Quantum Information Processing Protocols and Entanglement

By

Sumit Nandi

PHYS07201104005

Institute of Physics, Bhubaneswar

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DECLARATION

I, Sumit Nandi, hereby declare that the investigations presented in the thesis have been carried out by me. The matter embodied in the thesis is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

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[1]A part of the paper has contributed to the thesis.

To my beloved grandparents..

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Contents

Summary

Quantum mechanics exhibits an inherent correlation while describing a composite system. Suppose two distant observers namely, Alice and Bob share a singlet state $|singlet\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle|1_B\rangle - |1_A\rangle|0_B\rangle)$. The particle labelled as "A" is with Alice and same for the other particle. Now Alice measures her subsystem in computational basis and obtains the particle either in $|0\rangle$ state or in $|1\rangle$ state. If she obtains $|0\rangle$ state then she can be sure about the status of Bob's particle which is in $|1\rangle$ state. This kind of quantum correlation is known as $entanglement$ which has been a central issue of research since its birth. Endowed with a vast theoretical perspective, entanglement has also caught attention due to its usage in quantum information processing protocols like teleportation, quantum key distribution, etc.

Sometime it may require to have an exact replica of an unknown quantum state. But quantum mechanics prohibits cloning of an arbitrary state. Bennett *et al.* proposed a way to reproduce an unknown quantum state in a distant lab. In the original teleportation protocol, maximally entangled state was used as a resource. But later it was found that the protocol can be realized with partially entangled states as well as with mixed states. In these cases, teleportation is not perfect *i.e.* teleportation fidelity is less than $unity$. A general two-qubit mixed state has $fifteen$ real parameters and it has both classical and quantum properties. So teleportation fidelity should depend on those parameters. Previously it has been shown that teleportation fidelity depends on the mixedness of the resource state. We have used some of those results and found relations between teleporation fidelity, concurrence and mixedness for a certain class of states, X-states. Surprisingly we have observed that apart from concurrence and mixedness, teleportation fidelity depends on more such functions of parameters. These functions may characterize nonlocal properties of the state.

Entanglement theory becomes more rich when the system has macroscopic number of subsystems. In a multipartite scenario, entanglement theory is more complex qualitatively as well as quantitatively. Nevertheless these scenarios are interesting as the existing information processing protocols can be generalized with multipartite entangled states. A multipartite state can be grouped into bipartition to realise teleportation. It is also possible to teleport a multi-qubit state with a suitable resource state. For example, cluster state can be used to teleport a two-qubit state. Due to its complex structure, it is not always obvious which states are suitable for teleportation. We have addressed this issue and come up with a novel answer about the suitability of a resource state for teleportation. We have found that a state will be useful for telporting a n-qubit state with m-terms if the entropy of receiver's subsystem is $\log_2(m)$.

We have also used multipartite states to extend the scope of teleportation and quantum key distribution protocols. Suppose a tripartite state is distributed among three observers namely, Charlie, Alice and Bob. Now Charlie measures his subsystem and communicates the outcome to others who then perform measurements on their subsystems. In this way, maximum amount of entanglement can be localized between Alice and Bob so that they can establish secret key between them or carry out teleportation protocol. It may happen that Charlie can choose a measurement basis such that Alice and Bob share a mixed state or a partially entangled state. So, it is the Charlie who is controlling the success of the protocol. If he does not act wisely then the protocol cannot be carried out perfectly. The well-known GHZ state is useful for such co-operative protocols. We have found a larger class of tripartite states which can be used for the protocol. We have also analyzed quantum bit error rate of QKD and average fidelity of teleportation with the proposed resource state. We have also presented an entropic criterion about the suitability of the resource state for co-operative protocols. It states that a resource state would be suitable for the protocol iff the entropy of at least two subsystems are one.

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Chapter 1

Introduction

One of the distinguishing features of quantum mechanics is the notion of Hilbert space which replaces the concept of phase space in classical physics [1]. The total state space, in classical physics is defined by the cartesian product of n individual particles spaces, whereas quantum states live in Hilbert space and Hilbert space dimension of such a system is described by the tensor product of individual subsystem spaces. The state of a composite system in Hilbert space can be written as superposition of product states. It leads to the phenomenon like 'entanglement'. This word is derived from the German phrase *Verschränkung*. Entanglement is a sort of nonlocal correlation which has no classical counterpart. Nevertheless the nonlocality arising from entanglement has been found of paramount importance as it is the key ingredient in many quantum information processing protocols (QIPPs).

1.1 Entanglement

Let us start with a composite system consisting of two subsystems which are located in two different labs. The physical states of two subsystems are described in Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$ with dimension d_A and d_B respectively. The composite system of both parties is described by state vectors in a joint Hilbert space $\mathcal{H}_{AB}$ which is the tensor-product of the two spaces $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Any vector in $\mathcal{H}_{AB}$ can be written as

$$|\psi\rangle = \sum_{i,j} c_{ij} |\phi_i\rangle_A |\eta_j\rangle_B, \tag{1.1}$$

where c_{ij} constitutes a complex matrix C of dimension $d_A \times d_B$. If the state can be represented by only one term i.e if

$$|\psi\rangle = |\phi\rangle_A |\eta\rangle_B, \tag{1.2}$$

then it is called a separable state or a product state; otherwise it is entangled state. This means two subsystems are not correlated. If Alice and Bob measure some observable then their outcomes would be uncorrelated in the case of the state (1.2) where we can see that marginal statistics of one subsystem is independent of another subsystem state.

1.2 Maximally entangled states: Bell states

Naturally it makes sense to figure out maximally entangled states as entanglement is used as resource for QIPPs. These are the states where individual subsystems are perfectly correlated. In a bipartite scenario, we definitely have such states which are known as Bell states. Many authors prefer to call them EPR states. The explicit forms of these states in two-qubit scenario in computational basis are given as

$$\begin{aligned} |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \\ |\phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ |\phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle). \end{aligned} \tag{1.3}$$

In the case of these states, the state of individual subsystem is maximally mixed. If we use one of the measures of entanglement, as discussed below, we will find these states to be maximally entangled. If we take one of these states, say $|\psi^+\rangle$, and distribute the qubits between two observers conventionally called as Alice and Bob such that each of the observers hold one qubit. Now Alice makes a measurement in computational basis on her subsystem. If her measurement outcome is $|0\rangle$, then Bob finds his particle in $|1\rangle$ state and the other way round. So, we can see that corresponding to each outcome of Alice, Bob's state collapses into a specific configuration, even if they are separated by thousands of miles. This sort of perfect correlation, which is of nonlocal origin, is what entanglement leads to and provides a physical resource for various quantum information processing protocols.

1.3 Two-qubit density matrix

In a more general situation, we have limited knowledge of the original state, so we have to deal with an ensemble of quantum states. We wish to find out expectation value of an observable described by $\mathcal{Q}$ for this ensemble. Suppose, the system exists in a state $|\psi_1\rangle$ with probability p_1 and $|\psi_2\rangle$ with corresponding probability p_2 and so on. Then the the expectation value of $\mathcal{Q}$ can be written as

$$\begin{aligned} \langle \mathcal{Q} \rangle &= \sum_i p_i \langle \psi_i \mid \mathcal{Q} \mid \psi_i \rangle \\ &= \mathrm{Tr}(\mathcal{Q}\rho). \end{aligned} \tag{1.4}$$

In the last line the ensemble has been described by a density matrix ρ and is defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{1.5}$$

ρ would be a valid density matrix if it satisfies some prerequisite conditions:

• ρ has to be hermitian $i.e.$ $\rho = \rho^\dagger$.

• $\mathrm{Tr}(\rho) = 1$.

• All eigenvalues of ρ are positive or zero *i.e* ρ is positive-semidefinite.

1.3.1 Pure state

We can think of a simple case when the ensemble of our composite system is describe by one state vector $|\psi\rangle$. Then one can write,

$$\rho = \mid \psi\rangle\langle\psi \mid . \tag{1.6}$$

It follows that $\rho^2 = \rho$ *i.e.* ρ is idempotent. The state of the above form is called pure state. An ensemble of a pure state has only one eigenvector corresponding to one non-vanishing eigenvalue which is equal to one. We also note that $\mathrm{Tr}(\rho^2) = 1$. Following the definition (1.2) of separable state we call a state to be separable if it can be recast as a convex combination of tensor product of individual subsystem states, *i.e.*, $\rho = \sum_i p_i \rho_A \otimes \rho_B$, otherwise it is an entangled state.

1.3.2 Schmidt decomposition

A pure state of the form 1.1 can be recast into a simpler form using singular value decomposition, the matrix C can be written as $C = UDV$, where D is a positive diagonal matrix and U, V are unitary matrices. We rewrite 1.1 as

$$|\psi\rangle \;=\; \sum_{i,j,k} U_{ji} D_{ii} V_{ik} |\phi_j\rangle_A |\eta_k\rangle_B$$

Now we define $\sum_j U_{ji}|\phi_j\rangle = |i_A\rangle$, $\sum_k V_{ik}|\eta_k\rangle = |i_B\rangle$ and $D_{ii} = \lambda_i$. Thus the state vector takes the following form:

$$|\psi\rangle = \sum_{i=1}^{min\{d_A,d_B\}} \sqrt{\lambda_i}|i_A\rangle|i_B\rangle, \tag{1.7}$$

where $|i\rangle_{A(B)}$ constitutes an orthonormal basis in the individual Hilbert space $\mathcal{H}_{A(B)}$. This is known as Schmidt decomposition of a pure state [2]. Here λ_i's are known as Schmidt numbers with the property $\sum_i \lambda_i = 1$. It has some nice consequences. Schmidt numbers completely determine nonlocal aspects of a pure state. Naturally it makes pure states more tractable mathematically: once we know the eigenvalues of the density matrix, its nonlocality is understood qualitatively as well as quantitatively. As an example, Schmidt decomposition suffices to say whether a given pure state is entangled or not. If it has only one non-vanishing Schmidt number then the state can be recast as a product of two states, so it is a separable state. A necessary and sufficient condition to be a entangled state is that the state must have at least two non-vanishing Schmidt numbers.

1.3.3 Mixed state

In a practical situation, we have to deal with an ensemble of states where the state may be ρ_i with probability p_i, ρ_j with probability p_j and so on. Thus more generally, we can write down density matrix of a mixed state as

$$\rho_{mix} = \sum_i p_i \rho_i^{pure}. \tag{1.8}$$

We consider a mixture of orthogonal states $i.e.$ $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. Remarkably we find that

$$\rho_{mix}^2 = \sum_{i,j} p_i p_j |\psi_i\rangle\langle\psi_i \mid \psi_j\rangle\langle\psi_j| = \sum_{i,j} p_i p_j \delta_{ij} |\psi_i\rangle\langle\psi_j| = \sum_i |p_i|^2 |\psi_i\rangle\langle\psi_i| \neq \rho_{mix}. \tag{1.9}$$

It follows that $\mathrm{Tr}(\rho_{\mathrm{mix}}^2) \neq \mathrm{Tr}(\rho_{\mathrm{mix}})$, $\mathrm{Tr}(\rho_{\mathrm{mix}}^2) < 1$. This is a simple way to distinguish a mixed state from a pure state. In a realistic situation, a pure state evolves into a mixed state. So mixed states are found to be very frequent in any experimental set-up. A bipartite mixed state can also be realized as a part of a larger system consisting of more than two subsystems. So, study of mixed bipartite states has a profound importance for better understanding of entanglement theory. Unfortunately, there exists no notion of Schmidt decomposition for mixed states analogous to pure states. We need $fifteen$ real parameters to specify a general density matrix of a mixed state of two qubits. Naturally, it leads to a richer theory than of pure states containing only one nonlocal parameter. Let us examine the general form of a two-qubit state. Hilbert-Schmidt representation is given as [3]

$$\rho = \frac{1}{4}\Big(I \otimes I + \vec{r}.\vec{\sigma} \otimes I + I \otimes \vec{s}.\vec{\sigma} + \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j\Big), \tag{1.10}$$

where I stands for 2×2 identity operator, $\mathbf{r}$ and $\mathbf{s}$ are local Bloch vectors containing $three$ parameters each in $\mathbf{R^3}$ and as usual σ's represent Pauli matrices. The parameters can be obtained as

$$\begin{aligned} r_i &= \mathrm{Tr}[\rho.(\sigma_i \otimes I)], \\ s_j &= \mathrm{Tr}[\rho.(I \otimes \sigma_i)], \\ \text{and} \quad t_{ij} &= \mathrm{Tr}[\rho\sigma_i \otimes \sigma_j], \end{aligned} \tag{1.11}$$

where $i, j = (1, 2, 3)$. The elements t_{ij} constitute a 3×3 real matrix and often known as correlation matrix T. So, we need $fifteen$ real parameters to deal with this mixed state. However, we may reduce this number by choosing appropriate local reference frame so that correlation matrix can be diagonalized. It reduces six parameters leaving only $nine$ real parameters to characterize it [4]. Still we have more than one parameter to determine all of its nonlocal aspects as Schmidt numbers do the job for pure state.

1.4 Purity of a state

The degree of mixedness of a state is quantified by its purity and it is defined as $\mathcal{P} = \mathrm{Tr}(\rho^2)$. It is normalized to one for pure states, whereas it is $\frac{1}{4}$ for maximally mixed states. A maximally mixed state is an ensemble of an orthonormal basis with uniform probability distribution. In general, in a $d \times d$ dimensional Hilbert space, purity of maximally mixed state is $\frac{1}{d}$. A general one-qubit state can be represented as a point in Bloch sphere [5],

$$\rho = \frac{I + \vec{r}.\vec{\sigma}}{2},$$

where $|\vec{r}|$ defines the length of the Bloch vector. Purity of this state is related to the length of Bloch vector as $\mathcal{P} = \frac{1+|\vec{r}|^2}{2}$. Now if the state is pure then length of the vector is unity *i.e.* $|\vec{r}|^2 = 1$ and $\mathcal{P} = 1$, but it is less than one for mixed state. Pure states happen to lie on the surface of the Bloch sphere whereas mixed states lie inside the Bloch sphere. Many authors prefer another quantity to define the degree of mixedness. They use linear entropy which can be constructed from purity through the expression $\frac{4}{3}(1 - \mathcal{P})$. It is normalized to one for maximally mixed state and zero for pure states. In entanglement theory, purity can be considered as a quantifier of classicality of the state as it depends on the local Bloch vectors. It seems to affect nonlocal phenomena substantially. For example, an increase in purity can enhance the efficiency of teleportation protocol under proper conditions.

1.5 Detection of entanglement

1.5.1 Peres-Horodecki criterion

We can expand any density matrix (1.5) of a composite quantum system in a chosen product basis as

$$\rho = \sum_{i,j,k,l} \rho_{ij,kl} |i\rangle\langle j| \otimes |k\rangle\langle l|. \tag{1.12}$$

Now partial transposition with respect to the first subsystem ρ^{T_A} is described as

$$\rho^{T_A} = \sum_{i,j,k,l} \rho_{ji,kl} |i\rangle\langle j| \otimes |k\rangle\langle l|. \tag{1.13}$$

Similarly transposition with respect to second subsystem is realized by interchanging k and l. We call a state positive-partial transpose (PPT) state if all eigenvalues of $\rho^{T_A}(\rho^{T_B})$ are positive i.e $\rho^{T_A}(\rho^{T_B})$ is positive semidefinite. According to Peres-Horodecki criterion [6], a state is separable if $\rho^{T_A}(\rho^{T_B})$ is positive semidefinite. The statement is necessary and sufficient for 2×2 and 2×3 dimensions and sufficient otherwise. If a state is negative partial transposed (NPT) then entanglement can be distilled from that state. Although this criterion is found to be violated in higher dimensions say for example, in 3×3, 2×4 systems [9], it has importance due to its operational simplicity. Not only it helps us to characterize bipartite systems which are more useful for experimental purposes, it has been shown that the amount of the violation of PPT condition can be used to quantify entanglement [12].

1.5.2 Entanglement witness

In the Peres-Horodecki criterion, we have knowledge of the state and we are performing an operation on it to find if it is entangled. In another method, we use an observable $\mathcal{W}$ [7] [8] by which we can detect entanglement. For every entangled state, there exists an observable $\mathcal{W}$ such that

$$\mathrm{Tr}[\mathcal{W}\rho] < 0. \tag{1.14}$$

If $\mathrm{Tr}(\mathcal{W}\rho) > 0$ then the state is separable and $\mathcal{W}$ is called a witness operator. The concept comes from the fact that the set of separable states is convex and closed. So, if there is an entangled state then there must be a hyperplane described by $\mathrm{Tr}(\mathcal{W}\rho) = 0$ such that it separates the two regions. It is a geometrical interpretation of witness operators.

1.6 Measures of entanglement

Entanglement has been well recognized as a resource for carrying out some specific tasks which cannot be performed classically. Some entangled states may have low efficiency for a protocol which is of no practical use. Then a natural question arises, to have a decent value of efficiency factor such as teleportation fidelity or secure key rate, how much entanglement is needed. Its answer may require a suitable measure of entanglement. Any good measure of entanglement must have following properties [10]:

• An entanglement measure E when acts on a density matrix should produce a positive real number, *i.e* $E(\rho) \in \mathbf{R}_+$.

• $E(\rho) = 0$ whenever ρ is separable.

• One cannot increase entanglement of the system by means of local operation and classical communication (LOCC). So, it must be non-increasing under LOCC.

We now discuss some operational measures of entanglement which satisfy above properties and are popular in the context of entanglement theory.

1.6.1 von Neumann entropy

The entanglement of a pure state is quantified by von Neumann entropy which corresponds to the number of singlets that can be extracted from N numbers of partially entangled pure states. It is defined quite analogous to Shannon entropy in classical information theory. If ρ_A represents reduced density matrix of $|\psi\rangle\langle\psi|$ then Von Neumann entropy of $\rho_{A(B)}$ [5] is given by

$$S(\rho_A) = -\mathrm{Tr}(\rho_\mathrm{A}\mathrm{log}(\rho_\mathrm{A})) = -\mathrm{Tr}(\rho_\mathrm{B}\mathrm{log}(\rho_\mathrm{B})), \tag{1.15}$$

where $0\mathrm{log}0 \equiv 0$. It represents how much the subsystem A is entangled with rest of the system. It takes a simpler form when expressed in terms of Schmidt coefficients

$$S(\rho_A) = -\sum_i \lambda_i \mathrm{log}_2\lambda_\mathrm{i}, \tag{1.16}$$

where λ_i are Schmidt numbers. It has an interesting property because Schmidt numbers of individual subsystem reduced density matrices of a pure state are same. Simply it implies $S(\rho_A) = S(\rho_B)$. Unfortunately, von Neumann entropy can not be generalized to mixed states as it does not distinguish between classical and quantum correlations. However this definition can be extended to multipartite pure states. It has some important mathematical properties which are quite useful in information theory.

• It is non-negative and vanishes if the state is a product state.

• It is invariant under local unitary transformation, *i.e* $S(\rho) = S(U^\dagger \rho U)$.

• If ρ has d non-vanishing eigenvalues then $S(\rho) \leq \mathrm{log}_2\mathrm{d}$.

1.6.2 Concurrence

Suppose, Alice and Bob wants to create an ensemble of state associated with the density matrix (1.5). They are sharing maximally entangled state and no classical communication is allowed between them. Under this scenario they need to know how many EPR states they need asymptotically to create the ensemble. Entanglement of formation of ρ [11] gives the answer and it is defined as:

$$E_f(\rho) = inf \sum_i p_i E(|\psi_i\rangle). \tag{1.17}$$

As we know a mixed state can have infinite many pure state decompositions, hence infimum has to be taken over all decompositions. For a bipartite mixed state there exists a closed expression of E_f which can be written as a function of a quantity known as concurrence. Among many measures of entanglement of a two-qubit system, concurrence [11] is extensively used so far in the resource theory of entanglement. Concurrence $\mathcal{C}(|\psi\rangle)$ of a pure state $|\psi\rangle$ is defined as $\mathcal{C}(|\psi\rangle) = \langle\psi|\tilde{\psi}\rangle$, where $|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y)|\psi^*\rangle$. Here $(*)$ is complex conjugate of $|\psi\rangle$ in computational basis and σ_y is a Pauli matrix. The concurrence of a pure state of the form $|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ is given by

$$|\psi\rangle = |c_{00}c_{11} - c_{01}c_{10}|, \tag{1.18}$$

and for a two-qubit mixed state ρ, one can deduce

$$\mathcal{C} = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{1.19}$$

where λ_i's are the eigenvalues, in descending order of the matrix $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$. Here $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where '$*$' denotes the conjugate of ρ in computational basis. For Bell states it is 1 while for separable states $C(\rho) = 0$. It is monotonic and we cannot increase concurrence on average by means of LOCC only.

1.6.3 Negativity

Partial transposition operation has been found important to detect entanglement in a given state and according to Peres-Horodecki criterion, entanglement can be distilled if ρ does not fulfill the condition of positive partial transposition. It leads to a quantitative measure of entanglement namely, negativity [12] which determines how far the state is from 'to be a positive partial transposed' state. The expression of negativity is

$$\mathcal{N}(\rho) = \frac{||\rho^{T_A}|| - 1}{2}, \tag{1.20}$$

where $||\rho^{T_A}||$ denotes the sum of the eigenvalues of ρ^{T_A}. As like other entanglement measures, it does not increase under LOCC and yields $zero$ for unentangled states while giving $unity$ for Bell states.

1.7 Fidelity of quantum states

Often it is convenient to have a quantity which specifies similarity or closeness of two density matrices. For given two density matrices ρ and σ, Uhlmann fidelity [5] is defined as follow:

$$F(\rho, \sigma) = \mathrm{Tr}\sqrt{(\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}})}. \tag{1.21}$$

It is a distance measure and can be used to find distance between two states. Although in original paper, square root of this quantity has been described as transition probability between two states by Uhlamann [13]. Despite its different interpretation, it is found to be an important tool in quantum information theory as we shall find in the next chapter. An important consequence of the above definition can be seen if we assume one of the states to be pure. In this case the above equation reduces to

$$F(\psi, \sigma) = \sqrt{\langle\psi|\rho|\psi\rangle}, \tag{1.22}$$

which is the overlap between $|\psi\rangle$ and ρ. It is to be noted that $F(\rho, \sigma)$ is invariant under unitary transformation.

1.8 A few important classes of two-qubit states

In this section, some important classes of two-qubit states will be discussed. These are often encountered in the context of entanglement theory.

1.8.1 Werner state

Werner states named after its discoverer [14] are a mixture of a maximally mixed state and a maximally entangled state (Bell state). It has the nice property that makes it invariant under any arbitrary $U \times U^*$ unitary transformation. We shall use the following representation of 2×2 Werner state

$$\rho_W = \frac{1-\alpha}{4} I + \alpha |\phi^+\rangle\langle\phi^+|, \tag{1.23}$$

where $\alpha \in [0, 1]$. For $\alpha = 0$, it reduces to maximally mixed state whereas for $\alpha = 1$, it represents maximally entangled state. It can be shown that ρ_W is entangled whenever $\frac{1}{3} \leq \alpha \leq 1$; it is unentangled otherwise. Concurrence of Werner states can be computed easily by the prescription given in (1.6.2):

$$\mathcal{C}(\rho_W) = \frac{1}{2}(3\alpha - 1), \tag{1.24}$$

where $\frac{1}{3} \leq \alpha \leq 1$. On the other hand it is easy to verify using the criterion as suggested by the authors in [15] that ρ_W violates Bell-CHSH inequality whenever $\alpha \geq \frac{1}{\sqrt{2}}$. The relationship between entanglement and nonlocality has been a puzzling issue. Although it is well understood for pure states: whenever a state is entangled it also violates Bell-CHSH inequality and converse is also true. But it is not the case for mixed states and Werner states serve as a representative class of states where the above argument does not hold. We can find a range of the parameter α where ρ_W does not violate any Bell-CHSH inequality but it is still entangled. So for a mixed state, entanglement does not necessarily imply Bell nonlocality.

1.8.2 Maximally entangled mixed state

It is desirable to have as much amount of entanglement as possible in a given state to perform quantum information processing protocols. In this regard, Bell states are found to be the most efficient one. But from experimental perspective, one has to consider local noise which turns pure Bell states into a mixed state. Naturally a question arises, how much entanglement a given mixed state contains. Maximally entangled mixed states (MEMs) have been found with the interesting property that they have maximal amount of entanglement for a fixed linear entropy [16]. The authors have presented two sub-classes of such states,

$$\rho_{MEM_i} = \begin{pmatrix} \frac{g}{2} & 0 & 0 & \frac{g}{2} \\ 0 & 1-g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{g}{2} & 0 & 0 & \frac{g}{2} \end{pmatrix}, \quad \frac{2}{3} \leq g \leq 1,$$

$$\rho_{MEM_{ii}} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{g}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{g}{2} & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad 0 \leq g \leq \frac{2}{3}. \tag{1.25}$$

They have shown that for a given mixedness as quantified by linear entropy, these classes of states have maximal entanglement as measured by concurrence. Quite independently Ishizaka *et al.* [17] have provided another class of maximally entangled mixed states for which amount of entanglement of the states cannot be increased by any unitary operation. We write the following expression of MEM states following the authors,

$$\rho_M = p_1|\psi^-\rangle\langle\psi^-\rangle + p_2|00\rangle\langle 00| + p_3|\psi^+\rangle\langle\psi^+| + p_4|11\rangle\langle 11|, \tag{1.26}$$

where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ and $\sum_i p_i = 1(p1 \geq p_2 \geq p_3 \geq p_4)$. Theoretical importance of such class of states motivated people to realize it experimentally. The authors in [18] have presented novel approach to produce a MEM in a laboratory.

