Some Aspects of Holography and Black Holes

By

Partha Paul

PHYS07201404015

Institute of Physics, Bhubaneswar

A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

July, 2019

Certificate will be Upload soon.

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Partha Paul

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Partha Paul

List of Publications arising from the thesis:

1. Shamik Banerjee and **Partha Paul**, *"Black Hole Singularity, Generalized (Holographic) c-Theorem and Entanglement Negativity"*, *JHEP 1702 (2017) 043*.
2. Srijit Bhattacharjee, Bidisha Chakrabarty, David D. K. Chow, **Partha Paul** and Amitabh Virmani, *"On Late Time Tails in an Extreme Reissner-Nordström Black Hole: Frequency Domain Analysis"*, *Class. Quant. Grav. 35 (2018) no.20, 205002*.
3. **Partha Paul** and Pratik Roy, *"Linearized Einstein's Equations around pure BTZ from Entanglement Thermodynamics"*, *arXiv:1803.06484[hep-th]*.

- **Talks given in Conferences**

1. **Conformal Properties of Soft Operators: Use of Null States**
   Stringy Days IV: SOFT Holography – IISER Pune, India (03/2019).
2. **Horizon Instability and Late Time Tails in an Extremal Reissner-Nordström Black Hole**
   Indian Strings Meeting – IISER, Thiruvananthapuram, India (12/2018).
3. **Generalized (Holographic) c-Theorem and Entanglement Negativity**
   School and Workshop on String Field Theory and String Phenomenology – HRI, Allahabad, India (02/2018).
4. **Linearized Einstein's Equations around pure BTZ from Entanglement Thermodynamics**
   National Strings Meeting – NISER, Bhubaneswar, India (12/2017).

Partha Paul

DEDICATIONS

To my family

ACKNOWLEDGEMENTS

I am thankful to my Ph.D supervisor Dr. Shamik Banerjee for his continuous guidance, and encouragement throughout the course of my doctoral studies. I am also thankful to Prof. Amitabh Virmani for his collaboration and continuous support during my doctoral studies. I thank my parents, my sisters and my friends for their support in my ups and downs. I would like to thank all my collaborators, working with whom have been a good learning experience for me. I wish to thank the other faculty members at IOP for useful discussions and support during my academic pursuit. I acknowledge the help of IOP community and the fellowship from the administration during the course of my Ph.D. Finally I would like to acknowledge the people of India for their continuous support for the basic sciences in India.

Contents

List of Figures

Summary

One of the important element in gauge-gravity duality is the interpretation of the extra radial direction in the bulk as the inverse energy scale of the boundary field theory. The radial evolution of the bulk fields share many features with the renormalization group (RG) flow of the boundary field theory. One of the beautiful fact about the RG flow is that one can always define a c-function which monotonically decreases along the RG flow from the UV fixed point to the IR fixed point and becomes equal to the central charges of the corresponding CFT at the fixed points. Based on the holographic dictionary people have constructed the c-function using the dual metric and proved the monotonicity property using the null energy condition. Here we have constructed such an holographic c-function using the thermodynamics of the causal horizon. We take an empty AdS_5 black brane geometry and construct the $c-$function as the Bekenstein-Hawking entropy density of the spacelike slices of the future bulk light cone of a boundary point. The UV value of the $c-$function is a_{UV} and then it decreases monotonically to zero at the curvature singularity. In field theory, logarithmic entanglement negativity is an entanglement measure for mixed states. In two dimensions we have shown that the negativity has the same UV and IR limit as our holographic $c-$ function. So it could be potential candidate for our $c-$function. We have also discussed another important application of the holography in the thesis. It is known that the linearized Einstein's equation around the pure AdS can be obtained from the constraint $\Delta S = \Delta \langle H \rangle$, known as the first law of entanglement, on the boundary CFT. The corresponding dual state in the boundary CFT is the vacuum state around which the linear perturbation is taken. We revisit this question, in the context of AdS_3/CFT_2, with the state of the boundary CFT_2 as a thermal state. The corresponding dual geometry is a planar BTZ black hole. By considering the linearized perturbation around this black brane we show that Einstein's equation follows from the first law of entanglement. In the last part of the thesis we study the stability properties of the extremal black holes. We also study the leading order late time decay tails of massless scalar perturbations outside an extreme Reissner-Nordström black hole. We first consider initial perturbations with generic regular behaviour across the horizon on characteristic surfaces and present decay results at timelike infinity, near future null infinity, and near the future horizon. Along the way, using the inversion symmetry of the extreme Reissner-Nordström spacetime, we relate the higher multipole Aretakis and Newman-Penrose constants for a massless scalar in this background.

Chapter 1

Introduction

1.1 Black Hole Thermodynamics: A Hint of Holography

The idea of holography in quantum theory of gravity has emerged from the physics of black hole thermodynamics. Bekenstein [1] in 1972 proposed that a black hole has an entropy proportional to its horizon area. Later Hawking [3] showed that the proportionality constant is actually $\frac{1}{4}$ in Planck units. The fact that entropy of a black hole is proportional to its horizon area implies that when gravitational interaction is considered there is a radical reduction in the degrees of freedom of the system. To understand this circle of ideas better, let us consider a quantum system in a three dimensional space with volume V and boundary area A. Let us discretize the space with the Planck length (l_P) as the smallest length scale. For concreteness let us assume that the quantum system under consideration consists of spin $1/2$ particles with two degrees of freedom at each lattice site with lattice spacing l_P. The total number of independent quantum states Ω, the system can have, is $2^{V/l_P^3}$. Thus the entropy is given by the Boltzman formula, $S = \ln \Omega = \frac{V}{l_P^3} \ln 2$[1]. Thus we see that the entropy, which is a count of the degrees of freedom of a local quantum system, grows with the volume. But from the spherical entropy bound [5], we know that,

[1] We have set the Boltzman constant $k_B = 1$

when gravitational interaction is taken into account, the entropy of a system in a region with boundary area A, must satisfy the inequality $S \leq \frac{A}{4}$. The bound is saturated when the region is completely occupied by a black hole. Thus as soon as we incorporate gravity into the picture we are in a conflict. The conflict can be solved in the following way. Most of the states of the quantum system in the region with volume V will have the energy so large that they will eventually form a black hole giving an entropy proportional to its horizon area. Thus there was an over-counting of the degrees of freedom for our lattice system. Hence from the above analysis we learnt that whenever gravity is strong the total number of degrees of freedom of a system in a region with a boundary is always proportional to the boundary area not the volume of the region. Based on these observations 't Hooft [6] and Susskind [5] proposed that quantum theory gravity should be holographic, i.e., the physics of a gravitational system in some region can be described by a theory living on the boundary of that region with no more than one degree of freedom per Planck area. The Anti-de Sitter-space/Conformal Field Theory (AdS/CFT) [109] correspondence is the concrete realization of this holographic principle.

1.2 AdS/CFT Correspondence

Let us start by describing Anti de-Sitter (AdS) space-time geometrically. AdS spacetime is a maximally symmetric solution of Einstein's equation with negative cosmological constant. The isometry group of AdS_{d+1} is $SO(2,d)$ which is also the symmetry group of any d-dimensional conformal field theory. In the Poincare coordinates the metric of AdS_{d+1} can be written as,

$$ds^2 = L_{\mathrm{AdS}}^2 \frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2}, \quad 0 < z < \infty \tag{1.1}$$

where L_{AdS} is the AdS-radius. The (conformal) boundary of AdS is located at $z = 0$ and the boundary coordinates are given by (t, x_i). The coordinate z is also known as the radial coordinate of AdS. In the Poincare coordinates the induced metric on the boundary $z = 0$ is given by the Minkowski metric $\eta_{\mu\nu} = diag(- + + ...+)$. Let us now state the AdS/CFT correspondence.

According to the AdS/CFT correspondence or the gauge - gravity duality, any quantum theory of gravity on AdS_{d+1} is holographically dual to a d-dimensional conformal quantum field theory (CFT) living on the (conformal) boundary of AdS_{d+1}. One of the most well studied examples of AdS/CFT correspondence is between the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions and type-IIB string theory on $\text{AdS}_5 \times S^5$ with Ramond-Ramond self-dual five form flux. In the 't hooft large-N_c and large $\lambda = g_{YM}^2 N_c$ limit [2], the correspondence simplifies drastically and the bulk dual theory is effectively given by type-IIB supergravity. This is called the (Einstein) gravity limit and all the results of this thesis are obtained in this limit. Now we would like to emphasise that the existence of the gravity limit does not necessarily mean that the dual CFT is a gauge theory. For example, in the $\text{AdS}_3/\text{CFT}_2$ correspondence the bulk dual theory reduces to classical gravity when the central charge of the dual CFT_2 is large [3]. In the rest of the thesis we will assume that in the boundary field theory such a limit has already been taken so that the bulk theory is given by classical Einstein gravity.

1.3 Holographic Renormalization Group Flow

An exact equivalence between two theories implies that one should be able to extract the information of the one theory from the other. A precise recipe was given to compute the correlation functions of the large N conformal field theory from the classical supergravity on $AdS_5 \times S^5$ [110]. One of the interesting features of the gauge/gravity duality is the ultraviolet/infrared (UV/IR) connection. In the quantum field theory the correlation func-

[2] Here N_c is the rank of the gauge group $SU(N_c)$ and $\lambda = g_{YM}^2 N_c$ is the 't hooft coupling.

[3] The central charge of the dual theory is given by $c = \frac{3L_{\text{AdS}}}{2l_{pl}}$ where l_{pl} is the Planck length in the bulk. Therefore $c >> 1$ implies the classical limit $L_{\text{AdS}} >> l_{pl}$.

tions suffer from UV divergences. The UV/IR connection of the gauge/gravity duality means that the UV divergences of the field theory are related to IR divergences of the gravity theory. On the gravity side the IR divergences arise due to the infinite volume of the AdS space-time. One of the nice things about AdS gravity is that the IR divergences can be regulated by adding counterterms to the bulk action which are all localized near the boundary of AdS and also local in the boundary direction. Hence the IR counterterms can also be thought of as the countertems in the dual field theory which regulate the UV divergence. The method of holographic renormalization is how to renormalize the field theory correlation functions by dealing with the IR divergences of the gravity theory. Although the full correlation function depends on the dynamics of the boundary field theory, but renormalization of the UV divergences does not depend on the IR physics. In the gravity side the corresponding statement is that only near boundary analysis is enough to control the IR divergences.

In order to have an intuitive picture of the holographic RG flow it will be useful to have the Wilsonian picture in mind. In the Wilsonian picture one starts with a Lagrangian whose parameters are defined at some high energy scale Λ which is assumed to be much larger than the energy scale and masses of our interest. Now as we integrate over the high energy degrees of freedom, new interactions appear which renormalize the bare values of the parameters in the Lagrangian. This gives rise to a flow in the space of all possible couplings and this is known as the renormalization group flow. Conformal field theories appear as the fixed points of the renormalization group flow. Now suppose we start from a CFT defined at the UV and deform it by adding various operators to the Lagrangian. The operators whose coupling constants grow as we move towards the IR are called relevant. If the coupling constant decreases then the operator is called irrelevant and the rest of the operators are called marginal. Now suppose some of the operators in the UV are relevant and as a result we reach a fixed point in the IR described by a new CFT. In this context, one of the most beautiful results of quantum field theory is the Zamolodchikov c-theorem in $(1 + 1)$ dimensions [7]. This theorem states that along the RG flow one can define a function which decreases monotonically from the UV to the IR and whose values at the fixed points are given by the central charges of the UV and the IR CFTs. Since then there are many attempts to extend the c-theorem in higher dimensions. This resulted in F-theorem [13] in three dimensions and a-theorem [9] in four dimensions.

1.3.1 Gravity Description of Field Theory RG Flow

The holographic description of the RG flow of boundary field theories has been extensively studied in the literature [20–25, 29–32]. In this subsection we will briefly review the basic ideas [33]. Let us take a $d+1$-dimensional Einstein's gravity coupled to a bulk scalar field ϕ. The potential $V(\phi)$ is chosen to have one or more critical points, i.e., $\frac{\partial V(\phi)}{\partial\phi}\Big|_{\phi=\phi^i_{\text{crit.}}} = 0$. At each critical point the solution of our model becomes AdS_{d+1} geometry of radius L^i_{crit} with constant scalar field $\phi^i_{\text{crit.}}$. Away from the critical points we need more general solutions with $d-$dimensional Poincare isometry as the dual field theory has Poincare symmetry in $d-$dimensions. The most general ansatz is

$$\begin{aligned} ds^2 &= e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 \\ \phi &= \phi(r) \end{aligned} \tag{1.2}$$

This is known as domain wall solution [33]. r represents the radial coordinate and x^μ's are the d transverse coordinates. By manipulating Einstein's equation one can get the following equation for the scale factor $A(r)$:

$$A''(r) = \frac{1}{d-1}\left(T^t{}_t - T^r{}_r\right) = -\frac{1}{d-1}\left(\phi'(r)\right)^2 \tag{1.3}$$

where $'$ denotes derivative with respect to r. Thus for our system we have $A''(r) \le 0$. At each critical point ϕ becomes trivial and we have $A(r) = \frac{r}{L^i_{\text{crit}}} + a_0$. The integration constant a_0 can be absorbed by rescaling x^μ. If we change the radial coordinate as $z = L^i_{\text{crit}} e^{-\frac{r}{L^i_{\text{crit}}}}$ we get equation (1.1). Thus AdS_{d+1} geometries are the critical solutions of our gravity-scalar field system.

Near the maximum of the potential the mass of the scalar fluctuation satisfies $m^2 < 0$. From the AdS/CFT correspondence we know that the scalar fluctuation in the bulk is dual to a scalar operator of the boundary CFT with scaling dimension $\Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right)$. Now from the boundary asymptotics ($r \to \infty$ behavior) of the scalar fluctuation and using the AdS/CFT dictionary one can show that there is a term in the boundary QFT Lagrangian which is proportional to $\int d^d\vec{x}\, O_\Delta(\vec{x})\, \bar{\phi}$, where $\bar{\phi}$ is the boundary value of the domain wall scalar field. Thus the boundary CFT is deformed by a scalar operator of dimension Δ with coupling $\bar{\phi}$. If $0 > m^2 > -\frac{d^2}{4}$, then $d > \Delta > \frac{d}{2}$, and the boundary field theory is described by a **relevant deformation** of UV CFT. Near the minimum of the potential we have $m^2 > 0$ so $\Delta > d$. This critical point can be reached for large negative values of r and the field theory is described by an **irrelavent deformation** of IR CFT. Thus we see that domain wall geometries interpolate between two critical points, i.e., they approach the boundary region of an AdS space with scale L_{UV} as $r \to \infty$ and the deep interior of another AdS with scale $L_{\rm IR}$ as $r \to -\infty$. Such geometries are dual to field theories with RG flow with the AdS$_{UV}$ and AdS$_{IR}$ being dual to the UV and IR CFTs. Now one can define [24]

$$a(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2)\,(l_P A'(r))^{d-1}} \tag{1.4}$$

$$a'(r) = -\frac{(d-1)\pi^{d/2}}{\Gamma(d/2) l_P^{d-1} A'(r)^d} A''(r) \geq 0 \tag{1.5}$$

where in the second line we have used equation (1.3). In the more general case when gravity is coupled to more general matter fields using the null energy energy condition $T^t{}_t - T^r{}_r \leq 0$ one can show that $A''(r) \leq 0$. Thus for the domain wall geometry interpolating between two AdS spaces with radius of curvatures L_{UV} and L_{IR}, a monotonically decreasing $c-$function $a(r)$ can be constructed holographically [20–25] using the metric components. For even d, a_{UV} and a_{IR} indeed match with the A-type trace anomaly of the dual CFT stress tensor in a curved background.

There is an alternative way to construct the holographic $c-$function using the causal horizon of the domain wall geometry [27, 28]. A causal horizon is the boundary of the past(future) of any timelike curve of infinite proper length in the future(past) direction. The second law of causal horizon thermodynamics states that the area of (past)future causal horizon does not (increase)decrease [34, 35]. Using this monotonicity property one can construct a holographic $c-$function [27, 28] as the Bekenstein-Hawking entropy density associated with the causal horizon. This concept also generalizes the notion of the Zamolodchikov's $c-$theorem. This will be discussed in great detail in the present thesis.

1.4 Information Theory and Holography

Quantum information theory has been playing a very important role in understanding the fundamental issues of quantum field theory as well as quantum gravity. The different measures of the quantum entanglement, e.g. entanglement entropy, relative entropy etc, have seen wide range of applications in both of these fields. For example the entanglement entropy has been used extensively to prove the higher dimensional $c-$theorem [19, 38]. Recently they have also found importance in the context of holographic duality. For example the holographic formula for entanglement entropy proposed by Ryu and Takayanagi [39], has been used to prove the holographic c-theorem [24]. Thus the holographic $c-$function constructed from the metric components of the dual geometry has an interpretation in terms of the entanglement entropy of a subsystem of the boundary field theory. We will show in this thesis that when the holographic CFT is in a thermal state the holographic c-function [26] that we have constructed using the causal horizon thermodynamics can not be interpreted in terms of the entanglement entropy of the boundary field theory. Our holographic $c-$function has the finite UV limit proportional to the central charge of the UV CFT, then it monotonically decreases and goes to 0 in the deep IR [26]. The entanglement entropy of the boundary field theory has the finite UV limit which matches with our holographic $c-$function but it does not go to 0 in the IR limit [38]. The reason being that when the temperature of the field theory is high, entanglement entropy mixes the quantum entanglement with the classical counterpart [38]. Thus it fails to capture the pure quantum correlation between the two subsystems at finite temperature. Being unable to identify our holographic $c-$function with the entanglement entropy of the boundary field theory we will move to another entanglement measure known as the logarithmic entanglement negativity which is used to measure the entanglement between two subsystems in a mixed state [43–45,51–55]. Let us consider a two dimensional CFT living on an infinite line at finite temperature $T = \beta^{-1}$. The entanglement negativity for a single interval of length L is given by [52]

$$E = \frac{c}{2}\ln\Big[\frac{\beta}{\pi a}\sinh\Big(\frac{\pi L}{\beta}\Big)\Big] - \frac{\pi c L}{2\beta} + f(e^{-\frac{2\pi L}{\beta}}) + 2\ln c_{\frac{1}{2}} \tag{1.6}$$

where a is the short distance cutoff, c is the central charge of the CFT and $c_{\frac{1}{2}}$ is a constant. $f(x)$ is a universal scaling function which depends on the full operator content of the CFT such that $f(1) = 0$ and $f(0) =$ constant. Now one can define the renormalized entanglement negativity as, $E_R = L\frac{d}{dL}\Big|_{\beta}E$. Using (1.6) we can see that the UV($\beta >> L$) and IR($a << \beta << L$) limits of E_R are given by $\frac{c}{2}$ and 0 respectively. In four dimensions we expect the same things to happen in the UV limit as the structure of the UV divergences of the negativity is the same as that of entanglement entropy in the same dimension [54]. In the IR limit, we can expect the negativity to go to 0 given that there is a finite correlation length of order β. Thus if it satisfies the monotonicity condition it could be a potential candidate for our holographic $c-$function. We discuss these in detail in section 2.4.1.

There is another important application of the entanglement entropy in the context of AdS/CFT correspondence. For a linearized perturbation around a fixed state of a quantum field theory the entanglement entropy, S of a spatial region satisfies the relation

$$\Delta S = \Delta\langle H\rangle \tag{1.7}$$

known as the first law of entanglement [112–114], where H is the modular Hamiltonian associated with that region. For a holographic CFT, the entanglement entropy (S) associated with a spatial region (A) can be computed using the Ryu-Takayanagi formula [39],

$$S_A = \frac{\text{Length}(\gamma_A)}{4G_N}, \tag{1.8}$$

where γ_A is a space-like geodesic in the bulk homologous to A, and G_N is Newton's constant. When the spacetime is not static, one needs to use its covariant generalization [40]. The modular Hamiltonian (H_A) is defined in terms of the reduced density matrix σ_A associated with a spatial region A as, $\sigma_A = \frac{e^{-\beta H}}{\text{Tr}\left(e^{-\beta H}\right)}$, where β is the inverse of the temperature. Modular Hamiltonian in general is a non-local operator and thus hard to compute. However there are certain special cases where it can be expressed as the integral of some local operator. For example, let us consider the ground state of a CFT living in a $d+1$ dimensional Minkowski spacetime. In this case, the modular Hamiltonian associated with a ball-shaped region A of radius R can be written as [129]

$$H_A = 2\pi \int_A d^d x \, \frac{R^2 - r^2}{2R} \, T^{00}(x) \tag{1.9}$$

where $T^{00}(x)$ is the time-time component of the stress energy tensor and r is the radial coordinate from the center of the ball shaped region A. The boundary stress tensor can be found from the asymptotic form of the asymptotically AdS (AAdS) bulk metric [130–132]. Now if we perturb the initial state of a holographic CFT and the dual geometry corresponding to the perturbed state is an AAdS geometry then both sides of (1.7) can be computed using the holographic dictionary. Considering perturbations around pure AdS and using the holographic tool it was shown [112–114] that to linear order in the perturbation the first law of entanglement is satisfied. In [112, 113] it was also shown that Einstein's equation linearized around pure AdS do follow from the first law of entanglement. In chapter 3 of this thesis we will extend this result. We take the thermal state of a holographic CFT as the fixed reference state in 1 + 1 dimensions and perturb it infinitesimally. We will show that for metric components of the dual geometry satisfying the linearized Einstein's equations, the first law of entanglement holds. Then we go the other direction, i.e, we show that the first law of entanglement fixes the metric uniquely if we demand that it holds in all frames of reference [115].

1.5 Extremal Black Holes

Black holes are the solutions of the Einstein's equation of general relativity with the gravitational attraction so large that even light can't escape from it. Black holes are characterized by their macroscopic parameters like mass (M), charge (Q) and angular momentum (J). Classically a black hole is surrounded by an event horizon, nothing can go to the outside of the evnt horizon. Quantum mechanically this picture gets modified and it was shown that black holes do radiate with a definite temperature and entropy [3, 4]. Let us consider a non-rotating black hole with mass M and charge Q. To get a physically sensible definition of temperature and entropy for this black hole we must have $M^2 \geq Q^2$. This bound is known as the BPS bound. When this bound is saturated, i.e., $M = |Q|$ the temperature of the black hole vanishes and thus emits no Hawking radiation. These black holes are called extremal black holes. For rotating extremal black holes with angular momentum J we also have $M = \sqrt{|J|}$. Extremal black holes play an important role in the literature of string theory, e.g., supersymmetric extremal black holes have been used to give a microscopic explanation [145] of the Bekenstein-Hawking entropy formula. Thus it is important to ask whether they are stable classically.

