarize the Chern numbers and windings of the stacks $O_5, \ldots, O_8$, as well.

The four stacks $O_5, \ldots, O_8$ satisfy the supersymmetry condition:

$$\frac{87}{8} - \frac{1}{2}[J_1 J_2 - 25 J_2 J_3 + J_1 J_3] = 0, \tag{4.42}$$

for

$$\alpha^2 = \frac{99}{4} \times \frac{1431}{1131}, \tag{4.43}$$

consistently with the inequality (4.40). For this value of α, the positivity conditions (4.18) and (4.21) for the U_5 and U_1 stacks are also satisfied by J_i's of the form (4.40). On the other hand, using the flux components (4.23) from Table 4.3, the positivity condition for the four new stacks takes the following form:

$$J_1 J_2 J_3 + \frac{3}{4} J_1 + \frac{29}{4} J_2 + \frac{41}{4} J_3 > 0, \tag{4.44}$$

Stack no. a	No. of branes: N_a	Windings $(\hat{n}_1^a, \hat{n}_2^a, \hat{n}_3^a)$	Chern no. $(\hat{m}_1^a, \hat{m}_2^a, \hat{m}_3^a)$	Fluxes $[\frac{(\hat{m}_1^a+\hat{n}_1^a/2)}{\hat{n}_1^a}, \frac{(\hat{m}_2^a+\hat{n}_2^a/2)}{\hat{n}_2^a}, \frac{(\hat{m}_3^a+\hat{n}_3^a/2)}{\hat{n}_3^a}]$
Stack-A	1	$(1,1,1)$	$(147,0,0)$	$[\frac{295}{2}, \frac{1}{2}, \frac{1}{2}]$
Stack-B	1	$(1,1,1)$	$(1,16,0)$	$[\frac{3}{2}, \frac{33}{2}, \frac{1}{2}]$

Table 4.4: A and B branes

and is again obviously satisfied, as is the positivity condition (4.37) for stacks $O_1, \ldots, O_4$. The final uncanceled tadpoles from these stacks are:

$$Q^9 = 4\,, \quad Q^5_{x^1y^1} = -3\,, \quad Q^5_{x^2y^2} = -29\,, \quad Q^5_{x^3y^3} = -41\,, \tag{4.45}$$

while the chiral fermion degeneracy from the intersections $U_5 - O_a$ and $U_5 - O_a^*$ is given by:

$$I_{U_5 O_a} = -14\,, \quad I_{U_5 O_a^*} = 14\,, \quad a = 5, \ldots, 8\,. \tag{4.46}$$

4.2.6 Tadpole cancellation

We now collect the tadpole contribution from different stacks to find out how the total RR charges cancel in our model by adding two extra stacks of single branes, A and B. The tadpole contributions from stacks $O_1, \ldots, O_4$ with oblique fluxes, are given in eq. (4.39), while those from stacks $O_5, \ldots O_8$ are given in eq. (4.45). In addition, the stacks U_5 and U_1 together contribute:

$$Q^9 = 6\,, \quad Q^5_{x^1y^1} = -\frac{1}{2}\,, \quad Q^5_{x^2y^2} = -\frac{9}{2}\,, \quad Q^5_{x^3y^3} = \frac{3}{2}\,, \tag{4.47}$$

where we used the flux components (4.10) and (4.16). These tadpoles are then saturated by the brane stacks A and B of Table 4.4.

Their contributions to the tadpoles are:

$$Q^9 = 2\,, \quad Q^5_{x^1y^1} = \frac{34}{4}\,, \quad Q^5_{x^2y^2} = \frac{298}{4}\,, \quad Q^5_{x^3y^3} = \frac{394}{4}\,, \tag{4.48}$$

which precisely cancel the contributions from eqs. (4.39), (4.45) and (4.47). Moreover, chiral fermion multiplicities from the intersections of stacks A and B with U_5 vanish, as well:

$$I_{U_5A} = I_{U_5A^*} = I_{U_5B} = I_{U_5B^*} = 0\,. \tag{4.49}$$

We have thus obtained fluxes for the twelve stacks, saturating both $D9$ and $D5$ tadpoles. However, for supersymmetry, we have only discussed the conditions for nine of the twelve brane stacks, namely U_5 and $O_1, \ldots, O_8$. The status of supersymmetry for the brane stacks U_1, A and B will be studied later, in section 4.4.

Before closing this section, we also examine briefly whether it would be possible to manage tadpole cancellation without adding the extra stacks A and B, within the context of our construction specified by the choice (4.15) of intersection numbers. Note that the nine stacks U_5 and $O_1, \ldots, O_8$ were the minimal ones needed for Kähler moduli stabilization, since the use of the U_1 brane for this purpose was ruled out, as we discussed in section 4.2.2. The U_1 stack on the other hand is needed to get the right $SU(5)$ particle spectrum. Thus, in order to avoid the use of stacks A and B, one needs to examine whether there are solutions, other than the one found in eq. (4.16), for fluxes along the stack-U_1 such that tadpole cancellations are possible, while a scalar VEV charged under this $U(1)$ may have to be turned on in order to maintain supersymmetry. In such a situation, one needs a winding number three ($\det W = 3$) for the stack U_1 to saturate the $D9$ tadpole. Moreover, all oblique fluxes along the U_1 stack have to vanish, otherwise they would give rise to uncanceled tadpoles in oblique directions. Then, by writing the tadpole contributions of three diagonal fluxes f_i satisfying the constraint (4.15), it can be easily seen that one is not able to cancel the combined tadpoles from stacks U_5 and $O_1, \ldots, O_8$. Such a possibility is therefore ruled out. Of course, one could try to find a solution that satisfies the constraint (4.15) but not necessarily (4.12), as we discussed already in section 4.2.1. Alternatively, one can possibly attempt to manage with just two stacks U_1 and A, by using winding number two in one of them. These are straight-forward exercises which can be examined easily.

4.2.7 Non-chiral spectrum

The degeneracies of non-chiral states coming from intersections of the stack U_5 with O_a and O_a^* are already given in eqs. (4.38) and (4.46), leading to $4 \times (23 + 14) = 148$ pairs of $(\mathbf{5} + \bar{\mathbf{5}})$ representations of $SU(5)$. They follow from the degeneracy formulae (2.56), when the net numbers of left- and right-handed fermions are equal. In our case, this is insured since $I_{U_5O_a} = -I_{U_5O_a^*}$. However, non-chiral particle spectrum also follows from eqs. (2.56), (2.57) and (2.58), when any of I_{ab}, I_{ab^*}, $I^A_{aa^*}$ and $I^S_{aa^*}$ are zero, as explained at the end of section 2.6. This occurs because for instance $\prod_i(\tilde{\hat{m}}^a_i\hat{n}^b_i \pm \hat{n}^a_i\tilde{\hat{m}}^b_i)$ vanishes along one or more of the 2-tori, T^2_j. Similarly for $I^A_{aa^*}$ or $I^S_{aa^*}$, this occurs because of the vanishing of fluxes along one or more of the T^2's. Given the fluxes in stack U_5, which are non-zero along all three T^2's, non-chiral states can come only from various intersections of the U_5 stack with other branes.

For example, the intersection numbers between stacks U_5 and U_1 are given in eq. (4.15). One sees that $I_{U_5U_1}$ is zero as $(\tilde{\hat{m}}^{U_5}_i\hat{n}^{U_1}_i - \hat{n}^{U_5}_i\tilde{\hat{m}}^{U_1}_i)$ vanishes along T^2_1 and T^2_3. However, in this case there exists a non-zero intersection number in $d = 8$ dimensions corresponding to the T^2_2 compactification of the $d = 10$ theory, given by:

$$I_{U_5U_1}|_{T^2_1,T^2_3} = (\tilde{\hat{m}}^{U_5}_2\hat{n}^{U_1}_2 - \hat{n}^{U_5}_2\tilde{\hat{m}}^{U_1}_2) = -2, \tag{4.50}$$

with the subscripts T^2_1, T^2_3 of $I_{U_5U_1}|$ standing for those tori along which the intersection number vanishes. This implies two negative chirality (right-handed) fermions in $d = 8$, in the fundamental representation of $SU(5)$. Under further compactification along T^2_1 and T^2_3, we get four Dirac spinors in $d = 4$, or equivalently four pairs of $(\mathbf{5} + \bar{\mathbf{5}})$ Weyl fermions, shown already in the massless spectrum of Table 4.2. They give rise to four pairs of electroweak higgses, having non-vanishing tree-level Yukawa couplings with the down-type quarks and leptons, as it can be easily seen.

A similar analysis for the remaining stacks A and B gives chiral spectra in $d = 6$ with degeneracies:

$$I_{U_5A}|_{T^2_3} = (\tilde{\hat{m}}^{U_5}_1\hat{n}^A_1 - \hat{n}^{U_5}_1\tilde{\hat{m}}^A_1) \times (\tilde{\hat{m}}^{U_5}_2\hat{n}^A_2 - \hat{n}^{U_5}_2\tilde{\hat{m}}^A_2) = 149\,, \tag{4.51}$$

and

$$I_{U_5A^*}|_{T^2_2} = (\tilde{\hat{m}}^{U_5}_1\hat{n}^A_1 + \hat{n}^{U_5}_1\tilde{\hat{m}}^A_1) \times (\tilde{\hat{m}}^{U_5}_2\hat{n}^A_2 + \hat{n}^{U_5}_2\tilde{\hat{m}}^A_2) = 146\,. \tag{4.52}$$

They give rise to $149 + 146 = 295$ pairs of $(\mathbf{5} + \bar{\mathbf{5}})$. Similarly, we obtain for the stack B:

$$I_{U_5B}|_{T^2_3} = (\tilde{\hat{m}}^{U_5}_1\hat{n}^B_1 - \hat{n}^{U_5}_1\tilde{\hat{m}}^B_1) \times (\tilde{\hat{m}}^{U_5}_2\hat{n}^B_2 - \hat{n}^{U_5}_2\tilde{\hat{m}}^B_2) = 51\,, \tag{4.53}$$

and

$$I_{U_5B^*}|_{T^2_1} = (\tilde{\hat{m}}^{U_5}_2\hat{n}^B_2 + \hat{n}^{U_5}_2\tilde{\hat{m}}^B_2) \times (\tilde{\hat{m}}^{U_5}_3\hat{n}^B_3 + \hat{n}^{U_5}_3\tilde{\hat{m}}^B_3) = 16\,, \tag{4.54}$$

leading to $51 + 16 = 67$ pairs of $(\mathbf{5} + \bar{\mathbf{5}})$. All these non chiral states become massive by displacing appropriately the branes A and B in directions along the tori T^2_3, T^2_2 and T^2_3, T^2_1, respectively.

In addition to the states above, there are several $SU(5)$ singlets coming from the intersections among the branes $O_1, \ldots, O_8$, U_1, A and B. Since they do not play any particular role in physics concerning our analysis, we do not discuss them explicitly here. However, such scalars from the non-chiral intersections among U_1, A and B will be used in section 4.4 for supersymmetrizing these stacks, by cancelling the corresponding non-zero FI parameters upon turning on non-trivial VEVs for these fields. The corresponding non-chiral spectrum will be therefore discussed below, in section 4.4.

4.3 Moduli stabilization

Earlier, we have found fluxes along the nine brane stacks U_5, $O_1, \ldots, O_8$, given in Tables 4.1, 4.2, 4.3, 4.4 and in Appendix A, consistent with our ansatz (4.3) for the complex structure and (4.6) for the geometric Kähler moduli. We now prove our ansatz by showing that both Ω and J are uniquely fixed to the values (4.3), (4.6) and (4.40), (4.43). To show this, we make use of the full supersymmetry conditions for the U_5 stack as well as for the stacks $O_1, \ldots, O_8$.

For the complex structure moduli stabilization, we make use of the $F^a_{(2,0)}$ condition (2.41) implying that purely holomorphic components of fluxes vanish. Then, by inserting the flux components $p_{x^i x^j}$, $p_{x^i y^j}$ $p_{y^i y^j}$, as given in Table 4.1 and Table 4.3, as well as in Appendix A, along the U_5 and $O_1, .., O_8$ stacks, we obtain a set of conditions on the complex structure matrix Ω, given explicitly in Appendix B in eqs. (B.1) - (B.47). These equations imply the final answer (4.3). The details can be found in Appendix B.

For Kähler moduli stabilization, we make use of the D-flatness condition in stacks U_5, $O_1, \ldots O_8$ which amounts to using the last two equations in (2.40). Explicit stabilization of the geometric Kähler moduli to the diagonal form, $J_{i\bar{j}} = 0$, $(i \neq j)$ is given in eqs. (C.2) - (C.26) of Appendix C. For the stabilization of the diagonal components, the relevant equations are: (4.17), (4.18), (4.36), (4.37), (4.42), (4.44). The final solution for the stabilized moduli is given in eqs. (4.40) and (4.43). The numerical values of J_i's can also be approximated as:

$$J_1 \sim 63.96\,, \quad J_2 \sim 5.59\,, \quad J_3 \sim 4.42\,. \tag{4.55}$$

4.4 Supersymmetry of stacks U_1, A and B

We now discuss the supersymmetry of the remaining stacks U_1, A and B by making use of the D-flatness conditions (2.44), (2.45) and (2.46). From these equations, suppressing the superscript a, we obtain the FI parameters ξ as:

$$\xi = \frac{F^3_{(1,1)} - J^2 F_{(1,1)}}{J^3 - J F^2_{(1,1)}} \,, \tag{4.56}$$

where we have made use of eq. (2.25) and the canonical volume normalization (4.4). Then, using the values of the magnetic fluxes in stacks U_1, A and B from Tables 4.1 and 4.4, the explicit form of the FI parameters in terms of the moduli J_i (that are already completely fixed to the values (4.55)) is given by:

$$\xi^{U_1} = \frac{-\frac{9}{8} - \frac{1}{2}(J_1J_2 - 3J_2J_3 + 3J_1J_3)}{J_1J_2J_3 - \frac{1}{4}(3J_1 - 3J_2 - 9J_3)} \,, \tag{4.57}$$

$$\xi^{A} = \frac{\frac{295}{8} - \frac{1}{2}(J_1J_2 + 295J_2J_3 + J_1J_3)}{J_1J_2J_3 - \frac{1}{4}(J_1 + 295J_2 + 295J_3)} \,, \tag{4.58}$$

$$\xi^{B} = \frac{\frac{33}{8} - \frac{1}{2}(J_1J_2 + 3J_2J_3 + 33J_1J_3)}{J_1J_2J_3 - \frac{1}{4}(33J_1 + 3J_2 + 99J_3)} \,, \tag{4.59}$$

leading to the numerical values:

$$\xi^{U_1} \sim -0.366 \,, \quad \xi^{A} \sim -4.753 \,, \quad \xi^{B} \sim -5.173 \,. \tag{4.60}$$

On the other hand, the charged scalar VEVs v_ϕ entering in the modified D-flatness conditions (2.44) and (2.45) are related to the modified FI parameters ξ^a/G^a, as it can be easily seen from the expressions (2.42) and (2.43), that are also relevant for the perturbativity criterion: $v_\phi << 1$ in string units. Their knowledge needs determination of the matter field metric G^a on the branes U_1, A and B. For this purpose, we make use of eq. (2.48) with the angles θ_i defined in eq. (2.47). One finds the following values for the metric G in the three stacks [17]:

$$G^{U_1} \sim 2.815 \,, \quad G^{A} \sim 50.45 \,, \quad G^{B} \sim 94.551 \,, \tag{4.61}$$

that lead to the modified FI parameters:

$$\frac{\xi^{U_1}}{G^{U_1}} \sim -0.130 \,, \quad \frac{\xi^{A}}{G^{A}} \sim -0.094 \,, \quad \frac{\xi^{B}}{G^{B}} \sim -0.057 \,. \tag{4.62}$$

[17] The matter metric G_ϕ is diagonal to the leading order in α' but its elements have a non-trivial (torus) moduli dependence due to the magnetic fluxes, that we calculated explicitly and the values are given in equation number (4.61).

