Quantum Correlations in Multiparticle Systems and Its Applications

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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In dreams, we enter a world that's entirely our own. – Albus Dumbledore

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Synopsis

In this thesis, we will mainly focus on two important traits of composite quantum systems – quantum entanglement and quantum correlations that go beyond entanglement (QCsbE), and their applications in some quantum information processing (QIP) tasks. There exists many aspects of QCsbE in literature. We will mainly talk about information theoretic measure, quantum discord and its one particular generalization for multiparticle systems.

The detection and the quantification of the quantum entanglement in a quantum state are of immense importance. This is due to their importance in many quantum information processing (QIP) tasks as well as unravelling the true nature of such states. There exist several detection and quantification methods for quantum entanglement in literature. For pure bipartite state entanglement is properly quantifiable. Although we have some knowledge about entanglement of mixed bipartite states, but things are not straightforward here. Conclusive detection methods exists for $2 \otimes 2$ and $2(3) \otimes 3(2)$ states only. Furthermore, there does not exist any unique measure for mixed bipartite entanglement beyond two qubit systems. Different ways of characterizing entanglement exist in this case. These measures are usually very hard to compute in general. Situation becomes even worse when multipartite entangled states are considered. In this case, no universal detection and quantification technique exist. Entanglement witness is a good tool to detect entanglement in arbitrary entangled states, but its construction depends on the class of states one is detecting. Even the notion of maximally entangled states becomes murky here; it depends on the measure one is considering. We provide another detection method of bipartite entanglement using a modified form of *quantum covariance*. The detection procedure involves measurement of two local observables of the type $\mathcal{O}_1 \otimes \mathbb{I}$ and $\mathbb{I} \otimes \mathcal{O}_2$. By construction it will be able to detect bipartite arbitrary dimensional entanglement. Also this detection procedure is experimentally feasible in present day.

The measure, quantum discord is very hard to compute in general and there exist analytical expressions for a few classes of states. Note that discord reduces to von-Neumann entropy for pure bipartite states. Moreover, it is non-zero for separable states which make it questionable whether it is a measure of quantum correlations in usual sense. It can even increase under local operations. Then one can ask the following question. *Does discord captures quantum entanglement only?* The answer is not straight forward and is negative. It doesn't capture quantum entanglement only. It captures *local quantumness* also. Let me be clear what is local quantumness in this context. It is like local superposition (our idea is nonlocal superposition gives rise to entanglement type correlation) – eg., consider the discordant separable state, $\frac{1}{2}(|00\rangle\langle 00| + |++\rangle\langle ++|)$ ($|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$), the state has local superposition in any basis, one cannot mask its superposition. Hence, we can conclude that the discord captures local quantumness also. This idea is supported by some illustrations where we have specifically shown that the discord depends on local quantumness parameter – a parameter which is responsible for local superposition.

Originally discord was defined for bipartite systems using the quantum generalization of classical mutual information. They used the notion that mutual information captures correlations between two random variables. Hence, quantum version may capture total correlations in a bipartite state. It should be noted that there are different equivalent expressions of classical mutual information which become distinct in quantum regime. This is because their exist no unique definition of quantum conditional entropy. If we analogically extend the classical definition to quantum regime, it can be negative for entangled states. It was shown that the negativity of conditional entropy can be resolved if we consider measurement on the subsystem on which the conditional is there. The idea was: To gain information about the subsystem one has to perform measurement. If we perform such a measurement in place of conditioning, the equivalent expressions of classical mutual information become distinct when it is generalized to quantum case. This difference is actually called the quantum discord. In this sense quantum discord is supposed to capture quantumness of correlations in the state. The conditional version of mutual information is considered as a measure of classical correlations noticing its resemblance with Holevo quantity.

This raises many important questions. Does quantum discord put a clear boundary between classical and quantum world? Then immediately one can ask: Is quantum generalization of classical mutual information physically motivated? We know these generalizations are based on analogy – it doesn't come from the theory itself unlike its classical counterpart. Can we consider quantum mutual information as a measure of total correlations? It has been extensively discussed in the literature. It has been argued that quantum mutual information may capture total correlations. However, it is not clear if it does. Also there exists several definitions of multiparticle mutual information which may differ even when no conditional entropies are involved. This makes it even more difficult to generalize. Moreover, the notion of classical correlations in quantum state is not clear from their definition.

In this thesis, we have generalized the idea of quantum discord to multi-qubit systems. We adopt the Venn diagram approach to define the classical mutual information which is then generalized to quantum case. Its classical version is also known as *interaction information*. We didn't assume that this version of mutual information captures total correlations of the multi-qubit state to begin with. Using chain rule of mutual information, one can obtain many other expressions of mutual information. All these expressions differ from each other in quantum regime. Thus we can define many inequivalent expressions of quantum discord like quantities which we call *dissensions*. These quantities capture some form of quantumness of the multi-qubit states. We illustrate this fact with some multi-qubit states. The analysis reveals that one can characterize the states if appropriate vectors of these measures are considered.

To qualify as a measure of correlations a quantity should satisfy some properties listed in. One drawback of our measures is that it can be negative for some states. However, it may not be a drawback as classical mutual information can also be negative. We find that our method not only gives some signatures of quantum correlations present in the multi-qubit states but also it characterizes the states. Use of a vector measure to characterize the quantumness of the state helps us. It characterizes the nonlocal and local quantum resources of the state more completely. So, we argue that one number is not sufficient, and one needs a set of numbers to characterize the quantumness of a state.

Another important question is whether quantum states are useful in some tasks unlike classical one or in improving performance of some existing classical tasks? This basic and important question started the field of the quantum information science. The discovery of BB84 protocol, quantum teleportation, quantum algorithm have had immense impact on the development of the subject itself. The teleportation was the first protocol where correlations like entanglement was utilized as a resource. Teleportation is a quantum information processing task which enables two spatially separated parties to communicate quantum information (quantum states) if they share an entangled state between them. Then naturally one can ask: Which entangled states will be useful for teleportation? This question is very much related to the query: Can every entangled state be used as a resource for faithful teleportation (teleportation with unit fidelity and unit probability)? It depends on their structure of entanglement. The performance of an entangled state in teleportation is decided by the quantity *teleportation fidelity*. For two-qubit systems there is conclusive relation between entanglement and teleportation fidelity but this is not the case for arbitrary dimensional states. In this thesis, we have tried to answer this query to some extent. We have established a relation between teleportation fidelity and concurrence monotones (Monotones are the one of the ways to characterize entanglement of higher dimensional systems.) for upto Schmidt rank 3 states. We have shown that how much quantum entanglement is needed for a two-qudit entangled state to be useful as a resource for teleportation.

To create multipartite entangled states researchers often invoke *entanglement swapping*. Entanglement swapping is a process in which two remote parties become entangled though initially they were not. Using such protocol one can create a quantum network (QNet). We considered such a qubit-network and studied their entanglement properties as well as the teleportation fidelity and superdense coding capacity. We showed that how much classical and quantum information can be sent through such a network.

It would be very interesting to see what properties of a state are captured by QCsbE. One possible way is to analize its usefulness in some QIP tasks. There are some proposals that it may be the key factor in the speedup of performance of the DQC1 model. Also in state merging protocol, it is related to the cost of the performance of the task. It is not obvious most of the times what role QCsbE is playing in such protocols. In this respect more studies are indeed needed. We have studied broadcasting of quantum correlations in two qubit states to shed some light on this issue. We showed that under unital channel, there will be no-optimal broadcasting of the quantum discord unlike entanglement whereas one can have *task oriented* broadcasting for both the discord and the entanglement. Moreover, under non-unital channel the meaning of optimal broadcasting is not clear . Therefore, for any kind of local and nonlocal universal or state dependent cloning machines, the optimal broadcasting of QCsbE is impossible and thus it indicates that the entanglement and QCsbE are two distinct notions of correlations in quantum states. In another work , we have shown that better we clone (delete) a state, more difficult it will be to bring the state back to its original form by the process of deleting (cloning). This analysis gives new kind of two complementarity relations between the quantum correlations generated during the processes – (a) cloning $\rightarrow$ deletion and (b) deletion $\rightarrow$ cloning. Such a study may be useful in designing *quantum recycle bin* in future.

List of Publications

- ***Published***

  1. *"A Study of Teleportation and Super Dense Coding Capacity in Remote Entanglement Distribution", Sk Sazim and I. Chakrabarty, *Euro. Phys. J. D* **67**,174 [8 pages] (2013).
  2. *"Quantification of Entanglement of teleportation in Arbitrary Dimensions", by Sk Sazim, Satyabrata Adhikari, Subhashish Banerjee, and Tanumoy Pramanik, *Quantum Information Processing* **13**, 863 – 880 (2014).
  3. *"Complementarity of Quantum Correlation in Cloning and Deleting of Quantum State", Sk Sazim, Indranil Chakrabarty, Annewsa Datta, and Arun K. Pati, *Phys. Rev. A* **91**, 062311 [11 pages] (2015).
  4. "Retrieving and Routing Quantum Information in a Quantum Network", Sk Sazim, Indranil Chakrabarty, Chiranjeevi Vanarasa, and Kannan Srinathan, *Quantum Information Processing* **14**, 4651 – 4664 (2015).
  5. "Quantum Coherence Sets The Quantum Speed Limit For Mixed States", Debasis Mondal, Chandan Datta, and Sk Sazim, *Phys. Lett. A* **380**, 689 – 695 (2016).
  6. *"Broadcasting of Quantum Correlations: Possibilities & Impossibilities", Sourav Chatterjee, Sk Sazim, and Indranil Chakrabarty, *Phys. Rev. A* **93**, 042309 [13 pages] (2016).
  7. *"Quantum Discord has local and nonlocal quantumness" by Pankaj Agrawal, Sk Sazim, Indranil Chakrabarty, and Arun K. Pati, *Int. J. of Quantum Inform.* **14**, *1640034 (2016).*

- ***Communicated***

  1. *"Quantum mutual information and dissension vectors for multiqubit systems", Sk Sazim and Pankaj Agrawal.

A (*) indicates papers on which this thesis is based.

List of Figures

1.1 *Venn diagram*: The red lined circle depicts the information (uncertainty) about X i.e., $H(X)$ and black one $H(Y)$ for Y. The conditional entropies $H(X|Y)$ & $H(Y|X)$ show how much information we can still get if we have the knowledge about Y and X respectively. The middle inner space contains the common information about both X and Y, this information is called mutual information. 3

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Chapter 1

Introduction

"Quantum mechanics is old but new: because it is still developing." – Lev Vaidman @Patna 2015

Quantum mechanics is the most complete description of the nature so far but that does not exclude any possibility of finding other description which can replace quantum mechanics. Formulation of quantum mechanics is based on some axioms and rules [1]. These axioms and rules are not solely physically motivated like the axioms of special theory of relativity [2]. However, the beauty of quantum mechanics lies in its description of the behaviour of microscopic systems. Most of these descriptions can be experimentally verified, which establishes it as valid science[1].

Due to many facets of the quantum mechanics, understanding it clearly is an immense task. Many bizarre effects have been observed by the physicists throughout the history of development of quantum mechanics. One such characteristic of quantum mechanics is quantum correlations (QCs) [3]. This phenomenon is observed in quantum world when two or more parties are involved in the system.

One of the basic problems in quantum physics is to understand the nature of correlations present between different particles in a composite system. The existence of non-factorizable states play important role in the existence of many exotic features of quantum information theory. In the last decade, various measures of correlations [3, 4] have been introduced. It is believed that none of these measures can alone be sufficient to describe all the facets of quantum correlations. However, each of these measures unveils some aspects of quantum correlations.

Entanglement [3] is the key concept which alters the notion of reality in the microscopic system. Not only that, it is also responsible for the metamorphosis of the meaning of correlation as we move from classical systems to quantum systems. For pure states the situation is quite comprehensible as entanglement describes all about the correlation present. However, the situation is not so clear in the case of mixed states. In the case of mixed states, because of certain issues, researchers began to have a hunch that there

[1]"[Quantum mechanics] describes nature as absurd from the point of view of common sense. And yet it fully agrees with experiment. So I hope you can accept nature as She is - absurd."–Richard Feynman

might be something beyond the entanglement that actually quantifies the amount of correlation present in the system. Recently, many measures have been proposed to quantify the amount of correlation present in a mixed state [3, 4]. These measures have a unique feature that all of them seem to predict the presence of quantum correlation beyond the domain of entanglement. However, the nature of these correlation, as one goes beyond the entanglement, is far from understood.

Before moving to quantum correlations, we will discuss some elements of classical and quantum information sciences in the following sections.

1.1 Entropy and information – classical case

Classical information theory [5] deals with how much information[2] one can encode in a set of random variables. The smallest unit of information one can encode in is *bit*. In his seminal work [6], Shannon proposed a measure of information, called *entropy*.

1.1.1 Shannon entropy

Consider a random variable X. Shannon showed that if X takes value x with probability p_x then the information content in the random variable is

$$H(X) = -\sum_x p_x \log_2 p_x. \tag{1.1}$$

Consider events like unbiased single coin tossing, double coin tossing and rolling of dice. The entropy of these events are respectively 1, 2 and $\log_2 6$. The physical interpretation of Shannon entropy is given by *data compression*. Given a set of data, the limit upto which one can compress it is its entropy. Note that if $p_i = 0$, an event which does not occur should not contribute to the entropy i.e., $0 \log_2 0 = 0$. Mathematically, $\lim_{x \to 0} x \log_2 x = 0$.

1.1.2 Relative entropy

Relative entropy captures the *closeness*[3] between two probability distributions. Let us consider two probability distributions, $p(x)$ and $q(x)$ then the relative entropy between them is

$$H(p(x) \parallel q(x)) = \sum_x p(x) \log_2 \frac{p(x)}{q(x)} = -H(X) - \sum_x p(x) \log_2 q(x), \tag{1.2}$$

with $-0 \log_2 0 = 0$ and $-p(x) \log_2 0 = +\infty$ if $p(x) > 0$. Relative entropy is non-negative i.e., $H(p(x) \parallel q(x)) \geq 0$ with equality iff $p(x) = q(x)$, $\forall x$. The importance of relative

[2] The knowledge you don't have, you gain it – is information, e.g., "The sun rises in the east" is not a information whereas "Tomorrow, there will be snowfall in Delhi" is.

[3] Relative entropy captures distance between two probability distributions but it is not a metric as it is not symmetric. Intuitively, the uncertainty discrepancy in mistaking a fair dice to be unfair is not the same as the opposite situation.

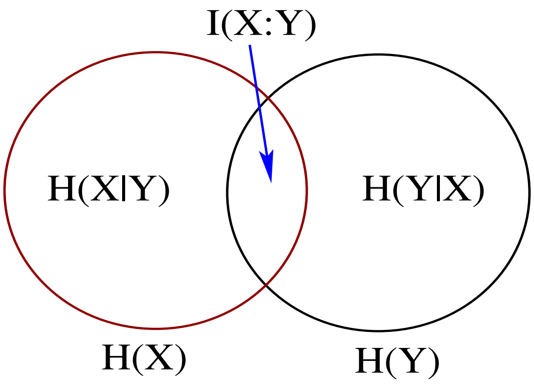


Figure 1.1: *Venn diagram*: The red lined circle depicts the information (uncertainty) about X i.e., $H(X)$ and black one $H(Y)$ for Y. The conditional entropies $H(X|Y)$ & $H(Y|X)$ show how much information we can still get if we have the knowledge about Y and X respectively. The middle inner space contains the common information about both X and Y, this information is called mutual information.

entropy is that one can derive a number of other entropic quantities as a special case of it. For example, suppose $p(x)$ is a probability distribution of X (X is a random variable with d outcomes) and $q(x) = \frac{1}{d}$, the uniform probability distribution over those d outcomes. Then

$$H(p(x) \parallel \frac{1}{d}) = \log_2 d - H(X), \tag{1.3}$$

which can be rewritten using the non-negative property of relative entropy as $H(X) \leq \log_2 d$ with equality iff X is also a uniform distribution.

1.1.3 Joint entropy, conditional entropy and mutual information

Let us consider two random variables X and Y which have joint probability distribution $p(x, y)$ then joint information (uncertainty) can be defined as

$$H(X, Y) = -\sum_{x,y} p(x, y) \log_2 p(x, y). \tag{1.4}$$

Here, $H(X, Y)$ is called the joint entropy. Now, if someone knows the uncertainty (information) in one random variable (say, Y) then the uncertainty of the other variable (X) may be affected. This is characterized by

$$H(X|Y) = H(X, Y) - H(Y), \tag{1.5}$$

where $H(X|Y)$ is the conditional entropy. If the two random variables are correlated (here by correlation, we mean ‘knowing one will diminish the uncertainty of other) then $H(X|Y) \leq H(X)$, where equality holds if they are disjoint distributions (see Fig 1.1).

Hence, in the presence of correlation $H(X,Y) \leq H(X) + H(Y)$ (again equality holds for two disjoint random variables). This inequality[4] can be rewritten as $H(X) + H(Y) - H(X,Y) \geq 0$. Naturally, the l.h.s of the inequality is used for the correlation measure between two random variables and is termed as mutual information, i.e.,

$$I(X:Y) = H(X) + H(Y) - H(X,Y). \tag{1.6}$$

In the Venn deiagram (see Fig 1.1), the overlap region is the mutual information, $I(X:Y)$. Using Eqn.(1.5), one can have different equivalent expressions for mutual information which we will discuss in detail later. All these quantities are very important in developing classical information theory. Mutual information quantifies the correlations between two random variables, capacity of quantum channel etc. In the later part of this thesis, we will discuss possible generalization of these quantities in quantum domain and their usefulness.

1.2 Entropy and information – quantum case

R. Feynman and D. Deutsch realized the advantages of any finite machines which obey the laws of quantum mechanics in computing over a classical computer. The BB84 protocol improved the secrecy in key distribution protocols [7]. Scientist also found the power of quantum principles in communication science. Bennett et al introduced quantum teleportation [8], superdense coding [9] etc., where they utilized the power of entangled states to show the advantages of these protocols over its classical counterparts. Thus scientists realized the power of quantum systems in information science, computational physics, communication science, cryptography [10], to name a few. Hence, the journey of "Quantum Information Science (QIS)" [11] started.

In QIS, the quantum mechanical systems are the resource. Holevo showed that to encode n bits of classical information one requires at least n bits classical resources [12]. Due to superposition principle, quantum mechanical systems are supposed to have a better encoding efficiency than their classical counterparts. This is the one of the motivations in developing QIS. However, we will not discuss the history of development of QIS here. We will quickly introduce some of the properties of quantum states before discussing few important elements of QIP.

1.2.1 Quantum states and density matrix formalism

In QIP, analogous to a bit we have a *qubit*. It is a pure two-level quantum state. Like classical bits 0 and 1, one can have quantum systems in $|0\rangle$ and $|1\rangle$ states, which form a single-particle basis in two dimensional Hilbert space. It is usually called as *computational basis*. Hence, mathematical expression for qubit ($|\psi\rangle$) is,

$$|\psi\rangle = \alpha|0\rangle + \beta|0\rangle, \tag{1.7}$$

[4]The inequality, $H(X,Y) \leq H(X) + H(Y)$ is called *subadditivity of Shannon entropy* and can easily be derived from relative entropy, $H(p(x,y) \parallel p(x)p(y)) \geq 0$.

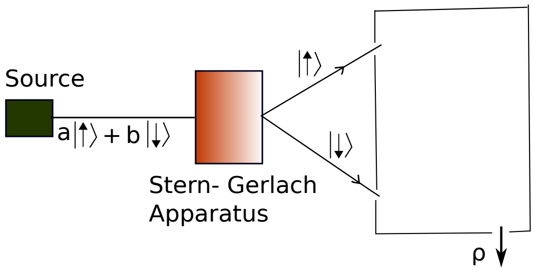


Figure 1.2: *Experimental preparation of mixed states*: The experimental set up for preparing single-particle mixed states is depicted here. Source emits particles in superposition state, $a|\uparrow\rangle + b|\downarrow\rangle$, which goes through the Stern-Gerlach apparatus. As a result spin up $|\uparrow\rangle$ particles go through the upper path and spin down $|\uparrow\rangle$ through the down path. Instead of putting a screen if we put a container and collect the particles which come out of the Stern-Gerlach apparatus we will not be able to tell which one is in spin up and which one is in spin down. Therefore, the particles will end up in a mixed state, $\rho = |a|^2|\uparrow\rangle\langle\uparrow| + |b|^2|\downarrow\rangle\langle\downarrow|$.

where α, $\beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Infinite number of bases are possible even in two dimensional Hilbert space and so infinite number of pure single qubit representations. These different basis are unitarily connected to each other.

Like single-particle quantum states, the multiparticle pure qubit states can be expressed in $2 \otimes 2 \otimes .... \otimes 2$ dimensional Hilbert space as

$$|\psi\rangle_n = \sum_{i_1, i_2, ..., i_n} \alpha_{i_1 i_2, ..., i_n} |i_1, i_2, ..., i_n\rangle, \tag{1.8}$$

where $n > 1$ is the number of particles and i_j can take the value 0 or 1 (in principle you can set any basis for the particles and the corresponding indices will take the value as the two levels). These pure single-particle and multiparticle superposition states are very difficult to prepare and even harder to preserve. So, in practice we always end up with mixed states. All these can be generalized for d dimensional Hilbert space.

Unlike the pure states, we don't have complete knowledge about the mixed states. In quantum mechanics, mixed states are usually represented by *density matrices*. Let us consider an ensemble of particles in states $\{|\phi_i\rangle\}$. If all the particles are in the same state then the ensemble can be expressed by pure states. If p_i is the probability to find the i^{th} particle in the state $|\phi_i\rangle$, then the ensemble is represented by density matrix as,

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|. \tag{1.9}$$

A valid density matrix should satisfy the following properties, (a) ρ is Hermitian i.e., $\rho^\dagger = \rho$ (b) ρ is semi-positive i.e., $\rho \geq 0$, (c) $\mathrm{Tr}[\rho] = 1$, normalization condition and (d) $\mathrm{Tr}[\rho^2] \leq 1$, equality holds iff ρ is pure.

How can one prepare a mixed state? One can prepare a mixed state by just classically mixing two or more pure states (see the Fig.(1.2)). In case of entangled pure states, the reduced density matrices of the subsystems are generally mixed states. Any mixed state can be expressed as a convex combination of pure states. But these decompositions for mixed states are not unique in general. There can be infinite numbers of them. For example, consider the density matrix $\rho_{mix} = p_1\rho_1 + p_2\rho_2 + p_3\rho_3 + p_4\rho_4$, where ρ_i may be pure states and $\sum_i p_i = 1$. Now it can be re-expressed in the following ways

$$\begin{aligned}
\rho_{mix} &= (p_1+p_2)\frac{p_1\rho_1+p_2\rho_2}{p_1+p_2} + (p_3+p_4)\frac{p_3\rho_3+p_4\rho_4}{p_3+p_4} \\
&= q_1\eta_1 + q_2\eta_2, \qquad \text{or,} \\
\rho_{mix} &= (p_1+p_3)\frac{p_1\rho_1+p_3\rho_3}{p_1+p_3} + (p_2+p_4)\frac{p_2\rho_2+p_4\rho_4}{p_2+p_4} \\
&= r_1\chi_1 + r_2\chi_2,
\end{aligned} \tag{1.10}$$

where $q_1 = p_1+p_2$, $q_2 = p_3+p_4$, $r_1 = p_1+p_3$, $r_2 = p_2+p_4$, $\eta_1 = \frac{p_1\rho_1+p_2\rho_2}{p_1+p_2}$, $\eta_2 = \frac{p_3\rho_3+p_4\rho_4}{p_3+p_4}$, $\chi_1 = \frac{p_1\rho_1+p_3\rho_3}{p_1+p_3}$, and $\chi_2 = \frac{p_2\rho_2+p_4\rho_4}{p_2+p_4}$. Here, the states η_i and χ_j are generally mixed states which have infinitely many pure state decompositions also. This tells us how complicated it is to characterize a mixed state.

Bloch sphere – Any single qubit can be geometrically represented by a Bloch sphere (see Fig.(1.3)) which is a unit sphere in polar coordinate (r, θ, ϕ). This is because most general pure qubit $|\psi\rangle$ can be written as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$. In Bloch representation, one can easily visualize the effects of quantum operations on single qubit. If we apply $\frac{1}{2}[\mathbb{I}_2 + (-1)^a\sigma_z]$ ($a = 0, 1$) operations on the qubit $|\psi\rangle$, either one will end up with state $|0\rangle$ (with probability $\cos^2\frac{\theta}{2}$) or with $|1\rangle$ (with probability $\sin^2\frac{\theta}{2}$). Here, $\mathbb{I}_k$ is the identity matrix of order k and $\{\sigma_i; i = x, y, z\}$ are Pauli matrices. Most general state of single qubit is

$$\rho = \frac{1}{2}(\mathbb{I} + \vec{\sigma}.\vec{r}), \tag{1.11}$$

where $\vec{r}$ is called Bloch vector with constraint $r = \parallel \vec{r} \parallel \leq 1$ with equality only holds for pure qubits. However, this beautiful representation is painfully limited to single qubit systems as there is no known generalization for multiqubit systems let alone higher dimensional systems.

1.2.2 Quantum entropies and mutual information

What is the information in QIS? We have already discussed that quantum mechanical systems are main resource in QIS. Because of the quantum superposition principle, an unknown pure qubit state, $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$ apparently possesses infinite amount of information as θ can be any point on the Bloch sphere (Number of points on unit sphere is infinite.). But to gain the information one needs to measure (which gives either outcome $|0\rangle$ or $|1\rangle$) and that yields one *bit* (classical bit) of information. The main issue becomes that although we can superpose many data to process in quantum computer but what amount of information we can extract from it.

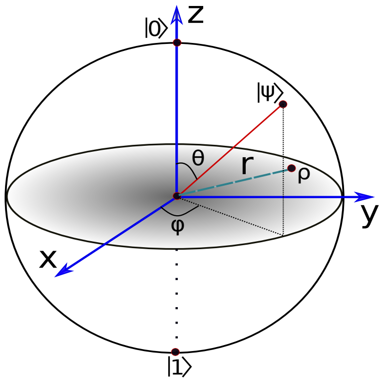


Figure 1.3: *Bloch sphere*: Geometric representation of single qubit states. The qubit $|\psi\rangle$ is the superposition of $|0\rangle$ and $|1\rangle$ which is at the surface of the sphere where all the pure single qubit states live. ρ represents a generic mixed single qubit which is inside the sphere with $\| \vec{r} \| < 1$ and $\mathbb{I}_2$ being maximally mixed state lives at the centre of sphere.

1.2.2.1 Von Neumann entropy

In the same spirit as Shannon entropy, Von Neumann defined the quantum entropy (popularly known as Von Neumann entropy) with density matrix ρ replacing probability distribution as

$$S(\rho) = -\operatorname{Tr}[\rho \log_2 \rho]. \tag{1.12}$$

If λ_i are the eigenvalues of ρ then the Von Neumann entropy is $S(\rho) = \sum_i \lambda_i \log_2 \lambda_i$. The Von Neumann entropy captures the uncertainty[5] in the state ρ. For example, Von Neumann entropy of the state $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ is $H_2(p)$, where $H_2(x) = -x \log_2 x - (1-x)\log_2(1-x)$. Like Shannon data compression theory, Schumacher gave quantum data compression theory which says that given quantum data, the limit upto which one can compress it is determined by its Von Neumann entropy. It gives physical meaning to Von Neumann entropy.

1.2.2.2 Quantum relative entropy

Let ρ and σ are two quantum density matrices, then the quantum relative entropy between two is

$$S(\rho \parallel \sigma) = \operatorname{Tr}[\rho \log_2 \rho] - \operatorname{Tr}[\rho \log_2 \sigma]. \tag{1.13}$$

Note that the relative entropy is defined to be $+\infty$ if the kernel of σ has non-trivial intersection with the support of ρ, and is finite otherwise. The quantum relative entropy is non-negative i.e., $S(\rho \parallel \sigma) \geq 0$. Again this is not a good metric as it is not symmetric in ρ and σ.

[5] It is the information we don't know about the preparation of ρ.

1.2.2.3 Quantum conditional entropy and quantum mutual information

In analogy to the classical case, one can define quantum joint entropy, quantum conditional entropy and quantum mutual information. If ρ_{AB} is the density matrix of the joint system AB, then the quantum joint entropy is $S(A,B) = -\operatorname{Tr}[\rho_{AB} \log_2 \rho_{AB}]$. One can prove *sub-additivity of Von Neumann entropy* using quantum relative entropy i.e., $S(A,B) \leq S(A) + S(B)$. Here, $S(A)$ and $S(B)$ are the Von Neumann entropy of the subsystems, A ($\rho_A = \operatorname{Tr}_B[\rho_{AB}]$) and B ($\rho_B = \operatorname{Tr}_A[\rho_{AB}]$). Here, Tr_B operation means we are ignoring the subsystem B completely. One can translate this inequality as $S(A) + S(B) - S(A,B) \geq 0$. The quantum mutual information is then defined as

$$I^q(A:B) = S(A) + S(B) - S(A,B). \tag{1.14}$$

By definition, the quantum mutual information should capture the total correlations between quantum sub-systems[6].

In the same spirit as classical conditional entropy, one can in analogy define quantum conditional entropy as

$$S(A|B) = S(A,B) - S(A). \tag{1.15}$$

Can the inequality, $S(A,B) \geq S(A)$ hold?[7] For the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $S(A|B) = -1$, which means $S(A|B)$ may not be always positive. There are many works in the literature about what could this negative value of quantum conditional entropy mean [13–17]. In the work [16], authors found that *the more an observer knows about the system, the less it costs to erase it.* They have argued it by noticing that if the quantum conditional entropy between the system and the memory (it is with the observer) is negative then the cost of erasing the system is negative.[8] In another work [17], authors were addressing the question, "If two parties A and B share a quantum state ρ_{AB} and party B wants to learn about the system A, then how much information party A has to send him?" They found that the partial information required to be sent to B is $S(A|B)$. If the two parties are correlated then quantum conditional entropy can be negative. Hence, B can obtain the information about the system A by only using classical communication with A. Moreover, the negative conditional entropy gives B the ability to receive future quantum information for free. Like in quantum teleportation, if two parties share an correlated state one party can gain quantum information just by receiving classical communication from other.

Using Eqn.(1.15), one can derive other expressions of quantum mutual informations. These expressions of mutual information are not equivalent, unlike their classical counterpart. This discrepancy was then interpreted by many physicists [4], and thus evolved a new area of quantum information theory.

Note that the quantum mutual information can exceed entire uncertainty of the individual

[6] This statement is really debatable, it is yet to be established whether it captures total correlations. Many papers in the literature exist about this debate. We will discuss some of it in the later part of this thesis.

[7] As in the case of two random variables, X and Y, intuitively, one cannot be more uncertain about X than he is about joint state of X and Y i.e., $H(X) \leq H(X,Y)$.

[8] In quantum regime, it is the violation of the Landauer's erasure principle. An observer (memory) can extract work from a system while erasing it, thus by cooling the environment if the observer (memory) is strongly correlated with the system.

subsystems i.e.,

$$I^q(A:B) \leq 2\min[S(A), S(B)]. \tag{1.16}$$

This is precisely because of the negativity of conditional entropy for quantum entangled states and thus forbidden for classical case where *classical mutual information cannot exceed entire uncertainty of the source ensemble*, i.e., $I(A:B) \leq \min[H(A), H(B)]$.

1.3 Quantum correlations

Superposition in quantum theory has been the resource for quantum information theory. As it was argued earlier that apparently a single qubit potentially contains infinite amount of information. Let us think about the information a multiparticle state can posses if they are superposed with each other. Harnessing this power is the primary goal of quantum computation and information. Let us consider a two-qubit pure state

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle, \tag{1.17}$$

where α, β, γ & $\delta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. Let us examine the properties of this two particle state. If one parameter has value 1, then the state becomes $\sim |\chi\rangle \otimes |\phi\rangle$, otherwise the state may not be written in this form. The states which satisfy this property are called *product states*. These states contain no correlations at all.

In the classical world, we have a clear perspective on 'correlation', e.g. 'due to draught in West Bengal, the price of rice may rise in India'. The statement clearly states the correlation between two events. So, two objects/events are correlated means they are connected in such a way, the change in one will effect the other. In quantum world, situation is a bit messy. It is also not clear what we mean by 'correlations' in quantum regime. Clearly, one will interpret it as that on doing something on one particle, one will observe some change in the other. In quantum framework what do we mean by 'doing something'? Maybe measuring some properties, or evolving the system of the particle or, maybe applying local operations. Here we have to be specific about the operations which we will perform.

A quantum state (more than one particle) can possess different types of quantum correlations. These correlations may be responsible for nonlocality [18], steering [19], non-zero entanglement [3] and non-zero discord [4]. Here lies the difficulty of describing correlations present in a quantum state. On top of that a quantum state can posses classical correlations also. What we mean by all these? By 'nonlocality' we mean mostly Bell nonlocality [20,21]. Consider a situation where Alice and Bob share a quantum state. Alice and Bob make two dichotomic measurements on their parts which are $\{A_1, A_2\}$ and $\{B_1, B_2\}$ respectively. If their measurement outcomes violate the following inequality i.e.

$$|\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle| \leq 2, \tag{1.18}$$

which is the famous Bell-CHSH inequality [21], then the state is nonlocal i.e., it posses Bell nonlocality. Here $\langle X\rangle = \mathrm{Tr}[X\rho]$. Violation of this inequality within quantum regime sufficiently means that the state is both steerable and entangled.

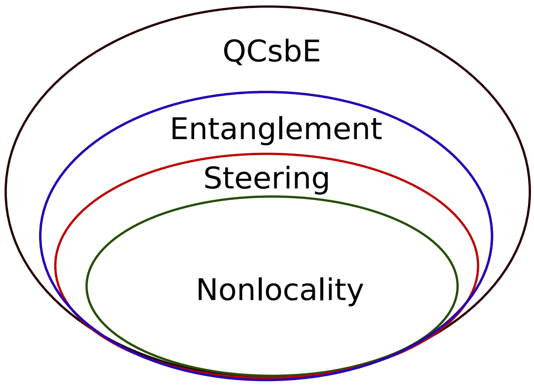


Figure 1.4: *Correlation hierarchy*: Pictorial depiction of quantum correlations present in a multiparticle quantum state.

Now, in the case of steering the role of Alice and Bob are different. If we say using the shared state Alice can steer Bob, then it is called Alice to Bob steering i.e., it is an asymmetric phenomenon. Alice will make measurement on her particle and by doing that she can remotely change (i.e., steer) the state of Bob's subsystem in such a way that would be impossible if their systems were only classically correlated. That means if a state is steerable, then it is 'sufficiently' entangled.

Entanglement is weaker than the steering and nonlocality. Here, one has to trust both parties to be quantum. Mathematically, if a state ρ_{AB} is entangled, then it cannot be expressed as $\sum_i p_i \rho_A^i \otimes \rho_B^i$ where $\rho_{A(B)}^i$ is a pure state for subsystem $A(B)$. A non-entangled state i.e., a separable state is the convex sum of product states. An entangle state may or may not violate Bell inequality i.e., entanglement can emerge from local-realistic model [22, 23]. But recently, Buscemi [24] established that all entangled state might display some kind of nonlocality (Note that his notion of nonlocality may not resemble with the Bell type). The connection between entanglement and nonlocality is not well understood, although some regards it as different sides of same coin [25–27]. Many works are available in the literature regarding this issue [28–31].

The discord is a measure of 'quantum correlations that go beyond entanglement' (QCsbE) (see Fig.(1.4)) because separable states also have non-zero discord values. And exactly for this reasons it is still debatable whether it is a correlation measure or not. And on top of that it not a LOCC (local operations and classical communications) [32] monotone.

We know that if a composite state violates Bell-CHSH inequality then it is entangled and steerable but reverse is not true [25, 28]. For example, consider the Werner state

$$\rho_w = \frac{1-p}{4}\mathbb{I}_4 + p|\psi^-\rangle\langle\psi^-|, \tag{1.19}$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. The state is entangled iff $p > \frac{1}{3}$, steerable $p > \frac{1}{2}$ and Bell nonlocal for $p > \frac{1}{\sqrt{2}}$. However, it has been argued that all the entangled states show some form of nonlocality [24, 26].

1.3.1 Quantum entanglement

If a multiparticle state (density matrix) can be represented by the convex sum of product states, then the state is said to be *separable state*, i.e., if $\rho_{12\ldots n}$ is a separable state then $\rho_{12\ldots n} = \sum_i p_i \otimes_{j=1}^n \rho_j^i$, where $\otimes_{j=1}^n \rho_j^i$ is a product state. Otherwise, the state is entangled. Let us consider two-particle scenario with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The state $|\psi\rangle_A \otimes |\phi\rangle_B$ is a product state with no correlation. However, $|\psi_1\rangle_A \otimes |\phi_1\rangle_B + |\psi_2\rangle_A \otimes |\phi_2\rangle_B$ (linear combination of two-particle states with proper normalization) is in general an entangled state. Now two important questions immediately follow – a) How to test entanglement of a given state? and b) Given an entangled state, how much entanglement is there in the state? Detection and quantification of entanglement [33] are important from the perspective of resource theory [3]. What do we mean by that is, whether an entangled state is useful for some specific tasks which otherwise is impossible using classical states or have advantages over classical resource. Quantum teleporation [8] and superdense coding [9] are two such examples.

The detection and quantification of entanglement are somehow settled (not well-settled!) in the case of two-qubit arbitrary states – which we will elaborate later. First we have to understand why 'the entangled states' are so important in the first place. *No-cloning theorem* [34] forbids the perfect copying of an unknown quantum state – which is a diversion from known classical theory. But one can ask, "Can one send an unknown state to a remote place?" Yes, it is possible – the 'quantum teleportation'. In teleportation if one uses an entangled state then only using maximum two-bits of classical communication one can send an unknown qubit faithfully [35]. There is no need to send the state itself. It provides extra security also. There are many aspects of entangled states, we will give some essence about some of them later. Let us now discuss the cases of two-particle and multiparticle systems.

1.3.1.1 Detection of entanglement: Two-particle systems

Before going into more complex systems, let us introduce simple two-particle systems in $\mathcal{H}_A^2 \otimes \mathcal{H}_B^2$ – two qubits. A general two-qubit state (density matrix) can be expressed in canonical form as

$$\rho_{AB} = \frac{1}{4}\Big(\mathbb{I}_4 + \vec{\sigma}.\vec{r} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{\sigma}.\vec{s} + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j\Big), \tag{1.20}$$

where $\vec{r}$ and $\vec{s}$ are local Bloch vectors of party A and B respectively. The 3×3 matrix $T = [t_{ij}]$ is the correlation matrix and for ρ_{AB} to be a valid density matrix, we should have $\sum_{i=1}^3 (r_i^2 + s_i^2) + \sum_{ij=1}^3 t_{ij}^2 \leq 3$. One can easily prove that the state (1.20) is a product state if $t_{ij} = r_i s_j$ but it is not a necessary condition. There are many such conditions by which one can easily detect its separability. We will introduce some of them.

A. Peres-Horodecki criteria – In the seminal papers [36, 37], Peres & Horodecki discovered that a separable state under *partial transposition* operations remains a valid state whereas an entangled state does not. Under partial transposition, the entangled state transforms to a matrix which has negative eigen values. Let ρ_{AB} be a bipartite state, then its partial transposition is defined as $\rho^T_{m\mu,n\nu} = \rho_{m\nu,n\mu}$ and

$$\rho_{m\mu,n\nu} = \langle e_m| \langle f_\mu|\rho|e_n\rangle |f_\nu\rangle \,, \tag{1.21}$$

where $|e_m\rangle$ ($|f_\mu\rangle$) denote the orthonormal basis in the Hilbert space of the first (second) subsystem of the composite system. Hence, *if at least one of the eigenvalues of a partially transposed density operator for a bipartite state, turns out to be negative then we can say that the state is inseparable.* This criteria is necessary and sufficient for the systems in the Hilbert space dimensions $2 \otimes 2$ and $2 \otimes 3$. This criteria fails for the higher dimensional systems because of the presence of 'bound entangled states'[9] [38]. Then how do we detect higher dimensional entanglement? The experimental way to detect any bipartite entanglement is via 'entanglement witness' [39].

B. Entanglement witness – To construct a witness operator [40–42], one uses the fact that the convex sum of separable states is a separable state i.e., the state space of separable state is convex. Not only that it is also a compact set. Therefore, one can find a hyperplane which will separate an entangle state from a separable state (see the Fig.1.5). From the figure it is clear that a particular witness will be able to detect a class of entangled states – hence it is a state dependent witness. Given an unknown state, witnessing its entanglement is a hard problem.

An observable, W will be an entanglement witness if

$$\begin{aligned}\mathrm{Tr}[W\rho_{sep}] &\geq 0 \;\forall \text{ separable states } (\rho_{sep}) \text{ and}\\ \mathrm{Tr}[W\rho_{ent}] &< 0 \text{ for at least one entangle state } (\rho_{ent})\end{aligned} \tag{1.22}$$

hold. So, if one measures $\mathrm{Tr}[W\rho] < 0$, the state ρ is entangled for sure and the entanglement of the state is detected by the witness W. The fact that all the witness operators are experimentally measurable makes them very useful tool for detecting entangled states. Note that the operators have also a clear geometrical meaning. It is well-known that the expectation value of the observable depends linearly on the state and hence the set of states with $\mathrm{Tr}[W\rho] = 0$ lives in a hyperplane in the set of all states, separating the whole set into two parts. The parts where $\mathrm{Tr}[W\rho] > 0$ lies all the separable states, the other part with $\mathrm{Tr}[W\rho] < 0$ is the set of states detected by the witness W.

C. Schmidt rank – The most general bipartite pure states can be expressed as $|\psi\rangle_{AB} = \sum_{i,j=0}^{d_A-1,d_B-1} \alpha_{ij}|ij\rangle$, where $d_A(d_B)$ is the Hilbert space dimension of subsystem $A(B)$. This state can be re-expressed by using 'singular value decomposition' in some other basis in the same Hilbert space as

$$|\psi\rangle_{AB} = \sum_{\ell=0}^{\min[d_A-1,d_B-1]} \sqrt{\lambda_\ell}|\ell\ell\rangle. \tag{1.23}$$

This decomposition is called Schmidt decomposition [43] and λ_ℓs are Schmidt coefficients which are real and positive with $\sum_\ell \lambda_\ell = 1$. The number of non-zero Schmidt

[9]The states which become negative after application of partial transposition are called negative partial transposition states i.e., NPT states and which remain positive are positive partial transposition states, in short PPT states. In higher dimension (above $2 \otimes 2$ & $2 \otimes 3$), some PPT states are also entangled but they are not distillable. We call these states bound entangled states.

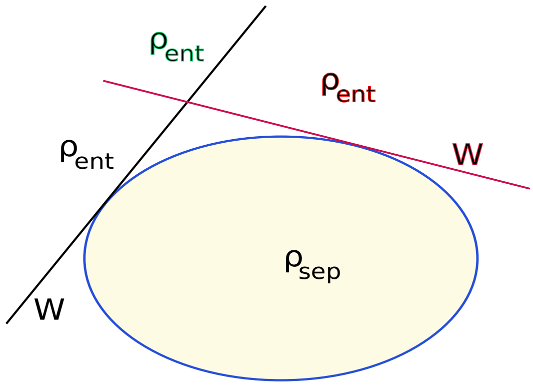


Figure 1.5: *Entanglement witness*: All the separable states are in the ellipsoid. The Ws (red and black) are two witness operators which can detect the entangled states (red & green) and (black & green) respectively. From the figure it is obvious that the witness operators are oblivious to some entangled states.

coefficients of a pure state is called its Schmidt rank ($SR(|\psi\rangle)$). Iff $SR(|\psi\rangle) = 1$, then the state is separable otherwise entangled.

For a bipartite mixed state $\rho_{AB} = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ in finite dimension, the Schmidt rank is defined as [44] $SR(\rho_{AB}) = \inf_{\{p_i,|\psi_i\rangle\}} \sup_i SR(|\psi_i\rangle)$ where, the infimum is taken over all the pure state decompositions. But finding Schmidt rank in this way is very difficult due to the optimization. Note that for more than two-party case, there exists no unique Schmidt decomposition and hence, this method loses its importance [45, 46].

1.3.1.2 Quantification of entanglement: Two-particle systems

Given a state the question we were posing: is it entangled? Now we will investigate, given an entangled state how much entanglement is there – quantification of entanglement. It has importance in many applications where entanglement is a resource or to improve the performance of protocols. For example, in the case of quantum teleporation where entanglement is the resource. The quantification of entanglement has direct connection with one more aspect 'given an entangled state, how much entanglement we can extract from it' — distillation of entanglement. Now to compare entanglement of two states, we usually compare them in terms of their performance/utility in a protocol where entanglement is the main resource. For example, in quantum teleportation using a Bell state one can faithfully teleport unknown single qubit pure state. Now take the other entangled states like $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$ ($\alpha, \beta \in \mathcal{C}$ and $|\alpha|^2 + |\beta|^2 = 1$), the teleportation will be inexact. The states which give exact teleportation have unit entanglement and entanglement of other states will be less than that.

Another intuition will be, given n pure states like $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$, can we distil $m(< n)$ Bell states ($\frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$)? The allowed transformation will be local operations and classical communications (LOCC) which will not increase the entanglement of the total system. Bennett et al. [47] showed that $m = nS(\rho_A)$, where ρ_A is the subsystem of

A. Then the quantification of entanglement of $|\psi\rangle$ is then

$$E(|\psi\rangle) = \frac{m}{n} = S(\rho_A) \tag{1.24}$$

in the unit of a maximally entangled state (Bell state). This idea is closely related with entanglement distillation [48, 49]. Now what about the reverse process, "How many Bell states are required to prepare a pure entangled state?" This idea is popularly termed as 'Entanglement of Formation' (EOF) [48, 50]. Before going into all these measures, we would list some properties which a good measure of entanglement [3] ($E(\rho)$) should have. These are

1. $E(\rho) = 0$ for all separable states.
2. It is invariant under Local Unitary (LU) i.e., $E(\rho) = E(\rho^{LU})$, where ρ^{LU} is the state after application of local unitary on ρ.
3. Under LOCC entanglement must not increase i.e., $E(\rho) \geq E(\rho^{LOCC})$ i.e., $E(\rho)$ is monotone under LOCC.
4. (a) Additivity, i.e., $E(\rho \otimes \sigma) = E(\rho) + E(\sigma)$ and (b) partial additivity i.e., $E(\rho^{\otimes n}) = nE(\rho)$.
5. Convexity, i.e., $E(\sum_i p_i \rho_i) \leq \sum_i p_i E(\rho_i)$ for the mixed state $\rho_{mix} = \sum_i p_i \rho_i$ with $\sum_i p_i = 1$,.
6. Continuity[10], i.e., if $\langle \psi^{\otimes n} | \rho_n | \psi^{\otimes n} \rangle \to 1$ for $n \to \infty$, then $\frac{1}{n} |E(|\psi^{\otimes n}\rangle) - E(\rho_n)| \to 0$.

Out of 6, first three postulates are the most important in the resource theoretic perspective, all others are needed but are not necessary. One good example of entanglement measure which satisfies all the above properties is *squashed entanglement* [51–53], but notoriously hard to compute. We will introduce some of the well known entanglement measures, namely, Von Neumann entropy, negativity [54], entanglement of formation, concurrence [50, 55], and concurrence monotones [56] which are relevant in this thesis.

A. Von Neumann entropy – It is a measure of entanglement for pure entangled states. It is well known that von Neumann entropy captures the uncertainty in a quantum state (which one can call as preparation uncertainty also). Consider the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The state has zero Von Neumann entropy but its individual part has maximal entropy i.e., one. This property was used to characterize the entanglement of the bipartite states. As a whole we know the preparation of bipartite entangled states, but about individual parts we have less knowledge. The more entangled a state is, the less knowledge we have about its parts. Hence, Von Neumann entropy of individual systems is a good measure of entanglement of the pure entangled states. The Von Neumann entropy also satisfies the three main properties to be a good measure.

Consider a pure bipartite state in the Schmidt decomposition form $|\psi_d\rangle = \sum_i \sqrt{\lambda_i} |ii\rangle$ in $d \otimes d$, then the reduced density matrix of the state is $\rho_{A(B)} = \sum_i \lambda_i |i\rangle\langle i|$ and hence the Von Neumann entropy, $S(\rho_{A(B)}) = -\sum_i \lambda_i \log_2 \lambda_i$. If $\lambda_r = 1$, then $S(\rho_{A(B)}) = 0$

[10] It tells us that if two states are close to each other then so are their entanglements per particle pair in a particular entanglement measure.

i.e., product state and if $\lambda_i = \frac{1}{d}$ $(\forall i)$ then $S(\rho_{A(B)}) = \log_2 d$ which is the maximum. The state with $S(\rho_{A(B)}) = \log_2 d$ are called maximally entangled states, e.g., Bell states $\{|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), |\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)\}$.

B. Negativity – Negativity [54, 57] quantifies how strongly the partial transpose of a density operator fails to become positive. The negativity, $N(\rho_{AB})$, of a bipartite state ρ_{AB} in $d \otimes d$, is defined as the absolute value of the sum of the negative eigenvalues of $\rho_{AB}^{T_A}$, where partial transposition is applied on particle A, i.e.,

$$N(\rho_{AB}) = \frac{1}{d-1}(||\rho_{AB}^{T_A}||_1 - 1), \tag{1.25}$$

where $||X||_1$ is called trace-norm of X and it is defined as $||X||_1 = \mathrm{Tr}[\sqrt{X^\dagger X}]$. The two main advantages of this measure are: It is very easy to compute and it is convex. But it is not additive, although one can consider *logarithmic negativity* [58] i.e., $E_N(\rho_{AB}) = \log_2[||\rho_{AB}^{T_A}||_1]$ which is additive but not convex. For all NPT states, a non-zero negativity implies that the state is entangled and distillable, whereas a vanishing logarithm negativity implies that the state may be separable. By construction, for $2 \otimes 2$ and $2 \otimes 3$ dimensional Hilbert space, it is if and only if condition.

Hence, in this case the main goal is to find the negative eigenvalues of the partial transposed states. Then what is the situation beyond $2 \otimes 2$ and $2 \otimes 3$ dimensions. In a recent work [59], S. Rana has found that for $m \otimes n$ dimensional state, it's partial transposition can have at most $(m-1)(n-1)$ number of negative eigenvalues. And not only that he also showed that all the eigenvalues of the partial transposed matrix of any $m \otimes n$ dimensional states must always lie within $[-\frac{1}{2}, 1]$ [59]. However, many issues are still open in this case.

C. Entanglement of Formation (EoF) – It is defined as the *convex roof extension*[11] of Von Neumann entropy of entanglement [50] i.e.,

$$E_F(\rho_{AB}) = \inf_{p_i, |\psi_i\rangle} \sum_i p_i E(|\psi_i\rangle), \tag{1.26}$$

where infimum is taken over all possible pure state decomposition of ρ_{AB} and $E(|\psi_i\rangle) = S(\rho_A^i)$. These optimizations are very hard to compute. As we were discussing, physically it says how many singlets one needs to create a single copy of the state. Another important aspect of EoF is that it is not additive [60].

D. Concurrence – Concurrence for pure state is the overlap between the state and its 'spin flipped version'. While calculating EoF for mixed state Wootters and Hill [50, 55] found that EoF is the monotonic function of concurrence (C) i.e.,

$$E_F(\rho_{AB}) = E(C(\rho_{AB})), \tag{1.27}$$

[11] **Convex roof extension**: Let g be a continuous real function on the space of pure states Ω^P, this function can be extended to the space of mixed states in the following way. Let Ω be the convex (and compact) set of normalized density operators. A state $\rho \in \Omega$ can be written as a convex combination $\rho = \sum_i p_i \rho_i$, where $\rho_i \in \Omega^P$ are pure states, are the extremal points of Ω. Then the real function $G : \Omega \to R$ is a convex roof extension of $g : \Omega^P \to R$ if G coincides with g on Ω^P and $G(\rho) := \inf \sum_i p_i g_i$, where minimization is taken over all pure state decomposition of ρ.

where $E(C) = H_2(\frac{1+\sqrt{1-C^2}}{2})$. Hence, the concurrence itself can also be regarded as an entanglement measure. Moreover, the relation is true only for $2 \otimes 2$ only i.e., the physical meaning of concurrence in higher dimension is not so clear [61]. The concurrence for two-qubit mixed state, ρ_{AB} is defined as

$$C(\rho_{AB}) = \max[0, \mu_1 - (\mu_2 + \mu_3 + \mu_4)], \tag{1.28}$$

where μ_i's are the eigenvalues (in decreasing order) of the matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)$ and complex conjugation $(*)$ is in the standard basis. This measure is not additive.

This measure for pure two-qubit state reduces to $C(\psi) = |\langle\psi|\tilde{\psi}\rangle|$, where $|\tilde{\psi}\rangle = (\sigma_2 \otimes \sigma_2)|\psi^*\rangle$. It can also be expressed as $C(|\psi_{AB}\rangle) = \sqrt{2(1 - \mathrm{Tr}[\rho_A^2])}$. These results can be extended for arbitrary dimensional states [62,63] and for multiparticle states [64–67]. For $d \otimes d$ dimensional pure states, the concurrence is defined as $C(|\psi_d\rangle) = \sqrt{\frac{2d}{d-1}\sum_{i<j}\lambda_i\lambda_j}$, where λ_i are Schmidt coefficients. If the state is of Schmidt rank r, then its maximal entanglement will be $\sqrt{\frac{d(r-1)}{r(d-1)}}$, for asymptotic case $d \to \infty$, the maximal value is $\sqrt{\frac{r-1}{r}}$, and for $d = r$ the maximal value is 1. For mixed $d \otimes d$ states $\rho_d = \sum_i p_i |\psi_d^i\rangle\langle\psi_d^i|$, it can be generalized using convex roof extension i.e., $C(\rho_d) = \inf \sum_i p_i C(|\psi_d^i\rangle)$, where minimization is taken over all possible pure state decompositions. Performing this minimization is very hard but some lower bounds have been derived in [66,67] generalizing the actual Wootters formula in higher dimensions.

E. Concurrence monotone – In the seminal work [68], G. Vidal developed the theory of *entanglement monotone*. In that work, he addresses a question aspect that if we have access to the finite number of bipartite entangled states then how one can characterize how good a resource it is. For a bipartite system in $d \otimes d$ dimensions, Vidal introduced a set of entanglement monotones [56] i.e.,

$$E_k(|\psi_d\rangle) = \sum_{i=k}^{d-1} \lambda_i, \quad k \in (0, d-1), \tag{1.29}$$

where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{d-1}$ are the Schmidt coefficients of the state $|\psi_d\rangle$. The entanglement monotones (Eq.1.29) play an important role in the transformation of states under LOCC [56,69,70]. All these monotones are also important for their distinct role in some particular information processing tasks. For example, $E_2 = 1 - \lambda_0$ quantifies the possibilities of faithful teleportation using partial entangled states [56,71].

Many such entanglement monotones can be constructed for entangled states. But question will be then whether they are important for characterizing resources for some protocols. One such monotone was introduced to study the *remote entanglement distribution* (RED) in a quantum network [71,72]. They are called *concurrence monotones* [71–73] and for $d \otimes d$ bipartite pure state $|\psi_d\rangle$,

$$C_k(|\psi_d\rangle) = \left(\frac{S_k(\lambda_0, \lambda_1, ..., \lambda_{d-1})}{S_k(\frac{1}{d}, \frac{1}{d}, ..., \frac{1}{d})}\right)^{\frac{1}{k}},$$

where, S_k is k^{th} elementary symmetric function of λ_k, the Schmidt's coefficients. $S_1(\lambda) = \sum_i \lambda_i$, $S_2(\lambda) = \sum_{i<j} \lambda_i\lambda_j$, $S_3(\lambda) = \sum_{i<j<k} \lambda_i\lambda_j\lambda_k$, ..., and $S_d(\lambda) = \prod_i \lambda_i$. If we carefully look at the C_2, it is just the bipartite concurrence. Note that these monotones do not play any role in transformation of states under LOCC [69,70]. For the bipartite mixed state $\rho_d = \sum_i p_i|\psi_d^i\rangle\langle\psi_d^i|$, using the convex roof extension, the concurrence monotone can be generalized to

$$C_k(\rho_d) = \inf \sum_i p_i C_k(|\psi_d^i\rangle), \tag{1.30}$$

where minimization is taken over all decomposition of ρ_d.