1.8.3 X-states

X-states were introduced in the context of evolution of entanglement in the presence of noise. Each system has an inevitable interaction with local environment as we know isolated system is an idealized situation. Study of an entangled system under noise has immense practical importance. The authors in [19] presented a family of states which is known as X-state due to its structural resemblance with the alphabet 'X'. The density matrix of a X-state has non-vanishing diagonal and anti-diagonal terms and it is given as:

$$\rho_x = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}, \tag{1.27}$$

where a, b, c, d ≥ 0 and satisfy $a + b + c + d = 1$. A X-state is said to be symmetric if we have $b = c$. In this case we have identical reduced density matrix for individual subsystem *i.e.* $\rho_1 = \rho_2$ [22]. The importance of X-states is due to the following interesting properties:

• Entanglement of a X-state is robust when two subsystems are exposed to local noise [19].

• As a consequence of the robustness, the authors in [20] have argued that whole concurrence-purity($\mathcal{CP}$) region can be covered by considering only X-states. It is to be noted here that their entire analysis uses different parameterization of X-states. It will be discussed in the next chapter.

• X-states can also be realized by different parameterization as provided in [20]. It gives us advantage to study entanglement properties of a two-qubit system in a more rigorous way as it has fewer number of parameters. In this case, one has to deal with only five real parameters instead of nine in (1.8).

Many well known classes of states like Werner states, maximally-entangled-mixed states, Bell-diagonal states belong to this family of states. Even Bell states have this kind of structure. For all these reasons, the properties of X-states have been extensively discussed in literature [21]. We shall use X-states to study local and nonlocal aspects of a two-qubit density matrix in the next chapter.

1.9 Multipartite entanglement

Most of the time, we deal with a physical system comprised of large number of subsystems such as Ising chain or some interacting many body system in other branch of physics. So entanglement theory would never be complete without its extension beyond bipartite system. We may consider a correlated system composed of N two-level quantum systems. A pure multipartite entangled state of such a system is described in a Hilbert space

$$\mathcal{H} = \bigotimes_i \mathcal{H}_i.$$

Here each particle is associated with a finite dimensional Hilbert space. A multipartite state $|\psi\rangle$ is said to be completely separable if it has the following form

$$|\psi\rangle_{seperable} = |\psi_1\rangle \otimes |\psi_2\rangle \otimes ..|\psi_n\rangle. \tag{1.28}$$

If a state cannot be written in this way then it is said to be an entangled state. For mixed entangled states, it cannot be written as a convex sum of product states. Without loss of generality, we write a multipartite entangled state as

$$|\psi\rangle_N = \sum_{i_1,i_2,..,i_N} D_{i_1..i_N}|\theta_{i_1}\rangle|\theta_{i_2}\rangle..|\theta_{i_N}\rangle, \tag{1.29}$$

where $i_p \in 0,..,d-1$ and $\{|\theta_{i_p}\rangle\}$ is an orthonormal basis of dimensionality d of an individual subsystem. Let us assume that each subsystem is a two level system such as photons or electrons. Then any state of the above form has 2^{N+1} real parameters characterizing local and nonlocal properties of the state $|\psi\rangle_N$. However, one can have smaller number of parameters describing nonlocal feature of the state. We know that local unitary transformations cannot affect the nonlocal properties of a state. These properties are invariant under the action of the group $U(2) \otimes U(2).... \otimes U(2)$. Again we can decompose $U(2) \otimes U(2).... \otimes U(2)$ as $U(1) \otimes SU(2)^N$ which is characterized by $3N+1$ real parameters because any $SU(2)$ has three real parameters. So we are left with the rest $2^{N+1}-(3N+1)$ real parameters, among these one can be reduced further by imposing normalization constraint. This number of parameters are needed to describe nonlocal properties of inequivalent classes of entangled states [23]. Naturally, it leads to a richer entanglement theory than that of bipartite entanglement. Even for the simpler case $N=3$, it is difficult to answer which is the maximally entangled state, like Bell states for two-qubit systems.

Let us begin with the simplest tripartite entangled system comprising of 3 two-level systems. In tripartite scenario there are six SLOCC (discussed below) inequivalent states [24] among which only two classes are genuinely entangled. By genuinely entangled states we mean those states which are entangled in all possible bipartitions. As an example, a state of the form $|0\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is not genuinely entangled state as it is separable between 1-23 cut. However one is usually more interested in using genuinely entangled states for QIP protocols. The well known GHZ and W states are examples of this category in tripartite scenario.

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \tag{1.30}$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{1.31}$$

These states represent two classes of genuine tripartite entangled states. These two states exhibit different kinds of entanglement properties. If we trace out one of the subsystems then the resultant state for GHZ is $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. It is straightforward to check that no entanglement can be distilled from it as it is a separable state. So this kind of state is more fragile under decoherence. On the other hand bipartite entanglement is more robust in W-type states, but as we shall see later, this state is not so useful in quantum information processing tasks.

As we have seen the arena of entanglement in multipartite scenario is vast. In multipartite scenario, the states belong to different inequivalent classes and behavior of each class under LOCC may differ considerably. As we proceed further, we shall encounter different inequivalent classes of quadripartite entangled states. Importantly equivalence can be established between two states of the same class. Here we discuss briefly equivalence of multipartite pure entangled states under LU (Local Unitary) transformation and LOCC operation.

1.9.1 LU equivalence of multipartite pure states

In this section, we shall briefly discuss local unitary equivalence of multipartite states. It is important because local unitary transformation leaves nonlocal properties invariant. So this is a very useful tool to gain insight in the nonlocal properties of a multipartite state. It also implies that if a state is useful for a particular information processing task, then any state which is unitarily equivalent to the former would be capable of accomplishing the same task. So given two arbitrary states $|\psi\rangle$ and $|\phi\rangle$, they would be inter convertible to each other iff $|\psi\rangle \simeq_{LU} |\phi\rangle$. B. Krauss [25] has given a formalism to investigate whether two given states are

LU equivalent or not. In this regard it would be beneficial to introduce the notion of standard form of multipartite state. A standard form of a multipartite state has the property that all single qubit reduced density matrices ρ_i are diagonal. We can write any ρ_i as:

$$\rho_i = V_i D_i V_i \dagger,$$

where D_i is a diagonal matrix whose nonzero elements are eigenvalues of ρ_i in ascending order:

$$D_i \equiv diag(\lambda_1, \lambda_2), \lambda_1 \leq \lambda_2.$$

Sorted trace decomposition form [26] of a multipartite state is thus defined as

$$|\psi_{st}\rangle = \bigotimes_i V_i |\psi\rangle,$$

when $\rho_i \neq I$. This is unique upto a global phase for a given $|\psi\rangle$ and represents the standard form of $|\psi\rangle$.

The following theorem gives necessary and sufficient criterion for LU equivalence of two given states $|\psi\rangle$ and $|\phi\rangle$.

Theorem: Two states $|\psi\rangle$ and $|\phi\rangle$ with $\rho_i \neq I \; \forall i$, are LU equivalent iff the standard forms of $|\psi\rangle$ and $|\phi\rangle$ are equivalent, i.e, $|\psi_{st}\rangle = |\phi_{st}\rangle$ [27].

We shall now discuss the procedure to find out local unitaries which can inter convert two LU equivalent states. Let ρ^r and $\rho^{r'}$ represent reference forms of two states ρ_1 and ρ_2

$$\rho^r = \bigotimes_{i=1}^{n} {V_i}^{\dagger} \rho_1 \bigotimes_{i=0}^{n-i} V_i,$$
$$\rho'^r = \bigotimes_{j=1}^{n} {V'_j}^{\dagger} \rho_2 \bigotimes_{j=0}^{n-j} V'_j.$$

Here V_i, V'_i are unitary matrix which diagonalize ρ_1 and ρ_2 respectively.

Theorem: Two states ρ_1 and ρ_2 will be LU equivalent iff there exists unitary U_i, such that [28]

$$\rho'^r = \bigotimes_{i=1}^{n} {U_i}^{\dagger} \rho^r \bigotimes_{i=0}^{n-i} U_i.$$

U_i are given by $U_i \equiv diag(e^{-\alpha_i}, e^{\alpha_i})$ when $\rho_i \neq I\ \forall i$. We know that each state with $\rho_i \neq I$ has a unique reference form which is unitarily related to it. If two states are unitarily equivalent then there exist unitary transformations which also relate their reference forms. Using this theorem we can find the unitaries $\bar{U}$ which can convert ρ into ρ'

$$\bar{U} = V_i' U_i V_i^{\dagger}.$$

Let us explicitly calculate the unitaries to check whether two following states are LU equivalent or not

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \qquad (1.32)$$

$$|\Sigma\rangle = \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle) - \frac{1}{2\sqrt{3}}(|101\rangle + |110\rangle - |000\rangle). \qquad (1.33)$$

The former state is quite popular in literature. It has genuine entanglement among three of its subsystem with vanishing tangle [29]. We want to see whether we can obtain the later by suitable unitary operation on the former state subsystems. It is easy to verify that we can achieve the reference form of $|W\rangle$ with the aid of following unitary :

$$\tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

i.e. $\tau \otimes \tau \otimes \tau |W\rangle = |W_r\rangle$, where we denote its reference form as $|W_r\rangle$. Similarly we can obtain reference form of $|\Sigma_r\rangle$ with υ and τ, where

$$\upsilon = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},$$

and $\upsilon \otimes \tau \otimes \tau |\Sigma\rangle = |\Sigma_r\rangle$. If $|W\rangle$ and $|\Sigma\rangle$ are LU equivalent state, then according to the theorem in the previous paragraph, we must find $\bar{U}_1$, $\bar{U}_2$ and $\bar{U}_3$ connecting those states. We have

$$\bar{U}_1 = \upsilon U_1 \tau = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \bar{U}_2 = \bar{U}_3 = \tau U_2 \tau = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Indeed it turns out that $\bar{U}_1 \otimes \bar{U}_2 \otimes \bar{U}_3 |W\rangle = |\Sigma\rangle$. We can also obtain U_i by putting $\alpha_1 = 0$, $\alpha_2 = \alpha_3 = \frac{\pi}{2}$ which connect corresponding reference forms. Sometime it may be necessary to find unitary equivalent state of a given unknown state to analyze its nonlocal properties as these remain invariant under local unitary operation. We can follow aforementioned procedure. In the above example, we have seen that $|\Sigma\rangle$ is LU equivalent to $|W\rangle$ state. So we can learn the nonlocal properties of $|\Sigma\rangle$ which was not so obvious from its complicated structure.

1.9.2 LOCC equivalence of multipartite pure states

In quantum information processing tasks, 'equivalent entanglement' is an important concept in the sense that it is helpful to understand usefulness of a particular multipartite state under LOCC. It is known that bipartite pure entangled states are inter-convertible, at least in asymptotic regime under LOCC transformations [30]. Once the state takes the form of the target state, the protocol can be carried out. So all bipartite pure entangled states can be used to perform the same quantum-information task in the asymptotic regime. To figure out equivalent entanglement property between the states, conversion of these states under LOCC is a convenient tool. The authors [31] have given mathematical tool to find out LOCC convertibility of two bipartite states. This does not hold in multipartite case. As the number of subsystems increases, the amount of information required to describe a N-party quantum state grows exponentially. It has been shown, for $N > 3$, it leads to a uncountable number of classes of equivalent multipartite states [32]. Till now there exists no mathematical algorithm to determine whether two states are LOCC equivalent or not even in asymptotic limit in multipartite scenario. Luckily we have a way out at hand, Bennett $et\ al.$ [33] introduced the notion of stochastic LOCC (SLOCC) transformation with non-zero success probability. Two states are said to be SLOCC equivalent if these are connected by invertible local operators (ILO). This leads to the notion of SLOCC classes of multipartite states with the property that the states belonging to the same class can be connected by ILOs. The states from different classes cannot be converted into each other, even probabilistically.

We say that two states $|\psi\rangle$ and $|\phi\rangle$ belong to the same SLOCC class if both transformations

$$|\psi\rangle \sim |\phi\rangle \quad \text{and} \quad |\phi\rangle \sim |\psi\rangle \tag{1.34}$$

exist by means of LOCC with non-zero probability. Once we know that two states are inter-convertible by LOCC we can learn nonlocal properties of the two states, their usefulness in certain QIPPs or compare their entanglement nature qualitatively as well as quantitatively.

1.9.3 Monogamy of entanglement

Entanglement differs in many other aspects from that of classical correlation. One of such important properties is that entanglement is monogamous in nature. If observers B and C share a maximally entangled two-qubit state with A, then set-up can be exploited to clone an unknown quantum state. A can teleport an unknown state $|\psi\rangle$ to B and to C. Thus, this tripartite network has succeeded in copying the state $|\psi\rangle$. Indeed, this is a violation of no-cloning theorem. So we see that there are severe restrictions on the sharing of the entanglement. These restrictions are known as monogamy of entanglement.

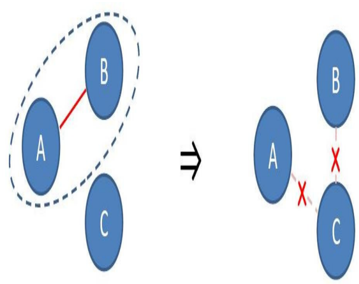

Figure 1.1: Monogamy of a three qubit state: if subsystems A and B are in singlet state, there cannot be any entanglement between AC and BC.

Let ρ_{ABC} be a tripartite qubit state shared by A, B and C. Let $E_{A|B}, E_{A|C}, E_{A|BC}$ denote the shared entanglement between the subsystems A and B, A and C, A and BC respectively. The monogamy relation as given by Coffman-Kundu-Wootters [34] is as follows:

$$E_{A|BC} \geq E_{A|B} + E_{A|C}. \tag{1.35}$$

This inequality conveys the monogamy of entanglement principle. The distribution of entanglement is constrained by the above inequality. If $E_{A|B} = E_{A|BC} = 1$, then A and B share maximal entanglement. So, inequality (1.35) implies $E_{A|C} = 0$. Thus there cannot be any entanglement between A and C. The entanglement measure E can be taken to be the square of concurrence $\mathcal{C}$ and when expressed in terms of concurrence above eq. (1.35) takes the following form:

$$\mathcal{C}^2_{A|BC} \geq \mathcal{C}^2_{A|B} + \mathcal{C}^2_{A|C}. \tag{1.36}$$

For example, let us take tripartite GHZ state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Here $\mathcal{C}^2(\rho_{A|BC}) = 1$, $\mathcal{C}^2(\rho_{A|B}) = 0$ and $\mathcal{C}^2(\rho_{A|C}) = 0$. So we see that for GHZ state the inequality (1.36) holds. As an example, let us consider tripartite W state:

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

In this case, we have $\mathcal{C}^2(\rho_{A|BC}) = \frac{8}{9}$, $\mathcal{C}^2(\rho_{A|B}) = \frac{4}{9}$ and $\mathcal{C}^2(\rho_{A|C}) = \frac{4}{9}$. So,

$$\mathcal{C}^2(\rho_{A|BC}) = \mathcal{C}^2(\rho_{A|B}) + \mathcal{C}^2(\rho_{A|C}).$$

We see that for W state, the inequality (1.36) saturates. The monogamy relation (1.35) has also been generalized for any N(≥ 3) by the authors [35].

$$E_{A_1|A_2 \ldots A_n} \geq E_{A_1|A_2} + E_{A_1|A_3} + \ldots . + E_{A_1|A_n} \tag{1.37}$$

1.10 Entanglement measures for multipartite states

1.10.1 4-Tangle measure

In a two-qubit scenario tangle of a bipartite pure state $|\psi_{AB}\rangle$ is defined as

$$\tau(|\psi_{AB}\rangle) = 2\big(1 - \mathrm{Tr_A}[\rho_\mathrm{r}^2]\big), \tag{1.38}$$

where ρ_r is reduced density matrix i.e $\rho_r = Tr_B|\psi_{AB}\rangle\langle\psi_{AB}|$. Furthermore, it is just the square of the concurrence. Later it has been extended to three-qubit system [34]. Tangle of a tripartite pure state $|\psi_{ABC}\rangle \in \mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$ is defined as

$$\tau(|\psi_{ABC}\rangle) = \tau_{A(BC)} - \tau_{AB} - \tau_{AC}. \tag{1.39}$$

$\tau_{A(BC)}$ is tangle between qubit systems A and BC and $\tau_{AB} = \tau(\rho_{AB})$, where $\rho_{AB} = \mathrm{Tr_C}(\rho_\mathrm{ABC})$. For three-qubit states, it has been called residual entanglement between the subsystems A and

BC which is independent of entanglement between AB and AC. Three-qubit GHZ and the states belonging to GHZ class have non-vanishing tangle whereas W class states have vanishing tangle. This measure has been found very useful to distinguish between these two types of genuine three qubit entanglement. The authors in [36] have shown the square of this quantity is entanglement monotone. Another interesting feature of this measure is that it is invariant under qubit permutations. Wong et al. [37] have generalized the definition of tangle to any n-qubit state and shown square of this quantity is an entanglement monotone. Thus, the 4-tangle of a 4-qubit state $|\psi_{ABCD}\rangle \in \mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2 \otimes \mathcal{C}^2$ is defined as

$$\tau_{|\psi_{ABCD}\rangle} = \langle\psi|\sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y|\psi^*\rangle. \tag{1.40}$$

It is an entanglement monotone and invariant under qubit permutation. it is worth mentioning that τ is not defined for odd $n > 3$. Below we have listed τ for some well known 4-qubit states in Table I:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \tag{1.41}$$
$$|\Omega\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle), \tag{1.42}$$
$$|W\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle), \tag{1.43}$$
$$|Q_4\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle), \tag{1.44}$$
$$|Q_5\rangle = \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle). \tag{1.45}$$

States	4-Tangle
$\|GHZ\rangle$	1
$\|\Omega\rangle$	0
$\|W\rangle$	0
$\|Q_4\rangle$	0
$\|Q_5\rangle$	0

Table I: Tangle of some well known 4-qubit states.

Clearly this measure works only for a restricted classes of states.

1.11 Entanglement as a resource for QIPPs

Nonlocal correlations emerging from entanglement have interesting theoretical aspects. However these are more appealing as they allow us to perform certain quantum information processing tasks [38, 39, 45–47] which are impossible otherwise. Ekert first realized the usage of entanglement in quantum key distribution which provides unconditional security in cryptography unlike classical cryptography schemes. Since then many variants of quantum cryptography protocols have been proposed as well as practically implemented. Another nice protocol namely, quantum teleportation is well known. We shall discuss these protocols as presented in original works and some variants of it in the next few sections.

1.11.1 Teleportation

Suppose, Alice and Bob with whom we have met in the previous sections live in two different labs which are far away from each other. Alice wants to send an unknown state to Bob's laboratory. The unknown state is described by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle), \tag{1.46}$$

where α and β are in general a complex quantities and satisfy $|\alpha|^2 + |\beta|^2 = 1$.

A simple way would be to copy the state $|\psi\rangle$ and then send it to Bob over a quantum channel. But nature forbids to copy an unknown state as it contradicts superposition principle which is one of the fundamental features of quantum mechanics [50]. Bennett $et\ al.$ [39] proposed a solution to overcome it and formulated teleportation which utilizes entanglement shared between Alice and Bob. Suppose, both of them are provided a maximally entangled state say, a singlet state $|R\rangle$ such that one particle is held by Alice and another one is held by Bob. They also have a classical channel at their disposal. We write the combined state as:

$$|\psi\rangle|R\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)_A \otimes (|01\rangle - |10\rangle)_{AB}. \tag{1.47}$$

Here the subscripts denote that first two qubits are with Alice and the last one is with Bob. Now Alice performs a joint measurement in Bell basis (1.3) on the composite system consisting of the system in unknown state and the subsystem of the entangled state. We can rewrite the above equation as:

$$
\begin{aligned}
|\psi\rangle|R\rangle &= \frac{1}{\sqrt{2}}(\alpha|00\rangle_A|1\rangle_B - |01\rangle_A|0\rangle_B + \beta|10\rangle_A|1\rangle_B - |11\rangle_A|0\rangle_B) \\
&= \frac{1}{2}\big(\alpha(|\phi_+\rangle + |\phi_-\rangle)_A|1\rangle_B - (|\psi_+\rangle + |\psi_-\rangle)_A|0\rangle_B + \\
&\quad \beta(|\psi_+\rangle - |\psi_-\rangle)_A|1\rangle_B - (|\phi_+\rangle - |\phi_-\rangle)_A|0\rangle_B\big).
\end{aligned}
$$

Alice obtains either of the $|\phi_\pm\rangle, |\psi_\pm\rangle$ as outcome with equal probability $\frac{1}{4}$. Bob's subsystem collapses to one of the four following states:

$$
\begin{aligned}
|\phi_+\rangle &\Rightarrow & |\psi\rangle_B &= \alpha|1\rangle - \beta|0\rangle, \\
|\phi_-\rangle &\Rightarrow & |\psi\rangle_B &= \alpha|1\rangle + \beta|0\rangle, \\
|\psi_+\rangle &\Rightarrow & |\psi\rangle_B &= -\alpha|0\rangle + \beta|1\rangle, \\
|\psi_-\rangle &\Rightarrow & |\psi\rangle_B &= -(\alpha|0\rangle + \beta|1\rangle).
\end{aligned} \tag{1.48}
$$

Then Alice broadcasts the outcome of her measurement over the classical channel. We can now easily verify that Bob can recover the original unknown state by applying $\sigma_x\sigma_z$, σ_x, σ_z and σ_0 respectively on his qubit. Since then many variants of this protocol have been considered conceptually as well as experimentally [40]. Now we discuss some of the variants of this protocol.

1.11.2 Teleportation with multiqubit bipartite states

A multipartite state can be partitioned into two different subsystems. This situation can be visualized by taking tripartite GHZ state into consideration [41]. Suppose, a GHZ state is distributed in such a way that Alice holds two qubits and Bob has the remaining one. So, we write the state as:

$$
|GHZ\rangle = \frac{1}{\sqrt{2}}(|00\rangle_A 0_B + |11\rangle_A 1_B). \tag{1.49}
$$

Alice can teleport an unknown one-qubit state to Bob using the same procedure as discussed in the previous section. We can write the joint state as:

$$
\begin{aligned}
|\psi\rangle|GHZ\rangle &= \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)_A \otimes (|00\rangle_A|0\rangle_B + |11\rangle_A|1\rangle_B) \\
&= \frac{1}{\sqrt{2}}(\alpha|000\rangle_A|0\rangle_B + \alpha|011\rangle_A|1\rangle_B + \beta|100\rangle_A|0\rangle_B + \beta|111\rangle_A|1\rangle_B).
\end{aligned}
$$

Now Alice uses the measurement basis $|3GHZ_1^{\pm}\rangle$ and $|3GHZ_2^{\pm}\rangle$ which are

$$\begin{aligned} |3GHZ_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), \\ |3GHZ_2^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|011\rangle \pm |100\rangle). \end{aligned} \tag{1.50}$$

Then (1.50) can be rewritten as

$$|\psi\rangle|GHZ\rangle = \frac{1}{2}(\alpha(|3GHZ_1^{+} + 3GHZ_1^{-}\rangle)|0\rangle + \alpha(|3GHZ_2^{+} + 3GHZ_2^{-}\rangle)|1\rangle + \beta(|3GHZ_2^{+} - 3GHZ_2^{-}\rangle)|0\rangle + \beta(|3GHZ_1^{+} - 3GHZ_1^{-}\rangle)|1\rangle).$$

According to each measurement outcome as obtained by Alice, Bob's state collapses into one of the following states:

$$\begin{aligned} |3GHZ_1^{+}\rangle &\Rightarrow |\psi\rangle_B = \alpha|0\rangle + \beta|1\rangle, \\ |3GHZ_1^{-}\rangle &\Rightarrow |\psi\rangle_B = \alpha|0\rangle - \beta|1\rangle, \\ |3GHZ_2^{+}\rangle &\Rightarrow |\psi\rangle_B = \beta|0\rangle + \alpha|1\rangle, \\ |3GHZ_2^{-}\rangle &\Rightarrow |\psi\rangle_B = -\beta|0\rangle + \alpha|1\rangle. \end{aligned} \tag{1.51}$$

Depending upon the outcome of Alice, Bob can retrieve the original state by applying local unitary transformation on his subsystem – σ_0, σ_z, σ_x and $\sigma_z\sigma_x$ respectively. In chapter three, we will revisit teleportation with multipartite entangled states under this scheme with few more examples.