Note that the positivity conditions (2.46), giving positive gauge couplings through eq. (2.43) for the stacks U_1, A and B, hold as well. These expressions appear also in the FI parameters ξ^a as the denominators in the rhs of eqs. (4.57) - (4.59).

The last part of the exercise is to cancel the FI parameters (4.62) with VEVs of specific charged scalars living on the branes U_1, A and B, in order to satisfy the D-flatness condition (2.44). For this we first compute the chiral fermion multiplicities on their intersections:

$$I_{U_1A} = (F^{U_1} - F^A)^3 = 0\ , \quad I_{U_1B} = (F^{U_1} - F^B)^3 = 0\ , \quad I_{AB} = (F^A - F^B)^3 = 0\,. \tag{4.63}$$

Since they all vanish, there are equal numbers of chiral and anti-chiral fields in each of these intersections. In order to determine separately their multiplicities, we follow the method used in section 4.2.7 and compute:

$$I_{U_1A}|_{T_3^2} = -149\ , \quad I_{U_1B}|_{T_3^2} = 45\ , \quad I_{AB}|_{T_3^2} = -2336\,. \tag{4.64}$$

These correspond to chiral fermion multiplicities in six dimensions generating upon compactification to $D = 4$ pairs of left- and right-handed fermions. We also have:

$$\begin{aligned} &I_{U_1A^*} = (F^{U_1} + F^A)^3 = 292\ , \quad I_{U_1B^*} = (F^{U_1} + F^B)^3 = 0 \\ &I_{AB^*} = (F^A + F^B)^3 = 149 \times 17\,, \end{aligned} \tag{4.65}$$

which gives zero net chirality for the $U_1 - B^*$ intersection. Computing

$$I_{U_1B^*}|_{T_1^2} = 18\,, \tag{4.66}$$

one then finds 18 pairs of left- and right-handed fermions in $D = 4$ from this intersection.

As a result, we have the following non-chiral fields, where the superscript refers to the two stacks between which the open string is stretched and the subscript denotes the charges under the respective $U(1)$'s : $(\phi^{U_1A}_{+-},\ \phi^{U_1A}_{-+})$, $(\phi^{U_1B}_{+-},\ \phi^{U_1B}_{-+})$, $(\phi^{AB}_{+-},\ \phi^{AB}_{-+})$, $(\phi^{U_1B^*}_{++},\ \phi^{U_1B^*}_{--})$, with fields in the brackets having multiplicities 149, 45, 2336 and 18, respectively. Restricting only to possible VEVs for these fields, eq. (2.44) takes the following form for the stacks U_1, A and B:

$$\frac{\xi^{U_1}}{G^{U_1}} + |\phi^{U_1A}_{+-}|^2 - |\phi^{U_1A}_{-+}|^2 + |\phi^{U_1B}_{+-}|^2 - |\phi^{U_1B}_{-+}|^2 + |\phi^{U_1B^*}_{++}|^2 - |\phi^{U_1B^*}_{--}|^2 = 0\,, \tag{4.67}$$

$$\frac{\xi^{A}}{G^{A}} + |\phi^{U_1A}_{-+}|^2 - |\phi^{U_1A}_{+-}|^2 + |\phi^{AB}_{+-}|^2 - |\phi^{AB}_{-+}|^2 = 0\,, \tag{4.68}$$

$$\frac{\xi^{B}}{G^{B}} + |\phi^{U_1B}_{-+}|^2 - |\phi^{U_1B}_{+-}|^2 + |\phi^{AB}_{-+}|^2 - |\phi^{AB}_{+-}|^2 + |\phi^{U_1B^*}_{++}|^2 - |\phi^{U_1B^*}_{--}|^2 = 0\,. \tag{4.69}$$

These equations can also be written as:

$$\frac{\xi^{U_1}}{G^{U_1}} + (v^{U_1})^2 = 0 \quad \Rightarrow \quad (v^{U_1})^2 = -\frac{\xi^{U_1}}{G^{U_1}}\,, \tag{4.70}$$

$$\frac{\xi^{A}}{G^{A}} + (v^{A})^2 = 0 \quad \Rightarrow \quad (v^{A})^2 = -\frac{\xi^{A}}{G^{A}}\,, \tag{4.71}$$

$$\frac{\xi^{B}}{G^{B}} + (v^{B})^2 = 0 \quad \Rightarrow \quad (v^{B})^2 = -\frac{\xi^{B}}{G^{B}}\,, \tag{4.72}$$

following the notation of eq. (2.45), where we defined:

$$\begin{aligned}
(v^{U_1})^2 &= |\phi^{U_1A}_{+-}|^2 - |\phi^{U_1A}_{-+}|^2 + |\phi^{U_1B}_{+-}|^2 - |\phi^{U_1B}_{-+}|^2 + |\phi^{U_1B^*}_{++}|^2 - |\phi^{U_1B^*}_{--}|^2 \\
&\equiv (v^{U_1A})^2 + (v^{U_1B})^2 + (v^{U_1B^*})^2\,, \qquad (4.73) \\
(v^{A})^2 &= |\phi^{U_1A}_{-+}|^2 - |\phi^{U_1A}_{+-}|^2 + |\phi^{AB}_{+-}|^2 - |\phi^{AB}_{-+}|^2 \\
&\equiv -(v^{U_1A})^2 + (v^{AB})^2\,, \qquad (4.74) \\
(v^{B})^2 &= |\phi^{U_1B}_{-+}|^2 - |\phi^{U_1B}_{+-}|^2 + |\phi^{AB}_{-+}|^2 - |\phi^{AB}_{+-}|^2 + |\phi^{U_1B^*}_{++}|^2 - |\phi^{U_1B^*}_{--}|^2 \\
&\equiv -(v^{U_1B})^2 - (v^{AB})^2 + (v^{U_1B^*})^2\,, \qquad (4.75)
\end{aligned}$$

with for instance $(v^{AB})^2 = |\phi^{AB}_{+-}|^2 - |\phi^{AB}_{-+}|^2$ and similarly for the others.

Since we have three equations and four unknowns, we choose to obtain a special solution by setting $(v^{U_1B})^2 = 0$. Equations (4.73) - (4.75) then give:

$$(v^{U_1A})^2 + (v^{U_1B^*})^2 = -\frac{\xi^{U_1}}{G^{U_1}} \sim 0.130\,, \tag{4.76}$$

$$-(v^{U_1A})^2 + (v^{AB})^2 = -\frac{\xi^{A}}{G^{A}} \sim 0.094\,, \tag{4.77}$$

$$-(v^{AB})^2 + (v^{U_1B^*})^2 = -\frac{\xi^{B}}{G^{B}} \sim 0.057\,, \tag{4.78}$$

that can be solved to obtain:

$$(v^{U_1A})^2 = -0.011\,, \quad (v^{U_1B^*})^2 = 0.141\,, \quad (v^{AB})^2 = 0.084\,. \tag{4.79}$$

Recalling from eqs. (4.73) - (4.75) that

$$\begin{aligned}&(v^{U_1A})^2 = |\phi^{U_1A}_{+-}|^2 - |\phi^{U_1A}_{-+}|^2\,, \quad (v^{U_1B^*})^2 = |\phi^{U_1B^*}_{++}|^2 - |\phi^{U_1B^*}_{--}|^2\,,\\ &(v^{AB})^2 = |\phi^{AB}_{+-}|^2 - |\phi^{AB}_{-+}|^2\,,\end{aligned} \tag{4.80}$$

and comparing with the results of eq. (4.79) (taking into account the different signs), VEVs for the fields $\phi^{U_1A}_{-+}$, $\phi^{U_1B^*}_{++}$ and ϕ^{AB}_{+-} are switched on. Moreover, as required by the validity of the approximation, the values of the charged scalar VEVs satisfy the condition $v^a << 1$ in string units.

4.5 Mass generation for non-chiral fermions

In this section, we briefly discuss one of the applications of the results derived in *chapter 3*, for giving mass to the non-chiral gauge non-singlet states of the magnetized brane model discussed in previous sections. We have constructed a three generation $SU(5)$ supersymmetric grand unified (GUT) model in simple toroidal compactifications of type I string theory with magnetized $D9$ branes. The final gauge group is just $SU(5)$ and the chiral gauge non-singlet spectrum consists of three families with the quantum numbers of quarks and leptons, transforming in the $\mathbf{10} + \bar{\mathbf{5}}$ representations of $SU(5)$. Brane stacks with oblique fluxes played a central role in this construction, in order to stabilize all close string moduli, in a manner restricting the chiral matter content to precisely that of $SU(5)$ GUT. Another interesting feature of this model is that it is free from any chiral exotics that often appear in such brane constructions. However, the model contains extra non-chiral matter that is expected to become massive at a high scale, close to that of $SU(5)$ breaking.

The results obtained in *chapter 3* can be used for examining the issue of the mass generation for these non-chiral multiplets in a supersymmetric ground state. The aim is to analyze the D and F term conditions, and show that a ground state allowing masses for the above matter multiplets is possible. The exercise will further fine tune our $SU(5)$ GUT model to the ones used in conventional grand unification.

Although, we will not be evaluating any of the Yukawa couplings explicitly, which using our results is in principle possible to do, the aim of the exercise below is to show that indeed one can give masses to non-chiral matter. Our procedure involves the analysis of both the F and D-term supersymmetry conditions. As discussed in section 4.4 certain charged scalar vacuum expectation values (VEVs) were turned on in order to restore supersymmetry in some of the “hidden" branes sector. These charged scalar VEVs gave a nontrivial solution to the D-term conditions, but left the F-terms identically zero in the vacuum. In the following, on the other hand, our aim is to find out the possibility for a large number of scalars in various chiral multiples to acquire expectation values. For this, we need to examine both the F and D conditions, as already mentioned.

As we discussed in the previous sections, the model is described by twelve stacks of branes, namely U_5, U_1, $O_1 \ldots, O_8$, A, and B. The magnetic fluxes are chosen to generate the required spectrum, to stabilize all the geometric moduli and to satisfy the RR-tadpole conditions as well. The fluxes for all the stacks are summarized in Appendix A. The fluxes for stacks U_5, U_1, A, B are purely diagonal whereas stacks $O_1 \ldots, O_8$ carry in general both oblique and diagonal fluxes. All 36 closed string moduli are fixed in a $\mathcal{N} = 1$ supersymmetric vacuum, apart from the dilaton, in a way that the T^6-torus metric becomes diagonal with the six internal radii given in terms of the integrally quantized magnetic fluxes.

Moreover, from our discussion in section 4.2, the two brane stacks U_5 and U_1 give the particle spectrum of $SU(5)$ GUT. We solve the condition $I_{U_5U_1} + I_{U_5U_1^*} = -3$ for the presence of three generations of chiral fermions transforming in $\bar{\mathbf{5}}$ of $SU(5)$ and continue with the solution $I_{U_5U_1} = 0$, $I_{U_5U_1^*} = -3$. The intersection of U_5 with U_1 is non-chiral since $I_{U_5U_1}$ vanishes. The corresponding non-chiral massless spectrum consists of four pairs of $\mathbf{5} + \bar{\mathbf{5}}$, which we would like to give mass. Obviously, we would like to keep massless at least one pair of electroweak higgses but this requires a detailed phenomenological analysis that goes beyond the scope of this work. Here, we would like only to show how to use the results obtained in *chapter 3* in order to give masses to unwanted non chiral states that often appear in magnetized brane constructions.

So, we have the following non-chiral fields where the superscript refers to the two stacks between which the open string is stretched and the subscript denotes the charges under the respective $U(1)$'s :($\phi^{U_5U_1}_{+-}$,$\phi^{U_5U_1}_{-+}$, 4), with numbers in the brackets denoting the corresponding multiplicities. Similarly, the intersections of the U_5 stack with the two extra branes A, B and their images are non-chiral, giving rise to the extra $\mathbf{5}+\bar{\mathbf{5}}$ pairs: ($\phi^{U_5A}_{+-}$,$\phi^{U_5A}_{-+}$, 149), ($\phi^{U_5A^*}_{++}$,$\phi^{U_5A^*}_{--}$, 146), ($\phi^{U_5B}_{+-}$,$\phi^{U_5B}_{-+}$, 51), ($\phi^{U_5B^*}_{++}$,$\phi^{U_5B^*}_{--}$, 16). A common feature of all these states is that they arise in non-chiral intersections, where the two brane stacks involved have diagonal fluxes and are parallel in one of the three tori. It is then straightforward to give masses by moving, say, the U_5 stack away from the others along these tori. In the language of $D9$ branes, this amounts to turn on corresponding open string Wilson lines.

