Nonequilibrium aspects of gauge/gravity duality

By

Ayan Mukhopadhyay
Harish-Chandra Research Institute, Allahabad

A Thesis submitted to the
Board of Studies in Physical Science Discipline
In partial fulfilment of requirements
For the degree of
DOCTOR OF PHILOSOPHY
of
Homi Bhabha National Institute

June 2010

Certificate

This is to certify that the Ph.D. thesis titled "Nonequilibrium aspects of gauge/gravity duality" submitted by Ayan Mukhopadhyay is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Date: **Professor Rajesh Gopakumar**

Thesis Adviser

Declaration

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

This work was done under guidance of Professor Rajesh Gopakumar, at Harish-Chandra Research Institute, Allahabad.

Date:

Ayan Mukhopadhyay

Ph.D. Candidate

To my dadu and dida, my parents, my Gurujis,
Dhrupad, Darjeeling tea and
Srinwanti Chakrabarti

Acknowledgments

I would firstly like to thank my mentor Prof. Rajesh Gopakumar for his comprehensive guidance and nurturing that has made me grow from an inquisitive student to an independent research worker. It is because of him that I have been able to inculcate in myself the zeal to make myself better at every moment, the ability to self criticize, and the discipline that makes thinking, calculating, reading and speaking fruitful. I also thank him for interacting with me very extensively and patiently on various areas of physics while giving me valuable insights, for encouraging my freedom of thought and also for encouraging me to take lessons in Dhrupad while doing my research.

I would like to thank Prof. Ashoke Sen for his generous guidance and support which also have been equally invaluable in my development as an independent research worker. It is because of him that I have learnt to believe in myself. It would also be ungrateful of me as well, to deny his profound influences on my approach to science and many other aspects of my personality, which have made me successfully deal with various challenges in my research career so far and will hopefully be my assets for the future.

I would like to thank all students, visiting fellows and professors of Harish-Chandra Research Institute, where I have been affiliated for the entire course of my doctoral studies, for the stimulating academic atmosphere where one always finds opportunities to learn new things and discuss one's own ideas with others. I would particularly like to thank Ted Erler, Chethan Gowdigere, Rajesh Gupta and Girish Kulkarni in this regard. It has also been a wonderful experience to have worked in the string group here for the last six years, as it has been a second family to me. Finally, I would like to thank my collaborator and friend Ramakrishnan Iyer for the invaluable support and help that he has offered me in the search for my identity as an independent research worker.

Synopsis

Gauge/Gravity duality is a broad framework in theoretical physics in which strong coupling phenomena in gauge theories for large rank of the gauge groups are mapped to the dynamics of classical gravity in one higher dimensional space-times with prescribed asymptotic boundary conditions. This thesis explores how this duality helps us to understand strongly coupled nonequilibrium physics and improves our understanding of the phenomenology of nonequilibrium phenomena in general. Though at present we do not know whether the gauge/gravity duality could apply to an arbitrary consistent four dimensional gauge theory, we can indeed hope to achieve a deep understanding of universal phenomena in strongly coupled conformal gauge theories with gravity duals. We can also argue that we can use this understanding to create useful phenomenological models for general nonequilibrium phenomena like decoherence, relaxation and hydrodynamics at sufficiently strong coupling.

The broader significance of the study comes from the fact that the gravity description allows us to obtain the exact nonequilibrium evolution in the microscopic theory where averaging is done only over the environmental degrees of freedom or the boundary conditions but no approximation is done for the microscopic degrees of freedom and their dynamics. Typically, we do not know how to refine the kinetic description systematically even when it is valid, for instance we do not know how to evaluate the corrections to the relativistic semiclassical Boltzmann equation systematically in case of gauge theories. Therefore, the gauge/gravity duality indeed gives us an opportunity to gain novel understanding into the origin of irreversibility in microscopic theories in terms of five-dimensional geometry.

Under the gauge/gravity duality five dimensional static black holes with appropriate masses and charges in the classical theory of gravity map to equilibrium states of the dual gauge theories at corresponding temperatures and chemical potentials. The temperature of the equilibrium configuration in the gauge theory is actually identified with the Hawking temperature of the black hole horizon and the chemical potential is precisely related to the corresponding charge of the black hole. The thermodynamics of the black holes gives us the phase diagrams of the dual gauge theories. The gauge/gravity duality also implies that any solution of the classical theory of gravity which has a regular future horizon maps to an appropriate non-equilibrium state of the dual gauge theory with the final equilibrium condition being given by the asymptotic static black hole geometry.

The fundamental requirement for setting up an instance of gauge/gravity duality is a well-defined holographic renormalization prescription. In this prescription the fifth dimension in the theory of gravity is interpreted as the scale of the dual gauge theory and a formulation is developed through which we can extract the space-time dependent expectation values of gauge-invariant local operators of the dual (nonequilibrium) state from the corresponding five-dimensional geometry. This further allows us to define a scheme for calculating how the anomalous dimensions of these operators and various couplings run with the scale. Particularly, if the dual gauge theory is conformal, the geometry in five dimensions is required to be asymptotically locally AdS. Under the holographic renormalization scheme, the boundary metric becomes the four dimensional metric in which the dual gauge theory lives while the boundary stress tensor becomes the energy-momentum tensor of the corresponding state. The classical equation of motion of gravity allows us to calculate the conformal (or Weyl) anomaly and also implies the conservation of the energy-momentum tensor.

Gauge/gravity duality for conformal gauge theories defines a universal sector for large rank of the gauge group and strong 't Hooft coupling. This follows from the fact that any two derivative theory of classical gravity which admits $AdS_5 \times X$ as a solution with X being a Sasaki-Einstein manifold, has a consistent truncation to just pure Einstein's gravity in AdS_5. Moreover, if the five-dimensional solution of pure gravity has a regular future horizon, it also has a regular future horizon when lifted to the corresponding solution of the untruncated theory. The universal sector in gauge gravity duality can be defined as the dual of pure Einstein's gravity in AdS_5. Here the dynamics is universal though the embedding of this sector in the dual gauge theory depends on the details of matter content and couplings. It has been shown that solutions of pure gravity in AdS_5 captures a huge range of strongly coupled nonequilibrium phenomena in the dual gauge theories like decoherence, early time nonhydrodynamic behavior and also late time hydrodynamic behavior in the deconfined plasma phase. The underlying equations governing these phenomena have to be universal for conformal gauge theories with gravity duals. However, since gauge fields in the theory of gravity are turned off the dual states have zero chemical potentials throughout their evolution.

In the first work described in this study, I have explored with Rajesh Gupta which data in the gauge theory determines the dual five-dimensional spacetime geometry in the universal sector. We have shown that when the boundary metric is flat, the boundary stress tensor uniquely determines the regular geometry. Holographic renormalization then implies that if the dual conformal gauge theory is living in flat space, all states in the universal sector will be determined uniquely by the expectation value of the energy-momentum tensor and its time evolution. This has not been completely obvious since the boundary stress tensor is not Cauchy data from the geometric point of view. However, we have proved that in the Fefferman-Graham coordinates the boundary stress tensor determines the dual geometry in a power series expansion in the radial coordinate. Further, there can be two distinct type of pathological boundary stress tensors. The first type which we call *abcd* (asymptotic boundary condition destroying) stress tensors are those where in a radial tube the power series expansion can have zero radius of convergence. In the other case, the power series expansion in the radial coordinate has a finite radius of convergence but has a naked singularity where the Fefferman-Graham coordinate system breaks down. When we have a regular geometry, the metric has only coordinate singularity in the Fefferman-Graham coordinates such that at late times the limit of the domain of validity of these coordinates coincides with the future horizon. This coordinate singularity can be sufficiently removed by translating to an appropriate coordinate system, such that the full geometry now has a regular future horizon, which thus gets determined uniquely by the energy-momentum tensor of the dual state.

In earlier works, it had been shown that one can get non-linear hydrodynamics in the universal sector of the dual gauge theory from pure gravity in AdS_5. This had been achieved by constructing "tubewise black brane solutions" in gravity, in which in a given radial tube ending at a given patch in the boundary the geometry locally is a boosted black-brane parametrized by the local hydrodynamic variables and this solution is constructed in the derivative expansion where the expansion parameter is the ratio between the typical spatio-temporal scale of variation of the hydrodynamic variables and the mean free path given by the temperature of final equilibrium. The constraints in Einstein's equations gives us the non-linear hydrodynamic equations of the gauge theory which contain systematic corrections to the conformal relativistic Navier-Stokes equation. The transport coefficients appearing at all orders in the expansion can be fixed by requiring that these solutions have regular future horizons. These solutions had been constructed in the Eddington-Finkelstein coordinates where they are manifestly regular at the future horizon.

In our work, we reproduce these solutions in the Fefferman-Graham coordinates. The comparative advantage is that the constraints simplify and the construction of these solutions is also manifestly Lorentz covariant. Moreover, one can construct these solutions for an arbitrary conformal purely hydrodynamic energy-momentum tensor which can be shown to be free of *abcd* type of pathology. One can also show by translating our solutions to Eddington-Finkelstein coordinates in a manifestly Lorentz covariant way that for a unique choice of transport coefficients at every order in the derivative expansion, the solutions are free of naked singularities at the future horizon. Thus one can conclude that we have states in the universal sector which are purely hydrodynamic such that the energy-momentum tensor can be parametrized by the hydrodynamic variables which thus uniquely specify the state and the dynamics even far away from equilibrium is described by hydrodynamic equations of motion alone. Our method has been elegantly adapted by others in some non-conformal versions of gauge/gravity duality.

The second work described here and done in collaboration with Ramakrishnan Iyer, has been motivated by the question how does the energy-momentum tensor alone determines the states and their dynamics in the universal sector. We first try to construct such states in the weakly coupled regime using well-established techniques. It has been shown earlier that perturbative nonequilibrium dynamics in gauge theories at high temperatures can be reproduced exactly by a relativistic semiclassical Boltzmann equation whose collision kernel involves fragmentation and two-body scattering phenomena. We show that we can indeed construct special solutions of the Boltzmann equation which are determined exactly by the energy-momentum tensor and its time evolution. We call these *conservative solutions*. In fact these are generalizations of *normal solutions* which are determined by hydrodynamic variables alone and had been constructed earlier in the case of the Boltzmann equation and in more refined kinetic theories.

We then argue that these conservative solutions should exist even nonperturbatively and also when we refine our description such that we are averaging only over the environmental degrees of freedom and the boundary conditions specifying the radiation at infinity but not doing any approximation for the microscopic degrees of freedom and their dynamics. We also naturally identify the conservative solutions with the universal sector at large 't Hooft coupling and rank of the gauge group as that explains why the states there are determined by energy-momentum tensor alone. We also confirm that using gauge/gravity duality one can indeed construct states which are determined by the energy-momentum tensor alone perturbatively in $1/\sqrt{\lambda}$ and $1/N$, where λ is the 't Hooft coupling and N is the rank of the gauge group, away from the universal sector limit.

We further reinterpret the tubewise black brane solutions as the duals of normal solutions at large 't Hooft coupling and rank of the gauge group. Using this, we are then able to build a complete framework for the entire range of phenomena constituting the universal sector by systematically constructing the equation of motion for the energy-momentum tensor which supplementing the conservation condition gives its complete evolution. Any solution to this equation which is conserved can be claimed to give geometries in the bulk which have future horizons regular up to given orders in two expansion parameters which measures generic departure from equilibrium for conservative solutions. This framework has sufficient predictive power to determine all states in the universal sector (even beyond hydrodynamics) given that the purely hydrodynamic states up to second order in the derivative expansion are known. We also make preliminary studies on how irreversibility emerges through the gravity description for long time scales of observation.

In the third work, described here we shift our focus to a novel nonrelativistic limit of the gauge/gravity duality. Nonrelativistic versions of the gauge/gravity duality had been proposed earlier with the hope of being able to design or simulate simple systems where gauge/gravity duality may actually work concretely. In the same vein, it may be useful to obtain some universal features of nonequilibrium nonrelativistic dynamics in these systems. Here we focus on a particular nonrelativistic scaling limit of the relativistic conformal group which retains the same number of generators, permits an infinite dimensional extension and is called the Galilean Conformal Algebra (GCA). This algebra had been obtained in this way earlier and some attempts had been made to construct Newton-Cartan like gravity which may give duals of GCA invariant microscopic theories.

In this work, we show how one can construct higher derivative hydrodynamics which is covariant under GCA. From our analysis of the dependence of the shear viscosity and the higher hydrodynamic transport coefficients on the temperature and pressure, we are also able to glean important insights into the structure of GCA invariant theories which allow usual thermalization and how they may be obtained from relativistic conformal theories or their dual classical theories of gravity. In the future, the analogue of the universal sector in these dynamical systems obtained by taking the limit of gauge/gravity duality correctly could be easier to solve due to the appearance of infinite dimensional symmetry, so this work may be an important step in this direction.

After presenting these works, we would mention future directions of research which could be immediately attempted and where we may hope to gain valuable insights into broader aspects of the origins of irreversibility based on the novel questions and more sharply defined older questions unraveled by our investigations.

List of publications discussed in the thesis

1. R. K. Gupta and A. Mukhopadhyay, "On the universal hydrodynamics of strongly coupled CFTs with gravity duals," JHEP **0903**, 067 (2009) [arXiv:0810.4851 [hep-th]].

2. R. Iyer and A. Mukhopadhyay, "AdS/CFT Connection between Boltzmann and Einstein equations : Kinetic Theory and pure gravity in AdS space," Phys. Rev. D **81**, 086005 (2010) [arXiv:0907.1156 [hep-th]].

3. A. Mukhopadhyay, "A Covariant Form of the Navier-Stokes Equation for the Galilean Conformal Algebra," JHEP **1001**, 100 (2010) [arXiv:0908.0797 [hep-th]].

Other Publications

1. A. Mukhopadhyay and T. Padmanabhan, "Holography of gravitational action functionals," Phys. Rev. D **74**, 124023 (2006) [arXiv:hep-th/0608120].

2. J. R. David, R. Gopakumar and A. Mukhopadhyay, "Worldsheet Properties of Extremal Correlators in AdS/CFT," JHEP **0810**, 029 (2008) [arXiv:0807.5027 [hep-th]].

Contents

I Introduction 1

1 Introduction 3

1.1 Motivation . . . 3

1.2 Gauge/gravity duality : Dictionary and holographic renormalization . . . 6

1.2.1 Asymptotically anti-de Sitter spacetimes . . . 7

1.2.2 Fields in asymptotically AdS_5 spacetimes . . . 9

1.2.3 The dictionary of gauge/gravity duality . . . 11

1.2.4 Holographic renormalization : General procedure and results . . . 12

1.2.5 Trace anomaly and the Ward identity for energy-momentum conservation . . . 16

1.3 Gauge/gravity duality at finite temperature : Equilibrium, quasinormal modes and transport coefficients . . . 20

1.4 Universal nonequilibrium phenomena in gauge/ gravity duality . . . 26

1.5 Plan of the rest of the thesis . . . 30

II On universal nonequilibrium phenomena in gauge/gravity duality and general phenomenological description of some ba-

sic nonequilibrium processes 33

2 The energy-momentum tensor, the universal sector and purely hydrodynamic states 35

2.1 Introduction 35

2.2 How the boundary stress tensor fixes the solution 39

2.3 Mutual translation between Eddington-Finkelstein and Fefferman-Graham coordinates 46

2.4 The derivative expansion in Fefferman-Graham coordinates 52

2.5 Getting rid of naked singularities 60

3 Phenomenological description of basic nonequilibrium processes and pure gravity 65

3.1 Introduction 65

3.2 The conservative solutions of the Boltzmann equation 68

3.2.1 The conservative solutions in brief 69

3.2.2 Relativistic and semiclassical corrections to conservative solutions . 75

3.2.3 Multicomponent systems 76

3.3 Regularity condition of solutions of pure gravity in asymptotically *AdS* spacetimes 77

3.3.1 The energy-momentum tensor and nonequilibrium physics 79

3.3.2 The complete framework 82

3.3.3 Checks, comparisons and comments 85

III Hydrodynamics in a nonrelativistic limit of gauge/gravity

duality **89**

4 The covariance of Navier-Stokes equation and invariance of theories under the infinite dimensional Galilean Conformal Algebra **91**

4.1 Introduction . . . 91

4.2 Covariant kinematics for the infinite Galilean Conformal Algebra . . . 95

4.2.1 Arbitrary time dependent rotations . . . 97

4.2.2 Arbitrary time dependent boosts . . . 100

4.2.3 Spatially correlated time reparametrization . . . 101

4.2.4 Summing all up . . . 102

4.3 Covariantizing the Navier-Stokes equation . . . 105

4.3.1 The kinematic term . . . 106

4.3.2 The pressure term . . . 107

4.3.3 The viscous term . . . 107

4.3.4 Summing all up . . . 108

4.4 The influence of the continuity equation . . . 109

4.5 GCA covariance and the viscosity . . . 110

4.6 Possible GCA covariant corrections to the Navier-Stokes equation . . . 112

4.7 Discussion . . . 115

IV Conclusion **117**

5 Concluding remarks **119**

6 Appendices **123**

6.1 Appendix A : Proof of the power series solution for AdS_5 asymptotics . . . 123

6.2 Appendix B: On fixing η/s by calculating curvature invariants . . . 128

6.3 Appendix C: Proof of existence of conservative solutions of the Boltzmann equation 132
6.3.1 C.1 A short description of the Boltzmann equation 132
6.3.2 C.2 The moment equations 136
6.3.3 C.3 Formal Proof of Existence of Conservative Solutions 137
6.4 Appendix D: A Simple Mathematical Interpretation of the GCA 151
6.5 Appendix E: $G = MLR$ 153

Part I

Introduction

Chapter 1

Introduction

1.1 Motivation

String theory is a very promising candidate for quantum theory of gravity. Probably, more correctly string theory promises us a complete, unified and simplified microscopic foundation for the relatively macroscopic theories like the standard model and Einstein's theory of gravity which explain most of the phenomena observed today. Over the last two decades we have indeed tested the mathematical framework of string theory very stringently, for instance we have successfully counted the degrees of freedom in order to explain the entropy of certain black holes in terms of an appropriate ensemble of states. However, we have yet to understand what *quantizing gravity* really means or how string theory solves conceptual problems like *information loss through Hawking radiation* which occur when quantum mechanics confronts Einstein's theory of gravity.

String theory, has however expanded the horizons of theoretical physics, by giving us new frameworks to deal with a variety of many longstanding problems in quantum field theories and has also given us new insights into nonperturbative aspects of quantum field theories. Recently, we have found out that there are also sufficient grounds to believe that string theory will give us new insights into a variety of new and old open problems like understanding non-Fermi liquids, high temperature superconductivity and turbulence. In this thesis, we will be concerned with how string theory gives us new frameworks and tools to deal with many fundamental problems of nonequilibrium physics, for instance the origin of irreversibility. We will be mostly concerned with nonequilibrium physics of a certain class of gauge theories, however we will argue that we would be able to obtain a broader picture of the phenomenology of nonequilibrium phenomena through our study.

Nonequilibrium phenomena of Quantum Chromodynamics (QCD), the microscopic theory underlying nuclear physics, has recently been accessible experimentally through Relativistic Heavy Hadron Collider (RHIC). There are strong indications, as we will discuss later that string theory will be able to give us the underlying fundamental equations for nonequilibrium processes that have been observed here and will be observed in future experiments. At present, in fact, there is no other tool to even model real time phenomena in gauge theories in the regime of strong coupling. The quark-gluon plasma that has been experimentally observed at RHIC at a temperature of about 4 trillion degrees Celsius, which is higher than the temperature needed to melt protons and neutrons into a soup of quarks and gluons, behaves like a strongly coupled nearly conformal fluid with surprisingly small viscosity. The tools offered by string theory are certainly relevant for these and some other future experiments.

There is, in fact, an even more deeper reason to take the tools offered by string theory to study nonequilibrium phenomena very seriously. For very few systems in nature, we can develop successful quantum kinetic descriptions to describe nonequilibrium processes even if the microscopic constituents of the system are weakly coupled to each other. In particular, for gauge theories we do not know how to refine the kinetic description of nonequilibrium dynamics beyond the relativistic semiclassical Boltzmann equation, which can be shown to be exactly equivalent to the perturbative microscopic description. At strong coupling, a kinetic theory becomes even harder to construct because often we do not know the quasiparticles which are the sufficiently stable weakly interacting microscopic states, in terms of which we can obtain such a description. String theory, through gauge/gravity duality gives us exact equations underlying nonequilibrium processes of a large class of conformal gauge theories. We will also argue that we can use this to construct phenomenological equations underlying nonequilibrium processes such that they will be equivalent to an exact microscopic treatment for a broader class of theories like the QCD.

By being *equivalent to exact microscopic treatment* we mean that we will only average over environmental degrees of freedom or boundary conditions specifying exchange of energy, momentum and charge, but not over the microscopic degrees of freedom or dynamics like in the case of kinetic approximations. It is therefore, no surprise that we get new insights and also able to sharply define some old questions pertaining to the origin of irreversibility in general, through our study.

Gauge/gravity duality is one of the major developments in string theory in the last decade. Gauge/gravity duality is a concrete realization of the *holographic principle* which states that the degrees of freedom of quantum gravity reside at the boundary of spacetime and has been invoked earlier to solve many puzzles of quantum gravity conceptually. When the 't Hooft coupling and rank of the gauge group of the gauge theory become large, the gauge theory becomes intractable by traditional methods of quantum field theory. However, by gauge/gravity duality, states in the gauge theory get mapped to smooth solutions of classical gravity in one higher dimensional spacetimes with specific asymptotic behavior. This theory of classical gravity is given by Einstein's equation with usually minimally coupled matter, the content of which depends on the details of the dual gauge theory.

We will use the framework of gauge/gravity duality to study nonequilibrium physics of a class of strongly coupled conformal gauge theories. We will be able to obtain a geometric description of nonequilibrium processes of these theories in terms of nonequilibrium dynamics of black holes in certain class of five dimensional spacetimes.

The plan for rest of this chapter is as follows. We will first introduce the reader to gauge/gravity duality, in particular how it enables us to obtain the dynamics of the energy-momentum tensor exactly in nonequilibrium states. Then we will briefly discuss how equilibrium states are described by stationary black holes and obtain first-order transport coefficients in the gauge theory through long wavelength and low frequency fluctuations of these black holes. We will follow this by a discussion on how we can obtain a universal sector in this class of gauge theories and describe how the dynamics in this universal sector of states features almost all basic nonequilibrium processes. Finally, we will outline the rest of the thesis in which we will study the universal sector in detail to obtain the fundamental equations underlying basic nonequilibrium processes in these and other theories.

1.2 Gauge/gravity duality : Dictionary and holographic renormalization

Gauge/gravity duality in the regime of strong 't Hooft coupling and for large rank of the gauge group maps states in the field theory to classical solutions of gravity in one higher dimension. Holographic renormalization gives us a scheme for defining the spacetime dependent expectation values of operators in the dual states along with the sources which couple these operators to the parent theory. This is necessary and sufficient to define the dual gauge theory through gravity in this regime.

In the setting of holographic renormalization, one may visualize the field theory state to be *living* in the boundary of space-time with the radial direction *being* the scale of the field theory. Moreover, as we get closer to the boundary we go towards the UV regime of the field theory while going inwards would be approaching the IR regime of the field theory. In fact this simple picture often serves as a useful guide for modeling classical geometries dual to the vacua of some confining gauge theories including QCD, when we lack a fundamental derivation. Such a picture is also useful in the context of nonrelativistic versions of gauge/gravity duality. In terms of the on-shell action of classical gravity which typically diverges, putting a cutoff in the radial coordinate near the boundary includes more volume of the spacetime, so it follows from the identification of the radial coordinate with the scale of the field theory that UV divergences of the field theory correspond to IR divergences of the dual theory of gravity and vice versa. So, gauge/gravity duality is an instance of UV/IR duality between two theories.

Here we will confine our discussion mainly to conformal gauge theories with gravity duals. For conformal gauge/gravity duality the classical solutions of gravity dual to field theory states are required to be asymptotically AdS spacetimes. This conformal version of the gauge/gravity duality, which historically had been developed first, is also called as the *AdS/CFT correspondence.*

A complete prescription for holographic renormalization has now been developed for these asymptotically *AdS* spacetimes which are dual to the CFT states at strong 't Hooft coupling and large rank of the gauge group. In conformal field theories, scale transformation along with Poincaré transformations form a larger group which is called the conformal group. At the quantum level, the conformal symmetries could be realized in the theory in a subtle manner which involves adding central charges to the conformal algebra and allowing anomalous transformation of the energy-momentum tensor under the conformal algebra. We will also see how holographic renormalization allows us to obtain these quantum effects in the boundary conformal field theory through solutions of classical gravity. In our discussion we will focus mostly on the dynamics of the energy-momentum tensor operator.