1.11.3 Cooperative teleportation

Another generalization using multipartite state would be cooperative teleportation. The idea originates from a nice property of entanglement. In a multipartite system, we can localize maximal amount of entanglement by performing a series of local operation and classical communication. This idea is popular as localization of entanglement [42]. In this protocol a genuine multipartite entangled state is distributed in such a way that each of the parties has at least one qubit; some of them may have more than one qubit. Let us consider the following example. Suppose, a tripartite GHZ state is distributed between Alice, Bob and Charlie. Alice has the unknown qubit which she wishes to teleport to Bob's lab with the help of Charlie. The combined state can be written as

$$\sum_i \frac{1}{\sqrt{8}} |\phi^i\rangle_A |\pm\rangle_C U_{i,\pm}(\alpha|0\rangle + \beta|1\rangle)_B. \tag{1.52}$$

Alice makes a join measurement in Bell basis and conveys the outcome to Charlie. He then makes another measurement on his subsystem in $|\pm\rangle$ basis and informs the outcome to Bob. Bob then recovers the original state by applying suitable unitary operation on his qubit. This collaborative effort to teleport an unknown qubit has been named as cooperative teleportation. An immediate generalization of this protocol is to any N-qubit state. It is not obvious that the protocol can be carried out with any genuinely entangled multipartite state. Certain properties of the resource state can guarantee its success. We shall discuss this in more details in chapter four and find out the resource state structure for this protocol in multiqubit scenario.

1.11.4 Probabilistic teleportation

Suppose, we have a non-maximally entangled state which we wish to use to teleport an unknown state. Alice and Bob share the pair between them and have a classical channel. It is known that teleportation protocol cannot be carried out with unit fidelity and unit probability under this scenario. Indeed teleportation protocol can be accomplished with unit fidelity but $probabilistically$ [43]. In this case Alice uses a generalised Bell basis (GBS) to perform measurement and proceeds as in the original protocol. Let non-maximally entangled state shared between Alice and Bob is

$$|\Psi\rangle = N(|00\rangle + n|11\rangle), \tag{1.53}$$

where $N = \frac{1}{\sqrt{1+|n|^2}}$ and n is a complex quantity. The GBS that Alice chooses can be represented as

$$|\chi_m^+\rangle = \frac{|00\rangle + m|11\rangle}{\sqrt{1+|m|^2}},$$
$$|\chi_m^-\rangle = \frac{m^*|00\rangle - |11\rangle}{\sqrt{1+|m|^2}},$$

$$
\begin{aligned}
|\zeta_m^+\rangle &= \frac{|01\rangle + m|10\rangle}{\sqrt{1+|m|^2}}, \\
|\zeta_m^-\rangle &= \frac{m^*|01\rangle - |10\rangle}{\sqrt{1+|m|^2}},
\end{aligned} \tag{1.54}
$$

where m is in general a complex quantity. Now proceeding as before in (1.11.1), she finds one of these GBS as outcome with probability

$$
\begin{aligned}
P_{\chi_m^+} &= \frac{|\alpha|^2 + |mn\beta|^2}{(1+|m|^2)(1+|n|^2)}, \\
P_{\chi_m^-} &= \frac{|m\alpha|^2 + |n\beta|^2}{(1+|m|^2)(1+|n|^2)}, \\
P_{\zeta_m^+} &= \frac{|n\alpha|^2 + |m\beta|^2}{(1+|m|^2)(1+|n|^2)}, \\
P_{\zeta_m^-} &= \frac{|mn\alpha|^2 + |\beta|^2}{(1+|m|^2)(1+|n|^2)},
\end{aligned} \tag{1.55}
$$

and Bob's state collapses accordingly into one of the following state:

$$
\begin{aligned}
|\chi_m^+\rangle \quad &\Rightarrow \quad |\psi\rangle_B \quad = \quad \frac{\alpha|0\rangle + nm^*\beta|1\rangle}{\sqrt{|\alpha|^2 + |mn\beta|^2}}, \\
|\chi_m^-\rangle \quad &\Rightarrow \quad |\psi\rangle_B \quad = \quad \frac{m\alpha|0\rangle + n\beta|1\rangle}{\sqrt{|m\alpha|^2 + |n\beta|^2}}, \\
|\zeta_m^+\rangle \quad &\Rightarrow \quad |\psi\rangle_B \quad = \quad \frac{m^*\beta|0\rangle + n\alpha|1\rangle}{\sqrt{n|\alpha|^2 + |n\beta|^2}}, \\
|\zeta_m^-\rangle \quad &\Rightarrow \quad |\psi\rangle_B \quad = \quad \frac{mn\alpha|0\rangle + \beta|1\rangle}{\sqrt{|mn\alpha|^2 + |\beta|^2}}.
\end{aligned} \tag{1.56}
$$

Without loss of generality we may assume m and n to be real. Now, we can see if we choose entanglement of the measuring basis carefully such that $m = n$, then Bob's state reduces to the original state corresponding to GBM $|\chi_m^-\rangle$ and $|\zeta_m^+\rangle$. Hence the state can be teleported with unit fidelity but probability of successful teleportation reduces to

$$
P_{successfull} = (P_{\chi_m^-} + P_{\zeta_m^+})|_{m=n} = \frac{2|n|^2}{(1+|n|^2)^2}. \tag{1.57}
$$

1.11.5 Teleportation fidelity

At this stage, we focus on a practical aspect of this protocol. We have learned already that Bob can recover the original state by applying suitable unitary operation on his subsystem, but this is not always the case. The quality of the teleported state depends on the entanglement of the resource state. In case of a maximally entangled state, Bob can recover the original state as it was. However, if the resource state is partially entangled or a mixed entangled, then the teleported state does not resemble the original one. In the context of teleportation, fidelity plays an important role in determining the quality of the teleported state. Fidelity of a teleported state using an entangled state ρ in Hilbert space dimension $d \times d$ is [48]

$$\mathcal{F}(\rho) = \frac{df + 1}{d + 1}, \tag{1.58}$$

where f is singlet fraction and it is defined as the maximal overlap of the state with a maximally entangled state

$$f(\rho) = \max_{|\psi\rangle = ME} |\langle\psi|\rho|\psi\rangle|. \tag{1.59}$$

However Horodecki [49] *et al.* obtained a general expression for optimal teleportation fidelity of a general state. Maximum teleportation fidelity of a two-qubit state ρ_{AB} can be expressed as

$$\mathcal{F}_{\text{max}} \leq \frac{1}{2}\big[1 + \frac{1}{3}\text{Tr}\sqrt{T^{\dagger}T}\big]. \tag{1.60}$$

Furthermore, they have shown that for a state ρ with $\text{Tr}\sqrt{T^{\dagger}T} > 1$, the inequality in (1.60) can be replaced by an equality,

$$\mathcal{F}_{\text{max}} = \frac{1}{2}\big[1 + \frac{1}{3}\text{Tr}\sqrt{T^{\dagger}T}\big]. \tag{1.61}$$

From the (1.61) we can see that teleportation fidelity depends on the correlation matrix. We note the fact that $\frac{2}{3} \leq \mathcal{F}_{\text{max}} \leq 1$, where $\frac{2}{3}$ is the optimal classical teleportation fidelity. As teleportation is a nonlocal phenomenon, it is quite natural that this quantity would depend on the entanglement of the resource state. Indeed it is true for any pure state- more entangled state yields larger fidelity. However, for a mixed entangled state, it is not always obvious, as we will see in the second chapter. There are other parameters upon which this nonlocal quantity is dependent.

1.12 Quantum Key Distribution

Quantum key distribution (QKD), an elegant application of entanglement has usage in the field of cryptography. Although the history of cryptography goes back to the ancient ages, its importance is only growing. It is an art of communicating a secret message between two persons which can not be revealed to any unauthorized person. One can achieve it by constructing a suitable algorithm, sometime known as cipher to write the secret message with some additional information. This process is known as encryption which lies at the heart of any cryptosystem. An advanced encryption process involves sophisticated technique so that it can be hidden from any adversary. However a classical encryption process has its own limitation: it can be broken in principle. A remarkable idea along this direction was first proposed by Bennett and Brassard in 1984 when they utilized quantum mechanics while encrypting the message. In quantum mechanics, as we know there are some fundamental axioms. One of the consequences of these axioms is:

An unknown state cannot be cloned [50].

Suppose there exists a quantum cloning machine which can clone two unknown states – $|\psi\rangle$ and $|\phi\rangle$. Then there exists a unitary operator $\mathcal{U}_{cl}$ such that

$$\mathcal{U}_{cl}|\psi\rangle \rightarrow |\psi\rangle|\psi\rangle,$$

and,

$$\mathcal{U}_{cl}|\phi\rangle \rightarrow |\phi\rangle|\phi\rangle.$$

Now taking the inner product of above two equations we obtain

$$\langle\phi|\psi\rangle = |\langle\phi|\psi\rangle|^2.$$

So we can infer, either $|\phi\rangle = |\psi\rangle$ or $|\phi\rangle = |\psi\rangle^{\perp}$. So a general cloning machine cannot be devised. This basic inference turns out to be very useful to secure BB84 protocol.

1.12.1 BB84 Protocol

In this protocol [51], Alice uses a bit string which can be polarized photons and randomly chooses two bases (Vertical-Horizontal or diagonal). To encode information they use the convention: '0' for $|\updownarrow\rangle$ and $|\nearrow\rangle$, '1' for $|\longleftrightarrow\rangle$ and $|\nwarrow\rangle$. Alice measures in one of these two basis sets and sends the photons one by one through a quantum channel to Bob. If Eve has access to that quantum channel and she keeps the incoming photons with her, resulting in no photons at Bob's disposal. In this case, they can track the presence of Eve easily and abort the process. So to get information, Eve is more likely to send an arbitrary photon to Bob taking the original one. Now Eve can copy the original photon to have the knowledge of the secret but here quantum mechanics plays its role. We have already seen that exact cloning of an arbitrary unknown state is not possible. Now Bob chooses those same bases randomly as Alice. They have a classical channel like telephone or internet to communicate publicly. As Bob is also measuring his photons randomly, half of the time in asymptotic limit it happens that their measurement bases coincide and obtain correlated outcome. They use the classical channel to communicate publicly the order of the bases they have used for measurement but do not reveal the measurement outcomes. They keep those bits for which their bases match. In this way they generate what is known as raw key which they can use to establish a secret key.

1.12.2 Ekert Protocol

In 1991, Ekert approached the issue of establishing secret key by incorporating entanglement which has been another mystery of quantum mechanics. Einstein, Podolosky and Rozen raised their strong view about the completeness of quantum theory and formulated EPR paradox [52] in 1935. Assuming local realism as one of the fundamental requirement of any physical theory, they argued that quantum mechanics has to be incomplete. In 1964, John Bell [53] starting from this key assumption proved that quantum mechanics indeed violates local realism. Assuming local realism, he constructed a Bell inequality and showed that it is violated by a maximally entangled state.

Suppose there is an electron gun capable of ejecting a pair of electrons and each of these electrons are being transmitted to Alice and Bob who are spatially separated. Alice has access to two different measurement apparatus A_1 and A_2 which represents the physical property of the electron and both of these measurement produce binary outcomes. Similarly, Bob has an experimental set up and his apparatus are labeled as B_1 and B_2. Without loss of generality, we may assume that both of them perform measurement at the same time so that measurement of

Alice does no affect Bob's and vice-versa. We are interested to find out the mean value of the following polynomial $\langle A_1B_1 + A_2B_1 + A_2B_2 - A_1B_2\rangle$. It is straightforward to verify that the value of the polynomial cannot exceed 2. *i.e*

$$\langle A_1B_1 + A_2B_1 + A_2B_2 - A_1B_2\rangle \leq 2. \tag{1.62}$$

This is known as Bell-CHSH inequality. Now we do the same experiment but this time replacing the electron gun by a more advanced electron gun which is capable of emanating a pair in a singlet state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

Alice and Bob perform measurement of the following observables:

$$A_1 = \sigma_z, A_2 = \sigma_x, B_1 = -\frac{\sigma_z + \sigma_x}{\sqrt{2}}, B_2 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}.$$

We can compute the expectation values of the following combination of observables for the singlet state and obtain

$$\langle A_1B_1\rangle = \langle A_2B_1\rangle = \langle A_2B_2\rangle = \frac{1}{\sqrt{2}}, \langle A_1B_2\rangle = -\frac{1}{\sqrt{2}}.$$

Putting these values into the polynomial we get

$$\langle A_1B_1 + A_2B_1 + A_2B_2 - A_1B_2\rangle = 2\sqrt{2} \nleq 2. \tag{1.63}$$

So quantum mechanics violates the above Bell-CHSH inequality. At this point we will put aside the debate about the completeness of quantum theory and discuss the usefulness of the above inequality violation in Ekert's quantum key distribution protocol.

According to this protocol [38] entanglement of the shared states is used as a resource. A large number of pairs of qubits in singlet state are distributed between Alice and Bob. Then they choose three different coplanar axis to make measurements on their incoming particles. Three different axes is labeled as a_i(i=1,2,3) for Alice and b_j(j=1,2,3) for Bob and further we are assuming that axes are confined to X-Y plane. Now there is $\frac{1}{3}$ probability that they would find their measurements have been done in compatible basis set. In these cases when their measurement bases match, their outcomes are also correlated. It means if Alice finds its particle in spin-up configuration, then it is spin-down for Bob and vice versa. They retain these keys to establish secret key. Whenever they measure in incompatible basis set, the results are not of interest for establishing key but it may be useful to verify the security of the protocol.

After the measurements are done, they publicly announce the basis they chose for each measurement. If the bases matches for both Alice and Bob, they keep the data otherwise they don't keep. The former set of bit strings known as sifted key is used to establish the secret key between them. To detect any intruder they choose later set of bit strings for which measurement bases are different. Now they can easily check the value of the polynomial (1.62) with their data. If it violates Bell inequality, then they can be sure about the absence of any intruder as any measurement by intruder will make the state unentangled and hence Bell inequality would not be violated. Thus a secure key can be established by using entanglement of the resource state and its security is assured by the fundamental law of quantum mechanics.

1.13 Outline of the thesis

This brief introduction covers the background materials for the following chapters of this thesis. The thesis is organized as follow:

• In the second chapter, we shall discuss teleportation protocol using mixed states as resource and find that concurrence and purity are not enough to characterize teleportation fidelity.

• Teleportation can be generalised to multipartite entangled states also. We shall see some examples of this protocol in the third chapter and find the criterion for a resource state to be suitable for $perfect$ teleportation. We will also see that sometime less entanglement can be more effective for QIPPs.

• In the last chapter, we will propose cooperative quantum key distribution (co-QKD) protocol using multipartite entangled states. This chapter will be devoted to find the suitable structure of the resource state for cooperative QKD and some of their properties.

• We will conclude the thesis with a brief summary where we discuss how this thesis can be useful for better understanding of some of the aspects of QIPPs.

Chapter 2

Teleportation fidelity of two-qubit mixed states

2.1 Introduction

Quantum teleportation [39] is a process of transmitting an unknown state to a distant lab by means of shared entanglement and classical communication. We have studied the protocol using mixed entangled state [55, 56] as a resource. The original protocol as proposed by Bennett *et al.* was to distribute each particle of a maximally entangled state to Alice and Bob. A similar kind of situation can arise if Alice has the source and she sends a subsystem of an entangled pair to Bob while keeping another subsystem in her possession. Such practical situation like distribution of quantum particles leads to entanglement to be lost or degraded. If the situation is like that, a maximally entangled state evolves into a mixed state due to interaction with environment. It has been shown that Werner states [57] can also be useful for teleportation for a certain range of its parameter [54]. Popescu [58] had shown in the context of Bell violation and nonlocality, mixed states can also be useful as a resource with fidelity larger than that of classical case. However with mixed states, we cannot obtain perfect teleportation *i.e.*, the teleported state does not resemble the original state. Teleportation fidelity is no longer unity. A unit teleportation fidelity is achievable when we have maximally entangled pure states a resource. Verstraete *et al.* [62] have found an upper bound on fidelity in terms of its entanglement as measured by concurrence or negativity.

As discussed in the last chapter, a two-qubit mixed state is characterized by 15 real parameters. A mixed state has both classical and quantum properties. Any nonlocal property of a state should remain invariant under local unitary transformation. So, for the sake of simplicity, we may apply local unitary transformation on individual subsystems so that correlation matrix becomes diagonal [3] and density matrix is now specified by nine parameters. Nevertheless it is not obvious which parameters specify classical and quantum aspects of a general state. Nonlocal quantum properties are characterized by an entanglement measure, such as concurrence, or negativity and teleportation fidelity has explicit relation with such quantities. Usually, classicality of a state is characterized by how pure a state is. Here, by classicality we mean the classical correlations in the state due to the mixing parameters. We shall use the purity as a measure of classical aspects. It attains its maximum value one for a pure state and $\frac{1}{4}$ for a two-qubit maximally mixed state. Its complementary $1 - \mathrm{Tr}[\rho^2]$ is referred as 'mixedness'.

One may expect that the teleportation fidelity may depend not only on the entanglement of a mixed state, but also classicality. Życzkowski *et al.* [59] have shown that states with purity less than $\frac{1}{3}$ are separable. It has also been proved [60] that if a state exceeds a certain degree of mixedness as quantified by von Neumann entropy, then it cannot be used as a teleportation channel. Likewise in [61], the authors have obtained rank dependent lower bound on concurrence and upper bound on purity of a state for the success of the teleportation protocol. However, it would be legitimate to consider functions of state parameters other than concurrence/negativity and purity. As a state may have more than two parameters, teleportation fidelity seems to depend on more than these two functions. We shall compute teleportation fidelity for some well known classes of states and show that concurrence and purity are not enough to fully characterize the nonlocal properties of a mixed state.

2.2 Teleportation fidelity of a pure state

Before proceeding further let us find out the expression of teleportation fidelity for a pure state of the form

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle. \tag{2.1}$$

It is straightforward to find the optimal fidelity using the prescription (1.60) and it turns out to be

$$\mathcal{F} = \frac{2}{3}(1 + |\alpha||\beta|). \tag{2.2}$$

We can also find concurrence of (2.1) using Wooter's recipe and it is given by

$$\mathcal{C} = 2|\alpha||\beta|. \tag{2.3}$$

We can relate above two equations by eliminating α and β and we obtain

$$3\mathcal{F} = 2 + \mathcal{C}. \tag{2.4}$$

We can see that for any pure entangled state teleportation fidelity is larger than $\frac{2}{3}$ and fidelity increases linearly with entanglement of the state. A two qubit pure state has only one nonlocal parameter, the so called Schmidt number and it seems reasonable that fidelity depends on that nonlocal parameter. But it is not the case always for a mixed state as we shall see in the subsequent sections.

2.3 Teleportation fidelity of Werner state

Werner state is a representative class of states which interpolates between maximally entangled state and maximally mixed state. We write a 2×2 bipartite Werner state [57] as:

$$\rho_W = \frac{1-\alpha}{4} I + \alpha|\phi^+\rangle\langle\phi^+|, \tag{2.5}$$

where $\alpha \in (0, 1)$. Using positive partial transpose criterion, we can easily verify that ρ_W is entangled whenever $\frac{1}{3} < \alpha \leq 1$, it is unentangled otherwise. We obtain the concurrence:

$$\begin{aligned} \mathcal{C}(\rho_W) &= \frac{1}{2}(3\alpha - 1) \quad \text{when} \quad \frac{1}{3} < \alpha \leq 1, \\ \mathcal{C}(\rho_W) &= 0 \quad \text{when} \quad 0 \leq \alpha \leq \frac{1}{3}. \end{aligned} \tag{2.6}$$

We now find expressions for purity and teleportation fidelity for Werner states. We find,

$$\mathcal{P} = \frac{1+3\alpha^2}{4}, \tag{2.7}$$

$$\mathcal{F} = \frac{1+\alpha}{2}. \tag{2.8}$$

Eliminating α from (2.6) and (2.8), we find

$$3\mathcal{F} = 2 + \mathcal{C}. \tag{2.9}$$

This is same as eq.(2.4), teleportation fidelity increases with the increase of concurrence. We can compare (2.7) and (2.8), as purity increases teleportation fidelity increases. This is not so surprising because increase in purity means α increases and α determines nonlocality of ρ_W (2.6).

2.4 Maximally entangled mixed state

We introduced maximally entangled mixed states (MEM) with some interesting properties in the introduction chapter. It has maximal amount of entanglement for a fixed mixedness. However this class of states can be found useful for implementing teleportation. Authors in [67] considered MEM as a resource for the protocol. In the next subsections, we shall find teleportation fidelity using MEMs as resource states. MEMs are characterized by $three$ real parameters. We write the general expression of MEM as [17]:

$$\rho_M = p_1|\psi^-\rangle\langle\psi^-\rangle + p_2|00\rangle\langle 00| + p_3|\psi^+\rangle\langle\psi^+| + p_4|11\rangle\langle 11|, \tag{2.10}$$

where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ and $\sum_i p_i = 1(p1 \geq p_2 \geq p_3 \geq p_4)$. We can obtain rank-2 and rank-3 MEMs by setting $p_3 = p_4 = 0$ and $p_4 = 0$ respectively. We shall compute purity, concurrence and fidelity to study comparative behavior of these three quantities for different rank MEMs.

2.4.1 Rank-2 MEMs

For a rank-2 MEM state, we have $p_3 = p_4 = 0$. It has two non-vanishing eigenvalues namely, p_1 and p_2. Purity, concurrence and fidelity for this state are given as

$$\mathcal{P} = p_1^2 + p_2^2, \tag{2.11}$$

$$\mathcal{C} = p_1 \text{ and} \tag{2.12}$$

$$\mathcal{F} = \frac{1}{3}(1 + 2p_1). \tag{2.13}$$

Therefore, $\mathcal{F} = \frac{1}{3}(1 + 2\mathcal{C})$. If $p_1 > \frac{1}{2}$ or $\mathcal{P} > \frac{1}{2}$ then $\mathcal{F} > \frac{2}{3}$. So all rank-2 MEMs with purity larger than $\frac{1}{2}$ are useful for teleportation. It is obvious from the relation that teleportation fidelity increases with entanglement of the resource state which is characterized by the parameter p_1. Here we find an interesting fact that ρ_M is no longer a good resource for teleportation if purity is less than $\frac{1}{2}$ although it is entangled. So it seems that purity of the resource state is a vital quantity in this context.

2.4.2 Rank-3 MEMs

Rank-3 MEM states can be written as:

$$\rho_3 = p_1|\psi^-\rangle\langle\psi^-\rangle + p_2|00\rangle\langle 00| + p_3|\psi^+\rangle\langle\psi^+|, \tag{2.14}$$

where $p_i \neq 0$, $p_1 + p_2 + p_3 = 1$, $p_1 \geq p_2$, and $p_2 \geq p_3$. Taking normalization constraint into account, here we have two independent parameters. Purity, concurrence and fidelity of ρ_3 is given by:

$$\begin{aligned} \mathcal{P} &= p_1^2 + p_2^2 + p_3^2. \\ \mathcal{C} &= p_1 - p_3. \\ \mathcal{F} &= \frac{1}{3}(1 + 2p_1). \end{aligned} \tag{2.15}$$

From the expression of fidelity, we find $p_1 > \frac{1}{2}$ implies $\mathcal{F} > \frac{2}{3}$. After doing some calculation we eliminate p_i and using the trace condition of the density matrix we obtain the following relation

$$\mathcal{F} = \frac{1}{3}\Big[\frac{5}{3} + \mathcal{C} + f(\mathcal{P}, \mathcal{C})\Big], \tag{2.16}$$

where $f(\mathcal{P},\mathcal{C}) \equiv \frac{1}{3}\sqrt{6\mathcal{P}-3\mathcal{C}^2-2}$ and $\mathcal{P} \in (\frac{1}{3},1)$. Now we have a nice relation which involves all of these three quantities. Fig.(2.1) shows comparative behavior between these quantities. It reflects the fact that not only teleportation fidelity increases with concurrence, fidelity increases with purity when entanglement is fixed at certain value.

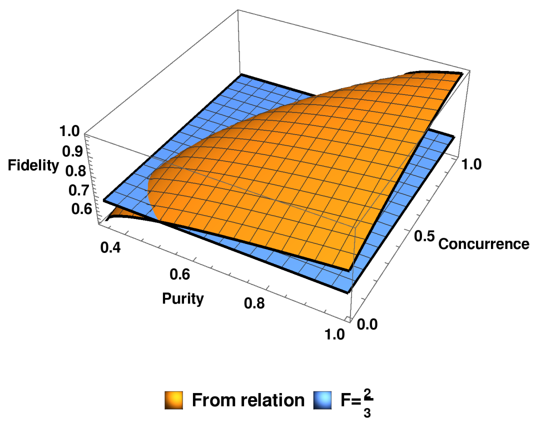

Figure 2.1: MEM states: Comparative behavior of teleportation fidelity with concurrence and purity for rank-3 states.