F. Squashed entanglement – For bipartite state ρ_{AB}, squashed entanglement [53,74] is defined as

$$E_{sq}(\rho_{AB}) = \frac{1}{2} \inf_{\rho_{ABC}} I^q(A:B|C),$$

where ρ_{ABC} is such that $\rho_{AB} = \mathrm{Tr}_C[\rho_{ABC}]$ and the conditional mutual information, $I^q(A:B|C) = S(A,C)+S(B,C)-S(A)-S(A,B,C)$. It is an additive measure which satisfies all the properties that one entangle measure should have. However, computing it is a NP hard problem.

1.3.1.3 Detection of entanglement: Multiparticle scenario

In the case of multiparticle systems, there are huge complexities in finding a good detection method and a good measure of entanglement. Many unique features arise and addressing all in one is very difficult. For example, how one will define separability here? Then we have to address the biseparability, triseparability etc and finally the fully separability opposite of which is the existence of genuine multiparticle entanglement [75]. The sharing of entanglement is also important here. There is a restriction on that, which is popularly known as *monogamy of entanglement* [76]. It leads to the frustration in the systems. Also the meaning of maximally entangled states also becomes dicey here. We really don't know how to characterize them. Many researchers have tried to address the question and many intriguing features have been unravelled. The 'absolutely maximally entangled states' [77,78], 'task oriented maximally entangled states' [79] etc. are some of them.

Over the past few years, there have been many attempts to characterize entanglement of the multiparticle systems [80]. Some have extended the idea of entanglement witness to witness the entanglement of the multiparticle states [33]. Others extended the idea of Von Neumann entropy of subsystems where they have considered the vector type entropic measure to characterize maximally, absolutely maximally entangled states. Recently, there have been some attempts to use the idea of concurrence for multiparticle systems – where the lower bound of the quantity has been considered. In this context, we will discuss some important concepts like entanglement witness, entropy vector method, concurrence, and monogamy of entanglement.

Entanglement witness – We know that we need specific witness operator to detect entanglement of specific class of states. In multiparticle case, there are many such classes. For pure three-qubit states, there are two well known witnesses i.e., W_{GHZ} and W_3 which

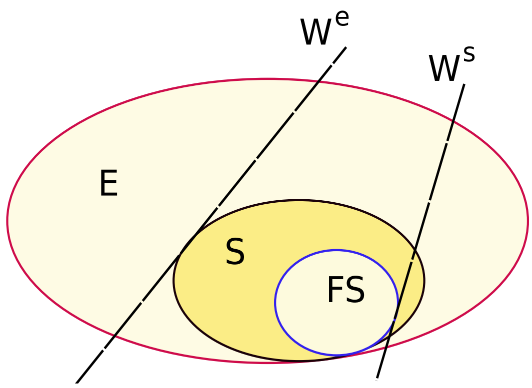


Figure 1.6: *Multiqubit states*: The pictorial view of multiqubit states: entangled states (E), separable states (S) and fully separable states (FS). The straight lines are the entanglement witnesses, W^e will witness entangled states and W^s separates out the fully separable states.

detect the GHZ states and genuine three-particle entangled states including the W-states [81] respectively. Any witness operator can be cast into the following form

$$W = \alpha\mathbb{I} - |\psi\rangle\langle\psi|, \tag{1.31}$$

where α is the overlap between state $|\psi\rangle$ and the biseparable state or the fully separable state (see Fig.1.6) [3][12]. The value of α for GHZ witness is $\frac{3}{4}$ which is exactly the overlap between GHZ state and W-state. Hence if the fidelity of GHZ witness is larger than $\frac{3}{4}$ then one can faithfully conclude that the state is in GHZ class. The calculation of α is crucial and depends on which class of states one wants to exclude. The calculation of α is straightforward if one takes into consideration of biseparable states only. This is because α is given by just the square of maximal Schmidt coefficient over all bipartitions [82, 83].

For more than three qubit pure entangled states, the standard procedure is to follow the above method of calculating the required overlap α and for many well known classes of states, the value of α is known. For graph states, the value of α is $\frac{1}{2}$ when biseparable states are considered. Hence, the witness will be $W_{G_n} = \frac{1}{2}\mathbb{I} - |G_n\rangle\langle G_n|$ for any n particle graph state $|G_n\rangle$ [84].

1.3.1.4 Quantification of entanglement: Multiparticle scenario

A. Von Neumann entropy vector – In case of pure bipartite systems, the Von Neumann entropy of one of the subsystems is enough to characterize the entanglement of the total system. But for multiparticle systems, it is not the case [78]. For example, let us consider the state $|0\rangle \otimes |\psi^+\rangle$. The entropies of three subsystems are 0, 1 and 1. Thus to characterize the entanglement of pure tri-partite systems one need at least three such num-

[12] A k-separable pure state is defined as $|\psi\rangle_{12\ldots n} = |\phi\rangle_1 \otimes |\phi\rangle_1 \otimes \cdots\cdots\cdots \otimes |\phi\rangle_k \otimes |\phi\rangle_{n-k}$, where $1 \leq k \leq n$ with $|\phi\rangle_{n-k}$ is $n-k$ partite non-separable state.

bers. Therefore, to characterize the entanglement of multi-qubit states one needs several[13] numbers of bipartition entropies.

In this context, one has to remember that to characterize the maximally entangled state one needs more numbers. The notion of maximally entangled states is not so clear in multiparticle scenario. Hence, the idea of *absolutely maximally entangled states* (AMES) came into the picture. Let us first define an AMES state. *'A multiparticle state with maximal possible subsystem entropies'* is called AMES. In the case of three qubits, GHZ state is the AMES with subsystem entropies being 1 which is the maximum possible. For four qubit case there exists no AME. Later, in this section we will discuss this notion in more detail.

B. Concurrence – A generalization of bipartite concurrence was discussed in [63, 64]. They derived the concurrence for pure n-particle arbitrary dimensional systems

$$C(|\psi\rangle_{12..n}) = 2^{1-\frac{n}{2}}\sqrt{(2^n-2)-(\sum_{\alpha}\mathrm{Tr}[\rho_\alpha^2])}, \tag{1.32}$$

where α denotes all different possible reduced density matrices of the state $|\psi\rangle_{12..n}$. Note that there are $\binom{n}{n_1}$ number of ρ_α which can be obtained by tracing over n_1 different subsystems. This measure is zero for fully separable states. It was later shown that this quantity can be experimentally measurable by recasting it using one factorizable observable while one has access to only two copies of the state [65]. The only drawback of this measure is that it gives maximal value for GHZ state which is not thought to be maximally entangled state for more than three-particle systems. To extend it for mixed states one needs to use the convex roof extension method i.e., $C(\rho_{12\ldots n}) = \inf \sum_i p_i C(|\psi\rangle_{12..n})$, where the infimum is taken over all possible pure state decomposition of mixed state $\rho_{12\ldots n}$. Performing this optimization is NP hard.

To bypass this optimization problem, many researchers have calculated the upper and lower bound on the concurrence of mixed n-particle systems. In [66], authors gave a lower bound using the Wootters method generalizing it for multiparticle systems. In another work [67], authors have directly derived a lower bound to the concurrence defined above by just manipulating the subsystems.

C. Monogamy of entanglement – Sharing of entanglement between parties cannot be arbitrary. In the seminal paper Coffman, Kundu and Wootters found out that there is a restriction on sharing of entanglement between parties – which is popularly known as monogamy of entanglement or CKW inequality [76]. They derived the relation for three qubit systems and using the concurrence. Let $|\psi\rangle_{ABC}$ be a three-qubit system then the inequality reads

$$C^2_{A|BC} \geq C^2_{A|B} + C^2_{A|C}, \tag{1.33}$$

where $C_{X|Y}$ is the concurrence of the bipartition $X|Y$. This relation means if the entanglement between the subsystems A and B is maximal i.e., unity, then there should not

[13] For n-particle systems one can have $\binom{n}{k}$ numbers of potential bipartition of the type $k|n-k$, where k can be atmost $\lfloor \frac{n}{2} \rfloor$ (the floor function $\lfloor\cdot\rfloor$ is introduced to take care of odd and even n). Hence, the number of subsystem entropies required to characterize the entanglement of the system are $\sum_k \binom{n}{k}$. Out of this number of entropies many will be repetitive, i.e., actual number will be smaller.

be any entanglement between A and C. This relation was then generalized to n-qubit systems which is $C^2_{1|23\ldots n} \geq \sum_{i=2}^{n} C^2_{1|i}$ [85]. The relation is a generic feature of entangled states and using this relation one can define a proper entanglement measure called *tangle*. The tangle for a three-qubit system is called the three-tangle [76]. The three-tangle [86] for a pure three-qubit system is defined as $\tau(|\psi\rangle_{ABC}) = C^2_{A|BC} - (C^2_{A|B} + C^2_{A|C})$.

The CKW type relations have been derived for many other entangled measures like squashed entanglement [87], negativity [88], etc., and not only that many new type of monogamy inequality have been discovered recently [89, 90]. These inequalities tell us that the entanglement shared between the parties in a multiparticle systems are restrictive and gives rise to a new phenomenon – frustration.

1.3.1.5 Multiparticle entangled states

An arbitrary bipartite pure entangled state in arbitrary dimension can be expressed as

$$|\psi\rangle_{AB} = \sum_{i,j} C_{ij}|ij\rangle, \tag{1.34}$$

where $C = [C_{ij}]$ in general is neither unitary nor Hermitian. The state can be a maximally entangled state iff its reduced density matrix, $\rho_A = \mathrm{Tr}_B[\rho_{AB}] = C^\dagger C = \frac{1}{d}\mathbb{I}$ or vice versa. If there exist two maximally entangled states $|\psi\rangle_{max}$ and $|\phi\rangle_{max}$, then $|\psi\rangle_{max} = U_A \otimes U_B|\phi\rangle_{max}$ i.e., maximally entangled states are same upto some local unitary transformations. Note that if the state $|\psi\rangle_{AB}$ is a maximally entangled state then U_A should be equal to $\frac{1}{\sqrt{d}}C$. Broadly, there exists two distinct classes – the *isotropic states* and the *Werner states*. All $U \otimes U^*$ invariant states have two fixed points – $\mathbb{I}$ and the projector P^+. These states are called isotropic states [91] and are expressed as

$$\rho_{iso} = \alpha\mathbb{I} + \beta P^+, \tag{1.35}$$

where $\alpha + \beta = 1$ and $P^+ = \frac{1}{\sqrt{d}}\sum_i |ii\rangle\langle ii|$. These states are either separable or distillable, no bound entanglement exists here. The $U \otimes U$ invariant states have two fixed points also – $\mathbb{I}$ and the swap operator V. These states are called Werner state [22] and are expressed as

$$\rho_{wer} = \alpha\mathbb{I} + \beta V, \tag{1.36}$$

$V(|\phi\rangle \otimes |\chi\rangle) = |\chi\rangle \otimes |\phi\rangle$. In this class bound entangled states may exist. In case of $2 \otimes 2$ dimensional states, isotropic states and Werner states are equivalent.

In the case of multiparticle systems, the classification of states require a detailed analysis and we still don't know the complete picture. Here, we will discuss properties of some important states using which we will gain some information about their diversity. There are some important notions regarding the maximally(?) entangled states in multiparticle systems ($n > 2$) [92]. The maximally entangled states (MES), absolutely maximally entangled states (AMES), and task oriented maximally entangled states (TOMES) to name a few. All these states are genuine entangled states [93].

A. MES and AMES – Maximally entangled states are the pure n-particle states where single party subsystems are in maximally mixed states. Hence, in this case the entropy of the single party will be $\log_2 d$, where d is the Hilbert space dimension of that part. The above one is a not a satisfactory definition of maximally entangled states. The more appropriate one will be: A n partite state $|\psi\rangle_n = \sum_{i_1,i_2,...,i_n} C_{i_1 i_2 ... i_n} |i_1, i_2, ..., i_n\rangle$, is called absolutely maximally entangled state iff $C = [C_{i_1 i_2 ... i_n}]$ is a perfect tensor[14] [78]. This means that for all AMES states the entropy of all possible subsystems (single-particle, two-particle, ... etc) will be maximum possible entropy i.e., $\log_2 d$, $2\log_2 d$, ... etc.

In three-qubit case, GHZ state ($|g_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$) is the maximally entangled state and also AMES. But when the four-qubit systems are considered the situation is not clear. For four-qubit case, there exists no AMES states. In this respect we will investigate the all possible subsystem entropies of the important states,

$$\begin{aligned}
|g_4\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \\
|W\rangle &= \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle), \\
|S_1\rangle &= \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle), \\
|S_2\rangle &= \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle), \\
|HS\rangle &= \frac{1}{\sqrt{6}}(|0011\rangle + |1100\rangle + \omega(|0101\rangle + |1010\rangle) + \omega^2(|0110\rangle + |1001\rangle)), \\
|C_4\rangle &= \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle), \\
|L\rangle &= \frac{1}{2\sqrt{3}}((1+\omega)(|0000\rangle + |1111\rangle) + (1-\omega)(|0011\rangle + |1100\rangle) \\
&\quad +\omega^2(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle)), \\
|B_4\rangle &= \frac{1}{2\sqrt{2}}(|0110\rangle + |1011\rangle + i(|0010\rangle + |1111\rangle) + (1+i)(|0101\rangle + |1000\rangle)), \\
|YC\rangle &= \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle), \\
|HD\rangle &= \frac{1}{\sqrt{6}}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle + \sqrt{2}|1111\rangle), \qquad (1.37)
\end{aligned}$$

where $\omega = e^{\frac{2i\pi}{3}}$. From Table (1.1), it is clear that all the states listed above are highly entangled states yet none of them satisfy the criteria to be the AMES [78]. This way of characterizing the multiparticle states might reveal more information about their structure and the nature of their entanglement [80]. *Note that the states $|HS\rangle$ and $|HD\rangle$ have same subsystem purities as measured by $\mathrm{Tr}[\rho^2]$ but have different subsystem entropies* [78]. We will now explain the notion of task oriented maximally entangled states (TOMES) in the context of three and four qubit states.

B. TOMES – In [79], authors noticed that the notion of maximally entangled states for bipartite pure entangled systems is unique but in case of multiparticle systems there is no clear evidence of such notion. This is because every entanglement measures for pure

[14] If C is a perfect tensor, then it will be multiunitary according to all possible permutations.

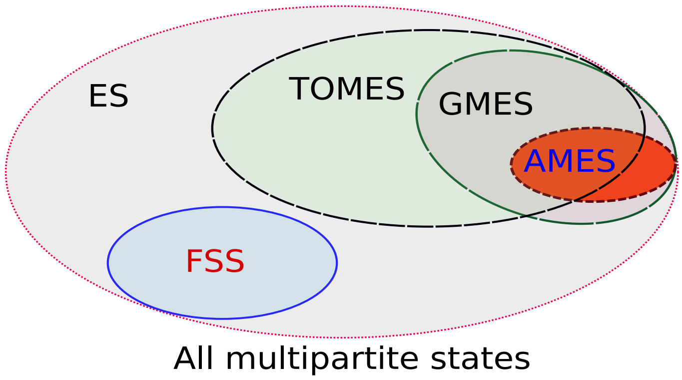


Figure 1.7: *Pure multiqubit states*: The picture depicts hypothetical distribution of multipartite states: the fully separable states (FSS), the multiparticle entangled states (ES), genuine multiparticle entangled states (GMES), task oriented maximally entangled states (TOMES), and absolutely maximally entangled states (AMES). Notice that TOMES can contain even some k-separable multiparticle states while it may not include all GMES and even some AMES.

bipartite systems behave similarly whereas for multiparticle systems, this is not the case. Hence, the notion of maximally entangled multiparticle states may not be universal. There may exist maximally entangled states with respect to some multiparticle entangled measure, but such a state may not be suitable for most tasks that one may envision.

They defined the TOMES for some specific tasks like quantum teleportation [8], superdense coding [9], quantum cryptography [7, 10, 94], secret sharing [95], telecloning [96], and or, violating Bell inequalities [18], violating steering equalities and inequalities [19], etc. *If the multiparticle state is useful in performing such tasks maximally then the state is called the TOMES with respect to that specific task.* Then they considered teleportation and superdense coding for illustrating their notion for n-qubit states.

An n-qubit state would allow one to teleport an unknown arbitrary $\frac{n}{2}$-qubit state faithfully (i.e., with unit fidelity and probability), when n is even and $\frac{n-1}{2}$-qubit state, when n is odd. A n-qubit state would allow one to transmit maximally n-bits of information by sending $\frac{n}{2}$ qubits when n is even and $\frac{n+1}{2}$ qubits when n is odd [97]. For example, the four qubit GHZ-state (see Eq.(1.37)) is not a TOMES in the context of teleportation because one cannot teleport an unknown arbitrary two-qubit state. Furthermore, using four qubit GHZ-state, although one can transmit 2 bits by sending 1 qubit and 3 bits by sending two qubits, one cannot transmit 4 bits by sending 2 qubits from Alice to Bob [98]. Therefore, the state is not suitable for the maximal superdense coding also. The subclass of three qubit W-state given in [99], $|W\rangle = \frac{1}{\sqrt{2+2n}}(|001\rangle + \sqrt{n}|010\rangle + \sqrt{n+1}|001\rangle)$ is suitable for maximal teleportation of unknown 1 qubit state as well as suitable for sending maximal 3 bits by sending two qubits from Alice to Bob for some specific bipartitions not all bipartions will serve the perpose of maximal superdense coding. As in the definition of TOMES, this type of restriction is not there, therefore, the state is a TOMES with respect to the tasks of teleportation and superdense coding [79]. Note that the product state $|\phi^+\rangle \otimes |0\rangle$ can also be useful for maximal superdense coding for specific bipartition [79].

For four qubit systems, the prototype states, $|C_4\rangle$ and $|YC\rangle$ are the TOMES with respect to the tasks of teleportation and superdense coding, but GHZ-state, $|g_4\rangle$ and $|HS\rangle$ are not. Also if we consider quantum secret sharing then the states $|g_4\rangle$, $|C_4\rangle$, and $|YC\rangle$ can be TOMES while $|HS\rangle$ cannot (for these states, see Eq.(1.37)).

Therefore, the nature of entanglement of multi-qubit systems are very diverse and not easy at all to explain by one approach (e.g. see the Fig.(1.7)). For example, we know that there exist multi-qubit AMES for 5, 6-qubit systems and it is numerically verified that there does not exist AMES for 7-qubit systems. Also, for $n \geq 8$ qubits, there exists no AMES [78]. This complexity increases if higher dimensional states are considered. Note that in this context we never talked about the mixed states. In general, the mixed states are notoriously complex to characterize.

states	$S(\rho_1)$	$S(\rho_2)$	$S(\rho_3)$	$S(\rho_4)$	$S(\rho_{12})$	$S(\rho_{13})$	$S(\rho_{14})$
$\|g_4\rangle$	1	1	1	1	1	1	1
$\|W\rangle$	0.81	0.81	0.81	0.81	1	1	1
$\|S_1\rangle$	0.81	1	0.81	0.81	1.5	1.22	1.22
$\|S_2\rangle$	0.81	1	1	1	1.5	1.5	1.5
$\|HS\rangle$	1	1	1	1	1.79	1.79	1.79
$\|C_4\rangle$	1	1	1	1	1	2	2
$\|L\rangle$	1	1	1	1	1.58	1.58	1.58
$\|HD\rangle$	1	1	1	1	1.58	1.58	1.58
$\|B_4\rangle$	1	1	1	1	1.6	1.6	2
$\|YC\rangle$	1	1	1	1	2	2	1

Table 1.1: *Entropy structure*: The table depicts the all possible subsystem entropies of the states given in Eq.(1.37).

1.3.2 Total, classical, and quantum correlations

Intuitively, a quantum state may posses two types of correlations – quantum correlations and classical correlations [100, 101]. The intuition is very straight forward – consider a classical state, $\rho_{cl} = p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11|$. The state ρ_{cl} can also be expressed in terms of joint probability P_{ab}. Any state which can be written as $\sum_{a,b} p_{ab}|ab\rangle\langle ab|$ is called classically correlated state if $\{|ab\rangle\}$ form a product basis. The state has no quantum correlations at all. Whereas the state of the form $\rho_q = p|00\rangle\langle 00| + (1-p)|++\rangle\langle ++|$ which is a mixture of non-orthogonal parts, can possess quantum correlations as well as classical correlations.

To capture total correlations in a quantum state, the quantifier $\mathcal{T}(\rho)$ should posses few requirements [102]

1. $\mathcal{T}(\rho) = 0$ for product states.
2. $\mathcal{T}(\rho) \geq 0$, i.e., positive always.
3. $\mathcal{T}(\rho)$ is invariant under local unitary.
4. $\mathcal{T}(\rho)$ is non-increasing under local operations.

Note that the classical correlation and quantum correlation quantifier should also satisfy the above criteria. Having set such criteria, one can now look for the suitable quantifier for total correlations. In classical information theory mutual information captures the correlations between two random variables. It was then argued that quantum generalizations of mutual information might be a good quantifier of total correlations [103]. But it is not the case in general [104]. We will argue it clearly in the next part of this section.

In 1989, Bernett and Phoenix introduced the quantum version of Pearson's coefficients[15] (which they termed as *index of correlation* as the measure of total correlations in the quantum states [105]. Later, many authors introduced distance based measures for total correlations, like distance between the bonafide state ρ and the closest product state ρ_{pro} i.e., $\mathcal{T}(\rho) = \inf ||\rho - \rho_{pro}||_p$, where $||.||_p$ is the Scattern-p norm, or relative entropy between the states i.e., $\mathcal{T}(\rho) = \inf S(\rho||\rho_{pro})$. All the above quantifier are hard to compute analytically and become more and more difficult with the increase in the number of particles. After minimization the quantifier, the relative entropy becomes a particular form of mutual information in qubit case [106]. Hence it is worth studying quantum mutual information in more details and investigate whether it is a good candidate to capture total correlations.

1.3.2.1 Quantum mutual information and total correlations

Quantum mutual information might capture total correlations in a quantum state. But problem is that there are many different inequivalent forms of quantum mutual information while generalizing them from classically equivalent expressions. Before going into the discussion we will collect all those forms of mutual informations which might be the potential quantifier of total correlations. For bipartite case, the relative entropy type quantifier and Venn diagram approach mutual information coincide with each other, i.e., $\inf_{\{\sigma_A,\sigma_B\}} S(\rho_{AB}||\sigma_A \otimes \sigma_B) \equiv I^q(A:B) = S(A) + S(B) - S(A,B)$, where $\sigma_{A(B)}$ may not be the marginals for most general case[16] [106]. Moreover, the bipartite quantum mutual information satisfies all the properties of total correlations measure listed above.

In the case of bipartite systems, the quantum generalization of mutual information in terms of Von Neumann entropy is $I^q(A:B) = S(A) + S(B) - S(A,B)$. Many arguments and numerical findings exist in literature to support the idea that quantum mutual information is a good quantifier of total correlations. Which are

1. Classical mutual information captures all the correlations between two random variables. Being a straight forward generalization from classical one, quantum mutual information should capture total correlations in a state.

2. Groisman *et al.* [107] showed that the amount of randomness required to erase all the correlations in a quantum state is exactly equal to its mutual information. This

[15] Quantum version of Pearson's coefficients for the observables $\mathcal{O}_A$ and $\mathcal{O}_B$ is defined as $\mathcal{T}_{\mathcal{O}_A\mathcal{O}_B} = \frac{\mathrm{Cov}(\mathcal{O}_A,\mathcal{O}_B)}{\sqrt{\mathrm{Var}(\mathcal{O}_A)}\sqrt{\mathrm{Var}(\mathcal{O}_B)}}$. where $\mathrm{Cov}(\mathcal{O}_A,\mathcal{O}_B) = \mathrm{Tr}[\rho\mathcal{O}_A \otimes \mathcal{O}_B] - \mathrm{Tr}[\rho_A\mathcal{O}_A]\,\mathrm{Tr}[\rho_B\mathcal{O}_B]$ and $\mathrm{Var}(\mathcal{O}) = \mathrm{Tr}[\mathcal{O}^2\rho] - (\mathrm{Tr}[\mathcal{O}\rho])^2$.

[16] If one considers relative Rènyi and Tsallis entropies instead of relative Von Neumann one.

gives the operational justification to Landaurer erasure principle [108].

3. If Alice shares a maximally entangled state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ with Bob, then by manipulating her part and sending the qubit to Bob, she can communicate 2 bits of information to him – which is called superdense coding [9]. As 1 qubit count for the encoding of 1 bit of classical communication, the additional communication power is related to the negative conditional entropy [13, 15]. Also, the mutual information of the maximally entangled state is 2.

4. The above fact is supported by Schumacher and Westmoreland further: For a one-time-pad cryptographic communication system if one uses a bipartite quantum state as the key, then the maximum amount of information that can be sent securely is the quantum mutual information of the state [109].

These arguments and facts support bipartite quantum mutual information as a good quantifier of total correlations. Note that there is no direct proof of this fact unlike its classical counterpart. These arguments may not be true for multiparticle quantum systems. Before delving deeper in the discussions we will try to define and quantify classical correlations in a quantum state.

1.3.2.2 Classical correlations in a quantum state

In classical word, intuitively we know what we mean by classical correlations. In case of two random variables, we calculate the Pearson's coefficient ($C_{X,Y} = C_{Y,X}$) (or co-variance)[17] to find the correlation between them. If Pearson coefficient is $+ve$ then it implies the correlation between them and if $-ve$, it is the anti-correlation. Another important measure in classical information science is mutual information, $I(X : Y) = H(X) + H(Y) - H(X, Y)$ which captures all possible correlations (obviously classical) between two random variables. This is possibly more general in the sense that Shannon entropy, $H(X)$ captures all possible moments while in Pearson coefficients it is only second order. Although quantum generalization of mutual information is often used as the quantifier of total correlations, quantum version of Pearson's coefficients involve the quantum observables and quantum state. It is not well established as a good quantifier of correlations in quantum states [e.g. see [110]]. Hence, we will not discuss Pearson's coefficient further.

Given all this background we can now focus on classical correlations in a quantum state. For example, the state $\rho_{cc} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ has maximal classical correlations in it. The mutual information of the state, $\rho_{cc} = 1$ is 1. Now if we take the state $p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11|$, the classical correlations in it is $H_2(p)$. $H_2(p) = 1$ if $p = \frac{1}{2}$. Hence, the classical correlations in a quantum state depends on the classical mixing parameter p in some sense. For example, the equal mixture of two Bell states ($|\phi^+\rangle$ & $|\phi^-\rangle$ or $|\psi^+\rangle$ & $|\psi^-\rangle$) are the maximally classical correlated state. Although it is too early to say

[17]For two random variables X and Y, the Pearson's coefficient is defined as $C_{X,Y} = \frac{\mathrm{Cov}(XY)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$, where $\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(X)$ is the covariance and $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ is the variance with $\mathrm{E}(\cdot)$ as the expectation value.

that the classical correlations in a quantum state is due to our lack of knowledge about their preparations, it may be the case. If indeed this is the case then a pure quantum state may not contain any classical correlation at all. But Groiseman *et al.* [107] argued that a maximally entangled state contains maximal amount of classical correlation. Their argument goes like: in order to erase the information content in $|\psi^+\rangle$, Alice flip a unbiased coin and if head comes she does nothing on her qubit and if tail comes she performs σ_z. In this whole process, Bob will not do anything on his part. Then the final state is $\frac{1}{2}(|\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|) = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, which contains 1 bit of classical information. But in order to do so they are introducing 1 bit of classicality (noise) in the state already through the random operation.

According to Modi [101], *"A state is said to be classically correlated if and only if it can be fully determined without disturbing it with the aid of local measurements and classical communication"*. According to him, a classically correlated states should be diagonal in an orthonormal product basis. The general structure for these type of states are $\sum_{a,b} p_{ab}|ab\rangle\langle ab|$, where $\{|ab\rangle\}$ forms an orthogonal product basis. It has been argued that the separable states are the *shadows* of classical states [111–113]. For a separable state $\rho \in \mathcal{H}_a \otimes \mathcal{H}_b$, there exists always a classical state χ in larger Hilbert space $\mathcal{K}_a \otimes \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{K}_b$ such that $\rho = \mathrm{Tr}_{\mathcal{K}_a, \mathcal{K}_b}[\chi]$.

Before concluding anything about the classical correlations, quantum correlations and total correlations in quantum states, let us review some of the existing literature regarding these.

1.3.2.3 Classical correlations and conditional quantum mutual information

In 2001, Henderson & Vedral [100] tried to split the total correlations in a state into classical and quantum one consistently. They argued that the quantum generalization of mutual information may capture total correlations of the state like its classical counterparts. They also considered relative entropy of entanglement as a measure of quantum correlations. This intuition they got from the definition of mutual information which captures the distance between the state and its marginals i.e., $I^q(A:B) = S(\rho_{AB}||\rho_A \otimes \rho_B)$, so as relative entropy of entanglement is the minimum distance of the state from its nearest separable state, i.e., $E_{RE}(\rho_{AB}) = \inf_{\sigma_{AB}} S(\rho_{AB}||\sigma_{AB})$ [114, 115]. Hence, $E_{RE}(\rho_{AB}) \leq I^q(A:B)$ in general. According to them a classical correlation measure, $(C_\ell(\rho))$, should satisfy the following properties: – (i) $C_\ell(\rho) = 0$ for product state, (ii) it is invariant under local unitary operations, (iii) it is non-increasing under local operations, and (iv) for pure state, it is just equal to the entropy of the subsystem, like the quantum correlations (here entanglement). Then they suggest a measure which satisfy all these criterion and the proposed measure is

$$C_{\ell_B}(\rho_{AB}) = \max_{\pi_i^B}[S(A) - \sum_i p_i S(A|\pi_i^B)], \tag{1.38}$$

where $S(A|\{\pi_i^B\}) = S(\rho_{A|\pi_i^B})$ with $\rho_{A|\pi_i^B} = \frac{\pi_i^B \rho_{AB} (\pi_i^B)^\dagger}{p_i}$ (where $p_i = \mathrm{Tr}(\pi_i^B \rho_{AB})$ is the probability of obtaining the i^{th} outcome). Here, $S(A|\{\pi_i^B\})$ is the Von Neumann entropy of the qubit A, when the POVM is applied on B and the measurement result is i.

Alternatively,

$$C_{\ell_A}(\rho_{AB}) = \max_{\pi_i^A}[S(B) - \sum_i p_i S(B|\pi_i^A)], \tag{1.39}$$

where measurement is performed on subsystem A. Then they conjectured that $C_{\ell_B}(\rho_{AB}) = C_{\ell_A}(\rho_{AB})$ if $S(A) = S(B)$. But we know this is not true in general. The measure is the generalized version of the classical mutual information, $I(A : B) = H(A) - H(A|B)$ in the quantum regime. Moreover, the quantity is similar to the *Holevo bound*[18] which measures the capacity of quantum states for classical communication. They used three examples to illustrate the measure of classical correlations. First, they considered the state,

$$\rho_{AB}^1 = p|\phi^+\rangle\langle\phi^+| + (1-p)|\phi^-\rangle\langle\phi^-|, \tag{1.40}$$

where $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $\frac{1}{2} \leq p \leq 1$. For the state, the mutual information is $I^q(A : B) = 2 - H_2(p)$, relative entropy of entanglement, $E_{RE}(\rho_{AB}^1) = 1 - H_2(p)$, and classical correlations, $C_{\ell_A}(\rho_{AB}^1) = C_{\ell_B}(\rho_{AB}^1) = 1 = C_\ell(\rho_{AB}^1)$. Therefore, in this case $I^q(A : B) = E_{RE}(\rho_{AB}^1) + C_\ell(\rho_{AB}^1)$. Note that $C_\ell(\rho_{AB}^1) \geq E_{RE}(\rho_{AB}^1)$. Next, they considered the Werner state

$$\rho_{AB}^w = \frac{1-p}{4}\mathbb{I}_4 + p|\phi^+\rangle\langle\phi^+|, \tag{1.41}$$

with $\frac{1}{2} \leq p \leq 1$. The mutual information for the state is given by $I^q(A : B) = 2 - H_2(f) + (1-f)\log_2 3$ and relative entropy of entanglement is $E_{RE}(\rho_{AB}^w) = 1 - H_2(f)$, where $f = \frac{3p+1}{4}$. As the state is symmetric, its classical correlations will be $C_{\ell_A}(\rho_{AB}^w) = C_{\ell_B}(\rho_{AB}^w) = C_\ell(\rho_{AB}^w)$. They numerically showed that in this case $C_\ell(\rho_{AB}^w) + E_{RE}(\rho_{AB}^w) < I^q(A : B)$ and interestingly $C_\ell(\rho_{AB}^w) \geq E_{RE}(\rho_{AB}^w)$. This may indicate that the mutual information may not be a good measure of total correlations, or the correlations are not additive at all. They noticed that in asymptotic limit measurements on many copies might yield larger value of classical correlations as classical correlations are superadditive i.e., $C_\ell(\rho \otimes \rho) \geq 2C_\ell(\rho)$, while relative entropy of entanglement is subadditive i.e., $E_{RE}(\rho\otimes\rho) \leq 2E_{RE}(\rho)$ and total correlation measure, mutual information is additive, i.e., $I^q(\rho \otimes \rho) = 2I^q(\rho)$. Also they raised the question whether $C_\ell(\rho) \leq I^q(A : B)$ is true in general. Then they provided another possible measure of classical correlations which is based on relative entropy, $C_{\ell_{RE}}(\rho_{AB}) = S(\sigma_{AB}^*||\rho_A \otimes \rho_B)$, where σ_{AB}^* is the closest separable state to ρ_{AB}. For the state ρ_{AB}^1, $C_{\ell_{RE}}(\rho_{AB}^1) = 1 = C_\ell(\rho_{AB}^1)$, for the state (this is their third example) $\rho_{AB}^q = p|00\rangle\langle 00| + (1-p)|++\rangle\langle ++|$ (where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$), $C_{\ell_{RE}}(\rho_{AB}^1) = I^q(A : B)$. For Werner state, $C_{\ell_{RE}}(\rho_{AB}^w) = 0.2075\ \forall p \in \{\frac{1}{2}, 1\}$, therefore for smaller values of p, $C_{\ell_{RE}}(\rho_{AB}^w) > E_{RE}(\rho_{AB}^w)$, but for higher values $C_{\ell_{RE}}(\rho_{AB}^w) < E_{RE}(\rho_{AB}^w)$. So, in general $I^q(A : B) > C_{\ell_{RE}}(\rho_{AB}^w) + E_{RE}(\rho_{AB}^w)$. It is also unclear whether $C_{\ell_{RE}}(\rho_{AB})$ is non-increasing under local operations.

[18] Let say, Alice has the random variable X which can take value x with probability p_x. She encodes X in an quantum ensemble $\{p_i, \rho_i\}$ and sends to Bob. Now, Bob can extract information about X by performing POVM operations $\{E_Y\}$ on the ensemble. He denotes the classical outcomes as Y. Therefore, the accessible information for Bob is the maximum possible mutual information $I(X : Y)$ which is upper bounded by Holevo bound (or popularly known as Holevo χ quantity), i.e., $I(X : Y) \leq \chi(\{p_i, \rho_i\})$, where $\chi(\{p_i, \rho_i\}) = S(\sum_i p_i\rho_i) - \sum_i p_i S(\rho_i)$.

1.3.2.4 Quantum discord

The quantum discord was introduced by Olivier and Zurek (2002) [116] as a measure of the “quantumness of correlations”. It is defined in terms of the mutual information. Classically, the mutual information is a measure of common information in two random variables. Therefore, it was natural to generalize it to the quantum domain and express quantum correlation in terms of this object. As discussed before, however, the definition of the mutual information in quantum domain is not straightforward. This is because there are more than one classical expressions to define the mutual information. These different expressions admit different generalizations. In the quantum discord, this difference is used to characterize the quantum correlations. Classically, one can write the mutual information in two alternate ways,

$$\begin{aligned} J(X:Y) &= H(X) - H(X|Y), \text{ and} \\ I(X:Y) &= H(X) + H(Y) - H(X,Y). \end{aligned} \tag{1.42}$$

Here $H(X), H(X,Y)$ and $H(X|Y)$ are the entropy, joint entropy, and conditional entropy for the random variables X and Y. The Joint entropy and conditional entropy are related by the chain rule, $H(X|Y) = H(X,Y) - H(Y)$.

These expressions for the entropies can be generalized to the quantum domain by substituting random variables by density matrices and Shannon entropies by Von Neumann entropies. For example, $H(X) \to S(\rho_X) = -\mathrm{Tr}[\rho \log_2(\rho)]$. The generalization of the mutual information will also involve the generalization of the conditional entropy. They used the generalization as suggest in the Ref [100]. Using this generalization to the quantum domain, one gets for the state ρ_{XY},

$$\begin{aligned} I^q(X:Y) &= S(X) + S(Y) - S(X,Y) \\ J^q(X:Y) &= S(X) - S(X|\{\pi_i^Y\}). \end{aligned} \tag{1.43}$$

where $S(X|\{\pi_i^Y\}) = \sum_i p_i S(\rho_{X|\pi_i^Y})$ with $\rho_{X|\pi_i^Y} = \frac{\pi_i^Y \rho_{XY} \pi_i^Y}{p_i}$ (where $p_i = \mathrm{Tr}(\pi_i^Y \rho_{XY})$ is the probability of obtaining the i^{th} outcome). Here, $S(X|\{\pi_i^Y\})$ is the Von Neumann entropy of the qubit X, when the one dimensional projective measurement is done on Y. The quantum discord function is then defined as,

$$D(X:Y) = I^q(X:Y) - J^q(X:Y) = S(Y) - S(X,Y) + S(X|\{\pi_i^Y\}) \tag{1.44}$$

This is to be minimized over the set of all one dimensional projectors $\{\pi_i^Y\}$. We shall call $D(X:Y)$ as discord function and its minimum value as the quantum discord i.e., $\delta_Y = \max_{\{\pi_i^Y\}} D(X:Y)$. The states for which the quantum discord vanishes are of the form

$$\rho_{cq} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i, \tag{1.45}$$

where $\{|i\rangle\}$ form a orthogonal basis. These types of states are called *classical-quantum* states. Therefore, the separability criteria and vanishing discord criteria are different, e.g. for Werner state, discord is non-zero for $p > 0$ while the state is separable for $p \leq \frac{1}{3}$. This means separable states may contain quantum correlations too.

Notice that the quantity, the quantum mutual information with conditional entropy, $J(X : Y)$ is related with the measure of classical correlations [100], i.e.,

$$C_{\ell_Y}(\rho_{XY}) = \max_{\{\pi_i^Y\}} J^q(X : Y). \tag{1.46}$$

Hence, the quantum discord is equivalent to the difference of 'total correlations−classical correlations' i.e.,

$$\delta_Y = I^q(X : Y) - C_{\ell_Y}(\rho_{XY}). \tag{1.47}$$

So, quantum discord can be a good quantifier of quantum correlations in a quantum state.

It is evident that the discord function is not symmetric in X and Y. In the above definition, we are making a measurement on the system Y. Let us call it Y-discord. Similarly, we can define X-discord, when the measurement is made on the system X, $D(Y : X) = S(X) - S(X, Y) + S(Y|\{\pi_i^X\})$. Here $S(Y|\{\pi_i^X\}$ is defined in the same way as $S(X|\{\pi_i^Y\})$. For a bipartite state, X-discord and Y-discord may have different values. They will have identical values when the state is symmetric in X and Y. But, they are always non-negative. For a pure bipartite state, both discords reduce to the Von Neumann entropy. Note that calculating discord is a NP hard problem [117].

This discovery got much attentions immediately for its simplicity and tremendous implications. It opens up once again the discussions of total correlations and splitting it into quantum and classical one. Many tried to generalize these results for multiparticle states [118–122], Gaussian states [123,124], higher dimensional states [125,126] but there are still some issues. Geometric approach was also employed to characterize the quantum correlations in the state – this quantity is popularly known as *geometric discord* [127].

1.3.2.5 Correlations and work extraction

Oppenheim *et al.* [128] argued that the correlations captured by quantum mutual information in a bipartite quantum state can be split into two parts – the classical correlations and quantum correlations. In case of a pure bipartite state $|\psi\rangle_{AB}$, these two give equal contributions to mutual information i.e., both are equal to the entropy of a subsystem – $S(\rho_{A(B)})$ bits. Hence, as a conjecture, they put forward that mutual information is a measure of total correlations. These arguments were supported by introducing local and nonlocal information which are complimentary in nature. Locally accessible information is called local information [129–131] and the information which is required to perform tasks which have no classical counterpart is called nonlocal information [132]. They defined these quantities in terms of work extractions from a bipartite state. Given a bipartite state one can extract total work, W_t from the state using a subclass of LOCC operations, CLOCC[19]. We know total extractable work from the state ρ_{AB} is

$$W_t = \log_2 d_{AB} - S(\rho_{AB}), \tag{1.48}$$

[19] We know that purer the state is the more work can be extracted from it locally. Hence simply introduction of pure ancilla may increase the extraction of work. Here, we will consider those LOCC operations which will not increase the number of particles of the systems. These operations, they termed as close LOCC (CLOCC) operations, e.g. local unitary operations, sending particle through dephasing channel.

where d_{AB} is the total Hilbert space dimension of the state. According to them the total information content of the system is $I_t = W_t$. Now if one extracts work from the subsystems $\{\rho_A = \mathrm{Tr}_B[\rho_{AB}], \rho_B\}$, then the total extractable work is

$$W_l = \log_2 d'_A - S(\rho'_A) + \log_2 d'_B - S(\rho'_B), \tag{1.49}$$

where the 'prime' signifies that in the process of localizing the information, CLOCC operations were applied on ρ_{AB} which transformed to ρ'_{AB}. This work is equivalent to the localizable information (I_l) i.e., $I_l = W_l$. The state will be classically correlated if $I_t = I_l$. If we subtract the localizable information from the total, the remaining one is the nonlocal information which in turn is nothing but the *work deficit* [128, 132] i.e.,

$$\triangle_q = I_t - I_l. \tag{1.50}$$

The quantity I_t can be identified with the mutual information I. Although this analysis is true for pure entangled states, for mixed bipartite states the quantity $\triangle_q$ exceeds the entanglement and is recognised as a measure of quantum correlations beyond entanglement [133]. They observed that local and nonlocal information taken together is equal to the mutual information. We quote them here, *"It is natural to suppose that the best defined information is the one defined operationally"*. Operationally, local and nonlocal information are useful in two complimentary processes – 1) to perform physical work, and 2) to perform useful logical quantum work (teleportation) respectively. Later, Horodecki *et al.* [128, 129] further investigated the above formalism and established that the work deficit is equivalent to entanglement for pure state but in general it identifies with the QCsbE [133]. They also defined two new concepts, *quantum deficit* and *classical deficit* to capture the quantum correlations and the classical correlations respectively. The quantum deficit is the difference between total (I_t) and localizable information (I_l) and the classical deficit is defined as the difference between the localizable information (I_l) and the local information of the initial state (I_{LO}), where $I_{LO} = \log_2 d_{AB} - S(\rho_A) - S(\rho_B)$, i.e.,

$$\triangle_{cl} = I_l - I_{LO}. \tag{1.51}$$

Hence, intuitively, $I = \triangle_q + \triangle_{cl}$. Then, they asked the question: Can quantum correlations be more than classical one? As two deficits sum up to the total correlations (quantum mutual information), can we distribute them arbitrarily? For pure state it is not the case, i.e., they both are equal to half of the mutual information. This implies that in the case of a pure state quantum correlations never exceeds classical one. But for mixed states, we have no conclusive proof why the quantum correlations will be less than the classical correlations. Note that according to [134] classical deficit can increase under local operations and hence does not qualify as a measure of classical correlations. Moreover, this analysis can easily be extended for multiparticle systems.

1.3.2.6 Erasing correlations – new perspective

In an operational approach, Groisman *et al.* [107] showed that the two-qubit maximally entangled states $|\psi^+\rangle$ contains 1 bit of classical and 1 bit of quantum correlations. They defined these correlations in terms of work (or, noise) required to erase the correlations in the state. For total correlations, they considered total work as a quantifier i.e., we have to erase the total correlations completely. In order to erase correlations, they considered a special type of operations – introduction of noise in the state by local operations. Let Alice and Bob share an entangled state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Now suppose Alice wants to erase the entanglement of the state. She will be able to do this by introducing 1 bit of randomness or noise in the state: she can apply one of the two local unitaries $\mathbb{I}$ and σ_3 on her qubit with equal probability. With this operations the state becomes

$$\rho_{er1} = \frac{1}{2}(|\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|), \tag{1.52}$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. This state has no entanglement as it is equivalent to the classically correlated state

$$\rho_{cl} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|). \tag{1.53}$$

Despite that its entanglement is gone, the state now contains 1 bit of classical correlations. To erase this correlations, Alice will apply random bit flip (i.e., applying either $\mathbb{I}$ or σ_1 at random). The operations will bring the state to

$$\rho_{er2} = \frac{1}{4}\mathbb{I}_A \otimes \mathbb{I}_B. \tag{1.54}$$

This state contains no correlations at all. Hence, they concluded that the maximally entangled state contains total 2 bits of correlations – 1 bit classical and 1 bit quantum. Later, they theorised that the total cost of erasing is equivalent to the quantum mutual information $I^q(A:B)$ of the state i.e., we need $I^q(A:B)$ bits of noise to erase total amount of correlations in the state. For example, notice that the quantum mutual information of the state $|\psi^+\rangle$ is 2 which is exactly equal to the amount of noise/randomness needed to erase the total correlations. Therefore, mutual information can be a good quantifier of total correlations for quantum states. They further introduced multiparticle quantum mutual information $I^q(A_1 : A_2 : ... : A_n) = \sum_i S(A_i) - S(A1, A_2, ..., A_n)$ which may capture total correlations in a multiparticle state.

1.3.2.7 Do all entanglement measures qualify as consistent measure of quantum correlations?

Li and Luo [103] examined whether the existing entanglement measures like the distillable entanglement [47–49], the entanglement of formation [50], the entanglement cost[20] [135], the squashed entanglement, and the relative entropy of entanglement [114, 115] qualify as a good measure of quantum correlations when we consider quantum mutual information as a measure of total correlations. They noticed that any pure two-qubit entangled state contains $I^q(A:B)$ bits information, e.g., as already we know $I^q(|\psi^+\rangle\langle\psi^+|) = 2$ bits. Although it is not yet clear how to separate quantum correla-

[20] It is a dual measure to the entanglement of distillation. Entanglement cost is an entanglement measure that quantifies how many maximally entangled states are required to prepare a copy of a state using only LOCC operations.

tions from classical one but we can gain some insight about them using the following argument. In case of maximally entangled state we know that it has 1 bit of entanglement in it. In this context many authors thought that it will be a paradox if we think that the maximally entangled state has only quantum correlations [13, 15, 107, 129]. Below we will list their arguments –

1. In superdense coding, Alice can communicate 2 bits of classical information to Bob by just applying local operations on her qubit if they share a maximally entangled state.
2. In a maximally entangled state, the rest one bit of information (apart from its entanglement) is related to the negative conditional entropy [13, 15]. According to them this negative information is giving the power to Alice to perform jobs which are impossible classically.
3. Groisman *et al.* showed that for a maximally entangled state, 2 bits of mutual information is equal to the 1 bit of quantum correlations + 1 bit of classical correlations [107].
4. Horodecki *et al* [129] gave same argument as Groisman *et al* but said that these two correlations are complementary to each other making them harder to utilize at the same time i.e., 2 is not equal to $1 + 1$ rather it is equal to either 1 or 1.

For pure bipartite states, the above arguments can be generalized easily i.e., it will contain $E(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B[|\psi\rangle\langle\psi|])$ bits of quantum correlations and $I - E(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B[|\psi\rangle\langle\psi|])$ bits of classical correlations – i.e., it contains equal amount of both the correlations. Then what about mixed states? Can the relation

$$I^q(\rho) = C_\ell(\rho) + Q(\rho), \tag{1.55}$$

hold? (Here $C_\ell(\rho)$ is the classical correlations and $Q(\rho)$ quantum correlations in the state ρ.) Since, the classical mixedness increases the classical correlations, and the mixing generally decreases the quantum correlations, it is plausible to assume that $C_\ell(\rho) \geq Q(\rho)$ [103]. This argument is also supported by the authors [103]. They argued that a quantum state may contain classical correlations without any quantum one but not vice versa. Then from Eq.(1.55), we can intuitively postulate that

$$Q(\rho) \leq \frac{1}{2} I^q(\rho). \tag{1.56}$$

According to them quantum mutual information is a well-established measure of total correlations in a quantum state. They are assuming it from the earlier literature that we have discussed in this section. On top of that they assume,

> “For any quantum state the classical correlation should not be less than the quantum one".

The justifications are

1. For pure states, the amount of classical correlations is equal to quantum one.
2. For mixed states, it is intuitive that with the increase of mixedness in a state its classicality will increase and its quantumness will reduce. Importantly if a pure

state which has equal quantum and classical correlations, becomes more and more mixed, its classicality will dominate over quantum one.

3. According to them, the separable states contain only classical correlations.

The above assumption was first considered by Henderson and Vedral [100] and later conjectured by Groisman *et al* [107]. Then, they considered a bunch of quantum correlations (Here, they took it as same as entanglement.) measures to see whether they satisfy the above assumption. The distillable entanglement and the squashed entanglement satisfy the Eq.(1.56) and the relative entropy of entanglement and entanglement cost satisfy the relation $Q(\rho) \leq I^q(\rho)$, hence are the good measures of quantum correlations. The distillable entanglement and the squashed entanglement are the better measures as they satisfy the assumption also. Although in some cases entanglement cost satisfy the Eq.(1.56) but in general it is not true. But the entanglement of formation does not satisfy any of the Eq.(1.56) or $Q(\rho) \leq I^q(\rho)$, it may exceed total correlations also. Hence it is not a good measure of quantum correlations although it has a clear physical meaning. It was conjectured that the entanglement of formation is additive on tensor product [136, 137] and hence will coincide with entanglement cost [138]. However, if it is true then this may lead to contradictions. Later, it was shown that the entanglement of formation is not additive [60]. However, it has been argued that the regularized version of entanglement of formation is equal to the entanglement cost [138].

1.3.2.8 Does mutual information capture all possible correlations?

In the work [104], the author reasoned that the quantum generalization of mutual information may not capture all types of correlations in the quantum states. Mainly in the context of classical states, he illustrated that the mutual information was not able to capture all possible classical correlations. Then he further suggested that a known information theoretic quantity might capture the total correlations in the quantum states. He observed that the assumptions

1. Total correlations, $\mathcal{T}(\rho_{AB}) = I^q(\rho_{AB})$.
2. $\mathcal{T}(\rho_{AB}) = C_\ell(\rho_{AB}) + Q(\rho_{AB})$.
3. $C_\ell(\rho_{AB}) \geq Q(\rho_{AB})$.

are contradicted by some results in the literature. For example,

- The statements (1) and (2) cannot be true simultaneously, if we measure quantum correlations by relative entropy of entanglement and classical one by *a measure based on the maximum information that could be extracted from one system by making a POVM measurement on the other one* [100].
- If one assumes that entanglement of formation is a measure of quantum correlation then for some states $Q(\rho_{AB}) \geq \frac{1}{2} I^q(\rho_{AB})$. Therefore, if statements (2) and (3) hold then $\mathcal{T}(\rho_{AB}) \geq I^q(\rho_{AB})$ [103]. For certain states, the quantum correlations measured by entanglement of formation may even exceed the quantum mutual information, i.e., $Q(\rho_{AB}) \geq I^q(\rho_{AB})$ [103].
- Moreover it is very evident that if one considers entanglement of formation as a measure of quantum correlations and if statements (1) and (2) hold then $C_\ell(\rho_{AB}) \leq Q(\rho_{AB})$ [103, 139].

Author then argued that the mutual information might not be a measure of total correlations as it could not accommodate the physically motivated and well established measures of quantum correlations (quantum entanglement). Moreover, if the statement (1) is true indeed then for classically correlated quantum states

$$\mathcal{T}(\rho_{AB}) = C_\ell(\rho_{AB}) = I^q(\rho_{AB}). \tag{1.57}$$

But according to him, for some classically correlated states this is not the case, mutual information alone might not capture all correlations in the states. For example, consider the state

$$\rho_{cl}^{AB} = p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11|, \tag{1.58}$$

where $0 \le p \le 1$. Suppose Alice measures observable $\mathcal{O}_A = a_1|0\rangle\langle 0| + a_2|1\rangle\langle 1|$ and gets the outcome a_i, then the post measurement the system state will be $\rho_i^{AB} = \frac{1}{p_i^A}(|i\rangle\langle i| \otimes \mathbb{I})\rho_{cl}^{AB}(|i\rangle\langle i| \otimes I)$, where $p_i^A = \mathrm{Tr}[(|i\rangle\langle i|)\rho^A]$. Next, Bob will measure $\mathcal{O}_B = b_1|0\rangle\langle 0| + b_2|1\rangle\langle 1|$, then the conditional probability that the Bob's outcome will be b_j given that Alice's is a_i, is $p_{j|i}^{B|A} = \mathrm{Tr}[(\mathbb{I} \otimes |j\rangle\langle j|)\rho_i^{AB}]$ and the joint probability p_{ij}^{AB} of measurement outcome a_i and b_j is given by $p_{ij}^{AB} = \mathrm{Tr}[(|i\rangle\langle i| \otimes |j\rangle\langle j|)\rho_{cl}^{AB}]$. It can be shown that

$$p^{B|A} = [p_{j|i}^{B|A}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad p^{AB} = p_{ij}^{AB} = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}, \tag{1.59}$$

with $p^A = p^B = (p, 1-p)$. One can see that random variables A and B (with alphabet $\mathcal{A} = \{a_1, a_2\}$ and $\mathcal{B} = \{b_1, b_2\}$ respectively) are not independent and thus there exists classical correlations between them. Now, mutual information of that is just equivalent to the mutual information between the random variables A and B, which is

$$I(A:B) = \sum_{i,j} p_{ij}^{AB} \log_2 \left(\frac{p_{ij}^{AB}}{p_i^A p_j^B} \right) = H_2(p). \tag{1.60}$$

Therefore, the classical correlation of the state can be arbitrarily small as measured by quantum mutual information as $p \in \{0, 1\}$ though the state is perfectly correlated in the information theoretic sense (see Eq.(1.59)). If the measurement outcome of Alice is a_i then Bob's will be b_i i.e., B is one-to-one function of A – which reflects the correlations between the two random variables. Therefore, mutual information may not be a good quantifier for total quantum correlations. Note that mutual information $I(A:B)$ does not capture all the correlations between random variables A and B, except for $p = \frac{1}{2}$. We know that Alice has *a priori* uncertainty about the measurement outcome of $\mathcal{O}_B$ which is given by $H(B) = -\sum_j p_j^B \log_2 p_j^B$. If the measurement outcome of $\mathcal{O}_A$ is a_i, then the uncertainty about the measurement outcome of $\mathcal{O}_B$ will reduce to $H(B|A = a_i) = -\sum_j p_{j|i}^{B|A} \log_2 p_{j|i}^{B|A}$. Therefore, the information she gain about the measurement outcome of $\mathcal{O}_B$ due to the measurement outcome of $\mathcal{O}_A$ is $H(B) - H(B|A = a_i)$. Hence, the average gain of information about the measurement outcome of $\mathcal{O}_B$ due to the knowledge of the measurement outcome of $\mathcal{O}_A$ is $\sum_i p_i^A(H(B) - H(B|A = a_i)) = H(B) - H(B|A) = I(A:B)$, which can be arbitrarily small. Thus it is clear that classical mutual information does not capture classical correlations between two random variables, rather it captures mutual dependency between them. Then what is the information theoretic measure of correlations between two random variables? He took the known quantity $\tilde{I}(A:B) = \frac{I(A:B)}{H(A)}$ [5] as a measure of correlations between two random variables which satisfies the required properties[21] to be one. For example, the correlations, $\tilde{I}(A:B) = 1$ (notice that $p^A = p^B$ and $I(A:B) = H(A)$), which is independent of p and clearly describes the perfect correlations between A and B. If the probability mass functions are not equal for both the random variables, then the above definition can be extended [5] to

$$\tilde{I}(A:B) = \max\left[\frac{I(A:B)}{H(A)}, \frac{I(A:B)}{H(B)}\right] = \frac{I(A:B)}{\min[H(A), H(B)]}. \tag{1.61}$$

It was pointed out that this quantity is symmetric and lies between 0 and 1 – 0 value depicts no correlations while 1 for perfect correlations between A and B. Let us consider

$$|\psi\rangle = \sqrt{\alpha}|00\rangle + \sqrt{1-\alpha}e^{i\phi}|11\rangle, \tag{1.62}$$

where $0 \leq \alpha \leq 1$ and ϕ is the phase. If we calculate the probability mass functions, joint probabilities and conditional probabilities for the same measurement observables ($\mathcal{O}_A$ and $\mathcal{O}_B$) performed by Alice and Bob, then from Eq.(1.61) one can show that the state $|\psi\rangle$ has perfect correlations $\forall$ (α and ϕ). One can generalize, by analogy, Eq.(1.61) to quantum regime by just replacing Shannon entropy with Von Neumann entropy, i.e., the total correlation measure for quantum states is

$$\mathcal{T}(\rho_{AB}) = \frac{I^q(\rho_{AB})}{\min[S(\rho_A), S(\rho_B)]}. \tag{1.63}$$

The Eq.(1.63) tells us that the total correlations of the state $|\psi\rangle$ is 2 $\forall$ (α and ϕ), which may not contradict the earlier results by Goisman [107], because to erase the correlations of two maximally entangled states may require different amount of randomness. For instance, to erase the correlations in the state ρ_{cl}^{AB} one needs 1 bit of randomness if $p = \frac{1}{2}$, and 2 bits if $p \neq \frac{1}{2}$.