1.2.1 Asymptotically anti-de Sitter spacetimes

Five dimensional anti-de Sitter (AdS_5) spacetime is a maximally symmetric solution of Einstein's equation in presence of a negative cosmological constant,

$$R_{MN} - \frac{1}{2}RG_{MN} = \Lambda G_{MN}, \tag{1.1}$$

in five spacetime dimensions. It has constant negative curvature such that,

$$R_{PQRS} = \frac{1}{l^2}(G_{PR}G_{QS} - G_{PS}G_{QR}), \tag{1.2}$$

where l is called the radius of AdS_5 and is related to the cosmological constant through the relation

$$\Lambda = -\frac{6}{l^2}. \tag{1.3}$$

The AdS_5 metric can be written in the Fefferman-Graham coordinates as follows

$$ds^2 = \frac{l^2}{\rho^2}\left(d\rho^2 + \eta_{\mu\nu}dz^\mu dz^\nu\right). \tag{1.4}$$

In these coordinates AdS_5 space is *conformal* to the five dimensional upper half plane (UHP) with the radial coordinate ρ satisfying $\rho \geq 0$ and $\rho = 0$ being the *boundary.* It is conformal to the UHP, in the following sense that the metric has a second order pole at the boundary and does not yield an induced metric at the boundary. However one can define a conformal structure at the boundary by using a *defining function* $r(\rho, z)$ which is positive definite in the interior but has a first order zero at the boundary. Since this defining function could be otherwise arbitrary, one can define a *boundary metric* $g_{(0)}$ up to conformal transformations through the relation

$$g_{(0)} = r^2 G|_{\rho=0}. \tag{1.5}$$

For instance, if one chooses $r = \rho$, then $g_{(0)\mu\nu}(z) = \eta_{\mu\nu}$. On the other hand, $\rho e^{w(z)}$ also satisfies the property of a defining function and yields $g_{(0)\mu\nu}(z) = e^{2w(z)}\eta_{\mu\nu}$. We can thus see that we can define the boundary metric only up to conformal transformations.

We define asymptotically AdS_5 spacetime [1, 2] as a spacetime where the metric takes the following form in the Fefferman-Graham coordinates,

$$ds^2 = \frac{l^2}{\rho^2}\left(d\rho^2 + g_{\mu\nu}(z,\rho)dz^\mu dz^\nu\right), \tag{1.6}$$

such that this metric is non-singular upto a *finite* radial distance from the boundary. Here non-singular implies even absence of coordinate singularities. Further, $g_{\mu\nu}(z,\rho)$ should have a smooth limit at the boundary where $\rho \to 0$ and should take the following form,

$$g_{\mu\nu}(z,\rho) = g_{(0)\mu\nu}(z) + g_{(2)\mu\nu}(z)\rho^2 + g_{(4)\mu\nu}(z)\rho^4 + \overline{g}_{(4)\mu\nu}\rho^4 log\ (\rho^2) + \ldots\ . \tag{1.7}$$

It can be shown that the form of the metric above yields a solution to Einstein's equation in presence of a negative cosmological constant in five dimensions.

The above definition of asymptotically AdS_5 spacetime can be motivated from the fact that in such spacetimes one can make a precise one-to-one correspondence [3, 4] between conformal transformations at the *boundary* and *bulk* diffeomorphisms which preserve the form of the metric given by Eqs. (1.6) and (1.7). Under such diffeomorphisms, the $g_{\mu\nu}$ in the five dimensional Fefferman-Graham metric (1.6) transform infinitesimally as

$$\delta g_{\mu\nu}(z,\rho) = 2\sigma(z)(1-\rho\partial_\rho)g_{\mu\nu}(z,\rho) + \nabla_\mu a_\nu(z,\rho) + \nabla_\nu a_\mu(z,\rho), \tag{1.8}$$

where ∇ is the covariant derivative constructed from the metric g and $a_\mu = g_{\mu\nu}a^\nu$ is determined by $\sigma(z)$ through the following relation

$$a^\mu(z,\rho) = \frac{1}{2}\int_0^\rho d\rho' g^{\mu\nu}(z,\rho')\partial_\nu\sigma(z,\rho'). \tag{1.9}$$

We can check that under these bulk diffeomorphisms, the boundary metric $g_{(0)}$ undergoes a Weyl transformation given by

$$\delta g_{(0)\mu\nu}(z) = 2\sigma(z)g_{(0)\mu\nu}(z). \tag{1.10}$$

Therefore in asymptotically AdS_5 spacetimes we can realize the $SO(4,2)$ conformal symmetries of the boundary theory kinemetically as the asymptotic symmetry group. For instance, the uniform scale transformation $z \to \lambda z$, at the boundary with λ being a constant gets lifted to $\rho \to \lambda\rho, z \to \lambda z$ in the bulk.

Henceforth, we will also choose our units such that $l = 1$, for the sake of convenience.

1.2.2 Fields in asymptotically AdS_5 spacetimes

We begin to study gauge/gravity duality at the dynamical level by looking at behavior of fields in asymptotically AdS_5 spacetimes. In the Fefferman-Graham coordinates introduced in the previous subsections any field $\Phi(\rho, z)$ in the bulk is required to have an asymptotic expansion of the following form

$$\begin{aligned}\Phi(z,\rho) = \rho^\alpha(\Phi_{(0)}(z) + \Phi_{(2)}(z)\rho^2 + ... + \Phi_{(2n)}(z)\rho^{2n} \\ +\overline{\Phi}_{(2n)}(z)\rho^{2n}\log(\rho^2) + ...).\end{aligned} \tag{1.11}$$

We note that the above simply generalizes the asymptotic form of $g_{\mu\nu}(z,\rho)$ in the full five dimensional metric.

Now we may impose the equations of motion on the bulk fields $\Phi(z,\rho)$ which include $g_{\mu\nu}(\rho,z)$ in the Fefferman-Graham metric. These equations of motion could be just linearized perturbation about the AdS_5 background or it could be the full nonlinear equations of classical gravity. In either case, the field equations of motion are second order differential equations in ρ, so we have two independent solutions which have asymptotic behaviors as ρ^α and $\rho^{\alpha+2n}$ respectively at the leading orders. In most of the examples we will be discussing here, n and 2α are non-negative integers, though we can readily generalize beyond these cases.

The asymptotic form (1.11) of the bulk fields $\Phi(z,\rho)$ including $g_{\mu\nu}(z,\rho)$ in the Fefferman-Graham coordinate system is consistent with the field equations of motion whether we consider the linearized perturbations about the AdS_5 background or the full nonlinear equations of the theory of gravity. The universal features are

- All the coefficients in the asymptotic expansion (1.11), $\Phi_{(2k)}(z)$, for $0 < k < n$ are determined *algebraically* in terms of the coefficients $\Phi_{(0)}(z)$ and their finitely many derivatives up to order $2k$.

- The coefficient $\Phi_{(2n)}(z)$ of the asymptotic expansion remains undetermined by the equations of motion. For the linearized equations of motion, this simply follows from it being the leading term in the solution which is linearly independent of the one whose leading behavior is ρ^α.

- The coefficient $\overline{\Phi}_{(2n)}(z)$ is also an algebraic function of $\Phi_{(0)}(z)$ and its derivatives.

Henceforth, we will call

- $\Phi_{(0)}(z)$ as the non-normalizable mode,

- $\Phi_{(2n)}(z)$ as the normalizable mode, and

- $\overline{\Phi}_{(2n)}(z)$ as the anomaly coefficient.

We have made slight abuse of notation here because we are using the word *mode* for coefficients of an expansion and not for solutions of the linearized equation of motion, but indeed the non-normalizable and the normalizable modes give the leading terms of the asymptotic expansion of two linearly independent solutions of linear perturbations about AdS_5 spacetime.

In the case of $g_{\mu\nu}(\rho, z)$ in the Fefferman-Graham metric, for instance, the boundary metric $g_{(0)\mu\nu}(z)$ is the non-normalizable mode, $g_{(4)\mu\nu}(z)$ is the normalizable mode and $\overline{g}_{(4)\mu\nu}(z)$ is the anomaly coefficient.

The other very important general observation is that, *for linearized perturbation around AdS_5, the normalizable mode gets fixed in terms of the non-normalizable mode if we demand regularity of the solution in the interior of AdS_5.* However, the normalizable mode is not a local functional of the non-normalizable mode. This is also true if we solve the equations of motion order by order in perturbation in some bulk coupling constant. There is an appropriate generalization of this result when we solve the full non-linear equations of motion by perturbing around a non-trivial background like the AdS black brane, but we will come back to this later.

1.2.3 The dictionary of gauge/gravity duality

The fundamentals of the gauge/gravity dictionary were developed originally in [5, 6, 7]. This dictionary can be built around the following tenets :

1. For every bulk field Φ there exists a corresponding gauge invariant local operator in the gauge theory, which we denote as O_Φ. In particular, the metric in the bulk corresponds to the energy-momentum tensor of the gauge theory and the bulk gauge fields correspond to the boundary symmetry currents. This one-to-one correspondence between the bulk field and the local gauge-invariant operator can be made from pure kinematic considerations by studying how they belong to representations of the $SO(4, 2)$ group, and the specific identification of conformal transformations at the boundary with the appropriate generators of asymptotic symmetry group of asymptotically AdS_5 spacetimes discussed earlier.

2. The value of the non-normalizable mode $\Phi_{(0)}(z)$ of the asymptotic expansion of the bulk field $\Phi(\rho, z)$ is identified with the source which couples the operator O_Φ with the parent gauge theory. The boundary metric, for instance being the non-normalizable mode for $g_{\mu\nu}(\rho, z)$ in Fefferman-Graham metric, is identified as the metric on $\mathcal{R}(3, 1)$ where the gauge theory *lives*.

3. The partition function of the theory of gravity with the non-normalizable modes $\Phi_{(0)}(z)$ specified as boundary condition for all the bulk fields Φ, corresponds to the generating functional of correlation functions of all local gauge-invariant operators O_Φ. We can put this compactly as

$$Z_{String}[\Phi_{(0)}] = \int_{\Phi\approx\Phi_{(0)}} D\Phi \; exp(-S[\Phi]) =< exp\left(-\int \Phi_{(0)}O_\Phi\right) >_{QFT} . \quad (1.12)$$

Further at strong 't Hooft coupling and for large rank of the gauge group, the dual string theory can be approximated by a classical theory of gravity which is usually a supergravity theory, so that the relation above in this limit reduces to

$$S_{Supergravity,on-shell}[\Phi_{(0)}] \approx - < exp\left(-\int \Phi_{(0)}O_\Phi\right) >_{QFT} . \quad (1.13)$$

Since this saddle point approximation becomes exact at strong 't Hooft coupling and large rank of the group, any smooth asymptotically AdS_5 solution of the equations of motion of gravity will be dual to an appropriate state in the gauge theory.

These tenets, however, are not sufficient for gauge/gravity duality. One important reason for insufficiency is that the on-shell bulk action of gravity is usually divergent. To give meaning to the gauge/gravity duality we have to now implement holographic renormalization.

1.2.4 Holographic renormalization : General procedure and results

We now outline the general procedure for holographic renormalization [8, 9, 10, 11, 12] which makes the gauge/gravity dictionary mentioned in the previous subsection concrete (please see [13] for a review). This procedure consists of the following steps.

1. **Regularization :** The most convenient way to regularize the on-shell classical action is to evaluate the Lagrangian density and the volume density of spacetime in Fefferman-Graham coordinates and restrict the range of integration of the radial coordinate ρ, for $\rho \geq \epsilon$, with ϵ/l being a small parameter. We would like to remind

the reader that the classical action of gravity also should contain boundary terms so that we can define a variational principle for the bulk fields $\Phi(z,\rho)$ with the Dirichlet boundary condition at the boundary schematically denoted as $\Phi \approx \Phi_{(0)}$ and has been mentioned in the previous subsection as a basic requirement of the gauge/gravity dictionary which relates the on-shell bulk action of gravity with the generating functional of QFT correlators. For example, in case of the bulk metric we require the Gibbons-Hawking term. These boundary terms have to be evaluated at $\rho = \epsilon$ for this regularization procedure.

The full classical on-shell action with the boundary terms will now have a finite number of pieces which will diverge as $\epsilon \to 0$ and these can be organized in the form

$$\begin{aligned} S_{reg,on-shell}[\Phi_{(0)}(z),\epsilon] = \int_{\rho=\epsilon} & d^4 z \sqrt{g_{(0)}(z)}[\epsilon^{-\nu} a_{(0)}(z) \\ & +\epsilon^{-\nu+1} a_{(2)}(z) + ... - \log\epsilon \; a_{(2\nu)}(z) + O(\epsilon^0)], \end{aligned} \tag{1.14}$$

where ν is a positive number that only depends on the scaling dimensions of the dual operators. Further,
(a) $a_{(2k)}(z)$ are *algebraic* functions of the non-normalizable modes $\Phi_{(0)}(z)$ and their finitely many derivatives,
(b) these divergences do not depend on the normalizable modes $\Phi_{(2n)}(z)$.
The logarithmically divergent term can be shown to be related to the conformal anomaly of the dual field theory.

2. **Functional inversion :** In this step we invert the asymptotic series (1.11) functionally to obtain the non-normalizable mode $\Phi_{(0)}(z)$ as a functional of $\Phi(z,\epsilon)$. This is clearly possible only up to certain orders of ϵ as $\Phi(z,\epsilon)$ is also determined by the normalizable mode $\Phi_{(2n)}(z)$. Up to certain orders of ϵ the functional inversion will always be possible and further all the coefficients of expansion in ϵ will be algebraic functions of the non-normalizable modes $\Phi_{(0)}(z)$ and their finitely many derivatives.

3. **Getting counterterms :** We can finally remove all the pieces of the on-shell action which diverge as $\epsilon \to 0$ by rewriting all the divergent coefficients $a_{(2k)}[\Phi_{(0)}(z)]$ as

$a_{(2k)}[\Phi(z,\epsilon),\epsilon]$ and then adding the counterterm action to the regularized action which removes all the divergent pieces. This counterterm action should simply be

$$S_{ct}[\Phi(z,\epsilon),\epsilon] = -divergent\ terms\ of\ S_{reg,on-shell}[\Phi_{(0)}(z);\epsilon]. \tag{1.15}$$

This counterterm action thus *lives* on the regulated surface $\rho = \epsilon$ where the induced metric is $\gamma_{\mu\nu} = g_{\mu\nu}(z,\epsilon)/\epsilon$. Further this action by construction is covariant and can be expanded in powers and logarithm of the *scale* ϵ with the coefficients being local functionals of the fields *living* on the regulated surface. This exactly captures the nature of the counterterm action of the dual field theory which can also be expanded into powers and logarithm of the scale with the coefficients being local covariant functionals of the operators defined at the same scale. The scheme dependence also arises in both cases through the freedom of adding local covariant terms of $O(\epsilon^0)$ such that they give finite contributions when the cutoff is removed by taking the limit $\epsilon \to 0$.

4. **Defining the renormalized action by taking limit and removing cutoff :** The final step of the procedure is to define the renormalized on-shell bulk action which is now identified with the generating functional of QFT correlators giving the expectation values of local gauge-invariant operators in CFT states and correlation functions in a well defined scheme. This renormalized action can be defined as follows. We first denote the cutoff-dependent action subtracted of the divergent pieces as S_{sub} such that

$$S_{sub}[\Phi(z,\epsilon);\epsilon] = S_{reg,on-shell}[\Phi_{(0)}(z);\epsilon] + S_{ct}[\Phi(z,\epsilon);\epsilon]. \tag{1.16}$$

Now the renormalized action can be defined as

$$S_{ren}[\Phi_{(0)}] = \lim_{\epsilon\to 0} S_{sub}[\Phi(z,\epsilon);\epsilon]. \tag{1.17}$$

To obtain expectation values of operators and correlation functions we actually need both S_{sub} which is a functional of the bulk fields and S_{ren} which is a functional of the sources because we need to perform functional differentiations before taking $\epsilon \to 0$ limit. The precise gauge/gravity dictionary is now

$$S_{ren}[\Phi_{(0)}] = - < exp\left(-\int \Phi_{(0)} O_{\Phi}\right) >_{QFT} . \tag{1.18}$$

The general results for expectation values of operators and n-point correlation functions obtained through holographic renormalization are.as follows.

1. **Expectation values of local gauge-invariant operators :** It follows from (1.18) and the first basic tenet of the gauge/gravity dictionary that

$$< O_{\Phi} >= \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{ren}}{\delta \Phi_{(0)}}. \tag{1.19}$$

Using the relation (1.17) it follows that

$$< O_{\Phi} >= \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon^{4-\alpha}} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{sub}}{\delta \Phi(z, \epsilon)} \right). \tag{1.20}$$

Explicit evaluation yields

$$< O_{\Phi} >= Z \Phi_{(2n)}(z) + C(\Phi_{(0)}(z)). \tag{1.21}$$

Here Z is a numerical coefficient which is scheme independent and $C(\Phi_{(0)})$ is a scheme dependent local functional of the sources $\Phi_{(0)}$ so that it yields contact terms to multiple point correlation functions which can be obtained by further functional differentiation with respect to the sources $\Phi_{(0)}$. This is true whether we are studying linearized fluctuations in response to sources about the AdS_5 spacetime corresponding to the vacuum of the dual theory, or an arbitrary smooth asymptotically AdS_5 solution of classical gravity dual to a state in the CFT.

In particular, it turns out that by an appropriate choice of scheme [9],

$$\begin{aligned} < t_{\mu\nu} >= g_{(4)\mu\nu} + \frac{3}{2}\overline{g}_{(4)\mu\nu} - \frac{1}{8} g_{(0)\mu\nu} [(Tr\ g_{(2)})^2 - Tr\ g_{(2)}^2] \\ -\frac{1}{2}(g_{(2)}^2)_{\mu\nu} + \frac{1}{4} g_{(2)\mu\nu} Tr\ g_{(2)}, \end{aligned} \tag{1.22}$$

where all traces involve appropriate raising or lowering with the boundary metric $g_{(0)\mu\nu}$ or its inverse. Using equations of motion $g_{(2)\mu\nu}$ and $\overline{g}_{(4)\mu\nu}$ get determined as a local covariant functional of $g_{(0)\mu\nu}$, the explicit forms of which will be given soon. The above also needs to be multiplied by $1/4\pi G_N$, but we will drop this prefactor. This will contribute to the dependence of the energy-momentum tensor on the rank of the gauge group. We observe that $< t_{\mu\nu} >$ indeed conforms with the general form of $< O_\Phi >$ as given in (1.21).

2. **Correlation functions :** The correlation functions can be obtained from the exact one-point function by functional differentiation. Using (1.21),we obtain,

$$< O_\Phi(x_1)...O_\Phi(x_n) >= (-1)^n \frac{1}{Z} \frac{\delta\Phi_{(2n)}(z)}{\delta\Phi_0(x_1)...\delta\Phi_{(0)}(x_n)}|_{\Phi_0=0} + \textit{ contact terms}, \quad (1.23)$$

   where the contact terms are scheme dependent.

   We can determine the vacuum expectation value of $\Phi_{(2n)}$ as a functional of the source $\Phi_{(0)}$ using bulk perturbation theory through the solution for $\Phi(\rho, z)$ which is determined in the bulk using an arbitrary source $\Phi_{(0)}(z)$ and requiring it to be regular in the interior. At each order in the perturbation expansion, the asymptotic form of $\Phi(\rho, z)$ takes the same form as in (1.11), however the functional dependence of $\Phi_{(2n)}$ on $\Phi_{(0)}$ is truncated up to a given polynomial order in $\Phi_{(0)}$. To obtain two-point function for example, it would be sufficient to know the linear dependence of the expectation value of the operator on the source. In fact, the two point function would basically be the ratio of the normalizable and the non-normalizable mode in the solution of the equations of linearized fluctuation about AdS_5.

1.2.5 Trace anomaly and the Ward identity for energy-momentum conservation

The dynamics of the energy-momentum tensor operator in conformal field theories has a lot of subtleties at the quantum level. At the classical level, the energy-momentum tensor transforms homogeneously under conformal transformations but at the quantum level it picks up inhomogeneous pieces which are just c-numbers (spacetime functions). Also, the energy-momentum tensor in conformal field theories is classically traceless, but at the quantum level it is not and its trace is called the trace anomaly. These have their origins in the fact that the conformal algebra at the quantum level itself gets modified by central charges. The central charges fix the transformation of the energy-momentum tensor. Conversely we can know the central charges by obtaining the trace anomaly. The conformal algebra for four dimensional quantum field theories can have two central charges. In four dimensions, further, the inhomogeneous pieces in the transformation of energy-momentum tensor and the trace anomaly occur only when appropriate curvature invariants constructed from the background metric do not vanish. It is remarkable that all these subtleties can be reproduced by classical gravity in five dimensions holographically. The conservation of energy and momentum, however, usually does not suffer from anomalies in quantum field theories and this is also reproduced unaltered holographically.

We will first obtain the trace anomaly and the Ward identity for energy-momentum conservation holographically. These follow from solving the equations of motion of $g_{\mu\nu}$ in the Fefferman-Graham metric *asymptotically*. The general result is that the Ward identities, with or without anomalies, can always be determined by solving the equations of motion asymptotically. In the previous subsection we have mentioned that we can always find a scheme such that

$< t_{\mu\nu} >= g_{(4)\mu\nu} + a \; local \; covariant \; functional \; of \; the \; boundary \; metric.$

We have also mentioned in subsection 2 that solving the equations of motion asymptotically do not determine $g_{(4)\mu\nu}$ completely, but they do determine the trace and divergence of $g_{(4)\mu\nu}$. Thus we obtain the trace anomaly and the Ward identity for conservation of energy and momentum respectively.

As we will see in detail in Chapter 3, equations of motion of classical gravity for $g_{\mu\nu}$ in the Fefferman-Graham metric can be decomposed into a equation for the radial evolution in the boundary metric, which is two-derivative with respect to the radial coordinate and a pair of constraints which are scalar and vector in structure and single-derivative with respect to the radial coordinate. We use the power series ansatz (1.7) for $g_{\mu\nu}$ and expand the constraint equations order by order in powers of the radial coordinate. If the equations of motion are just Einstein's equations with a negative cosmological constant, the first non-trivial equations in the Taylor expansion of the radial evolution equation yields [9]

$$g_{(2)\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6}Rg_{(0)\mu\nu}\right). \tag{1.24}$$

The radial equation also determines the anomaly coefficient $\overline{g}_{(4)}$ as [9]

$$\begin{aligned}\overline{g}_{(4)\mu\nu} = \frac{1}{8}R_{\mu\nu\rho\sigma}R^{\rho\sigma} + \frac{1}{48}\nabla_\mu\nabla_\nu R - \frac{1}{16}\nabla^2 R_{\mu\nu} \\ -\frac{1}{24}RR_{\mu\nu} + (\frac{1}{96}\nabla^2 R + \frac{1}{96}R^2 - \frac{1}{32}R_{\rho\sigma}R^{\rho\sigma})g_{(0)\mu\nu}.\end{aligned} \tag{1.25}$$

The radial equation leaves $g_{(4)\mu\nu}$ undetermined. The first non-trivial equations in the Taylor expansion of the vector and scalar constraints give the divergence and trace of $g_{(4)\mu\nu}$ respectively. Using (1.22) we can convert these to obtain the divergence and trace of $t_{\mu\nu}$ which are [8, 9]

$$\nabla^\mu t_{\mu\nu} = 0, \tag{1.26}$$

$$Tr\ t = \frac{1}{4}[(Tr\ g_{(2)})^2 - Tr\ g_{(2)}^2], \tag{1.27}$$

where Tr implies appropriate raising or lowering with the boundary metric $g_{(0)\mu\nu}$ or its inverse, ∇ is also the covariant derivative constructed from $g_{(0)\mu\nu}$ and $g_{(2)\mu\nu}$ is as given by (1.24). The first equation above implies the Ward identity for energy-momentum conservation. We can rewrite the second equation as,

$$Tr\ t = \frac{1}{2}\left(E_4 + I_4\right), \tag{1.28}$$

where

$$E_4 = \frac{1}{64}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right), \tag{1.29}$$

is the Euler density in four dimensions and

$$I_4 = -\frac{1}{64}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2\right), \tag{1.30}$$

is a four dimensional conformal invariant. The coefficients of each of these terms yield the appropriate central charges which turn out to be exactly the same as in weakly coupled $\mathcal{N} = 4$ super Yang-Mills theory, indicating a non-renormalization theorem for protection of these central charges for this theory, and universality for all conformal field theories with gravity duals at strong 't Hooft coupling and large rank of the gauge group.

To obtain the transformation of the energy-momentum tensor under $S0(4,2)$ transformations we first simply calculate how the coefficients of asymptotic expansion of $g_{\mu\nu}$ transform under bulk diffeomorphisms which preserve the asymptotic form of the Fefferman-Graham metric (with (1.8) and (1.9) as consequences) and then obtain the transformation of $< t_{\mu\nu} >$ using (1.22). Doing these we obtain [9],

$$\begin{aligned}\delta < t_{\mu\nu} > &= -2\sigma < t_{\mu\nu} > -2\sigma\overline{g}_{(4)\mu\nu} + \frac{1}{4}\nabla^{\rho}\sigma[\nabla_{\rho}R_{\mu\nu} - \frac{1}{2}(\nabla_{\mu}R_{\nu\rho} + \nabla_{\nu}R_{\mu\rho}) \\ &-\frac{1}{6}Rg_{(0)\mu\nu}] + \frac{1}{48}(\nabla_{\mu}\sigma\nabla_{\nu}R + \nabla_{\nu}\sigma\nabla_{\mu}R) + \frac{1}{12}R(\nabla_{\mu}\nabla_{\nu}\sigma - \nabla^{2}\sigma g_{(0)\mu\nu}) \\ &+\frac{1}{8}(R_{\mu\nu}\nabla^{2}\sigma - (R_{\mu}{}^{\rho}\nabla_{\rho}\nabla_{\nu}\sigma + R_{\nu}{}^{\rho}\nabla_{\rho}\nabla_{\mu}\sigma) + g_{(0)\mu\nu}R_{\rho\kappa}\nabla^{\rho}\nabla^{\kappa}\sigma).\end{aligned} \quad (1.31)$$

Finally, substituting (1.25) above in (1.31), we obtain the conformal transformation of the energy-momentum tensor. We readily see that if the boundary metric is flat, the energy-momentum tensor transforms homogeneously under conformal transformations. The above transformation is just the four-dimensional generalization of the relatively well known result for the two dimensional conformal energy-momentum tensor in conformal field theories given by

$$\delta < t_{\mu\nu} > = \frac{c}{12}(\nabla_{\mu}\nabla_{\nu}\sigma - g_{(0)\mu\nu}\nabla^{2}\sigma), \quad (1.32)$$

where c is the central charge of the Virasoro algebra and $g_{(0)\mu\nu}$ is the background metric on which the two dimensional theory *lives*.