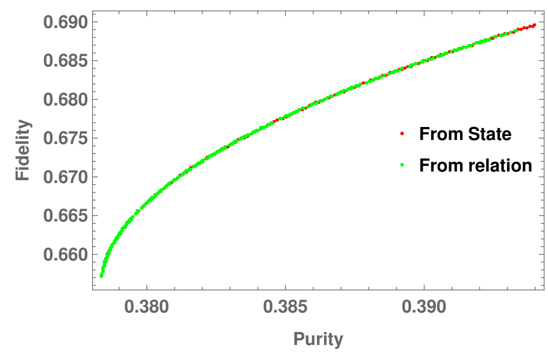

Figure 2.2: MEM states: Variation of teleportation fidelity with purity for a fixed $\mathcal{C} = 0.3$.

From Fig.(2.1) we can see that all those states lying above the plane described by $\mathcal{F} = \frac{2}{3}$ produce nonlocal fidelity. The states lying below that plane are entangled but cannot produce fidelity better than the classical value. The classicality of the states suppress its nonlocal aspect. So there must exist a threshold purity above which mixed entangled states exhibit nonlocal behavior. We shall compute that value for this class of states. Mathematically our problem reduces to the following:

Minimize: $\mathcal{P} = p_1^2 + p_2^2 + p_3^2$, with the constraints (g_j):

$p_1 \geq p_2, p_2 \geq p_3, \mathcal{F} \geq \frac{2}{3}$, and $p_1 + p_2 + p_3 = 1$.

We shall use Kun-Tucker condition in this particular case to solve the minimization problem. The Lagrangian is

$$\mathcal{L}(p_1,p_2,p_3,\lambda_1,\lambda_2,\lambda_3) = p_1^2 + p_2^2 + p_3^2 + \lambda_1(2p_1 - 1) + \lambda_2(p_1 - p_2) + \lambda_3(p2 - p3), \quad (2.17)$$

where $\lambda_j \geq 0$. We have to solve $\frac{\partial L}{\partial p_i} = 0$ and $\lambda_j g_j = 0$, which produces the following set of equations:

$$2p_1 + 2\lambda_1 + \lambda_2 = 0, \tag{2.18a}$$

$$2p_2 - \lambda_2 + \lambda_3 = 0, \tag{2.18b}$$

$$2p_3 - \lambda_3 = 0, \tag{2.18c}$$

$$\lambda_1(1 + 2p_1 - 2) = 0, \tag{2.18d}$$

$$\lambda_2(p_1 - p_2) = 0, \tag{2.18e}$$

$$\lambda_3(p_2 - p_3) = 0. \tag{2.18f}$$

The above set of equations is overdetermined system due to the fact that $\sum p_i = 1$. It reduces the number of independent variables from six to five, where we have six equations to solve. In general no solution exists for overdetermined system. But in some cases when equations are of polynomial type it may exist. To find the solutions we set $\lambda_1 \neq 0$. It implies from (2.18d) that $p_1 = \frac{1}{2}$. From (2.18e) we see $\lambda_2 = 0$, otherwise we would have $p_1 = p_2$. This implies $p_3 = 0$ and ρ_3 is of rank-2. By letting $\lambda_3 \neq 0$ we obtain $p_2 = p_3$ from (2.18f). As $p_2 = p_3$, we have $p_2 = p_3 = \frac{1}{4}$. Finally we obtain the set of solutions: $p_1 = \frac{1}{2}$, $p_2 = p_3 = \frac{1}{4}$. Substituting the value of p_i's into the expression of purity we obtain $\mathcal{P}_{min} = \frac{3}{8}$. All mixed states with this value of purity lie on $\mathcal{F} = \frac{2}{3}$ plane in CP diagram.

2.4.3 Rank-4 MEMs

In this section, we shall continue our discussion for more general MEM states. Purity, concurrence and fidelity of the state Eq.(2.10) are

$$\mathcal{P} = p_1^2 + p_2^2 + p_3^2 + p_4^2, \tag{2.19}$$

$$\mathcal{C} = p_1 - p_3 - 2\sqrt{p_2 p_4}, \tag{2.20}$$

$$\mathcal{F} = \frac{1}{3}(1 + 2p_1). \tag{2.21}$$

In the same way as in the previous case, we find the following expression for teleportation fidelity

$$\mathcal{F} = \frac{1}{3}\Big[1 + 2f(\mathcal{P}, \mathcal{C}, p_3)\Big], \tag{2.22}$$

where we define $f(\mathcal{P}, \mathcal{C}, p_3)$ as

$$f(\mathcal{P}, \mathcal{C}, p_3) = \frac{4 - 2\mathcal{C} - 6p_3 + \sqrt{(4 - 2\mathcal{C} - 6p_3)^2 - 12(2 + 3p_3^2 - \mathcal{C}^2 - 2p_3\mathcal{C} - 4p_3 - 2\mathcal{P})}}{6}. \tag{2.23}$$

To find the behavior of fidelity with purity, we have plotted it keeping other quantities like concurrence and p_3 fixed. Eventually we obtain same comparative behavior between fidelity and purity in fig.(2.3) as we found for rank-3 states.

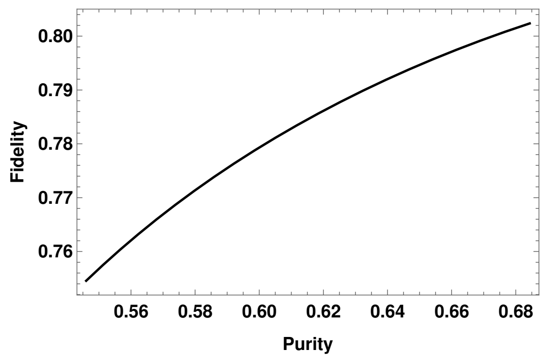

Figure 2.3: MEM states: Variation of fidelity with purity for a fixed $\mathcal{C} = 0.2$ and $p_3 = .005$ for rank-4 MEM-states.

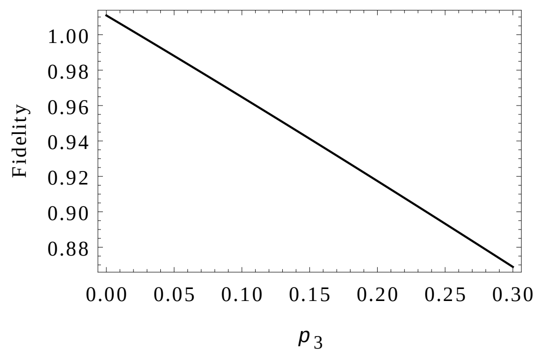

Figure 2.4: MEM states: Variation of teleportation fidelity with p_3 for a fixed $\mathcal{C} = 0.2$ and $\mathcal{P} = .7$.

So far we have seen from (2.16) and (2.22) that teleportation fidelity is a monotonically increasing function of purity of the state. In Introduction, we briefly mentioned that we are defining classicality of a state by its purity in a way such that more pure state is less classical. As purity is reduced, non-local aspects of a state as visualized by teleportation fidelity declines in value. So teleportation fidelity does depend not only on concurrence, but purity also plays a significant role. However, we find from (2.22) that teleportation fidelity depends on another state parameter p_3 apart from purity and concurrence. From Fig.(2.4) we can see that teleportation fidelity decreases monotonically with the state parameter p_3 for a fixed value of entanglement and purity. Evidently, concurrence and purity are not enough to characterize fidelity completely.

2.5 X-states

In this section, we shall discuss comparative relations between these three quantities namely, $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{P}$ for a larger class of states. We have seen earlier in the introduction that X-states are characterized by $five$ real parameters and generalize many well-known classes of states. Mendonça *et al.* [20] have shown that for every two-qubit state there exists a corresponding X state with the same purity and entanglement, as measured by concurrence, negativity or entropy of entanglement. We shall exploit the parameterization of X-states for different ranks [20] to find out the comparative relations for teleportation fidelity. We also emphasize the importance of parameters other than purity and concurrence in characterizing the nonlocal properties of a two-qubit mixed state.

The parametric form of an arbitrary two-qubit X-state of a bipartite system can be represented as follow [20]:

$$\rho_{AB} = \begin{pmatrix} \cos^2\theta & 0 & 0 & \sqrt{x}e^{i\mu} \\ 0 & \sin^2\theta\cos^2\phi & \sqrt{y}e^{i\nu} & 0 \\ 0 & \sqrt{y}e^{-i\nu} & \sin^2\theta\sin^2\phi\cos^2\psi & 0 \\ \sqrt{x}e^{-i\mu} & 0 & 0 & \sin^2\theta\sin^2\phi\sin^2\psi \end{pmatrix}. \tag{2.24}$$

with $\theta, \phi, \psi \in [0, \frac{\pi}{2}]$, $x, y \geq 0$ and $\mu, \nu \in [0, 2\pi]$. However these conditions are not enough to make Eq. (2.24) a valid density matrix. Further constraints $x \in [0, \mathcal{H}]$ and $y \in [0, \mathcal{G}]$ are required to make it positive semidefinite. Here we define $\mathcal{H} = \sin^2\theta\cos^2\theta\sin^2\phi\sin^2\psi$ and $\mathcal{G} = \sin^4\theta\cos^2\phi\sin^2\phi\cos^2\psi$. Mendonça *et al.* [20] have given a parameterization of two-qubit X-states of different ranks. A two-qubit X-state will be of rank one if $(x = \mathcal{H}, y = 0, \mathcal{A} = 0)$ or $(x = 0, y = \mathcal{G}, \mathcal{B} = 0)$, where $\mathcal{A} = \sin^2\theta(1 - \sin^2\phi\sin^2\psi)$ and $\mathcal{B} = 1 - \mathcal{A}$. A rank two X-state can be parameterized as $(x < \mathcal{H}, y = 0, \mathcal{A} = 0)$ or $(x = 0, y < \mathcal{G}, \mathcal{B} = 0)$ or $(x = \mathcal{H}, y = \mathcal{G}, \mathcal{AB} > 0)$. Conditions $(x < \mathcal{H}, y = \mathcal{G}, \mathcal{A} > 0)$ or $(x = \mathcal{H}, y < \mathcal{G}, \mathcal{B} > 0)$ parameterize a two-qubit X-state of rank three. A rank four X-state can be parameterized as $(x < \mathcal{H}, y < \mathcal{G}, \mathcal{AB} > 0)$. They have also given the concurrence and purity for an arbitrary X-state. We will be using those expressions along with the expression of fidelity to obtain functional relationship between fidelity, concurrence and purity. This relationship will give us the pattern how fidelity changes with purity and concurrence. In addition we will find dependence on other parameters of the state. We underscore the importance of these other parameters to characterize the nonlocal properties of a two-qubit mixed state.

It is straightforward to find the following expressions for purity and concurrence for an arbitrary X-state of the form(2.24),

$$\mathcal{P} = 1 - 2(\mathcal{AB} + \mathcal{G} - y + \mathcal{H} - x) \tag{2.25}$$

$$\mathcal{C} = 2 \text{ max } [\sqrt{x} - \sqrt{\mathcal{G}}, \sqrt{y} - \sqrt{\mathcal{H}}] \tag{2.26}$$

Using Eq. (1.60), we have evaluated the expression for teleportation fidelity for the general X-state as,

$$\mathcal{F} = \frac{1}{6}\Big[3 + 2\sqrt{(\sqrt{x} + \sqrt{y})^2} + 2\sqrt{(\sqrt{x} - \sqrt{y})^2} + \sqrt{\Big(\cos^2\theta - \sin^2\theta\big(\cos^2\phi + \sin^2\phi\cos 2\psi\big)\Big)^2}\Big]. \tag{2.27}$$

Now to remove the last square root in the above expression, we write Eq. (2.27) as

$$\mathcal{F} = \frac{1}{6}\Big[3 + 2\sqrt{(\sqrt{x} + \sqrt{y})^2} + 2\sqrt{(\sqrt{x} - \sqrt{y})^2} + \text{sgn}\big(k(\theta, \phi, \psi)\big)k(\theta, \phi, \psi)\Big], \tag{2.28}$$

where $k(\theta, \phi, \psi) = \Big(\cos^2\theta - \sin^2\theta\big(\cos^2\phi + \sin^2\phi\cos 2\psi\big)\Big)$ and 'sgn' represents the sign function. Eq. (2.28) will give two different expressions depending on the choice, $x > y$ or $y > x$. Depending on that fidelity expression will involve x or y and will be

$$\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{a} + \text{sgn}\big(k(\theta, \phi, \psi)\big)k(\theta, \phi, \psi)\Big], \tag{2.29}$$

where $a = \text{max}[x, y]$. As stated earlier, depending upon the parameters values and ranges, we can categorize a X-state as second rank, third rank, or fourth rank state. We will deal with the case of each rank separately as the complexity of the functional relationship will grow with rank.

2.5.1 Rank-2 X-states of first kind

We first consider the second rank states of first type, i.e $x < \mathcal{H}, y = 0, \mathcal{A} = 0$. First putting $y = 0$, in the general expression for purity and concurrence we get,

$$\mathcal{P} = 1 + 2x - 2p\sin^2\theta - 2q\sin^4\theta, \tag{2.30}$$

$$\mathcal{C} = 2(\sqrt{x} - f\sin^2\theta), \tag{2.31}$$

where,

$$f = \cos\phi\sin\phi\cos\psi, \tag{2.32}$$

$$p = \sin^2\phi\sin^2\psi, \tag{2.33}$$

$$q = \sin^2\phi(\cos^2\phi\cos^2\psi - \sin^2\psi). \tag{2.34}$$

And fidelity is given by,

$$\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{x} \pm \cos^2\theta \mp e\sin^2\theta\Big], \tag{2.35}$$

where,

$$e = \cos^2\phi + \sin^2\phi\cos 2\psi. \tag{2.36}$$

Now, situation for the rank two state will be simpler because we also have $\mathcal{A} = \sin^2\theta(1 - \sin^2\phi\sin^2\psi) = 0$. That will make either $\theta = 0$ or $\phi = \psi = \pi/2$. No other solutions are possible. So it will suffice to consider these two cases. But for $\theta = 0$, we have $\mathcal{H} = 0$. As $x < \mathcal{H}$, it should be negative (because with the other restrictions for the first kind of second rank X-states, x must be less than $\mathcal{H}$, otherwise it won't be a second rank X-state). But by definition of X-states, x is non-negative. So, if we allow x to be at most equal to $\mathcal{H}$ (which is not allowed anyway), which is zero in this case, we just get a pure state. So, $\theta = 0$ is not a valid solution here. Now, when $\phi = \psi = \pi/2$,

$$p = 1, \; q = -1, \; \text{and } e = -1. \tag{2.37}$$

Putting these in the expressions for purity, fidelity and concurrence, we get,

$$\mathcal{P} = 1 + 2x - 2\sin^2\theta + 2\sin^4\theta, \tag{2.38}$$

$$\mathcal{C} = 2\sqrt{x} \;\text{ and} \tag{2.39}$$

$$\mathcal{F} = \frac{1}{3}\Big[2 + 2\sqrt{x}\Big]. \tag{2.40}$$

Using Eq.(2.39) and Eq.(2.40) we obtain

$$\mathcal{F} = \frac{1}{3}(2 + \mathcal{C}). \tag{2.41}$$

Again, it is evident that Fidelity is greater than classical value whenever the state is entangled. This relation is same as that for pure states.

2.5.2 Rank-2 X-states of second kind

For this we have the parameterization, $\mathcal{B} = 0, x = 0, y < \mathcal{G}$. All the expressions for purity, concurrence and fidelity will be same as before, just x will be replaced by y.

$$\mathcal{P} = 1 + 2y - 2p\sin^2\theta - 2qsin^4\theta, \tag{2.42}$$

$$\mathcal{C} = 2(\sqrt{y} - f'\sin\theta\cos\theta) \text{ and} \tag{2.43}$$

$$\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{y} \pm \cos^2\theta \mp e\sin^2\theta\Big]. \tag{2.44}$$

Here, $f' = \sin\phi\sin\psi$. Now the condition $\mathcal{B} = 0$, i.e $\mathcal{A} = \sin^2\theta(1 - \sin^2\phi\sin^2\psi) = 1$ will make either $\theta = \pi/2$ and $\phi = 0$, or $\theta = \pi/2$ and $\psi = 0$. No other solutions are possible. But for the first choice, we have, $\mathcal{G} = 0$. So y should be negative. So this is not a valid solution by the same arguments as above. Let us see what happens for the other solution i.e, $\theta = \pi/2$ and $\psi = 0$. In this case, we have

$$p = 0,\ q = \sin^2\phi\cos^2\phi,\ f' = 0 \text{ and } e = 1. \tag{2.45}$$

We obtain,

$$\mathcal{P} = 1 + 2y - 2\sin^2\phi\cos^2\phi, \tag{2.46}$$

$$\mathcal{C} = 2\sqrt{y} \text{ and} \tag{2.47}$$

$$\mathcal{F} = \frac{1}{3}\Big[2 + 2\sqrt{y}\Big]$$

So finally we have

$$\mathcal{F} = \frac{1}{3}(2 + \mathcal{C}). \tag{2.48}$$

Situation is same as before, i.e. fidelity is independent of purity and we will get teleportation fidelity always greater than classical value as long as the state is entangled.

2.5.3 Rank-2 X-states of third kind

This is characterized by, $x = \mathcal{H}, y = \mathcal{G}, 0 \leq A \leq 1$. The expressions change accordingly. Maximum concurrence can be $2\sqrt{x} - 2\sqrt{y}$ or $2\sqrt{y} - 2\sqrt{x}$. We begin with the first choice such that $x > y$. We get,

$$\mathcal{P} = 1 + 2r\sin^2\theta + 2r^2\sin^4\theta, \tag{2.49}$$

$$\mathcal{C} = 2\sqrt{x} - 2\sqrt{y} \text{ and} \tag{2.50}$$

$$\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{x} \pm \cos^2\theta \mp e\sin^2\theta\Big]. \tag{2.51}$$

Here,

$$r = -1 + \sin^2\phi\sin^2\psi. \tag{2.52}$$

From Eq. (2.49) we solve for $\sin^2\theta$ and get,

$$\sin^2\theta = \frac{-1 \pm \sqrt{2\mathcal{P} - 1}}{2r} = V(\mathcal{P}, r). \tag{2.53}$$

Here we choose $\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{x} + \cos^2\theta - e\sin^2\theta\Big]$. From Eq. (2.50) we write, $4\sqrt{x} = 2\mathcal{C} + 4\sqrt{y}$ and using these,

$$\mathcal{F} = \frac{1}{6}\Big[4 + 2\mathcal{C} + 4\sqrt{y} - (1 + e)\sin^2\theta\Big]. \tag{2.54}$$

As $e + 1 = -2r$, we get

$$\mathcal{F} = \frac{1}{6}\Big[4 + 2\mathcal{C} + 4\sqrt{y} + (-1 \pm \sqrt{2\mathcal{P} - 1})\Big]. \tag{2.55}$$

To make it optimum, we choose the plus sign and hence, $\mathcal{F} = \frac{1}{6}\Big[4 + 2\mathcal{C} + 4\sqrt{y} + (-1 + \sqrt{2\mathcal{P} - 1})\Big]$. Now one can take the other sign of fidelity as well i.e., $\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{x} - \cos^2\theta + e\sin^2\theta\Big]$. In this case by substituting the value of $\sin^2\theta$ we get

$$\mathcal{F} = \frac{1}{6}\Big[4 + 2\mathcal{C} + 4\sqrt{y} - (-1 \pm \sqrt{2\mathcal{P} - 1})\Big]. \tag{2.56}$$

To make it optimum, we choose the minus sign in the expression of $\sin^2\theta$, i.e., in Eq. (2.53). Both these expressions are same but depending on the situation, we need to choose the sign of $\sin^2\theta$ properly. The plus or minus sign in fidelity expression can be fixed by the sign of Eq.(2.53). Therefore, without loosing generality, we can take the fidelity expression as

$$\mathcal{F} = \frac{1}{6}\Big[4 + 2\mathcal{C} + 4\sqrt{y} + (-1 + \sqrt{2\mathcal{P} - 1})\Big]. \tag{2.57}$$

We will encounter similar kind of situation for other ranks of X-states as well. By giving similar argument and without loosing generality we can take fidelity as

$$\mathcal{F} = \frac{1}{6}\Big[4 + 4\sqrt{a} - (1+e)\sin^2\theta\Big], \tag{2.58}$$

where, $a = \max[x, y]$. We will use this expression throughout the manuscript.

Now, as the minimum value of $(1+e)$ can be zero, in the expression for $\sin^2\theta$ we have to choose the minus sign for the optimum fidelity. For this choice, it is evident from Eq.(2.53) that $\sin^2\theta = V(P, e)$, decreases as P increases for any ϕ, ψ. Hence from Eq.(2.57) fidelity will also increase as the purity increases keeping concurrence and other parameters fixed at any values. Also the fidelity changes monotonically with respect to parameters other than purity, or concurrence, here y. This is one of the main points of this paper and as we will show in the following that the same conclusion holds for 3rd and 4th rank X-states also.

The expression of $V(P, e)$ shows a very interesting feature. As fidelity is always a real quantity, we must have $\mathcal{P} \geq 1/2$. So, this physical constraint also restricts the minimum purity a second rank X-state can have. This fact is also evident from the expression of purity, i.e., Eq.(2.49). It can be shown that minimum value that $\mathcal{P}$ can take is $1/2$. As stated earlier, from Eq.(2.54) and (2.55), it is evident that for a fixed value of y and $\mathcal{C}$, fidelity increases with the increment of purity. We emphasize this fact by plotting fidelity with respect to purity for $y = 0.001$ and $\mathcal{C} = 0.2$.

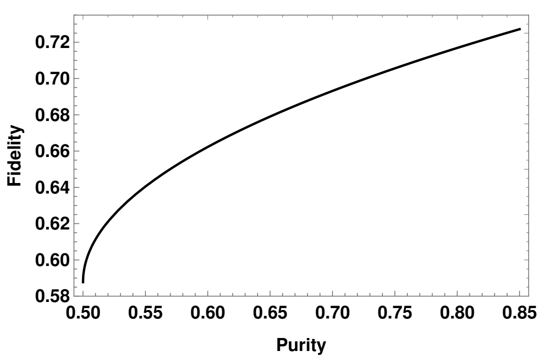

Figure 2.5: X-state: Variation of fidelity with purity for $y = 0.001$ and $\mathcal{C} = 0.2$ of third kind second rank state.

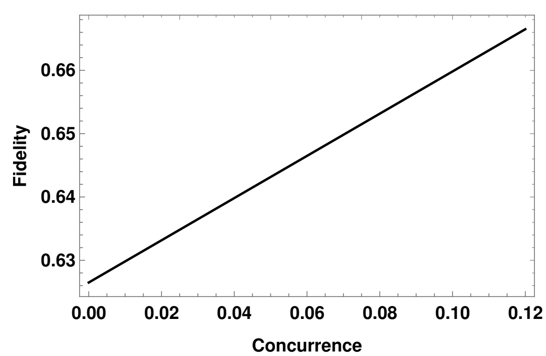

Figure 2.6: X-state: Variation of fidelity with concurrence for $y = 0.001$ and $\mathcal{P} = 0.7$ of third kind second rank state.

It is clear from FIG.(2.5), that if a state is not suitable for teleportation, by increasing its purity, one can make it effective for teleportation. Hence, fidelity not only depends on entanglement but also on the purity of the state. In FIG. (2.6) we have plotted fidelity for a fixed value of purity to show its variation with entanglement of the state and indeed fidelity is increasing with concurrence. Here we emphasize that it may happen a state with less entanglement but higher value of purity can achieve higher fidelity. We see it explicitly in the following example. Let us consider a state with $\mathcal{P} = 0.6$, $\mathcal{C} = 0.2$ and $y = 0.001$. For this state $\theta \approx 0.5809$, $\phi \approx 0.3124$ and $\psi \approx 1.2036$. Fidelity for this state is $\mathcal{F} \approx 0.6623$. Let us take another state with $\mathcal{P} = 0.7, \mathcal{C} = 0.15$ and $y = 0.001$. For this state, we have $\theta \approx 1.4515$, $\phi \approx 1.4953$ and $\psi \approx 1.1304$. In this case fidelity is $\mathcal{F} \approx 0.6765$. So a state that has less entanglement but more purity can provide higher fidelity. Another parameter on which fidelity depends on is y. From the fidelity expression, it is clear that it increases monotonically with y. This is the first example where we see the dependence of fidelity on properties other than purity and concurrence. This parameter also seems to characterize the nonlocal properties of the state.

Now if we consider the second choice of concurrence i.e., $\mathcal{C} = 2\sqrt{y} - 2\sqrt{x}$ when $y > x$. Everything will remain same except x in Eq. (2.51) will change to y and y in Eq.(2.54) and (2.55) will change to x. All the arguments and results remain the same.

2.5.4 Rank-3 X-states of first kind

Third rank X-state of first kind is characterized by, $x < \mathcal{H}, y = \mathcal{G}, A > 0$. Given that we get,

$$\mathcal{P} = 1 + 2x - 2\sin^2\theta + 2d\sin^4\theta, \tag{2.59}$$

$$\mathcal{C} = 2\ \max\ [\sqrt{x} - f\sin^2\theta, \sqrt{y} - f'\sin\theta\cos\theta] \tag{2.60}$$

$$\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{a} + \cos^2\theta - e\sin^2\theta\Big], \tag{2.61}$$

where, e, f, f', a are same as before and,

$$d = 1 - \sin^2\phi\sin^2\psi + \sin^4\phi\sin^4\psi. \tag{2.62}$$

We first take $x > y$. As $x > y$ implies $x > \mathcal{G}$ and $x < \mathcal{H}$, we have $\mathcal{H} > x > \mathcal{G}$ and for this choice, $\mathcal{C}$ is $2(\sqrt{x} - f\sin^2\theta)$ and $\mathcal{F}$ is $\frac{1}{6}\Big[3 + 4\sqrt{x} + \cos^2\theta - e\sin^2\theta\Big]$.