On the other hand, analysis of the particle spectrum on the intersections of the stack U_5 with the oblique branes O_a and O_a^* , satisfying the condition $I_{U_5a} + I_{U_5a^*} = 0$, for $a = 1, .., 8$, leads to $4 \times (23 + 14) = 148$ pairs of $(\mathbf{5} + \bar{\mathbf{5}})$ representations of $SU(5)$ (eqs. (4.38) and (4.46)):

$$I_{U_5O_a} = -23\,, \quad I_{U_5O_a^*} = 23\,, \quad a = 1, \ldots, 4\,,$$

$$I_{U_5O_a} = -14\,, \quad I_{U_5O_a^*} = 14\,, \quad a = 5, \ldots, 8\,.$$

We then have the following chiral multiplets, ($\phi_{-+}^{U_5O_a}$, 23), ($\phi_{++}^{U_5O_a^*}$, 23), ($\phi_{-+}^{U_5O_b}$, 14), ($\phi_{++}^{U_5O_b^*}$, 14) ($a = 1, \ldots, 4$, $b = 5, \ldots, 8$). In order to examine the mass generation for these fields, one needs to write down the superpotential terms involving the above chiral multiplets, as well as those coming from the brane stacks $O_1, \cdots, O_8$ and their orientifold images. The list of the later, involving purely oblique stacks, is given in Appendix A.

Now, using the results in Appendix A in eqs. (A.45) and (A.46), one can analyze the associated superpotential and D-terms and look for supersymmetric minima. The relevant superpotential reads:

$$\begin{aligned} W = &\sum_{ijk} W_{O_1}^{ijk}\, (\phi_{+-}^{O_1U_5})^i\, (\phi_{++}^{U_5O_3^*})^j\, (\phi_{--}^{O_3^*O_1})^k + \sum_{ijk} W_{O_2}^{ijk}\, (\phi_{+-}^{O_2U_5})^i\, (\phi_{++}^{U_5O_4^*})^j\, (\phi_{--}^{O_4^*O_2})^k \\ &+ \sum_{ijk} W_{O_3}^{ijk}\, (\phi_{+-}^{O_3U_5})^i\, (\phi_{++}^{U_5O_8^*})^j\, (\phi_{--}^{O_8^*O_3})^k + \sum_{ijk} W_{O_4}^{ijk}\, (\phi_{+-}^{O_4U_5})^i\, (\phi_{++}^{U_5O_7^*})^j\, (\phi_{--}^{O_7^*O_4})^k \\ &+ \sum_{ijk} W_{O_5}^{ijk}\, (\phi_{+-}^{O_5U_5})^i\, (\phi_{++}^{U_5O_6^*})^j\, (\phi_{--}^{O_6^*O_5})^k + \sum_{ijk} W_{O_7}^{ijk}\, (\phi_{+-}^{O_7U_5})^i\, (\phi_{++}^{U_5O_8^*})^j\, (\phi_{--}^{O_8^*O_7})^k \end{aligned} \tag{4.81}$$

where the sum over i, j, k runs over the “flavor" indices. The couplings $W_{O_i}^{ijk}$, given in eq. (4.81), can be read off from our results in the previous sections. In addition to the complex structure, these also depend on the first Chern numbers of the branes in each triangle.

The F-flatness conditions $\langle \mathrm{F}_i \rangle = \langle \mathcal{D}_{\phi_i} W \rangle = 0$ (at zero superpotential, $W = 0$), imply that for each “triangle” at least two fields must have a zero VEV in order to form a supersymmetric vacuum [121]. In this theory, there exists indeed a supersymmetric vacuum where six charged fields remain unconstrained by the F-flatness conditions. Let’s choose them to be ($\phi_{--}^{O_3^*O_1}$), ($\phi_{--}^{O_4^*O_2}$), ($\phi_{--}^{O_8^*O_3}$), ($\phi_{--}^{O_7^*O_4}$), ($\phi_{--}^{O_6^*O_5}$), ($\phi_{--}^{O_8^*O_7}$) (they are neutral under the $U(1)$ of the $U(5)$). The remaining fields appearing in the superpotential acquire a mass from the F-term potential only if these unconstrained scalars possess a non-vanishing VEV.

Indeed, their masses read:

$$\begin{aligned}&M^2_{\phi_{u_5o_1}} \sim M^2_{\phi_{u_5o_3^*}} \sim \langle|\phi_{o_3^*o_1}|^2\rangle\,,\ M^2_{\phi_{u_5o_2}} \sim M^2_{\phi_{u_5o_4^*}} \sim \langle|\phi_{o_4^*o_2}|^2\rangle\,,\\ &M^2_{\phi_{u_5o_7'}} \sim M^2_{\phi_{u_5o_8^*}} \sim \langle|\phi_{o_8^*o_7'}|^2\rangle\,,\ M^2_{\phi_{u_5o_4}} \sim M^2_{\phi_{u_5o_7^*}} \sim \langle|\phi_{o_7^*o_4}|^2\rangle\,,\\ &M^2_{\phi_{u_5o_5}} \sim M^2_{\phi_{u_5o_6^*}} \sim \langle|\phi_{o_6^*o_5}|^2\rangle\,,\end{aligned} \tag{4.82}$$

where $\phi_{u_5o_7'}$ denotes linear combinations of $\phi_{u_5o_7}$ with $\phi_{u_5o_3}$ and $\phi_{o_8^*o_7'}$ denotes linear combinations of $\phi_{o_8^*o_7}$ with $\phi_{o_8^*o_3}$. Thus, the leftover massless states from the intersection of U_5 with the oblique branes are 60 pairs of $\mathbf{5}+\bar{\mathbf{5}}$: $\phi_{u_5o_a^*}$ for $a = 1,2,5$ of positive chirality together with the negative chirality states $\phi_{u_5o_a}$ for $a = 6,7$, as well as 23 linear combinations of $\phi_{u_5o_3}$ with $\phi_{u_5o_7}$, and 14 $\phi_{u_5o_4}$.

However, switching on non-zero VEVs for these fields, modifies the existing D-term conditions for the stacks of branes $O_1, \ldots O_8$. As it is described in section 4.4, the stacks U_5, $O_1 \ldots O_8$ satisfy the supersymmetry conditions in the absence of charged scalar VEVs, but VEVs for the fields $\phi^{U_1A}_{-+}$, $\phi^{U_1B^*}_{++}$ and ϕ^{AB}_{+-} are switched on, for the same supersymmetry to be preserved by the stacks U_1, A and B. The D-terms for each $U(1)$ factor of the eight branes $O_1, \ldots O_8$ read

$$\begin{aligned}&D_{O_1} = -|\phi^{O_1O_3^*}|^2\,,\ D_{O_2} = -|\phi^{O_2O_4^*}|^2\\ &D_{O_3} = -|\phi^{O_1O_3^*}|^2 - |\phi^{O_3O_8^*}|^2\,,\ D_{O_4} = -|\phi^{O_2O_4^*}|^2 - |\phi^{O_4O_7^*}|^2\\ &D_{O_5} = -|\phi^{O_5O_6^*}|^2\,,\ D_{O_6} = -|\phi^{O_5O_6^*}|^2\\ &D_{O_7} = -|\phi^{O_4O_7^*}|^2 - |\phi^{O_7O_8^*}|^2\,,\ D_{O_8} = -|\phi^{O_3O_8^*}|^2 - |\phi^{O_7O_8^*}|^2\end{aligned} \tag{4.83}$$

We can regain the supersymmetry conditions $D_a = 0$, $\forall a = 1,\ldots,8$ with $\xi_a(F^a, J) = 0$, by switching on VEVs for the following fields: $(\phi^{O_1O_5^*}_{++})$, $(\phi^{O_2O_7^*}_{++})$, $(\phi^{O_3O_7^*}_{++})$, $(\phi^{O_3O_4^*}_{++})$, $(\phi^{O_4O_8^*}_{++})$, $(\phi^{O_6O_8^*}_{++})$, provided these fields do not modify the superpotential (4.81). The modified D-terms read:

$$\begin{aligned}D_{O_1} &= -|\phi^{O_1O_3^*}|^2 + |\phi^{O_1O_5^*}|^2\\ D_{O_2} &= -|\phi^{O_2O_4^*}|^2 + |\phi^{O_2O_7^*}|^2\\ D_{O_3} &= -|\phi^{O_1O_3^*}|^2 - |\phi^{O_3O_8^*}|^2 + |\phi^{O_3O_4^*}|^2 + |\phi^{O_3O_7^*}|^2\\ D_{O_4} &= -|\phi^{O_2O_4^*}|^2 - |\phi^{O_4O_7^*}|^2 + |\phi^{O_3O_4^*}|^2 + |\phi^{O_4O_8^*}|^2\\ D_{O_5} &= -|\phi^{O_5O_6^*}|^2 + |\phi^{O_1O_5^*}|^2\\ D_{O_6} &= -|\phi^{O_5O_6^*}|^2 + |\phi^{O_6O_8^*}|^2\\ D_{O_7} &= -|\phi^{O_4O_7^*}|^2 - |\phi^{O_7O_8^*}|^2 + |\phi^{O_2O_7^*}|^2 + |\phi^{O_3O_7^*}|^2\\ D_{O_8} &= -|\phi^{O_3O_8^*}|^2 - |\phi^{O_7O_8^*}|^2 + |\phi^{O_6O_8^*}|^2 + |\phi^{O_4O_8^*}|^2\end{aligned} \tag{4.84}$$

The supersymmetry conditions $D_a = 0$, $\forall a = 1, \ldots, 8$ with $\xi_a(F^a, J) = 0$ can be simultaneously satisfied if and only if the VEVs for all these fields appearing in the expressions (4.84), have the same value, say v^2. Moreover we can restrict $v << 1$ in string units, as required by the validity of the approximation for inclusion of charged scalar fields in the D-term.

We have therefore shown the mass generation for a large set of non-chiral fields as given in eq. (4.82). It is possible, that remaining ones can also be made massive by incorporating non perturbative instanton contributions to the superpotential. We also do not give any superpotential couplings, in terms of fluxes, as given explicitly in *chapter 3*.

4.6 Discussion

In this chapter, we have constructed a three generation $SU(5)$ supersymmetric GUT in simple toroidal compactifications of type I string theory with magnetized $D9$-branes. All 36 closed string moduli are fixed in a $\mathcal{N} = 1$ supersymmetric vacuum, apart from the dilaton, in a way that the T^6-torus metric becomes diagonal with the six internal radii given in terms of the integrally quantized magnetic fluxes. Supersymmetry requirement and RR tadpole cancellation conditions impose some of the charged open string scalars (but $SU(5)$ singlets) to acquire non-vanishing VEVs, breaking part of the $U(1)$ factors. The rest become massive by absorbing the RR scalars which are part of the Kähler moduli supermultiplets. Thus, the final gauge group is just $SU(5)$ and the chiral gauge non-singlet spectrum consists of three families with the quantum numbers of quarks and leptons, transforming in the $\mathbf{10} + \bar{\mathbf{5}}$ representations of $SU(5)$. It is of course desirable to study the physics of this model in detail and perhaps to construct other more 'realistic' variations, using the same framework which has an exact string description.

As discussed in the last section, giving a mass to the non-chiral gauge non-singlet states with the quantum numbers of higgses transforming in pairs of $\mathbf{5}+\bar{\mathbf{5}}$ representations, keeping massless only one pair needed to break the electroweak symmetry is one of the obvious questions to be examined. Breaking the $SU(5)$ GUT symmetry down to the Standard Model is another important issue to be studied. This can be in principle realized at the string level separating the $U(5)$ stack into $U(3) \times U(2)$ by parallel brane displacement. However, one would like to realize at the same time the so-called doublet-triplet splitting for the Higgs $\mathbf{5} + \bar{\mathbf{5}}$ pair, i.e. giving mass to the unwanted triplets which can mediate fast proton decay and invalidate gauge coupling unification, while keeping the doublets massless. One possibility would be to deform the model by introducing angles, in realizing the $SU(5)$ breaking.

A general defect of the present construction is the absence of up-type Yukawa couplings. The recent developments in writing the instanton induced superpotential terms are also encouraging, for the purpose of examining the up-quark mass generations in a GUT setting [81, 130, 131]. In this context, it has been shown that the magnetized branes too can give rise to interesting superpotentials through the lift of fermion zero modes when fluxes are turned on.

Supersymmetry breaking is of course an important issue in model building. Though generally, for magnetized branes, one encounters instabilities in such a situation, it should be however possible to obtain non-supersymmetric magnetized brane constructions for a rich variety of fluxes accompanied by orientifold planes which can possibly project out tachyons that may be generated during the process of supersymmetry breaking. In order to study the supersymmatry breaking in the $SU(5)$ model, an attractive direction would be to start with a supersymmetry breaking vacuum in the absence of charged scalar VEVs for the extra branes needed to satisfy the RR tadpole cancellation, $U(1)\times U(1)_A\times U(1)_B$. This 'hidden sector' could then mediate supersymmetry breaking, which is mainly of D-type, to the Standard Model via gauge interactions. Gauginos can then acquire Dirac masses at one loop without breaking the R-symmetry, due to the extended supersymmetric nature of the gauge sector [132].

5

Summary

String theory provides us an exciting avenue for research. It brings together different aspects of our world in a very natural and compelling manner. To mention a few, it provides us with a ultraviolet finite theory of gravity, allows us to understand the holographic nature of the gravitational interactions and unifies all the four fundamental forces in nature. Supersymmetry appears as a consistency requirement of this theory. It is our hope that this theory will, in the future, exhibit a mechanism producing the $SU(3) \times SU(2) \times U(1)$ gauge group, the exact particle content of our world with broken supersymmetry at low scale. As we have discussed in the beginning of this thesis, enormous efforts have gone in this direction with partial success. This thesis can perhaps be considered as a small step in this direction. We have presented a detailed study of building some phenomenological models, with an exact chiral fermion spectrum and gauge group, where some/all moduli are stabilized and space-time supersymmetry is partially broken. This is done within a simple framework of toroidal compactification of type I string theory with magnetized D-branes. In the next few paragraphs we provide a summary of the wrk done in the thesis.

In *chapter 2*, we have briefly discussed the compactification of type I strings on a torus with additional background gauge flux on the D9-branes and summarize the necessary constraints needed for constructing semi-realistic models in such a framework. We reviewed the main properties of the six-dimensional toroidal compactification and its moduli space. We considered the open string propagation in the presence of constant internal magnetic fields and summarized the conditions for unbroken supersymmetry. We have discussed the closed string moduli stabilization by analyzing the conditions for the unbroken supersymmetry in the presence of stacks of magnetized D9-branes. In order to stabilize all 36 closed string geometric moduli of the torus T^6, one needs to include both diagonal and oblique fluxes. We have also studied the tadpole cancellation conditions which are required for consistency of type I string vacua. Then we discussed the low-energy spectrum of the effective theory within this compactification scheme. Since a crucial step in a three generation model building is the introduction of a NS-NS B-field background, the effects of non-zero B on the chirality and the tadpoles is summarized.