Specializing to scale transformations so that σ appearing in (1.31) is a constant, we obtain the renormalization group flow of the energy-momentum tensor. The same procedure can be applied to all other operators. That we need to solve the equations of motion only asymptotically should have been obvious in hindsight, because we only need the ultraviolet behavior of the theory to obtain renormalization group flow and the ultraviolet in the field theory corresponds to the asymptotic region in the bulk.

1.3 Gauge/gravity duality at finite temperature : Equilibrium, quasinormal modes and transport coefficients

Gauge/gravity duality at finite temperature is, in principle, derivable from the dictionary at zero temperature discussed in the previous section. However, it is possible to approach this more intuitively from the gravity point of view and then verify or test the consistency of the rules at finite temperature. We can also generalize to the case where we have finite chemical potential however we will not have much to say about this here.

We know that stationary black holes classically behave like thermodynamic objects such that one can define thermodynamic functions and identities on the space of stationary black hole solutions [14, 15]. For stationary black holes which are solutions of Einstein-Maxwell theory, the surface gravity on the black hole horizon can be identified with the temperature, the area of the horizon can be identified with the entropy, the mass of the black hole can be identified with the total internal energy. Moreover, these black holes also behave very much like equilibrium states dynamically. For instance, for all dynamical processes in which the black hole interacts with gravitational radiation or minimally coupled well behaved matter, the event horizon can only increase in area monotonically, in perfect agreement with the fact that it can be identified with the entropy. Similarly, for any dynamical process at sufficiently late times the black hole hole event horizon should have uniform surface gravity, which we can reword as coming to thermal equilibrium. These thermodynamic properties also belong to black holes in asymptotically AdS_5 spacetimes which are solutions of Einstein's gravity with a negative cosmological constant, minimally coupled to well behaved matter. It would thus be natural to identify these five-dimensional black holes with equilibrium states of the dual gauge theory.

Five dimensional black holes in asymptotically AdS_5 spacetimes can have many horizon topologies. It turns out that the horizon topology is the same as the topology of the boundary. Gauge/gravity duality fixes the topology of the boundary to be $\mathcal{R}(3,1)$ and boundary metric to be the Minkowski metric $\eta_{\mu\nu}$ because here we are interested in studying the dual gauge theory in Minkowski space time. So, the appropriate black hole solution is

$$ds^2 = -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2(dx^2 + dy^2 + dz^2), \quad f(r) = 1 - \frac{r_0^4}{r^4}, \tag{1.33}$$

where $r = r_0$ is the location of the three dimensional planar horizon at a given moment of time. One can consider a larger family of solutions by boosting the boundary coordinates (t, x, y, z). We can replace dt by $u_\mu dx^\mu$, where u^μ is a timelike unit vector in Minkowski space such that $u^\mu u^\nu \eta_{\mu\nu} = -1$, so that we achieve Lorentz covariant parametrization. We can construct a covariant projection tensor, $P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}$, which projects in the spatial slice orthogonal to u^μ. Using these, we can readily construct a larger family of solutions given by

$$ds^2 = -r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu + \frac{dr^2}{r^2 f(r)} + r^2 P_{\mu\nu} dx^\mu dx^\nu, \quad f(r) = 1 - \frac{r_0^4}{r^4}. \tag{1.34}$$

This family of solutions for reasons we will not mention here, are known as *boosted black brane solutions*. One can easily see why such a family of solutions parametrized by three velocities in the boost u^μ and r_0 exist in the following way. For large r which is the asymptotic region, $f(r) \approx 1$ and the solution (1.33) becomes pure AdS_5 space. The full $SO(4,2)$ asymptotic symmetries are broken to $SO(3)$, the group of spatial rotations in x, y, z coordinates in the interior. Among the symmetries broken by the solution, the boost and the scale transformation form the generators of the *maximally commuting subgroup of broken symmetries*. We can thus form a family of solutions by applying boost and scale transformations to the solution (1.33). Applying scale transformations $(r \to r/\lambda, x^\mu \to \lambda x^\mu)$, we generate a solution which takes the same form as (1.33) with the new horizon at r_0/λ. On applying boost to the boundary coordinates, we arrive at the most general form of the metric given by (1.34).

We can convert the boosted black brane metric (1.34) into Fefferman-Graham coordinate system, in which the metric has no coordinate singularity till the horizon. We will give explicit form of this metric in Chapter 3. We also note that when the boundary metric is $\eta_{\mu\nu}$, the expectation value of the energy-momentum tensor as given by (1.22) is $t_{\mu\nu} = g_{(4)\mu\nu}$. From the metric in Fefferman-Graham coordinates we can extract $g_{(4)\mu\nu}$. After doing an appropriate uniform scaling which preserves the Fefferman-Graham form of metric, we obtain from (1.34),

$$t_{\mu\nu} = (\pi T)^4 (4u_\mu u_\nu + \eta_{\mu\nu}), \quad T = \frac{r_0}{\pi}. \tag{1.35}$$

In the above form, the energy-momentum tensor is exactly the form as that of thermal blackbody radiation and in fact, in any conformal theory, one can define the temperature T such that it takes this form. When $u^\mu = (1,0,0,0)$, i.e. when we consider (1.33), we have $t_{\mu\nu} = diag(\epsilon, P, P, P)$, with the energy density $\epsilon = 3(\pi T)^4$ and the pressure $P = (\pi T)^4$. This gives support to our contention that (1.34) is the gravity dual of the thermal equilibrium state. We emphasize here that in this solution the normalizable and non-normalizable modes of all fields except the non-normalizable dilaton which is set equal to a constant equal to the Yang-Mills coupling, vanishes by the holographic renormalization scheme discussed earlier.

We will describe the elegant prescription suggested in [16] for obtaining thermal retarded correlators. Retarded correlators, as we know, measures the causal response to a source in a field theory such that it vanishes outside the future light cone. The most intuitive way for ensuring causal response in the theory of gravity is to replace the condition for regularity of solutions in the interior of AdS_5 with the *incoming wave boundary condition at the horizon.* We know that black holes should indeed only let probe waves to fall inside realistically and never come out, if we choose the direction of time [1] such that the gravitating system of the black hole and the waves reach thermal equilibrium corresponding to a stationary black hole for large times. This boundary condition at the horizon now determines the normalizable mode in terms of the non-normalizable mode. The two point function then by the application of the remaining usual rules to be applied to the asymptotic form of the solution as described before, turns out to be the ratio of the normalizable and the non-normalizable mode. To be more concrete, the general solution for a bulk field $\Phi(r,t,x,y)$ corresponding to a linearized perturbation around (1.33) is,

$$\begin{aligned}\Phi(r,t,x,y) = {} & \mathcal{A}(\omega,\mathbf{q}) \exp(-i\omega t + i\mathbf{q}.\mathbf{x}) r^{-\Delta_-}(1+...) + \\ & \mathcal{B}(\omega,\mathbf{q}) \exp(-i\omega t + i\mathbf{q}.\mathbf{x}) r^{-\Delta_+}(1+...),\end{aligned} \tag{1.36}$$

[1] The time reverse of this situation will correspond to a white hole.

with $\Delta_- < \Delta_+$ and $\Delta_+ > 0$. In Fefferman-Graham coordinates $\rho = 1/r + \mathcal{O}(1/r^2), z^\mu = x^\mu + \mathcal{O}(1/r)$, this implies that $\mathcal{A}(\omega, \mathbf{q})$ is the non-normalizable mode or the source and $\mathcal{B}(\omega, \mathbf{q})$ is the normalizable mode or the response. The incoming wave boundary condition at the horizon uniquely determines $\mathcal{B}(\omega, \mathbf{q})$ as a function of $\mathcal{A}(\omega, \mathbf{q})$. Further the two point retarded thermal correlator in Fourier space is

$$< O_\Phi O_\Phi >_R = Z \frac{\mathcal{B}(\omega, \mathbf{q})}{\mathcal{A}(\omega, \mathbf{q})} + \textit{ contact terms}, \tag{1.37}$$

where Z is a scheme independent constant. One can also prove that the retarded correlator has a pole *only when* $\mathcal{A}(\omega, \mathbf{q})$ vanishes [20].

From the point of solution of linearized perturbation about the black brane, the vanishing of the non-normalizable mode $\mathcal{A}(\omega, \mathbf{q})$ and the incoming wave boundary condition at the horizon give very special solutions for $\Phi(r, t, x, y, z)$. Such solutions are called quasinormal modes. Therefore, by the prescription mentioned above, the poles of the retarded correlators of the boundary theory occur *if and only if* the dispersion relations corresponding to quasinormal modes are satisfied.

We will concentrate here on the quasinormal modes of the metric with the underlying bulk theory of gravity being just Einstein's equation of motion with a negative cosmological constant. The features we are going to describe will persist even when we couple matter minimally to Einstein's equation for pure gravity. In any physical system capable of equilibration, the long wavelength and low frequency perturbations about equilibrium correspond to hydrodynamic dispersion relations, leading order terms of which can be obtained from the Navier-Stokes equation. The hydrodynamic dispersion relations are of two types, one corresponds to the sound branch which is longitudinal to the direction of propagation and is given by

$$\omega = \pm v_s q + i\Gamma_s q^2 + \mathcal{O}(q^3), \tag{1.38}$$

and the other one corresponds to the shear branch which is transverse to the direction of propagation and is given by

$$\omega = -i\gamma_\eta q^2 + \mathcal{O}(q^3). \tag{1.39}$$

Here, v_s is the speed of sound, Γ_s is called the sound diffusion constant and γ_η is called the momentum diffusion constant. Γ_s and γ_η are related to the bulk viscosity ζ and shear viscosity η through

$$\Gamma_s = \frac{1}{2}\frac{1}{\epsilon + P}(\zeta + \frac{4}{3}\eta), \quad \gamma_\eta = \frac{\eta}{\epsilon + P}. \tag{1.40}$$

Now the quasinormal modes of the metric, using symmetries preserved by the black brane, tracelessness, and conservation of energy and momentum can be divided into a scalar channel, a longitudinal sound channel and a transverse shear channel [20]. Each channel corresponds to a linear combination of various combinations of the components of metric perturbation such that they are invariant under diffeomorphisms at the linearized level. The scalar channel has no branch which can have both low frequencies and long wavelengths. The sound and shear channels remarkably have a branch each of the forms (1.38) and (1.39) respectively [17], with

$$v_s = \frac{1}{\sqrt{3}}, \quad \Gamma_s = \frac{1}{6\pi T}, \quad \gamma_\eta = \frac{1}{4\pi T}. \tag{1.41}$$

We can identify these quasinormal modes with the hydrodynamic branches of the dual conformal field theory. The speed of sound is $1/\sqrt{3}$ of the speed of light in any conformal field theory and simply follows from the Euler equation obtained by taking the divergence of the energy-momentum tensor (1.35) at local equilibrium where T and u_μ are functions of space and time and demanding that it should vanish owing to energy-momentum conservation. The energy-momentum tensor should receive higher derivative corrections but this does not alter the speed of sound. Using (1.40) further we obtain from (1.41) that

$$\zeta = 0, \quad \frac{\eta}{s} = \frac{1}{4\pi}, \tag{1.42}$$

where s is the entropy density defined through the thermodynamic relation $s = \partial P/\partial T = 4\pi^4 T^3$. The vanishing of the bulk viscosity again follows from conformal invariance. The result $\eta/s = 1/4\pi$, conveniently for a dimensionless quantity, can in principle depend on the gravity theory in the bulk concerned. However, one can prove that $\eta/s = 1/4\pi$ is true for any two-derivative theory of gravity [21]. This result for η/s is thus universal for all strongly coupled gauge theories with gravity duals for large ranks of the gauge group.[2] The charge diffusion constants and all other first order transport coefficients can be similarly obtained from appropriate low frequency and long wavelength quasinormal modes of bulk gauge fields and metric.

The quasinormal modes of the metric in all the three channels have infinite branches of higher overtones in the lower half plane, i.e. with negative imaginary values of frequencies, as well [20]. The presence of infinite tower of overtones is a generic feature of quasinormal modes. The meaning of the higher overtones is not clear. We will argue in chapter 3 that these higher overtones should be excised and have no interpretation as constituting the spectrum of the boundary theory with the thermal open boundary condition. However, we will also argue that they could be interpreted as peculiar kind of resonances in a certain precise sense that needs explicit confirmation in the future.

In any case, the hydrodynamic branches of excitations, can be obtained from solutions of the full non-linear equations of motion, just like the equilibrium energy density and pressure. This follows from the obvious generalization of gauge/gravity duality to the case of finite temperature. Any solution of the equations of motion of the classical theory of gravity which has a regular future horizon and is an asymptotic AdS_5 space-time with appropriate behavior for all fields will be dual to a specific non-equilibrium state of the gauge theory at strong 't Hooft coupling and large rank of the gauge group, where the saddle point approximation in gravity becomes exact. The final temperature of the horizon is identified with the temperature of the gauge theory. We will describe the solutions which reproduce hydrodynamic behavior in the boundary in the next section. The transport coefficients which can be defined through linear response theory calculated from solutions of the full non-linear equations of motion of gravity, agree with the results obtained from dispersion relations of quasinormal modes. We will investigate the reason for this agreement in chapter 2.

[2] There is also a conjecture [18], which can be motivated from the uncertainty principle, that $1/4\pi$ is the lowest possible value for η/s. For a review, please see [19].

1.4 Universal nonequilibrium phenomena in gauge/ gravity duality

In all instances of gauge/gravity duality which have concrete embedding in string theory, the gauge theory is supersymmetric with at least $\mathcal{N} = 1$ supersymmetry and the theory of gravity in the bulk is a ten dimensional supergravity which needs to be reduced to five dimensions. If the gauge theory is conformal as well, the Poincaré algebra gets enhanced to superconformal algebra in which the $\mathcal{R}$ symmetry subgroup needs to be included for closure. This superconformal algebra needs now to be the super isometries of the ten dimensional background dual to the vacuum of the gauge theory generalizing the realization of the conformal algebra as the asymptotic symmetry group of asymptotically AdS_5 spacetimes. Such backgrounds are of the form of $AdS_5 \times X$, with X being a Sasaki-Einstein manifold, whose isometries generate the $\mathcal{R}$ symmetry subgroup.

When the supergravity is reduced to AdS_5, the five dimensional reduced theory is a gauged supergravity, where the $\mathcal{R}$ symmetry group forming the group of isometries of X, is gauged. The gauge transformations in the bulk get mapped to global symmetries in the boundary in the usual way. We will be mainly interested in such ten dimensional supergravities which contains $AdS_5 \times X$ as a solution, with X being a Sasaki-Einstein manifold. We will show that the class of superconformal gauge theories which can be holographically defined at strong 't Hooft coupling and large rank of the gauge group through five dimensional gauged supergravities obtained by dimensional reduction of such ten dimensional supergravities having $AdS_5 \times X$ as a solution, contains a universal sector of dynamics [3].

The universal sector can be obtained as follows [4]. One can prove that all ten dimensional supergravities mentioned above has a consistent truncation of their equations of motion, to pure gravity described by Einstein's equation with a negative cosmological

[3]The universality can happen even when we have nonconformal versions of AdS/CFT or consider specific $1/N$ and finite coupling corrections in the conformal case [22, 23].

[4]This should not be confused with the same term used in some other contexts in string theory, like the universal sector in open string field theories which are studied in the context of tachyon condensation [24].

constant in five dimensions. However, this by itself is not enough. We know by the gauge/gravity dictionary we need solutions of gravity to have regular future horizons in order to map to states in the gauge theory. So, we require that all such smooth solutions of pure gravity in five dimensions must lift to smooth solutions in ten dimensions. This indeed turns out to be the case as the lift of the metric from five to ten dimensions do not involve any warping. Thus we can define a universal sector for the entire class of superconformal gauge theories mentioned at strong 't Hooft coupling and large rank of the gauge group as the dual of pure gravity described by Einstein's equation with a negative cosmological constant in five dimensions. The embedding of the universal sector in the full theory will depend on the details of the theory but the dynamics for all states within this sector is exactly the same in all theories within this class.

Solutions of five dimensional pure gravity described by Einstein's equation with a negative cosmological constant, with regular future horizons, have been shown to capture a whole range of nonequilibrum behavior like hydrodynamics, relaxation and decoherence (for a recent review please see [29].). The most studied class of solutions are boost-invariant and it has recently been shown that such solutions qualitatively capture both the non-hydrodynamic early time evolution [28] and the late time hydrodynamic evolution [25] of the QCD quark-gluon plasma formed at RHIC. However, we still lack precise tools to decode nonequilibrium behavior in the gauge theory from the five dimensional geometry mainly because the phenomenology of generic nonequilibrium processes is not well developed. One of the aims of this thesis is to make concrete progress in this direction by postulating the most general phenomenological equations of nonequilibrium processes in the universal sector. At the intuitive level, we know for instance, horizon formation in the bulk should describe decoherence in the gauge theory [5], but we do not know how to demonstrate this concretely in a very generic way. The most remarkable advance recently made has been to understand how *generic* hydrodynamic behavior in the gauge theory

[5] An example of interest is the formation of quark-gluon plasma through collisions of gold nuclei at RHIC, which happens spontaneously without the direct influence of environment. Strictly speaking, such systems are never isolated since they may lose energy through low energy radiation and we need to average out the boundary conditions determining the radiation escaping to infinity appropriately, so that the final equilibrium state is specified.

can be reproduced by solutions of pure gravity (for a review please see [27].). We will now describe these solutions briefly. We will call these *tubewise black brane solutions*.

These solutions, first obtained in [26], are manifestly regular at the horizon in ingoing Eddington-Finkelstein coordinates and can be constructed in the so-called derivative expansion. We start from the boosted black brane solution in Eddington-Finkelstein coordinates and then make the four-velocity u^μ and the temperature T functions of the boundary coordinates. This metric no longer is solution of Einstein's equation however we can correct the metric order by order in derivatives of the four-velocity and temperature with respect to the boundary coordinates. These solutions thus approximate a boosted black brane at every radial tube emanating from a patch in the boundary and the local values of the velocity and temperature parametrize local equilibrium of the boundary fluid. The dimensionless parameter controlling the derivative expansion is the ratio of typical spatio-temporal scale of variation of the hydrodynamic variables and the temperature of final equilibrium.

At the first order in the derivative expansion, the solution in ingoing Eddington-Finkelstein coordinates is

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr + G_{\mu\nu}dx^\mu dx^\nu, \qquad (1.43)\\ G_{\mu\nu} &= r^2 P_{\mu\nu} + \left(-r^2 + \frac{1}{b^4 r^2}\right) u_\mu u_\nu + 2r^2 bF(br)\sigma_{\mu\nu} - \\ &\quad r\left((u.\partial)u_\mu u_\nu - \frac{2}{3}u_\mu u_\nu(\partial.u)\right), \\ F(x) &= \frac{1}{4}\left(log\left(\frac{(x+1)^2(x^2+1)}{x^4}\right) - 2\arctan(x) + \pi\right), \end{aligned}$$

with $b(x) = 1/(\pi T(x)) = 1/r_0(x)$ and $\sigma_{\mu\nu}$ being the relativistic hydrodynamic shear-stress tensor whose explicit form is

$$\sigma_{\mu\nu} = \frac{1}{2}P_\mu{}^\alpha P_\nu{}^\beta(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \frac{1}{3}P_{\mu\nu}\partial_\alpha u^\alpha. \qquad (1.44)$$

Further, the hydrodynamic variables constituted by the four-velocity and temperature should satisfy the relativistic Navier-Stokes equation and can be written in terms of the expectation value of the energy-momentum tensor as

$$\begin{aligned} \partial^\mu t_{\mu\nu} &= 0, \\ t_{\mu\nu} &= \frac{\eta_{\mu\nu} + 4u_\mu u_\nu}{4b^4} - \frac{1}{2b^3}\sigma_{\mu\nu}. \end{aligned} \tag{1.45}$$

The relativistic Navier-Stokes equation above follow from components of Einstein equations which could be interpreted as constraints, while the dynamical equation is a two-derivative equation describing evolution of $G_{\mu\nu}$ in (1.43), purely in the radial coordinate. The metric and the fluid dynamical equations receive systematic higher derivative corrections order by order in the derivative expansion. The metric and the fluid mechanical equations are explicitly known up to second order in the derivative expansion. At each order in the derivative expansion, the dynamical equation of pure radial evolution remains the same, however the source term in the right hand side of this equation changes. We will have more to say about the structure of the metric and higher derivative Weyl covariant hydrodynamics we thus obtain later.

We readily see from (1.45) that the bulk viscosity of the hydrodynamics in the gauge theory vanishes and the dimensionless ratio of the shear viscosity and the entropy density, η/s is $1/4\pi$. We thus reproduce the first order hydrodynamic transport coefficients obtained in the previous subsection. However, this method of obtaining higher derivative hydrodynamics from the *tubewise black brane solutions* is more powerful as we can generate the full non-linear equations of hydrodynamics of the gauge theory while obtaining those higher order transport coefficients which cannot be defined through linear response theory. Further, these hydrodynamic equations are universal for this class of gauge theories at strong 't Hooft coupling and large rank of the gauge group, as they have been obtained from solutions of pure gravity holographically describing the universal sector.

We also find a very generic feature here which we will prove in the case of hydrodynamic stress tensors in the next chapter. The phenomenological transport coefficients appearing in the expansion of the energy-momentum tensor in the most general hydrodynamic ansatz consistent with Weyl covariance, get fixed uniquely by requiring the solution to have a regular future horizon. We will make a more general claim for solutions dual to nonequilibrium states which end up in thermal equilibrium in chapter 3 which will apply to even nonhydrodynamic states.

The dimensionless quantity η/s for the QCD quark-gluon plasma formed at RHIC is about 0.3 which is tantalizingly close to $1/4\pi$. It remains to be seen if the higher order transport coefficients are close to the values of the same for QCD quark-gluon plasma formed at RHIC.

1.5 Plan of the rest of the thesis

The plan for rest of the thesis is as follows. In Chapter 2, we will prove some general results about states in the universal sector. Here we will mainly use methods of gravity. We will be able to prove that the states in the universal sector and their dynamics can be completely determined by the expectation value of the energy-momentum tensor alone. Further, studying the structure of tubewise black brane solutions we will show that there are purely hydrodynamic states in the universal sector which can be completely characterized by hydrodynamic variables, and their dynamics can also be completely determined by the equations of fluid mechanics, even far away from equilibrium.

In chapter 3, we will try to obtain effective equations of motion for all states in the universal sector and argue that these equations are sufficient to model basic nonequilibrium processes even for states outside of the universal sector. We will show that we can construct these equations systematically in two expansion parameters. Here, we will use results from chapter 2, however we will mainly be using kinetic theory methods. We will also obtain conjectures for the regularity condition on all asymptotically AdS_5 solutions of pure gravity such that they will have regular future horizons.

In chapter 4 we will investigate a novel non-relativistic limit of gauge/gravity duality for conformal cases particularly to find out if hydrodynamics can be contained in this limit. In the process, we will obtain valuable clues of how to take this limit dynamically so that we get hydrodynamic behavior after the limit is taken. This chapter is a slight departure from the general theme of the thesis, but has been included with the hope that it can have some relevance for a tabletop experiment or a simulated system in the future. It may also turn out that the universal sector in this dynamical limit can be solved sometime in the future by appropriately exploiting the infinite-dimensional symmetry which appears in this limit, so the results here could be important steps taken in this direction as well.

Finally, we end with discussions on future work that needs to be done based on the results of chapters 2 and 3. In particular, we will focus on how we can investigate some novel questions and sharply defined older questions regarding the origin of irreversibility unraveled by our investigations.

Part II

On universal nonequilibrium phenomena in gauge/gravity duality and general phenomenological description of some basic nonequilibrium processes

Chapter 2

The energy-momentum tensor, the universal sector and purely hydrodynamic states

2.1 Introduction

Using gauge/gravity duality, we have defined the universal sector of conformal gauge theories with strong 't Hooft coupling and large rank of the gauge group as the set of states dual to solutions of pure gravity which are asymptotically AdS_5 spacetimes and have regular future horizons. The states constituting this sector are generically nonequilibrium states. Moreover, all basic nonequilibrium phenomena like decoherence, relaxation and hydrodynamics feature in the dynamics of these states. The strongly coupled nonequilibrium dynamics for states constituting this sector is the same for all theories in the class of conformal gauge theories with gravity duals.