Now, solving for $\sin^2\theta$ from Eq. (2.59) and (2.60) we get,

$$\begin{aligned}\sin^2\theta &= V_1(\mathcal{P},\mathcal{C},d,f)\\ &= \frac{(1-f\mathcal{C})\pm\sqrt{(1-f\mathcal{C})^2-(1-\mathcal{P}+\mathcal{C}^2/2)(2d+2f^2)}}{(2d+2f^2)}\end{aligned} \tag{2.63}$$

Using the expression for $\mathcal{C}$ and the evaluated $\sin^2\theta$, we now write fidelity $\mathcal{F}$ in terms of $\mathcal{C},\mathcal{P},d,e$ and f as,

$$\mathcal{F} = \frac{1}{6}\big[4+2\mathcal{C}-(1+e-4f)V_1(\mathcal{P},\mathcal{C},d,f)\big]. \tag{2.64}$$

As stated for the 2nd rank case, to get optimum fidelity, we have to choose the minus sign of $V_1(\mathcal{P},\mathcal{C},d,f)$ as the minimum value of $(1+e-4f)$ can be zero. Then it is evident from the expression that for any values of ϕ,ψ and $\mathcal{C}$, $\mathcal{F}$ increases with $\mathcal{P}$, as $V_1(\mathcal{P},\mathcal{C},d,f)$ decreases with the increase of P. So as before the same result holds for 3rd rank X-states of first kind.

To illustrate this behavior graphically, we set $\phi=\frac{\pi}{4}$, $\psi=\frac{\pi}{2}$ and then the Eq. (2.64) reduces to the following form,

$$\mathcal{F} = \frac{1}{18}\left(10+6\mathcal{C}+\sqrt{6\mathcal{P}-3\mathcal{C}^2-2}\right). \tag{2.65}$$

We plot this expression for fidelity $\mathcal{F}$ with purity $\mathcal{P}$ for a fixed entanglement, i.e concurrence $\mathcal{C}$. FIG. (2.7), shows that for a fixed entanglement, the fidelity increases with purity. Like second rank X-state, fidelity of these states also increases with concurrence for fixed purity as shown in FIG. (2.8).

Moreover the right hand side of Eq.(2.64) involves few more parameters. Here we are giving a very interesting example. Consider a state with $\mathcal{P}=0.6,\mathcal{C}=0.2$, $\phi=\frac{\pi}{4}$ and $\psi=\frac{\pi}{2}$. For this state, we have $\mathcal{F}\approx 0.6898$. These values of ψ and ϕ give $d=\frac{3}{4}$, $e=0$ and $f=0$. For another state with $\mathcal{P}=0.64,\mathcal{C}=0.22$, $\phi=\frac{\pi}{2}$ and $\psi=\frac{2\pi}{25}$, we get $\mathcal{F}\approx 0.6612$. For this state $d\approx 0.9419$, $e=0.87563$ and $f=0$. So with less entanglement and purity one can have more fidelity for different values of ϕ and ψ or d, e and f. Now, let us see the ranges of d, e and f. We obtain

$$\frac{3}{4}\le d<1 \tag{2.66}$$

$$-1<e<1 \tag{2.67}$$

$$0\le f<\frac{1}{2} \tag{2.68}$$

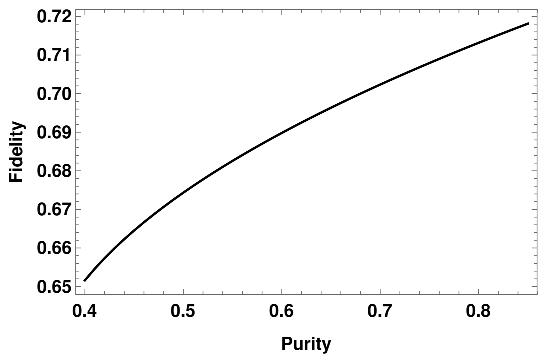

Figure 2.7: X-state: Variation of fidelity with purity for $d = \frac{3}{4}$, $e = 0$ and $f = 0$ with $\mathcal{C} = 0.2$ of third rank X-state of first kind.

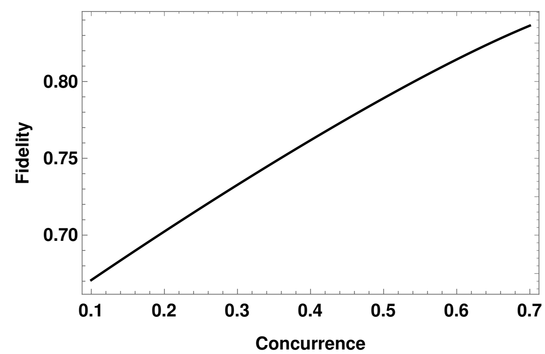

Figure 2.8: X-state: Variation of fidelity with concurrence for $d = \frac{3}{4}$, $e = 0$ and $f = 0$ with $\mathcal{P} = 0.7$ of third rank X-state of first kind.

The ranges of d , e and f have been obtained by maximizing and minimizing the functions independently. However, as they all are functions of ϕ and ψ, they can not be varied independently. As fidelity depends on those parameters also rather than only depending on purity and concurrence, we have plotted variations of fidelity with those parameters in Fig.(2.9) and Fig.(2.10) showing fidelity decreases monotonically with e, whereas it increases monotonically with f. In the figures the ranges of the parameters e and f have been appropriately modified.

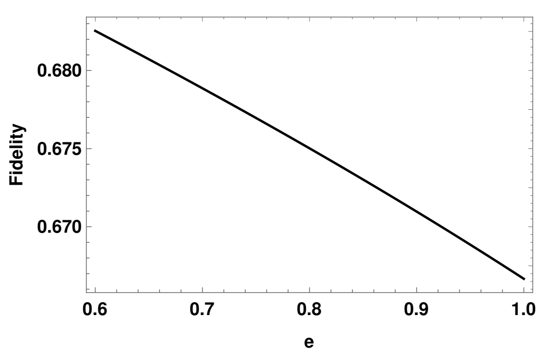

Figure 2.9: X-state: Variation of fidelity with 'e' for $f = 0$, $\mathcal{P} = .7$ and $\mathcal{C} = 0.2$ of third rank X-state of first kind.

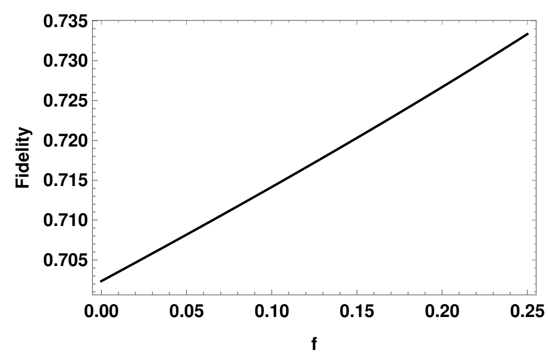

Figure 2.10: X-state: Variation of fidelity with 'f' for $e = 0$, $\mathcal{P} = .7$ and $\mathcal{C} = 0.2$ of third rank X-state of first kind.

Now, we are left with the situation when $y > x$. In this case the concurrence will be

$2(\sqrt{y} - f' \sin\theta\cos\theta)$ and $\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{y} + \cos^2\theta - e\sin^2\theta\Big]$. Doing similar kind of calculation one can show that

$$\begin{aligned}\mathcal{F} = \quad & \frac{1}{6}\Big[4 + 2\mathcal{C} + 4f'\sqrt{V_2(\mathcal{P}, x, \phi, \psi)(1 - V_2(\mathcal{P}, x, \phi, \psi))} \\ & -(1+e)V_2(\mathcal{P}, x, d)\Big], \end{aligned} \tag{2.69}$$

where $V_2(\mathcal{P}, x, d) = \frac{1 \pm \sqrt{1-2d(1+2x-\mathcal{P})}}{2d}$. Here also one can easily verify a similar kind of trend as before.

2.5.5 Rank-3 X-states of second kind

This class of states are characterized by $x = \mathcal{H}$, $y < \mathcal{G}$, $\mathcal{A} < 1$. We get,

$$\mathcal{P} = 1 + 2y - 2t\sin^2\theta + 2u\sin^4\theta, \tag{2.70}$$

$$\mathcal{C} = 2\ \text{max}\ [\sqrt{x} - f\sin^2\theta, \sqrt{y} - f'\sin\theta\cos\theta] \tag{2.71}$$

$$\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{a} + \cos^2\theta - e\text{sin}^2\theta\Big], \tag{2.72}$$

where, $t = 1 - p$. p, a are same as before and,

$$u = 1 + \sin^4\phi\sin^4\psi - \cos^2\phi\cos^2\psi\sin^2\phi - 2\sin^2\phi\sin^2\psi. \tag{2.73}$$

First we take $x > y$. As $x > y$ implies $\mathcal{H} > y$ and $y < G$, concurrence will be $2(\sqrt{x} - f\sin^2\theta)$ and $\mathcal{F}$ is $\frac{1}{6}\Big[3 + 4\sqrt{x} + \cos^2\theta - e\sin^2\theta\Big]$. After doing a calculation as above, we get

$$\mathcal{F} = \frac{1}{6}\Big[4 + 2\mathcal{C} - (1 + e - 4f)V_3(\mathcal{P}, t, u, y)\Big], \tag{2.74}$$

where $V_3(\mathcal{P}, t, u, y) = \frac{t \pm \sqrt{t^2 - 2u(1+2y-\mathcal{P})}}{2u}$. So, the situation is similar as first kind and we would be getting similar results.

Now, we will consider the situation $y > x$ i.e., $\mathcal{G} > y > \mathcal{H}$. Here concurrence will be $2(\sqrt{y} + \sin\theta\cos\theta f')$ and $\mathcal{F}$ to be $\frac{1}{6}\Big[3 + 4\sqrt{y} + \cos^2\theta - e\text{sin}^2\theta\Big]$. In this situation calculation will be slightly different. The reason is that now the expression for purity $\mathcal{P}$ involves y, not x. So, in this case we will be getting a 4th order equation of $\sin^2\theta$ from the expression of $\mathcal{P}$ and $\mathcal{C}$. We will not do this in this section as in the next section for 4th rank X-states we will discuss a similar situation.

2.5.6 Analysis for Rank-4 X-states

General fourth rank X-states will be characterized by $x < \mathcal{H}$, $y < \mathcal{G}$, $\mathcal{AB} > 0$. Putting the values of $\mathcal{A}, \mathcal{B}, \mathcal{G}, \mathcal{H}$ we get the values of P and C as,

$$\mathcal{P} = 1 + 2x + 2y - 2\sin^2\theta + 2\sin^4\theta g, \tag{2.75}$$

$$\mathcal{C} = 2\,\text{max}[\sqrt{x} - f\sin^2\theta, \sqrt{y} - f'\sin\theta\cos\theta] \tag{2.76}$$

$$\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{a} + \cos^2\theta - e\sin^2\theta\Big], \tag{2.77}$$

where,

$$g = \frac{1}{64}[53 + 4\cos 2\phi + 7\cos 4\phi + 8\cos 4\psi \sin^4\phi], \tag{2.78}$$

$$f = \sqrt{\sin^2\phi\cos^2\phi\cos^2\psi} \text{ and} \tag{2.79}$$

$$f' = \sqrt{\sin^2\phi\sin^2\psi}. \tag{2.80}$$

First, we choose $x > y$ and also $\sqrt{x} - f\sin^2\theta > \sqrt{y} - f'\sin\theta\cos\theta$. So, we take $\mathcal{C}$ to be $2(\sqrt{x} - f\sin^2\theta)$ and $\mathcal{F}$ to be $\frac{1}{6}\Big[3 + 4\sqrt{x} + \cos^2\theta - e\sin^2\theta\Big]$. Now, from Eq. (2.75) and (2.76) we get,

$$\sin^2\theta = V_4(\mathcal{P}, \mathcal{C}, f, g, y) = \frac{(1 - f\mathcal{C}) \pm \sqrt{(1 - f\mathcal{C})^2 - (1 - \mathcal{P} + 2y + \mathcal{C}^2/2)(2g + 2f^2)}}{(2g + 2f^2)}. \tag{2.81}$$

Using the expression for $\mathcal{C}$ and the evaluated $\sin^2\theta$, we now write fidelity $\mathcal{F}$ in terms of $\mathcal{C}, \mathcal{P}, f, g$ and y as,

$$\mathcal{F} = \frac{1}{6}\big[4 + 2\mathcal{C} - (1 + e - 4f)V_4(\mathcal{P}, \mathcal{C}, f, g, y)\big]. \tag{2.82}$$

In similar fashion here also we can argue that as purity increases keeping others constant, fidelity also increases and also it is evident from the FIG. 2.11. For this plot we choose $\psi = \phi = \frac{\pi}{4}$ or $\psi = 2\tan^{-1}(\sqrt{5 - 2\sqrt{6}})$, $\phi = \frac{\pi}{3}$ and $y = 0$. For these values of ϕ and ψ,

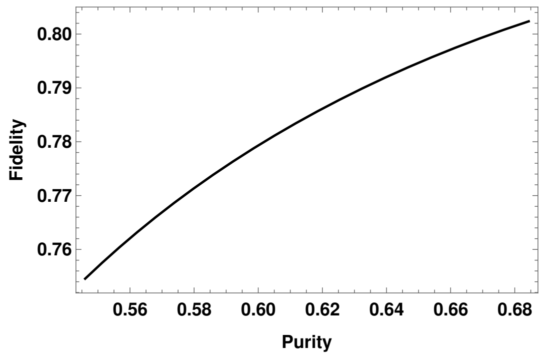

Figure 2.11: X-state: Variation of fidelity with purity for $g = \frac{11}{16}$, $e = \frac{1}{2}$, $f = \frac{1}{2\sqrt{2}}$ and $y = 0$ with $\mathcal{C} = 0.2$ of rank-4 state.

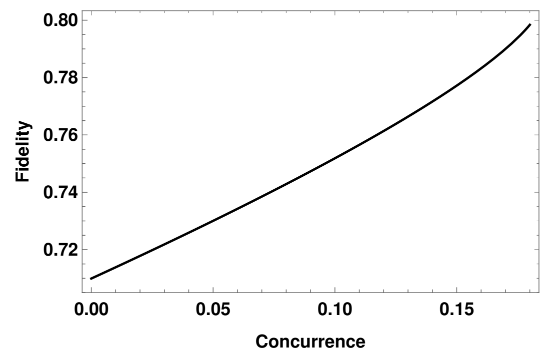

Figure 2.12: X-state: Variation of fidelity with purity for $g = \frac{11}{16}$, $e = 0$, $f = \frac{1}{2\sqrt{2}}$ and $y = 0$ with $\mathcal{P} = 0.7$ of rank-4 state.

$g = \frac{11}{16}$, $f = \frac{1}{2\sqrt{2}}$ and $e = \frac{1}{2}$. FIG. 2.12 shows the variation of fidelity with concurrence for a fixed value of purity.

Now for $x > y$, we could have $\sqrt{x} - f\sin^2\theta < \sqrt{y} - f'\sin\theta\cos\theta$. So $\mathcal{C} = 2(\sqrt{y} - f'\sin\theta\cos\theta)$. In this case from the expression of $\mathcal{P}$ and $\mathcal{C}$ we will get a fourth order equation for $\sin^2\theta$, which will not involve y In principle we will get four solutions of $\sin^2\theta$ from this equation as a function of ϕ, ψ and x. Putting these solutions of $\sin^2\theta$ in the expression for $\mathcal{F}$, we will have $\mathcal{F}$ as a function of C, P, ϕ, ψ and x. The fourth order equation will be similar like we will derive in the following for the case of $y > x$ and $\sqrt{x} - f\sin^2\theta < \sqrt{y} - f'\sin\theta\cos\theta$.

So, let us consider the case when $y > x$ and $\sqrt{x} - f\sin^2\theta < \sqrt{y} - f'\sin\theta\cos\theta$, we have the value of concurrence to be $2(\sqrt{y} - f'\sin\theta\cos\theta)$. In this case we need to replace x by y in the fidelity expression given in Eq. (2.77). Here also we will get a fourth order equation of $\sin^2\theta$. Using the expression for purity $\mathcal{P}$ and concurrence $\mathcal{C}$, we get the following equation,

$$\begin{aligned}
&\alpha^2\sin^8\theta + 2\alpha\beta\sin^6\theta + [\beta^2 + 2\alpha(1 + 2x + \mathcal{C}^2/2 - \mathcal{P}) \\
&+4\mathcal{C}^2 f'^2]\sin^4\theta + [2\beta(1 + 2x + \mathcal{C}^2/2 - \mathcal{P}) - 4C^2 f'2] \\
&\sin^2\theta + (1 + 2x + \mathcal{C}^2/2 - \mathcal{P})^2 = 0,
\end{aligned} \tag{2.83}$$

where,

$$\alpha = 2g - 2f'^2 \text{ and } \beta = 2f'^2 - 2. \tag{2.84}$$

From this equation, in principle one can get four solutions for $\sin^2\theta$ and using that one can get the expression for fidelity $\mathcal{F}$ in terms of $\mathcal{C}$, $\mathcal{P}$, x, ϕ and ψ. As solving this equation will be very involved, we avoid that and get some plots for some particular values of the parameters showing the pattern. We choose $x = 0$, $\phi = \psi = \frac{\pi}{4}$ or $\phi = \frac{\pi}{3}$, $\psi = -2\tan^{-1}(\sqrt{2} - \sqrt{3})$ and $\mathcal{C} = 0.2$. In FIG. 2.13, we see a trend as before, i.e, fidelity increases with purity for a fixed concurrence.

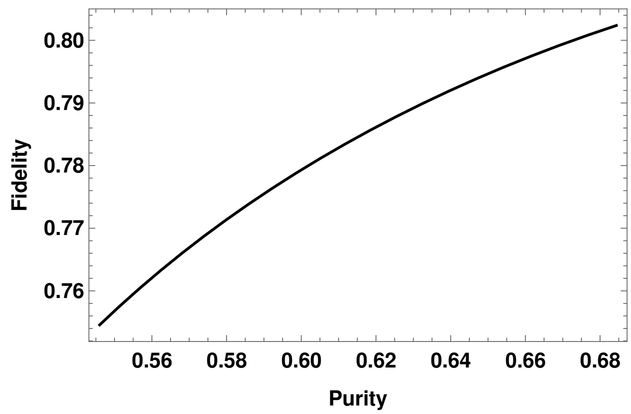

Figure 2.13: X-state: Variation of fidelity with purity for $g = \frac{11}{16}$, $f' = \frac{1}{2}$ and $x = 0$ with $\mathcal{C} = 0.2$ of fourth rank X-state.

Now finally we are left with $y > x$ and $\sqrt{x} - f\sin^2\theta > \sqrt{y} - f'\sin\theta\cos\theta$. In this case $\mathcal{C} = 2(\sqrt{x} - f\sin^2\theta)$ and $\mathcal{F} = \frac{1}{6}\Big[3 + 4\sqrt{y} + \cos^2\theta - e\sin^2\theta\Big]$. After few steps of calculation, we find

$$\mathcal{F} = \frac{1}{6}\Big[4 + 4\sqrt{y} - (1+e)V_5(\mathcal{P}, \mathcal{C}, y, f, g)\Big], \tag{2.85}$$

where,

$$V_5(\mathcal{P}, \mathcal{C}, y, f, g) = \sin^2\theta = \\ \frac{(1 - f\mathcal{C}) \pm \sqrt{(1 - f\mathcal{C})^2 - (2g + 2f^2)(1 + \mathcal{C}^2/2 + 2y - \mathcal{P})}}{(2g + 2f^2)}. \tag{2.86}$$

From this expression also one can verify that the trends are similar as above.

We shall end this section with few important remarks. We have observed through our analysis that sometime less entanglement but higher value of purity can produce better fidelity as exemplified in section(2.5.3). We also note that fidelity also depends on some other parameters, or functions of the parameters, as it has been shown in (2.5.4). It would be interesting to investigate the physical significance of these new functions of the state parameters.

2.6 Uhlmann Fidelity

As we have seen, the teleportation fidelity changes monotonically with parameters, or functions of parameters, of states. Question is apart from purity and concurrence what other physical quantities these functions of parameters may be related to. In this subsection, we consider one such physical quantity – Uhlmann Fidelity. It is known that the closeness of two states can be characterized by Uhlmann fidelity [71]. For two arbitrary quantum states ρ and σ, the Uhlmann fidelity is defined as [71]

$$\mathcal{R} = \left[\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right]^2. \tag{2.87}$$

It is a relevant quantity that describes how far apart two states are. Here we will compute the Uhlmann fidelity of a class of X-states with Bell states. As there are four Bell states, we take the maximum of the values. So we choose σ of Eq.(2.87) as density matrices of Bell states. Then Uhlmann fidelity of the rank four X-states as in Eq. (2.24) is

$$\begin{aligned}\mathcal{R} = \max\Big[\quad & \frac{1}{4}(1+e)\sin^2\theta + \sqrt{y}\cos\nu, \\ & \frac{1}{4}(1+e)\sin^2\theta - \sqrt{y}\cos\nu, \\ & \frac{1}{4}(2-(1+e)sin^2\theta) + \sqrt{x}\cos\mu, \\ & \frac{1}{4}(2-(1+e)sin^2\theta) - \sqrt{x}\cos\mu\Big].\end{aligned} \tag{2.88}$$

There are four Uhlmann fidelities, one for each of the Bell states. Uhlmann fidelity also corresponds to the transition probability of one state to another state. We take the maximum among four, because the maximum is the most probable state. As fidelity is independent of μ and ν, so without loosing generality we can choose $\mu = \nu = 0$. We find out $\mathcal{R}$ explicitly with $\mathcal{C} = 0.2$, $\mathcal{P} = 0.7$, $f = \frac{1}{2\sqrt{2}}$ and $y = 0$. For this state one can check that the Uhlmann fidelity is

$$\mathcal{R} = \frac{1}{4}(2-(1+e)sin^2\theta) + \sqrt{x}. \tag{2.89}$$

It is quite obvious from this expression that $\mathcal{R}$ is monotonically decreasing function of e. We have also seen in Eq.(2.82) that teleportation fidelity also monotonically decreases with e. To

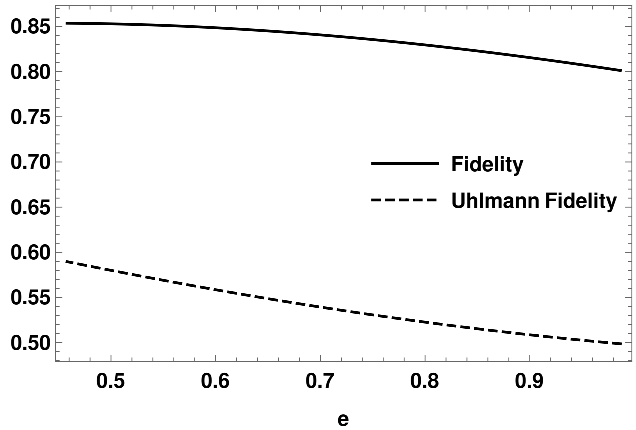

Figure 2.14: Variation of fidelity and Uhlmann fidelity with e for $f = \frac{1}{2\sqrt{2}}$, $y = 0$, $\mathcal{C} = 0.2$ and $\mathcal{P} = 0.7$ of rank-4 X-state.

visualize it we have plotted $\mathcal{R}$ and $\mathcal{F}$ as a function of e in Fig. 2.14. One interpretation of the Fig. 2.14 is that the increment of parameter e is somehow introducing classicality in the system. With increment of e, Uhlmann fidelity is decreasing i.e., the state is going far away from the Bell states and as a result fidelity is also decreasing. To emphasize the importance of this quantity, we consider the states with same values of concurrence and purity that have different fidelity. Let's see it explicitly with two following states:

$$\gamma_1 = \frac{1}{3}|\psi^+\rangle\langle\psi^+| + \frac{2}{3}|\psi^-\rangle\langle\psi^-| \quad \text{and}$$
$$\gamma_2 = \frac{1}{3}|\psi^+\rangle\langle\psi^+| + \frac{2}{3}|00\rangle\langle 00|, \tag{2.90}$$

where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle \pm |10\rangle\big)$. These states have same purity $\frac{5}{9}$ and concurrence $\frac{1}{3}$, but fidelity is different, $\mathcal{F}(\gamma_1) = \frac{7}{9}$ and $\mathcal{F}(\gamma_2) = \frac{2}{3}$ respectively. Uhlmann fidelity of these states are given by $\frac{2}{3}$ and $\frac{1}{3}$ respectively i.e, higher Uhlmann fidelity corresponds to larger teleportation fidelity also. It seems plausible as Uhlmann fidelity is associated with distance between two density matrices. So larger distance of a state from maximally entangled state implies larger deviation from nonlocality which in turn degrades its teleportation fidelity.