1.3.2.9 Correlations and uncertainties

In [140], Luo discussed the possible decomposition of total uncertainty in a general state in terms of pure quantum and classical uncertainties. He showed that indeed the total uncertainty measured by variance can be split into two such distinct parts, i.e., *total uncertainty = classical part + quantum part*. Let us consider the following decomposition of state ρ and observable $\mathcal{O}_A$

$$\mathrm{Var}(\mathcal{O}_A)_\rho = C_u(\rho, \mathcal{O}_A) + Q_u(\rho, \mathcal{O}_A), \tag{1.64}$$

where $C_u(\rho, \mathcal{O}_A)$ is the classical part of uncertainty and $Q_u(\rho, \mathcal{O}_A)$ is the quantum part. These uncertainties should have the following properties

[21] Properties of $\tilde{I}(A:B)$: – (i) $\tilde{I}(A:B) = \tilde{I}(B:A)$, (ii) $0 \leq \tilde{I}(A:B) \leq 1$, (iii) $\tilde{I}(A:B) = 0$ if A and B are totally independent, and (iv) $\tilde{I}(A:B) = 1$ if A and B are perfectly correlated.

- $Q_u(\rho, \mathcal{O}_A)$ should not increase under classical mixing of states i.e., for the mixed state $\rho_m = \sum_i p_i \rho_i$, where ρ_i are pure states and $\sum_i p_i = 1$, $Q_u(\sum_i p_i \rho, \mathcal{O}_A) \leq \sum_i p_i Q_u(\rho, \mathcal{O}_A)$ whereas $C_u(\sum_i p_i \rho, \mathcal{O}_A) \geq \sum_i p_i C_u(\rho, \mathcal{O}_A)$.
- If ρ is pure then $Q_u(\rho, \mathcal{O}_A) = \mathrm{Var}(\mathcal{O}_A)_\rho$ and $C_u(\rho, \mathcal{O}_A) = 0$ because there is no classical mixing.
- If $[\rho, \mathcal{O}_A] = 0$, then $Q_u(\rho, \mathcal{O}_A) = 0$ and $C_u(\rho, \mathcal{O}_A) = \mathrm{Var}(\mathcal{O}_A)_\rho$. In this case we can work in the eigenbasis of ρ and $\mathcal{O}_A$. In this basis both the state and the observable are classical.

For quantum part of uncertainty, he chose the *square root quantum Fisher information* [141] and proposed a new quantity which will serve as the measure of classical uncertainty i.e.,

$$Q_u(\rho, \mathcal{O}_A) = -\frac{1}{2}\,\mathrm{Tr}[\sqrt{\rho}, \mathcal{O}_A]^2 \ \text{ and } \ C_u(\rho, \mathcal{O}_A) = \mathrm{Tr}[\sqrt{\rho}\mathcal{O}_{A_0}\sqrt{\rho}\mathcal{O}_{A_0}], \tag{1.65}$$

where $\mathcal{O}_{A_0} = \mathcal{O}_A - \mathrm{Tr}[\rho\mathcal{O}_A]$. It is clear that all these quantities can be measured experimentally [142, 143]. These two quantities satisfy the following properties also –

1. Under unitary operations U, $Q_u(U\rho U^\dagger, U\mathcal{O}_A U^\dagger) = Q_u(\rho, \mathcal{O}_A)$.
2. $Q_u(\rho_1 \otimes \rho_2, \mathcal{O}_{A_1} \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{O}_{A_2}) = Q_u(\rho_1, \mathcal{O}_{A_1}) + Q_u(\rho_2, \mathcal{O}_{A_2})$.
3. $Q_u(\rho, \mathcal{O}_{A_1} \otimes \mathbb{I}) \geq Q_u(\rho_1, \mathcal{O}_{A_1})$, where $\rho_1 = \mathrm{Tr}_2[\rho]$.

All these properties also hold for $C_u(\rho, \mathcal{O}_A)$ with the inequality reversed. For illustration let us consider following two states

$$\rho_p = |\psi\rangle\langle\psi| \ \text{ and } \ \rho_m = \frac{1}{N}\sum_{i_1, i_2} |i_1 i_2\rangle\langle i_1 i_2|, \tag{1.66}$$

where $|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{i_1,i_2} |i_1 i_2\rangle$ and $\{|i_1\rangle\}$ and $\{|i_1\rangle\}$ form orthogonal bases. Let us consider two observables $\mathcal{O}_1 = \sum_{i_1} \alpha_i |i_1\rangle\langle i_1|$ and $\mathcal{O}_2 = \sum_{i_2} \beta_i |i_2\rangle\langle i_2|$. Then the uncertainties of observable $\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2$ are

$$Q_u(\rho_p, \mathcal{O}) = \mathrm{Var}(\mathcal{O})_{\rho_m} = \frac{1}{N}\left[\sum_i \alpha_i^2\beta_i^2 - \left(\sum_i \alpha_i\beta_i\right)^2\right] \ \& \ Cu(\rho_p, \mathcal{O}) = 0,$$

$$Q_u(\rho_m, \mathcal{O}) = 0 \ \& \ Cu(\rho_p, \mathcal{O}) = \mathrm{Var}(\mathcal{O})_{\rho_m} = \frac{1}{N}\left[\sum_i \alpha_i^2\beta_i^2 - \left(\sum_i \alpha_i\beta_i\right)^2\right].$$

The formalism is giving more information about the states which the conventional variance cannot. Taking together quantum and classical uncertainties as two dimensional vector quantity, $\{Q_u(\rho, \mathcal{O}_A), C_u(\rho, \mathcal{O}_A)\}$ one can get more information about a quantum state.

From this perspective, he defined two new correlations measures, measure of classical and quantum correlations, directly motivated from the idea of classical and quantum uncertainties. The classical and quantum correlation measures are

$$\begin{aligned} Q(\mathcal{O}_A, \mathcal{O}_B)_\rho &= \mathrm{Tr}[\rho \mathcal{O}_{A_0}^\dagger \mathcal{O}_{B_0}] - \mathrm{Tr}[\sqrt{\rho}\mathcal{O}_{A_0}^\dagger \sqrt{\rho}\mathcal{O}_{A_0}], \\ \text{and } C_\ell(\mathcal{O}_A, \mathcal{O}_B)_\rho &= \mathrm{Tr}[\sqrt{\rho}\mathcal{O}_{A_0}^\dagger \sqrt{\rho}\mathcal{O}_{B_0}]. \end{aligned} \tag{1.67}$$

Hence, $Q(\mathcal{O}_A, \mathcal{O}_B)_\rho + C_\ell(\mathcal{O}_A, \mathcal{O}_B)_\rho = \mathrm{Cov}(\mathcal{O}_A, \mathcal{O}_B)_\rho$, where $\mathrm{Cov}(\mathcal{O}_A, \mathcal{O}_B)_\rho = \mathrm{Tr}[\rho \mathcal{O}_{A_0}^\dagger \mathcal{O}_{B_0}]$. Here conventional covariance seems to capture total correlations in the state ρ. How good are these measures? Let us consider the same states, ρ_p and ρ_m and observables, $\mathcal{O}_A = \mathcal{O}_1 \otimes \mathbb{I}$ and $\mathcal{O}_B = \mathbb{I} \otimes \mathcal{O}_2$. Then

$$Q(\mathcal{O}_A, \mathcal{O}_B)_{\rho_p} = \mathrm{Cov}(\mathcal{O}_A, \mathcal{O}_B)_{\rho_p} = \frac{1}{N}\sum_i (\alpha_i - \bar{\alpha})(\beta_i - \bar{\beta}) \ \ \& \ \ C_\ell(\mathcal{O}_A, \mathcal{O}_B)_{\rho_p} = 0,$$

$$C_\ell(\mathcal{O}_A, \mathcal{O}_B)_{\rho_m} = \mathrm{Cov}(\mathcal{O}_A, \mathcal{O}_B)_{\rho_m} = \frac{1}{N}\sum_i (\alpha_i - \bar{\alpha})(\beta_i - \bar{\beta}) \ \ \& \ \ Q(\mathcal{O}_A, \mathcal{O}_B)_{\rho_m} = 0,$$

where $\bar{\alpha} = \frac{1}{N}\sum_i \alpha_i$ and $\bar{\beta} = \frac{1}{N}\sum_i \beta_i$. Hence, classical correlations and quantum correlations distinguish the states ρ_p and ρ_m, while the conventional covariance fails to do so. As ρ_m is a classical state it only has classical correlations whereas the pure quantum state ρ_m has only quantum correlations. Note that to capture correlations the observables should be of the form $\mathcal{O}_A = \mathcal{O}_1 \otimes \mathbb{I}$ and $\mathcal{O}_B = \mathbb{I} \otimes \mathcal{O}_2$ i.e., locally applied to the subsystems and have less than unit norm. To capture entanglement like correlations author introduced another quantity $\tilde{E}(\rho) = \sup_{\mathcal{O}_A, \mathcal{O}_B} Q(\mathcal{O}_A, \mathcal{O}_B)_\rho$, where supremum is taken over all possible $\mathcal{O}_A = \mathcal{O}_1 \otimes \mathbb{I}$ and $\mathcal{O}_B = \mathbb{I} \otimes \mathcal{O}_2$. Therefore, $Q(\mathcal{O}_A, \mathcal{O}_B)_\rho$ may capture correlations beyond entanglement [4]. This idea then was perused by the author and his collaborator in their next work, where they have established that these type of measures may capture quantum correlations in a quantum state [144].

1.3.2.10 Measurement induced disturbance and correlations

Entanglement and *quantumness* are two distinct fundamental features of the quantum theory. While the entanglement arises from *superposition*, the quantumness is due to the *noncommutativity* of operators representing the states, the observables and the measurements. As these two ideas are closely intertwined, it is sometimes difficult to separate them out. While for pure states both the ideas can be identical, it become quite complex when mixed states are considered.

Luo in [145], tried to address this issue through the concept of measurement induced disturbance (MID). He employed a simple idea: *'In classical theory, measurements can reveal properties of the systems without disturbing it. But for quantum systems, measurements necessarily disturb it. If a state can be measured locally by some projective measurements without disturbing it, then the state is a classical state'*. Let ρ_{AB} be a bipartite state and $\{\pi_i^A\}$ and $\{\pi_j^B\}$ are complete projective measurements for the particles A and B respectively. Then after measurements, the state ρ_{AB} changes to

$$\Pi(\rho_{AB}) = \sum_{ij} \pi_i^A \otimes \pi_j^B \rho_{AB} \pi_i^A \otimes \pi_j^B. \tag{1.68}$$

Therefore, if $\Pi(\rho_{AB}) = \rho_{AB}$, then ρ_{AB} is a classical state. It can be proved that if ρ_{AB} is classical then its reduced density matrices are $\rho_A = \sum_i p_i \pi_i^A$ and $\rho_B = \sum_i p_i \pi_i^B$. Which implies that the classical state can always be expressed as

$$\rho_{AB}^C = \sum_{ij} p_{ij} \pi_i^A \otimes \pi_j^B. \tag{1.69}$$

It means $[\rho_{AB}^C, \pi_i^A \otimes \pi_j^B] = 0$. Moreover, $p_i = \sum_j p_{ij}$ and $p_j = \sum_i p_{ij}$. From these findings one can easily conclude that a classical-quantum state can be re-expressed as $\rho_{AB}^{CQ} = \sum_i p_i \pi_i^A \otimes \rho_B^i$, where ρ_B^i are local non-orthogonal states for B.

Then he asked, *which measurement will induce the closest classical state to the original quantum state while keeping the reduced states invariant?* This lead him to define a quantity which can capture the quantum correlations of the original state. One such quantity is

$$Q_D(\rho_{AB}) = \inf_{\Pi} D(\rho_{AB}||\Pi(\rho_{AB})), \tag{1.70}$$

where $D(\cdot||\cdot)$ may be any suitable distance on quantum states.

If we accept quantum mutual information, $I^q(A:B) = S(A) + S(B) - S(A,B)$ as a measure of total correlations in a quantum state, then one can define another quantity

$$Q(\rho_{AB}) = I^q(A:B) - I^q(\Pi(\rho_{AB})), \tag{1.71}$$

where $I^q(\Pi(\rho_{AB}))$ captures the total correlations in the classical state $\Pi(\rho_{AB})$ and thus $I^q(\Pi(\rho_{AB}))$ quantifies the total classical correlations in the state ρ_{AB}. Therefore, naturally $Q(\rho_{AB})$ captures the quantum correlations in ρ_{AB}. Note that $Q(\rho_{AB})$ is symmetric in parties. For pure bipartite states it reduces to the Von Neumann entropy of the subsystems, and is invariant under local unitary transformations i.e., $Q(U_1 \otimes U_2 \rho U_1^\dagger \otimes U_2^\dagger) = Q(\rho)$.

It is shown that MID is non zero for classical states also [146]. In the [147], this problem was addressed by invoking optimization over general local measurements (POVMs) i.e., $Q'(\rho_{AB}) = I^q(A:B) - \max_{\{E_A \otimes E_B\}} I^q(\Pi(\rho_{AB}))$. Note that a mathematically similar quantity as given in Eq. (1.71), was also introduced in [148] under the name *deficit*, which is different from work deficit.

1.3.2.11 Relative entropic approach

Modi *et al.* [118] discussed the problem of separating total correlations into classical correlations, dissonance, and entanglement using the relative entropy of distance in case of multiparticle quantum systems. They introduced the new concept of dissonance, which together with entanglement roughly sum up to quantum correlations. In other words, ‘dissonance is the quantum correlations without entanglement’. The relative entropic measures for total correlations, classical correlations, quantum discord, entanglement, and dissonance for the multiparticle state ρ are respectively,

$$\mathcal{T}(\rho) = \inf_{\pi_\rho} S(\rho||\pi_\rho) \qquad \text{(Total correlations)}, \tag{1.72}$$

$$C_\ell(\rho) = \inf_{\pi_\rho} S(\chi_\rho||\pi_\rho) \qquad \text{(Classical correlations)}, \tag{1.73}$$

$$Q(\rho) = \inf_{\chi_\rho} S(\rho||\chi_\rho) \qquad \text{(Quantum correlations)}, \tag{1.74}$$

$$E(\rho) = \inf_{\sigma_\rho} S(\rho||\sigma_\rho) \qquad \text{(Quantum entanglement)}, \tag{1.75}$$

$$L(\rho) = \inf_{\chi_\rho} S(\sigma_\rho||\chi_\rho) \qquad \text{(Quantum dissonance)}, \tag{1.76}$$

where π_ρ, χ_ρ, and σ_ρ are the closest product, classical, and separable states respectively to the state ρ. Now to compute these correlation one has to compute these states from the initial states ρ. According to them the closest product state to ρ is $\otimes_i^n \rho_i$, where ρ_i is the i^{th} subsystem of the state. In this case the total correlations reduces to the mutual information $I^q(A_1 : A_2 : ... : A_n) = \sum_i S(A_i) - S(A_1, A_2, ..., A_n)$ [107]. And the closest classical state to ρ is given by

$$\chi_\rho = \sum_{\vec{k}} |\vec{k}\rangle\langle\vec{k}|\rho|\vec{k}\rangle\langle\vec{k}| \tag{1.77}$$

where $\{|\vec{k}\rangle\}$ forms the eigen basis for χ_ρ. Note that the definition of entanglement measure is just the relative entropy of entanglement. As set of all separable states (within the Hilbert space of ρ) are convex and compact (see, e.g. [149]), one can find the closest separable state to ρ and compute $E(\rho)$ by employing convex optimization techniques (see, e.g. [150]). The measure $E(\rho)$ is called the relative entropy of entanglement [115]. Also the definition of quantum correlations here is somewhat similar to the information theoretic discord [116], only difference is here we consider the projective measurement only. From Fig.(1.8), it is easy to visualize that the correlations may give following additive relations,

$$\mathcal{T}(\rho) = Q(\rho) + C_\ell(\chi_\rho) - \omega_\rho \ \text{ and } \ \mathcal{T}(\sigma) = L(\sigma) + C_\ell(\chi_\sigma) - \omega_\sigma, \tag{1.78}$$

where $\omega_\rho = S(\pi_\rho||\pi_{\chi_\rho})$. Now the important question is: How do quantum correlations, entanglement, dissonance, and classical correlations compare to the total correlations. To illustrate these they took following examples. For Bell-diagonal states, $\rho_b = \sum_i \lambda_i |\psi_i^b\rangle\langle\psi_i^b|$, where λ_i are ordered in non-increasing size and $|\psi_i^b\rangle$ are four Bell states, the correlations are subadditive i.e., $\mathcal{T}(\rho_b) \geq E(\rho_b) + L + C_\ell$. It was earlier shown that the closest separable state to a bipartite entangle state is a classical state [151]. Hence, it leads to a additive relation for pure entangled states, $\mathcal{T}(\rho) = E(\rho) + C_\ell(\chi_\rho)$. Is it also true for multiparticle states? Consider the W-state, $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. For W-state, we get subadditivity relation, i.e., $\mathcal{T}(|W\rangle\langle W|) > E(|W\rangle\langle W|) + L + C_\ell$ and in this case $Q < E + L$. To shed further light into the matter, they considered cluster state, $|C_4\rangle = \mathcal{N}(|0+0+\rangle + |1+1+\rangle + |1-0-\rangle + |0-1-\rangle)$. For this state surprisingly the additive relation, $\mathcal{T} = E + C_\ell$ holds. Hence, on the basis of these illustration they conjectured that the subadditivity relation, $\mathcal{T} \geq E + L + C_\ell$ may hold for multiparticle states.

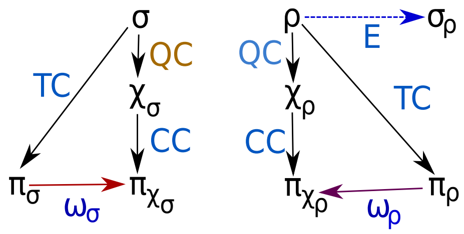


Figure 1.8: *Correlations in quantum states*: In the figure an arrow, $i \to j$, indicates j is the closest state to i as measured by the relative entropy $S(i||j)$. The states, ρ, σ, χ, and π represent the entangled states, separable states, classical states, and product states. The distances are entanglement (E), quantum correlations that go beyond entanglement (QC), classical correlations (CC), total correlations (TC), and dissonance (*To separate it out from QC of entangled states we coloured it differently*.). The quantities ω_ρ and ω_σ will be helpful to prove the additivity of the correlations.

1.3.2.12 Correlations and multiparticle states

Generalizing the concept of bipartite correlations to multiparticle case is complex and challenging. The structure of the multiparticle states is often not clear. Another challenge is to handle the mixed states. However, many authors have tried to generalize the concept to multiparticle states and address the unique phenomenon of *genuine multiparticle correlations* for multiparticle states.

In [152], authors proposed a set of criteria which every measure of genuine multiparticle correlations, $\mathcal{G}(\rho)$ should satisfy:

1. $\mathcal{G}(\rho) \geq 0 \; \forall$ quantum states ρ.
2. $\mathcal{G}(\rho) = 0$ for bi-product states i.e., $\rho_i \otimes \rho_{\bar{i}}$.
3. $\mathcal{G}(\rho)$ is invariant under local unitary operations i.e., $\mathcal{G}(\otimes_i U_i \rho \otimes_i U_i^\dagger) = \mathcal{G}(\rho)$.
4. $\mathcal{G}(\rho \otimes \sigma_L) = \mathcal{G}(\rho)$, where $\sigma_L = \otimes_i \sigma_L^i$ is a auxillary state.
5. $\mathcal{G}(\rho)$ is non-increasing under local operations i.e., $\mathcal{G}(\mathcal{E}_L(\rho)) = \mathcal{G}(\rho)$, where $\mathcal{E}_L = \otimes_i \mathcal{E}_L^i$ is local CPTP quantum operations.

They also proposed a measure in terms of cumulants of the n-particle states. A n-qubit state can be expressed in canonical form as

$$\rho_{12..n} = \frac{1}{2^n} \sum_{\mu_1,\mu_2,\ldots,\mu_n=0}^{3} a_{\mu_1,\mu_2,\ldots,\mu_n} \otimes_{\ell=1}^{n} \sigma_{\mu_\ell}, \tag{1.79}$$

where $a_{\mu_1,\mu_2,\ldots,\mu_n} = \mathrm{Tr}[\rho \otimes_{\ell=1}^{n} \sigma_{\mu_\ell}]$ with the condition $\mathrm{Tr}[\rho] = 1$. Here the quantity $T_{i_1,i_2,\ldots,i_n} = \mathrm{Tr}[\rho \otimes_{\ell=1}^{n} \sigma_{i_\ell}]$ are the correlations (they are usually called correlation coefficients) with $i_\ell \in \{1,2,3\}$. For such a n-party density matrix $\rho_{12..n}$, its cumulant is defined as

$$C_M(\rho_{12..n}) = \rho_{12..n} - \tilde{\rho}_{12..n}, \tag{1.80}$$

where $\tilde{\rho}_{12..n}$ is the pseudo-n party density matrix constructed in such a way that $\rho_{12..n} = \rho_{s_1}\rho_{s_2} + C_M(\rho_{12..n})$. Here, $\{s_1, s_2\}$ is a partition of correlations, T_n. For this construction, if the state is product state i.e., $\rho_{12..n} = \rho_{s_1}\rho_{s_2}$, then $C_M(\rho_{12..n}) = 0$. Then a measure of genuine multiparticle correlations is

$$\mathcal{G}_M(\rho_{12..n}) = \frac{1}{2}\,\mathrm{Tr}[|C_M(\rho_{12..n})|], \tag{1.81}$$

where $|X| = \sqrt{X^\dagger X}$. With some illustrations, authors claimed that this measure satisfies most of the above criteria. But it is not the case in general for general multiparticle mixed states.

Later, D. Kaszlikowski *et al.* [153, 154] showed that such a quantification is not sufficient as the state

$$\rho_W^{nc} = \frac{1}{2}(|W\rangle\langle W| + |\bar{W}\rangle\langle\bar{W}|), \tag{1.82}$$

where $|W\rangle = \frac{1}{\sqrt{n}}(|00...01\rangle + |00...10\rangle + \cdots\cdots\cdots + |10...00\rangle$ is a n-qubit W-state and $|\bar{W}\rangle = \sigma_1^{\otimes n}|W\rangle$, has vanishing correlation coefficients and cumulant for odd n yet contains genuine quantum correlations. Hence, they conclude that the cumulant or correlations [105] cannot quantify genuine quantum correlations in a state faithfully. They further suggested that the zero value of correlation coefficients (it is equivalent to covariance if we consider traceless operators) is a signature of lack of genuine classical correlations present in the state. Hence the existence of genuine multiparticle correlation without the genuine classical correlations in case of ρ_W^{nc}. They argued that if $\mathrm{Cov}(\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n) = \mathrm{Tr}[\rho\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes ... \otimes \mathcal{O}_n]$ (here $\mathcal{O}_i$ are traceless observables) is non-zero then state contains non-zero genuine classical correlations. Z. Walczak [155] proved that it is not true in general. According to the author the state ρ_W^{nc} has genuine classical correlations because the measurement outcomes of$\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n$ are not independent for all possible combination of them although the covariance is zero. Hence pointing out a wellknown fact that if the measurement outcomes of $\{\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n\}$ are independent then $\mathrm{Cov}(\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n) = 0$ but the converse is not true. Recently, it was observed that for such a state quantum entanglement has non-zero value although the state has vanishing correlation coefficients [156].

The above conclusions raise many questions. Bennett *et al.* [110] postulated three conditions which a good measure of genuine multiparticle correlations should satisfy:

- If a n-partite state does not contain genuine n-partite correlations then adding one more party in product state will also have no genuine $n+1$-partite quantum correlations.
- $\mathcal{G}(\rho) = 0 \Rightarrow \mathcal{G}(\Lambda_L(\rho)) = 0$, where $\Lambda_L = \otimes_i \Lambda_L^i$ are local *trace-nonincreasing* operations containing general local quantum operations and *unanimous postselection*.
- $\mathcal{G}(\rho) = 0 \Rightarrow \mathcal{G}(\rho_{split}) = 0$, ρ_{split} is the same state ρ but with the systems of some parties split into more parties, i.e., *splitting subsystems into more parties should not create genuine multiparticle correlations*.

It turns out that the cumulant and covariance don't satisfy all the above postulates and hence don't qualify as a measure of genuine multiparticle correlations. In this context, authors proposed a new measure of genuine multiparticle classical correlations using the concept of *classical deficit*. The idea is: *"If parties can extract more work with CLOCC (CC across any bipartite cut) than with CLOCC, and without sending CC across at least one cut, then the state has genuine multiparticle classical correlations"*. The calculations indicates that the state in Eq.(1.82) has non-zero genuine classical correlations for $n = 3$, confirming the claim of Walczak [155].

In another work, Giorgi *et al.* [157] have discussed a possible route to generalize the concept of information theoretic measures [116] to genuine correlation measures. They employed the relative entropic type measures to capture genuine tripartite total, classical and quantum correlations. They noticed that mutual information $\mathcal{T}(\rho_{ABC}) = S(\rho_{ABC}||\rho_A \otimes \rho_B \otimes \rho_C) = S(A) + S(B) + S(C) - S(A, B, C)$ is a well define measure to capture the total correlations in the state ρ_{ABC}. Then they define tripartite classical correlation measures implementing the result of [118],

$$C_\ell(\rho_{ABC}) = \max_{\pi_j^{\ell_1}, \pi_i^{\ell_3},} [S(\ell_1) + S(\ell_2) - S(\ell_1|\{\pi_i^{\ell_3}\}) - S(\ell_2|\{\pi_j^{\ell_1}, \pi_i^{\ell_3}\})], \tag{1.83}$$

where $S(\ell_1|\{\pi_i^{\ell_3}\})$ is defined in the subsection.(1.3.2.4) and $S(\ell_2|\{\pi_j^{\ell_1}, \pi_i^{\ell_3}\})$ is defined in the same way but the POVM will be applied locally on the subsystems ℓ_1 and ℓ_3. Notice that the quantity $C_\ell(\rho_{ABC})$ captures the distance between the closest classical state to the state ρ_{ABC} and its product state [118]. Hence the tripartite discord can be defined as $Q(\rho_{ABC}) = \mathcal{T}(\rho_{ABC}) - C_\ell(\rho_{ABC})$ which might capture quantum correlations in the state. Moreover to capture the genuine correlations in tripartite state they defined the following quantities

$$\begin{aligned}
\mathcal{T}^{(3)} &= \mathcal{T}(\rho_{ABC}) - \mathcal{T}^{(2)} \text{ (Genuine total correlations)}, && (1.84)\\
C_\ell^{(3)} &= C_\ell(\rho_{ABC}) - C_\ell^{(2)} \text{ (Genuine classical correlations)}, && (1.85)\\
Q^{(3)} &= Q(\rho_{ABC}) - Q^{(2)} \text{ (Genuine quantum correlations)}, && (1.86)
\end{aligned}$$

where $\mathcal{T}^{(2)}$ is the maximum among the bipartite mutual information i.e., $\mathcal{T}^{(2)} = \max[I^q(A : B), I^q(A : C), I^q(B : C)]$ and $C_\ell^{(2)} = \max[C_\ell(\rho_{AB}), C_\ell(\rho_{AC}), C_\ell(\rho_{BC})]$ and $Q^{(2)} = \min[Q(\rho_{AB}), Q(\rho_{AC}), Q(\rho_{BC})]$. Here $\mathcal{T}^{(3)}$ is defined as $\mathcal{T}^{(3)} = \inf[S(\rho_{ABC}||\rho_{ij} \otimes \rho_k)]$ i.e., it is the shortest distance to a state with no tripartite correlations. Hence, if $\mathcal{T}^{(3)} = 0$ means state is $\rho_{ij} \otimes \rho_k$. Generally for pure n-particle states the relation $C_\ell^{(n)} = Q^{(n)} = \frac{1}{2}\mathcal{T}^{(n)}$ holds. However note that all these correlation measures don't satisfy all the postulates proposed by Bennett *et al.* [110].

1.3.2.13 Quantum secrecy monotones and correlations in multiparticle states

In [158], Cerf *et al.* introduced the concept of quantum *secrecy monotones* in the context of quantum cryptography [159], to detect the shared secret correlations between the parties. These secrecy monotones are considered as the possible quantum generalizations of the multi-variate classical mutual information. For n-partite states, one of the secrecy monotone is

$$I_s^q(A_1:...:A_n) = \sum_{i=1}^{n} S(A_1,...,A_{i-1},A_{i+1},...,A_n) - (n-1)S(A_1,...,A_n). \quad (1.87)$$

In literature, the classical version is known as *dual total correlation*, or *binding information*. Note that for $n = 2$, $I_s^q(A_1 : A_2) = S(A_1) + S(A_2) - S(A_1, A_2)$. Important point is the secrecy monotone defined here is non-zero always. Later, the above quantity is proposed as the measure of total correlations in the multiparticle states and termed as *operational quantum mutual information* [160].

In [160], Kumar pointed out some important differences between the secrecy monotones and the relative entropy mutual information, $I^q(A_1 : A_2 : ... : A_n) = \sum_i S(A_i) - S(A_1, A_2, ..., A_n)$ and the quantum version of *interaction information*. The interaction information is a direct generalization of classical mutual information for multivariate classical systems using venn-diagram approach and is defined as for quantum case, $I_0^q(A_1 : A_2 : ... : A_n) = \sum_{p=1}^{n}(-1)^{p-1}\sum_{\{l_p\}}^{n} S(A_{l_1}, A_{l_2}, .., A_{l_p})$, where $\{l_p\}$ in the sum denotes $l_1 < l_2 < l_3... < l_p$ and l_i varies from 1 to n. Most generalizations of mutual information are positive except the quantum interaction information. In the Table.(1.2), the comparison between the interaction information I_0^q and the secrecy monotone I_s^q has been illustrated. While I_0^q can be negative, the I_s^q is always positive. It can also be shown that for pure multiparticle states, $I_s^q = I^q$ [159, 160]. For mixed state this is not the case in general. Hence, the secrecy monotone can be treated as a measure of total correlations [160, 161].

Kumar [160] defined the measures of quantum correlations (quantum discord) using the secrecy monotones, for the three-particle state ρ_{ABC},

$$\begin{aligned}
\delta_A(\rho_{ABC}) &= I_s^q(A:B:C) - \max_{\Phi_A} I_s^q(\Phi_A(\rho_{ABC})), \\
\delta_{AB}(\rho_{ABC}) &= I_s^q(A:B:C) - \max_{\Phi_{AB}} I_s^q(\Phi_{AB}(\rho_{ABC})), \\
\delta_{ABC}(\rho_{ABC}) &= I_s^q(A:B:C) - \max_{\Phi_{ABC}} I_s^q(\Phi_{ABC}(\rho_{ABC})), \quad (1.88)
\end{aligned}$$

where $\Phi_X(\rho_{ABC}) = \sum_i \Phi_X^i \rho_{ABC} \Phi_X^i$ ($X \in \{A, AB, ABC\}$) and $\Phi_A^i = \pi_i^A \otimes \mathbb{I} \otimes \mathbb{I}$, $\Phi_{AB}^i = \pi_i^A \otimes \pi_i^B \otimes \mathbb{I}$, and $\Phi_{ABC}^i = \pi_i^A \otimes \pi_i^B \otimes \pi_i^C$. Here, π_i are all possible one dimensional local projectors and $I_s^q(\Phi_X(\rho_{ABC}))$ depicts the secrecy monotone of the conditional state $\Phi_X(\rho_{ABC})$. Notice that the quantity $\delta_{ABC}(\rho_{ABC})$ is a symmetric quantity. Then he plotted the three correlation measures in Eq.(1.88) for the three qubit GHZ-state and W-state mixed with white noise ($\frac{1}{8}\mathbb{I}$) and observed that $\delta_A \leq \delta_{AB} \leq \delta_{ABC}$, i.e., quantum correlations increases when measurements are performed on larger number of subsystems. *This is in contradiction with the notion of classical correlations because measurement on more subsystems should yield larger value of classical correlations and hence less quantum correlations.* So, the quantity $I_s^q(\Phi_X(\rho_{ABC}))$ may not qualify as measure of classical correlations. However, the same situation also occur if one employs relative entropy based quantum mutual information. Then it is a serious question whether this is the usual feature of secrecy monotone and relative entropy based quantum mutual information.

State	I_0	I_s	State	I_c	I_s
$\lvert g_2\rangle$	2	2	$\lvert D_3^1\rangle$	0	2.75
$\lvert g_3\rangle$	0	3	$\lvert D_4^1\rangle$	0.49	3.25
$\lvert g_4\rangle$	2	4	$\lvert D_4^2\rangle$	0.49	4
$\lvert C_4\rangle$	-2	4	$\lvert As\rangle$	0	4.75

Table 1.2: I_0^q vs I_s^q: Values of interaction information I_0^q and secrecy monotone I_s^q for the states, GHZ state, $|g_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$, Dicke state, $|D_n^r\rangle = \mathcal{N}\sum_{\mathcal{P}}\mathcal{P}[|0\rangle^{\otimes n-r}|1\rangle^{\otimes r}]$, cluster state, $|C_4\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$, and totally asymmetric three qutrit state $|As\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle)$, where $\mathcal{P}$ denotes all possible permutations and $\mathcal{N}$ is the normalization. While for pure states, I_s^q is equivalent to the relative entropy quantum mutual informations, for mixed state they may differ. (This table is directly taken from A. Kumar's paper.)

1.4 Applications of quantum correlations

One may asks, "What is the importance of the correlated states?" The discovery of many quantum information processing (QIP) tasks demonstrate the power of correlated states [162]. Quantum teleportation is one of the first QIP protocols, discovered by Bennett *et al.* They showed that one can send an unknown quantum state to a remote party if they share an entangled state [8]. Remote state preparation [163, 164], superdense coding [9], quantum secret sharing [95], quantum key distribution [159], bit commitment [165], quantum state merging [17] etc. to name a few, are the QIP protocols which use entanglement as a resource for the success of the protocols. We will discuss some of them in the following subsections.

While there are many protocols in existence where there is clear evidence of importance of entangled states, for QCsbE, it is not the case. While analysing the role of quantum correlations in *deterministic quantum computation with single qubit* (DQC1) model, it was found that entanglement generated during the process is very negligible but the presence of QCsbE is significant [166]. This gave some physical importance of QCsbE. Later, researchers discovered that in protocols like remote state preparation [167–169], state merging [170] where QCsbE plays some significant role. Before discussing all these we will revisit some elements of quantum gates.

Quantum gates – Quantum gates are the building blocks of quantum circuits. These gates are mostly represented by unitary matrices. For example, for qubit systems, the X, Y and Z gates are nothing but three Pauli matrices i.e., σ_1, σ_2 and σ_3. The Hadamard gate (H) is represented by the matrix

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The Hadamard gate applied on $|0\rangle$ will transform it to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ i.e., it introduces rotation of π about the axis $\frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$. The operation of H gate is: $|i\rangle \rightarrow (-1)^i|i\rangle + |1-i\rangle$. The Hadamard gate is useful in creating superposition of single qubits. One important gate, the controlled NOT (CNOT) gate acts on two qubits. It is a global gate and can create entanglement. The operation of CNOT gate is: $|i\rangle|j\rangle \to |i\rangle|i+j \mod 2\rangle$. The another important gate is control-U ($C-U$) gate which maps $|0\rangle|j\rangle \to |0\rangle|j\rangle$ but $|1\rangle|j\rangle \to |1\rangle(U|j\rangle)$. Importantly, these gates are very useful in simulating multiqubit unitary operations [171].

1.4.1 Teleportation

Teleportation is one of the first QIP protocols where one can see the importance of quantum entanglement. It was discovered by Bennett et al. [8, 172]. The protocol enables one to send an unknown quantum state to a distant party if they share an entangled state. Two parties, Alice and Bob share an entangled state (Bell state) and live in distant places. Alice wants to send an unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. To do so, She will make joint measurement on her qubit and the unknown qubit in Bell basis and send the measurement outcome to Bob through classical channel. Upon receiving the message, Bob will make appropriate unitary operations on his qubit to retrieve the state $|\psi\rangle$. Hence, the protocol utilizes '1 ebit of entanglement $+$ 2 bits of classical communication' to send '1 qubit' [35].

Here, we have considered the perfect scenario where one can perfectly teleport an qubit – this is called *perfect* teleportation. If the shared state is not a maximally entangled or is a mixed entangled state then we may have *imperfect* or *probabilistic* teleportation [173]. The performance of an entangled state as teleportation resource is usually captured by its *singlet fraction* (f) which is defined as

$$f(\rho_{AB}) = \max_U \langle\psi^+|U^\dagger \rho_{AB} U|\psi^+\rangle, \tag{1.89}$$

where $U = U_A \otimes U_B$ are local unitary operations and ψ^+ is maximally entangled state. This quantity tells us how close a entangled state is to the set of maximally entangled states in the same Hilbert space. Whether a quantum state is perfectly teleported or not is captured by the quantity called *teleportation fidelity* i.e., $F(\rho_{AB}) = |\langle\psi|\phi\rangle|$, where $|\psi\rangle$ is intended as a message but instead $|\phi\rangle$ is recovered. These two quantities are closely related, for $d \otimes d$ resource states $F = \frac{df+1}{d+1}$ [174]. For perfect teleportation $F = 1$. Using a separable state one can reach upto $F = \frac{2}{d+1}$, i.e., $f = \frac{1}{d}$, which Bob can achieve just by random guessing + some classical communication [175]. But for entangled state $\frac{2}{d+1} \leq F \leq 1$.

1.4.2 Superdense coding

Like quantum teleporation, superdense coding was discovered by Bennett *et al.* [9]. The protocol uses entanglement as a resource. Here, a third party Eve prepares an entangled state (Bell state for perfect superdense coding) and sends one qubit to Alice and Bob each. Now Alice can send two bits to Bob. Alice has the four possible classical messages $\{00, 01, 10, 11\}$. Alice will apply local operations $\{\mathbb{I}_2, X, Z, XZ\}$ on her qubit according to the message she wants to communicate to Bob (see Fig.1.10), e.g. she will do nothing ($\mathbb{I}_2$) if the message is 00, she will apply X-gate if the message is 01 etc. Then she will send

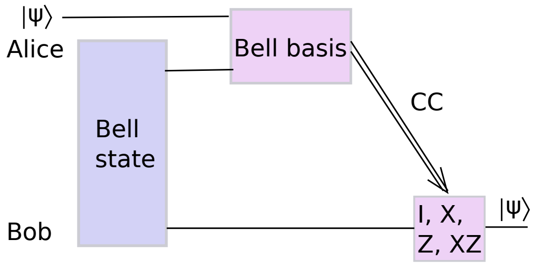


Figure 1.9: *Quantum teleporation*: The figure shows the schematic view of teleportation. The parties Alice and Bob initially share an entangled state. In order to send an unknown state $|\psi\rangle$ to Bob, Alice will make measurement in Bell basis jointly on her qubit and the unknown quantum state and send the measurement outcome (CC) to Bob. Bob then makes suitable operation on his qubit to retrieve $|\psi\rangle$.

her qubit to Bob. Now Bob will possess two qubits. Bob will apply suitable procedure to retrieve the message that Alice intended to send.

Now, if we carefully analyse the figure, we can explain it in details. Eve prepares a Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Then Alice and Bob will acquire one qubit each from Eve. After applying the local operations (accordingly) on her qubit, Alice will send the qubit to Bob via a quantum channel (ideal). Now Bob will have either of the state according to Alice's message $00 : \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle); 01 : \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle); 10 : \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \& 11 : \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$. This is the problem of state discrimination. Bob will apply a suitable state discrimination protocol to figure out what is the state after Alice's encryption i.e., the classical message. In the Fig.(1.10), he uses the CNOT-gate then Hadamard H and measurement in computational basis to know the message. Hence in order to communicate '2 bits of classical informations we need '1 ebit of entanglement + 1 qubit of communication' [98].

If the shared state is not a maximally entangled state, then what amount of classical communication will be possible? For a non-maximally or a mixed entangled state, the amount of classical communications will be smaller. It is captured by the quantity, the 'superdense coding capacity' [176] which is defined as

$$\mathcal{C}(\rho_{AB}) = \log_2 d_A + S(\rho_A) - S(\rho_{AB}) \text{ bits}, \tag{1.90}$$

where the amount $\log_2 d_A$ bits is the classical limit. Here d_A is the Hilbert space dimension of subsystem A. Hence, the quantum advantage is $S(\rho_A) - S(\rho_{AB})$ bits i.e., the $-S(A|B)$. More state is entangled more will be the quantum advantage.

1.4.3 Quantum secret sharing

Secret sharing is a procedure in which a secret is splitted into several parts and shared among different parties such that no subset of it is sufficient to read the secret, but the entire set is. Classically, there are many ways to do it but none of these are very efficient in detecting the presence of an eavesdropper/dishonest party, while the sharing of secret is going on. But in quantum regime, this is not the case, one will be able to detect if an

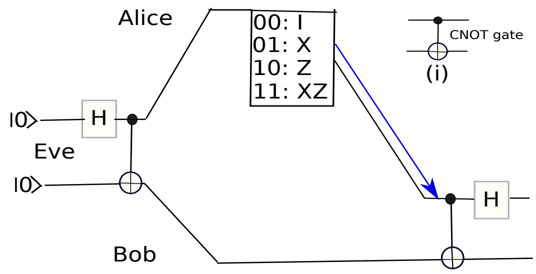


Figure 1.10: *Superdense coding*: The figure shows the schematic view of superdense coding. This QIP enables Alice to send two bits of classical information to Bob by sending just one qubit. Eve prepares a two qubit entangled state and sends one qubit of the state to Alice and Bob each. Alice will encode her classical bits by applying quantum operations, and send her qubit to Bob. Then Bob will decode the message.

eavesdropper tries to sabotage the message. Mainly two types of secret sharing protocols are available in quantum world – splitting of the message and teleportation type [95, 177].

In 1999, Hillery *et al.* introduced the idea of secret sharing using three particle quantum entangled states [95]. Later, Karlsson *et al.* studied the same protocol using two-particle entangled states [178]. The main idea is: A dealer Alice wants to send a message to Bob and Charlie but she suspects one of them is dishonest and does not know which one. She will split the message into two and sends one part to each of them so that to retrieve the message both have to collaborate. Let Alice has access to a two qubit entangled state. Now she will encode her message in the state by just locally applying some quantum operations and then she sends one particle each to Bob and Charlie. Now Bob and Charlie will retrieve the secret collaborating between themselves. In another work [179], authors discussed another protocol where Alice, Bob and Charlie initially share an entangled state. First Alice does single qubit measurement on her qubit in order to encode the secret and sends the qubit to Bob. Now Bob and Charlie's job will be to retrieve the secret collaboratively. Bob will make two-qubit measurement and sends the outcome to Charlie so that he will be able to read the message.

Hillery *et al.* [95] also discussed another protocol where Alice wants to send a qubit state to Charlie introducing another party Bob, without his help Charlie will not be able to retrieve the message. In this case Alice, Bob and Charlie share an entangled state (at least three-particle state) to begin with. Then Alice makes a measurement jointly on her message qubit and the particle from the entangled state. According to Alice's measurement outcomes, Bob and Charlie's state will collapse into an entangled state which will contain the information about the message (see the Fig.(1.11)). At this point nobody will be able to retrieve the message alone. If Bob makes measurement on his particle and sends the outcome to Charlie, then he can retrieve it.

Gottesman [180] developed the theory of quantum secret sharing where he has discussed sharing of both the classical (e.g. see the ref. [9]) and quantum secret using quantum resource state. Basically security of all these protocols depends on the entanglement of the states. But recently, there are many studies where authors have discussed quantum secret sharing protocols which need no quantum entanglement (e.g. see [181]). These

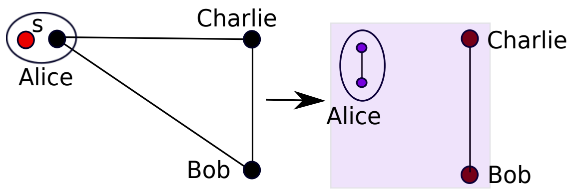


Figure 1.11: *Quantum secret sharing*: Here Alice, Bob and Charlie share an entangle state. In order to send the quantum message, Alice will make joint measurement on her qubit from the state and the message qubit s and send the measurement outcomes to the receiver Charlie. Charlie will retrieve the message by some local operations on his qubit with the help of Bob. Notice that Charlie and Bob end up with entangled state which contains the information about the secret s.

protocols are easily implementable as we can avoid practical hazards of creating and moreover of preserving and handling of entangled states [162].

1.4.4 Quantum recycle bin

Impossibility of perfect deleting [182] prevents from building perfect quantum recycle bin. In classical world, a recycle bin is a set of operations which will enable one to store deleted information in a virtual place called *trash* and one can retrieve the information, if needs be, at a later time from the trash. In classical world, information can be deleted perfectly and one can retrieve it perfectly from trash also. But this is not the case in quantum world. Let us consider the case where one wants to delete (erase the information in) the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. But one should be able to retrieve the state after some point of time if needs be. Then a ideal chain of operations for the quantum recycle bin will be

$$|\psi\rangle\mathcal{R}(|b\rangle) \xrightarrow{delete/eraze} |b\rangle\mathcal{R}(|\psi\rangle) \tag{1.91}$$
$$\mathcal{R}(|\psi\rangle)|b\rangle \xrightarrow{\mathcal{U}_r} |\psi\rangle\mathcal{R}(|b\rangle), \tag{1.92}$$

where $\mathcal{R}(|b\rangle)$ is a quantum trash state with blank state $|b\rangle$ inside it and $\mathcal{U}_r$ is suitable reverse operations acting on both the trash and the blank state to restore the quantum state from the trash.

An practical quantum recycle bin can be implemented in following two ways –

1. First apply deleting operations on the quantum state to bring it into the trash state. If one wants to delete it permanently just apply partial trace on the trash (assuming one will have multiple copies of trash states). Now, if it is required to restore the state from trash at a later time, a reverse operations will be applied on the joint state of blank and trash state. Question is: What will be the reverse operations? Technically after applying deletion on the state in presence of trash state make them correlated (as one can apply only imperfect deletion [183, 184]) – hence the performance of quantum recycle bin will depend on the fidelity of deletion [185]. Therefore, it

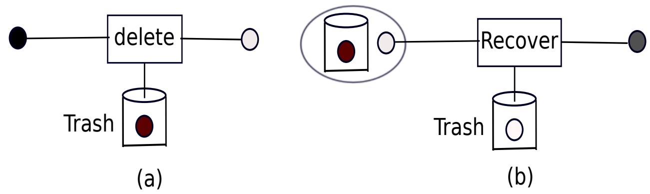


Figure 1.12: *Hypothetical quantum recycle bin*: A two step processes depicts the quantum recycle bin. (a) The quantum information (black dot) is deleted by using standard quantum deleter/eraser. Now trash contains the information. To recover it one should use suitable recovery operations which will restore the information. Here quantum information is synonymous to quantum states.

is difficult to choose the appropriate reverse operations. Some kind of controlled SWAP operations[22] are needed to restore the state.

2. One can erase superposition of the quantum state (e.g. see the [186] and references within) such that it becomes either $|0\rangle$ or $|1\rangle$ – which is nothing but a classical bits and apply classical deletion (sending) operations to send them to trash. Now if one needs to get back the state, it will be difficult to get back the desired superposition – unless trash has the memory of their exact superposition parameter. Here, trash state should keep the memory of the exact basis states and their superposition parameter to be an ideal recycle bin. It has been shown that to erase the quantum state one can SWAP the state with the blank state and then trash the original state into the environment [182]. This whole operation can equivalently be thought as erasure. However, the operation is irreversible.

1.4.5 DQC1

In 1998, Knill and Laflamme introduced the a deterministic quantum computation with one quantum bit (DQC1) model where they showed if one qubit was in non-maximally mixed state while rest are in maximally mixed state then one can achieve an exponential improvement in efficiency over the classical computers for some limited number of tasks [187]. This started to throw doubt on entanglement being responsible for all quantum speedups, since a computer register which is so mixed as to have only one non-maximally qubit is unlikely to be entangled [188]. The inception of QCsbE showed the fact that entanglement does not account for all non-classical correlation, even separable states contains some form of quantum correlations. Soon after this work, Laflamme et al., [189] gave some intuition that the QCsbE may be connected with the performance of DQC1 model. Later Datta et al., put this argument firmly [166]. Here, we will discuss the role of QCsbE behind the performance of DQC1 model that in turn gives some physical ground for QCsbE. The model is for estimating the trace of a unitary matrix. The model uses two resources – an ancilla $\rho_A = \frac{1}{2}(\mathbb{I}_2 + \mu\sigma_z)$ with purity μ and a register of n qubits

[22] A SWAP operation (V) is defined as $V(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle$.

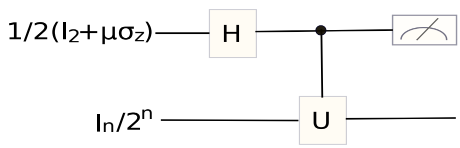


Figure 1.13: *DQC1 model*: The picture depicts the famous DQC1 model with n-qubit maximally mixed state and an ancilla of purity μ. Measuring σ_x and σ_y on the ancilla will return real and imaginary parts of $\mathrm{Tr}[U]$.

in maximally mixed state, $\rho_B = \frac{1}{2^n}\mathbb{I}_n$. They are initially in product state $\rho_i = \rho_A \otimes \rho_B$. A Hadamard gate ($H$) is applied on ancilla first, followed by a global control-U operation. Then the spin measurements (σ_x & σ_y) are made on the ancilla. From the outcome one can retrieve the value of trace of the unitary matrix U (see the Fig.1.13).

The output state, before the spin measurement in the basis of ancilla is

$$\rho_f = \frac{1}{2^{n+1}} \begin{bmatrix} \mathbb{I}_n & \mu U^\dagger \\ \mu U & \mathbb{I}_n \end{bmatrix}. \tag{1.93}$$

The reduced density matrix for ancilla is

$$\rho_A^f = \frac{1}{2} \begin{bmatrix} 1 & \mu\,\mathrm{Tr}[U^\dagger] \\ \mu\,\mathrm{Tr}[U] & 1 \end{bmatrix}. \tag{1.94}$$

It then immediately follows that measuring σ_x and σ_y will return the real and imaginary part of $\mathrm{Tr}[U]$, i.e., $\langle\sigma_x\rangle_{\rho_A^f} = \mu\mathrm{Re}[\mathrm{Tr}[U]]$ and $\langle\sigma_y\rangle_{\rho_A^f} = \mu\mathrm{Im}[\mathrm{Tr}[U]]$. It is evident that the efficiency of the protocol solely depends on the the polarization of the ancilla qubit, not on the dimensionality of unitary matrix. One can investigate the discord content of the final state state ρ_f in the bipartition ancilla and register which is given by [166] (see the Fig.(1.14))

$$D(A|B)_{DQC1} = 2 - H(\frac{1-\mu}{2}) - \log_2 \ell_+ - \ell_- \log_2 e, \tag{1.95}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ and $\ell_\pm = 1 \pm \sqrt{1-\mu^2}$. If $\mu = 1$, then $D(A|B)_{DQC1} = 2 - \log_2 e$, a constant fraction of maximal possible value of discord. Therefore discord scales like the efficiency of the model. *This is the first quantitative evidence that QCsbE play a part in the speedup associated with a quantum algorithm [166].*

1.4.6 Entanglement swapping and RED

Entanglement swapping [190, 191] is a protocol by which two remote parties can get entangled even if they initially don't share any entanglement. In entanglement swapping

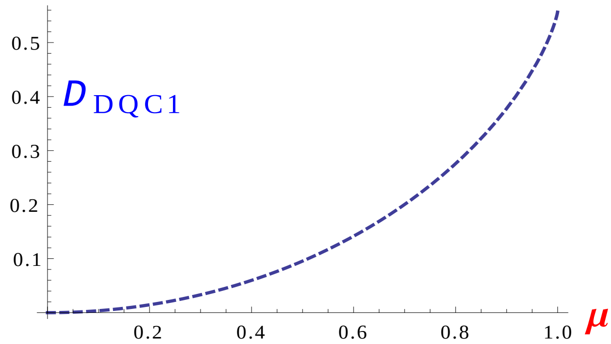


Figure 1.14: The figure shows how correlations generated in DQC1 model depend on the initial polarization (μ) of the ancilla qubit.

protocol, at least three parties are involved. In Fig.(1.15-a), we consider a simple swapping protocol. Initially two entangled states $|\psi\rangle_{12}$ & $|\psi\rangle_{12}$ are shared between the nodes $(1, 2)$ and $(3, 4)$ respectively. The particle 1 is with first party, particle 2 & 3 with second party and 4 with third party. Now second party will make Bell measurement on his/her two particles and communicate the outcome to the other parties. Irrespective of the measurement outcome, the other two parties will become entangled i.e., node 1 and 4 will become entangled although they share no entanglement initially. This process is called entanglement swapping if the initial shared states have same entanglement.

In general a distribution of bipartite entangled states between any two nodes will include states that do not have the same entanglement; thus we name this general process remote entanglement distribution (RED). The entanglement swapping with partially entangled states is a particular class of remote entanglement distribution protocols [72]. In case of mixed state RED, the second party will apply LOCC [72].

In quantum information science creating entangled multiparticle systems have great importance. But creating and maintaining them in lab have many practical issues. But, in [192], authors showed that one can create higher multi-particle states from lower particle states (see Fig.(1.15-b)). Consider two n-particle GHZ states, $\frac{1}{\sqrt{2}}\sum_{i=0}^{1}|i\cdots i\rangle$, if any party makes Bell measurement jointly on two particles one each from the two states, then the resultant states are – a $2(n-1)$-particle state and a two-particle state. So, finally one can create more particle state from two less particle states. This way one can introduce more states to create a single multiparticle state which may possess more particle. These states will later be useful in some multiparticle QIP protocols [193].

1.4.7 Broadcasting of quantum correlations

Classical theory permits perfect broadcasting (copy and send) of information, whereas this is not the case for quantum world. It is what demarcates the boundary between two [194]. However, one can have imperfect broadcasting of quantum information. Quantum broadcasting is a protocol by which one can create two or more copies of lesser correlated

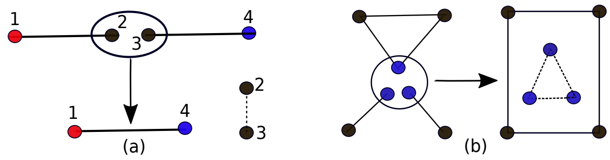


Figure 1.15: *Entanglement swapping*: It is a pictorial depiction of entanglement swapping. In (a) three nodes are involved where 2^{nd} node possessing particles 2 & 3 makes measurement to entangle the remotely separated nodes 1 and 3. In (b), from a three qubit and two two-qubit states one can produce one four-qubit and one three-qubit states. In this way one can produce more particle entangled states from fewer particle entangled states.

states from a correlated state. Let us consider a correlated state ρ_{12} shared between parties A and B. Now we apply local cloning operations [195], U_{cl} on each particles. As a result, a four-particle state will be created (here we will ignore the machine states and ancillas) i.e., ρ_{1324}. The local states created with party A and B are $\{\rho_{13}, \rho_{24}\}$ respectively and two shared less correlated nonlocal states are $\{\rho_{13}, \rho_{24}\}$ (see the Fig.(1.16)). To broadcast optimally the correlations in the created nonlocal states, the local states must have some minimum correlations in them.