In this chapter, we will try to address what kind of data characterizes the states of the universal sector when the gauge theory is *living* in Minkowski spacetime. This question is equivalent to asking what kind of boundary data uniquely characterizes the solutions of pure gravity in asymptotically AdS_5 spacetimes with regular future horizons, when the boundary metric is flat. We will prove that all such solutions are determined uniquely by the boundary stress tensor, or equivalently, the expectation value of the energy-momentum tensor of the dual field theory state. This is not so obvious as the boundary stress-tensor is not Cauchy data for pure gravity in asymptotically AdS_5 spacetimes. In fact, we will need to use regularity in a perturbative sense as a very crucial input to show that the boundary stress tensor uniquely determines the solution. This result is even more surprising from the field-theoretic point of view because the expectation value of a single operator, namely the energy-momentum tensor determines the dual state in the universal sector and its dynamics. We will try to attain a field-theoretic understanding of this result in the next chapter.

A theorem due to Fefferman and Graham [1] states that for any solution of Einstein's equation which is an asymptotically AdS_5 spacetime, we can always use the Fefferman-Graham coordinate system previously defined within a finite radial distance from the boundary. We will prove that the solution in these coordinates is given by a power series with no log terms when the boundary metric is flat. We will also argue using gauge/gravity duality that regular solutions should be given by a power series with no log terms in Fefferman-Graham coordinates whenever the Weyl anomaly of the gauge theory vanishes. A special case of this claim for even dimensional asymptotically AdS spacetimes has already been obtained by Fefferman and Graham [1]. Our argument suggests that asymptotically AdS_5 spacetimes which are regular solutions of even a higher derivative gravity theory with holographic CFT desciption, should also be given by a power series with no log terms in the Fefferman-Graham coordinates, when the boundary metric is flat.

The power series solution for asymptotically AdS_5 spacetime, as we will also prove, exists for any traceless boundary stress tensor where energy and momentum are conserved. However, any arbitrary traceless and conserved stress tensor will not correspond to the expectation value of the energy-momentum tensor of a CFT state. Gauge/gravity duality implies this can happen if and only if the solution is regular. Mathematically speaking the solution has to be regular in the five-dimensional UHP with possible real singularities only at infinity, that is for infinite value of the radial coordinate.

We will show that in the gravity solutions either of two distinct pathologies can occur. For stress tensors with pathology of the first kind the reverse question of finding the corresponding gravity solution will be ill posed. For such stress tensors, the formal power series solution of the metric in Fefferman-Graham coordinates will exist but this power series will have zero radius of convergence in the radial coordinate along radial tubes starting from certain patches in the boundary. These pathological stress tensors will be of the "asymptotic boundary condition destroying", or, in short, of "abcd" type. The other distinct set of pathological stress tensors will produce naked singularities in the bulk. [1]

We will argue that "abcd" type of stress tensors can be avoided if we construct the solution in a perturbation expansion about the final stationary late-time configuration. In pure gravity, it is expected that any solution at late time will settle down to a stationary single black brane. Multi black brane static solutions will not occur if there are no p-form gauge fields as is the case in pure gravity. Further, metastable configurations like small black holes do not occur if the boundary metric is flat. The perturbation expansion about the final stationary configuration can be more general than the derivative expansion used for constructing gravity duals of purely hydrodynamic states, which we have mentioned in the previous chapter and will soon have more to say about. In the following chapter, we will make claims about the most general nature of this perturbative expansion.

We will show that for such perturbative solutions the solution in Fefferman-Graham coordinates will break down for a particular value of the radial coordinate which is the location of the late-time horizon. Here we may either have a coordinate singularity or a real curvature singularity. Further, if it is a curvature singularity, this singularity is naked, i.e. not covered by a horizon. The naked singularity happens for pathological boundary stress tensors of the second type.

We will examine these issues by constructing gravity solutions with boundary stress tensors which can be parametrized purely hydrodynamically. We will construct these

[1] We expect that such pathologies will occur for stress tensors which are physically not viable, for instance, $t_{\mu\nu} = (\pi T_1)^4(4u_{1\mu}u_{1\nu} + \eta_{\mu\nu}) + (\pi T_2)^4(4u_{2\mu}u_\nu + \eta_{2\mu\nu})$, such that the flows $u_{1\mu}$ and $u_{2\mu}$ are not parallel to each other. This example represents a stress tensor of two fluids at different temperatures flowing in different directions without reaching mutual thermal and mechanical equilibrium. This is certainly impossible in any realistic theory. On the other hand, the second type of pathology will occur for stress tensors which are physically viable but have wrong values of various parameters, for instance, a hydrodynamic stress tensor for which η/s is different from $1/4\pi$.

solutions in the derivative expansion, about which we have mentioned in the previous chapter. We will see that indeed all these solutions are free of *abcd* type of pathology. We can systematically transform these solutions to Eddington-Finkelstein coordinate system order by order in the derivative expansion and then the regularity or the irregularity of these solutions will become manifest. We will see that when all the transport coefficients of the purely hydrodynamic Weyl covariant stress tensor of the most general form are correctly chosen order by order in the derivative expansion, the solution is free of naked singularities and coincides with the *tubewise black brane solutions* originally obtained in [26]. We will explicitly demonstrate this by constructing the solution in Fefferman-Graham coordinates up to first order in the derivative expansion for an arbitrary η/s, then transforming this solution to Eddington-Finkelstein coordinate system such that the regularity at the future horizon is manifest when η/s is $1/4\pi$ and the irregularity is manifest otherwise.

The derivative expansion in the Fefferman-Graham has some advantages over the same expansion in Eddington-Finkelstein coordinates developed in [26]. The first advantage is that the constraints simplify and just reduce to the tracelessness of the energy-momentum tensor and conservation of energy and momentum. Tracelessness simply follows from the construction of the energy-momentum tensor, and the conservation of energy and momentum just gives us the desired equations of fluid dynamics. The dynamical equation is, as usual an ultralocal equation and is basically just the radial evolution of the boundary metric along a radial tube emanating from a patch on the boundary given by a second order ordinary differential equation. The second advantage over the perturbation in Eddington-Finkelstein coordinates is that here the whole procedure will be Lorentz-covariant, whereas in the Eddington-Finkelstein coordinates we had to decompose all terms into tensors, vectors and scalars of $SO(3)$. In fact, the translation to Eddington-Finkelstein coordinates also preserves manifest Lorentz covariance. The third advantage is that we can also manifestly preserve the asymptotic boundary conditions and is therefore suited to generalizations like in non-conformal cases. In fact, this feature has already been exploited in the literature [23]. Given these features, one can think of the Fefferman-Graham coordinate system as the "Coulomb gauge" in the context of finding out metrics corresponding to arbitrary hydrodynamic stress tensors.

Our method will be shown to be equivalent to the procedure discussed in [26] in Eddington-Finkelstein coordinates. However, because our method allows us to construct gravity solutions corresponding to arbitrary hydrodynamic energy-momentum tensors and also prove that for a unique choice of transport coefficients at every order in the derivative expansion these solutions are regular, we can demonstrate that
(a) there are states with energy-momentum tensors which are purely hydrodynamic in the universal sector of the dual gauge theory, such that these can be determined by hydrodynamic variables alone even far away from equilibrium, and
(b) regularity of the future horizon determines all the hydrodynamic transport coefficients.

The organization of this chapter is as follows. In section 2, we show how the metric is determined by the boundary stress tensor. In section 3, we translate some known solutions like the tubewise black brane solutions in Fefferman-Graham coordinate system in a power series expansion as an illustration of general properties of the metric near the boundary. In section 4, we will set up and elucidate the derivative expansion in the Fefferman-Graham coordinates and establish that all hydrodynamic stress tensors preserve asymptotic AdS_5 boundary condition. In section 5, we will do the regularity analysis of our solutions. The proof of the existence of power series solutions and the technical issue of the futility of using curvature invariants to determine the regularity of solutions in perturbative expansion will be discussed in the Appendices A and B, which will be referred appropriately in this Chapter.

2.2 How the boundary stress tensor fixes the solution

In this section we will restrict our attention mainly to a five dimensional asymptotically AdS space with flat boundary metric, though we will indicate in the end that our results may be sufficiently generalized. We will soon explain what is meant by the boundary metric for asymptotically AdS spaces.

The Einstein-Hilbert action on 5-dim manifold M, with an appropriate counterterm to have a well defined variational principle with Dirichlet boundary condition is

$$S = \frac{1}{16\pi G_N}[-\int_M d^5x\sqrt{G}(R + \frac{12}{l^2}) - \int_{\partial M} d^4x\sqrt{\gamma}2K], \tag{2.1}$$

where K is the extrinsic curvature and γ is the induced metric on the boundary. We are using the convention of [9] in which the cosmological constant Λ of AdS_{d+1} is normalized to be $-\frac{d(d-1)}{2l^2}$, hence for AdS_5 we have $\Lambda = -\frac{6}{l^2}$.

We want to solve Einstein's equation

$$R_{MN} - \frac{1}{2}RG_{MN} = \frac{6}{l^2}G_{MN}, \tag{2.2}$$

subject to the condition that the solution is asymptotically AdS with a given conformal structure at the boundary. Fefferman and Graham have shown that for such solutions we can use a specific coordinate system called the Fefferman-Graham coordinate system near the boundary. In this coordinate system, the metric takes the following form,

$$ds^2 = G_{MN}dx^M dx^N = \frac{l^2}{\rho^2}[d\rho^2 + g_{\mu\nu}(\rho, z)dz^\mu dz^\nu]. \tag{2.3}$$

In the expression above the indices (M,N) run over all AdS coordinates and the indices (μ, ν) run over the four field theory coordinates. The boundary metric $g_{(0)\mu\nu}$ is defined as

$$g_{(0)\mu\nu}(z) = \lim_{\rho\to 0} g_{\mu\nu}(z, \rho). \tag{2.4}$$

Let this boundary metric have a conformal structure. Then it can be shown that any conformal transformation of the boundary coordinates (z) can be lifted to a bulk diffeomorphism of the Fefferman-Graham coordinates which preserves the form of the metric (2.3) [3, 4]. Under this bulk diffeomorphism, the boundary metric undergoes the same conformal transformation. The simplest case for instance will be a scale transformation, $z \to \lambda z$, of the boundary coordinates for which the corresponding bulk diffeomorphism will be $\rho \to \lambda\rho$ (note that in the case of the bulk diffeomorphism, the field theory coordinates z do not transform at all so that the boundary metric $g_{(0)\mu\nu}$ scales like $g_{(0)\mu\nu}(z) \to \lambda^{-2}g_{(0)\mu\nu}(z)$).

In the Fefferman-Graham coordinate system the various components of Einstein's equation reads as [9]: [2]

$$\frac{1}{2}g'' - \frac{3}{2\rho}g' - \frac{1}{2}g'g^{-1}g' + \frac{1}{4}Tr(g^{-1}g')g' - Ric(g) - \frac{1}{2\rho}Tr(g^{-1}g')g = 0, \tag{2.5}$$
$$\nabla_\mu Tr(g^{-1}g') - \nabla^\nu g'_{\mu\nu} = 0,$$
$$Tr[g^{-1}g''] - \frac{1}{\rho}Tr[g^{-1}g'] - \frac{1}{2}Tr[g^{-1}g'g^{-1}g'] = 0.$$

Here "($'$)" denotes a derivative with respect to ρ and ∇_μ is the covariant derivative constructed from the metric $g_{\mu\nu}$. Also in the above equations we have set our units such that l, the radius of AdS is set to unity.

When the boundary metric is flat, we will argue that we can expand $g_{\mu\nu}(z,\rho)$ in a simple integer power Taylor series of ρ with coefficients which are functions of z. Since we have chosen the boundary metric to be flat, the leading term has to be $\eta_{\mu\nu}$. Our power series ansatz will be

$$g_{\mu\nu}(z,\rho) = \eta_{\mu\nu} + \Sigma_{n=2}^{\infty} g_{(2n)\mu\nu}(z)\rho^{2n}. \tag{2.6}$$

We have written down only even powers of ρ in the above expansion because it follows from a result due to Fefferman and Graham [1] that the power series (2.6) should be an even function of ρ. [3] The only even term which is absent is $g_{(2)\mu\nu}(z)$ which follows as an easy consequence of the equations of motion (2.5).

It is not obvious that this power series ansatz will indeed provide us a solution, so we will give a simple argument why this works.This argument will hold for smooth solutions of classical gravity which are dual to the states in the CFT at zero or finite temperature. This argument will also apply when the theory of classical gravity receives

[2]The (minor) difference with the system of equations given in this reference will be that we will use the original Fefferman-Graham radial coordinate ρ, whereas there the radial coordinate is chosen to be the square root of ours. Also, the reference uses a definition of the Riemann tensor such that the scalar curvature of AdS comes out to be positive.

[3]The existence of power series solution has been proved by Fefferman and Graham for all even dimensional asymptotic AdS solutions and in case of odd dimensional asymptotic AdS solutions they also argued that if the solution is a power series it should be even. The Fefferman Graham coordinates are however unique only up to diffeomorphisms which are the lifts of the boundary conformal transformations into the bulk. Although, it is not obvious, it can also be shown [1] that the evenness of the series (2.6) is independent of the choice of any particular Fefferman-Graham coordinate system.

higher derivative corrections through which we take into account the corrections in square-root of the inverse 't Hooft coupling and inverse of the rank of the gauge group.

By AdS/CFT correspondence any solution of the bulk equations of motion would give us a state in the CFT, so the coefficients of the Taylor series expansion in (2.6) should be functions of the expectation values of the local operators in the dual CFT state. We will explicitly see below that all these coefficients are just functions of the expectation value of the energy-momentum tensor in the CFT state. It is possible to see the effect of space-time independent scale transformation on the CFT operators from $g_{\mu\nu}(z,\rho)$. To do this we have to lift the scale transformation to a bulk diffeomorphism so that the form of the metric (2.3) remains the same and the boundary metric also remains flat. This lift, as mentioned before, is achieved by $\rho \to \lambda\rho$. In the most general case it has been shown [13] that the form of the ansatz (2.6) should be modified by terms like $\rho^n(log(\rho))^m$ with non-negative n and m. To illustrate our argument we will consider just two such possible terms

$$g_{(n)}(z)\rho^n + h_{(n)}(z)\rho^n log(\rho).$$

Under the bulk scaling transformation $\rho \to \lambda\rho$,

$$g_{(n)}(z) \to \lambda^{n-2} g_{(n)}(z) - log(\lambda)\lambda^{n-2} h_{(n)}(z). \tag{2.7}$$

We find the above transformation by checking the new coefficient of ρ^n in $g_{\mu\nu}$ after the scale transformation. In a CFT any local operator simply scales like a power of λ, the power being given by the conformal dimension of the operator. A $log(\lambda)$ term is present only when the Weyl anomaly doesn't vanish. In flat space the Weyl anomaly vanishes and since we have chosen the boundary metric to be flat the log term in (2.7) should not be present as $g_{(n)\mu\nu}$ is a function of the expectation values of local operators. The absence of the $log(\lambda)$ term in a scale transformation applies not only to primary operators but also to their descendents. So we can argue that terms like $\rho^n(log(\rho))^m$ should be absent and $g_{\mu\nu}$ should be given by a simple power series of ρ.

This argument for why the power series ansatz should work will also apply when the theory of classical gravity receives higher derivative corrections through which we take into account the corrections in square-root of the inverse 't Hooft coupling and inverse of the rank of the gauge group. We have just used the fact that a conformal transformation in the boundary should have an appropriate lift to a bulk diffeomorphism consistent with the transformation of CFT operators. The transformation of the CFT operators under conformal transformations, as well, is independent of the value of the coupling or the rank of the gauge group. In fact one can readily check that exact static black hole solutions of Gauss-Bonnet gravity which are asymptotically AdS [31] or tubewise black brane solutions in higher dervative gravity [32] have power series expansion when written in Fefferman-Graham coordinates.

However, our argument, of course, breaks down when we consider an arbitrary solution of Einstein's equation with a negative cosmological constant, i. e. if the boundary stress tensor does not correspond to any CFT state. In Appendix A, we have given the general proof of the existence of the power series solution for AdS_5 asymptotics, so that even for such cases we can state that the solution, is indeed, a power series. In fact we will explicitly see, that for all hydrodynamic stress tensors, whether they do or do not correspond to CFT states, the solutions are always power series.

Now we will substitute our ansatz (2.6) in the equations of motion (2.5) and solve them order by order in powers of ρ. It is known from earlier work of Skenderis et.al. [9] that the first term $g_{(4)\mu\nu}(z)$ is just the expectation value of the stress tensor. Briefly this is how it comes about to be so. Upto this order the first equation (the tensor equation) identically vanishes while the second and third equation of motion give

$$\begin{aligned} Tr(g_{(4)}) &= 0, \\ \partial^{\mu} g_{(4)\mu\nu} &= 0. \end{aligned} \tag{2.8}$$

Since the equations of motion by themselves cannot specify $g_{(4)}$ we need a data from the CFT to specify it subject to the above constraints. Most naturally $g_{(4)}$ is the traceless conserved stress tensor of the CFT. However we can also explicitly check this. An explicit calculation shows that $g_{(4)}$ is indeed the Balasubramanian-Kraus stress tensor [**?**] which could be defined for any asymptotically AdS space. Hence we may write

$$g_{(4)\mu\nu} = t_{\mu\nu}. \tag{2.9}$$

With our ansatz (2.6) it turns out that all the other coefficients $g_{(2n)}$ $(n > 2)$ are fixed uniquely by the equations of motion in terms of $g_{(4)}$ and its derivatives (or in other words the stress tensor and its derivatives). We observe that the first and the third of the equations of motion (2.5) (i.e. the tensor and the scalar equations) are sufficient to solve for $g_{(n)}$. All the higher powers of the second of the equations of motion (2.5) (i.e the vector equation) identically vanishes on imposing the constraints (2.8) i.e. by imposing the tracelessness and the conservation of the stress tensor. It is not difficult to argue that this should be the case because it can be shown [9] that the second (i.e the vector) equation of motion simply implies the conservation of the Brown-York stress tensor (which when regulated becomes the holographic boundary stress tensor discussed in the Introduction) for an arbitrary constant ρ hypersurface. Now the conservation of the Brown-York stress tensor at a given hypersurface is not independent of the same requirement for another hypersurface, because in the ADM-like formulation of the Einstein's equations if we satisfy our constraints at a given hypersurface in which our initial conditions are given the evolution (here in the radial coordinate ρ) automatically satisfies the constraints. The conservation of the Brown-York stress tensor at the boundary is already forced at leading order in ρ of the vector equation of motion through (2.8). Hence we should expect that the vector equation should not impose any new constraints on the stress tensor given that the tensor and scalar equations specify all the coefficients uniquely and this is exactly what is borne out. In our proof in Appendix A, we show how the tensor, vector and scalar equations of motion turn out to be consistent with each other when we employ the power series ansatz.

Below we give the a few of the the coefficients $g_{(n)\mu\nu}$

$$g_{(6)\mu\nu} = -\frac{1}{12}\Box t_{\mu\nu},$$

$$g_{(8)\mu\nu} = \frac{1}{2}t_{\mu}{}^{\rho}t_{\rho\nu} - \frac{1}{24}\eta_{\mu\nu}(t^{\alpha\beta}t_{\alpha\beta}) + \frac{1}{384}\Box^{2}t_{\mu\nu},$$

$$
\begin{aligned}
g_{(10)\mu\nu} = & -\frac{1}{24}(t_\mu{}^\alpha \Box t_{\alpha\nu} + t_\nu{}^\alpha \Box t_{\alpha\mu}) \\
& + \frac{1}{180}\eta_{\mu\nu} t^{\alpha\beta}\Box t_{\alpha\beta} + \frac{1}{360} t^{\alpha\beta}\partial_\mu\partial_\nu t_{\alpha\beta} \\
& - \frac{1}{120} t^{\alpha\beta}(\partial_\mu\partial_\alpha t_{\beta\nu} + \partial_\nu\partial_\alpha t_{\beta\mu}) \\
& + \frac{1}{60} t^{\alpha\beta}\partial_\alpha\partial_\beta t_{\mu\nu} - \frac{1}{180}\partial_\mu t^{\alpha\beta}\partial_\nu t_{\alpha\beta} \\
& + \frac{1}{720}\eta_{\mu\nu}\partial_\alpha t^{\beta\gamma}\partial^\alpha t_{\beta\gamma} \\
& + \frac{1}{120}(\partial_\mu t^{\alpha\beta}\partial_\alpha t_{\beta\nu} + \partial_\nu t^{\alpha\beta}\partial_\alpha t_{\beta\mu}) \\
& - \frac{1}{60}\partial_\alpha t^\beta_\mu \partial_\beta t^\alpha_\nu - \frac{1}{23040}\Box^3 t_{\mu\nu},
\end{aligned}
$$

$$
g_{(12)\mu\nu} = \frac{1}{6} t_\mu{}^\alpha t_\alpha{}^\beta t_{\beta\nu} - \frac{1}{72} t_{\mu\nu}(t^{\alpha\beta}t_{\alpha\beta}) + \ldots\ldots\ldots \quad . \tag{2.10}
$$

Here, as before in (2.5) the boundary indices are raised and lowered by $\eta_{\mu\nu}$ and $\Box$ is the Laplacian in flat space. Let us observe and explain certain simple features of the results above. The first observation is that every term in the RHS of the above equations contain only even number of derivatives. This is so because the terms containing derivatives originate only from Ric(g) in the first of the equations of (2.5). The second observation is that the terms independent of the derivatives appear only for $g_{(4n)}$. This is so because if we omit Ric(g) in the first of the equations of (2.5), then the solution is a power series in ρ^{4n} as the first non-trivial term in the series is $g_{(4)}$. So for a solution where the stress tensor is uniform (like in the case of a static black brane solution), g has an expansion containing only ρ^{4n} terms.

With our argument that the ansatz (2.6) should give us a consistent solution, it is obvious that the stress tensor, which appears as $g_{(4)}$ in g uniquely specifies the solution because all the higher coefficients are fixed uniquely in terms of $g_{(4)}$ with no new constraints like (2.8) appearing for $g_{(4)}$. This completes the argument that when the boundary metric is flat we should have a solution uniquely specified locally by the stress tensor alone. This statement readily generalizes to other dimensions in the case of a flat boundary metric and most likely also generalizes when the boundary metric is not flat. The general validity could be argued for on the basis of the equations of motion (2.5) which are second order (specifically in derivatives of ρ). Intuitively the boundary metric and the stress tensor specifies all the initial data we need for a unique solution, however a concrete demonstration of this would probably require methods beyond what we have employed here.

The argument we have given above, however, cannot be reversed to argue that a solution with asymptotic AdS_5 boundary conditions exists for any arbitrary stress tensor. The reason that we can't reverse the argument is that the series (2.6) for $g_{\mu\nu}$ exists only formally. The coefficients $g_{(n)}$ may not be well behaved at large n, for an arbitrary stress tensor. We will give a simple example to show what can go wrong. For a specific choice of stress tensor, we may find that $g_{(n)\mu\nu} = f(n)s_{\mu\nu}$ plus other terms. Here $s_{\mu\nu}$ is a specific term in the stress tensor. If, for instance, the series $\Sigma_n f(n)\rho^n$ has zero radius of convergence, $g_{\mu\nu}$ will not be a meaningful series of ρ as it will also have zero radius of convergence in ρ. Such boundary stress tensors, for which $g_{\mu\nu}$ has zero radius of convergence in ρ, could be appropriately called, "asymptotic boundary condition destroying" stress tensor or in short "abcd" stress tensor. We will have more to say about such stress tensors in section 4.[4]

2.3 Mutual translation between Eddington-Finkelstein and Fefferman-Graham coordinates

In the previous section, we have seen that, the Fefferman-Graham coordinate system is good for finding a solution to Einstein's equation with a negative cosmological constant when the corresponding boundary stress tensor is specified. However the solutions are usually found in other coordinate systems. For instance, the static black brane solution is usually described in the Schwarzchild-like coordinate system and the hydrodynamic metric of [26] has been found in the Eddington-Finkelstein coordinate system. It would be useful to see how we can rewrite these solutions in the Fefferman-Graham coordinate

[4]Interestingly, Fefferman and Graham have shown in [1] that for even dimensional asymptotic AdS solutions, $g_{\mu\nu}$ always has a finite radius of convergence in ρ. However their argument does not readily generalize to the odd dimensional case.

system asymptotically. We will demonstrate a novel technique towards this end for the boosted black brane and the hydrodynamic metrics. In both cases we will see that we can achieve a mutual translation between Eddington-Finkelstein coordinate system and Fefferman-Graham coordinate system by using a power series ansatz similar to (2.6) and we can solve this ansatz algebraically order by order. We expect this method to work for all solutions in which the boundary metric is flat, or more generally when the Weyl anomaly vanishes.

