# The Non-Relativistic Limit of the AdS/CFT Conjecture

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# Certificate

This is to certify that the Ph.D. thesis titled "The Non-Relativistic Limit of the AdS/CFT Conjecture" submitted by Arjun Bagchi is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Ashoke Sen

**Professor Ashoke Sen**
Thesis Adviser

Date: 03.05.2010

# Declaration

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

This work was done under guidance of Professor Ashoke Sen, at Harish Chandra Research Institute, Allahabad.

Date: 03.05.2010

**Arjun Bagchi**

Ph.D. Candidate

**To My Wife and My Parents ....**

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*There is a theory which states that if ever anybody discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.*

– **Douglas Adams**

*"The Hitchhiker's Guide to the Galaxy."*

# Synopsis

String Theory has been the leading candidate for a theory of quantum gravity over the last thirty years. One of the most interesting theoretical developments of String theory which emerged in the past decade has been the AdS/CFT conjecture. This is a conjectured duality between a theory of gravity or string theory living in a negatively curved Anti de-Sitter spacetime and a conformal field theory living on the boundary of the spacetime. A conformal field theory is a quantum field theory that is invariant under the group of conformal transformations. In its strongest avatar, the correspondence connects Type IIB superstring theory in $AdS_5 \times S^5$ to $\mathcal{N} = 4$ Supersymmetric $SU(N)$ Yang-Mills theory on the boundary of $AdS_5$. This is a specific example of the holographic principle which states that the description of the volume of a region should be encoded on the boundary of the region. This is a duality in the usual sense, viz. when one description is weakly coupled, the dual description is strongly coupled. Thus it allows the use of weak-coupling perturbative physics in one theory to learn about the dual strongly coupled theory. Though the AdS/CFT correspondence has not been proved directly, there have been a large number of checks on both sides of the duality which have convinced physicists of its correctness. This remarkable conjecture thus allows us to learn about gravity from the theory without gravity and vice-versa.

Particular focus has been given in trying to use the conjecture to improve our understanding of the theory of strong interactions. One of the newer applications, called the Anti de Sitter-Condensed Matter Theory (AdS/CMT) correspondence has opened up a new way to study strongly interacting systems, which remains one of the stumbling blocks of theoretical physics. This also has been instrumental in shedding new light in the study of high $T_c$ superconductivity.

This thesis aims at looking at the non-relativistic limit of the AdS/CFT conjecture. There has been a substantial progress in understanding AdS/CFT from a non-relativistic point of view recently. This is particularly important because of the potential application of the gauge-gravity duality to real life strongly interacting condensed matter systems which are inherently non-relativistic. The non-relativistic sector might also be a tractable sub-sector where we could learn more about the systematics of AdS/CFT.

The focus of the thesis would be on a systematic non-relativistic limit of the AdS/CFT conjecture formulated and further investigated by the author and his collaborators. The bosonic part of the story will be dealt with at length here. The thesis would discuss in detail the non-relativistic field theory on the boundary and also look at the proposed modifications of the bulk theory in this limit.

The study of non-relativistic AdS/CFT has been primarily discussed in terms of the Schrodinger symmetry group, the group of symmetries of the free Schrodinger equation. This is a non-relativistic version of the conformal symmetry relevant to the study of cold atoms at unitarity. We would however be interested in an alternative non-relativistic realization of the conformal symmetry by considering the contraction of the relativistic conformal group $SO(d+1,2)$ in $d+1$ space-time dimensions. The resulting Galilean conformal symmetry has the same number of generators as the relativistic symmetry group and thus is different from the Schrodinger group (which has fewer). The Galilean Conformal Algebra, unlike the Schrodinger algebra, does not allow a mass term which is a central extension between momenta and boosts. This implies that the GCA is the symmetry of massless or gapless non-relativistic systems.

One of the interesting features of the Galilean Conformal Algebra is that it admits an extension to an *infinite* dimensional symmetry algebra in all space-time dimensions which can potentially be dynamically realized. The latter contains a Virasoro-Kac-Moody subalgebra. We discuss the physical significance of the vector fields generating the GCA and comment on realizations of this extended symmetry in a boundary field theory. We show that a sub-algebra of the infinite GCA is a symmetry of the Navier-Stokes equation of hydrodynamics and the full finite algebra becomes a symmetry when the viscosity is put to zero, i.e. for the Euler equations.

The representations of the Galilean Conformal Algebra are looked at next. Like in the relativistic case, one defines primary operators and builds the representations by acting with raising operators on these primaries. The global part of the algebra is used to construct the novel two and three point correlation functions. We find non-trivial exponential pieces in the correlators and discuss the points of difference of these with the relativistic correlators and the correlation functions of the Schrodinger algebra.

The case of two space-time dimensions is special as the relativistic conformal algebra becomes infinite here. The question naturally arises if one can learn more about the non-relativistic systems in this case. Thus, we make a detailed study of the infinite dimensional Galilean Conformal Algebra in the case of two spacetime dimensions. Classically, this algebra is precisely obtained from a contraction of the generators of the relativistic conformal symmetry in two dimensions. Here we find quantum mechanical realizations of the (centrally extended) GCA by considering scaling limits of certain two dimensional CFTs. These parent CFTs are non-unitary and have their left and right central charges become large in magnitude and opposite in sign.

We therefore develop, in parallel to the usual machinery for two dimensional CFT, many of the tools for the analysis of the quantum mechanical GCA. These include the representation theory based on GCA primaries, Ward identities for their correlation functions and a nonrelativistic Kac table. The analysis mostly is carried out in two ways, viz. in the intrinsically non-relativistic way by using only the GCA and by taking the above mentioned limit of the quantities in relativistic conformal field theory. The answers are consistent with each other and this provides a non-trivial check of our analysis.

The gravity duals of systems with non-relativistic Schrodinger symmetry have been found in two higher dimensions i.e. for a field theory living in three space-time dimensions, the gravity theory lives in five dimensions. One of the motivations of the systematic limiting procedure was to develop a non-relativistic bulk theory in the standard one higher dimension. We find that the above limit leads to a degenerate metric and one might think that this is cause for concern as it might lead to singular gravitational dynamics. But a look back at the Newtonian limit of Einstein gravity provides us with an answer for what is to be done in this context. There exists a well defined geometric non-relativistic theory called the Newton-Cartan theory in this limit, where the dynamical variables are the Christoffel symbols and there is no non-degenerate space-time metric. We propose a variant of this theory as the bulk gravity dual to any realization of the GCA. Taking this Newton-Cartan like limit of Einstein's equations in anti de Sitter space singles out an $AdS_2$ comprising of the time and radial direction. The structure of the non-relativistic bulk is thus $AdS_2 \times S^{d-1}$. We comment on a possible $AdS_2/CFT_1$ duality in this context.

The GCA is obtained by contracting the bulk killing vectors and again we find that the algebra can be given an infinite dimensional lift. We comment on the physical significance of the vector fields generating this infinite bulk algebra. The infinite dimensional Virasoro extension is identified with the asymptotic isometries of the $AdS_2$. We show that it is natural to consider the above mentioned Newton-Cartan like geometry when we look at the action of this infinite dimensional symmetry algebra.

# List of Publications

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# Part I

# Introduction

# Chapter 1

# The AdS/CFT correspondence: A Poor Man's Introduction

## 1.1 Quantum Gravity

Gravity is the most ubiquitous of the four fundamental forces and also the least understood. Einstein's celebrated theory of General Relativity has been one of the grand successes of physics in the last century. It has correctly predicted small departures from the Newtonian theory and has essentially changed the very fabric of our imagination towards gravity and spacetime. However, the theory is inherently classical and hence can only be viewed as a limit of a complete theory. The road to the complete theory is through the unification of gravity with the other grand success of the twentieth century, quantum mechanics. That we need to consider a higher theory is manifest from general relativity itself. The theory predicts singularities in black hole spacetimes and the initial big bang singularity and signals its own breakdown in such places of extreme strong gravitational fields at extreme short distances.

On the other hand, we seem to understand the physics of fundamental particles rather well. The Standard Model of particle physics, based on the quantum theory of fields, has been spectacularly successful and has matched experimental data to an unbelievable accuracy. But it does not contain gravity. Then how do we get such accurate answers? This apparent puzzle is resolved when we look at the difference in orders of magnitude of the strengths of the other forces compared to gravity. Let us compare the ratio of the electromagnetic and gravitational forces between two protons separated by a distance $r$:

$$\frac{F_{grav}}{F_{em}} = \frac{G_N m_p^2/r^2}{e_p^2\ r^2} \simeq 10^{-36} \tag{1.1.1}$$

In usual circumstances, for the physics of fundamental particles, we can safely disregard the contribution due to gravity. But it would be erroneous to presume that the gravitational interaction of fundamental particles always remains small. Suppose the protons mentioned above are moving in opposite directions with large energy and colliding with each other. In that case, the effective mass of the protons becomes $(E/m_pc^2)m_p$. As a result, the gravitational force is enhanced by a factor of $(E/m_pc^2)^2$. The electromagnetic interaction also gets enhanced, but only logarithmically $\sim \ln E/m_pc^2$. If the factor $E/m_pc^2 \sim 10^{19}$, then the forces become of the same order. Predicting the result of this experiment is beyond the scope of the Standard Model. Our present day accelerators are nowhere close to such energy scales, but practical limitations cannot be the criteria for ruling out such experiments. Indeed, such high energies are thought to be present in the early universe, moments after the big bang. Any complete theory must be able to predict the result of all possible experiments and we see that one needs a theory which contains a quantum theory of gravity and the Standard Model of Particle Physics.

The most straight forward method to marry quantum mechanics to gravity is to use the framework of perturbative quantum field theory. But this runs into the problem of unresolvable infinities. The usual techniques of renormalization do not work and the ultraviolet divergences get worse at each order in perturbation theory. Naive power counting tells us that this would be the case and there are no miracles which invalidate this argument.

## 1.2 String Theory

### 1.2.1 Some basic features

String theory [1, 2, 3], over the last three decades, has emerged as the leading candidate for a quantum theory of gravity. The theory is based on fundamental one-dimensional extended strings instead of point-like particles. These strings have a characteristic length-scale which is the Planck length $l_p = 1.6 \times 10^{-33}$cm. This is much much smaller than the length scales that can be probed by present day experiments and hence for present-day energy scales, strings can be treated as point particles and quantum field theory turns out to be very successful in describing physics. In string theory, we proceed to quantize relativistic strings in a way analogous to the treatment of relativistic point particles and find surprises straight away. Strings can be either closed strings or open strings. Using special relativity and quantum mechanics as inputs, string theory gives automatically a quantum general relativistic theory. All string theories contain closed strings and the graviton appears as a massless mode of these closed strings. Open strings on the other hand need to satisfy Neumann or Dirichlet boundary conditions and hence need to end on hypersurfaces called D-branes. String theory uses this to generate non-abelian gauge theories like the $SU(3) \times SU(2) \times U(1)$ Standard Model, but it is far from clear, as of now, why this should be the choice for the physical vacuum.

All realistic string theories need fermions and they are introduced by supersymmetry, a symmetry that relates boson to fermions. One of the predictions of string theory indeed is the existence of space-time supersymmetry[1]. Supersymmetry has not been experimentally detected yet. This is an indication that the symmetry is broken and the scale of breaking is at an energy still unaccessible to experiments.

String theory also predicts additional dimensions of space-time. For quantum consistency of the theory, superstring theories need to exist in ten or eleven dimensions. Of them, four are our usual observable dimensions. The rest are curled up or compactified on an internal manifold. Again the internal consistencies of the theory dictate that the compactification needs to occur on special type of manifolds called Calabi-Yau manifolds [2]. The topological properties of the internal manifold determines the particle content of the theory in four dimensions. The condition that the manifold needs to be Calabi-Yau is not stringent enough to get a unique theory. In fact there are a huge number of possible vacua of string theory and as mentioned before, we don't have an understanding why the Standard Model like vacuum is selected.

There are five types of consistent superstring theories in ten dimensions. $\mathcal{N} = 2$ SUSY in ten dimensions is obtained by arranging that both left and right moving sectors of the string world-sheet theory give rise to supersymmetry generators. Here there are two possibilities. If the left and right movers have the opposite handedness, the theory is called Type IIA and if they have same handedness, it is Type IIB. A third possibility, called Type I, can be obtained by modding out Type IIB by its left-right symmetry. Quantum anomaly cancellations in ten dimensions single out the gauge groups $SO(32)$ and $E_8 \times E_8$ when we look at string theories with $\mathcal{N} = 1$ SUSY. Type I theory realizes the $SO(32)$ gauge group. The most intriguing of the string theories is the so-called Heterotic string theory. This combines 26-dimensional bosonic strings on the left moving side with 10-dimensional superstrings on the right moving side. This has $\mathcal{N} = 1$ space-time SUSY and realizes both $SO(32)$ and $E_8 \times E_8$ gauge groups.

### 1.2.2 Dualities in String Theory

The various superstring theories are linked by a web of dualities. T-duality relates different geometries of extra dimensions. In the simplest case, T-duality implies that a circle of radius $R$ is equivalent to a circle of radius $l_s/R$, where $l_s$ is the fundamental string scale. T-duality can relate either the same theory for different values of the circle radius, or two different theories, e.g. the two type II theories and the two heterotic theories are T-dual to each other.

[1] However, there exist some exotic theories like the $O(16) \times O(16)$ string theory which are tachyon-free and non-supersymmetric.

[2] This is not strictly necessary. We shall encounter string theory on $AdS_5 \times S^5$ spacetimes below when we talk of the AdS/CFT correspondence. $S^5$ is not a Calabi-Yau but one can get a consistent 5D theory on $AdS_5$ by compactifying on $S^5$. Calabi-Yau's probably form the most widely studied examples.

Another duality that manifests itself in several places in string theories is the strong-weak duality or the so-called S-duality. This relates the string coupling constant $g_s$ in one theory to the inverse $1/g_s$ of the coupling constant in another theory. Type I string theory is known to be S-dual to $SO(32)$ heterotic string theory and Type IIB is S-dual to itself.

Through S-duality then, we know how three of the five superstring theories behave at strong coupling. It is natural to wonder what happens to the two other,- type IIA and $E_8 \times E_8$ at large values of the string coupling. The surprising answer is that they grow an extra dimension and this is a circle in type IIA and a line interval in the heterotic theory. Hence a new type of quantum theory called M-theory emerges in 11 dimensions, which at low energies can be approximated by 11-dimensional supergravity. Now we have a unified picture of all superstring theories. All the five string theories together with 11d SUGRA form different corners of a single theory and are related to each other by various duality transformations.

An entirely different class of dualities was discovered in the late 1990s after Maldacena's pioneering work [4]. These relate conventional quantum field theories without gravity to string theories and M-theory. The most well studied among them is the $AdS/CFT$ correspondence [4, 5, 6]. This states that type IIB superstring theory in a background which is $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions with a gauge group $SU(N)$. $AdS_5$ is the maximally symmetric five dimensional space-time with a negative cosmological constant. Super-Yang Mills is a conformal field theory which lives on the boundary of $AdS_5$. In the following sections we motivate and elaborate on the conjecture, which at first glance seems suspiciously non-obvious.

## 1.3 Gauge-Gravity Duality

In this section, we shall begin by trying to understand the origin of the gauge-gravity duality [7, 8] without recourse to any complicated machinery from string theory. We start off with the rather curious statement that some non-gravitational quantum field theories are in a deep way quantum theories of gravity. By a quantum theory of gravity we mean a theory with a dynamic metric whose linearized equations of motion generate a massless spin 2 graviton. So this would indicate that a quantum field theory would contain a spin-2 degree of freedom. This seems to be in direct contradiction with the Weinberg-Witten theorem [9]. The theorem states that any QFT with a Poincare invariant conserved stress tensor does not allow any massless particles carrying momentum with spin greater than one.

The way around this apparently insurmountable hurdle is to understand that the graviton need not live in the same spacetime as the QFT. Further evidence why this might be true comes from the Holographic Principle[12, 13]. The maximum amount of matter one can pack inside a given volume of space-time is the amount needed to create a black hole which fills the entire volume. Now, the theory of black hole thermodynamics says that the entropy of a black hole is the area of its boundary in Planck units [11]. This defines a limit of the maximum entropy in a theory of quantum gravity. In an ordinary QFT, the entropy scales like the volume of the space-time and so a quantum gravity theory has many degrees of freedom less than a QFT without gravity in the same dimensions. This suggests that if the theory of gravity is to be equivalent to a QFT, it must live in more dimensions than the QFT.

Wilsonian approach to renormalization group gives us an elegant answer to what this extra dimension might be. A QFT at a particular length or energy scale is related to the same QFT but at a different scale by the renormalization group equation:

$$u\partial_u g = \beta(g(u)) \tag{1.3.2}$$

This is remarkably a local equation: we don't need to know the behaviour of the coupling at any other energy scale to know how it is going to change and this is basically a consequence of the locality in space-time. This leads us to view the extra dimension as the energy scale of the QFT. And clearly, the QFT in question should also be strongly coupled.

Now, we wish to find specific examples to the above and to this end, we shall make the simplifying assumption of $\beta = 0$. In this case, we have an extra dilatation symmetry added to the usual Lorenz invariant theory. So, $x^\mu \longrightarrow \lambda x^\mu$ is now a symmetry. For the energy scale u, which now is treated as the extra dimension, this by dimensional arguments leads to $u \rightarrow \lambda^{-1}u$. The most general $d$+1 dimensional metric with $d$ dimensional Poincare invariance having this symmetry is

$$ds^2 = (\frac{u'}{L'})^2\eta_{\mu\nu}dx^\mu dx^\nu + L^2\frac{du'^2}{u'^2} = (\frac{u}{L})^2\eta_{\mu\nu}dx^\mu dx^\nu + L^2\frac{du^2}{u^2} \tag{1.3.3}$$

where we have used $u = \frac{L}{L'}u'$ to bring the metric into a more familiar form. The metric is easily recognised to be $AdS_{d+1}$, and $L$ here is the AdS radius.

So, we see that conformal field theories in $d$ dimensions should be related to a theory of gravity in $AdS_{d+1}$. Different conformal theories would correspond to different theories of gravity with different field contents and bulk actions. The best studied example is the duality of $\mathcal{N} = 4$ Super Yang-Mills with a gauge group $SU(N)$ with type IIB superstring on $AdS_5 \times S^5$. In the next sub-section, we shall briefly look at the usual way of getting to the duality via string theory methods.

## 1.4 The Maldacena Correspondence

### 1.4.1 The D3 brane picture

In this section, as indicated before, we look at the usual route of obtaining the Maldacena correspondence which connects type IIB string theory compactified on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ super-Yang-Mills theory [4]. Our discussion here would closely follow the very well-known and comprehensive AdS/CFT review [10]. We will start with type IIB string theory in flat, ten dimensional Minkowski space. This theory contains stable odd dimensional D branes. Consider $N$ parallel D3 branes that are sitting together. The D3 branes are extended along a $(3 + 1)$ dimensional plane in $(9 + 1)$ dimensional spacetime. String theory on this background contains two kinds of perturbative excitations, closed strings and open strings. As we saw before, the closed strings are the excitations of empty space and the open strings end on the D-branes and describe excitations of the D-branes. If we consider the system at energies lower than the string scale $1/l_s$ then only the massless string states can be excited, and we can write an effective Lagrangian describing their interactions. The closed string massless states give a gravity supermultiplet in ten dimensions, and their low-energy effective Lagrangian is that of type IIB supergravity. The open string massless states give an $\mathcal{N} = 4$ vector supermultiplet in $(3 + 1)$ dimensions, and their low-energy effective Lagrangian is that of $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory.

The complete effective action of the massless modes is

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}. \tag{1.4.4}$$

$S_{\text{bulk}}$ is the action of ten dimensional supergravity, plus some higher derivative corrections. The Lagrangian is a Wilsonian one, got by integrating out the massive modes. Hence it is not renormalizable. The brane action $S_{\text{brane}}$ is defined on the $(3+1)$ dimensional brane worldvolume, and it contains the $\mathcal{N} = 4$ super-Yang-Mills Lagrangian plus some higher derivative corrections. $S_{\text{int}}$ describes the interactions between the brane modes and the bulk modes.

We can expand the bulk action as a free quadratic part describing the propagation of free massless modes (including the graviton), plus some interactions which are proportional to positive powers of the square root of the Newton constant. Schematically we have

$$S_{bulk} \sim \frac{1}{2\kappa^2}\int \sqrt{g}\mathcal{R} \sim \int (\partial h)^2 + \kappa(\partial h)^2 h + \cdots , \tag{1.4.5}$$

where we have written the metric as $g = \eta + \kappa h$. We indicate explicitly the dependence on the graviton, but the other terms in the Lagrangian, involving other fields, can be expanded in a similar way. Similarly, the interaction Lagrangian $S_{int}$ is proportional to positive powers of $\kappa$. If we take the low energy limit, all interaction terms proportional to $\kappa$ drop out.

We keep the energy fixed and send $l_s \to 0$ equivalently $\alpha' \to 0$ keeping all the dimensionless parameters including the string coupling constant and $N$ fixed. In this limit we see more clearly what happens at low energies. Here the coupling $\kappa \sim g_s\alpha'^2 \to 0$, so that the interaction Lagrangian relating the bulk and the brane vanishes. In addition all the higher derivative terms in the brane action vanish, leaving just the pure $\mathcal{N} = 4$ $U(N)$ gauge theory in $3+1$ dimensions. This is known to be a conformal field theory, even quantum mechanically. Again, in this low energy limit the supergravity theory in the bulk becomes free. So, we have two decoupled systems: free gravity in the bulk and the four dimensional gauge theory.

Next, we consider the same system from a different point of view. D-branes are massive charged objects which act as a source for the various supergravity fields. We can find a D3 brane solution of supergravity, of the form

$$\begin{aligned} ds^2 &= f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \ , \\ F_5 &= (1+*)dt dx_1 dx_2 dx_3 df^{-1} \ , \\ f &= 1 + \frac{R^4}{r^4} \ , \qquad R^4 \equiv 4\pi g_s \alpha'^2 N \ . \end{aligned} \tag{1.4.6}$$

Since $g_{tt}$ is non-constant, the energy $E_p$ of an object as measured by an observer at a constant position $r$ and the energy $E$ measured by an observer at infinity are related by the redshift factor

$$E = f^{-1/4} E_p \ . \tag{1.4.7}$$

So, if we move an object closer to $r = 0$, it would appear to have lower energy as measured by an observer at infinity. Now we take the low energy limit in the background described by equation (1.4.6). There are two kinds of excitations from the point of view of an observer at infinity. These are massless particles propagating in the bulk region with wavelengths that becomes very large, or any kind of excitation brought closer and closer to $r = 0$. In the low energy limit these two types of excitations decouple from each other. The low energy theory hence consists of two decoupled pieces, one is free bulk supergravity and the second is the near horizon region of the geometry. In the near horizon region, $r \ll R$, we can approximate $f \sim R^4/r^4$, and the geometry becomes $AdS_5 \times S^5$

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2, \tag{1.4.8}$$

So, from both points of view there are two decoupled theories in the low-energy limit and one of the decoupled systems in both cases is supergravity in flat space. Hence we compare the two other theories and are led to the conjecture that $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in $3+1$ dimensions is the same as (or dual to) type IIB superstring theory on $AdS_5 \times S^5$ [4].

The reader would have no doubt noted the fact that we are talking here of a $U(N)$ theory as opposed to the $SU(N)$ theory that was advertised earlier. Let us comment briefly on this. A $U(N)$ gauge theory is essentially equivalent to a free $U(1)$ vector multiplet times an $SU(N)$ gauge theory. In the string theory all modes interact with gravity, so there are no decoupled modes. Therefore, the bulk $AdS$ theory describes the $SU(N)$ part of the gauge theory. The $U(1)$ degrees of freedom are actually some zero modes which live in the region connecting the throat to the bulk and are related to the centre of mass motion of the $D3$ branes.

### 1.4.2 Matching both sides

Anti-de-Sitter space can be viewed as a hyperboloid embedded in a six-dimensional flat space-time. This by construction has a large group of isometries, which is $SO(4,2)$. This is the same group as the conformal group in $3+1$ dimensions. We will have more to say about this in the later chapters. Moving on to supersymmetries, we see that the number of supersymmetries is twice that of the full solution (1.4.6) containing the asymptotic region. This doubling of supersymmetries is viewed in the field theory as a consequence of superconformal invariance, since the superconformal algebra has twice as many fermionic generators as the corresponding Poincare superalgebra. In the bulk, there is also an $SO(6)$ symmetry which rotates the $S^5$. This can be identified with the $SU(4)_R$ R-symmetry group of the field theory. Thus we see the whole supergroup is the same for the $\mathcal{N}=4$ field theory and the $AdS_5 \times S^5$ geometry, so both sides of the conjecture have the same spacetime symmetries. In this thesis, we would be focussing on the bosonic part of the symmetry on both sides.

Another important fact about the duality is that the rank of the gauge group corresponds to the flux of the five-form Ramond-Ramond field strength through the five-sphere

$$\int_{S^5} F_5 = N. \tag{1.4.9}$$

The extremal $D3$-brane construction started with $N$ coincident $D3$-branes, which carry a total of $N$ units of $D3$-brane charge. This charge is measured by enclosing the $D3$-branes with a five-sphere and computing the five-form flux. Thus, the parameter N, which labels the gauge-theory group, corresponds to the five-form flux in the dual type $IIB$ description.

We spoke briefly about S-duality. In flat ten-dimensional space-time the S-duality group of type IIB superstring theory is $SL(2,Z)$ under which

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{1.4.10}$$

where $a, b, c, d$ are integers with $ad - bc = 1$. In particular, the complex scalar field

$$\tau_{IIB} = C_0 + ie^{-\Phi} \tag{1.4.11}$$

transforms as a modular parameter under $SL(2,Z)$ transformations. This $SL(2,Z)$ is a conjectured duality of type IIB in flat space, and it should also be a symmetry in the present context since all the fields that are being turned on in the $AdS_5 \times S^5$ background (the metric and the five form field strength) are invariant under this symmetry.

It is also known that $U(N)$ $\mathcal{N} = 4$ super Yang Mills theory exhibits a $SL(2, Z)$ S-duality. If one includes a topological $\theta$ term in the Lagrangian, one can define a complexified gauge theory coupling

$$\tau_{SYM} \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi} \tag{1.4.12}$$

The identification $g_{YM}^2 = 4\pi g_s$, where $e^{-\Phi} = g_s$ for the extremal D3-brane solution along with $\theta = 2\pi C_0$ leads to

$$\tau_{YM} = \tau_{IIB} \tag{1.4.13}$$

This can be looked at in two ways. The existing evidence in support of the S-duality of $\mathcal{N} = 4$ super Yang-Mills theory can be regarded as support for the AdS/CFT conjecture. On the other hand, if we believe the conjecture, then the S-duality of the gauge theory can be seen as being induced by the S-duality of the string theory. Since the AdS/CFT correspondence requires the S-duality of the gauge theory, any test of this S-duality is also a test of the correspondence.

The AdS/CFT conjecture is a duality in the usual sense. When the theory on one side of the duality is weakly coupled, the other side is strongly coupled. So, one can use perturbative methods in the weakly coupled theory to find answers to strong coupling physics on the other side. To see this let us look at the validity of various approximations. The analysis of loop diagrams in the field theory shows that we can trust the perturbative analysis in the Yang-Mills theory when

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1. \tag{1.4.14}$$

Note that we need $g_{YM}^2 N$ to be small and not just $g_{YM}^2$. On the other hand, the classical gravity description becomes reliable when the radius of curvature $R$ of $AdS$ and of $S^5$ becomes large compared to the string length,

$$\frac{R^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1. \tag{1.4.15}$$

We see that the gravity regime (1.4.15) and the perturbative field theory regime (1.4.14) are perfectly incompatible. Hence, we avoid any obvious contradiction due to the fact that the two theories look very different. This is the reason that this correspondence is called a duality. The two theories are conjectured to be exactly the same, but when one side is weakly coupled the other is strongly coupled and vice versa. This makes the correspondence both hard to prove and useful, as we can solve a strongly coupled gauge theory via classical supergravity.

### 1.4.3 Holography

We have already encountered the Bekenstein bound, which states that the maximum entropy in a region of space is $S_{max} = \text{Area}/4G_N$ [11], where the area is that of the boundary of the region and noted that this bound implies that the number of degrees of freedom inside some region grows as the area of the boundary of a region and not like the volume of the region. This is not the case in standard quantum field theories and lead to the holographic principle, which states that in a quantum gravity theory all physics within some volume can be described in terms of some theory on the boundary which has less than one degree of freedom per Planck area [12, 13].

In the AdS/CFT correspondence, physics in the bulk of $AdS$ space was described in terms of a field theory of one less dimension living on the boundary. This, indeed, looks like holography. However, it is hard to check through explicit counting what the number of degrees of freedom per Planck area is. The theory is conformal and hence has an infinite number of degrees of freedom, and the area of the boundary of AdS space is also infinite. Hence we need to introduce a cutoff on the number of degrees of freedom in the field theory and see what it corresponds to in the gravity theory. To this end we will write the metric of $AdS$ as

$$ds^2 = R^2 \left[ -\left(\frac{1+r^2}{1-r^2}\right)^2 dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\Omega^2) \right]. \tag{1.4.16}$$

In these coordinates the boundary of $AdS$ is at $r = 1$. Let us look at $r = 1 - \delta$ and take the limit of $\delta \to 0$. It is clear by studying the action of the conformal group on Poincare coordinates that the radial position plays the role of some energy scale, since we approach the boundary when we do a conformal transformation that localizes objects in the CFT. We also saw this in the initial discussion where we had motivated the correspondence. The limit $\delta \to 0$, in this case, corresponds to going to the UV of the field theory. When we are close to the boundary we could also use the Poincare coordinates[3]

$$ds^2 = R^2 \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2}, \tag{1.4.17}$$

in which the boundary is at $z = 0$. If we consider a particle or wave propagating in (1.4.17) or (1.4.16) we see that its motion is independent of $R$ in the supergravity approximation. Furthermore, if we are in Euclidean space and we have a wave that has some spatial extent $\lambda$ in the $\vec{x}$ directions, it will also have an extent $\lambda$ in the $z$ direction. This can be seen from (1.4.17) by eliminating $\lambda$ through the change of variables $x \to \lambda x$, $z \to \lambda z$. This implies that a cutoff at

$$z \sim \delta \tag{1.4.18}$$

[3]Here we would like to emphasise that the Poincare co-ordinates are just another way of describing AdS and is not an approximation except for the fact that it does not cover the whole of the spacetime like the global co-ordinates described earlier.

corresponds to a UV cutoff in the field theory at distances $\delta$, with no factors of $R$ ($\delta$ here is dimensionless, in the field theory it is measured in terms of the radius of the $S^4$ or $S^3$ that the theory lives on). Equation (1.4.18) is called the UV-IR relation [14].

Consider the case of $\mathcal{N} = 4$ SYM on a three-sphere of radius one. We can estimate the number of degrees of freedom in the field theory with a UV cutoff $\delta$. We get

$$S \sim N^2 \delta^{-3}, \tag{1.4.19}$$

since the number of cells into which we divide the three-sphere is of order $1/\delta^3$. In the gravity solution (1.4.16) the area in Planck units of the surface at $r = 1 - \delta$, for $\delta \ll 1$, is

$$\frac{\text{Area}}{4G_N} = \frac{V_{S^5} R^3 \delta^{-3}}{4G_N} \sim N^2 \delta^{-3}. \tag{1.4.20}$$

Thus, we see that the AdS/CFT correspondence saturates the holographic bound [14].

## 1.5 Non-relativistic AdS/CFT

Even after more than a decade, the AdS-CFT conjecture continues to throw up rich, new avenues of investigation. One such recent direction has been to consider extensions of the conjecture from its original relativistic setting to a non-relativistic context. This opens the door to potential applications of the spirit of gauge-gravity duality to a variety of real-life strongly interacting systems [19]. It was pointed out in [20] that the Schrödinger symmetry group [21, 22], a non-relativistic version of conformal symmetry, is relevant to the study of cold atoms. A gravity dual possessing these symmetries was then proposed in [25, 26] (see also [30, 31] for a somewhat different bulk realization). An incomplete list of references for further developments along this line is [32] –[47]. For a more comprehensive list see e.g. the references in [19].

The Schrödinger group is the largest group of symmetries of the *free Schrödinger equation*. There is no reason to believe that all non-relativistic systems would have the symmetry of the free Schrödinger equation and it is worthwhile to look at systems with other symmetries. Driven by this motivation, in this thesis we will consider an *alternative* non-relativistic realization of conformal symmetry and study its consequences and realizations in the context of the AdS/CFT conjecture [27, 28, 29]. This symmetry will be obtained by considering the nonrelativistic group contraction of the relativistic conformal group $SO(d+1,2)$ in $d+1$ space-time dimensions. [4] The process of group contraction leads, in $d = 3$, for instance, to a fifteen parameter group (like the parent $SO(4,2)$ group)

[4] The process of group contraction is, of course, standard and may have been applied by many people to the relativistic conformal group. To the best of our knowledge, [48] is a recent reference with the explicit results of this contraction of the relativistic conformal group. This reference goes on to study a realization of the $2+1$ dimensional case, which has some special features.

which contains the ten parameter Galilean subgroup. This Galilean conformal group is to be contrasted with the twelve parameter Schrödinger group (plus central extension) with which it has in common only the Galilean subgroup. The Galilean conformal group is, in fact, different from the Schrödinger group in some crucial respects, which we will describe in more detail later. For instance, the dilatation generator $\tilde{D}$ in the Schrödinger group scales space and time differently $x_i \to \lambda x_i, t \to \lambda^2 t$. Whereas the corresponding generator $D$ in the Galilean Conformal Algebra (GCA) scales space and time in the *same* way $x_i \to \lambda x_i, t \to \lambda t$. Relatedly, the GCA does *not* admit a mass term as a central extension. Thus, in some sense, this symmetry describes "massless" or "gapless" non-relativistic theories, like the parent relativistic group but unlike the Schrödinger group.

We would also see how the GCA can be naturally extended to an infinite dimensional symmetry algebra in a way somewhat analogous to the extension of the finite conformal algebra of $SL(2, C)$ in two dimensions to two copies of the Virasoro algebra. One important distinction between the two is that the infinite extension for the relativistic conformal algebra occurs only in two dimensions, whereas the extension for the GCA is valid in all spacetime dimensions. We will see that it is natural to expect this extended non-relativistic symmetry to be dynamically realized (perhaps partially) in actual systems possesing the finite dimensional Galilean conformal symmetry. Our infinite algebra contains one copy of a Virasoro together with an $SO(d)$ current algebra (on adding the appropriate central extension). We would look explicitly at the case of $d = 2$, as it is special in the relativistic algebra and learn important lessons about the quantum mechanical nature of the GCA.