The protocol was first described for quantum correlations like entanglement which is nonlocal in nature [196, 197]. For broadcasting of entanglement they used Pares-Horodecki criteria to check in which range the local states are not entangled. And that range is supposed to be the range of broadcasting. It is obvious that *the better the cloner is, the broader will be the broadcasting range* [198, 199].

We know that there exist QCsbE. Recently, many researchers have shown that *the correlations in a single bipartite state can be locally or unilocally broadcasted if and only if the states are classical (i.e., having classical correlations) or classical-quantum respectively* [200–203].

This protocol is enabling parties to share more entangled/correlated states between them only through local operations. Obviously, this procedure is very important for those processing tasks where small amount of entanglement/correlations is/are required. Here there is no need to prepare the state directly. However, loss of correlations must be counted during the process [199].

1.5 Plan of the thesis

The plan of the thesis is as follows. In the chapter 2, we discuss the quantum correlations in two-qubit systems. We will argue that the information theoretic measure of QCsbE, quantum discord captures local and nonlocal quantumness. The local quantumness is like local superposition and nonlocal quantumness is synonymous with the quantum entanglement. We argue that because of the presence of local quantumness, QCsbE increases under local noise. We generalize this idea to n-qubit systems in chapter 3. There exist atleast

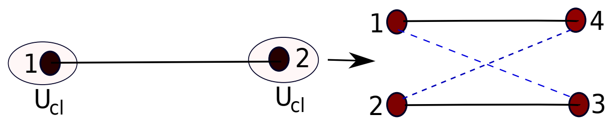


Figure 1.16: *Broadcasting*: In the broadcasting of correlations from a single two-particle state (ρ_{12}), two two-particle less correlated states are created. For that purpose one uses local cloning operations U_{cl} on each particles. As a result two less correlated nonlocal $\{\rho_{14}, \rho_{23}\}$ and two local $\{\rho_{13}, \rho_{24}\}$ two-particle states are created. As we are interested in nonlocal states, we will simply ignore (trace out) the local states.

three possible generalizations of the bivariate mutual information to multi-variables. Out of three, the interaction information has been generalized using the Venn diagram approach. We have discussed the generalization of the mutual information to quantum systems. We use the quantum version of interaction information for n-qubit systems to define discord type quantities which we call as dissensions. We consider two tracks approach to quantify the quantumness in a multi-qubit system. The Track-I generalization is discord type, while the Track-II generalization is based on all possible measurements. We give expressions for these generalization for n-qubit systems, and explore some of their properties. We consider a vector of dissensions – which we call as dissension vector. We also consider a set of three-qubit and four-qubit states to illustrate the usefulness of the dissension vector. We argue that our approach is useful in revealing quantumness in multi-qubit systems.

As an application of entanglement, we discuss the usefulness of two-qudit rank three entangled states in quantum teleportation in chapter 4. Due to the presence of bound entangled states, negativity sometimes fails in detecting such states. We have established relations between concurrence monotones and teleportation fidelity. These relations will tell whether a state will be useful in teleportation. We have considered a two-qutrit rank two mixed state to illustrate our results. In the next chapter, we consider RED (remote entanglement distribution) network and show how much classical as well as quantum information one can send through the network. We establish the relation between the teleportation fidelities and superdense coding capacities of final and initial states to address this issue. Next, we will discuss two applications of QCsbE.

In chapter 6, we study the correlations generations in the processes of “cloning then deleting" and “deleting then cloning" of quantum states. We showed that the better one clone (delete) a state, the more difficult it will be to bring the state back to its original form by the reverse process. In chapter 7, we examine whether one can broadcast QCsbE optimally. We argue that it is not possible under both unital and non-unital channel. However, one can have task oriented broadcasting for QCsbE. Moreover, we argue that if one applies $1 \rightarrow n$ ($n >> 2$) cloning machines, then there may arise a possibility of optimal broadcasting of QCsbE under unital channel. Finally, we conclude.

Chapter 2

Local, nonlocal quantumness, and quantum discord

2.1 Introduction

There are two aspects of the quantum mechanical formalism that play important role in quantum information processing. The one aspect may be referred to as nonlocal quantumness. This is due to superposition of the states of two or more particles. We will refer it as nonlocal superposition. This nonlocal superposition leads to entanglement [3]. We will call the other aspect as local quantumness. The local quantumness appears due to local superposition of the states. Every extant correlation measures which is non-zero for separable states will show such quantumness [204]. For the sake of illustration and simplicity, we will focus here on the information theoretic measure – quantum discord [100, 116]. Here, in this chapter, we demonstrate that such quantities probe not only the nonlocal quantumness but also the local quantumness. That is the prime reason why such measures are non-zero for mixed states even when there is no entanglement present in the system.

The organization of the chapter is as follows. In the section 2.2, we discuss the notion of quantum covariance to characterize the correlations. In the section 2.3, we give a brief introduction to the quantum discord vector. In the section 2.4, we discuss the phenomenon of local and nonlocal quantumness. In the section 2.5, we consider few states to exemplify the difference between the classical and separable states in the context of local and nonlocal quantumness. In the section 2.6, we introduce the parametric representation of local and nonlocal quantumness and show that the discord function depends on both of them. Finally, we conclude in the last section.

2.2 Quantum Covariance

The covariance for a bipartite state ρ_{XY} is defined as

$$\mathrm{Cov}(\rho_{XY}, \mathcal{O}_X, \mathcal{O}_Y) = \mathrm{Tr_{XY}}(\rho_{\mathrm{XY}}\mathcal{O}_{\mathrm{X}}\mathcal{O}_{\mathrm{Y}}) - \mathrm{Tr_X}(\rho_{\mathrm{X}}\mathcal{O}_{\mathrm{X}})\mathrm{Tr_Y}(\rho_{\mathrm{Y}}\mathcal{O}_{\mathrm{Y}}), \tag{2.1}$$

where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are observables acting on the part X and Y respectively. Unlike its classical counterpart, this covariance is not a measure of quantum entanglement (or quantum correlations). However, we can use it to detect quantum correlations. (See also the discussion below about nonlocal quantumness.) Using the intuitive meaning of quantum correlation, one can argue that a bipartite pure state has no quantum correlations, if the covariance vanishes for any two arbitrary observables X and Y. Clearly, covariance vanishes for the product states $\rho_{XY} = \rho_X \otimes \rho_Y$. Here ρ_X and ρ_Y are reduced density matrices. For a mixed state, one can minimize the magnitude of quantum covariance over all possible decompositions. We can then define covariance for the system with the density matrix $\rho_{XY} = \sum_i p_i \rho^i_{XY}$ as

$$\Lambda(\rho_{XY}) = \min \sum_i p_i |\mathrm{Cov}(\rho^i_{XY}, \mathcal{O}_X, \mathcal{O}_Y)|. \tag{2.2}$$

Here ρ_{XY} is a convex combination of ρ^i_{XY}. To avoid the negative value of covariance, we have considered its magnitude. In case the $\Lambda(\rho_{XY}, \mathcal{O}_X, \mathcal{O}_Y)$ is non-zero, then the state will have quantum correlations.

Lemma 2.1 *For all bipartite two-qubit separable states, $\Lambda(\rho_{XY}) = 0$.*

Proof: Here we can use the fact that (a) all the separable states can be decomposed in terms of product states and (b) for product states $\Lambda = 0$.

Hence the lemma (2.1) is important to identify bipartite correlated states.

2.3 Quantum discord vector

For a bipartite quantum states, X-discord and Y-discord may have different values. They will have identical values when the state is symmetric in X and Y. But, they are always non-negative. When one of the discord is zero, then the state would be separable. However it still may not be completely classical state and may exhibit quantum behaviour. For the state to be completely classical, both discords must vanish. As we shall see below, there exist states for which only one of these discords is zero. Therefore, for the complete characterization of the quantumness, one should know both discords. For our convenience, we define a vector quantity, $\vec{\delta}$, which is an array containing both discord as,

$$\vec{\delta}(\rho_{XY}) = \{\delta(X:Y), \delta(Y:X)\}. \tag{2.3}$$

where $\delta(X:Y)$ and $\delta(Y:X)$ are the X-discord and Y-discord respectively after minimization over measurement parameters.

Observation 2.2 *A two-qubit state is either classically correlated or is a product state iff $\vec{\delta} = \vec{0}$.*

In the literature, there exit witness operators for discord (cf. [205, 206]) but we will not discuss them here. Observation (2.2) is enough for our analysis and it also gives us information about the structure of the states.

2.4 Quantumness – local and nonlocal

A state of a bipartite quantum system may exhibit nonclassical behaviour due to either the local superposition ("local quantumness") or due to the nonlocal superposition, i.e. entanglement, ("nonlocal quantumness"). Usually, one is more concerned about the entanglement and its characterization and quantification – in part due to its mysterious nature and to use it as a resource. However, local quantumness can also be important if we can exploit the superposition as a resource in general. It is the superposition, local or nonlocal, that gives advantage in many quantum information processing protocols.

In the case of quantum discord, therefore we have $D(X:Y) = D(\varphi_L(X:Y), \varphi_{NL}(X:Y))$ where $\varphi_L(X:Y)$ characterizes the local quantumness, and $\varphi_{NL}(X:Y)$ characterizes the nonlocal quantumness of the state. We don't know yet if $D(X:Y) = D(\varphi_L(X:Y) + D(\varphi_{NL}(X:Y)$. Their properties are – 1) both $\varphi_L(X:Y)$ and $\varphi_{NL}(X:Y)$ are invariant under local unitary operation; 2) $\varphi_L(X:Y)$ may increase under local operations, but $\varphi_{NL}(X:Y)$ would not; 3) under global operations, $D(X:Y)$ may increase or decrease; 4) if the state is separable then $D(X:Y) = D(\varphi_L(X:Y))$ and $D(X:Y) = D(\varphi_L(X:Y), \varphi_{NL}(X:Y))$ for an entangled state. We now discuss these features of a quantum state in a bit more detail.

2.4.1 Local Quantumness

A bipartite separable quantum state may not have entanglement, but it is a quantum state and can exhibit quantum features. This quantum feature may be called as the local quantumness. What we mean by local quantumness can be seen by following three examples. Consider the density operators

$$\begin{aligned} \rho_a &= |++\rangle\langle++|, \\ \rho_b &= p\,|++\rangle\langle++| \;+\; (1-p)\,|--\rangle\langle--|, \\ \rho_c &= q\,|++\rangle\langle++| \;+\; (1-q)\,|00\rangle\langle 00|, \end{aligned} \tag{2.4}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The state ρ_a is a pure quantum state with no entanglement. Now one can argue that this state shows local quantumness. However, this local quantumness can be masked in the case of a pure product state. If we make a local measurement on the particle 'A' in Hadamard basis, we will get the particle in the state $|+\rangle$ with unit probability and the state would not change after the measurement. So, the local quantumness may not apparent. However, if we make a measurement in the computational basis $\{|0\rangle, |1\rangle\}$, then the particle 'A' can be found in any of the computational basis state with equal probability and the state would change after the measurement. This can be easily seen if we think of the state as a local superposition of the computational basis states. The state ρ_b is what is known as classical mixed state. Its behavior will be similar to ρ_a with respect to the measurements. We can mask its local quantumness. The state ρ_c is also a separable state. However, in this state we cannot mask the local quantumness, irrespective of the measurement basis. This is because one particle state is not orthogonal and the state in one of mixture component can be written in terms of the superposition of the state in the other component and the rest of the measurement basis. Therefore, irrespective of the measurement basis, local quantumness (local superposition) cannot be hidden. So, we see that a separable state which is not completely classical, will have local superposition which can be exploited. This is what has been showing up as a resource in the case of, eg, the model deterministic quantum computational with one quantum bit (DQC1) [166,187].

Lemma 2.3 *A two qubit state* ρ_{XY} *has only local quantumness iff* $\Lambda(\rho_{XY}) = 0$ *and* $\vec{\delta}(\rho_{XY}) \neq \vec{0}$.

Proof: The proof follows from the observations (a) for all two-qubit separable states $\Lambda(\rho_{AB}) = 0$ and, (b) only for product states, or, classical states $\vec{\delta}(\rho_{XY}) = \vec{0}$.

2.4.1.1 Local noise can enhance Discord

Since discord probes also local quantumness, therefore it can even increase by local operations. However, the local operation should be such that it changes the relative local quantumness of the mixture components. Quantum noise can be a good candidate for such a local operation. However standard local noise such as bit flip and phase flip noise cannot change discord, because no relative local superposition is introduced. In Ref [207], a set of Krass operators are given which can convert a classical mixed state, like ρ_1, given below in Eq. (2.5), to a classical-quantum mixed state, like ρ_2 or ρ_3, given below. This local noise can convert one separable state to another separable state, but not to an entangled state. This noise is only changing the local quantumness properties of a bipartite state.

2.4.2 Nonlocal Quantumness

In this paper, we shall mean the existence of quantum correlations in a state as equivalent to the state showing "nonlocal quantumness". It will also be synonymous with the existence of entanglement. If there is a system made of two subsystems, and there are quantum correlations, then the properties of the one subsystem, say A, would depend on the properties of the other subsystem, say B. The states of the subsystems are interdependent. This is the intuitive meaning of correlations. One can give a criteria for a pure bipartite state to possess quantum correlations. This criteria can then be generalized to a mixed state. This has been discussed in the section II.

Observation 2.4 *A two qubit state* ρ_{XY} *has nonlocal quantumness iff* $\Lambda(\rho_{XY}) \neq 0$.

This just follows from the lemma (2.1).

We have discussed above the quantumness of a state goes beyond entanglement. We suggest that discord characterizes the quantumness of a state. This quantumness has both local and nonlocal components. A separable state can have local quantumness, but no nonlocal quantumness. An individual system may also show quantum, i.e., non-classical behaviour. So quantum behaviour of any system encompasses quantumness due to correlation and quantumness of an individual system in the absence of correlation. Essence of the local quantumness is due to the superposition property of the state of a subsystem of a composite system. We can visualize the classification of bipartite states as in Fig.(2.1).

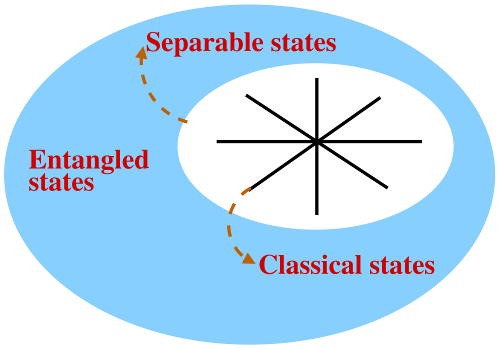


Figure 2.1: The large ellipse represents all two-qubit states. The small ellipse represents all separable states (i.e., $\Lambda = 0$). The lines represent set of product states (end points of the lines) and classical states (in different basis). The point where the lines meet is the maximally mixed state. The outer annular space contains all entangled or, nonlocal states (i.e., $\Lambda \neq 0$ & $\vec{\delta} \neq \vec{0}$), inner ellipse (except the lines) contains all separable states with local quantumness (i.e., $\Lambda = 0$ & $\vec{\delta} \neq \vec{0}$) and line depicts all product states and classical states ($\Lambda = 0$ & $\vec{\delta} = \vec{0}$).

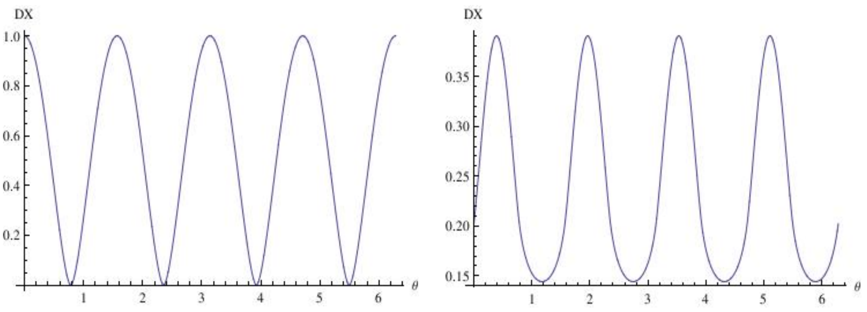
2.5 Simple Examples

2.5.1 Separable and Classical States

In order to exemplify our argument, we consider separable mixtures and examine the discord function for them. In the mixed state domain, a state is said to be separable if it can be expressed as convex combination of product states. So, in principle a product state is a separable state while the converse is not always true. Therefore, separable states do not possess entanglement. However, as is known, not all separable states have zero discord. As a paradigm, we start with following mixed states,

$$\begin{aligned}
\rho_1 &= p\,|00\rangle\langle 00| \;+\; (1-p)\,|11\rangle\langle 11|, \\
\rho_2 &= p\,|++\rangle\langle ++| \;+\; (1-p)\,|0-\rangle\langle 0-|, \\
\rho_3 &= p\,|++\rangle\langle ++| \;+\; (1-p)\,|-0\rangle\langle -0|, \\
\rho_4 &= p\,|++\rangle\langle ++| \;+\; (1-p)\,|00\rangle\langle 00|,
\end{aligned} \tag{2.5}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ are the Hadamard states. These density matrices represent four different categories of separable states. Neither of these states have entanglement. However, these states differ in important ways. ρ_1 belongs to the category of completely classical states. ρ_2 and ρ_3 are not completely classical, because, in the mixture, the states of only one of the particles are orthogonal. In the case of ρ_1, both X-discord and Y-discord are zero. For ρ_2, X-discord is zero, while for ρ_3, Y-discord is zero. For ρ_4, both discords are non-zero. If we make a measurement in computational basis, then the discord function is nonzero for ρ_1 and ρ_2. But we have to minimize the discord function to obtain the discord, the discord is zero for ρ_1, but not for ρ_2. For ρ_1, the discord function is zero in the Hadamard basis. This is the basis formed out of the states, of which the ρ_1 is a mixture. In this basis conditional entropy is zero, while entropies of the individual and



(i) (ii)

Figure 2.2: Dependence of X-Discord function on measurement basis with classical mixing parameter p = 0.5 for (i) ρ_1 and (ii) ρ_4.

composite system cancel. These facts are illustrated in Figs.(2.2 and 2.3), where discord functions are plotted as a function of the angle θ that characterizes the measurement basis. In these plots, $DX = D(Y : X)$ and $DY = D(X : Y)$.

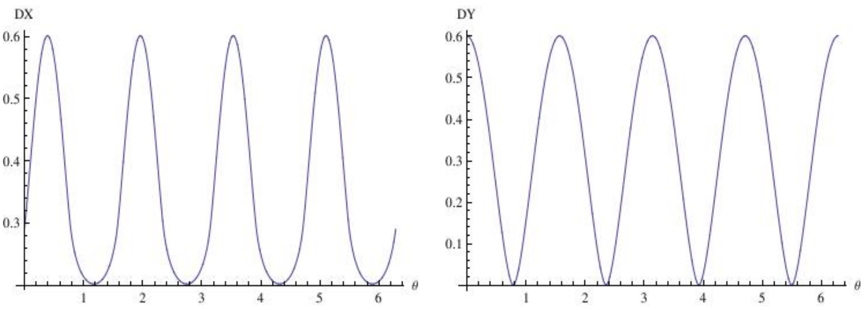
This is in accordance with the fact that while the density operator ρ_1 represents a classical mixture, i.e., a mixture of orthogonal states, the density mixture ρ_4 represents a mixture of non-orthogonal states. In the case of ρ_4, unlike ρ_1, states in one of the component, $|+\rangle$ is a linear superposition of the computational basis states $\{|0\rangle, |1\rangle\}$. This is the case of local superposition. Therefore, the discord is non-zero for ρ_4 because it also probes local quantumness (apart from nonlocal quantumness due to entanglement). One can say that a mixture of non-orthogonal separable state has local quantumness, i.e., local superposition, which cannot be washed away by writing down another decomposition of the density matrix.

2.5.2 Werner State

In this subsection, we show the importance of local quantumness for non vanishing value of the quantum discord with the aid of the Werner state. This state is given by

$$\rho_{\rm w} = (1-p)\,\frac{\mathbb{I}}{4} \;\;+\;\; p\,|\Phi^+\rangle\langle\Phi^+|, \tag{2.6}$$

where, $|\Phi^+\rangle \;=\; \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a Bell state, $\mathbb{I}$ is the identity operator and p is the classical mixing parameter. Naively, one may think that this state is not separable and has quantum correlations for all values of classical mixing parameter p. However, it is known that this state is not entangled when $p < \frac{1}{3}$ (using Peres-Horodecki criterion [36,37], e.g.,). It is also known that this state (pseudo pure state) is useful for information processing. If we look at the plot of the discord and the concurrence in the Fig.(2.4), we see that concurrence is zero, when the state is not entangled, but the discord is non-zero.



(i) (ii)

Figure 2.3: Dependence of (i) X-Discord and (ii) Y-Discord functions on measurement basis with classical mixing parameter p = 0.5 for ρ_2. For ρ_3 the X-Discord and Y-Discord are interchanged.

At this point we ask this question: Does it necessarily mean that Werner state has quantum correlations that, in some sense, go beyond entanglement ? We claim that the answer to this question is no. Our argument is that one can always rewrite Werner state in such a way that this state is a valid mixture of non-orthogonal states whenever $p < \frac{1}{3}$ [148]. Therefore the discord is nothing but just revealing the local quantumness. Rewriting the Werner state in that form, we have

$$\begin{aligned}\rho_{\rm w} \;=\; & (1-3p)\,\frac{\mathbb{I}}{4} \;+\; \frac{p}{2}\,(|++\rangle\langle++| \;+\; |--\rangle\langle--| \;+|00\rangle\langle 00| \\ & +\; |11\rangle\langle 11| \;+\; |\tilde{+}\tilde{-}\rangle\langle\tilde{+}\tilde{-}| \;+\; |\tilde{-}\tilde{+}|\rangle\langle|\tilde{-}\tilde{+}|),\end{aligned} \tag{2.7}$$

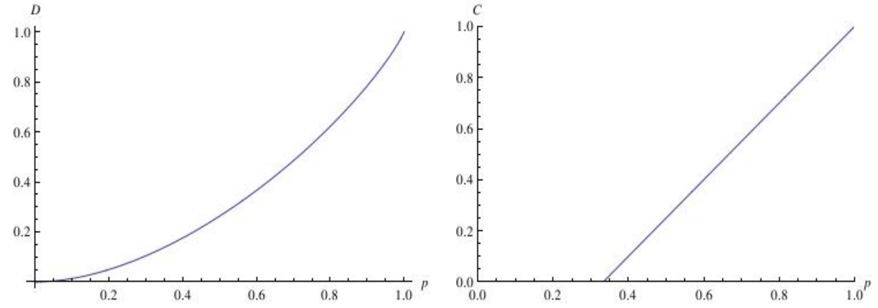
where $|\tilde{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. This is a valid density operator when $p \leq \frac{1}{3}$. This is precisely the region of p, where Werner state is not entangled. Since $\langle +|0\rangle \neq 0$ and $\langle +|\tilde{+}\rangle \neq 0$, this state is a mixture of separable non-orthogonal states; so it is expected to have non-zero discord due to local quantumness.

2.6 Generalized Werner State: A comparative analysis of Entanglement and Discord

In this section we generalize the Werner state to investigate the interdependence of local quantumness, nonlocal quantumness and classical mixedness by parametrization of each of these quantities. The major thrust of our claim lies in this part where we are able to see that the measures of entanglement like concurrence are independent of local quantumness, where as discord is a function of all these quantities. The generalized Werner state is defined as

$$\rho_{\rm GW} = (1-p)\,\frac{\mathbb{I}}{4} \;+\; p\,|\Phi^{+}_{nk}\rangle\langle\Phi^{+}_{nk}|, \tag{2.8}$$

where, $|\Phi^{+}_{nk}\rangle = N_{nk}\,(|+\rangle_n|+\rangle_n \;+\; k|-\rangle_n|-\rangle_n)$, $|+\rangle_n = N\,(|0\rangle + n|1\rangle)$ and $|-\rangle_n = N\,(-n^{*}|0\rangle + |1\rangle)$. Here N_{nk} and N are normalization constants. We can think of n as a local superposition parameter; k as a nonlocal superposition parameter and p as the



(i) (ii)

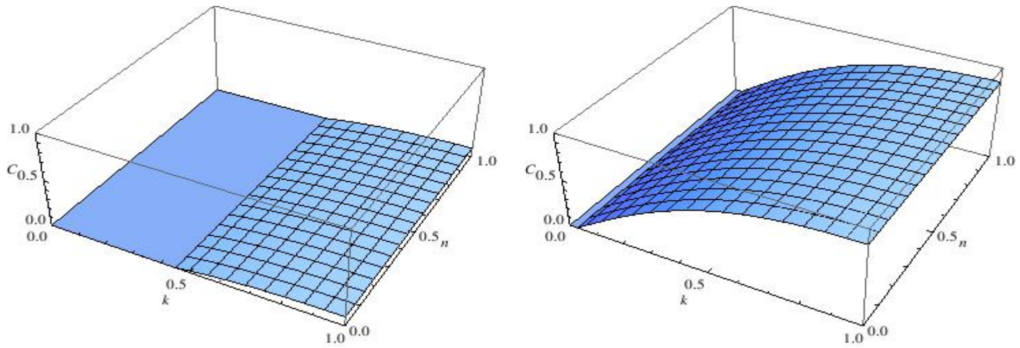
Figure 2.4: (i) Quantum discord (D) and (ii) Concurrence (C) for the Werner State as a function of the classical mixing parameter p.

classical mixing parameter. We note that this state becomes a separable state as $k \to 0$. Furthermore, there is no local superposition as $n \to 0$. To study the behavior of the state with respect to these parameters, we compute concurrence and discord for this state. To see how the discord and concurrence change as we vary p, n and k, in the following Figs.(2.5, 2.6, and 2.7), we have plotted these functions. In the Fig.(2.5), we have plotted concurrence for two different values of p as a function of the parameters n and k. We observe that concurrence is independent of the local superposition parameter n. It is important because discord depends on n. It is expected that measures of entanglement are independent of local superposition parameter (n), while the measures of correlations which claim to go beyond entanglement will depend on it. Coming back to these, we see that concurrence vanishes when mixing is small and the state is not entangled. It is also noteworthy to see that larger the value of p, larger is the concurrence. Concurrence also vanishes when nonlocal superposition parameter is small. In other words, if one is small then other has to be large for the state to be entangled. In fact, we find that this generalized Werner state is entangled, i.e., the concurrence is non-zero when

$$p > \frac{(1+k^2)}{(1+k^2+4k)}. \tag{2.9}$$

This requirement is independent of n. And it reduces to familiar condition $p > \frac{1}{3}$ for the Werner state $(n = 0, k = 1)$ in order that it is entangled.

Let us now see how discord varies with respect to changes in p, n and k. Similar to the concurrence, the discord is plotted in Figs.(2.6 and 2.7). With the increase of the value of mixing parameter, the value of discord increases. Even for very small values of mixing, when there is expected to be no entanglement, the discord is non-zero. When there is no local superposition, i.e., $n = 0$, the discord value increases as mixing becomes stronger. The value also increases, as the value of nonlocal parameter k increases, i.e., entangled component of the mixture becomes more entangled, as expected. When there is no nonlocal superposition, i.e., $k = 0$, and the mixture is separable, the discord is non-zero. Its value increases, as the mixing parameter increases, or local quantumness becomes stronger. Here important point is that the concurrence is independent of the



(i) (ii)

Figure 2.5: Variation of concurrence (C) for the Generalized Werner State with local superposition parameter n and nonlocal superposition parameter k for the two values of classical mixing parameter (i) for p = 0.4 and (ii) p = 0.9.

local superposition parameter n, while the discord increases with an increase in the value of n.

2.7 Conclusion

We have proposed that quantum discord (and other similar measures) as a measure of quantum correlations for a bipartite system contains both the local and the nonlocal quantumness. A quantum states with nonzero value of discord does not mean existence of quantum correlations beyond entanglement. In the absence of entanglement, there can be local quantumness that can make the discord nonzero. We have illustrated our proposal using a generalized Werner state to demonstrate the interplay of local quantumness, nonlocal quantumness, and classical mixedness by computing concurrence and quantum discord. To characterize the quantumness of a state, one also needs to compute both X-discord and Y-discord. Both discords have to be zero to mask the local quantumness.

We hope the present findings will help in understanding the nature of quantumness that goes beyond entanglement.

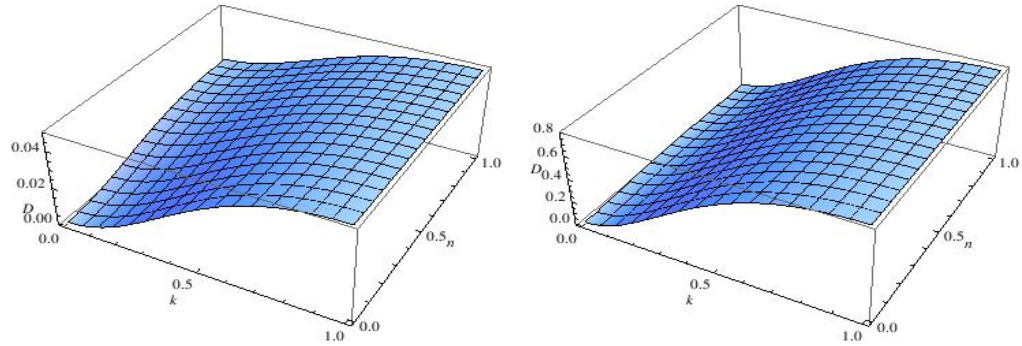
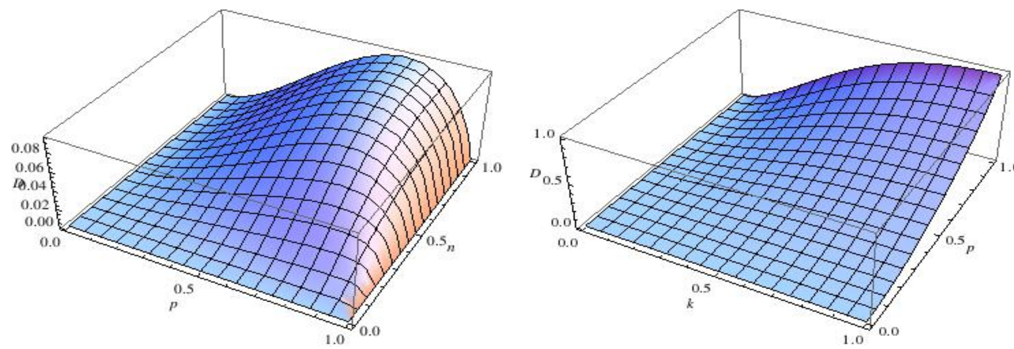


Figure 2.6: Variation of quantum discord (D) for the Generalized Werner State with local superposition parameter n and nonlocal superposition parameter k for the two values of classical mixing parameter (i) for p = 0.2 and (ii) p = 0.9.



(i) (ii)

Figure 2.7: Quantum discord (D) for the Generalized Werner State (i) for k = 0 as a function of p and n and (ii) for n = 0 as a function of p and k.

Chapter 3

Quantumness vectors for multi-qubit systems

3.1 Introduction

It has recently been argued that the quantum discord, and similar information theoretic measures, actually not only quantifies entanglement, i.e., nonlocal quantumness but also local quantumness [204]. For example, it the presence of local quantumness, which leads to increase in discord by applying certain kinds of local noise [207]. It is still to be established that a state with zero entanglement and non-zero discord can act as a resource for a nonlocal task. In this sense, the phrase "quantum correlations beyond entanglement" may be a misnomer. However information theoretic measures like discord do seem to characterize quantum properties of a state beyond entanglement, in particular local quantumness. Such measures appear to characterize the quantum properties of a state more completely. Therefore, it will be useful to generalize the measures like quantum discord to multiparticle systems. There have been several attempts in this direction [4, 118, 122, 157]. We will use multivariate mutual information for our generalization.

One important point that we emphasize in this chapter is the usefulness of a vector-like quantity to characterize and quantify the quantumness of a state [208]. The correlations in mixed states of a system, or even pure states of a multiparticle system are multifaceted. They can not be characterized by just one number. We first illustrate it by considering two-qubit mixed states. We introduce a quantumness vector for characterizing these mixed states. This idea is then extended to multiparticle states. For generalization of quantum discord to n-qubit case, we use multivariate mutual information [122]. Classically this mutual information characterizes genuine multivariate correlations in n random variables. It is based on a Venn-diagram type approach. There exist many expressions for this n-variable mutual information, all of which are same classically but differ when conditional entropies are generalized to quantum level. For a multiparticle system, one can make measurement on one-particle, or on more than one-particle to probe the different aspects of the quantum correlations. This would lead to multiple quantities that can eventually characterize the correlations present in the system. Some such physical quantities, called "quantum dissension", were introduced in our previous work [122]. Here we extend the notion of dissension to two different tracks. In the first track we proceed in the usual way by which quantum discord was defined as difference of classical information from total amount of information present in the system. Then we extend the definition to multiparticle case. In the second approach we extend the notion of quantum correlation from the perspective of quantifying the maximum amount of correlation induced in the system as a result of measurement. All possible measurements are included. In each track, to characterize multiparticle correlations we will have $n-1$ quantities based on $(n-1)$ types of measurements we can have. For example, in the tripartite case, in each track we shall have two quantities that will characterize the correlations. Interestingly, these values can be negative because a measurement on a subsystem can enhance the correlations in the rest of the system. This approach emphasizes the fact that a single quantity alone is not sufficient to characterize the quantum properties of a state. This paves the way of defining quantum correlation as a vector quantity.

The organization of the chapter is as follows. In section 3.2, we discuss classical mutual information and its extension to quantum regime. In section 3.3, we discuss correlations and quantumness. In section 3.4, we extend the notion of discord along two different tracks and give expressions for dissension vectors for n-qubit case. In section 3.5, we analyze these measures with examples for two, three, and four-qubit cases. In sections 3.6 and 3.7, we address a few related issues. Finally we conclude in section 3.8.

3.2 Mutual information and its generalization to Quantum regime

Let us consider two random variables X and Y. The common information that they possess is characterized by mutual information

$$I(X:Y) = H(X) + H(Y) - H(X,Y), \tag{3.1}$$

where $H(X)$ is Shannon entropy of X and $H(X,Y)$ is the joint entropy. There are many uses of mutual information. Our interest is in its ability to capture correlations between two probability distributions. Using chain rule, one can express mutual information also as,

$$\begin{aligned} I(X:Y) &= H(X) - H(X|Y), \\ &= H(Y) - H(Y|X), \\ &= H(X,Y) - (H(X|Y) + H(Y|X)), \end{aligned} \tag{3.2}$$

where $H(X|Y) = H(X,Y) - H(X)$ is the conditional entropy.

In quantum regime, mutual information is written in terms of Von Neumann entropy of density matrices. Intuitively this quantity solely should measure the correlations between two subsystems of a bipartite density matrix. But in reality it does not. It is sometimes suggested that the mutual information quantifies the total correlations of a bipartite system [107]. However, in general what it characterizes about the state is somewhat elusive [103, 104]. Also the generalization of this quantity to quantum regime leads to many new features and complexities. One way of generalization is that of replacing the probability distributions with density matrices and another is using relative entropy, i.e., for a bipartite state ρ_{xy},

$$\begin{aligned} I^q(X:Y) &= S(X)+S(Y)-S(X,Y), \\ &= S(\rho_{XY} \parallel \rho_X \otimes \rho_Y), \end{aligned} \tag{3.3}$$

where $S(X) = -\operatorname{Tr}(\rho_X \log_2 \rho_X)$ represents Von Neumann entropy and $S(\rho \parallel \sigma) = \operatorname{Tr} \rho(\log_2 \rho - \log_2 \sigma)$ is relative entropy.

3.2.1 Quantum conditional entropy and mutual information

The generalization of Eq. (3.2) for the bipartite quantum state ρ_{XY} are,

$$\begin{aligned} I^q_Y(X:Y) &= S(X)-S(X|Y), \\ I^q_X(X:Y) &= S(Y)-S(Y|X), \\ I^q_a(X:Y) &= S(X,Y)-(S(X|Y)+S(Y|X)), \end{aligned} \tag{3.4}$$

where $S(X|Y)$ is the quantum conditional entropy. If we directly extend the classical conditional entropy expression to quantum domain, then $S(X|Y) = S(X,Y) - S(Y)$, which is negative for pure entangled state. This negativity of conditional entropy was explained in the references [13–17]. However, there is an alternate view which says that to know a state we have to make a measurement [100]. This is then the meaning of “conditional". So, conditional entropy can also be expressed as,

$$S(X|Y) = \sum_i p_i S(\rho_{X|\pi_i^Y}), \tag{3.5}$$

where $\rho_{X|\pi_i^Y} = \frac{1}{p_i} \operatorname{Tr}_Y(\mathbb{I}_2 \otimes \pi_i^Y)\rho_{XY}(\mathbb{I}_2 \otimes \pi_i^Y)$ with $p_i = \operatorname{Tr}(\mathbb{I}_2 \otimes \pi_i^Y)\rho_{XY}(\mathbb{I}_2 \otimes \pi_i^Y)$. $\mathbb{I}_p$ is the identity matrix of order p and $\{\pi_i^Y;\ i=1,2\}$ are, in general, the rank one positive operator valued measure (POVM) on part Y.

3.2.2 Multiparticle mutual information

Our goal in this paper is to examine multiparticle systems. So we need a generalization of the bipartite mutual information to a multiparticle situation. We will use the usual generalization based on Venn diagram approach. In this approach, three variable mutual information for three variables X, Y and Z is defined as

$$I_0(X:Y:Z) = I(X:Y) - I(X:Y|Z), \tag{3.6}$$

where $I(X:Y|Z) = H(X|Z)+H(Y|Z)-H(X,Y|Z)$ is conditional mutual information [5]. This can be immediately generalized to n-variate mutual information. Using chain rules, this generalization will lead to the multivariate mutual information as,

$$I_0(x_1:x_2:...:x_n) = \sum_{p=1}^{n}(-1)^{p-1}\sum_{\{l_p\}} H(x_{l_1}, x_{l_2}, ..., x_{l_p}). \tag{3.7}$$

In literature, this quantity is also known as the ‘interaction information. By analogy, one can write the multiparticle mutual information of the state $\rho_{x_1x_2..x_n}$ as,

$$I_0^q(x_1 : x_2 : ... : x_n) = \sum_{p=1}^{n}(-1)^{p-1}\sum_{\{l_p\}}^{n} S(x_{l_1}, x_{l_2}, .., x_{l_p}), \tag{3.8}$$

where $\{l_p\}$ in the sum denotes $l_1 < l_2 < l_3... < l_p$ and l_i varies from 1 to n. This generalization has not been explored much. In this paper, we will use this generalization and define a vector type correlation measure to characterize and quantify multiparticle correlations.

However, there exist at least two more mutual information like quantities in literature. First one is known as ‘total correlation’. The total correlation for three variables is

$$I(X : Y : Z) = I(X : Y) + I(XY : Z), \tag{3.9}$$

where $I(XY : Z) = I(X : Z) + I(Y : Z|X)$. This quantity can be generalized for multi-variables i.e.,

$$I(x_1 : x_2 : ... : x_n) = \sum_{i=1}^{n} H(x_i) - H(x_1, x_2, ..., x_n). \tag{3.10}$$

This can easily be generalized to quantum regime for the state $\rho_{x_1x_2...x_n}$

$$\begin{aligned} I^q(x_1 : x_2 : ... : x_n) &= \sum_{p=1}^{n} S(x_i) - S(x_1, x_2, ..., x_n) \\ &= S(\rho_{x_1,x_2,...,x_n} \parallel \otimes_{i=1}^{n}\rho_{x_i}). \end{aligned} \tag{3.11}$$

The second line of the Eq.(3.11) shows that it is a distance between the state and tensor products of its marginals. This generalization has been used in literature [107] to capture total correlations in a multiparticle quantum state. Note that the above generalization is always positive.

Another possible quantity is the ‘dual total correlation’, or ‘binding information’ or, sometime known as ‘secrecy monotone’ [158]. For three random variables it is expressed as

$$I_s(X : Y : Z) = I(X : YZ) + I(Y : Z|X), \tag{3.12}$$

where $I(X : YZ) = I(X : Y) + I(X : Z|Y)$. The above quantity can be generalized for multi-variables i.e.,

$$I_s(x_1 : x_2 : ... : x_n) = \sum_{i=1}^{n} H(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) - (n-1)H(x_1, ..., x_n). \tag{3.13}$$

This quantity can easily be extended to the quantum state, $\rho_{x_1x_2...x_n}$

$$I_s^q(x_1 : x_2 : ... : x_n) = \sum_{i=1}^{n} S(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) - (n-1)S(x_1, ..., x_n). \tag{3.14}$$

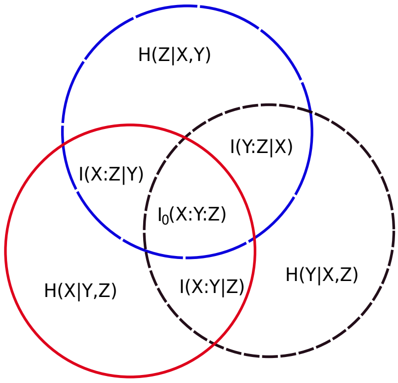


Figure 3.1: Venn diagram: The information theoretic quantities for three random variables, X, Y, and Z. The total correlation, $I(X:Y:Z) = I_s(X:Y:Z) + I_0(X:Y:Z)$, and the binding information, $I_s(X:Y:Z) = I(X:Y|Z) + I(X:Z|Y) + I(Y:Z|X) + I_0(X:Y:Z)$, where $I_0(X:Y:Z)$ is the interaction information.

Note that the above quantity is also always positive [160] and for pure states $I^q(x_1:x_2:...:x_n) = I_s^q(x_1:x_2:...:x_n)$. This quantity has been used in literature for capturing correlations in a quantum state [160] and to detect the shared secret correlations between the parties [158]. The total correlation, $I^q(x_1:x_2:...:x_n)$ and the binding information, $I_s^q(x_1:x_2:...:x_n)$ are monotones under LOCC (local operation and classical communication) [158]. Moreover, the Eqs.(3.8, 3.11 and 3.14) reduce to $I^q(x_1:x_2)$. The Fig.(3.1) depicts the relations between the possible generalizations of multivariate mutual information. These relations may not hold for quantum case. From the diagram, it is clear that only I_0 characterizes genuine multipartite correlations. Other two generalizations, I and I_s also contain bipartite correlations. We are using generalization of I_0 to quantum domain.

3.2.3 Can Mutual Information be negative?

One feature of the multivariate mutual information, as given by Venn diagram approach, is that it can be negative. Sometimes, it is considered a negative aspect of this approach. However, as we will see, the negative value characterizes a very special type of correlations. For this we consider mutual information [5] of three variables X, Y and Z, as given in Eq.(3.6). In this definition, both $I(X:Y)$ and $I(X:Y|Z)$ are non-negative, but $I(X:Y:Z)$ can be negative, when $I(X:Y) < I(X:Y|Z)$. This situation will occur when knowing Z enhances the correlation between X and Y. Let us take a well known example of 'modulo 2 addition ($\oplus$) of two binary random variables (XOR-gate)'. Suppose, $X \oplus Y = Z$. If X and Y are independent then $I(X:Y) = 0$. However, once we know the value of Z, knowing the value of X uniquely determines the value of Y. Hence the knowledge of Z enhances the correlation between X and Y, i.e., $I(X:Y|Z)$ is non-zero. This implies when $I(X:Y:Z)$ is negative, it captures certain aspect of the correlations among the variables X, Y and Z.

The generalization of Eq. (3.6) in the quantum regime, for the state ρ_{XYZ} is

$$I_0^q(X:Y:Z) = I^q(X:Y) - I^q(X:Y|Z), \tag{3.15}$$

where $I^q(X:Y|Z) = S(X|Z)+S(Y|Z)-S(XY|Z)$ is conditional mutual information. Let us consider the case of a three-qubit GHZ state $|g_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. If we trace out any one qubit from the state then the reduced density matrix is a mixture of product states, i.e., $\rho_r = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. For this state, the mutual information is $I^q(X:Y) = 1$. Next, we have to compute $I^q(X:Y|Z)$ for $\rho_{g_3} = |g_3\rangle\langle g_3|$. Its value will depend on the measurement basis. We know $S(XY|Z) = 0$ in any measurement basis but other two terms $S(X|Z)$ and $S(Y|Z)$ depend on the measurement basis. If we do measurement in computational basis $\{|0\rangle, |1\rangle\}$ on qubit Z, the conditional mutual information $I^q(X:Y|Z) = 0$ because both $S(X|Z)$ and $S(Y|Z)$ are zero. So the total mutual information is $I_0^q(X:Y:Z) = 1$, i.e., positive. It is not surprising because the state of remaining two qubits, after measurement on one qubit, does not have enhanced entanglement. But if we do measurement on one qubit, say Z, in Hadamard basis $\{|+\rangle, |-\rangle\}$, the mutual information, $I^q(X:Y|Z) = 2$, which means, total mutual information is $I_0^q(X:Y:Z) = -1$, i.e., negative. This is expected, since now the state of two qubits XY is a Bell state; so measurement on Z qubit has enhanced the entanglement in XY subsystem. The essence of this discussion is that in both classical and quantum regime multivariate mutual information can be negative, characterizing a special type of correlations.

3.3 Correlations and Quantumness

Whether a quantum state (of more than one particle) has correlations or not, is often far from obvious. This is because the meaning of the word ‘correlation’, as often used in literature, is quite fluid. We know the meaning of correlation in classical world but in the case of a quantum state there are classicality and quantumness. This makes the nature of correlations very complex. If we take the intuitive meaning of correlations [3], then quantum correlations are nonlocal in nature, and can be taken as due to entanglement of the state only. They exist due to the nonlocal quantumness of a state. A state can also have classical correlations [100] and local quantumness [122]. When we speak of quantumness of a state, it can be local or nonlocal in character. Information theoretic measures like quantum discord, and its generalization like dissension, characterize and quantify both types of quantumness. Next we emphasize the need of a vector measure to characterize the quantumness of a state. We then expand on local and nonlocal quantumness.

3.3.1 Quantum Discord: Is one number sufficient?

In the reference [100, 116], authors have given a way of quantifying quantum correlations present in bipartite two-qubit states through quantum discord. To do so they used different generalizations of the mutual information to quantum regime. Let us consider the bipartite state ρ_{xy}. Then using Eqs. (3.3 and 3.4), the discord is defined in the following way,

$$\delta_j(\rho_{xy}) = \inf_{\pi^j}\{I_0^q(x:y) - I_j(i:j)\}, \tag{3.16}$$

where $I_j(i:j) = S(i) - S(i|j)$ with $\{i, j;\ i \neq j\} = x, y$. Obviously the above definition is not symmetric in the parties. When $j = y$, it is usual discord and for $j = x$ it is $\delta_x(\rho_{xy})$. Sometimes one of the discords is zero even when other is nonzero. If we take that the quantum discord captures 'quantumness' present in a state, it is quite clear that we need both the discords to know the exact quantumness of the state.

One can define 'another discord' as,

$$\delta_a(\rho_{xy}) = \inf_{\{\pi^x, \pi^y\}} \{I_0^q(x:y) - I_a(x:y)\}. \tag{3.17}$$

This definition is symmetric in the parties. This quantity is nothing but the sum of the two discords $\delta_x(\rho_{xy})$ and $\delta_y(\rho_{xy})$.

Let us compute the above quantities in the following examples. For this purpose we introduce a vector-type quantity $\{\delta_x, \delta_y\}$ instead of using δ_x and δ_y separately. It is a quantumness vector – discord vector.

(E1) *Product and classical states:* Consider two-qubit states

$$\begin{aligned} \rho_p &= |++\rangle\langle++|, \\ \rho_c &= \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|). \end{aligned} \tag{3.18}$$

Here, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. These states are purely classical and the values are $\{\delta_x, \delta_y\} = \{0, 0\}$, $\delta_a = 0$.

(E2) *Classical-quantum (CQ)/quantum-classical (QC) states:* We now consider two types of states, one classical-quantum state and other quantum-classical state

$$\begin{aligned} \rho_{cq} &= \frac{1}{2}(|++\rangle\langle++| + |-0\rangle\langle -0|), \\ \rho_{qc} &= \frac{1}{2}(|++\rangle\langle++| + |0-\rangle\langle 0-|). \end{aligned} \tag{3.19}$$

The values are $\{\delta_x(\rho_{cq}), \delta_y(\rho_{cq})\} = \{0, 0.2\}$, $\delta_a(\rho_{cq}) = 0.2$ and $\{\delta_x(\rho_{qc}), \delta_y(\rho_{qc})\} = \{0.2, 0.0\}$, $\delta_a(\rho_{qc}) = 0.2$. We clearly see, that to characterize ρ_{cq} and ρ_{qc} completely, we need discord vector. δ_a is same for both the states.

(E3) *Separable quantum-quantum (QQ) states:* A QQ-state is

$$\rho_{qq} = \frac{1}{2}(|00\rangle\langle 00| + |++\rangle\langle++|). \tag{3.20}$$

For this state, $\{\delta_x(\rho_{qq}), \delta_x(\rho_{qq})\} = \{0.15, 0.15\}$, $\delta_a(\rho_{qq}) = 0.3$.

(E4) *Pure entangle states:* A pure entangle state is

$$|g_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{3.21}$$

For this state, $\{\delta_x(\rho_{g_2}), \delta_x(\rho_{g_2})\} = \{0.15, 0.15\}$, $\delta_a(\rho_{g_2}) = 0.3$, where $\rho_{g_2} = |g_2\rangle\langle g_2|$.

The above vector type quantification of correlation reveals more information about the correlation of a state than δ_x or δ_y alone.

3.4 Dissension vectors

From the discussion of the last section, it is clear that a vector-type correlation measure is better in describing the quantum properties of a state. Using multivariate mutual information, we will now generalize the quantum discord to n-qubit system, calling it dissension. We will introduce two types of quantumness vectors – called dissension vectors.

Let us consider a state $\rho_{x_1 x_2 ... x_n}$ in Hilbert space $H_2 \otimes H_2 \otimes ... \otimes H_2$ where x_i qubit is with i^{th} party. The mutual information for this state is

$$I_0^q(x_1 : x_2 : ... : x_n) = \sum_{p=1}^{n} (-1)^{p-1} \sum_{\{l_p\}}^{n} S(x_{l_1}, x_{l_2}, .., x_{l_p}), \tag{3.22}$$

where $\{l_p\}$ in the sum denotes $l_1 < l_2 < l_3 ... < l_p$. Using chain rule, we can now introduce conditional entropies. These conditional entropies are to be understood in terms of measurements. In this way, one can introduce one party, two party, ..., $(n-1)$-party measurements in the above expression of mutual information (see Eq.(3.22)) and each leads one to a expression for new mutual information. When more than one party is involved, joint measurement is to be implemented. Following reference [122], one can have mutual information with all possible conditionals which we called Track-II type definition, but following [100, 116] one can have mutual information with smaller number of conditionals which we call Track-I (or discord track) definition of mutual information.

3.4.1 Track-I

Let us consider the most general situation where we have state $\rho_{x_1 x_2 ... x_n}$ with n number of qubits. On the basis of m-party joint measurement (One can employ local measurements simultaneously.), we will have $(n-1)$ expressions for mutual information $\{I_m^1(x_1 : x_2 : ... : x_n); m = 1, 2, ..., (n-1)\}$,

$$\begin{aligned} I_m^1(x_1 : x_2 : ... : x_n) &= \sum_{k=1}^{m-1} (-1)^{k-1} \sum_{\{l_k\}}^{n} S(x_{l_1}, x_{l_2}, .., x_{l_k}) \\ &+ (-1)^{m-1} \sum_{\{k_{m-1}\};k_1=2} S(x_1, x_{k_1}, ..., x_{k_{m-1}}) \\ &+ \sum_{p=m+1}^{n} (-1)^{p-1} \sum_{\{l_p\}}^{n} S(x_{l_1}, .., x_{l_{p-m}} | x_{l_{p-m+1}}, .., x_{l_p}), \end{aligned} \tag{3.23}$$

where $S(x_{l_1}, .., x_{l_{p-m}} | x_{l_{p-m+1}}, .., x_{l_p})$ denotes conditional entropy where joint measurement are to be done on parties $x_{l_{p-m+1}}, .., x_{l_p}$. We can define dissension function, $D_m^1 = (-1)^n (I_0^q - I_m^1)$, and then the dissension,

$$\delta_m^1 = \inf_{\pi_m} [(-1)^n (I_0^q - I_m^1)], \tag{3.24}$$

where minimization is done over m-party measurement. The expressions of mutual information in Eq.(3.23) are not symmetric under interchange of parties. For example, if we take $m = 1$, I_1^1 can have n number of different expressions which are very different from one another. In Eq. (3.23), if we put $m = 1$, we can have one type of I_1^1; let us name it $I_{x_n}^1$. Now exchanging x_n with $x_1, x_2, ..., x_{n-1}$ respectively one can have others. So we have n number of δ_1^1. We label them as $\delta_{x_p}^1 = (-1)^n(I_0^q - I_{x_p}); p = 1, 2, ..., n$. This leads us to define dissension vector

$$\vec{\delta}_1^1 = \{\delta_{x_p}^1; p = 1, 2, ..., n\}. \tag{3.25}$$

In this way with some particular choice of entries one can have $n - 1$ vectors i.e., $\vec{\delta}_i^1; i = 1, 2, ..., (n - 1)$.

3.4.2 Track II

Next we extend the definitions of mutual information in this track for all possible m party conditionals and we have the expression for mutual information i.e., $I_m^2; m = 1, 2, ..., (n - 1)$,

$$\begin{aligned} I_m^2 &= f_m \Bigg[\sum_{\{k_{m-1}\};k_1=2}^{n-1} \{S(x_1, x_{k_1}, ..., x_{k_{m-1}}, x_{k_{m-1}+1}) - S(x_2|x_1, x_{k_{n-m+2}}, ..., x_{n-1}, x_n) \\ &- S(x_{k_{m-1}+1}|x_1, x_{k_1}, ..., x_{k_{m-1}})\} + \cdots + S(x_1, x_2, x_{k_{n-m+2}}, x_{k_{n-m+3}}, ..., x_{n-1}, x_n)\Bigg] \\ &+ \sum_{k=1}^{m-1} f_k \sum_{\{l_k\}}^{n} S(x_{l_1}, x_{l_2}, .., x_{l_k}) + \sum_{p=m+1}^{n} f_p \sum_{\{l_p\}}^{n} S(x_{l_1}, .., x_{l_{p-m}}|x_{l_{p-m+1}}, .., x_{l_p}), \end{aligned} \tag{3.26}$$

where the function, $f_k = (-1)^{k-1}$. The dissension function in this track is defined as $D_m^2 = (-1)^n(I_0^q - I_m^2)$. Therefore, the dissensions are

$$\delta_m^2 = \inf_{\pi_m}[(-1)^n(I_0^q - I_m^2)]. \tag{3.27}$$

If we interchange parties, the mutual information in the Eq. (3.26) will not remain same except for $m = (n - 1)$. For example if we consider $m = 1$ in the Eq. (3.26), we will get one I_1^2; let us call it as $I_{x_n}^2$. Now interchanging x_n with $x_1, x_2, ..., x_{n-1}$ respectively we will get others. In this way, we will have n numbers of δ_1^2. If we label them as $\delta_{x_p}^2 = (-1)^n(I_0^q - I_{x_p}^2); p = 1, 2, ..., n$, we have dissension vector

$$\vec{\delta}_1^2 = \{\delta_{x_p}^2; p = 1, 2, ..., n\}. \tag{3.28}$$

With some particular choice of entries one can have $n-2$ vectors i.e., $\vec{\delta}_i^2; i = 1, 2, ..., (n-2)$ and one symmetric quantity δ_{n-1}^2 which we call dissension vectors in Track-II.