2.7 Discussion and Conclusions

We have studied teleportation fidelity for a number of different classes of states including Werner state, maximally entangled mixed states and X-states. In all these cases, we obtain a general conclusion that teleportation fidelity depends on the purity of the state. Remarkably, using X-states we have found the dependence of optimal teleportation fidelity on the functions of the state parameters. All the relations for optimal fidelity indicate that noise can reduce the effectiveness of state as a teleportation channel. In concurrence-purity region, we can find some entangled states for each rank which have optimal fidelity less than $\frac{2}{3}$. Below a certain value of purity, optimal fidelity does not increase if only concurrence is increased. Also concurrence can not be changed arbitrarily keeping purity fixed. Moreover the amount of variation of optimal fidelity with purity, for fixed amount of concurrence, depends on the rank of the states. Our result also agrees with the work in reference [61]. Higher rank X-states give larger optimal fidelity for a fixed value of purity and concurrence.

Our investigations suggest that the nonlocal character of a two-qubit mixed state is more involved, and require several quantities to fully characterize it. For example, for a rank-3 X-state, the optimal fidelity depends not only on purity and concurrence, but also on the functions e and f. All of these quantities are functions of state parameters. By choosing a specific set of values for the functions e and f, we find that the optimal fidelity changes monotonically with concurrence and purity. However, the optimal fidelity also changes monotonically with functions e and f. This has been illustrated in a number of plots. Thus, these quantities also characterize the nonlocal (classical or quantum) properties of the mixed states. Purity and concurrence are not enough. They may characterize some average nonlocal properties. At some level, it is not surprising. Unlike a two-qubit pure state, a two-qubit mixed state can have several independent parameters. However, we have found some specific functions of the state parameters, which in addition to concurrence, also determine the nonlocal properties of the state. The optimal teleportation fidelity changes monotonically with respect to these functions. Interestingly, Bell violation by a X-state varies in the same way with respect to these functions e and f. These extra functions of parameters, should be related with other properties of the states which are not captured by purity, or concurrence. We have considered one such quantity, Uhlmann fidelity, and shown its importance. There should be many more such quantities which are still to be found to fully characterize the nonlocal properties of two-qubit mixed states.

Chapter 3

Teleportation of an arbitrary n-qubit state

Entanglement provides nonlocal correlations which can be used to transfer information between the parties by local operation and classical communication (LOCC). In a general scenario, multipartite states are frequently encountered and entanglement of such states is also a potential resource for QIPPs. The nature of multipartite entanglement has more promise than bipartite entanglement. Unlike bipartite entanglement, it has been less understood qualitatively as well as quantitatively [75]. A multipartite state provide many possibilities. For example, a quadripartite state can be distributed among four observers or it may be distributed among three observers such that one of them has two qubits or it may be distributed among two observers. We can realize the original teleportation protocol [76] with multipartite state which has been divided into two different subsystems [77, 78]. Teleportation using GHZ-state and W-state have been shown by the authors [79, 80]. Cluster state has been found to be useful for the protocol as shown in [81]. Five-qubit Brown state [82] can also be used for the protocol [83]. It also appears that we can teleport more than one qubit with multipartite entangled state. A n-qubit state can have 2^n terms. One can teleport a two-qubit state with two terms with a quadripartite GHZ. The authors in [84] have shown perfect teleportation of an arbitrary two-qubit state with a quadripartite entangled state.

The nature of multipartite entanglement is more versatile than that of bipartite entanglement. In bipartite scenarios, Bell states are maximally entangled, hence are most suitable for QIPPs. But the notion of maximally entangled states in multipartite scenarios is more complex. Instead we have different inequivalent SLOCC classes of multipartite states. These are inequivalent because they cannot be converted into each other by LOCC, not even probabilistically. Different states belonging to different classes may not exhibit same kind of entanglement properties. On the other hand, different partitions of a state will allow teleportation of different unknown states. Indeed existing protocols can be implemented and improvised if we use a multipartite entangled state as resource. It may be possible to teleport a multiqubit unknown state with varying number of terms with a multipartite resource state.

Therefore it is natural to investigate the necessary criterion of a multipartite entangled state for teleportation of an arbitrary n-qubit state. We have addressed the issue in this chapter and presented a novel approach to find out the property of a resource state to teleport a n-qubit state with m-terms.

3.1 Teleporatation with a few quadripartite entangled states

In this section, we shall discuss teleporatation of a single-qubit state between two distant laboratories with quadripartite entangled states [85]. All these states are SLOCC inequivalent, *i.e.* entanglement properties are quite different. Here we shall consider only perfect teleporatation, *i.e.* with unit fidelity and unit probability. This restricts our choice of resource states. There is no universal notion of maximally entangled state in higher qubit system, rather there exists classes of states in which each state exhibits similar kind of properties under LOCC operation. In the subsequent subsections, we shall discuss some states belonging to different SLOCC classes and its usage in teleportation.

3.1.1 Teleportation with a GHZ state

Let us suppose, Alice wants to teleport an unknown qubit state given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{3.1}$$

to Bob. Here α and β are complex quantities and satisfy $|\alpha|^2 + |\beta|^2 = 1$. They share a GHZ state which has been distributed in such a manner that first three qubits and the unknown one are held by Alice and Bob has fourth qubit at his disposal. Now Alice performs a von Neumann measurements on her subsystem and conveys the results to Bob. Bob then performs a suitable unitary operation on his qubit to retrieve the unknown qubit. We can write the combined state as:

$$
\begin{aligned}
|\psi\rangle|GHZ\rangle &= \frac{1}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle)\otimes(|0000\rangle+|1111\rangle) \\
&= \frac{1}{\sqrt{2}}(\alpha|0000\rangle|0\rangle+\alpha|0111\rangle|1\rangle+\beta|1000\rangle|0\rangle+\beta|1111\rangle|1\rangle).
\end{aligned}
$$

Now Alice chooses the measurement basis $|4GHZ_1^{\pm}\rangle$ and $|4GHZ_3^{\pm}\rangle$ which are defined as

$$
\begin{aligned}
|4GHZ_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle \pm |1111\rangle), \\
|4GHZ_2^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0111\rangle \pm |1000\rangle). \qquad (3.2)
\end{aligned}
$$

So Eq.(3.2) can be rewritten as

$$
\begin{aligned}
|\psi\rangle|GHZ\rangle = \frac{1}{2}(&\alpha(|4GHZ_1^{+}+4GHZ_1^{-}\rangle)|0\rangle+\alpha(|4GHZ_2^{+}+4GHZ_2^{-}\rangle)|1\rangle+ \\
&\beta(|4GHZ_2^{+}-4GHZ_2^{-}\rangle)|0\rangle+\beta(|4GHZ_1^{+}-4GHZ_1^{-}\rangle)|1\rangle).
\end{aligned}
$$

According to each measurement outcome, Bob's state collapses into any one of the following states:

$$
\begin{aligned}
|4GHZ_1^{+}\rangle &\Rightarrow |\psi_B\rangle = \alpha|0\rangle+\beta|1\rangle, \\
|4GHZ_1^{-}\rangle &\Rightarrow |\psi_B\rangle = \alpha|0\rangle-\beta|1\rangle, \\
|4GHZ_2^{+}\rangle &\Rightarrow |\psi_B\rangle = \beta|0\rangle+\alpha|1\rangle, \\
|4GHZ_1^{+}\rangle &\Rightarrow |\psi_B\rangle = \beta|0\rangle-\alpha|1\rangle. \qquad (3.3)
\end{aligned}
$$

Bob can retrieve the original state(3.1) by applying σ_0, σ_z, σ_x and $\sigma_x\sigma_z$ respectively. So the protocol works and it consumes the entanglement of $|GHZ\rangle$ and two cbits of classical information. Here we observe that the protocol would work for any distribution of the resource state, $i.e.$ if Bob is given any other qubit than the fourth one he will be able to retrieve the unknown state. It seems reasonable also from the structure of the resource state as it is invariant under the permutation of the qubits. GHZ state is also an useful resource for teleporting a subclass of a two-qubit states. Let us write a two-qubit state with only two terms:

$$
|\psi_2\rangle = \alpha|00\rangle+\beta|11\rangle, \qquad (3.4)
$$

where α and β are complex number as before. Alice has to teleport this state to Bob with GHZ state which is distributed in a way such that Alice has first two qubits whereas Bob has rest of the qubits. We can rewrite the combined state as below:

$$\begin{aligned}
|\psi_2\rangle_a|GHZ\rangle_{1234} &= \frac{1}{\sqrt{2}}(\alpha|0000\rangle_{a12}|00\rangle_{34} + \alpha|0011\rangle_{a12}|11\rangle_{34} + \\
&\quad \beta|1100\rangle_{a12}|00\rangle_{34} + \beta|1111\rangle_{a12}|11\rangle_{34}) \\
&= \frac{1}{2}\big(|4GHZ_1^+\rangle_{a12}(\alpha|00\rangle_{34} + \beta|11\rangle_{34}) + \\
&\quad |4GHZ_1^-\rangle_{a12}(\alpha|00\rangle_{34} - \beta|11\rangle_{34}) + \\
&\quad |4GHZ_3^+\rangle_{a12}(\alpha|11\rangle_{34} + \beta|00\rangle_{34}) + \\
&\quad |4GHZ_3^-\rangle_{a12}(\alpha|11\rangle_{34} - \beta|00\rangle_{34})\big)
\end{aligned}$$

Here the subscripts denote the ordinal number of the qubits and a refers to the unknown qubit. We have defined $|4GHZ_1^\pm\rangle$ and $|4GHZ_3^\pm\rangle$ are given as:

$$|4GHZ_3^\pm\rangle = \frac{1}{\sqrt{2}}(|0011\rangle \pm |1100\rangle). \tag{3.5}$$

Having measured in any one of these basis states, Bob has one of the following collapsed states:

$$\begin{aligned}
|4GHZ_1^+\rangle &\Rightarrow |\psi_B\rangle = \alpha|00\rangle + \beta|11\rangle, \\
|4GHZ_1^-\rangle &\Rightarrow |\psi_B\rangle = \alpha|00\rangle - \beta|11\rangle, \\
|4GHZ_3^+\rangle &\Rightarrow |\psi_B\rangle = \alpha|11\rangle + \beta|00\rangle, \\
|4GHZ_3^-\rangle &\Rightarrow |\psi_B\rangle = \alpha|11\rangle - \beta|00\rangle.
\end{aligned} \tag{3.6}$$

Evidently Bob would find suitable unitary to obtain the original one. For example he would apply $\sigma_0 \otimes \sigma_z$ for measurement outcome $|4GHZ_1^-\rangle$ and $\sigma_z\sigma_x \otimes \sigma_x$ for measurement outcome $|4GHZ_3^+\rangle$.

3.1.2 Teleportation with a cluster state

Cluster state has been introduced in [86] in the context of relatively large persistent entanglement. In computational basis it has the following form:

$$|\Omega\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle). \tag{3.7}$$

As we know $|GHZ\rangle$ states are very fragile by nature, entanglement is lost if any one of the subsystems are traced out whereas W-state exhibits bipartition entanglement under these circumstances. Similarly cluster state exhibits persistence entanglement under local operations. It can be created by Ising interaction between two-particle system in a lattice configuration which can be found in various condensed matter system. Cluster states have been found a convenient resource for QIPPs. It is a useful resource for teleportation of a single qubit state as can be seen,

$$\begin{aligned} |\psi\rangle|\Omega\rangle &= \frac{1}{2\sqrt{2}}\Big((|4GHZ_1^+\rangle + |4GHZ_3^-\rangle)(\alpha|0\rangle - \beta|1\rangle) + \\ &\quad (|4GHZ_1^-\rangle + |4GHZ_3^+\rangle)(\alpha|0\rangle + \beta|1\rangle) + \\ &\quad (-|4GHZ_2^-\rangle + |4GHZ_4^+\rangle)(\alpha|1\rangle + \beta|0\rangle) + \\ &\quad (-|4GHZ_2^+\rangle + |4GHZ_4^-\rangle)(\alpha|1\rangle - \beta|0\rangle)\Big), \end{aligned} \tag{3.8}$$

where $|4GHZ_4^{\pm}\rangle$ are defined as:

$$|4GHZ_4^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0100\rangle \pm |1011\rangle). \tag{3.9}$$

According to each measurement outcome, Bob obtains any one of the following states:

$$\begin{aligned} |4GHZ_1^+\rangle + |4GHZ_3^-\rangle &\Rightarrow |\psi_B\rangle = \alpha|0\rangle - \beta|1\rangle \\ |4GHZ_1^-\rangle + |4GHZ_3^+\rangle &\Rightarrow |\psi_B\rangle = \alpha|0\rangle + \beta|1\rangle \\ -|4GHZ_2^-\rangle + |4GHZ_4^+\rangle &\Rightarrow |\psi_B\rangle = \beta|0\rangle + \alpha|1\rangle \\ -|4GHZ_2^+\rangle + |4GHZ_4^-\rangle &\Rightarrow |\psi_B\rangle = -\beta|0\rangle + \alpha|1\rangle. \end{aligned} \tag{3.10}$$

Bob can obtain the original state (3.1) by applying σ_z, σ_0, σ_x and $\sigma_z\sigma_x$ respectively. Cluster state is not invariant under qubit permutations, still one can teleport an unknown qubit perfectly to any qubit. Like GHZ state, one can teleport a subclass of a two-qubit unknown state with cluster state if the receiver has the qubit ‘14’. It is more useful than the GHZ-state because one can also teleport an arbitrary two- qubit state with four terms. Let Alice has the following two-qubit state:

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle. \tag{3.11}$$

So the combined state is:

$$|\psi_2\rangle|\Omega\rangle = \frac{1}{2}(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle)(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle). \tag{3.12}$$

Now Bob has qubit '12' at his disposal while Alice has qubit '34' of the resource state along with the unknown qubit. We can rewrite the above equation as:

$$\begin{aligned}|\psi_2\rangle_a|\Omega\rangle = \frac{1}{2}\big(&\alpha(|0000\rangle_{a34}|00\rangle_{12} + |0010\rangle_{a34}|01\rangle_{12} + |0001\rangle_{a34}|10\rangle_{12} - |0011\rangle_{a34}|11\rangle_{12}) + \\ &\beta(|0100\rangle_{ab34}|00\rangle_{12} + |0110\rangle_{a34}|01\rangle_{12} + |0101\rangle_{a34}|10\rangle_{12} - |0111\rangle_{a34}|11\rangle_{12}) + \\ &\gamma(|1000\rangle_{a34}|00\rangle_{12} + |1010\rangle_{a34}|01\rangle_{12} + |1001\rangle_{a34}|10\rangle_{12} - |1011\rangle_{a34}|11\rangle_{12}) + \\ &\delta(|1100\rangle_{ab34}|00\rangle_{12} + |1110\rangle_{a34}|01\rangle_{12} + |1101\rangle_{a34}|10\rangle_{12} - |1111\rangle_{a34}|11\rangle_{12})\big).\end{aligned} \tag{3.13}$$

Here the subscript a refers to the unknown qubit. Alice can use following orthogonal measurement basis

$$|\chi_0\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle), \tag{3.14}$$

$$|\chi_1\rangle = \frac{1}{2}(-|0000\rangle + |0110\rangle + |1001\rangle + |1111\rangle, \tag{3.15}$$

$$|\chi_2\rangle = \frac{1}{2}(|0000\rangle - |0110\rangle + |1001\rangle + |1111\rangle), \tag{3.16}$$

$$|\chi_3\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle - |1001\rangle + |1111\rangle). \tag{3.17}$$

Here we note that $\langle\chi_i|\chi_j\rangle = \delta_{ij}$. For each measurement outcome, Bob's state collapses to any of these four states:

$$\begin{aligned}|\chi_0\rangle \;\Rightarrow\; |\psi_B\rangle &= \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle, \\ |\chi_1\rangle \;\Rightarrow\; |\psi_B\rangle &= -\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle - \delta|11\rangle, \\ |\chi_2\rangle \;\Rightarrow\; |\psi_B\rangle &= \alpha|00\rangle - \beta|01\rangle + \gamma|10\rangle - \delta|11\rangle, \\ |\chi_3\rangle \;\Rightarrow\; |\psi_B\rangle &= \alpha|00\rangle + \beta|01\rangle - \gamma|10\rangle - \delta|11\rangle.\end{aligned} \tag{3.18}$$

In all these cases, Bob finds a suitable unitary transformation to retrieve the original state. For example, He does nothing if measurement outcome is $|\chi_0\rangle$ whereas he applies $\sigma_z \otimes \sigma_z$ for the outcome $|\chi_1\rangle$. Similarly he applies $\sigma_0 \otimes \sigma_z$ and $\sigma_z \otimes \sigma_0$ for remaining two outcomes respectively.

3.1.3 Teleportation with a HD state

Next we shall be using another useful class of state known as hyperdeterminant state. In computational basis it can be written as

$$|HD\rangle = \frac{1}{\sqrt{6}}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle + \sqrt{2}|1111\rangle). \tag{3.19}$$

Another way of writing this state is

$$|HD\rangle = \frac{1}{\sqrt{3}}(|\psi^+00\rangle + |00\psi^+\rangle + |1111\rangle), \tag{3.20}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

It maximizes the four-qubit hyperdeterminant [87] which remains invariant under local transformations. The entropy structure of different subsystems of this state is interesting. All single qubit subsystems are maximally mixed and three distinct two-qubit subsystems have entropy $\mathrm{Log}_2 3$. We know that for four qubits, there exists no states where all bipartite subsystem of which are maximally mixed [88]. All subsystems of HD state still contain relatively large entropy in each bipartition. Naturally, it renders this state as an useful resource for many QIPPs. One can readily verify that a single qubit state can be teleported with this state perfectly if Alice holds first three qubits and Bob has the last qubit. It may also work for any other possible distribution of the qubits among Alice and Bob.

HD state can be useful for teleportation of two-qubit state with three terms. For example, we take the following two-qubit state:

$$|\psi_2\rangle = \alpha|00\rangle + \beta|\psi^+\rangle + \gamma|11\rangle. \tag{3.21}$$

We shall use the particular form as given by (3.20). The combined state can be rewritten as:

$$\begin{aligned}
|\psi\rangle_2|HD\rangle &= \frac{1}{\sqrt{3}}(\alpha|00\rangle + \beta|\psi^+\rangle + \gamma|11\rangle)(|\psi^+00\rangle + |00\psi^+\rangle + |1111\rangle) \\
&= \frac{1}{\sqrt{3}}\big(\alpha(|00\psi^+\rangle|00\rangle + |0000\rangle|\psi^+\rangle + |0011\rangle|11\rangle) \\
&\quad +\beta(|\psi^+\psi^+\rangle|00\rangle + |\psi^+00\rangle|\psi^+\rangle + |\psi^+11\rangle|11\rangle \\
&\quad +\gamma(|11\psi^+\rangle|00\rangle + |1100\rangle|\psi^+\rangle + |1111\rangle|11\rangle))\big).
\end{aligned} \tag{3.22}$$

The first four qubits are with Alice and the last two qubits are held by Bob. Alice makes a measurement on her subsystem. She uses the following orthogonal measurement basis:

$$|h_0\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1111\rangle), \tag{3.23}$$

$$|h_1\rangle = \frac{1}{2}(|0001\rangle - |0010\rangle + |0100\rangle - |1111\rangle, \tag{3.24}$$

$$|h_2\rangle = \frac{1}{2}(-|0001\rangle + |0010\rangle + |0100\rangle - |1111\rangle), \tag{3.25}$$

$$|h_3\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle - |0100\rangle - |1111\rangle). \tag{3.26}$$

She may use also other set of orthogonal measurement basis. For each measurement outcome $|h_i\rangle$, Bob finds suitable unitary to get back the original state. For example, measurement outcome $|h_0\rangle$ yields $|\psi_B\rangle = \alpha|00\rangle + \beta|\psi^+\rangle + \gamma|11\rangle$ which is the original state. Measurement outcome $|h_1\rangle$ corresponds to $|\psi_B\rangle = \alpha|00\rangle + \beta|\psi^+\rangle - \gamma|11\rangle$. Bob then applies a unitary $\sigma_0 \otimes \sigma_z$ to obtain the original state.

3.1.4 Teleportation with a W-state

Next we consider the W-state as a source of entanglement for teleportation. Like GHZ state it is also invariant under the permutation of the qubits. Unfortunately, we shall see here that this state can not be used as a resource for perfect teleportation. We write the combined system of W-state and (3.1)

$$\begin{aligned} |\psi\rangle|W\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle) \\ &= \frac{1}{2}(\alpha|0100\rangle|0\rangle + \alpha|0010\rangle|0\rangle + \alpha|0001\rangle|0\rangle + \alpha|0000\rangle|1\rangle + \\ &\quad \beta|1100\rangle|0\rangle + \beta|1010\rangle|0\rangle + \beta|1001\rangle|0\rangle + \beta|1000\rangle|1\rangle). \end{aligned} \tag{3.27}$$

Unlike the other cases, here we cannot find any suitable orthogonal basis to measure Alice's subsystem. In [93] it was shown that instead of the W state, one can consider the state $|W_n\rangle$, which is $\frac{1}{\sqrt{2+2n}}(|100\rangle + \sqrt{n}|010\rangle + \sqrt{n+1}|001\rangle)$. This state can be used for the perfect teleportation of an unknown qubit. We construct the state for the case of four qubits. We can consider the state

$$|W_{mn}\rangle = \frac{1}{\sqrt{2m+2n+2}}(|1000\rangle + me^{i\rho}|0100\rangle + ne^{i\eta}|0010\rangle + \sqrt{m+n+1}e^{i\sigma}|0001\rangle). \tag{3.28}$$

Here m and n are real numbers. For sake of simplicity, we set the phases to unity and choose $m = n = 1$,

$$|W_{11}\rangle = \frac{1}{\sqrt{6}}(|1000\rangle + |0100\rangle + |0010\rangle + \sqrt{3}|0001\rangle). \tag{3.29}$$

With this quantum resource, the combined state of five particles would be

$$\begin{aligned} |\psi\rangle|W_{11}\rangle &= \frac{1}{\sqrt{6}}\big(\alpha(|01000\rangle + |00100\rangle + |00010\rangle + \sqrt{3}|00001\rangle) + \\ &\quad \beta(|11000\rangle + |10100\rangle + |10010\rangle + \sqrt{3}|10001\rangle)\big) \\ &= \frac{1}{2}\big((|\eta^+\rangle + |\eta^-\rangle)\alpha|0\rangle + (|\zeta^+\rangle - |\zeta^-\rangle)\alpha|1\rangle + \\ &\quad (|\zeta^+\rangle + |\zeta^-\rangle)\beta|0\rangle + (|\eta^+\rangle - |\eta^-\rangle)\beta|1\rangle\big) \\ &= \frac{1}{2}\big(|\eta^+\rangle(\alpha|0\rangle + \beta|1\rangle) + |\eta^-\rangle(\alpha|0\rangle - \beta|1\rangle) + \\ &\quad |\zeta^+\rangle(\alpha|1\rangle + \beta|0\rangle) + |\zeta^-\rangle(-\alpha|1\rangle + \beta|0\rangle)\big) \end{aligned} \tag{3.30}$$

where,

$$\begin{aligned} |\eta^\pm\rangle &= \frac{1}{\sqrt{6}}(|0100\rangle + |0010\rangle + |0001\rangle \pm \sqrt{3}|1000\rangle), \\ |\zeta^\pm\rangle &= \frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |1001\rangle \pm \sqrt{3}|0000\rangle). \end{aligned} \tag{3.31}$$

Now according to different outcome as obtained by Alice, Bob's state collapses into one of the following states

$$\begin{aligned} |\eta^+\rangle &\Rightarrow |\psi_B\rangle = \alpha|0\rangle + \beta|1\rangle, \\ |\eta^-\rangle &\Rightarrow |\psi_B\rangle = \alpha|0\rangle - \beta|1\rangle, \\ |\zeta^+\rangle &\Rightarrow |\psi_B\rangle = \beta|0\rangle + \alpha|1\rangle, \\ |\zeta^-\rangle &\Rightarrow |\psi_B\rangle = \beta|0\rangle - \alpha|1\rangle. \end{aligned} \tag{3.32}$$

Bob now obtains the unknown qubit by applying suitable unitary rotation on his qubit. From the table (3.2) we see that the entropy of the last qubit is one. This state is a found to be useful resource for teleporatation if Bob is given the last qubit, for any other distribution it would not work. Thus it is quite evident that entropy of a subsystem plays an important role in teleporatation.