1.6 Are Extremal Black Holes Stable?

The fact that supersymmetric extremal black holes saturate the BPS bound and minimizes the energy does not imply their classical stability. For example, the classical stability of

Minkowski spacetime does not follow from the positive energy theorem. Thus to study the stability problem of extremal black holes one first needs to understand the problem of dynamics for Einstein's equation which is formulated as Cauchy problem. Formulating the Cauchy problem is not an easy task in general theory of relativity as the notion of global hyperbolicity is not clear. Choquet-Bruhat [146] and Choquet-Bruhat-Geroch [147] first formulated the well-posed Cauchy problem for the vacuum Einstein's equation. Using the language provided by them one can then study the dynamical stability problem for any spacetime, i.e. the solution of Einstein's equation. We are interested in the stability analysis of extremal black holes. So the question that we are going to ask is whether a small perturbation changes the parameters of an extremal black hole by a small amount such that the modified parameters still belong to the same family of black holes. All the extremal black holes against all kind of perturbations are unstable in this sense [160, 164–166]. We will only study the scalar perturbation on an extremal Reissner-Nordström (RN) black hole [163] in this thesis.

An Outline of the Thesis

In chapter 2 of the thesis we will study various aspects of the holographic RG flow in the holographic set up and relate it to the quantum entanglement in the boundary theory. We take an empty AdS_5 black brane and construct a c-function using the causal horizon thermodynamics of the black brane and generalized the notion of the c-theorem. We will discuss the renormalized entanglement negativity (REN), which is a measure for the entanglement of the mixed state of the boundary CFT and show that it shares similar properties with our generalized c-function in the UV and IR limit.

In chapter 3 We will discuss using the holographic tool how the quantum entanglement in the boundary theory imposes constraint on the bulk geometry. we take the thermal state of a holographic CFT in $1 + 1$ dimensions as the fixed reference state and perturb it infinitesimally. The change in the entanglement entropy and the modular Hamiltonian of a spatial region will satisfy the first law of entanglement. Based on the holographic dictionary we then compute each side of this relation using metric components of the dual geometry. We show that for metric components of the dual geometry satisfying the linearized Einstein's equations, the first law of entanglement holds. Then we go the other direction, i.e, we show that the first law of entanglement fixes the metric uniquely if we demand that it holds in all frames of reference. We end this chapter by discussing possible future directions.

In chapter 4 of the thesis we will study the stability analysis of extremal black holes. First we will present a detail introduction on late time perturbations of black holes and Aretakis instability for the extremal black holes. Then we reproduce and extend some of the previous results on late time perturbations of extreme RN black hole, including the exact coefficient, using rather simple Fourier methods. Along the way, using the Couch-Torrence symmetry, we also relate higher multipole Aretakis and Newman-Penrose constants for a massless scalar in an extreme RN black hole background. We conclude this chapter with a summary and possible future directions.

Chapter 2

Generalized Holographic *c*-Theorem and Entanglement Negativity

2.1 Introduction

In AdS-CFT black brane is a thermal state of the boundary conformal field theory living on the Minkowski space-time. This is not a relevant deformation of the CFT Hamiltonian and there is no renormalization group flow in the ordinary sense. Therefore the question of the existence of a c-function, in the sense of Zamolodchikov [7–9], does not naturally arise in this situation. Moreover Zamolodchikov c-function is constant at a fixed point and independent of the state of the CFT. The purpose of this chapter is to point out that AdS-CFT duality and the thermodynamic nature of classical gravity allows us to introduce a generalized notion of c-function, at least for large-N theories with classical gravity dual. This generalized c-function cannot be interpreted as an off-shell central charge. Rather it can be interpreted as a measure of quantum entanglement that exists at different energy scales in the given state. We will construct this c-function holographically when the CFT is in thermal state and the gravity dual is an empty black brane geometry. We focus on four dimensional field theories only. Our choice of the thermal state is motivated by the fact that the gravity dual has a curvature singularity and the Lorentz invariance is broken everywhere except near the UV boundary of AdS. So it can teach us some lessons about RG-flow interpretation of more general geometries.

Throughout this chapter we will assume that the bulk theory is Einstein gravity coupled minimally to a set of matter fields.

2.2 Holographic View

The holographic picture is based on the fact that the gravity dual of c-theorem is the second law of causal horizon thermodynamics in asymptotically AdS spaces [27, 28]. In a nutshell, second law for causal horizons say that if we consider the future bulk light-cone of a boundary point then the expansion of the null geodesic generators of the light-cone is negative [34, 35]. Now one can assign Bekenstein-Hawking entropy to the causal horizon. The fact that the expansion is negative then implies that as we move away from the boundary the entropy density decreases monotonically. This is essentially holographic c-theorem [20–25] if we specialize to a domain-wall geometry. The bulk future light-cone interpolates between the UV-AdS and the IR-AdS and the monotonically decreasing Bekenstein-Hawking entropy density gives the holographic c-function [27, 28].

If we focus on domain-wall geometry then the second law has the interpretation of holographic c-theorem. But what about other asymptotically AdS (AAdS) geometries ? Second law of causal horizon thermodynamics holds in any AAdS geometry and in fact holographic RG [23] applies to any such setup. It has been argued that the holographic RG in the bulk is dual to the Wilsonian RG in the boundary [29–32]. So is it possible to associate a notion of irreversibility to any classical AAdS geometry ? It seems that the existence of second law for classical gravity allows us to do precisely this thing. In the field theory side its interpretation will require us to generalize the concept of Zamolodchikov-type c-function. We will have almost nothing to say on it in this chapter.

To gain some experience with such generalized c-functions we will work out a reasonably simple but interesting example of empty black brane geometry in AdS_5 [1]. It has curvature singularity hidden behind the black brane horizon and the geometry is not Lorentz invariant except near the AdS_5 boundary. The c-function we construct is just the Bekenstein-Hawking entropy density of a causal horizon in the black brane geometry [27, 28]. The causal horizon originates at some point of the AdS boundary and terminates at the curvature singularity. Nothing depends on the choice of the boundary point where the causal horizon originates because of space-time translation invariance. Second law guarantees that our function monotonically decreases as we move away from the boundary along the null geodesic generators of the causal horizon. We will see that the c-function monotonically decreases from a_{UV} to zero at the curvature singularity.

2.3 Calculation and Results

The causal horizon is just the future bulk light-cone of a boundary point Figure 2.1. We take the boundary point, p, to have coordinates, $x^\mu = z = 0$.

The metric of the five-dimensional black brane is,

$$ds^2 = \frac{1}{z^2}\Big[-(1-z^4)dt^2 + \frac{dz^2}{1-z^4} + d\vec{x}^2\Big] \tag{2.1}$$

Here we have set the AdS radius to 1. Our job is to construct the ingoing null geodesics in this geometry which originate from the boundary point p.

Let us define the ingoing Eddington-Finkelstein coordinate as, $v = t + z^*$ where $z^* = \frac{1}{2}\tan^{-1} z + \frac{1}{4}\log\left(\frac{1+z}{1-z}\right)$, so that the metric takes the form

[1]Construction of holographic c-function by viewing the black brane background as RG flow, was also considered in [63–65]. c-function for attractor flows were considered in [86].

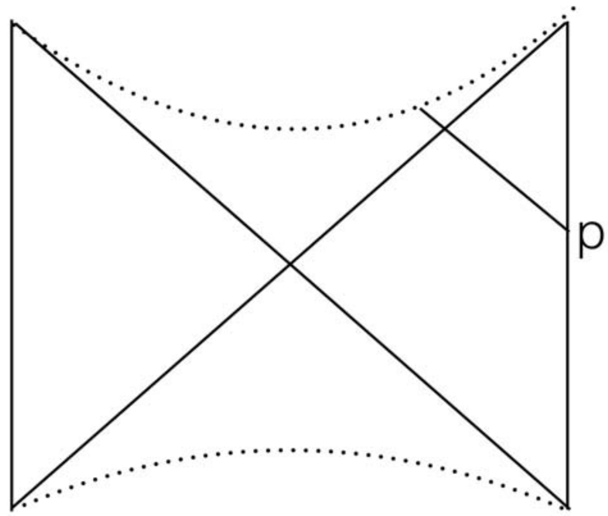

Figure 2.1: Penrose diagram of the maximally extended AdS_5 black brane [67]. We have shown only the radial null geodesic coming out from the boundary point p. Other non-radial null geodesics from p are not shown here. In this chapter we do not need the setup of the two sided black brane. We have drawn it for the sake of completeness.

$$ds^2 = \frac{1}{z^2}\Big[-(1-z^4)dv^2 - 2dvdz + d\vec{x}^2\Big] \tag{2.2}$$

There is no singularity at the horizon, $z = 1$, and so we can follow the null geodesics all the way to the curvature singularity at, $z = \infty$. Since we want to find out the null geodesics we can as well work with the conformally transformed metric [2] given by,

$$d\tilde{s}^2 = -(1-z^4)dv^2 - 2dvdz + d\vec{x}^2 \tag{2.3}$$

Let λ denote the affine parameter along a null geodesic in the conformally transformed

[2]Null geodesics are invariant under conformal transformation of the metric. In other words if g and g' are two conformally related metrics then the null geodesics of g and g' are the same. What changes under the conformal transformation is the parametrization of a specific null geodesic. For example if $x^\mu(\alpha)$ is an affinely parametrized null geodesic in metric g, then $x^\mu(\alpha)$ is also a null geodesic for the conformally related metric g', but α is not necessarily an affine parameter in the new metric g'. For a detailed derivation and explanation of this fact please see [89].

metric $d\tilde{s}^2$. [3]

So we have,

$$\tilde{g}_{AB}\frac{dx^A}{d\lambda}\frac{dx^B}{d\lambda} = 0 \tag{2.4}$$

We also have four conserved charges corresponding to the translations in v and the x^i's.

$$\begin{aligned} -(1-z^4)\frac{dv}{d\lambda} &= -E + \frac{dz}{d\lambda}, \\ \frac{dx^i}{d\lambda} &= -p^i, \end{aligned} \tag{2.5}$$

where $i = 1, 2, 3$. E and $\vec{p}$ are the conserved charges along a null geodesic. Here we are assuming that the affine parameter λ increases as we move away from the boundary at $z = 0$. We will be working with the future bulk light-cone of the boundary point p and so with our convention for the affine parameter, $\frac{dt}{d\lambda} \geq 0$. So $E \geq 0$.

Now using (2.4) and (2.5) we get,

$$\left(\frac{dz}{d\lambda}\right)^2 = E^2 - p^2(1 - z^4) \tag{2.6}$$

So we can see that the null geodesics which can reach the boundary point must satisfy the constraint, $E^2 - p^2 \geq 0$. This constraint together with the constraint $E \geq 0$, allow us to parametrize the conserved charges as,

[3]We could not solve the equation for $z(\lambda)$ by using the affine parameter corresponding to the original geometry. But this does not affect the physics. This is just a change in scheme. It will of course be better to solve this in terms of the original affine parameter.

$$
\begin{aligned}
E &= \alpha \cosh \eta \\
p^i &= \alpha \sinh \eta \, \hat{n}^i
\end{aligned}
\tag{2.7}
$$

where $\alpha > 0$, $0 \leq \eta \leq \infty$ and $\hat{n}$ is a unit vector in R^3. Now it is easy to see that α is redundant because it can be absorbed by an affine reparametrization, $\lambda \to \alpha\lambda$. Therefore we will set $\alpha = 1$.

So the equation for z simplifies to,

$$
\frac{dz}{d\lambda} = \sqrt{1 + z^4 \sinh^4 \eta} \tag{2.8}
$$

We have chosen the positive root because our convention is $\frac{dz}{d\lambda} \geq 0$. So we can write,

$$
\lambda = \int_0^z \frac{dz'}{\sqrt{1 + z'^4 \sinh^2 \eta}} \tag{2.9}
$$

where the boundary condition, $z(0) = 0$ has been imposed.

The solution of this equation is, [4]

$$
z^2(\lambda) = \frac{1}{\sinh \eta} \frac{1 - cn(2\lambda \sqrt{\sinh \eta}, 1/\sqrt{2})}{1 + cn(2\lambda \sqrt{\sinh \eta}, 1/\sqrt{2})} \tag{2.10}
$$

where cn is one of the Jacobian elliptic functions. Its properties are well studied although a closed form expression in terms of elementary functions does not exist.

Given the solution for $z(\lambda)$ we can in principle determine $v(\lambda)$ from (2.5), but we were unable to do so in any convenient way. In any case the complete set of solutions can be

[4] We are using the convention of Gradshteyn and Ryzhik. In Mathematica, $\frac{1}{\sqrt{2}}$ in the argument of cn should be replaced by $\frac{1}{2}$.

written as,

$$
\begin{aligned}
z(\lambda,\eta) &= \sqrt{\frac{1}{\sinh\eta}\frac{1 - cn(2\lambda\sqrt{\sinh\eta}, 1/\sqrt{2})}{1 + cn(2\lambda\sqrt{\sinh\eta}, 1/\sqrt{2})}} \\
v(\lambda,\eta) &= \int_0^{\lambda} d\lambda' F(\lambda',\eta) \\
x^i(\lambda,\eta,\hat{n}^i) &= -\lambda \sinh\eta\, \hat{n}^i
\end{aligned}
\tag{2.11}
$$

where we have defined,

$$
F(\lambda,\eta) = \sinh^2\eta \frac{(1+cn)^2\cosh\eta - \sqrt{2}\sqrt{1+cn^2}(1+cn)}{(1+cn)^2\sinh^2\eta - (1-cn)^2} \tag{2.12}
$$

and $cn \equiv cn(2\lambda\sqrt{\sinh\eta}, \frac{1}{\sqrt{2}})$. We have imposed boundary conditions such that, $z(0,\eta) = v(0,\eta) = x^i(0,\eta,\hat{n}^i) = 0$ for all values of η and $\hat{n}^i$. This corresponds to the fact that the null geodesics are all coming out of the point p with coordinates $x^i = v = z = 0$. Note that $v = t$ at the boundary $z = 0$.

For any fixed values of η and n^i, the above equation (2.11) reduces to the equation of the null geodesic parametrised by the affine parameter λ and coming out of the fixed boundary point $p(x^i = t = z = 0)$. As we vary η and n^i, we scan over all the geodesics coming out of the point p. All these null geodesics form a null hyper surface whose parametric equation is given by (2.11). The intrinsic coordinates on the null hyper surface are $(\lambda,\eta,\hat{n}^i)$. $(\eta,\hat{n}^i)$ are comoving coordinates along a null geodesic parametrised by λ. This null hyper surface is the sought for bulk future light-cone or the past causal horizon of the point p.

Our next job is to find out the induced metric on the null-hypersurface (2.11). To find out the induced metric we have to use the original black brane metric (2.1). Using this we get,

$$ds^2_{ind} = \frac{1}{z^2}\Big[-(1-z^4)\Big(\frac{\partial v}{\partial \eta}\Big)^2 - 2\frac{\partial v}{\partial \eta}\frac{\partial z}{\partial \eta} + \lambda^2 \cosh^2\eta\Big] d\eta^2 + \frac{1}{z^2}\lambda^2 \sinh^2\eta d\Omega_2^2 \tag{2.13}$$

where $d\Omega_2^2$ is the metric of a unit two-sphere parametrised by $\hat{n}^i$. The induced metric is degenerate as it should be because (2.11) is a null-hypersurface. (2.13) is the metric on a λ = constant space-like slice of the causal horizon (2.11), parametrised by the coordinates $(\eta, \hat{n}^i)$.

The volume form can be written as,

$$dV_{ind} = c(\lambda, \eta)\, dV_{H^3} \tag{2.14}$$

where we have defined,

$$c(\lambda, \eta) = \frac{\lambda^2}{z^3}\sqrt{\Big[-(1-z^4)\Big(\frac{\partial v}{\partial \eta}\Big)^2 - 2\frac{\partial v}{\partial \eta}\frac{\partial z}{\partial \eta} + \lambda^2 \cosh^2\eta\Big]} \tag{2.15}$$

dV_{H^3} is the volume form on a unit three dimensional hyperbolic space given by,

$$\begin{aligned} ds^2_{H^3} &= d\eta^2 + \sinh^2\eta d\Omega_2^2 \\ dV_{H^3} &= \sinh^2\eta \sin\theta d\eta d\theta d\phi \end{aligned} \tag{2.16}$$

where we have parametrised $\hat{n}^i$ as $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The fact that c is a function only of λ and η is a consequence of the rotational symmetry of the metric. In the more standard domain-wall geometry c is function only of λ because of the Lorentz invariance of the metric. In the black brane geometry Lorentz invariance is broken down to the spatial rotation group and so the η dependence is non-trivial.

Now second law for causal horizons is the statement that,

$$\boxed{\frac{\partial}{\partial\lambda}\Big|_{\eta} c(\lambda,\eta) \leq 0} \tag{2.17}$$

Here we have used the fact that dV_{H^3} is a comoving volume element and η is a comoving coordinate i.e, η is constant along a null geodesic generator of the causal horizon.

The Bekenstein-Hawking entropy density associated to the volume element dV_{ind} is,

$$dS_{BH} = \frac{dV_{ind}}{4G_N} = \frac{c(\lambda,\eta)}{4G_N} dV_{H^3} \tag{2.18}$$

We can put in the AdS radius L by replacing $dV_{ind} \to L^3 dV_{ind}$. This gives,

$$dS_{BH} = \frac{dV_{ind}}{4G_N} = \frac{L^3}{4G_N} c(\lambda,\eta) dV_{H^3} \tag{2.19}$$

So our c-function is,

$$\boxed{c_{\eta}(\lambda) = \frac{L^3}{4G_N} c(\lambda,\eta)} \tag{2.20}$$

We get a family of c-functions parametrised by η Figure 2.2. We check in the appendix using perturbation theory for small λ that $c(\lambda,\eta) \to 1$ as $\lambda \to 0$ for **all** values of η i.e, $c(0,\eta) = 1$. It will be true for any AAdS geometry, not just the black brane. Note that $\lambda = 0$ is the AdS boundary and λ increases as we move away from the boundary along the null geodesics. So for any fixed value of η the c-function $c_{\eta}(\lambda)$ starts at the UV value a_{UV} and decreases monotonically as a result of the second law (2.17). It turns out that in the case of the black brane the c-function becomes zero at the curvature singularity for **all** values of η. So for black brane in five dimensions, the c-function monotonically decreases from the UV central charge to zero at the curvature singularity. It does not show any characteristic behavior while crossing the black brane horizon.

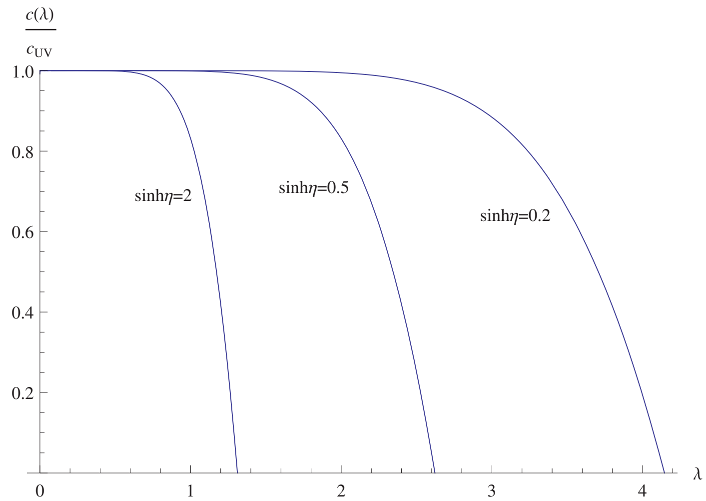

Figure 2.2: We have plotted the c-function for three different values of η. All of them start at the UV value $a_{UV}(= c_{UV})$ and monotonically decreases to zero at the curvature singularity. The values of λ at the singularity for different values of η can be obtained from (2.9) by setting $z = \infty$. Of course from a physical point of view going to the singularity with GR is meaningless. But if we forget about any stringy physics for the time being, then as a classical theory GR holds everywhere except at the singularity.

We would like to emphasise that the fact that we have obtained a family of c-functions parametrized by η, instead of just one, is no cause for concern. c-function is not unique. For example in two dimensions one can construct the standard Zamolodchikov c-function [7] and also the entanglement entropy c-function due to Casini and Huerta [9]. It is known that they are not the same, but they both monotonically interpolate between the UV and the IR central charges. In fact if we can construct one c-function then we can construct an infinite family all of which contain the same physical information [8].

The plot of the c-function in Figure 2.2 shows that it is not stationary at the singularity. This is not a problem because strictly speaking the function is not analytic there. We do not know how to extend the function beyond the singularity. But the fact that it is zero at the singularity shows that the flow comes to an end at the singularity. The c-function is an element of area of the causal horizon and so it is positive semidefinite by construction. So the flow saturates the lower bound at the singularity. This is similar to what sometimes happens in case of the c-function constructed out of entanglement entropy. For example in three dimensions, the entropic c-function for a massive scalar is not stationary at the UV fixed point [58]. This is attributed to the fact that a scalar field with negative mass squared is pathological and the entropic c-function knows about that [62]. Also another is that in our case the geometry is not Lorentz-invariant anywhere except near the boundary and so our standard intuition about c-function may need some modification.

Before we conclude we would like to mention an important point. In Einstein gravity one cannot really distinguish between the a and c central charges. In order to do that one has to include higher-derivative terms in the bulk gravity action. In the presence of higher derivative terms instead of Bekenstein-Hawking entropy we have to use the entropy expression which satisfies the second law in the bulk and reduces to the Wald entropy when evaluated on a Killing horizon [89]. If we do this we will recover the a-charge at the asymptotic UV boundary as was shown in [28]. That means the a-function will start decreasing from a_{UV}. The important point is the fate of this a-charge in the deep IR i.e, when the causal horizon reaches the singularity. We expect it to go to zero because the thermal state has a finite correlation length even in the presence of the higher-derivative terms, but proving this in general seems to be a complicated thing.