In *chapter 3*, we have obtained the close form expressions for Yukawa couplings in such magnetized brane constructions. We summarized the results for the fermion (scalar) wave functions and the Yukawa interaction for factorized tori and diagonal fluxes. In this case, the fermion wavefunctions are given by Jacobi Theta functions. The Yukawas are obtained by performing the overlap integrals of these wavefunctions and using certain identity satisfied by Jacobi theta functions. We have presented a proof of the identity. We then generalized the results to write down the expression for the Yukawa interaction when oblique fluxes are present. In order to perform this task, fermion (scalar) wavefunctions on toroidally compactified spaces are presented for general fluxes. The wavefunctions, so obtained, are given by general Riemann Theta functions with matrix valued modular parameter. We have also given explicit mappings among fermion wavefunctions, of different internal chiralities on the tori, which interchange the role of the flux components with the complex structure of the torus. By evaluating the overlap integral of the wave functions, the expressions for Yukawa couplings among chiral multiplets are obtained. This essentially leads us to construct certain mathematical identities for general Riemann theta functions. We generalized the theta identity for Riemann theta functions and presented a proof of this. We then used this new mathematical relation for writing down the expression for the Yukawa interaction when oblique fluxes consistent with supersymmetry and 'Riemann condition' requirements are present. In order to relax the later, the results are further generalized to include the wavefunctions of the other internal chiralities, in order to accommodate general fluxes consistent with supersymmetry restrictions.

Finally, in *chapter 4*, we have presented a minimal example of a supersymmetric grand unified model in a toroidal compactification of type I string theory with magnetized $D9$-branes. We obtain general solutions for fluxes along magnetized $D9$-branes yielding the chiral spectrum and gauge group of a three generation SU(5) GUT model, with no extra chiral matter nor U(1) factors. The gauge symmetry is just $SU(5)$ and the gauge non-singlet chiral spectrum contains only three families of quarks and leptons transforming in the $\mathbf{10} + \bar{\mathbf{5}}$ representations. Moreover, all geometric moduli are stabilized in terms of the background internal magnetic fluxes. Another interesting feature of this model is that it is free from any chiral exotics that often appear in such brane constructions. The flux solutions also satisfy the RR tadpole cancellation conditions resulting the model to be consistent. However, the model contains extra non-chiral matter that is expected to become massive at a high scale, close to that of $SU(5)$ breaking. We presented a brief analysis of the superpotential and D-terms for the model in order to show the mass generation for several non-chiral fermion multiplets in a supersymmetric ground state. Using the results for Yukawa couplings, we showed that a ground state allowing masses for the above matter multiplets is possible. This exercise further fine tunes our $SU(5)$ GUT model to the ones used in conventional grand unification.

Thus, the framework of toroidal string compactification, with magnetized branes, offers a possible self-consistent setup for string phenomenology, in which one can build simple calculable models of particle physics with stabilized moduli and implement low energy supersymmetry breaking that can be studied directly at the string level.

So, finally where are we? It is evidently true that, in spite of remarkable progress, we still lack a complete understanding of string theory. It is yet to produce $SU(3) \times SU(2) \times U(1)$ gauge group, the exact particle content of our world and a mechanism to break supersymmetry at low energy scale. However, we believe that pursuance will surely bring in success and conclude with an encouraging remark by Ashoke Sen, "I think we have an extremely strong candidate for the basic constituents of matter and this theory needs to be explored much more than it has been so far."[18]

[18] As appeared in http://parsareport.blogspot.com/2006/12/i-cannot-talk-about-others-but-i-am-as.html.

A

Fluxes for the stacks U_5, U_1 ,A, B, $O_1, \ldots, O_8$

In this Appendix, we write all the fluxes in the complex coordinate basis $(z, \bar{z})$ with $z = x + iy$. Then, for the windings and 1st Chern numbers of Table 4.1, we obtain:

$$F^{U_5}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} -\frac{3}{2} & & \\ & -\frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}. \tag{A.1}$$

Below, we sometimes suppress the subscript $(1,1)$ to keep the expressions simpler. The fluxes of the 8 stacks $O_1, \ldots, O_8$ can also be written in the same coordinate basis:

$$F^{O_1}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} \frac{5}{2} & 4 & 3 \\ 4 & \frac{1}{2} & 1 \\ 3 & 1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}. \tag{A.2}$$

From eq. (A.2) we get

$$|F^{U_5} + F^{O_1}| = 23\,, \quad |F^{U_5} - F^{O_1}| = -23\,, \quad |F^{O_1}| = \frac{195}{8}\,, \tag{A.3}$$

where we have used the notation $|F^{U_5} + F^{O_1}| \equiv \det(F^{U_5} + F^{O_1})$ etc. The oblique $D5$ tadpoles are:

$$Q^{O_1}_{1\bar{2}} = 3 + 2\,, \quad Q^{O_1}_{2\bar{3}} = 12 - \frac{5}{2}\,, \quad Q^{O_1}_{3\bar{1}} = 4 - \frac{3}{2}\,, \tag{A.4}$$

while the diagonal ones are:

$$Q^{O_1}_{1\bar{1}} = -\frac{5}{4}\,, \quad Q^{O_1}_{2\bar{2}} = -\frac{41}{4}\,, \quad Q^{O_1}_{3\bar{3}} = -\frac{59}{4}\,. \tag{A.5}$$

In real coordinates, the fluxes are:

$$\begin{aligned} &p^{O_1}_{x^1y^1} = \frac{5}{2},\ p^{O_1}_{x^2y^2} = -p^{O_1}_{x^3y^3} = \frac{1}{2},\ p^{O_1}_{x^1y^2} = p_{x^2y^1} = 4, \\ &p^{O_1}_{x^1y^3} = p^{O_1}_{x^3y^1} = 3,\ p^{O_1}_{x^2y^3} = p^{O_1}_{x^3y^2} = 1. \end{aligned} \tag{A.6}$$

The 1st Chern numbers given in Table 4.4 can then be read directly from the values of fluxes given above. We now give similar data for the stacks $O_2, \ldots, O_8$:

$$F^{O_2}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} \frac{5}{2} & 4 & -3 \\ 4 & \frac{1}{2} & -1 \\ -3 & -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \tag{A.7}$$

leading to:

$$|F^{U_5} + F^{O_2}| = 23\,, \quad |F^{U_5} - F^{O_2}| = -23\,, \quad |F^{O_2}| = \frac{195}{8}\,. \tag{A.8}$$

The oblique tadpoles are:

$$Q^{O_2}_{1\bar{2}} = 3 + 2\,, \quad Q^{O_2}_{2\bar{3}} = -12 + \frac{5}{2}\,, \quad Q^{O_2}_{3\bar{1}} = -4 + \frac{3}{2}\,, \tag{A.9}$$

while the diagonal tadpoles are:

$$Q^{O_2}_{1\bar{1}} = -\frac{5}{4}\,, \quad Q^{O_2}_{2\bar{2}} = -\frac{41}{4}\,, \quad Q^{O_2}_{3\bar{3}} = -\frac{59}{4}\,. \tag{A.10}$$

The fluxes in the real basis are:

$$\begin{aligned} &p^{O_2}_{x^1y^1} = \frac{5}{2},\ p^{O_2}_{x^2y^2} = -p^{O_2}_{x^3y^3} = \frac{1}{2},\ p^{O_2}_{x^1y^2} = p^{O_2}_{x^2y^1} = 4, \\ &p^{O_2}_{x^1y^3} = p^{O_2}_{x^3y^1} = -3,\ p^{O_2}_{x^2y^3} = p^{O_2}_{x^3y^2} = -1. \end{aligned} \tag{A.11}$$

$$F^{O_3}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} \frac{5}{2} & -4 & -3i \\ -4 & \frac{1}{2} & i \\ 3i & -i & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \tag{A.12}$$

leading to

$$|F^{U_5} + F^{O3}| = 23\,, \quad |F^{U_5} - F^{O3}| = -23\,, \quad |F^{O3}| = \frac{195}{8}\,. \tag{A.13}$$

The oblique tadpoles are:

$$Q^{O_3}_{1\bar{2}} = -3 - 2\,, \quad Q^{O_3}_{2\bar{3}} = -12i + \frac{5i}{2}\,, \quad Q^{O_3}_{3\bar{1}} = -4i + \frac{3i}{2}\,, \tag{A.14}$$

and the diagonal ones are:

$$Q^{O_3}_{1\bar{1}} = -\frac{5}{4}\,, \quad Q^{O_3}_{2\bar{2}} = -\frac{41}{4}\,, \quad Q^{O_3}_{3\bar{3}} = -\frac{59}{4}\,. \tag{A.15}$$

The fluxes in the real basis are:

$$\begin{aligned} &p^{O_3}_{x^1y^1} = \frac{5}{2},\ p^{O_3}_{x^2y^2} = -p^{O_3}_{x^3y^3} = \frac{1}{2},\ p^{O_3}_{x^1y^2} = p^{O_3}_{x^2y^1} = -4, \\ &p^{O_3}_{x^3x^1} = p^{O_3}_{y^3y^1} = 3,\ p^{O_3}_{x^2x^3} = p^{O_3}_{y^2y^3} = 1. \end{aligned} \tag{A.16}$$

$$F^{O_4}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & -4 & 3i \\ -4 & \frac{1}{2} & -i \\ -3i & i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \tag{A.17}$$

leading to

$$|F^{U_5} + F^{O_4}| = 23\,, \quad |F^{U_5} - F^{O_4}| = -23\,, \quad |F^{O_4}| = \frac{195}{8}\,. \tag{A.18}$$

The oblique tadpoles are:

$$Q^{O_4}_{1\bar{2}} = -3 - 2\,, \quad Q^{O_4}_{2\bar{3}} = 12i - \frac{5i}{2}\,, \quad Q^{O_4}_{3\bar{1}} = 4i - \frac{3i}{2}\,, \tag{A.19}$$

and the diagonal tadpoles are:

$$Q^{O_4}_{1\bar{1}} = -\frac{5}{4}\,, \quad Q^{O_4}_{2\bar{2}} = -\frac{41}{4}\,, \quad Q^{O_4}_{3\bar{3}} = -\frac{59}{4}\,. \tag{A.20}$$

The fluxes in the real basis are:

$$\begin{aligned} &p^{O_4}_{x^1y^1} = \frac{5}{2},\ p^{O_4}_{x^2y^2} = -p^{O_4}_{x^3y^3} = \frac{1}{2},\ p^{O_4}_{x^1y^2} = p^{O_4}_{x^2y^1} = -4, \\ &p^{O_4}_{x^3x^1} = p^{O_4}_{y^3y^1} = -3,\ p^{O_4}_{x^2x^3} = p^{O_4}_{y^2y^3} = -1. \end{aligned} \tag{A.21}$$

The stacks $O_1, \ldots, O_4$, given above, satisfy the supersymmetry conditions (4.36). We now give the set of four stacks, $O_5, \ldots, O_8$, which satisfy the supersymmetry condition (4.42) for the values of J_i given in eqs. (4.40), (4.43):

$$F^{O_5}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} -\frac{25}{2} & -2i & -i \\ 2i & \frac{1}{2} & 1 \\ i & 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}; \tag{A.22}$$

$$|F^{U_5} + F^{O_5}| = 14\,, \quad |F^{U_5} - F^{O_5}| = -14\,, \quad |F^{O_5}| = \frac{87}{8}\,; \tag{A.23}$$

$$Q^{O_5}_{1\bar{2}} = i - i\,, \quad Q^{O_5}_{2\bar{3}} = 2 + \frac{25}{2}\,, \quad Q^{O_5}_{3\bar{1}} = -2i + \frac{i}{2}\,, \tag{A.24}$$

$$Q^{O_5}_{1\bar{1}} = -\frac{3}{4}\,, \quad Q^{O_5}_{2\bar{2}} = -\frac{29}{4}\,, \quad Q^{O_5}_{3\bar{3}} = -\frac{41}{4}\,; \tag{A.25}$$

$$\begin{aligned} &p^{O_5}_{x^1y^1} = -\frac{25}{2},\ p^{O_5}_{x^2y^2} = p^{O_5}_{x^3y^3} = \frac{1}{2},\ p^{O_5}_{x^1x^2} = p^{O_5}_{y^1y^2} = -2, \\ &p^{O_5}_{x^3x^1} = p^{O_5}_{y^3y^1} = 1,\ p^{O_5}_{x^2y^3} = p^{O_5}_{x^3y^2} = 1. \end{aligned} \tag{A.26}$$

$$F^{O_6}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} -\frac{25}{2} & -2i & i \\ 2i & \frac{1}{2} & -1 \\ -i & -1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}; \tag{A.27}$$

$$|F^{U_5} + F^{O_6}| = 14\,, \quad |F^{U_5} - F^{O_6}| = -14\,, \quad |F^{O_6}| = \frac{87}{8}; \tag{A.28}$$

$$Q^{O_6}_{1\bar{2}} = i - i\,, \quad Q^{O_6}_{2\bar{3}} = -2 - \frac{25}{2}\,, \quad Q^{O_6}_{3\bar{1}} = 2i - \frac{i}{2}\,, \tag{A.29}$$

$$Q^{O_6}_{1\bar{1}} = -\frac{3}{4}\,, \quad Q^{O_6}_{2\bar{2}} = -\frac{29}{4}\,, \quad Q^{O_6}_{3\bar{3}} = -\frac{41}{4}\,; \tag{A.30}$$

$$p^{O_6}_{x^1y^1} = -\frac{25}{2},\ p^{O_6}_{x^2y^2} = p^{O_6}_{x^3y^3} = \frac{1}{2},\ p^{O_6}_{x^1x^2} = p^{O_6}_{y^1y^2} = -2,$$
$$p^{O_6}_{x^3x^1} = p^{O_6}_{y^3y^1} = -1,\ p^{O_6}_{x^2y^3} = p^{O_6}_{x^3y^2} = -1. \tag{A.31}$$

$$F^{O_7}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} -\frac{25}{2} & 2i & -1 \\ -2i & \frac{1}{2} & i \\ -1 & -i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}; \tag{A.32}$$

$$|F^{U_5} + F^{O_7}| = 14\,, \quad |F^{U_5} - F^{O_7}| = -14\,, \quad |F^{O_7}| = \frac{87}{8}\,; \tag{A.33}$$

$$Q^{O_7}_{1\bar{2}} = -i + i\,, \quad Q^{O_7}_{2\bar{3}} = -2i - \frac{25i}{2}\,, \quad Q^{O_7}_{3\bar{1}} = -2 + \frac{1}{2}\,, \tag{A.34}$$

$$Q^{O_7}_{1\bar{1}} = -\frac{3}{4}\,, \quad Q^{O_7}_{2\bar{2}} = -\frac{29}{4}\,, \quad Q^{O_7}_{3\bar{3}} = -\frac{41}{4}\,; \tag{A.35}$$

$$p^{O_7}_{x^1y^1} = -\frac{25}{2},\ p^{O_7}_{x^2y^2} = p^{O_7}_{x^3y^3} = \frac{1}{2},\ p^{O_7}_{x^1x^2} = p^{O_7}_{y^1y^2} = 2,$$
$$p^{O_7}_{x^1y^3} = p^{O_7}_{x^3y^1} = -1,\ p^{O_7}_{x^2x^3} = p^{O_7}_{y^2y^3} = 1. \tag{A.36}$$

$$F^{O_8}_{(1,1)} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} -\frac{25}{2} & 2i & 1 \\ -2i & \frac{1}{2} & -i \\ 1 & i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}; \tag{A.37}$$

$$|F^{U_5} + F^{O_8}| = 14\,, \quad |F^{U_5} - F^{O_8}| = -14\,, \quad |F^{O_8}| = \frac{87}{8}\,; \tag{A.38}$$

$$Q^{O_8}_{1\bar{2}} = -i + i\,, \quad Q^{O_8}_{2\bar{3}} = 2i + \frac{25i}{2}\,, \quad Q^{O_8}_{3\bar{1}} = 2 - \frac{1}{2}\,, \tag{A.39}$$

$$Q^{O_8}_{1\bar{1}} = -\frac{3}{4}\,, \quad Q^{O_8}_{2\bar{2}} = -\frac{29}{4}\,, \quad Q^{O_8}_{3\bar{3}} = -\frac{41}{4}\,; \tag{A.40}$$

$$p^{O_8}_{x^1y^1} = -\frac{5}{2},\ p^{O_8}_{x^2y^2} = p^{O_8}_{x^3y^3} = \frac{1}{2},\ p^{O_8}_{x^1x^2} = p^{O8}_{y^1y^2} = 2,$$
$$p^{O_8}_{x^1y^3} = p^{O_8}_{x^3y^1} = 1,\ p^{O_8}_{x^2x^3} = p^{O_8}_{y^2y^3} = -1. \tag{A.41}$$