In addition to possible applications to non-relativistic systems, one of the motivations for studying the contracted $SO(d + 1, 2)$ conformal algebra is to examine the possibility of a new tractable limit of the parent AdS/CFT conjecture. In fact, the BMN limit [57] of the AdS/CFT conjecture is an instance where, as result of taking a particular scaling limit, one obtains a contraction of the original $SO(4, 2) \times SO(6)$ (bosonic) global symmetry [5]. In our case, the non-relativistic contraction is obtained by taking a similar scaling limit on the parent theory. Like in the BMN case, taking this limit would isolate a closed subsector of the full theory. The presence of an enhanced symmetry in our scaling limit raises interesting possibilities about the solvability of this subsector, though we shall not discuss this aspect in any detail in this thesis.

There are, however, some important differences here from a BMN type limit which have to do with the nature of taking the scaling. Normally the BMN type scaling leads to a Penrose limit of the geometry in the vicinity of some null geodesic. These are typically pp-wave like geometries whose isometry is the same as that of the contracted symmetry

[5]An algebraically equivalent contraction to ours, of the isometries of $AdS_5 \times S^5$, was studied in [64] as an example of a non-relativistic string theory. However, the embedding of this contraction in $AdS_5$ is not manifestly such that it corresponds to a non-relativistic CFT on the boundary. We will comment further on this at a later stage.

group on the boundary. The non-relativistic scaling limit that leads to the GCA on the boundary is at first sight more puzzling to implement in the bulk. This is because, under the corresponding scaling, the bulk metric degenerates in the spatial directions $x_i$. Thus one might think one has some kind of singular limit in the bulk description. However, this degeneration is a feature common to all non-relativistic limits. It arises, for instance, in taking the Newtonian gravity limit of Einstein's equations in asymptotically flat space. However, in this case, there is a well defined geometric description of the limit despite the degeneration of the metric. This description, originally due to Cartan, and studied fairly thoroughly by geometers describes Newtonian gravity in terms of a non-dynamical metric but with a dynamical *non-metric* connection[6]. The Einstein's equations reduce to equations determining the curvature of this connection in terms of the matter density. These are nothing but the Poisson equations for the Newtonian gravitational potential. In geometric terms the spacetime takes the form of a vector bundle with fibres as the spatial $R^d$ over a base $R$ which is time, together with an affine connection related to the gradient of the Newtonian potential.

We propose a similar limiting description for the bulk geometry in our case. The main difference is that the time and the radial direction together constitute an $AdS_2$ with a non-degenerate metric. Thus one has a geometry with an $AdS_2$ base and the spatial $R^d$ fibred over it. Once again there is no overall spacetime metric. The dynamical variables are affine connections determined by the limiting form of Einstein's equations. As a check of this proposal, we will see that the infinite dimensional GCA symmetries are realized in this bulk geometry as asymptotic isometries. In fact the Virasoro generators of the GCA are precisely the familiar generators of asymptotic global isometries of $AdS_2$. These generators will also be seen to reduce to the generators of the GCA on the boundary. This construction is to be contrasted with the metric theory for the Schrödinger space-times. Our bulk theory is also in a standard one higher dimension as opposed to the two extra dimensions of the Schrödinger bulk.

## 1.6 Outline of the Thesis

As advertised, this thesis deals with the systematic non-relativistic limit of the above described AdS/CFT correspondence. The original results of this thesis appear in the papers [27, 28, 29]. The author has, in some places, taken the liberty to follow closely some well-known and well established reviews and text books in the field to make the thesis a self-contained review on its own and the thesis is meant to be accessible to beginning graduate students or infact to anyone who intends to have a quick look at this new line of development in AdS/CFT. The main body of the thesis is divided into two parts, viz. the Non-relativistic Boundary Theory and the Non-Relativistic Bulk.

[6]See the textbook [71] Chap.12 for a basic discussion.

The Non-relativistic boundary theory begins in Chapter 2 with a review of conformal field theory in the relativistic setting in arbitrary space-time dimensions. We mainly focus our attention on the symmetry generators. We then describe the process of group contractions in some generality. The technique described is then utilized to obtain a non-relativistic limit of the relativistic conformal algebra. The resulting new algebra is named the Galilean Conformal Algebra (GCA). We also review the other non-relativistic algebra, the Schrödinger Algebra (SA) which has gained a lot of interest in recent studies of non-relativistic AdS/CFT. The GCA and SA are compared and contrasted.

In Chapter 3, we describe the natural extension of the finite GCA to an infinite dimensional algebra, which we also refer to as the GCA. The physical interpretations of the vector fields generating the GCA are discussed. We would like to look at physical systems which realize this symmetry algebra. To that end, we discuss the equations of fluid dynamics and describe how the GCA emerges as the symmetry algebra there. The end of the chapter is devoted to a short discussion on a similar infinite extension of the Schrödinger Algebra.

Chapter 4 is devoted to the study of representations and correlation functions of the GCA. We first describe how to label operators using the dilatation and boost eigenvalues and then introduce the notion of primary operators in this context. The representations are built by acting with raising operators on these primaries. We then focus our attention on the finite GCA and show that this, as in the relativistic case, is enough to fix the forms of the two and three point functions. The construction of these novel correlators is described in detail and then the similarities and differences between these and the two and three-point correlators of the relativistic theory and Schrödinger Algebra are pointed out.

The case of two space-time dimensions is special in the relativistic case as the conformal algebra there gets lifted to two copies of the infinite dimensional Virasoro algebra. It is natural then to look at this case in the non-relativistic limit and see what lessons we can draw from it. This is the focus of Chapters 5, 6 and 7. We see how the GCA emerges as the linear combination of the two copies of the Virasoro, thus providing a map between the infinite algebras. The analysis of the GCA in the earlier chapters was mainly classical. Here we find quantum mechanical realizations of the (centrally extended) GCA by considering scaling limits of certain 2d CFTs. These parent CFTs are non-unitary and have their left and right central charges become large in magnitude and opposite in sign. We develop, in parallel to the usual machinery for $2d$ CFT, many of the tools for the analysis of the quantum mechanical GCA. These include the representation theory based on GCA primaries, Ward identities for their correlation functions and a nonrelativistic Kac table. The analysis mostly is carried out in two ways, viz. in the intrinsically non-relativistic way by using only the GCA and by taking the above mentioned limit of the quantities in relativistic conformal field theory. The answers are consistent with each other and this provides a non-trivial check of our analysis. Chapter 5 focuses on the algebra, the representations and the two and three point correlation functions. Chapter 6 deals with the GCA null vectors and fusion rules. Finally, in Chapter 7 we look at the various technicalities of the four-point function.

The Non-relativistic Bulk theory is discussed in Chapter 8. The non-relativistic scaling of the metric leads to a degenerate space-time metric. One might think that this may lead to singular gravitational dynamics. But we point the reader to the non-relativistic limit of usual Einstein gravity and even there there is no non-degenerate space-time metric. There is however a well-defined geometric formulation of non-relativistic gravity called the Newton-Cartan formulation. Here the dynamical variables are the Christoffel symbols in terms of which the theory is formulated. We review the Newton-Cartan theory first and then propose a variant of this theory as the bulk gravity dual to any realization of the GCA. Taking this Newton-Cartan like limit of Einstein's equations in anti de Sitter space singles out an $AdS_2$ comprising of the time and radial direction. The structure of the non-relativistic bulk is thus $AdS_2 \times S^{d-1}$. We comment on a possible $AdS_2/CFT_1$ duality in this context.

The GCA is obtained by contracting the bulk killing vectors and again we find that the algebra can be given an infinite dimensional lift. We comment on the physical significance of the vector fields generating this infinite bulk algebra. The infinite dimensional Virasoro extension is identified with the asymptotic isometries of the $AdS_2$. We show that it is natural to consider the above mentioned Newton-Cartan like geometry when we look at the action of this infinite dimensional symmetry algebra. At the end of the chapter we present briefly the Schrödinger bulk theory to contrast with the radically different bulk theory proposed for the GCA

We conclude with a summary of the results discussed in this thesis. We briefly discuss some other related pieces of work in the field. We also comment on possible exciting directions of future research.

# Part II

# The Non-Relativistic Boundary Theory

# Chapter 2

# Non-Relativistic Conformal Algebras

Conformal symmetry has gradually moved from being a mere mathematical curiosity (e.g. as an invariance of certain classical equations of motion like the Maxwell and more generally Yang-Mills equations) to playing a central dynamical role governing the quantum behaviour of field theories through RG fixed points. Thus an understanding of CFTs in various dimensions [15] has had a major impact on physics, starting with the study of critical phenomena all the way through to the more recent AdS/CFT correspondence. In all this, the focus has largely been on relativistic conformal field theories. In this chapter, after looking at relativistic conformal field theories in arbitrary dimensions, we describe a process of group contraction which would be a systematic process of taking a non-relativistic limit of the conformal algebra. This generates the finite version of the Galilean Conformal Algebra (GCA) which would be the focal point of this thesis. There are other conformal non-relativistic algebras which have been looked at in more detail in the literature. The Schrödinger Algebra, the symmetry of free Schrödinger equations, has been the one which has attracted the greatest attention. We discuss and compare the Schrödinger Algebra with the GCA.

## 2.1 Relativistic Conformal Symmetry

Any relativistic field theory is assumed to be invariant under a general co-ordinate transformation. This implies a metric dependence of the theory[1]. The metric $g_{\mu\nu}$ defines an invariant line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.1.1}$$

[1]This is not the case for topological field theories, but we shall not be concerned with them in this thesis.

Under a finite transformation $x^\mu \to \tilde{x}^\mu$ the metric transforms as a rank-two symmetric tensor

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\frac{\partial x^\rho}{\partial \tilde{x}^\nu} g_{\lambda\rho}(x) \tag{2.1.2}$$

A *conformal transformation* is defined as a restricted general co-ordinate transformation under which the metric is invariant upto a *scale factor*:

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \Omega(x) g_{\mu\nu}(x), \quad \Omega(x) \equiv e^{w(x)} \tag{2.1.3}$$

It is clear that these transformations form a group and this is the *conformal group.* To make connections with the better known definition of conformal transformation being angle preserving transformations, let us remind ourselves that the angle ($\theta$) between two vectors $A^\mu$ and $B^\nu$ is

$$\cos\theta = \frac{A.B}{|A||B|} = \frac{g_{\mu\nu}A^\mu B^\nu}{\sqrt{g_{\mu'\nu'}A^{\mu'}A^{\nu'}g_{\mu''\nu''}B^{\mu''}A^{\nu''}}} \tag{2.1.4}$$

From the above, it is straight-forward to see that under a conformal transformation (2.1.3), the cosine of the angle between the two vectors remains unchanged and hence angles are preserved under conformal transformations.

### 2.1.1 Conformal Group in Flat Space

The conformal transformations contain as a sub-group the Poincare group consisting of Lorentz transformations and translations. This is just the group with $\Omega = 1$ and $g_{\mu\nu} = \eta_{\mu\nu}$. We wish to look at the full conformal group in flat space-time and for this let us look at infinitesimal transformations of co-ordinates $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$. From (2.1.2), we see that under this infinitesimal transformation the metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{2.1.5}$$

If we require this transformation to be conformal, (2.1.3) implies the above equation reduces to

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = w(x) g_{\mu\nu}. \tag{2.1.6}$$

We can take the trace of (2.1.6) to get the form of $w(x)$

$$w(x) = \frac{2}{d}\partial_\alpha \epsilon^\alpha \tag{2.1.7}$$

We shall assume that we are in flat Euclidean spacetime $g_{\mu\nu} = \eta_{\mu\nu} = diag(1, 1, \ldots, 1)$. Taking a derivative of (2.1.6), we get

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\mu\rho}\partial_\nu w + \eta_{\nu\rho}\partial_\mu w - \eta_{\mu\nu}\partial_\rho w$$

$$2\partial^2\epsilon_\mu = (2-d)\partial_\mu w \tag{2.1.8}$$

where in the second line, we have contracted with $\eta^{\mu\nu}$. Using (2.1.8) and (2.1.6) we get

$$\begin{aligned}(2-d)\partial_\mu\partial_\nu w &= \eta_{\mu\nu}\partial^2 w \\ \Rightarrow (d-1)\partial^2 w &= 0\end{aligned} \tag{2.1.9}$$

In $d = 1$, all smooth transformations are conformal since there are no constraints. This is to be expected as there is no notion of angle in one dimension. The case of $d = 2$ is special and we would have more to say about it in later chapters. For now, we focus our attention on $d > 2$. From the above equations, it is clear that the most general form that the undetermined parameters $w(x)$ and $\epsilon_\mu(x)$ are given by

$$w = A + B_\mu x^\mu, \quad \epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{2.1.10}$$

where $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. The constriants hold for all values of $x$ and we can treat each power of $x$ in the expansion (2.1.10) separately and plug into the above equations. The constant term $a_\mu$ is unconstrained and corresponds to translations. The linear term put in (2.1.6) yields

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}b_\alpha{}^\alpha\eta_{\mu\nu} \tag{2.1.11}$$

This implies that $b_{\mu\nu}$ is the sum of an anti-symmetric and a trace part.

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \quad \text{where} \quad m_{\mu\nu} = -m_{\nu\mu} \tag{2.1.12}$$

The anti-symmetric part is an infinitesimal rotation while the trace part is an infinitesimal dilatation. The quadratic part yields the following relation:

$$c_{\mu\nu\rho} = \eta_{\mu\rho}f_\nu + \eta_{\nu\rho}f_\mu - \eta_{\mu\nu}f_\rho \tag{2.1.13}$$

where $f_\mu = \frac{1}{d}c^\nu{}_{\nu\mu}$. This corresponds to an infinitesimal special conformal transformation

$$x'^\mu = x^\mu + 2(x.b)x^\mu - b^\mu x^2 \tag{2.1.14}$$

The finite transformations and the generators of these transformations are listed below:

| | Finite Transformation | Generator |
|---|---|---|
| Translations: | $x'^\mu = x^\mu + a^\mu$ | $P_\mu = \partial_\mu$ |
| Dilatations: | $x'^\mu = \alpha x^\mu$ | $D = x^\mu\partial_\mu$ |
| Rotations: | $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ | $M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$ |
| Special Conformal: | $x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2(x.b)+b^2x^2}$ | $K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu$ |

### 2.1.2 Relativistic Conformal Algebra

We are going to use slightly different types of conventions in the non-relativistic case, so let us re-phrase the generators in a way that would be useful later.

$$\begin{aligned} P_\mu &= P_i \ (\mu \neq 0), \quad P_0 = H \\ M_{\mu\nu} &= J_{ij} \ (\mu, \nu \neq 0), \quad M_{\mu 0} = B_i \ (\mu \neq 0) \\ K_\mu &= K_i \ (\mu \neq 0), \quad K_0 = K \end{aligned} \tag{2.1.15}$$

The Poincare sub-group, consisting of $\{P_i, H, J_{ij}, D\}$ has the following commutation relations:

$$\begin{aligned} [J_{ij}, J_{rs}] &= so(d) \\ [J_{ij}, B_r] &= -(B_i\delta_{jr} - B_j\delta_{ir}) \\ [J_{ij}, P_r] &= -(P_i\delta_{jr} - P_j\delta_{ir}), \quad [J_{ij}, H] = 0 \\ [B_i, B_j] &= -J_{ij}, \quad [P_i, P_j] = 0, \quad [B_i, P_j] = \delta_{ij} H \\ [H, P_i] &= 0, \quad [H, B_i] = -P_i \end{aligned} \tag{2.1.16}$$

The remaining conformal generators make up the rest of the relativistic conformal algebra:

$$\begin{aligned} [K, K_i] &= 0, \quad [K, B_i] = K_i, \quad [K, P_i] = 2B_i \\ [J_{ij}, K_r] &= -(K_i\delta_{jr} - K_j\delta_{ir}), \quad [J_{ij}, K] = 0, \quad [J_{ij}, D] = 0 \\ [K_i, K_j] &= 0, \quad [K_i, B_j] = \delta_{ij} K, \quad [K_i, P_j] = 2J_{ij} + 2\delta_{ij} D \\ [H, K_i] &= -2B_i, \quad [D, K_i] = -K_i, \quad [D, B_i] = 0, \quad [D, P_i] = P_i, \\ [D, H] &= H, \quad [H, K] = -2D, \quad [D, K] = -K. \end{aligned} \tag{2.1.17}$$

Of special interest would be the following generators:

$$\begin{aligned} [B_i, B_j] &= -J_{ij} \quad [B_i, P_j] = \delta_{ij} H \\ [K_i, B_j] &= \delta_{ij} K, \quad [K_i, P_j] = 2J_{ij} + 2\delta_{ij} D \end{aligned} \tag{2.1.18}$$

We shall elaborate on this in the coming sections.

## 2.2 Inönu-Wigner Group Contractions

Our basic aim is to take a systematic non-relativistic limit of the relativistic conformal algebra which was presented in the last section. In order to do that we would take recourse in a procedure called contraction [16, 17] which is a transformations of a Lie Algebra that alters the structure of the parent algebra.

Since the structure constants of a Lie algebra transform like tensors under non-singular basis changes, we need to do something more drastic. We would thus be looking at basis transformations which become singular in a particular limit. When the transformed structure constants approach a well-defined limit as the transformation becomes singular, a new Lie algebra emerges. This algebra is called the contracted limit of the parent Lie algebra. The contracted algebra is always non-semisimple, as we shall prove later in this section. We shall look at a co-ordinate dependent version of the procedure of contraction, known as Inönu-Wigner contractions.

### 2.2.1 Some Generalizations

Let $X_i$ for $i = 1, 2, \ldots n$ be a set of basis vectors of the Lie algebra $\mathsf{g}$. We make a co-ordinate transformation to a new set of basis vectors $Y_i$. The new basis is related to the old one by

$$Y_j = U(\epsilon)_j{}^i X_i \tag{2.2.19}$$

$$\text{where} \quad U(\epsilon = 1)_j{}^i = \delta_j{}^i \quad \text{and} \quad det||U(\epsilon = 0)|| = 0. \tag{2.2.20}$$

We see that for $\epsilon = 0$ we have a singular basis transformation. Let us look at the structure constants of the new algebra and relate them to the old one.

$$\begin{aligned} &[Y_i, Y_j] = c_{ij}{}^k(\epsilon) Y_k \\ &[U(\epsilon)_i{}^r X_r, U(\epsilon)_j{}^s X_s] = c_{ij}{}^k(\epsilon) U(\epsilon)_k{}^t X_t \\ &\Rightarrow c_{ij}{}^k(\epsilon) = U(\epsilon)_i{}^r U(\epsilon)_j{}^s c_{rs}{}^t(\epsilon = 1) U^{-1}(\epsilon)_t{}^k \end{aligned} \tag{2.2.21}$$

The structure constants have usual tensorial properties for non-zero $\epsilon$. When $\epsilon \to 0$, $c_{ij}{}^k(\epsilon)$ may or maynot exist. When the limit exists and is well defined, i.e.

$$\lim_{\epsilon \to 0} c_{ij}{}^k(\epsilon) = c_{ij}{}^k(\epsilon = 0) = d_{ij}{}^k \tag{2.2.22}$$

the new structure constants define a Lie algebra which may not be isomorphic to the old algebra. The new algebra which emerges is the contracted algebra.

### 2.2.2 A Particular Basis Transformation

To further elucidate the process of contraction let us look at a particular form of the singular basis transformation. Let $\mathsf{g} = V_R \oplus V_N$ and

$$U(\epsilon) = \begin{pmatrix} I & 0 \\ 0 & \epsilon I \end{pmatrix} \tag{2.2.23}$$

where U is identity in the subspace $V_R$ and is $\epsilon I$ in $V_N$. The basis for $V_R$ is taken to be $X_\alpha, X_\beta, \ldots$ and for $V_N$, it is chosen as $X_i, X_j, \ldots$. The basis transformation then has the form

$$U(\epsilon) X_\alpha = X_\alpha = Y_\alpha, \quad U(\epsilon) X_i = \epsilon X_i = Y_i. \tag{2.2.24}$$

The change of basis leads to the following transformation of the structure constants:

$$c_{\mu\nu}^{\ \ \lambda}(\epsilon) = \epsilon^p c_{\mu\nu}^{\ \ \lambda}(\epsilon = 1) \tag{2.2.25}$$

where $p =$ number of covariant Latin indices $-$ number of contravariant Latin indices. In order to avoid convergence problems, we need $c_{\alpha\beta}^{\ \ k}(\epsilon = 1) = 0$. To see this consider the following commutator:

$$[Y_\alpha, Y_\beta] = c_{\alpha\beta}^{\ \ k}(\epsilon) Y_k, \quad c_{\alpha\beta}^{\ \ k}(\epsilon) = \epsilon^{-1} c_{\alpha\beta}^{\ \ k}(\epsilon = 1). \tag{2.2.26}$$

This particular structure constant converges if and only if $c_{\alpha\beta}^{\ \ k}(\epsilon = 1) = 0$. All other $c$'s converge in the limit. They are given by

$$\begin{aligned} d_{\alpha\beta}^{\ \ \gamma} &= c_{\alpha\beta}^{\ \ \gamma}, \qquad d_{\alpha\beta}^{\ \ k} = c_{\alpha\beta}^{\ \ k} = 0 \\ d_{\alpha j}^{\ \ \gamma} &= \epsilon c_{\alpha j}^{\ \ \gamma} \to 0, \quad d_{\alpha j}^{\ \ k} = c_{\alpha j}^{\ \ k} \\ d_{ij}^{\ \ \gamma} &= \epsilon^2 c_{ij}^{\ \ \gamma} \to 0, \quad d_{ij}^{\ \ k} = \epsilon c_{ij}^{\ \ k} \to 0 \end{aligned} \tag{2.2.27}$$

From the above, we see that $X_\alpha$ and $Y_\alpha$ are closed under commutation and thus span subalgebras of $\mathsf{g}$ and $\mathsf{g}'$. So, $V_R$ is closed under commutation in $\mathsf{g}$ and also forms a sub-algebra in $\mathsf{g}'$. We also see that $Y_i$ span an invariant sub-algebra of $\mathsf{g}'$ and this sub-algebra is also abelian. So, $V_N$ is an abelian invariant sub-algebra of $\mathsf{g}'$.

So, in summary, if $\mathsf{g}^{(0)} = V_R \oplus V_N$ is a Lie algebra and $U(\epsilon)$ is a singular transformation in the limit $\epsilon \to 0$ such that $U(0)V_R = V_R$ and $U(0)V_N = 0$, then $\mathsf{g}^{(0)}$ can be contracted if and only if $V_R$ is closed under commutation. $V_R$ forms a sub-algebra in $\mathsf{g}^{(0)}$ and $\mathsf{g}^{(1)}$ and $V_N$ is an abelian invariant sub-algebra of $\mathsf{g}^{(1)}$. $\mathsf{g}^{(1)}$ is thus necessarily nonsemisimple.

### 2.2.3 A Simple Example

We would want to use this formalism to investigate the non-relativistic limit of the conformal algebra. But before going into that, to have a clearer understanding of the procedure, it is instructive to look at a simple example. We would look at the case of $SO(3)$. As is well known, $SO(3)$ maps the surface of the sphere ($S^2$) embedded in $R_3$ to itself. Below we list some of the familiar facts about the algebra $so(3)$.

$$\text{Equation for } S^3: \qquad x_1^2 + x_2^2 + x_3^2 = R^2. \tag{2.2.28}$$

$$\text{Infinitesimal generators:} \qquad X_{ij} = x_i \partial_j - x_j \partial_i \tag{2.2.29}$$

$$\text{Algebra:} \quad [X_{ij}, X_{rs}] = X_{is}\delta_{jr} + X_{jr}\delta_{is} - X_{ir}\delta_{js} - X_{js}\delta_{ir} \tag{2.2.30}$$

Now we wish to look at the case where we take the radius of the sphere to be very large i.e. take the limit $R \to \infty$. Let us look at the north pole: $x_{1,2} = 0$ and $x_3 = R$. Physically, it is clear that in this limit we would be looking at a flat two-dimensional plane. But we are interested to know how to make this implementation from an algebraic standpoint. To that end, let us redefine the generators in the following way:

$$Y_{12} = \lim_{R\to\infty} X_{12} = x_1\partial_2 - x_2\partial_1 \tag{2.2.31}$$

$$P_i = \lim_{R\to\infty} \frac{1}{R} X_{i,3} = \lim_{R\to\infty} \frac{1}{R}(x_i\partial_3 - x_3\partial_i) \to -\partial_i \tag{2.2.32}$$

The principle followed above is to make the generators finite in the limit by dividing out the divergent generators by a factor of R and then take the limit. In terms of these redefined generators, the redefined or contracted algebra is:

$$[Y_{12}, P_i] = P_1\delta_{2i} - P_2\delta_{1i}, \quad [P_1, P_2] = 0 \tag{2.2.33}$$

This is the $ISO(2)$ group and is as we expected! At North Pole, with $R \to \infty$, $S^2$ looks like $R_2$.

## 2.3 Galilean Conformal Algebra

### 2.3.1 The Non-relativistic Limit

In this section, we construct the finite dimensional Galilean Conformal Algebra as a Inönu-Wigner contraction of the relativistic conformal algebra [27]. As is well known, the Galilean algebra $G(d,1)$ arises as a contraction of the Poincare algebra $ISO(d,1)$. Physically this comes from taking the non-relativistic scaling

$$t \to \epsilon^r t \qquad x_i \to \epsilon^{r+1} x_i \tag{2.3.34}$$

with $\epsilon \to 0$. This is equivalent to taking the velocities $v_i \sim \epsilon$ to zero (in units where $c = 1$). We have allowed for a certain freedom of scaling through the parameter $r$, since we might have other scales in the theory with respect to which we would have to take the above nonrelativistic limit. We will later consider the example of nonrelativistic fluid mechanics, in which we have a scale set by the temperature. In this case the natural scaling corresponds to $r = -2$. However, for the process of group contraction the parameter $r$ will play no role apart from modifying an over all factor which is unimportant. Hence we will mostly take $r = 0$.

### 2.3.2 The Contracted Algebra

Starting with the expressions for the Poincare generators ($\mu, \nu = 0, 1 \dots d$)

$$J_{\mu\nu} = -(x_\mu\partial_\nu - x_\nu\partial_\mu) \qquad P_\mu = \partial_\mu, \tag{2.3.35}$$

the above scaling gives us the Galilean vector field generators

$$\begin{aligned} J_{ij} &= -(x_i\partial_j - x_j\partial_i) \qquad P_0 = H = -\partial_t \\ P_i &= \partial_i \qquad J_{0i} = B_i = t\partial_i. \end{aligned} \tag{2.3.36}$$

The symmetry algebra is given by

$$\begin{aligned} [J_{ij}, J_{rs}] &= so(d) \\ [J_{ij}, B_r] &= -(B_i\delta_{jr} - B_j\delta_{ir}) \\ [J_{ij}, P_r] &= -(P_i\delta_{jr} - P_j\delta_{ir}), \quad [J_{ij}, H] = 0 \\ [B_i, B_j] &= 0, \quad [P_i, P_j] = 0, \quad [B_i, P_j] = 0 \\ [H, P_i] &= 0, \quad [H, B_i] = -P_i. \end{aligned} \tag{2.3.37}$$

Here $J_{ij}$ $(i, j = 1 \dots d)$ are the usual $SO(d)$ generators of spatial rotations. $P_r$ are the $d$ generators of spatial translations and $B_j$ those of boosts in these directions. Finally $H$ is the generator of time translations.

To obtain the Galilean Conformal Algebra, we simply extend the scaling (2.3.34) to the rest of the generators of the conformal group $SO(d+1, 2)$. Namely to

$$D = -(x \cdot \partial) \qquad K_\mu = -(2x_\mu(x \cdot \partial) - (x \cdot x)\partial_\mu) \tag{2.3.38}$$

where $D$ is the relativistic dilatation generator and $K_\mu$ are those of special conformal transformations. The non-relativistic scaling in (2.3.34) now gives (see also [48])

$$\begin{aligned} D &= -(x_i\partial_i + t\partial_t) \\ K &= K_0 = -(2tx_i\partial_i + t^2\partial_t) \\ K_i &= t^2\partial_i. \end{aligned} \tag{2.3.39}$$

Note that the dilatation generator $D = -(x_i\partial_i + t\partial_t)$ is the *same* as in the relativistic theory. It scales space and time in the same way $x_i \to \lambda x_i, t \to \lambda t$. The generators of the Galilean Conformal Algebra are $(J_{ij}, P_i, H, B_i, D, K, K_i)$.

The other non-trivial commutators of the GCA are [48]

$$\begin{aligned} [K, K_i] &= 0, \quad [K, B_i] = K_i, \quad [K, P_i] = 2B_i \\ [J_{ij}, K_r] &= -(K_i\delta_{jr} - K_j\delta_{ir}), \quad [J_{ij}, K] = 0, \quad [J_{ij}, D] = 0 \\ [K_i, K_j] &= 0, \quad [K_i, B_j] = 0, \quad [K_i, P_j] = 0, \quad [H, K_i] = -2B_i, \\ [D, K_i] &= -K_i, \quad [D, B_i] = 0, \quad [D, P_i] = P_i, \\ [D, H] &= H, \quad [H, K] = -2D, \quad [D, K] = -K. \end{aligned} \tag{2.3.40}$$

Note the differences of the GCA commutators (2.3.37) and (2.3.40) with the relativistic ones (2.1.16) and (2.1.17). The number of generators remain the same, as is expected in a process of group contraction. But the right-hand side of the commutators in (2.1.18) have gone from non-zero in the relativistic algebra to zero in the non-relativistic one.

## 2.4 Schrödinger Symmetry

### 2.4.1 Another non-relativistic limit

The Schrödinger symmetry group in $(d+1)$ dimensional spacetime (which we will denote as $Sch(d,1)$) has been studied as a non-relativistic analogue of conformal symmetry. It's name arises from being the group of symmetries of the free Schrödinger wave operator in $(d+1)$ dimensions. In other words, it is generated by those transformations that commute with the operator $S = i\partial_t + \frac{1}{2m}\partial_i^2$. However, this symmetry is also believed to be realized in interacting systems, most recently in cold atoms at criticality.

This can also be thought about as a non-relativistic limit of the original relativistic algebra. Let us elaborate how. For massive systems, consider cases where the rest energy is much higher than the kinetic energy. There we make the following replacement:

$$\partial_0 \to -im_0 + \partial_t; \quad m_0 \to \frac{m}{\epsilon^2}; \quad x_i \to \epsilon x_i. \tag{2.4.41}$$

In this limit, it is easy to see that the Klein Gordon equation reduces to the Schrödinger equation

$$(\partial_0^2 - \partial_i^2 + m_0^2)\phi = 0 \to (i\partial_t + \frac{1}{2m}\partial_i^2)\phi = 0. \tag{2.4.42}$$

The parameter $\epsilon \sim \frac{v}{c}$ signifies taking the nonrelativistic limit.

### 2.4.2 Schrödinger Algebra

We are interested in the algebraic aspects of the symmetry group. It contains the usual Galilean group $G(d,1)$, now with its central extension, between momenta and boosts.

$$[B_i, P_j] = m\delta_{ij} \tag{2.4.43}$$

The parameter $m$ is the central extension and has the interpretation as the non-relativistic mass (which also appears in the Schrödinger operator $S$).

In addition to these Galilean generators there are *two* more generators which we will denote by $\tilde{K}, \tilde{D}$. $\tilde{D}$ is a dilatation operator, which unlike the relativistic case, scales time and space differently. As a vector field $\tilde{D} = -(2t\partial_t + x_i\partial_i)$ so that

$$x_i \to \lambda x_i, \qquad t \to \lambda^2 t. \tag{2.4.44}$$

$\tilde{K}$ acts something like the time component of special conformal transformations. It has the form $\tilde{K} = -(tx_i\partial_i + t^2\partial_t)$ and generates the finite transformations (parametrised by $\mu$)

$$x_i \to \frac{x_i}{(1+\mu t)}, \qquad t \to \frac{t}{(1+\mu t)}. \tag{2.4.45}$$

These two additional generators have non-zero commutators

$$\begin{aligned} [\tilde{K}, P_i] &= B_i, \quad [\tilde{K}, B_i] = 0, \quad [\tilde{D}, B_i] = -B_i \\ [\tilde{D}, \tilde{K}] &= -2\tilde{K}, \quad [\tilde{K}, H] = -\tilde{D}, \quad [\tilde{D}, H] = 2H. \end{aligned} \tag{2.4.46}$$

The generators $\tilde{K}, \tilde{D}$ are invariant under the spatial rotations $J_{ij}$. We also see from the last line that $H, \tilde{K}, \tilde{D}$ together form an $SL(2, R)$ algebra. The central extension term of the Galilean algebra is compatible with all the extra commutation relations.

Note that there is no analogue in the Schrödinger algebra of the spatial components $K_i$ of special conformal transformations. Thus we have a smaller group compared to the relativistic conformal group. In $(3+1)$ dimensions the Schrödinger algebra has twelve generators (ten being those of the Galilean algebra) and the additional central term. Whereas the relativistic conformal group has fifteen generators.

### 2.4.3 SA v/s GCA

In this section, we have devoted our attention to two different realizations of conformal non-relativistic symmetries, the GCA and the Schrödinger algebra. Just to emphasize the differences between them, let us again revisit the points of distinction.

The GCA, as we saw, is obtained by a direct contraction from relativistic theory. Hence it has the same number of generators as the parent theory for example, both have fifteen in four dimensions. The Schrödinger algebra, on the other hand has an additional invariance under the Schrödinger wave operator $S$. Hence it has less number of generators, twelve in D=4 and there is no spatial analogue of Special Conformal transformations. Both share the non-centrally extended Galilean sub-group. Rest of the algebra is different.

Dilatation operator scales space and time differently in the Schrödinger algebra, with time scaling twice as fast as space. This is what is referred to as $z = 2$ in the literature. The GCA Dilatation operator, like the relativistic one, scales space and time in the same way (both are $z = 1$ systems). The most important difference between the two is that the Schrödinger algebra allows a mass term, a central extension between the momenta and the boosts. In the GCA, Jacobi identities don't allow for such mass like central extensions. The GCA is thus in some sense the symmetry group of massless or gapless non-relativistic systems.

# Chapter 3

# Infinite Non-Relativistic Conformal Symmetries

It is well known that the relativistic finite conformal algebra of $SL(2,C)$ extends to two copies of the Virasoro algebra and becomes infinite dimensional in the case of two space-time dimensions. We shall first revisit this fact in this chapter. The interesting and curious thing about the non-relativistic limit is that the GCA recovered in the limit is naturally extendable to an infinite dimensional symmetry algebra (which we will also often denote as GCA when there is no risk of confusion ). This is true for all space-time dimensions. We shall have more to say about the relation between the two infinite algebras in $D = 2$ in the later chapters.