Now, we have defined the "dissension function" D_m^t, where m is for 'on how many qubit measurements are done' and t is for the 'track you are taking'. All the dissension functions can be expressed in terms of 'usual discord' as,

$$\begin{aligned}
D_m^1 &= \sum_{p=m+1}^{n} (-1)^{n+p} \sum_{\{l_p\}}^{n} D(x_{l_1}, ..., x_{l_{p-m}} : x_{l_{p-m+1}} ... x_{l_p}), \\
D_m^2 &= \sum_{p=m+1}^{n} (-1)^{n+p} \sum_{\{l_p\}}^{n} D(x_{l_1}, ..., x_{l_{p-m}} : x_{l_{p-m+1}} ... x_{l_p}) \\
&+ (-1)^{2n+1} \left[\sum_{(l_n)}^{n} D(x_{l_1}, ..., x_{l_{n-m}} : x_{l_{n-m+1}} ... x_{l_n}) \right. \\
&- \left. D(x_1, ..., x_{n-m} : x_{n-m+1} ... x_n) \right], \qquad (3.29)
\end{aligned}$$

where (l_n) is abbreviation of $l_1, l_2, ..., l_n$ with each term taking any values from 1 to n. Note that the quantity, $D(X : Y) = I^q(X : Y) - I_Y(X : Y)$ is the bipartite discord function. Hence, one may get different dissensions just by computing the bipartite discords.

3.5 Simple illustrations

In this section, we will present our numerical results for a set of two-qubit, three-qubit, and four-qubit states. It will illustrate the usefulness of the discord and dissension vectors. We will consider track-I and track-II dissension vectors, as defined in the last section. We will see that both tracks are most of the time useful.

3.5.1 Two-qubit states

Correlation (or, quantumness) present in a bipartite two-qubit system have been explored extensively in literature. Discord was one of them, Our modified vectorial approach will capture the complete quantumness of the states. We have already discussed the quantumness properties by finding the discord vectors for several states in section 3.3. Here, we consider the Wener state,

$$\rho_{wer} = \frac{(1-p)}{4} \mathbb{I}_4 + p\rho_{g_2}. \qquad (3.30)$$

The Fig.(3.2) depicts the behaviour of the dissension vectors of the states in Eq. (3.30). The plot of dissension vectors, δ_x^1 and δ_y^1 shows that it contains both local and nonlocal quantumness because for $p \leq \frac{1}{3}$, the Werner state is separable.

3.5.2 Three-qubit states

The dissension vectors for three qubit states in track-I are,

$$\vec{\delta}_1^1 = \{\delta_{1k}^1;\ k = x, y, z\} \text{ and } \vec{\delta}_2^1 = \{\delta_{2k}^1;\ k = x, y, z\},$$

where $\delta_{1k}^1 = I_{1k}^1 - I_0^q$ and $\delta_{2k}^1 = I_{2k}^1 - I_0^q$ with $I_{1k}^1 = S(i) - (S(i|j) + S(i|k) + S(j|k)) + S(ij|k)$ and $I_{2k}^1 = S(i) + S(j) + S(k) - (S(ik) + S(jk)) + S(k|ij)$. And $I_0^q = S(x) +$

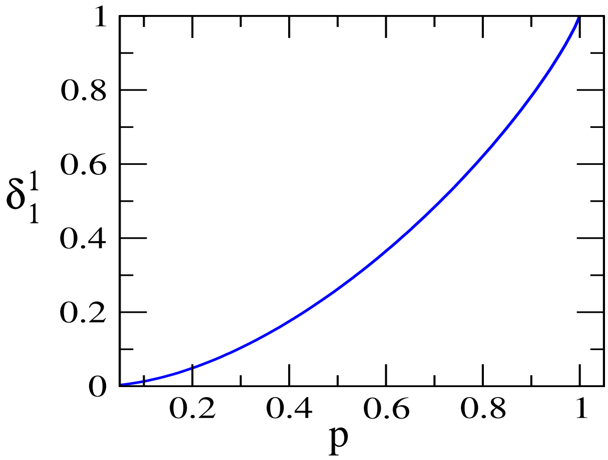


Figure 3.2: The figure shows how the $\vec{\delta_1^1}$ (For Werner state, $\delta_x^1 = \delta_y^1$.) behaves as a function of mixing parameter p for the Werner state, ρ_{wer}.

$S(y) + S(z) - (S(x,y) + S(x,z) + S(y,z)) + S(x,y,z)$. The mutual information I_{1k}^1 is not symmetric in $\{i, j, k;\ (i \neq j \neq k)\}$ i.e., interchanging i, j, or k one can get many expressions but all will give the same value after the maximization over measurement. So there will be three inequivalent expression for I_{1k}^1. And similarly for I_{2k}^1.

In track-II, the dissension vectors are,

$$\vec{\delta_1^2} = \{\delta_{1k}^2;\ k = x, y, z\},\ \text{and}\ \delta_2^2,$$

where $\delta_{1k}^2 = I_{1k}^2 - I_0^q$ and $\delta_2^2 = I_2 - I_0^q$ with $I_{1k}^2 = S(ij) - (S(i|j) + S(j|i) + S(i|k) + S(j|k)) + S(ij|k)$ and $I_2 = S(x) + S(y) + S(z) + (S(x|yz) + S(y|xz) + S(z|xy)) - 2S(x,y,z)$. The mutual information I_{1k} can have many forms for a fixed k but after maximization on measurement those expression give same result. The quantity, δ_2^2 is the equal to $\sum_k \delta_{2k}^1$ i.e., symmetric discord.

To see the usefulness of these vectors, let us consider the following three-qubit states. For convenience we will divide the states in following categories:

I) **Product states**: We consider here a simple product state i.e.,

$$\rho_{pro} = |000\rangle\langle 000|. \tag{3.31}$$

This state has no quantum and classical correlations (see the tables (3.1 and 3.2)).

II) **Classical states**: A generic example of classical state in $\{|0\rangle, |1\rangle\}$ basis is

$$\rho_{ccc} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|). \tag{3.32}$$

The non-zero dissension vectors for this state are $\vec{\delta_1^1}$ in track-I and $\vec{\delta_1^2}$ in track-II. These are given in the tables (3.1 and 3.2).

III) **Separable states**: We have picked up some important separable states for illustration. These three different states represent three different structures. The states are

$$\begin{aligned}\rho_{ccq} &= \frac{1}{2}(|00+\rangle\langle 00+| + |110\rangle\langle 110|),\\ \rho_{qqc} &= \frac{1}{2}(|++0\rangle\langle ++0| + |001\rangle\langle 001|),\\ \rho_{qqq} &= \frac{1}{2}(|+++\rangle\langle +++| + |000\rangle\langle 000|),\end{aligned} \tag{3.33}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. These separable states have one or more parts as *quantum*. We can also write down states ρ_{qcc}, ρ_{cqc} with structure similar to the first state of (see, Eq.(3.33)). Also, the states ρ_{cqq}, ρ_{qcq} have structure similar to that of ρ_{qqc}. The 'quantumness' in these states is 'local quantumness'. The non-zero dissension vectors (see the tables (3.1 and 3.2)) of these states are $\vec{\delta}_1^1$ in track-I and $\vec{\delta}_1^2$ in track-II, but have different values than that of classical states.

IV) **Pure entangled states**: Here, we take two examples of pure entangled states and one pure biseparable one. The states are

$$\begin{aligned}|g_3\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),\\ |\psi\rangle_{i|jk} &= |0\rangle_i |g_2\rangle_{jk}, \text{ where } i \neq j \neq k \in (x, y, z)\\ |w\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle).\end{aligned} \tag{3.34}$$

Out of these states $|\psi\rangle_{i|jk}$ are biseparable states of the structure $1|23$, $2|13$ and $3|12$. For a specific bipartition, these states are product states, but if three parties share the state then two parties will be entangled with each other while the third one will be completely separable with the rest. The GHZ state $|g_3\rangle$ is a maximally entangled state. The W-state, $|w\rangle$, belongs to a different class. The dissension vectors for these states are given in tables (3.1 and 3.2), which characterize the state very well.

V) **Mixed entangled states**: In this category, we consider two types of states, one biseparable and other mixed entangled. The biseparable states are

$$\begin{aligned}\rho^c_{x|yz} &= \frac{1}{2}(|0\psi^+\rangle\langle 0\psi^+| + |1\phi^+\rangle\langle 1\phi^+|),\\ \rho^q_{x|yz} &= \frac{1}{2}(|+\psi^+\rangle\langle +\psi^+| + |0\phi^+\rangle\langle 0\phi^+|),\\ \rho_m &= \frac{1}{2}(|+\psi^+\rangle\langle +\psi^+| + |\phi^+0\rangle\langle \phi^+0|),\end{aligned} \tag{3.35}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.
The mixed entangled states are

$$\begin{aligned}\rho_{wg} &= (1-p)\rho_w + p\rho_{g_3},\\ \rho_{wwc} &= (1-p)\rho_{wc} + p\rho_w,\\ \rho_{wer} &= \frac{(1-p)}{8} I + p\rho_{g_3},\end{aligned} \tag{3.36}$$

where $|wc\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle)$. Out of these states, ρ_{wer} is the Werner state in three qubit scenario.

State	$\vec{\delta}_1^1$	$\vec{\delta}_2^1$
ρ_{pro}	$\{0,0,0\}$	$\{0,0,0\}$
ρ_{ccc}	$\{-2,-2,-2\}$	$\{0,0,0\}$
ρ_{ccq}	$\{-1.2,-1.2,-1.6\}$	$\{0,0,0\}$
ρ_{cqc}	$\{-1.2,-1.6,-1.2\}$	$\{0,0,0\}$
ρ_{qcc}	$\{-1.6,-1.2,-1.2\}$	$\{0,0,0\}$
ρ_{qqc}	$\{-0.99,-0.99,-0.78\}$	$\{0,0,0.22\}$
ρ_{qcq}	$\{-0.99,-0.78,-0.99\}$	$\{0,0.22,0\}$
ρ_{cqq}	$\{-0.78,-0.99,-0.99\}$	$\{0.22,0,0\}$
ρ_{qqq}	$\{-0.67,-0.67,-0.67\}$	$\{0.15,0.15,0.15\}$
ρ_{g_3}	$\{-2,-2,-2\}$	$\{1,1,1\}$
$\rho_{x\mid yz}$	$\{-1,0,0\}$	$\{0,1,1\}$
$\rho_{y\mid xz}$	$\{0,-1,0\}$	$\{1,0,1\}$
$\rho_{z\mid xy}$	$\{0,0,-1\}$	$\{1,1,0\}$
$\rho^c_{x\mid yz}$	$\{-1,0,0\}$	$\{0,1,1\}$
$\rho^q_{x\mid yz}$	$\{-0.71,0,0\}$	$\{0,0.6,0.6\}$
ρ_m	$\{-0.26,-0.03,-0.26\}$	$\{0.52,0.8,0.52\}$
ρ_w	$\{-1.08,-1.08,-1.08\}$	$\{0.92,0.92,0.92\}$

Table 3.1: Track-I dissension vectors for a few three-qubit states. Here, $\rho_{g_3} = |g_3\rangle\langle g_3|$, $\rho_w = |w\rangle\langle w|$, $\rho_{wc} = |wc\rangle\langle wc|$, and $\rho_{i\mid jk} = |\psi\rangle\langle\psi|_{i\mid jk}$ (see Eq.(3.34)).

State	$\vec{\delta}_1^2$	$\vec{\delta}_2^2$
ρ_{pro}	$\{0,0,0\}$	0
ρ_{ccc}	$\{-3,-3,-3\}$	0
ρ_{ccq}	$\{-1.8,-1.8,-2.6\}$	0
ρ_{cqc}	$\{-1.8,-2.6,-1.8\}$	0
ρ_{qcc}	$\{-2.6,-1.8,-1.8\}$	0
ρ_{qqc}	$\{-1.6,-1.6,-1.17\}$	0.22
ρ_{qcq}	$\{-1.6,-1.17,-1.6\}$	0.22
ρ_{cqq}	$\{-1.17,-1.6,-1.6\}$	0.22
ρ_{qqq}	$\{-1.06,-1.06,-1.06\}$	0.45
ρ_{g_3}	$\{-3,-3,-3\}$	3
$\rho_{x\mid yz}$	$\{-2,0,0\}$	2
$\rho_{y\mid xz}$	$\{0,-2,0\}$	2
$\rho_{z\mid xy}$	$\{0,0,-2\}$	2
$\rho^c_{x\mid yz}$	$\{-2,0,0\}$	2
$\rho^q_{x\mid yz}$	$\{-1.8,0,0\}$	1.2
ρ_m	$\{-0.66,-0.15,-0.66\}$	1.84
ρ_w	$\{-1.75,-1.75,-1.75\}$	2.76

Table 3.2: Track-II dissension vectors for a few three-qubit states.

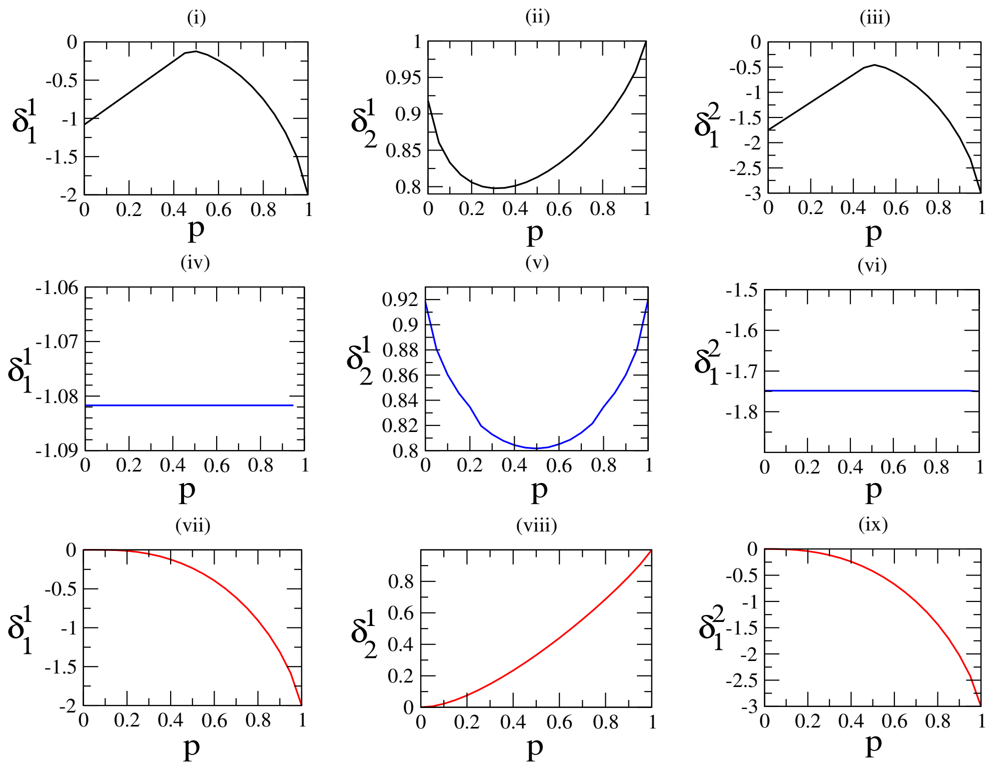


Figure 3.3: The figure shows how the dissension vectors behave as a function of mixing parameter p for the three qubit mixed states ρ_{gw}, ρ_{wwc} and ρ_{wer}. The subfigures $[(i)-(iii)]$ depict the behaviour of the dissensions for the state ρ_{gw}, $[(iv)-(vi)]$ for ρ_{wwc} and $[(vii)-(ix)]$ for ρ_{wer} where subfigures (i), (iv) and (vii) depict $\vec{\delta}_1^1$; (ii), (v) and (vii) depict $\vec{\delta}_2^1$ and (iii), (vi) and (ix) depict $\vec{\delta}_1^2$. (Note that we have plotted one of the elements from each dissension vectors. This is because within a vector each elements are same as the states are symmetric.)

The tables (3.1 and 3.2) and Fig.(6.7) show that using the dissension vectors one can clearly distinguish each state from the rest.

3.5.3 Four-qubit states

In case of four-qubit system, there are three dissension vectors, in track-I and two vectors and one symmetric discord in track-II. The exact form of these vectors in track-I are,

$$\vec{\delta}_1^1 = \{\delta_{1l}^1;\ l = x, y, z, w\},\ \ \vec{\delta}_2^1 = \{\delta_{2l}^1;\ l = x, y, z, w\} \text{ and } \vec{\delta}_3^1 = \{\delta_{3l}^1;\ l = x, y, z, w\},$$

where $\delta_{1l}^1 = I_0^q - I_{1l}^1$, $\delta_{2l}^1 = I_0^q - I_{2l}^1$ and $\delta_{3l}^1 = I_0^q - I_{3l}^1$ with I_{1l}^1, I_{2l}^1 and I_{3l}^1 are conditional mutual informations with conditionals on one qubit, two qubits and three qubits respectively and I_0^q is mutual information of a four-qubit states without conditional. Now consider I^1_{1l}, for fixed l, we have three different qubits (i, j, k). Permuting them one can have many inequivalent expressions but on maximization over measurement they will give same value. Same goes for I^1_{2l} and I^1_{3l}.

In track-II the dissension vectors are,

$$\vec{\delta}^2_1 = \{\delta^2_{1l};\ k = x, y, z, w\},\ \vec{\delta}^2_2 = \{\delta^2_{2l};\ l = x, y, z, w\} \text{ and } \delta^2_3,$$

where $\delta^2_{1l} = I^q_0 - I^2_{1l}$, $\delta^2_{2l} = I^q_0 - I^2_{2l}$ and $\delta^2_{3l} = I^q_0 - I^2_{3l}$ with I^2_{1l}, I^2_{2l} and I^2_{3l} are conditional mutual informations with conditionals (all possible) on one qubit, two qubits and three qubits respectively and I^q_0 is mutual information of a four-qubit states without conditional. Now consider I^2_{1l}, for fixed l, we have three different qubits (i, j, k). Permuting them on can have many inequivalent expressions but on maximization over measurement they will give same value. Same goes for I^2_{2l}. And δ^2_{3l} is symmetric discord i.e., $\sum_l \delta^1_{2l}$.

With the above dissension vectors, we will quantify the quantumness present in certain classes of states. We consider the following categories:

I) **Product states**: A multiqubit state is said to be a product state if it can be expressed in the form $\rho = \otimes^n_i \rho_i$, where ρ_i are single qubit pure states. For our convenience we are taking $\rho_{pro} = |\psi\rangle\langle\psi|_{pro}$, where $|\psi\rangle_{pro} = |0000\rangle$ as a prototype product state. From tables (3.3 and 3.4), we can see that the dissension vectors for the state is a null vector i.e., $\vec{0}$.

II) **Classical states**: A multiqubit classical state can be expressed as $\rho = \sum_{j=1}^2 p_j \otimes^n_i \rho^j_i$, where ρ^j_i are single qubit pure states and ρ^k_i is orthogonal to ρ^ℓ_i. Note, it is not necessary that all the component qubits should be same, it can be different as long as they constitute orthogonal sets. A simple example of this type of state is $\rho_{cl} = \frac{1}{2}(|0000\rangle\langle 0000| + |1111\rangle\langle 1111|)$. From tables (3.3 and 3.4), we notice that only non-zero dissension vectors are $\vec{\delta}^1_2$ (track-I) and $\vec{\delta}^2_2$ (track-II).

III) **Separable states**: Here, we consider separable states which have the form $\rho = \sum_j p_j \otimes^n_i \rho^j_i$ with restriction that some part will be *quantum*. Here by *quantum*, we mean that there will be some local superposition in that part i.e., $\mathrm{Tr}[\rho^k_i.\rho^\ell_i] \neq 0$ for $k \neq \ell$. These can be of four types e.g.,

$$\begin{aligned}
\rho_{cccq} &= \frac{1}{2}(|000+\rangle\langle 000+| + |1110\rangle\langle 1110|),\\
\rho_{qqcc} &= \frac{1}{2}(|++00\rangle\langle ++00| + |0011\rangle\langle 0011|),\\
\rho_{qqqc} &= \frac{1}{2}(|+++0\rangle\langle +++0| + |0001\rangle\langle 0001|),\\
\rho_{qqqq} &= \frac{1}{2}(|++++\rangle\langle ++++| + |0000\rangle\langle 0000|),
\end{aligned} \tag{3.37}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Depending on the position of the 'q' and 'c' in the first three states, one can have many different states. Again one can replace $|+\rangle$ with more general state $|n\rangle = \frac{1}{\sqrt{1+|n|^2}}(|0\rangle + n|1\rangle)$ with $n \in \mathbb{C}$. Some asymmetry can be introduced just by assigning different n values for different parts in the last three states.

The dissension vectors in track-I and track-II reveal their correlation contents as well as their identity, e.g., by looking at their dissension vectors one can easily identify the structure of these states separately. One can see the numerical values of the vectors $\vec{\delta}_1^1$, $\vec{\delta}_2^1$ and $\vec{\delta}_3^1$ in table (3.3) and $\vec{\delta}_1^2$, $\vec{\delta}_2^2$ and $\vec{\delta}_3^2$ in table (3.4). These two tracks are revealing the same type of information for these states.

IV) **Pure entangled states**: Here, we will consider three famous states, GHZ state $|\psi\rangle_{g_4}$, W-state $|w\rangle$ and Omega-state $|\Omega\rangle$ i.e.,

$$\begin{aligned} |\psi\rangle_{g_4} &= \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \\ |w\rangle &= \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle), \\ |\Omega\rangle &= \frac{1}{\sqrt{2}}(|0\psi^+0\rangle + |1\psi^-1\rangle), \end{aligned} \tag{3.38}$$

where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. Out of these states, $|\Omega\rangle$ is particularly robust state. The dissension vectors in tables (3.3 and 3.4), indicates that these states are different from each other.

V) **Mixed entangled states**: In this category, we consider two states, Werner state, ρ_{wer} and mixture of W-state and GHZ state, ρ_{wg}, i.e.,

$$\begin{aligned} \rho_{wer} &= \frac{(1-p)}{16} I + p\rho_{g_4}, \\ \rho_{wg} &= (1-p)\rho_w + p\rho_{g_4}. \end{aligned} \tag{3.39}$$

The dissension vectors are plotted in Fig.(3.4). The plots show how the vectors are changing with mixing parameter p.

If we introduce some noise (colored or white) in a state, what will be the behaviour of its dissension vectors? To study it, we have picked up the state $|\Omega\rangle$. After introduction of noises, the states will look like,

$$\begin{aligned} \rho_{wh} &= \frac{(1-p)}{16} I + p\rho_\Omega, \\ \rho_{col} &= (1-p)|0000\rangle\langle 0000| + p\rho_\Omega, \end{aligned} \tag{3.40}$$

where ρ_{wh} & ρ_{col} are the Omega-states mixed with white and colored noises respectively. In Fig.(3.5), we observe that white noise has less affect on almost all dissension vectors.

$State$	$\vec{\delta}_1^1$	$\vec{\delta}_2^1$	$\vec{\delta}_3^1$
ρ_{g_4}	$\{1,1,1,1\}$	$\{-3,-3,-3,-3\}$	$\{1,1,1,1\}$
ρ_{pro}	$\{0,0,0,0\}$	$\{0,0,0,0\}$	$\{0,0,0,0\}$
ρ_{cccc}	$\{0,0,0,0\}$	$\{-3,-3,-3,-3\}$	$\{0,0,0,0\}$
ρ_{qqqq}	$\{0.35,0.35,0.35,0.35\}$	$\{-1.35,-1.35,-1.35,-1.35\}$	$\{0.11,0.11,0.11,0.11\}$
ρ_{cccq}	$\{0.2,0.2,0,0.2\}$	$\{-3,-3,-2.6,-1.8\}$	$\{0,0,0,0\}$
ρ_{ccqc}	$\{0.2,0,0.2,0.2\}$	$\{-3,-2.6,-1.8,-3\}$	$\{0,0,0,0\}$
ρ_{cqcc}	$\{0,0.2,0.2,0.2\}$	$\{-2.6,-1.8,-3,-3\}$	$\{0,0,0,0\}$
ρ_{qccc}	$\{0.2,0.2,0.2,0\}$	$\{-1.8,-3,-3,-2.6\}$	$\{0,0,0,0\}$
ρ_{ccqq}	$\{0.35,0.14,0.14,0.35\}$	$\{-2.8,-2.4,-1.6,-1.8\}$	$\{0,0,0,0\}$
ρ_{cqcq}	$\{0.14,0.35,0.2,0.35\}$	$\{-2.4,-1.8,-2.4,-1.8\}$	$\{0,0,0,0\}$
ρ_{cqqc}	$\{0.14,0.14,0.35,0.35\}$	$\{-2.4,-1.6,-1.8,-2.8\}$	$\{0,0,0,0\}$
ρ_{qcqc}	$\{0.35,0.2,0.35,0.14\}$	$\{-1.8,-2.4,-1.8,-2.4\}$	$\{0,0,0,0\}$
ρ_{qccq}	$\{0.35,0.35,0.14,0.14\}$	$\{-1.8,-2.8,-2.4,-1.6\}$	$\{0,0,0,0\}$
ρ_{qqcc}	$\{0.14,0.35,0.35,0.14\}$	$\{-1.6,-1.8,-2.8,-2.4\}$	$\{0,0,0,0\}$
ρ_{cqqq}	$\{0.26,0.26,0.26,0.46\}$	$\{-2.1,-1.5,-1.5,-1.7\}$	$\{0.18,0,0,0\}$
ρ_{qcqq}	$\{0.46,0.26,0.26,0.26\}$	$\{-1.7,-2.1,-1.5,-1.5\}$	$\{0,0.18,0,0\}$
ρ_{qqcq}	$\{0.26,0.46,0.26,0.26\}$	$\{-1.5,-1.7,-2.1,-1.5\}$	$\{0,0,0.18,0\}$
ρ_{qqqc}	$\{0.26,0.26,0.46,0.26\}$	$\{-1.5,-1.5,-1.7,-2.1\}$	$\{0,0,0,0,0.18\}$
ρ_w	$\{0.88,0.88,0.88,0.88\}$	$\{-1.75,-1.75,-1.75,-1.75\}$	$\{0.81,0.81,0.81,0.81\}$
ρ_Ω	$\{-2,-2,-2,-2\}$	$\{-5,-5,-5,-5\}$	$\{1,1,1,1\}$

Table 3.3: Track-I dissension vectors for a few four-qubit states. Here, $\rho_{g_4} = |g_4\rangle\langle g_4|$, $\rho_w = |w\rangle\langle w|$ and $\rho_\Omega = |\Omega\rangle\langle\Omega|$.

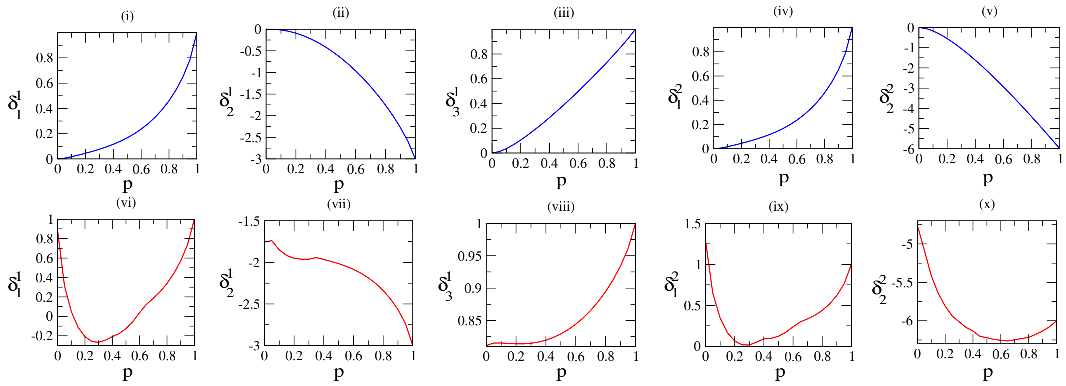


Figure 3.4: The figure shows how the dissension vectors behave as a function of mixing parameter, p for the four qubit states ρ_{wer} and ρ_{gw}. The subfigures $[(i)-(v)]$ depict the behaviour of the dissensions for the state, ρ_{wer} and $[(vi)-(x)]$ for ρ_{gw} where subfigures (i) and (vi) depict $\vec{\delta}_1^1$; (ii) and (vii) depict $\vec{\delta}_2^1$; (iii) and $(viii)$ depict $\vec{\delta}_3^1$; (iv) and (ix) depict $\vec{\delta}_1^2$ and (v) and (x) depict $\vec{\delta}_2^2$. (Note that we have plotted one of the elements from each dissension vectors. This is because within a vector each elements are same as the states are symmetric.)

$State$	$\vec{\delta}_1^2$	$\vec{\delta}_2^2$	$\vec{\delta}_3^2$
ρ_{g_4}	$\{1, 1, 1, 1\}$	$\{-6, -6, -6, -6\}$	4
ρ_{pro}	$\{0, 0, 0, 0\}$	$\{0, 0, 0, 0\}$	0
ρ_{cccc}	$\{0, 0, 0, 0\}$	$\{-6, -6, -6, -6\}$	0
ρ_{qqqq}	$\{0.5, 0.5, 0.5, 0.5\}$	$\{-3.35, -3.35, -3.35, -3.35\}$	0.44
ρ_{cccq}	$\{0.2, 0.2, 0.2, 0.2\}$	$\{-6, -6, -5.6, -4.8\}$	0
ρ_{ccqc}	$\{0.2, 0.2, 0.2, 0.2\}$	$\{-6, -5.6, -4.8, -6\}$	0
ρ_{cqcc}	$\{0.2, 0.2, 0.2, 0.2\}$	$\{-5.6, -4.8, -6, -6\}$	0
ρ_{qccc}	$\{0.2, 0.2, 0.2, 0.2\}$	$\{-4.8, -6, -6, -5.6\}$	0
ρ_{ccqq}	$\{0.35, 0.29, 0.35, 0.35\}$	$\{-5.6, -5.2, -4.4, -4.6\}$	0
ρ_{cqcq}	$\{0.35, 0.35, 0.35, 0.35\}$	$\{-5.2, -4.6, -5.2, -4.6\}$	0
ρ_{cqqc}	$\{0.29, 0.35, 0.35, 0.35\}$	$\{-5.2, -4.4, -4.6, -5.6\}$	0
ρ_{qcqc}	$\{0.35, 0.35, 0.35, 0.35\}$	$\{-4.6, -5.2, -4.6, -5.2\}$	0
ρ_{qccq}	$\{0.35, 0.35, 0.29, 0.35\}$	$\{-4.6, -5.6, -5.2, -4.4\}$	0
ρ_{qqcc}	$\{0.35, 0.35, 0.35, 0.29\}$	$\{-4.2, -4.6, -5.6, -5.2\}$	0
ρ_{cqqq}	$\{0.4, 0.4, 0.46, 0.46\}$	$\{-4.56, -3.92, -3.92, -4.14\}$	0.18
ρ_{qcqq}	$\{0.46, 0.4, 0.4, 0.46\}$	$\{-4.14, -4.56, -3.92, -3.92\}$	0.18
ρ_{qqcq}	$\{0.46, 0.46, 0.4, 0.4\}$	$\{-3.92, -4.14, -4.56, -3.92\}$	0.18
ρ_{qqqc}	$\{0.4, 0.46, 0.46, 0.4\}$	$\{-3.92, -3.92, -4.14, -4.56\}$	0.18
ρ_w	$\{1.3, 1.3, 1.3, 1.3\}$	$\{-4.75, -4.75, -4.75, -4.75\}$	3.24
ρ_Ω	$\{-2, -2, -2, -2\}$	$\{-10, -10, -10, -10\}$	4

Table 3.4: Track-II dissension vectors for a few four-qubit states.

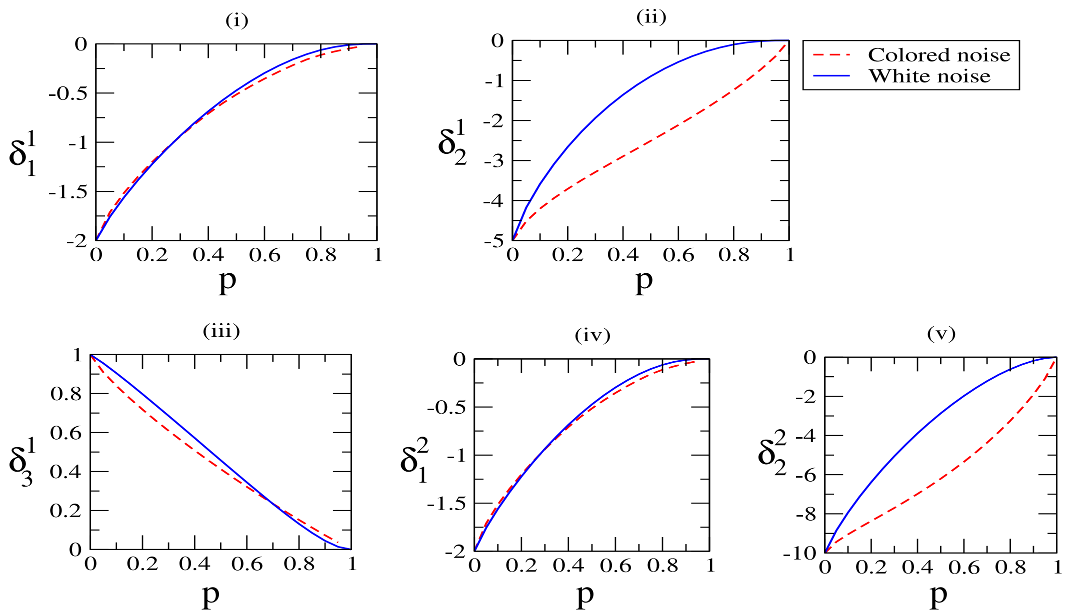


Figure 3.5: The figure shows how the dissension vectors behave as a function of mixing parameter, p for the four-qubit state, ρ_Ω when exposed with colored (red dashed line), and white noise (black solid line). The subfigures $[(i)-(v)]$ depict the behaviour of the dissension with subfigures (i) depicts $\vec{\delta}_1^1$; (ii) depicts $\vec{\delta}_2^1$; (iii) depicts $\vec{\delta}_3^1$; (iv) depicts $\vec{\delta}_1^2$ and (v) depicts $\vec{\delta}_2^2$. (Note that we have plotted one of the elements from each dissension vectors. This is because within a vector each elements are same as the state is symmetric.)

3.5.4 Why track-II is required?

Consider the states

$$
\begin{aligned}
|\psi\rangle_{i|jkl} &= |0\rangle|g_3\rangle, \\
\rho^c_{i|jkl} &= \frac{1}{2}(|0g_3\rangle\langle 0g_3| + |1g_3\rangle\langle 1g_3|), \\
\rho^q_{i|jkl} &= \frac{1}{2}(|0g_3\rangle\langle 0g_3| + |+g_3\rangle\langle +g_3|),
\end{aligned} \tag{3.41}
$$

where (i,j,k,l) can take any of (x,y,z,w) but $i \neq j \neq k \neq l$. The values of dissension vectors for the above states (see, Eq.(3.41)) in track-I and track-II are in tables (3.5 and 3.6). We notice that for all the states (see, Eq.(3.41)) dissension vectors are equal except $\vec{\delta}_2^2$. In particular, let us look at the states $\rho_{x|yzw} = |\psi\rangle\langle\psi|_{x|yzw}$, $\rho^c_{x|yzw}$, and $\rho^q_{x|yzw}$. For these states, all track-I dissension vectors are identical. Only one vector in track-II is different (see, table (3.6)). That's why we need all the tracks to charactarise the quantumness present in a state.

$State$	$\vec{\delta}_1^1$	$\vec{\delta}_2^1$	$\vec{\delta}_3^1$
$\rho_{x\vert yzw}$	$\{-1,0,0,0\}$	$\{0,-3,-3,-2\}$	$\{0,1,1,1\}$
$\rho_{y\vert xzw}$	$\{0,-1,0,0\}$	$\{-2,0,-3,-3\}$	$\{1,0,1,1\}$
$\rho_{z\vert xyw}$	$\{0,0,-1,0\}$	$\{-3,-2,0,-3\}$	$\{1,1,0,1\}$
$\rho_{w\vert xyz}$	$\{0,0,0,-1\}$	$\{-3,-3,-2,0\}$	$\{1,1,1,0\}$
$\rho^c_{x\vert yzw}$	$\{-1,0,0,0\}$	$\{0,-3,-3,-2\}$	$\{0,1,1,1\}$
$\rho^c_{y\vert xzw}$	$\{0,-1,0,0\}$	$\{-2,0,-3,-3\}$	$\{1,0,1,1\}$
$\rho^c_{z\vert xyw}$	$\{0,0,-1,0\}$	$\{-3,-2,0,-3\}$	$\{1,1,0,1\}$
$\rho^c_{w\vert xyz}$	$\{0,0,0,-1\}$	$\{-3,-3,-2,0\}$	$\{1,1,1,0\}$
$\rho^q_{x\vert yzw}$	$\{-1,0,0,0\}$	$\{0,-3,-3,-2\}$	$\{0,1,1,1\}$
$\rho^q_{y\vert xzw}$	$\{0,-1,0,0\}$	$\{-2,0,-3,-3\}$	$\{1,0,1,1\}$
$\rho^q_{z\vert xyw}$	$\{0,0,-1,0\}$	$\{-3,-2,0,-3\}$	$\{1,1,0,1\}$
$\rho^q_{w\vert xyz}$	$\{0,0,0,-1\}$	$\{-3,-3,-2,0\}$	$\{1,1,1,0\}$

Table 3.5: Track-I dissension vectors for a few specific four qubit states. The table shows that one will not be able to distinguish the states from the dissension vectors.

$State$	$\vec{\delta}_1^2$	$\vec{\delta}_2^2$	$\vec{\delta}_3^2$
$\rho_{x\vert yzw}$	$\{-1,0,0,0\}$	$\{-3,-6,-6,-5\}$	3
$\rho_{y\vert xzw}$	$\{0,-1,0,0\}$	$\{-5,-3,-6,-6\}$	3
$\rho_{z\vert xyw}$	$\{0,0,-1,0\}$	$\{-6,-5,-3,-6\}$	3
$\rho_{w\vert xyz}$	$\{0,0,0,-1\}$	$\{-6,-6,-5,-3\}$	3
$\rho^c_{x\vert yzw}$	$\{-1,0,0,0\}$	$\{-6,-7,-7,-6\}$	3
$\rho^c_{y\vert xzw}$	$\{0,-1,0,0\}$	$\{-6,-6,-7,-7\}$	3
$\rho^c_{z\vert xyw}$	$\{0,0,-1,0\}$	$\{-7,-6,-6,-7\}$	3
$\rho^c_{w\vert xyz}$	$\{0,0,0,-1\}$	$\{-7,-7,-6,-6\}$	3
$\rho^q_{x\vert yzw}$	$\{-1,0,0,0\}$	$\{-4.8,-6.6,-6.6,-5.6\}$	3
$\rho^q_{y\vert xzw}$	$\{0,-1,0,0\}$	$\{-5.6,-4.8,-6.6,-6.6\}$	3
$\rho^q_{z\vert xyw}$	$\{0,0,-1,0\}$	$\{-6.6,-5.6,-4.8,-6.6\}$	3
$\rho^q_{w\vert xyz}$	$\{0,0,0,-1\}$	$\{-6.6,-6.6,-5.6,-4.8\}$	3

Table 3.6: Track-II dissension vectors for a few specific four qubit states. The table shows that one will be able to distinguish the states from the dissension vectors.

3.6 Behaviour of quantumness under local noise

For almost all quantum processing devices, effect of noise is inevitable. This leads us to examine the behaviour of our dissension vector under local noise. From a property of a measure of quantum correlations, e.g. Q, for the bipartite state ρ_{12},

$$Q(\rho_{12}) \geq Q(\Lambda_{12}[\rho_{12}]), \tag{3.42}$$

where $\Lambda_{12} = \Lambda_1 \otimes \Lambda_2$ are local channels. Under global operations, the situation may be different. One can create or increase entanglement under such operations.

It is evident that our measures are also affected by the local noise. In this respect we can define two important classes of channels- a unital/semiclassical channel $\Lambda_{u/sc}$ is defined as

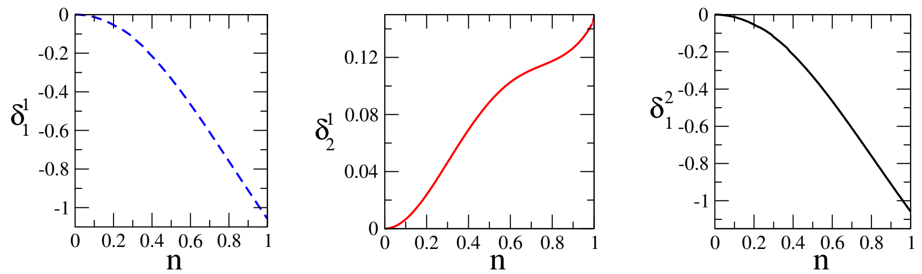


Figure 3.6: The figure depicts the behaviour of the "dissension vectors" of ρ_{cl} under non-unital channel parameter n. Here, the non-unital channel (Λ_{nu}) is applied on the first qubit. The dissension vectors, $\vec{\delta}_1^1$ (blue dashed line), $\vec{\delta}_2^1$ (red solid line), and $\vec{\delta}_1^2$ (black solid line) are nonzero for finite value of n. (Note that we have plotted one of the elements from each dissension vectors. This is because within a vector each elements are same.)

$\Lambda_{u/sc}(\frac{\mathbb{I}}{2}) = \frac{\mathbb{I}}{2}$ while for a non-unital channel Λ_{nu}, $\Lambda_{nu}(\frac{\mathbb{I}}{2}) \neq \frac{\mathbb{I}}{2}$. Streltsov et al. [207] have shown that a local quantum channel acting on a single qubit can create 'quantumness' in a multiqubit system iff it is neither semiclassical nor unital. This results holds for the dissension vector also. In our vector type measure, at least one of the elements will be affected. For example, let us consider a classical state $\rho_{cl} = \frac{1}{2}(|0000\rangle\langle 0000| + |1111\rangle\langle 1111|)$. Now, application of non-unital channel $\{E_1 = |0\rangle\langle 0|, E_2 = |n\rangle\langle 1|\}$ with $|n\rangle = \frac{1}{1+n^2}(|0\rangle + n|1\rangle)$ ($n \in \mathbb{C}$) on any subsystem will make the state non-classical and will have non-zero element in the vector (see, Fig.(3.6)).

3.7 Average quantumness of multiqubit states

A vector measure characterizes a state in a fine-grained manner. Sometime, one may be interested in average correlation properties. For some quantum tasks, average properties may be relevant. For such tasks, two states with different vector measures, but same 'average' properties may both be suitable. Therefore, in this section, we consider average of the dissension vectors. We will investigate if our measures are good in characterizing the states if we take average in a particular dissension quantity. Let us define the average dissension quantities,

$$\langle \delta_1^\ell \rangle = \frac{1}{n} \sum_{k=1}^{n} \delta_{1k}^\ell, \tag{3.43}$$

where $\ell = 1, 2$ denotes the track in which we are calculating them. Similarly, we can have different quantities like $\{\langle \delta_i^\ell \rangle; i = 1, 2, ..., n-1\}$, except the quantity, δ_{n-1}^2 which is a symmetric quantity and sum of all bipartite discord. Here, we will illustrate these measures particularly for some three qubit states. As expected, once we look at the average properties, some states cannot be distinguished (see, tables (3.7 and 3.7). For example ρ_{ccq}, ρ_{cqc}, and ρ_{qcc} have same average quantumness.

State	$\langle\delta_1^1\rangle$	$\langle\delta_2^1\rangle$
ρ_{g_3}	-2	1
$\rho_{i\shortmid jk}$	$-\frac{1}{3}$	$\frac{2}{3}$
ρ_{pro}	0	0
ρ_{ccc}	-2	0
ρ_{ccq}	$-\frac{4}{3}$	0
ρ_{qqc}	-0.92	0.07
ρ_{qqq}	-0.67	0.15
$\rho^c_{i\shortmid jk}$	$-\frac{1}{3}$	$\frac{2}{3}$
$\rho^q_{x\shortmid yz}$	-0.24	0.4
ρ_w	-1.08	0.92

Table 3.7: Track-I average dissensions for few three-qubit states.

State	$\langle\delta_1^2\rangle$	δ_2^2
ρ_{g_3}	-3	3
$\rho_{i\shortmid jk}$	$-\frac{2}{3}$	2
ρ_{pro}	0	0
ρ_{ccc}	-3	0
ρ_{ccq}	-2.06	0
ρ_{qqc}	-1.46	0.22
ρ_{qqq}	-1.06	0.45
$\rho^c_{x\shortmid yz}$	$-\frac{2}{3}$	2
$\rho^q_{x\shortmid yz}$	$-0.6\}$	1.2
ρ_w	-1.75	2.76

Table 3.8: Track-II average dissensions for few three-qubit states.

3.8 Conclusion

By considering the quantum discord as a measure of the quantumness of a two-qubit state, we argued that a vector quantity does a better job in characterizing a state. We generalized the discord to n-qubit systems – dissension. It is based on n-variable mutual information. We argued that though multivariate mutual information can be negative, it may not be a drawback. For a n-qubit state, one can introduce $(n-1)$ vector measures to characterize the state. We considered two tracks of these measures for a two-qubit, three-qubit, and four-qubit systems. We showed how various classes of states can be distinguished and characterized using these measures. More work is still required to understand these measures and how useful they are beyond what we have considered. For example: Can they characterize the resources of a state better for a specific task?.

Chapter 4

Quantum teleportation in $d \otimes d$: A special case

4.1 Introduction

For two-qutrit Schmidt rank two states, there exist bound entangled states [209]. Which means all the states in $3 \otimes 3$ may not be useful in teleportation. Although, the entangled states with Schmidt rank three would, presumably, be useful for teleportation. Hence, it is very hard in general to find which states in $d \otimes d$ are useful for teleportation, specially the low rank (Schmidt rank) states. We know, in general, negativity [36, 37, 210] fails to distinguish separable states from PPT entangled states, that is, bound entangled states. However, this difficulty can be overcome by the use of convex-roof extension of negativity (CREN) [211]. We have shown that sometimes it may also fail to detect which state is useful for teleportation. Hence, we choose concurrence monotones [71, 73] to characterize the entanglement of the state. Concurrence monotones have added advantage that it depends on the rank of the states and can usually able to determine the rank of the state. Also, sometime a number of quantities have advantages in characterizing entanglement of the higher dimensional states.

We establish various relations between teleportation fidelity and concurrence monotones depending upon the Schmidt rank of the states [212]. These relations and bounds help us to answer the above issues. Given an arbitrary two-qudit state with Schmidt rank upto three, we showed that one can predict its utility as a resource for teleportation. We quantify the amount of entanglement present in the resource state to find out the bounds within which these states can be useful for teleportation. Our results are obtained for arbitrary dimensional bipartite states with at most three non vanishing Schmidt coefficients. We implement our results to detect mixed states useful for teleportation.

The chapter is planned as follows. In Section 4.2, we study the relation between negativity and teleportation fidelity for pure as well as mixed systems. Based on our conclusions from Section 4.2, we establish a relation between singlet fraction and different concurrence monotones for arbitrary dimensional pure two qudit system with a maximum of three Schmidt coefficients in Section 4.3. Then we study the bounds of teleportation fidelity and the monotones for two special cases, i) arbitrary dimensional bipartite states with two Schmidt coefficients, and ii) arbitrary dimensional bipartite statse with three

Schmidt coefficients. In section 4.4, we illustrate our results. Finally, we conclude in section 4.5.

4.2 Relation between negativity and teleportation fidelity for $d \otimes d$ systems

4.2.1 Pure Systems

Let H_A and H_B be two Hilbert spaces each with dimension d. In $d \otimes d$ system, any pure state $|\psi\rangle$ can be expressed as $|\psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j}|j\rangle|j\rangle$. The negativity of the state $|\psi\rangle$ is defined as

$$N(|\psi\rangle) = \frac{2}{d-1} \sum_{i<j} \sqrt{\lambda_i \lambda_j}. \tag{4.1}$$

The singlet fraction for any pure state in $d \otimes d$ system is given by

$$f(|\psi\rangle) = \frac{1}{d} \left(\sum_{i=1}^{d} \sqrt{\lambda_i} \right)^2 . \tag{4.2}$$

The relation between negativity and singlet fraction is given by [174]

$$N(|\psi\rangle) = \frac{df(|\psi\rangle) - 1}{d-1}. \tag{4.3}$$

In terms of teleportation fidelity, Eq. (4.3) reduces to

$$F(|\psi\rangle) = \frac{2}{d+1} + \frac{(d-1)N(|\psi\rangle)}{d+1}. \tag{4.4}$$

Therefore, it follows that every entangled pure state in a $d \otimes d$ system is useful for teleportation.

4.2.2 Mixed Systems

A bipartite mixed state described can be described by the density operator ρ

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{4.5}$$

The negativity of the mixed state ρ can be extended from the pure state by means of convex roof, that is, convex-roof extended negativity (CREN) [211]:

$$N(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i N(|\psi_i\rangle). \tag{4.6}$$

The upper bound of the negativity of the mixed state ρ can be expressed in terms of the singlet fraction as

$$\begin{aligned} N(\rho) &= \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i N(|\psi_i\rangle) \\ &\leq \sum_i p_i N(|\psi_i\rangle) = \frac{d}{d-1}\sum_i p_i f(|\psi_i\rangle) - \frac{1}{d-1}. \end{aligned} \tag{4.7}$$

In terms of teleportation fidelity, the bound on negativity is

$$N(\rho) \leq \frac{d+1}{d-1}\sum_i p_i F(|\psi_i\rangle) - \frac{2}{d-1}. \tag{4.8}$$

The above inequality (4.8) measures the upper bound of entanglement contained in the mixed state ρ. From this, it is clear that CREN detects both PPT bound entangled states as well as states useful for teleportation. However, it is not clear how to distinguish between these two classes of states. Further, there exists a strong conjecture in the literature [209] that all PPT entangled states, in $3\otimes 3$ systems, have Schmidt rank two. This motivates us to develop measures capable of identifying states useful for teleportation and dependent on the Schmidt number.

4.3 Relation between singlet fraction and concurrence monotones for $d\otimes d$ systems with Schmidt rank three

In this section we obtain an explicit relation that will connect entanglement monotones with singlet fraction for a two qudit system of arbitrary dimension. We obtain results in $d\otimes d$ systems with two and three non zero Schmidt coefficients.

4.3.1 Pure two qudit systems

Let us consider a bipartite $d\otimes d$ system in which three Schmidt coefficients are non zero. Without any loss of generality we assume that the first three Schmidt coefficients are non zero. Any pure two qudit system with three non zero Schmidt coefficients λ_1, λ_2 and λ_3 can be written in Schmidt decomposition form as, $|\psi^d\rangle = \sqrt{\lambda_1}|00\rangle + \sqrt{\lambda_2}|11\rangle + \sqrt{\lambda_3}|22\rangle$, with the Schmidt coefficients summing to one, i.e., $\lambda_1+\lambda_2+\lambda_3 = 1$. To quantify the amount of entanglement in $|\psi^d\rangle$ we consider two different concurrence monotones $C_2(|\psi^d\rangle)$ and $C_3(|\psi^d\rangle)$ which can be defined as [71],

$$C_2(|\psi^d\rangle) = \sqrt{\frac{2d}{d-1}(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3)}, \tag{4.9}$$

$$C_3(|\psi^d\rangle) = \left(\frac{6d^2}{(d-1)(d-2)}\right)^{\frac{1}{3}} (\lambda_1\lambda_2\lambda_3)^{\frac{1}{3}}. \tag{4.10}$$

We note that for a Schmidt rank two state, $C_3(|\psi^d\rangle) = 0$ but $C_2(|\psi^d\rangle) \neq 0$. Now the singlet fraction $f(|\psi^d\rangle)$ for $|\psi^d\rangle$ can also be expressed in terms of Schmidt coefficients [91] as

$$f(|\psi^d\rangle) = \frac{1}{d}\Big(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}\Big)^2. \tag{4.11}$$

Expanding the the right hand side part of Eq.(4.11) and using $\lambda_1 + \lambda_2 + \lambda_3 = 1$, we get

$$\sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_2\lambda_3} + \sqrt{\lambda_1\lambda_3} = \frac{df(|\psi^d\rangle) - 1}{2}. \tag{4.12}$$

Also, we have the following identity

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = (\sqrt{\lambda_1\lambda_2} + \sqrt{\lambda_2\lambda_3} + \sqrt{\lambda_1\lambda_3})^2 - 2\sqrt{\lambda_1\lambda_2\lambda_3}\sqrt{df(|\psi^d\rangle)}. \tag{4.13}$$

Using (4.9), (4.10), (4.11), (4.12) and (4.13) we have

$$(C_2(|\psi^d\rangle))^2 = \frac{d^3}{2(d-1)}\left(f(|\psi^d\rangle) - \frac{1}{d}\right)^2 - \ell_d(C_3(|\psi^d\rangle))^{\frac{3}{2}}\sqrt{f(|\psi^d\rangle)}. \tag{4.14}$$

where, $\ell_d = \frac{4}{d-1}\sqrt{\frac{d(d-1)(d-2)}{6}}$

This establishes the required relationship between the concurrence monotones $C_2(|\psi^d\rangle)$ and $C_3(|\psi^d\rangle)$ with the singlet fraction $f(|\psi^d\rangle)$ for a pure two qudit system $|\psi^d\rangle$ with three non vanishing Schmidt coefficients.

Next, we will consider separately the cases of states of Schmidt ranks two and three, respectively.

4.3.1.1 States with Schmidt rank two

When one of the Schmidt coefficients (say, λ_3) is zero, i.e., $C_3(|\psi^d\rangle) = 0$, from Eq. (4.14), we have

$$C_2(|\psi^d\rangle) = \sqrt{\frac{d^3}{2(d-1)}}\left(f(|\psi^d\rangle) - \frac{1}{d}\right), \tag{4.15}$$

where, $f(|\psi^d\rangle)$ denotes the singlet fraction of Schmidt rank two state, and $f(|\psi^d\rangle) > \frac{1}{d}$. If $F(|\psi^d\rangle)$ denotes the teleportation fidelity of Schmidt rank two states, then $C_2(|\psi^d\rangle)$ can be expressed in terms of $F(|\psi^d\rangle)$ as

$$C_2(|\psi^d\rangle) = \sqrt{\frac{d^3}{2(d-1)}}\left[\frac{(d+1)F(|\psi^d\rangle) - 2}{d}\right]. \tag{4.16}$$

This establishes the relation between the entanglement monotone and teleportation fidelity of Schmidt rank two states. If the state $|\psi^d\rangle$ has Schmidt number two and useful for teleportation, then we have [44]

$$\frac{1}{d} < f(|\psi^d\rangle) \leq \frac{2}{d}. \tag{4.17}$$

Eq. (4.17) can be recast in terms of teleportation fidelity as

$$\frac{2}{d+1} < F(|\psi^d\rangle) \leq \frac{3}{d+1}. \tag{4.18}$$

Using Eq. (4.18), $C_2(|\psi^d\rangle)$ can be seen to be bounded as

$$0 < C_2(|\psi^d\rangle) \leq \sqrt{\frac{d}{2(d-1)}}. \tag{4.19}$$

When the amount of entanglement lies in the above range we can use the state for teleportation. This quantifies the entanglement required for teleportation for a pure qudit state with two non-vanishing Schmidt coefficients.

4.3.1.2 States with Schmidt rank three

Next we take up sates where none of the three Schmidt coefficients are zero, i.e., $C_3(|\psi^d\rangle) \neq 0$.

Using the well known result of arithmetic mean (AM) being greater than or equal to geometric mean (GM) on three real quantities $\sqrt{\lambda_1\lambda_2}$, $\sqrt{\lambda_1\lambda_3}$ and $\sqrt{\lambda_2\lambda_3}$, we have

$$\frac{\sqrt{\lambda_1\lambda_2}+\sqrt{\lambda_1\lambda_3}+\sqrt{\lambda_1\lambda_3}}{3} \geq \Big(\lambda_1\lambda_2\lambda_3\Big)^{\frac{1}{3}}. \tag{4.20}$$

Using Eqs. (4.10), and (4.12), we have

$$f(|\psi^d\rangle) \geq \frac{6}{d}\left[\Big(\frac{(d-1)(d-2)}{6d^2}\Big)^{\frac{1}{3}} C_3(|\psi^d\rangle)\right] + \frac{1}{d}. \tag{4.21}$$

Since, the singlet fraction $f(|\psi^d\rangle)$ attains its maximum value unity at $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{d}$, we have

$$\frac{6}{d}\left[\Big(\frac{(d-1)(d-2)}{6d^2}\Big)^{\frac{1}{3}} C_3(|\psi^d\rangle)\right] + \frac{1}{d} \leq f(|\psi^d\rangle) \leq 1. \tag{4.22}$$

In terms of teleportation fidelity $F(|\psi^d\rangle)$, the above inequality can be expressed as

$$\frac{2}{d+1} + \frac{6}{d+1}\Big(\frac{(d-1)(d-2)}{6d^2}\Big)^{\frac{1}{3}} C_3(|\psi^d\rangle) \leq F(|\psi^d\rangle) \leq 1. \tag{4.23}$$

Hence, pure entangled states with $C_3(|\psi^d\rangle)$ satisfying Eq. (4.23) and teleportation fidelity $F(|\psi^d\rangle) > \frac{2}{d+1}$ are Schmidt rank three states useful for teleportation.