2.4 Towards A Physical Interpretation

Empty black brane in AdS is dual to a thermal state of the boundary CFT [36]. This is not a relevant deformation of the CFT Hamiltonian and there is no renormalization group flow in the ordinary sense. So it is unlikely that the holographic c-function is an off-shell central charge. To make further progress, it will be useful to take note of the fact that a thermal state is effectively massive with a gap set by the temperature. There is a finite correlation length of the order of inverse temperature. The IR behavior of the holographic c-function that we have constructed shows the presence of this effective mass gap. It is monotonically decreasing from the central charge of the UV-CFT, a_{UV}, to zero at the curvature singularity which is in the deep IR and space-time ends there. *Therefore the causal-horizon c-function faithfully quantifies the amount of pure quantum correlation or the effective number of "quantum degrees of freedom" that exists at different scales in the thermal state.*

Can this be related to renormalized entanglement entropy in the boundary theory ? First of all space-like slices of the causal horizon are not in general extremal surfaces in the bulk [39]. In the field theory side, suppose we consider a ball in R^3 of radius R. This is our subsystem for which we want to compute the renormalized entanglement entropy [38] when the field theory is in the thermal state. Since the the theory is scale invariant the renormalized entanglement entropy will have the functional form $S_{REE}(RT)$, where T is the temperature. It is known that as $T \rightarrow 0$, $S_{REE} \rightarrow a_{UV}$ [38]. This matches with the behavior of our c-function in the same limit. In the opposite limit of $T \rightarrow \infty$ on the other hand the renormalized entanglement entropy S_{REE} is nonzero and dominated by thermal entropy of the system [38]. This does not match with the behavior of the c-function. This is not surprising because entanglement entropy is not an entanglement measure in a mixed state. In the high temperature limit it is contaminated by classical correlations and fails to capture the quantum part, which should go to zero. On the contrary the behavior of the causal horizon c-function shows that it is sensitive only to quantum correlations. Is there a candidate for such a quantity in the field theory ?

2.4.1 Is Finite Temperature Entanglement Negativity A Generalized *c*-function ?

As we have discussed entanglement entropy at finite temperature is not a candidate for this generalized c-function because it is not an entanglement measure in a mixed state. One such measure which can be calculated in field theory is entanglement negativity [43–45, 51–55]. Entanglement negativity was studied from a holographic point of view in [53], but to the best of our knowledge a geometric prescription of computing this in gravity does not exist so far.

Entanglement negativity at finite temperature in a two dimensional CFT was computed in [52] [5] . They calculated this for a single interval of length L when the total system lives on an infinite line and the temperature is $T = \beta^{-1}$. In this case the answer is given by,

$$E = \frac{c}{2} \ln \Big[\frac{\beta}{\pi a} \sinh \Big(\frac{\pi L}{\beta} \Big) \Big] - \frac{\pi c L}{2\beta} + f(e^{-\frac{2\pi L}{\beta}}) + 2 \ln c_{\frac{1}{2}} \tag{2.21}$$

where a is the short distance cutoff, c is the central charge of the CFT and $c_{\frac{1}{2}}$ is a constant. $f(x)$ is a universal scaling function which depends on the full operator content of the CFT such that $f(1) = 0$ and $f(0)$ =constant. Given this we can calculate its value in the UV and the IR. UV is the region where $\beta >> L$ and we get,

$$E_{UV} = \frac{c}{2} \ln \frac{L}{a} + 2 \ln c_{\frac{1}{2}} \tag{2.22}$$

which is the correct zero temperature result. Similarly in the IR, $a << \beta << L$ and we get,

$$E_{IR} = \frac{c}{2} \ln \frac{\beta}{2\pi a} + f(0) + 2 \ln c_{\frac{1}{2}} \tag{2.23}$$

So in the IR this becomes a non-universal constant independent of the length L of the subsystem [52]. The second term in (2.21) is very important in the high temperature limit because it cancels the contribution to the negativity which is extensive in L. This is the principal difference from entanglement entropy which is useful for us. Now if we define a renormalized negativity, E_R, just like renormalized entanglement entropy [18, 38], as,

[5] See also [51].

$$E_R = L\frac{d}{dL}\Big|_{\beta} E \tag{2.24}$$

then we get,

$$\begin{aligned} E_R(UV) &= \frac{c}{2} \\ E_R(IR) &= 0 \end{aligned} \tag{2.25}$$

E_R is a UV-finite quantity. Therefore we can see that the renormalized entanglement negativity at least satisfies the asymptotic conditions, i.e, in the UV it is given by the central charge of the theory and in the IR this is zero. The reason that it is going to zero in the IR or in the high temperature limit is that it is an entanglement measure and at very high temperature quantum entanglement goes to zero because the system should crossover to a classical one [52] . This is a non-trivial constraint. Anything that is sensitive to classical correlations may fail to satisfy the IR-condition. Therefore the question is does it satisfy the monotonicity condition, i.e,

$$T\frac{d}{dT}\Big|_{L} E_R \leq 0 \ ? \tag{2.26}$$

If this condition is satisfied then it is a generalized c-function. In four dimensions we expect the same thing to happen in the UV. We have to compute the logarithmic negativity for a ball of radius R when the field theory is in a thermal state with temperature T. The structure of the UV divergences of the negativity is the same as that of entanglement entropy in the same dimension [54]. So if we apply the Liu-Mezei operator then we will get a UV finite quantity. The main question is what happens in the IR. Does the renormalized negativity go to zero ? This will be the case if negativity becomes independent of the size of the ball in the high temperature limit. This is a reasonable thing to expect given that there is a finite correlation length of order β. So we expect the same thing to happen but we cannot prove this right now. It will be fascinating to prove the monotonicity of the negativity at least in two dimensions.

In the large c limit we expect some simplifications [56]. In fact negativity in the large c limit was considered in [55]. Their calculation was for the vacuum sector of the CFT. It will be fascinating to extend the calculation to the thermal state using technology of [55, 56].

Before we end this section we would like to emphasize that we are not saying that the causal horizon entropy density is computing some entanglement measure in a thermal state. That may turn out to be the case but our calculation does not show that. What we can infer from this is the existence of such a monotonic function in field theory which is most likely an entanglement measure. In two dimensional CFT we have shown a potential candidate for this. Causal horizon entropy density represents that quantity in the bulk but perhaps in a different choice of scheme. So numerically they can be different but they will have the same physical content just like in more conventional c-theorem.

2.4.2 Black Hole Singularity From Loss Of quantum correlation

There is a different aspect to this problem. Our results can be thought of as a realization of the paradigm that space-time is built out of entanglement [69], but in a different setting. In the IR there is no quantum correlation or entanglement because of the effective mass gap in the thermal state. In the bulk our holographic c-function is monotonically decreasing and nonzero everywhere except at the curvature singularity. The curvature singularity is the end of space-time and represents the extreme IR of the dual field theory. Therefore the behavior of our c-function correlates the two facts : loss of quantum correlation/entanglement in the IR field theory and the end of geometry which in this case is the formation of curvature singularity behind the horizon. In fact this is one of our main motivations for interpreting the c-function as an effective bulk measure of quantum correlation or quantum entanglement between the field theory degrees of freedom at different scales.

There is another thing which we would like to point out is that since the causal horizon goes behind the black brane horizon and reaches the singularity, the holographic c-function is affected by things behind the horizon. Therefore the corresponding boundary c-function knows something about physics behind the horizon. If it turns out that the entanglement negativity indeed satisfies the monotonicity condition then this function will have some information about the interior. [6] At infinite temperature when the negativity is zero we are on the singularity because there is no quantum entanglement . As we lower the temperature we are moving away from the singularity but space-time is still very curved because there is only a very small amount of entanglement. So high temperature expansion is an expansion around the singularity. This is a difficult expansion because negativity depends on the full operator content of the theory, but this may be a virtue of the function for many purposes.

In [66–68] behind the horizon physics was explored using the analytically continued correlation functions in the CFT. The entanglement negativity (or any candidate thermal c-function) does not seem to have any simple expression in terms of thermal correlators. It is a highly non-local object. It will be interesting to see if there are more fine-grained characterisations of RG-flow which can tell us about the physics behind the horizon.

2.4.3 An Infalling Observer ?

Let us now go back to the issue of irreversibility associated to a particular geometry. In a black hole geometry there is a natural notion of irreversibility, which is crossing the

[6] We would like to clarify that we are not talking about a two-sided eternal AdS black hole. We have in mind a black hole, at sufficiently late time, which has formed out of collapsing matter and so the other part of the geometry does not exist. We are making the approximation of a thermal state because the CFT correlators at sufficiently late time are well approximated by thermal correlators.

horizon or falling into the the black hole. Anything that goes into the black hole does not come out. Nothing comes out of the black hole singularity. How is that irreversibility encoded in the field theory ? This is a very difficult question and so we will only try to make a guess. First of all, our c-function does not show any particular sharp feature which can be used to predict the existence of horizon. [7] So a natural guess will be that this is a quantity which is associated with an infalling observer. In GR an infalling observer does not see anything special happening while crossing the horizon. So let us make the assumption that the RG-flow or coarse-graining of the thermal state of the CFT describes an infalling observer. We cannot make this statement more precise right now. This assumption together with the fact that this coarse-graining is an irreversible process due to the existence of the c-theorem seem to imply that the observer can never come out of the black hole. The coarse graining starts in the UV when the observer is near the AdS boundary. As we lower the energy scale the observer moves deeper into the bulk. In the extreme IR when the c-function hits zero the observer hits the singularity. This is consistent with the fact that our holographic c-function reaches zero at the curvature singularity. Things cannot come out of the black hole singularity because in the field theory there is no unitary RG-flow which starts at $c = 0$ and go to $c = a_{UV}$. This is forbidden by c-theorem [8]. No unitary RG-flow can start at $c = 0$ because along the RG-flow c has to decrease. So in RG-time there is an ordering in which the $c = 0$ theory always lives in the future. This is also the ordering of time for the infalling observer for whom the black hole singularity is always in the future. This is not quantitative and many things need to be checked before one can say anything conclusive, but at least it is clear that the existence of the c-theorem imposes an ordering among different scales in the field theory which, it looks like, can be translated to the bulk under certain assumption and does not immediately produce a contradiction.

[7] We do not know how this picture will change if there is a firewall.

[8] We have in mind the generalization of the c-theorem to the thermal state. We have proved such a theorem only in the bulk.

2.4.4 Tensor Network

There is another reason to suspect that this may be a correct interpretation. This is related to the tensor network representation of the thermofield double of a scale invariant theory after time evolution. This representation was proposed by Hartman and Maldacena [77]. In this picture the tensor network has a scale-invariant UV region and a gapped IR region. The gapped region arises due to the effective mass gap of the thermal state and this represents the interior of the black brane. This resonates well with the behavior of our holographic c-function because it shows the extreme thinning of the "effective number of degrees of freedom" near the curvature singularity. A better understanding of this will probably require a more covariant formulation of tensor network ideas. Overall, it seems that MERA [78, 79] might be a proper framework to think about such generalized holographic c-functions. The function we have constructed measures the quantum correlation that exists at different scales in the thermal density matrix. MERA does a coarse-graining of the wave function and the generalized c-theorem seems to be associated to the irreversibility of that coarse graining procedure.

Chapter 3

Linearized Einstein's Equation Around Pure BTZ from Entanglement Thermodynamics

3.1 Introduction

After the realization of a connection between gravity and thermodynamics [2, 4, 105] various attempts have been made to understand gravitational dynamics from horizon thermodynamics [106–108]. The discovery of AdS/CFT correspondence [109, 111] led to the new idea that the dynamics of spacetime can be understood from some sort of entanglement between the degrees of freedom of the boundary CFT [112–114, 116–121]. See also [69, 70, 122]. In this chapter, following [112–114], we explore this idea further.

Linear perturbations around a fixed reference state in the continuum field theory satisfy the first law of entanglement, $\Delta S = \Delta \langle H \rangle$ [112–114], where S is the entanglement entropy of a spatial region and H is the modular Hamiltonian associated with that region. In the AdS/CFT framework, each side of this first law can be computed using the dual geometry. In [112, 114], the vacuum of a holographic CFT, with corresponding dual geometry pure AdS, was chosen as a fixed reference state. Considering perturbations around pure AdS, [112, 114] calculated ΔS and $\Delta \langle H \rangle$ hologrpahically and showed that to linear order in the perturbation the first law of entanglement is satisfied, while inclusion of higher order contributions gives the constraint $\Delta \langle H \rangle \geq \Delta S$ [112, 114][1]. In [112, 113] linear perturbations were considered and it was shown that Einstein's equations linearized around pure AdS do follow from the first law of entanglement, thus showing their equivalence at first order.

In this chapter, we take the thermal state of a holographic CFT as the fixed reference state and perturb it infinitesimally. The change in the entanglement entropy and the modular Hamiltonian of a spatial region will satisfy the first law of entanglement. Based on the holographic dictionary we then compute each side of this relation using metric components of the dual geometry. We show that for metric components of the dual geometry satisfying the linearized Einstein's equations, the first law of entanglement holds. Then we go the other direction, i.e, we show that the first law of entanglement fixes the metric uniquely if we demand that it holds in all frames of reference.

Entanglement entropy for a holographic field theory can be computed by applying Ryu-Takayanagi formula [39] and its covariant generalization [40]. See also [128] for some useful discussions on holographic enetanglement entropy in case of warped AdS_3 geometries. Computing modular Hamiltonian in the field theory side is not an easy task. There are only few cases where it can be expressed as the integral of some local quantity, mainly the stress tensor [104, 129]. The modular Hamiltonian for a spatial interval of a two dimensional CFT at finite temperature was calculated in [129]. Using the holographic dictionary one can obtain the boundary stress tensor from the asymptotic behavior of the metric components [130–132], which can then be used to compute modular Hamiltonian. See also [133–135] for relevant discussions on holographic stress tensor.

[1]For related discussions one can also see, e.g, [123–127]

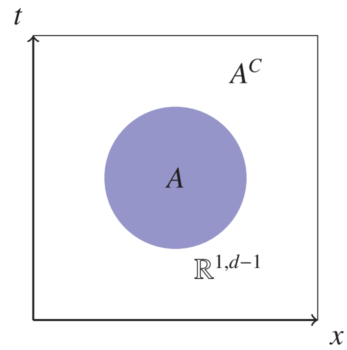

Figure 3.1: Spatial region A defined on a constant time slice $t = t_0$ of $d-$dimensional Minkowski spacetime $\mathbb{R}^{1,d-1}$. A^C, the compliment of A, denotes the rest of the spacetime. σ_A denotes the reduced density matrix defined for this spatial region A, where σ is a state (represented as a density matrix) of a QFT defined on $\mathbb{R}^{1,d-1}$.

3.2 First Law of Entanglement $\Delta S = \Delta \langle H \rangle$

In this section we briefly review the first law of entanglement. Relative entropy quantifies distinguishability between two states of a quantum field theory in the same Hilbert space [136][2]. Let us consider a d-dimensional Minkowski spacetime $\mathbb{R}^{1,d-1}$. Also consider a spatial region A on a fixed time slice $t = t_0$ on this spacetime.

Let density matrices σ and ρ define two states of a QFT defined on $\mathbb{R}^{1,d-1}$. Then the relative entropy of ρ with respect to σ for the spatial region A is defined as

$$\begin{aligned} S_A(\rho_A|\sigma_A) &= -Tr(\rho_A \log \sigma_A) - S_A(\rho_A) \\ &= Tr(\rho_A \log \rho_A) - Tr(\rho_A \log \sigma_A), \end{aligned} \tag{3.1}$$

where ρ_A and σ_A are the reduced density matrices associated with the spatial region A.

Choose σ as the reference state. With the definitions of entanglement entropy of a region A, $S_A(\rho_A) = -Tr(\rho_A \log \rho_A)$, and of the modular Hamiltonian associated with that region,

[2] For relative entropy of excited states in two dimensional CFT see [137, 138].

$H_A = -\log \sigma_A$, one can rewrite (3.1) as

$$S_A(\rho_A|\sigma_A) = \Delta \langle H_A \rangle - \Delta S_A, \tag{3.2}$$

where

$$\Delta S_A = S_A(\rho_A) - S_A(\sigma_A), \tag{3.3}$$

and

$$\Delta \langle H_A \rangle = Tr(\rho_A H_A) - Tr(\sigma_A H_A). \tag{3.4}$$

It has the nice positivity property that $S_A(\rho_A|\sigma_A) \geq 0$ with the inequality saturated if and only if $\rho_A = \sigma_A$. Using this property one can show easily that for ρ_A very close to σ_A the relative entropy vanishes at linear order of perturbation giving rise to the constraint [112, 114]

$$\Delta S = \Delta \langle H \rangle \,. \tag{3.5}$$

This is known as the first law of entanglement.

Below, we will be interested in the entanglement entropy of a single spatial interval $A = (-R, +R)$ for a holographic state of a CFT in 1+1-dimensions. We choose the initial holographic state to be a thermal state with temperature $T = \frac{1}{2\pi}$. In this case the modular Hamiltonian is given by [104, 139][3]

$$H_A = \frac{4\pi}{\sinh R} \int_{-R}^{+R} dx \left[\sinh\left(\frac{R+x}{2}\right) \sinh\left(\frac{R-x}{2}\right)\right] T_{00}(x,t) \tag{3.6}$$

where $T_{00}(x,t)$ is the time-time component of the field theory stress tensor.

[3] We thank David Blanco for poiting out reference [139] to us.

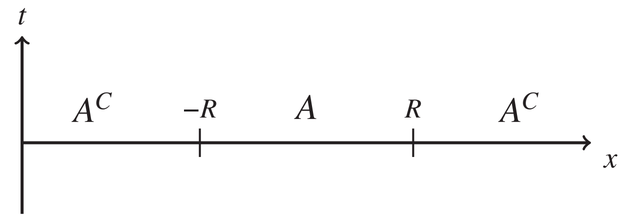

Figure 3.2: Spatial region $A \equiv (-R, R)$ on a fixed time slice $t = t_0$ of the boundary spacetime $\mathbb{R}^{1,1}$. Our modular Hamiltonian (3.6) is defined for this region in a thermal state of a CFT with temperature $T = 1/2\pi$.

3.3 Holographic Computation of ΔS and $\Delta \langle H \rangle$

We consider a thermal state of 1+1-dimensional holographic CFT with temperature $T = \frac{1}{2\pi}$ as a fixed reference state. We know that the dual geometry of a thermal state of a holographic CFT is a black hole in AdS. In 2+1 bulk dimensions, it is the BTZ black hole. We would consider the BTZ black brane instead of black hole because the field theory is defined on $\mathbb{R}^{1,1}$. The static BTZ black brane metric is[4] [140]

$$ds^2 = \frac{1}{z^2}\left[\frac{dz^2}{1-z^2} - (1-z^2)dt^2 + dx^2\right] \tag{3.7}$$

Here we have set the AdS radius of curvature, $L_{\text{AdS}} = 1$. With this convention one can check that the inverse temperature of this black brane is $\beta = 2\pi$.

3.3.1 Holographic Computation of ΔS

Let us consider a spatial interval $A = (-R, +R)$ at a fixed time $t = t_0$ in the boundary spacetime $\mathbb{R}^{1,1}$. The entanglement entropy of this interval can be computed using the Ryu-Takayanagi formula [39]:

$$S_A = \frac{\text{Length}(\gamma_A)}{4G_N}, \tag{3.8}$$

[4]Our convention is that we use indices K, L for the three bulk coordinates $\{z, t, x\}$ and indices μ, ν for two boundary coordinates $\{t, x\}$.

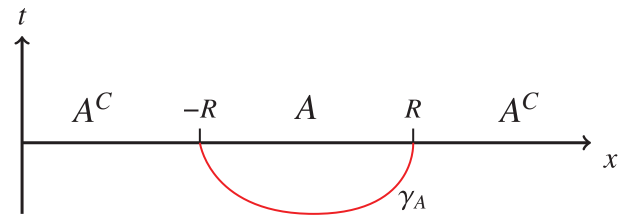

Figure 3.3: Pictorial representation of the Ryu-Takayanagi proposal. Area of the minimal area curve, i.e., length of the geodesic, γ_A gives the holographic entanglement entropy for the spatial region A in the boundary.

where γ_A is a geodesic in the bulk homologous to A, and G_N is Newton's constant. When the spacetime is not static, one needs to use its covariant generalization [40]. We will deal with that in the case of boosted black brane.

Let us now concentrate on our specific situation. The equation for the geodesic, with prescribed boundary conditions $t = t_0 = 0,\ z = 0,\ x = \pm R$, is

$$z^2 \cosh^2 R + \cosh^2 x = \cosh^2 R. \tag{3.9}$$

Performing a coordinate transformation to Fefferman-Graham(FG) coordinates [141],

$$z^2 = \left(1 + \frac{\tilde{z}^2}{4}\right)^{-2} \tilde{z}^2, \tag{3.10}$$

we can write down the metric (3.7) as

$$ds^2 = \frac{1}{\tilde{z}^2}\left[d\tilde{z}^2 - \left(1 - \frac{\tilde{z}^2}{4}\right)^2 dt^2 + \left(1 + \frac{\tilde{z}^2}{4}\right)^2 dx^2\right]. \tag{3.11}$$

The advantage of writing down any asymptotically AdS spacetime in this coordinate is to identify the boundary stress tensor easily using the holographic prescription [130, 132]. One can easily check that in case of pure BTZ (3.11), the boundary stress tensor has the non-vanishing components $T_{tt} = T_{xx} = \frac{1}{16\pi G_N}$.

In terms of FG coordinates, the extremal surface equation (3.9) transforms as

$$\frac{\tilde{z}^2}{\left(1+\frac{\tilde{z}^2}{4}\right)^2} = \frac{\cosh^2 R - \cosh^2 x}{\cosh^2 R}$$
$$\Rightarrow \qquad \tilde{z}^2 = 4\frac{\cosh R - \cosh x}{\cosh R + \cosh x}\,. \tag{3.12}$$

We can treat x as the intrinsic coordinate on the geodesic. The induced metric on the geodesic before perturbation is

$$g^{(0)}_{xx} = G^{(0)}_{KL}\frac{dx^K}{dx}\frac{dx^L}{dx}, \tag{3.13}$$

where $G^{(0)}_{KL}$ are the metric components of pure BTZ (3.11). The length functional is

$$A = \int_{-R}^{+R} dx\,\sqrt{g^{(0)}_{xx}}. \tag{3.14}$$

Now we add some pure metric perturbation to the BTZ metric (3.11)[5]. Any such perturbation in the FG coordinates can be written as

$$ds^2 = \frac{1}{\tilde{z}^2}\left[d\tilde{z}^2 - \left(1-\frac{\tilde{z}^2}{4}\right)^2 dt^2 + \left(1+\frac{\tilde{z}^2}{4}\right)^2 dx^2 + \tilde{z}^2 H_{\mu\nu}(\tilde{z},x,t)dx^\mu dx^\nu\right] \tag{3.15}$$

From now on (in the case of static BTZ black brane) everything will be done in FG coordinates so we will drop the ˜ sign over $\tilde{z}$ and simply write z.