In this chapter, after looking at the infinite extension of the relativistic case in two dimensions, we would see how to extend the finite non-relativistic algebra to the infinite one in arbitrary dimensions. The vector fields generating the infinite algebra are interpretted. We will also see that it is natural to expect this extended symmetry to be dynamically realized (perhaps partially) in actual systems possesing the finite dimensional Galilean conformal symmetry. Specifically, we would be looking at non-relativistic fluid dynamics and see how the GCA emerges as symmetry algebra in the fluid equations.

Indeed, it has been known (see [58] and references therein) that there is a notion of a "Galilean isometry" which encompasses the so-called Coriolis group of arbitrary time dependent (but spatially homogeneous) rotations and translations. In this language, our infinite dimensional algebra is that of "Galilean conformal isometries". As we will see, it contains one copy of a Virasoro together with an $SO(d)$ current algebra (on adding the appropriate central extension).

## 3.1 Infinite Extension of the Relativistic Conformal Algebra

We have seen in Chapter 2 how the conformal group is constructed in flat space. We had also remarked at the end of (2.1.9) that the case of $d = 2$ is special [18]. From Eq. (2.1.6) and (2.1.7), we can readily see that for $g_{\mu\nu} = \delta_{\mu\nu}$ and $d = 2$, we recover the Cauchy-Riemann equations for holomorphic functions (and its anti-holomorphic counterpart)

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \tag{3.1.1}$$

The theory is best formulated in terms of the complex co-ordinates on the plane and there the solution the the C-R equation is simply any holomorphic (and antiholomorphic) map $z \to f(z)$. The conformal group in two dimensions is thus the set of all analytic maps and since an infinite number of parameters are required to specify all analytic maps in a neighbourhood, the group is infinite dimensional. It is not correct, however to call this a group, as only a finite sub-set of these maps would be invertible and well-defined everywhere. To exhibit the generators, let us consider infinitesimal conformal transformations of the form

$$z \to z' = z - \epsilon_n z^{n+1} \text{ and } \bar{z} \to \bar{z}' = \bar{z} - \bar{\epsilon}_n \bar{z}^{n+1} \tag{3.1.2}$$

The corresponding infinitesimal generators are

$$\mathcal{L}_n = -z^{n+1}\partial_z\,, \quad \bar{\mathcal{L}}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\,. \tag{3.1.3}$$

These generators satisfy the classical Virasoro algebra, also called the Witt algebra

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \;\; [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n} \;\; [\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0. \tag{3.1.4}$$

In the quantum case the Virasoro algebra can acquire a central charge and then the algebra has the form

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12}\delta_{m+n,0}m(m^2-1) \tag{3.1.5}$$

In a two-dimensional conformal field theory the Virasoro operators are the modes of the energy-momentum tensor. This is the operator that generates conformal transformations. The central charge is a constant term can be understood to multiply the unit operator, which is adjoined to the Lie algebra. This is also called the conformal anomaly and can be interpreted as signaling a quantum mechanical breaking of the classical conformal symmetry. The algebra contains a finite-dimensional subalgebra generated by $\mathcal{L}_{0,\pm1}, \bar{\mathcal{L}}_{0,\pm1}$. These are the generators that are well defined all over the complex plane and form the global conformal group in two dimensions. The rest of the generators are local. The global generators have the following interpretation physically,- $\mathcal{L}_{-1}$ and $\bar{\mathcal{L}}_{-1}$ generate translations, $(\mathcal{L}_0 + \bar{\mathcal{L}}_0)$ generates scalings, $i(\mathcal{L}_0 - \bar{\mathcal{L}}_0)$ generates rotations and $\mathcal{L}_1$ and $\bar{\mathcal{L}}_1$ generate special conformal transformations. The finite form of the group transformations is

$$z \to \frac{az+b}{cz+d} \text{ with } a,b,c,d \in C, \text{ and } ad-bc=1. \tag{3.1.6}$$

This is the group $SL(2;C)/Z_2$. The division by $Z_2$ accounts for the freedom to replace the parameters $a,b,c,d$ by their negatives, leaving the transformations unchanged[1]. In the Lorentzian case the group is replaced by $SL(2,R)\times SL(2,R) = SO(2,2)$ where one factor of $SL(2,R)$ pertains to left-movers and the other to right-movers. This is the two-dimensional case of $SO(D,2)$, which is the conformal group for $D>2$.

# 3.2 The Infinite Dimensional Extended GCA

## 3.2.1 Extending the Algebra

We have looked at how the finite dimensional GCA is constructed out of the contraction of the relativistic conformal algebra in the previous chapter. The most interesting feature of the GCA is that it admits a very natural extension to an infinite dimensional algebra of the Virasoro-Kac-Moody type. To see this we denote

$$\begin{aligned} L^{(-1)} &= H, \qquad L^{(0)} = D, \qquad L^{(+1)} = K, \\ M_i^{(-1)} &= P_i, \qquad M_i^{(0)} = B_i, \qquad M_i^{(+1)} = K_i. \end{aligned} \tag{3.2.7}$$

The finite dimensional GCA which we had in the previous chapter ((2.3.37), (2.3.40))can now be recast as

$$\begin{aligned} [J_{ij}, L^{(n)}] &= 0, \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)} \\ [J_{ij}, M_k^{(m)}] &= -(M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik}), \qquad [M_i^{(m)}, M_j^{(n)}] = 0, \\ [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)}. \end{aligned} \tag{3.2.8}$$

The indices $m,n = 0,\pm1$ We have made manifest the $SL(2,R)$ subalgebra with the generators $L^{(0)}, L^{(\pm1)}$. In fact, we can define the vector fields

$$L^{(n)} = -(n+1)t^n x_i\partial_i - t^{n+1}\partial_t, \quad M_i^{(n)} = t^{n+1}\partial_i \tag{3.2.9}$$

with $n = 0,\pm1$. These (together with $J_{ij}$) are then exactly the vector fields in (2.3.36) and (2.3.39) which generate the GCA (without central extension).

If we now consider the vector fields of (3.2.9) for *arbitrary* integer $n$, and also define

$$J_a^{(n)} \equiv J_{ij}^{(n)} = -t^n(x_i\partial_j - x_j\partial_i) \tag{3.2.10}$$

[1]We need to consider the conformal transformations to be real and cannot treat $z$ and $\bar{z}$ as independent variables.

then we find that this collection obeys the current algebra

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} \qquad [L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)} \\ [J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \end{aligned} \tag{3.2.11}$$

The index $a$ labels the generators of the spatial rotation group $SO(d)$ and $f_{abc}$ are the corresponding structure constants. We see that the vector fields generate a $SO(d)$ Kac-Moody algebra without any central terms. In addition to the Virasoro and current generators we also have the commuting generators $M_i^{(n)}$ which function like generators of a global symmetry. We can, for instance, consistently set these generators to zero. The presence of these generators therefore do not spoil the ability of the Virasoro-Kac-Moody generators to admit the usual central terms in their commutators.

### 3.2.2 Physical Interpretation of GCA Vector-fields

What is the meaning of this infinite dimensional extension? Do these additional vector fields generate symmetries?

There is a relatively simple interpretation for the generators $M_i^{(n)}, L^{(n)}, J_a^{(n)}$. We know that $P_i = M_i^{(-1)}, B_i = M_i^{(0)}, K_i = M_i^{(1)}$ generate uniform spatial translations, velocity boosts and accelerations respectively. In fact, it is simple to see from (3.2.9) that the $M_i^{(n)}$ generate arbitrary time dependent (but spatially independent) accelerations.

$$x_i \to x_i + b_i(t). \tag{3.2.12}$$

Similarly the $J_{ij}^{(n)}$ in (3.2.10) generate arbitrary time dependent rotations (once again space independent)

$$x_i \to R_{ij}(t)x_j \tag{3.2.13}$$

These two set of generators together generate what is sometimes called the Coriolis group: the biggest group of "isometries" of "flat" Galilean spacetime [58].

Recall that in the absence of gravity Galilean spacetime is characterised by a degenerate metric. The time intervals are much larger than any space-like intervals in the nonrelativistic scaling limit (2.3.34). We thus have an absolute time $t$ and spatial sections with a flat Euclidean metric. We can, in a precise sense, describe the analogue of the isometries in this Galilean spacetime. The Coriolis group by virtue of preserving the spatial slices (at any given time) are the maximal set of isometries. See the appendix at the end of Chap. 8 for more details. This realisation of the current algebra in our context is a bit like the occurrence of a loop group.

The generators $L^{(n)}$ have a more interesting action in acting both on time as well as space. We can read this off from (3.2.9)

$$t \to f(t), \qquad x_i \to \frac{df}{dt}x_i. \tag{3.2.14}$$

Thus it amounts to a reparametrisation of the absolute time $t$. Under this reparametrisation the spatial coordinates $x_i$ act as vectors (on the worldline $t$). It seems as if this is some kind of "conformal isometry" of the Galilean spacetime, rescaling coordinates by the arbitrary time dependent factor $\frac{df}{dt}$.

With this interpretation of the infinite extension of the GCA, one might expect that it ought to be partially or fully dynamically realized in physical systems where the finite GCA is (partially or fully) realized. We will see below an example which lends support to this idea. We will also see in Chap. 8 that the bulk geometry which we propose as the dual has the extended GCA among its *asymptotic* isometries. An analogy might be two dimensional conformal invariance where the Virasoro algebra is often a symmetry when the finite conformal symmetry of $SL(2, C)$ is realized. And the (two copies of the) Virasoro generators are reflected in the bulk $AdS_3$ as asymptotic isometries.

## 3.3 Realization of the GCA: NR Hydrodynamics

Given that the Galilean limit can be obtained by taking a definite scaling limit within a relativistic theory, we expect to see the GCA (and perhaps its extension) as a symmetry of some subsector within every relativistic conformal field theory. For instance, in the best studied case of $\mathcal{N} = 4$ Yang-Mills theory, we ought to be able to isolate a sector with this symmetry. One clue is the presence of the $SL(2, R)$ symmetry together with the preservation of spatial rotational invariance. One might naively think this should be via some kind of conformal quantum mechanics obtained by considering only the spatially independent modes of the field theory. But this is probably not totally correct for the indirect reasons explained in the next paragraph.

Recently, the nonrelativistic limit of the relativistic conformal hydrodynamics (for a recent review of the fluid-gravity correspondence, see []), which describes the small fluctuations from thermal equilibrium, have been studied [50, 51, 52]. One recovers the non-relativistic incompressible Navier-Stokes equation in this limit. The symmetries of this equation were then studied by [51] (see also [52]). One finds that all the generators of the finite GCA are indeed symmetries[2] except for the dilatation operator $D$ [3]. In particular it has the $K_i$ as symmetries. It is not surprising that the choice of a temperature should break the scaling symmetry of $D$ [4]. The interesting point is that the arbitrary accelerations $M_i^{(n)}$ are also actually a symmetry [53] (generating what is sometimes called the Milne group [58]). Thus we have a part of the extended GCA as a symmetry of the non-relativistic Navier-stokes equation which should presumably describe the hydrodynamics in every nonrelativistic field theory. In particular, the closed non-relativistic subsector

[2] For a realization of the Schrödinger symmetry in the context of the Navier-Stokes equation see [54, 55].
[3] The generator $K$ acts trivially.
[4] However, one can define an action of the $\tilde{D}$ as in (2.4.44) to be a symmetry.

within every relativistic conformal field theory should have a hydrodynamic description governed by the Navier-Stokes equation. This might seem to suggest that this sector ought to have more than just the degrees of freedom of a conformal quantum mechanics.

Coming back to the Navier-Stokes equation, if the viscosity is set to zero, one gets the incompressible Euler equations

$$\partial_t v_i(x,t) + v_j \partial_j v_i(x,t) = -\partial_i P(x,t) \tag{3.3.15}$$

In this case one has the entire finite dimensional GCA being a symmetry since $D$ is now also a symmetry. (In the following subsection, we explicitly show the symmetries of the Euler equation.) It is the viscous term which breaks the symmetry under equal scaling of space and time. This shows that one can readily realise "gapless" non-relativistic systems in which space and time scale in the same way! [5]

### 3.3.1 Symmetries of the Euler equations

As we had been discussing above, the motion of an inviscid, incompressible fluid is described by the continuity equation,

$$\partial_i v_i = 0 \tag{3.3.16}$$

where $v_i$ are the components of the velocity field, and the Euler equations,

$$\partial_t v_i(x,t) + v_j \partial_j v_i(x,t) = -\partial_i P(x,t) \tag{3.3.17}$$

where $P$ is the pressure and we have put the constant density of the fluid to unity. Equations (3.3.16) and (3.3.17) are a set of four equations to be solved for the four unknown functions, $v_i$ and $P$. This would describe classical fluids in situations where its compressibility and viscosity can be neglected (e.g. very long wavelength ocean waves like tsunamis). The quantum version of the system describes the dynamics of superfluids.

We shall consider the transformation of the equations under the GCA [56]. Let us start off with the Galilean sub-algebra. Equations (3.3.16) and (3.3.17) are manifestly invariant under space-time translations and spatial rotations. The Galilean boosts correspond to the co-ordinate transformations,

$$t' = t, \quad x'_i = x_i + u_i t. \tag{3.3.18}$$

which in turn implies

$$\partial_t = \partial_{t'} + u_i \partial_{x'_i}, \quad \partial_{x_i} = \partial_{x'_i} \tag{3.3.19}$$

[5] Inonu and Wigner [60] have considered representations of the Galilean group without the mass extension and concluded that a particle interpretation of states of the irreducible representations is subtle. In particular such states are not localisable. Just as in the case of relativistic conformal group it is likely that observables such as the S-matrix are ill-defined. We thank Sean Hartnoll for bringing this reference to our attention.

If the velocities and the pressure transform as,

$$v_i'(x_i', t') = v_i(x_i, t) - u_i, \quad P(x_i', t') = P(x_i, t). \tag{3.3.20}$$

Then we have,

$$\begin{aligned} \partial_t v_i = (\partial_{t'} + u_j \partial_j')(v_i' - u_i) = \partial_{t'} v_i + u_j \partial_j' v_i' \\ v_j \partial_j v_i = v_j' \partial_j' v_i' - u_j \partial_j' v_i', \ \ \partial_i P = \partial_i' P' \end{aligned} \tag{3.3.21}$$

The above three equations establish the invariance of the Euler equations. The invariance of the continuity equation is also straightforward.

We then move on to time dependent boosts ($M_n$) and consider them for arbitrary $n$. As we shall see the Euler equations are also invariant under time dependent boosts. i.e. transformations to arbitrary moving frames,

$$x'^i = x^i + b^i(t), \quad t' = t \tag{3.3.22}$$

We denote the velocity and the acceleration of the moving frame as,

$$u^i = \frac{db^i}{dt}, \quad a^i = \frac{du^i}{dt} \tag{3.3.23}$$

If the velocities and the pressure transform as,

$$v^i(x', t') = v^i(x, t) - u^i, \quad P'(x', t') = P(x, t) - a^i x^i \tag{3.3.24}$$

Then,

$$\begin{aligned} \partial_t v^i = (\partial_{t'} + u^j \partial_j')(v'^i - u^i) = \partial_{t'} v^i + u^j \partial_j' v'^i - a^i \\ v_j \partial_j v_i = v_j' \partial_j' v_i' - u_j \partial_j' v_i', \quad \partial_i P = \partial_i' P' - a_i \end{aligned} \tag{3.3.25}$$

The above three equations establish the invariance of the Euler equations. The invariance of the continuity equation is again straightforward. Physically, in the moving frame the fluid will experience pseudo-forces. In the case of time dependent boosts, since the frame is being rigidly translated, the pseudo-force is spatially constant. Such forces (like gravity) can always be absorbed in the pressure term. So if an infinite volume of fluid is moved around, it will only experience time dependent changes in pressure. The flow pattern will be unchanged. Of course if the fluid is in a container, then the boundary conditions will break this invariance and the above statement will not be true.

Lastly we look at dilatations. Under the scale transformations,

$$x_i' = \lambda x_i, \ \ t' = \lambda t \tag{3.3.26}$$

If,

$$v_i'(x_i', t') = v_i(x_i, t), \quad P(x_i', t') = P(x_i, t) \tag{3.3.27}$$

then the Euler equations and the continuity equation are manifestly Invariant- ant since every term involves exactly one spatial or time derivative. The viscosity term breaks the scale invariance so the Navier-Stokes equations- are not scale invariant.

We can also see that the special conformal transformation $K$ acts trivially on the equation and hence is a symmetry. So, as remarked before, the entire finite-dimensional algebra is a symmetry here. Added to this, arbitrary time dependent boosts are also a symmetry giving a realization of a part of the infinite dimensional GCA.

## 3.4 Infinite extension of the Schrödinger Algebra

We have spoken at length about the infinite extension of the GCA and not touched upon the Schrödinger algebra in this chapter. Interestingly, an infinite extension for the Schrödinger algebra has also been known for a long time [23], but has not been much exploited in the context of the non-relativistic limit of the AdS/CFT duality. See however [75] We here mention this infinite extension of the Schrödinger algebra. We wish to rewrite the Schrödinger algebra in a different form and in a representation from which we would be able to see the infinite lift.

The generators of the Schrödinger algebra can be thought of as vector fields defined on $d$ dimensional spacetime with the following representation

$$\begin{aligned} &J_{ij} = -(x_i\partial_j - x_i\partial_i), \quad P_i = -\partial_i, \quad H = -\partial_t, \quad m = -M \\ &B_i = -(t\partial_i + x_i M), \quad \tilde{D} = -(t\partial_t + \frac{1}{2}x_i\partial_i), \quad \tilde{K} = -(t^2\partial_t + tx_i\partial_i + \frac{1}{2}x^2 M). \end{aligned}$$

Following [23] one can define the generators of the corresponding infinite dimensional algebra in $d-1$ dimensions as follows

$$\begin{aligned} L_n &= -t^{n+1}\partial_t - \frac{n+1}{2}t^n x_i\partial_i - \frac{n(n+1)}{4}t^{n-1}x^2 M, \\ Q_{i\hat{n}} &= -t^{\hat{n}+1/2}\partial_i - (\hat{n} + \frac{1}{2})t^{\hat{n}-1/2}x_i M, \quad T_n = -t^n M. \end{aligned} \tag{3.4.28}$$

Here $n \in Z$ and $\hat{n} \in Z + \frac{1}{2}$. It is straightforward to see that the above generators satisfy the following commutation relations

$$\begin{aligned} &[L_n, L_m] = (n-m)L_{n+m}\,, \quad [Q_{i\hat{n}}, Q_{j\hat{m}}] = (\hat{n} - \hat{m})\delta_{ij}T_{\hat{n}+\hat{m}}\,, \\ &[L_n, Q_{i\hat{m}}] = (\frac{n}{2} - \hat{m})Q_{i(n+\hat{m})}\,, \quad [L_n, T_m] = -mT_{n+m}. \end{aligned} \tag{3.4.29}$$

Due to non-trivial contribution of number operator to the Galilean boost the Schrödinger algebra does not allow an infinite dimensional extension for rotations $J_{ij}$, unlike the case of the GCA [75].

It is not known if this infinite algebra is indeed realized even partially in any systems. It would be interesting to look at this infinite extension and try out things analogous to what we have discussed and will go on to describe in the remainder of the thesis.

# Chapter 4

# Representations and Correlation Functions of the GCA

We have seen in the previous chapters how one obtains the Galilean Conformal Algebra from the relativistic case and how this algebra naturally sits in an infinite dimensional algebra for all space-time dimensions. In this chapter we first look at the representations of this algebra. The difference from the usual relativistic case, where one looks to represent the operators by their conformal weight, is that here we would have another label for the operators, their boost eigenvalue. We look at the consequences of this novel representation. We construct primary operators and the representations are built by acting with raising operators. We then focus on the finite part of the algebra and look at correlation functions of operators which are primary with respect to this algebra. We find that the finite part of the algebra is sufficient to fix the two and three-point correlation functions upto constant numbers, like in the relativistic case. The structure of the correlators are unique in the sense that there are exponential pieces. We comment on the differences of the GCA correlation functions with the relativisitic ones and also the ones from Schrödinger Algebra.

## 4.1 Representations of the GCA

Along the lines of usual relativistic conformal field theories, we can look to build the representations of the infinite dimensional algebra that we have obtained before. The representations are build by considering local operators which have well defined scaling dimension and "boost" eigenvalue (which we will call the *rapidity*). We introduce the notion of local operators as

$$\mathcal{O}(t,x) = U\mathcal{O}(0)U^{-1}, \quad \text{where} \quad U = e^{tH - x_i P^i} = e^{tL_{-1} - x_i M^i_{-1}}\,. \tag{4.1.1}$$

We note that $[L_0, M_0^i] = 0$. The representations should thus be labeled by the eigenvalues of both the operators, as we stated before. Let us define local operators as those which are simultaneous eigenstates of $L_0, M_0^i$ :

$$[L_0, \mathcal{O}] = \Delta\mathcal{O}, \quad [M_0^i, \mathcal{O}] = \xi^i\mathcal{O}. \tag{4.1.2}$$

Let us look at the set of all local operators $\mathcal{O}_a(t, x)$. These operators, put at $t = 0$ and $x = 0$, form a representation of the contracted subalgebra of the GCA which leaves the spacetime origin $\{t = 0, x = 0\}$ invariant : for any operator $A$ in the subalgebra

$$[A,\, \mathcal{O}_a(0)] = A_{ab}\mathcal{O}_b(0)\,. \tag{4.1.3}$$

We would describe the irreducible representations of the infinite algebra. We can use the Jacobi identities to show that

$$\begin{aligned} [L_0, [L_n, \mathcal{O}]] &= (\Delta - n)[L_n, \mathcal{O}]\,, \\ [L_0, [M_n^i, \mathcal{O}]] &= (\Delta - n)[M_n^i, \mathcal{O}]\,. \end{aligned} \tag{4.1.4}$$

The $L_n, M_n^i$ thus lower the value of the scaling dimension while $L_{-n}, M_{-n}^i$ raise it. Demanding that the dimension of the operators be bounded from below defines the primary operators in the theory to have the following properties :

$$[L_n, \mathcal{O}_p] = 0\,, \quad [M_n^i, \mathcal{O}_p] = 0\,, \tag{4.1.5}$$

for all $n > 0$.

Starting with a primary operator $\mathcal{O}_p$, one can build up a tower of operators by taking commutators with $L_{-n}$ and $M_{-n}^i$. The operators built from a primary operator in this way form an irreducible representation of the contracted algebra. It is also possible to show that the full set of all local operators can be decomposed into irreducible representations, each of which is built upon a single primary operator. The task of finding the spectrum of dimensions of all local operators reduces to finding the $(\Delta\,,\,\xi^i)$ eigenvalues of the primary operators.

Curiously, the odd behaviour of the "current algebra" commutation spoils the diagonalisation of this representation. To see this, let us look at how $M_0^i$ labels $[L_n, \mathcal{O}]$. Again, we use the Jacobi identity

$$[M_0^i, [L_n, \mathcal{O}]] = \xi^i[L_n, \mathcal{O}] + n[M_n^i, \mathcal{O}]\,, \tag{4.1.6}$$

which tells us that $[L_n, \mathcal{O}]$ is not an eigenstate of $M_0^i$. In a particular representation labelled by $(\Delta, \xi^i)$, when one moves up by acting with $L_n$, the significance of the rapidity, which is a vector quantum number, is not clear. But

$$[M_0^i, [M_n^j, \mathcal{O}]] = \xi^i[M_n^j, \mathcal{O}]\,. \tag{4.1.7}$$

So, moving up with $M_n^i$ preserves rapidity. We will have more to say about representations when we focus on the case of two spacetime dimensions in the later chapters.

## 4.2 Non-Relativistic Conformal Correlation Functions

We wish to find the form of the correlation functions of the Galilean Conformal Algebras. To this end, let us define quasi-primary operators in this section.

We want to find the explicit form of the commutator $[L_n, \mathcal{O}(x,t)]$ for $\mathcal{O}(x,t)$ primary (i.e., obeying (5.2.14)) and $n \geq 0$.

$$\begin{aligned}[L_n, \mathcal{O}(x,t)] &= [L_n, U\mathcal{O}(0)U^{-1}] \\ &= [L_n, U]\mathcal{O}(0)U^{-1} + U\mathcal{O}(0)[L_n, U^{-1}] + U[L_n, \mathcal{O}(0)]U^{-1} \\ &= U\{U^{-1}L_nU - L_n\}\mathcal{O}(0)U^{-1} + U\mathcal{O}(0)\{L_n - U^{-1}L_nU\}U^{-1} + \delta_{n,0}\Delta\mathcal{O}(x,t)\,,\end{aligned} \tag{4.2.8}$$

where $U$ is as defined in (4.1.1).
Using the Baker-Campbell-Hausdorff (BCH) formula, we get

$$\begin{aligned} U^{-1}L_n\,U &= \sum_{k=0}^{n+1}\frac{(n+1)!}{(n+1-k)!k!}(t^kL_{n-k} - kx_it^{k-1}M^i_{n-k})\,, \\ U^{-1}M^i_n\,U &= \sum_{k=0}^{n+1}\frac{(n+1)!}{(n+1-k)!k!}t^kM^i_{n-k}\,. \end{aligned}$$

Using the above, (5.3.24) gives us

$$\begin{aligned}[L_n, \mathcal{O}(x,t)] &= U[t^{n+1}L_{-1} - (n+1)t^nx_iM^i_{-1}, \mathcal{O}(0)]U^{-1} \\ &\quad + U(n+1)(t^n\Delta - nt^{n-1}x_i\xi^i\mathcal{O}(0)U^{-1} \\ &= [t^{n+1}L_{-1} - (n+1)t^nx_iM^i_{-1}, \mathcal{O}(x,t)\,] + (n+1)(t^n\Delta - nt^{n-1}x_i\xi^i)\mathcal{O}(x,t)\end{aligned} \tag{4.2.9}$$

Here, in the second line, we have used the fact that $U$ commutes with $L_{-1}$, $M^i_{-1}$ and any function of $(x,t)$. Now $[L_{-1}, \mathcal{O}(x,t)] = \partial_t\mathcal{O}(x,t)\,, [M^i_{-1}, \mathcal{O}(x,t)] = -\partial_i\mathcal{O}(x,t)\,.$
Hence we finally obtain for $n \geq 0$ :

$$[L_n, \mathcal{O}(x,t)] = [t^{n+1}\partial_t + (n+1)t^nx^i\partial_i + (n+1)(t^n\Delta - nt^{n-1}x_i\xi^i)]\,\mathcal{O}(x,t) \tag{4.2.10}$$

By an analogous procedure, one can show that (again for $n \geq 0$)

$$[M^i_n, \mathcal{O}(x,t)] = [-t^{n+1}\partial_i + (n+1)t^n\xi^i]\,\mathcal{O}(x,t). \tag{4.2.11}$$

Borrowing notation from the usual relativistic CFT, we would call those operators *quasi-primary operators* which transform as (4.2.10) and (5.3.25) with respect to the finite subalgebra $\{L_0, L_{\pm1}, M_0, M_{\pm1}\}$.[1] We would examine the correlation functions of these

[1] We do not look at the action of rotation because we would be labeling the operators with their boost eigenvalues, and boosts and rotations do not commute.

quasi-primary operators. For any set of $n$ operators, one can define an $n$-point correlation function,

$$G_n(t_1, x_1, t_2, x_2, ....., t_n, x_n) = \langle 0|T\mathcal{O}_1(t_1, x_1)\mathcal{O}_2(t_2, x_2).....\mathcal{O}_n(t_n, x_n)|0\rangle, \tag{4.2.12}$$

where $T$ is time ordering.

The exponentiated version of the action of the dilatation operator is

$$e^{-\lambda L_0}\mathcal{O}(t, x)e^{\lambda L_0} = e^{\lambda\Delta_{\mathcal{O}}}\mathcal{O}(e^\lambda t, e^\lambda x). \tag{4.2.13}$$

If all $\mathcal{O}_i$ have definite scaling dimensions then the correlation function (4.2.12) has a scaling property

$$G_n(e^\lambda t_i, e^\lambda x_i) = \exp\left(-\lambda\sum_{i=1}^{n}\Delta_{\mathcal{O}_i}\right)G_n(t_i, x_i), \tag{4.2.14}$$

which follows from Eq. (4.2.13) and $e^{\lambda L_0}|0\rangle = |0\rangle$.

### 4.2.1 Two Point Function of the GCA

We wish to first look at the details of how the two point function is constrained by the GCA. For that, we begin by considering two quasi-primary local operators $\mathcal{O}_1(x_1^i, t_1)$ and $\mathcal{O}_2(x_2^i, t_2)$ of conformal and rapidity weights $(\Delta_1, \xi_1^i)$ and $(\Delta_2, \xi_2^i)$ respectively. The two point function is defined as

$$G^{(2)}(x_1^i, x_2^i, t_1, t_2) = \langle 0|T\mathcal{O}_1(t_1, x_1^i)\mathcal{O}_2(t_2, x_2^i)|0\rangle\,. \tag{4.2.15}$$

We require the vacuum to be translationally invariant. The correlation functions can thus only depend on differences of the co-ordinates. From the fact that we are dealing with quasi-primaries, we get four more equations which would constrain the form of the 2-point function. Let us label $\tau = t_1 - t_2$ and $r^i = x_1^i - x_2^i$ and call $\Delta = \Delta_1 + \Delta_2$ and $\xi^i = \xi_1^i + \xi_2^i$. We first look at the action under the Galilean boosts :

$$\begin{aligned}&\langle 0|[M_0^i, G^{(2)}|0\rangle = 0 \Rightarrow (-\tau\partial_{r_i} + \xi_i)G^{(2)} = 0\\ \Rightarrow\quad &G^{(2)} = C(\tau)\exp\left(\frac{\xi_i r^i}{\tau}\right)\,,\end{aligned} \tag{4.2.16}$$

where $C(\tau)$ is an arbitrary function of $\tau$.
The behaviour under dilatations gives :

$$(\tau\partial_\tau + r_i\partial_{r_i} + \Delta)G^{(2)}(r_i, \tau) = 0\,. \tag{4.2.17}$$

This fixes the two point function to be

$$G^{(2)}(r_i, \tau) = C^{(2)}\tau^{-\Delta}\exp\left(\frac{\xi_i r^i}{\tau}\right)\,, \tag{4.2.18}$$

where $C^{(2)}$ is an arbitrary constant. We still have two conditions left. These are:

$$\langle 0|[L_1, G^{(2)}]|0\rangle = 0, \quad \langle 0|[M_1^i, G^{(2)}]|0\rangle = 0\,, \tag{4.2.19}$$

which give respectively

$$\Delta_1 = \Delta_2, \quad \xi_1^i = \xi_2^i\,. \tag{4.2.20}$$

So, in its full form, the two-point function for the GCA reads:

$$G^{(2)}(r_i,\tau) = C^{(2)}\delta_{\Delta_1,\Delta_2}\delta_{\xi_1,\xi_2}\tau^{-2\Delta_1}\exp\left(\frac{2\xi_1^i r_i}{\tau}\right)\,, \tag{4.2.21}$$

where $C^{(2)}$ is, like before, an arbitrary constant.