4.3.2 Mixed two qudit systems

In this section we would like to answer the following questions : (i) What is the minimum amount of entanglement needed to perform teleportation with the mixed $d \otimes d$ states with Schmidt rank two? (ii) What is the minimum amount of entanglement needed to perform teleportation with the mixed $d \otimes d$ state with Schmidt rank three?

Let us consider a mixed qudit state described by the density operator $\rho = \sum_{i=1}^{n} p_i \rho_i$, where $\sum_{i=1}^{n} p_i = 1$ and ρ_i $(= |\psi_i^d\rangle\langle\psi_i^d|)$ are composite pure states. The singlet fraction $f(\rho)$ of the state ρ is defined as $f(\rho) = \max_U \langle\psi^+|U^\dagger \otimes \mathbb{I} \rho U \otimes \mathbb{I}|\psi^+\rangle$, where U is the unitary matrix, $\mathbb{I}$ is the identity matrix and $|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kk\rangle$ represents a pure maximally entangled state.

The entanglement measure $C_2(|\psi^d\rangle)$ and $C_3(|\psi^d\rangle)$ given in Eqs. (4.9) and (4.10) for pure states can also be defined for a mixed state ρ as

$$C_2(\rho) = \min \sum_{i=1}^{n} p_i C_2(\rho_i), \text{ and } C_3(\rho) = \min \sum_{i=1}^{n} p_i C_3(\rho_i). \tag{4.24}$$

Here the minimum is taken over all pure state decompositions of ρ. Now one may wonder whether, like concurrence monotones, the singlet fraction $f(\rho)$ also have the same property [213], i.e.,

$$f(\rho) = \min \sum p_i f(\rho_i), \tag{4.25}$$

where the minimum is taken over all decomposition of ρ. Unfortunately, the answer is no.

4.3.2.1 Two qudit mixed state with Schmidt rank two

From Eq. (4.14), $C_2(\rho_i)$ for any bipartite pure qudit state with Schmidt rank two ρ_i whose $f(\rho_i) = \frac{1}{d}$, i.e., for states not useful for teleportation, we have

$$C_2(\rho_i) = 0. \tag{4.26}$$

In general for any bipartite pure qudit state with Schmidt rank two ρ_i useful for teleportation, the entanglement C_2 is

$$C_2(\rho_i) = \sqrt{\frac{d^3}{2(d-1)}}\Big(f(\rho_i) - \frac{1}{d}\Big). \tag{4.27}$$

Using Eqs. (4.24) and (4.27), we have

$$\begin{aligned} C_2(\rho) &= \min \sum_i p_i \sqrt{\frac{d^3}{2(d-1)}} \left(f(\rho_i) - \frac{1}{d}\right) \\ &\leq \sum_i p_i \sqrt{\frac{d^3}{2(d-1)}} \left(f(\rho_i) - \frac{1}{d}\right) < \sqrt{\frac{d}{2(d-1)}}, \end{aligned} \tag{4.28}$$

where the last inequality follows from an application of Eq.(4.17). Hence, if the mixed state ρ with Schmidt rank two in a $d \otimes d$ system is useful for teleportation then

$$0 < C_2(\rho) < \sqrt{\frac{d}{2(d-1)}}. \tag{4.29}$$

4.3.2.2 Two qudit mixed state with Schmidt rank three

Using, once again, the result of arithmetic mean (AM) being greater than or equal to geometric mean (GM) on three real quantities $\lambda_1\lambda_2$, $\lambda_1\lambda_3$ and $\lambda_2\lambda_3$ and Eqs. (4.9), (4.10) we obtain the following bound on $C_3(\rho)$ for two qudit mixed states with Schmidt rank three:

$$0 < C_3(\rho) < \left[\frac{d(d-1)}{6}\right]^{\frac{1}{6}} \frac{1}{(d-2)^{1/3}}. \tag{4.30}$$

Comparing Eqs. (4.30) and (4.29), we can see that if the entanglement lies in the range $\sqrt{\frac{d}{2(d-1)}}$ to $\left[\frac{d(d-1)}{6}\right]^{\frac{1}{6}} \frac{1}{(d-2)^{1/3}}$ it can be concluded that the state is of Schmidt rank three.

4.4 An example of two qutrit mixed states with Schmidt rank two

We consider a two qutrit mixed state with Schmidt rank two given by

$$\rho_{m3} = \frac{5p}{p+2}\rho_c + \frac{2(1-2p)}{p+2}|\phi\rangle\langle\phi|; \qquad 0 \leq p \leq \frac{1}{2}, \tag{4.31}$$

where, $\rho_c = \frac{1}{2}(|\chi_0\rangle\langle\chi_0| + |\chi_1\rangle\langle\chi_1|)$. This decomposition for state ρ_f is optimal. Here, $|\chi_0\rangle$ and $|\chi_1\rangle$ are of the form $|\chi_0\rangle = \sqrt{\frac{3}{5}}|\psi\rangle + \sqrt{\frac{2}{5}}|\phi\rangle$ and $|\chi_1\rangle = \sqrt{\frac{3}{5}}|\psi\rangle - \sqrt{\frac{2}{5}}|\phi\rangle$, respectively, and the states $|\psi\rangle, |\phi\rangle$ are given by, $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle - e^{\frac{i\pi}{3}}|22\rangle)$ and $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Also, p is the classical probability of mixing. For the above state, C_2 (see Eq. (4.28)) becomes

$$\begin{aligned} C_2(\rho_{m3}) &= \frac{3\sqrt{3}}{2}\left(\min \sum_i p_i f(\rho_i)\right) - \frac{\sqrt{3}}{2} \\ &= \frac{3\sqrt{3}}{2}\left(\min_{\{p\}}\left[\frac{1+p}{2+p}\right]\right) - \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{4}; \quad \text{for } p = 0. \end{aligned} \tag{4.32}$$

In this calculation we have used the appropriate maximally entangled basis given in [214]. From Eqs. (4.32) and (4.19), it can be seen that the state (in Eq. (4.31)) is useful for teleportation.

4.5 Conclusion

We have made a study of entanglement of teleportation for arbitrary $d \otimes d$ dimensional states having Schmidt rank upto three. We found that there is a simple relation between negativity and teleportation fidelity for pure states but for mixed states, an upper bound was obtained for negativity in terms of teleportation fidelity using convex-roof extension negativity (CREN). The existence of a strong conjecture in the literature concerning all PPT entangled states, in $3 \otimes 3$ systems, having Schmidt rank two, motivated us to develop measures capable of identifying states useful for teleportation and dependent on the Schmidt number. This enabled a classification of entanglement as a function of teleportation fidelity, the "Entanglement of Teleportation". These results were then extended to mixed two qudit states, which we illustrated on specific examples of a two qutrit mixed state with Schmidt rank two. This work thus brings into focus the utility of studying higher dimensional entangled states using measures like "Entanglement of Teleportation" along with negativity.

Chapter 5

Teleportation and super dense coding in RED

5.1 Introduction

In classical information theory we have seen networking plays a key role in sending information from one node to a distant node. In quantum networking one of the most challenging problem is to know how much classical and quantum information one can send from one node to a distant node which are not initially entangled. In this work we claim to provide a solution to this problem by finding out the teleportation fidelity and super dense coding capacity of the remotely prepared state in terms of teleportation fidelities and super dense coding capacities of the resource states. In particular, we find out these relations in the context of the entanglement distribution between distant nodes by the standard swapping of the entangled resource states [215]. But before that we find such relations involving the amount of entanglement of the resource states with the final state in terms of two different measures of entanglement namely concurrence [50,55,62,216] and entanglement entropy. Then by using these relations we establish the relations involving the teleportation fidelity and super dense coding capacity of the entangled channels that can be produced with remote entanglement distribution (RED) protocols [72].

The organization of the chapter is as follows. In section 5.2, we start with two pure entangled resource states shared by three parties and obtain the relations involving the concurrences of the resource states with the final state obtained in the process of RED by swapping. In section 5.3, we extend our results where we have more than two resource states and three parties. In section 5.4, we provide strong results connecting the teleportation fidelity and super dense coding capability of the resource states with the state engineered by the process of entanglement swapping. Finally we conclude in section 5.5.

5.2 Study of enatnglement (concurence) RED

In this section, we consider the most simplest situation where Alice and Bob share a pure entangled state $|\psi\rangle_{12}$ between them. Similarly, Bob and Charlie also share another entangled state $|\psi\rangle_{23}$ between them. This is equivalent of saying, we have entanglement between the nodes 1 and 2 as well as between the nodes 2 and 3. Our aim is to establish the entanglement between the remote nodes 1 and 3 which are not initially entangled. We adopt the procedure of entanglement swapping to carry out the remote entanglement distribution between the nodes 1 and 3. In order to swap the entanglement, Bob carries out measurement on his qubits which are at the node 2. Interestingly, we find an important relationship between the concurrences of the entangled states before and after swapping. The most remarkable aspect of this relationship is that this tells us about the amount of entanglement that can be created in a remote entanglement distribution (RED) via swapping.

5.2.1 For two qubit pure states

In this subsection we start with two entangled resource states in $2 \otimes 2$ dimensions. These states are given by $|\psi\rangle_{12} = \sum_{i,j} a_{ij}|ij\rangle$ and $|\psi\rangle_{23} = \sum_{p,q} b_{pq}|pq\rangle$ respectively. Here, $a_{ij}, b_{pq} \in \mathcal{C}$ $(i,j,p,q = 0,1)$ are the probability amplitudes satisfying the normalization conditions $\sum_{ij} a_{ij}^2 = 1$ and $\sum_{pq} b_{pq}^2 = 1$. We consider a situation, where we take into account a general measurement strategy. Here, Bob carries out measurement in a non-maximally Bell-type entangled basis given by the basis vectors, $|\phi_G^{rh}\rangle = \frac{1}{\sqrt{B_{rh}}} \sum_{t=0}^{1} e^{\pi Irt} R_t^{rh} |t\rangle|t \oplus h\rangle$, where $B_{rh} = \sum_t (R_t^{rh})^2$ and the coefficients R_j^{rh} are defined as

$$R_j^{rh} = \begin{cases} n & \text{if } (r,h,j) = (0,0,1) \text{ or } (1,0,0), \\ m & \text{if } (r,h,j) = (0,1,1) \text{ or } (1,1,0), \\ 1 & \text{otherwise.} \end{cases} \tag{5.1}$$

Here the indices $n, m (\in \mathcal{C})$ are the entangling parameters and $0 \le (n,m) \le 1$. And $t \oplus h$ means the sum of t and h modulo 2. Now according to general measurements done by Bob on his qubits, we have four possible states between the nodes 1 and 3 at Alice and Charlie's locations respectively. These four possible states based on Bob's measurement outcomes $|\phi_G^{rh}\rangle$ $(r,h = 0,1)$ are given by,

$$|\chi^{rh}\rangle_{13} = \frac{1}{\sqrt{M_{rh}}} \sum_{i,q=0}^{1} \Big(\sum_{j=0}^{1} e^{-I\pi rj} R_j^{rh} a_{ij} b_{j\oplus h,q}\Big)|iq\rangle_{13}. \tag{5.2}$$

The modulo sum $j \oplus h$ represents the sum of j and h modulo 2 and the normalization factors are given by $M_{rh} = \sum_{i,q=0}^{1} (\sum_{j=0}^{1} e^{-I\pi rj} R_j^{rh} a_{ij} b_{j\oplus h,q})^2$. Interestingly, here we obtain an important relation between the concurrences of the initial and final states,

$$C(|\chi^{rh}\rangle_{13}) = \frac{F_{rh}}{2M_{rh}} C(|\psi\rangle_{12}) C(|\psi\rangle_{23}), \tag{5.3}$$

where the coefficients F_{rh} are given by,

$$F_{rh} = \begin{cases} n & \text{if } (r,h) = (0,0) \text{ or } (1,0), \\ m & \text{if } (r,h) = (0,1) \text{ or } (1,1). \end{cases} \tag{5.4}$$

This relation (5.3) shows that we can always determine the amount of entanglement to be created between the unentangled nodes depending upon the choice of the resource states.

5.2.2 For two qudit pure states

In this subsection we extend our result to the situation where we have entangled states in $d \otimes d$ dimension instead of states in $2 \otimes 2$ dimension. If we know the state properly then we can always rewrite it in the Schmidt decomposed form. If we have a pure two-qudit state in the form $|\psi\rangle = \sum_{i,j=0}^{d-1} a_{ij}|ij\rangle$ where $\sum_{i,j=0}^{d-1} a_{ij}^2 = 1$, then the Schmidt decomposition form for this state will be $|\psi\rangle = \sum_{\tilde{i}=0}^{d-1} \lambda_{\tilde{i}}|\tilde{i}\tilde{i}\rangle$, where $\sum_{\tilde{i}=0}^{d-1} \lambda_{\tilde{i}}^2 = 1$ and $\lambda_{\tilde{i}}$ are real and non-negative, and $\{|\tilde{i}\rangle\}$ is an orthonormal basis of the corresponding Hilbert space. The concurrence for two-qudit state $|\psi\rangle$ can be written in the form [50, 62]

$$C(|\psi\rangle) = \sqrt{\frac{2d}{d-1}(\sum_{\tilde{i},\tilde{j}=0(\tilde{i}<\tilde{j})}^{d-1} \lambda_{\tilde{i}}^2 \lambda_{\tilde{i}}^2)}. \tag{5.5}$$

For $d = 2$, this equation reduces to $C = 2 \mid \lambda_0\lambda_1 \mid$.

Let us consider a two-qudit pure state shared by parties Alice and Bob $|\psi\rangle_{12} = \sum_{i=0}^{d-1} \lambda_i|ii\rangle$ and Bob and Charlie shares the pure two-qudit state $|\psi\rangle_{23} = \sum_{j=0}^{d-1} \mu_j|jj\rangle$, where $\sum_{i=0}^{d-1} \lambda_i^2 = 1 = \sum_{j=0}^{d-1} \mu_j^2$. In other words, $|\psi\rangle_{12}$ is the entanglement shared between the nodes 1 and 2, whereas $|\psi\rangle_{23}$ is the entanglement between the nodes 2 and 3. Now Bob carries out Bell measurements on his qudits. These basis vectors on which the Bell measurements are carried out are given by,

$$|\phi^{rh}\rangle = \frac{1}{\sqrt{d}} \sum_{t=0}^{d-1} e^{\frac{2\pi Irt}{d}} |t\rangle|t \oplus h\rangle, \tag{5.6}$$

where $t \oplus h$ means the sum of t and h modulo d. The indices r and h can take integer values between 0 and $d-1$. We can revert the above equation to obtain

$$|ij\rangle = \frac{1}{\sqrt{d}} \sum_{r,h=0}^{d-1} e^{\frac{-2\pi Ijr}{d}} \delta_{i,i\oplus h}|\phi^{rh}\rangle. \tag{5.7}$$

Hence, the combined state of Alice, Bob and Charlie is

$$\begin{aligned}
|\Phi\rangle_{1223} &= |\psi\rangle_{12} \otimes |\psi\rangle_{23} \\
&= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} \lambda_i\mu_j|ij\rangle_{13}|ij\rangle_{22} \\
&= \frac{1}{\sqrt{d}} \sum_{i,j}^{d-1}\sum_{r,h}^{d-1} e^{\frac{-2\pi Irj}{d}} \lambda_i\mu_j|ij\rangle_{13}\delta_{j,i\oplus h}|\phi^{rh}\rangle_{22} \\
&= \frac{1}{\sqrt{d}} \sum_{i,r,h}^{d-1} e^{\frac{-2\pi Irj}{d}} \lambda_i\mu_{i\oplus h}|i, i \oplus h\rangle_{13}|\phi^{rh}\rangle_{22}.
\end{aligned} \tag{5.8}$$

According to measurement outcomes $|\phi^{rh}\rangle_{22}$ on Bob's side, the states created between the nodes 1 and 3 are given by

$$|\chi^{rh}\rangle_{13} = \frac{1}{\sqrt{N_{rh}}}\sum_{i=0}^{d-1} e^{\frac{-2\pi Irj}{d}}\lambda_i\mu_{i\oplus h}|i, i\oplus h\rangle_{13} \tag{5.9}$$

where $N_{rh} = \sum_{i=0}^{d-1}\lambda_i^2\mu_{i\oplus h}^2$ are normalization factors. We can construct unitary operators of the form $U_{st} = \sum_{r=0}^{d-1} e^{\frac{2\pi isr}{d}}|r\rangle\langle r\oplus t|$ which can transform states in Eq.(5.9) into its diagonal form. Hence, the concurrence of the final two-qudit state is given by,

$$C(|\chi^{rh}\rangle_{13}) = \frac{1}{N_{rh}}\sqrt{\frac{2d}{d-1}(\sum_{i<f}^{d-1}(\lambda_i^2\lambda_f^2)(\mu_{i\oplus h}^2\mu_{f\oplus h}^2))}. \tag{5.10}$$

To understand the terms in Eq.(5.10) we have to split it in the following way

$$\sum_{i<f}^{d-1}(\lambda_i^2\lambda_f^2)(\mu_{i\oplus h}^2\mu_{f\oplus h}^2) = \sum_{i<f}^{d-1}\lambda_i^2\lambda_f^2\sum_{i<f}^{d-1}\mu_{i\oplus h}^2\mu_{f\oplus h}^2 - \sum_{i<f}^{d-1}(\lambda_i^2\lambda_f^2\sum_{l<m}^{d-1}\Theta_{lm}^{if}\mu_{l\oplus h}^2\mu_{m\oplus h}^2), \tag{5.11}$$

where function Θ_{lm}^{if} is defined as

$$\Theta_{lm}^{if} = \begin{cases} 1 & \text{if } (l,m) \neq (i,f) \text{ for } d \geq 3, \\ 0 & \text{if } (l,m) = (i,f) \text{ or } d \leq 2. \end{cases} \tag{5.12}$$

It is evident that in case of $d\otimes d$ dimensions, we have no direct relationship as we have obtained in the multiqubit case. However we consider a special situation where we have only two non-vanishing Schmidt coefficients, then we have the concurrence of the state $|\psi\rangle$ as

$$C_{ij}(|\psi\rangle) = \sqrt{\frac{2d}{d-1}}(\lambda_i\lambda_j). \tag{5.13}$$

The C_{ij} are the concurrences of $|\psi\rangle$, when two of the Schmidt coefficients are present only. Then we have the relation with the concurrences of the initial and final entangled states as

$$C^2(|\chi^{rh}\rangle_{13}) = \frac{(d-1)}{2dN_{rh}^2}[C^2(|\psi\rangle_{12})C^2(|\psi\rangle_{23}) - K_d^h], \tag{5.14}$$

where $K_d^h = \sum_{i<f}^{d-1}(C_{if}^2(|\psi\rangle_{12})\sum_{l<m}^{d-1}\Theta_{lm}^{if}C_{l\oplus h,m\oplus h}^2(|\psi\rangle_{23}))$ is a term that depends on dimension d and $K_2^q = 0$ only when $d = 2$. Hence for $d = 2$, Eq.(5.14) becomes

$$C(|\chi^{pq}\rangle_{13}) = \frac{1}{2N_{rh}}C(|\psi\rangle_{12})C(|\psi\rangle_{23}). \tag{5.15}$$

This relation involving the concurrences also reflects that the amount of entanglement that can be created between the remote nodes is solely a function of the amount of entanglement of the resource states.

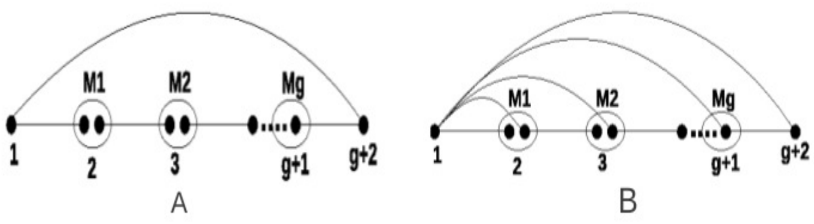


Figure 5.1: In this Figure entanglement swapping is done with simultaneous measurements (A) and sequential measurements (B). Here g number of measurements $M1, M2, ..., Mg$ are carried out simultaneously (A) and sequentially (B) at the nodes $(2, 3, 4, ..., g+1)$ to obtain an entangled state between the first and last node (i.e., $1, g+2$)).

5.3 Study of concurrence in RED: Multiparty scenario

Let us assume that we have $(g+1)$ entangled states in $2 \otimes 2$ dimensions with $g+2$ nodes. In order to obtain an entangled state between 1st and last node we carry out g number of entanglement swappings. We separately study two different types of measurement strategies in the entire swapping procedure. First of all we consider the case where we carry out simultaneous measurement in a non maximally entangled basis in each of these intermediate nodes to obtain an entangled state between the qubits in the first and the last node. Secondly, we consider sequential measurements to create successive entanglements between the nodes $(1,3)$, $(1,4)$ and finally between the nodes $(1, g+2)$. In each of these cases we obtain the extension of the relationships involving the concurrences of initial and final entangled states.

5.3.1 Simultaneous and sequential measurement

In this subsection we start with $(g+1)$ entangled states in the most general form, $|\psi\rangle = \sum_{i_k, j_k=0}^{1} a_{i_k j_k} |i_k j_k\rangle$, where k denotes the index for the number of entangled states and varies from 0 to g. Here for a fixed k, $a_{i_k j_k}$ denotes the corresponding coefficients of the given entangled state. Then we create an entangled state between the qubits at the nodes 1 and $g+2$ by entanglement swapping. In other words we carry out simultaneous measurements $M1, M2, ..., Mg$ at the nodes $2, 3, 4, ..., g+1$ respectively to obtain an entangled state between the qubits at the nodes 1 and $g+2$ or we carry out measurements $M1, M2, ..., Mg$ one after the other to create successive entanglement between the pair of nodes $(1,3), (1,4), ..., (1, g+2)$ respectively [see Fig.(5.1)]. After evaluating the concurrences for the initial states and final state we find them to be related by,

$$C(|\chi^{r_1 h_1, ..., r_g h_g}\rangle_{1(g+2)}) = \frac{\Pi_{i=1}^{g} F_{r_i h_i}}{2^g M_{r_1 h_1, ..., r_g h_g}} C(|\psi\rangle_{12}) C(|\psi\rangle_{23}) \cdots C(|\psi\rangle_{(g+1)(g+2)}). \quad (5.16)$$

Here the indices r and h take the values 0 and 1 and the subscript $i(=1,2,...,g)$ denotes the number of measurements that have taken place. The normalization factors are given by $M_{r_1h_1,...,r_gh_g} = \sum_{i_0,j_g=0}^{1}(\sum_{j_1,j_2,...,j_{g-1}=0}^{1} e^{-I\pi r_1 j_0} e^{-I\pi r_2 j_1}....e^{-I\pi r_g j_{g-1}} R_{j_0}^{r_1h_1} R_{j_1}^{r_2h_2} ...R_{j_{g-1}}^{r_gh_g} a_{i_0j_0} a_{j_0\oplus h_1 j_1} a_{j_1\oplus h_2 j_2}...a_{j_{g-1}\oplus h_g j_g})^2$. The superscripts $r_i, h_i \in [0,1], i=1,2,...,g$ comes from the measurement of g parties (i.e., if their measurement results are $|\phi^{r_1h_1}\rangle \otimes |\phi^{r_2h_2}\rangle \otimes .... \otimes |\phi^{r_gh_g}\rangle = \otimes_{i=1}^{g}|\phi^{r_ih_i}\rangle$). The resultant states obtained after swapping are given by,

$$\begin{aligned}|\chi^{r_1h_1,...,r_gh_g}\rangle_{1(g+2)} &= \frac{1}{\sqrt{M_{r_1h_1,...,r_gh_g}}}\sum_{i_0,j_g}(\sum_{j_1,j_2,...,j_{g-1}} e^{-I\pi r_1 j_0} e^{-I\pi r_2 j_1}...e^{-I\pi r_g j_{g-1}} R_{j_0}^{r_1h_1} \\ &\quad R_{j_1}^{r_2h_2}...R_{j_{g-1}}^{r_gh_g} a_{i_0j_0} a_{j_0\oplus h_1 j_1} a_{j_1\oplus h_2 j_2}...a_{j_{g-1}\oplus h_g j_g})|i_0, j_g\rangle_{1(g+2)}.\end{aligned}$$

The coefficients, $F_{r_ih_i}$ and $R_{j_i}^{r_ih_i}$ are defined as

$$F_{r_ih_i} = \begin{cases} n & \text{if } (r_i,h_i)=(0,0) \text{ or } (1,0), \\ m & \text{if } (r_i,h_i)=(0,1) \text{ or } (1,1), \end{cases} \tag{5.17}$$

$$R_{j_i}^{r_ih_i} = \begin{cases} n & \text{if } (r_i,h_i,j_i)=(0,0,1) \text{ or } (1,0,0), \\ m & \text{if } (r_i,h_i,j_i)=(0,1,1) \text{ or } (1,1,0), \\ 1 & \text{otherwise.} \end{cases} \tag{5.18}$$

5.4 Teleportation fidelity and superdense coding capacity in RED

In this section, we have obtained the relations for teleportation fidelity and super dense coding capacity of a remotely prepared entangled states with that of the resource states. These relations are very much important and relevant in the context of quantum networking. A quantum network, is a collection of nodes interconnected by entangled states that allow sharing of resources and information. Here we ask the question that in an quantum network what is the amount of quantum information and classical information one can send between the initial and final nodes. We find that indeed there are certain relations, which determine the amount of information one can send from the initial and final node after creating an entangled state between the initial and final node through the process of remote entanglement distribution (RED). In particular, we showed that how the teleportation capability of a remotely prepared state is linked up with the fidelity of teleportation of the initial resource states. Similarly, we analyzed the super dense coding capacity of the remotely prepared state in terms of the capacity of the initial entangled states. In other words, these analysis both in the case of teleportation and super dense coding shows that the amount of information both quantum and classical one can send between two unentangled nodes is dependent on the choice of resource states. These results may be useful in determining the path in an arbitrary quantum network through which we can send maximal possible quantum information between any two unentangled nodes.

5.4.1 Study on teleportation fidelity in RED

Let us begin with very simplistic situation where there are two parties Alice, Bob share an entangled state $|\psi\rangle_{12} = \sum_{i,j=0}^{1} a_{ij}|ij\rangle$ between them, where as Bob and Charlie share another state $|\psi\rangle_{23} = \sum_{p,q=0}^{1} b_{pq}|pq\rangle$. This is equivalent of saying that we have considered the parties and nodes to be synonymous, then $|\psi\rangle_{12}$ and $|\psi\rangle_{23}$ are the respective entangled states between the nodes $(1,2)$ and $(2,3)$. We then propose the following theorem connecting the teleportation capability of the resource states with that of the entangled state obtained between the non connected nodes as a result of swapping.

Theorem 5.1 *For the initial resource states written in the form* $|\psi\rangle_{12} = \sum_{i,j=0}^{1} a_{ij}|ij\rangle$ *and* $|\psi\rangle_{23} = \sum_{p,q=0}^{1} b_{pq}|pq\rangle$*, the teleportation fidelities of the initial states and final state* $|\chi^{rh}\rangle_{13}$ *obtained after the measurement in the general basis* $|\phi_G^{rh}\rangle$*, are related by,* $3F(|\chi^{rh}\rangle_{13}) - 2 = \frac{F_{pq}}{2M_{pq}}[3F(|\psi\rangle_{12}) - 2][3F(|\psi\rangle_{23}) - 2]$*, where* F_{pq} *is a function of the measurement parameters,* M_{pq} *are the normalization constants and* (r,h) *are the indices to denote the measurement outcomes.*

Proof: We can write fidelities of initial resource states and final remotely prepared state as $F(|\psi\rangle_{12}) = \frac{1}{3}(2 + C(|\psi\rangle_{12}))$, $F(|\psi\rangle_{23}) = \frac{1}{3}(2 + C(|\psi\rangle_{23}))$ and $F(|\chi^{rh}\rangle_{13}) = \frac{1}{3}(2 + C(|\chi^{rh}\rangle_{13}))$ respectively, then just by substituting the values of concurrences in terms of teleportation fidelities in Eq.(5.3) one can have the relation concerning teleportation fidelities of the initial resource states with the final remotely prepared state.
This gives the more generalized version of the expression relating the teleportation fidelities of thee initial resource states with the final remotely prepared states.

Then we consider a complicated situation where we have $(g+1)$ entangled states distributed among hypothetical parties in $g+2$ nodes. These entangled states are shared between consecutive nodes. We consider two types of measurement namely simultaneous and consecutive measurements $M1, M2, ..., Mg$ at g number of nodes [see Fig.(5.1)]. As we have seen in the previous section that both of these measurements create entanglement between the first and final node. Here we prove a theorem, quite analogous to previous theorems relating the teleportation capability of the resource states with the final state obtained as a result of swapping in the process of remote entanglement distribution (RED).

Theorem 5.2 *If we start with* $(g+1)$ *entangled states in the most general form,* $|\psi\rangle_{12}$*,* $|\psi\rangle_{23}$*, ...,* $|\psi\rangle_{(g+1)(g+2)}$*, between the nodes* $(1,2), (2,3), ...(g+1, g+2)$ *with respective teleportation fidelities* $F(|\psi\rangle_{12}), F(|\psi\rangle_{23}), ..., F(|\psi\rangle_{(g+1)(g+2)})$*, then the teleportation fidelity of the state* $|\chi^{r_1h_1,...,r_gh_g}\rangle_{1(g+2)}$ *is given by*

$$\begin{aligned} 3F(|\chi^{r_1h_1,...,r_gh_g}\rangle_{1,(g+2)}) - 2 \ &= \ \frac{\prod_{i=1}^{g} F_{r_ih_i}}{2^g M_{r_1h_1,...,r_gh_g}} [3F(|\psi\rangle_{12}) - 2]\,[3F(|\psi\rangle_{23}) - 2] \\ &\quad ... \left[3F(|\psi\rangle_{(g+1)(g+2)}) - 2\right] \end{aligned} \tag{5.19}$$

Proof: Just by substituting the values of the concurrences in terms of the teleportation fidelities in the Eq.(5.16) we finally obtain the relation involving the teleportation fidelities of the initial resource states with the final remotely prepared state.

5.4.2 Study on superdense coding capacity in RED

It is quite well known that if we have a maximally entangled state in $H_d \otimes H_d$ as our resource, then we can send $2\log_2 d$ bits of classical information. In the asymptotic case, we know one can send $\log_2 d + S(\rho)$ amount of bit when one considers non-maximally entangled state as resource [176]. It had been seen that the number of classical bits one can transmit using a non-maximally entangled state in $H_d \otimes H_d$ as a resource is $(1 + \lambda_0 \frac{d}{d-1}) \log_2 d$, where λ_0 is the smallest Schmidt coefficient. However, when the state is maximally entangled in its subspace then one can send up to $2\log_2(d-1)$ bits [217]. In particular, super dense coding capacity (see Eq.(1.90)) for pure states, is given by,

$$\mathcal{C}(\rho_{AB}) = \log_2 d + S(\rho_B) = \log_2 d + E(\rho_{AB}), \tag{5.20}$$

where, $E(\rho_{AB})$ is the entanglement entropy of the pure state ρ_{AB}.

In this subsection we find how the super dense coding capacities of the resource states are related with the super dense coding capacity of the entangled state obtained as a result of entanglement swapping. Here, we consider only the simplest situation where we have two resource states at our disposal and we want to send classical information from one node to another which are not initially entangled. Let us once again begin with a situation where two parties Alice, Bob sharing an entangled state $|\psi\rangle_{12} = \sum_i \lambda_i |ii\rangle$ between them, where as Bob and Charlie share another state $|\psi\rangle_{23} = \sum_j \mu_j |jj\rangle$ (where λ_i, μ_j, $(i, j = 0, 1, ..., d)$ are the Schmidt coefficients, satisfying $\sum_i \lambda_i^2 = 1, \sum_j \mu_j^2 = 1$) with each other. Then Bob carries out the Bell state measurement on his qubits at the node 2 and according to measurement outcomes $|\phi^{rh}\rangle_{22}$ on Bob's side, the resultant entangled pairs generated between the nodes 1 and 3 are $|\chi^{rh}\rangle_{13}$ (given in Eq.(5.9)). The entanglement entropy of these states are given by,

$$E(|\chi^{rh}\rangle_{13}) = -\frac{1}{N_{rh}} \sum_i \lambda_i^2 \mu_{i\oplus h}^2 \log_2 \left[\frac{\lambda_i^2 \mu_{i\oplus h}^2}{N_{rh}} \right]. \tag{5.21}$$

Then there arises three situations depending upon the choice of the Schmidt coefficients of the resource states.

Case I: First of all we consider the case when both the resource states are maximally entangled i.e., when all the Schmidt coefficients are equal to $\frac{1}{\sqrt{d}}$. Then the super dense coding capacity of the resource state is related with the super dense coding capacity of the remotely prepared entangled states $|\chi^{rh}\rangle_{13}$ (where r, h are the indices indicating the measurement outcomes) as, $\mathcal{C}(|\chi^{rh}\rangle_{13}) = \mathcal{C}(|\psi\rangle_{12}) = \mathcal{C}(|\psi\rangle_{23}) = \mathcal{C}$(say). Hence if we have a network consisting of $g+1$ number of maximally entangled states then the super dense coding capacity of final state between the nodes 1 and $g+2$ [see Fig.(5.1)] will be,

$$\mathcal{C}(|\chi^{rh}\rangle_{1,(g+2)}) = \mathcal{C}, \tag{5.22}$$

where, $\mathcal{C} = \mathcal{C}(|\psi\rangle_{12}) = \mathcal{C}(|\psi\rangle_{23}) = ... = \mathcal{C}(|\psi\rangle_{(g+1),(g+2)})$.

Case II: In this particular case we consider the situation when one of the entangled state is maximally entangled and the rest is non maximally entangled i.e $\lambda_i = \frac{1}{\sqrt{d}}$, $\mu_j \neq \frac{1}{\sqrt{d}}$, then we have, $\mathcal{C}(|\chi^{rh}\rangle_{13}) = \mathcal{C}(|\psi\rangle_{23}) = \mathcal{C}_2$(say) and if $\lambda_i \neq \frac{1}{\sqrt{d}}$, $\mu_j = \frac{1}{\sqrt{d}}$, then we have, $\mathcal{C}(|\chi^{rh}\rangle_{13}) = \mathcal{C}(|\psi\rangle_{12}) = \mathcal{C}_1$(say). Now if we consider a network consisting of $g+1$ number of entangled bipartite qudit states out of which n number of states are non-maximally entangled and $g+1-n$ number of states are maximally entangled then for the strategies in Fig.(5.1), the super dense coding capacity of final state between the nodes 1 and $g+2$ will be,

$$\mathcal{C}(|\chi^{rh}\rangle_{1,(g+2)}) < \mathcal{C}_p^{\max}, \tag{5.23}$$

where, $\mathcal{C}_p^{\max}$ is the maximum out of n number of super dense coding capacities [$\mathcal{C}_p$; $p = 1, 2, 3, ..., n$] of non-maximally entangled pure two-qudit resource states.

Case III: Finally, we consider the case when both the entangled states are not maximally entangled i.e $\lambda_i \neq \frac{1}{\sqrt{d}}$, $\mu_j \neq \frac{1}{\sqrt{d}}$, then the super dense coding capacity of the swapped state is given by, $\mathcal{C}(|\chi^{rh}\rangle_{13}) < \max[\mathcal{C}_1, \mathcal{C}_2]$. And hence easily we can write for a network consisting of $g+1$ number of non-maximally entangled pure two-qudit states, the super dense coding capacity of the final state (as a result of the strategies in Fig.(5.1)) between the nodes 1 and $g+2$ will be,

$$\mathcal{C}(|\chi^{rh}\rangle_{1,(g+2)}) < \mathcal{C}_i^{\max}, \tag{5.24}$$

where, $\mathcal{C}_i^{\max}$ is the maximum out of $g+1$ number of super dense coding capacities [$\mathcal{C}_i$; $i = 1, 2, 3, ..., (g+1)$] of non-maximally entangled pure two-qudit resource states.

5.5 Conclusion

In a nutshell, here in this chapter, we established an important relationship connecting the fidelities of teleportation of the resource states with the fidelity of the final state obtained as a result of entanglement swapping. Similarly, we also connected the super dense coding capacities of the resource states with that of the final state. All these relations are very much important and relevant in the context of quantum networking. These relations actually determine the amount of information both classical and quantum, one can send from one node to a desired node in a quantum network. In other words, in an arbitrary network when two nodes are not connected, our result shows how much information both quantum and classical can be sent from one node to other. In fact the amount of transferable information depends on the capacities of the inter connecting entangled resources. Depending upon the inter connecting entangled resources, we can choose the optimal path in a quantum network to send the maximal possible information.

Chapter 6

Correlations generated in cloning and deletion

6.1 Introduction

In quantum information theory the no-cloning theorem plays a fundamental role [34,218, 219]. This theorem states how nature prevents us from amplifying an unknown quantum state. However, in principle it is always possible to construct a quantum cloning machine that replicates an unknown quantum state approximately [34, 195, 220–232]. These approximate quantum cloning machines can be of two types. One is a state-dependent quantum cloning machine, for example, the Wootters-Zurek (WZ) quantum cloning machine, whose copying quality depends on the input state [34,197,220,232]. The other type is a universal quantum copying machine, for example, the Buzek-Hillery (BH) quantum cloning machine [195], whose copying quality remains the same for all input states. In addition, the performance of the universal BH quantum cloning machine is, on the average, better than that of the state-dependent WZ cloning machine. The fidelity of cloning of the BH universal quantum copying machine is $\frac{5}{6}$ - the optimal fidelity for the universal quantum cloning machines [220, 233]. Although it is impossible to copy a state perfectly, one can probabilistically clone a quantum state, secretly chosen from a certain set of linearly independent states [229, 234]. Also, it is possible to have linear superposition of multiple clones and obtain a probabilistic cloning machine as a special case of the former [235]. Quantum deletion [182] on the other hand, is about the impossibility of deleting an arbitrary quantum state. More specifically, it states that the linearity of quantum theory precludes deleting an unknown quantum state from two identical copies in either a reversible or an irreversible manner. The principle behind quantum deletion will be clearer, if we compare the deletion operation with the Landauer erasure operation [236]. Erasure of classical or quantum information cannot be performed reversibly. The erasure principle says that a single copy of some classical information can be erased at the cost of some energy. Thermodynamically, it is an irreversible operation. In quantum theory the erasure of a single unknown state is considered as swapping it with some standard state and then trashing it into the environment. In contrast, quantum deletion [182] is more of reversible 'uncopying' of an unknown quantum state. It has been shown that in addition to the linear structure of quantum mechanics, other principles like unitarity, nosignalling, incomparability and conservation of entanglement are not congruous to the concept of perfect deletion [69, 237–240]. However, if one tries to delete an unknown quantum state probabilistically, then it is possible with a success probability of less than unity [241]. It has also been shown that using these probabilistic deletion machines one cannot send superluminal signals probabilistically [242]. Since perfect deletion is not possible, it is interesting to see whether one can delete an unknown state imperfectly. Researchers have devised various approximate deletion machines. These deletion machines are either state dependent or state independent [185, 243–247]. Recent explorations have revealed that one can construct a universal quantum deletion machine [244], and its fidelity can be further enhanced by the application of suitable unitary transformation [245]. These deletion machines can have various applications in quantum information theory [248–250]. However, the optimal quantum deletion machine has not been found yet.

At this point one might ask an important question whether quantum correlations are responsible for our inability to produce high fidelity states in the approximate cloning or deleting a quantum state? Note that initially there are no correlations between the input states. This is because they are the individual systems which are in a product state. However, at the output port we always obtain a combined state, which is usually correlated. A priori, it is not clear whether this correlations play an important role in deciding the fidelity of cloning and in deleting an unknown quantum state. In order to find an answer to this question, we consider a particular type of cloning machine, the BH cloning machine, and try to quantify the amount of correlations present in the mixed two qubit output state. Similarly, for the deletion operation we consider a state-dependent quantum deleting machine to find out the correlations in the output modes. The basic motivation is to see how the correlations regulates the fidelity of the cloning and deletion processes. We find that the more the output modes are correlated the less is the fidelity in either cases. In other words, the process of cloning and deletion will be more perfect if the output modes are poorly correlated, i.e., the correlations generated in the processes of cloning and deletion behave in complimentary way [184].

The problem of complementarity or mutually exclusive aspects of quantum phenomena arose with the birth of quantum mechanics, soon after, Heisenberg discovered the uncertainty principle for the momentum and the position [251–253]. A year later, Bohr proposed the concept of complementarity [254, 255]. Even in the domain of quantum information theory, the idea of complementarity is not new, as some authors have shown that there does exist the complementarity between the local and nonlocal information of quantum systems [128]. In this work we observe a new kind of complementarity in terms of successive correlations generated in the system when a state undergoes deletion after the cloning or the cloning after the deletion.

We quantify the correlations in the cloning and deleting processes with three different kind of measures to make this observation more precise. These measures are (i) negativity [54], (ii) quantum discord [100, 116, 256] and (iii) geometric discord [127]. Each of them represents three different classes of measures. We would like to see how generic the complementarity is for cloning and deleting if we use different measures of quantum correlations.

The chapter is organized in the following manner. In section 6.2, we provide a short introduction to the geometric discord. In section 6.3, we analyse the correlations content of the output of the Buzek-Hillery quantum cloning machine. We also analyse how the correlations content of the output modes plays a pivotal role in determining the fidelity of cloning. In section 6.4, we study the standard approximate deleting machine to obtain a correspondence between the fidelity of deletion and the amount of correlations generated in the process. In section 6.5, we obtain a new kind of complementarity relation between the correlations generated in the system for the process of successive cloning and deletion, and also for the case when we clone the state after deletion. This complementarity gives a new bounds to the total correlations generated in the context of quantum correlation measures. Finally, we conclude in section 6.6.

6.2 Quantum correlations beyond entanglement: Geometric discord

It had been argued that the difficulty experienced in calculating quantum discord can be minimized, for a general two-qubit state, by defining its geometrical version [127]. Distance-based discord is defined as the minimal distance between a quantum state and all other states with zero discord [127, 257, 258]. It is similar to the geometric measure of quantum entanglement [82]. It is well known that almost all (entangled or separable) states are disturbed by the measurement. However, there are certain states which are invariant under the measurement performed on the sub-system A. These states are the so called classical-quantum (CQ) states. A CQ density matrix is of the form

$$\rho = \sum_i p_i |i\rangle\langle i| \otimes \rho_i, \tag{6.1}$$

where p_i is a probability distribution, $\{|i\rangle\}$ is an orthonormal set of vectors for A and ρ_i are the elements of B. A classical-quantum state is not affected by a measurement on A in any case. One can show that the state ρ is of zero-discord if and only if there exists a von Neumann measurement $\{\Pi_k = |\psi_k\rangle\langle\psi_k|\}$ such that [166]

$$\sum_k (\Pi_k \otimes \mathbb{I}_B)\rho(\Pi_k \otimes \mathbb{I}_B) = \rho. \tag{6.2}$$

It had been seen in Ref. [127], that these two states in Eq.(6.1) and (6.2) are identical. Let S be the set consisting of all classical–quantum two qubit states, and let us assume that χ is a generic element of this set. Then the geometric discord DG of an arbitrary two-qubit state ρ_{AB} is given by the distance between the state ρ_{AB} and the closest classical-quantum state. Geometric discord has been introduced as

$$D_G(\rho_{AB}) = 2 \min_{\chi \in S} ||\rho_{AB} - \chi||_2^2, \tag{6.3}$$

where the coefficient 2 on the right hand side is the normalization factor and $||X - Y||_2 = \mathrm{Tr}(X - Y)^2$ is the square norm in the Hilbert-Schmidt space. For the geometric discord of the state ρ_{AB} to have a nice closed form, one needs to express the state in terms of the Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$ as $\rho_{AB} = \frac{1}{4}(\mathbb{I}_4 + \sum_{i=1}^{3} x_i\sigma_i \otimes \mathbb{I}_2 + \sum_{j=1}^{3} y_j \mathbb{I}_2 \otimes \sigma_j + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j)$, where $t_{ij} = \mathrm{Tr}[\rho(\sigma_i \otimes \sigma_j)]$, $\mathbb{I}_n$ is the identity matrix of order n, $\vec{x} = \{x_i\}, \vec{y} = \{y_i\}$ represent the three-dimensional Bloch column vectors and $t = [t_{ij}]$ is the correlation matrix. Then, we can rewrite the geometric discord as [259]

$$D_G(\rho_{AB}) = \frac{1}{2}(\|\vec{x}\|_2^2 + \|t\|_2^2 - 4k_{\max}) = 2\mathrm{Tr}[S] - 2k_{\max}, \tag{6.4}$$

with $k_{\max}$ being the largest eigenvalue of the matrix $S = \frac{1}{4}(\vec{x}\vec{x}^{\mathsf{T}} + tt^{\mathsf{T}})$ where 'T' denotes transposition. There are other approaches to define the geometric discord, however we focus only on the above presented one.

6.3 Analysis of the correlations content of the output of Buzek-Hillery copying machine

In this section we consider the universal Buzek-Hillery cloning machine and quantify the correlations present in the output copies of the Buzek-Hillery cloning machine [195]. But before that we give a short description of the Buzek-Hillery cloning machine. We recall that the action of the Buzek-Hillery quantum cloning machine [195] is given by the transformations

$$\begin{aligned} |0\rangle_a|0\rangle_b|Q\rangle_x &\longrightarrow |00\rangle_{ab}|Q_0\rangle_x + [|01\rangle_{ab} + |10\rangle_{ab}]|Y_0\rangle_x, \\ |1\rangle_a|0\rangle_b|Q\rangle_x &\longrightarrow |11\rangle_{ab}|Q_1\rangle_x + [|01\rangle_{ab} + |10\rangle_{ab}]|Y_1\rangle_x, \end{aligned} \tag{6.5}$$

where a, b and x denote qubits corresponding to input state port, blank state port and the machine state port. The unitarity and the orthogonality of the cloning transformation demand the following conditions to be satisfied:

$$\langle Q_i|Q_i\rangle_x + 2\langle Y_i|Y_i\rangle_x = 1 \;\;\&\;\; \langle Y_0|Y_1\rangle_x = \langle Y_1|Y_0\rangle_x = 0 \quad (i = 0, 1). \tag{6.6}$$

Here, we assume the machine state vectors $|Y_i\rangle_x$ and $|Q_i\rangle_x$ to be mutually orthogonal. This is also true for the state vectors $\{|Q_0\rangle, |Q_1\rangle\}$.
The unknown quantum state which is to be cloned is given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{6.7}$$

where α, β are complex numbers satisfying, $|\alpha|^2 + |\beta|^2 = 1$. After using the cloning transformation (6.5) on the quantum state and then tracing out the machine state, the reduced density operator describing the two qubit output modes of the original and the cloned state is given by

$$\begin{aligned} \rho_{ab}^{clone} &= (1-2\xi)(\alpha^2|00\rangle_{ab}\langle 00| + \beta^2|11\rangle_{ab}\langle 11|) + \frac{\alpha\beta}{\sqrt{2}}(1-2\xi)(|00\rangle_{ab}\langle\psi^+| \\ &+ |\psi^+\rangle_{ab}\langle 00| + |\psi^+\rangle_{ab}\langle 11| + |11\rangle_{ab}\langle\psi^+|) + 2\xi|\psi^+\rangle_{ab}\langle\psi^+|, \end{aligned} \tag{6.8}$$

where we have used the following notations $\langle Y_0|Y_0\rangle_x = \langle Y_1|Y_1\rangle_x = \xi$, $\langle Y_0|Q_1\rangle_x = \langle Q_0|Y_1\rangle_x = \langle Q_1|Y_0\rangle_x = \langle Y_1|Q_0\rangle_x = \frac{\eta}{2}$, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Here, $\eta = 1 - 2\xi$ with ξ being the machine parameter determining the nature of the cloning transformations. The output state ρ_{ab}^{clone} is of prime importance as we will investigate the amount of

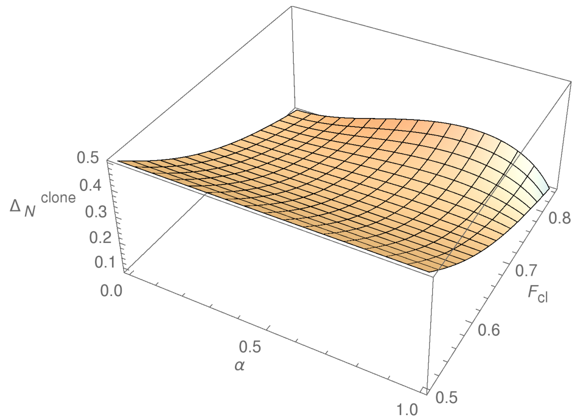


Figure 6.1: The plot shows how the correlation measure negativity (Δ_N^{clone}), vary with the input parameter α and the fidelity of cloning F_{cl}.

correlations present in it. Cloning fidelity is given by the overlap between the real output state ρ_b^{clone} with the desired output state $|\psi\rangle$. It can be seen that that the cloning fidelity $F_{cl} = \mathrm{Tr}[\rho_b^{clone}|\psi\rangle\langle\psi|] = 1-\xi$ is dependent on the machine parameter ξ. It has been shown that the BH cloning machine should satisfy the inequality $\eta \leq 2(\xi - 2\xi^2)^{\frac{1}{2}}$. The relation $\eta = 1-2\xi$ reduces the inequality $\eta \leq 2(\xi-2\xi^2)^{\frac{1}{2}}$ to the inequality $\frac{1}{6} \leq \xi \leq \frac{1}{2}$. Henceforth, we study the different measures of quantum correlations in this range of the machine parameter to see how it behaves with the cloning fidelity. The amount of correlations generated in the process of cloning is given by the difference between the amount of correlations in the output modes and the amount of correlations in the same two modes before the application of cloning operations. We will denote this difference of correlations as $\Delta_K^{clone} = K(\rho_{ab}^{final}) - K(\rho_{ab}^{initial})$, for a correlation measure $K(\rho_{ab})$. Here we compute three different correlation measures, namely (i) negativity (N), (ii) discord (D) and (iii) geometric discord (DG) for both the initial input state ($\rho_{ab}^{initial}$) and the final output state (ρ_{ab}^{final}). Since the Buzek-Hillary cloning machine we start with product states, the respective differences $\Delta_N^{clone}, \Delta_D^{clone}$ and Δ_{DG}^{clone} of correlations are nothing but the amount of correlations $N(\rho_{ab}^{final}), D(\rho_{ab}^{final})$ and $DG(\rho_{ab}^{final})$ in the output modes. Our first motivation is to see how these different measures of correlation behave with the fidelity of cloning. For this purpose, we first express these different measures of the correlation $\Delta_N^{clone}, \Delta_D^{clone}$ and Δ_{DG}^{clone} in terms of the fidelity F_{cl} of cloning. We rewrite these measures as a function of a variable like the fidelity of cloning F_{cl} and input state parameter α. The expression for Δ_N^{clone} is given by

$$\Delta_N^{clone} = \frac{1}{2}\left[2\left\{g_1 + \frac{1}{2}g_2 f_1\right\}^{\frac{1}{2}} + \{g_1 + g_2 f_2\}^{\frac{1}{2}} + \{g_1 + g_2 f_3\}^{\frac{1}{2}} - 1\right], \tag{6.9}$$

where $f_1 = |\alpha|^2\beta^2$, $f_2 = (1 + \frac{1}{2}|\alpha|^2 - \alpha^{*2})\beta^2$, $f_3 = |\alpha|^2(|\alpha|^2 + \frac{1}{2}\beta^2)$ (here, $|.|$ denotes absolute value and $*$ the complex conjugation), $g_1 = (F_{cl}-1)^2$ and $g_2 = (2F_{cl}-1)^2$.

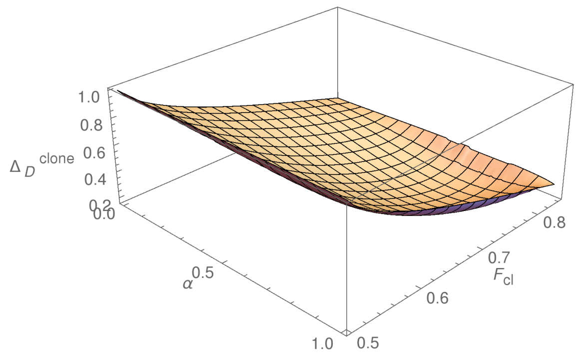


Figure 6.2: The plot shows how the correlation measure discord (Δ_D^{clone}) vary with the input parameter α and the fidelity of cloning F_{cl}.

Similarly, the expression for Δ_D^{clone} is given by

$$\Delta_D^{clone} \quad = \quad H_2(F_{cl}) + mH_2(X_+) - nH_2(Y_+), \tag{6.10}$$

where $H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$, $X_+ = \frac{1}{2}(1+\frac{1}{m}\{1+C_+\}^{\frac{1}{2}})$, $Y_+ = \frac{1}{2}(1+\frac{1}{n}\{4+F_{cl}(7-5F_{cl})+C_-\}^{\frac{1}{2}})$, $C\pm = F_{cl}[-2-10\alpha^2+F_{cl}(1+8\alpha^2)] \pm 3\alpha^2$, $m = n-1$, and $n = \alpha^2 + (1-2\alpha^2)F_{cl}$.

Lastly, the corresponding expression for the geometric discord is given by

$$\Delta_{DG}^{clone} = 2(\lambda + \lambda_+ + \lambda_- - \max[\lambda, \lambda_+, \lambda_-]), \tag{6.11}$$

where $\lambda = (1-F_{cl})^2$, $\lambda_\pm = \frac{1}{2}(3.5 - 9F_{cl} + 6F_{cl}^2 \pm \sqrt{p - \alpha^2\beta^2 q})$, (here $p = 2.25 - 15F_{cl} + 37F_{cl}^2 - 40F_{cl}^3 + 16F_{cl}^4$ and $q = 5 - 36F_{cl} + 96F_{cl}^2 - 112F_{cl}^3 + 48F_{cl}^4$).

To have a better insight, we plot these expressions Δ_N^{clone}, Δ_D^{clone} and Δ_{DG}^{clone} of the correlations generated in terms of the fidelity F_{cl} of cloning and the input state parameter α in the Figs.(6.1, 6.2 & 6.3). Since ξ lies in the range $\frac{1}{6} \leq \xi \leq \frac{1}{2}$, we have the range of the fidelity $\frac{1}{2} \leq F_{cl} \leq \frac{5}{6}$ and the range of the input parameter α from 0 to 1. In Figs.(6.1, 6.2 & 6.3), we find that the more correlated are states, the less is the fidelity of cloning. In other words, when we have a cloning machine that performs better, the joint output mode will be poorly correlated. Altogether, these plots indicate that the amount of correlations generated in the process of cloning plays a vital role in determining the fidelity of cloning. As is evident from these figures, the more the amount of correlations present in the original and the cloned copy in the output, the more difficult it is to copy the information of the original copy in the blank state, because the information gets hidden in the correlations between the copies. Though we have considered a particular type of cloning machine to illustrate this phenomenon, we believe that this phenomenon is independent of the transformation we choose, and is true in general for the process of imperfect quantum cloning.

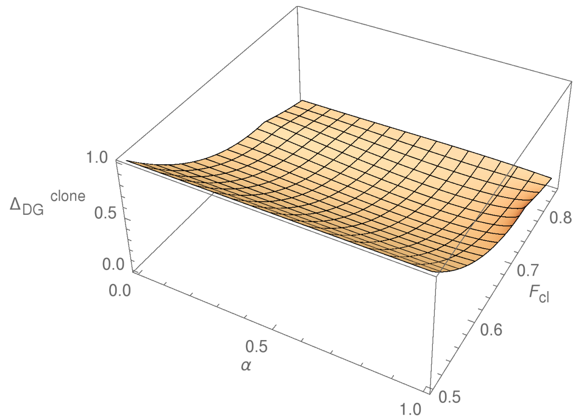


Figure 6.3: The plot shows how the correlation measure geometric discord (Δ_{DG}^{clone}) vary with the input parameter α and the fidelity of cloning F_{cl}.