To linear order in perturbation $H_{\mu\nu}(z,x,t)$, the change in the length functional (3.14) is

$$\Delta A = \int dx \frac{1}{2}\sqrt{g^{(0)}}g^{(0)xx}\delta g_{xx},$$
$$= \frac{1}{\sinh 2R}\int dx[\sinh(R+x)\sinh(R-x)]H_{xx}(z(x),x,t). \tag{3.16}$$

[5]Linear perturbations around BTZ black brane were also considered in [142] and the first order correction to holographic entanglement entropy was calculated. They have also discussed the dynamics of the shift of holographic entanglement entropy.

Hence to first order in the perturbation, the change in the entanglement entropy is

$$\Delta\hat{S}_A = \int dx[\sinh(R+x)\sinh(R-x)]H_{xx}(z(x),x,t). \tag{3.17}$$

Here $\Delta\hat{S}_A = 4G_N \sinh(2R)\Delta S_A$.

3.3.2 Holographic Computation of $\Delta\langle H\rangle$

Equation (3.6) gives the modular Hamiltonian for a spatial interval $A = (-R,+R)$ of a thermal state with temperature $T = \frac{1}{2\pi}$. If we perturb this state infinitesimally, the change in the modular Hamiltonian is

$$\Delta\langle H_A\rangle = \frac{4\pi}{\sinh R}\int_{-R}^{+R} dx\left[\sinh\left(\frac{R+x}{2}\right)\sinh\left(\frac{R-x}{2}\right)\right]\Delta\langle T_{00}(x,t)\rangle, \tag{3.18}$$

where $\Delta\langle T_{00}(x,t)\rangle$ is the change in the expectation value of the time-time component of the field theory stress tensor due to infinitesimal perturbations.

We have already mentioned that the boundary stress tensor can be found from the asymptotic form of the asymptotically AdS bulk metric. From [130–132] we know that for a $d+1$-dimensional asymptotically AdS bulk metric written in FG coordinates,

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu + z^d g_{\mu\nu}(z,x)dx^\mu dx^\nu\right], \tag{3.19}$$

the boundary stress tensor can be found from the following asymptotic relation,

$$\left\langle T_{\mu\nu}(x)\right\rangle = \frac{d}{16\pi G_N}g_{\mu\nu}(z=0,x). \tag{3.20}$$

Using the formula in our case, we have

$$\left\langle T_{\mu\nu}(x,t)\right\rangle = \frac{1}{8\pi G_N}\left(\frac{1}{2} + H_{\mu\nu}(z=0,x,t)\right), \tag{3.21}$$

where $\frac{1}{16\pi G_N}$ is the background boundary stress tensor. Hence the change in the modular Hamiltonian is

$$\Delta\left\langle \hat{H}_A \right\rangle = 4\cosh R \int_{-R}^{+R} dx \left[\sinh\left(\frac{R+x}{2}\right)\sinh\left(\frac{R-x}{2}\right)\right] H_{tt}(x,t) \tag{3.22}$$

where $\Delta\left\langle \hat{H}_A \right\rangle = 4G_N \sinh(2R)\Delta\left\langle H_A \right\rangle$.

3.4 Proof That Einstein's Equations Imply $\Delta S = \Delta\langle H\rangle$

We now have the expressions for both ΔS and $\Delta\langle H\rangle$ to linear order in the bulk perturbation. In this section we will show that the solutions of linearized Einstein's equations satisfy the relation $\Delta S = \Delta\langle H\rangle$.

In $d+1$-dimensions with a cosmological constant $\Lambda = -\frac{d(d-1)}{2L_{\text{AdS}}^2}$, Einstein's equations read (recall that we have set $L_{\text{AdS}} = 1$)

$$R_{AB} - \frac{1}{2}G_{AB}(R + d(d-1)) = 0. \tag{3.23}$$

Using the metric (3.15) to linear order in $H_{\mu\nu}$, different components of equations (3.23) (with $d = 2$) read

$$32zH_{tt}(z,x,t) - (4-z^2)\left[-(12+z^2)\partial_z H_{tt}(z,x,t) - z(4-z^2)\partial_z^2 H_{tt}(z,x,t)\right] = 0 \tag{3.24}$$

$$-32zH_{xx}(z,x,t) + (4+z^2)\left[(12-z^2)\partial_z H_{xx}(z,x,t) + z(4+z^2)\partial_z^2 H_{xx}(z,x,t)\right] = 0 \tag{3.25}$$

$$(48+z^4)\partial_z H_{tx}(z,x,t) + z(16-z^4)\partial_z^2 H_{tx}(z,x,t) = 0 \tag{3.26}$$

$$-2(4-z^2)^2\partial_t H_{tx}(z,x,t) + 2(16+z^4)\partial_x H_{tt} - z(16-z^4)\,\partial_z\left(\partial_t H_{tx}(z,x,t) - \partial_x H_{tt}(z,x,t)\right) \;=\; 0 \tag{3.27}$$

$$2(4+z^2)^2\partial_x H_{tx}(z,x,t) - 2(16+z^4)\partial_t H_{xx} - z(16-z^4)\,\partial_z\left(\partial_t H_{xx}(z,x,t) - \partial_x H_{tx}(z,x,t)\right) \;=\; 0 \tag{3.28}$$

$$\begin{aligned} &-2(4+z^2)^2 H_{tt}(z,x,t) + 2(4-z^2)^2 H_{xx}(z,x,t) + z\Big[16z\partial_t^2 H_{xx}(z,x,t) \\ &-32z\partial_t\partial_x H_{tx}(z,x,t) + 16z\partial_x^2 H_{tt}(z,x,t) - (16-z^4)\partial_z H_{tt}(z,x,t) + (16-z^4)\partial_z H_{xx}(z,x,t)\Big] \;=\; 0 \end{aligned} \tag{3.29}$$

Setting $z = 0$ in (3.29) one can see that

$$-H_{tt}(z=0,x,t) + H_{xx}(z=0,x,t) = 0. \tag{3.30}$$

This holds because the boundary theory is a conformal field theory and (3.30) is the tracelessness condition of the boundary field theory stress tensor. Now demanding the smoothness condition at $z = 0$ and using (3.30) one arrives at the following solutions for the perturbations:

$$H_{tt}(z,x,t) = (4-z^2)H(x,t), \tag{3.31}$$

$$H_{xx}(z,x,t) = (4+z^2)H(x,t), \tag{3.32}$$

$$H_{tx}(z,x,t) = h(t,x) \tag{3.33}$$

with $h(x,t)$ and $H(x,t)$ restricted by the conditions,

$$\partial_t h(x,t) = 4\partial_x H(x,t), \tag{3.34}$$

$$\partial_x h(x,t) = 4\partial_t H(x,t), \tag{3.35}$$

$$(\partial_t^2 - \partial_x^2)H(x,t) = 0 \tag{3.36}$$

Now we consider the expressions for $\Delta \hat{S}_A$ and $\Delta \left\langle \hat{H}_A \right\rangle$.

$$\begin{aligned}
\Delta \hat{S}_A &= \int_{-R}^{R} dx \sinh(R+x)\sinh(R-x) H_{xx}(z,x,t_0) \\
&= \int_{-R}^{R} dx \sinh(R+x)\sinh(R-x)(4+z^2)H(x,t_0) \\
&= 16\cosh R \int_{-R}^{+R} dx \sinh\left(\frac{R+x}{2}\right)\sinh\left(\frac{R-x}{2}\right) H(x,t_0) \qquad (3.37)
\end{aligned}$$

and

$$\begin{aligned}
\Delta \langle \hat{H}_A | =\rangle\, 4\cosh R \int_{-R}^{R} dx \sinh\left(\frac{R+x}{2}\right)\sinh\left(\frac{R-x}{2}\right) \\
\times H_{tt}(0,x,t_0) \\
= 16\cosh R \int_{-R}^{R} dx \sinh\left(\frac{R+x}{2}\right)\sinh\left(\frac{R-x}{2}\right) H(x,t_0) \qquad (3.38)
\end{aligned}$$

Comparing (3.37) and (3.38) we see that indeed $\Delta S_A = \Delta \langle H_A \rangle$.

3.5 Proof That First Law of Entanglement Implies Einstein's Equations

In this section we go the other direction. We will show that the constraint (3.5) with the boundary condition (3.21) fixes the metric uniquely. We would follow the strategy of [112] for the proof .

Let $H^{EE}_{\mu\nu}$ be the metric that solves Einstein's equations (3.24 − 3.29) with the boundary conditions (3.21). We will show that there is no metric other than $H^{EE}_{\mu\nu}$ with the boundary conditions (3.21) which satisfies the same relation (3.5). Suppose we have another metric $H_{\mu\nu}$ satisfying (3.5) with the same boundary condition (3.21). We will show that

$$\Delta_{\mu\nu}(z,x,t) \equiv H^{EE}_{\mu\nu} - H_{\mu\nu} = 0, \qquad (3.39)$$

for all z. Just by demanding that both the metrics satisfy the same boundary condition, we already have

$$\Delta_{\mu\nu}(0, x, t) = 0. \tag{3.40}$$

Hence (3.22) tells us that

$$\Delta(\Delta \langle H_A \rangle) = 0. \tag{3.41}$$

Now the constraint (3.5) together with the expression (3.17) tells us that in a fixed frame of reference (say, the $t = t_0$ frame) we have

$$0 = \int dx\, \Delta_{xx}(z(x), x + x_0, t_0)[\sinh(R + x) \sinh(R - x)]. \tag{3.42}$$

In the integral (3.42) we have shifted the origin from $x = 0$ to $x = x_0$. Now we expand $\Delta_{\mu\nu}$ as

$$\begin{aligned} \Delta_{\mu\nu}(z(x), x + x_0, t_0) &= \sum_{n=0}^{\infty} z^n \Delta_{\mu\nu}^{(n)}(x + x_0, t_0) \\ &= \sum_{n,m} z^n \frac{x^{2m}}{(2m)!} \partial_x^{2m} \Delta_{\mu\nu}^{(n)}(x_0, t_0). \end{aligned} \tag{3.43}$$

Equation (3.42) thus becomes

$$\sum_{n,m} \frac{1}{(2m)!} \partial_x^{2m} \Delta_{xx}^{(n)}(t_0, x_0) \int_{-R}^{+R} dx[z^n x^{2m} \sinh(R + x) \sinh(R - x)] = 0. \tag{3.44}$$

Substituting for z from the expression (3.12) and performing some simplifications, we finally get

$$\sum_{n,m} \frac{2^{n+1}}{(2m)!} \partial_x^{2m} \Delta_{xx}^{(n)}(t_0, x_0) I_{n,m}(R) = 0, \tag{3.45}$$

where we write

$$I_{n,m} = \int_0^R dx \left[x^{2m} (\cosh R + \cosh x)^{-\frac{n}{2}+1} (\cosh R - \cosh x)^{\frac{n}{2}+1} \right]. \tag{3.46}$$

We now need to expand (3.45) in powers of R and set each coefficient to zero to see what constraints do they impose on $\Delta_{xx}(t_0, x_0)$. Thus expanding (3.45) around $R = 0$, we get

$$\sum_{n,m,j} \frac{2^{n+1}}{(2m)!} \partial_x^{2m} \Delta_{xx}^{(n)}(t_0, x_0) \frac{R^j}{j!} \left[\partial_R^j I_{n,m}(R)\right]_{R=0} = 0. \tag{3.47}$$

Vanishing of the $R^{J+3}-$th term requires

$$\sum_{n,m} \frac{2^{n+1}}{(2m)!} \partial_x^{2m} \Delta_{xx}^{(n)}(t_0, x_0) \left[\partial_R^{J+3} I_{n,m}(R)\right]_{R=0} = 0 \tag{3.48}$$

The LHS of (3.48) contains a summation over n, m for a fixed J. In appendix B.1 we explicitly show that for both odd and even J (for odd J numerical analysis is presented in appendix B.1), all the terms in the summation vanish except when $n \leq J$. For $n = J$, we have $m = 0$ and for $n < J$, we need $m \neq 0$ (Please see appendix B.1 for details). The last non-vanishing term in the summation (3.48) with a fixed J is $n = J$, $m = 0$. This term can be expressed as a linear combination of the lower order terms which establishes the result that

$$\Delta_{xx}^{(J)}(t_0, x_0) = 0 \tag{3.49}$$

for $J = 0, 1, 2, \cdots$

Having shown that the entanglement first law fixes the solution to linearized Einstein's equations in the rest frame, we now consider a boosted frame and try to repeat the analysis above. Consider then the boosted BTZ black brane. The coordinate transformations to go from the static BTZ black brane to the boosted BTZ black brane are

$$t = \gamma(t' - \beta x'), \tag{3.50}$$

$$x = \gamma(x' - \beta t'). \tag{3.51}$$

Here β, γ have their usual meanings from the special theory of relativity. The metric of the boosted black brane is given by

$$ds^2 = \frac{1}{z^2}\left[\frac{dz^2}{1-z^2} + (-1+\gamma^2 z^2)dt'^2 + (1+\beta^2\gamma^2 z^2)dx'^2 - 2\beta\gamma^2 z^2 dt' dx'\right], \tag{3.52}$$

Notice that here, z denotes the original radial coordinate and not the FG coordinate $\tilde{z}$. Below, we will explicitly write $\tilde{z}$ for the FG coordinate. We need to solve the spacelike geodesic equations in this geometry with boundary conditions $z = 0$, $t = 0$, $x = \pm R$. Working in the geometry (3.52) with these boundary conditions is equivalent to working in the geometry (3.7) with the following boundary conditions

$$\begin{aligned} z = 0, \quad t = -\beta\gamma R, \quad x = \gamma R, \\ z = 0, \quad t = \beta\gamma R, \quad x = -\gamma R. \end{aligned} \tag{3.53}$$

Hence we will solve for the spacelike geodesics with metric (3.7) and boundary conditions (3.53). Let s be the proper length along the geodesic. Then we have the following equations for the spacelike geodesics

$$\frac{1}{z^2}\left[\frac{1}{1-z^2}\left(\frac{dz}{ds}\right)^2 - (1-z^2)\left(\frac{dt}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2\right] = 1, \tag{3.54}$$

$$\frac{1}{z^2}\left[\frac{dx}{ds}\right] = p, \tag{3.55}$$

$$-\frac{1}{z^2}\left[(1-z^2)\left(\frac{dt}{ds}\right)\right] = e, \tag{3.56}$$

where p and e are two constants of motion along the geodesic. We want to write down the geodesic in a parametric form $z = z(x)$, $t = t(x)$. Hence we write down the above three equations in the following form

$$pz\frac{dz}{dx} = \sqrt{1-(1-e^2+p^2)z^2 + p^2 z^4}\,, \tag{3.57}$$

$$\frac{dt}{dx} = -\frac{e}{p(1-z^2)}\,. \tag{3.58}$$

With the prescribed boundary conditions (3.53), the solutions are

$$\tanh(\gamma R)\tanh t + \tanh(\beta\gamma R)\tanh x = 0, \tag{3.59}$$

and

$$\begin{aligned}\cosh^2(\gamma R) z^2 + \left[1 - \frac{\sinh^2(\beta\gamma R)}{\sinh^2(\gamma R)}\right]\cosh^2 x \\ = \left[1 - \frac{\sinh^2(\beta\gamma R)}{\sinh^2(\gamma R)}\right]\cosh^2(\gamma R).\end{aligned} \tag{3.60}$$

In terms of the FG coordinates, (3.60) becomes

$$\tilde{z}^2 = 4\frac{\cosh(\gamma R) - \sqrt{(1-r)\cosh^2(\gamma R) + r\cosh^2 x}}{\cosh(\gamma R) + \sqrt{(1-r)\cosh^2(\gamma R) + r\cosh^2 x}}, \tag{3.61}$$

where

$$r = \left[1 - \frac{\sinh^2(\beta\gamma R)}{\sinh^2(\gamma R)}\right].$$

The induced metric on the spacelike geodesic is

$$ds^2_{ind} = \frac{\sinh^2(2\gamma R)}{[\cosh(2\gamma R) - \cosh(2x)]^2} dx^2. \tag{3.62}$$

Change in the geodesic length is

$$\Delta A = \int_{-\gamma R}^{+\gamma R} dx \, \frac{\sinh(\gamma R + x)\sinh(\gamma R - x)}{\sinh(2\gamma R)} \left[\left(\frac{dt}{dx}\right)^2 H_{tt} + 2\left(\frac{dt}{dx}\right) H_{tx} + H_{xx}\right].$$

An argument similar to that leading to (3.42) gives the following equation for the boosted case

$$\int_{-\gamma R}^{+\gamma R} dx \sinh(\gamma R + x)\sinh(\gamma R - x)\left[\left(\frac{dt}{dx}\right)^2 \Delta_{tt} + 2\left(\frac{dt}{dx}\right)\Delta_{tx} + \Delta_{xx}\right] = 0. \tag{3.63}$$

Now assuming that $\Delta_{\mu\nu}(z, x, t)$ is an analytic function, we can expand it in the following form

$$\Delta_{\mu\nu}(\tilde{z}, t+t_0, x{+}x_0) = \sum_{n=0}^{\infty} \tilde{z}^n \Delta_{\mu\nu}^{(n)}(t+t_0, x+x_0)$$
$$= \sum_{n=0}^{\infty} \tilde{z}^n \left[\sum_{m_t,m_x} \frac{\partial_t^{m_t} \partial_x^{m_x} \Delta_{\mu\nu}^{(n)}(t_0, x_0)}{m_t! m_x!} \right] t^{m_t} x^{m_x} = \sum_{n,m_t,m_x} B_{\mu\nu}^{n,m_t,m_x} \tilde{z}^n t^{m_t} x^{m_x}. \quad (3.64)$$

(3.63) then becomes

$$\sum_{n,m_t,m_x} [B_{tt}^{n,m_t,m_x} I_{tt}^{n,m_t,m_x}(R) + B_{tx}^{n,m_t,m_x} I_{tx}^{n,m_t,m_x}(R) + B_{xx}^{n,m_t,m_x} I_{xx}^{n,m_t,m_x}(R)] = 0, \quad (3.65)$$

where

$$I_{tt}^{n,m_t,m_x}(\gamma R) = \int_{-\gamma R}^{+\gamma R} dx x^{m_x} [\sinh(\gamma R + x)\sinh(\gamma R - x)]\tilde{z}^n t^{m_t} \left(\frac{dt}{dx}\right)^2, \quad (3.66)$$
$$I_{tx}^{n,m_t,m_x}(\gamma R) = 2\int_{-\gamma R}^{+\gamma R} dx x^{m_x} [\sinh(\gamma R + x)\sinh(\gamma R - x)]\tilde{z}^n t^{m_t} \left(\frac{dt}{dx}\right), \quad (3.67)$$
$$I_{xx}^{n,m_t,m_x}(\gamma R) = \int_{-\gamma R}^{+\gamma R} dx x^{m_x} [\sinh(\gamma R + x)\sinh(\gamma R - x)]\tilde{z}^n t^{m_t}. \quad (3.68)$$

We now use the same technique of expanding around $R = 0$ as before, but the calculation is difficult even for even integer n. One can examine term by term the expansion series (3.65) and check which coefficients in a particular term are non zero. In the appendix B.1 we discuss the first few terms. Working in this manner we finally arrive, apart from some constant factors, at the following equations

$$\begin{aligned} \Delta_{xx}^{(0)} - 2\beta\Delta_{tx}^{(0)} + \beta^2\Delta_{tt}^{(0)} &= 0 \\ \Delta_{xx}^{(1)} - 2\beta\Delta_{tx}^{(1)} + \beta^2\Delta_{tt}^{(1)} &= 0 \\ \Delta_{xx}^{(2)} - 2\beta\Delta_{tx}^{(2)} + \beta^2\Delta_{tt}^{(2)} &= 0 \\ &\vdots \end{aligned} \quad (3.69)$$

One can take any arbitrary value of n and check that the pattern should be the same, although for high values of n computations become difficult. Now (3.69) is a polynomial in β and the coefficients of each β^k must vanish individually. From the coefficients of β^0, we get:

$$\Delta_{xx}^{(n)} = 0, \tag{3.70}$$

from the coefficients of β:

$$\Delta_{tx}^{(n)} = 0, \tag{3.71}$$

and from the coefficients of β^2:

$$\Delta_{tt}^{(n)} = 0, \tag{3.72}$$

where $n = 0, 1, 2, \ldots$

Thus we have shown that (3.39) is true for all z. Note that we have assumed that $\Delta_{\mu\nu}(z, x, t)$ is an analytic function.

3.6 Discussion

In this chapter we have shown that linearized Einstein's equations around the BTZ black brane can be obtained from the first law of entanglement thermodynamics, $\Delta S = \Delta \langle H \rangle$, where the reference state was taken to be a thermal state of the CFT which is dual to the black brane. It would be interesting to check if non-linear Einstein's equations can be obtained from some constraints on the entanglenemnt entropy of the thermal state of boundary CFT as well. In particular, in [143] the vacuum state of the boundary CFT was perturbed by some scalar primary or stress tensor operators and it was shown that for such excited states, up to second order in the perturbation, the entanglement entropy of all ball-shaped regions can be obtained using the covariant prescription for holographic entanglement entropy from the corresponding dual geometries. It was shown that the corresponding dual spacetimes must satisfy Einstein's equations up to second order in the perturbation around AdS. It would be interesting to extend their work for the thermal state of holographic CFTs.

One of the important points we would like to emphasize is that the Einstein equations that we have derived hold outside the horizon. This is because the spacelike geodesics used to compute the entanglement entropy do not see the region behind the horizon. It will be very interesting if a similar method can be used to prove Einstein's equations behind the horizon. It seems that generalization of this method to the two-sided eternal BTZ black hole may give us some insights into the derivation of Einstein's equations behind the horizon.

It would also be interesting to check it for higher dimensional black holes and also at non-linear order. We have only considered the metric perturbations in the bulk such that only the boundary stress tensor has a non-trivial expectation value. One can also consider other excitations as well, e.g., turning on scalar field in the bulk where the scalar operator will acquire non-trivial expectation values in the boundary theory.

Chapter 4

Late Time Perturbations of Extremal Black Holes and Aretakis Instability

4.1 Introduction

More than 45 years ago, Price [148], in his seminal analysis, showed that when a Schwarzschild black hole is perturbed by a massless scalar field, at late times the perturbation typically decays as an inverse power in the Schwarzschild coordinate t. Price's law has been rigorously proved in the mathematical general relativity literature by Dafermos and Rodnianski [149, 150]. This is a key result, as the problem of late time asymptotics for solutions to the wave equation finds important applications in the study of black hole stability [151–153] and the dynamics of black hole interiors [154–156]. The late time asymptotics to wave equations on extreme black holes have attracted exceptional interest in the last few years.