### 4.2.2 Three Point Function of the GCA

Along lines similar to the previous subsection, we wish to construct the three point function of three quasi-primary operators $\mathcal{O}_a(\Delta = \Delta^a, \xi^i = \xi_i^a)$, where $a = 1, 2, 3$. The three point function is defined as

$$G^{(3)}(x_1^i, x_2^i, x_3^i, t_1, t_2, t_3) = \langle 0|T\mathcal{O}_1(t_1, x_1^i)\mathcal{O}_2(t_2, x_2^i)\mathcal{O}_3(t_3, x_3^i)|0\rangle\,. \tag{4.2.22}$$

Again, remembering that we need to take only differences of co-ordinates into account, we define $\tau = t_1 - t_3$, $\sigma = t_2 - t_3$ and $r^i = x_1^i - x_3^i$, $s^i = x_2^i - x_3^i$ and call $\Delta = \Delta^1 + \Delta^2 + \Delta^3$ and $\xi_i = \xi_i^1 + \xi_i^2 + \xi_i^3$. The constraining equations are listed below. Using the equation for dilatations, one obtains

$$\begin{aligned} &\langle 0|[L_0,\, T\mathcal{O}_1(\vec{x^1}, t^1)\mathcal{O}_2(\vec{x^2}, t^2))\mathcal{O}_3(\vec{x^3}, t^3)]|0\rangle = 0 \\ \Rightarrow\ &(\tau\partial_\tau + \sigma\partial_\sigma + r_i\partial_{r_i} + s_i\partial_{s_i} + \Delta)G^{(3)} = 0. \end{aligned} \tag{4.2.23}$$

The boost equations give the following constraints :

$$\begin{aligned} &\langle 0|[M_0^i,\, T\mathcal{O}_1(\vec{x^1}, t^1)\mathcal{O}_2(\vec{x^2}, t^2))\mathcal{O}_3(\vec{x^3}, t^3)]|0\rangle = 0 \\ \Rightarrow\ &(\tau\partial_{r_i} + \sigma\partial_{s_i} - \xi_i)G^{(3)} = 0\,. \end{aligned} \tag{4.2.24}$$

The non-relativistic spatial analogues of the special conformal transformations give

$$\begin{aligned} &\langle 0|[M_1^i,\, T\mathcal{O}_1(\vec{x^1}, t^1)\mathcal{O}_2(\vec{x^2}, t^2))\mathcal{O}_3(\vec{x^3}, t^3)]|0\rangle = 0 \\ \Rightarrow\ &[(t^a)^2\partial_{x_i^a} - 2t\xi_i^a]G^{(3)} = 0\,, \end{aligned} \tag{4.2.25}$$

which simplify to

$$[\tau\sigma(\partial_{r_i} + \partial_{s_i}) + \tau(\xi_i^1 - \xi_i^2 - \xi_i^3) + \sigma(-\xi_i^1 + \xi_i^2 - \xi_i^3)]G^{(3)} = 0\,. \tag{4.2.26}$$

And finally, the last constraining equation comes from the temporal non-relativistic special conformal generator. Using $\langle 0|[L_1, T\mathcal{O}_1(\vec{x^1}, t^1)\mathcal{O}_2(\vec{x^2}, t^2))\mathcal{O}_3(\vec{x^3}, t^3)]|0\rangle = 0$, and equations for $L_0$ and $M_0^i$, one obtains :

$$\begin{aligned}&[\tau\sigma(\partial_\tau + \partial_\sigma) + (\sigma r_i + \tau s_i)(\partial_{r_i} + \partial_{s_i}) - \tau(\Delta^1 - \Delta^2 - \Delta^3)\\ &+\sigma(\Delta^1 - \Delta^2 + \Delta^3) + r_i(\xi_i^1 - \xi_i^2 - \xi_i^3) - s_i(\xi_i^1 - \xi_i^2 + \xi_i^3)]G^{(3)} = 0.\end{aligned} \quad (4.2.27)$$

Motivated by the form of the the two-point function and examining (4.2.24), we make the following ansatz for the three-point function of the GCA :

$$G^{(3)} = f(\tau, \sigma, \Delta^a) \exp\left(\frac{a_i r^i}{\tau} + \frac{b_i s^i}{\sigma} + \frac{c_i(r^i - s^i)}{(\tau - \sigma)}\right) \Psi(r_i, s_i, \tau, \sigma)\,, \quad (4.2.28)$$

where $f(\tau, \sigma, \Delta^a)$ is a function of $\tau$, $\sigma$, and the $\Delta^a$'s; $\Psi(r_i, s_i, \tau, \sigma)$ is a function of $r_i, s_i, \tau, \sigma$, and independent of the $\Delta^a$'s and $\xi_i^a$'s; and $a_i, b_i, c_i$ (for $i = 1, 2, 3$) are constants to be determined in terms of the $\xi_i^a$'s. Plugging this form of $G^{(3)}$ in (4.2.24) and remembering that $\Psi$ does not depend on the $\xi_i^a$'s, we get the constraints:

$$a_i + b_i + c_i = \xi_i, \quad \text{and} \quad (\tau\partial_{r_i} + \sigma\partial_{s_i})\Psi = 0. \quad (4.2.29)$$

Using (4.2.23) and employing the $\Delta^a$-independence of $\Psi$, we get the equations :

$$(\tau\partial_\tau + \sigma\partial_\sigma) f(\tau, \sigma, \Delta^a) = -\Delta\,, \quad (4.2.30)$$

and

$$(\tau\partial_\tau + \sigma\partial_\sigma + r_i\partial_{r_i} + s_i\partial_{s_i})\Psi = 0\,. \quad (4.2.31)$$

Eq.(4.2.30) dictates that $f(\tau, \sigma, \Delta^a)$ should have the form

$$f(\tau, \sigma, \Delta^a) = \tau^{-A}\sigma^{-B}(\tau - \sigma)^{-C}\,, \quad (4.2.32)$$

where $A, B, C$ depend on the $\Delta^a$'s and obey the condition:

$$A + B + C = \Delta\,. \quad (4.2.33)$$

Now the constraints in Eq.(4.2.26) fix $a_i, b_i$ to be

$$a_i = \xi_i^1 + \xi_i^3 - \xi_i^2\,, \quad b_i = \xi_i^2 + \xi_i^3 - \xi_i^1\,, \quad (4.2.34)$$

and restrict $\Psi$ to obey :

$$\tau\sigma(\partial_{r_i} + \partial_{s_i})\Psi = 0\,. \quad (4.2.35)$$

Finally, Eq.(4.2.27) fixes $A, B$ to be

$$A = \Delta^1 + \Delta^3 - \Delta^2\,, \quad B = \Delta^2 + \Delta^3 - \Delta^1\,, \quad (4.2.36)$$

and constrains $\Psi$ to obey :

$$\{\tau\sigma(\partial_\tau + \partial_\sigma) + (\sigma r_i + \tau s_i)(\partial_{r_i} + \partial_{s_i}))\}\Psi = 0\,. \tag{4.2.37}$$

Using (4.2.29), (4.2.34), (4.2.33) and (4.2.36), the remaining constants $c_i, C$ are found to be

$$c_i = \xi_i^1 + \xi_i^2 - \xi_i^3 \;\;, \quad C = \Delta^1 + \Delta^2 - \Delta^3\,. \tag{4.2.38}$$

The equations (4.2.29) and (4.2.35) make $\Psi$ a function of $\tau$ and $\sigma$ alone. Then applying (4.2.31) and (4.2.37), in a similar way, we find that $\Psi$ is restricted to a constant. Finally, the full three point function is:

$$\begin{aligned} &G^{(3)}(r_i, s_i, \tau, \sigma) \\ &= C^{(3)}\tau^{-(\Delta^1-\Delta^2+\Delta^3)}\sigma^{-(\Delta^2+\Delta^3-\Delta^1)}(\tau-\sigma)^{-(\Delta^1+\Delta^2-\Delta^3)} \\ &\quad\times \exp\left(\frac{(\xi_i^1+\xi_i^3-\xi_i^2)r^i}{\tau} + \frac{(\xi_i^2+\xi_i^3-\xi_i^1)s^i}{\sigma} + \frac{(\xi_i^1+\xi_i^2-\xi_i^3)(r^i-s^i)}{(\tau-\sigma)}\right)\,, \end{aligned} \tag{4.2.39}$$

where $C^{(3)}$ is an arbitrary constant.

So we see that like in the case of relativistic CFTs, the three point function is fixed upto a constant.

### 4.2.3 Comparing Correlation Functions

We want to understand the expressions for the two and three point functions better. To that end, we wish to recapitulate the two and three point functions in the usual relativistic CFTs and the Schrödinger algebra.

As is very well known, in the relativistic CFTs in any dimension, the conformal symmetry is enough to fix the two and three point functions upto constants. Framed in the language of 2D CFTs, one needs only the finite $SL(2,R)$ sub-group of the full Virasoro algebra to fix the 2 and 3 point function. Let us list them:
Two point function:

$$G^{(2)}_{CFT}(z_i, \overline{z_i}) = \delta_{h_1,h_2}\delta_{\overline{h}_1,\overline{h}_2}\frac{C_{12}}{z_{12}^{2h}\overline{z}_{12}^{2\overline{h}}} \tag{4.2.40}$$

Three point function:

$$\begin{aligned} G^{(3)}_{CFT}(z_i, \overline{z_i}) &= C_{(123)} z_{12}^{-(h_1+h_2-h_3)} z_{23}^{-(h_2+h_3-h_1)} z_{13}^{-(h_3+h_1-h_2)} \\ &\quad \times\text{non-holomorphic} \end{aligned} \tag{4.2.41}$$

One can find the two-point and three point functions for the quasi-primary operators in the Schrödinger Algebra as well [23]. We list the answers here:

Two point function:

$$G^{(2)}_{SA} = C_{(12)}\delta_{\Delta_1,\Delta_2}\delta_{m_1,m_2}(t_1-t_2)^{\Delta_1}\exp\left(\frac{m_1(x^i_1-x^i_2)^2}{2(t_1-t_2)}\right) \tag{4.2.42}$$

Three point function:

$$\begin{aligned} G^{(3)}_{SA} &= C_{(123)}\delta_{m_1+m_2,m_3}(t_1-t_3)^{\frac{-(\Delta_1+\Delta_3-\Delta_2)}{2}}(t_2-t_3)^{\frac{-(\Delta_2+\Delta_3-\Delta_1)}{2}} \\ &\quad (t_1-t_2)^{\frac{-(\Delta_1+\Delta_2-\Delta_3)}{2}}\exp\left(\frac{m_1(x^i_1-x^i_3)^2}{2(t_1-t_3)}+\frac{m_2(x^i_2-x^i_3)^2}{(t_2-t_3)}\right) \\ &\times\ \Psi\left(\frac{[(x^i_1-x^i_3)(t_2-t_3)-(x^i_2-x^i_3)(t_1-t_3)]^2}{(t_1-t_2)(t_2-t_3)(t_1-t_3)}\right) \end{aligned} \tag{4.2.43}$$

Some remarks are in order. We first note that all the two-point functions vanish if the quasi-primaries are of different weight. The non-relativistic ones are further constrained by the equality of the rapidity and mass eigenvalues for the GCA and SA respectively.
The exponential part is an inherently non-relativistic effect. This form comes from the generator of Galilean boosts.
The GCA and SA three point functions *crucially differ*. As we have seen above, the GCA one is completely fixed upto a constant factor, like in usual CFT. But for the Schrödinger symmetry, the three point function is fixed only upto an arbitrary function of a particular combination of variables. This is a result of the differing number of generators. The finite sub-algebra of the GCA has two more generators and this constrains the form of the three-point function further.
The Schrödinger algebra correlators also have an additional superselection rule of the mass. But there is no analogue of such a selection rule for the GCA. This is because the boosts, unlike the mass, acts non-trivially on the co-ordinates. The action of the boosts in this case is more like the dilation operator which also has a non-trivial action on the space-time.[2]

[2]We would like to thank Shiraz Minwalla for pointing this out and Ashoke Sen for valuable discussions on this point.

# Chapter 5

# GCA in 2d: I. Algebra and Representation

The most of the analysis of the Galilean Conformal Algebra in the previous chapters [27] was classical and one would like to understand the quantum mechanical realisation of these symmetries better. This is specially true if one would like to exploit this symmetry in the context of the nonrelativistic limit of AdS/CFT as proposed in [27]. A first step was taken in [28] (see also [77, 79]), as described in Chap. 4 where two and three point correlation functions (of primary fields) were obtained as solutions of the Ward identities for the finite part of the GCA (which arises as the contraction of $SO(d,2)$).[1] These were found to be completely determined in form just as in the relativistic case. One would therefore imagine that the infinite dimensional GCA should give much stronger constraints when realised on the quantum dynamics just as in the case of relativistic $2d$ conformal invariance.

With this in mind and given the centrality of $2d$ CFTs it is natural to specialise to two spacetime dimensions and study realisations of the infinite dimensional GCA in this case. In this process we might hope to extract dynamical information from the GCA comparable to that obtained from the Virasoro algebra. The main aim of this chapter and the two following ones is to show that this hope can be largely realised. We find, in fact, that there is a very tight relation between the GCA symmetry in 2d and the relativistic Virasoro symmetry. Classically, we see that the infinite dimensional GCA in 2d arises as a contraction of the generators of the usual holomorphic and antiholomorphic transformations. Quantum mechanically, in a 2d CFT, the latter get promoted to left and right moving Virasoro generators each with its central extension. We will find evidence that taking a scaling limit on a suitable family of 2d CFTs gives consistent quantum mechanical realisations of the GCA (now centrally extended), in what might be termed

[1] The expression for the two point function is actually contained in [78] as part of more general expressions derived for arbitrary dynamical exponents.

as 2d Galilean conformal field theories (or 2d GCFTs, for short).

The families of 2d CFTs we will need to consider are rather unusual in that their left and right central charges $c$ and $\bar{c}$ are scaled (as we take the nonrelativistic limit) such that their magnitudes go to infinity but are opposite in sign. The parent theories are thus necessarily non-unitary and, not unsurprisingly, this non-unitariness is inherited by the daughter GCFTs. Since non-unitary 2d CFTs arise in a number of contexts in statistical mechanics as well as string theory, one might expect that the 2d GCFTs realised here would also be interesting objects to study.

Our study of 2d GCFTs proceeds along two parallel lines. The first line of development is as described above and consists of taking carefully the non-relativistic scaling limit of the parent 2d CFT. We find that this limit, while unusual, appears to give sensible answers. Specifically, we will study in this way, the representation theory (including null vectors determined by a nonrelativistic Kac table), the Ward identities, fusion rules and finally the equations for the four point function following from the existence of level two null states. In all these cases we find that a non-trivial scaling limit of the 2d CFT's exists. This is not, *a priori*, obvious since, as we will see, the limit involves keeping terms both of $\mathcal{O}(\frac{1}{\epsilon})$ as well as of $\mathcal{O}(1)$(where $\epsilon$ is the scaling parameter which is taken to zero).

The second line of development obtains many of these same results by carrying out an autonomous analysis of the GCA i.e. independent of the above limiting procedure. In some cases we will see that the constraints from the GCA are slightly weaker than those arising from the scaling limit of parent relativistic CFTs. This is understandable since the limit of 2d CFTs is presumably only a particular way to realise GCFT's and there is no reason to expect it to be the only way to realise such GCFTs. Thus our dual considerations help to distinguish between some of the general results for any 2d GCFT from that obtained by taking the nonrelativistic limit. It is also an important consistency check of our investigation that these two strands of development agree whenever they do and that it is always the GCA that gives weaker constraints[2].

In this chapter we discuss how, at the classical level, the generators of the $2d$ GCA arise from a group contraction of (combinations of) the usual holomorphic and anti-holomorphic vector fields. We also discuss the scaling limit one should take in the quantum theory thus relating the central charges of the GCA to the Virasoro central charges $c$ and $\bar{c}$. Then we proceed to construct representations of the $2d$ GCA in a manner analogous to the Virasoro representation theory, defining primaries and descendants. The primaries are labelled by a conformal weight $\Delta$ and a boost eigenvalue $\xi$. We also show that the state space is generically non-unitary. The following section deals with the non-relativistic Ward identities focussing on the case of two and three point functions. We show how the answers there may also be obtained from the nonrelativistic scaling limit of the Virasoro

[2]Our analysis in this and the following chapters would only be upto a certain order (e.g. level two null states). There might be instances at higher order where pathologies occur. See a related discussion about null states for supersymmetric GCA in pg. 104.

algebra discussed earlier..

We go back to the representation theory in the next chapter to consider null vectors of the GCA. We explicitly find the conditions for having null states at level two and check that the resulting conditions are precisely those obtained from the scaling limit of the Virasoro algebra. We go on to take the scaling limit of the Kac table for null states at arbitrary level and find a sensible nonrelativistic Kac table. After that we focus on GCA primaries taking values in the nonrelativistic Kac table. We first derive the general differential equations for an $n$-point correlator which follow from the existence of level two null states and check that it is consistent with the form of the two point function of [28]. We proceed to derive the GCA fusion rules which follow from the GCA three point function using the differential equations of the previous section. These turn out to be only slightly weaker than what one would obtain from the scaling of the corresponding Virasoro fusion rules.

Finally, in Chap. 7 we consider the four point function which satisfies second order differential equations when one of the primaries has a null descendant at level two. We can find the solution in an explicit form both from this equation as well as from considering the limit of the solution of the corresponding Virasoro four point functions. The solution obeys various nontrivial conditions and seems to be consistent with the factorization into the three point function and the fusion rules. There is a final section with remarks of a general nature. An appendix contains various technical details relevant to the four point function in the chapter.

## 5.1 2d GCA from Group Contraction

We have seen in the previous chapters the details of the Galilean Conformal Algebra. Let us briefly remind ourselves of the basic features again. The maximal set of conformal isometries of Galilean spacetime generates the infinite dimensional GCA [27]. The notion of Galilean spacetime is a little subtle since the spacetime metric degenerates into a spatial part and a temporal piece. Nevertheless there is a definite limiting sense (of the relativistic spacetime) in which one can define the conformal isometries (see [76]) of the nonrelativistic geometry. Algebraically, the set of vector fields generating these symmetries are given by

$$\begin{aligned} L^{(n)} &= -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t \,, \\ M_i^{(n)} &= t^{n+1}\partial_i \,, \\ J_a^{(n)} \equiv J_{ij}^{(n)} &= -t^n(x_i\partial_j - x_j\partial_i)\,, \end{aligned} \tag{5.1.1}$$

for integer values of $n$. Here $i = 1\ldots(d-1)$ range over the spatial directions. These vector fields obey the algebra

$$[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)}, \qquad [L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)},$$

$$[J_a^{(n)}, J_b^{(m)}] = f_{abc}J_c^{(n+m)}, \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \tag{5.1.2}$$

We expect that there would be a central extension to the Virasoro and Current algebras in the quantum theory. In fact, we will see that in two dimensions the GCA with central charges can be realised by taking a special limit of a relativistic 2d CFT (albeit non-unitary).

### 5.1.1 GCA from Virasoro in 2d

As we have seen earlier, there is a finite dimensional subalgebra of the GCA (also sometimes referred to as the GCA) which consists of taking $n = 0, \pm 1$ for the $L^{(n)}, M_i^{(n)}$ together with $J_a^{(0)}$. This algebra is obtained by considering the nonrelativistic contraction of the usual (finite dimensional) global conformal algebra $SO(d,2)$ (in $d > 2$ spacetime dimensions).

However, in two spacetime dimensions, as is well known, the situation is special. The relativistic conformal algebra is infinite dimensional and consists of two copies of the Virasoro algebra. One expects this to be also related, now to the infinite dimensional GCA algebra. Indeed in two dimensions the non-trivial generators in (3.2.11) are the $L_n$ and the $M_n$ (where we have dropped the spatial index from the latter since there is only one spatial direction and instead restored the mode number $n$ to the conventional subscript) :

$$\begin{aligned} L_n &= -(n+1)t^n x\partial_x - t^{n+1}\partial_t\,, \\ M_n &= t^{n+1}\partial_x\,, \end{aligned} \tag{5.1.3}$$

which obey

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}\,, \qquad [M_m, M_n] = 0\,, \\ [L_m, M_n] &= (m-n)M_{m+n}\,. \end{aligned} \tag{5.1.4}$$

We will now show that the generators in (5.1.3) arise precisely from a nonrelativistic contraction of the two copies of the Virasoro algebra of the relativistic theory.[3] The non-relativistic contraction consists of taking the scaling

$$t \to t\,, \qquad x \to \epsilon x\,, \tag{5.1.5}$$

with $\epsilon \to 0$. This is equivalent to taking the velocities $v \sim \epsilon$ to zero (in units where $c = 1$).

Consider the vector fields which generate (two copies of) the centre-less Virasoro Algebra (or Witt algebra as it is often called) in two dimensions :

$$\mathcal{L}_n = -z^{n+1}\partial_z\,, \quad \bar{\mathcal{L}}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\,. \tag{5.1.6}$$

[3] This observation has also been independently made in [84]. As mentioned in footnote 1, a slightly different contraction of the Virasoro algebra was also made in [73] to obtain the same result.

In terms of space and time coordinates, $z = t + x$, $\bar{z} = t - x$. Hence $\partial_z = \frac{1}{2}(\partial_t + \partial_x)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_t - \partial_x)$. Expressing $\mathcal{L}_n, \bar{\mathcal{L}}_n$ in terms of $t, x$ and taking the above scaling (2.3.34) reveals that in the limit the combinations

$$\begin{aligned} \mathcal{L}_n + \bar{\mathcal{L}}_n &= -t^{n+1}\partial_t - (n+1)t^n x\partial_x + \mathcal{O}(\epsilon^2)\,, \\ \mathcal{L}_n - \bar{\mathcal{L}}_n &= -\frac{1}{\epsilon}t^{n+1}\partial_x + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{5.1.7}$$

Therefore we see that as $\epsilon \to 0$

$$\mathcal{L}_n + \bar{\mathcal{L}}_n \longrightarrow L_n\,, \quad \epsilon(\mathcal{L}_n - \bar{\mathcal{L}}_n) \longrightarrow -M_n\,. \tag{5.1.8}$$

Thus the GCA in 2d arises as the non-relativistic limit of the relativistic algebra. This was at the classical level of vector fields. At the quantum level the two copies of the Virasoro get respective central extensions

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \\ [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] &= (m-n)\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}\,. \end{aligned} \tag{5.1.9}$$

Considering the linear combinations (5.1.7) which give rise to the GCA generators as in (5.1.8), we find

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + C_1 m(m^2-1)\delta_{m+n,0}\,, \\ [L_m, M_n] &= (m-n)M_{m+n} + C_2 m(m^2-1)\delta_{m+n,0}\,, \\ [M_m, M_n] &= 0\,. \end{aligned} \tag{5.1.10}$$

This is the centrally extended GCA in 2d.[4] Note that the relation between central charges is

$$C_1 = \frac{c+\bar{c}}{12}\,, \qquad \frac{C_2}{\epsilon} = \frac{\bar{c}-c}{12}\,. \tag{5.1.11}$$

Thus, for a non-zero $C_2$ in the limit $\epsilon \to 0$ we see that we need $\bar{c} - c \propto \mathcal{O}(\frac{1}{\epsilon})$. At the same time requiring $C_1$ to be finite we find that $c + \bar{c}$ should be $\mathcal{O}(1)$. (Motivated by the second equation in (5.1.7), we will make the slightly stronger assumption that $\bar{c} - c = \mathcal{O}(1/\epsilon) + \mathcal{O}(\epsilon)$.) Thus (5.1.11) can hold only if $c$ and $\bar{c}$ are large (in the limit $\epsilon \to 0$) and opposite in sign. This immediately implies that the original 2d CFT on which we take the non-relativistic limit cannot be unitary. This is, of course, not a problem since there are many statistical mechanical models which are described at a fixed point by non-unitary CFTs.[5]

[4] One can check that a central extension in the commutator $[M_m, M_n]$ of the form $C_3(m)\delta_{m+n,0}$ is not allowed by the Jacobi identity.

[5] We note that modular invariance of a 2d CFT implies that $\bar{c} - c \equiv 0(\text{mod}24)$. We will implicitly assume that this is true for the CFT's we are considering. In other words, we will take $\frac{C_2}{\epsilon}$ to be an even integer.

## 5.2 Representations of the 2d GCA

With these requirements in mind we now turn to the representations of the 2d GCA. We will be guided in this by the representation theory of the Virasoro algebra.

### 5.2.1 Primary States and Descendants

We have seen the construction of representations in terms of operators in previous chapters. Now, let us start from a slightly different point of view and connect with the previous construction by a correspondence, at the cost of being slightly repeatative. We will construct the representations by considering the states having definite scaling dimensions :

$$L_0|\Delta\rangle = \Delta|\Delta\rangle\,. \tag{5.2.12}$$

Using the commutation relations (5.1.10), we obtain

$$L_0 L_n|\Delta\rangle = (\Delta - n)L_n|\Delta\rangle, \quad L_0 M_n|\Delta\rangle = (\Delta - n)M_n|\Delta\rangle. \tag{5.2.13}$$

Then the $L_n, M_n$ with $n > 0$ lower the value of the scaling dimension, while those with $n < 0$ raise it. If we demand that the dimension of the states be bounded from below then we are led to defining primary states in the theory having the following properties :

$$L_n|\Delta\rangle_p = 0\,, \quad M_n|\Delta\rangle_p = 0\,, \tag{5.2.14}$$

for all $n > 0$. Since the conditions (5.2.14) are compatible with $M_0$ in the sense

$$L_n M_0|\Delta\rangle_p = 0\,, \quad M_n M_0|\Delta\rangle_p = 0\,, \tag{5.2.15}$$

and also since $L_0$ and $M_0$ commute, we may introduce an additional label, which we will call "rapidity" $\xi$:

$$M_0|\Delta, \xi\rangle_p = \xi|\Delta, \xi\rangle_p\,. \tag{5.2.16}$$

Starting with a primary state $|\Delta, \xi\rangle_p$, one can build up a tower of operators by the action of $L_{-n}$ and $M_{-n}$ with $n > 0$. These will be called the GCA descendants of the primary. The primary state together with its GCA descendants form a representation of GCA. As in the Virasoro case, we have to be careful about the presence of null states. We will look at these in some detail later in Sec.6.1. An interesting property of the representation constructed above is that a generic secondary state will not be an eigenstate of $M_0$. Examples of the secondary states, which are eigenstates of $M_0$ are the ones which are constructed only with $M_{-n}$'s acting on the primary state. In the rest of the chapter and the following two, we omit the subscript "$p$" for the primary state.

The above construction is quite analogous to that of the relativistic 2d CFT. In fact, from the viewpoint of the limit (5.1.8) we see that the two labels $\Delta$ and $\xi$ are related to the conformal weights in the 2d CFT as

$$\Delta = \lim_{\epsilon\to 0}(h+\bar{h})\,, \qquad \xi = \lim_{\epsilon\to 0}\epsilon(\bar{h}-h)\,, \tag{5.2.17}$$

where $h$ and $\bar{h}$ are the eigenvalues of $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$, respectively. We will proceed to assume that such a scaling limit (as $\epsilon \to 0$) of the 2d CFT exists. In particular, we will assume that the operator state correspondence in the 2d CFT gives a similar correspondence between the states and the operators in the GCA :

$$\mathcal{O}(t,x) \leftrightarrow \mathcal{O}(0)|0\rangle\,, \tag{5.2.18}$$

where $|0\rangle$ would be the vacuum state which is invariant under the generators $L_0, L_{\pm 1}$, $M_0, M_{\pm 1}$. Indeed in rest of this and the couple of following chapters, we will offer several pieces of evidence that the scaling limit gives a consistent quantum mechanical system.

### 5.2.2 Unitarity

However already at level one of the GCA (using the usual terminology of CFT) we can see that the representation is generically non-unitary. For this we have to define hermiticity and an inner product. We will assume that this is what would be naturally inherited from the parent relativistic CFT. In other words, we take $L_n^\dagger = L_{-n}$ and $M_n^\dagger = M_{-n}$ for all $n$.

Let us now consider the tower above a primary state $|\Delta,\xi\rangle$. The states at level one are $L_{-1}|\Delta,\xi\rangle$ and $M_{-1}|\Delta,\xi\rangle$. The (hermitian) matrix of inner products of these states can be evaluated and the determinant is easily seen to be given by $-\xi^2$.

$$\begin{aligned}\mathcal{M} &= \begin{pmatrix}\langle\Delta,\xi|L_1L_{-1}|\Delta,\xi\rangle & \langle\Delta,\xi|M_1L_{-1}|\Delta,\xi\rangle\\ \langle\Delta,\xi|L_1M_{-1}|\Delta,\xi\rangle & \langle\Delta,\xi|M_1M_{-1}|\Delta,\xi\rangle\end{pmatrix} = \begin{pmatrix}2\Delta & \xi\\ \xi & 0\end{pmatrix}\\ &\Rightarrow det\mathcal{M} = -\xi^2\end{aligned} \tag{5.2.19}$$

Thus, for non-zero $\xi$, at least one of the eigenvalues must be negative. In such a case the Hilbert space clearly has negative norm states. Thus only for $\xi = 0$ might one have the possibility of a unitary representation though there are clearly null states even in this case. This is not too surprising given that the parent CFT is generically non-unitary as we argued above.

## 5.3 Non-Relativistic Ward Identities

In Chap. 4 [28], we saw the GCA transformation laws for primary operators were written down as:

$$\delta_{L_n}\mathcal{O}_p(t,x) \;=\; [L_n, \mathcal{O}_p(t,x)]$$

$$
\begin{aligned}
&= [t^{n+1}\partial_t + (n+1)t^n x\partial_x + (n+1)(\Delta t^n - n\xi t^{n-1}x)]\mathcal{O}(t,x)\,, \\
\delta_{M_n}\mathcal{O}_p(t,x) &= [M_n, \mathcal{O}_p(t,x)] = [-t^{n+1}\partial_x + (n+1)\xi t^n]\mathcal{O}(t,x)\,. \qquad (5.3.20)
\end{aligned}
$$

These transformation laws [28], in the special case of the global transformations (i.e. for $L_{0,\pm1}, M_{0,\pm1}$), were used to derive Ward Identities (see also [77, 79]) for two and three point functions. The functional dependence of these correlators was then completely fixed by the resulting equations (see below for the explicit expressions). The analysis in [28] is based on the argument directly coming from GCA. We will now see that all these statements can be also derived from the non-relativistic limit of the relativistic 2d CFT.

### 5.3.1 Transformation Laws

Consider first how the transformation laws (5.3.20) of GCA primaries arise from the transformation laws of primary operators in 2d CFT, which are given by

$$
\tilde{\mathcal{O}}(\tilde{z}, \tilde{\bar{z}}) = \mathcal{O}(z, \bar{z})(\frac{dz}{d\tilde{z}})^h (\frac{d\bar{z}}{d\tilde{\bar{z}}})^{\bar{h}}\,, \qquad (5.3.21)
$$

or infinitesimally under $z \to \tilde{z} = z + a(z)$ as

$$
\delta\mathcal{O}(z, \bar{z}) = -(h\partial_z a(z) + a(z)\partial_z)\mathcal{O}(z, \bar{z})\,, \qquad (5.3.22)
$$

and similarly for $\bar{z} \to \tilde{\bar{z}} = \bar{z} + \bar{a}(\bar{z})$ . We take $a(z) = -z^{n+1}$ for $\delta_{\mathcal{L}_n}$ and $\bar{a}(\bar{z}) = -\bar{z}^{n+1}$ for $\delta_{\bar{\mathcal{L}}_n}$,

Motivated from the relation (5.1.8), we may define the infinitesimal transformation $\delta_{L_n}$ and $\delta_{M_n}$ as

$$
\delta_{L_n}\mathcal{O} = \lim_{\epsilon\to 0}(\delta_{\mathcal{L}_n} + \delta_{\bar{\mathcal{L}}_n})\mathcal{O}\,, \qquad \delta_{M_n}\mathcal{O} = -\lim_{\epsilon\to 0}\epsilon(\delta_{\mathcal{L}_n} - \delta_{\bar{\mathcal{L}}_n})\mathcal{O}\,. \qquad (5.3.23)
$$

Then by taking the limits on the right hand sides (RHS's), we obtain

$$
\begin{aligned}
\delta_{L_n}\mathcal{O} &= \lim_{\epsilon\to 0}[(n+1)h(t+\epsilon x)^n + \frac{1}{2}(t+\epsilon x)^{n+1}(\partial_t + \frac{1}{\epsilon}\partial_x) \\
&\quad +(n+1)\bar{h}(t-\epsilon x)^n + \frac{1}{2}(t-\epsilon x)^{n+1}(\partial_t - \frac{1}{\epsilon}\partial_x)]\mathcal{O} \\
&= [t^{n+1}\partial_t + (n+1)t^n x\partial_x + (n+1)(\Delta t^n - n\xi t^{n-1}x)]\mathcal{O}\,, \qquad (5.3.24)
\end{aligned}
$$

and similarly,

$$
\delta_{M_n}\mathcal{O} = [-t^{n+1}\partial_x + (n+1)t^n\xi]\mathcal{O}\,. \qquad (5.3.25)
$$

where recall that $\Delta$ and $\xi$ are defined as (5.2.17). These are exactly those given in (5.3.20).

### 5.3.2 Two and Three Point Functions

We have already looked at the two and three point functions in detail in Chap. 4 (Sec. 4.2). Let us just review the construction in the light of two dimensions. The constraints from the Ward identities for the global transformations $L_{0,\pm1}, M_{0,\pm1}$ apply to primary GCA operators.

Therefore consider the two point function of primary operators $\mathcal{O}_1(t_1, x_1)$ and $\mathcal{O}_2(t_2, x_2)$ of conformal and rapidity weights $(\Delta_1, \xi_1)$ and $(\Delta_2, \xi_2)$ respectively.

$$G^{(2)}_{\rm GCA}(t_1, x_1, t_2, x_2) = \langle \mathcal{O}_1(t_1, x_1)\mathcal{O}_2(t_2, x_2)\rangle \,. \tag{5.3.26}$$

The correlation functions only depend on differences of the coordinates $t_{12} = t_1 - t_2$ and $x_{12} = x_1 - x_2$ because of the translation symmetries $L_{-1}$ and $M_{-1}$. The remaining symmetries give four more differential equations which constrain the answer to be [28]

$$G^{(2)}_{\rm GCA}(\{t_i, x_i\}) = C_{12}\delta_{\Delta_1,\Delta_2}\delta_{\xi_1,\xi_2} t_{12}^{-2\Delta_1} \exp\left(\frac{2\xi_1 x_{12}}{t_{12}}\right). \tag{5.3.27}$$

Here $C_{12}$ is an arbitrary constant, which we can always take to be one by choosing the normalization of the operators.

Similarly, the three point function of primary operators is given by

$$\begin{aligned} G^{(3)}_{\rm GCA}(\{t_i, x_i\}) &= C_{123} t_{12}^{-(\Delta_1+\Delta_2-\Delta_3)} t_{23}^{-(\Delta_2+\Delta_3-\Delta_1)} t_{13}^{-(\Delta_1+\Delta_3-\Delta_2)} \\ &\times \exp\left(\frac{(\xi_1+\xi_2-\xi_3)x_{12}}{t_{12}} + \frac{(\xi_2+\xi_3-\xi_1)x_{23}}{t_{23}} + \frac{(\xi_1+\xi_3-\xi_2)x_{13}}{t_{13}}\right), \end{aligned} \tag{5.3.28}$$

where $C_{123}$ is an arbitrary constant. So we see that like in the case of relativistic CFTs, the three point function is fixed upto a constant.[6]

### 5.3.3 GCA Correlation Functions from 2d CFT

We now show that these expressions for the GCA two and three point functions can also be obtained by taking an appropriate scaling limit of the usual 2d CFT answers. This limit requires scaling the quantum numbers of the operators as (5.2.17), along with the non-relativistic limit for the coordinates (2.3.34).

Let us first study the scaling limit of the two point correlator.[7]

$$\begin{aligned} G^{(2)}_{\rm 2d\,CFT} &= \delta_{h_1,h_2}\delta_{\bar h_1,\bar h_2} z_{12}^{-2h_1} \bar z_{12}^{-2\bar h_1} \\ &= \delta_{h_1,h_2}\delta_{\bar h_1,\bar h_2}\; t_{12}^{-2h_1}\left(1+\epsilon\frac{x_{12}}{t_{12}}\right)^{-2h_1} t_{12}^{-2\bar h_1}\left(1-\epsilon\frac{x_{12}}{t_{12}}\right)^{-2\bar h_1} \\ &= \delta_{h_1,h_2}\delta_{\bar h_1,\bar h_2}\; t_{12}^{-2(h_1+\bar h_1)} \exp\Big(-2(h_1-\bar h_1)\big(\epsilon\frac{x_{12}}{t_{12}} + \mathcal{O}(\epsilon^2)\big)\Big). \end{aligned} \tag{5.3.29}$$

[6]See, however, the explanation at the end of 5.3.3.

[7]This was obtained in discussion with S. Minwalla.

Now by taking the scaling limit as (5.2.17), we obtain the GCA two point function

$$\lim_{\epsilon\to 0} G^{(2)}_{\rm 2d\,CFT} = \delta_{\Delta_1,\Delta_2}\delta_{\xi_1,\xi_2}\, t_{12}^{-2\Delta_1}\exp\left(\frac{2\xi_1 x_{12}}{t_{12}}\right) = G^{(2)}_{\rm GCA}\,. \tag{5.3.30}$$

A similar analysis yields the three point function of the GCA from the relativistic three point function. The relativistic three point function is written as

$$G^{(3)}_{\rm 2d\,CFT} = C_{123} z_{12}^{-(h_1+h_2-h_3)} z_{23}^{-(h_2+h_3-h_1)} z_{13}^{-(h_1+h_3-h_2)} \times \text{ (anti-holomorphic)} \tag{5.3.31}$$

$$= C_{123} t_{12}^{-(h_1+h_2-h_3)} t_{12}^{-(\bar h_1+\bar h_2-\bar h_3)} \mathrm{e}^{-(h_1+h_2-h_3)(\epsilon\frac{x_{12}}{t_{12}}+\mathcal{O}(\epsilon^2))} \mathrm{e}^{(\bar h_1+\bar h_2-\bar h_3)(\epsilon\frac{x_{12}}{t_{12}}+\mathcal{O}(\epsilon^2))} \times \text{ (product of two permutations)}\,. \tag{5.3.32}$$

Then taking the non-relativistic limit, we obtain the GCA three point function (5.3.28). Note that the constant factor in (5.3.28) would be given by taking the limit of the constant in (5.3.31).