6.4 Analysis of the correlations content of the output of a state-dependent deleting machine

In this section we analyze the correlations generated in the process of quantum deletion which can be thought of as the opposite procedure of quantum cloning. As an example, we consider a state-dependent quantum deletion machine and study the amount of correlations present in the output modes. As in the previous section, we wish to determine the role of quantum correlations in regulating the fidelity of deletion. In order to do that, we consider three different correlation measures and indeed we see that the physical finding is no different from the cloning. The action of a state-dependent deleting machine as mentioned in reference [185, 260] is given by the unitary operation

$$|\psi\rangle_A|\psi\rangle_B|A\rangle_C \to \alpha^2|0\rangle_A|0\rangle_B|A_0\rangle_C + \beta^2|1\rangle_A|0\rangle_B|A_1\rangle_C + \sqrt{2}\alpha\beta|\psi^+\rangle_{AB}|A\rangle_C, \quad (6.12)$$

where we start with two copies of the unknown state $|\psi\rangle$ with the purpose of deleting one copy against the other. Here $|A\rangle_C$ is the initial state of the ancilla, $|A_0\rangle_C$ and $|A_1\rangle_C$ are the final states of the ancilla. Moreover, the unitarity of the transformation demands the states $|A\rangle$, $|A_0\rangle$ and $|A_1\rangle$ to be orthogonal to each other. After the application of the deletion transformation given in (6.12) on two copies of $|\psi\rangle$, the output reduced density matrix of these two modes takes the form

$$\rho_{ab}^{del} = |\alpha|^4|00\rangle\langle 00| + |\beta|^4|10\rangle\langle 10| + 2|\alpha|^2|\beta|^2|\psi^+\rangle\langle\psi^+|, \quad (6.13)$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The fidelity of the deletion for this machine is given by $F_{del} = 1 - |\alpha|^2|\beta|^2$. By expressing the input parameter $|\alpha|^2$ in terms of fidelity F_{del} we have $|\alpha|^2 = \frac{1}{2}(1 \pm \sqrt{4F_{del} - 3})$. However the feasible solution for $|\alpha|^2$ is $\frac{1}{2}(1 - \sqrt{4F_{del} - 3})$. Based on the range of $|\alpha|^2$ we find that F_{del} satisfies the relation $\frac{3}{4} \leq F_{del} < 1$. This is also consistent with the fact that if we are given two copies of an unknown qubit, and we perform optimal measurement on both the copies, then we can

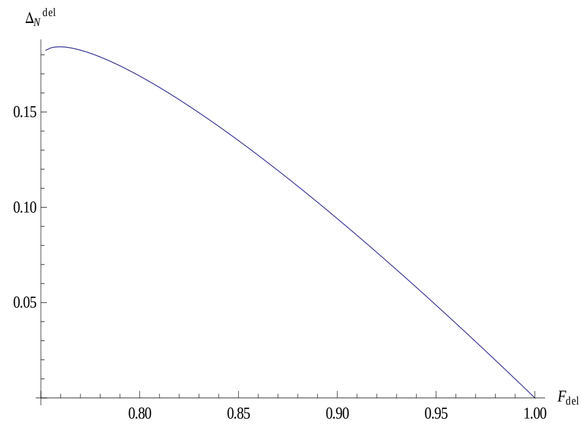


Figure 6.4: The figure shows how the correlation measure negativity (Δ_N^{del}) vary with the fidelity of deletion (F_{del}).

estimate the state with a fidelity $3/4$ [227] which is also the lower bound of the deletion machine. In a similar way, we define the amount of correlations generated in the process of deletion. This is given by the difference between the amount of correlations in the output modes after the process of deletion, and the amount of correlations in those two modes before the application of the deletion operation. We denote this difference of correlations for any correlation measure $K(\rho_{ab})$ as $\Delta_K^{del} = K(\rho_{ab}^{final}) - K(\rho_{ab}^{initial})$. We compute various correlation measures for both the initial input states and the final output states. Since we start with product states having no initial correlation, the amount of correlations generated in the process of deletion is the same as the amount of correlations between the output modes. We denote these correlations for three different measures (i) negativity (N), (ii) discord (D) and (iii) geometric discord (DG) by the notations Δ_N^{del}, Δ_D^{del} and Δ_{DG}^{del}, respectively. The expression for Δ_N^{del} is given by

$$\Delta_N^{del} = \frac{1}{2}\left[\frac{(1-a)}{4}\{(1+a)^2+1\}^{\frac{1}{2}} + (2-a)(1+a) - 1\right], \tag{6.14}$$

where $a = \sqrt{4F_{del}-3}$. Similarly, the expression for Δ_D^{del} is given by

$$\Delta_D^{del} = \left(\frac{6}{5}\right)^2 \left[H_2(c) + H_2(T_+) - h(T_+^2) - h(S_+, S_-)\right], \tag{6.15}$$

where $h(x,y) = -x\log_2 x - y\log_2 y$, $h(x) = -x\log_2 x$, $c = \frac{1}{2F_{del}}(a+1)$, $S_\pm = \frac{1}{4}(3 - 2F_{del} + a \pm \{14 - 2a + 4F_{del}(a + 5F_{del} - 8)\}^{\frac{1}{2}})$ and $T_+ = \frac{1}{2}(1-a)$.

Lastly, the corresponding expression for the geometric discord is given by

$$\Delta_{DG}^{del} = 2(\lambda_0 + 2\lambda_1 - \max[\lambda_0, \lambda_1]), \tag{6.16}$$

where, $\lambda_0 = \frac{1}{4}[l_-^2 + l_+^2]$, $\lambda_1 = K_+^2$, $a = \sqrt{4F_{del}-3}$, $l_\pm = K_- \pm K_+ - 1$ and $K_\pm = \frac{1}{2}(1-a)\{1 \pm \frac{1}{2}(a-1)\}$.

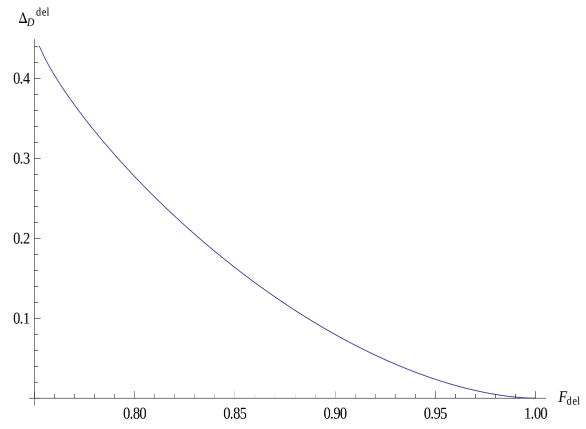


Figure 6.5: The figure shows how the correlation measure discord (Δ_D^{del}) vary with the fidelity of deletion (F_{del}).

Now, our aim is to see how correlations generated in the process controls the fidelity of achieving it. For this, we plot these measures with respect to the fidelity of deletion F_{del} in Figs.(6.4, 6.5 & 6.6). These figures show that the amount of correlations generated in the process varies inversely with the efficiency of carrying out the deletion process successfully. This is very similar to the behavior that we have observed in the process of cloning. Our conjecture is that this is independent of the machine we select. This agrees with our physical intuition that the amount of information not available for the deletion process is hidden in the correlations between the two modes.

6.5 Concatenation of Cloning and Deletion – Correlation Complementarity

In this section, we consider the successive action of cloning and deletion on an arbitrary quantum state to see that the total amount of correlations generated as a result of these two processes is bounded. Here also we find that a similar thing happens even in the opposite case where cloning is followed by the deletion. These bounds actually show a new aspect of quantum correlations, i.e., the “complementarity". We analytically obtain these bounds for different measures and exemplify for a particular measure with the help of cloning and deletion machines.

6.5.1 Deleting imperfect cloned copies

In this subsection, we consider the case where we start with the state to be cloned along with a blank state. The initial state is a product state having no correlation at all. After the cloning operation these two states are no longer uncorrelated and they are given by joint density matrix ρ_{ab}^{final}. The amount of correlations generated in the process of cloning for a given correlation measure K is given by $\Delta_K^{clone} = K(\rho_{ab}^{final}) - K(|\psi\rangle \otimes |\Sigma\rangle)$.

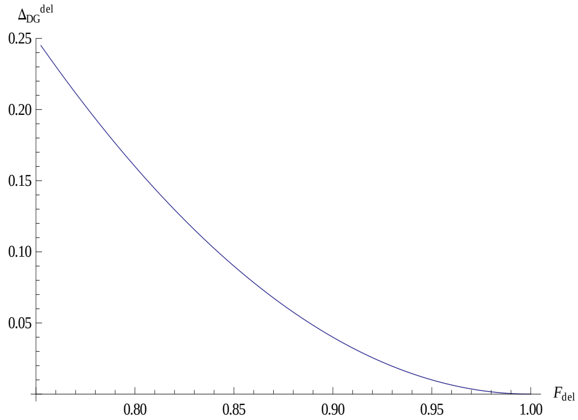


Figure 6.6: The figure shows how the correlation measure geometric discord (Δ_{DG}^{del}) vary with the fidelity of deletion (F_{del}).

Since the initial states are product states, we have $K(|\psi\rangle \otimes |\Sigma\rangle) = 0$ and consequently $\Delta_K^{clone} = K(\rho_{ab}^{final})$. K being any correlation measure, is bounded by its maximum and minimum values K_{max} and K_{min} respectively. Now if we delete these imperfect cloned copies in order to get back to its original product form $|\psi\rangle \otimes |\Sigma\rangle$, we get a new combined state ρ'_{ab} at the output mode. Then the amount of correlations generated in the process is given by $\Delta_K^{del} = K(\rho'_{ab}) - K(\rho_{ab}^{final})$ for a particular correlation measure K. It can be seen that by combining the correlations generated in the cloning and deleting process we have

$$\Delta_K^{clone} + \Delta_K^{del} = K(\rho'_{ab}). \tag{6.17}$$

Since the correlation measure K is always bounded by its maximum value K_{max} for any arbitrary state, we have

$$\Delta_K^{clone} + \Delta_K^{del} \leq K_{max}. \tag{6.18}$$

Thus, for the different correlation measures like negativity (N), discord (D) and geometric discord (DG) we have various bounds for the correlations as given below

$$\begin{aligned} \Delta_N^{clone} + \Delta_N^{del} &\leq \frac{1}{2}, \\ \Delta_D^{clone} + \Delta_D^{del} &\leq 1, \\ \Delta_{DG}^{clone} + \Delta_{DG}^{del} &\leq 1, \end{aligned} \tag{6.19}$$

respectively. These bounds together tell us about an intriguing property of quantum correlations which is " complementarity". The amount of correlations generated in the process of cloning is complementary to the amount of correlations generated in the process of deletion. Thus, we can say that when the amount of correlations generated in the cloning process is more (less), the amount of correlations for the deletion process is less (more).

The above result can be stated differently: it tells us that the better we clone the worse we delete. Thus, our conjecture is that this complementarity is not only true for the correlations generated but also true for the fidelity of achieving the cloning and deletion process successively.

6.5.1.1 Complementarity for 1→ 2 cloning, 2 → 1 deleting

Next we exemplify our result with the help of a particular cloning and deleting transformation in the context of a specific correlation measure such as the geometric discord (DG). We start with an arbitrary quantum state $|\psi\rangle$ and a blank state $|\Sigma\rangle$ initially in the product state. Then, we apply the universal Buzek-Hillery quantum cloning machine defined by the transformations (6.5) on $|\psi\rangle$ and on the output of BH copying machine we apply the deletion operations defined by

$$\begin{gathered}|0\rangle|0\rangle|Q_0\rangle \rightarrow |0\rangle|0\rangle|A_0\rangle, \;\; |1\rangle|1\rangle|Q_1\rangle \rightarrow |1\rangle|0\rangle|A_1\rangle, \\ (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_i\rangle \rightarrow (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_i\rangle \;\; (i = 0, 1),\end{gathered} \tag{6.20}$$

to obtain the final output state [243, 260]

$$\rho'_{ab} \;\; = \;\; \frac{1}{1+2\xi}(\alpha^2|00\rangle\langle 00| + \beta^2|10\rangle\langle 10| + 2\xi|\psi^+\rangle\langle\psi^+|), \tag{6.21}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $\langle A_i|Y_i\rangle = 0$. The fidelity of deleting imperfect cloned copies is given by $F_3 = \frac{1+\xi}{1+2\xi}$ [243] and it ranges from $\frac{3}{4}$ to $\frac{7}{8}$. The total correlations generated in the successive process of cloning and deletion is given by the sum of the respective correlations

$$\Delta^T_{DG} = \Delta^{clone}_{DG} + \Delta^{del}_{DG} = DG(\rho'_{ab}). \tag{6.22}$$

The expression for the Δ^T_{DG}, i.e., $DG(\rho'_{ab})$ is given by

$$\Delta^T_{DG} \;\; = \;\; 2(\lambda_0 + 2\lambda_1 - \max[\lambda_0, \lambda_1]), \tag{6.23}$$

where $\lambda_0 = \frac{1}{2} + \sqrt{2}\alpha^4(1-2F_3)^2 + 2\alpha^2 F_3(1-2F_3) - F_3(1-F_3)$ and $\lambda_1 = (1-F_3)^2$. Thus, we see that the total correlations generated in the process is given by the correlation content of the final state, and that it is bounded by its maximum value. Since we adopt geometric discord as a measure of correlation, the total correlations content is bounded by one, i.e., $\Delta^T_{DG} < 1$. In Fig.(6.7) we plot the total correlations with respect to the machine parameter ξ and the input state parameter α and clearly find that this is always bounded by its maximum value one.

6.5.1.2 Complementarity for 1→ N cloning, N → M deleting

Next we extend our result to a more general situation, where we first create N copies from a single copy with the help of a "$1 \mapsto N$"-cloning machine. Then we use a "$N \mapsto M$ ($N > M$) "deleting machine to produce M distorted copies of the input state at the output

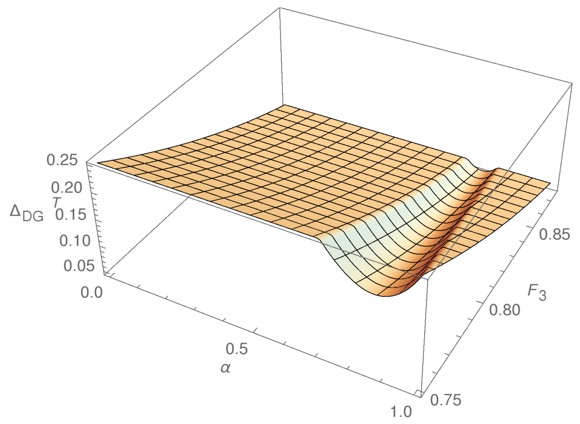


Figure 6.7: The figure shows how the total correlations (Δ_{DG}^{T}) for the scheme "1$\rightarrow$ 2 cloning then $2 \rightarrow 1$ deleting", which varies with the input parameter α and the fidelity of deletion F_3.

port. First of all, we apply "$1 \mapsto N$" Cloning machine on an arbitary input state $|\psi\rangle$. We use the result of Gisin and Massar who first generalized the Buzek-Hillery's $1 \mapsto 2$ cloning machine to $M' \mapsto N$ ($M' < N$) [226]. Now for $M' = 1$, the unitary operator (U) for $1 \mapsto N$ cloning machine is given by

$$U|0\rangle|R\rangle = \sum_{j=0}^{N-1} \alpha_j |\chi(N,j)\rangle|R_j\rangle, \;\; U|1\rangle|R\rangle = \sum_{j=0}^{N-1} \alpha_{N-1-j}|\chi(N,j+1)\rangle|R_j\rangle, \quad (6.24)$$

where R denotes initial combined state of the copying machine and $(N-1)$ blank copies. Here R_j are orthonormalized internal states of the quantum cloning machine. Here, $\alpha_j = \sqrt{2(N-j)/N(N+1)}$ and we have denoted $|\chi(N,j)\rangle = |(N-j)0, j1\rangle$ as the symmetric and normalized state. After "$1 \mapsto N$"-cloning operation is over, we use the ouput of cloning machine as an input to a "$N \mapsto M$" deleting machine . The action of the deleting machine is given by the transformations [185, 260],

$$\begin{aligned} &|0\rangle^{\otimes N}|R_0\rangle \mapsto |0\rangle^{\otimes N}|A_0\rangle, \;\; |\chi(N,j)\rangle|R_j\rangle \mapsto |\chi(N,j)\rangle|R_j\rangle, \;\; j \neq 0, \\ &\qquad |1\rangle^{\otimes N}|R_{N-1}\rangle \mapsto |1\rangle^{\otimes M}|0\rangle^{\otimes(N-M)}|A_1\rangle, \end{aligned} \quad (6.25)$$

where $|A_0\rangle$, $|A_1\rangle$ are machine states at the output port of the deleting machine. Combining these two machine, the complete transformation of $|\psi\rangle$ is given by

$$\begin{aligned} |\psi\rangle \;\; &\mapsto \;\; \beta\left[\sum_{j=0}^{N-2} \alpha_{N-1-j}|(N-1-j)0,(j+1)1\rangle \otimes R_j + \alpha_0|M1(N-M)0\rangle \otimes A_1\right] \\ &+ \;\; \alpha\left[\alpha_0|N0\rangle \otimes A_0 + \sum_{j=1}^{N-1}\alpha_j|(N-j)0, j1\rangle \otimes R_j\right], \end{aligned} \quad (6.26)$$

where $\langle A_i|R_j\rangle = 0$, $\langle A_i|A_j\rangle = \delta_{ij}$, $\langle R_i|R_j\rangle = \delta_{ij}$ (δ_{ij} is Kronecker delta). The density matrix at the output port after tracing out the machine states is given by

$$\begin{aligned}\rho &= \alpha^2\left[\alpha_0^2|N0\rangle\langle N0| + \sum_{j=1}^{N-1}\alpha_j^2|\chi(N,j)\rangle\langle\chi(N,j)|\right]\\ &+ \beta^2\left[\sum_{j=0}^{N-2}\alpha_{N-1-j}^2|\chi(N,j+1)\rangle\langle\chi(N,j+1)| + \alpha_0^2|\chi(N,N-M)\rangle\langle\chi(N,N-M)|\right]\\ &+ \sum_{i,j}\alpha_i\alpha_{N-1-j}(\alpha\beta^*|\chi(N,i)\rangle\langle\chi(N,j+1)| + \alpha^*\beta|\chi(N,j+1)\rangle\langle\chi(N,i)|)\delta_{ij}.\end{aligned} \tag{6.27}$$

After tracing out rest of modes, the reduced density matrix of the first mode is is given by

$$\rho^a = \sum_{i=0}^{N-1}\vartheta(N-i,i)\alpha_i^2|0\rangle\langle 0| + \sum_{i=0}^{N-1}\vartheta(i,N-i)\alpha_i^2|1\rangle\langle 1|,$$

where $\vartheta(x,y) = \frac{x}{N}C(N,x)\alpha^2 + \frac{y}{N}C(N,y)\beta^2$ and $C(x,y) = \frac{x!}{y!(x-y)!}$.

Since, we know that for a multiqubit state, there is no unique way to quantify quantum correlations present in the state. For that reason we have taken a simple approach and have considered bipartite discord as a measure of quantum correlations. The bipartite discord of the N-qubit state $\rho_{1,\dots,N}$ (for the partition $(i,\bar{i})$) is defined as,

$$D(i|\bar{i}) = \min_{\Pi_{\bar{i}_j}}\{S(\rho_{\bar{i}}) + S(\rho_{i|\bar{i}}) - S(\rho_{1,\dots,N})\}, \tag{6.28}$$

where $S(\rho_{i|\bar{i}}) = \sum_j p_j S(\rho_{i|j})$ is the average of the entropies of states $\rho_{i|j} = \frac{1}{p_j}\mathrm{Tr}_{\bar{i}}[(\mathbb{I}_i\otimes\Pi_{\bar{i}_j})\rho_{1,\dots,N}(\mathbb{I}_i\otimes\Pi_{\bar{i}_j})]$ with corresponding probability $p_j = \mathrm{Tr}[(\mathbb{I}_i\otimes\Pi_{\bar{i}_j})\rho_{1,\dots,N}(\mathbb{I}_i\otimes\Pi_{\bar{i}_j})]$. Here $\Pi_{\bar{i}_j}$'s are all possible $(N-1)$ qubits projective measurement operators. The bipartite discord given in Eq.(6.28) is to be minimized over all possible projective measurements $\Pi_{\bar{i}_j}$. In this paper we have performed projective measurements upto three qubits. For single qubit measurement we have used the measurement bases given in [100, 116]. For two and three qubit measurements we have followed the reference [122]. The bipartite discord (see Eq.(6.28)) is not symmetric under the exchange of qubit. So, the average correlations present in the N-qubit state is given by,

$$\delta(\rho_{1,\dots,N}) = \frac{1}{N}\sum_{i=1}^{N}D(i|\bar{i}). \tag{6.29}$$

So the total correlations generated in this process is $\Delta_\delta^T = \Delta_\delta^{clone} + \Delta_\delta^{del} = \delta(\rho)$. In Fig.(6.8), we have plotted the total correlations Δ_δ^T generated during cloning and deleting against the input state parameter (α) to show the complementary nature of correlations production in this two processes, i.e., $\Delta_\delta^{clone} + \Delta_\delta^{del} \leq 1$. In the figure, we have looked for the correlations generated in each of the dual processes: (a) $1 \mapsto 3$ cloning then $3 \mapsto 1$ deleting, (b) $1 \mapsto 3$ cloning then $3 \mapsto 2$ deleting, (c) $1 \mapsto 4$ cloning then $4 \mapsto 1$ deleting, (d) $1 \mapsto 4$ cloning then $4 \mapsto 2$ deleting and (e) $1 \mapsto 4$ cloning then $4 \mapsto 3$ deleting. It is evident from the figure itself, that the total correlations generated as a consequence of

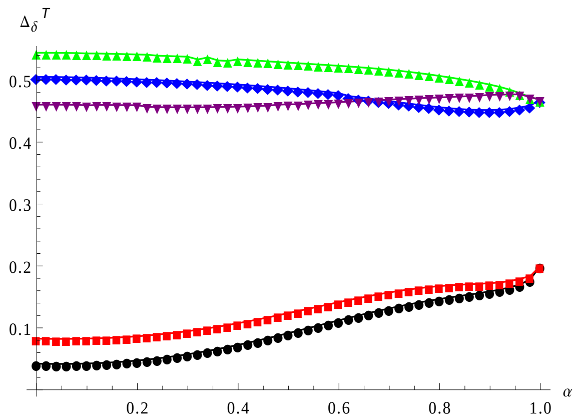


Figure 6.8: Quantum correlations Δ_δ^T versus input state parameter α for the processes (a) $1 \mapsto 3$ cloning then $3 \mapsto 1$ deleting (●), (b) $1 \mapsto 3$ cloning then $3 \mapsto 2$ deleting (■), (c) $1 \mapsto 4$ cloning then $4 \mapsto 1$ deleting (♦), (d) $1 \mapsto 4$ cloning then $4 \mapsto 2$ deleting (▲) and (e) $1 \mapsto 4$ cloning then $4 \mapsto 3$ deleting (▼).

dual processes in each of these cases is bounded. Thus even in a most general setting of multiple qubits the correlations generated in each of cloning and deleting process are complementary in nature.

6.5.2 Cloning of imperfect deleted copies

In this subsection we carry out the reverse process where we perform deleting first and then clone the imperfect deleted copies. We start with two identical copies of an unknown quantum state $|\psi\rangle$. Initially, there is no correlation between these two states as they are in the product form. Consequently, we can write the correlation content of these states for a given correlation measure K as $K(|\psi\rangle \otimes |\psi\rangle) = 0$. However, after the deletion operation they are no longer uncorrelated. Instead, we obtain a correlated two qubit state ρ_{ab}^{del}. The amount of correlations generated in the process of deletion is given by the difference of the correlations of the final and the initial states, i.e., $\Delta_K^{del} = K(\rho_{ab}^{del}) - K(|\psi\rangle \otimes |\psi\rangle) = K(\rho_{ab}^{del})$. Next, we apply the cloning transformations on the combined state ρ_{ab}^{del} in order to get back to the initial identical copies of the state $|\psi\rangle$. However, due to the imperfectness of the process we get a mixed state ρ_{clone}^{del} at the output port. The amount of correlations generated in the process is given by the difference of the correlations of the states ρ_{ab}^{del} and ρ_{clone}^{del}, i.e., $\Delta_K^{clone} = K(\rho_{clone}^{del}) - K(\rho_{ab}^{del})$. The total correlations generated in the process of cloning and deletion is given by

$$\Delta_K^{del} + \Delta_K^{clone} = K(\rho_{clone}^{del}). \tag{6.30}$$

Since for a given correlation measure K the correlations of a particular state is always bounded by its maximum and minimum value K_{max} and K_{min}, we will get back the same bound on the total correlations generated, i.e.,

$$\Delta_K^{del} + \Delta_K^{clone} \leq K_{max}, \tag{6.31}$$

irrespective of whether we delete and then clone or we clone first and then delete. This once again establishes the same complementarity in terms of the correlations generated in the process of cloning and deletion. The complementarity of quantum correlations are independent of whether we apply cloning or deletion first.

6.5.2.1 Complementarity for 2→ 1 deleting, 1 → 2 cloning

Next, we give an example of the complementarity phenomenon in this case with the help of a particular deleting and cloning machine in the context of a specific correlation measure, namely geometric discord (DG). Here we start with two identical copies of the state $|\psi\rangle$ and we apply the quantum deletion machine defined in Eq.(6.12) which results in a two qubit state ρ_{ab}^{del} (see Eq.(6.13)). Then, we apply BH cloning operation on the state ρ_{ab}^{del} which will give us two output states as $\rho_{aa'} = \mathrm{Tr}_b[(U_{BH} \otimes I)(\rho_{ab}^{del}|0\rangle_{a'}\langle 0|)(U_{BH} \otimes I)^\dagger]$ and $\rho_{bb'} = \mathrm{Tr}_a[(I \otimes U_{BH})(\rho_{ab}^{del}|0\rangle_{b'}\langle 0|)(I \otimes U_{BH})^\dagger]$. The density operators $\rho_{aa'}$ and $\rho_{bb'}$ are given by

$$\begin{aligned} \rho_{aa'} &= (1-2\xi)(\alpha^2|00\rangle\langle 00| + \beta^2|11\rangle\langle 11|) + 2\xi|\psi^+\rangle\langle\psi^+|, \text{ and} \\ \rho_{bb'} &= (1-2\xi)\{(1-\alpha^2\beta^2)|00\rangle\langle 00| + \alpha^2\beta^2|11\rangle\langle 11|\} + 2\xi|\psi^+\rangle\langle\psi^+|. \end{aligned} \tag{6.32}$$

The total correlations generated in the successive process of deletion and cloning is given by the sum of the respective correlations. Here, we obtain the total correlations in terms of the measure geometric discord (DG) as

$$\Delta_{DG}^T = \Delta_{DG}^{del} + \Delta_{DG}^{clone} = DG(\rho_{clone}^{del}). \tag{6.33}$$

In this case ρ_{clone}^{del} are $\{\rho_{aa'}, \rho_{bb'}\}$. Hence, the total correlations for the state $\rho_{aa'}$ is given by

$$\Delta_{DG}^T = 2(\lambda_0 + 2\lambda_1 - \max[\lambda_0, \lambda_1]), \tag{6.34}$$

where $\lambda_0 = \frac{1}{4}[L^2 + (L-2\xi)^2]$, $\lambda_1 = \xi^2$ and $L = (1-2\xi)(\alpha^2 - \beta^2)$.

Similarly, for $\rho_{bb'}$ we find

$$\Delta_{DG}^T = 2(\lambda_0 + 2\lambda_1 - \max[\lambda_0, \lambda_1]), \tag{6.35}$$

where $\lambda_0 = \frac{1}{4}[J^2 + (1-4\xi)^2]$, $\lambda_1 = \xi^2$ and $J = (1-2\xi)(1-2\alpha^2\beta^2)$.

As in the previous process, here too the total correlations are given by the correlations of the final state. In Figs.(6.9 & 6.10), we plot the total correlations Δ_{DG}^T against the input parameter α to find that this is always bounded by its maximum value one, i.e., $\Delta_{DG}^T \leq 1$.

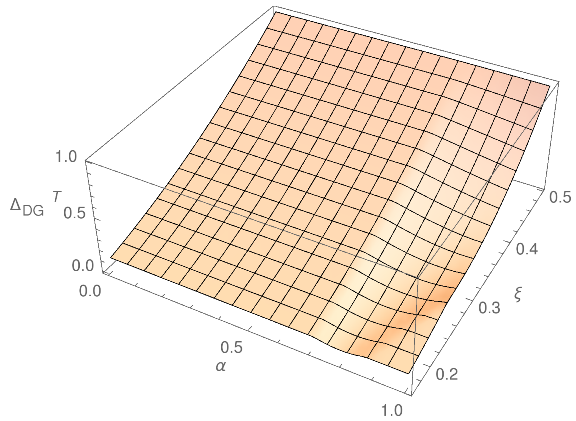


Figure 6.9: The figure shows how the total correlations (Δ_{DG}^{T}) of Eq.(6.34) for the scheme "2$\rightarrow$ 1 deleting then 1 $\rightarrow$ 2 cloning" varies with input parameter α and the cloning machine parameter ξ.

6.5.2.2 Complementarity for N$\rightarrow$ 1 deleting, 1 $\rightarrow$ M cloning

Further we move on to much more general setting where we start with the application of "$N \mapsto 1$" deleting machine on N copies of the state $|\psi\rangle$ to produce a distorted state at the output port. Let say, at the output port we will have the state $\rho^{del}_{a_1,..,a_N}$ after tracing out the machine states, where a_1 is the 'undeleted mode' and $a_2,..,a_N$ are the 'deleted modes'. In the next step, we take the state ($\rho^{del}_{a_i}$; $a_i \neq a_1$) of $\rho^{del}_{a_1,..,a_N}$ as an input to "$1 \mapsto M$" cloning process. Initially, after applying $N \mapsto 1$ deleting machine (6.25) on the state $|\psi\rangle^{\otimes N}$ we will have $\rho^{del}_{a_1,a_2,...,a_N}$ as

$$\rho^{del}_{a_1,..,a_N} = \beta^{2N}|1(N-1)0\rangle\langle 1(N-1)0| + \sum_{k=0}^{N-1} g(k)\varpi(k)|\chi(N,k)\rangle\langle\chi(N,k)|, \quad (6.36)$$

where $\varpi(k) = C(N-k,k)\alpha^{2(N-k)}\beta^{2k}$ and $g(k) = 2$ iff $k = 0$ otherwise $g(k) = 1$. Then the reduced density matrix ($\rho^{del}_{a_i}$; $a_i \neq a_1$) of the state in Eq.(6.36) is given by,

$$\rho^{del}_{a_i} = \eta_0|0\rangle\langle 0| + \eta_1|1\rangle\langle 1|, \quad (6.37)$$

where η_0 and η_1 are

$$\eta_0 = \sum_{i=0}^{N-1}\frac{(N-i)}{N}\varpi(i) + \varpi(N), \;\; \& \;\; \eta_1 = \sum_{i=1}^{N-1}\frac{iC(N,i)}{NC(N-i,i)}\varpi(i), \quad (6.38)$$

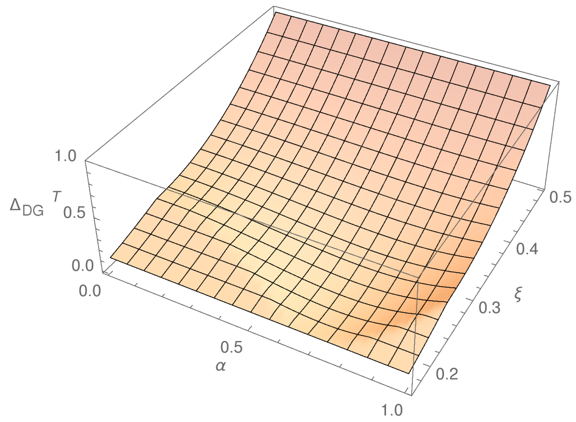


Figure 6.10: The figure shows how the total correlations (Δ^T_{DG}) of Eq.(6.35) for the scheme "2→ 1 deleting then 1 → 2 cloning" varies with input parameter α and the cloning machine parameter ξ.

respectively. Now the state $\rho^{del}_{a_i}$ (in Eq.6.37) is taken as input to $1 \mapsto M$ cloning machine. After the overall dual transformation the final reduced density matrix is,

$$\rho_f = \eta_0 \sum_{j=0}^{M-1} \alpha_j^2 |\chi(M,j)\rangle\langle\chi(M,j)| + \eta_1 \sum_{j=0}^{M-1} \alpha^2_{M-1-j} |\chi(M,j+1)\rangle\langle\chi(M,j+1)| \tag{6.39}$$

Finally, the reduced density matrix at first mode is given by,

$$\rho^a_f = \sum_{j=0}^{M-1} \bar{\vartheta}(M-j,j)\alpha_j^2 |0\rangle\langle 0| + \sum_{j=0}^{M-1} \bar{\vartheta}(j,M-j)\alpha_j^2 |1\rangle\langle 1| \tag{6.40}$$

where $\bar{\vartheta}(x,y) = \frac{x}{M}C(M,x)\eta_0 + \frac{y}{M}C(M,y)\eta_1$. Here also, we use bipartite quantum discord to quantify multiqubit quantum correlations in the dual physical process. The total correlations generated in this process is $\Delta^T_\delta = \Delta^{del}_\delta + \Delta^{clone}_\delta = \delta(\rho_f)$. In Fig.(6.11) we once again have plotted the total correlations Δ^T_δ generated in the dual physical process of deletion followed by cloning against the state parameter α of the input state $|\psi\rangle$. We have considered several cases and interestingly plots which show that the total correlations are always bounded. More precisely, the correlations generated in individual processes are complementary in nature, i.e., $\Delta^{del}_\delta + \Delta^{clone}_\delta \leq 1$.

6.6 Conclusions

Complementarity is a fundamental feature of the quantum world which manifests in the dual physical nature of quantum particles. In this chapter, we have shown a new kind of complementarity between two different physical processes such as approximate quantum cloning and the deleting. We have shown that there is a relationship between quantum cor-

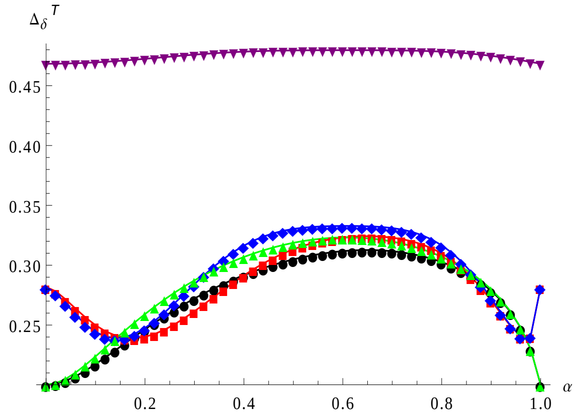


Figure 6.11: Quantum correlations Δ_δ^T varses input state parameter α for the processes (a) $3 \mapsto 1$ deleting then $1 \mapsto 2$ cloning (■), (b) $3 \mapsto 1$ deleting then $1 \mapsto 3$ cloning (●) , (c) $4 \mapsto 1$ deleting then $1 \mapsto 2$ cloning (♦), (d) $4 \mapsto 1$ deleting then $1 \mapsto 3$ cloning (▲) and (e) $4 \mapsto 1$ deleting then $1 \mapsto 4$ cloning (▼).

relations generated in the process of cloning and deleting and the fidelity of the process in question. This has been illustrated using various measures of quantum correlations such as the geometric discord (DG), entropic quantum discord (D) and negativity (N). To bring out the generic nature of the complementarity, we have chosen three different classes of measure and irrespective of these measures we find that fidelity decreases with increase of correlations for both the processes of cloning and deletion. This is well exhibited in terms of the amount of correlations generated in the successive processes of cloning and deletion (and vice versa). Moreover, we have witnessed an important property of quantum correlations called ”complementarity" property in dual physical processes. We have shown that the total correlations change in the cloning and the deleting is bounded by the maximum value of the measure of quantum correlations. We have illustrated complementarity for a particular choice of cloning and deleting machine as well as for a particular measure of correlations. We believe that this phenomenon is true for all classes of correlation measures and is independent of the choice of measure. It will be interesting to see if other quantum correlations display some complementary behavior in dual physical processes.

Chapter 7

Broadcasting of quantum correlations

7.1 Introduction

Quantum entanglement [3] is one of the key factor for deciding the fidelity of QCMs [184]. Atleast in the context of quantum information processing, purer the entanglement, more valuable is the given two qubit state. Therefore, extraction of pure quantum entanglement from a partially entangled state is considered to be an important task. Consequently, there have been a lot of work on purification procedures by many researchers over the last few years showing how one can compress the amount of quantum entanglement locally [47, 261]. The possibility of compression of quantum correlations naturally raises the question if the opposite i.e. decompression of correlations is realizable or not? Many researchers have answered this query using the process known as “Broadcasting of Inseparability" [196–198]. This question becomes important when there is an exigency in increasing the number of available entangled pairs rather than the purity of it. In simple sense, broadcasting here refers to local or nonlocal copying of quantum correlations [196, 245].

In general, the term broadcasting can be used in different contexts. Classical theory permits broadcasting of information, however that is not the case for all states in quantum theory. Cloning and broadcasting principles demarcate the boundary between classical and quantum worlds. In this context, Barnum *et al.* were the first to show that non-commuting mixed states do not meet the criteria of broadcasting [194].

It is impossible to have a process which will perfectly copy (clone and broadcast) an arbitrary quantum state [34, 194, 196]. By referring to perfect broadcasting of correlations we mean that the correlations in a two qubit state ρ^{ab} are locally broadcastable if there exist two operations, Σ^a: $S(\mathbb{H}^a) \rightarrow S(\mathbb{H}^{a_1} \otimes \mathbb{H}^{a_2})$ and Σ^b: $S(\mathbb{H}^b) \rightarrow S(\mathbb{H}^{b_1} \otimes \mathbb{H}^{b_2})$ such that $I^q(\rho^{a_1b_1}) = I^q(\rho^{a_2b_2}) = I^q(\rho^{ab})$. Here, $I^q(\rho^{ab})$ is the quantum mutual information, $\rho^{a_1a_2b_1b_2} := \Sigma^a \otimes \Sigma^b(\rho^{ab})$ and $\rho^{a_ib_i} := \mathrm{Tr}_{a_{\bar{i}}b_{\bar{i}}}(\rho^{a_1a_2b_1b_2})$ [201]. Quite recently, many authors showed that correlations in a single bipartite state can be locally or unilocally broadcast if and only if the states are classical (i.e. having classical correlations) or classical-quantum respectively [200–203].

In the previous cases, one generally discussed about broadcasting of a general quantum state or perfect broadcasting of correlations. But when we refer broadcasting of an entangled state, we generally talk about creating more pairs of lesser entangled states from a given entangled state where $I^q(\rho^{a_1b_1})$ and $I^q(\rho^{a_2b_2})$ are less than $I^q(\rho^{ab})$. This is done via the application of local cloning operation on each qubit of the given entangled state, or sometimes by applying global cloning operations on the total input entangled state itself [196, 198, 221]. Bandyopadhyay *et al.* [198] showed that only UQCMs having fidelity over $\frac{1}{2}(1+\frac{1}{\sqrt{3}})$ can broadcast entanglement and further that entanglement in the input state is optimally broadcast only if the quantum cloners used for local copying are optimal. However, the fact that if local cloners are used then broadcasting of entanglement into more than two entangled pairs is impossible. Ghiu *et al.* addressed the question of broadcasting of entanglement by using local universal optimal asymmetric Pauli cloning machines. They presented that if one employs symmetric cloners instead of asymmetric ones, then only optimal broadcasting of inseparability is achievable [262]. In other works, authors investigated the problem of secretly broadcasting of three-qubit entangled state between two distant partners with universal quantum cloning machine and then the result is generalized to generate secret entanglement among three parties [245].

In this chapter, we mainly investigate the problem of broadcasting of quantum correlations (QCs) [199]. Traditionally, by QCs we refer to entanglement. First part of our study is about broadcasting of quantum entanglement for general two qubit mixed states. For the first time in the existing research on broadcasting, we provide the broadcasting range for general two qubit state in terms of Bloch vectors. To do this we apply the Buzek-Hillery (B-H) QCM, both locally and nonlocally. We separately provide broadcasting ranges for Werner-like and Bell-diagonal states as illustration. In the second part of our work, while exploring the possibility of broadcasting of quantum correlations that go beyond entanglement (QCsbE), remarkably we find that it is impossible to broadcast optimally such correlations with the help of any local or nonlocal cloners. We analytically prove this by first taking the B-H state dependent and independent cloners and then by logically extending our result for the other cloners as well. This is indeed one such result which highlights how fundamentally two approaches, QCsbE and entanglement, are different. However, we can broadcast QCsbE if we relax the optimality conditions.

In section 7.2, we first introduce the quantum cloning machines, more specifically the state independent and dependent versions of B-H cloners, which we will later use for our local as well as nonlocal cloning processes. In section 7.3, we define broadcasting of entanglement via local cloning operations as well as nonlocal cloning operation and then obtain the generalized optimal broadcasting range for any two qubit state in terms of Bloch vectors. In each of the two above cases, we exemplify our results for two types of mixed states: namely the Werner-like and the Bell-diagonal states. In section 7.4, we give the definition for broadcasting of QCsbE and explicitly discuss the possibilities and impossibilities of such broadcasting. Lastly, in section 7.5, we conclude with a small conjecture by which broadcasting of correlations beyond entanglement might be possible.

7.2 Quantum cloning machines beyond No-cloning theorem

Quantum cloning transformations can be viewed as a completely positive (CP) trace preserving map between two quantum systems, supported by an ancilla [220, 232]. In this section, we briefly describe the Buzek-Hillery (B-H) QCM which we will later use for analysing the possibility and impossibility of broadcasting of entanglement as well as correlations beyond entanglement respectively.

B-H cloning machine (U_{bh}) is a M-dimensional quantum copying transformation acting on a state $|\Psi_i\rangle_{a_0}$ (i = 1, ..., M). This state is to be copied on a blank state $|0\rangle_{a_1}$. The copier is initially prepared in state $|X\rangle_x$ which subsequently get transformed into another set of state vectors $|X_{ii}\rangle_x$ and $|Y_{ij}\rangle_x$ as a result of application of the cloner. Here a_0, a_1 and x represent the input, blank and machine qubits respectively. In this case, these transformed state vectors belong to the orthonormal basis set in the M-dimensional space. The transformation scheme U_{bh} is given by [221],

$$U_{bh}\,|\Psi_i\rangle_{a_0}|0\rangle_{a_1}|X\rangle_x \rightarrow c\,|\Psi_i\rangle_{a_0}|\Psi_i\rangle_{a_1}|X_{ii}\rangle_x + d\sum_{j\neq i}^{M}\left|\Psi_{ij}^{+}\right\rangle_{a_0a_1}|Y_{ij}\rangle_x\,, \tag{7.1}$$

where $i,\ j$ = $\{1, ..., M\}$, $\left|\Psi_{ij}^{+}\right\rangle_{a_0a_1} = |\Psi_i\rangle_{a_0}|\Psi_j\rangle_{a_1} + |\Psi_j\rangle_{a_0}|\Psi_i\rangle_{a_1}$, and the coefficients c and d are real.

7.2.1 State independent cloning transformations

An optimal state independent version of the B-H cloner (U_{bhsi}) can be obtained from Eq.(7.1) by imposing the unitarity and normalization conditions which give rise to the following constraints,

$$\langle X_{ii}|X_{ii}\rangle = \langle Y_{ij}|Y_{ij}\rangle = \langle X_{ii}|Y_{ji}\rangle = 1, \tag{7.2}$$

when $\langle X_{ii}|Y_{ij}\rangle = \langle Y_{ji}|Y_{ij}\rangle = \langle X_{ii}|X_{jj}\rangle = 0$, with $i \neq j$ and $c^2 = \frac{2}{M+1}$, $d^2 = \frac{1}{2(M+1)}$. Here, we consider $M = 2^m$ where m is the number of qubits in a given quantum register. In the above transformation, by demanding the independence of the scaling (shrinking) property on input state parameters it is ensured that the quality of the cloning (fidelity of the output copies) doesn't depend on the input state [220, 221].

7.2.1.1 Local state independent cloner

The above optimal cloner U_{bhsi} with $M = 2$ becomes a local copier (U_{bhsi}^{l}). From Eq. (7.2) it can be easily observed that the corresponding values of coefficients c and d become $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{1}{6}}$ respectively. By substituting these values of the coefficients in Eq. (7.1), we can obtain the optimal state independent cloner which can be used for local copying purposes [196].

7.2.1.2 Nonlocal state independent cloner

When $M = 4$ the above optimal cloner U_{bhsi} becomes a nonlocal copier (U^{nl}_{bhsi}). Then the corresponding values of the coefficients c and d in Eq. (7.2) become $\sqrt{\frac{2}{5}}$ and $\sqrt{\frac{1}{10}}$ respectively. By substituting these coefficients in U_{bh} given by Eq. (7.1), we can obtain the optimal state independent cloner used for nonlocal copying purposes [221].

7.2.2 State dependent cloning transformations

The B-H state dependent cloner (U_{bhsd}) was developed from this B-H state independent cloning transformation (U_{bhsi}), given in Eq. (7.1) with $U_{bh} = U_{bhsi}$, by relaxing the universality condition: $\frac{\partial D}{\partial < inp >} = 0$; where $< inp >$ represents all the parameters of the input state. The distortion D describes the distance between the input and output states of the cloner [197].

With c = d = 1, the unitarity constraints on the B-H cloning transformation in Eq. (7.1) give rise to the following conditions on the output states, which are no longer necessarily orthonormal,

$$\langle X_{ii}|X_{ii}\rangle + \sum_{j\neq i}^{M} 2\,\langle Y_{ij}|Y_{ij}\rangle = 1,\ \langle Y_{ij}|Y_{kl}\rangle = 0 \tag{7.3}$$

where $i \neq j$ and $ij \neq kl$ for $i, j, k, l = \{1, ..., M\}$. We assume that, $\langle X_{ii}|Y_{jk}\rangle = \frac{\mu}{2}$, $\langle Y_{ij}|Y_{ij}\rangle = \lambda$, $\langle X_{ii}|X_{jj}\rangle = \langle X_{ii}|Y_{ij}\rangle = 0$, where again $i \neq j$ for $i, j, k = \{1, ..., M\}$; μ and λ are the machine parameters. By equating the dependence of the distortion D on the machine parameter λ to zero, in each of the cases, we can calculate the value of λ for which the B-H state dependent cloners become optimal with respect to that ensemble of input states.

7.2.2.1 Local state dependent cloner

For the case of a local state dependent cloner (U^{l}_{bhsd}), the distortion D is D_{ab} = $\mathrm{Tr}[\rho^{(out)}_{ab} - \rho^{(id)}_{a} \otimes \rho^{(id)}_{b}]^2$. If $|\psi^{(id)}_{a(b)}\rangle = \alpha|0\rangle_{a(b)} + \beta|1\rangle_{a(b)}$ be an arbitrary pure state of one qubit in mode "a" or "b", where α, β represents the input state parameters with $\alpha^2 + \beta^2 = 1$ being the normalization condition; then $\rho^{(id)}_{a} = |\psi^{(id)}_{a}\rangle\langle\psi^{(id)}_{a}|$ and $\rho^{(id)}_{b} = |\psi^{(id)}_{b}\rangle\langle\psi^{(id)}_{b}|$ represents output modes in case of an ideal copy. However, in a more realistic situation when cloning fidelity is non-ideal then the output state of the cloner is given by $\rho^{(out)}_{ab}$. Solving the equation $\frac{\partial D_a}{\partial \alpha^2} = 0$, where D_a = $\mathrm{Tr}[\rho^{(out)}_{a} - \rho^{(id)}_{a}]^2$; with $\rho^{(out)}_{a} = \mathrm{Tr}_b[\rho^{(out)}_{ab}]$, we can derive the relation between the parameters λ and μ. It turns out to be $\mu = 1 - 2\lambda$. So the permitted range of λ is bounded by $\{0, \frac{1}{2}\}$ in this case. However, it can be noted that here the value $\lambda = \frac{1}{6}$ is restricted, since for such values it reduces to the B-H optimal state independent local cloner U^{l}_{bhsi} and consequently looses the input state dependence property.

7.2.2.2 Nonlocal state dependent cloner

For the case of a nonlocal state dependent cloner (U^{nl}_{bhsd}), the distortion D is D_{abcd} = $\mathrm{Tr}[\rho^{(out)}_{abcd} - \rho^{(id)}_{ab} \otimes \rho^{(id)}_{cd}]^2$. If $|\phi^{(id)}_{ab(cd)}\rangle = \alpha|00\rangle_{ab(cd)} + \beta|11\rangle_{ab(cd)}$ be the non-maximally entangled state of two qubits in mode "ab" or "cd"; then $\rho^{(id)}_{ab} = |\psi^{(id)}_{ab}\rangle\langle\psi^{(id)}_{ab}|$ and $\rho^{(id)}_{cd} = |\psi^{(id)}_{cd}\rangle\langle\psi^{(id)}_{cd}|$ represents output modes in case of an ideal copy. However, in a more realistic situtation when cloning fidelity is non-ideal then the output state of the cloner is given by $\rho^{(out)}_{abcd}$. Solving the equation $\frac{\partial D_{ab}}{\partial \alpha^2} = 0$, where D_{ab} = $\mathrm{Tr}[\rho^{(out)}_{ab} - \rho^{(id)}_{ab}]^2$; with $\rho^{(out)}_{ab} = \mathrm{Tr}_{c,d}[\rho^{(out)}_{abcd}]$, we can derive the relation between the parameters λ and μ. Here, it turns out to be $\mu = 1 - 4\lambda$. So the permitted range of λ is bounded by $\{0, \frac{1}{4}\}$ in this case. However, it can be noted that the value $\lambda = \frac{1}{10}$ is restricted, since for such values it reduces to the B-H optimal state independent nonlocal cloner U^{nl}_{bhsi} thereby loosing the input state dependence property.

7.3 Broadcasting of Quantum Entanglement

In this section, we consider broadcasting of quantum entanglement (inseparability) with the help of both local and nonlocal cloning operations. Let us begin with a situation where we have two distant parties A and B and they share a two qubit mixed state ρ_{12} which can be canonically expressed as [233]:

$$\rho_{12} = \frac{1}{4}[\mathbb{I}_4 + \sum_{i=1}^{3}(x_i\sigma_i \otimes \mathbb{I}_2 + y_i\mathbb{I}_2 \otimes \sigma_i) + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j] = \{\vec{x}, \vec{y}, T\} \quad \text{(say)}, \tag{7.4}$$

where $x_i = \mathrm{Tr}[\rho_{12}(\sigma_i \otimes \mathbb{I}_2)]$, $y_i = \mathrm{Tr}[\rho_{12}(\mathbb{I}_2 \otimes \sigma_i)]$ and $t_{ij} = \mathrm{Tr}[\rho_{12}(\sigma_i \otimes \sigma_j)]$ with $[\sigma_i;\ i = \{1,2,3\}]$ are $2 \otimes 2$ Pauli matrices and $\mathbb{I}_n$ is the identity matrix of order n. And $\vec{x} = \{x_1,\ x_2,\ x_3\}$, $\vec{y} = \{y_1,\ y_2,\ y_3\}$ are Bloch column vectors and $T = [t_{ij}]$ is the correlation matrix.

In order to test the separability as well as inseparability for the bipartite states, we generally use Peres-Horodecki criteria [36, 37]. This is a necessary and sufficient condition for detection of entanglement for bipartite systems with dimension $2 \otimes 2$ and $2 \otimes 3$. The criteria can be equivalently expressed by the condition that at least one of the two determinants

$$W_3 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} & \rho_{00,10} \\ \rho_{00,01} & \rho_{01,01} & \rho_{00,11} \\ \rho_{10,00} & \rho_{11,00} & \rho_{10,10} \end{vmatrix},\ W_4 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} & \rho_{00,10} & \rho_{01,10} \\ \rho_{00,01} & \rho_{01,01} & \rho_{00,11} & \rho_{01,11} \\ \rho_{10,00} & \rho_{11,00} & \rho_{10,10} & \rho_{11,10} \\ \rho_{10,01} & \rho_{11,01} & \rho_{10,11} & \rho_{11,11} \end{vmatrix} \tag{7.5}$$

is negative; with $W_2 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} \\ \rho_{00,01} & \rho_{01,01} \end{vmatrix}$ being simultaneously non-negative, then the state ρ is inseparable. Where $\rho_{m\mu,n\nu}$ are the matrix elements of bipartite state ρ.

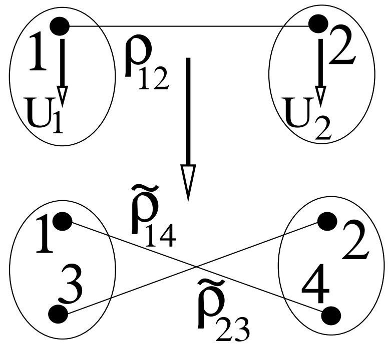


Figure 7.1: The figure shows the broadcasting of the state ρ_{12} into $\tilde{\rho}_{14}$ and $\tilde{\rho}_{23}$ through application of local cloning unitaries U_1 and U_2 on both sides.

7.3.1 Broadcasting of entanglement via local and nonlocal cloning operations

Local cloning: Each of the parties now individually apply a local copying operation on their own qubit i.e., $U_1 \otimes U_2$ to produce the state $\tilde{\rho}_{1234}$. The B-H state independent symmetric optimal cloning transformation (U^{l}_{bhsi}) used for local copying is obtained by putting $M = 2$ in Eq. (7.1) with $c = \sqrt{\frac{2}{3}}$ and $d = \sqrt{\frac{1}{6}}$. The corresponding basis vectors are $|\Psi_1\rangle = |0\rangle$ and $|\Psi_2\rangle = |1\rangle$. After we obtain the composite system $\tilde{\rho}_{1234}$, we trace out the qubits 2, 4 and 1, 3 to obtain the local output states $\tilde{\rho}_{13}(= \mathrm{Tr}_{24}[U_1 \otimes U_2(\rho_{12})])$ on A's side and $\tilde{\rho}_{24}(= \mathrm{Tr}_{13}[U_1 \otimes U_2(\rho_{12})])$ on B's side respectively. Similarly, after tracing out the local output states from the composite system, we have the nonlocal output states $\tilde{\rho}_{14}(= \mathrm{Tr}_{23}[U_1 \otimes U_2(\rho_{12})])$ and $\tilde{\rho}_{23}(= \mathrm{Tr}_{14}[U_1 \otimes U_2(\rho_{12})])$ [see FIG. (7.1)].