The problem of late time decay of a scalar perturbation in four-dimensional extreme Reissner-Nordström black hole was first analysed by Bičák [157]. He observed that the effective potential for a massless scalar in an extreme Reissner-Nordström black hole has the same asymptotic form near the horizon as near infinity. Couch and Torrence [158] later showed that not only the effective potential has the same asymptotic form, it is in fact symmetric under r_* going to $-r_*$, where r_* is the tortoise coordinate for the extreme Reissner-Nordström metric. This surprising symmetry allows one to relate scattering dynamics near the horizon to the asymptotic region. This symmetry adds several novel features to the late time dynamics of a massless scalar field in an extreme Reissner-Nordström black hole background compared to a Schwarzschild black hole. This richness is one of the reasons that several authors have studied this problem [159–162].

Another reason the problem has attracted attention in the last few years is that Aretakis [164–167] has shown that a massless scalar has an instability at the future horizon of an extreme Reissner-Nordström black hole. More precisely, Aretakis showed that a massless scalar field decays at late time on and outside the future horizon, however, generically on the horizon its first radial derivative does not decay. This implies an instability. Since the first radial derivative of the scalar decays away from the horizon but not on the horizon, it follows that the second-derivative must blow up at late times *on the horizon.* The Aretakis instability was studied numerically in detail in [160]. They found excellent agreement with Aretakis' results. Using the Couch-Torrence symmetry, the Aretakis instability has been related to the similar growth in the behaviour of the derivatives of the massless scalar field at null infinity [160, 168]. Motivated by these developments, more recently, Ori and Sela [161, 162] have re-analysed analytically the problem of late time decay of scalar perturbations outside an extreme Reissner-Nordström black hole along the lines of Price's analysis. These questions are currently being explored in the mathematical general relativity literature [169–171] as well.

In this chapter we revisit this problem. While the previous authors [159–162] have analysed the problem in the time domain, we analyse the problem in the frequency domain. Our analysis brings a different perspective. We reproduce and extend some of the previous results. Using the Couch-Torrence symmetry, an initial data with regular behaviour across the horizon on the $v := t + r_* = 0$ surface can be mapped to an initial data on the $u := t - r_* = 0$ surface. Analysing this inverted initial data, Sela [162] has argued that there is a contribution to the late time tail in an extreme Reissner-Nordström background that is *not* due to the curvature of the spacetime. This contribution can be obtained from a flat space analysis of the inverted initial data. We first reproduce these results, including the exact coefficients, using rather simple Fourier methods.

Application of the frequency domain Green's function technique requires knowing initial data on a t = const surface. Obtaining a precise relationship between characteristic initial data specified on $u = 0$ and $v = 0$ null surfaces and initial data specified on a t = const Cauchy surface is a difficult problem. However, to the extent the above mentioned flat space analysis is valid, it can be easily done. We use the solution of the flat space wave equation to obtain the correct fall off on the $t = 0$ surface near spatial infinity. To this, we add a sub-leading term (slower fall-off) proportional to the "initial static moment" and compute its contribution to the late time tail. This contribution arises due to backscattering from the weakly curved asymptotic region of the spacetime.

Along the way, using the Couch-Torrence symmetry [158], we also relate higher multipole Aretakis and Newman-Penrose constants [172] for a massless scalar in an extreme Reissner-Nordström black hole background.

The rest of the chapter is organised as follows. In section 4.2 we review various interesting features that this problem has, namely, the Couch-Torrence symmetry and the construction of Aretakis and Newman-Penrose constants. In section 4.3 we analyse the late time dynamics of a massless scalar in the frequency domain. We end with a brief summary and a discussion of open problems in section 4.4.

4.2 Massless scalar in 4d extreme Reissner-Nordström spacetime

The massless scalar wave equation in a 4d extreme Reissner-Nordström spacetime has a number of rich features. In this section we review some of these features. Along the way, we relate higher multipole Aretakis and Newman-Penrose constants.

4.2.1 Couch-Torrence discrete conformal isometry

The extreme Reissner-Nordström solution has a discrete conformal isometry [158]. A similar discrete conformal isometry also exists for the extreme D1-D5 string and for the extreme D3 brane; see comments and references in [173] and for recent discussions see [174, 175]. We will use this symmetry in an important way in later sections, so we start with a brief review of this symmetry following [160, 168]. In static coordinates the extreme Reissner-Nordström metric takes the form,

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2, \tag{4.1}$$

where r is the area radial coordinate and $d\Omega^2$ is the line element of the unit 2-sphere. The Couch-Torrence symmetry is

$$\mathcal{T} : (t, r, \theta, \varphi) \to \left(t, M + \frac{M^2}{r - M}, \theta, \varphi\right). \tag{4.2}$$

It has number of interesting properties. It is an involution, i.e., $\mathcal{T}^2 = 1$. Its pull-back on the Reissner-Nordström metric acts by a conformal transformation

$$\mathcal{T}_*(g) = \Omega^2 g, \qquad \text{where} \qquad \Omega = \frac{M}{r - M}. \tag{4.3}$$

On the tortoise coordinate r_* defined by

$$r_* = r - M + 2M \log\left(\frac{|r-M|}{M}\right) - \frac{M^2}{r-M}, \tag{4.4}$$

so that $\frac{dr_*}{dr} = \left(1 - \frac{M}{r}\right)^{-2}$, it acts as $\mathcal{T} : r_* \to -r_*$. This last property implies that it interchanges the ingoing and the outgoing Eddington-Finkelstein coordinates:

$$\text{ingoing:} \quad v = t + r_*, \qquad \text{outgoing:} \quad u = t - r_*, \qquad \mathcal{T} : \quad u \leftrightarrow v. \tag{4.5}$$

Since the Ricci scalar of the extreme Reissner-Nordström metric vanishes, the conformally covariant operator is simply the box operator:

$$L_g = \Box_g - \frac{1}{6}R = \Box_g. \tag{4.6}$$

Recall that under a conformal transformation $\tilde{g}_{ab} = \omega^2 g_{ab}$, (see e.g., Wald's Appendix D, discussion around equation (D.13) [89]),

$$L_{\omega^2 g}(\omega^{-1}\Phi) = \omega^{-3} L_g(\Phi) = \omega^{-3}\Box_g \Phi. \tag{4.7}$$

Moreover, from tensor transformation properties, it follows that

$$L_{\mathcal{T}_*(g)}(\mathcal{T}_*(\Phi)) = \mathcal{T}_*(L_g(\Phi)). \tag{4.8}$$

Combining the two in the following way, it follows that if $\Box_g \Phi = 0$, then

$$0 = \Box_g \Phi = \mathcal{T}_*(L_g(\Phi)) = L_{\mathcal{T}_*(g)}(\mathcal{T}_*(\Phi)) = L_{\Omega^2 g}(\Omega^{-1}\Omega(\mathcal{T}_*(\Phi)) = \Omega^{-3}\Box_g(\Omega \mathcal{T}_*(\Phi)). \tag{4.9}$$

That is, if Φ is a solution then,

$$\tilde{\Phi} = \Omega \mathcal{T}_*(\Phi) \tag{4.10}$$

is also a solution. We will use mapping (4.10) to map solutions near the horizon to solutions near future null infinity and vice versa.

4.2.2 Aretakis constants, Newman-Penrose constants, and initial static moments

We now briefly review the construction of Aretakis and Newman-Penrose constants in an extreme Reissner-Nordström background, and relate them via the Couch-Torrence symmetry (4.10).

Previous studies have related Aretakis and Newman-Penrose constants for $l = 0$ modes [160, 168, 175]. To the best of our knowledge, details for the $l \neq 0$ have not been written out. In this subsection we write out those details explicitly.

In ingoing Eddington-Finkelstein coordinates the extreme Reissner-Nordström metric is

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2dvdr + r^2 d\Omega^2. \tag{4.11}$$

Expanding the scalar in spherical harmonics in ingoing Eddington-Finkelstein coordinates as

$$\Phi(v, r, \theta, \varphi) = \sum_{lm} \phi_l(v, r) Y_{lm}(\theta, \varphi), \tag{4.12}$$

we get equations for the mode functions $\phi_l(v, r)$,

$$2r\partial_v\partial_r(r\phi_l) + \partial_r((r - M)^2\partial_r\phi_l) - l(l + 1)\phi_l = 0. \tag{4.13}$$

Applying ∂_r^l on this equation, we see that

$$A_l[\phi_l] = \frac{M^l}{(l + 1)!}\partial_r^l[r\partial_r(r\phi_l)]\Big|_{r=M} \tag{4.14}$$

is conserved, i.e., independent of v along the horizon. These constants are called Aretakis constants. For a solution of the wave equation of the form near the horizon

$$\phi_l(v,r) = \frac{1}{r}\sum_{k=0}^{\infty} c_k(v)\left(\frac{r}{M}-1\right)^k \tag{4.15}$$

the Aretakis constants are [161, 162]

$$A_l = c_{l+1} + \frac{l}{l+1}c_l. \tag{4.16}$$

Note the factor of $\frac{1}{r}$ in equation (4.15).

In outgoing Eddington-Finkelstein coordinates the extreme Reissner-Nordström metric is

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 du^2 - 2dudr + r^2 d\Omega^2. \tag{4.17}$$

Expanding the scalar in spherical harmonics in these coordinates as

$$\Phi(u,r,\theta,\varphi) = \sum_{lm} \phi_l(u,r) Y_{lm}(\theta,\varphi), \tag{4.18}$$

we get equations for the mode functions

$$-2r\partial_u\partial_r(r\phi_l) + \partial_r((r-M)^2\partial_r\phi_l) - l(l+1)\phi_l = 0. \tag{4.19}$$

We now construct the Newman-Penrose constants. Consider the solution of the wave equation near infinity of the form

$$\phi_l(u,r) = \frac{1}{r}\sum_{k=0}^{\infty} d_k(u)\left(\frac{M}{r}\right)^k. \tag{4.20}$$

Inserting this expansion into equation (4.19) and looking at successive inverse powers of r gives equations that can be expressed concisely in terms of matrices, whose components are labelled by indices $i, j = 0, \ldots, l$. We label the components of the vector $\mathbf{d}$ by d_i,

$i = 0, \ldots, l$, and the $(l+1)$-dimensional vectors $\mathbf{d}_+$ and $\mathbf{c}_+$ have respective components $(d_+)_i = d_{i+1}$ and $(c_+)_i = c_{i+1}$. We obtain

$$M\mathsf{N}_l\dot{\mathbf{d}}_+ = [\tfrac{1}{2}l(l+1) - \mathsf{P}_l]\mathbf{d}, \tag{4.21}$$

where N_l is the diagonal matrix of natural numbers with components $(N_l)_{ij} = (i+1)\delta_{ij}$ and P_l is a lower-triangular matrix with components $(P_l)_{ij} = \frac{1}{2}i(i+1)\delta_{ij} - i^2\delta_{i,j+1} + \frac{1}{2}i(i-1)\delta_{i,j+2}$, and over-dots denote u-derivatives. We can diagonalize P_l as

$$\mathsf{P}_l = \mathsf{L}_l\mathsf{T}_l\mathsf{L}_l^{-1}, \tag{4.22}$$

where T_l is the diagonal matrix of triangular numbers, $(T_l)_{ij} = \frac{1}{2}i(i+1)\delta_{ij}$, and L_l is the lower-triangular Pascal matrix (see, e.g., [176])

$$(L_l)_{ij} = \binom{i}{j}, \tag{4.23}$$

whose inverse is

$$(L_l^{-1})_{ij} = (-1)^{i+j}\binom{i}{j}. \tag{4.24}$$

It follows that

$$M\mathsf{L}_l^{-1}\mathsf{N}_l\dot{\mathbf{d}}_+ = [\tfrac{1}{2}l(l+1) - \mathsf{T}_l]\mathsf{L}_l^{-1}\mathbf{d}, \tag{4.25}$$

whose last component implies conservation of

$$N_l = \frac{1}{l+1}\sum_{i=1}^{l+1}(-1)^{l+i-1}i\binom{l}{i-1}d_i, \tag{4.26}$$

at null infinity, i.e., $\partial_u N_l = 0$. These are examples of Newman-Penrose constants.

How are these constants related to Aretakis constants? Recall that, applying the mapping (4.10), we can construct a solution near null infinity from a given solution near the horizon. Let us apply this mapping on the solution of the form (4.15) to get

$$\phi_l = \frac{M}{r-M}\left(M + \frac{M^2}{r-M}\right)^{-1}\left(c_0(u) + c_1(u)\left(\frac{M}{r-M}\right) + c_2(u)\left(\frac{M}{r-M}\right)^2 + \dots\right) \tag{4.27}$$

$$= \frac{1}{r}\left(c_0 + c_1\frac{M}{r} + (c_1+c_2)\left(\frac{M}{r}\right)^2 + (c_1 + 2c_2 + c_3)\left(\frac{M}{r}\right)^3 + \dots\right). \tag{4.28}$$

Expanding this solution in inverse powers of r, we find the coefficients in (4.20) to be

$$\mathbf{d}_+ = \mathsf{L}_l \mathbf{c}_+. \tag{4.29}$$

Then we have

$$M\mathsf{Q}_l\dot{\mathbf{c}}_+ = [\tfrac{1}{2}l(l+1) - \mathsf{T}_l]\mathsf{L}_l^{-1}\mathbf{d}, \tag{4.30}$$

where

$$\mathsf{Q}_l = \mathsf{L}_l^{-1}\mathsf{N}_l\mathsf{L}_l, \tag{4.31}$$

has components $(Q_l)_{ij} = (i+1)\delta_{ij} + i\delta_{i-1,j}$. The Newman-Penrose constant N_l arising from the last component of (4.25) and equivalently (4.30) is expressed in terms of c_i as

$$N_l = c_{l+1} + \frac{l}{l+1}c_l, \tag{4.32}$$

which is nothing but the Aretakis constant A_l, cf. (4.16).

The diagonalizations (4.22) and (4.31) are straightforwardly proved using induction on l, by checking that the columns of L_l and L_l^{-1} are eigenvectors of P_l and Q_l respectively. As an explicit example, the Pascal matrices for $l = 4$ are

$$\mathsf{L}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}, \qquad \mathsf{L}_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}, \tag{4.33}$$

which diagonalize

$$\mathsf{P}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 \\ 0 & 3 & -9 & 6 & 0 \\ 0 & 0 & 6 & -16 & 10 \end{pmatrix}, \qquad \mathsf{Q}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 4 & 5 \end{pmatrix}, \tag{4.34}$$

whose corresponding diagonal matrices T_4 and N_4 are simply their diagonal entries. The Newman-Penrose constant is

$$N_4 = \tfrac{1}{5}(\mathsf{L}_4^{-1}\mathsf{N}_4\mathbf{d}_+)_4 = \tfrac{1}{5}(d_1 - 8d_2 + 18d_3 - 16d_4 + 5d_5). \tag{4.35}$$

The term static moment is often used in the literature [148] to discuss time independent solutions of the wave equations. In the static coordinates, the mode expansion,

$$\Phi = \frac{1}{r}\sum_{lm}\psi_l(t,r)Y_{lm}(\theta,\varphi), \tag{4.36}$$

results in the equations

$$[\partial^2_{r_*} - \partial^2_t]\psi_l = V_l(r)\psi_l, \tag{4.37}$$

with potential $V_l(r)$

$$V_l(r) = \left(1 - \frac{M}{r}\right)^2\left[\frac{2M}{r^3}\left(1 - \frac{M}{r}\right) + \frac{l(l+1)}{r^2}\right]. \tag{4.38}$$

This equation has two time independent solutions

$$\psi_l = r(r-M)^l, \tag{4.39}$$

and

$$\psi_l = \frac{r}{(r-M)^{l+1}}. \tag{4.40}$$

Under the mapping (4.10) one static solution goes to the other (up to normalisation):

$$\tilde{\Phi} = \Omega \mathcal{T}_* \left[\Phi(t, r, \theta, \varphi) = (r - M)^l Y_{lm} \right] = \frac{M^{2l+1}}{(r - M)^{l+1}} Y_{lm}. \tag{4.41}$$

4.3 Late time behavior of scalar perturbation

We are now in position to analyse the problem of late time decay of scalar field outside the horizon in an extreme Reissner-Nordström background.

4.3.1 Late time tails for non-compact initial data in flat space

We first reproduce some of the key results of Ori [161] and Sela [162] from a relatively simple Fourier analysis. In the next subsection we look at the contributions due to backscattering from the weakly curved asymptotic region.

To begin with, we are interested in the characteristic initial value of the field ψ_l specified at two intersecting null surfaces, $u = 0$ and $v = 0$, for equation (4.37). The initial data is thus composed of two functions

$$\psi_l^v(v) = \psi_l(u = 0, v), \qquad \psi_l^u(u) = \psi_l(u, v = 0). \tag{4.42}$$

See figure 4.1. Due to the linearity of the problem, we can analyse the two functions $\psi_l^u(u)$ and $\psi_l^v(v)$ separately. More precisely, we can split the characteristic initial value problem into two parts: (i) non-vanishing data on the $u = 0$ surface $\psi_l^v(v) = \psi_l(u = 0, v)$, along with vanishing data on the $v = 0$ surface $\psi_l^u(u) = 0$, (ii) non-vanishing data on the $v = 0$ surface $\psi_l^u(u) = \psi_l(u, v = 0)$, along with vanishing data on the $u = 0$ surface $\psi_l^v(v) = 0$. We can analyse the two parts separately and add the late time behaviour to obtain the final answer. In the following, this is how we will think of the evolution problem. For extreme Reissner-Nordström this logic has been employed by several authors in the past [159, 161, 162].

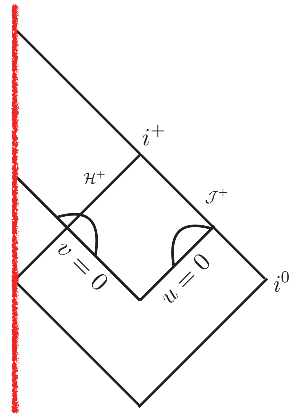

Figure 4.1: Initial data for the characteristic initial value problem for a scalar field in an extreme Reissner-Nordström black hole background. The initial data is composed of two functions $\psi_l^v(v) = \psi_l(u = 0, v)$ and $\psi_l^u(u) = \psi_l(u, v = 0)$.

Sela [162] considered initial data of "compact support" — an initial data for which the function $\psi_l^v(v)$ vanishes beyond a certain value of v. In his analysis, the function $\psi_l^u(u)$ is taken to be supported near the event horizon $r = M$. Furthermore, this function is taken to admit a Taylor expansion near $r = M$ as

$$\psi_l^u(u) = c_0 + c_1\left(\frac{r}{M} - 1\right) + c_2\left(\frac{r}{M} - 1\right)^2 + \ldots, \tag{4.43}$$

where r is to be thought of as function of u on the $v = 0$ surface. We also take our initial data of this form for the function $\psi_l^u(u)$. For the function $\psi_l^v(v)$ we consider a slightly more general behaviour than considered in [162]. We allow for an initial static moment, i.e., as $r \to \infty$ the function $\psi_l^v(v)$ is taken to behave as

$$\psi_l^v(v) = \hat{d}_l \frac{R^l}{r^l} + \text{compactly supported data}, \tag{4.44}$$

where, now, r is to be thought of as function of v on the $u = 0$ surface, and R is an arbitrary scale we have introduced. The coefficient $\hat{d}_l$ is called the static moment.

For the initial data on the $u = 0$ surface, it is believed that the late time tail arises due to backscattering from the weakly curved asymptotic region [148, 177, 178]. The tail does not depend on the exact nature of the central object. For compactly supported initial data the solution at late times decays as t^{-2l-3} and for initial data with an initial static moment it decays as $\hat{d}_l t^{-2l-2}$ [148].

For the function $\psi^u_l(u)$, following [159, 161, 162], we use the Couch-Torrence symmetry to map the problem from near the horizon to near infinity. The problem near infinity can be analysed again using the well developed techniques mentioned above. The map of the initial data is (4.2):

$$\psi^v_l(v) = c_0 + c_1\left(\frac{M}{r-M}\right) + \ldots + c_l\left(\frac{M}{r-M}\right)^l + c_{l+1}\left(\frac{M}{r-M}\right)^{l+1} + \ldots \tag{4.45}$$

where now r is to be regarded as a function of v along the $u = 0$ surface. Expanding in powers of r results in an expansion

$$\psi^v_l(v) = \hat{c}_0 + \hat{c}_1\frac{R}{r} + \hat{c}_2\frac{R^2}{r^2} + \ldots + \hat{c}_l\frac{R^l}{r^l} + \hat{c}_{l+1}\frac{R^{l+1}}{r^{l+1}} + \ldots . \tag{4.46}$$

In this expansion there is a term that decays as the static moment. There are terms that decay more slowly than the static moment and there are also terms that decay more quickly than the static moment. The coefficients $\hat{c}_k$ receive contributions from c_m, $m \leq k$.

Again using linearity of the problem, the effective problem that we need to analyse is therefore,

$$\psi^v_l(v) = \hat{c}_0+\hat{c}_1\frac{R}{r}+\hat{c}_2\frac{R^2}{r^2}+\ldots+(\hat{c}_l+\hat{d}_l)\frac{R^l}{r^l}+\hat{c}_{l+1}\frac{R^{l+1}}{r^{l+1}}+\ldots+\text{compactly supported data}, \tag{4.47}$$

with $\psi^u_l(u) = 0$. Generically, if the Aretakis and the Newman-Penrose constants are non-zero, the coefficients of $\frac{1}{r^l}$ and $\frac{1}{r^{l+1}}$ would be non-zero in equation (4.47).

Sela [162], building upon the work of Barack [179], argued that for the initial data of the form (4.47), there is a contribution to the late time in an extreme Reissner-Nordström background that is not due to the curvature of the spacetime. The term $\hat{c}_{l+1}\frac{R^{l+1}}{r^{l+1}}$ in the expansion of the data (4.47) results in a leading order tail as it disperses in flat space.[1] We reproduce Sela's results from a relatively simple Fourier analysis.

The Fourier transform of the field $\psi_l(t, r)$,

$$\psi_l(\omega, r) = \int_{-\infty}^{\infty} e^{i\omega t}\psi_l(t, r)dt, \tag{4.48}$$

satisfies the equation

$$\left(-\omega^2 - \partial_r^2 + \frac{l(l+1)}{r^2}\right)\psi_l(\omega, r) = 0. \tag{4.49}$$

The general solution to this equation is

$$\psi_l(\omega, r) = A(\omega)\sqrt{r}J_{l+1/2}(\omega r) + B(\omega)\sqrt{r}Y_{l+1/2}(\omega r). \tag{4.50}$$

To obtain regular solutions at $r = 0$ we must set $B(\omega) = 0$. Thus, we get

$$\psi_l(\omega, r) = A(\omega)\sqrt{r}J_{l+1/2}(\omega r). \tag{4.51}$$

The solution in the time domain is simply the inverse Fourier transform,

$$\psi_l(t, r) = \frac{1}{2\pi}\sqrt{r}\int_{-\infty}^{\infty} A(\omega)J_{l+1/2}(\omega r)e^{-i\omega t}d\omega. \tag{4.52}$$

If we know the function $A(\omega)$, we can do this integral and would know the full solution for the field $\psi_l(t, r)$, in particular its late time behaviour. Due to linearity of the problem we can consider each term in the expansion (4.47) separately.