The general procedure is as follows. In the Eddington-Finkelstein coordinates (x^μ, r) the metric takes the form

$$ds^2 = -2u_\mu(x)dx^\mu dr + G_{\mu\nu}(x,r)dx^\mu dx^\nu. \tag{2.11}$$

Here we are using ingoing Eddington-Finkelstein coordinate system, so that u^μ is a four-velocity (hence $u_\mu u_\nu \eta^{\mu\nu} = -1$) such that it is directed forward in time. We will express the general structure of coordinate transformation from the Eddington-Finkelstein coordinates (x^μ, r) to Fefferman-Graham coordinates (z^μ, ρ) as below

$$d\rho = p_\mu(r,x)dx^\mu + q(r,x)dr, \tag{2.12}$$

$$dz^\mu = m^\mu{}_\nu(r,x)dx^\nu + n^\mu(r,x)dr \ . \tag{2.13}$$

We substitute the above in the Fefferman-Graham form of the metric (2.3) to get

$$\begin{aligned} ds^2 = \frac{1}{\rho^2}[&(p_\mu p_\nu + g_{\eta\xi}(\rho,z)m^\eta{}_\mu m^\xi{}_\nu)dx^\mu dx^\nu + 2(p_\mu q + g_{\xi\sigma}(\rho,z)m^\xi{}_\mu n^\sigma)dx^\mu dr \\ &+ (q^2 + g_{\mu\nu}(\rho,z)n^\mu n^\nu)dr^2]. \end{aligned} \tag{2.14}$$

Comparing the above with the Eddington-Finkelstein form of the metric (2.11), we get the following set of equations,

$$\begin{aligned} (q(x,r))^2 + g_{\mu\nu}(\rho,z)n^\mu(x,r)n^\nu(x,r) = 0, \\ 2p_\mu(x,r)q(x,r) + g_{\alpha\beta}(\rho,z)(m^\alpha{}_\mu(x,r)n^\beta(x,r) + m^\beta{}_\mu(x,r)n^\alpha(x,r)) \\ = -2u_\mu(x)(\rho(x,r))^2, \\ p_\mu(x,r)p_\nu(x,r) + g_{\alpha\beta}(\rho,z)m^\alpha{}_\mu(x,r)m^\beta{}_\nu(x,r) \\ = G_{\mu\nu}(x,r)(\rho(x,r))^2. \end{aligned} \tag{2.15}$$

So we have a scalar, a vector and a tensor equation and three unknowns to solve for. The unknowns are a scalar $\rho(x,r)$, a vector $z^\mu(x,r)$ and the tensor $g_{\mu\nu}(z,\rho)$ which appear in the Fefferman-Graham metric (2.3). It is clear from the definitions (2.12) of q, etc. that they are just various partial derivates of (ρ, z), for instance $q = \partial_r \rho$, etc. We will make the following general ansatz to solve the above equations. The ansatz for ρ and z^μ will be that they will be an integer power series of the inverse of the Eddington-Finkelstein radial coordinate r.

$$\begin{aligned} \rho &= \frac{1}{r} + \frac{\rho_2(x)}{r^2} + \frac{\rho_3(x)}{r^3} + \text{.........} , \\ z^\mu &= x^\mu + \frac{z_1^\mu(x)}{r} + \frac{z_2^\mu(x)}{r^2} + \text{.....} . \end{aligned} \tag{2.16}$$

To solve the equations of transformation (4.53), the above should be supplemented with the ansatz (2.6) for the $g_{\mu\nu}(z,\rho)$ in the Fefferman Graham metric. The expressions for the partial derivatives like q, etc. then turn out to be as below:

$$\begin{aligned} q = \partial_r \rho &= -\frac{1}{r^2} - \frac{2\rho_2}{r^3} - \frac{3\rho_3}{r^4} - \text{.....} , \\ p_\mu = \partial_\mu \rho &= \frac{\partial_\mu \rho_2}{r^2} + \frac{\partial_\mu \rho_3}{r^3} + \text{.....} , \\ n^\mu = \partial_r z^\mu &= -\frac{z_1^\mu}{r^2} - \frac{2z_2^\mu}{r^3} - \text{....} , \\ m^\mu{}_\nu = \partial_\nu z^\mu &= \delta^\mu_\nu + \frac{\partial_\nu z_1^\mu}{r} + \frac{\partial_\nu z_2^\mu}{r^2} + \text{....} . \end{aligned} \tag{2.17}$$

One thing to be kept in mind is that when we substitute our ansatz (2.16) to solve the equations of transformation (4.53), $g_{\mu\nu}(\rho, z)$ should be re-expressed as functions of (x,r). Below, we just give the first three terms which appear after it is rewritten as functions of (x,r).

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{t_{\mu\nu}(x)}{r^4} + \frac{(4\rho_2 t_{\mu\nu} + (z_1.\partial)t_{\mu\nu})(x)}{r^5} + \text{....} . \tag{2.18}$$

We now consider a boosted black brane metric in Eddington-Finkelstein coordinate

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu, \tag{2.19}$$

where

$$f(r) = 1 - \frac{1}{r^4}, \tag{2.20}$$

$$u^0 = \frac{1}{\sqrt{1-\beta_i^2}}, \tag{2.21}$$

$$u^i = \frac{\beta_i}{\sqrt{1-\beta_i^2}}. \tag{2.22}$$

and the temperature is $T = \frac{1}{\pi b}$ and the three-velocity β_i are all constants, and

$$P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}, \tag{2.23}$$

is the projector onto the spatial hypersurface orthogonal to the four velocity u^μ. This metric can be obtained by applying a boost parameterized by the three-velocity β_i and a scaling by b to the usual AdS black hole with unit temperature where the time coordinate t is itself a Killing vector. In this case actually the exact transformation from Eddington-Finkelstein to Fefferman-Graham coordinate system can be exactly worked out easily and it is given by:

$$\begin{aligned} \rho &= \frac{\sqrt{2}b}{\sqrt{b^2r^2 + \sqrt{b^4r^4 - 1}}}, \\ z^\mu &= x^\mu + u^\mu b k(br), \\ k(y) &= \frac{1}{4}(log(\frac{y+1}{y-1}) - 2arctan(y) + \pi). \end{aligned} \tag{2.24}$$

The solution for $g_{\mu\nu}$ in the Fefferman-Graham metric (2.3) for the boosted black brane is given by:

$$g_{\mu\nu}(z,\rho) = (1 + \frac{\rho^4}{4b^4})\eta_{\mu\nu} + \frac{4\rho^4}{4b^4+\rho^4}u_\mu u_\nu. \tag{2.25}$$

The boundary stress tensor could be easily read off by looking at the coefficient of ρ^4 after Taylor expanding the RHS of the above expression. The stress tensor turns out to be that of an ideal conformal fluid (like that of a gas of photons)

$$t_{0\mu\nu} = {g_{(4)}}_{\mu\nu} = \frac{1}{4b^4}[4u_\mu u_\nu + \eta_{\mu\nu}], \tag{2.26}$$

where the temperature is $T = \frac{1}{\pi b}$. The horizon in the Fefferman-Graham coordinates is at $\rho = \sqrt{2}b$ and at the horizon $g_{\mu\nu}$ given by (2.25) is not invertible as $g_{\mu\nu}(\rho = \sqrt{2}b, z) = 2P_{\mu\nu}$.

So clearly the Fefferman-Graham coordinate system has a coordinate singularity at the horizon. Also it is easy to check from (2.24) that the change of coordinates also becomes singular at the horizon.

Now we turn to the hydrodynamic metric found in [26] which is a solution to Einstein's equation upto first order in the derivative expansion and has a regular horizon. Here the "maximally commuting Goldstone parameters" of the boosted black brane solution, the velocities β^i and the temperature T are functions of the field theory coordinates (x). The $G_{\mu\nu}$ in the Eddington-Finkelstein form of the metric (2.11) is:

$$G_{\mu\nu} = r^2 P_{\mu\nu} + (-r^2 + \frac{1}{b^4 r^2}) u_\mu u_\nu + 2r^2 bF(br)\sigma_{\mu\nu} - r((u.\partial)u_\mu u_\nu - \frac{2}{3} u_\mu u_\nu (\partial.u)), \quad (2.27)$$

with

$$F(x) = \frac{1}{4}(log(\frac{(x+1)^2(x^2+1)}{x^4}) - 2arctan(x) + \pi). \quad (2.28)$$

In this case we will solve the set of equation (4.53) by putting in our ansatz (2.16). We solve order by order for each power n in r^{-n}. At each order we have to solve algebraic equations and remarkably the equations can be consistently solved at each order. It is important to throw away all the terms which have two x-derivatives or more and solve the series for ρ and z^μ given in (2.16) and the series for $g_{\mu\nu}$ given in (2.6) only up to first derivative order. This is justified because the hydrodynamic metric above in Eddington-Finkelstein form is a solution to Einstein's equation only up to first order in x-derivatives and hence it can have a Fefferman-Graham expansion near the boundary only upto first derivative order. The results of the non-vanishing terms in the expansion for ρ and z^μ in (2.16) upto r^{-9} order are given below:

$$\rho_2 = \frac{1}{3}(\partial.u), \rho_5 = \frac{1}{8b^4}, \rho_6 = \frac{13(\partial.u)}{120b^4}, \rho_9 = \frac{7}{128b^8}, \quad (2.29)$$
$$z_1^\mu = u^\mu, z_2^\mu = \frac{1}{3}u^\mu(\partial.u), z_5^\mu = \frac{u^\mu}{5b^4},$$
$$z_6^\mu = \frac{9u^\mu(\partial.u) + 7(u.\partial)u^\mu}{60b^4}, z_9^\mu = \frac{u^\mu}{9b^8}.$$

We can easily observe some patterns in the results above. Firstly the terms without any derivatives only appear as coefficients of r^{-4n-1}. These are precisely the terms that appear in the expansion for the case of the boosted black brane as given in (2.24). This is because the original black brane solution in Fefferman-Graham coordinates as we know from (2.25) is a series with "gaps" of four (which means only the fourth next term is non-zero). So the solution of (4.53) should provide a series for ρ and z^μ in gaps of four as well. Secondly, it also turns out that the terms which have first derivative pieces occur for $\rho_2, \rho_6, z_2^\mu, z_6^\mu$, etc. again in gaps of four. We obtain the coefficients of the series for $g_{\mu\nu}$ given in (2.6) which was part of our ansatz. The second non-zero term in the series gives us the boudary stress tensor

$$t_{\mu\nu} = g_{(4)\mu\nu} = \frac{\eta_{\mu\nu} + 4u_\mu u_\nu}{4b^4} - \frac{1}{2b^3}\sigma_{\mu\nu}, \tag{2.30}$$

where

$$\sigma_{\mu\nu} = P_\mu{}^\alpha P_\nu{}^\beta \partial_{(\alpha} u_{\beta)} - \frac{1}{3} P_{\mu\nu} \partial_\alpha u^\alpha. \tag{2.31}$$

This is stress tensor for a relativistic conformal fluid satisfying Navier-Stokes' equation and with $\eta/s = 1/4\pi$. The next non vanishing term in the series for $g_{\mu\nu}$ is:

$$g_{(8)\mu\nu} = -\frac{u_\mu u_\nu}{4b^8} - \frac{\sigma_{\mu\nu}}{8b^7}. \tag{2.32}$$

We can check that the expression for $g_{(8)}$ is given by the general results of the the previous section when we substitute the dissipative stress tensor (2.30) in (2.10).

In this section we have worked out the case for a specific "hydrodynamic metric" given in [26]. This metric has no naked singularities and this corresponds to the choice of $\eta/s = 1/4\pi$ in the dissipative stress tensor (2.31). However we will see in section 5 that our ansatz (2.16) for translation between the Eddington-Finkelstein and Fefferman-Graham coordinates will work even when the above is not the case, i.e the metric contains naked singularities. In what follows we will reverse the translation. That is, we will work out the Fefferman-Graham form of the metric exactly upto first order in derivatives first and then find out the Eddington-Finkelstein form of the metric also exactly upto first order in derivatives. We will see that the power series ansatz (2.16) is consistent for any metric corresponding to an arbitrary hydrodynamic stress tensor.

2.4 The derivative expansion in Fefferman-Graham coordinates

We have already seen that the Fefferman-Graham form of the metric is the ideal one to use if we are asking given a boundary stress tensor what the corresponding solution of Einstein's equations of motion should be. The most general hydrodynamic stress tensor for a conformal fluid (in the Landau gauge) upto first order in derivatives is as below

$$t_{\mu\nu}(z) = \frac{\eta_{\mu\nu} + 4u_\mu(z)u_\nu(z)}{4b(z)^4} - \frac{\gamma}{2b(z)^3}\sigma_{\mu\nu}(z), \tag{2.33}$$

with $\sigma_{\mu\nu}(z)$ given by (2.31), b related to the temperature through $b = 1/\pi T$ and γ an arbitrary constant. However here, unlike in the case of the specific solution (without naked singularities) we considered in the previous section, $\eta/s = \gamma/4\pi$ and hence is arbitrary. We now ask what would be the corresponding solution for this arbitrary case.

Before we get into this specific case, we will show that we can get some insights into the reverse question from some generally known facts and our previous results given in section 2. We have seen, briefly, at the end of section 2 that the reverse question is ill posed for an *abcd* (asymptotic boundary condition destroying) stress tensor, for which the formal power series (2.6) for $g_{\mu\nu}$ has zero radius of convergence in ρ. One must devise a strategy in which such stress tensors do not appear at all. To this end we may always exploit a general property of solutions of Einstein's equation that in the long run the solution always becomes stationary. For the moment let us further restrict to those solutions which have no (ADM) angular momentum or any other (ADM) conserved charges (like the R-charge). These will, in the long run, settle down to the known boosted black brane solution (2.19). Static multi black brane like solutions do not appear if we turn off p-form gauge fields, so if more than one black brane are present they eventually will collapse to form a single black brane. A good strategy to recover all solutions will be to perturb around the late-time static black brane and build up all solutions in a systematic derivative expansion. Since any solution would eventually become static (or equilibrate) this strategy should always work at sufficiently late times.

Since the approach to equilibrium at long time scales and length scales can be naturally described by hydrodynamics, one can intuitively expect that the late time behavior of the solutions will correspond to a hydrodynamic description in terms of the boundary theory *if the equilibrium can be described in terms of a perfect fluid.* The boundary stress tensor of a boosted black brane indeed corresponds to that of a perfect conformal fluid like that of photons in pure QED. Our expectation is indeed borne out by the fact that all solutions in the derivative expansion correspond to a traceless conserved hydrodynamic boundary stress tensor, but with arbitrary number of derivatives. We will see that in the derivative expansion at each order the solutions always have finite radius of convergence away from the boundary, so we can conclude that all hydrodynamic stress tensors are asymptotic boundary condition preserving.

In fact, it is also easy to argue that whenever we construct the solutions of the full non-linear equations of motion of gravity perturbatively such that the dynamical equation at each order in the expansion will become ultralocal, i.e. an ordinary differential equation in the radial coordinate, we should have the feature that these solutions at every order in the expansion will be free of *abcd* type of pathology. This will remain true even beyond the hydrodynamic regime. The dynamical equation will be the same at every order in the expansion but the source term will differ. If the source is well behaved, the solution has a singularity at the location of the unperturbed horizon. This singularity can be just a coordinate singularity or a true curvature singularity. If it is not a true singularity, it has to be naked singularity, because at late time the singularity coincides with the original horizon, which should have coincided with the actual future event horizon in case the solution had a smooth future event horizon. In the next chapter we will argue that all such solutions can be constructed in two expansion parameters, one of which will be the derivative expansion parameter of hydrodynamics.

The fact that all hydrodynamic stress tensors preserve the asymptotic AdS boundary condition should have a certain measure of validity even for solutions with net angular momentum. In fact in [33], it has been shown that a large class of rotating black holes in AdS can be described by perfect fluid hydrodynamics. However, we do not know how general the result is. For any solution if the hydrodynamic description holds for the stationary solution to which a given solution eventually equilibrates, it can be expected to hold for sufficiently late times as well. Hence solutions can be constructed in the derivative expansion. Therefore, certainly a large class of solutions even in the sector with net angular momentum which can be constructed by perturbing around certain stationary solutions, will have a hydrodynamic description at least at late times.

To build up a solution corresponding to an arbitrary hydrodynamic stress tensor, we will work in the Fefferman-Graham coordinate system as we have said before and we will construct the solution exactly order by order in the derivative expansion. To develop the derivative expansion we follow the same method which the authors of [26] followed but now in the Fefferman-Graham coordinate system. In fact, based on the results of section 2, we will see that their method simplifies in these coordinates. We take the boosted black brane solution with $g_{\mu\nu}$ of the form of (2.25), but now the "maximally commuting Goldstone parameters" (u^μ, b) are arbitrary functions of z. We will call this the zeroth order metric g_0 which is no more a solution to Einstein's equation, so we need to correct this with g_1 which will now depend on the first derivatives of the "maximally commuting Goldstone parameters" (u^μ, b). This correction g_1 can be found substituting $g = g_0 + g_1$ in our equations of motion (2.5) and retaining only terms which have no more than one derivative of z.

The first of the equations of motion (2.5), i.e the tensor equation gives us a source free linear equation for g_1 which is second order in the derivatives of ρ and has no z-derivatives.

$$\begin{aligned}&\frac{1}{2}g_1^{''} - \frac{3}{2}\frac{g_1^{'}}{\rho} - \frac{1}{2}g_1^{'}g_0^{-1}g_0^{'} - \frac{1}{2}g_0^{'}g_0^{-1}g_1^{'} + \frac{1}{2}g_0^{'}g_0^{-1}g_1g_0^{-1}g_0^{'} \\ &+ \frac{1}{2}\left(Tr(g_0^{-1}g_1^{'}) - Tr(g_0^{-1}g_1g_0^{-1}g_0^{'})\right)(\frac{g_0^{'}}{2} - \frac{g_0}{\rho}) + \frac{1}{2}Tr(g_0^{-1}g_0^{'})(\frac{g_1^{'}}{2} - \frac{g_1}{\rho}) = 0.\end{aligned} \tag{2.34}$$

At the first order in derivative expansion, the only term which can provide a source term is Ric(g) since it has no derivatives of ρ. However Ric(g) contains at least two derivatives of z, so at this order the source vanishes.

At the first order the second of the equations of motion, which is a vector equation gives us the following:

$$\nabla_{0\mu} Tr(g_0^{-1} g_0') - \nabla_0^{\nu} g_{0\mu\nu}^{'} = 0, \tag{2.35}$$

where ∇_0 is the covariant derivative constructed from g_0.The major simplification which occurs in the Fefferman-Graham coordinates is the general observation in section 2, that this gives us nothing but the conservation of the stress tensor. It may be checked that if we choose to solve this vector fluctuation equation order by order in powers of ρ, like we did in section 2, at the leading order we would get $\partial^{\mu} t_{o\mu\nu} = 0$, where $t_{0\mu\nu}$ is the perfect fluid stress tensor (2.26) and all the coefficients of the higher powers of ρ will vanish identically once the leading order condition is imposed. This simplification will happen at every order in the derivative expansion, which means that if t_{n-1} is the stress tensor upto n-1 th order in the derivative expansion, at the n-th order the second equation will simply imply the conservation of t_{n-1}.

At the first order in the derivative expansion the third equation of motion vanishes identically. It is easy to see why this will happen. Again we go back to the general observations of section 2. If $t_{\mu\nu} = t_{0\mu\nu} + t_{1\mu\nu}$ with $t_{0\mu\nu}$ given by the perfect fluid stress tensor (2.26) and $t_{1\mu\nu}$ is the first order correction to the stress tensor satisfying the tracelessness and the Landau gauge $u^{\mu} t_{1\mu\nu} = 0$ conditions, then the correction to the coefficients of the power series expansion $g_{(n)\mu\nu}$ (some of which are listed in (2.10)) is simply proportional to $t_{1\mu\nu}$. The first order derivatives of $t_{0\mu\nu}$ doesn't appear because, as we have observed the general expressions for $g_{(n)}$ must contain even number of derivatives of $t_{0\mu\nu}$. It follows that the correction to the zeroth order metric, g_1, is proportional to t_1. It also follows from the the tracelessness of t_1 and the Landau gauge condition that the third equation vanishes identically as all traces appearing in the equation vanish. We will soon see that, this simplifying feature also, remarkably generalizes to all orders in the derivative expansion.

In the Fefferman-Graham coordinates the first order correction to the metric g_1 is, therefore, proportional to the first order correction to the stress tensor which is proportional to $\sigma_{\mu\nu}$ and therefore g_1 takes the form of $\gamma^{'} b \sigma_{\mu\nu} f(\rho)$, where $\gamma^{'}$ is an arbitrary constant. Substituting this in the tensor equation (2.34), we find that $f(\rho)$ satisfies the following differential equation

$$f'' - f'\frac{(12b^4-\rho^4)(4b^4+3\rho^4)}{\rho(16b^8-\rho^8)} + f\frac{128\rho^6 b^4}{(4b^4+\rho^4)(16b^8-\rho^8)} = 0. \tag{2.36}$$

We already know that the solution is a power series in ρ^4, so we change our variable ρ to $x = \rho^4$. The equation now reads

$$f'' - f'\frac{8b^4}{16b^8-x^2} + f\frac{8b^4}{(4b^4+x)(16b^8-x^2)} = 0. \tag{2.37}$$

The solution of this differential equation which vanishes at the boundary (after re-substituting x with ρ^4) [5] is

$$(1+\frac{\rho^4}{4b^4})log\left(\frac{1-\frac{\rho^4}{4b^4}}{1+\frac{\rho^4}{4b^4}}\right). \tag{2.38}$$

The metric in Fefferman-Graham coordinates up to first order then is

$$ds^2 = \frac{d\rho^2 + g_{\mu\nu}(\rho,z)dz^\mu dz^\nu}{\rho^2},$$

$$g_{\mu\nu}(\rho,z) = (1+\frac{\rho^4}{4b^4})\eta_{\mu\nu} + \frac{4\rho^4}{4b^4+\rho^4}u_\mu u_\nu + \gamma' b\sigma_{\mu\nu}(1+\frac{\rho^4}{4b^4})log\left(\frac{1-\frac{\rho^4}{4b^4}}{1+\frac{\rho^4}{4b^4}}\right). \tag{2.39}$$

To read off the stress tensor upto first order, we simply need the ρ^4 term in the Taylor expansion of $g_{\mu\nu}$. We get

$$t_{\mu\nu} = \frac{\eta_{\mu\nu}+4u_\mu u_\nu}{4b^4} - \frac{\gamma'}{2b^3}\sigma_{\mu\nu}. \tag{2.40}$$

Comparing with (2.33) we get that we must set $\gamma' = \gamma$ in the first order metric (2.39) to get the desired solution corresponding to the boundary stress tensor.

One very interesting feature of our solution at the first order can be found out by putting $\gamma' = \gamma = 0$. This implies that our zeroth order solution itself, now with velocities and temperatures satisfying the relativistic Euler equation, is an exact solution of Einstein's equations up to first order. Such is never the case in Eddington-Finkelstein coordinate system where as we will see we need to correct the zeroth order solution even for a dissipation-less stress tensor so that the solution is exact up to first order. We do not understand any deep reason for this feature of our solution.

[5] The other solution is $f_2 = 1 + \frac{\rho^4}{4b^4}$

Now we can proceed to examine the higher orders in the derivative expansion. Though we will postpone explicit solutions beyond the first order for a future publication, here we will show that it is trivial to satisfy the vector and scalar constraints at each order in perturbation theory. The tensor equation takes the following form at each order in perturbation theory:

$$\begin{aligned} &D_1 g_{n\mu\nu} + D_2(g_{n\mu\rho}u^\rho u_\nu + g_{n\nu\rho}u^\rho u_\mu) + D_3(g_{n\rho\sigma}\eta^{\rho\sigma})\eta_{\mu\nu} + D_4(g_{n\rho\sigma}\eta^{\rho\sigma})u_\mu u_\nu \\ &+ D_5(g_{n\rho\sigma}u^\rho u^\sigma)\eta_{\mu\nu} + D_6(g_{n\rho\sigma}u^\rho u^\sigma)u_\mu u_\nu = s_{n\mu\nu}(z,\rho), \end{aligned} \tag{2.41}$$

where D_1, D_2, etc. are linear differential operators involving derivatives in the radial coordinate only and $s_{n\mu\nu}(z,\rho)$ is the source term which is a (nonlinear) function of the corrections to the metric up to n-1 th order in the derivative expansion. The left hand side of the above equation is in fact the same as in (2.34) with g_1 replaced by the n-th order correction to the metric g_n, but now source terms are present on the right hand side. Also the differential operator D_1 is the same as the operator which acts on f in (2.36) at every order in the derivative expansion. We dropped the operators D_2, D_3, etc. at the first order, i.e. for g_1, because as we saw the general results of section 2 (equations in (2.10) for instance) forced it to be proportional to be stress tensor and hence be traceless and vanish when contracted with the four velocity. However, from the second order in the derivative expansion onwards, the general results of section 2 do not imply this to be true for the correction to the metric and in fact the source terms which appear on the right hand side of the equation indeed do not have this property. All the other operators except D_1, however, involve no more than one derivative in the radial coordinate.