We should also mention here the issue of singlevaluedness of the correlation functions. The holomorphic part, or the anti-holomorphic part, of the correlation functions in the 2d CFT have branch cuts for generic values of the conformal dimensions $h_i$ and $\bar h_i$. Then the requirement for the singlevaluedness of the two point functions gives rise to a condition that the difference $h_i - \bar h_i$ of every operator must be an integer or a half integer. This is the usual spin statistics theorem.

We see that this theorem need not hold in a generic GCFT since there is no such singlevaluedness requirement. However, GCFT's arising as limits of 2d CFTs would inherit this relation.

A word regarding the overall phase of correlation functions: If we pay attention to the overall phase in the derivation (5.3.29), it would be like ($x = 0$ for simplicity) $z^{-2h}\bar z^{-2\bar h} \to (\pm 1)^{2(h-\bar h)}|t|^{-2(h+\bar h)}$. Here the plus (the minus) sign is for $t > 0$ ($t < 0$), and the exponent $2(h - \bar h)$ is the twice of the spin as mentioned in the main text. Note that in the derivation of (5.3.27), the differential equation is solved separately for the two segments $t < 0$ and $t > 0$. So far we have no argument, from the GCA side, to fix the relative coefficient of the solutions for these segments. However, when we take the non-relativistic limit from 2d CFT, this ambiguity is fixed as we saw above. This relative phase plays an important role in the discussion of the four point function in Chap. 7.

# Chapter 6

# GCA in 2d: II. Null Vectors and Fusion Rules

In this chapter, we investigate first the conditions under which the GCA representations contain null vectors. The explicit null vectors for level two are constructed in the intrisic GCA and reproduced by the limit way as we have seen in the previous chapter. We go on to write down the differential equations for the GCA correlators for the null states and then have a detailed look at the fusion rules of the GCA.

## 6.1 GCA Null Vectors

Just as in the representation of the Virasoro algebra, we will find that there are null states in the GCA tower built on a primary $|\Delta, \xi\rangle$ for special values of $(\Delta, \xi)$. These are states which are orthogonal to all states in the tower including itself. We can find the null states at a given level by writing the most general state at that level as a combination of the $L_{-n}, M_{-n}$'s $(n > 0)$ acting on the GCA primary and then imposing the condition that all the positive modes $L_n, M_n$ (with $n > 0$) annihilate this state. Actually one needs to only impose this condition for $n = 1$ and $n = 2$ since the others are given as the commutators of these modes. This will give conditions that fix the relative coefficients in the linear combination as well as give a relation between $\Delta, \xi$ and the central charges $C_1, C_2$.

Thus at level one we have only the states $L_{-1}|\Delta, \xi\rangle$ and $M_{-1}|\Delta, \xi\rangle$. It is easy to check that one has a null state only if either $\Delta$ (and) or $\xi$ is zero. At level two things are a little more non-trivial. Let us consider the most general level two state of the form

$$|\chi\rangle = (a_1 L_{-2} + a_2 L_{-1}^2 + b_1 L_{-1} M_{-1} + d_1 M_{-1}^2 + d_2 M_{-2})|\Delta, \xi\rangle. \tag{6.1.1}$$

We now impose the condition that $L_{1,2}, M_{1,2}$ annihilate this state.

A little algebra using (5.1.10) gives us the conditions :

$$3a_1 + 2(2\Delta + 1)a_2 + 2\xi b_1 = 0; \quad (4\Delta + 6C_1)a_1 + 6\Delta a_2 + 6\xi b_1 + (6C_2 + 4\xi)d_2 = 0;$$
$$2(\Delta + 1)b_1 + 4\xi d_1 + 3d_2 = 0; \quad (4\xi + 6C_2)a_1 = 0;$$
$$3a_1 + 2a_2 + 2\xi b_1 = 0; \quad \xi a_2 = 0. \tag{6.1.2}$$

We will now separately consider the two cases where $C_2 \neq 0$ and $C_2 = 0$.

### 6.1.1 The Case of $C_2 \neq 0$

Here we will first consider the case where $\xi \neq 0$. In this case we have two further options. Either $a_1 = 0$ or $a_1 \neq 0$. In the former case, $b_1 = 0$ as well. We then find that for a nontrivial solution, $\xi = -\frac{3C_2}{2}$ and $d_1 = -\frac{3}{4\xi}d_2$. Thus there is a null state of the form

$$|\chi^{(1)}\rangle = (M_{-2} - \frac{3}{4\xi}M_{-1}^2)|\Delta, \xi\rangle. \tag{6.1.3}$$

We can also consider solutions for which $a_1 \neq 0$. In this case $b_1 = -\frac{3}{2\xi}a_1$. We have once again $\xi = -\frac{3C_2}{2}$. But then we also have to satisfy the consistency condition $\Delta = \frac{(9-6C_1)}{4}$. For $\Delta \neq -1$, we also must have at least one of $d_1, d_2$ nonzero for such a solution. By taking a suitable linear combination with $|\chi^{(1)}\rangle$ we can choose $d_2 = 0$ and then we get another null state of the form

$$|\chi^{(2)}\rangle = (L_{-2} - \frac{3}{2\xi}L_{-1}M_{-1} + \frac{3(\Delta+1)}{4\xi^2}M_{-1}^2)|\Delta, \xi\rangle. \tag{6.1.4}$$

We can also have the case where $\xi = 0$. Then we must have $a_1 = a_2 = d_2 = 0$ and $d_1$ is again undetermined corresponding again to $|\chi^{(1)}\rangle = M_{-1}^2|\Delta, 0\rangle$. However, it is only for $\Delta = -1$ that one gets a second null state. Here $b_1$ is undetermined and corresponds to $|\chi^{(2)}\rangle = L_{-1}M_{-1}|\Delta = -1, 0\rangle$. Note, however, that both these states are descendants of a level *one* null state $M_{-1}|\Delta, 0\rangle$.

In any case, for both $\xi = 0$ as well as $\xi \neq 0$, the null states obey

$$M_0|\chi^{(2)}\rangle = \xi|\chi^{(2)}\rangle + \alpha|\chi^{(1)}\rangle, \qquad M_0|\chi^{(1)}\rangle = \xi|\chi^{(1)}\rangle, \tag{6.1.5}$$

where $\alpha$ is either of $1, 2$. That is, they are eigenstates of $M_0$ (upto null states). More generally, this is a reflection of the fact that the operator $M_0$ is not diagonal on the states at a given level. In fact, it can be easily seen to take an upper triangular form in terms of a suitable ordering of the basis elements at a given level. This is reminiscent of the behaviour in logarithmic CFTs.

At a general level $K$ one will find in parallel with $|\chi^{(1)}\rangle$ a null state formed only from the $M_{-n}$'s taking the form

$$|\chi\rangle = (M_{-K} + \eta_1 M_{-K+1}M_{-1} + \ldots)\,|\Delta, \xi\rangle. \tag{6.1.6}$$

The action of $L_K$ gives the constraint as

$$L_K|\chi\rangle = ([L_K, M_{-K}] + \ldots)\,|\Delta,\xi\rangle \quad \rightarrow \quad 2K\xi + C_2K(K^2-1) = 0. \tag{6.1.7}$$

Here we have used the fact that all other commutators generate $M_m$ with $m > 0$ and these go through and annihilate the state $|\Delta,\xi\rangle$. Thus there are null states at level $K$ if $\xi$ obeys the above relation with respect to $C_2$.

### 6.1.2 The Case of $C_2 = 0$

In this case, it is easy to see that we have a non-zero null state only if $\xi = 0$. Then from (6.1.2) we find that for $(\Delta \neq -\frac{3}{2}C_1)$ that both $a_2$ and $a_1 = 0$ and we have a relation $d_2 = -\frac{2(\Delta+1)}{3}b_1$ and $d_1$ is undetermined. Therefore the two independent null states are now

$$|\chi^{(1)}\rangle = M_{-1}^2|\Delta,0\rangle, \tag{6.1.8}$$

(which is just a descendant of the level one null state) and

$$|\chi^{(2)}\rangle = (L_{-1}M_{-1} - \frac{2(\Delta+1)}{3}M_{-2})|\Delta,0\rangle. \tag{6.1.9}$$

Note that there is no constraint on the values of $C_1, \Delta$ apart from the fact that $(\Delta \neq -\frac{3}{2}C_1)$. In fact, in the case of $(\Delta = -\frac{3}{2}C_1)$ we find that we can have $a_1, a_2 \neq 0$ only if $\Delta = C_1 = 0$, which is a trivial case. So we continue to take $a_1 = a_2 = 0$ even when $(\Delta = -\frac{3}{2}C_1)$ and we have the two null states $|\chi^{(1,2)}\rangle$ for all values of $\Delta, C_1$.

We can actually say more about these states. Note that the states $M_{-1}^2|\Delta,0\rangle$ and $L_{-1}M_{-1}|\Delta,0\rangle$ are descendants of the level one null state $M_{-1}|\Delta,\xi\rangle$. Thus if we consistently set the null state at level one to zero together with its descendants[1], then the new null state at level two is given by $M_{-2}|\Delta,0\rangle$. Continuing this way it is easy to see that we have a new null state given by $M_{-K}|\Delta,0\rangle$ at level $K$ if we set all the null states (and their descendants) at lower levels to zero. In this case, the GCA tower precisely reduces to the Virasoro tower given by the Virasoro descendants of the primary. As noted for level two, there is generically no condition on $C_1, \Delta$ : we only require $\xi = C_2 = 0$. Thus we can consider a truncation of the Hilbert space to this Virasoro module. We can, of course, have the Virasoro tower reducible by having Virasoro null vectors. These are analysed in the usual way. Note that we can have unitary representations if $\Delta$ and $C_1$ obey the conditions familiar from the study of the Virasoro algebra.

Unfortunately, this sector is relatively uninteresting from the point of view of its spacetime dependence. The correlation functions of operators are ultralocal depending

[1] Even though there are some small differences with the case of the 2d CFT such as $M_0$ not being Hermitian, it is not difficult to verify that we can consistently set the descendants of null states to zero.

only on time. This is because all $x$ dependence arises in combination with the $\xi$ dependence so as to survive the nonrelativistic limit. Setting $\xi = 0$ thus removes the spatial dependence of correlators.

### 6.1.3 GCA Null Vectors from 2d CFT

Here we will show that the level two GCA null vectors can alternatively be obtained by taking the nonrelativistic scaling limit of the familiar level two null vectors of 2d CFT.

The null vector at level two in a Virasoro tower is given by

$$|\chi_L\rangle = (\mathcal{L}_{-2} + \eta\mathcal{L}_{-1}^2)|h\rangle \otimes |\bar{h}\rangle \,, \tag{6.1.10}$$

with

$$\eta = -\frac{3}{2(2h+1)}, \tag{6.1.11}$$

$$h = \frac{1}{16}(5 - c \pm \sqrt{(1-c)(25-c)}). \tag{6.1.12}$$

One has a similar null state for the antiholomorphic Virasoro obtained by replacing $\mathcal{L}_n \to \bar{\mathcal{L}}_n$, $h \to \bar{h}$ and $c \to \bar{c}$.

Using these expressions let us first take the limit of the relation in (6.1.12) (together with its antiholomorphic counterpart). Recall that the nonrelativistic scaling limit for the central charges and conformal weights are given by (5.1.11) and (5.2.17). Since the central charges $c, \bar{c}$ are opposite in sign as $\epsilon \to 0$, on taking the positive sign for the negative central charge part,[2] and vice versa, for the square root in (6.1.12), we get

$$\xi = -\frac{3C_2}{2}, \qquad \Delta = \frac{(9 - 6C_1)}{4}\,. \tag{6.1.13}$$

These are precisely the relations we obtained in the previous section if we require the existence of both the GCA null states $|\chi^{(1)}\rangle, |\chi^{(2)}\rangle$ at level two.

These states themselves can be obtained by taking the nonrelativistic limit on appropriate combinations of the relativistic null vectors $|\chi_L\rangle$ and its antiholomorphic counterpart $|\chi_R\rangle$. Consider

$$|\chi^{(1)}\rangle = \lim_{\epsilon\to 0} \epsilon(-|\chi_L\rangle + |\chi_R\rangle)\,, \qquad |\chi^{(2)}\rangle = \lim_{\epsilon\to 0}(|\chi_L\rangle + |\chi_R\rangle). \tag{6.1.14}$$

From the expressions (5.2.17), we obtain $\eta = \frac{3\epsilon}{2\xi}(1 + \frac{(\Delta+1)\epsilon}{\xi})$ and $\bar{\eta} = -\frac{3\epsilon}{2\xi}(1 - \frac{(\Delta+1)\epsilon}{\xi})$ upto terms of order $\epsilon^2$. Substituting this into (6.1.14), using the relations (5.1.8) and taking the limit $\epsilon \to 0$, we obtain

$$|\chi^{(1)}\rangle \;=\; (M_{-2} - \frac{3}{4\xi}M_{-1}^2)|\Delta, \xi\rangle,$$

[2] If we take the negative sign in the square root we get in the same limit where $C_2 \neq 0$ that $\xi = 0$ and $\Delta = -1$. This is precisely what we obtained in Sec. 5.1.

$$|\chi^{(2)}\rangle = (L_{-2} - \frac{3}{2\xi}L_{-1}M_{-1} + \frac{3(\Delta+1)}{4\xi^2}M_{-1}^2)|\Delta,\xi\rangle, \tag{6.1.15}$$

which are exactly what we found from the intrinsic GCA analysis in (6.1.3) and (6.1.4).

Similarly the case mentioned in footnote 2 is also easily seen to correspond to the null states constructed in Sec. 5.1 for $C_2 \neq 0$. Finally, there is the case when $C_2 = 0$ and hence $c = \bar{c}$. Therefore it follows from (6.1.12) and its antiholomorphic counterpart that $h = \bar{h}$, i.e., $\xi = 0$ and $\Delta = 2h$. It is then easy to verify that the pair of states in (6.1.10) and its antiholomorphic counterpart reduce in the non-relativistic limit to the states constructed in Sec. 5.2 (see (6.1.8) and (6.1.9)). It is satisfying that the limiting process gives answers consistent with the intrinsic GCA analysis.

### 6.1.4 Non-Relativistic Limit of the Kac Formula

More generally if we want to examine the GCA null states at a general level, we would have to perform an analysis similar to that in the Virasoro representation theory. A cornerstone of this analysis is the Kac determinant which gives the values of the weights of the Virasoro Primaries $h(\bar{h})$ for which the matrix of inner products at a given level has a zero eigenvalue :

$$\det M^{(l)} = \alpha_l \prod_{1\leq r,s;rs\leq l} (h - h_{rs}(c))^{p(l-rs)}\,, \tag{6.1.16}$$

where $\alpha_l$ is a constant independent of $(h,c)$; $p(l-rs)$ is the number of partitions of the integer $l-rs$. The functions $h_{r,s}(c)$ are expressed in a variety of ways. One convenient representation is :

$$h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2\,, \tag{6.1.17}$$

$$h_0 = \frac{1}{24}(c-1)\,, \tag{6.1.18}$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}\,. \tag{6.1.19}$$

One can write a similar expression for the antiholomorphic sector. The values $h_{r,s}$ are the ones for which we have zeroes of the determinant and hence null vectors (and their descendants).

One could presumably generalise our analysis for GCA null vectors at level two and directly obtain the GCA determinant at a general level. This would give us a relation for $\Delta$ and $\xi$ in terms of $C_1, C_2$ for which there are null states, generalising the result

$$\xi = -\frac{3C_2}{2}\,, \qquad \Delta = \frac{(9-6C_1)}{4}\,, \tag{6.1.20}$$

at level two. However, here instead of a direct analysis we will simply take the non-relativistic limit of the Kac formula and see that one obtains sensible expressions for the $\Delta$ and $\xi$ at which the GCA determinant would vanish.

In taking the non-relativistic limit, let us first consider the case where $C_2 \neq 0$ and chosen to be positive. Therefore (from (5.1.11)) we need to take $c \ll -1$ and $\bar{c} \gg 1$ as $\epsilon \to 0$. We then find

$$h_{r,s} = \frac{1}{24}c(1-r^2) + \frac{1}{24}(13r^2 - 12rs - 1) + \mathcal{O}(\epsilon), \tag{6.1.21}$$

$$\bar{h}_{r',s'} = \frac{1}{24}\bar{c}(1-r'^2) + \frac{1}{24}(13r'^2 - 12r's' - 1) + \mathcal{O}(\epsilon). \tag{6.1.22}$$

Then eq. (5.2.17) gives the values of $\Delta$ and $\xi$ in terms of the RHS of (6.1.21) and (6.1.22) which in turn can be expressed in terms of $C_1$ and $C_2$. In the simple case where we take $(r,s) = (r',s')$[3]

$$\Delta_{r,s} = \lim_{\epsilon\to 0}(h_{r,s} + \bar{h}_{r,s}) = \frac{1}{2}C_1(1-r^2) + \frac{1}{12}(13r^2 - 12rs - 1)\,, \tag{6.1.23}$$

$$\xi_{r,s} = -\lim_{\epsilon\to 0}\epsilon(h_{r,s} - \bar{h}_{r,s}) = \frac{1}{2}C_2(1-r^2)\,. \tag{6.1.24}$$

In the case of $r = 2$, $s = 1$ we have two null states at level two built on the primary $|\Delta_{2,1}, \xi_{2,1}\rangle$. We see from (6.1.21) and (6.1.22) that the values of $\Delta_{2,1}$ and $\xi_{2,1}$ are exactly those given in (6.1.20). This is also what we explicitly constructed in the previous subsection.

We can also construct a pair of null states at level two for $r = 1, s = 2$. In this case, we see that $\Delta_{1,2} = -1$ and $\xi_{1,2} = 0$. Such a null state for $C_2 \neq 0$ was found in Sec. 5.1 and we see that it corresponds to the case mentioned in footnote 2.

Finally, as a last consistency check, we note that at level $K$, we can have a null state with $r = K, s = 1$. The condition on $\xi_{K,1}$ given by (6.1.24) is exactly the same as given in (6.1.7).

## 6.2 Differential Equations for GCA Correlators from Null States

The presence of the null states gives additional relations between correlation functions which is at the heart of the solvability of relativistic (rational) conformal field theories. To obtain these relations one starts with differential operator realisations $\hat{\mathcal{L}}_{-k}$ of the $\mathcal{L}_{-k}$

[3] Requiring that $\Delta$ should not have a $\frac{1}{\epsilon}$ piece immediately implies that $r = r'$. The choice $s = s'$ is merely to simplify expressions.

with ($k \geq 1$). Thus one has

$$\langle(\mathcal{L}_{-k}\phi(z,\bar{z}))\phi_1(z_1,\bar{z}_1)\cdots\phi_p(z_n,\bar{z}_n)\rangle = \hat{\mathcal{L}}_{-k}\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\cdots\phi_p(z_n,\bar{z}_n)\rangle\,, \qquad (6.2.25)$$

where

$$\hat{\mathcal{L}}_{-1} = \partial_z\,, \quad \hat{\mathcal{L}}_{-k} = \sum_{i=1}^{n}\left\{\frac{(k-1)h_i}{(z_i-z)^k} - \frac{1}{(z_i-z)^{k-1}}\partial_{z_i}\right\} \quad (\text{for } k \geq 2). \qquad (6.2.26)$$

We can obtain similar differential operators for the GCA generators by taking appropriate non-relativistic limits (2.3.34) of these relativistic expressions. Expanding the operators $\hat{\mathcal{L}}_{-k}$ as

$$\hat{\mathcal{L}}_{-k} = \epsilon^{-1}\hat{\mathcal{L}}_{-k}^{(-1)} + \hat{\mathcal{L}}_{-k}^{(0)} + \mathcal{O}(\epsilon)\,, \qquad (6.2.27)$$

and similarly for the anti-holomorphic part. Using (5.1.8), we obtain the expressions for the differential operators $\hat{M}_{-k}$ and $\hat{L}_{-k}$ as :

$$\begin{aligned} \hat{M}_{-k} &= \sum_{i=1}^{n}[\frac{(k-1)\xi_i}{t_{i0}^k} + \frac{1}{t_{i0}^{k-1}}\partial_{x_i}]\,, \\ \hat{L}_{-k} &= \sum_{i=1}^{n}[\frac{(k-1)\Delta_i}{t_{i0}^k} + \frac{k(k-1)\xi_i}{t_{i0}^{k+1}}x_{i0} - \frac{1}{t_{i0}^{k-1}}\partial_{t_i} + \frac{k-1}{t_{i0}^k}x_{i0}\partial_{x_i}]\,, \end{aligned} \qquad (6.2.28)$$

where $x_{i0} = x_i - x$ and $t_{i0} = t_i - t$. For $k = 1$ we have the simpler expressions $\hat{M}_{-1} = -\partial_x$ and $\hat{L}_{-1} = \partial_t$.

Therefore, correlation functions of GCA descendants of a primary field are given in terms of the correlators of the primaries by the action of the corresponding differential operators $\hat{M}_{-k}$ and $\hat{L}_{-k}$. In Sec. 6.3 we illustrate how this works and give the analogue of the conformal blocks for the non-relativistic case.

Now we will study the consequences of having null states at level two. We will consider the two null states $|\chi^{(1)}\rangle$ , $|\chi^{(2)}\rangle$ of Sec. 6.1.1, or rather correlators involving the corresponding fields $\chi^{(1,2)}(t,x)$. Setting the null state and thus its correlators to zero gives rise to differential equations for the correlators involving the primary $\phi_{\Delta,\xi}(t,x)$ with other fields. Using the forms (6.1.3) and (6.1.4), we find that the differential equations take the form

$$(\hat{M}_{-2} - \frac{3}{4\xi}\hat{M}_{-1}^2)\langle\phi_{\Delta,\xi}(t,x)\phi_1(t_1,x_1)\cdots\phi_n(t_n,x_n)\rangle = 0\,, \qquad (6.2.29)$$

$$(\hat{L}_{-2} - \frac{3}{2\xi}\hat{L}_{-1}\hat{M}_{-1} + \frac{3(\Delta+1)}{4\xi^2}\hat{M}_{-1}^2)\langle\phi_{\Delta,\xi}(t,x)\phi_1(t_1,x_1)\cdots\phi_n(t_n,x_n)\rangle = 0\,, \qquad (6.2.30)$$

with $\hat{L}_{-2}$ and $\hat{M}_{-2}$ as given in equation (6.2.28).

Acting on a two point function

$$G^{(2)}_{\mathrm{GCA}}(t,x) = \langle \phi_{\Delta,\xi}(t,x)\phi_{\Delta',\xi'}(0,0)\rangle\,, \tag{6.2.31}$$

we have the simple differential equations

$$[\frac{\xi'}{t^2} + \frac{1}{t}\partial_x - \frac{3}{4\xi}\partial_x^2]G^{(2)}_{\mathrm{GCA}}(t,x) = 0\,, \tag{6.2.32}$$

$$[\frac{\Delta'}{t^2} + 2\frac{\xi' x}{t^3} - \frac{1}{t}\partial_t + \frac{x}{t^2}\partial_x + \frac{3}{2\xi}\partial_t\partial_x + \frac{3(\Delta+1)}{4\xi^2}\partial_x^2]G^{(2)}_{\mathrm{GCA}}(t,x) = 0\,. \tag{6.2.33}$$

It is not difficult to check that the GCA two point function [28] given in (5.3.27) $G^{(2)}_{\mathrm{GCA}}(t,x) \propto t^{-2\Delta}\mathrm{e}^{\frac{2\xi x}{t}}$ (together with $\xi = \xi'$ and $\Delta = \Delta'$) identically satisfies, as it should, both these differential equations.

## 6.3 Descendants and GCA Conformal Blocks

### 6.3.1 GCA Descendants

By means of the differential operators $\hat{L}_{-k}, \hat{M}_{-k}$ (with $k > 0$) in (6.2.28), we may express the correlation function including a general GCA descendant with the correlation function of its primary $\phi_{\Delta\xi}(t,x)$. We had in fact already used this for the simple cases of these descendants corresponding to null states as (6.2.29) and (6.2.30). The general expression can be written as

$$\begin{aligned}&\langle \phi^{\{\vec{k},\vec{q}\}}_{\Delta\xi}(t,x)\phi_1(t_1,x_1)\cdots\phi_n(t_n,x_n)\rangle \\ &\qquad = \hat{L}_{-k_i}\cdots\hat{L}_{-k_1}\hat{M}_{-q_j}\cdots\hat{M}_{-q_1}\langle\phi_{\Delta\xi}(t,x)\phi_1(t_1,x_1)\cdots\phi_n(t_n,x_n)\rangle\,,\end{aligned} \tag{6.3.34}$$

where $\vec{k} = (k_1, k_2, \cdots, k_i)$ and $\vec{q} = (q_1, q_2, \cdots, q_j)$ are sequences of positive integers such that $k_1 \leq k_2 \cdots \leq k_i$ and similarly for the $q$'s. Also note that $\phi^{\{0,0\}}_{\Delta\xi}(t,x)$ denotes the primary $\phi_{\Delta\xi}(t,x)$ itself.

### 6.3.2 The OPE and GCA Blocks

Just as in the relativistic case, the OPE of two GCA primaries can be expressed in terms of the GCA primaries and their descendants as

$$\phi_1(t,x)\phi_2(0,0) = \sum_p \sum_{\{\vec{k},\vec{q}\}} C^{p\{\vec{k},\vec{q}\}}_{12}(t,x)\phi^{\{\vec{k},\vec{q}\}}_p(0,0)\,. \tag{6.3.35}$$

We should mention that, unlike in the case of a 2d CFT such an expansion is not analytic (see (6.3.40) below). The form of the two and three point function clearly exhibit essential singularities. Nevertheless we will go ahead with the expansion assuming it makes sense in individual segments such as $x, t > 0$. One can find the first few coefficients $C_{12}^{p\{\vec{k},\vec{q}\}}(t,x)$ by considering the three point function of the primary fields $\langle\phi_3\phi_1\phi_2\rangle$ in the situation where $\phi_1$ approaches $\phi_2$. In such a situation one can replace $\phi_1\phi_2$, in the three point function, with the RHS of (6.3.35) and obtain

$$\langle\phi_3(t',x')\phi_1(t,x)\phi_2(0,0)\rangle = \sum_{p,\{\vec{k},\vec{q}\}} C_{12}^{p\{\vec{k},\vec{q}\}}(t,x)\langle\phi_3(t',x')\phi_p^{\{\vec{k},\vec{q}\}}(0,0)\rangle. \tag{6.3.36}$$

We can find $C_{12}^{p\{0,0\}}$, $C_{12}^{p\{1,0\}}$ and $C_{12}^{p\{0,1\}}$ by expanding the left hand side (LHS) of (6.3.36) with respect to the small parameter $\frac{t}{t'}$ while keeping $\frac{x'}{t'}$ and $\frac{x}{t}$ finite, and comparing the $(t',x')$-dependence of the both sides. To make the final formulae simple, we concentrate on the case with $\Delta_1 = \Delta_2 = \Delta$ and $\xi_1 = \xi_2 = \xi$.

The expansion of the LHS is given as

$$\begin{aligned}
&\langle\phi_3(t',x')\phi_1(t,x)\phi_2(0,0)\rangle \\
&= C_{312}t^{-2\Delta+\Delta_3}t'^{-\Delta_3}(t'-t)^{-\Delta_3}\exp[\xi_3\tfrac{x'-x}{t'-t} + (2\xi-\xi_3)\tfrac{x}{t} + \xi_3\tfrac{x'}{t'}] \\
&= C_{312}t'^{-2\Delta_3}e^{2\xi_3\frac{x'}{t'}}\cdot t^{-2\Delta+\Delta_3}e^{(2\xi-\xi_3)\frac{x}{t}}[1 + \{\Delta_3 - \xi_3(\tfrac{x}{t}-\tfrac{x'}{t'})\}\tfrac{t}{t'} + \mathcal{O}((t/t')^2)],
\end{aligned} \tag{6.3.37}$$

while the RHS is given by

$$\begin{aligned}
&\sum_{p,\{\vec{k},\vec{q}\}} C_{12}^{p\{\vec{k},\vec{q}\}}(t,x)\langle\phi_3(t',x')\phi_p^{\{\vec{k},\vec{q}\}}(0,0)\rangle \\
&= [C_{12}^{3\{0,0\}}(t,x) + C_{12}^{3\{1,0\}}(t,x)\hat{L}_{-1} + C_{12}^{3\{0,1\}}(t,x)\hat{M}_{-1} + \ldots](t'^{-2\Delta_3}e^{2\xi_3\frac{x'}{t'}}) \\
&= t'^{-2\Delta_3}e^{2\xi_3\frac{x'}{t'}}[C_{12}^{3\{0,0\}} + C_{12}^{3\{1,0\}}(2\Delta_3 + 2\xi_3\frac{x'}{t'})t'^{-1} + C_{12}^{3\{0,1\}}(2\xi_3)t'^{-1} + \cdots].
\end{aligned} \tag{6.3.38}$$

One can easily read off the coefficients by comparing (6.3.37) and (6.3.38) :

$$\begin{aligned}
C_{12}^{3\{0,0\}} &= C_{312}t^{-2\Delta+\Delta_3}e^{(2\xi-\xi_3)\frac{x}{t}}, \\
C_{12}^{3\{1,0\}} &= \frac{1}{2}C_{312}t^{-2\Delta+\Delta_3+1}e^{(2\xi-\xi_3)\frac{x}{t}}, \\
C_{12}^{3\{0,1\}} &= -\frac{1}{2}C_{312}xt^{-2\Delta+\Delta_3}e^{(2\xi-\xi_3)\frac{x}{t}}.
\end{aligned} \tag{6.3.39}$$

So, in this case we have the GCA OPE as:

$$\begin{aligned}
&\phi_1(t,x)\phi_2(0,0) \\
&= \textstyle\sum_p C_{p12}t^{-2\Delta+\Delta_p}e^{(2\xi-\xi_p)\frac{x}{t}}\left(\phi_p(0,0) + \frac{t}{2}\phi_p^{\{1,0\}}(0,0) - \frac{x}{2}\phi_p^{\{0,1\}}(0,0) + \ldots\right).
\end{aligned} \tag{6.3.40}$$

## 6.4 GCA Fusion Rules

Analogous to the relativistic case [15], we can derive "*Fusion rules*",

$$[\phi_1] \times [\phi_2] \simeq \sum_p [\phi_p] \ ,$$

for the GCA conformal families, that determine which families $[\phi_p]$ have their primaries and descendants occurring in an OPE of any two members of the families $[\phi_1]$ and $[\phi_2]$. Here we have denoted a family $[\phi_i]$ by the corresponding primary $\phi_i$.

We illustrate how the fusion rules can be obtained for the families $[\phi_{\Delta,\xi}]$ and $[\phi_{\Delta_1,\xi_1}]$, where both fields are members of the GCA Kac table as specified by (6.1.21) (6.1.22). As mentioned in footnote 3, we need to take $r = r'$. The resulting $\Delta, \xi$ are thus labelled by a triple $\{r(s,s')\}$. In particular, we will consider below the case of $\Delta = \Delta_{2(1,1)}$ and $\xi = \xi_{2(1,1)}$. (In Sec. 6.4.1 , we consider the case of the fusion rule following from the case where $\Delta = \Delta_{1(2,2)}$ and $\xi = \xi_{1(2,2)}$. This case is interesting in that the GCA limit generally gives a weaker constraint than that following from the nonrelativistic limit of the 2d CFT.)

The fusion rules are derived from applying the condition that $\phi_{\Delta,\xi}$ has a null descendant at level two. For $(\Delta_1, \xi_1)$ we will consider a general member $r(s,s')$ of the GCA Kac table. Thus we have from (6.1.13), (6.1.21) and (6.1.22) :

$$\Delta = \Delta_{2(1,1)} = \frac{1}{4}(9 - 6C_1)\,, \quad \xi = \xi_{2(1,1)} = -\frac{3C_2}{2}\,; \tag{6.4.41}$$

$$\Delta_1 = \Delta_{r(s,s')} = \frac{C_1}{2}(1 - r^2) + \frac{1}{12}\{13r^2 - 6r(s+s') - 1\}\,, \tag{6.4.42}$$

$$\xi_1 = \xi_{r(s,s')} = \frac{C_2}{2}(1 - r^2)\,. \tag{6.4.43}$$

We need to consider the conditions (6.2.29) and (6.2.30) for the case of the three point function, i.e., $n = 2$. With $G^{(3)}_{\rm GCA}(t,x,\{t_i,x_i\}) = \langle \phi_{\Delta,\xi}(t,x)\phi_{\Delta_1,\xi_1}(t_1,x_1)\phi_{\Delta_2,\xi_2}(t_2,x_2)\rangle$, these give the constraints :

$$[\sum_{i=1}^{2}(\frac{\xi_i}{t_{i0}^2} + \frac{1}{t_{i0}}\partial_{x_i}) - \frac{3}{4\xi}\partial_x^2]\ G^{(3)}_{\rm GCA} = 0\,, \tag{6.4.44}$$

$$[\sum_{i=1}^{2}(\frac{\Delta_i}{t_{i0}^2} + \frac{2\xi_i}{t_{i0}^3}x_{i0} - \frac{1}{t_{i0}}\partial_{t_i} + \frac{1}{t_{i0}^2}x_{i0}\partial_{x_i}) + \frac{3}{2\xi}\partial_x\partial_t + \frac{3}{4}\frac{\Delta+1}{\xi^2}\partial_x^2]G^{(3)}_{\rm GCA} = 0 \tag{6.4.45}$$

respectively. Now by using (5.3.28), these translate into

$$\frac{1}{2}\xi - (\xi_1 + \xi_2) + \frac{3}{2\xi}(\xi_1 - \xi_2)^2 = 0\,,$$

$$4(2\Delta_1 - \Delta_2 + \Delta) + \frac{3(\xi_2 - \xi - \xi_1)^2}{\xi^2}(\Delta + 1) - \frac{6(\xi_2 - \xi - \xi_1)}{\xi}(\Delta_2 - \Delta_1 - \Delta - 1) = 0\,.$$

Solving the above equations, we get two simple sets of solutions :

$$\xi_2 = \frac{C_2}{2}[1-(r\pm1)^2]\,, \quad \Delta_2 = \frac{1}{2}C_1\{1-(r\pm1)^2\}+\frac{1}{12}[13(r\pm1)^2-6(r\pm1)(s+s')-1]. \tag{6.4.46}$$

Comparing with (6.4.42) and (6.4.43), we see that

$$\Delta_2 = \Delta_{r\pm1(s,s')}\,, \qquad \xi_2 = \xi_{r\pm1(s,s')}\,, \tag{6.4.47}$$

which is exactly what the relativistic fusion rules imply, namely

$$[\phi_{2(1,1)}] \times [\phi_{r(s,s')}] = [\phi_{r+1(s,s')}] + [\phi_{r-1(s,s')}]\,. \tag{6.4.48}$$

Thus once again we see evidence for the consistency of the GCA limit of the 2d CFT. In this case the GCA analysis gives as strong a constraint as the relativistic CFT. However, we will in the next subsection give an example where the GCA analysis gives a weaker constraint than what can be extracted from the 2d CFT.