Moreover,

$$F^{U_1} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} -\frac{3}{2} & & \\ & \frac{3}{2} & \\ & & \frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \tag{A.42}$$

$$F^{A} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} \frac{295}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \tag{A.43}$$

$$F^{B} = -\frac{i}{2}\begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix}\begin{pmatrix} \frac{3}{2} & & \\ & \frac{33}{2} & \\ & & \frac{1}{2} \end{pmatrix}\begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}. \tag{A.44}$$

Using the above fluxes, one can find out the chiral multiplets in the model. This has been done for the brane intersections involving stacks - U_5, U_1. A computation of the chiral fermion multiplicities on the intersections $O_i - O_j$ and $O_i - O_j^*$,for $i, j = 1, \ldots 8$, implies the existence of following fields in the non-chiral spectrum of the model. They are:
$(\phi^{O_1O_2}_{+-},\ \phi^{O_1O_2}_{-+},\ 40)$, $(\phi^{O_1O_3}_{+-},\ \phi^{O_1O_3}_{-+},\ 84)$, $(\phi^{O_1O_4}_{+-},\ \phi^{O_1O_4}_{-+},\ 84)$, $(\phi^{O_1O_5}_{+-},\ 20)$, $(\phi^{O_1O_6}_{+-},\ \phi^{O_1O_6}_{-+},\ 49)$, $(\phi^{O_1O_7}_{+-},\ 6)$, $(\phi^{O_1O_8}_{+-},\ 14)$, $(\phi^{O_2O_3}_{+-},\ \phi^{O_2O_3}_{-+},\ 84)$, $(\phi^{O_2O_4}_{+-},\ \phi^{O_2O_4}_{-+},\ 84)$, $(\phi^{O_2O_5}_{+-},\ \phi^{O_2O_5}_{-+},\ 49)$, $(\phi^{O_2O_6}_{+-},\ 20)$, $(\phi^{O_2O_7}_{+-},\ 14\ )$, $(\phi^{O_2O_8}_{+-},\ 6)$, $(\phi^{O_3O_4}_{+-},\ \phi^{O_3O_4}_{-+},\ 40)$, $(\phi^{O_3O_5}_{+-},\ 14)$, $(\phi^{O_3O_6}_{+-},\ 6)$, $(\phi^{O_3O_7}_{+-},\ 20)$, $(\phi^{O_3O_8}_{+-},\ \phi^{O_3O_8}_{-+},\ 49)$, $(\phi^{O_4O_5}_{+-},\ 6)$, $(\phi^{O_4O_6}_{+-},\ 14)$, $(\phi^{O_4O_7}_{+-},\ \phi^{O_4O_7}_{-+},\ 49)$, $(\phi^{O_4O_8}_{+-},\ 20)$, $(\phi^{O_5O_6}_{+-},\ \phi^{O_5O_6}_{-+},\ 8)$, $(\phi^{O_5O_7}_{+-},\ \phi^{O_5O_7}_{-+},\ 20)$, $(\phi^{O_5O_8}_{+-},\ \phi^{O_5O_8}_{-+},\ 20)$, $(\phi^{O_6O_7}_{+-},\ \phi^{O_6O_7}_{-+},\ 20)$, $(\phi^{O_6O_8}_{+-},\ \phi^{O_6O_8}_{-+},\ 20)$, $(\phi^{O_7O_8}_{+-},\ \phi^{O_7O_8}_{-+},\ 8)$, $(\phi^{O_1O_2^*}_{++},\ 59)$, $(\phi^{O_1O_3^*}_{--},\ 33)$, $(\phi^{O_1O_4^*}_{--},\ 33)$, $(\phi^{O_1O_5^*}_{++},\ 86)$, $(\phi^{O_1O_6^*}_{--},\ 10)$, $(\phi^{O_1O_7^*}_{++},\ 24)$, $(\phi^{O_1O_8^*}_{++},\ 52)$, $(\phi^{O_2O_3^*}_{--},\ 33)$, $(\phi^{O_2O_4^*}_{--},\ 33)$, $(\phi^{O_2O_5^*}_{--},\ 10)$, $(\phi^{O_2O_6^*}_{++},\ 86)$, $(\phi^{O_2O_7^*}_{++},\ 52)$, $(\phi^{O_2O_8^*}_{++},\ 24)$, $(\phi^{O_3O_4^*}_{++},\ 59)$, $(\phi^{O_3O_5^*}_{++},\ 52)$, $(\phi^{O_3O_6^*}_{++},\ 24)$, $(\phi^{O_3O_7^*}_{++},\ 86)$, $(\phi^{O_3O_8^*}_{--},\ 10)$, $(\phi^{O_4O_5^*}_{++},\ 24)$, $(\phi^{O_4O_6^*}_{++},\ 52)$, $(\phi^{O_4O_7^*}_{--},\ 10)$, $(\phi^{O_4O_8^*}_{++},\ 86)$, $(\phi^{O_5O_6^*}_{--},\ 41)$, $(\phi^{O_5O_7^*}_{++},\ 23)$, $(\phi^{O_5O_8^*}_{++},\ 23)$, $(\phi^{O_6O_7^*}_{++},\ 23)$, $(\phi^{O_6O_8^*}_{++},\ 23)$, $(\phi^{O_7O_8^*}_{--},\ 41)$.

(A.45)

As a result of a similar analysis for the remaining stacks A and B, we have also the following fields:
$(\phi^{U_5A}_{+-},\ \phi^{U_5A}_{-+},\ 149)$, $(\phi^{U_5A^*}_{++},\ \phi^{U_5A^*}_{--},\ 146)$, $(\phi^{U_5B}_{+-},\ \phi^{U_5B}_{-+},\ 51)$, $(\phi^{U_5B^*}_{++},\ \phi^{U_5B^*}_{--},\ 16)$, $(\phi^{U_1A}_{+-},\ \phi^{U_1A}_{-+},$

$$149), (\phi^{U_1B}_{+-}, \phi^{U_1B}_{-+}, 45), (\phi^{AB}_{+-}, \phi^{AB}_{-+}, 2336), (\phi^{U_1B^*}_{++}, \phi^{U_1B^*}_{--}, 18), (\phi^{U_1A^*}_{+-}, 292), (\phi^{AB^*}_{+-}, 149). \tag{A.46}$$

B

Complex structure moduli stabilization

For each stack of magnetized $D9$-branes, we have three complex conditions for the moduli of the complex structure derived from eq. (2.41).
From stack-O_1 :

$$\begin{aligned} 4\Omega_{11} + \frac{1}{2}\Omega_{21} + \Omega_{31} &= \frac{5}{2}\Omega_{12} + 4\Omega_{22} + 3\Omega_{32}, && \text{(B.1)} \\ 3\Omega_{11} + \Omega_{21} - \frac{1}{2}\Omega_{31} &= \frac{5}{2}\Omega_{13} + 4\Omega_{23} + 3\Omega_{33}, && \text{(B.2)} \\ 3\Omega_{12} + \Omega_{22} - \frac{1}{2}\Omega_{32} &= 4\Omega_{13} + \frac{1}{2}\Omega_{23} + \Omega_{33}. && \text{(B.3)} \end{aligned}$$

From stack-O_2 :

$$\begin{aligned} 4\Omega_{11} + \frac{1}{2}\Omega_{21} - \Omega_{31} &= \frac{5}{2}{}_{12} + 4\Omega_{22} - 3\Omega_{32}, && \text{(B.4)} \\ -3\Omega_{11} - \Omega_{21} - \frac{1}{2}\Omega_{31} &= \frac{5}{2}\Omega_{13} + 4\Omega_{23} - 3\Omega_{33}, && \text{(B.5)} \\ -3\Omega_{12} - \Omega_{22} - \frac{1}{2}\Omega_{32} &= 4\Omega_{13} + \frac{1}{2}\Omega_{23} - \Omega_{33}. && \text{(B.6)} \end{aligned}$$

From stack-O_3 :

$$-3\Omega_{11}\Omega_{32} + \Omega_{21}\Omega_{32} + 3\Omega_{31}\Omega_{12} - \Omega_{31}\Omega_{22} + 4\Omega_{11} - \frac{1}{2}\Omega_{21} + \frac{5}{2}\Omega_{12} - 4\Omega_{22} = 0, \quad \text{(B.7)}$$

$$-3\Omega_{11}\Omega_{33} + \Omega_{21}\Omega_{33} + 3\Omega_{13}\Omega_{31} - \Omega_{31}\Omega_{23} + \frac{1}{2}\Omega_{31} + \frac{5}{2}\Omega_{13} - 4\Omega_{23} - 3 = 0, \quad \text{(B.8)}$$

$$-3\Omega_{12}\Omega_{33} + \Omega_{22}\Omega_{33} + 3\Omega_{13}\Omega_{32} - \Omega_{23}\Omega_{32} + \frac{1}{2}\Omega_{32} - 4\Omega_{13} + \frac{1}{2}\Omega_{23} + 1 = 0. \quad \text{(B.9)}$$

From stack-O_4 :

$$3\Omega_{11}\Omega_{32} - \Omega_{21}\Omega_{32} - 3\Omega_{31}\Omega_{12} + \Omega_{31}\Omega_{22} + 4\Omega_{11} - \frac{1}{2}\Omega_{21} + \frac{5}{2}\Omega_{12} - 4\Omega_{22} = 0, \quad (B.10)$$

$$3\Omega_{11}\Omega_{33} - \Omega_{21}\Omega_{33} - 3\Omega_{13}\Omega_{31} + \Omega_{31}\Omega_{23} + \frac{1}{2}\Omega_{31} + \frac{5}{2}\Omega_{13} - 4\Omega_{23} + 3 = 0, \quad (B.11)$$

$$3\Omega_{12}\Omega_{33} - \Omega_{22}\Omega_{33} - 3\Omega_{13}\Omega_{32} + \Omega_{23}\Omega_{32} + \frac{1}{2}\Omega_{32} - 4\Omega_{13} + \frac{1}{2}\Omega_{23} - 1 = 0. \quad (B.12)$$

From stack-O_5 :

$$-2\Omega_{11}\Omega_{22} - \Omega_{11}\Omega_{32} + 2\Omega_{21}\Omega_{12} + \Omega_{31}\Omega_{12} - \frac{1}{2}\Omega_{21} - \Omega_{31} - \frac{25}{2}\Omega_{12} - 2 = 0, \quad (B.13)$$

$$-2\Omega_{11}\Omega_{23} - \Omega_{11}\Omega_{33} + 2\Omega_{21}\Omega_{13} + \Omega_{31}\Omega_{13} - \Omega_{21} - \frac{1}{2}\Omega_{31} - \frac{25}{2}\Omega_{13} - 1 = 0, \quad (B.14)$$

$$-2\Omega_{12}\Omega_{23} - \Omega_{12}\Omega_{33} + 2\Omega_{22}\Omega_{13} + \Omega_{32}\Omega_{13} - \Omega_{22} - \frac{1}{2}\Omega_{32} + \frac{1}{2}\Omega_{23} + \Omega_{33} = 0. \quad (B.15)$$

From stack-O_6 :

$$-2\Omega_{11}\Omega_{22} + \Omega_{11}\Omega_{32} + 2\Omega_{21}\Omega_{12} - \Omega_{31}\Omega_{12} - \frac{1}{2}\Omega_{21} + \Omega_{31} - \frac{25}{2}\Omega_{12} - 2 = 0, \quad (B.16)$$

$$-2\Omega_{11}\Omega_{23} + \Omega_{11}\Omega_{33} + 2\Omega_{21}\Omega_{13} - \Omega_{31}\Omega_{13} + \Omega_{21} - \frac{1}{2}\Omega_{31} - \frac{25}{2}\Omega_{13} + 1 = 0, \quad (B.17)$$

$$-2\Omega_{12}\Omega_{23} + \Omega_{12}\Omega_{33} + 2\Omega_{22}\Omega_{13} - \Omega_{32}\Omega_{13} + \Omega_{22} - \frac{1}{2}\Omega_{32} + \frac{1}{2}\Omega_{23} - \Omega_{33} = 0. \quad (B.18)$$

From stack-O_7 :

$$2\Omega_{11}\Omega_{22} - 2\Omega_{21}\Omega_{12} + \Omega_{21}\Omega_{32} - \Omega_{22}\Omega_{31} - \frac{1}{2}\Omega_{21} - \frac{25}{2}\Omega_{12} - \Omega_{32} + 2 = 0, \quad (B.19)$$

$$2\Omega_{11}\Omega_{23} - 2\Omega_{21}\Omega_{13} + \Omega_{21}\Omega_{33} - \Omega_{23}\Omega_{31} + \Omega_{11} - \frac{1}{2}\Omega_{31} - \frac{25}{2}\Omega_{13} - \Omega_{33} = 0, \quad (B.20)$$

$$2\Omega_{12}\Omega_{23} - 2\Omega_{22}\Omega_{13} + \Omega_{22}\Omega_{33} - \Omega_{23}\Omega_{32} + \Omega_{12} - \frac{1}{2}\Omega_{32} + \frac{1}{2}\Omega_{23} + 1 = 0. \quad (B.21)$$