We have to choose a particular solution to the above equation. We can always choose the particular solution to be such that it vanishes at the boundary like ρ^6 so that it doesn't contribute to the stress tensor (as the coefficient of its ρ^4 term vanishes). One can explicitly check this, however, more efficiently we can prove it as follows. The source term for the n-th order correction clearly is determined by various terms of the stress tensor up to n-1 th order, so it follows from the general results of section 2 that the particular solution can be chosen to be independent of $t_{n\mu\nu}$, which is the n-th order correction to the stress tensor. In that case the ρ^4 term should be absent. For instance, based on the results like those in (2.10), we can write down the Taylor series expansion in the radial coordinate for the particular solution for g_2 as below.

$$\begin{aligned}
g_{2\mu\nu} &= -\frac{\rho^6}{12}\Box t_{0\mu\nu} + \rho^8[\frac{1}{2}t_{1\mu}^{\ \ \rho}t_{1\rho\nu} - \frac{1}{24}\eta_{\mu\nu}(t_1^{\rho\sigma}t_{1\rho\sigma})] \\
&+ \rho^{10}[-\frac{1}{24}(t_{0\mu}^{\ \ \alpha}\Box t_{0\alpha\nu} + t_{0\nu}^{\ \ \alpha}\Box t_{0\alpha\mu}) \\
&+ \frac{1}{180}\eta_{\mu\nu}t_0^{\alpha\beta}\Box t_{0\alpha\beta} + \frac{1}{360}t_0^{\alpha\beta}\partial_\mu\partial_\nu t_{0\alpha\beta} - \frac{1}{120}t_0^{\alpha\beta}(\partial_\mu\partial_\alpha t_{0\beta\nu} + \partial_\nu\partial_\alpha t_{0\beta\mu}) \\
&+ \frac{1}{60}t_0^{\alpha\beta}\partial_\alpha\partial_\beta t_{0\mu\nu} - \frac{1}{180}\partial_\mu t_0^{\alpha\beta}\partial_\nu t_{0\alpha\beta} + \frac{1}{720}\eta_{\mu\nu}\partial_\alpha t_0^{\beta\gamma}\partial^\alpha t_{0\beta\gamma} + \frac{1}{120}(\partial_\mu t_0^{\alpha\beta}\partial_\alpha t_{0\beta\nu} + \partial_\nu t_0^{\alpha\beta}\partial_\alpha t_{0\beta\mu}) \\
&- \frac{1}{60}\partial_\alpha t_{0\mu}^{\ \ \beta}\partial_\beta t_{0\nu}^{\alpha}] + ... \ .
\end{aligned} \tag{2.42}$$

More generally, the particular solution for g_n is uniquely determined once we specify that it vanishes at the boundary like $-(1/12)\rho^6\Box t_{n-2}$. Then it follows that it is independent of t_n and doesn't contribute to the stress tensor at the n th order.

Now the particular solution at every order in the derivative expansion should by itself satisfy the scalar constraint. Let us see it explicitly for the particular solution for g_2. The particular solution chosen to vanish at the boundary like $-(1/12)\rho^6\Box t_0$ has an expansion of the above form (42). So by this choice, the coefficients of the Taylor expansion (now fixed by the source) will automatically agree with the general formula, like those in (2.10). These general formula are automatically consistent with the scalar constraint. The scalar constraint also will be a linear differential equation for g_n with a source term. The source term again is a (nonlinear) function of the corrections to the metric up to n-1 th order in the derivative expansion. The particular solution by itself will satisfy this equation. So the homogeneous solution of the tensor equation for g_n must also be a homogeneous solution of the scalar constraint.

The homogeneous solution of the tensor equation for g_n which will be consistent with the scalar constraint is simply $-2b^4 f(\rho)t_{n\mu\nu}$, with $f(\rho)$ being given by (2.38) and $t_{n\mu\nu}$ being an arbitrarily chosen correction to the hydrodynamic stress tensor involving n derivatives of the field theory coordinates z. However $t_{n\mu\nu}$ must be traceless and also satisfy the Landau gauge condition. Let us illustrate again by explicitly doing the Taylor series expansion of the homogeneous solution to g_2 which is $-2b^4 f(\rho)t_{n\mu\nu}$. The Taylor expansion is as below

$$g_{2\mu\nu} = t_{2\mu\nu}(\rho^4 + \frac{\rho^8}{4b^4} + \frac{\rho^{12}}{48b^8} + ...). \tag{2.43}$$

Using the tracelessness and Landau gauge condition for t_2, one can check from the general formula like those in (2.10) that this is just the part of the metric determined by t_2 at the second order. Hence this should be the only homogeneous solution that is consistent with the scalar constraint. Similarly at each order one can see that the part of the solution for g_n which contains t_n is proportional to t_n and since the particular solution by choice contains all other terms, the homogeneous solution should be always proportional to t_n. Then the tensor equation fixes the radial part of the homogeneous solution so that it should be $-2b^4 f(\rho) t_{n\mu\nu}$.

The vector constraint, at the n-th order in the derivative expansion, as we have argued before simply implies the conservation of the stress tensor up to n-1 th order.

To summarize, these are the features of the derivative expansion in the Fefferman-Graham coordinates.

- At every order in the derivative expansion, the tensor equation for g_n is a linear differential equation of the form of (2.41) involving derivatives in the radial coordinate only. The operators D_1, D_2, etc are the same at every order, while the source term s_n is a nonlinear function of the various corrections to the metric up to n-1 th order.

- The particular solution to the tensor equation for g_n can be chosen to vanish at the boundary like $-(1/12)\rho^6 \Box t_{n-2}$. With this choice the particular solution automatically satisfies the scalar constraint.

- The homogeneous solution to the tensor equation which is consistent with the scalar constraint is $-2b^4 f(\rho) t_{n\mu\nu}$ at very order, with f being given by (2.38) and $t_{n\mu\nu}$ being an arbitrary n th order correction to the stress tensor which satisfy the tracelessness and the Landau gauge condition conditions.

- The vector constraint at the n-th order just implies the conservation of n-1 th order stress tensor.

- We can keep manifest Lorentz covariance at each order in the derivative expansion.

- We can construct a solution corresponding to an arbitrary stress tensor because the homogeneous solution of the tensor equation for g_n at the n-th order is simply proportional to an arbitrarily chosen n-th order correction to the stress tensor. *At every order in the derivative expansion for any choice of the hydrodynamic stress tensor, the solution has finite radius of convergence away from the boundary, so all hydrodynamic stress tensors preserve the asymptotic AdS boundary condition.*

2.5 Getting rid of naked singularities

The comparative advantage of solving Einstein's equation of pure gravity in Fefferman Graham coordinates in the derivative expansion over doing the same in Eddington-Finkelstein coordinate system is that the constraints simplify dramatically and also we do not need to split the terms into tensors, vectors and scalars of SO(3), thus preserving manifest Lorentz covariance. The comparative disadvantage of the Fefferman-Graham coordinate system is that the regularity analysis is not straightforward. At the first order in the derivative expansion, the metric in Fefferman-Graham coordinates (2.39) has a singularity at $\rho = \sqrt{2}b$. This is the location of the horizon at the zeroth order and the zeroth order metric itself is not invertible here.

The first order perturbation has a *log* piece which also blows up here. This singularity could be just a coordinate singularity in which case it could be removed by going to a different coordinate system as it happened for the boosted black brane, or it could be a real singularity. If it is a real singularity, it is naked because it coincides with the original horizon at late time. At late times the solution approaches a boosted black brane but since the horizon coincides with a real singularity, no infalling observer can continue life after reaching the horizon.

To analyse the singularity in the Fefferman-Graham coordinates we will simply translate the metric to Eddington-Finkelstein coordinates (r, x). It will be of course suffice to change our coordinates near $\rho = \sqrt{2}b$, however, for the sake of completeness and better general understanding we will do the change of coordinates exactly up to first order in the derivative expansion. The Eddington-Finkelstein metric which we will get as a result of this translation will also be an exact solution of Einstein's equation up to first order in x-derivatives. We now return to the equations (4.53) in section 3 which gives the translation between the two coordinate systems. We still treat the Fefferman-Graham coordinates ($\rho(x,r)$, $z^\mu(x,r)$) as unknowns, but the third unknown is now the $G_{\mu\nu}(x,r)$ which appears in the Eddington-Finkelstein metric (2.11). The zeroth order solutions to these three are known and are given in (2.19) and (2.24). To find the corrected solutions due to change in the Fefferman-Graham metric at first order it is straightforward to perturb these equations and solve them exactly at first order. The complete solutions to the three unknowns exact up to first order are

$$\begin{aligned}
\rho &= \frac{\sqrt{2}b}{\sqrt{b^2r^2+\sqrt{b^4r^4-1}}}\left(1+bk(br)\frac{\partial.u}{3}\right),\\
z^\mu &= x^\mu + u^\mu bk(br) + u^\mu\frac{\partial.u}{3}b^2k_A(br) + (u.\partial)u^\mu b^2k_B(br), \qquad (2.44)\\
G_{\mu\nu} &= r^2P_{\mu\nu} + \left(-r^2+\frac{1}{b^4r^2}\right)u_\mu u_\nu + 2r^2bF(br)\sigma_{\mu\nu} - r\left((u.\partial)(u_\mu u_\nu) - \frac{2}{3}u_\mu u_\nu(\partial.u)\right)\\
&\quad + \frac{(\gamma-1)b}{4}r^2 log\left(1-\frac{1}{b^4r^4}\right)\sigma_{\mu\nu},
\end{aligned}$$

where,

$$\begin{aligned}
k(x) &= \frac{1}{4}\left(log\left(\frac{x+1}{x-1}\right) - 2arctan(x) + \pi\right), \qquad (2.45)\\
F(x) &= \frac{1}{4}\left(log\left(\frac{(x+1)^2(x^2+1)}{x^4}\right) - 2arctan(x) + \pi\right),
\end{aligned}$$

and $k_A(x), k_B(x)$ satisfy the following differential equations

$$\begin{aligned}
\frac{dk_A}{dx} &= -\frac{x^2}{x^4-1}\left(k(x)+\frac{x}{\sqrt{x^4-1}}\right), \qquad (2.46)\\
\frac{dk_B}{dx} &= \frac{1}{x\sqrt{x^4-1}} - \frac{k(x)x^2}{x^4-1}.
\end{aligned}$$

with the boundary condition that they vanish at $x=\infty$. One may easily check that if we do the Taylor series expansion of ρ, z^μ in 1/r, we can reproduce the results (2.29) of section 3 in which we have solved these equations using a power series ansatz.

The crucial point, as realized by authors of [26] is that in the Eddington-Finkelstein coordinates if there is a blow-up in $G_{\mu\nu}(x, r)$ it should be a real singularity. For a general conformal fluid at first order with $\eta/s = \gamma/4\pi$, the corresponding solution in Eddington-Finkelstein coordinates has $G_{\mu\nu}(x, r)$ given by (2.44). Except for the *log* term which appears in the last line, all other terms are well behaved for $r > 0$ and the *log* term blows up at $r = 1/b$, the location of the unperturbed black brane horizon. Only when $\gamma = 1$, the coefficient of the *log* term vanishes and so the naked singularity at $r = 1/b$ is absent. For this value of γ we have in fact reproduced the $G_{\mu\nu}$ of the Eddington-Finkelstein metric given by the authors of [26].

We learn the following general facts. The translation to Eddington-Finkelstein coordinates exists for an arbitrary solution in the Fefferman-Graham coordinates irrespective of whether there is any naked singularity or not. Also the Fefferman-Graham coordinates have a power series expansion in terms of the inverse of the radial Eddington-Finkelstein coordinates for all cases. For all cases, the change of coordinates also become singular at the location of the original horizon in the Eddington-Finkelstein coordinates which is $r = 1/b$.

We can continue the regularity analysis to higher orders in the derivative expansion by solving the equations (4.53) for translating the solution from the Fefferman-Graham coordinates to Eddington-Finkelstein coordinates order by order in the derivative expansion as well. In this way at each order we will be able to determine what values the coefficients in the terms of the hydrodynamic stress tensor should have so that a naked singularity is avoided. It would be interesting to see if we can understand the values of these coefficients of the hydrodynamic stress tensor, more directly in terms of the geometry of the unperturbed boosted black brane horizon.

We will conclude this section by emphasizing certain points.

- We can think of translating to outgoing Eddington-Finkelstein coordinates also as an attempt to remove the singularity and then as expected the situation will be time-reversed. We will now need $\gamma = -1$ for regularity. In the boundary theory, all fluid dynamical solutions will then be time-reversed and our gravity solutions will be perturbed white-hole solutions exact up to first order in the derivative expansion.

- We could have attempted to fix γ by studying regularity at the horizon by computing curvature invariants (like $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$). However, we do not know, if for these space-times, checking that a finite number of curvature invariants do not blow up at the horizon will suffice to demonstrate regularity. So the best strategy is to translate to a coordinate system where the solution is explicitly regular up to first order in the derivative expansion and this is what we have done here. For the sake of completeness, however, we have studied a few curvature invariants and have found that the leading singularity of $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ at second derivative order vanishes for the right choices of γ which are 1 and -1, the details of which are presented in Appendix B. Here we have also pointed out the dangers of using curvature invariants for regularity analysis at higher orders in the derivative expansion.

- The manifest regularity in the ingoing Eddington-Finkelstein coordinates can be thought of as generalization of the incoming wave boundary condition at the horizon in case of the linearized solution, at the non-linear level. This in fact is the underlying reason why the transport coefficients obtained in this more robust method agrees with the values obtained from the dispersion of long wavelength and low frequency quasinormal modes. For nonhydrodynamic configurations, the regularity of the solutions need not manifest in the ingoing Eddington-Finkelstein coordinates, so the dispersion relations of nonhydrodynamic quasinormal modes may at best be only approximate regular solutions at the non-linear level.

- A special case of our metrics are the solutions corresponding to the Bjorken flow found in [25]. With our method we find the solutions for arbitrary slowly varying velocity configurations at each order in the derivative expansion. Our method clarifies the issues raised in [34] regarding finding the solutions in Fefferman-Graham coordinates by implementing a systematic derivative expansion.

- The derivative expansion in Fefferman-Graham coordinates is equivalent to the same in Eddington-Finkelstein coordinates to all orders in the derivative expansion even when the solutions do not have a regular horizon. This is so because the equations (4.53) for translating Fefferman-Graham coordinates to Eddington-Finkelstein coordinates can always be solved order by order in the derivative expansion as well. In fact, this is natural, because any asymptotic AdS solution can be written in the Fefferman Graham coordinates.

- Using our method we can construct solutions in the Fefferman-Graham coordinates, corresponding to an arbitrary hydrodynamic energy-momentum tensor with arbitrary first order and higher order transport coefficients. It is also possible to translate these solutions to ingoing Eddington-Finkelstein coordinates up to any given order in the derivative expansion. At any given order the solution is linear in the highest order transport coefficients. The solution is also manifestly regular or irregular in Eddington-Finkelstein coordinates. If the transport coefficients at the lower orders are fixed to values such that the solutions at all lower orders are regular, then the solution at the highest order when singular, contains the same log singularities as in the case of the first-order solution when η/s is different from $1/4\pi$, because the singularity comes from translating the homogeneous solution in the Fefferman-Graham coordinates which alone can depend on the highest order transport coefficients. These log divergences being linear in the highest order transport coefficients, can always be fixed to values such that they disappear. Therefore, we conclude, *all hydrodynamic transport coefficients can be determined by requiring the regularity of the solution up to required orders in the derivative expansion.* Moreover, the solutions, when regular, corresponds to a purely hydrodynamic energy-momentum tensor. So, we can demonstrate, that *purely hydrodynamic states which can be characterized by hydrodynamic variables alone and whose evolution can be completely determined by a higher derivative hydrodynamic equation, exist at strong 't Hooft coupling and large rank of the gauge group, in the universal sector of conformal gauge theories with gravity duals.*

Chapter 3

Phenomenological description of basic nonequilibrium processes and pure gravity

3.1 Introduction

In the previous chapter, we have established that the expectation value of the energy-momentum tensor uniquely characterizes all states in the universal sector of gauge/gravity duality and determines their dynamics. A generic conformal energy momentum-tensor has nine independent components, therefore should require nine field variables for being parametrized. We have also established in the previous chapter that there are purely hydrodynamic states in the universal sector, which can be characterized by four hydrodynamic variables, namely the local velocity and temperature even far away from equilibrium. The four equations of conservation of energy and momentum give us the equations of fluid dynamics and completely determine the dynamics of the purely hydrodynamic states. Regularity of the future horizon in the dual gravity solutions require systematic corrections to the energy-momentum tensor in the derivative expansion giving systematic Weyl covariant higher derivative corrections to the relativistic Navier-Stokes' equation. Further, all the higher order transport coefficients also get determined by requirement of regularity of the future horizon in the dual gravity solutions order by order in the derivative expansion. The derivative expansion is under control when the hydrodynamic variables are slowly varying both spatially and temporally with respect to the temperature in the final configuration.

In this chapter, we will attempt a field-theoretic understanding of how the expectation value of the energy-momentum tensor characterizes states in the universal sector and determines their dynamics. Moreover, we will also seek to understand the whole range of phenomena in the universal sector. Clearly, the four equations of energy-momentum conservation will not suffice generically to determine the evolution of the nine independent components of the energy-momentum tensor. In this chapter, we will use field-theoretic insights to understand how pure gravity in asymptotically AdS_5 spacetime gives us the evolution of a generic energy-momentum tensor for a state in the universal sector through the requirement of regularity of the future horizon. This will lead us to make conjecture about regularity condition of asymptotically AdS_5 spacetimes which are solutions of Einstein's equation. This conjecture can be verified by constructing solutions in a general perturbation expansion and doing analysis of regularity for the given orders of expansion.

The broader result of this study is that we will be able to develop a complete framework for the whole range of phenomena in the universal sector including decoherence, relaxation and hydrodynamics. We will argue that all these phenomena even beyond the universal sector can be effectively described by just nine phenomenological equations giving the evolution of the energy-momentum tensor which can be systematically constructed in two expansion parameters and reduce to fluid dynamical equations in special cases. The description of the states in terms of these equations become exact for states in the universal sector. We will be also be able to argue that a generic state even away from the regime of strong 't Hooft coupling and large rank of the gauge group can be approximated at sufficiently late time by an appropriate state where the dynamics will be exactly given by the energy-momentum tensor alone. We will also see how we will be able to use this framework to establish concretely which features in the five dimensional geometry determine nonequilibrium phenomena like decoherence and attain a deeper understanding of irreversibility.

Here we will address the question of how energy-momentum tensor characterizes the states; first in the regime of weak coupling, so that we can employ the quasiparticle description and also use kinetic theories, which are coarse-grained descriptions of microscopic laws. Specifically, we use the Boltzmann equation which has proven useful [35, 36] in determining the shear viscosity, higher order hydrodynamic transport coefficients and the relaxation time in weakly coupled gauge field theories. It has also been shown that an effective Boltzmann equation can be used to study nonequilibrium phenomena in high temperature QCD and is equivalent to an exact perturbative treatment [35]. Despite being a coarse-grained description, the Boltzmann equation retains the power to describe nonequilibrium phenomena far away from the hydrodynamic regime and at length scales and time scales shorter than the mean-free path and the relaxation time respectively. However it is not applicable to phenomena at microscopic length and time scales.

We prove that there exist very special solutions of the Boltzmann equation which are functionally determined by the energy-momentum tensor alone. We call such solutions "conservative solutions". These solutions, although very special, constitute phenomena far away from equilibrium and well beyond the hydrodynamic regime. The existence of conservative solutions can be conveniently proven for nonrelativistic monoatomic gases using some basic structural properties of the Boltzmann equation and can be easily extended to include relativistic and semiclassical corrections. We show that these solutions can be constructed even for multicomponent systems relevant for relativistic quantum gauge theories.

It will thus be natural to make the assumption that the conservative solutions constitute the universal sector of strongly coupled gauge theories with gravity duals. This will explain why the states in the universal sector are determinable functionally by the energy-momentum tensor alone. This assumption, through the gauge/gravity duality, will have powerful consequences for gravity. The same condition on the energy-momentum tensor,required to make the state in the field theory a conservative solution, will now be required to make the dual solution in gravity have a smooth future horizon. In other words, the *conservative* condition on the energy-momentum tensor in field theory should now transform into the *regularity* condition in gravity.

The plan of this chapter is as follows. In Section 2 we outline the conservative solutions in the Boltzmann equation. We then state and investigate our proposal for the regularity condition on the energy-momentum tensor for pure gravity in AdS_5 in Section 3. The proof of existence of the conservative solutions in the Boltzmann equation is slightly technical and elaborate. So, this proof will be presented in full details in the Appendix C in a self-contained manner and will referred appropriately in this Chapter.

3.2 The conservative solutions of the Boltzmann equation

The study of equilibrium and transport properties of dilute gases through the dynamics of one-particle phase space distribution functions was pioneered by Maxwell [38] and further developed by Boltzmann [39] in the 19th century. The Boltzmann equation provides a successful description of nonequilibrium phenomena in rarefied monoatomic gases. It is an equation for the evolution of the one-particle phase space distribution function. It can successfully describe nonequilibrium phenomena in rarefied gases, even at length scales between the microscopic molecular length scale and the mean-free path, and time scales between the time it takes to complete binary molecular collisions [1] and the average time between intermolecular collisions.

The Boltzmann equation is neither microscopic nor phenomenological, but a result of averaging the dynamics over microscopic length scales and time scales. Unlike phenomenological equations, it has no undetermined parameters and is completely fixed once the intermolecular force law is known. The structural details of the molecules are however ignored and effectively they are taken to be pointlike particles. The hydrodynamic equations with all the transport coefficients can be determined from the Boltzmann equation.

We start with a brief description of the conservative solutions of the Boltzmann

[1] Classically this is just the typical time it takes the trajectories of the molecules to straighten out after collision; a good estimate of this is r/c_s, where r is the range of the force and c_s is the thermal speed (the average root mean square velocity of the particles).

equation for a system of pointlike classical nonrelativistic particles interacting via a central force. As mentioned in the Introduction, the proof of existence and uniqueness of such solutions is detailed in the Appendix in a self-contained manner. This is followed by a discussion on how to generalize our construction of conservative solutions to the semi-classical and relativistic versions of the Boltzmann equation. Finally we show how our results apply to multicomponent systems relevant for relativistic gauge theories. These generalizations are straightforward and the discussion on the nonrelativistic Boltzmann equation will be convenient for a first understanding of the conservative solutions.

3.2.1 The conservative solutions in brief

A generic solution of Boltzmann equation (6.19) is characterized by infinite number of local variables. In general, these could be chosen to be the infinite local velocity moments ($f^{(n)}(\mathbf{x})$'s) of the one-particle phase space distribution $f(x, \xi)$, given by

$$f^{(n)}_{i_1 i_2 \ldots . i_n}(\mathbf{x}, t) = \int d\xi \;\; c_{i_1} c_{i_2} \ldots .. c_{i_n} \; f(\mathbf{x}, \xi) \quad . \tag{3.1}$$

where $c_i = \xi_i - u_i(\mathbf{x}, t)$ with $u_i(\mathbf{x}, t)$ being the local average velocity.

However the first ten velocity moments suffice to parametrize the energy-momentum tensor. The conservative solutions, which are determined by the energy-momentum tensor alone, are thus a very special class of solutions obtained when the initial value data satisfy certain constraints.

Another special class of solutions to the Boltzmann equation is actually well known in the literature. These are the *normal* solutions, where the local hydrodynamic variables given by the first five velocity moments of f suffice to describe the solution even when it is far from equilibrium. Our conservative solutions are a generalization of these normal solutions. We review the normal solutions below before describing the conservative solutions.

The hydrodynamic equations and normal solutions

It is well-known that the first five velocity moments of the Boltzmann equation (6.19), obtained by multiplying with $(1, \xi_i, \xi^2)$ and integrating over ξ, give the hydrodynamic equations as

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}(\rho u_r) &= 0 \quad , \\ \frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho}\frac{\partial (p\delta_{ir} + p_{ir})}{\partial x_r} &= 0 \quad , \\ \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r}(u_r p) + \frac{2}{3}(p\delta_{ir} + p_{ir})\frac{\partial u_i}{\partial x_r} + \frac{1}{3}\frac{\partial S_r}{\partial x_r} &= 0 \quad , \end{aligned} \tag{3.2}$$

where the hydrodynamic variables (ρ, u_i, p) are respectively the local density, components of local average molecular velocity and the local pressure of the gas defined in terms of the average root mean square kinetic energy. In terms of the velocity moments

$$\begin{aligned} \rho(\mathbf{x}, t) &= \int f d\xi \quad , \\ u_i(\mathbf{x}, t) &= \frac{1}{\rho}\int \xi_i f d\xi \quad , \\ p(\mathbf{x}, t) &= \frac{1}{3}\int \xi^2 f d\xi \quad . \end{aligned} \tag{3.3}$$

The local temperature is defined through the local equation of state, $(RT = p/\rho)$ [2]. The shear-stress tensor p_{ij} and the heat flow vector S_i (defined through $S_i = S_{ijk}\delta_{jk}$) are related to the velocity moments by

$$\begin{aligned} p_{ij} &= \int (c_i c_j - RT\delta_{ij}) f d\xi \quad , \\ S_{ijk} &= \int c_i c_j c_k f d\xi \quad , \end{aligned} \tag{3.4}$$

where $c_i = \xi_i - u_i$. It can be easily seen from the definition that $p_{ij}\delta_{ij} = 0$.

The collision term $J(f, f)$ (as defined in (6.20)) does not contribute when deriving the hydrodynamic equations (3.2) from the Boltzmann equation. The first five velocity

[2] If we refine the kinetic description beyond the Boltzmann equation, we need to refine this equation of state which holds for ideal gases. The gaseous equation of state assumes that the potential energy density is negligible compared to the kinetic energy density which could be true only if the number density of particles is sufficiently small.

moments of $J(f,f)$ are zero owing to particle number, momentum and energy conservation as proven in the Appendix.