### 6.4.1 Fusion Rule for $\phi_{1(2,2)}$

Here we consider the level two null state corresponding to the primary

$$\Delta = \Delta_{1(2,2)} = -1\,, \quad \xi = \xi_{1(2,2)} = 0\,, \tag{6.4.49}$$

and study the fusion rule for the product $[\phi_{\Delta,\xi}]\times[\phi_{\Delta_1,\xi_1}]$, where $\Delta_1, \xi_1$ are given by (6.4.42) and (6.4.43) with a generic triple $r(s,s')$.

From the GCA in this case we have level two null states (see below (6.1.4)) given by $|\chi^{(1)}\rangle = M_{-1}^2|\Delta, 0\rangle$ and $|\chi^{(2)}\rangle = L_{-1}M_{-1}|\Delta = -1, 0\rangle$. However, these null states are actually just descendants of the level one null state $M_{-1}|\Delta = -1, 0\rangle$. The imposition of the GCA constraint

$$\begin{aligned}&\hat{M}_{-1}\langle\phi_{\Delta,\xi}(t,x)\phi_{\Delta_1,\xi_1}(t_1,x_1)\phi_{\Delta_2,\xi_2}(t_2,x_2)\rangle\\ &\qquad = -\partial_x\langle\phi_{\Delta,\xi}(t,x)\phi_{\Delta_1,\xi_1}(t_1,x_1)\phi_{\Delta_2,\xi_2}(t_2,x_2)\rangle = 0\,,\end{aligned} \tag{6.4.50}$$

simply gives $\xi_1 = \xi_2$. Examination of (6.4.42) and (6.4.43) immediately implies that the $r$ value of $\phi_{\Delta_2,\xi_2}(t_2,x_2)$ is the same as that of $\phi_{\Delta_1,\xi_1}(t_1,x_1)$ but its $(s,s')$ values are undetermined. As we will see below, in taking the nonrelativistic limit of the 2d CFT we can extract the fusion rule obeyed in the relativistic CFT.

Thus, let us take the nonrelativistic limit of the relativistic fusion rule involving the primary $\phi_{2(1,1)}$. The conformal weights corresponding to $2(1,1)$ and $r(s,s')$ are expanded as

$$\begin{aligned} h &= h_{1,2} = -\frac{1}{2} + \frac{3\epsilon}{4C_2} + \mathcal{O}(\epsilon^2)\,, \quad \bar{h} = \bar{h}_{1,2} = -\frac{1}{2} + \frac{3\epsilon}{4C_2} + \mathcal{O}(\epsilon^2)\,; \\ h_1 &= h_{r,s} = \frac{C_2}{4\epsilon}(r^2-1) + (\frac{C_1}{4}(1-r^2) + \frac{13}{24}r^2 - \frac{rs}{2} - \frac{1}{24}) - \frac{\epsilon}{4C_2}(r^2-s^2) + \mathcal{O}(\epsilon^2)\,, \\ \bar{h}_1 &= \bar{h}_{r,s'} = \frac{C_2}{4\epsilon}(1-r^2) + (\frac{C_1}{4}(1-r^2) + \frac{13}{24}r^2 - \frac{rs'}{2} - \frac{1}{24}) + \frac{\epsilon}{4C_2}(r^2-s'^2) + \mathcal{O}(\epsilon^2)\,, \end{aligned}$$

respectively. Here in addition to the terms in (6.1.21) and (6.1.22), if we maintain the terms of order $\mathcal{O}(\epsilon)$ then we can extract the information in the relativistic fusion rule.

The differential equation for the three point function $\langle\phi_{1(2,2)}\phi_1\phi_2\rangle$ now implies that the $h$'s obey the equation

$$\frac{3}{2}(h_2-h-h_1)(h_2-h-h_1-1) + (h_2-h-2h_1)(2h+1) = 0\,, \tag{6.4.51}$$

and the same for $\bar{h}$'s. Solving these equations, keeping the $\mathcal{O}(\epsilon)$ terms, we get the solutions as

$$h_2 = h_{r,s\pm1}\,, \quad \bar{h}_2 = \bar{h}_{r,s'\pm1}\,,$$

which tell us that we have four possible families appearing in the OPE between $[\phi_{1(2,2)}]$ and $[\phi_{r(s,s')}]$. Note that the $(s,s')$ values are now constrained. In terms of the GCA weights, the four families are specified as

$$\Delta_2 = \Delta_{r(s\pm1,s'\pm1)}\,, \qquad \xi_2 = \xi_{r(s\pm1,s'\pm1)}\,, \tag{6.4.52}$$

where both the $\pm$ signs are taken independently.

# Chapter 7

# GCA in 2d: III. The Four Point Function

In this chapter we make the most nontrivial check, as yet, of the consistency of the scaling limit on the 2d CFT. First we consider the GCA differential equation for the four point function of GCA primaries one of which is $\phi_{2(1,1)}$ which has a level two null descendant. The general solution of this equation consistent with the crossing symmetry of the problem is discussed. We then proceed to rederive this result by taking the scaling limit of the corresponding four point function in the 2d CFT. One finds a solution in the scaling limit which is of the general form inferred from the GCA but now further constrained by the requirements of monodromy invariance present in the parent CFT. Finally, as a zeroth order consistency check of the full theory we briefly examine the factorisation of the four point answer into three point functions and find results in agreement with the fusion rules derived in the previous section.

## 7.1 GCA Four Point Function

We study the correlation function of four GCA primary fields (in the nonrelativistic Kac table)

$$G^{(4)}_{\rm GCA}(\{t_i,x_i\}) = \langle \phi_{r0(s_0,s'_0)}(t_0,x_0)\phi_{r1(s_1,s'_1)}(t_1,x_1)\phi_{r2(s_2,s'_2)}(t_2,x_2)\phi_{r3(s_3,s'_3)}(t_3,x_3)\rangle. \quad (7.1.1)$$

By solving the Ward identities coming from the symmetries $L_{0,\pm1}$, $M_{0,\pm1}$, the form of the four point function is restricted to

$$G^{(4)}_{\rm GCA}(\{t_i,x_i\}) = \prod_{0\le i<j\le 3} t_{ij}^{\frac{1}{3}\Sigma_{k=0}^{3}\Delta_k-\Delta_i-\Delta_j}\, {\rm e}^{-\frac{x_{ij}}{t_{ij}}(\frac{1}{3}\Sigma_{k=0}^{3}\xi_k-\xi_i-\xi_j)}\mathcal{G}_{\rm GCA}(t,x). \quad (7.1.2)$$

Here $\Delta_i$ and $\xi_i$ are defined by (6.4.42) and (6.4.43) with replacing $\{r(s,s')\}\to\{r_i(s_i,s'_i)\}$.

The non-relativistic analogues of the cross ratio $t$ and $x$, which are defined by

$$t=\frac{t_{01}t_{23}}{t_{03}t_{21}}, \qquad \frac{x}{t}=\frac{x_{01}}{t_{01}}+\frac{x_{23}}{t_{23}}-\frac{x_{03}}{t_{03}}-\frac{x_{21}}{t_{21}}, \tag{7.1.3}$$

are invariant under the coordinate transformation $L_{0,\pm1}$, $M_{0,\pm1}$. Hence the function $\mathcal{G}_{\rm GCA}(t,x)$ is not determined from these symmetries. As explained in Sec.6.2, for null states we have differential equations which can be used to further restrict the four point function. In the following we consider the differential equations coming from the primary $\{r_0(s_0,s_0')\}=\{2(1,1)\}$. So, in the following $(\Delta_0,\xi_0)=(\Delta,\xi)$ given in (6.4.41).

The differential equations coming from the null states $|\chi^{(1)}\rangle$ and $|\chi^{(2)}\rangle$ are given by

$$[\frac{1}{2}\partial_{x_0}^2+\sum_{i=1}^{3}\kappa^{(-1)}(\frac{\xi_i}{t_{i0}^2}+\frac{1}{t_{i0}}\partial_{x_i})]G^{(4)}_{\rm GCA}=0, \tag{7.1.4}$$

$$[\,\partial_{x_0}\partial_{t_0}-\frac{1}{2}\frac{\kappa^{(0)}}{\kappa^{(-1)}}\partial_{x_0}^2-\sum_{i=1}^{3}\kappa^{(-1)}(\frac{2\xi_i x_{i0}}{t_{i0}^3}+\frac{\Delta_i}{t_{i0}^2}+\frac{x_{i0}}{t_{i0}^2}\partial_{x_i}-\frac{1}{t_{i0}}\partial_{t_i})\,]G^{(4)}_{\rm GCA}=0 \tag{7.1.5}$$

respectively. Here we have introduced the notation

$$\kappa^{(-1)}=-\frac{2}{3}\xi_0, \quad \kappa^{(0)}=\frac{2}{3}\Delta_0+\frac{2}{3}. \tag{7.1.6}$$

By substituting (7.1.2) into (7.1.4) and (7.1.5), we obtain differential equations for $\mathcal{G}_{\rm GCA}(t,x)$. We can set $t_1=0$, $t_2=1$, $t_3=\infty$ and $x_1=x_2=x_3=0$, by using the finite part of GCA (we can use the $L_{0,\pm1}$ to fix the $t$'s and then the $M_{0,\pm1}$ to fix the $x$'s), so that we have $t_0=t$, $x_0=x$. Further introducing $H(t,x)$ as

$$|t|^{\Sigma_{i=0}^{3}\frac{\Delta_i}{3}-\Delta_0-\Delta_1}|1-t|^{\Sigma_{i=0}\frac{\Delta_i}{3}-\Delta_0-\Delta_2}\mathcal{G}_{\rm GCA}(t,x)=e^{\frac{x}{t}(\Sigma_{i=0}^{3}\frac{\xi_i}{3}-\xi_0-\xi_1)-\frac{x}{1-t}(\Sigma_{i=0}^{3}\frac{\xi_i}{3}-\xi_0-\xi_2)}H(t,x),$$

the differential equations for $H(t,x)$ are given by

$$\partial_x^2H+2\kappa^{(-1)}\tfrac{1-2t}{t(1-t)}\partial_xH+2\kappa^{(-1)}\left(\tfrac{\xi_1}{t^2}+\tfrac{\xi_2}{(1-t)^2}+\tfrac{\xi_0+\xi_1+\xi_2-\xi_3}{t(1-t)}\right)H=0, \tag{7.1.7}$$

$$\begin{aligned}\partial_t\partial_xH-\tfrac{1}{2}\tfrac{\kappa^{(0)}}{\kappa^{(-1)}}\partial_x^2H+\kappa^{(-1)}\Bigg\{&\tfrac{1-2t}{t(1-t)}\partial_tH-x\tfrac{1-2t+2t^2}{t^2(1-t)^2}\partial_xH\\&-\left(\tfrac{\Delta_1}{t^2}+\tfrac{\Delta_2}{(1-t)^2}+\tfrac{\Delta_0+\Delta_1+\Delta_2-\Delta_3}{t(1-t)}\right)H\\&-x\left(2\tfrac{\xi_1}{t^3}-2\tfrac{\xi_2}{(1-t)^3}+(\xi_0+\xi_1+\xi_2-\xi_3)\tfrac{1-2t}{t^2(1-t)^2}\right)H\Bigg\}=0.\end{aligned} \tag{7.1.8}$$

The first equation can be easily solved to fix the $x$-dependence:

$$H_\pm(t,x)=(D(t))^{-\frac{1}{2}}|t(1-t)|^{-\kappa^{(0)}+1}\mathcal{H}_\pm(t)\exp\left\{\frac{xC_2}{t(1-t)}\big(-1+2t\pm\sqrt{D(t)}\big)\right\}, \tag{7.1.9}$$

where we have extracted some $t$-dependence just for later convenience. The function $D(t)$ is defined by

$$D(t) = r_1^2(1-t) + r_2^2 t - r_3^2 t(1-t), \tag{7.1.10}$$

and it satisfies $D(t) \geq 0$ because of the triangle inequality for $(r_1, r_2, r_3)$, which follows from the fusion rule (see Appendix B, pg. 81).

Now in terms of $\mathcal{H}_\pm(t)$, the second equation for $H(t,x)$ is simplified as

$$\partial_t \log \mathcal{H}_\pm = \pm \frac{1}{\sqrt{D}} \left\{ \frac{\mathcal{C}_1}{t} + \frac{\mathcal{C}_2}{1-t} - \mathcal{C}_3 \right\}. \tag{7.1.11}$$

Here $t$-independent constants $\mathcal{C}_i$ are defined by

$$\begin{aligned} \mathcal{C}_i &= \frac{1}{3} r_i^2 \Delta_0 + \frac{1}{3} r_i^2 + \Delta_i + \frac{1}{3}\Delta_0 - \frac{2}{3} &(7.1.12)\\ &= \pm r_i (\Delta_{r_i \pm 1 (s_i, s_i')} - \Delta_0 - \Delta_i + \kappa^{(0)} - 1). &(7.1.13) \end{aligned}$$

The second line is quite a remarkable simplification. In fact it has a simple physical interpretation with $\Delta_{r_i \pm 1(s_i, s_i')}$ corresponding to the conformal dimension of the allowed intermediate states. We will return to this point later in Sec. 7.4.

Finally the differential equation (7.1.11) can be solved if we notice the relation

$$\partial_t \log |r_1 + r_3 t + \sqrt{D}| = \frac{1}{2\sqrt{D}} \frac{1}{t} (-r_1 + r_3 t + \sqrt{D}). \tag{7.1.14}$$

Notice that this equation still holds after a flip of the signs in front of $r_1$ and $r_3$, and it also gives an additional relation by replacing $r_1 \leftrightarrow r_2$ and $t \to 1-t$. By taking linear combinations of these equations, we obtain the solutions of the above differential equations which are given by

$$\mathcal{H}_+(t) = \mathcal{I}_{1,2,3}(t), \quad \mathcal{H}_-(t) = \mathcal{I}_{2,1,3}(1-t), \tag{7.1.15}$$

with

$$\begin{aligned} \mathcal{I}_{1,2,3}(t) &= \left| \frac{r_1 + r_2 + r_3}{r_1 + r_2 - r_3} \frac{-r_1 + r_3 t + \sqrt{D}}{r_1 + r_3 t + \sqrt{D}} \right|^{\frac{\mathcal{C}_1}{r_1}} \left| \frac{r_1 + r_2 + r_3}{r_1 + r_2 - r_3} \frac{r_2 - r_3(1-t) + \sqrt{D}}{r_2 + r_3(1-t) - \sqrt{D}} \right|^{\frac{\mathcal{C}_2}{r_2}} \\ &\quad \times \left| \frac{r_1 + r_2 - r_3}{-r_1 + r_2 + r_3} \frac{r_2 + r_3(1-t) + \sqrt{D}}{r_2 - r_3(1-t) + \sqrt{D}} \right|^{\frac{\mathcal{C}_3}{r_3}}. \end{aligned} \tag{7.1.16}$$

The overall constant factor is chosen for the later convenience.

Thus the four point function is given by a general linear combination of the two solutions $H_\pm(t,x)$, which are defined by (7.1.9) with the functions (7.1.15). In particular, we may allow different linear combinations for the different segments $t < 0$, $0 < t < 1$ and $1 < t$, since the differential equations are solved independently for each segment. It may be worth pointing out that the term in each absolute value sign takes a definite signature for the each segment. In the relativistic 2d CFT, we would determine the particular linear combination by requiring the four point function to be singlevalued on the complex $z$ plane. In the case of the GCA such an argument is not available. Thus we will discuss only the constraint coming from crossing symmetry.

First of all, it is clear that the following linear combination is invariant under the exchange $1 \leftrightarrow 2$ and $(t, x) \leftrightarrow (1-t, -x)$:

$$H(t,x) = H_+(t,x) + H_-(t,x). \tag{7.1.17}$$

Next, by using the following property of the function $\mathcal{I}_{1,2,3}(t)$:

$$\mathcal{I}_{3,2,1}(1/t) \;=\begin{cases}\mathcal{I}_{1,2,3}(t)\,, \\ \mathcal{I}_{2,1,3}(1-t)\,,\end{cases} \qquad \mathcal{I}_{2,3,1}(1-1/t)=\begin{cases}\mathcal{I}_{2,1,3}(1-t)\,, & (t>0)\,, \\ \mathcal{I}_{1,2,3}(t)\,, & (t<0)\,.\end{cases} \tag{7.1.18}$$

It is easy to show that $H(t,x)$ behaves correctly under the exchange $1 \leftrightarrow 3$ and $(t,x) \leftrightarrow (1/t, -x/t^2)$, namely,

$$H(t,x) \to |t|^{2\Delta_0} \mathrm{e}^{-2\xi_0 \frac{x}{t}} H(t,x)\,. \tag{7.1.19}$$

This would be the GCA analogue of the transformation $G(z,\bar{z}) \to z^{2h}\bar{z}^{2\bar{h}} G(z,\bar{z})$ in 2d CFT.

Then we may ask whether (7.1.17) is the unique linear combination which has these properties. In fact since we should allow independent coefficients for different segments (see the discussion at the end of 5.3.3), we have infinitely many combinations which have the correct property of crossing symmetry. Here we will not pursuit the most general form of such possibilities, but just write down a class of such combinations:

$$H(t,x) = \begin{cases} f_{123} H_+(t,x) + f_{321} H_-(t,x), & t<0, \\ f_{132} H_+(t,x) + f_{231} H_-(t,x), & 0<t<1, \\ f_{312} H_+(t,x) + f_{213} H_-(t,x), & 1<t. \end{cases} \tag{7.1.20}$$

Here $f_{123}$ is an arbitrary function of the quantum numbers of the three fields $\phi_{r_i(s_i,s_i')}$ $(i=1,2,3)$. It is obvious that this combination has the same property as (7.1.17). Also, by changing the relative sign of $H_+$ and $H_-$, we would obtain the combination which flips the overall sign under the exchange $1 \leftrightarrow 2$, $(t,x) \leftrightarrow (1-t,-x)$ or/and $1 \leftrightarrow 3$, $(t,x) \leftrightarrow (1/t, -x/t^2)$. We see no reason to forbid these combinations.

Thus, we find that the requirement of crossing symmetry does not fix the GCA four point function uniquely. In the next section, we consider the corresponding four point function in the 2d CFT and take the non-relativistic limit. In this case, the original four point function is fixed uniquely by the requirement of singlevaluedness. It would be interesting to study whether further conditions coming directly from GCA context would fix the four point function completely or not. We will leave this issue as a problem for the future.

## 7.2 GCA Four Point Function from 2d CFT

Next we discuss the four point function starting from the 2d CFT and taking the limit. We take the holomorphic part of the four point function in 2d CFT as[1]

$$G^{(4)}_{\mathrm{2d\,CFT}}(\{z_i\}) = \langle \phi_0(z_0)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \prod_{0\le i<j\le 3} z_{ij}^{\frac{1}{3}\sum_{k=0}^{3} h_k - h_i - h_j}\mathcal{G}_{\mathrm{2d\,CFT}}(z), \quad (7.2.21)$$

where the cross ratio is defined by $z = (z_{01}z_{23})/(z_{03}z_{21})$. By taking $(r_0, s_0) = (2, 1)$, we obtain the differential equation of the form

$$(\hat{\mathcal{L}}_{-2} + \eta\hat{\mathcal{L}}_{-1}^2)G^{(4)}_{\mathrm{2d\,CFT}}(\{z_i\}) = 0, \quad (7.2.22)$$

where $\eta$ is introduced at (6.1.11) and the differential operators $\hat{\mathcal{L}}_{-n}$ are defined in (6.2.26).

We introduce the function $K(z)$ as

$$z^{\beta_1}(1-z)^{\beta_2}K(z) \qquad = z^{\Sigma_{i=0}^{3}\frac{h_i}{3} - h_0 - h_1}(1-z)^{\Sigma_{i=0}^{3}\frac{h_i}{3} - h_0 - h_2}\mathcal{G}_{\mathrm{2d\,CFT}}(z) \quad (7.2.23)$$

$$= \lim_{z_3\to\infty} z_3^{2h_3} G^{(4)}_{\mathrm{2d\,CFT}}(z_0 = z, z_1 = 0, z_2 = 1, z_3)\,, \quad (7.2.24)$$

with $\beta_i$ defined as any one of the solutions of the quadratic equation

$$\beta_i(\beta_i - 1) + \kappa\beta_i - \kappa h_i = 0, \qquad (\kappa = -\eta^{-1}). \quad (7.2.25)$$

The second expression (7.2.24) is upto a constant phase. Then the differential equation is rewritten into the standard form of the Hypergeometric differential equation:

$$z(1-z)\partial_z^2 K(z) + (\gamma - (\alpha+\beta+1)z)\partial_z K(z) - \alpha\beta K(z) = 0. \quad (7.2.26)$$

The parameters $\alpha$, $\beta$ and $\gamma$ are given by

$$\gamma = 2\beta_1 + \kappa, \quad \alpha + \beta + 1 = 2(\beta_1 + \beta_2 + \kappa), \quad (7.2.27)$$

$$\alpha\beta = \beta_1(\beta_1 - 1) + 2\beta_1\beta_2 + \beta_2(\beta_2 - 1) + \kappa(2\beta_1 + 2\beta_2 + h_0 - h_3). \quad (7.2.28)$$

The two independent solutions of (7.2.26) which diagonalize the Monodromy transformation $z \to e^{2\pi i}z$ around $z = 0$ are given by $K_1(\alpha,\beta,\gamma,z) = F(\alpha,\beta,\gamma,z)$ and $K_2(\alpha,\beta,\gamma,z) = z^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z)$, where $F(\alpha,\beta,\gamma,z)$ is defined as the Hypergeometric series and its analytic continuation. The argument so far would be applied in a parallel way to the anti-holomorphic part with replacing parameters as $h_i \to \bar{h}_i$, $\beta_i \to \bar{\beta}_i$, $\kappa \to \bar{\kappa}$ and $(\alpha,\beta,\gamma) \to (\bar{\alpha},\bar{\beta},\bar{\gamma})$, along with the replacement of the coordinate $z \to \bar{z}$.

Now we construct the full four point function by multiplying the holomorphic and the anti-holomorphic contributions and taking an appropriate linear combination. A standard

[1]Hereafter, to avoid complexity, we use the notation such as $\phi_i$ and $h_i$ for the objects corresponding to the member $(r_i, s_i)$ on the 2d CFT Kac table.

argument (see for e.g. [15]) for choosing the combination is based on the singlevaluedness of the full four point function on the complex plane. In fact, by considering the behaviour under the Monodromy transformations $z \to e^{2\pi i} z$ $(1-z \to e^{2\pi i}(1-z))$ around $z = 0$ $(z = 1)$, we find that the following combination:

$$I(z,\bar{z}) = K_1(\alpha,\beta,\gamma,z)K_1(\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{z}) + AK_2(\alpha,\beta,\gamma,z)K_2(\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{z})\,, \tag{7.2.29}$$

with

$$A = -\frac{\Gamma(\gamma)\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\frac{\Gamma(\bar{\gamma})\Gamma(\bar{\alpha}-\bar{\gamma}+1)\Gamma(\bar{\beta}-\bar{\gamma}+1)}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})\Gamma(2-\bar{\gamma})}, \tag{7.2.30}$$

is the unique singlevalued combination, provided $\alpha-\bar{\alpha}$, $\beta-\bar{\beta}$ and $\gamma-\bar{\gamma}$ are all integers. To see this one may use the formula (7.4.77).

Next we take the non-relativistic limit. For the purpose, we use the asymptotic form of the Hypergeometric functions for large values of $\alpha$, $\beta$ and $\gamma$, which can be derived by the saddle point analysis of the integral formula:

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 dw w^{\beta-1}(1-w)^{\gamma-\beta-1}(1-zw)^{-\alpha}, \tag{7.2.31}$$

where the conditions $\gamma > \beta > 0$ and that $z$ is not a real value greater than 1 are assumed. We expand every parameter as

$$\beta_i = \frac{1}{\epsilon}\beta_i^{(-1)} + \beta_i^{(0)} + \cdots\,, \quad \alpha = \frac{1}{\epsilon}\alpha^{(-1)} + \alpha^{(0)} + \cdots\,, \quad \text{etc.} \tag{7.2.32}$$

Then, the quadratic equations for $\beta_i$'s are solved by

$$\beta_i^{(-1)} = -\frac{1}{2}C_2(1+\delta_i r_i), \quad \beta_i^{(0)} = -\frac{1}{2}\delta_i\ell_i - \frac{1}{2}(\kappa^{(0)}-1), \quad (\delta_i = \pm 1), \tag{7.2.33}$$

with

$$\ell_i = \frac{(\kappa^{(0)}-1)\kappa^{(-1)} + 2\kappa^{(-1)}h_i^{(0)} + 2\kappa^{(0)}h_i^{(-1)}}{C_2 r_i} = \Big(\frac{13}{6} - C_1\Big)r_i - s_i. \tag{7.2.34}$$

Since $\beta_i$ are auxiliary parameters introduced in order to change the form of the differential equation, we take $(\delta_1,\delta_2) = (-1,+1)$ without loss of generality. Also, since the Hypergeometric function is symmetric under the exchange $\alpha \leftrightarrow \beta$ we take $\alpha^{(-1)} < \beta^{(-1)}$. Then the solutions $(\alpha,\beta,\gamma)$ of Eqs.(7.2.27) (7.2.27) are expanded as

$$\begin{aligned} \alpha^{(-1)} &= \frac{C_2}{2}(r_1-r_2-r_3), & \beta^{(-1)} &= \frac{C_2}{2}(r_1-r_2+r_3), & \gamma^{(-1)} &= C_2 r_1, \\ \alpha^{(0)} &= \frac{1}{2}(\ell_1-\ell_2-\ell_3+1), & \beta^{(0)} &= \frac{1}{2}(\ell_1-\ell_2+\ell_3+1), & \gamma^{(0)} &= \ell_1+1, \end{aligned} \tag{7.2.35}$$

where $\ell_3$ is defined by (7.2.34). As for the expansion of the parameters of the anti-holomorphic part, we first choose the leading term of $\bar{\beta}_i$ as $\bar{\beta}_i^{(-1)} = -\beta_i^{(-1)}$. This can be achieved by taking appropriate branch of the quadratic equation for $\bar{\beta}_i$. Then the leading order terms of the rest of the parameters are given just by flipping the sign as $(\bar{\alpha}^{(-1)}, \bar{\beta}^{(-1)}, \bar{\gamma}^{(-1)}) = -(\alpha^{(-1)}, \beta^{(-1)}, \gamma^{(-1)})$, while $\mathcal{O}(1)$ term is independent from those of the holomorphic part, since we consider different $s$ and $s'$ as (6.1.21) and (6.1.22). So, we introduce the notation $\bar{\ell}_i$, which are defined by (7.2.34) with changing $s_i \to s'_i$. Then the $\mathcal{O}(1)$ terms are given by replacing $\ell_i$ with $\bar{\ell}_i$ as $\bar{\gamma}^{(0)} = \bar{\ell}_1 + 1$, and the same for $\bar{\alpha}^{(0)}$ and $\bar{\beta}^{(0)}$.

For this parameter choice, the integral (7.2.31) can be evaluated by taking account of a single saddle point located on the segment $0 < w < 1$, and the result is given by

$$K_1(\alpha,\beta,\gamma,z) \to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\gamma-\alpha)} K_+(z), \quad (z = t + \epsilon x, \quad t < 1), \tag{7.2.36}$$

where the function $K_+$ is defined by (7.4.58) and (7.4.59) ($\delta_{C_2}$ in (7.4.58) is the sign of $C_2$, which we take $+1$ in the main text). See the Appendix A for a detailed discussion on the saddle point analysis, where the asymptotic form of the rest of the region $1 < t$, the other solution $K_2 = z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z)$, and the anti-holomorphic counterparts of them are also discussed.

In particular, for the region $t < 0$, the asymptotic form of the basis functions are given by Eqs.(7.4.65)–(7.4.67) in terms of the functions $K_\pm(z)$ and $\bar{K}_\pm(\bar{z})$. Inserting these into (7.2.29), it is easy to show that the "cross term" $K_+(z) \times \bar{K}_-(\bar{z})$ cancels and we obtain the following asymptotic form of the function $I(z,\bar{z})$ for $z = t + \epsilon x$ and $\bar{z} = t - \epsilon x$:

$$\begin{aligned} I(z,\bar{z}) &\to K_+(z)\bar{K}_+(\bar{z}) \\ &+ (-1)^{n_\beta + n_\gamma} z^{-\frac{C_2}{\epsilon} r_1 - \ell_1} \bar{z}^{\frac{C_2}{\epsilon} r_1 - \bar{\ell}_1} (1-z)^{\frac{C_2}{\epsilon} r_2 + \ell_2} (1-\bar{z})^{-\frac{C_2}{\epsilon} r_2 + \bar{\ell}_2} K_-(z)\bar{K}_-(\bar{z}) \,. \end{aligned} \tag{7.2.37}$$

Here we have omitted the overall $(t,x)$-independent constant. $n_\gamma = \bar{\gamma} - \gamma$ and $n_\beta = \bar{\beta} - \beta$ are the integers (see the next subsection). The asymptotic form of the other regions are also derived in the same way and we obtain the same result (7.2.37) (See Appendix A).

Now remember that the four-point function is given by multiplying the additional factor $z^{\beta_1}(1-z)^{\beta_2}$ of (7.2.23) (and also the anti-holomorphic factor) to $I(z,\bar{z})$. By further expanding with respect to $\epsilon$ which appears in $z = t + \epsilon x$ and $\bar{z} = t - \epsilon x$, all singular exponents cancel among the holomorphic and the anti-holomorphic contribution. Finally we obtain the following form of the four point function:[2]

$$H(t,x) = \begin{cases} (-1)^{n_1} H_+(t,x) + (-1)^{n_2+n_3} H_-(t,x)\,, & t < 0\,, \\ H_+(t,x) + (-1)^{n_1+n_2+n_3} H_-(t,x)\,, & 0 < t < 1\,, \\ (-1)^{n_2} H_+(t,x) + (-1)^{n_1+n_3} H_-(t,x)\,, & 1 < t\,. \end{cases} \tag{7.2.38}$$

[2]We have used that $C_2/\epsilon$ is an even integer.

Here the functions $H_{\pm}(t,x)$ are the ones introduced in the previous subsection and $n_i$ are defined by

$$n_i = \frac{C_2}{\epsilon} r_i + \frac{1}{2}(s_i' - s_i). \tag{7.2.39}$$

As we will explain in the next subsection, the differences $s_i' - s_i$ are now even integers. Hence the factors like $(-1)^{n_1}$ in (7.2.38) are just signs.

The behaviour of the function (7.2.38) under the exchange $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$ can be easily found as

$$H(t,x) \to \begin{cases} (-1)^{n_1+n_2+n_3} H(t,x)\,, & (1 \leftrightarrow 2, (t,x) \to (1-t,-x))\,, \\ (-1)^{n_2} |t|^{2\Delta_0} \mathrm{e}^{-2\xi_0 \frac{x}{t}} H(t,x)\,, & (1 \leftrightarrow 3, (t,x) \to (1/t, -x/t^2))\,. \end{cases} \tag{7.2.40}$$

Here we should notice that the above monodromy argument does not fix the overall factor $C^{(4)}_{1,2,3}$ of the four-point function, which should depend on the quantum numbers of the fields $\phi_i$ $(i = 1,2,3)$. For example, if it satisfies the condition $C^{(4)}_{2,1,3} = (-1)^{n_1+n_2+n_3} C^{(4)}_{1,2,3}$ and $C^{(4)}_{3,2,1} = (-1)^{n_2} C^{(4)}_{1,2,3}$ then the four-point function including the overall constant belongs to the class which is written down in (7.1.20), with $f_{123} = (-1)^{n_1} C^{(4)}_{1,2,3}$. Deriving the explicit form of the overall constant needs the three point functions of the theory whose overall factor we have also not determined here.

In summary, contrary to the discussion in the previous section, the asymptotic form of the four point function (7.2.38) is determined upto overall constant factor.

## 7.3 Singlevaluedness Condition

Now we study the issue of the singlevaluedness in detail. It contains two aspects. As we mentioned below the equation (7.2.30), the singlevaluedness of the function $I(z,\bar{z})$ requires $\bar{\alpha} - \alpha$, $\bar{\beta} - \beta$ and $\bar{\gamma} - \gamma$ are all integers. Another condition comes from the singlevaluedness of the factor $z^{\beta_1} \bar{z}^{\bar{\beta}_1} (1-z)^{\beta_2} (1-\bar{z})^{\bar{\beta}_2}$, which requires $\bar{\beta}_1 - \beta_1$ and $\bar{\beta}_2 - \beta_2$ to be integers.

Before studying these conditions, let us mention the singlevaluedness of the two point functions of the members on the Kac table, which is given by

$$h_{r,s} - \bar{h}_{r,s'} = \frac{1}{2} \times \text{integer}\,. \tag{7.3.41}$$

From (6.1.21) and (6.1.22), we have

$$h_{r,s} - \bar{h}_{r,s'} = -\frac{1}{2}\frac{C_2}{\epsilon}(1-r^2) - \frac{1}{2} r(s-s') + \mathcal{O}(\epsilon)\,. \tag{7.3.42}$$

Then the above condition requires that $-\frac{C_2}{2\epsilon}(1-r^2)+\mathcal{O}(\epsilon)$ is an integer or a half integer. Although it is still not very clear in this parameterization, more detailed study[3] tells us that it is satisfied only in the strict limit $\epsilon = 0$ with a "large" integer $C_2/\epsilon$. This final point is always satisfied in the modular invariant 2d CFT, in which case $C_2/\epsilon$ is required to be an even integer (see the footnote 5).

Now from (7.2.34) and (7.2.35), we first notice that $\bar{\gamma}-\gamma$ is an integer because of the same reason as above. Then in order for the differences $\bar{\beta}-\beta$ and $\bar{\alpha}-\alpha$ to be integers, $\Sigma_{i=1}^{3}(s'_i - s_i)$ must be an even integer.

On the other hand the requirement for $\bar{\beta}_i-\beta_i$ to be integers give rise to the condition that each $s'_i - s_i$ $(i=1,2)$ is an even integer.

In fact, the latter condition is directly related to the singlevaluedness of the three point function which includes the primary field $\phi_{2(1,1)}$, namely, the single valuedness of the three point function $\langle\phi_{2(1,1)}\phi_{r(s,s')}\phi_{r\pm 1(s,s')}\rangle$ give rise to the condition that $s'-s$ is an even integer.[4] This just means that the primary field $\phi_{2(1,1)}$ interacts only with the primary field having an even $s'-s$. For an odd $s'-s$, the three point function, and hence the four point function, vanishes. Then for the non-vanishing four point function, $s'_i - s_i$ $(i=1,2,3)$ are automatically even integers and the singlevaluedness condition is satisfied. A relation between the parameter $\beta_i$ and the three point function is further clarified in the next section.