3.1.5 Teleportation with a $|Q_4\rangle$-state

The state $|Q_4\rangle$ can be expressed in computational basis as follow:

$$|Q4\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle). \tag{3.33}$$

This state is not invariant under the permutations of the qubits. But the states obtained on permutation would also belong to the same SLOCC class. Surprisingly this state can be used for teleporatation if Bob has the second qubit at his disposal. For this particular distribution we can write the joint state comprising of (3.1) and (3.33) as

$$\begin{aligned}|\psi\rangle|Q4\rangle &= \frac{1}{2}[\alpha|0000\rangle_{a134}|0\rangle_2 + \alpha|0001\rangle_{a134}|1\rangle_2 + \alpha|0100\rangle_{a134}|0\rangle_2 + \alpha|0011\rangle_{a134}|1\rangle_2 + \\ &\quad \beta|1000\rangle_{a134}|0\rangle_2 + \beta|1001\rangle_{a134}|1\rangle_2 + \beta|1100\rangle_{a134}|0\rangle_2 + \beta|1011\rangle_{a134}|1\rangle_2].\end{aligned} \tag{3.34}$$

Alice can use one of the following set of basis vectors to make four-particle von-Neumann measurements. One set of basis vectors are

$$\begin{aligned}|\rho_1^{\pm}\rangle &= \frac{1}{2}[(|0000\rangle + |0100\rangle) \pm (|1001\rangle + |1011\rangle)] \\ |\rho_2^{\pm}\rangle &= \frac{1}{2}[(|0001\rangle + |0011\rangle) \pm (|1001\rangle + |1100\rangle)],\end{aligned} \tag{3.35}$$

Using the basis (3.35), we can rewrite the combined state as

$$\begin{aligned}|\psi\rangle|Q4\rangle &= \frac{1}{2}[|\rho_1^{+}\rangle_{a134}(\alpha|0\rangle_2 + \beta|1\rangle_2) + |\rho_1^{-}\rangle_{a134}(\alpha|0\rangle_2 - \beta|1\rangle_2) + \\ &\quad |\rho_2^{+}\rangle_{a134}(\alpha|1\rangle_2 + \beta|0\rangle_2) + |\rho_2^{-}\rangle_{a134}(\alpha|1\rangle_2 - \beta|0\rangle_2)].\end{aligned} \tag{3.36}$$

Irrespective of the outcome of Alice's measurement Bob can always find suitable unitary to convert his qubit to the original state. Now if Bob is given any other qubit except the second one, Alice would not be able to find suitable measurement basis. So $|Q_4\rangle$ is useful for teleportation for certain distribution of its qubits.

3.1.6 Teleportation with a $|Q_5\rangle$-state

Let's focus our attention to an interesting four-qubit state given by

$$|Q5\rangle = \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle). \tag{3.37}$$

It has the property that concurrence vanishes if any two particles are traced out but has non-vanishing tangle for any three-qubit subsystem. This entangled state can also be used as a suitable quantum resource for teleportation under certain distribution of the qubits among the sender and the receiver. For example, we consider the distribution such that the fourth qubit is held by Bob while Alice has remaining qubits. The combined state can be written as:

$$
\begin{aligned}
|\psi\rangle|Q5\rangle &= \frac{1}{2}[|\varphi_1^+\rangle_{a123}(\alpha|0\rangle+\beta|1\rangle)_b+|\varphi_1^-\rangle_{a123}(\alpha|0\rangle-\beta|1\rangle)_b+ \\
&\quad |\varphi_2^+\rangle_{a123}(\beta|0\rangle_4+\alpha|1\rangle)_b+|\varphi_2^-\rangle_{a123}(\alpha|0\rangle-\beta|1\rangle)_b],
\end{aligned} \tag{3.38}
$$

Here we have,

$$
|\varphi_1^\pm\rangle = \frac{1}{2}[(|0000\rangle+|0111\rangle)\pm(|1101\rangle+|1110\rangle)], \tag{3.39}
$$

$$
|\varphi_2^\pm\rangle = \frac{1}{2}[(|0101\rangle+|0110\rangle)\pm(|1000\rangle+|1111\rangle)]. \tag{3.40}
$$

It turns out that Alice can teleport the state $|\psi\rangle$, if Alice makes a four-particle von Neumann measurement using the above basis vectors. Similarly same conclusion follows if Bob is given second and third qubit respectively. The protocol would not work if Bob has the first qubit at his disposal. So the success of the protocol relies on the proper distribution of the resource state qubits between Alice and Bob.

3.2 Entropy and teleportation

Some observations are in order. We have seen that GHZ state can teleport a one-qubit state and a two-qubit state with two terms irrespective of the distributions of the qubit among the sender and the receiver. Likewise $|\Omega\rangle$ is also a useful resource for one qubit state teleportation. We have also seen that one can also teleport a two-qubit state with four terms with this state if Bob has particle 12 and 13 with him. Entropy of the subsystem comprising particles 12 and 13 are 2. It is found that W-state, although belongs to a genuinely multipartite entangled state is not suitable for teleportation whereas generalized W-state can be used for specific distribution of the particles. $|Q_4\rangle$ is useful when Bob has second qubit while $|Q_5\rangle$ can be used if any one except the first qubit is held by Bob. Thus we can see that there may be a strong connection between the entropy of the resource state subsystem and a state's usefulness for the protocol.

From the structure of the resource states, it is not always easy to predict the suitability of a state for teleportation. In this regard, we will present an entropic criterion, a state must satisfy to be an useful resource for teleportation.

At this point, it will be useful to catalogue the von Neumann entropy (henceforth, called entropy) of all the bipartite partitions of theses states. The entropy for a state ρ is defined as $S(\rho) = -\mathrm{Tr}(\rho \log_2(\rho))$. We have four-qubit states. We can label these qubits as "$1, 2, 3, 4$". Then ρ_1 is the reduced density matrix of the qubit with label '1'; ρ_{12} is the reduced density matrix of the qubits with labels '1' and '2' and so on. In a bipartite partition, both subsystems will have identical entropies. For example, $S(\rho_1) = S(\rho_{234})$, $S(\rho_{12}) = S(\rho_{34})$ etc. This can be seen by considering the Schmidt decomposition of the states.

In Table I, we list the entropies of the subsystems for the above five states.

States	$S(\rho_1)$	$S(\rho_2)$	$S(\rho_3)$	$S(\rho_4)$	$S(\rho_{12})$	$S(\rho_{13})$	$S(\rho_{14})$
$\|GHZ\rangle$	1	1	1	1	1	1	1
$\|\Omega\rangle$	1	1	1	1	2	2	1
$\|HD\rangle$	1	1	1	1	1.58	1.58	1.58
$\|W\rangle$	0.81	0.81	0.81	0.81	1	1	1
$\|Q_4\rangle$	0.81	1	0.81	0.81	1.5	1.22	1.22
$\|Q_5\rangle$	0.81	1	1	1	1.5	1.5	1.5

Table I: Entropies of the subsystems

When the four-qubit system is partitioned into '1' and '234' parts, then each subsystem will have entropy $S(\rho_1)$; for the partition '14' and '23', each subsystem will have entropy $S(\rho_{14})$. Other entries of the table can be understood in the similar manner. In our analysis of above four-qubit states, these entropies have played an important role. In fact, the success of various protocols depends on the values of these entropies. In the next section we will formalize it in a theorem.

3.3 General formalism for teleporatation

We saw in the last section that the entropy of the subsystems of the given resource entangled state plays an important role. Now we ask the question that if we are given a n-qubit state with m terms to teleport, what kind of resource state is needed ? In a reverse way, the question can be posed as: given a resource state, what are the states that can be teleported using this resource? The answer to this will also tell us that why sometimes a more entangled state is less suitable. We also nothe that we are interested in perfect teleportation.

Theorem: A resource state is useful to teleport an unknown n-qubit state which has m terms if and only if the resource states qubits could be distributed in such a way that the receiver's n qubits have entropy $\log_2 m$.

Proof: Let us consider an unknown n-qubit state with m-terms that Alice wishes to teleport to Bob

$$|\Psi\rangle_n = \sum_{k=1}^{m} \alpha_k |\eta_k\rangle_n. \tag{3.41}$$

The state is normalized and the basis set is orthonormal

$$\langle \eta_k | \eta_l \rangle = \delta_{kl}, \qquad \sum_{k=1}^{m} |\alpha_k|^2 = 1. \tag{3.42}$$

Let the resource state be a N-qubit state. For this resource state to be useful, we should be able to write it as,

$$|R\rangle_N = \frac{1}{\sqrt{m}} \sum_{l=1}^{m} |\chi_l\rangle_{N-n} |\eta_l\rangle_n. \tag{3.43}$$

Here the states $|\chi_l\rangle_{N-n}$ may not be orthonormal. The combined state can be written as

$$\begin{aligned} |\Psi\rangle_n |R\rangle_N &= \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \sum_{i=1}^{m} \alpha_i |\eta_i\rangle_n |\chi_k\rangle_{N-n} |\eta_k\rangle_n \\ &= \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \sum_{i=1}^{m} |\eta_i\rangle_n |\chi_k\rangle_{N-n} \alpha_i |\eta_k\rangle_n. \end{aligned} \tag{3.44}$$

Alice will now make a measurement in an orthonormal basis $|\theta_l\rangle_p$. Therefore, we should be able to write

$$|\eta_i\rangle_n |\chi_k\rangle_{N-n} = \frac{1}{\sqrt{m}} \sum_{l=1}^{m^2} C_{ik,l} |\theta_l\rangle_N. \tag{3.45}$$

The $C_{ik,l}$ is an interesting object. For each l, it is a $m \times m$ matrix in 'ik' space. It is also a $m^2 \times m^2$ matrix with row label as ik. We need to find a condition on $C_{ik,l}$ such that we indeed have a suitable measurement basis. Using the above equation

$$\begin{aligned} |\Psi\rangle_n |R\rangle_N &= \frac{1}{m} \sum_{i=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m^2} C_{ik,l} |\theta_l\rangle_N \alpha_i |\eta_k\rangle_n \\ &= \frac{1}{m} \sum_{l=1}^{m^2} |\theta_l\rangle_N \sum_{k=1}^{m} \sum_{i=1}^{m} C_{ik,l} \alpha_i |\eta_k\rangle_n. \end{aligned}$$

For the teleportation to succeed, we should have,

$$\sum_{k=1}^{m} \sum_{i=1}^{m} C_{ik,l} \alpha_i |\eta_k\rangle_n = V^l \sum_{n=1}^{m} \alpha_n |\eta_n\rangle_n. \tag{3.46}$$

Taking its adjoint and scalar product with itself, we get,

$$\sum_{k=1}^{m} \sum_{i=1}^{m} \sum_{i'=1}^{m} C_{ik,l} C^*_{i'k,l} \alpha_i \alpha^*_{i'} = 1. \tag{3.47}$$

This condition is true for each l,

$$\sum_{i=1}^{m} \sum_{i'=1}^{m} (CC^\dagger)_{ii',l} \alpha_i \alpha_{i'} = 1. \tag{3.48}$$

For this equation to be satisfied, we must have

$$(CC^\dagger)_{ii',l} = \delta_{ii'}. \tag{3.49}$$

This suggests that C is unitary in 'ik' space for each l for teleportation to succeed. Let us now see what it means for the resource state.

Taking the adjoint and the scalar product,

$$\langle \eta'_i | \eta_i \rangle \langle \chi'_k | \chi_k \rangle = \frac{1}{m} \sum_{l=1}^{m^2} \sum_{l'=1}^{m^2} C_{ik,l} C^*_{i'k',l'} \langle \theta_l | \theta'_l \rangle, \tag{3.50}$$

$$\delta_{ii'} \langle \chi'_k | \chi_k \rangle = \frac{1}{m} \sum_{l=1}^{m^2} C_{ik,l} C^*_{i'k',l}. \tag{3.51}$$

Multiplying by $\delta_{ii'}$ and summing over i' and i, we get,

$$\langle \chi'_k | \chi_k \rangle = \frac{1}{m^2} \sum_{l=1}^{m^2} (C^\dagger C)_{kk',l}. \tag{3.52}$$

Since C is unitary in the subspace, we get

$$\langle\chi'_k|\chi_k\rangle = \delta_{kk'}. \tag{3.53}$$

Therefore $|\chi_k\rangle$ should be orthonormal for the exact teleportation. Thus the resource state should have the form

$$|R\rangle_N = \frac{1}{\sqrt{m}}\sum_{l=1}^{m}|\chi_l\rangle_{N-n}|\eta_l\rangle_n, \tag{3.54}$$

with both $|\chi_k\rangle$ and $|\eta_l\rangle$ being orthonormal. So, the entropy of the reduced density matrix of the Bob's qubits would be $\log_2 m$. Conversely, if the entropy of the reduced density matrix of the Bob's qubits is $\log_2 m$, then we can always find a measurement basis given in (3.45) so that one can do faithful teleportation of n-qubit state with m-terms. We have shown is that if we wish to teleport a n-qubit state with m terms, then we should be able to distribute resource states qubits in such a way such that Bob's n qubits have entropy $\log_2 m$. Given a resource state, we can compute entropy of all the partitions. If there is a partition where Bob's n qubits have entropy as $\log_2 m$, then the theorem tells that the state with m terms can be teleported with that partition. In general, the structure of a multipartite state may be quite complex. So it seems difficult to find its suitability for teleportation all the time. But this theorem makes our life simple, as an example we may consider the following state:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{4\sqrt{2}}\big(|0000\rangle + (2-\sqrt{3})(|0001\rangle + |1110\rangle) + (2+\sqrt{3})(|0010\rangle - |1101\rangle + \\ &\quad |0011\rangle + |1100\rangle - |1111\rangle)\big).\end{aligned} \tag{3.55}$$

Although it looks a humongous state but we can easily compute its subsystem entropy and it turns out to be $S(\rho_i) = 1$, $i = 1, 2, 3, 4$. So from the theorem we can conclude that $|\Psi\rangle$ is a useful resource for perfect teleportation.

3.3.1 Less entanglement may sometime be more useful

In the preceding discussion, we have observed an interesting fact. We have seen that GHZ state is useful to teleport a two-qubit state with two terms. W-state is also found to be suitable resource for teleporting a subclass of states. For example, we consider the following state:

$$|\psi\rangle = \alpha|00\rangle + \beta|\psi_+\rangle, \tag{3.56}$$

where $|\psi_+\rangle$ is Bell state. We assume that Bob has last two-qubits of the resource state. The combined state can be written as:

$$|\psi\rangle|W\rangle = \alpha(|0000\rangle|\psi_+\rangle + |00\psi_+\rangle|00\rangle) + \beta(|\psi_+00\rangle|\psi_+\rangle + |\psi_+\psi_+\rangle|00\rangle)$$

Now Alice can use following measurement basis:

$$\begin{aligned} |G_1^\pm\rangle &= \frac{1}{\sqrt{2}}(|00\psi_+\rangle \pm |\psi_+00\rangle), \\ |G_2^\pm\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle \pm |\psi_+\psi_+\rangle). \end{aligned} \tag{3.57}$$

For each measurement outcome, Bob can obtain the original state by suitable unitary operation on his subsystem.

All bipartite subsystems of these two states have entropy one. Bipartite entanglement, as quantified by entropy of the subsystems of the states like $|Q_4\rangle$ and $|Q_5\rangle$ are more than one. Although it is surprising that $|Q_4\rangle$ state is not useful for the protocol, even a subclass of two-qubit state cannot be teleported. Same thing happens for quantum key distribution protocol also. We can rewrite GHZ state as:

$$\begin{aligned} |GHZ\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \\ &= \frac{1}{\sqrt{2}}(|\phi_+\phi_+\rangle + |\phi_-\phi_-\rangle). \end{aligned} \tag{3.58}$$

$|\phi_\pm\rangle$ are the Bell states. So a secret key can be established between Alice and Bob if former has qubit '12' and later has qubit '34'. One can use $|\phi_\pm\rangle$ measurement basis for the purpose. But we cannot find any measurement basis in the case of $|Q_4\rangle$ state for QKD. Same is true for $|Q_5\rangle$ state also. Entropy is used to quantify entanglement between subsystems of a pure state and bipartite entropy of $|Q_4\rangle$ and $|Q_5\rangle$ are more than W-state and GHZ state. So we can argue that more entanglement would not necessarily help in performing QIPPs perfectly. Sometime less entanglement is more suitable for the tasks.

3.4 Discussion and Conclusion

Multipartite states allow many variations of the communication protocols that were introduced for bipartite states. We have considered a number of different genuine quadripartite entangled states as quantum resources for exact teleportation. With a multipartite state as a resource, one can consider possibilities of teleporting multiple qubit states with different number of terms. We find that in such scenarios the phenomenon of less is more may occur. It means that depending upon the value of entropy, the resource state is either suitable or not suitable for the teleportation of an unknown state. The condition that we have obtained for teleportation helps us in understanding this. In particular, we find that to teleport a m-term n-qubit state, a subsystem of the resource state must have entropy $\log_2 m$. So to be able to teleport a two-term two-qubit state, a subsystem needs to have entropy as one, but for a most general two-qubit teleportation, the required entropy is two. Therefore, a four-qubit GHZ state can be used to teleport a two-term two-qubit state but not the most general two-qubit state. Similarly HD state can be used to teleport a two-qubit state with three terms as all bipartitions have entropy $\log_2 3$. Some states may be suitable for teleportaion, but in suitable condition such as a specific distribution of the resource state particles. The system has to be distributed in such a manner that the receiver must hold the subsystem with appropriate value of entropy. Under this scenario we have shown that states like $|Q4\rangle$ and $|Q5\rangle$ or generalised-W state may be suitable.

Chapter 4

Resource state structure for cooperative QKD

4.1 Introduction

As discussed in the previous chapters, quantum entanglement leads to nonlocal correlations which can be used as prime ingredient for implementing existing QIPPs such as teleportation [39], QKD [38] in many different ways. A simple yet interesting example in this regard would be to use a three-qubit GHZ state which is distributed among Alice, Charlie and Bob for QKD. If Charlie makes a measurement on his subsystem in Hadamard basis, then the collapsed state between Alice and Bob would be a Bell state. Alice and Bob can then establish a secret key with the collapsed state. Instead of this, if Charlie chooses computational basis for the measurement, then the collapsed state between Alice and Bob would be a product state which would not be suitable for establishing a key. However, if Charlie decides not to make any measurement then Alice and Bob are left with a maximally mixed state which is again not useful for establishing a secret key. So a secret key can be established by involvement of all and we call this joint venture as *cooperative quantum key distribution*. In the same spirit, we can generalize teleportation to implement *cooperative teleportation* which has been discussed in the introduction chapter.

Multipartite entanglement lies at the heart of this protocol. In the similar spirit, the idea of controlled teleportation was introduced earlier [89, 90]. In this protocol, a third party is involved and measurement performed by this party is very crucial to make the protocol successful. It is often regarded as if the third party is controlling the protocol. A GHZ state is always a useful resource for these protocols. However, a partially entangled state with less control power can also be used [91]. In a tripartite scenario, there are two inequivalent SLOCC classes of genuinely entangled states. So a natural question arises – how do we determine if a tripartite state is suitable for a particular coopeartive quantum information processing task? In this chapter, we will discuss a larger class of tripartite states which is useful for cooperative quantum information tasks. GHZ state is a perfect resource for cooperative QKD, whereas W-state is not useful for this purpose. We must have a Bell state after the measurement by the third party. However, if the collapsed state is a partially entangld state, then a secret key cannot be established efficiently. The authors in [92] have introduced a novel protocol of quantum key distribution with partially entangled state. This idea is based on probabilistic teleportation [43] and hence key is established with full secrecy with a probabilistic success rate. We have exploited the idea to discuss the success rate of coopertaive QKD with partially entangled tripartite state. Moreover we know that there is no notion of maximally entangled state in multipartite scenario. A state which is useful for a specific task may be found not suitable for another protocol. Hence there is a concept of task oriented maximally entangled state [95] for multipartite states. We have found such states for our proposed protocol and shown how to realize it.

4.2 Resource state structure for three qubits

In the previous section, we have seen how our protocol works and it is evident that the structure of the resource state plays an important role. We start discussion of cooperative QKD with three-qubit resource state which is distributed among Alice, Bob and Charlie. We know that to establish a secret key between two parties namely Alice and Bob, they must share a Bell state. So, for a cooperative scheme, after Charlie's measurement the collapsed state between Alice and Bob must be a Bell state. Any three qubit pure state can be written using five orthogonal product states [97] and it is given by

$$|\Phi\rangle_{CAB} = \lambda_0|000\rangle_{CAB} + \lambda_1 \exp^{i\phi} |100\rangle_{CAB} + \lambda_2|101\rangle_{CAB} + \lambda_3|110\rangle_{CAB} + \lambda_4|111\rangle_{CAB}, \quad (4.1)$$

where $\lambda_i \geq 0$, $\sum_i |\lambda_i^2| = 1$, $0 \leq \phi \leq \pi$ and the subscripts denote the distribution of qubits between Charlie, Alice and Bob. For the sake of simplicity we can take ϕ as zero. We perform most general projective measurement on any of its qubit, say on first qubit and seek that the collapsed state be a Bell state. We use general measurement basis

$$|+\rangle_\theta = \cos\frac{\theta}{2}|0\rangle + e^{i\alpha}\sin\frac{\theta}{2}|1\rangle, \tag{4.2}$$

$$|-\rangle_\theta = -\sin\frac{\theta}{2}e^{-i\alpha}|0\rangle + \cos\frac{\theta}{2}|1\rangle. \tag{4.3}$$

After the measurement the state collapses to $\rho_\pm$ with the corresponding probability $p_\pm$, where $p_\pm$ are given by

$$p_+ = \cos^2\frac{\theta}{2}\lambda_0^2 + \cos\alpha\sin\theta\lambda_0\lambda_1 + \sin^2\frac{\theta}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2), \tag{4.4}$$

$$p_- = \sin^2\frac{\theta}{2}\lambda_0^2 - \cos\alpha\sin\theta\lambda_0\lambda_1 + \cos^2\frac{\theta}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2). \tag{4.5}$$

We discuss the case for ρ_+. If we trace out Alice's qubit, the elements of the reduced density matrix of Bob's qubit (ρ_B^+) are given by

$$\rho^B_{+,00} = \frac{\lambda_0^2\cos^2\frac{\theta}{2} + \cos\alpha\sin\theta\lambda_0\lambda_1 + \sin^2\frac{\theta}{2}(\lambda_1^2+\lambda_3^2)}{p_+}, \tag{4.6}$$

$$\rho^B_{+,01} = \frac{\sin\frac{\theta}{2}\left(e^{i\alpha}\cos\frac{\theta}{2}\lambda_0\lambda_2 + \sin\frac{\theta}{2}(\lambda_1\lambda_2+\lambda_3\lambda_4)\right)}{p_+}, \tag{4.7}$$

$$\rho^B_{+,10} = \frac{\sin\frac{\theta}{2}\left(e^{-i\alpha}\cos\frac{\theta}{2}\lambda_0\lambda_2 + \sin\frac{\theta}{2}(\lambda_1\lambda_2+\lambda_3\lambda_4)\right)}{p_+}, \tag{4.8}$$

$$\rho^B_{+,11} = \frac{\sin^2\frac{\theta}{2}(\lambda_2^2+\lambda_4^2)}{p_+}. \tag{4.9}$$

For a given set of measurement parameters, we require ρ_B^+ to be maximally mixed $i.e.$

$$\rho_B^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

We set $\alpha = 0$ and $\theta = \frac{\pi}{4}$. With this set of parameters the measurement basis of Charlie are

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{4.10}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle). \tag{4.11}$$

Now equating $\rho^B_{+,01} = \rho^B_{+,10} = 0$, we obtain the following conditions:

$\Rightarrow \lambda_2 = 0$ and $\lambda_3 = 0$ or $\lambda_4 = 0$ and rest of the coefficients of the state eq.(4.1) are non zero. But with this choice of parameters the collapsed state after Charlie's measurement outcome $|+\rangle$ would be a product state instead of a Bell state.

$\Rightarrow$ Another set of conditions might be $\lambda_0 = 0$, $\lambda_1 = 0$ and λ_3 or $\lambda_4 = 0$. If $\lambda_0 = 0$, then the state given by the eq.(4.1) is a biseparable state between Charlie and Alice-Bob and it would not help cooperative QKD.

$\Rightarrow \lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$ and rest of the coefficients are non zero. With this parameter values we have $\rho^B_{+,00} = \lambda_0^2$ and $\rho^B_{+,11} = \lambda_4^2$. Comparing the elements with the maximally mixed state we have $\lambda_0 = \lambda_4 = \frac{1}{\sqrt{2}}$. Substituting these values into eq.(4.1) we obtain $|\Phi\rangle_{CAB} = \frac{1}{\sqrt{2}}(|000\rangle_{CAB} + |111\rangle_{CAB})$ which is a GHZ state. If Charlie makes measurement in $|\pm\rangle$ basis, the collapsed state between Alice and Bob is a Bell state. Hence secret key can be established perfectly between them with the cooperation of Charlie. However, there is another class of states which can also be useful for the purpose. In the next subsection, we shall find a larger class of resource states for cooperative QKD.