It must be emphasized that, in the hydrodynamic equations (3.2), the shear-stress tensor p_{ij} and the heat flow vector S_i are functionally independent of the hydrodynamic variables. However there exist unique algebraic solutions to these and all the higher moments $f^{(n)}_{i_1 \ldots i_n}(\mathbf{x}, t)$, which are functionals of the hydrodynamic variables. These functional forms contain only spatial derivatives of the hydrodynamic variables and can be systematically expanded in the so-called derivative expansion discussed below. This leads to the construction of the normal or purely hydrodynamic solutions of the Boltzmann equation, which we discuss below. For a generic solution of the Boltzmann equation, the higher moments of f will have explicit time-dependent parts which are functionally independent of the hydrodynamic variables.

The *normal* solutions of the Boltzmann equation [40, 41, 42] have been extensively discussed in [43]. These solutions can be determined in terms of the five hydrodynamic variables (ρ, u_i, p) *alone*. They describe situations far away from equilibrium, such that observables which vanish at equilibrium do not vanish anymore but are functionally determined in terms of the hydrodynamic variables and their spatial derivatives. The existence of such solutions follows from the existence of unique algebraic solutions (as functionals of the hydrodynamic variables) to the equations of motion of the higher moments. The functional forms of the shear-stress tensor and the heat flow vector, for instance, are given by

$$
\begin{aligned}
p_{ij} &= \eta\sigma_{ij} + \beta_1 \frac{\eta^2}{p}(\partial\cdot u)\sigma_{ij} + \beta_2 \frac{\eta^2}{p}\left(\frac{D}{Dt}\sigma_{ij} - 2\left(\sigma_{ik}\sigma_{kj} - \frac{1}{3}\delta_{ij}\sigma_{lm}\sigma_{lm}\right)\right) \\
&\quad + \beta_3 \frac{\eta^2}{\rho T}\left(\partial_i\partial_j T - \frac{1}{3}\delta_{ij}\Box T\right) + \beta_4 \frac{\eta^2}{p\rho T}\left(\partial_i p \partial_j T + \partial_j p \partial_i T - \frac{2}{3}\delta_{ij}\partial_l p \partial_l T\right) \\
&\quad + \beta_5 \frac{\eta^2}{p\rho T}\left(\partial_i T \partial_j T - \frac{1}{3}\delta_{ij}\partial_l T \partial_l T\right) + \ldots \quad , \qquad (3.5) \\
S_i &= \chi \partial_i T + \ldots \quad ,
\end{aligned}
$$

with the convective derivative $D/Dt = \partial/\partial t + u_i\partial_i$, and

$$\begin{aligned} \sigma_{ij} &= \partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial\cdot u \quad , \\ \eta &= \frac{p}{B^{(2)}} + ... \quad , \qquad \chi = \frac{15R}{2}\eta + ... \quad , \end{aligned}$$

where η and ξ, appearing as in the Navier-Stokes equation and the Fourier's law of heat conduction, are the shear viscosity and heat conductivity respectively. $B^{(2)}$ is a specific function of the local thermodynamic variables determined by the collision kernel of the Boltzmann equation. The β_i's are pure numbers that can be determined from the Boltzmann equation. The time derivative in D/Dt can be converted to spatial derivatives using the hydrodynamic equations of motion; in fact, up to the orders shown above, we can assume that the Euler equation is valid and that the heat conduction is adiabatic.

The functional forms can be expanded systematically in the derivative expansion, which counts the number of spatial derivatives present in the expansion. The expansion parameter is the ratio of the typical length scale of variation of the hydrodynamic variables with the mean-free path. This is true for all the higher moments of f. The functional forms of p_{ij} and S_i (3.5), when substituted into the hydrodynamic equations (3.2), give us systematic corrections to the Navier-Stokes equation and Fourier heat conduction respectively which can be expanded in the derivative expansion scheme.

The hydrodynamic equations are now the only dynamical equations. The higher moments are given algebraically in terms of the hydrodynamic variables and their spatial derivatives. The phase space distribution function f is completely determined by the hydrodynamic variables through its velocity moments. The hydrodynamic equations thus form a closed system of equations and any solution of this system can be lifted to a unique solution of the full Boltzmann equation.

Stewart has shown [44] that such normal solutions exist even for the relativistic and semiclassical Boltzmann equations.

The conservative solutions

We are able to prove that a more general class of special solutions to the Boltzmann equation - which we call *conservative solutions* - exist. Here we outline these solutions, leaving the details of the proof to the Appendix. These solutions can be completely determined in terms of the energy-momentum tensor, analogous to the *normal* solutions being completely determined in terms of the hydrodynamic variables. The energy-momentum tensor (as shown later) can be parametrized by the first ten moments of f :

- i) the five hydrodynamic variables (ρ, u_i, p), and
- ii) the five components of the shear-stress tensor p_{ij} in a comoving locally inertial frame.

Importantly, for a generic conservative solution the shear-stress tensor is an *independent variable* unlike the case of normal solutions, where it is a functional of the hydrodynamic variables.

These ten independent variables satisfy the following equations of motion

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}(\rho u_r) &= 0 \quad , \\
\frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho}\frac{\partial (p\delta_{ir} + p_{ir})}{\partial x_r} &= 0 \quad , \\
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r}(u_r p) + \frac{2}{3}(p\delta_{ir} + p_{ir})\frac{\partial u_i}{\partial x_r} + \frac{1}{3}\frac{\partial S_r}{\partial x_r} &= 0 \quad , \\
\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{\partial S_{ijr}}{\partial x_r} - \frac{1}{3}\delta_{ij}\frac{\partial S_r}{\partial x_r} & \\
+\frac{\partial u_j}{\partial x_r}p_{ir} + \frac{\partial u_i}{\partial x_r}p_{jr} - \frac{2}{3}\delta_{ij}p_{rs}\frac{\partial u_r}{\partial x_s} & \\
+p(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_r}{\partial x_r}) &= \sum_{p,q=0;p\geq q;(p,q)\neq(2,0)}^{\infty} B^{(2,p,q)}_{ij\nu\rho}(\rho, T) f^{(p)}_{\nu} f^{(q)}_{\rho} \\
&\quad +B^{(2)}(\rho, T) p_{ij} \quad .
\end{aligned} \tag{3.6}$$

where $B^{(2,p,q)}_{ij\nu\rho}$ are determined by the collision kernel in the Boltzmann equation. ν and ρ indicate abstractly all the p and q indices of the moments $f^{(p)}$ and $f^{(q)}$, respectively.

The above equations are now a closed system of equations, just like the hydrodynamic equations were in case of the normal solutions. All the higher moments appearing in the above equations are given as functionals of the hydrodynamic variables *and* the stress tensor. These functional forms are unique and special algebraic solutions of the higher moments of f. For instance, the heat flow vector can be determined from

$$S_i = \frac{15pR}{2B^{(2)}}\frac{\partial T}{\partial x_i} + \frac{3}{2B^{(2)}}\left(2RT\frac{\partial p_{ir}}{\partial x_r} + 7Rp_{ir}\frac{\partial T}{\partial x_r} - \frac{2p_{ir}}{\rho}\frac{\partial p}{\partial x_r}\right) + ... \quad . \tag{3.7}$$

The functional forms of all the higher moments, as for the heat flow vector above, can be expanded systematically in two expansion parameters ϵ and δ. The parameter ϵ is the old derivative expansion parameter – the ratio of the typical length scale of spatial variation to the mean-free path. The new parameter δ is an amplitude expansion parameter, defined as the ratio of the typical amplitude of the nonhydrodynamic shear-stress tensor with the hydrostatic pressure in the final equilibrium.

The closed system of ten equations (3.6) are thus the only dynamical equations and any solution of this system can be lifted to a full solution of the Boltzmann equation through the unique functional forms of the higher moments.

The normal solutions,being independent of the stress tensor, are clearly a special class of conservative solutions. There is another interesting class of conservative solutions which are homogeneous or invariant under spatial translations. The phase space distribution function f is a function of $\mathbf{v}$ only for these homogeneous solutions and the hydrodynamic variables are constants both over space and time [this can be easily seen from (3.6)]. The shear-stress tensor and consequently all the higher moments are functions of time alone. Such solutions have dynamics in velocity space only and describe relaxation processes.

In a generic solution of the Boltzmann equation, the dynamics at short time scales is more like the homogeneous class, where the initial one-particle distribution relaxes to a local equilibrium given by a local Maxwellian distribution parametrized by the local values of the hydrodynamic variables. At long time scales the dynamics is more like the normal solutions, where the system goes to global equilibrium hydrodynamically. Thus conservative solutions, despite being mathematically special, capture both relaxation and hydrodynamics which constitute generic nonequilibrium processes in a phenomenological manner. In other words, the dynamics of the energy-momentum tensor alone given by (3.6) captures both relaxation and hydrodynamics in a systematic fashion.

3.2.2 Relativistic and semiclassical corrections to conservative solutions

The proof for existence of conservative solutions in the nonrelativistic classical Boltzmann equation can be readily generalized to its semiclassical and relativistic versions. This is because all the properties of the collision term J required for the proof of the existence of conservative solutions carry over to the semiclassical and relativistic versions as well.

Let us consider the semiclassical version of the collision term which takes into account quantum statistics. This was first obtained by Uehling and Uhlenbeck [45] to be

$$\begin{aligned} J(f,g) &= \int \mathcal{J}(\xi,\xi^*)B(\theta,V)d\xi^* d\epsilon d\theta \quad , \\ \mathcal{J}(\xi,\xi^*) &= \left[f(\mathbf{x},\xi^{'})g(\mathbf{x},\xi^{*'})\mathcal{F}(\xi)\mathcal{G}(\xi^*) - f(\mathbf{x},\xi)g(\mathbf{x},\xi^*)\mathcal{F}(\xi^{'})\mathcal{G}(\xi^{*'})\right] \quad , \\ \mathcal{F}(\xi) &= \left(1 \pm \frac{h^3 f(\xi^{'})}{(2s+1)}\right), \quad \mathcal{G}(\xi) = \left(1 \pm \frac{h^3 g(\xi)}{(2s+1)}\right) \quad , \end{aligned} \tag{3.8}$$

where the + sign applies for bosons, the − sign for fermions and s is the spin of the particles comprising the system. The final velocities $\xi^{'}$ and $\xi^{*'}$ are determined by the velocities ξ and ξ^* before the binary molecular collision according to the intermolecular force law. Importantly, now $J(f,f)$ vanishes if and only if f is the Bose-Einstein or the Fermi-Dirac distribution in velocity space for bosons and fermions respectively, instead of being the Maxwellian distribution [3].

The proof for the existence of conservative solutions in the nonrelativistic classical case does not require any explicit form of the collision kernel J. Only certain key properties suffice, as will be evident from the proof. We can pursue the same strategy with the

[3] The proof follows along exactly the same lines as shown in Appendix. The Boltzmann equation still takes the same form as in (6.19). The semiclassical form of J then readily follows from (6.27), which in turn follows from (6.25) and (6.26), all of which are true for the semiclassical form of J too. The hydrodynamic equations will take the same form as before and the shear-stress tensor, p_{ij} and the heat flow vector S_i can be defined as before.

semiclassical corrections as well [4].

One has to employ the Sonine polynomials, which are generalizations of Hermite polynomials, to find solutions of the required algebraic solutions of the higher moments as in [46]. The main objection could be that for the proof of existence of solutions, we use a theorem due to Hilbert which is explicitly stated for the nonrelativistic classical J. However the details are exactly the same as that for constructing the normal solutions. It has been seen that normal solutions can indeed be constructed in the semiclassical case [44], so there ought to be no obstruction to the construction of conservative solutions also. Indeed, our proof shows that we can construct the conservative solutions given that the normal solutions exist.

The generalization in the relativistic case again holds on similar grounds as above. It is more convenient to use a covariant description now. Normal solutions of the semiclassical relativistic Boltzmann equation have also been constructed [44]. So there should be no obstruction in constructing conservative solutions as well.

In fact the same arguments could be used to state that any solution of the relativistic semiclassical Boltzmann equation at sufficiently late times can be approximated by an appropriate conservative solution, since the maximum speed of propagation of linearized modes increases monotonically [47] as more and more higher moments are included.

3.2.3 Multicomponent systems

So far we have pretended as if our system is composed of only one component or particle. However gauge theories have many species of particles and internal degrees of freedom, hence we need to understand how to extend our results to multicomponent systems.

Let us consider the example of $\mathcal{N} = 4$ super Yang-Mills theory. In the weakly coupled description we need to deal with all the adjoint fermions and scalars along with the gauge bosons; all these particles form a SUSY multiplet. We note that in the universal sector all charge densities or currents corresponding to local (gauge) and global $[SO(6)_R]$

[4] The explicit solutions in the recursive expansion series will be more complicated now. In the nonrelativistic proof, one uses the Hermite polynomials which can no longer be conveniently employed.

charges are absent. Similarly we should not have any multipole moments of local or global charge distributions, because in the gravity side we have pure gravity only. Therefore, most naturally we should have that all members of the $\mathcal{N} = 4$ SUSY multiplet, distinguished by their spin, global charge and color, should be present in equal density at all points in phase space. So we are justified in our analysis in dealing with a single phase space distribution f. The Boltzmann equation we have considered above is obtained after summing over interactions in all possible spin, charge and color channels.

The situation should be similar in any other conformal gauge theory. We can still treat the spin, color and charge as internal degrees of freedom owing to mass degeneracy even though the particles do not form a SUSY multiplet. In the absence of any chemical potential, there should be equipartition at all points in phase space over these internal degrees of freedom. This should be the most natural weak coupling extrapolation of the situation in the universal sector, dual to pure gravity, where gravity is blind to all the internal degrees of freedom of the particles.

3.3 Regularity condition of solutions of pure gravity in asymptotically AdS spacetimes

We will now argue that conservative solutions should exist even in the exact microscopic theory. In the exact microscopic theory, we do not make any approximation over the microscopic degrees of freedom and their dynamics unlike the Boltzmann equation, though an appropriate averaging over the environmental degrees of freedom is required to get the final equilibrium configuration.

To begin with, consider the BBGKY heirarchy of equations [48] which describes a hierarchy of coupled semiclassical nonrelativistic equations for the evolution of multiparticle phase space distributions. This hierarchy is useful for developing kinetic theory of liquids. If the hierarchy is not truncated, then it is equivalent to the exact microscopic description. It has been shown that normal or purely hydrodynamic solutions to the untruncated hierarchy exist. These solutions lead to the determination of viscosity of liquids which behave correctly as a function of density and temperature [49]. It is therefore likely that the conservative solutions also exist for this system which means they are likely to exist for the microscopic nonequilibrium theory of nonrelativistic classical systems constituted by pointlike particles.

Experiments at the Relativistic Heavy Ion Collider (RHIC) suggest that the evolution of quark-gluon plasma (QGP) can be well approximated by hydrodynamic equations, very soon after its formation from the fireball [50]. Given that the perturbative nonequilibrium dynamics of hot QCD for temperatures greater than the microscopic scale Λ is equivalent to a relativistic semiclassical Boltzmann equation [35], we know perturbatively normal or purely hydrodynamic solutions exist for these microscopic theories. In fact any generic solution of the relativistic semiclassical Boltzmann equation can be approximated by an appropriate normal solution at a sufficiently late time. The quick approach to almost purely hydrodynamic behavior for the strongly coupled QGP in RHIC suggests that even nonperturbatively a normal solution should exist which could approximate the late-time behavior for any generic nonequilibrium state. It is also true that not all transport coefficients of generic conformal higher derivative hydrodynamics can be defined through linear response theory. The plausible route of defining these higher order transport coefficients could be through the construction of normal solutions in nonequilibrium quantum field theories. Extremely fast relaxation dynamics in quark-gluon plasma similarly suggest that conservative solutions should capture generic nonequilibrium behavior. This is because in such systems the approach to the conservative regime, where the dynamics is given in terms of the energy-momentum tensor alone, should be faster than in weakly coupled systems, where even the corrections to Navier-Stokes hydrodynamics are hard to determine experimentally.

If we accept that conservative solutions exist in the exact microscopic theory, it is only natural to identify the conservative solutions with the universal sector at large N and strong coupling in gauge theories with gravity duals. Such an identification explains the dynamics in the universal sector being determined by the energy-momentum tensor alone. We emphasize, however, that the conservative solutions become universal only at strong coupling and large N.

An appropriate AdS/CFT argument can also be provided for the existence of conservative solutions for finite N and coupling. In such cases, we need to consider higher derivative corrections to Einstein's equation and the full $ten - dimensional$ equations of motion. There is no guarantee of a consistent truncation to pure gravity anymore. However, we can use holographic renormalization with Kaluza-Klein reduction to five dimensions [51] to argue that we can readily extend the solutions in the universal sector, perturbatively in the string tension ($\approx 1/\sqrt{\lambda}$ in appropriate units) and string coupling (whose N dependence is $1/N$). This can be done by turning off the normalizable mode of the dilaton while keeping its non-normalizable mode constant, turning off the normalizable and non-normalizable modes of all other fields while keeping the boundary metric flat and perturbatively correcting the energy-momentum tensor to appropriate orders of the string tension and string coupling, so that the gravity solution still has a future horizon regular up to desired orders in the perturbation expansion. These solutions, again by construction, are determined by energy-momentum tensor alone. Our claim that the conservative solutions exist in the exact microscopic theory at any value of coupling and N is therefore validated.

The identification of conservative solutions with the universal sector at strong coupling and large N for conformal gauge theories with gravity duals allows us to create a framework for solutions of pure gravity in AdS with regular future horizons. We first study the parametrization of the boundary stress tensor which will allow us to make the connection with nonequilibrium physics. Then we will proceed to give a framework for regular solutions, with the only assumption being the identification of conservative solutions with the universal sector at strong coupling and large N. Finally we will make some connections with known results.

3.3.1 The energy-momentum tensor and nonequilibrium physics

A general parametrization of the energy-momentum tensor allows us to connect gravity to the nonequilibrium physics of conformal gauge theories. This parametrization has been first applied in the AdS/CFT context in[37]. The energy-momentum tensor is first written as

$$t_{\mu\nu} = t_{(0)\mu\nu} + \pi_{\mu\nu} \quad , \tag{3.9}$$

where $t_{(0)\mu\nu}$ is the part of the energy-momentum tensor in local equilibrium. It can be parametrized in conformal theories by the hydrodynamic variables, the timelike velocity (u^μ) and the temperature (T), as

$$t_{(0)\mu\nu} = (\pi T)^4 (4u_\mu u_\nu + \eta_{\mu\nu}) \quad , \tag{3.10}$$

and $\pi_{\mu\nu}$ is the nonequilibrium part of the energy-momentum tensor.

If we define the four velocity u^μ to be the local velocity of energy transport and the temperature T such that $3(\pi T)^4 = u^\mu u^\nu t_{\mu\nu}$ is the local energy density, then in the local frame defined through u^μ, the energy-momentum tensor must receive nonequilibrium contributions in the purely spatial block orthogonal to the four velocity. This means

$$u^\mu \pi_{\mu\nu} = 0 \quad . \tag{3.11}$$

The constraints in Einstein's equations impose the tracelessness and conservation condition on the energy-momentum tensor so that

$$\begin{aligned} \partial^\mu t_{\mu\nu} = 0 \quad \Rightarrow \quad \partial^\mu \left((\pi T)^4 (4u_\mu u_\nu + \eta_{\mu\nu}) \right) &= -\partial^\mu \pi_{\mu\nu} \quad , \\ Tr(t) = 0 \Rightarrow Tr(\pi) &= 0 \quad . \end{aligned} \tag{3.12}$$

In the second equation above, the implication for the tracelessness for $\pi_{\mu\nu}$ comes from the fact that the equilibrium energy-momentum tensor as given by (3.10) is by itself traceless.

In the dual theory these conditions are satisfied automatically owing to the full $SO(4, 2)$ conformal invariance. Note that the first of the equations above is just the forced Euler equation and can be thought of as the equation of motion for the hydrodynamic variables.

We can reinterpret a class of known solutions of pure gravity in AdS as the duals of the normal solutions in the exact microscopic theory at strong coupling and large N. These solutions are the "tubewise black-brane solutions" [26] which, in any radial tube ending in a patch at the boundary, are approximately boosted black brane solutions corresponding to local equilibrium and can be parametrized by the hydrodynamic variables corresponding to the patch at the boundary. These solutions can be constructed perturbatively in the derivative expansion. The expansion parameter, being the ratio of length and time scale of variation of the local hydrodynamic parameters and the mean-free path in final equilibrium, simply counts the number of boundary derivatives. We can identify these solutions as duals of normal solutions because the nonequilibrium part of the energy-momentum tensor $\pi_{\mu\nu}$ can be parametrized by the hydrodynamic variables and their derivatives alone.

The complete parametrization of the purely hydrodynamic $\pi_{\mu\nu}$ in any conformal theory is known up to second order in the derivative expansion.In this parametrization, aside from the shear viscosity four higher order transport coefficients appear [37, 26], which can be fixed by requiring the regularity of the future horizon giving us the tubewise black-brane solutions [26].

Let us define the projection tensor $P_{\mu\nu}$ which projects on the spatial slice locally orthogonal to the velocity field, so that

$$P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu} \quad .$$

The hydrodynamic shear strain rate $\sigma_{\mu\nu}$ is defined as

$$\sigma^{\mu\nu} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right) - \frac{1}{3} P^{\mu\nu} (\partial \cdot u) \quad . \tag{3.13}$$

We also introduce the hydrodynamic vorticity tensor,

$$\omega^{\mu\nu} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} (\partial_\alpha u_\beta - \partial_\beta u_\alpha) \quad . \tag{3.14}$$

The purely hydrodynamic $\pi_{\mu\nu}$ up to second order in the derivative expansion, for the tubewise black-brane solutions, with all nonvanishing transport coefficients fixed by regularity is

$$\begin{aligned}\pi^{\mu\nu} &= -2(\pi T)^3\sigma^{\mu\nu} \\ &+(2-\ln 2)(\pi T)^2\left[(u\cdot\partial)\sigma^{\mu\nu}+\frac{1}{3}\sigma^{\mu\nu}(\partial\cdot u)-(u^\nu\sigma^{\mu\beta}+u^\mu\sigma^{\nu\beta})(u\cdot\partial)u_\beta\right] \\ &+2(\pi T)^2\left(\sigma^{\alpha\mu}\sigma_\alpha{}^\nu-\frac{1}{3}P^{\mu\nu}\sigma_{\alpha\beta}\sigma^{\alpha\beta}\right) \\ &+(\ln 2)(\pi T)^2(\sigma^{\alpha\mu}\omega_\alpha{}^\nu+\sigma^{\alpha\mu}\omega_\alpha{}^\nu)+O(\partial^3 u) \quad . \end{aligned} \tag{3.15}$$

Having identified the normal solutions in the universal sector with a class of solutions which could in principle be constructed up to any order in the derivative expansion, we will now naturally extend this observation to a framework which captures all regular solutions in certain expansion parameters.

3.3.2 The complete framework

In the hydrodynamic case we had four hydrodynamic variables, so the conservation of the energy-momentum tensor alone is sufficient to determine the evolution in the boundary. However in the generic case we need an independent equation of motion for $\pi_{\mu\nu}$.

The regularity condition must be an equation for the evolution of $\pi_{\mu\nu}$ similar to the last equation of (3.6), This is because, as per our argument, the conservative solutions should be identified with the universal sector at large N and strong coupling. However Eq. (3.6) came from an underlying Boltzmann equation. At strong coupling, we have no kinetic equation to guide us because a valid quasiparticle description at strong coupling is not known even for $\mathcal{N}=4$ supersymmetric Yang-Mills theory. Moreover an entropy current cannot be probably constructed beyond the class of purely hydrodynamic solutions, hence we cannot use any formalism like the Israel-Stewart-Muller formalism [52] to guess an equation for $\pi_{\mu\nu}$. This is because we should not expect a monotonic approach to equilibrium, as in the case of the Boltzmann equation, when we go to the exact microscopic description [5].

[5]Even in the purely hydrodynamic context of "tubewise black-brane solutions," the Israel-Stewart-Muller formalism is not valid. This has been discussed with references later in the text.

The safest strategy therefore, will be to use only the following basic inputs without resorting to guesswork.

- The first input is that the equation for $\pi_{\mu\nu}$ has to be conformally covariant because the dual gauge theory is conformal.

- The second input is that the solutions in the purely hydrodynamic sector are known exactly up to second order in the derivative expansion and, being identified with the normal solution, should be special cases of our complete framework. The equation for $\pi_{\mu\nu}$ must therefore have (3.15), the purely hydrodynamic energy-momentum tensor known up to second order in the derivative expansion, as a solution up to those orders.

With only these inputs, we will be able to propose the equation for $\pi_{\mu\nu}$ only up to certain orders of expansion in both the hydrodynamic and nonhydrodynamic expansion parameters about the equilibrium state. However we should consider the most general equation for $\pi_{\mu\nu}$ which satisfies the above criteria. The expansion parameters are again the derivative expansion parameter (as in the purely hydrodynamic sector, but with the spatio-temporal variation of $\pi_{\mu\nu}$ taken into account additionally) and the amplitude expansion parameter, which is the ratio of a typical value of $\pi_{\mu\nu}$ divided by the pressure in final equilibrium.