To determine $A(\omega)$ corresponding to the r^{-k} term, we use the fact that the initial data

[1] In some sense, this result is the "Couch-Torrence dual" of the results of section 4 of [160], where they have obtained such tails from a purely $AdS_2 \times S^2$ analysis. $AdS_2 \times S^2$ is conformal to flat space, and the massless scalar wave equation is conformally invariant.

behaves as $\hat{c}_k \frac{R^k}{r^k}$ on the $u = 0$ surface. We make the ansatz $A(\omega) = 2\pi A_0\ \omega^p$ to get from (4.52)

$$\psi_l(t,r) = A_0 \sqrt{r} \int_{-\infty}^{\infty} \omega^p J_{l+1/2}(\omega r) e^{-i\omega t} d\omega \tag{4.53}$$

$$= 2\, A_0\ \sqrt{r}\, e^{i(p+l+1/2)\frac{\pi}{2}} \int_0^{\infty} \omega^p\ J_{l+1/2}(\omega r)\ \cos\left[(p+l+1/2)\frac{\pi}{2} + \omega t\right] d\omega, \tag{4.54}$$

where we have used the appropriate symmetry property of $J_{l+1/2}(\omega r)$ under ω to $-\omega$ to convert the integral to one along the positive ω axis. This last integral can be easily done using the identity (6.699-1) or (6.699-2) of Gradshteyn and Ryzhik [180].

Matching the resulting answer at $u = 0$ with

$$\psi_l(u = 0, r) = \hat{c}_k \frac{R^k}{r^k}, \tag{4.55}$$

gives

$$p = k - 1/2, \tag{4.56}$$

and fixes the constant A_0. Substituting the constant A_0 in terms of $\hat{c}_k$ gives a final answer

$$\psi_l(t,r) = -\frac{\hat{c}_k R^k 2^{k+1} \Gamma(k+1)}{\pi(2l+1)!!} \sin(k\pi)\ \Gamma(l-k+1)\ r^{l+1}\ t^{-(k+l+1)}\ F\left(\frac{l+k+2}{2}, \frac{l+k+1}{2}; l+\frac{3}{2}; \frac{r^2}{t^2}\right), \tag{4.57}$$

where $F(a, b; c; z)$ is the standard hypergeometric function. For $k \leq l$ this expression vanishes due to the $\sin(k\pi)$ factor. However, for $k \geq l+1$, the $\Gamma(l-k+1)$ factor develops a pole that exactly cancels with the zero of the sin function and gives a finite result. At timelike infinity, i.e., in the limit $t \gg r$, (4.57) becomes

$$\psi_l(t,r) \sim -\frac{\hat{c}_k R^k 2^{k+1} \Gamma(k+1)}{\pi(2l+1)!!}\ \sin(k\pi)\ \Gamma(l-k+1)\ r^{l+1}\ t^{-(k+l+1)}. \tag{4.58}$$

The leading contribution to the late time tail comes from $k = l + 1$. We get

$$\psi(t, r | t \gg r) \sim 2\hat{c}_{l+1} R^{l+1} (-1)^{l+1} (4r)^{l+1} \frac{[(l+1)!]^2}{(2l+2)!} t^{-(2l+2)}. \tag{4.59}$$

This expression matches with Sela's equation (6.18), including the pre-factors.

We can use solution (4.57) to obtain the tail behaviour near future null infinity. In order to achieve the limit, we must take $r \to \infty$ together with $u := t - r$ finite, i.e., $u \ll r$. The leading contribution to the tail comes once again from $k = l + 1$. In this limit we find, using equation (9.131-2) of Gradshteyn and Ryzhik [180],

$$\psi_l(t, r | u \ll r) \sim 2^{l+2} \hat{c}_{l+1} R^{l+1} (-1)^{l+1} \frac{[(l+1)!]^2}{(2l+2)!} u^{-l-1}. \tag{4.60}$$

This expression matches with Sela's equation (6.11), including the pre-factors, provided we relate our u to Sela's retarded time u_s: $u_s = u/2$.

Setting $k = l + 1$, equation (4.57) simplifies to

$$\psi_l(t, r) = 2\hat{c}_{l+1} R^{l+1} \frac{[(l+1)!]^2}{(2l+2)!} (-1)^{l+1} (4r)^{l+1} \left(1 - \frac{r^2}{t^2}\right)^{-l-1} t^{-2l-2}. \tag{4.61}$$

We can use solution (4.61) to obtain the fall off behaviour at spatial infinity,

$$\psi_l(t = 0, r) \sim 2^{2l+3} \hat{c}_{l+1} R^{l+1} \frac{[(l+1)!]^2}{(2l+2)!} r^{-l-1}, \tag{4.62}$$

together with $\partial_t \psi_l(t = 0, r) = 0$.

There are other contributions to the $t^{-(2l+2)}$ late time tail. They arise due to backscattering from the curvature of spacetime. For initial data (4.47), these contributions come from r^{-k} terms for $k < l + 1$. It is expected that they should decay as

$$\text{(pre-factor) } M^{l+1-k} \hat{c}_k t^{-(2l+2)}. \tag{4.63}$$

We look at a related problem for $k = l$ in the next subsection.

4.3.2 Contributions due to asymptotic curvature of spacetime

Equation (4.62) can be interpreted as initial data on the $t = 0$ surface in the extreme Reissner-Nordström background; see figure 4.2. We conclude that for an initial data on the $t = 0$ surface with r_*^{-l-1} decay near spatial infinity, there is a contribution to a t^{-2l-2} tail at late times. This contribution is due to decay of a massless scalar field in flat space, not due to backscattering from the curvature of the spacetime. More precisely, if

$$\psi_l(t = 0, r) = \mu_{l+1} R^{l+1} r_*^{-l-1}, \tag{4.64}$$

then the late time tail is

$$\psi(t, r_* | t \gg r_* \gg M) \sim (-1)^{l+1} 2^{-l-1} \mu_{l+1} R^{l+1} (r_*)^{l+1} t^{-(2l+2)}, \tag{4.65}$$

and

$$\psi_l(t, r | u \ll r_*) \sim (-1)^{l+1} 2^{-2l-2} \mu_{l+1} R^{l+1} u^{-l-1}. \tag{4.66}$$

From our discussion above, it is clear that such an initial data generically will have a non-zero Newman-Penrose constant, and its Couch-Torrence reflection will have a non-zero Aretakis constant.

Now we address the question of if, in addition to (4.64), there is an r_*^{-l} term present at the $t = 0$ surface near spatial infinity, how does it contribute to the late time tail? The contribution arises due to backscattering from the curvature of spacetime. If the initial data has non-zero static moment, such a term would be present.

To compute this contribution, fortunately, we do not need to do much. Since it is believed that the late time tail arises due to backscattering from the weakly curved asymptotic region [148, 177, 178], this computation is exactly the same as in the Schwarzschild back-

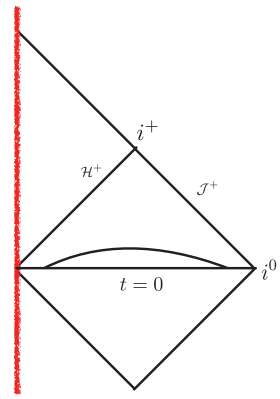

Figure 4.2: Time-symmetric initial data for a scalar field in an extreme Reissner-Nordström black hole background. The initial data consists of a function specified on the $t = 0$ surface. $\partial_t \psi_l(t = 0, r)$ is taken to be zero.

ground.

We very briefly review the Green's function approach to late time tails following [181] and supplement it with a discussion for an extended source of the type:

$$\psi_l(t = 0, r) = \mu_l R^l r_*^{-l}. \tag{4.67}$$

The retarded Green's function for the wave operator appearing in (4.37) satisfies

$$\left[\partial_t^2 - \partial_{r_*}^2 + V(r_*)\right] G(r_*, r_*'; t) = \delta(t)\, \delta(r_* - r_*') \tag{4.68}$$

with the boundary condition

$$G(r_*, r_*'; t) = 0, \qquad \text{for} \qquad t < 0. \tag{4.69}$$

We are interested in analysing the Green's function in the frequency domain. Therefore, we do a Fourier transform via

$$\tilde{G}(r_*, r'_*; \omega) = \int_0^\infty G(r_*, r'_*; t)\, e^{i\omega t}\, dt. \tag{4.70}$$

The range of the r_* coordinate for black hole spacetimes is $-\infty$ to ∞. In the frequency domain the solutions to the wave equations we are interested in should satisfy outgoing boundary conditions at infinity, and ingoing boundary conditions at the horizon. In terms of the r_* coordinate, these become

$$\tilde{\psi}_l(r_*, \omega) \to e^{i\omega r_*} \qquad \text{as} \qquad r_* \to \infty, \tag{4.71}$$

$$\tilde{\psi}_l(r_*, \omega) \to e^{-i\omega r_*} \qquad \text{as} \qquad r_* \to -\infty. \tag{4.72}$$

The Fourier transform of the Green's function $\tilde{G}(r_*, r'_*; \omega)$ satisfies

$$\left[-\omega^2 - \partial^2_{r_*} + V(r_*)\right] \tilde{G}(r_*, r'_*; \omega) = 0, \tag{4.73}$$

and is analytic in the upper half plane. Now recall that for a second order ODE with homogeneous boundary conditions, the Green's function can be uniquely constructed simply using two auxiliary functions $f(r_*, \omega)$ and $g(r_*, \omega)$, where $f(r_*, \omega)$ satisfies the left boundary condition and $g(r_*, \omega)$ satisfies the right boundary condition. We adopt normalisations such that

$$g(r_*, \omega) \to e^{i\omega r_*} \qquad \text{as} \qquad r_* \to \infty, \tag{4.74}$$

$$f(r_*, \omega) \to e^{-i\omega r_*} \qquad \text{as} \qquad r_* \to -\infty. \tag{4.75}$$

Then the Green's function is given by

$$\tilde{G}(r_*, r'_*; \omega) = \begin{cases} \frac{f(r_*,\omega) g(r'_*,\omega)}{W(\omega)}, & \text{if} \quad r_* < r'_* \\ \frac{f(r'_*,\omega) g(r_*,\omega)}{W(\omega)}, & \text{if} \quad r_* > r'_* \end{cases} \tag{4.76}$$

where $W(\omega)$ is the Wronskian of the two solutions $f(r_*, \omega)$ and $g(r_*, \omega)$: $W(\omega) = g\partial_{r_*} f - f\partial_{r_*} g$. The Wronskian is independent of r_*. The late time tails come from the branch cut along the negative imaginary axis of the Green's function $\tilde{G}(r_*, r'_*; \omega)$ in the complex ω place [182].

Andersson [181] has presented a very clear computation of the branch cut of the Green's function in the low-frequency asymptotic expansion using some results from [183]. Instead of reviewing those details here, we simply write equation (40) of that reference (which has a typo of an overall minus sign) [181]:

$$G^C(r_*, r'_*, t) = -2\pi i M \sqrt{r_* r'_*} \int_0^{-i\infty} \omega \, J_{l+1/2}(\omega r_*) \, J_{l+1/2}(\omega r'_*) \, e^{-i\omega t} d\omega. \tag{4.77}$$

The late time solution using this Green's function is simply (see e.g., equation (7.3.5) of [184] or [182])

$$\psi_l^C(r_*, t) = \int_0^\infty G^C(r_*, r'_*, t) \, \partial_t \psi_0(r'_*, 0) \, dr'_* - \int_0^\infty \partial_t G^C(r_*, r'_*, t) \, \psi_0(r'_*) \, dr'_*, \tag{4.78}$$

where we have implicitly used the fact that the leading contribution only comes from the asymptotic region, and our non-compact initial data has support only in the $r_* \gg M$ asymptotic region.

Inserting initial data (4.67) together with $\partial_t \psi_0(r_*, 0) = 0$ in equation (4.78), we get

$$\psi_l(r_*, t) = 2\pi \mu_l R^l M \sqrt{r_*} \int_0^{-i\infty} d\omega \, \omega^2 \, e^{-i\omega t} \, J_{l+1/2}(\omega r_*) \int_0^\infty dr'_* \, {r'_*}^{-l+1/2} \, J_{l+1/2}(\omega r'_*). \tag{4.79}$$

We can evaluate the second integral in equation (4.79) using identity (6.561-14) of Gradshteyn and Ryzhik [180] to get

$$\psi_l(r_*, t) = \frac{2\sqrt{2\pi}}{(2l-1)!!} \mu_l R^l M \sqrt{r_*} \int_0^{-i\infty} \omega^{l+1/2} \, e^{-i\omega t} \, J_{l+1/2}(\omega r_*) \, d\omega. \tag{4.80}$$

To compute the tail at timelike infinity, we approximate $\omega r_* \ll 1$ to get

$$\psi_l(t, r_* | t \gg r_* \gg M) \sim \frac{4\mu_l R^l M r_*^{l+1}}{(2l-1)!!(2l+1)!!} \int_0^{-i\infty} \omega^{2l+1}\, e^{-i\omega t}\, d\omega \tag{4.81}$$

$$= (-1)^{l+1} 4\mu_l R^l M \frac{(2l)!!}{(2l-1)!!} r_*^{l+1} t^{-2l-2}. \tag{4.82}$$

This expression can be compared with equation (69) of reference [185] and equations IV-1 and IV-2 of reference [186]. In those papers, computations are done differently, and in different contexts.

To compute the tail near null infinity, we approximate $\omega r_* \gg 1$. A similar calculation then gives

$$\psi_l(t, r_*) \sim (-1)^{l+1} 2\mu_l R^l M \frac{l!}{(2l-1)!!} u^{-l-1}. \tag{4.83}$$

This expression can be compared with equation (68) of reference [185].

Equations (4.82) and (4.83) are contributions proportional to $\mu_l M$ to the late time tails in an extreme Reissner-Nordström black hole background.

4.4 Discussion

In this chapter we have revisited the study of the leading order late time decay tails for massless scalar perturbations outside an extreme Reissner-Nordström black hole. While previous studies have analysed this problem in the time domain, we analysed the problem in the frequency domain. A systematic time domain analysis was reported by Sela [162].[2] Sela's analysis is quite involved. The merit of our work lies in its simplicity. We are able to obtain most of the key results of Sela's analysis, including all pre-factors, using rather straightforward Fourier methods.

We find that initial perturbations with generic regular behaviour across the horizon decays

[2] For electromagnetic and gravitational perturbations see [160, 187].

at late times as t^{-2l-2} near timelike infinity ($t \gg r_*$). It decays as u^{-l-1} near future null infinity. The inversion map (4.10) maps the decay behaviour near future null infinity to the decay behaviour v^{-l-1} near the horizon.

For initial data of the form (4.47) at the $u = 0$ surface, there are other contributions to the $t^{-(2l+2)}$ late time tail. They arise due to backscattering from the curvature of spacetime, from terms r^{-k} for $k < l + 1$. These contributions should go as

$$\text{(pre-factor)}\ M^{l+1-k}\hat{c}_k t^{-(2l+2)}. \tag{4.84}$$

We have not addressed these contributions in this chapter. For $k = l$ we considered a related problem with initial data on the $t = 0$ surface of the form (4.67) near spatial infinity. It corresponds to a term proportional to the static moment. Equations (4.82) and (4.83) are the contributions to the late time tails due to these terms. From the Couch-Torrence symmetry, it follows that such a term, if present near the bifurcation surface, will contribute to the v^{-l-1} tail near the horizon. It seems likely that the iterative scheme of [179] can be adopted in the frequency domain to compute tail contributions from $k < l$ terms.

In section 4.2 using the Couch-Torrence symmetry we also related higher multipole Aretakis and Newman-Penrose constants for a massless scalar in an extreme Reissner-Nordström black hole background. Although a number of relations involving Pascal matrices are known in the literature, the identities (4.22) and (4.31) seem to be new. We used these matrix relations to explain relations of functions, but from a mathematical perspective it would be more interesting to turn the logic around. Namely, one could seek interpretations, e.g., through combinatorics or functional methods, of these and more general matrix relations.

All of our analysis is only valid in the asymptotic regions, either near infinity or near the horizon $|r_*| \gg M$. We have not attempted to compute the correct radial dependence of the coefficient of the tail in full generality. From general results in the literature, we do expect the correct radial dependence of the tail at timelike infinfity to be the static solution to the extreme Reissner-Nordström potential [148, 161, 162], cf. (4.40)

$$\frac{r}{M}\left(\frac{r}{M}-1\right)^{-l-1} \tag{4.85}$$

with a constant pre-factor. We expect that the constant pre-factor gets contributions from the Newman-Penrose constant as well as from the Aretakis constant. This has been observed in numerical simulations [160]. The proportionality to the Aretakis constant is briefly discussed in [161, 162], but details have not been presented. Together with the suggestion of references [160, 188] that "initial static moments" are more precisely thought of as initial data with non-zero Newman-Penrose constants, it is natural to conjecture that the total tail coefficient is proportional to the sum of (appropriately normalised) Aretakis and Newman-Penrose constants. It will be interesting to understand this circle of ideas better in the future.

In a series of papers Casals, Gralla, and Zimmerman [189–191] have analyzed the Aretakis instability and related questions in the frequency domain. They have obtained late time decay results on and off the horizon from the AdS_2 perspective. Their analysis is restricted to perturbations with vanishing Aretakis constants. When adapted to an extreme Reissner-Nordström black hole, and extended to perturbations with non-vanishing Aretakis constants, their analysis could be compared to ours through the Couch-Torrence duality. It will be useful to relate our work to their work in detail. It will be very interesting to reproduce the late time tails from a microscopic CFT analysis for extreme black holes.

In a recent paper [192], Camps, Hadar, and Manton studied moduli space scattering of two extreme Reissner-Nordström black holes. They obtained the asymptotic gravitational radiation field wave-form at "moderately" late times, when the two black holes have not merged. They found that the asymptotic radiation field exhibits a quadrupolar late time tail of the form t^{-2l-2} for $l = 2$. It will be interesting to understand how their results relate to our analysis. We hope to report on some of these problems in our future work.

Appendix A

Appendix for Chapter 2

A.1 Perturbation Near $\lambda \to 0$

We have

$$
\begin{aligned}
z^2(\lambda,\eta)\sinh\eta &= \frac{1 - cn(2\lambda\sqrt{\sinh\eta}, 1/\sqrt{2})}{1 + cn(2\lambda\sqrt{\sinh\eta}, 1/\sqrt{2})} \\
\Rightarrow z(\lambda,\eta) &= \lambda + \frac{\sinh^2\eta}{10}\lambda^5 + \frac{\sinh^4\eta}{120}\lambda^9
\end{aligned}
\tag{A.1}
$$

where we have kept terms upto order λ^9. Thus we get the solution for $v(\eta,\lambda)$ as

$$
v(\lambda,\eta) = (-1+\cosh\eta)\lambda - \frac{2}{5}\sinh^4(\eta/2)\lambda^5 - \frac{1}{45}((-7+3\cosh\eta)\sinh^6\eta/2)\lambda^9 \tag{A.2}
$$

Using these solutions and the solutions for x^i we get

$$c(\lambda, \eta) = 1 - \frac{\sinh \eta^4}{75}\lambda^8 + ... \quad \text{(A.3)}$$

Thus

$$\begin{aligned} \left.\frac{\partial c}{\partial \lambda}\right|_{\eta} &= -\frac{8 \sinh \eta^4}{75}\lambda^7 \leq 0 \\ c(0, \eta) &= 1 \end{aligned} \quad \text{(A.4)}$$

Appendix B

Appendix for Chapter 3

B.1 Analysis to show that $\Delta^{(n)}_{\mu\nu}(t_0, x_0) = 0$ for all n

The j-th derivative of $I_{n,m}(R)$ (3.46) gives

$$\partial_R^j I_{n,m}(R) = \sum_{i=0}^{j-1} \partial_R^{j-i-1} \left(\partial_R^i f_{n,m}(x, R)|_{x=R} \right) + \int_0^R \partial_R^j f_{n,m}(x, R) dx \tag{B.1}$$

where

$$f_{n,m}(x, R) = x^{2m} (\cosh R + \cosh x)^{-\frac{n}{2}+1} (\cosh R - \cosh x)^{\frac{n}{2}+1}. \tag{B.2}$$

Thus from (3.48) we have

$$\sum_{n,m} \frac{2^{n+1}}{(2m)!} \partial_x^{2m} \Delta_{xx}^{(n)}(t_0, x_0) \left[\sum_{i=0}^{J+2} \partial_R^{J-i+2} \left(\partial_R^i f_{n,m}(x, R)|_{x=R} \right) \right]_{R=0} = 0. \tag{B.3}$$

First we will show that $\Delta_{xx} = 0$ for even integers. For odd integers, we check it numerically below. Writing $2l$ for the even integer n, and simplifying a bit, we get

$$\sum_{l=0}^{J+1} \sum_m \frac{(l+1)!\, 2^l}{(2m)!} \partial_x^{2m} \Delta_{xx}^{(2l)}(t_0, x_0) C^{J,l,m}(R=0) = 0, \tag{B.4}$$

where

$$C^{J,l,m}(R=0) = \left[\partial_R^{J-(l-1)}\left(R^{2m}\cosh^{1-l}R\,\sinh^{1+l}R\right)\right]_{R=0}. \tag{B.5}$$

Notice that we have restricted l to range from 0 to $J+1$. This is because the terms with $l > J+1$ all vanish at $x = R$. Now, (B.4) can be written as

$$\sum_l (l+1)!2^l\Delta_{xx}^{(2l)}(t_0,x_0)C^{J,l,0}(R=0) = -\sum_{l,m\neq 0}\frac{(l+1)!\,2^l}{(2m)!}\partial_x^{2m}\Delta_{xx}^{(2l)}(t_0,x_0)C^{J,l,m}(R=0). \tag{B.6}$$

Setting $J = 0$, we get

$$\Delta_{xx}^{(0)}(t_0,x_0) = 0. \tag{B.7}$$

$J = 2$ gives

$$\begin{aligned} \Delta_{xx}^{(2)}(t_0,x_0) &= -\frac{1}{2}\Delta_{xx}^{(0)}(t_0,x_0) + \frac{3}{8}\partial_x^2\Delta_{xx}^{(0)}(t_0,x_0), \\ \Rightarrow \qquad \Delta_{xx}^{(2)}(t_0,x_0) &= 0. \end{aligned} \tag{B.8}$$

$J = 4$ gives

$$\begin{aligned} \Delta_{xx}^{(4)}(t_0,x_0) = &-\frac{2}{9}\Delta_{xx}^{(2)}(t_0,x_0) + \frac{1}{3}\partial_x^2\Delta_{xx}^{(2)}(t_0,x_0) \\ &-\frac{1}{9}\Delta_{xx}^{(0)}(t_0,x_0) + \frac{5}{18}\partial_x^2\Delta_{xx}^{(0)}(t_0,x_0) \\ &+\frac{5}{144}\partial_x^4\Delta_{xx}^{(0)}(t_0,x_0) \\ \Rightarrow \qquad \Delta_{xx}^{(4)}(t_0,x_0) = &\ 0 \end{aligned} \tag{B.9}$$

Let's check it for $J = 2N$. It is easy to see that for all $l \geq N+1$ the coefficients C are zero for all m. For $l = N$

$$\begin{aligned} C^{2N,N,0}(R=0) &= [\partial_R^{N+1}(\cosh^{1-N}R\,\sinh^{1+N}R)]_{R=0} \\ &= (N+1)! \end{aligned} \tag{B.10}$$

So the last non vanishing term in the l series in (B.6) with $m = 0$ is $\Delta_{xx}^{(2N)}(t_0, x_0)$ and with $m \neq 0$ it will be a lower order term. Now notice that this term is a linear combination of all the lower order terms and their derivatives, which are zero. Hence $\Delta_{xx}^{(2N)}(t_0, x_0)$ is also zero for $N = 0, 1, \ldots$.