Nonlocal cloning: Here, the basic idea is that the entire state ρ_{12} (given in Eq. (7.4)) is in the same lab and the intention is to have more than one copy of it. In that process, we apply a global unitary operation U_{12} to produce $\tilde{\rho}_{1234}$. The B-H state independent optimal cloning transformation (U^{nl}_{bhsi}) used for nonlocal copying is obtained by substituting $M = 4$ in Eq. (7.1) with $c = \sqrt{\frac{2}{5}}$ and $d = \sqrt{\frac{1}{10}}$. In this case, the corresponding basis vectors are $|\Psi_1\rangle = |00\rangle$, $|\Psi_2\rangle = |01\rangle$, $|\Psi_3\rangle = |10\rangle$ and $|\Psi_4\rangle = |11\rangle$. Once we have the composite system $\tilde{\rho}_{1234}$, we trace out the qubits 3 and 4 to obtain the output state $\tilde{\rho}_{12}(= \mathrm{Tr}_{34}[U_{12}\rho_{12}])$ or the qubits 1 and 2 to obtain $\tilde{\rho}_{34}(= \mathrm{Tr}_{12}[U_{12}\rho_{12}])$. Next, proceeding in similar manner, we obtain the remaining states $\tilde{\rho}_{13}(= \mathrm{Tr}_{24}[U_{12}\rho_{12}])$ and $\tilde{\rho}_{24}(= \mathrm{Tr}_{13}[U_{12}\rho_{12}])$ by tracing out the qubits 2, 4 and 1, 3 from $\tilde{\rho}_{1234}$ respectively. We could have also chosen the diagonal pairs ($\tilde{\rho}_{14}$ & $\tilde{\rho}_{23}$) instead of choosing the pairs: $\tilde{\rho}_{12}$ & $\tilde{\rho}_{34}$ as our desired pairs. However, we refrain ourselves from choosing the pairs $\tilde{\rho}_{13}$ & $\tilde{\rho}_{24}$ as the desired pairs [221] [see FIG. (7.2)]. In principle, to broadcast the amount of entanglement between the desired pairs $(1, 4)/(1, 2)$ and $(2, 3)/(1, 4)$ we just maximize the entanglement between the output pairs, regardless of the states between $(1, 3)$ and $(2, 4)$. However, for optimal broadcasting of entanglement across parties we require to minimize the amount of entanglement within parties. This is because the total amount of entanglement (E) produced is the sum of the entanglement within parties (E_l) and the entanglement across the parties (E_{nl}), i.e

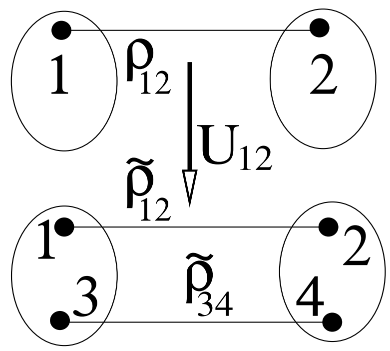


Figure 7.2: The figure shows the broadcasting of the state ρ_{12} into $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$ through application of a nonlocal (global) cloning unitary U_{12}.

$E = E_l + E_{nl}$. The amount of entanglement (E) is strictly less or equal to the total entanglement of the input state. To maximize E_{nl}, we must have $E_l = 0$. In other words, for optimal broadcasting we should have no entanglement between the qubits $(1, 3)$ and $(2, 4)$.

Definition 2.1: An entangled state ρ_{12} is said to be broadcast after the application of local cloning operation ($U_1 \otimes U_2$), if for some values of the input state parameters, the states $\{\tilde{\rho}_{14}, \tilde{\rho}_{23}\}$ are inseparable.

Definition 2.2: An entangled state ρ_{12} is said to be broadcast after the application of nonlocal cloning operation (U_{12}), if for some values of the input state parameters, the desired output states $\{\tilde{\rho}_{12}, \tilde{\rho}_{34}\}$ are entangled.

Definition 2.3: An entangled state ρ_{12} is said to be broadcast optimally after the application of local cloning operation ($U_1 \otimes U_2$), if for some values of the input state parameters, the states $\{\tilde{\rho}_{14}, \tilde{\rho}_{23}\}$ are inseparable and the states $\{\tilde{\rho}_{13}, \tilde{\rho}_{24}\}$ are separable.

Definition 2.4: An entangled state ρ_{12} is said to be broadcast optimally after the application of nonlocal cloning operation (U_{12}), if for some values of the input state parameters, the desired output states $\{\tilde{\rho}_{12}, \tilde{\rho}_{34}\}$ are entangled, and the remaining output states $\{\tilde{\rho}_{13}(= \mathrm{Tr}_{24}[U_{12}\rho_{12}]), \tilde{\rho}_{24}(= \mathrm{Tr}_{13}[U_{12}\rho_{12}])\}$ are separable.

If we consider the non-optimal broadcasting then the broadcasting range will increase whereas for optimal one the broadcasting range will be small. Let us consider a general pure two-qubit state in Schmidt form $|\psi_{12}\rangle = \sqrt{\lambda}|00\rangle\langle 00| + \sqrt{1-\lambda}|11\rangle\langle 11|$, where λ is Schmidt coefficient and $0 \le \lambda \le 1$. Now if we apply B-H local cloning operation ($U_1 \otimes U_2$) on this state, the local output states will only be separable when $L_- < \lambda < L_+$, where $L_\pm = \frac{1}{16}(8 \pm \sqrt{39})$ [196] and hence it is the optimal broadcasting range. If we relax the optimality condition i.e., $E_l \neq 0$ then we can easily conclude that the broadcasting of entanglement may be possible for greater range of λ. The same analysis is applicable for nonlocal cloning and same type of feature will appear. Next, we will discuss the optimal broadcasting of entanglement [196] in detail.

7.3.2 Optimal broadcasting of entanglement via local cloning

In this subsection, we deal with the problem of broadcasting of quantum entanglement by using local cloning transformation.

The local output states $\tilde{\rho}_{13}$ on A's side and $\tilde{\rho}_{24}$ on B's side respectively and are given in canonical representation by,

$$\tilde{\rho}_{13} = \left\{ \frac{2}{3}\vec{x}, \frac{2}{3}\vec{x}, \frac{1}{3}\mathbb{I}_3 \right\}, \; \& \; \tilde{\rho}_{24} = \left\{ \frac{2}{3}\vec{y}, \frac{2}{3}\vec{y}, \frac{1}{3}\mathbb{I}_3 \right\}, \tag{7.6}$$

where $\vec{x}$, $\vec{y}$ are the Bloch vectors of the initial state ρ_{12}.

Next, we apply Peres-Horodecki criterion to investigate whether these local output states on either side of these two parties are separable or not. After evaluating determinants W_2, W_3 and W_4 (as given in Eq. (7.5)) we obtain a range involving input state parameters within which the local outputs, $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$, are separable. These ranges for $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$ are

$$\begin{aligned} 0 \;\; &\leq \;\; \|\vec{x}\| \leq \frac{3}{4} \; \& \; \|\vec{x}\| \leq 1 + x_3 + x_3^2, \\ 0 \;\; &\leq \;\; \|\vec{y}\| \leq \frac{3}{4} \; \& \; \|\vec{y}\| \leq 1 + y_3 + y_3^2 \end{aligned} \tag{7.7}$$

respectively. Here $\|\vec{a}\| = \text{Tr}\left(a^\dagger a\right)$ with $\dagger$ denoting the Hermitian conjugate.

We have the nonlocal output states $\tilde{\rho}_{14}$ and $\tilde{\rho}_{23}$ as

$$\tilde{\rho}_{14} = \tilde{\rho}_{23} = \left\{ \frac{2}{3}\vec{x}, \frac{2}{3}\vec{y}, \frac{4}{9}T \right\}, \tag{7.8}$$

where $\vec{x}$, $\vec{y}$ are the Bloch vectors and T is the correlation matrix of the initial state ρ_{12}.

Again with the help of Peres-Horodecki criterion we find out the condition under which the nonlocal output states will be inseparable. This condition for inseparability of the states $\tilde{\rho}_{14}$ and $\tilde{\rho}_{23}$ involving input state parameters is given as,

$$\left(W_3^l < 0 \text{ or } W_4^l < 0\right) \text{ and } W_2^l \geqslant 0. \tag{7.9}$$

Here the explicit expressions of W_2^l, W_3^l and W_4^l are given by Eqs. (A-1), (A-2) and (A-3) in Appendix-1.

Now combining these two ranges determining the separability of the local states given by Eq. (7.7) and inseparability of the nonlocal states given by Eq. (7.9), we obtain the range for broadcasting of entanglement.

To exemplify our above study with a local cloner, we next consider two different classes of mixed entangled states, namely: (a) werner-like states [263] and (b) Bell-diagonal states [264] and then separately analyse their broadcasting ranges.

7.3.2.1 Example 2.1: Werner-like States

First of all, we consider the example of werner-like states. These states can more formally be expressed as,

$$\rho_{12}^{w} = \{\vec{x}^{w}, \vec{x}^{w}, T^{w}\}, \tag{7.10}$$

where $\vec{x}^{w} = \{0,\ 0,\ p\,(\alpha^2 - \beta^2)\}$ is the Bloch vector and the correlation matrix is $T^{w} = \mathrm{diag}(2p\alpha\beta, -2p\alpha\beta, p)$ with the condition $\alpha^2 + \beta^2 = 1$ and $0 \leqslant p \leqslant 1$. (Please note that whenever we use $M = \mathrm{diag}(.,.,.)$, we mean M is a diagonal matrix with diagonal elements given inside the first bracket.)

The local output states obtained after applying cloning operation on both the qubits 1 and 2 are given by,

$$\tilde{\rho}_{13} = \tilde{\rho}_{24} = \left\{\frac{2}{3}\vec{x}^{w},\ \frac{2}{3}\vec{x}^{w},\ \frac{1}{3}\mathbb{I}_3\right\}, \tag{7.11}$$

where $\vec{x}^{w}$ is the Bloch vector of the state ρ_{12}^{w}.

From Peres-Horodecki theorem, if follows that by using Eq. (7.5) the local output states will be separable if either of the following two conditions are satisfied,

$$\begin{aligned} 0 &\leqslant p \leqslant \frac{\sqrt{3}}{2} \ \&\ 0 \leqslant \alpha^2 \leqslant 1, \text{ Or,} \\ \frac{\sqrt{3}}{2} < p &\leqslant 1 \ \&\ \frac{2p - \sqrt{3}}{4p} \leqslant \alpha^2 \leqslant \frac{\sqrt{3} + 2p}{4p}. \end{aligned} \tag{7.12}$$

Similarly after cloning, we have the nonlocal output states as,

$$\tilde{\rho}_{14} = \tilde{\rho}_{23} = \left\{\tfrac{2}{3}\vec{x}^{w},\ \tfrac{2}{3}\vec{x}^{w},\ \tfrac{4}{9}T^{w}\right\}, \tag{7.13}$$

where $\vec{x}^{w}$ is Bloch vector and T^{w} is the correlation matrix of the state ρ_{12}^{w}.

Using Peres-Horodecki theorem, the inseparability range of these nonlocal output states turn out to be,

$$\frac{3}{4} < p \leq 1 \ \&\ N_{-} < \alpha^2 < N_{+}, \tag{7.14}$$

where $N_{\pm} = \frac{1}{16}\{8 \pm (48 - \frac{81}{p^2} + \frac{72}{p})^{\frac{1}{2}}\}$. On merging this inseparable zone along with the separable zone given by Eq. (7.12) we discover that the broadcasting range is exactly same as the inseparability range given by Eq. (7.14). In FIG. 7.3, we depict this broadcastable zone (given by Eq. (7.14)) among the allowed region of input state parameters p and α.

7.3.2.2 Example 2.2: Bell-diagonal States

Here our initial resources are Bell-diagonal states to the local cloner which can be formally expressed as,

$$\rho_{12}^{b} = \left\{\vec{0}, \vec{0}, T^{b}\right\}, \tag{7.15}$$

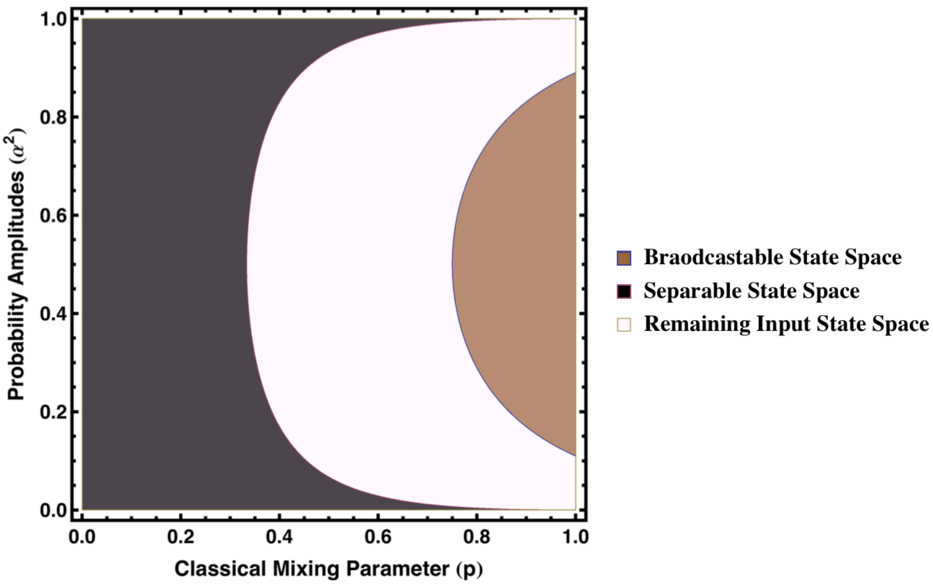


Figure 7.3: The figure illustates the states which can be used for broadcasting of entanglement via local cloning out of the total input state space of werner-like states ρ_{12}^{w}.

where $\vec{0}$ is the Bloch vector which is a null vector and the correlation matrix is $T^b = \text{diag}(c_1, c_2, c_3)$ with $-1 \leqslant c_i \leqslant 1$.

The above input Bell-diagonal state can be rewritten as [264], $\rho_{12}^{b} = \sum_{m,n} \lambda_{mn} |\gamma_{mn}\rangle \langle\gamma_{mn}|$ where the four Bell states $|\gamma_{mn}\rangle \equiv (|0,n\rangle + (-1)^m |1, 1 \oplus n\rangle) / \sqrt{2}$ represents the eigenstates of ρ_{12}^{b} with eigenvalues,

$$\lambda_{mn} = \frac{1}{4}\left[1 + (-1)^m c_1 - (-1)^{(m+n)} c_2 + (-1)^n c_3\right].$$

Also, for ρ_{12}^{b} to be a valid density operator, its eigenvalues have to be positive, i.e. $\lambda_{mn} \geqslant 0$.

Once again by applying local cloning and tracing out the qubits we get the local output states as:

$$\tilde{\rho}_{13} = \tilde{\rho}_{24} = \left\{\vec{0}, \vec{0}, \frac{1}{3}\mathbb{I}_3\right\}. \tag{7.16}$$

It turns out that for these local output states both W_3 as well as W_4 given by Eq. (7.5) are non-negative and independent of the input state parameters (c_i's). Hence, $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$ will always remain separable.

On the other hand, the nonlocal outputs are given by,

$$\tilde{\rho}_{14} = \tilde{\rho}_{23} = \left\{ \vec{0}, \vec{0}, \frac{4}{9} T^b \right\}, \tag{7.17}$$

where T^b is the correlation matrix of the state ρ_{12}^b.

The inseparability range for these nonlocal output states of the input Bell-diagonal state ρ_{12}^b in terms of c_i's, is given by

$$\begin{aligned} -1 \leq c_1 < -\frac{1}{4} \;\&\; \left(\gamma < -\frac{9}{4} \text{ or } \frac{9}{2} - c_- < c_2 \leq 1 \right) \\ \text{Or, } \frac{1}{4} < c_1 \leq 1 \;\&\; \left(c_- < c_2 \leq 1 \text{ or } -1 \leq c_2 < c_+ \right), \end{aligned} \tag{7.18}$$

along with the condition that $\lambda_{mn} \geqslant 0$, where $c_\pm = \mp\frac{9}{4} \pm (c_1 \pm c_3)$ and $\gamma = \mathrm{Tr}(T^b)$. It is evident that the broadcasting range of the Bell-diagonal state is same as the inseparability range in Eq. (7.18) since the local output states in this case are always separable.

In FIG. 7.4, we depict the above broadcastable zone (given by Eq. (7.18)) within the permissible region of the input state parameters, specified by the 3-tuple (c_1, c_2, c_3) from Eq. (7.15). Now for $-1 \leqslant c_i \leqslant 1$, where $i = \{1, 2, 3\}$, the condition that ρ_{12} is necessarily a positive operator, i.e. $\lambda_{mn} \geqslant 0$, results in giving a tetrahedral geometrical representation of Bell-diagonal states $\mathscr{T}$ whose four vertices are the four Bell states or the eigenstates $|\gamma_{mn}\rangle$. The separable part within the geometry of Bell-diagonal states $\mathscr{T}$ comes out to be an octahedron $\mathscr{O}$ which is specified by the relation $|c_1|+|c_2|+|c_3| \leqslant 1$ or $\lambda_{mn} \leqslant \frac{1}{2}$. Within the tetrahedron $\mathscr{T}$, the four entangled (inseparable) zones lie outside the octahedron $\mathscr{O}$, one from each vertex of $\mathscr{T}$ with the value of λ_{mn} being greatest at the vertex points for each of them [264]. Interestingly, we discover that the broadcastable zone procured by using the above broadcasting condition in Eq. (7.18) turns out to be cones $\mathscr{C}$s, fitting as small caps on these entangled zones of the tetrahedron $\mathscr{T}$. It is also consistent with the fact that the maximally entangled states $|\gamma_{mn}\rangle$ lie at the vertices of $\mathscr{T}$, so the broadcastable regions start from those and vanish on the way towards the separable part $\mathscr{O}$. This is because the amount of entanglement keeps decreasing in the same direction. In other words, the states beyond the conic regions ($\mathscr{C}$s) lack the amount of initial entanglement required to be able to broadcast the same by local cloning operations. It is interesting to observe that if $c_i = -1$ then $c_j = c_k$ and if $c_i = 1$ then $c_j = -c_k$ where for each case $-1 \leqslant c_j\ (c_k) < -\frac{5}{8}$ or $\frac{5}{8} < c_j\ (c_k) \leqslant 1$ with $i \neq j \neq k$ and $i, j, k = \{1, 2, 3\}$. This happens due to the symmetry of the Bell-diagonal states and that of the conic broadcasting zones as depicted in FIG. 7.4. For the same reason, we also find that the four $\mathscr{C}$s or the conic zones grow symmetrically and uniformly from c_i's = -1 (1) and ceases to exist for any value equal or beyond $-\frac{5}{8}$ ($\frac{5}{8}$).

7.3.3 Optimal broadcasting of entanglement via nonlocal cloning

In this subsection, we reconsider the problem of broadcasting of entanglement but this time by using nonlocal cloning transformation.

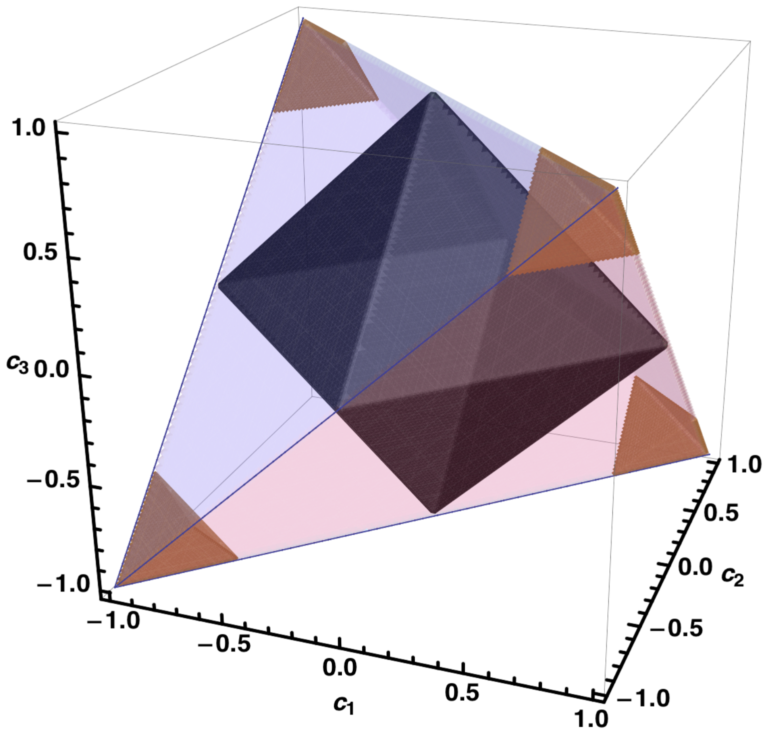


Figure 7.4: The figure illustates the broadcastable region obtained using local cloning operations within the geometry of Bell-diagonal states ρ^b_{12}. The translucent tetrahedron $\mathscr{T}$ hosts the Bell states $|\gamma_{mn}\rangle$ at the vertex tuples (-1,-1,-1), (1,1,-1), (1,-1,1) and (-1,1,1) from each of which a (brown) cone $\mathscr{C}$ emerges marking the broadcastable zones. The (black) octahedron $\mathscr{O}$ in the middle of the tetrahedron $\mathscr{T}$ depicts the separable region within the Bell-diagonal state space.

The obtained nonlocal output states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$ are identical and they can be represented as,

$$\tilde{\rho}_{12} = \tilde{\rho}_{34} = \left\{ \frac{3}{5}\vec{x}, \frac{3}{5}\vec{y}, \frac{3}{5}T \right\} \tag{7.19}$$

where $\vec{x}$, $\vec{y}$ are the Bloch vectors and T is the correlation matrix of the state ρ_{12}.

We apply the Peres-Horodecki criteria to find out the condition on input state parameters under which the above output states ($\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$) will be inseparable. This condition of inseparability turns out to be,

$$W_3^{nl} < 0 \text{ or } W_4^{nl} < 0 \;\&\; W_2^{nl} \geqslant 0, \tag{7.20}$$

where the explicit expressions of W_2^{nl}, W_3^{nl} and W_4^{nl} are given by Eqs. (A-5), (A-6) and (A-7) in Appendix-2.

Next, the remaining states $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$ are given by,

$$\tilde{\rho}_{13} = \left\{\tfrac{3}{5}\vec{x}, \tfrac{3}{5}\vec{x}, \tfrac{1}{5}\mathbb{I}_3\right\}, \ \& \ \tilde{\rho}_{24} = \left\{\tfrac{3}{5}\vec{y}, \tfrac{3}{5}\vec{y}, \tfrac{1}{5}\mathbb{I}_3\right\} \tag{7.21}$$

where, $\vec{x}$ and $\vec{y}$ are the Bloch vectors of the state ρ_{12}.

Similarly, here also we apply the Peres-Horodecki criterion to see whether these output states are separable or not. After evaluating determinants W_2, W_3 and W_4 (as given in Eq. (7.5)) we obtain a range involving input state parameters for which the output states, $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$, are separable. This range is given by,

$$\begin{aligned} 0 \leq \|\vec{x}\| \leq \frac{8}{9} \ \& \ \|\vec{x}\| - x_3^2 \leq \frac{4}{3}(1 + x_3), \\ 0 \leq \|\vec{y}\| \leq \frac{8}{9} \ \& \ \|\vec{y}\| - y_3^2 \leq \frac{4}{3}(1 + y_3) \end{aligned} \tag{7.22}$$

respectively.

Now, clubbing the two ranges given by Eq. (7.20) and Eq. (7.22), we obtain the range for broadcasting of entanglement for ρ_{12} via nonlocal copying.

Next, in order to exemplify our study with nonlocal cloner we look into the broadcasting ranges of two different classes of input states: (a) Werner-like states [263] and (b) Bell-diagonal states [264].

7.3.3.1 Example 3.1: Werner-Like State

Quite similar to the previous section, here we reconsider the class of werner-like states given earlier by Eq. (7.10) and apply nonlocal cloning operation on it.

After cloning, the desired output states are given by,

$$\tilde{\rho}_{12} = \tilde{\rho}_{34} = \left\{\frac{3}{5}\vec{x}^w, \frac{3}{5}\vec{x}^w, \frac{3}{5}T^w\right\}, \tag{7.23}$$

where, $\vec{x}^w$ is the Bloch vector and T^w is the correlation matrix of the state ρ_{12}^w. The inseparability range for these states is given by,

$$\frac{5}{9} < p \leqslant 1 \ \text{and} \ H_- < \alpha^2 < H_+, \tag{7.24}$$

where $H_\pm = \frac{1}{2} \pm \{\frac{1}{144p}(27p^2 + 30p - 25)\}^{\frac{1}{2}}$. The remaining output states are given by,

$$\tilde{\rho}_{13} = \tilde{\rho}_{24} = \left\{\frac{3}{5}\vec{x}^w, \frac{3}{5}\vec{x}^w, \frac{1}{5}\mathbb{I}_3\right\}, \tag{7.25}$$

where $\vec{x}^w$ is the Bloch vector of the state ρ_{12}^w. These output states will be separable if either of the following two conditions are satisfied,

$$\begin{aligned} 0 \leqslant p \leqslant d \,\&\, (0 &\leqslant \alpha^2 \leqslant \xi_-, \ \text{or} \ \xi_+ < \alpha^2 \leqslant 1), \\ \text{Or}, 0 &\leqslant p \leqslant 1 \,\&\, \xi_- < \alpha^2 \leqslant \xi_+, \end{aligned} \tag{7.26}$$

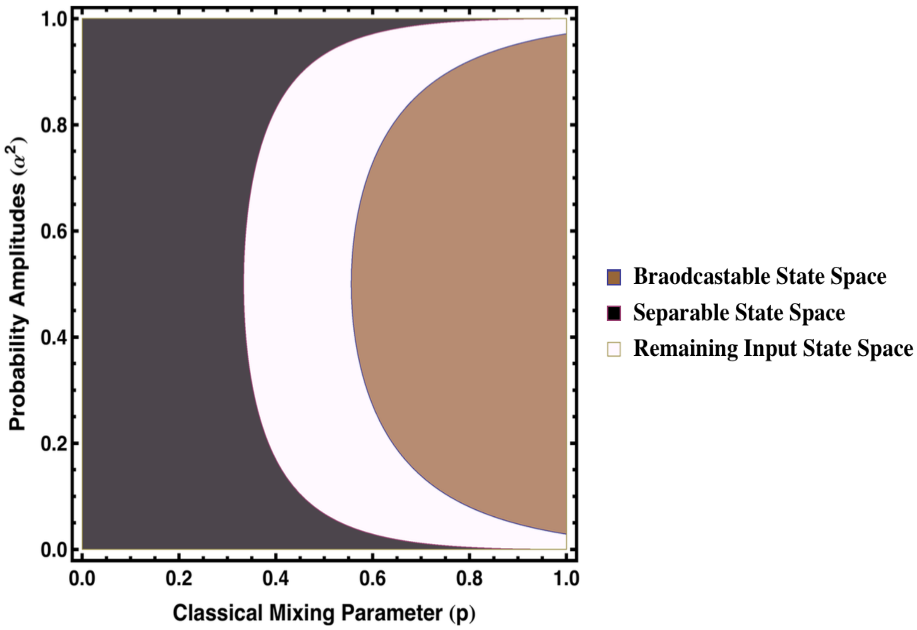


Figure 7.5: The figure illustates the states which can be used for broadcasting of entanglement via nonlocal cloning out of the total input state space of werner-like states ρ_{12}^{w}.

where $d = \sqrt{\frac{8}{9(1-2\alpha^2)^2}}\ \xi_{\pm} = \frac{1}{6}(3 \pm 2\sqrt{2})$

After merging the separability and inseparability conditions given by Eq. (7.26) and Eq. (7.24) respectively, the broadcasting range of the werner-like state turns out to be same as the inseparability range and is thus given by Eq. (7.24).

In FIG. 7.5, we demarcate this broadcastable zone, given by Eq. (7.24), amidst the prescribed region of input state space.

7.3.3.2 Example 3.2: Bell-diagonal states

In this example, we once again consider the Bell-diagonal states (given earlier by Eq. (7.15)) as our initial entangled state.

Once the nonlocal cloner is applied to it we have the desired output states as,

$$\tilde{\rho}_{12} = \tilde{\rho}_{34} = \left\{\vec{0}, \vec{0}, \frac{3}{5}T^b\right\}, \tag{7.27}$$

where T^b is the the correlation matrix of the state ρ_{12}^{b}.

The inseparability range of the desired output states is given by,

$$\begin{gathered}(6c_1 - 3\gamma + 5)(3\gamma - 6c_3 - 5)(3\gamma - 6c_2 - 5)(3\gamma + 5) < 0 \\ \text{Or, } (3c_3 + 5)\left((5 - 3c_3)^2 - 9(c_1 - c_2)^2\right) < 0\end{gathered} \tag{7.28}$$

where $\gamma = \mathrm{Tr}(T^b)$ along with the condition that $\lambda_{mn} \geqslant 0$ from the positivity of input density operator ρ_{12}.

The remaining output states are given by,

$$\tilde{\rho}_{13} = \tilde{\rho}_{24} = \left\{\vec{0}, \vec{0}, \frac{1}{5}\mathbb{I}_3\right\}. \tag{7.29}$$

These output states are independent of the input state parameter (c_i's) and will be always separable since for them the W_3 and W_4 from Eq. (7.5) comes out to be a positive number. Hence, the broadcasting range of the Bell-diagonal state is same as the inseparability range as given in Eq. (7.28).

Quite analogous to our geometric analysis in local copying case of the broadcasting region of Bell-diagonal state, in FIG. 7.6, we depict the above broadcastable zone (given by Eq. (7.28)) among the allowed region of the input state parameters, specified by the 3-tuple (c_1, c_2, c_3) from Eq. (7.15). Similarly as in the case with local cloners, here also we notice that if $c_i = -1$ then $c_j = c_k$ and if $c_i = 1$ then $c_j = -c_k$ where for each case $-1 \leqslant c_j\ (c_k) < -\frac{1}{3}$ or $\frac{1}{3} < c_j\ (c_k) \leqslant 1$ with $i \neq j \neq k$ and $i, j, k = \{1, 2, 3\}$. This happens due to the symmetry of the Bell-diagonal states and that of the conic broadcasting zones as depicted in FIG. 7.6. For the same reason, we also find that the four $\mathscr{C}$s or the conic zones grow symmetrically and uniformly from c_i's = -1 (1) and ceases to exist for any value equal or beyond $-\frac{1}{3}$ ($\frac{1}{3}$).

Interestingly, here we find for the above two cases that the use of a nonlocal cloner despite being difficult to implement gives us a much wider broadcasting range for entanglement. In nonlocal cloning of entanglement, the bipartite system as a whole gets entangled with a single cloning machine, whereas in local cloning each individual subsystem separately gets entangled with a cloning machine. A larger amount of entanglement transfer to the machine takes place in the local cloning case. So indeed it is not surprising that nonlocal cloning will produce a wider range for broadcasting of entanglement than the local cloning [198].

7.4 Broadcasting of Quantum Correlations Beyond Entanglement

In this section, we consider broadcasting of quantum correlations which go beyond the notion of entanglement. Here, we analyse the possibility of creating more number of lesser correlated quantum states from an intial quantum state having correlations using cloning operations. Here, we use geometric discord [127] particularly to quantify the amount of QCsbE present in between a pair of qubits although our results hold for any measures of discord (QCsbE). However, the whole analysis is not limited to gemetric discord only.

It is well known that geometric discord (GD) can increase under local unitary e.g., under a simple channel Λ: $\rho \to \rho \otimes \sigma$, i.e., a channel which introduces an ancilla only [265, 266]. In order to overcome this, it was suggested that we can use different distance measures

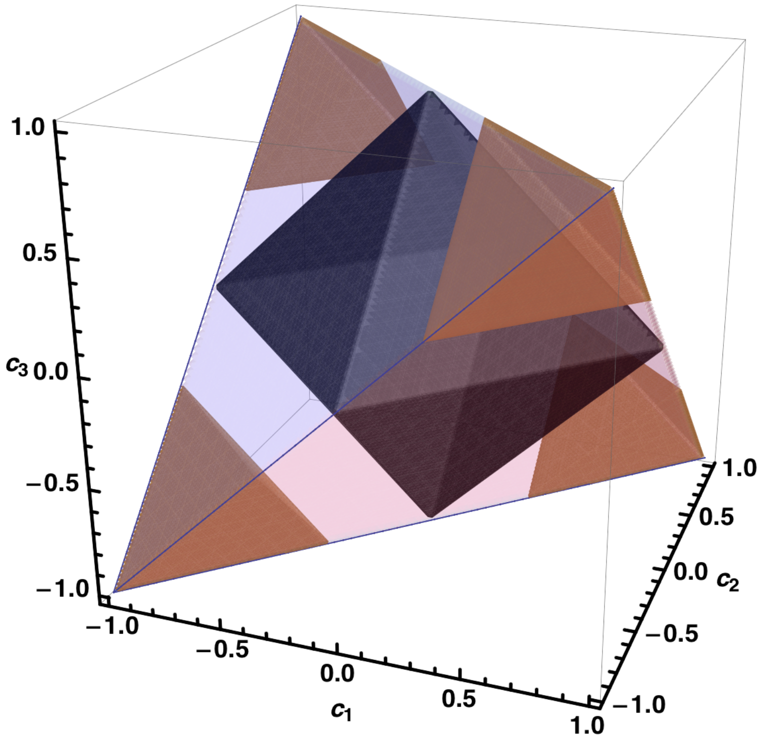


Figure 7.6: The figure illustates the broadcastable region obtained using nonlocal cloning operations within the geometry of Bell-diagonal states ρ_{12}^{b}. The translucent tetrahedron $\mathscr{T}$ hosts the Bell states $|\gamma_{mn}\rangle$ at the vertex tuples (-1,-1,-1), (1,1,-1), (1,-1,1) and (-1,1,1) from each of which a (brown) cone $\mathscr{C}'$ emerges marking the broadcastable zones. The (black) octahedron $\mathscr{O}$ in the middle of the tetrahedron $\mathscr{T}$ depicts the separable region within the Bell-diagonal state space. Interestingly enough, by the use of nonlocal cloner we find that the height broadcastable conic regions have increased considerably compared to that obtained in FIG. 7.4 with local cloners.

(norms) which will overcome this shortcoming [267–269]. Although information theoretic discord is invariant under local unitary, in general QCsbE are not monotone under any local operations [4, 207]. According to Streltsov *et al.* [207]: *A local quantum channel acting on a single qubit can create QCsbE in a multiqubit system if and only if it is not unital.*

Hence, we discuss the broadcasting of QCsbE under two types of channel a) unital channel (Λ_u): $\mathbb{I} \rightarrow \mathbb{I}$ and b) non-unital channel Λ_{nu}: $\mathbb{I} \nrightarrow \mathbb{I}$. We will call this type of operations on the bonafied states as 'processing': 'pre-pocessing' (applying the channel on the input state before broadcasting) or 'post-processing' (applying the channel on the output states after broadcasting).

7.4.1 Definition of broadcasting of QCsbE via. local and nonlocal cloning operations

Here, we define what we mean by the broadcasting of QCs by using state independent (optimal) and state dependent B-H cloning machines. These cloning machines are applied both locally and nonlocally.

The scenario of broadcasting of QCsbE is similar to that of broadcasting of entanglement (see Fig. (7.1 & 7.2)). Let Q be the total amount of QCsbE produced as a result of both local or non local cloning and the sum of the QCsbE within parties (Q_l) and the QCsbE across the parties (Q_{nl}) then $Q = Q_l + Q_{nl}$. To maximize Q_{nl}, we must have $Q_l = 0$.

Definition 3.3.1: A quantum correlated state ρ_{12} is said to be broadcast after the application of local cloning operation ($U_1 \otimes U_2$), if for some values of the input state parameters, the amount of QCsbE for the states $\{\tilde{\rho}_{14}, \tilde{\rho}_{23}\}$ are non-vanishing.

Definition 3.3.2: A quantum correlated state ρ_{12} is said to be broadcast after the application of nonlocal cloning operation (U_{12}), if for some values of the input state parameters, the QCsbE for the states $\{\tilde{\rho}_{12}, \tilde{\rho}_{34}\}$ are non-vanishing.

Definition 3.3.3: A quantum correlated state ρ_{12} is said to be optimally broadcast after the application of local cloning operation ($U_1 \otimes U_2$), if for some values of the input state parameters, the QCsbE for the states $\{\tilde{\rho}_{14}, \tilde{\rho}_{23}\}$ are non-vanishing and for the states $\{\tilde{\rho}_{13}, \tilde{\rho}_{24}\}$, the amount of QCsbE are zero.

Definition 3.3.4: A quantum correlated state ρ_{12} is said to be optimally broadcast after the application of nonlocal cloning operation (U_{12}), if for some values of the input state parameters, the QCsbE for the states $\{\tilde{\rho}_{12}, \tilde{\rho}_{34}\}$ are non-vanishing whereas for the states $\{\tilde{\rho}_{13}, \tilde{\rho}_{24}\}$, the QCsbE are zero.

7.4.2 Optimal Broadcasting of QCsbE via. local and nonlocal cloning operations under unital channel (Λ_u)

In this subsection, we investigate the problem of broadcasting of QCsbE by using state independent (optimal) and state dependent B-H cloning machines under the unital channel (Λ_u). These cloning machines are applied both locally and nonlocally. As QCsbE are non-incrasing under Λ_u, it is evident that we need not to mention it everytime.

7.4.2.1 Broadcasting of correlations using Buzek-Hillery (B-H) local cloners

Here we use B-H state independent optimal (U^l_{bhsi}) and state dependent (U^l_{bhsd}) cloning operation locally (given by Eq. (7.1)) and we find that it is possible to broadcast QCsbE by such methods but contrary to the broadcasting of entanglement, we will not have optimal one.

Theorem 7.1 *Given a two qubit general mixed state ρ_{12} and B-H local cloning transformations (state independent optimal U^l_{bhsi} or state dependent U^l_{bhsd}), it is impossible to broadcast the QCsbE optimally within ρ_{12} into two lesser quantum correlated states: $\{\tilde{\rho}_{14}, \tilde{\rho}_{23}\}$.*

Proof: When B-H state dependent cloning transformation U^{l}_{bhsd} (given by Eq. (7.1)) is applied locally to clone the qubits '$1 \rightarrow 3$' and '$2 \rightarrow 4$' of an input most general mixed quantum state ρ_{12}, then we have the local output states as, $\tilde{\rho}_{13} = \{\mu\vec{x},\ \mu\vec{x},\ T^{sd}_{l}\}$ and $\tilde{\rho}_{24} = \{\mu\vec{y},\ \mu\vec{y},\ T^{sd}_{l}\}$; where $T^{sd}_{l} = \mathrm{diag}(2\lambda, 2\lambda, 1-4\lambda)$ and the nonlocal output states, $\tilde{\rho}_{14} = \tilde{\rho}_{23} = \{\mu\vec{x},\ \mu\vec{y},\ \mu T\}$. Here $\mu = 1-2\lambda$; $\vec{x}$ and $\vec{y}$ represent the Bloch vectors and T denotes the correlation matrix of the input state ρ_{12}. The GD D_G, calculated using Eq. (6.4), of the local output states are given by $D_G(\tilde{\rho}_{13}) = \frac{1}{2}\left(1+\mu^2\|\vec{x}\| - 8\lambda + 20\lambda^2\right)$ and $D_G(\tilde{\rho}_{24}) = \frac{1}{2}\left(1+\mu^2\|\vec{y}\| - 8\lambda + 20\lambda^2\right)$ which always remains non-vanishing for $0 \leqslant \lambda \leqslant \frac{1}{2}$. This is because the minima of $D_G(\tilde{\rho}_{13})$ and $D_G(\tilde{\rho}_{24})$ come out to be $D^{min}_{G} = \frac{w}{2} - \frac{2}{5}$ at $\lambda = \frac{1}{5}$; where $w = 1+\mu^2\|\vec{x}\|$ or $w = 1+\mu^2\|\vec{y}\|$, giving $w \geqslant 1$ and ensuring always that $D^{min}_{G} > 0$.

Hence we will never have optimal broadcasting of QCsbE although it is possible that we can have task oriented one.

7.4.2.2 Broadcasting of correlations using Buzek-Hillery (B-H) nonlocal cloners

In this approach, we use symmetric B-H state independent optimal (U^{nl}_{bhsi}) as well as state dependent (U^{nl}_{bhsd}) nonlocal cloning operations (given by Eq. (7.1)) and we find that, here also it is possible to broadcast QCsbE by such approaches but not the optimal one.

Theorem 7.2 *Given a two qubit general mixed state ρ_{12} and B-H nonlocal cloning transformations (state independent optimal U^{nl}_{bhsi} or state dependent U^{nl}_{bhsd}), it is impossible to broadcast the QCsbE optimally within ρ_{12} into two lesser quantum correlated states: $\{\tilde{\rho}_{12},\ \tilde{\rho}_{34}\}$.*

Proof: When B-H state dependent nonlocal cloning transformation U^{nl}_{bhsd} (given by Eq. (7.1)) is applied to clone the qubits 1 & 2 of an input most general mixed two qubit state ρ_{12} (given in Eq. (7.4)), then we have the output states, $\tilde{\rho}_{13} = \{\mu\vec{x},\ \mu\vec{x},\ T^{sd}_{nl}\}$ and $\tilde{\rho}_{24} = \{\mu\vec{y},\ \mu\vec{y},\ T^{sd}_{nl}\}$; where $T^{sd}_{nl} = \mathrm{diag}(2\lambda, 2\lambda, 1-8\lambda)$ and the desired output states, $\tilde{\rho}_{12} = \tilde{\rho}_{34} = \{\mu\vec{x},\ \mu\vec{y},\ \mu\, T\}$; where $\mu = 1-4\lambda$. Here $\vec{x}$ as well as $\vec{y}$ represent the Bloch vectors and T denotes the correlation matrix of the input state. The GD D_G, calculated using Eq. (6.4), of the local output states are given by: $D_G(\tilde{\rho}_{13}) = \frac{1}{2}\left(1+\mu^2\|\vec{x}\| - 16\lambda + 68\lambda^2\right)$ and $D_G(\tilde{\rho}_{24}) = \frac{1}{2}\left(1+\mu^2\|\vec{y}\| - 16\lambda + 68\lambda^2\right)$ which always remains non-vanishing for $0 \leqslant \lambda \leqslant \frac{1}{4}$. This is because the minima of $D_G(\tilde{\rho}_{13})$ and $D_G(\tilde{\rho}_{24})$ come out to be $D^{min}_{G} = \frac{1+5w}{34+8w}$ at $\lambda = \frac{2+w}{17+4w}$; where $w = \|\vec{x}\|$ or $w = \|\vec{y}\|$, giving $0 \leqslant w \leqslant 1$ and ensuring always that $D^{min}_{G} > 0$. Hence we will never have optimal broadcasting of QCsbE although it is possible that we can have task oriented one.

Now moving beyond the realms of the above theorems, we claim that *if in the case of B-H state independent optimal cloners, when applied locally or nonlocally, we are unable to broadcast the QCsbE optimally then no other state independent deterministic cloner can do so*. It is mainly because of the recent result by Sazim *et al.* that for a given input state, the outputs of an optimal cloner are least correlated since as the fidelity of cloning increases the correlations transfer to the machine state also grows [184]. Again in 2003, Ghiu *et al.* showed that entanglement is optimally broadcast and maximal fidelities of the two final entangled states are obtained only when symmetric cloning machines are applied [262]. So by combining the above two results by Sazim *et al* and Ghiu *et al.*, we can logically infer that *even asymmetric Pauli cloning machines will be unable to broadcast QCsbE optimally since for those also the local outputs will always possess non-vanishing GD* [184, 262]. This enables us to comprehensively conclude that optimal broadcasting of QCsbE for any two qubit state via cloning operations is impossible.

7.4.3 Optimal Broadcasting of QCsbE via. local and nonlocal cloning operations under Nonunital channel (Λ_{nu})

In this subsection, we will discuss the possibilities and impossibilities of broadcasting QCsbE under non-unital channel (Λ_{nu}). Here many situations can occur depending on the free will of the parties: a) pre-possesing the state with unital channel & post-processing with non-unital channel, b) pre-processing with non-unital channel & post-processing with unital channel, and c) pre- & post-procesing with nonunital channel. All these situations are equivalent in the sense that QCsbE can increase under Λ_{nu}.

It is also evident that we can have task oriented broadcasting of QCsbE and can increase the QCsbE of the broadcasted states if needed. *And conceptually the notion of optimal broadcasting of QCsbE is not clear as we can have quantum correlated broadcast states although we start with totally classical correlated states.*

7.5 Conclusion

In literature, generalized approaches exist for purification or compression of entanglement procedures but no such generalization exists for broadcasting (decompression) of entanglement via cloning operations [196,270]. Such a study can aid in discovering operational meaning of quantifying the amount of entanglement [3]. In a nutshell, in this chapter, we present a holistic picture of broadcasting of quantum entanglement via cloning from any input two qubit state. We explicitly provide a set of ranges in terms of input state parameters for a most general representation of two qubit states for which broadcasting of entanglement will be possible. We exemplify our generalized results by examining them for two class of states: (a) Werner-like and (b) Bell-diagonal. We perform this study with both type of cloning techniques, local and nonlocal, to examine how the range of broadcasting increases under nonlocal cloning operations. Thereafter, we focus on the question whether broadcasting of QCsbE via cloning operations is possible or not. Contrary to the broadcasting of entanglement, we find that it is impossible to broadcast such QCsbE optimally via cloning operations, whether local or nonlocal, from a given quantum mechanically correlated pair to two lesser correlated pairs. But we can have task oriented broadcasting for QCsbE. We also explicitly reason out why the local outputs from cloner (state dependent or state independent) will never possess vanishing QCsbE which is imperative to broadcast QCsbE. However, we can intuitively conjecture that if one tries to broadcast QCsbE to more than two pairs, say N pairs, from an initial two qubit state then for some $N > 2$ pairs there is possibility of success in broadcasting such correlations optimally. This is because the nonlocal outputs become unentangled when $1 \to 3$ and $1 \to 7$ pairs are generated by the optimal local and nonlocal cloners respectively, which hints that the QCsbE in the output states decreases as more pairs are produced by the cloner [198].

Our findings brings out a fundamental difference between the correlation defined from the perspective of entanglement and the correlation measure which claims to go beyond entanglement.

Appendix-1: Inseparability range of nonlocal outputs obtained using local cloners

In this part, we evaluate the determinants W_2, W_3 and W_4 (as given in Eq. (7.5)) of the Peres-Horodecki criterion for the states $\tilde{\rho}_{14}$ and $\tilde{\rho}_{23}$ given by Eq. (7.8), and denote them as W_2^l, W_3^l and W_4^l respectively. The mathematical expressions of these determinants are given as follow,

$$W_2^l = -\frac{1}{6^4}\left[4\sum_{i=1}^{3}(-1)^{\delta_{3i}}\left(t_{3i}+3y_i\right)^2 + 9\left(2x_3+3\right)^2\right], \tag{A-1}$$

$$\begin{aligned} W_3^l = \frac{1}{3^6}\Bigg[& t_{33}\left(\sum_{i=1}^{3} t_{i3}^2 + \sum_{i=1}^{2} t_{3i}^2 - \sum_{i,j}^{2} t_{ij}^2\right) - \frac{9}{4}\left\{t_{33}\Upsilon_t + 3\sum_{i=1}^{3}\left(t_{i3}x_i + t_{3i}y_i\right)\right\} \\ & +\frac{3}{2}\left\{3\left(\Omega - \sum_{i=1}^{2}\left\{2t_{i3}x_i y_3 - \sum_{j=1}^{2}\left(t_{ij}x_j y_i - t_{ii}x_j y_j\right)\right\}\right) - \sum_{j=1}^{3} g_3\Gamma_j\right\} \\ & +3\sum_{i\neq j}^{2}\left\{(t_{ii}-t_{jj})\Phi_i + (t_{ij}-t_{ji})\Phi_j + \tilde{\Phi}_i t_{33}\right\} + 2\sum_{i,j}^{2} t_{ij}t_{i3}t_{3j}\Bigg] + L_f, \end{aligned} \tag{A-2}$$

$$\begin{aligned} W_4^l = \frac{1}{6^8}\Bigg[& K_2 + 6^4(4\Omega - \Upsilon_t) + \frac{2}{9}\Bigg\{\frac{1}{2}\Xi_t - \sum_{i,j}^{2}\sum_{p=j+1,q=i+1}^{3}\left(t_{ij}^2 t_{qp}^2 - 4t_{ij}t_{ip}t_{qj}t_{qp}\right) \\ & + \sum_{j=1}^{3}\sum_{p=j+1,q}^{3,j} S_q t_{1j}^2 t_{pq}^2 + 4\left\{\Pi - \sum_{i<j}^{3}\left(x_i x_j \sum_{p=1}^{3} t_{ip}t_{jp} + y_i y_j \sum_{p=1}^{3} t_{pi}t_{pj}\right)\right\} \\ & + \sum_{i=1}^{2} t_{2i}^2 t_{3i}^2 + \sum_{i\geqslant j}^{2}\sum_{p=j+1}^{3} t_{ij}^2 t_{ip}^2 + \frac{9}{4}\sum_{p=1}^{3}\sum_{i,j}^{3}\left\{(-1)^{\delta_{pi}}x_p + (-1)^{\delta_{pj}}y_p\right\}t_{ij}^2\Bigg\}\Bigg], \end{aligned} \tag{A-3}$$

where δ is the determinant of correlation matrix T of the initial state ρ_{12}, $\Phi_i = x_i t_{3i} + y_i t_{i3}$, $\Upsilon_t = \sum_{ij=1}^{3} t_{ij}^2$, $\Omega = \sum_{ij=1}^{3} t_{ij}x_i y_j$, $\Pi = \sum_{ij=1}^{3} x_i y_j \sum_{p\neq i,j}^{3}\left(t_{ji}t_{pp} - t_{jp}t_{pi}\right)$ $\Xi_t = \sum_{ij=1}^{3} t_{ij}^4$, $\tilde{\Phi}_i = x_i t_{i3} + y_i t_{3i}$, $\Gamma_j = \sum_{i=1}^{2}\{t_{ij}^2 - x_3 t_{3j}^2 - y_3 t_{j3}^2\}$, $L_f = \frac{1}{6^6}(3^6 + 2^6 L_\delta)$, $L_\delta = 2\delta + 3g_{33} + \frac{9}{4}L_5$, $g_3 = (x_3 + y_3)$, $L_1 = \gamma_+ - 2(x_3^2 + y_3^2)$, $L_2 = \frac{9}{4}(t_{33} + \gamma_+ - 2x_3 y_3)$, $L_3 = x_3 + y_3(\gamma_- + \frac{9}{4})$, $L_4 = -L_2 + \frac{3}{2}L_3$, $L_5 = t_{33}L_1 + L_4$, $g_{33} = (g_3 + \frac{3}{2})C_{33}$, $S_q = (-1)^{1-\delta_{jq}}$, $\gamma_\pm = \|\vec{x}\| \pm \|\vec{y}\|$, $K_1 = \gamma_-^2 + \frac{9}{4}\gamma_+$, $K_2 = 3^8 + 6^4 K_1 + \frac{8}{9}\delta$ and δ_{ij} is the kronekar delta. Here $\|\vec{a}\| = \mathrm{Tr}\left(a^\dagger a\right)$ with $\dagger$ denoting the Hermitian conjugate. These nonlocal outputs $\tilde{\rho_{14}}$ and $\tilde{\rho_{23}}$ will be inseparable when,

$$W_3^l < 0 \text{ or } W_4^l < 0 \text{ and } W_2^l \geqslant 0. \tag{A-4}$$

Appendix-2: Inseparability range of desired outputs obtained using nonlocal cloners

Here, we again evaluate the determinants W_2, W_3 and W_4 (as given in Eq. (7.5)) of the Peres-Horodecki criterion for the states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$ given by Eq. (7.19), and denote them as W_2^{nl}, W_3^{nl} and W_4^{nl} respectively. The mathematical expressions of these determinants turn out to be the following,

$$W_2^{nl} = \frac{1}{20^2}\left[5(5+6x_3) - 9\left(\sum_{i=1}^{3}\{t_{3i}^2 + y_i\,(2t_{3i}+y_i)\} - x_3^2\right)\right], \tag{A-5}$$

$$\begin{aligned} W_3^{nl} = \frac{9}{20^3}\Bigg[f_4 - \Bigg\{ \ell_- \sum_{i=1}^{3}(t_{i3}+x_i)^2 + \ell_+ \sum_{i,j,k,l}^{2} S_+ t_{ij}t_{kl} + 3\sum_{i,j}^{2}[t_{ij}^2 - (x_i - t_{i3})^2] \\ -6\Bigg(\sum_{i=1}^{2}\Bigg\{(x_i t_{i2} + y_i t_{2i}) + x_i(t_{1i}t_{3i} - t_{2i}t_{3i}) + \kappa_i \sum_{j=1}^{2}[t_{ij}t_{j3}t_{3j} + x_i y_i (t_{ii} - t_{2i})]\Bigg\} \\ + \sum_{i\neq j}^{2} t_{ij}(t_{i3}+t_{3j}) + \sum_{i,j}^{2}(-1)^{\delta_{ij}} y_1 t_{ij} t_{j3}\Bigg)\Bigg\} + f_{33}\sum_{i=1}^{3}(t_{3i}+y_i)^2\Bigg], \end{aligned} \tag{A-6}$$

$$\begin{aligned} W_4^{nl} = \frac{1}{20^4}\Bigg[f_5 - 18\Upsilon_t^{\ell} + 3^4\left(\sum_{i,j}^{3}\sum_{k,l}^{3} S_\delta t_{ij}^2 t_{lk}^2 + 8\sum_{i,j}^{2}\sum_{k=i+1,l=j+1}^{3} t_{ij}t_{il}t_{kj}t_{kl}\right) \\ +324\left\{\sum_{l=2}^{3}\mathcal{I}_l + 2\left(\sum_{i,j}^{3} x_i y_j C_{ij}^t - \sum_{i}^{2}\sum_{j\neq i,k}^{3}(t_{ik}t_{jk}x_i x_j + t_{ki}t_{kj}y_i y_j)\right)\right\}\Bigg], \end{aligned} \tag{A-7}$$

where $\Upsilon_t^{\ell} = \sum_{ij=1}^{3}\ell_{ij}t_{ij}^2, \mathcal{I}_l = \sum_{i=1}^{3} x_l^2(t_{1i}^2 - t_{li}^2) + y_l^2(t_{i1}^2 - t_{il}^2)$, $f_{33} = 3(x_3+y_3+t_{33}) - 5$, $f_3 = \frac{1}{9}(5+3x_3)^2$, $S_+ = (-1)^{i+j+k+l}$, $f_4 = 6y_3C_{33}^t - f_3 f_{33}$, $\kappa_i = (-1)^{i+1}$, $f_5 = 1080(\Omega - \delta) - 275$, $S_\delta = (-1)^{1-\max(\delta_{il},\delta_{jk})}$, $\ell_\pm = 5 \pm 3t_{33} + 3x_3$, C_{ij}^t is the co-factor of t_{ij} in correlation matrix T, and ℓ_{ij} are elements of coefficient matrix $[\ell_{ij}] = \begin{pmatrix} 43 & 25 & 25 \\ 25 & 7 & 7 \\ 7 & 7 & 7 \end{pmatrix}$. These desired output states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$ will be inseparable when,

$$W_3^{nl} < 0 \text{ or } W_4^{nl} < 0 \text{ and } W_2^{nl} \geqslant 0. \tag{A-8}$$

Summary

In this thesis, we have discussed a possible way of characterizing correlations in a multi-particle quantum state. To do so, we briefly investigated what actually the existing QCsbE measure, quantum discord captures in a two-qubit state. We have shown that discord actually captures local and nonlocal quantumness in the state. By local quantumness, we mean local superposition and nonlocal quantumness is synonymous to the quantum entanglement. We have illustrated this with the help of simple examples. We have also argued that due to the presence of local quantumness, local noise of certain types can enhance quantum discord. We have taken generalized Werner state to illustrate that while entanglement measure, concurrence, does not depend on local quantumness parameter, the quantum discord does. This makes quantum discord nonzero in the absence of entanglement. In the case of multiparticle systems, we extend this notion of quantum discord to characterize its quantumness. In order to do so, we generalize classical multivariate mutual information (Venn diagram type) to quantum regime. Then, we define multi-qubit discord type quantities – the dissensions. We show that these dissensions are required to reveal the total quantumness of a multi-qubit states. Moreover, our analysis emphasizes that a single quantity alone is not sufficient to characterize quantum properties of a state.

To give operational meaning to QCsbE, we have presented two applications where QCsbE have a role to play. We have shown that the better we clone (delete) a state, the more difficult it will be to bring the state back to its original form by the reverse process. This result shows that one cannot arbitrary have successive cloning and deleting or vice versa. This result may in future help in developing quantum recycle bin. Then, we investigated the possibility of broadcasting of QCsbE. We find that it is impossible to broadcast QCsbE optimally. However, one may have task oriented broadcasting.

We have discussed the possibilities of optimal and task oriented broadcasting of entanglement. Interestingly, we found that under both local and nonlocal cloning operations, it is possible to have task oriented broadcasting of the entanglement. We have also discussed the teleportation capability of low rank two-qudit entangled states. Using the relations between concurrence monotones and teleportation fidelity, one can characterize such states. Also, we have discussed how much classical and quantum information one can send through a quantum network.

We hope that our explorations regarding the characterization of quantum correlations will be useful in understanding the nature of multi-particle states.

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