4.3 Cooperative QKD with a partially entangled three-qubit state

As we know, establishment of a secret key requires a Bell state to be shared by the observers. So in our scheme it is possible when after Charlie's measurement the reduced state between Alice and Bob is a Bell state. In this regard we present following proposition:

Proposition : *Co-operative QKD will be successful if the resource state is such that all three single qubit reduced density matrices are maximally mixed or only two of them are maximally mixed, such that the partially mixed qubit is with Charlie who makes the first measurement.*

Proof : A successful establishment of a key between Alice and Bob requires that they share a Bell state. It means that, after Charlie's measurement, the reduced state between Alice and Bob has to be LU equivalent to a Bell state. Without loss of generality, we take that Charlie is doing measurement in computational basis. So, the state suitable for a successful cooperative QKD is,

$$|\Psi\rangle = \sqrt{1/2}\big[|0\rangle|\phi^+\rangle + |1\rangle(I\otimes U)|\phi^-\rangle\big] \tag{4.12}$$

Obviously, for each measurement outcome for Charlie the reduced state between Alice and Bob is a Bell state or its LU equivalent state. For this state the qubits of Alice and Bob both have entropy one and the qubit of Charlie has entropy less than one if, U is not identity. If U is identity, then all the qubits have entropy one and it is just LU equivalent to GHZ state. So, the state has the entropy structure as stated in the proposition. Now the proof will be complete if we can show that this is the only structure possible for all states which have the entropy structure as mentioned in the proposition. To show this we use the results the of [98, 99], which are about the local unitary equivalence of multipartite states. Firstly, they showed that for a three-qubit state, if all the single qubit reduced density matrices are maximally mixed, then they are LU equivalent to $\sqrt{1/2}\big[|0\rangle|\phi^+\rangle + |1\rangle|\phi^-\rangle\big]$, which is again LU equivalent to GHZ state. This proves our one case, where all the qubits have entropy one. Next case is when only two qubits have entropy one each. Now, any three-qubit pure state can be written as a Schmidt decomposition between 1-23 bipartition [109].

$$|\psi\rangle = \sqrt{p}|0\rangle_C|\psi_0\rangle_{AB} + \sqrt{1-p}|1\rangle_C|\psi_1\rangle_{AB}, \tag{4.13}$$

where, $|\psi_0\rangle$ and $|\psi_1\rangle$ both are normalized and $\langle\psi_0|\psi_1\rangle = 0$. One can choose different parametrization of these two orthonormal states, such that total number of parameters of the state $|\psi\rangle$ is five. One simple parametrization is the LPS [110] scheme. Now we want this state to have maximally mixed reduced density matrices for A and B. This makes the structure to be LU equivalent to,

$$|\psi\rangle = \sqrt{p}|0\rangle_C|\phi^+\rangle_{AB} + \sqrt{1-p}|1\rangle_C|\phi^-\rangle_{AB}, \tag{4.14}$$

with $p \neq 1/2$. Now, the last step is to show that eq.(4.12) and eq.(4.14) are LU equivalent. To show that we again use the result of [99]. For two non generic three-qubit states with two single qubit reduced density matrices maximally mixed are LU equivalent, if the third qubit has same entropy for the both of the states. Hence, eq.(4.12) and eq.(4.14) will be LU equivalent if we can just show that for every p we can choose a unitary U, such that the qubit with Charlie has the same entropy for both states. We take a very simple one parameter unitary,

$$U = \begin{bmatrix} \sqrt{a} & \sqrt{1-a} \\ -\sqrt{1-a} & \sqrt{a} \end{bmatrix}. \tag{4.15}$$

Now applying this unitary to eq.(4.12) and then tracing out Alice and Bob, we calculate the entropy for reduced density matrix for Charlie and found that for $p = 1/2(1-\sqrt{a})$, Charlie's qubit will have same entropy for both eq.4.12) and eq.(4.14) and hence they are LU equivalent. This completes our proof.

Now, from the equation (4.14) it is evident that if Charlie makes measurement in computational basis then the collapsed state between Alice and Bob will be a maximally entangled state. But if Charlie chooses an arbitrary basis then the collapsed state may be partially entangled state or separable. In the later case QKD cannot be carried out perfectly. So, let us consider that Charlie performs measurement in a general basis,

$$|+\rangle_n = \frac{|0\rangle + n|1\rangle}{\sqrt{1+|n|^2}}, \ \ |-\rangle_n = \frac{-n^*|0\rangle + |1\rangle}{\sqrt{1+|n|^2}}. \tag{4.16}$$

We are considering the state (4.14) as the resource state. as this is the more general form of suitable resource state for the protocol. Specifically, for $p = 1/2$, the state is a LU equivalent state of conventional GHZ state. From Eqn. (4.16) we can write,

$$|0\rangle = \frac{|+\rangle_n - n|-\rangle_n}{\sqrt{1+|n|^2}}, \ \ |1\rangle = \frac{n^*|+\rangle_n + |-\rangle_n}{\sqrt{1+|n|^2}}. \tag{4.17}$$

Putting these in Eqn. (4.14) we get,

$$|\psi\rangle = |+\rangle_n[N\sqrt{p}|\phi^+\rangle + Nn^*\sqrt{1-p}|\phi^-\rangle] + \tag{4.18}$$

$$|-\rangle_n[-Nn\sqrt{p}|\phi^+\rangle + N\sqrt{1-p}|\phi^-\rangle], \tag{4.19}$$

where $N = 1/\sqrt{1+|n|^2}$. So, when Charlie makes a measurement, the collapsed states between Alice and Bob could be

$$|\psi_+\rangle_{AB} = \frac{1}{\sqrt{p_+}}\big[N\sqrt{p}|\phi^+\rangle + Nn^*\sqrt{1-p}|\phi^-\rangle\big], \tag{4.20}$$

$$|\psi_-\rangle_{AB} = \frac{1}{\sqrt{p_-}}\big[N\sqrt{1-p}|\phi^-\rangle - Nn\sqrt{p}|\phi^+\rangle\big], \tag{4.21}$$

with probability $p_+ = N^2 p + N^2|n|^2(1-p)$ and $p_- = N^2 p|n|^2 + N^2(1-p)$ corresponding to measurement outcome $|+\rangle_n$ and $|-\rangle_n$ respectively. It is evident from the structure of the collapsed state that it can be separable, partially entangled or maximally entangled. As an example, when, $\sqrt{p} = n^*\sqrt{1-p}$, the collapsed state $|\psi_+\rangle$ is separable. When $\sqrt{p} \neq n^*\sqrt{1-p}$, state will be partially entangled, and when $n = 0$, the collapsed state will be maximally entangled.

We rewrite the states (4.20) and (4.21) as

$$|\psi_+\rangle_{AB} = N_1(|00\rangle + n_1|11\rangle), \tag{4.22}$$

$$|\psi_-\rangle_{AB} = N_2(|00\rangle + n_2|11\rangle), \tag{4.23}$$

where we define,

$$N_1 = \frac{N(\sqrt{p} + n^*\sqrt{1-p})}{\sqrt{2p_+}}, \tag{4.24}$$

$$n_1 = \frac{\sqrt{p} - n^*\sqrt{1-p}}{\sqrt{p} + n^*\sqrt{1-p}}, \tag{4.25}$$

$$N_2 = \frac{N(\sqrt{1-p} - n\sqrt{p})}{\sqrt{2p_-}}, \tag{4.26}$$

$$n_2 = \frac{-\sqrt{1-p} - n\sqrt{p}}{\sqrt{1-p} - n\sqrt{p}}. \tag{4.27}$$

With these partially entangled states, Alice and Bob can initiate an entanglement based QKD protocol like in ref. [100]. The original protocol [38] involves a maximally entangled state namely a Bell state. In the absence of an eavesdropper, *e.g.* Eve, the protocol is reminiscent of the Ekert protocol. Two parties hold one qubit each and agree on two sets of basis states in which they measure their own qubits randomly. After the measurement step, they announce their choices of the bases. Those outcomes are kept when the bases are matched and the rest are discarded. So, in half of the cases they get perfectly correlated results and hence can construct a secure key. The secure key rate is $1/2$ in this scenario. But often in practical scenario, the perfect correlation is not obtained, which indicates noise in the entanglement channel or imperfect measurement or possibly Eve's intervention. Alice and Bob use a part of the matched outcome to determine the Quantum Bit Error Rate (QBER) and the remaining part is used to build secure key after error correction and privacy amplification. To know

Eve's presence, Ekert originally used a third basis to test the violation of CHSH inequality. If it is maximally violated then there is no eavesdropper's attack. Non-maximal violation would indicate a non-zero QBER. In this protocol the key rate is changed from $1/2$ to $2/9$. We assume that there is no eavesdropper and measurement imperfection but only concentrate on the noise in the entanglement channel, which is a partially entangled state in our case. Obviously, perfect correlation will not be obtained, which will produce nonzero QBER. As stated, in this scenario, Alice and Bob measure their subsystems with two different bases. We assume that they have agreed to measure with the projectors $P_{0(1)}$ and $P_{+(-)}$ where

$$\begin{aligned} P_{0(1)} &= M^{\dagger}_{0(1)} M_{0(1)}, \\ P_{+(-)} &= M^{\dagger}_{+(-)} M_{+(-)}, \end{aligned} \tag{4.28}$$

and $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$, $M_+ = |+\rangle\langle +|$, $M_- = |-\rangle\langle -|$. Here, QBER is defined as the probability that they would obtain different outcome even if the measurement basis is same [101].

$$\begin{aligned} QBER &= \mathrm{Tr}(\mathrm{P}_0 \otimes \mathrm{P}_1.\rho) + \mathrm{Tr}(\mathrm{P}_1 \otimes \mathrm{P}_0.\rho) + \\ &\quad \mathrm{Tr}(\mathrm{P}_+ \otimes \mathrm{P}_-.\rho) + \mathrm{Tr}(\mathrm{P}_- \otimes \mathrm{P}_+.\rho), \end{aligned} \tag{4.29}$$

where $\rho = |\psi\rangle\langle\psi|$. Now Plugging $|\psi_+\rangle_{AB}$ into the above equation we find the expression for QBER:

$$QBER = \frac{n^2(1-p)}{2\big(n^2(1-p)+p\big)}. \tag{4.30}$$

We have plotted it with Charlie's measurement parameter n. It shows that error rate vanishes if measurement basis corresponds to $n = 0$ in which case the collapsed states are Bell states and hence measurement outcomes of Alice and Bob are perfectly correlated. But in more general scenario, *i.e.* when the resource state is not a Bell state, there exists error rate even in the absence of any eavesdropper. However Charlie can control QBER by choosing appropriate measurement basis. With information about QBER, they can employ some error correcting protocols to distill a secure key.

We can also design the protocol in such a way that key can be established perfectly but probabilistically [92]. The protocol is as follow: Each time Charlie makes a measurement, she announces publicly the parameters N_1, n_1 and N_2, n_2 which determines the degree of

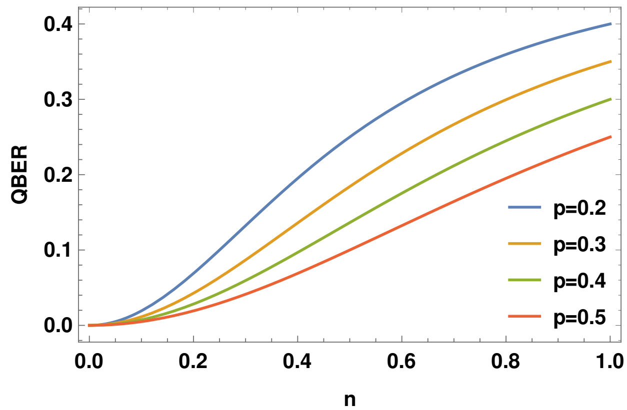

Figure 4.1: QBER with control power of Charlie. Lower one corresponds to GHZ state for which QBER is smaller than the others.

entanglement of the collapsed state between Alice and Bob. Now, Alice prepares a state $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ which she associates with the secret key and assigns the bit value 0 for $|+\rangle$ and 1 for $|-\rangle$ respectively. Both of them have pre-agreement about this assignment. Alice makes a joint measurement on her subsystem which consists of a qubit of the collapsed state and either of the $|\pm\rangle$. For this, she randomly chooses one of the states from the generalized orthonormal Bell basis given as,

$$|\chi_m^+\rangle = \frac{|00\rangle + m|11\rangle}{\sqrt{1+|m|^2}}, \tag{4.31}$$

$$|\chi_m^-\rangle = \frac{m^*|00\rangle - |11\rangle}{\sqrt{1+|m|^2}}, \tag{4.32}$$

$$|\zeta_m^+\rangle = \frac{|01\rangle + m|10\rangle}{\sqrt{1+|m|^2}}, \tag{4.33}$$

$$|\zeta_m^-\rangle = \frac{m^*|01\rangle - |10\rangle}{\sqrt{1+|m|^2}}. \tag{4.34}$$

After the measurement, Alice informs Bob the GBS (Generalized Bell state) she gets but does not disclose m. Probabilities of getting one of these GBS as outcome are,

$$P_{\chi_m^+} = \frac{N_1^2(1+n_1^2 m^2)}{2(1+|m|^2)}, \tag{4.35}$$

$$P_{\chi_m^-} = \frac{N_1^2(m^2+n_1^2)}{2(1+|m|^2)} \tag{4.36}$$

$$P_{\zeta_m^+} = \frac{N_1^2(m^2+n_1^2)}{2(1+|m|^2)} \tag{4.37}$$

$$P_{\zeta_m^-} = \frac{N_1^2(n_1^2 m^2+1)}{2(1+|m|^2)} \tag{4.38}$$

Similar expressions would be obtained if we take into account other collapsed state characterized by N_2 and n_2. Only N_1 and n_1 would be replaced by N_2 and n_2 respectively. After knowing about the GBS from Alice, Bob applies appropriate unitaries to his qubit and projects onto the $|+\rangle$ state or $|-\rangle$ state. Next, they discuss publicly the value of n (= n_1 or n_2, value Bob uses for GBS) and m (= n_1 or n_2, value Alice uses for the GBS). They only keep those cases when m matches with n and discard other outcomes. Subsequently within these matching cases they keep those data when measurement outcome is either $|\chi_m^-\rangle$ or $|\zeta_m^+\rangle$. Half of these successful runs are used to construct the secret key and the other half runs are used to detect Eve. If Eve tries to tamper the protocol, the shared entangled state between Alice and Bob will be disturbed and Eve's presence will be detected much like in BB84 protocol. Success of this protocol is given by

$$K_{success} = \frac{1}{2}\Big(\frac{N_1^2 n_1^2}{1+n_1^2} + \frac{N_2^2 n_2^2}{1+n_2^2}\Big). \tag{4.39}$$

$\frac{1}{2}$ factor appears due to the fact that they discard half of their data to check eavesdropping. We plot success rate of the protocol with Charlie's measurement basis n: From Fig.(4.2) we can see that success rate depends on the degree of entanglement of the collapsed states (4.20), (4.21) which is determined by Charlie's choice of measurement basis. So, whole key distribution protocol is being controlled by Charlie. It is evident that non-conditional control of Charlie is not attainable unless $p = \frac{1}{2}$ which turns the state (4.14) into GHZ state. By non-conditional control we mean that Charlie should be able to find one measurement basis such that the collapsed state is separable, thus no key can be generated by Alice and Bob.

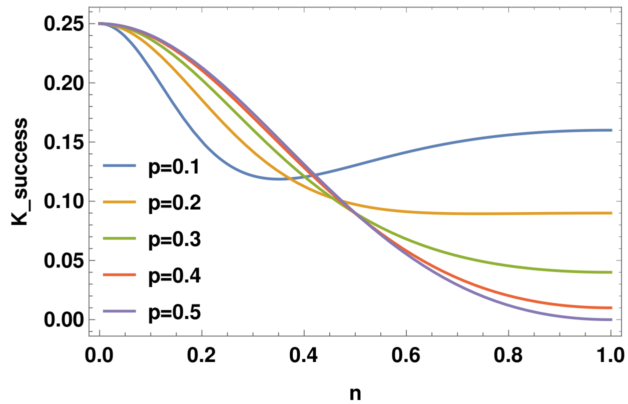

Figure 4.2: Success rate with control power of Charlie. Lower one is for GHZ state ($p = \frac{1}{2}$).

4.4 Comments on cooperative teleportation

In this section, we shall discuss usefulness of the resource states (4.14) in cooperative teleportation scheme. The protocol can be carried out in the same way as Co-QKD: Charlie makes a measurement on his subsystem and classically communicates his measurement outcome to Alice who has the unknown qubit to teleport to Bob. She then makes a joint measurement on the composite system conditioned on Charlie's measurement outcome and informs her measurement outcome to Bob who finds a suitable unitary transformation to retrieve the original state.

In tripartite scenario, the state (4.14) is suitable for the protocol. In this case depending on Charlie's choice of measurement basis $i.e.$ on $|n|$, fidelity $\mathcal{F}$ of teleportation would be determined and for $p = \frac{1}{2}$ Charlie would have full control over the protocol [105]. For a particular choice of Charlie's measurement basis, collapsed state between Alice and Bob is a separable state, resulting in no teleporatation. But in a more general scenario the collapsed state between Alice and Bob is partially entangled. Naturally telepotation fidelity is no longer unity and it depends on Charlie's choice of measurement basis. It would be interesting to find the dependence of average fidelity on the Charlie's measurement outcome which is characterized by n. Here we define average fidelity as

$$\mathcal{F}_{\text{av}} = p_+\mathcal{F}_+ + p_-\mathcal{F}_-, \tag{4.40}$$

where $\mathcal{F}_{+(-)}$ corresponds to the fidelity of $|\psi_{+(-)}\rangle_{AB}$ respectively. We calculate fidelity using the formula given by Horodecki *et al.* [49], $\mathcal{F} \leq \frac{1}{2}(1+\frac{1}{3}\mathrm{Tr}\sqrt{T^{\dagger}T})$, where the elements of the matrix T are defined as $t_{\alpha\beta} = \mathrm{Tr}\big[(\sigma_{\alpha}\otimes\sigma_{\beta})\rho\big]$ for a state ρ. Here we consider the optimal fidelity $\mathcal{F} = \frac{1}{2}(1+\frac{1}{3}\mathrm{Tr}\sqrt{T^{\dagger}T})$.

The variation of average fidelity with control parameter n has been depicted in the Fig.(4.3). It is interesting to point out that Charlie has full control of the protocol when the state is GHZ,

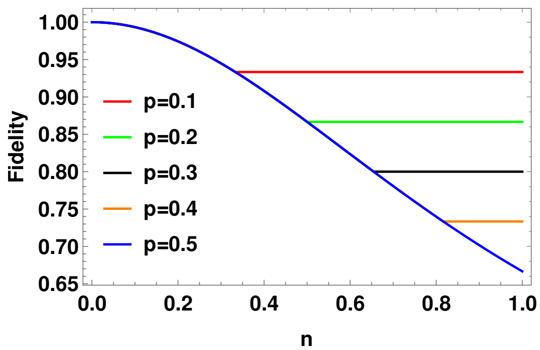

Figure 4.3: The plots show comparative behavior of average fidelity with the control parameter n.

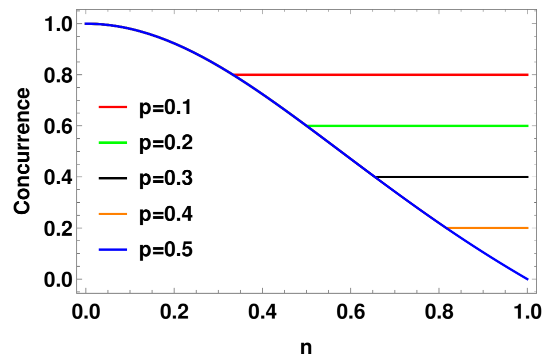

Figure 4.4: Variation of average concurrence with control parameter n.

i.e., $p = 0.5$. However, for other values of p and some range of n Charlie does not have any control as average fidelity remains constant. To describe this behavior we find concurrence of the states $|\psi_{+}\rangle_{AB}$ and $|\psi_{-}\rangle_{AB}$ and then average it to find out the average concurrence which can be expressed as

$$\mathcal{C}_{\mathrm{av}} = p_{+}\mathcal{C}_{+} + p_{-}\mathcal{C}_{-}. \tag{4.41}$$

We plot the average concurrence with control parameter n for some values of p. The nature of Fig. (4.4) is exactly same as the nature of Fig. (4.3). The reason is that for a pure state fidelity is related to the concurrence by the formula $\mathcal{F} = \frac{2}{3}(1+\mathcal{C})$. In our case average fidelity is $\mathcal{F}_{\mathrm{av}} = p_{+}\mathcal{F}_{+} + p_{-}\mathcal{F}_{-} = \frac{2}{3}(1+p_{+}\mathcal{C}_{+}+p_{-}\mathcal{C}_{-}) = \frac{2}{3}(1+\mathcal{C}_{\mathrm{av}})$.

4.5 Realization of the resource states of cooperative QIPPs

So far we have studied how to implement cooperative QKD and cooperative teleportation and we found suitable resource states for the same. These states are task oriented maximally entangled states (TMES) as introduced by the authors in [95]. For multipartite states, there is no unique notion of maximally entangled state like Bell states in two-qubit case. But we can construct a set of states which may be suitable for a particular protocol and those states that can execute the protocols maximally. We have seen apart from GHZ state, the state (4.14) is capable of performing cooperative QKD as well as cooperative teleportation. In the case of tripartite states, as shown in [95], one should be able obtain a TMES for teleportation by applying a suitable multinary transformation on the product state of a one-qubit state and a Bell state. We now present these transformations for the resource state. At the same time it would be interesting to see how these states can be realized. A single qubit unitary operation followed by a global unitary on a Bell state would suffice to produce this kind of state:

$$(U_{12} \otimes \sigma_3^0)|0\rangle_1|\phi^+\rangle_{23} = \sqrt{p}|0\rangle_1|\phi^+\rangle_{23} + \sqrt{1-p}|1\rangle_1|\phi^-\rangle_{23}, \tag{4.42}$$

where operator $U_{12} = \begin{bmatrix} \sqrt{p}\sigma^0 & \sqrt{1-p}\sigma^z \\ -\sqrt{1-p}\sigma^z & \sqrt{p}\sigma^0 \end{bmatrix}$ acts on first two qubits and σ_3^0 is σ^0 acting on the third qubit. σ^0 is the 2×2 identity matrix and σ^z is the Pauli-Z matrix.

4.6 Discussion and Conclusion

We have presented a novel protocol and found suitable resource states for the same. For a given multipartite state, it is not always obvious whether this state can be used for cooperative QKD or cooperative teleportation. In this chapter, we have constructed resource states for successful cooperative QKD and teleportation. The resource states we have discussed are exhaustive for three-qubit case. The efficiency of the protocols depends on the choice of the measurement basis. We have explicitly shown the dependence of the key rate of co-QKD protocol and fidelity of cooperative teleportation with Charlie's choice of measurement basis. Efficiency of the protocol is controlled by Charlie. If he chooses a basis set wisely then the protocol can be carried out maximally. Apart from this we have also found possible entropic structure of different subsystems of the resource states. This can be generalized to multipartite $(N > 3)$ scenario. We hope this criterion can be used for experimental observations of cooperative schemes.

Chapter 5

Conclusion

In this chapter, we shall summarize this thesis with few important remarks. In the second chapter, we have seen that the characterization of entanglement of mixed bipartite states is not as straightforward as for pure states. For pure states, Schmidt numbers are sufficient to characterize all of its nonlocal aspects. But for mixed bipartite states, we have $nine$ real parameters which specifies its quantum and classical aspects. We have studied maximal teleportation fidelity of Werner states, maximally-entangled mixed states and X-states and found comparative behavior of the fidelity with concurrence, purity and other functions of the state parameters. We have shown that merely specifying concurrence and purity would not help to explore all of a state's nonlocal features. This suggests that one needs to go beyond these two quantities (purity and concurrence) to characterize a states's nonlocal aspects. In this context, we have shown that Uhlmann fidelity may play a significant role in determining the teleportation fidelity of a general state. There may be other such functions which may be interesting to figure out in future.

As we go beyond bipartite scenario to multipartite entanglement, the theory becomes more enriched. Indeed, entanglement arising from multipartite state can be more useful. We have seen some examples of teleportation protocol with a multipartite system partitioned into two subsystems in the third chapter. We have also found suitable criterion for a multipartite state to be suitable as a resource state for teleportation. To teleport a n-qubit unknown state with m-terms, receiver subsystem's entropy must have the value $\log_2 m$. This criterion seems to be sufficient for the purpose. We have also studied QKD protocol with multipartite state for some specific states. Eventually we have found an interesting fact that sometime less entanglement is more useful for implementing QIPPs.

In the fourth chapter, we have utilized entanglement originating from tripartite states to implement cooperative QKD. This protocol can be carried out with participation of all individuals who are in possession of the qubits. We have also presented an exhaustive discussion about the resource state structure for the task in three-qubit scenario. A tripartite system would be perfect for cooperative QKD if any two subsystems have entropy one. These two subsystems can be employed to establish a secret key. This condition is also sufficient to concentrate maximal amount of entanglement between two subsystems. We have commented on the efficiency of the protocol which depends on the controlling power of the third party. It has been shown that the states useful for cooperative QKD are also suitable for cooperative teleportation. It would also be useful to extend the scope of this protocol for mixed states and continuous variable scenario. Thus our analysis may be found to be useful for further research regarding the nature of multiparticle entanglement and improvising QIPPs.

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