From stack-O_8 :

$$2\Omega_{11}\Omega_{22} - 2\Omega_{21}\Omega_{12} - \Omega_{21}\Omega_{32} + \Omega_{22}\Omega_{31} - \frac{1}{2}\Omega_{21} - \frac{25}{2}\Omega_{12} + \Omega_{32} + 2 = 0, \quad (B.22)$$

$$2\Omega_{11}\Omega_{23} - 2\Omega_{21}\Omega_{13} - \Omega_{21}\Omega_{33} + \Omega_{23}\Omega_{31} - \Omega_{11} - \frac{1}{2}\Omega_{31} - \frac{25}{2}\Omega_{13} + \Omega_{33} = 0, \quad (B.23)$$

$$2\Omega_{12}\Omega_{23} - 2\Omega_{22}\Omega_{13} - \Omega_{22}\Omega_{33} + \Omega_{23}\Omega_{32} - \Omega_{12} - \frac{1}{2}\Omega_{32} + \frac{1}{2}\Omega_{23} - 1 = 0. \quad (B.24)$$

Now, from stack-O_1 and stack-O_2 one obtains from eqs. (B.1) and (B.4):

$$\Omega_{31} = 3\Omega_{32}\,, \tag{B.25}$$

and

$$4\Omega_{11} + \frac{1}{2}\Omega_{21} = \frac{5}{2}\Omega_{12} + 4\Omega_{22}\,; \tag{B.26}$$

from eqs. (B.2) and (B.5):

$$3\Omega_{11} + \Omega_{21} = 3\Omega_{33}\,, \tag{B.27}$$

and

$$-\frac{1}{2}\Omega_{31} = \frac{5}{2}\Omega_{13} + 4\Omega_{23}\,; \tag{B.28}$$

and from eqs. (B.3) and (B.6):

$$3\Omega_{12} + \Omega_{22} = \Omega_{33}\,, \tag{B.29}$$

and

$$-\frac{1}{2}\Omega_{32} = 4\Omega_{13} + \frac{1}{2}\Omega_{23}\,; \tag{B.30}$$

Similarly, from stack-O_3 and stack-O_4 one has, by adding eqs. (B.7) and (B.10):

$$4\Omega_{11} - \frac{1}{2}\Omega_{21} + \frac{5}{2}\Omega_{12} - 4\Omega_{22} = 0\,; \tag{B.31}$$

by adding eqs. (B.8) and (B.11):

$$\frac{1}{2}\Omega_{31} + \frac{5}{2}\Omega_{13} - 4\Omega_{23} = 0\,; \tag{B.32}$$

and by adding eqs. (B.9) and (B.12):

$$\frac{1}{2}\Omega_{32} - 4\Omega_{13} + \frac{1}{2}\Omega_{23} = 0. \tag{B.33}$$

Use of eqs. (B.30) and (B.33) gives:

$$\Omega_{13} = 0\,, \tag{B.34}$$

and

$$\Omega_{32} + \Omega_{23} = 0\,. \tag{B.35}$$

Moreover, one has from eqs. (B.34) and (B.32):

$$\Omega_{31} = 8\Omega_{23}\,; \tag{B.36}$$

from eqs. (B.36) and (B.25):

$$3\Omega_{32} = 8\Omega_{23}\,; \tag{B.37}$$

from eqs. (B.37) and (B.35):

$$\Omega_{32} = \Omega_{23} = 0\,; \tag{B.38}$$

and from eqs. (B.38) and (B.36):

$$\Omega_{31} = 0\,. \tag{B.39}$$

Similarly, use of eqs. (B.26) and (B.31) implies:

$$\Omega_{21} = 5\Omega_{12}\,, \tag{B.40}$$

and

$$\Omega_{11} = \Omega_{22}\,; \tag{B.41}$$

while use of eq. (B.41) in eqs. (B.27) and (B.29) gives:

$$3\Omega_{11} + \Omega_{21} - 3\Omega_{33} = 0\,, \tag{B.42}$$

and

$$3\Omega_{11} + 9\Omega_{12} - 3\Omega_{33} = 0\,. \tag{B.43}$$

Eqs. (B.42) and (B.43) give:

$$\Omega_{21} = 9\Omega_{12}\,, \tag{B.44}$$

which comparing with eq. (B.40) implies:

$$\Omega_{21} = \Omega_{12} = 0\,. \tag{B.45}$$

Using the result of eq. (B.45) into eq. (B.42) then gives (using also eq. (B.41)),

$$\Omega_{11} = \Omega_{22} = \Omega_{33} \equiv \Omega\,. \tag{B.46}$$

The value of Ω is finally determined from any of the bilinear equations, such as eq. (B.8) or (B.9):

$$\Omega = i\,. \tag{B.47}$$

C

Kähler class moduli stabilization

For the stabilization of Kähler class, let us denote for definiteness the volume of the 4-cycles associated to $J \wedge J$ as

$$(J \wedge J)_{i\bar{j}} = V_{i\bar{j}}\,, \tag{C.1}$$

where the indices $i, \bar{j}$ correspond to the $(1,1)$-cycle perpendicular to the given 4-cycle. In the above notation, the supersymmetry conditions on the Kähler moduli for the various stacks read as follows :

From stack-O_1 using eq. (A.2):

$$\frac{195}{8} - \left[\frac{5}{2}V_{1\bar{1}} + \frac{1}{2}V_{2\bar{2}} - \frac{1}{2}V_{3\bar{3}} + 4V_{1\bar{2}} + 4V_{2\bar{1}} + 3V_{1\bar{3}} + 3V_{3\bar{1}} + V_{2\bar{3}} + V_{3\bar{2}}\right] = 0, \tag{C.2}$$

from stack-O_2 using eq. (A.7):

$$\frac{195}{8} - \left[\frac{5}{2}V_{1\bar{1}} + \frac{1}{2}V_{2\bar{2}} - \frac{1}{2}V_{3\bar{3}} + 4V_{1\bar{2}} + 4V_{2\bar{1}} - 3V_{1\bar{3}} - 3V_{3\bar{1}} - V_{2\bar{3}} - V_{3\bar{2}}\right] = 0, \tag{C.3}$$

from stack-O_3 using eq. (A.12):

$$\frac{195}{8} - \left[\frac{5}{2}V_{1\bar{1}} + \frac{1}{2}V_{2\bar{2}} - \frac{1}{2}V_{3\bar{3}} - 4V_{1\bar{2}} - 4V_{2\bar{1}} - 3iV_{1\bar{3}} + 3iV_{3\bar{1}} + iV_{2\bar{3}} - iV_{3\bar{2}}\right] = 0, \tag{C.4}$$

from stack-O_4 using eq. (A.17):

$$\frac{195}{8} - \left[\frac{5}{2}V_{1\bar{1}} + \frac{1}{2}V_{2\bar{2}} - \frac{1}{2}V_{3\bar{3}} - 4V_{1\bar{2}} - 4V_{2\bar{1}} + 3iV_{1\bar{3}} - 3iV_{3\bar{1}} - iV_{2\bar{3}} + iV_{3\bar{2}}\right] = 0, \tag{C.5}$$

from stack-O_5 using eq. (A.22):

$$\frac{87}{8} - \left[\frac{-25}{2}V_{1\bar{1}} + \frac{1}{2}V_{2\bar{2}} + \frac{1}{2}V_{3\bar{3}} - 2iV_{1\bar{2}} + 2iV_{2\bar{1}} - iV_{1\bar{3}} + iV_{3\bar{1}} + V_{2\bar{3}} + V_{3\bar{2}}\right] = 0, \tag{C.6}$$

from stack-O_6 using eq. (A.27):

$$\frac{87}{8} - \left[\frac{-25}{2}V_{1\bar{1}} + \frac{1}{2}V_{2\bar{2}} + \frac{1}{2}V_{3\bar{3}} - 2iV_{1\bar{2}} + 2iV_{2\bar{1}} + iV_{1\bar{3}} - iV_{3\bar{1}} - V_{2\bar{3}} - V_{3\bar{2}}\right] = 0, \tag{C.7}$$

from stack-O_7 using eq. (A.32):

$$\frac{87}{8} - \left[\frac{-25}{2}V_{1\bar{1}} + \frac{1}{2}V_{2\bar{2}} + \frac{1}{2}V_{3\bar{3}} + 2iV_{1\bar{2}} - 2iV_{2\bar{1}} - V_{1\bar{3}} - V_{3\bar{1}} + iV_{2\bar{3}} - iV_{3\bar{2}}\right] = 0, \tag{C.8}$$

from stack-O_8 using eq. (A.37):

$$\frac{87}{8} - \left[\frac{-25}{2}V_{1\bar{1}} + \frac{1}{2}V_{2\bar{2}} + \frac{1}{2}V_{3\bar{3}} + 2iV_{1\bar{2}} - 2iV_{2\bar{1}} + V_{1\bar{3}} + V_{3\bar{1}} - iV_{2\bar{3}} + iV_{3\bar{2}}\right] = 0. \tag{C.9}$$

Now, from stacks-O_1 and O_2, eqs. (C.2) and (C.3) give:

$$3\left(V_{1\bar{3}} + V_{3\bar{1}}\right) + \left(V_{2\bar{3}} + V_{3\bar{2}}\right) = 0; \tag{C.10}$$

from stacks-O_3 and O_4, eqs. (C.4) and (C.5) give:

$$-3i\left(V_{1\bar{3}} - V_{3\bar{1}}\right) + i\left(V_{2\bar{3}} - V_{3\bar{2}}\right) = 0; \tag{C.11}$$

from stacks-O_5 and O_6, eqs. (C.6) and (C.7) give:

$$-i\left(V_{1\bar{3}} - V_{3\bar{1}}\right) + \left(V_{2\bar{3}} + V_{3\bar{2}}\right) = 0; \tag{C.12}$$

and from stacks-O_7 and O_8, eqs. (C.8) and (C.9) give:

$$-\left(V_{1\bar{3}} + V_{3\bar{1}}\right) + i\left(V_{2\bar{3}} - V_{3\bar{2}}\right) = 0. \tag{C.13}$$

Eq. (C.13) implies

$$i\left(V_{2\bar{3}} - V_{3\bar{2}}\right) = \left(V_{1\bar{3}} + V_{3\bar{1}}\right), \tag{C.14}$$

which leads from eq. (C.10)

$$3i\left(V_{2\bar{3}} - V_{3\bar{2}}\right) + \left(V_{2\bar{3}} + V_{3\bar{2}}\right) = 0\,. \tag{C.15}$$

Similarly, eq.(C.12) implies

$$i\left(V_{1\bar{3}} - V_{3\bar{1}}\right) = \left(V_{2\bar{3}} + V_{3\bar{2}}\right), \tag{C.16}$$

which leads from eq. (C.11)

$$-3\,(V_{2\bar{3}} + V_{3\bar{2}}) + i\,(V_{2\bar{3}} - V_{3\bar{2}}) = 0\,. \tag{C.17}$$

Now eqs. (C.15) and (C.17) can be solved to give

$$V_{2\bar{3}} + V_{3\bar{2}} = 0, \tag{C.18}$$

and

$$V_{2\bar{3}} - V_{3\bar{2}} = 0, \tag{C.19}$$

implying

$$V_{2\bar{3}} = V_{3\bar{2}} = 0. \tag{C.20}$$

Then one has from eq. (C.10)

$$V_{1\bar{3}} + V_{3\bar{1}} = 0, \tag{C.21}$$

and from eq. (C.11)

$$V_{1\bar{3}} - V_{3\bar{1}} = 0, \tag{C.22}$$

implying

$$V_{1\bar{3}} = V_{3\bar{1}} = 0. \tag{C.23}$$

Using the obtained values, eqs. (C.2) - (C.4) give

$$V_{1\bar{2}} + V_{2\bar{1}} = 0, \tag{C.24}$$

while eqs. (C.8) - eq. (C.6) give

$$V_{1\bar{2}} - V_{2\bar{1}} = 0, \tag{C.25}$$

implying

$$V_{1\bar{2}} = V_{2\bar{1}} = 0. \tag{C.26}$$

D
More information on fluxes

In general, the $(1,1)$ form flux $F_{z^i\bar{z}^j}$ given by a hermitian matrix in eq. (2.24) is constrained by two equations (2.26) and (2.27) which mix the matrix components p_{xx}, p_{yy} and p_{xy} for general Ω. However, for a canonical complex structure, corresponding to orthogonal tori, the constraints simplify and are written in the matrix form:

$$p_{xx} = p_{yy}, \quad p_{xy}^T = p_{xy}. \tag{D.1}$$

Fluxes of such types have been used in [7] for constructing an $SU(5)$ GUT with stabilized moduli and in Section 4.5 we apply the Yukawa couplings computation results to show the mass generation for extra non-chiral states in the model of [7]. In this case, the $(1,1)$ form flux $F_{z^i\bar{z}^j}$, for $(\Omega = iI_3)$, reduces to:

$$F_{z^i\bar{z}^j} = \frac{1}{2}(p_{xy} - ip_{xx}) \tag{D.2}$$

Explicitly, the hermitian flux matrix F in eq. (3.17) is given as:

$$F = \begin{pmatrix} p_{x^1y^1} & p_{x^1y^2} + ip_{x^1x^2} & p_{x^1y^3} + ip_{x^3x^1} \\ p_{x^1y^2} - ip_{x^1x^2} & p_{x^2y^2} & p_{x^2y^3} + ip_{x^2x^3} \\ p_{x^3y^1} - ip_{x^3x^1} & p_{x^2y^3} - ip_{x^2x^3} & p_{x^3y^3} \end{pmatrix}. \tag{D.3}$$

For magnetized branes in [103, 7], we used the quantization rule for p's:

$$p_{x^iy^j} = \frac{m_{x^iy^j}}{n^{x^i}n^{y^j}}, \quad p_{x^ix^j} = \frac{m_{x^ix^j}}{n^{x^i}n^{x^j}}, \quad p_{x^iy^j} = \frac{m_{x^iy^j}}{n^{y^i}n^{y^j}}, \tag{D.4}$$

where $m_{x^iy^j}$, $m_{x^ix^j}$, $m_{y^iy^j}$ are the first Chern numbers along the corresponding 2-cycles and n^{x^i}, n^{y^i} etc. are the wrapping numbers along the 1-cycles x^i, y^i. However, for the model [7], we have used only integral fluxes corresponding to $n^{x^i} = n^{y^i} = 1$.