## 7.4 A Quick Look at Factorization and the Fusion Rule

In 2d CFT, the four point function is written as the summation over the various intermediate fields. Let us consider the four point function (7.2.21), with its anti-holomorphic part, and set $z_0 = z$, $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. Then for small $z$, the four point function is given by

$$\sum_{p,\{\vec{k},\vec{\bar{k}}\}} C_{01}^{p,\{\vec{k},\vec{\bar{k}}\}} z^{h_p-h_0-h_1+K}\bar{z}^{\bar{h}_p-\bar{h}_0-\bar{h}_1+\bar{K}}\langle\phi_p^{\{\vec{k},\vec{\bar{k}}\}}(0)\phi_2(1)\phi_3(\infty)\rangle. \tag{7.4.43}$$

Here $\phi_p^{\{\vec{k},\vec{\bar{k}}\}}$ is the intermediate primary field (for $\vec{k}=\vec{\bar{k}}=0$), and its descendants (for nontrivial $\vec{k}$ and $\vec{\bar{k}}$), which are given by acting, on the primary, series of the operators $\mathcal{L}_{-k_n}\cdots\mathcal{L}_{-k_1}$ and the same for the anti-holomorphic part. These intermediate states

[3]If we parameterize as $c = 13-6(\kappa+1/\kappa)$ and $\bar{c} = 13-6(\bar{\kappa}+1/\bar{\kappa})$ with $\kappa \gg 1$ and $\bar{\kappa} \ll -1$, then the singlevaluedness requires both $\kappa-\bar{\kappa}$ and $1/\kappa - 1/\bar{\kappa}$ are integer, which is satisfied only in the case with $1/\kappa = 1/\bar{\kappa} = 0$.

[4]It can be shown explicitly by using (6.1.21) and (6.1.22) and requiring that the summation of the spins of the three fields is an integer.

appear in the OPE of $\phi_0$ and $\phi_1$. This relation shows that the four point function is essentially written in terms of the three point functions. It would be interesting to study the similar property in detail in the case of the GCA. Here, as a first step in such an analysis, we study the contribution coming from the intermediate primary fields.

In the previous subsections, we studied the four point function of the primary fields with setting one of the field to have $\{r_0(s_0, s_0')\} = \{2(1,1)\}$ on the Kac table. In this case, the possible primary fields $\phi_p^{\{0,0\}}$ appearing in (7.4.43) are the ones with $r_p(s_p, s_p') = r_1 \pm 1(s_1, s_1')$. Let us see this from the equation (7.2.25). Remember that the holomorphic part of the four point function is given in terms of the Hypergeometric functions $K_1(z)$ and $K_2(z)$ as

$$\lim_{z_3\to\infty} z_3^{2h_3} G^{(4)}_{2\mathrm{d\,CFT}}(\{z_i\}) = z^{\beta_1}(1-z)^{\beta_2}\big(a_1K_1(z) + a_2K_2(z)\big), \qquad (7.4.44)$$

where $a_i$ are coefficients. Hence for small $z$, we have two channels contributing to the four point function, namely, one is $z^{\beta_1}K_1(z) \sim z^{\beta_1}$ and the other is $z^{\beta_1}K_2(z) \sim z^{\beta_1-\gamma+1} = z^{-\beta_1-\kappa+1}$. In terms of the intermediate states of (7.4.43), this means there are two intermediate conformal families. The conformal dimensions of the primary fields of these families are given by $h_{p_+} = h_0 + h_1 + \beta_1$ and $h_{p_-} = h_0 + h_1 - \beta_1 - \kappa + 1$. Now we may notice that these two exponents $\beta_1$ and $-\beta_1 - \kappa + 1$ are the two solutions of (7.2.25), which means that $h_{p_\pm}$ are indeed the conformal dimensions which are allowed from the relativistic version of the fusion rule, namely, $h_{p_\pm} = h_{r_1\pm 1}$ (or $h_{p_\pm} = h_{r_1\mp 1}$ depending on the choice of the branch of the quadratic equation for $\beta_i$).[5]

Next let us study the GCA four point function. In Sec. 7.1, we have seen that the GCA four point function is composed from, again, two solutions of the differential equation, which we call $H_\pm(t,x)$. The solution is given by (7.1.9), with $\mathcal{H}_\pm(t)$ satisfying the differential equation (7.1.11). In fact, the small $t$ behaviour of the four point function can be derived by using Eqs.(7.1.9)–(7.1.11) as

$$\begin{aligned} H_\pm(t,x) &\sim t^{-\kappa^{(0)}+1}\cdot t^{\pm\frac{\mathcal{C}_1}{r_1}}\cdot \exp\Big(\frac{x}{t}C_2(-1\pm r_1)\Big) \\ &= t^{\Delta_{r_1\pm1(s_1,s_1')}-\Delta_0-\Delta_1}\exp\Big(-(\xi_{r_1\pm1}-\xi_0-\xi_1)\frac{x}{t}\Big). \end{aligned} \qquad (7.4.45)$$

These are indeed the two leading behaviours which appear for the operator product expansion of the primary fields $\phi_{2(1,1)}$ and $\phi_{r_1(s_1,s_1')}$ in GCA. (See Sec. 6.3 for a preliminary examination of the GCA OPE.) Now it is clear from (7.1.11) and (7.1.13), that the other ordering, i.e., first taking the OPE between $\phi_0(z)$ and $\phi_2(1)$, gives the two intermediate states which are again allowed from the fusion rule.

[5]In 2d CFT, the differential equations for the three point function $\langle\phi_{2,(1,1)}\phi_{r_1(s_1,s_1')}\phi_{r_p(s_p,s_p')}\rangle$ give rise to the following constraint: $2(2h_{2,1}+1)(h_{2,1}+2h_{r_p,s_p}-h_{r_1,s_1}) = 3(h_{2,1}-h_{r_1,s_1}+h_{r_p,s_p})(h_{2,1}-h_{r_1,s_1}+h_{r_p,s_p}+1)$ and the same for the anti-holomorphic part. This is the same equation as (7.2.25).

# Appendix: Asymptotic Form of the Hypergeometric Function

## A. Saddle Point Analysis

In this section we perform the saddle point analysis of the integral formula for the Hypergeometric function:

$$F(\alpha,\beta,\gamma,z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 dww^{\beta-1}(1-w)^{\gamma-\beta-1}(1-zw)^{-\alpha}. \tag{7.4.46}$$

This formula is valid for the parameter $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0$, and we will use it for $\mathrm{Re}(z)<0$. The parameters $\alpha$, $\beta$ and $\gamma$ are related to the physical parameters as

$$\alpha+\beta+1=2(\beta_1+\beta_2+\kappa),\quad \gamma=2\beta_1+\kappa, \tag{7.4.47}$$

$$\alpha\beta=\beta_1(\beta_1-1)+2\beta_1\beta_2+\beta_2(\beta_2-1)+\kappa(2\beta_1+2\beta_2+h_0-h_3)\,, \tag{7.4.48}$$

with $\kappa=(2/3)(2h_0+1)$ and $\beta_i$ defined as solutions of the equations

$$\beta_i(\beta_i-1)+\kappa\beta_i-\kappa h_i=0,\quad (i=1,2). \tag{7.4.49}$$

We expand parameters as

$$\beta_i=\frac{1}{\epsilon}\beta_i^{(-1)}+\beta_i^{(0)}+\cdots\,,\quad \alpha=\frac{1}{\epsilon}\alpha^{(-1)}+\alpha^{(0)}+\cdots\,, \tag{7.4.50}$$

etc.. Then the quadratic equations for $\beta_i$ are solved as

$$\beta_i^{(-1)}=-\frac{1}{2}C_2(1+\delta_i r_i),\quad \beta_i^{(0)}=-\frac{\delta_i}{2}\ell_i-\frac{1}{2}(\kappa^{(0)}-1), \tag{7.4.51}$$

with

$$\begin{aligned}\ell_i &= \frac{(\kappa^{(0)}-1)\kappa^{(-1)}+2\kappa^{(-1)}h_i^{(0)}+2\kappa^{(0)}h_i^{(-1)}}{C_2 r_i}\\ &= \frac{1}{r_i}\left\{\frac{1}{3}\Delta_0-\frac{2}{3}+2h_i^{(0)}+\frac{1}{3}\Delta_0 r_i^2+\frac{1}{3}r_i^2\right\}.\end{aligned} \tag{7.4.52}$$

Since $\beta_i$ are auxiliary parameters introduced in order to change the form of the differential equation (see the explanation around (7.2.25)), we take $(\delta_1,\delta_2)=\delta_{C_2}(-1,+1)$ without loss of generality, where $\delta_{C_2}$ is the sign of $C_2$. Also, since the Hypergeometric function is symmetric under the exchange $\alpha\leftrightarrow\beta$, we take $\alpha^{(-1)}<\beta^{(-1)}$, then $\alpha$, $\beta$ and $\gamma$ are expanded as

$$\begin{aligned}&\alpha^{(-1)}=\frac{|C_2|}{2}(r_1-r_2-r_3),\qquad \beta^{(-1)}=\frac{|C_2|}{2}(r_1-r_2+r_3),\qquad \gamma^{(-1)}=|C_2|r_1,\\ &\alpha^{(0)}=\frac{\delta_{C_2}}{2}(\ell_1-\ell_2-\ell_3)+\frac{1}{2},\quad \beta^{(0)}=\frac{\delta_{C_2}}{2}(\ell_1-\ell_2+\ell_3)+\frac{1}{2},\quad \gamma^{(0)}=\delta_{C_2}\ell_1+1.\end{aligned} \tag{7.4.53}$$

Here $\ell_3$ is defined by same way as (7.4.52).

Now the integral (7.4.46) is written as

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 dw e^{\frac{1}{\epsilon}g^{(-1)}(w,t)+x\partial_t g^{(-1)}(w,t)+g^{(0)}(w,t)+\mathcal{O}(\epsilon)}, \tag{7.4.54}$$

where $g^{(n)}(w,z)$ are defined as the expansion of the exponent of the integrand as

$$(\beta-1)\log w + (\gamma-\beta-1)\log(1-w) - \alpha\log(1-zw) = \sum_n \epsilon^n g^{(n)}(w,z). \tag{7.4.55}$$

In (7.4.54), we have further expanded based on $z = t+\epsilon x$. The saddle points of the integrand are given by $\partial_w g^{(-1)}(w_*,t) = 0$ and they are located at

$$w_*^{\pm} = \frac{\gamma^{(-1)} - (\alpha^{(-1)}-\beta^{(-1)})t \mp |C_2|\sqrt{D(t)}}{2(\gamma^{(-1)}-\alpha^{(-1)})t}. \tag{7.4.56}$$

By using the fact that $r_i$'s satisfy the triangle inequality, which follows from the fusion rule (see the next section), we can show that both of the saddle points are on the real axis, namely $D(t)\geq 0$. In fact, for the parameterization explained above, only one ($w = w_*^+$) of the saddle points is located on the segment $0<w<1$, and the integral is evaluated by taking it. By taking the saddle point value and doing the Gaussian integral, we obtain the following asymptotic form of the Hypergeometric function for small $\epsilon$:

$$F(\alpha,\beta,\gamma,z) \to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\gamma-\alpha)}\mathcal{K}_+(t,x), \quad (t<1), \tag{7.4.57}$$

where the function $\mathcal{K}_+$ is defined by

$$\begin{aligned}
\mathcal{K}_+(t,x) &= |C_2|^{-\frac{1}{2}}\big(D(t)\big)^{-\frac{1}{4}}\left\{\frac{r_1+r_2+r_3}{r_1+r_2-r_3}\frac{-r_1+r_3t+\sqrt{D(t)}}{r_1+r_3t+\sqrt{D(t)}}\frac{1}{t}\right\}^{\frac{1}{\epsilon}\frac{|C_2|}{2}r_1+\frac{\delta_{C_2}}{2}\ell_1} \\
&\quad\times\left\{\frac{r_1+r_2+r_3}{r_1+r_2-r_3}\frac{r_2-r_3(1-t)+\sqrt{D(t)}}{r_2+r_3(1-t)-\sqrt{D(t)}}(1-t)\right\}^{\frac{1}{\epsilon}\frac{|C_2|}{2}r_2+\frac{\delta_{C_2}}{2}\ell_2} \\
&\quad\times\left\{\frac{r_1+r_2-r_3}{-r_1+r_2+r_3}\frac{r_2+r_3(1-t)+\sqrt{D(t)}}{r_2-r_3(1-t)+\sqrt{D(t)}}\right\}^{\frac{1}{\epsilon}\frac{|C_2|}{2}r_3+\frac{\delta_{C_2}}{2}\ell_3} \\
&\quad\times\exp\left\{\frac{x|C_2|}{2t(1-t)}\big(-r_1(1-t)-r_2t+\sqrt{D(t)}\big)\right\} \qquad (7.4.58)\\
&= K_+(t)\exp\left\{\frac{x|C_2|}{2t(1-t)}\big(-r_1(1-t)-r_2t+\sqrt{D(t)}\big)\right\}. \qquad (7.4.59)
\end{aligned}$$

The function $K_+(t) = \mathcal{K}_+(t,0)$ is introduced for convenience. The $x$-dependence appears only in the exponential form, which can be derived by expanding $K_+(z=t+\epsilon x)$ with respect to $\epsilon$, as it can be seen from (7.4.54). The over all factor $\epsilon^{\frac{1}{2}}$ on the RHS of (7.4.57) is canceled by the factor $\epsilon^{-\frac{1}{2}}$ when we expand the Gamma functions.

## B. Fusion Rule and Relation Between $r_i(s_i, s'_i)$'s

In the main text and this appendix, relations between three primary fields $\phi_1$, $\phi_2$ and $\phi_3$ in the four point function $\langle\phi_0\phi_1\phi_2\phi_3\rangle$ with $\phi_0 = \phi_{2(1,1)}$ are used in several places. These relations are derived from the generic fusion rule in the relativistic 2d CFT:

$$[\phi_{r_i(s_i,s'_i)}] \times [\phi_{r_j(s_j,s'_j)}] = \sum_{r_p=|r_i-r_j|+1}^{r_i+r_j-1} \sum_{s_p=|s_i-s_j|+1}^{s_i+s_j-1} \sum_{s'_p=|s'_i-s'_j|+1}^{s'_i+s'_j-1} [\phi_{r_p(s_p,s'_p)}]. \tag{7.4.60}$$

The variables $r_p$, $s_p$ and $s'_p$ are incremented by 2. Let us discuss only the channel in which we take the OPE between $\phi_0$ and $\phi_1$ first, then we take the OPE between the intermediate field $\phi_p$ and $\phi_2$ to give the remaining field $\phi_3$. Symbolically, we consider the order $\langle\phi_0\phi_1\phi_2\phi_3\rangle = \Sigma_p C_{01p}\langle\phi_p\phi_2\phi_3\rangle = \Sigma_{p,q} C_{01p}C_{p2q}\langle\phi_q\phi_3\rangle$. Here in the last step, because of the orthogonality of the conformal families, we have $r_q(s_q, s'_q) = r_3(s_3, s'_3)$. The other cases can be discussed similarly.

In this case, the possible intermediate state is given by $r_p(s_p, s'_p) = (r_1 \pm 1)(s_1, s'_1)$ and the fusion rule for the OPE between $[\phi_p]$ and $\phi_2$ is given by

$$r_3 = |r_1 \pm 1 - r_2| + 1,\ |r_1 \pm 1 - r_2| + 3,\ \cdots,\ r_1 \pm 1 + r_2 - 1\,, \tag{7.4.61}$$

$$s_3 = |s_1 - s_2| + 1,\ |s_1 - s_2| + 3,\ \cdots,\ s_1 + s_2 - 1\,, \tag{7.4.62}$$

$$s'_3 = |s'_1 - s'_2| + 1,\ |s'_1 - s'_2| + 3,\ \cdots,\ s'_1 + s'_2 - 1\,. \tag{7.4.63}$$

From the first equation (7.4.61), we see that three $r_i$'s satisfy the triangle inequality, which is important in the saddle point analysis in Appendix A. Also we notice that possible values for the summation of three $r_i$'s are even, while the summation of three $s_i$'s and $s'_i$'s are always odd numbers. It is consistent with the property that $s'_i - s_i$ are even integers (see Sec. 7.3).

## C. Validity of the Analysis

Let us now clarify the validity of the analysis in appendix 7A. As we mentioned, the integral formula (7.4.46) is valid for the parameter satisfying $\gamma > \beta > 0$. From the expansion (7.4.53) with (7.4.52), it is easy to see that for the case with $r_1 - r_2 + r_3 = 0$ or $r_1 + r_2 - r_3 = 0$, namely, $\beta_{-1} = 0$ or $\gamma_{-1} = \beta_{-1}$, the condition is violated for general $s_i$'s. Hence for these cases, the simple integral formula (7.4.46) is not available and we need to consider more generic formula, which we do not pursuit presently. Examples of such situations include the case when one of the fields $\phi_{1,2,3}$ is the identity operator. On the other hand, in the case with $r_1 - r_2 - r_3 = 0$, the above analysis itself is valid. However, for deriving the asymptotic form of the rest of the region $1 < t$ and also the other basis function in the next subsection, the case $r_1 - r_2 - r_3 = 0$ violates the conditions which are equivalent to the ones explained above. Hence our derivation of the asymptotic form of the four point function is applicable only for the cases with $r_1 < r_2 + r_3$, $r_2 < r_1 + r_3$ and $r_3 < r_1 + r_2$, i.e. the strict triangle inequality is satisfied.

The validity of the saddle point analysis also requires the second derivative

$$\partial_w^2 g^{(-1)}(w_*^+, t) = -\frac{|C_2|\sqrt{D}}{(1-tw_*^+)^2}\frac{(2\alpha_{-1}\beta_{-1} - \gamma_{-1}(\alpha_{-1}+\beta_{-1}))t + \gamma_{-1}^2 + \gamma_{-1}|C_2|\sqrt{D}}{2\beta_{-1}(\gamma_{-1}-\beta_{-1})}, \tag{7.4.64}$$

to be negative definite (with large $1/\epsilon$ in front of it). This condition is perfectly satisfied with the above assumption for the parameters.

## D. A Complete Set of the Solutions and Useful Formulae

Next we derive the other solution $(-z)^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z)$ for the region $t < 0$ by using (7.4.57) and the formula (7.4.76) (formula (I)). By changing the parameters of the formula (I) as $\alpha \to \alpha - \gamma + 1$, $\beta \to \beta - \gamma + 1$ and $\gamma \to 2 - \gamma$, and multiplying the factor $(-z)^{1-\gamma}$ to the both sides of the resulting equation, we obtain the similar formula (formula (II)) for $(-z)^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z)$ expressed in terms of the same two functions on the RHS of (7.4.76). Regarding the second function on the RHS, i.e., $(-1)^{-\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1, 1/z)$, the condition $\alpha < 0 < \beta < \gamma$, which we are now assuming, assures $\alpha' < 0 < \beta' < \gamma'$ with $\alpha' = \beta - \gamma + 1$, $\beta' = \beta$ and $\gamma' = \beta - \alpha + 1$. Then the asymptotic form (7.4.57) is applicable for this case with the parameter change $\alpha \to \alpha'$, $\beta \to \beta'$ and $\gamma \to \gamma'$ along with the coordinate change $z \to 1/z$. Note that the condition $\alpha^{(-1)} = \frac{|C_2|}{2}(r_1 - r_2 - r_3) < 0$ is necessary here in order for the condition $\beta' < \gamma'$ to be satisfied in general, as we mentioned in the previous subsection.

Then by using the two formulae (I) and (II), which are mentioned above, we can write down the asymptotic form of the function $(-z)^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z)$ in terms of the asymptotic forms of the functions $F(\alpha, \beta, \gamma, z)$ and $(-z)^{-\beta}F(\alpha', \beta', \gamma', 1/z)$. The explicit form is given by

$$\begin{aligned}(-z)^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z) \to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}}\bigg\{\frac{\Gamma(\beta)\Gamma(2-\gamma)}{\Gamma(\beta-\gamma+1)}K_+(z) \\ +\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma-1)}(-z)^{-\frac{1}{\epsilon}|C_2|r_1 - \delta_{C_2}\ell_1}(1-z)^{\frac{1}{\epsilon}|C_2|r_2 + \delta_{C_2}\ell_2}K_-(z)\bigg\}.\end{aligned} \tag{7.4.65}$$

Here again $z = t + \epsilon x$ with small $\epsilon$ is assumed. The function $K_-(z)$ is given from $K_+(z)$ by replacing as $1 \leftrightarrow 2$ and $(t, x) \leftrightarrow (1-t, -x)$, or equivalently $z \leftrightarrow 1 - z$.

The asymptotic form of the anti-holomorphic functions for the region $t < 0$ can be derived in the same way. In our parameter choices (see the explanation after (7.2.35) for the parameter choice of the anti-holomorphic part), the parameters in the anti-holomorphic part satisfy $\bar{\gamma} < \bar{\beta} < 0 < \bar{\alpha}$. By finding appropriate basis functions to which the asymptotic form (7.4.57) is applicable, and also by using the formula (7.4.76), we obtain the following result:

$$F(\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{z}) \to (2\pi\epsilon)^{\frac{1}{2}}\Bigg\{\frac{\Gamma(\bar{\alpha}-\bar{\gamma}+1)}{\Gamma(1-\bar{\gamma})\Gamma(\bar{\alpha})}\bar{K}_{+}(\bar{z})$$
$$+\frac{\Gamma(\bar{\alpha}-\bar{\gamma}+1)\Gamma(\bar{\gamma})}{\Gamma(\bar{\alpha})\Gamma(\bar{\gamma}-\bar{\beta})\Gamma(\bar{\beta}-\bar{\gamma}+1)}(-\bar{z})^{\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\bar{\ell}_1}(1-\bar{z})^{-\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\bar{\ell}_2}\bar{K}_{-}(\bar{z})\Bigg\}, \quad (7.4.66)$$

$$(-\bar{z})^{1-\bar{\gamma}}F(\bar{\alpha}-\bar{\gamma}+1,\bar{\beta}-\bar{\gamma}+1,2-\gamma,\bar{z})$$
$$\to (2\pi\epsilon)^{\frac{1}{2}}\frac{\Gamma(2-\bar{\gamma})}{\Gamma(1-\bar{\beta})\Gamma(\bar{\beta}-\bar{\gamma}+1)}(-\bar{z})^{\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\bar{\ell}_1}(1-\bar{z})^{-\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\bar{\ell}_2}\bar{K}_{-}(\bar{z}), \quad (7.4.67)$$

where $\bar{K}_{\pm}(\bar{z})$ are given from $K_{\pm}(z)$ with replacements $\epsilon \to -\epsilon$ and $\ell_i \to \bar{\ell}_i$. In the main text, we use Eqs.(7.4.65), (7.4.66) and (7.4.67) for deriving the asymptotic form of the four point function in the region $t < 0$.

In fact, by means of (7.4.57), we can derive the asymptotic forms of sets of two independent solutions for the remaining segments. Here is the list of the asymptotic forms:

1. The functions whose asymptotic forms are given in terms of $K_{+}(z)$:

- $t < 1$

$$F(\alpha,\beta,\gamma,z) \to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\gamma-\alpha)}K_{+}(z) \quad (7.4.68)$$

- $1 < t$

$$z^{-\beta}F(\beta-\gamma+1,\beta,\beta-\alpha+1,1/z) \to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\beta-\alpha+1)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha)}K_{+}(z) \quad (7.4.69)$$

2. The functions whose asymptotic forms are given in terms of $K_{-}(z)$:

- $t < 0$

$$(-z)^{-\beta}F(\beta-\gamma+1,\beta,\beta-\alpha+1,1/z)$$
$$\to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\beta-\alpha+1)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha)}(-z)^{-\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\ell_1}(1-z)^{\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\ell_2}K_{-}(z) \quad (7.4.70)$$

- $0 < t$

$$z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}F(1-\beta,1-\alpha,\gamma-\alpha-\beta+1,1-z)$$
$$\to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\gamma-\alpha-\beta+1)\Gamma(\beta)}{\Gamma(\gamma-\alpha)}z^{-\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\ell_1}(1-z)^{\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\ell_2}K_{-}(z) \quad (7.4.71)$$

3. The functions whose asymptotic forms are given in terms of $\bar{K}_+(\bar{z})$:

   - $t<0$

$$(-\bar{z})^{\bar{\beta}-\bar{\gamma}}(1-\bar{z})^{\bar{\gamma}-\bar{\alpha}-\bar{\beta}}F(\bar{\gamma}-\bar{\beta},1-\bar{\beta},\bar{\alpha}-\bar{\beta}+1,1/\bar{z}) \\ \to (2\pi\epsilon)^{\frac{1}{2}}\tfrac{\Gamma(\bar{\alpha}-\bar{\beta}+1)}{\Gamma(1-\bar{\beta})\Gamma(\bar{\alpha})}\bar{K}_+(\bar{z}) \tag{7.4.72}$$

   - $0<t$

$$F(\bar{\beta},\bar{\alpha},\bar{\alpha}+\bar{\beta}-\bar{\gamma}+1,1-\bar{z}) \to (2\pi\epsilon)^{\frac{1}{2}}\frac{\Gamma(\bar{\alpha}+\bar{\beta}-\bar{\gamma}+1)}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta}-\bar{\gamma}+1)}\bar{K}_+(\bar{z}) \tag{7.4.73}$$

4. The functions whose asymptotic forms are given in terms of $\bar{K}_-(\bar{z})$:

   - $t<1$

$$\bar{z}^{1-\bar{\gamma}}(1-\bar{z})^{\bar{\gamma}-\bar{\alpha}-\bar{\beta}}F(1-\bar{\alpha},1-\bar{\beta},2-\bar{\gamma},\bar{z}) \\ \to (2\pi\epsilon)^{\frac{1}{2}}\tfrac{\Gamma(2-\bar{\gamma})}{\Gamma(1-\bar{\beta})\Gamma(\bar{\beta}-\bar{\gamma}+1)}\bar{z}^{\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\bar{\ell}_1}(1-\bar{z})^{-\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\bar{\ell}_2}\bar{K}_-(\bar{z}) \tag{7.4.74}$$

   - $1<t$

$$\bar{z}^{\bar{\beta}-\bar{\gamma}}(1-\bar{z})^{\bar{\gamma}-\bar{\alpha}-\bar{\beta}}F(\bar{\gamma}-\bar{\beta},1-\bar{\beta},\bar{\alpha}-\bar{\beta}+1,1/\bar{z}) \\ \to (2\pi\epsilon)^{\frac{1}{2}}\tfrac{\Gamma(\bar{\alpha}-\bar{\beta}+1)}{\Gamma(\bar{\alpha})\Gamma(1-\bar{\beta})}\bar{z}^{\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\bar{\ell}_1}(1-\bar{z})^{-\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\bar{\ell}_2}\bar{K}_-(\bar{z}) \tag{7.4.75}$$

The functions on the LHS's are the solutions of the same Hypergeometric differential equation (7.2.26). This basis cover all the segments $t<0$, $0<t<1$ and $1<t$. Note that these functions satisfy the parameter condition which is required for the application of (7.4.57). On the RHS's, $z=t+\epsilon x$ and $\bar{z}=t-\epsilon x$ with the small parameter $\epsilon$ are assumed. Further expansion with respect to this $\epsilon$ gives the $x$-dependent exponential factors.

Finally, we summarize some useful formulae:

$$F(\alpha,\beta,\gamma,z) = \tfrac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1,1/z) \\ +\tfrac{\Gamma(\alpha-\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1,1/z), \tag{7.4.76}$$

$$= \tfrac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z) \\ +\tfrac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta-\gamma+1,1-z). \tag{7.4.77}$$

The functions appearing in (7.4.76) and (7.4.77) satisfy the Hypergeometric differential equation independently. The first form (7.4.76) is valid for any $z$ except for the positive real number, while the second (7.4.77) is valid for any $z$ except on the segment $[0,1]$ of the real axis. By using the following formula:

$$F(\alpha,\beta,\gamma,z) = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma,z), \tag{7.4.78}$$

the functions on the LHS of (7.4.68)–(7.4.75) can be converted into the functions appearing in (7.4.76) and (7.4.77) or $z^{1-z}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,z)$.

# Part III

# The Non-Relativistic Bulk

# Chapter 8

# Bulk Theory

We have looked at the Galilean Conformal Algebra as the symmetry of the boundary field theory. Now, in this chapter, we turn our attention to what the theory in the bulk should look like. We deal at length with our proposal of the gravity dual to systems with Galilean Conformal Algebra. First, we review the Newton-Cartan theory of gravity and then speak about our proposed modification for the Anti de Sitter spacetimes. We go on to describe how the GCA is realized in the bulk by contracting the Killing vectors of AdS. The infinite extension is then discussed and we make precise how the GCA is naturally realized as the asymptotic symmetry algebra of the proposed Newton-Cartan AdS. Near the end of the chapter, we look briefly at the bulk dual to the Schrödinger algebra and talk of the very different nature of the two bulk theories. The chapter concludes with an appendix on Galilean Isometries.

## 8.1 Newton-Cartan Theory of Gravity

Given a particular instance of an AdS/CFT duality, we should be able parametrically to take the non-relativistic scaling limit on both sides of the duality. On the field theory side, as we have discussed, the relativistic conformal invariance reduces to the GCA with a possible infinite dimensional dynamical extension. On the string theory side it should be possible to take a similar scaling limit along the lines of the non-relativistic limit studied in [62, 63, 64]. Below we will only consider features of this scaling limit when the parent bulk theory is well described by gravity. This will already involve some novel features. This has to with the fact that the usual pseudo-Riemannian metric degenerates when one takes a non-relativistic limit. Nevertheless, there is a well defined, albeit somewhat unfamiliar, geometric description of gravity in such a limit [71]. In the (asymptotically) flat space case this is known as the Newton-Cartan theory of gravity which captures Newtonian gravity in a geometric setting. This is a *non-metric* gravitational theory. The dynamical variables are affine connections. Einstein's equations reduce to equations for the curvature of these non-metric connections. One can generalise this to the case of a negative cosmological constant as well. A variant of this is what we propose below as the right framework for the gravity dual of systems with the GCA. In the next subsection we will briefly review features of the Newton-Cartan theory and then go onto describe the case with a negative cosmological constant.

### 8.1.1 A Quick Review of Newton-Cartan Theory

In the Newton-Cartan description of gravity, the $(d+1)$ dimensional spacetime $M$ has a time function $t$ on it which foliates the spacetime into $d$ dimensional spatial slices. Stated more precisely (see for example [72]): one defines a contravariant tensor $\gamma = \gamma^{\mu\nu}\partial_\mu \otimes \partial_\nu$ $(\mu, \nu = 0 \dots d)$ such there is a time 1-form $\tau = \tau_\mu dx^\mu$ which is orthogonal to $\gamma$ in the sense that $\gamma^{\mu\nu}\tau_\mu = 0$. The metric $\gamma$, which has three positive eigenvalues and one zero eigenvalue, will be the non-dynamical spatial metric on slices orthogonal to the worldlines defined by $\tau$. There is no metric on the spacetime as a whole. In fact, its geometric structure is that of a fibre bundle with a one dimensional base (time) and the $d$ dimensional spatial slices as fibres.

The dynamics is encoded in a torsion free affine connection $\Gamma^\mu_{\nu\lambda}$ on $M$. We will demand that this connection is compatible with both $\gamma$ and $\tau$ i.e.

$$\nabla_\rho \gamma^{\mu\nu} = 0 \qquad \nabla_\rho \tau_\nu = 0. \tag{8.1.1}$$

This enables one to define a time function $t$ ("absolute time") since we have $\tau_\mu = \nabla_\mu t$. Unlike the Christoffel connections which are determined by the spacetime metric in Einstein's theory, this Newton-Cartan connection is not fixed by just these conditions. One has to impose some additional relations. Defining $R^{\mu\nu}_{\lambda\sigma} = \gamma^{\nu\alpha} R^\mu_{\lambda\alpha\sigma}$, one can define a Newtonian connection as one which obeys the additional condition $R^{\mu\nu}_{\lambda\sigma} = R^{\nu\mu}_{\sigma\lambda}$.[1]

In the presence of matter sources specified by a contravariant second rank stress tensor $T^{\mu\nu}$, which is additionally required to be covariantly conserved $\nabla_\mu T^{\mu\nu} = 0$, we can write down the field equations which determine the connection in terms of the sources. This is best done by introducing a "time" like metric $g_{\mu\nu} = \tau_\mu \tau_\nu$ which is orthogonal to the spatial metric $\gamma_{\mu\nu}$. The field equations are then familiar in form

$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) \tag{8.1.2}$$

where $T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}$ and $T = g_{\alpha\beta} T^{\alpha\beta}$. Note that to define the Ricci tensor $R_{\mu\nu}$ (unlike the Ricci scalar $R$) one does not need a metric, only the affine connection.

When one chooses coordinates such that $\gamma = \delta^{ij}\partial_i \otimes \partial_j$ $(i, j = 1 \dots d)$, $\tau = dt$, the non-zero components of the Newtonian connection take the form (imposing appropriate

[1] In Einstein gravity with the Christoffel connection and $\gamma$ being the *nondegenerate* spacetime metric, this relation is identically satisfied but here it has to be imposed additionally.

boundary conditions at infinity) $\Gamma^i_{00} = \partial_i \Phi$. The field equations then reduce to Poisson's equations with $\Phi$ being the Newtonian gravitational potential and the source being the matter density $\rho = T^{00}$.

This is, of course, an intrinsic characterisation of Newtonian gravity. Not unsurprisingly, this geometric structure has also been shown to arise in the degenerating limit of a usual Einsteinian geometry [65]. Namely, one can study a one parameter ($\epsilon$) family of usual Lorentzian signature metrics, with the non-relativistic limit $\epsilon \to 0$ leading to a degenerate metric. The condition that this limiting geometry be a Newtonian spacetime is satisfied under fairly mild conditions on the $\epsilon$ dependence of the Lorentzian metric (and therefore the associated geometric objects such as the covariant derivative, curvature tensor etc.). This shows that the nonrelativistic scaling limit is a sensible one to take of a generic Einsteinian geometry.

### 8.1.2 Newtonian limit of Gravity on $AdS_{d+2}$

We would like to parametrically carry out the non-relativistic scaling on the bulk $AdS_{d+2}$ which would capture the physics of the nonrelativistic limit in the $(d+1)$ dimensional boundary theory. In the next section we will describe the bulk scaling in more detail. Here we will simply motivate its qualitative features and give the resulting Newton-Cartan like description of the bulk geometry.