Numerical analysis for odd n

Consider the expansion (3.47):

$$\sum_{n,m,j} \frac{2^{n+1}}{(2m)!} \partial_x^{2m} \Delta_{xx}^{(n)}(t, x_0) R^j C_{n,m}^j(0) = 0, \tag{B.11}$$

where

$$C_{n,m}^j(0) = \frac{1}{j!} \left[\partial_R^j I_{n,m}(R) \right]_{R=0} .$$

We have already seen that the first two terms in this series do not give any constraint and that the terms with odd powers of R (i.e, odd j) impose constraints on even n. Thus one should expect that terms with alternative powers of R (i.e, even j) should impose constraints on odd n. Let us check the term of order R^4. Using numerics one can see that the coefficient $C_{n,m}^4(0)$ is zero for all n, m except for $n = 1, m = 0$. The relevant plots are shown in figure *B*.1.

At order R^6, there are three non-zero coefficients, namely, $C_{3,0}^6, C_{1,0}^6, C_{1,1}^6$. This implies that $\Delta_{xx}^{(3)}(t_0, x_0)$ can be written in terms of $\Delta_{xx}^{(1)}(t_0, x_0)$ and $\partial_x^2 \Delta_{xx}^{(1)}(t_0, x_0)$, and hence $\Delta_{xx}^{(3)}(t_0, x_0)$ is also 0. Figure *B*.2 shows the behavior of coefficients at $R = 0$.

At order R^8, there are several non-zero coefficients which are shown in figure *B*.3. From the plot, one can deduce that $\Delta^{(5)}(t_0, x_0)$ can be written as a linear combination of the lower order terms and their derivatives, and hence is 0.

From this pattern, we conclude that in general the higher order coefficients can be written in terms of lower order coefficients. This means that

$$\Delta_{xx}^{(n)}(t_0, x_0) = 0$$

for all n.

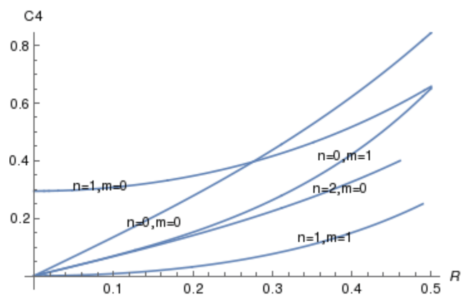

Figure B.1: $C^4_{n,m}$ vs R for different values of n and m.

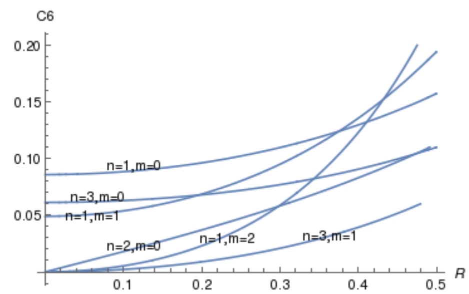

Figure B.2: $C^6_{n,m}$ vs R for different values of n and m.

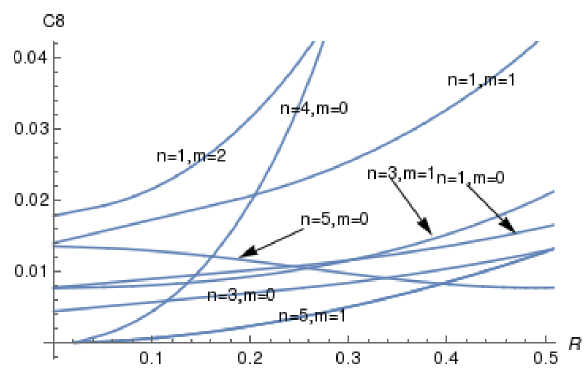

Figure B.3: $C^8_{n,m}$ vs R for different values of n and m.
These figures show for a particular j what coefficients $C^j_{n,m}$ are non-zero as $R \to 0$.

Some numerical computations for boosted black brane:

Here we have plotted different coefficients of R^4-th and R^5-th terms in the expansion series (3.65) around $R = 0$. They are denoted by $C^4_{\mu\nu}$ and $C^5_{\mu\nu}$ respectively.

Figures B.4, B.5 and B.6 show that all the coefficients of R^4-th term are zero as $R \to 0$ except when $n = 1$ and $m_t = m_x = 0$ which give the first equation in (3.69) with $\beta = 0.5$. Similarly for R^5-th term the only non-zero coefficient with maximum value of n as $R \to 0$

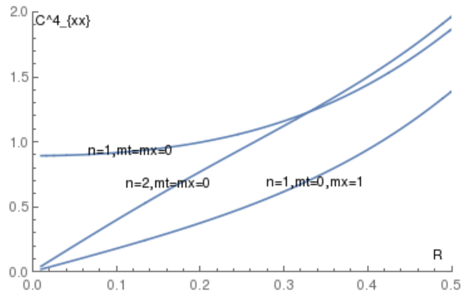

Figure B.4: C^4_{xx} vs R for different values of n, m_t and m_x.

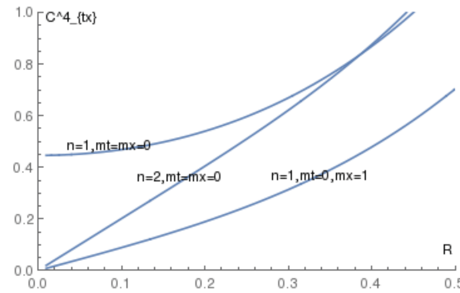

Figure B.5: C^4_{tx} vs R for different values of n, m_t and m_x.

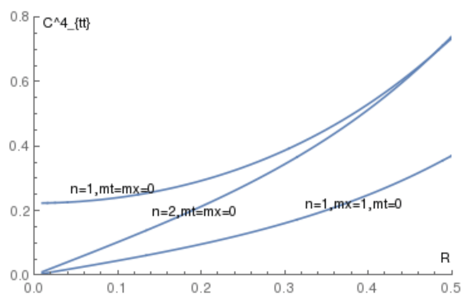

Figure B.6: C^4_{tt} vs R for different values of n, m_t and m_x.

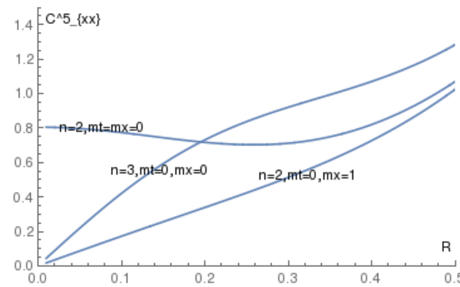

Figure B.7: C^5_{xx} vs R for different values of n, m_t and m_x.

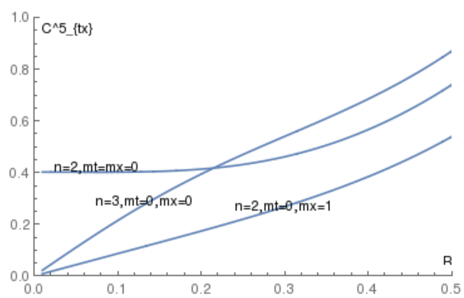

Figure B.8: C^5_{tx} vs R for different values of n, m_t and m_x.

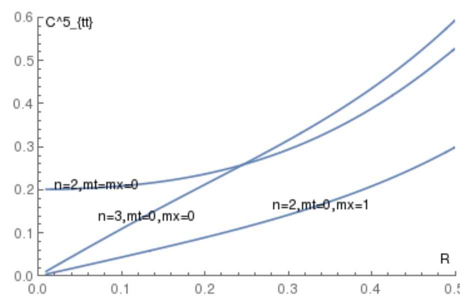

Figure B.9: C^5_{tt} vs R for different values of n, m_t and m_x.

is $n = 2$ and $m_t = m_x = 0$. This can be expressed as a linear combination of the terms with lower values of n which are already zero. These give the second equation in (3.69).

Bibliography

[1] J. D. Bekenstein, “Black holes and the second law,” Nuovo Cim. Lett. bf 4 (1972) 737 .

[2] J. D. Bekenstein, “Black Holes and Entropy,” Phys. Rev. D **7** (1973) 2333 .

[3] S. W. Hawking, “Black hole explosions,” Nature **248** (1974) 30 .

[4] S. W. Hawking, “Particle Creation by Black Holes,” Commun. Math. Phys. **43**, 199 (1975).

[5] L. Susskind, “The World as a Hologram,” J. Math. Phys. **36** (1995) 6377 , hep-th/9409089.

[6] G. ’t. Hooft, “Dimensional Reduction in Quantum Gravity,” gr-qc/9310026.

[7] A. B. Zamolodchikov, “Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory,” JETP Lett. **43** (1986) 730 [Pisma Zh. Eksp. Teor. Fiz. **43** (1986) 565], [INSPIRE].

[8] A. Cappelli, D. Friedan and J. I. Latorre, “C theorem and spectral representation,” Nucl. Phys. B **352**, 616 (1991).

[9] J. L. Cardy, “Is There a c Theorem in Four-Dimensions?,” Phys. Lett. B **215** (1988) 749.

[10] H. Osborn, “Derivation of a Four-dimensional c Theorem,” Phys. Lett. B **222** (1989) 97.

[11] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions," JHEP **1112** (2011) 099, [arXiv:1107.3987 [hep-th]].

[12] Z. Komargodski, "The Constraints of Conformal Symmetry on RG Flows," JHEP **1207**, 069 (2012), [arXiv:1112.4538 [hep-th]].

[13] D. L. Jafferis, I. R. Klebanov, S. S. Pufu and B. R. Safdi, "Towards the F-Theorem: N=2 Field Theories on the Three-Sphere," JHEP **1106**, 102 (2011), [arXiv:1103.1181 [hep-th]].

[14] I. Affleck and A. W. W. Ludwig, "Universal noninteger 'ground state degeneracy' in critical quantum systems," Phys. Rev. Lett. **67**, 161 (1991).

[15] D. Friedan and A. Konechny, "On the boundary entropy of one-dimensional quantum systems at low temperature," Phys. Rev. Lett. **93**, 030402 (2004), [hep-th/0312197].

[16] K. Jensen and A. O'Bannon, "A Constraint on Defect and Boundary Renormalization Group Flows," Phys. Rev. Lett. **116**, 091601 (2016), arXiv:1509.02160 [hep-th].

[17] J. Kaplan and J. Wang, "An Effective Theory for Holographic RG Flows," JHEP **1502**, 056 (2015), [arXiv:1406.4152 [hep-th]].

[18] H. Casini and M. Huerta, "A Finite entanglement entropy and the c-theorem," Phys. Lett. B **600**, 142 (2004), [hep-th/0405111].

[19] H. Casini and M. Huerta, "On the RG running of the entanglement entropy of a circle," Phys. Rev. D **85**, 125016 (2012), [arXiv:1202.5650 [hep-th]].

[20] E. T. Akhmedov, "A Remark on the AdS / CFT correspondence and the renormalization group flow," Phys. Lett. B **442**, 152 (1998), [hep-th/9806217].

[21] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, "Renormalization group flows from holography supersymmetry and a c theorem," Adv. Theor. Math. Phys. **3**, 363 (1999), [hep-th/9904017].

[22] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, "Novel local CFT and exact results on perturbations of N=4 superYang Mills from AdS dynamics," JHEP **9812**, 022 (1998), [hep-th/9810126].

[23] J. de Boer, E. P. Verlinde and H. L. Verlinde, "On the holographic renormalization group," JHEP **0008**, 003 (2000), [hep-th/9912012].

[24] R. C. Myers and A. Sinha, "Holographic c-theorems in arbitrary dimensions," JHEP **1101**, 125 (2011), [arXiv:1011.5819 [hep-th]].

[25] R. C. Myers and A. Sinha, "Seeing a c-theorem with holography," Phys. Rev. D **82**, 046006 (2010), [arXiv:1006.1263 [hep-th]].

[26] S. Banerjee and P. Paul, "Black Hole Singularity, Generalized (Holographic) c-Theorem and Entanglement Negativity," JHEP 1702 (2017) 043, [arXiv:1512.02232[hep-th]].

[27] S. Banerjee, "RG Flow and Thermodynamics of Causal Horizons in AdS," JHEP **1510**, 098 (2015), [arXiv:1508.01343 [hep-th]].

[28] S. Banerjee and A. Bhattacharyya, "RG Flow and Thermodynamics of Causal Horizons in Higher-Derivative AdS Gravity," JHEP **05**, 126 (2016), arXiv:1509.08475 [hep-th].

[29] I. Heemskerk and J. Polchinski, "Holographic and Wilsonian Renormalization Groups," JHEP **1106**, 031 (2011), [arXiv:1010.1264 [hep-th]].

[30] T. Faulkner, H. Liu and M. Rangamani, "Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm," JHEP **1108**, 051 (2011), [arXiv:1010.4036 [hep-th]].

[31] D. Elander, H. Isono and G. Mandal, “Holographic Wilsonian flows and emergent fermions in extremal charged black holes,” JHEP **1111**, 155 (2011), [arXiv:1109.3366 [hep-th]].

[32] D. Radicevic, “Connecting the Holographic and Wilsonian Renormalization Groups,” JHEP **1112**, 023 (2011), [arXiv:1105.5825 [hep-th]].

[33] E. D’Hoker and D. Z. Frredman, “Supersymmetric Gauge Theories and the AdS/CFT Correspondence,” arXiv:hep-th/0201253.

[34] G. W. Gibbons and S. W. Hawking, “Cosmological Event Horizons, Thermodynamics, and Particle Creation,” Phys. Rev. D **15**, 2738 (1977).

[35] T. Jacobson and R. Parentani, “Horizon entropy,” Found. Phys. **33** (2003) 323, [gr-qc/0302099].

[36] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. **2**, 505 (1998), [hep-th/9803131].

[37] J. M. Maldacena, “Eternal black holes in anti-de Sitter,” JHEP **0304**, 021 (2003), [hep-th/0106112].

[38] H. Liu and M. Mezei, “A Refinement of entanglement entropy and the number of degrees of freedom,” JHEP **1304** (2013) 162, [arXiv:1202.2070 [hep-th]].

[39] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. **96**, 181602 (2006), [hep-th/0603001].

[40] V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP **0707**, 062 (2007), [arXiv:0705.0016 [hep-th]].

[41] V. E. Hubeny, M. Rangamani and E. Tonni, “Global properties of causal wedges in asymptotically AdS spacetimes,” JHEP **1310**, 059 (2013), [arXiv:1306.4324 [hep-th]].

[42] V. E. Hubeny and M. Rangamani, "Causal Holographic Information," JHEP **1206**, 114 (2012), [arXiv:1204.1698 [hep-th]].

[43] G. Vidal and R. F. Werner, "Computable measure of entanglement," Phys. Rev. A **65**, 032314 (2002)

[44] G. Vidal, "Entanglement monotones," J. Mod. Opt. **47**, 355 (2000), [quant-ph/9807077].

[45] P. Calabrese, J. Cardy and E. Tonni, "Entanglement negativity in quantum field theory," Phys. Rev. Lett. **109**, 130502 (2012), [arXiv:1206.3092 [cond-mat.stat-mech]].

[46] P. Calabrese, J. Cardy and E. Tonni, "Entanglement negativity in extended systems: A field theoretical approach," J. Stat. Mech. **1302**, P02008 (2013), [arXiv:1210.5359 [cond-mat.stat-mech]].

[47] P. Calabrese, L. Tagliacozzo and E. Tonni, "Entanglement negativity in the critical Ising chain," J. Stat. Mech. **1305**, P05002 (2013), [arXiv:1302.1113 [cond-mat.stat-mech]].

[48] V. Alba, "Entanglement negativity and conformal field theory: a Monte Carlo study," J. Stat. Mech. **1305**, P05013 (2013), [arXiv:1302.1110 [cond-mat.stat-mech]].

[49] A. Coser, E. Tonni and P. Calabrese, "Partial transpose of two disjoint blocks in XY spin chains," J. Stat. Mech. **1508**, no. 8, P08005 (2015), [arXiv:1503.09114 [cond-mat.stat-mech]].

[50] A. Coser, E. Tonni and P. Calabrese, "Towards entanglement negativity of two disjoint intervals for a one dimensional free fermion," arXiv:1508.00811 [cond-mat.stat-mech].

[51] V. Eisler and Z. Zimboras, " Entanglement negativity in the harmonic chain out of equilibrium, " New J. Phys. 16 (2014) 123020, arXiv: 1406.5474 [cond-mat.stat-mech]

[52] P. Calabrese, J. Cardy and E. Tonni, "Finite temperature entanglement negativity in conformal field theory," J. Phys. A **48**, no. 1, 015006 (2015), [arXiv:1408.3043 [cond-mat.stat-mech]].

[53] M. Rangamani and M. Rota, "Comments on Entanglement Negativity in Holographic Field Theories," JHEP **1410**, 60 (2014), [arXiv:1406.6989 [hep-th]].

[54] E. Perlmutter, M. Rangamani and M. Rota, "Central Charges and the Sign of Entanglement in 4D Conformal Field Theories," Phys. Rev. Lett. **115**, no. 17, 171601 (2015), [arXiv:1506.01679 [hep-th]].

[55] M. Kulaxizi, A. Parnachev and G. Policastro, "Conformal Blocks and Negativity at Large Central Charge," JHEP **1409**, 010 (2014), [arXiv:1407.0324 [hep-th]].

[56] T. Faulkner, "The Entanglement Renyi Entropies of Disjoint Intervals in AdS/CFT," arXiv:1303.7221 [hep-th].

[57] T. Hartman, "Entanglement Entropy at Large Central Charge," arXiv:1303.6955 [hep-th].

[58] I. R. Klebanov, T. Nishioka, S. S. Pufu and B. R. Safdi, "Is Renormalized Entanglement Entropy Stationary at RG Fixed Points?," JHEP **1210**, 058 (2012), [arXiv:1207.3360 [hep-th]].

[59] T. Nishioka, "Relevant Perturbation of Entanglement Entropy and Stationarity," Phys. Rev. D **90**, no. 4, 045006 (2014), [arXiv:1405.3650 [hep-th]].

[60] S. Banerjee, Y. Nakaguchi and T. Nishioka, "Renormalized Entanglement Entropy on Cylinder," JHEP **03**, 048 (2016), arXiv:1508.00979 [hep-th].

[61] H. Casini and M. Huerta, “Entanglement and alpha entropies for a massive scalar field in two dimensions,” J. Stat. Mech. **0512**, P12012 (2005), [cond-mat/0511014].

[62] H. Casini, M. Huerta, R. C. Myers and A. Yale, “Mutual information and the F-theorem,” JHEP **1510**, 003 (2015), [arXiv:1506.06195 [hep-th]].

[63] M. F. Paulos, “Holographic phase space: c-functions and black holes as renormalization group flows,” JHEP **1105**, 043 (2011), [arXiv:1101.5993 [hep-th]].

[64] S. Cremonini and X. Dong, “Constraints on renormalization group flows from holographic entanglement entropy,” Phys. Rev. D **89**, no. 6, 065041 (2014), [arXiv:1311.3307 [hep-th]].

[65] J. T. Liu and Z. Zhao, “Holographic Lifshitz flows and the null energy condition,” arXiv:1206.1047 [hep-th].

[66] P. Kraus, H. Ooguri and S. Shenker, “Inside the horizon with AdS / CFT,” Phys. Rev. D **67**, 124022 (2003), [hep-th/0212277].

[67] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, “The Black hole singularity in AdS / CFT,” JHEP **0402**, 014 (2004), [hep-th/0306170].

[68] G. Festuccia and H. Liu, “Excursions beyond the horizon: Black hole singularities in Yang-Mills theories. I.,” JHEP **0604**, 044 (2006), [hep-th/0506202].

[69] M. Van Raamsdonk, “Building up spacetime with quantum entanglement,” Gen. Rel. Grav. **42**, 2323 (2010), [arXiv:1005.3035 [hep-th]].

[70] M. Van Raamsdonk, “Comments on quantum gravity and entanglement,” arXiv:0907.2939 [hep-th].

[71] J. Maldacena and L. Susskind, “Cool horizons for entangled black holes,” Fortsch. Phys. **61**, 781 (2013), [arXiv:1306.0533 [hep-th]].

[72] S. H. Shenker and D. Stanford, "Black holes and the butterfly effect," JHEP **1403**, 067 (2014), [arXiv:1306.0622 [hep-th]].

[73] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, "Black Holes: Complementarity or Firewalls?," JHEP **1302**, 062 (2013), [arXiv:1207.3123 [hep-th]].

[74] A. Almheiri, D. Marolf, J. Polchinski, D. Stanford and J. Sully, "An Apologia for Firewalls," JHEP **1309**, 018 (2013), [arXiv:1304.6483 [hep-th]].

[75] S. L. Braunstein, S. Pirandola and K. ?yczkowski, "Better Late than Never: Information Retrieval from Black Holes," Phys. Rev. Lett. **110**, no. 10, 101301 (2013), [arXiv:0907.1190 [quant-ph]].