An additional modification comes when nonzero NS-NS B-field background is turned on along some 2-cycle. In this case, the first Chern number along the particular 2-cycle (for $n^{x^i} = n^{y^i} = 1$) is shifted by:

$$m_{x^i y^j} \rightarrow \tilde{m}_{x^i y^j} = m_{x^i y^j} + \frac{1}{2}, etc. \tag{D.5}$$

In the model that we discussed in [7], we turn on nonzero NS-NS B-field, ($B = \frac{1}{2}$), along the 2-cycles diagonally in the three T^2's. Resulting fluxes are then half-integral. However, as already mentioned earlier, in writing the wavefunctions of chiral fermions χ_{ab} in bifundamentals, the relevant quantities are the difference of fluxes in the two stacks, or the two diagonal blocks in the gauge theory picture. In addition to the D-branes, an orientifold model also contains image D-branes with fluxes of opposite signature than the ones present in the original brane. In such cases, the corresponding wavefunctions χ_{ab^*} will obey similar equations as that of χ_{ab}, but with the addition of the gauge potentials $A^a + A^b$ rather than their difference as in eq. (3.22). The relevant matrix $\mathbf{N}$ which will now be the addition of fluxes in the two stacks, rather than their difference, will once again be integral.

We also learnt from the second equation in (3.26) that ($\mathbf{N}.Im\mathbf{\Omega}$) is a symmetric matrix. However, as explained in eqs. (2.24) in the general situation and in (D.2) for $\Omega = iI_3$, fluxes are in general hermitian when components of all types: p_{xx}, p_{yy} and p_{xy} are present.

Bibliography

[1] J. Scherk and J. H. Schwarz, *"Dual Models For Nonhadrons,"* Nucl. Phys. B **81**, 118 (1974).

[2] J. Polchinski, *"Dirichlet Branes and Ramond-Ramond charges,"* Phys. Rev. Lett. **75** (1995) 4724, hep-th/9510017.

[3] J. Polchinski, *" Tasi lectures on D-branes,"* hep-th/9611050.

[4] C. Bachas, *"A Way to break supersymmetry,"* hep-th/9503030.

[5] I. Antoniadis, A. Kumar and B. Panda, *"Fermion Wavefunctions in Magnetized branes: Theta identities and Yukawa couplings,"* Nucl. Phys. B **823**, 116 (2009) [arXiv:0904.0910 [hep-th]].

[6] D. Cremades, L. E. Ibanez and F. Marchesano, *"Computing Yukawaa Couplings from Magnetized Extra Dimensions,"* JHEP **0405** 079 (2004),. hep-th/0404229.

[7] I. Antoniadis, A. Kumar and B. Panda, *"Supersymmetric SU(5) GUT with stabilized moduli,"* Nucl. Phys. **B** (2008), arXiv: 0709.2799.

[8] M. B. Green, J. H. Schwarz and E. Witten, *"Superstring Theory. Vol. 1: Introduction"*; *Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).*

[9] M. B. Green, J. H. Schwarz and E. Witten, *"Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology"*; *Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics).*

[10] D. Bailin and A. Love, *"Supersymmetric Gauge Field Theory And String Theory"*; *Bristol, UK: IOP (1994) 322 p. (Graduate student series in physics).*

[11] J. Polchinski, *"String Theory, Volume 1: An Introduction to the Bosonic String"*; Cambridge University Press (1998) (Cambridge Monographs on Mathematical Physics).

[12] J. Polchinski, *"String Theory, Volume 2: Superstring Theory and Beyond"*; Cambridge University Press (1998) (Cambridge Monographs on Mathematical Physics).

[13] C. V. Johnson, *"D-brane primer,"* hep-th/0007170.

[14] M. B. Green and J. H. Schwarz, *"Anomaly Cancellation In Supersymmetric D=10 Gauge Theory And Superstring Theory,"* Phys. Lett. B **149**, 117 (1984).

[15] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *"The Heterotic String,"* Phys. Rev. Lett. **54**, 502 (1985).

[16] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *"Heterotic String Theory. 1. The Free Heterotic String,"* Nucl. Phys. B **256**, 253 (1985).

[17] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *"Heterotic String Theory. 2. The Interacting Heterotic String,"* Nucl. Phys. B **267**, 75 (1986).

[18] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *"Vacuum Configurations For Superstrings,"* Nucl. Phys. B **258**, 46 (1985).

[19] D. Nemeschansky and A. Sen, *"Conformal Invariance Of Supersymmetric Sigma Models On Calabi-Yau Manifolds,"* Phys. Lett. B **178**, 365 (1986).

[20] B. R. Greene, *"String theory on Calabi-Yau manifolds,"* hep-th/9702155.

[21] T. Hübsch, *"Calabi-Yau Manifolds: A Bestiary For Physicists,"*

[22] D. Joyce, *"Lectures on Calabi-Yau and special Lagrangian geometry,"* math.dg/0108088.

[23] B. R. Greene, K. H. Kirklin, P. J. Miron and G. G. Ross, *"A Three Generation Superstring Model. 1. Compactification And Discrete Symmetries,"* Nucl. Phys. B **278**, 667 (1986).

[24] B. R. Greene, K. H. Kirklin, P. J. Miron and G. G. Ross, *"A Three Generation Superstring Model. 2. Symmetry Breaking And The Low-Energy Theory,"* Nucl. Phys. B **292**, 606 (1987).

[25] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, *"Strings On Orbifolds,"* Nucl. Phys. B **261**, 678 (1985).

[26] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, *"Strings On Orbifolds. 2,"* Nucl. Phys. B **274**, 285 (1986).

[27] L. E. Ibáñez, J. E. Kim, H. P. Nilles and F. Quevedo, *"Orbifold Compactifications With Three Families Of $SU(3) \times SU(2) \times U(1)^N$,"* Phys. Lett. B **191**, 282 (1987).

[28] A. Font, L. E. Ibáñez, F. Quevedo and A. Sierra, *"The Construction Of 'Realistic' Four-Dimensional Strings Through Orbifolds,"* Nucl. Phys. B **331**, 421 (1990).

[29] L. E. Ibáñez, *"The Search For A Standard Model $SU(3) \times SU(2) \times U(1)$ Superstring: An Introduction To Orbifold Constructions"*; CERN-TH-4769-87. *Based on lectures given at the XVII GIFT Seminar on Strings and Superstrings, El Escorial, Spain, Jun 1-6, 1987 and Mt. Sorak Symposium, Korea, Jul 1987 and ELAF '87, La Plata, Argentina, Jul 6-24, 1987.*

[30] D. Gepner, *"Space-Time Supersymmetry In Compactified String Theory And Superconformal Models,"* Nucl. Phys. B **296**, 757 (1988).

[31] H. Kawai, D. C. Lewellen and S. H. Tye, *"Construction Of Fermionic String Models In Four-Dimensions,"* Nucl. Phys. B **288**, 1 (1987).

[32] I. Antoniadis, C. P. Bachas and C. Kounnas, *"Four-Dimensional Superstrings,"* Nucl. Phys. B **289**, 87 (1987).

[33] I. Antoniadis, J. R. Ellis, J. S. Hagelin and D. V. Nanopoulos, *"Supersymmetric Flipped SU(5) Revitalized,"* Phys. Lett. B **194** (1987) 231;

"GUT Model Building with Fermionic Four-Dimensional Strings," Phys. Lett. B **205** (1988) 459;

"An Improved $SU(5) \times U(1)$ Model from Four-Dimensional String," Phys. Lett. B **208** (1988) 209 [Addendum-ibid. B **213** (1988) 562];

"The Flipped $SU(5) \times U(1)$ String Model Revamped," Phys. Lett. B **231** (1989) 65.

[34] E. Witten, *"String theory dynamics in various dimensions,"* Nucl. Phys. B **443**, 85 (1995), hep-th/9503124.

[35] M. B. Green, C. M. Hull and P. K. Townsend, *"D-brane Wess-Zumino actions, t duality and the cosmological constant,"* Phys. Lett. B **382** (1996) 65, hep-th/9604119.

[36] G. Pradisi and A. Sagnotti, *"Open String Orbifolds,"* Phys. Lett. B **216**, 59 (1989).

[37] M. Bianchi and A. Sagnotti, *"On The Systematics Of Open String Theories,"* Phys. Lett. B **247**, 517 (1990).

[38] A. Sagnotti, in Cargese 87, *"Open Strings And Their Symmetry Groups,"* ed. G. Mack et al. (Pergamon Press, 1988) p. 521, hep-th/0208020.

[39] P. Horava, *"Strings On World Sheet Orbifolds,"* Nucl. Phys. B **327**, 461 (1989).

[40] P. Horava, *"Background Duality Of Open String Models,"* Phys. Lett. B **231**, 251 (1989).

[41] J. Dai, R. G. Leigh and J. Polchinski, *"New Connections Between String Theories,"* Mod. Phys. Lett. A **4**, 2073 (1989).

[42] E. G. Gimon and J. Polchinski, *"Consistency Conditions for Orientifolds and D-Manifolds,"* Phys. Rev. D **54**, 1667 (1996), hep-th/9601038.

[43] A. Dabholkar and J. Park, *"An Orientifold of Type-IIB Theory on $K3$,"* Nucl. Phys. B **472**, 207 (1996) hep-th/9602030.

[44] E. G. Gimon and C. V. Johnson, *"$K3$ Orientifolds,"* Nucl. Phys. B **477**, 715 (1996) hep-th/9604129.

[45] M. Berkooz and R. G. Leigh, *"A $D = 4$ $N = 1$ orbifold of type I strings,"* Nucl. Phys. B **483**, 187 (1997), hep-th/9605049.

[46] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y. S. Stanev, *"Chiral asymmetry in four-dimensional open- string vacua,"* Phys. Lett. B **385**, 96 (1996), hep-th/9606169.

[47] Z. Kakushadze, *"Aspects of $N = 1$ type I-heterotic duality in four dimensions,"* Nucl. Phys. B **512**, 221 (1998), hep-th/9704059.

[48] Z. Kakushadze and G. Shiu, *"A chiral $N = 1$ type I vacuum in four dimensions and its heterotic dual,"* Phys. Rev. D **56**, 3686 (1997), hep-th/9705163.

[49] Z. Kakushadze and G. Shiu, *"4D chiral $N = 1$ type I vacua with and without D5-branes,"* Nucl. Phys. B **520**, 75 (1998) hep-th/9706051.

[50] G. Zwart, *"Four-dimensional $N = 1$ $Z_N \times Z_M$ orientifolds,"* Nucl. Phys. B **526**, 378 (1998), hep-th/9708040.

[51] L. E. Ibáñez, *"A chiral $D = 4$, $N = 1$ string vacuum with a finite low energy effective field theory,"* JHEP **9807**, 002 (1998), hep-th/9802103.

[52] G. Aldazábal, A. Font, L. E. Ibáñez and G. Violero, *"$D = 4$, $N = 1$, type IIB orientifolds,"* Nucl. Phys. B **536**, 29 (1998), hep-th/9804026.

[53] Z. Kakushadze, *"A three-family SU(6) type I compactification,"* Phys. Lett. B **434**, 269 (1998), hep-th/9804110.

[54] Z. Kakushadze, *"A three-family $SU(4)_c \times SU(2)_w \times U(1)$ type I vacuum,"* Phys. Rev. D **58**, 101901 (1998), hep-th/9806044.

[55] J. Lykken, E. Poppitz and S. P. Trivedi, *"Branes with GUTs and supersymmetry breaking,"* Nucl. Phys. B **543**, 105 (1999), hep-th/9806080.

[56] L. E. Ibáñez, *"New perspectives in string phenomenology from dualities,"* hep-ph/9804236.

[57] I. Antoniadis, *"A Possible new dimension at a few TeV,"* Phys. Lett. B **246** (1990) 377.

[58] J. D. Lykken, *"Weak Scale Superstrings,"* Phys. Rev. D **54**, 3693 (1996), hep-th/9603133.

[59] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *"The hierarchy problem and new dimensions at a millimeter,"* Phys. Lett. B **429**, 263 (1998), hep-ph/9803315.

[60] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *"New dimensions at a millimeter to a Fermi and superstrings at a TeV,"* Phys. Lett. B **436**, 257 (1998), hep-ph/9804398.

[61] G. Shiu and S. H. Tye, *"TeV scale superstring and extra dimensions,"* Phys. Rev. D **58**, 106007 (1998), hep-th/9805157.

[62] K. Benakli, *"Phenomenology of low quantum gravity scale models,"* Phys. Rev. D **60** (1999) 104002 [arXiv:hep-ph/9809582].

[63] L. E. Ibáñez, C. Muñoz and S. Rigolin, *"Aspects of type I string phenomenology,"* Nucl. Phys. B **553**, 43 (1999), hep-ph/9812397.

[64] L. E. Ibáñez, *"The second string (phenomenology) revolution,"* Class. Quant. Grav. **17**, 1117 (2000), hep-ph/9911499.

[65] I. Antoniadis, E. Dudas and A. Sagnotti, *"Brane supersymmetry breaking,"* Phys. Lett. B **464**, 38 (1999), hep-th/9908023.

[66] G. Aldazábal and A. M. Uranga, *"Tachyon-free non-supersymmetric type IIB orientifolds via brane-antibrane systems,"* JHEP **9910**, 024 (1999), hep-th/9908072.

[67] G. Aldazábal, L. E. Ibáñez and F. Quevedo, *"Standard-like models with broken supersymmetry from type I string vacua,"* JHEP **0001**, 031 (2000), hep-th/9909172.

[68] G. Aldazábal, L. E. Ibáñez and F. Quevedo, *"A D-brane alternative to the MSSM,"* JHEP **0002**, 015 (2000), hep-ph/0001083.

[69] G. Aldazábal, L. E. Ibáñez, F. Quevedo and A. M. Uranga, *"D-branes at singularities: A bottom-up approach to the string embedding of the standard model,"* JHEP **0008**, 002 (2000), hep-th/0005067.

[70] M. Berkooz, M. R. Douglas and R. G. Leigh, *"Branes intersecting at angles,"* Nucl. Phys. B **480**, 265 (1996), hep-th/9606139.

[71] R. Blumenhagen, L. Görlich, and B. Körs, *"Supersymmetric orientifolds in 6D with D-branes at angles,"* Nucl. Phys. **B569**, 209 (2000), hep-th/9908130.