We know that the boundary metric degenerates in the nonrelativistic limit with the $d$ spatial directions scaling as $x_i \propto \epsilon$ while $t \propto \epsilon^0$. We expect this feature to be shared by the bulk metric. One expects that the geometry on constant radial sections to have such a scaling. Since the radial direction of the $AdS_{d+2}$ is an additional dimension, we have to fix its scaling. The radial direction is a measure of the energy scales in the boundary theory via the holographic correspondence. We therefore expect it to also scale like time i.e. as $\epsilon^0$. This means that in the bulk the time and radial directions of the metric *both* survive when performing the scaling. Together these constitute an $AdS_2$ sitting inside the original $AdS_{d+2}$.

What this implies for the dynamics is that we should have a Newton-Cartan like description but with the special role of time being replaced by an $AdS_2$. The geometric structure, in analogy with that of the previous section, is that of a fibre bundle with $AdS_2$ base and the $d$ dimensional spatial slices as fibres.

Accordingly, we will consider a ("spatial") metric $\gamma = \gamma^{\mu\nu}\partial_\mu \otimes \partial_\nu$ $(\mu, \nu = 0 \ldots d+1)$ which now has *two* zero eigenvalues corresponding to the time and radial directions. (In a canonical choice of coordinates these directions will correspond to $\mu = 0, d+1$). Mathematically the two null eigenvectors will be taken to span the space of left invariant 1-forms of $AdS_2$. These will also define the $AdS_2$ metric $g_{\alpha\beta}$ in the usual way (This is the analogue of the time metric defined in the previous subsection).

We will once again have dynamical, torsion free affine connections $\Gamma^\mu_{\nu\lambda}$ which are compatible with both the spatial and $AdS_2$ metrics

$$\nabla_\rho \gamma^{\mu\nu} = 0 \qquad \nabla_\rho g_{\alpha\beta} = 0. \tag{8.1.3}$$

There will also be Christoffel connections from the $AdS_2$ and spatial metrics which will not be dynamical if we do not allow these metrics, specifically $g_{\mu\nu}$, to fluctuate. We will also impose the condition below (8.1.1) on the Riemann tensor.

In standard Poincare coordinates where $\gamma = z^2\delta^{ij}\partial_i \otimes \partial_j$ $(i, j = 1 \ldots d)$ and $g_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{z^2}(dt^2 - dz^2)$, the non-zero components of the dynamical connection can be taken to be $\Gamma^i_{ab}(t, z, x_i) = \partial_i \Phi_{ab}(t, z, x_i)$ (with $a, b = 0, d+1$). There will be Christoffel components from $\gamma$ and $g$ as mentioned above.

The field equations are the expected modification of (8.1.2)

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{8.1.4}$$

where $T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}$ and $T = g_{\alpha\beta}T^{\alpha\beta}$ and $\Lambda$ is the cosmological constant. These are dynamical equations for the fields $\Phi_{ab}(t, z, x_i)$ once the stress tensor $T^{ab}(t, z, x_i)$ in the $AdS_2$ directions is specified.

### 8.1.3 Non-Relativistic holography and $AdS_2/CFT_1$

The non-relativistic limit of the bulk theory in $AdS_{d+2}$ described above singles out an $AdS_2$ when we look at the non-relativistic scaling in the Poincare patch. The $AdS_2$ describes the base manifold of the fibre bundle structure with $R^d$ fibres on it. We have also seen that the field equations, in the same limit, reduces to Einstein's equations on $AdS_2$. It is natural then to expect that the bulk-boundary dictionary in this case is some sort of an $AdS_2/CFT_1$, keeping in mind that the boundary field theory has one copy of an infinite dimensional Virasoro algebra.

Similar $AdS_2$ factors have also been noted while realising the Schrödinger Algebra as asymptotic isometries in [75]. So, it is plausible that all non-relativistic limits would isolate some sort of an $AdS_2/CFT_1$ duality.

We looked at correlation functions of these non-relativistic theories in Chapter 4. In the correlation functions obtained for the GCA and those of the Schrödinger algebra there is further evidence of this $AdS_2/CFT_1$. Let us explain how.

We have seen that both the two and three point function in the cases of the GCA and the Schrödinger algebra contains pieces which are reminiscent of the parent relativistic theory, but those pieces have been the pieces where only the time co-ordinate was present. The $AdS_2/CFT_1$ that we talk about separates out the radial $z$ and the temporal $t$ direction in the bulk. This time co-ordinate would be the one which leads to the $CFT_1$. What we get from the field theory side is precisely this conformal behaviour of the time piece.

But, there are hints that this more more than just an $AdS_2/CFT_1$. We saw that the correlators in the two theories differed in the structure of the exponential pieces which multiplied the purely temporal pieces. The exponential which contains both the time and space dependent terms probably arises out of the complicated fibre-bundle structure. The two non-relativistic theories would be realised as different fibre structures in the bulk and possibly, this is why these differences arise. However, if we restrict our attention to the sector where the boost eigenvalues $\xi_i$ of the operators are zero, we will avoid the complications of the fibre bundle and as shown in [77], the usual techniques of AdS/CFT can be applied and we, in that case, do end up with a non-relativistic $AdS_2/CFT_1$ correspondence starting out from a relativistic $AdS_{d+2}/CFT_{d+1}$.

## 8.2 GCA in the Bulk

In this section we will carry out the non-relativistic scaling limit on the $AdS_5$ piece of the bulk. We will also do this for the $SO(4,2)$ isometries of $AdS_5$ and obtain the same contracted algebra as in Sec.3[2]. We will then see, in the next section, how the infinite dimensional extension of this algebra is realised in the bulk. They will have the interpretation as being the generators of asymptotic isometries of the bulk Newton-Cartan like geometry described in the previous section. Since asymptotic isometries, under appropriate circumstances, act on the physical Hilbert space of the theory, one finds support for the assertion that the infinite extension can be dynamically realised.

Consider the metric of $AdS_5$ in Poincare coordinates

$$ds^2 = \frac{1}{z'^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz'^2) = \frac{1}{z'^2}(dt'^2 - dz'^2 - dx_i^2) \qquad (8.2.5)$$

The nonrelativistic scaling limit that we will be considering is, as motivated in the previous section

$$t', z' \to \epsilon^0 t', \epsilon^0 z' \qquad x_i \to \epsilon^1 x_i. \qquad (8.2.6)$$

In this limit we see that only the components of the metric in the $(t', z')$ directions survive to give the metric on an $AdS_2$. The $d$ dimensional spatial slices parametrised by the $x_i$ are fibred over this $AdS_2$. The Poincare patch has a horizon at $z' = \infty$ and to extend the coordinates beyond this we will choose to follow an infalling null geodesic,

[2]As mentioned earlier, an algebraically equivalent contraction of the bulk isometries was carried out in [64](see also [61]). However the actual embedding of this contracted algebra in $AdS$ is not manifestly that of a nonrelativistic CFT on the boundary. In particular, their foliation of the bulk corresponds to a boundary geometry which is a time dependent $AdS_2 \times S^2$. This is natural from the point of view of considering the worldvolume of a half BPS string. By considering Poincare coordinates and foliating the bulk in terms of $R^{3,1}$ slices, it is plausible that their limit will be related directly to ours. We thank Jaume Gomis for helpful communication in this regard.

in an analogue of the Eddington-Finkelstein coordinates. Therefore define $z = z'$ and $t = t' + z'$. In these coordinates

$$ds^2 = \frac{1}{z^2}(-2dtdz + dt^2) = \frac{dt}{z^2}(dt - 2dz). \tag{8.2.7}$$

### 8.2.1 Killing vectors of $AdS_{d+2}$ and Bulk Contraction

Here we list the killing vectors of $AdS_{d+2}$ in the $d+3$ dimensional Minkowskian embedding space and then rewrite them in intrinsic AdS coordinates. For taking the contraction, we will work in Poincare coordinates.

We denote flat $d+1$-dimensional space with co-ordinates $y_a$ with $a = 1, .., d+1$. Embedding equation:

$$uv + \eta_{ab}y^a y^b = 1, \quad ds^2 = dudv + \eta_{ab}dy^a dy^b \tag{8.2.8}$$

where $\eta_{ab}$ has a form $diag(1, -1, -1, \ldots - 1)$. Also note that we have rescaled the $AdS_{d+2}$ radius to 1.

We wish to write everything in explicit $SO(d+1, 2)$ notation. So we choose $u = y_0 + y_{d+2}$ and $v = y_0 - y_{d+2}$. Now we would be interested in the co-ordinates on the Poincare patch.

$$\begin{aligned} z &= v^{-1} = (y_0 - y_{d+2})^{-1} \\ t &= v^{-1}y_1 = (y_0 - y_{d+2})^{-1}y_1 \\ x_i &= v^{-1}y_i = (y_0 - y_{d+2})^{-1}y_i \end{aligned} \tag{8.2.9}$$

where we label $i = 2, 3 \ldots d+1$.

The constraint equation becomes:

$$u = z - (z)^{-1}(t^2 - x_i^2) \tag{8.2.10}$$

which gives the metric on the Poincare patch (8.2.5).

We can take the inverse transformations and express the derivatives of the Poincare co-ordinates in terms of derivatives of $y$'s and take various linear combinations to obtain

$$\begin{aligned} M_{01} &= -(y_0\partial_1 - y_1\partial_0) = -\frac{1}{2}(z^2 + 1 + t^2 + x_i^2)\frac{\partial}{\partial t} - zt\frac{\partial}{\partial z} - tx_i\frac{\partial}{\partial x_i} \\ M_{0i} &= -(y_0\partial_i + y_i\partial_0) \\ &= -\frac{1}{2}(z^2 + 1 - t^2 + +x_i^2 + x_j^2)\frac{\partial}{\partial x_i} + zx_i\frac{\partial}{\partial z} + tx_i\frac{\partial}{\partial t} + x_i^2\frac{\partial}{\partial x_i} + x_ix_j\frac{\partial}{\partial x_j} \quad (j \neq i) \\ M_{1i} &= -(y_1\partial_i + y_i\partial_1) = -t\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial t} \\ M_{ij} &= -(y_i\partial_j - y_j\partial_i) = -(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}) \end{aligned}$$

$$
\begin{aligned}
M_{0,d+2} &= -(y_0\partial_{d+2} + y_{d+2}\partial_0) = -z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} - x_i\frac{\partial}{\partial x_i} \\
M_{1,d+2} &= -(y_1\partial_{d+2} + y_{d+2}\partial_1) = -\frac{1}{2}(z^2+t^2+x_i^2-1)\frac{\partial}{\partial t} - zt\frac{\partial}{\partial z} - tx_i\frac{\partial}{\partial x_i} \\
M_{i,d+2} &= -(y_{d+2}\partial_i + y_i\partial_{d+2}) \\
&= -\frac{1}{2}(z^2-t^2+x_i^2+x_j^2-1)\frac{\partial}{\partial x_i} + zx_i\frac{\partial}{\partial z} + tx_i\frac{\partial}{\partial t} + x_i^2\frac{\partial}{\partial x_i} + x_ix_j\frac{\partial}{\partial x_j} \quad (j \neq i)
\end{aligned}
$$

In the above equations, the repeated indices $j$ are summed over in $M_{0i}$ and $M_{i,d+2}$.

To connect with our notation for the boundary generators, we define:

$$
\begin{aligned}
H &= M_{01} - M_{1,d+2}; \quad K = M_{01} + M_{1,d+2}; \quad D = M_{0,d+2} \\
P_i &= -M_{0i} + M_{i,d+2}; \quad K_i = M_{0i} + M_{i,d+2}; \quad B_i = -M_{1i} \\
M_{ij} &= M_{ij}
\end{aligned}
\tag{8.2.11}
$$

We would transform these into the infalling E-F co-ordinates that were previously discussed.

### 8.2.2 Contraction of the Bulk Isometries

In the infalling Eddington-Finkelstein coordinates, the Killing vectors of $AdS_5$ read as

$$
\begin{aligned}
P_i &= \partial_i, \ B_i = (t-z)\partial_i - x_i\partial_t \\
K_i &= (t^2 - 2tz - x_j^2)\partial_i + 2tx_i\partial_t + 2zx_i\partial_z + 2x_ix_j\partial_j \\
J_{ij} &= -(x_i\partial_j - x_j\partial_i) \\
H &= -\partial_t, \ D = -t\partial_t - z\partial_z - x_i\partial_i \\
K &= -(t^2 + x_i^2)\partial_t - 2z(t-z)\partial_z - 2(t-z)x_i\partial_i
\end{aligned}
\tag{8.2.12}
$$

Here we have used the same labeling for the bulk generators as on the boundary to facilitate easy comparison.

Carrying out the scaling (8.2.6) we obtain the contracted Killing vectors

$$
\begin{aligned}
P_i &= \partial_i, \ B_i = (t-z)\partial_i, \quad K_i = (t^2-2tz)\partial_i, \quad J_{ij} = -(x_i\partial_j - x_j\partial_i) \\
H &= -\partial_t, \quad D = -t\partial_t - z\partial_z - x_i\partial_i, \quad K = -t^2\partial_t - 2(t-z)(z\partial_z + x_i\partial_i)
\end{aligned}
\tag{8.2.13}
$$

We see that at the boundary $z = 0$ these reduce to the contracted Killing vectors of the relativistic conformal algebra. It can also be checked that these obey the same algebra as (2.3.37) and (2.3.40) or equivalently (3.2.8) after the relabeling of (3.2.7).

The interpretation of most of the generators is straightforward. We note that the $H, K, D$ are scalars under the spatial $SO(d-1)$ and generate, as before, an $SL(2,R)$. We identify this as the isometry group of the $AdS_2$ base of our Newton-Cartan theory.

We can again define an infinite family of vector fields in the bulk

$$\begin{aligned} M_i^{(m)} &= (t^{m+1} - (m+1)zt^m)\partial_i \\ J_{ij}^{(n)} &= -t^n(x_i\partial_j - x_j\partial_i) \\ L^{(n)} &= -t^{n+1}\partial_t - (n+1)(t^n - nzt^{n-1})(x_i\partial_i + z\partial_z) \end{aligned} \tag{8.2.14}$$

These vector fields reduce on the boundary to (3.2.9) and (3.2.10).

It is rather remarkable that these vector fields also obey the commutation relations of the Virasoro-Kac-Moody algebra, the same as in the boundary theory

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} \qquad [L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)} \\ [J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \end{aligned} \tag{8.2.15}$$

### 8.2.3 Bulk Vector Fields and Asymptotic Isometries

How do we interpret all these additional vector fields from the point of view of the bulk? Firstly, notice that the vector fields $M_i^{(n)}$ and $J_a^{(n)}$ only act on the spatial coordinates $x_i$ (with $t, z$ dependent coefficients). From the viewpoint of the fibre bundle structure, these are simply rotations and translations on the spatial slices which happen to be dependent on time as well as $z$. These are the isometries of the spatial metric $\gamma$ of the previous section. They are also trivially isometries of the $AdS_2$ metric since they do not act on those coordinates. In general, these transformations will have a non-trivial effect on the dynamical connection coefficient (though trivial action on the non-dynamical Christoffel coefficients). This is not unusual since it is only the vacuum configuration of the bulk theory (in which the dynamical connections vanish) which preserves the full symmetry.

Now we come to the action of the Virasoro generators, $L^{(n)}$ which act non-trivially on all coordinates. We have under its action (with infinitesimal parameter $a_n$)

$$\begin{aligned} z \to \tilde{z} &= z[1 + a_n(n+1)(t^n - nzt^{n-1})] \\ t \to \tilde{t} &= t[1 + a_n t^n] \\ x_i \to \tilde{x}_i &= x_i[1 + a_n(n+1)(t^n - nzt^{n-1})]. \end{aligned} \tag{8.2.16}$$

In other words,

$$\begin{aligned} dz \to \tilde{dz} &= dz[1 + a_n(n+1)(t^n - nzt^{n-1})] \\ &+ za_n n(n+1)t^{n-2}[(t-(n-1)z)dt - tdz] \\ dt \to \tilde{dt} &= dt[1 + (n+1)a_n t^n] \\ dx_i \to \tilde{dx}_i &= dx_i[1 + a_n(n+1)(t^n - nzt^{n-1})] \\ &+ n(n+1)a_n x_i t^{n-2}[(t-(n-1)z)dt - tdz]. \end{aligned} \tag{8.2.17}$$

To see how this acts on the Newton-Cartan structure, consider first the above action on the original Poincare metric on $AdS_5$ but transformed to the Eddington-Finkelstein coordinates (8.2.7). Only after that do we take the scaling limit (8.2.6). We find

$$\begin{aligned} ds^2 &= \frac{1}{z^2}(-2dtdz + dt^2 + dx_i^2) \to \frac{1}{z^2}(-2dtdz + dt^2 + dx_i^2) \\ &+ 2n(n^2-1)a_n t^{n-2}dt^2 - 2\frac{a_n n(n+1)}{z^2}x_i dx_i[(t-(n-1)z)dt - tdz]. \end{aligned} \quad (8.2.18)$$

We now see that on taking the scaling limit (8.2.6) we have

$$ds^2 = \frac{1}{z^2}(-2dtdz + dt^2) \to \frac{1}{z^2}(-2dtdz + dt^2 + 2n(n^2-1)a_n z^2 t^{n-2}dt^2). \quad (8.2.19)$$

As expected the $SL(2,R)$ subgroup $L^{(0)}, L^{(\pm1)}$ are exact isometries. The other $L^{(n)}$ are not exact isometries. However, they are asymptotic isometries (See [66] and references therein). Near the boundary $z = 0$ the diffeomorphisms generated by these vector fields leave the metric unchanged upto a factor which has a falloff like $z^2$. Thus these do not affect the non-normalizable mode of the metric.

One expects that when the charges for these asymptotic isometries are constructed, then just as in the Brown-Henneaux construction for $AdS_3$ [66](and recent generalisations to $AdS_2$ [67]), there will actually be a central term due to boundary contributions. Thus the Virasoro algebra will presumably act in a faithful way on the physical Hilbert space.

We also notice from the action of the $L^{(n)}$ (8.2.18) on the spatial metric that on the slices of constant $t, z$, the action is again an isometry. Thus the $L^{(n)}, J_a^{(n)}, M_i^{(N)}$ together generate (asymptotic) isometries of the spatial and $AdS_2$ metrics $\gamma^{ij}$ and $g_{ab}$. Therefore it seems natural to consider the action of these generators on the Newton-Cartan like geometry

## 8.3 The Schrödinger Bulk

Throughout this thesis, we have looked at the differences of the GCA with the Schrödinger Algebra and it is apt to carry on making this distinction in the case of the bulk theory. The differences in this case are much more fundamental and the theory for the bulk with Schrödinger symmetry that has been looked at in the literature is very different from the Newton-Cartan like gravity that we have discussed above. The primary difference is that the theory has a well-defined spacetime metric albeit in a rather non-conventional two higher dimensional bulk. Let us briefly discuss this.

We shall start with what we call the generalized Schrödinger algebra here, having a dynamical exponent $z$ which determines the relative scaling between the time coordinate and the spatial coordinates, $[t] = \text{length}^z$. The special case of $z = 2$ was discussed in the earlier chapters. The non-trivial commutators in this case are

$$[D, P_i] = iP_i, \;\; [D, B_i] = i(1-z)B_i, \;\; [D, H] = izH,$$

$$[D,N] = i(2z)N, \quad [P_i,B_j]=\delta_{ij}N \tag{8.3.20}$$

In the special case $z=2$, the algebra may be extended by an additional special conformal generator, $k$, whose non-trivial commutation relations are

$$[D,C]=2iC, \quad [H,C]=iD. \tag{8.3.21}$$

In this case $z=2$, both $D$ and $N$ may be diagonalized, so representations of the Schrodinger algebra are in general labeled by two numbers, a dimension $\Delta$ and a number $l$.

One wishes to construct the gravity dual to the above non-relativistic conformal symmetry. The most straight forward way is to look for a metric which would realize the Schrödinger algebra as an isometry. Here, since one has two simulataneously diagonalizable symmetry generators which label the representations as opposed to the $AdS$ case which has only one, it is expected that the bulk theory which realizes the algebra as its isometry would exist in two higher dimensions instead of the natural one higher dimension in AdS/CFT. In [25, 26], the authors constructed a $d+3$-dimensional metric realizing the $d+1$-dimensional Schrödinger group as its isometry group, and conjectured the associated gravitational system to be dual to systems with Schrödinger symmetry at zero temperature and zero density. The metric proposed was

$$ds^2 = L^2\left(-\frac{2\beta^2 dt^2}{r^{2z}}+\frac{2d\xi dt+d\vec{x}^2+dr^2}{r^2}\right) \tag{8.3.22}$$

It is interesting to note that for $z=1$, we have usual relativistic $AdS_{d+3}$. This is to be contrasted with the GCA, which has a $z=1$ non-relativistic symmetry.

Let us look at a few generators that are realized as isometries of the metric. The dilatation D is realized as the simultaneous scaling,

$$\{t,\xi,\vec{x},r\}\rightarrow\{\lambda^z t,\lambda^{2-z}\xi,\lambda\vec{x},\lambda r\}, \tag{8.3.23}$$

while a boost acts as,

$$\vec{x}\rightarrow\vec{x}\vec{v}t, \quad \xi\rightarrow\xi+\vec{v}\vec{x}\frac{v^2}{2}t. \tag{8.3.24}$$

A very important identification is

$$N=i\partial\xi. \tag{8.3.25}$$

The number operator in the Schrödinger symmetric CFT is gapped and hence $\xi$ must be periodic. Again, $\xi$ is a null direction in the bulk geometry and hence further analysis is in terms of a discrete light cone quantization (DLCQ). The fact that $\xi$ is a compact direction, seems to suggest that there is a singularity in the $r\rightarrow\infty$ limit. But this goes away (actually is lost behind a horizon) as soon as one turns on a temperature.

The metric (8.3.22) is not a solution of Einstein's equations in vacuum and hence one needs to couple the system to additional background fields. These extra fields provide the Schrödinger backgrounds with a natural embedding in string theory. More evidence for this conjectured duality has been found by matching correlation functions on both sides. We commented on the two and three point correlators from the boundary point of view in earlier chapters. These have been reproduced by calculations in the bulk as well [45, 46].

There has been a great deal of progress on issues related to the Schrödinger algebra from the point of view of the non-relativistic AdS/CFT correspondence, but we do not wish to go into further details in this thesis. The above discussion was a flavour of the radical differences between the bulk theory proposed by us for the GCA and the Schrödinger gravity theory. A few comments are in order. We have suggested above in Sec.(8.1.3) that the Schrödinger bulk also might be obtained by a Newton-Cartan like limit of the usual AdS bulk theory. This was due to the observation of [75] that the Schrödinger algebra could be realized as asymptotic isometries very much like what we have described above for the GCA. Conversely, one can also ponder if we could try to realize the GCA as the isometries of a higher dimensional bulk theory. There has been an attempt in this direction in [79]. The symmetry algebra is a version of the GCA in 2+1 dimensions with an additional central term for the boosts (i.e. the boosts on the plane don't commute). The authors of [79] propose a metric for the system in $AdS_7$, a three dimensional higher space-time. This, however, runs into the trouble of having an incurable (4,3) signature and thus, even though the authors carry out a bulk-boundary matching of the two-point function, it is would be unwise to trust this duality.

# Appendix: Galilean Isometries

In the Newton-Cartan spacetime described in this chapter, we do not have a spacetime metric. Therefore the usual notion of an isometry as generated by a Killing vector field of the metric is not applicable. There is consequently some ambiguity in the definition of an isometry. We will paraphrase here some of the different possibilities as outlined in [58].

1. **Galilei Algebra**: This consists of all vector fields $X$ satisfying

$$L_X \gamma^{\mu\nu} = 0 \qquad L_X \tau = 0 \qquad L_X \Gamma^{\alpha}_{\mu\nu} = 0. \tag{8.3.26}$$

   This gives rise to the usual set of vector fields which generate the finite dimensional Galilean algebra of uniform translations (in space and time), uniform velocity boosts and spatial rotations.

2. **Milne Algebra**: This consists of all vector fields $X$ satisfying

$$L_X \gamma^{\mu\nu} = 0 \qquad L_X \tau = 0 \qquad L_X \Gamma^{\nu\alpha}_{\mu} = 0 \tag{8.3.27}$$

where $\Gamma_{\mu}^{\nu\alpha} = \gamma^{\beta\nu}\Gamma_{\mu\beta}^{\alpha}$. The set of vector fields $X$ satisfying this condition is an infinite dimensional extension of the Galilean algebra, now involving arbitrary time dependent boosts/accelerations.

3. **Coriolis Algebra**: This consists of all vector fields $X$ satisfying

$$L_X\gamma^{\mu\nu} = 0 \qquad L_X\tau = 0 \qquad L_X\Gamma^{\mu\nu\alpha} = 0 \tag{8.3.28}$$

where $\Gamma^{\mu\nu\alpha} = \gamma^{\rho\mu}\gamma^{\beta\nu}\Gamma_{\rho\beta}^{\alpha}$. The set of vector fields $X$ satisfying this condition is a further infinite dimensional extension of the Milne algebra, now involving in addition to the arbitrary boosts or accelerations, arbitrary time dependent rotations as well.

# Part IV

# Conclusions

# Chapter 9

# Summary and Future Directions

## 9.1 A look back

In this thesis, we have discussed at length the first systematic non-relativistic limit of the AdS/CFT conjecture. We have concentrated on the bosonic part. Let us briefly summarize what we have seen in this thesis. We started out by motivating why we should look at non-relativistic conformal algebras different from the Schrödinger algebra. To that end, we reviewed the well-known process of Inönu-Wigner contractions and applied it to the relativistic conformal algebra. The Galilean Conformal Algebra thus emerged as a parametric contraction of the relativistic algebra. We saw that this algebra can be written in a suggestive way and can be embedded in an infinite dimensional algebra, which we also called the GCA. The interesting feature here was that as opposed to the relativistic case, where one gets the two infinite copies of the Virasoro algebra from the finite algebra only in two dimensions, the lift to an infinite algebra in the case of the GCA was in all space-time dimensions. The most relevant question to ask then was if this algebra is realized in any physical system. We answered this in the affirmative and showed that the GCA is the symmetry algebra of the equations of non-relativistic hydrodynamics.

We then moved on to construct the representations of this algebra. Here we saw that the representations are labelled by two quantum numbers, those corresponding to the dilatations and the boosts. We revisited the notion of primary operators in the context of the GCA and constructed the representations by acting on these GCA primaries with raising operators. It was then shown that the finite part of the algebra, like in the relativistic case, is enough to fix the two and three point correlation functions. We explicitly constructed these correlators starting with the non-relativistic Ward identities. The correlators were found to have novel exponential pieces and were compared to other known cases.

The case of two dimensions in the relativistic algebra is special as it is here that the finite algebra gets lifted to the infinite dimensional one. It is natural to expect that there would exist a map between these two infinite algebras when we look at $d = 2$. The expectation was met and we saw that one could obtain the classical infinite GCA in $2d$ as a contraction of linear combinations of the two Virasoro algebras. Then the quantum GCA was looked at and we found that 2d GCFT's with nonzero central charges $C_1, C_2$ can be readily obtained by considering a somewhat unusual limit of a non-unitary 2d CFT. While the resulting Hilbert space of the GCFT is again non-unitary, the theory seems to be otherwise well defined. We found that many of the structures parallel those in the Virasoro algebra and indeed arise from them when we realise the GCA by means of the scaling limit. But in most cases we could also obtain many of the same results autonomously from the definition of the GCA itself, showing that these are features of any realisation of this symmetry. This completed a rather detailed look at the boundary theory of the non-relativistic limit of the AdS/CFT correspondence.

We had a novel structure for the non-relativistic bulk that involved taking a Newton-Cartan like limit of the Anti de-Sitter space. The spacetime metric degenerated in the limit and the theory was formulated in terms of the dynamic Christoffel connections. The non-relativistic geometry took on a fibre bundle structure with a base manifold that was $AdS_2$ and flat d-dimensional fibres. We looked at how one arrives at the GCA in the bulk by contracting the original Anti de-Sitter killing vectors. The infinite GCA turned out to be realized as the asymptotic symmetry algebra of the above described Newton-Cartan structure.

## 9.2 Discussions and Open directions

We have seen that the nonrelativistic conformal symmetry obtained as a scaling limit of the relativistic conformal symmetry has several novel features which make it a potentially interesting case for further study. One of the major advantages is that our method of obtaining the non-relativistic physics is based on a systematic limiting procedure which makes the calculations much more tractable. The GCA, we have argued, is different from the Schrödinger group which has been studied recently. It also has the advantage of being embedded within the relativistic theory. Hence we ought to have realisations of the GCA in every interacting relativistic conformal field theory. The obvious question is to understand this sector in a particular case such as $\mathcal{N} = 4$ Super Yang-Mills theory. The analysis of the two dimensional theory tells us that we should be looking at the sector with operators having very large spin but finite conformal dimensions. One needs to see if this is the correct sector to look at in higher dimensions.

Again, all quantum field theories have a hydrodynamic regime and the non-relativistic limit of this is presumably described by the Navier-Stokes equation. That a part of the infinite GCA is realized as a symmetry of this Navier-Stokes equation could well indicate that the GCA is indeed realized in a non-relativistic sector of all relativistic QFTs.

We raised the question of integrability at the very beginning. The fact that we have an infinite algebra generating a possibly closed sub-sector within the relativistic theory is indeed indicative of the solvability of the sector. This is a question that has not been touched upon in this thesis, or in any related work, but seems a very tantalizing prospect to persue.

Returning to the case of two dimensions, it would also be nice to obtain the differential operator realisations $\hat{M}_{-k}, \hat{L}_{-k}$ given in (6.2.28) directly from the GCA. This would be crucial in trying to generalize the results here to higher dimensions where we do have the GCA but no analogue of the Virasoro Algebra. In this context it must be clear from the explicit solutions in the study of the four point functions that the lack of holomorphicity, while it makes the expressions more cumbersome, is not a real barrier to solving the theory provided one has GCA null vectors. It would be interesting if one can find an analogue of minimal models for the GCA for which there are a finite number of GCA primaries and for which the solution of the theory can be completely given. Though we must hasten to remind the reader that such theories (if they at all exist) would not, in any sense, arise from the Virasoro minimal models which have $0 < c < 1$.

As far as the scaling limit of the 2d CFT is concerned, it is important to make some further checks on its consistency with the requirements from the GCA. From the point of view of the 2d CFT it would be nice to get a better understanding of this scaling limit on the central charges. Sending their magnitude to infinity seems like some kind of large $N$ or classical limit. While it is somewhat dismaying that the theory is non-unitary, could there perhaps be an interesting "physical" subsector which is unitary? It is amusing to note in this context that in perturbative string theories we do have 2d CFTs with large (26 or even 15!) positive and negative values of $c$ and $\bar{c}$ in the matter and ghost sectors. It is important to understand whether there are concrete physical situations where such a scaling limit is actually realised. That would make the further study of these symmetries all the more exciting.

A straight-forward generalization of the topics discussed in the thesis is to look at the supersymmetric extensions of the GCA. These have been looked at in [80, 81, 82]. There exist two versions of the Super GCA. In [80], we looked at extensions of the GCA to $\mathcal{N} = 1$ supersymmetry in 4 dimensions by demanding an $SO(3)$ invariance of the resulting non-relativistic theory. We implemented the group contraction at the level of the co-ordinates in superspace. The supersymmetric extension in terms of the generators is for the sub-algebra which contains the bosonic commuting boosts. Interestingly, the algebra can be given an infinite dimensional extension, again for all dimensions of space-time. We also spoke of the generalization of our methods to higher supersymmetries, which seems to be natural in our formulation. In [81, 82], however, the authors have only dealt with the finite algebra and the crutial difference between the two finite algebras is that in [81, 82], the fermionic part of the algebra generates all the generators of the finite bosonic GCA. But the construction seems possible only in even $\mathcal{N}$. The quantum mechanical analysis for the two-dimensional *GCA* can also be analogously extended to the case with supersymmetry. This has been looked at in [85]. In the context of the supersymmetric GCA, it would be good to understand the differences between the two types of SUSY extensions better and see if one can construct an infinite version of the algebra constructed in [81, 82]. The questions that arose in the 2d GCA also rear their heads in the supersymmetric version [85]. It is also less clear in this case if the limit of the relativistic theory and intrinsic GCA analysis is perfectly compatible, as there seems to be additional null states arising in the intrinsic analysis. These seem to arise from states with positive norm which become null in the non-relativistic effect in analysis of the lowest levels. One needs to understand if we can understand all such pathological cases in terms of degenerations happening due to the limit.

The bulk description in terms of a Newton-Cartan like geometry is somewhat unfamiliar and it would be good to understand it better. In particular, one needs a precise bulk-boundary dictionary to characterise the duality. At least implicitly this is determined by taking the parametric limit of the relativistic duality.

Then there is the question of how such non-metric theories lift to string theories. This is something we have not touched upon at all in this thesis. One might hope to get some guidance from previous studies of nonrelativistic string theories, though in all these cases one had additional fields like the two form $B_{\mu\nu}$ turned on which made the sigma model well defined. It is therefore not completely clear how to define a string theory on these Newton-Cartan like geometries[1].

In the case of the Schrödinger symmetry the dual gravity theory is proposed to live in *two* higher dimensions than the field theory. This also provided the route for embedding the dual geometry in string theory. It is interesting to ask if there is something analogous in our case, whereby the GCA is realised as a standard isometry of a higher dimensional geometry (e.g. $(d+3)$ dimensional for a $(d+1)$ dimensional field theory). We discussed briefly an attempt to realize such a geometry for a centrally extended finite 2+1 dimensional GCA at the end of the last chapter and its drawbacks. It is possible that any similar construction for the finite GCA without central terms would also suffer from similar problems of signature of spacetime.

Interestingly, there has been a recent realization of the quantum 2d GCA in terms of a Cosmological Topologically Massive Gravity (CTMG) [86]. The GCA is realized by a Brown-Henneaux like analysis when the coupling constant of the gravitational Chern-Simons term $\mu$ scales like $\epsilon$. It would be interesting in this context to see how the correlation functions work out and if the limiting procedure implemented in the bulk gives the correct correlators in calculated in the boundary theory. Now that there is an

[1]There have been works attempting to quantise nonrelativistic theories of gravity directly in a canonical framework [68, 69]. It would be interesting to see if these can be generalised to the case with negative cosmological constant. This could have interesting implications for gauge-gravity dualities in the non-relativistic setting.

action that one can start off with, there seems to be a more concrete path to follow in order to get to the bulk-boundary correspondence in the case of the three dimensional gravity and two dimensional field theory.

Coming back to the boundary theory, it is interesting to ask whether there are intrinsically non-relativistic realisations of the GCA, perhaps in a real life system. It is encouraging in this context that the incompressible Euler equations concretely realise the GCA, providing an example of a gapless non-relativistic system.

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