[76] K. Papadodimas and S. Raju, "An Infalling Observer in AdS/CFT," JHEP **1310**, 212 (2013), [arXiv:1211.6767 [hep-th]].

[77] T. Hartman and J. Maldacena, "Time Evolution of Entanglement Entropy from Black Hole Interiors," JHEP **1305**, 014 (2013), [arXiv:1303.1080 [hep-th]].

[78] G. Vidal, "Entanglement Renormalization," Phys. Rev. Lett. **99**, no. 22, 220405 (2007), [cond-mat/0512165].

[79] B. Swingle, "Constructing holographic spacetimes using entanglement renormalization," arXiv:1209.3304 [hep-th].

[80] B. Swingle, "Entanglement Renormalization and Holography," Phys. Rev. D **86**, 065007 (2012), [arXiv:0905.1317 [cond-mat.str-el]].

[81] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico and M. M. Wolf, "Renormalization group transformations on quantum states," , Phys. Rev. Lett. **94**, 140601 (2005) [quant-ph/0410227].

[82] M. Nozaki, S. Ryu and T. Takayanagi, “Holographic Geometry of Entanglement Renormalization in Quantum Field Theories,” JHEP **1210**, 193 (2012), [arXiv:1208.3469 [hep-th]].

[83] Evenbly, G. and Vidal, G., "Tensor Network Renormalization Yields the Multiscale Entanglement Renormalization Ansatz," Phys. Rev. Lett. 115, 200401 (2015), arXiv:1502.05385 [cond-mat.str-el]

[84] B. Czech, G. Evenbly, L. Lamprou, S. McCandlish, X. L. Qi, J. Sully and G. Vidal, “Tensor network quotient takes the vacuum to the thermal state,” Phys. Rev. B **94**, 085101 (2016), arXiv:1510.07637 [cond-mat.str-el].

[85] R. M. Wald, *General relativity*, University of Chicago Press, 1984, doi:10.7208/chicago/9780226870373.001.0001.

[86] K. Goldstein, R. P. Jena, G. Mandal and S. P. Trivedi, “A C-function for non-supersymmetric attractors,” JHEP **0602**, 053 (2006), [hep-th/0512138].

[87] S. Kachru, N. Kundu, A. Saha, R. Samanta and S. P. Trivedi, “Interpolating from Bianchi Attractors to Lifshitz and AdS Spacetimes,” JHEP **1403**, 074 (2014), [arXiv:1310.5740 [hep-th]].

[88] A. Bhattacharyya, S. S. Haque, V. Jejjala, S. Nampuri and A. Veliz-Osorio, “Attractive holographic c-functions,” JHEP **1411**, 138 (2014), [arXiv:1407.0469 [hep-th]].

[89] R. M. Wald, “Black hole entropy is the Noether charge,” Phys. Rev. D **48** (1993) 3427, [gr-qc/9307038].

[90] T. Jacobson, G. Kang and R. C. Myers, “On black hole entropy,” Phys. Rev. D **49**, 6587 (1994), [gr-qc/9312023].

[91] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Phys. Rev. D **50**, 846 (1994), [arXiv:gr-qc/9403028].

[92] T. Jacobson and R. C. Myers, "Black hole entropy and higher curvature interactions," Phys. Rev. Lett. **70**, 3684 (1993), [hep-th/9305016].

[93] T. Jacobson, G. Kang, R. C. Myers, "Increase of black hole entropy in higher curvature gravity," Phys. Rev. **D52**, 3518-3528 (1995), [gr-qc/9503020].

[94] S. Sarkar and A. C. Wall, "Generalized second law at linear order for actions that are functions of Lovelock densities," Phys. Rev. **D88**, 044017 (2013), [arXiv:1306.1623 [gr-qc]].

[95] X. Dong, "Holographic Entanglement Entropy for General Higher Derivative Gravity," JHEP **1401**, 044 (2014), [arXiv:1310.5713 [hep-th]].

[96] J. Camps, "Generalized entropy and higher derivative Gravity," JHEP **1403**, 070 (2014), [arXiv:1310.6659 [hep-th]].

[97] S. Bhattacharjee, S. Sarkar and A. Wall, "The holographic entropy increases in quadratic curvature gravity," Phys. Rev. **D92**, 064006 (2015), arXiv:1504.04706 [gr-qc].

[98] A. C. Wall, "A Second Law for Higher Curvature Gravity," arXiv:1504.08040 [gr-qc].

[99] S. Bhattacharjee, A. Bhattacharyya, S. Sarkar and A. Sinha, "Entropy functionals and c-theorems from the second law," arXiv:1508.01658 [hep-th].

[100] A. Bhattacharyya, M. Sharma and A. Sinha, "On generalized gravitational entropy, squashed cones and holography," JHEP **1401**, 021 (2014), [arXiv:1308.5748 [hep-th]].

A. Bhattacharyya, A. Kaviraj and A. Sinha, "Entanglement entropy in higher derivative holography," JHEP **1308**, 012 (2013), [arXiv:1305.6694 [hep-th]].

A. Bhattacharyya and M. Sharma, "On entanglement entropy functionals in higher derivative gravity theories," JHEP **1410**, 130 (2014), [arXiv:1405.3511 [hep-th]].

[101] D. V. Fursaev, A. Patrushev and S. N. Solodukhin, "Distributional Geometry of Squashed Cones," Phys. Rev. D **88**, no. 4, 044054 (2013), [arXiv:1306.4000 [hep-th]].

[102] J. de Boer, M. Kulaxizi and A. Parnachev, "Holographic Entanglement Entropy in Lovelock Gravities," JHEP **1107**, 109 (2011), [arXiv:1101.5781 [hep-th]].

[103] L. Y. Hung, R. C. Myers and M. Smolkin, "On Holographic Entanglement Entropy and Higher Curvature Gravity," JHEP **1104**, 025 (2011), [arXiv:1101.5813 [hep-th]].

[104] J. Cardy and E. Tonni, "Entanglement Hamiltonians in two-dimensional conformal field theory," J.Stat.Mech (2016) 123103, [arXiv:1608.01283 [cond-mat.stat-mech]].

[105] J. M. Bardeen, B. Carter, and S. W. Hawking, "The Four Laws of Black Hole Mechanics," Commun. Math. Phys. **31**, 161 (1973).

[106] T. Jacobson, "Thermodynamics of Space-time: The Einstein Equation of State," Phys. Rev. Lett. **75**, 1260 (1995), [gr-qc/9504004].

[107] T. Padmanabhan, "Thermodynamical Aspects of Gravity: New insights," Rept. Prog. Phys. **73** 046901 (2010), [arXiv: 0911.5004 [gr-qc]].

[108] E. P. Verlinde, "On the Origin of Gravity and the Laws of Newton," JHEP **1104**, 029 (2011), [arXiv:1001.0785[hep-th]].

[109] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998), [Int. J. Theor. Phys. **38**, 1113 (1999)], [arXiv:hep-th/9711200].

[110] E. Witten, "Anti-de Sitter space and holography,“ Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[111] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N Field Theories, String Theory and Gravity," Phys. Rept. **323**, 183 (2000), [arXiv:hep-th/9905111].

[112] N. Lashkari, M. B. McDermott, M. V. Raamsdonk, "Gravitational Dynamics From Entanglement "Thermodynamics"," JHEP **1404**, 195 (2014), [arXiv:1308.3716 [hep-th]].

[113] T. Faulkner, M. Guica, T. Hartman, R. C. Myers, M. V. Raamsdonk, "Gravitation from entanglement in holographic CFTs," JHEP **1403**, 051 (2014), [arXiv:1312.7856[hep-th]].

[114] D. D. Blanco, H. Casini, L. Y. Hung, R. C. Myers, "Relative Entropy and Holography," JHEP **1308**, 060 (2014), [arXiv:1305.3182 [hep-th]].

[115] P. Paul and P. Roy, "Linearized Einstein's Equations around pure BTZ from Entanglement Thermodynamics," [arXiv:1803.06484[hep-th]].

[116] A. J. Speranza, "Entanglement Entropy of Excited States in Conformal Perturbation Theory and the Einstein Equation," JHEP **04** (2016) 105, [arXiv:1602.01380].

[117] B. Mosk, "Holographic Equivalence Between the First Law of Entanglement Entropy and the Linearized Gravitational Equations," Phys. Rev. D **94** (2016) 126001, [arXiv:1608.06292].

[118] B. Czech, L. Lamprou, S. McCandlish, B. Mosk and J. Sully, "Equivalent Equations of Motion for Gravity and Entropy," JHEP **02** (2017) 004, [arXiv:1608.06282].

[119] B. Swingle and M. V. Raamsdonk, "Universality of Gravity from Entanglement," arXiv:1405.2933.

[120] T. Jacobson, "Entanglement Equilibrium and the Einstein Equation," Phys. Rev. Lett. **116** (2016) no. 20, 201101, [arXiv:1505.04753].

[121] Xian-Hui Ge, Bin Wang, "Quantum computational complexity, Einstein's equations and accelerated expansion of the Universe," JCAP **1802** (2018) no.02, 047, [arXiv:1708.06811 [hep-th]].

[122] M. V. Raamsdonk, "Lectures on Gravity and Entanglement," arXiv:1609.00026[hep-th].

[123] J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, "Thermodynamical Property of Entanglement Entropy for Excited States," Phys. Rev. Lett. **110**, no. 9, 091602 (2013), [arXiv:1212.1164[hep-th]].

[124] M. Nozaki, T. Numasawa, and T. Takayanagi, "Holographic Local Quenches and Entanglement Density," JHEP **1308** 102 (2013), [arXiv:1302.5703 [hep-th]].

[125] M. Nozaki, T. Numasawa, A. Prudenziati, T. Takayanagi, "Dynamics of Entanglement Entropy from Einstein Equation," Phys. Rev. D **88** (2013) no. 2, 026012, [arXiv:1304.7100 [hep-th]].

[126] D. Allahbakhshi, M. Alishahiha and A. Naseh, "Entanglement Thermodynamics," JHEP **1308** (2013) 102, [arXiv: 1305.2728 [hep-th]].

[127] G. Wong, I. Klich, L. A. Pando Zayas and D. Vaman, "Entanglement Temperature and Entanglement Entropy of Excited States," JHEP **1312** (2013) 020, [arXiv:1305.3291 [hep-th]].

[128] I. Bakhmatov, N. S. Deger, J. Gutowski, E. Ó. Colgáin, H. Yavartanoo, "Calibrated Entanglement Entropy," JHEP **07** (2017) 117, [arXiv:1705.08319 [hep-th]]

[129] H. Casini, M. Huerta, R. C. Myers, "Towards a derivation of holographic entanglement entropy," JHEP **1105**, 036 (2016), [arXiv:1102.0440[hep-th]].

[130] R. C. Myers, "Stress tensors and Casimir energies in the AdS/CFT correspondence," Phys. Rev. D **60**, 046002 (1999), [hep-th/9903203].

[131] K. Skenderis, "Lecture Notes on Holographic Renormalization," Class.Quant.Grav. 19 (2002) 5849-5876, [arXiv:hep-th/0209067]

[132] S. de Haro, K. Skenderis, S. N. Solodukhin, "Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence," Commun. Math. Phys. **217**, 595 (2001), [hep-th/0002230].

[133] J. Erdmenger and H. Osborn, "Conserved Currents and the Energy Momentum Tensor in Conformally Invariant Theories for General Dimensions," Nucl. Phys. **B 483** (1997) 431, [hep-th/9605009].

[134] V. Balasubramanian and P. kraus, "A Stress Tensor For Anti-de Sitter Gravity," Commun. Math. Phys. **208** (1999) 413-428, [hep-th/9902121].

[135] K. Skenderis, "Asymptotically Anti-de Sitter Spacetimes and Their Stress Energy Tensor," Int. J. Mod. Phys. **A16** (2001) 740-749, [hep-th/0010138].

[136] V. Vedral, "Introduction to quantum information science," Oxford University Press, New York (2006).

[137] G. Sárosi and T. Ugajin,"Relative Entropy of Excited States in Two Dimensional Conformal Field Theories," JHEP **1607** (2016) 114, [arXiv:1603.03057[hep-th]]

[138] M.M. Sheikh-Jabbari and H. Yavartanoo, "Excitation Entanglement Entropy in 2d Conformal Field Theories," Phys. Rev. D **94**, 126006 (2016), [arXiv:1605.00341 [hep-th]]

[139] T. Hartman and N. Afkhami-Jeddi, "Speed Limits for Entanglement," arXiv:1512.02695 [hep-th].

[140] M. Bañados, C. Teitelboim and J. Zanelli, "The Black hole in three dimensional spacetime," Phys. Rev. Lett. **69** (1992) 1849-1851, [hep-th/9204099].

[141] I. Kanitscheider and K. Skenderis, "Universal hydrodynamics of non-conformal branes," JHEP **04** (2009) 062, arXiv:0901.1487 [hep-th].

[142] J. Bhattacharya and T. Takayanagi, "Entropic Counterpart of Perturbative Einstein Equation," JHEP **1310**, 219 (2013), [arXiv:1308.3792 [hep-th]].

[143] T. Faulkner, F. M. Haehl, E. Hijano, O. Parrikar, C. Rabideau and M. V. Raamsdonk, "Nonlinear Gravity from Entanglement in Conformal Field Theories," JHEP **08** (2017) 057, [arXiv:1705.03026[hep-th]].

[144] D. Blanco, M. Leston and G. PÃľrez-Nadal, "Gravity from entanglement for boundary subregions," JHEP 0618 (2018) 130, arXiv:1803.01874 [hep-th].

[145] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," Phys. Lett. **B379** (1996) 99-104, arXiv:hep-th/9601029.

[146] Y. Choquet-Bruhat, "Théoreme d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires," Acta Math. **88** (1952) 141-225.

[147] Y. Choquet-Bruhat and R. Geroch, "Global aspects of the Cauchy problem in general relativity," Comm. Math. Phys. **14** (1969) 329-335.

[148] R. H. Price, "Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations," Phys. Rev. D **5**, 2419 (1972).

[149] M. Dafermos and I. Rodnianski, "A proof of Price's law for the collapse of a self-gravitating scalar field," Invent. Math. **162**, 381 (2005), [gr-qc/0309115].

[150] M. Dafermos, “Price’s law, mass inflation, and strong cosmic censorship,” in “Relativity today: Proceedings of the Seventh Hungarian Relativity Workshop,” edited by I. Rácz (Akadémiai Kiadó, 2004), p. 79 gr-qc/0401121.

[151] M. Dafermos and I. Rodnianski, “Lectures on black holes and linear waves,” in “Evolution equations,” Clay Math. Proc. **17** edited by D. Ellwood, I. Rodnianski, G. Staffilani and J. Wunsch (American Mathematical Society, 2013), p. 97 [arXiv:0811.0354 [gr-qc]].

[152] M. Dafermos and I. Rodnianski, “The black hole stability problem for linear scalar perturbations,” in “XVIth International Congress on Mathematical Physics,” edited by P. Exner (World Scientific, London, 2010), p. 421, doi:10.1142/9789814374552–0008, 10.1142/9789814304634–0032 [arXiv:1010.5137 [gr-qc]].

[153] M. Dafermos, G. Holzegel and I. Rodnianski, “The linear stability of the Schwarzschild solution to gravitational perturbations,” arXiv:1601.06467 [gr-qc].

[154] E. Poisson and W. Israel, “Internal structure of black holes,” Phys. Rev. D **41**, 1796 (1990).

[155] M. Dafermos, “The interior of charged black holes and the problem of uniqueness in general relativity,” Commun. Pure Appl. Math. **58**, 445 (2005), [gr-qc/0307013].

[156] D. Marolf and A. Ori, “Outgoing gravitational shock-wave at the inner horizon: The late-time limit of black hole interiors,” Phys. Rev. D **86**, 124026 (2012), [arXiv:1109.5139 [gr-qc]].

[157] J. Bičák, “Gravitational collapse with charge and small asymmetries. I. Scalar perturbations,” Gen. Rel. Grav. **3**, 33 (1972).

[158] W. E. Couch and R. J. Torrence, “Conformal invariance under spatial inversion of extreme Reissner-Nordström black holes,” Gen. Rel. Grav. **16**, 789 (1984).

[159] C. J. Blaksley and L. M. Burko, “Late-time tails in the Reissner-Nordström spacetime revisited,” Phys. Rev. D **76**, 104035 (2007), [arXiv:0710.2915 [gr-qc]].

[160] J. Lucietti, K. Murata, H. S. Reall and N. Tanahashi, “On the horizon instability of an extreme Reissner-Nordström black hole,” JHEP **1303**, 035 (2013), [arXiv:1212.2557 [gr-qc]].

[161] A. Ori, “Late-time tails in extremal Reissner-Nordstrom spacetime,” arXiv:1305.1564 [gr-qc].

[162] O. Sela, “Late-time decay of perturbations outside extremal charged black hole,” Phys. Rev. D **93**, no. 2, 024054 (2016), [arXiv:1510.06169 [gr-qc]].

[163] S. Bhattacharjee, B. Chakrabarty, D. D. K. Chow, P. Paul and A. Virmani, “On Late Time Tails in an Extreme Reissner-Noerdstrom Black Hole: Frequency Domain Analysis,” Class. Quant. Grav. 35 (2018) no.20, 205002, [arXiv:1805.10655[gr-qc]].

[164] S. Aretakis, “Stability and instability of extreme Reissner-Nordstróm black hole spacetimes for linear scalar perturbations I,” Commun. Math. Phys. **307**, 17 (2011), [arXiv:1110.2007 [gr-qc]].

[165] S. Aretakis, “Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations II,” Annales Henri Poincaré **12**, 1491 (2011), [arXiv:1110.2009 [gr-qc]].

[166] S. Aretakis, “A note on instabilities of extremal black holes under scalar perturbations from afar,” Class. Quant. Grav. **30**, 095010 (2013), [arXiv:1212.1103 [gr-qc]].

[167] S. Aretakis, "Nonlinear instability of scalar fields on extremal black holes," Phys. Rev. D **87**, 084052 (2013), [arXiv:1304.4616 [gr-qc]].

[168] P. Bizoń and H. Friedrich, "A remark about wave equations on the extreme Reissner-Nordström black hole exterior," Class. Quant. Grav. **30**, 065001 (2013), [arXiv:1212.0729 [gr-qc]].

[169] Y. Angelopoulos, S. Aretakis and D. Gajic, "The trapping effect on degenerate horizons," Annales Henri Poincaré **18**, no. 5, 1593 (2017), [arXiv:1512.09094 [gr-qc]].

[170] Y. Angelopoulos, S. Aretakis and D. Gajic, "Asymptotic blow-up for a class of semilinear wave equations on extremal Reissner-Nordström spacetimes," arXiv:1612.01562 [math.AP].

[171] Y. Angelopoulos, S. Aretakis and D. Gajic, "Late-time asymptotics for the wave equation on spherically symmetric, stationary spacetimes," Adv. Math. **323**, 529 (2018), [arXiv:1612.01566 [math.AP]].

[172] E. T. Newman and R. Penrose, "New conservation laws for zero rest-mass fields in asymptotically flat space-time," Proc. Roy. Soc. Lond. A **305**, 175 (1968)

[173] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large *N* field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000), [hep-th/9905111].

[174] D. D. K. Chow, "Conformal isometry of the six-dimensional black string and dualities of null geodesics," arXiv:1711.11020 [hep-th].

[175] H. Godazgar, M. Godazgar and C. N. Pope, "Aretakis charges and asymptotic null infinity," Phys. Rev. D **96**, no. 8, 084055 (2017), [arXiv:1707.09804 [hep-th]].

[176] A. Edelman and G. Strang, "Pascal matrices," Amer. Math. Monthly **111**, 189 (2004).

[177] K. S. Thorne, “Nonspherical gravitational collapse: A short review,” in “Magic without magic: John Archibald Wheeler. A collection of essays in honor of his sixtieth birthday,” edited by J. R. Klauder (Freeman, 1972), p. 231.

[178] C. Gundlach, R. H. Price and J. Pullin, “Late-time behavior of stellar collapse and explosions. I. Linearized perturbations,” Phys. Rev. D **49**, 883 (1994), [gr-qc/9307009].

[179] L. Barack, “Late time dynamics of scalar perturbations outside black holes. I. A shell toy model,” Phys. Rev. D **59**, 044016 (1999), [gr-qc/9811027].

[180] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 7th edition, Elsevier/Academic Press, Amsterdam, 2007.

[181] N. Andersson, “Evolving test fields in a black-hole geometry,” Phys. Rev. D **55**, 468 (1997), [gr-qc/9607064].

[182] E. S. C. Ching, P. T. Leung, W. M. Suen and K. Young, “Wave propagation in gravitational systems: Late time behavior,” Phys. Rev. D **52**, 2118 (1995), [gr-qc/9507035].

[183] J. A. H. Futterman, F. A. Handler and R. A. Matzner, *Scattering from black holes*, Cambridge University Press, 1988.

[184] P. M. Morse and H. Feshbach, *Methods of theoretical physics, Part I* (International Series in Pure and Applied Physics), McGraw-Hill, 1953.

[185] E. W. Leaver, “Spectral decomposition of the perturbation response of the Schwarzschild geometry,” Phys. Rev. D **34**, 384 (1986).

[186] C. T. Cunningham, R. H. Price and V. Moncrief, “Radiation from collapsing relativistic stars. I . Linearized odd-parity radiation,” Astrophysical Journal **224**, 643 (1978).

[187] O. Sela, "Late-time decay of coupled electromagnetic and gravitational perturbations outside an extremal charged black hole," Phys. Rev. D **94**, no. 8, 084006 (2016), [arXiv:1609.00397 [gr-qc]].

[188] R. Gómez, J. Winicour and B. G. Schmidt, "Newman-Penrose constants and the tails of selfgravitating waves," Phys. Rev. D **49**, 2828 (1994).

[189] M. Casals, S. E. Gralla and P. Zimmerman, "Horizon instability of extremal Kerr black holes: Nonaxisymmetric modes and enhanced growth rate," Phys. Rev. D **94**, no. 6, 064003 (2016), [arXiv:1606.08505 [gr-qc]].

[190] P. Zimmerman, "Horizon instability of extremal Reissner-Nordström black holes to charged perturbations," Phys. Rev. D **95**, no. 12, 124032 (2017), [arXiv:1612.03172 [gr-qc]].

[191] S. E. Gralla and P. Zimmerman, "Scaling and universality in extremal black hole perturbations," JHEP **06**, 061 (2018), arXiv:1804.04753 [gr-qc].

[192] J. Camps, S. Hadar and N. S. Manton, "Exact gravitational wave signatures from colliding extreme black holes," Phys. Rev. D **96**, no. 6, 061501(R) (2017), [arXiv:1704.08520 [gr-qc]].