[72] R. Blumenhagen, L. Görlich, and B. Körs, *"Supersymmetric 4D orientifolds of type IIA with D6-branes at angles,"* JHEP **01**, 040 (2000) , hep-th/9912204.

[73] S. Förste, G. Honecker, and R. Schreyer, *"Supersymmetric $Z(N) \times Z(M)$ orientifolds in 4D with D-branes at angles,"* Nucl. Phys. **B593**, 127(2001), hep-th/0008250.

[74] G. Aldazabal, S. Franco, L. E. Ibáñez, R. Rabadán, and A. M. Uranga, *"Intersecting brane worlds,"* JHEP **02**, 047 (2001), hep-ph/0011132.

[75] G. Aldazabal, S. Franco, L. E. Ibáñez, R. Rabadán, and A. M. Uranga, *"$D = 4$ chiral string compactifications from intersecting branes,"* J. Math. Phys. **42**, 3103 (2001), hep-th/0011073.

[76] L. E. Ibáñez, F. Marchesano, and R. Rabadán, *"Getting just the standard model at intersecting branes,"* JHEP **11**, 002 (2001), hep-th/0105155.

[77] R. Blumenhagen, B. Körs, D. Lüst, and T. Ott, *"The standard model from stable intersecting brane world orbifolds,"* Nucl. Phys. **B616**, 3 (2001), hep-th/0107138.

[78] M. Cvetič, G. Shiu, and A. M. Uranga, *"Three-family supersymmetric standard like models from intersecting brane worlds,"* Phys. Rev. Lett. **87** (2001), hep-th/0107143.

[79] M. Cvetič, G. Shiu, and A. M. Uranga, *"Chiral four-dimensional $N = 1$ supersymmetric type IIA orientifolds from intersecting D6-branes,"* Nucl. Phys. **B615**, 3 (2001), hep-th/0107166.

[80] D. Cremades, L. E. Ibáñez and F. Marchesano, *"Standard model at intersecting D5-branes: Lowering the string scale,"* Nucl. Phys. B **643**, 93 (2002), hep-th/0205074.

[81] D. Cremades, L. E. Ibáñez and F. Marchesano, *"Yukawa couplings in intersecting D-brane models,"* JHEP **0307**, 038 (2003), hep-th/0302105.

[82] D. Cremades, L. E. Ibáñez and F. Marchesano, *"SUSY quivers, intersecting branes and the modest hierarchy problem,"* JHEP **0207**, 009 (2002), hep-th/0201205.

[83] D. Cremades, L. E. Ibáñez and F. Marchesano, *"Intersecting brane models of particle physics and the Higgs mechanism,"* JHEP **0207**, 022 (2002), hep-th/0203160.

[84] P. van Baal, *"SU(N) Yang-Mills Solutions With Constant Field Strength On T^4,"* Commun. Math. Phys. **94**, 397 (1984).

[85] Z. Guralnik and S. Ramgoolam, *"From 0-branes to torons,"* Nucl. Phys. B **521**, 129 (1998), hep-th/9708089.

[86] J. Troost, *"Constant field strengths on T(2n),"* Nucl. Phys. B **568**, 180 (2000) hep-th/9909187.

[87] J. Bogaerts, A. Sevrin, J. Troost, W. Troost and S. van der Loo, *"D-branes and constant electromagnetic backgrounds,"* Fortsch. Phys. **49**, 641 (2001), hep-th/0101018.

[88] R. Blumenhagen, L. Görlich, B. Körs and D. Lüst, *"Asymmetric orbifolds, noncommutative geometry and type I string vacua,"* Nucl. Phys. B **582**, 44 (2000), hep-th/0003024.

[89] C. Angelantonj, I. Antoniadis, E. Dudas and A. Sagnotti, *"Type-I strings on magnetised orbifolds and brane transmutation,"* Phys. Lett. B **489**, 223 (2000), hep-th/0007090.

[90] R. Blumenhagen, L. Görlich, B. Körs and D. Lüst, *"Magnetic flux in toroidal type I compactification,"* Fortsch. Phys. **49**, 591 (2001), hep-th/0010198.

[91] M. Bianchi, G. Pradisi and A. Sagnotti, *"Toroidal compactification and symmetry breaking in open string theories,"* Nucl. Phys. B **376** (1992) 365;
C. Angelantonj, *"Comments on open-string orbifolds with a non-vanishing B(ab),"* Nucl. Phys. B **566** (2000) 126, hep-th/9908064;

[92] C. Angelantonj and A. Sagnotti, *"Type-I vacua and brane transmutation,"* hep-th/0010279.

[93] R. Blumenhagen, L. Görlich, B. Körs and D. Lüst, *“Noncommutative compactifications of type I strings on tori with magnetic background flux,”* JHEP **0010**, 006 (2000), hep-th/0007024.

[94] R. Blumenhagen, B. Körs and D. Lüst, *“Type I strings with F- and B-flux,”* JHEP **0102**, 030 (2001), hep-th/0012156.

[95] C. Angelantonj and A. Sagnotti, *“Open strings,”* Phys. Rept. **371** (2002) 1 [Erratum-ibid. **376** (2003) 339] [arXiv:hep-th/0204089]; and references therein.

[96] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, *“Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes,”* Phys. Rept. **445** (2007) 1 [arXiv:hep-th/0610327]; and references therein.

[97] R. Donagi and M. Wijnholt, *“Model Building with F-Theory,”* arXiv:0802.2969 [hep-th]; R. Donagi and M. Wijnholt, *“Breaking GUT Groups in F-Theory,”* arXiv:0808.2223 [hep-th].

[98] C. Beasley, J. J. Heckman and C. Vafa, *“GUTs and Exceptional Branes in F-theory - I,”* JHEP **0901**, 058 (2009) [arXiv:0802.3391 [hep-th]];

C. Beasley, J. J. Heckman and C. Vafa, *“GUTs and Exceptional Branes in F-theory - II: Experimental Predictions,”* JHEP **0901**, 059 (2009) [arXiv:0806.0102 [hep-th]].

[99] S. B. Giddings, S. Kachru and J. Polchinski, *“Hierarchies from Fluxes in String Compactifications,”* Phys. Rev. D **66** (1997) 106006, [arXiv:hep-th/0105097].

[100] S. Kachru, M.B. Schulz and S. Trivedi, *“Moduli Stabilization from Fluxes in a Simple IIB Orientifold,”* JHEP **0310** 007 (2003), [arXiv: hep-th/0201028];

A. Frey and J. Polchinski, *“N=3 Warped Compactifications,”* Phys. Rev. D **65** (2002) 126009, [arXiv:hep-th/0201029].

[101] R. Kallosh, S. Kachru, A. Linde and S. Trivedi, *“de Sitter Vacua in String Theory,”* Phys. Rev. D **68** (2003) 046005, [arXiv:hep-th/0301240]; L. Gorlich, S. Kachru, P. K. Tripathy and S. P. Trivedi, *“Gaugino condensation and nonperturbative superpotentials in flux compactifications,”* JHEP **0412**, 074 (2004) [arXiv:hep-th/0407130].

[102] I. Antoniadis and T. Maillard,*“Moduli stabilization from magnetic fluxes in type I string theory,”* Nucl. Phys. B **716** (2005) 3, [arXiv:hep-th/0412008].

[103] I. Antoniadis, A. Kumar, T. Maillard, *“Magnetic fluxes and moduli stabilization”* Nucl. Phys. B **767** (2007) 139, [arXiv:hep-th/0610246].

[104] M. Bianchi and E. Trevigne, *"The open story of the magnetic fluxes,"* JHEP **0508** (2005) 034, [arXiv:hep-th/0502147] and *"Gauge thresholds in the presence of oblique magnetic fluxes,"* JHEP **0601** (2006) 092, [arXiv:hep-th/0506080].

[105] E. S. Fradkin and A. A. Tseytlin, *"Nonlinear Electrodynamics From Quantized Strings,"* Phys. Lett. B **163** (1985) 123;

A. Abouelsaood, C. G. . Callan, C. R. Nappi and S. A. Yost, *"Open Strings In Background Gauge Fields,"* Nucl. Phys. B **280** (1987) 599;

V. V. Nesterenko, *"The Dynamics Of Open Strings In A Background Electromagnetic Field,"* Int. J. Mod. Phys. A **4** (1989) 2627.

[106] R. Blumenhagen, D. Lust and T. R. Taylor, *"Moduli stabilization in chiral type IIB orientifold models with fluxes,"* Nucl. Phys. B **663** (2003) 319 [arXiv:hep-th/0303016];

[107] J. F. G. Cascales and A. M. Uranga, *"Chiral 4d N = 1 string vacua with D-branes and NSNS and RR fluxes,"* JHEP **0305** (2003) 011 [arXiv:hep-th/0303024].

[108] I. Antoniadis, J. P. Derendinger and T. Maillard, *"Nonlinear N=2 Supersymmetry, Effective Actions and Moduli Stabilization,"* Nucl. Phys. B **808**, 53 (2009) [arXiv:0804.1738 [hep-th]].

[109] M. Cvetic and I. Papadimitriou,*"Conformal Field Theory Couplings for Intersecting D-branes on Orientifolds,"* Phys. Rev. D **68**, 046001 (2003) [Erratum-ibid. D **70**, 029903 (2004)] [arXiv:hep-th/0303083].

[110] M. Cvetic and T. Weigand,*"Hierarchies from D-brane instantons in globally defined Calabi-Yau orientifolds,"* Phys.Rev.Lett.**100** 251601 (2008), arXiv:0711.0209 [hep-th].

[111] R. Blumenhagen, M. Cvetic, R. Richter and T. Weigand, *"Lifting D-Instanton Zero Modes by Recombination and Background Fluxes,"* JHEP **0710** 098 (2007), arXiv:0708.0403 [hep-th].

[112] R. Blumenhagen, M. Cvetic, D. Lust, R. Richter and T. Weigand, *"Non-perturbative Yukawa Couplings from String Instantons,"* Phys.Rev.Lett. **100** 061602 (2008), arXiv:0707.1871 [hep-th].

[113] M. Cvetic, R. Richter, T. Weigand,*"Computation of D-brane instanton induced superpotential couplings: Majorana masses from string theory,"* Phys.Rev. **D76** 086002 (2007), hep-th/0703028.

[114] R. Blumenhagen, M. Cvetic and T. Weigand, *"Spacetime instanton corrections in 4D string vacua: The Seesaw mechanism for D-Brane models,"* Nucl.Phys. **B771** 113 (2007), hep-th/0609191.

[115] L.E. Ibanez and A.M. Uranga, *"Instanton induced open string superpotentials and branes at singularities,"* JHEP **0802** 103 (2008), arXiv:0711.1316 [hep-th].

[116] L.E. Ibanez, A.N. Schellekens and A.M. Uranga, *"Instanton Induced Neutrino Majorana Masses in CFT Orientifolds with MSSM-like spectra,"* JHEP **0706** 011 (2007), arXiv:0704.1079 [hep-th].

[117] L.E. Ibanez and A.M. Uranga, *"Neutrino Majorana Masses from String Theory Instanton Effects,"* JHEP **0703** 052 (2007), [hep-th/0609213].

[118] Angel M. Uranga, *"D-brane instantons and the effective field theory of flux compactifications,"* JHEP **0901** 048 (2009), arXiv:0808.2918 [hep-th]

[119] D. Mumford, *Tata lectures on Theta I*, Berkhauser, Boston, 1983.

[120] M. Marino, R. Minasian, G. W. Moore and A. Strominger,*"Nonlinear instantons from supersymmetric p-branes,"* JHEP **0001** (2000) 005, [arXiv:hep-th/9911206].

[121] T. Maillard, *"Toward metastable string vacua from magnetized branes,"* arXiv:0708.0823 [hep-th].

[122] D. Lust, P. Mayr, R. Richter and S. Stieberger, *"Scattering of gauge, matter, and moduli fields from intersecting branes,"* Nucl. Phys. B **696** (2004) 205, [hep-th/0404134];
M. Bertolini, M. Billo, A. Lerda, J.F. Morales and R. Russo, *"Brane world effective actions for D-branes with fluxes,"* Nucl. Phys. B **743** (2006) 1, [hep-th/0512067].

[123] I. Antoniadis, S. Dimopoulos, *"Splitting supersymmetry in string theory,"* Nucl. Phys. B **715** (2005) 120, [arXiv:hep-th/0411032].

[124] R. Russo and S. Sciuto, *"The Twisted open string partition function and Yukawa couplings,"* JHEP **0704** (2007) 030, [arXiv: hep-th/0701292].

[125] D. Duo, R. Russo and S. Sciuto, *"New twist field couplings from the partition function for multiply wrapped D-branes,"* JHEP **0712** (2007) 042, arXiv:0709.1805 [hep-th].

[126] P. Di Vecchia, A. Liccardo, R. Marotta, F. Pezzella, *"Kahler Metrics and Yukawa Couplings in Magnetized Brane Models,"* JHEP **0903** (2009) 029, arXiv:0810.5509 [hep-th].

[127] P. Di Vecchia, A. Liccardo, R. Marotta, F. Pezzella, *"Kahler Metrics: String vs Field Theoretical Approach,"* Fortsch. Phys. **57** (2009) 718 arXiv:0901.4458 [hep-th].

[128] Y. Tenjinbayashi, H. Igarashi, T. Fujiwara,*"Dirac operator zero-modes on a torus,"* Annals Phys. **322** 460 (2007), [arXiv: hep-th/0506259].

[129] H. Abe, K. S. Choi, T. Kobayashi and H. Ohki, *"Higher Order Couplings in Magnetized Brane Models,"* JHEP **0906** (2009) 080, arXiv:0903.3800 [hep-th].

[130] R. Blumenhagen, M. Cvetic, D. Lust, R. Richter and T. Weigand, *"Non-perturbative Yukawa Couplings from String Instantons,"* Phys. Rev. Lett. **100** (2008) 061602 [hep-th/0707.1871].

[131] M. Billo, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, *"Instantons in N=2 magnetized D-brane worlds,"* JHEP **0710** (2007) 091 arXiv:0708.3806 [hep-th]; *"Instanton effects in N=1 brane models and the Kahler metric of twisted matter,"*JHEP **0712** (2007) 051, arXiv:0709.0245 [hep-th].

[132] I. Antoniadis, K. Benakli, A. Delgado, M. Quiros and M. Tuckmantel, *"Split extended supersymmetry from intersecting branes,"* Nucl. Phys. B **744** (2006) 156, arXiv:hep-th/0601003;
I. Antoniadis, K. Benakli, A. Delgado and M. Quiros, *"A new gauge mediation theory,"* Adv. Stud. Theor. Phys. **2** (2008) 645, arXiv:hep-ph/0610265.