Our proposal then amounts to the following equation of motion for $\pi_{\mu\nu}$, whose solutions should give all the regular solutions of pure gravity in AdS_5 :

$$
\begin{aligned}
(1-\lambda_3)\left[(u\cdot\partial)\pi^{\mu\nu}+\frac{4}{3}\pi^{\mu\nu}(\partial\cdot u)-\left(\pi^{\mu\beta}u^{\nu}+\pi^{\nu\beta}u^{\mu}\right)(u\cdot\partial)u_{\beta}\right]& \\
=-\frac{2\pi T}{(2-\ln 2)}\Bigg[\pi^{\mu\nu}+2(\pi T)^3\sigma^{\mu\nu}& \\
-\lambda_3(2-\ln 2)(\pi T)^2\left((u\cdot\partial)\sigma^{\mu\nu}+\frac{1}{3}\sigma^{\mu\nu}(\partial\cdot u)-\left(u^{\nu}\sigma^{\mu\beta}+u^{\mu}\sigma^{\nu\beta}\right)(u\cdot\partial)u_{\beta}\right)& \\
-\lambda_4(\ln 2)(\pi T)^2(\sigma^{\alpha\mu}{\omega_{\alpha}}^{\nu}+\sigma^{\alpha\mu}{\omega_{\alpha}}^{\nu})& \\
-2\lambda_1(\pi T)^2\left(\sigma^{\alpha\mu}{\sigma^{\nu}}_{\alpha}-\frac{1}{3}P^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta}\right)\Bigg]& \\
-(1-\lambda_4)\frac{ln2}{(2-\ln 2)}({\pi^{\mu}}_{\alpha}\omega^{\alpha\nu}+{\pi^{\nu}}_{\alpha}\omega^{\alpha\mu})& \\
-\frac{2\lambda_2}{(2-\ln 2)}\left[\frac{1}{2}(\pi^{\mu\alpha}{\sigma^{\nu}}_{\alpha}+\pi^{\nu\alpha}{\sigma^{\mu}}_{\alpha})-\frac{1}{3}P^{\mu\nu}\pi^{\alpha\beta}\sigma_{\alpha\beta}\right]& \\
+\frac{1-\lambda_1-\lambda_2}{(2-\ln 2)(\pi T)^3}\left(\pi^{\mu\alpha}{\pi^{\nu}}_{\alpha}-\frac{1}{3}P^{\mu\nu}\pi^{\alpha\beta}\pi_{\alpha\beta}\right)& \\
+O\left(\pi^3,\pi\partial\pi,\partial^2\pi,\pi^2\partial u,\pi\partial^2 u,\partial^2\pi,\partial^3 u,(\partial u)(\partial^2 u),(\partial u)^3\right)\quad ,&\qquad (3.16)
\end{aligned}
$$

where the $O(\pi^3, \pi\partial\pi, ...)$ term indicates the corrections which lie beyond our inputs.

The corrections can only include terms of the structures displayed or those with more derivatives or containing more powers of $\pi_{\mu\nu}$ or both. We cannot say much about these corrections because for purely hydrodynamic solutions, they contribute to the energy-momentum tensor at the third derivative order only and the general structure of the hydrodynamic energy-momentum tensor at third order in derivatives is not known. The four λ_i's ($i = 1, 2, 3, 4$) are pure numbers. Though we have not been able to specify their values, they are not free parameters. Once their values are fixed by regularity of the future horizon for certain configurations, they should give the complete framework for the whole class of regular solutions.

As we have already mentioned, this equation of motion (3.16) for the shear-stress tensor $\pi_{\mu\nu}$ has to be supplemented by the conservation of energy-momentum tensor in the form given in (3.12) so that we have nine equations for the nine variables (including the hydrodynamic variables) parameterizing the general nonequilibrium energy-momentum tensor. The tracelessness of the energy-momentum tensor begets the tracelessness of $\pi_{\mu\nu}$ as in (3.12) and this, as we have mentioned before, has led to the requirement that our equation of motion for $\pi_{\mu\nu}$ should be Weyl covariant.

This equation is thus a phenomenological framework for the universal sector as a whole up to certain orders in perturbation about the final equilibrium state. This framework governs both hydrodynamic and nonhydrodynamic situations and goes much beyond linear perturbation theory. This is however, only valid within the universal sector. Beyond this sector we need many other inputs other than the boundary energy-momentum tensor to specify the boundary state or the solutions in gravity.

3.3.3 Checks, comparisons and comments

We will begin with a couple of comments. The first comment is that our Eq. (3.16) does not hold well at early times in the generic case. At early times the terms with time derivatives of various orders coming from the higher order corrections to our equation would become important. We will soon see the effect of such time-derivative terms in a simple example. We give an argument why such terms with time-derivatives must appear in the higher order corrections [6]. Any data at early times in the bulk, which will result in smooth behavior in the future, should get reflected in terms of an infinite set of variables in the boundary. The only way we can represent this in terms of the energy-momentum tensor alone is to include its higher order time derivatives in the initial data, so the equation for evolution of the energy-momentum tensor should contain higher order time derivatives.

The second comment is that, in the particular case of boost-invariant flow, we have a better structural understanding of the hydrodynamic behavior at higher orders in the derivative expansion [53]. We can, in principle, use our procedure to give a framework for general boost-invariant flows at late times. However we will leave this for future work. Moreover, the basic logic of our proposal is to use the purely hydrodynamic behavior as an input and then extend this to the complete framework. So our proposal and its extension

[6] We thank Shiraz Minwalla for discussion on this point.

at higher orders, by construction, reproduce the hydrodynamic sound and shear branches of the quasinormal modes.

We now develop a straightforward strategy to check our proposal. We could look at simple nonhydrodynamic configurations first and construct the bulk solution perturbatively in the amplitude expansion parameter to determine some of the λ's. Once these have been determined, we can construct bulk solutions corresponding to a combination of hydrodynamic and nonhydrodynamic behaviors perturbatively in both the amplitude and derivative expansion parameters and then check if the regularity fixes those λ's to the same values.

The simplest nonhydrodynamic configurations are the analogs of homogenous conservative solutions of the Boltzmann equation we have mentioned before and which describe pure relaxation dynamics. Such configurations are homogeneous in space, but time dependent and satisfy the conservation equation trivially. In such configurations the flow is at rest, so that $u^\mu = (1,0,0,0)$ and the temperature T is also a spatiotemporal constant. The nonequilibrium part of the energy-momentum tensor satisfies the following conditions

(i) the time-time component π_{00} and the time-space components π_{0i} for $i = 1,2,3$ vanish and

(ii) the space-space components π_{ij} for $i, j = 1,2,3$ are dependent only on time.

The above conditions on $\pi_{\mu\nu}$ result in the conservation equation being trivially satisfied. It follows from our proposal (3.16) that regularity in the bulk implies that π_{ij} satisfy the following equation of motion :

$$(1-\lambda_3)\frac{d\pi_{ij}}{dt} + \frac{2\pi T}{(2-ln2)}\pi_{ij} - \frac{1-\lambda_1-\lambda_2}{(2-ln2)(\pi T)^3}\left(\pi_{ik}\pi_{kj} - \frac{1}{3}\delta_{ij}\pi_{lm}\pi_{lm}\right) = O(\frac{d^2\pi_{ij}}{dt^2}) \quad . \tag{3.17}$$

If we look at the linearized solution, we have

$$\pi_{ij} = \mathcal{A}_{ij} exp(-\frac{t}{\tau_\pi}), \quad \tau_\pi = (1-\lambda_3)\frac{2-ln\ 2}{2\pi T} \quad , \tag{3.18}$$

where $\mathcal{A}_{ij}$ is a spatiotemporally constant matrix such that $\mathcal{A}_{ij}\delta_{ij} = 0$. This implies that we have a nonhydrodynamic mode such that when the wave vector $\mathbf{k}$ vanishes, the frequency

ω becomes purely imaginary and equals $-i\tau_\pi^{-1}$, i.e $\omega = -i\tau_\pi^{-1}$ as $\mathbf{k} \to 0$. There is however, no such mode in the quasinormal spectrum of black branes [20]. This makes us conclude that $\lambda_3 = 1$, so that at the linearized level the only solution of (3.17) is $\pi_{ij} = 0$.

However, at the nonlinear level we still have nonhydrodynamic solutions given by

$$\frac{2\pi T}{(2 - ln2)}\pi_{ij} - \frac{1 - \lambda_1 - \lambda_2}{(2 - ln2)(\pi T)^3}\left(\pi_{ik}\pi_{kj} - \frac{1}{3}\delta_{ij}\pi_{lm}\pi_{lm}\right) = O(\frac{d^2\pi_{ij}}{dt^2}) \quad . \tag{3.19}$$

In fact, up to the orders explicitly given above, the equation is nondynamical and predicts that we should, at least perturbatively, have tensor hair on the black-brane solution in pure gravity in AdS. This gives us the simplest nontrivial test of our proposal and also a means of determining $\lambda_1 + \lambda_2$.

In this connection, we also note that the possible second order in the time-derivative correction in (3.19) implies that we need not have a monotonic approach to equilibrium as we should have in the presence of an entropy current.

We end here with some comments on the issue of connecting our proposal with physics of quasinormal modes of the black brane. The linearized limit of the conservation equation, along with our proposed Eq. (3.16), supports at most three branches of linearized fluctuations. We have further argued that the third branch giving pure relaxation dynamics is not present. However, we know that the quasinormal modes have infinite branches of higher overtones other than the hydrodynamic sound and shear branches. This naive comparison is somewhat misplaced, [7] because, as we know, nonlinearities do affect linearized propagation in quantum field theories. Since our equations are actually equivalent to the nonequilibrium field theory equation of motion of the state in the gauge theory, we must take into account nonlinearities of our equation in the propagation of the energy-momentum tensor before making any comparison. We leave this for future work.

We would also like to mention here that it is only natural that the higher overtones are more like resonances and are built out of the dynamics of the nine degrees of freedom of the conformal energy-momentum tensor, as it would have been surprising if infinite branches in the spectrum in the universal sector would have been blind to the

[7] We thank A. O. Starinets for discussion on this point.

microscopic details of the theory like the matter content and couplings. Our framework suggests that these infinite branches could be obtained from the nonlinear dynamics of the energy-momentum tensor. However we should exhibit caution here because although these nonhydrodynamic higher overtones of quasinormal modes are indeed regular linear perturbations of the black brane, it is yet to be demonstrated that these can be developed into complete regular solutions of Einstein's equation non-linearly.

Part III

Hydrodynamics in a nonrelativistic limit of gauge/gravity duality

Chapter 4

The covariance of Navier-Stokes equation and invariance of theories under the infinite dimensional Galilean Conformal Algebra

4.1 Introduction

In this chapter, we will investigate a novel non-relativistic limit of gauge/gravity duality for conformal cases particularly to find out if hydrodynamics can be contained in this limit. In the process, we will obtain valuable clues of how to take this limit dynamically so that we get hydrodynamic behavior after the limit is taken. This chapter is a slight departure from the general theme of the thesis, but has been included with the hope that it can have some relevance for a tabletop experiment or a simulated system in the future. It may also turn out that the universal sector in this dynamical limit can be solved sometime in the future owing to the infinite-dimensional symmetry which appears in this limit, so the results here could be important steps taken in this direction as well.

A new non-relativistic extension of gauge/gravity duality became possible when it was shown [56, 57] that a non-relativistic conformal algebra could be obtained as a parametric contraction of the relativistic conformal group. This contraction retained the same number of generators as the relativistic conformal group. It was also found out by the authors of [57] that an infinite-dimensional extension of the finite non-relativistic algebra was possible and following them, we call this algebra the Galilean Conformal Algebra, in short GCA. In the context of developing the version of gauge/gravity duality for this non-relativistic symmetry, important steps were also taken in [57] and later these have been extended in [58, 59] (for some related work, please also see [60] [1]). The development is still under progress [2], however it has been realized that this is different from the case of the non-relativistic Schrodinger group. The Schrodinger group, has the advantage that, it can be embedded in the relativistic conformal group of two higher dimensions, so gauge/gravity duality in this case, can be developed on lines closer to the conventional relativistic setting, though in two higher dimensions [66]. In the case of the Galilean Conformal Algebra, however, it seems that the dynamics in the bulk involves a degenerate limit, which is possibly a Newton-Cartan like gravity involving an AdS_2 factor [57] [3].

To get a better understanding, it will be useful to understand the pure gravity sector first and in this sector, the gravity duals of hydrodynamic flows ubiquitously plays a very special role, because of the conceptual clarity of their construction (for a review, see [27]). However, even before constructing gravity duals, it is important, to understand the role of the *full* Galilean Conformal Algebra as symmetries of the hydrodynamics of the boundary theory. In the original work [57], it was shown that the Euler equation for incompressible flows was invariant under some of the elements of the Galilean Conformal Algebra. However, the hydrodynamics in any physical theory, should have a non-zero viscosity and moreover there are typically higher derivative corrections to all orders. Here, we will investigate how the Galilean Conformal Algebra can act as symmetries of the Navier-Stokes equation and also its role in constraining higher derivative corrections.

The important point of our approach will be that we will be looking for *covariance* rather than *invariance*, in close analogy with the case of relativistic conformal hydrodynamics where the relativistic Navier-Stokes equation and its higher derivative corrections can be made covariant (not invariant) under the relativistic conformal group [37]. An ele-

[1] Superconformal extensions have been dealt with in [72].

[2] For 2D CFTs, this nonrelativistic limit has recently been taken even dynamically [61]. However, in this limit we do not obtain hydrodynamics.

[3] For an interesting earlier work, please look at [62] and a recent work in this direction is [63].

ment in GCA may take a Galilean inertial frame to a non-inertial one. After *covariantizing* under GCA, as expected, the equation will take its usual form in an inertial frame, but in a non-inertial frame it will assume a non-standard form. In our case, the covariantizing will involve novel features, like the absolute (time-dependent) acceleration and absolute (time-dependent) angular velocity of the non-inertial frame,*which are not non-relativistic degenerations of the relativistic covariant form.*. The basic reason for the appearance of novel features is straightforward, the infinite GCA has no relativistic analogue (for a lucid description of non-relativistic degenerations of relativistically covariant hydrodynamics, etc, please see [67]). Also, in non-relativistic dynamics, the absolute acceleration or the absolute angular velocity of a non-inertial frame are the more natural objects to be used for covariantizing rather than "connections". Since our approach involves covariantizing the usual Navier-Stokes equation for incompressible flows which holds in inertial frames, it is very different from that in [70][4].

We will divide the Navier-Stokes equation into three parts, namely, the kinematic term, the pressure term and the viscous term, and we will show that each term separately transforms covariantly, exactly like in the case of the covariance of the relativistic Navier-Stokes equation under the relativistic conformal group. The kinematic term, in an inertial frame, is just the Euler derivative acting on the velocity field. This term transforms just like the acceleration. Since, the GCA can transform an inertial frame to a non-inertial frame, as mentioned above, the covariantizing will naturally involve the absolute angular velocity and the absolute acceleration of the non-inertial frame. However, the covariance under the "spatially correlated time reparametrizations" will be possible only if the flow is incompressible[5]. Therefore, we would require the flow to be incompressible too.

The pressure term is just the gradient of the pressure divided by the density. We will show that this leads to the speed of sound being GCA invariant, essentially because the pressure transforms in the same way as the density under GCA.

[4]For some related work please also see [71].

[5]When a non-relativistic limit is taken by applying an appropriate scaling of the relativistic Navier-Stokes equation, the incompressibility of the flow is automatically obtained (please see the first two references of [70]. The GCA covariant form, however, cannot be obtained as a limit of the usual conformally covariant relativistic Navier-Stokes equation.

The viscous term is $(1/\rho)\partial_i\sigma_{ij}$, where ρ is the density and σ_{ij} is the hydrodynamic shear stress tensor given by, $\sigma_{ij} = \eta(\nabla_i v_j + \nabla_j v_i - (2/3)\delta_{ij}\nabla \cdot v)$, with η being the shear viscosity. Here the shear viscosity also transforms as a *field* only through its dependence on the thermodynamic variables which transform under GCA.

We will see that when all chemical potentials vanish (as in a gas of phonons), c_s, which denotes the speed of sound in an inertial comoving with the flow, is invariant under GCA. We will see that this implies that it must be a fundamental constant like the speed of light or given in terms of the microscopic parameters. We will see how each could be possible, in particular we will see that when the number of spatial dimensions is two, GCA admits a central charge with dimension $(1/speed)^2$. Then we will study the transformation of viscosity under GCA and see that in the absence of chemical potentials the transformation could be realized only if the microscopic theory contains a length scale, or a time scale, or both and if this is not possible, the viscosity should vanish.

We also find that the GCA also has the potential to restrict the possible corrections to the Navier-Stokes equation and we explicitly evaluate the possible three derivative corrections. It is intriguing that all these four possibilities correspond to the relativistic conformal case so that the relativistic terms reduce to our terms in the non-relativistic limit in inertial frames, when the flow is incompressible. The general lesson is that a phenomenological law can be covariantized under GCA only if its form in the inertial frame is sufficiently restricted.

The plan of the paper is as follows. In section 2, we arrive at a covariant description of the hydrodynamics for the GCA. In section 3, we use this to covariantize the Navier-Stokes equation. In scetion 4, we discuss how we can covariantize the continuity equation and how it influences the transformations of the density, pressure and viscosity. In section 5, we show how the GCA constrains higher derivative corrections to the Navier-Stokes equations. Then we conclude with some discussions on the implications of our results for the version of gauge/gravity duality with GCA as the conformal symmetry group. In appendices D and E, we elucidate some technical points and in particular, we also give a simple mathematical interpretation of the GCA, that could be useful for constructing

GCA invariant microscopic theories. We will refer to these appendices appropriately in this chapter.

4.2 Covariant kinematics for the infinite Galilean Conformal Algebra

The finite part of the Galilean Conformal Algebra can be obtained as a parametric contraction of the $SO(d+1,2)$ relativistic conformal group of $(d,1)$ dimensional Minkowskian space-time [56, 57]. This finite part forms a Lie group with exactly the same number of generators as the $SO(d+1,2)$ relativistic conformal group. The generators of this finite part consists of the following

$$\begin{aligned} H &= -\frac{\partial}{\partial t}, \\ P_i &= \nabla_i, \\ J_{ij} &= -(x_i\nabla_j - x_j\nabla_i), \\ B_i &= t\nabla_i, \\ D &= -(\mathbf{x}.\nabla + t\frac{\partial}{\partial t}), \\ K &= -(2t\mathbf{x}.\nabla + t^2\frac{\partial}{\partial t}), \\ K_i &= t^2\nabla_i. \end{aligned} \tag{4.1}$$

Clearly, H is the Hamiltonian, P_i are the momenta and J_{ij} are the angular momenta generating time translations, spatial translations and angular rotations respectively. The B_i's generate the Galilean boosts. The dilation operator D acts differently from the Schrodinger group as it scales all spatial coordinates and time in the same way. The other generators K and K_i can be thought of non-relativistic counterparts of relativistic special conformal transformations.

This finite algebra has an infinite extension which forms the full GCA, the generators of which can be labelled as below

$$L^{(n)} = -(n+1)t^n(\mathbf{x}.\nabla) - t^{n+1}\frac{\partial}{\partial t}, \tag{4.2}$$
$$M_i^{(n)} = t^{n+1}\nabla_i,$$
$$J_a^{(n)} \equiv J_{ij}^{(n)} = -t^n(x_i\nabla_j - x_j\nabla_i),$$

where n runs over all integers. The $SL(2,R)$ part of $L^{(n)}$'s belong to the finite group (as $H = L^{(-1)}, D = L^{(0)}, L^{(1)} = K$). Also, $P_i = M_i^{(-1)}, B_i = M_i^{(0)}, K_i = M_i^1$, while only $J_{ij}^{(0)}$ belong to the finite group. The full algebra is

$$[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)}, \tag{4.3}$$
$$[L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)},$$
$$[J_a^{(n)}, J_b^{(m)}] = f_{abc}J_c^{(n+m)},$$
$$[L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)},$$
$$[J_{ij}^{(n)}, M_k^{(m)}] = -(M_i^{(m+n)}\delta_{jk} - M_i^{(m+n)}\delta_{jk}),$$
$$[M_i^{(m)}, M_j^{(n)}] = 0.$$

The index a above form an alternative label corresponding to the spatial rotation group $SO(d)$ and f_{abc} are the structure constants of this group. Further $J_{(a)}^{(n)}$'s and $L^{(m)}$'s together form a Virasoro Kac-Moody algebra. The GCA admits the usual (dimensionless) central charges for the Virasoro Kac-Moody subalgebra as the $M_i^{(n)}$'s can be consistently put to zero [57]. Besides, these usual dimensionless, central charges, a special kind of central charge, is possible in the case of two spatial dimensions and it will be important for us because only in the case of two spatial dimensions we can have a dimensionful central charge. A dimensionful central charge, unlike a dimensionless one can appear in the Lagrangian description of the theory. A simple example is the central charge with dimension of mass in the Schrodinger group actually being the mass of the free particle. This central charge Θ, appears in the commutator of $M_i^{(m)}$'s in the GCA as below [59, 64]

$$[M_i^{(m)}, M_j^{(n)}] = I^{mn}\epsilon_{ij}\Theta, \tag{4.4}$$

where I^{mn} is the invariant tensor of the spin one representation of $SL(2,R)$. The central charge Θ has the dimension of $(1/speed)^2$. For possible physical interpretations of this term, please look at [59, 64, 65]. Further, in the case of the Schrodinger group, as mentioned above, there is another possible central charge (for any number of spatial dimensions) which has the dimension of mass (in units where the Planck's constant is set to unity, mass is basically time divided by square of length) and in fact has the interpretation of the mass scale in the corresponding theory. The absence of this central term in the GCA has been argued [57, 60] to reflect the absence of any mass scale in the microscopic theory and we will also hold to this point of view here.

The $J_a^{(n)}$'s actually generate arbitrary time dependent rotations, the $M_i^{(n)}$'s generate arbitrary time-dependent boosts and the $L^{(n)}$'s generate spatially correlated time reparametrization [57]. Each of these form a subalgebra by themselves. We now proceed to consider each of these categories of space-time transformations in detail to see how one can have a covariant description of kinematics for each of these categories. Finally, we will sum up by arriving at a kinematic description which will be covariant under the full set of transformations.

4.2.1 Arbitrary time dependent rotations

These transformations are

$$x_i^{'} = R_{ij}(t)x_j, \qquad t^{'} = t, \tag{4.5}$$

where R_{ij} is an arbitrary time dependent rotation matrix (so that $R_{ij}^{-1} = R_{ji}$). The velocity transforms in the following manner,

$$v_i = R_{ij}^{-1}(v_j^{'} - \frac{dR_{jk}}{dt'} R_{kl}^{-1} x_l^{'}). \tag{4.6}$$

Now we will show that from the above transformation one can extract a covariant time derivative. Let us define Ω_{ij} to be the absolute angular velocity of the non-inertial frame with respect to any inertial frame (note when the number of spatial dimensions is more than three this is actually a tensor, but by abuse of notation we will still call it absolute angular velocity, in three dimensions $\Omega_{ij} = \epsilon_{ikj}\Omega_k$). Suppose the unprimed coordinates are in the inertial frame and the primed ones are in the non-inertial frame. Then clearly the absolute angular velocity $\Omega_{ij} = -(dR_{ik}/dt)R^{-1}_{kj}$. Of course the absolute angular velocity of a frame is very much a physical quantity as it can be determined by an observer using that frame. The covariant time derivative in a given frame, can now be defined through its action on vectors as below,

$$\frac{D}{Dt}V_i = \frac{d}{dt}V_i + \Omega_{ij}V_j, \tag{4.7}$$

where $\mathbf{V}$ is an arbitrary vector. Note that in an inertial frame $D/Dt = d/dt$, so if the unprimed coordinates are inertial and primed coordinates non-inertial we may rewrite (4.6) as,

$$\frac{D}{Dt}x_i = R^{-1}_{ij}\frac{D}{Dt'}x^{'}_j. \tag{4.8}$$

In fact, we may replace the position vector x_i above with any arbitrary vector V_i which transforms like $V^{'}_i = R_{ij}V_j$, then it also follows that

$$\frac{D}{Dt}V_i = R^{-1}_{ij}\frac{D}{Dt'}V^{'}_j. \tag{4.9}$$

We now claim that the above relation is valid even when both the primed and unprimed coordinates are non-inertial. An easy way to prove this is as follows. Let us take two non-inertial frames $(\mathbf{x_{(1)}}, t_{(1)})$ and $(\mathbf{x_{(2)}}, t_{(2)})$ which are related to the inertial frame $(\mathbf{x}, t)$ through $x_{(1)i} = R_{(1)ij}x_j, t_{(1)} = t$ and $x_{(2)i} = R_{(2)ij}x_j, t_{(2)} = t$ respectively. Obviously the absolute angular velocities of the non-inertial frames are $\Omega_{(1)ij} = -(dR_{(1)ik}/dt)R^{-1}_{(1)kj}$ and $\Omega_{(2)ij} = -(dR_{(2)ik}/dt)R^{-1}_{(2)kj}$ respectively. Clearly,

$$\frac{D}{Dt}V_i = R^{-1}_{(1)ij}\frac{D}{Dt_1}V_{(1)j} = R^{-1}_{(2)ij}\frac{D}{Dt_2}V_{(2)j}. \tag{4.10}$$

Therefore,

$$\frac{D}{Dt_1}V_{(1)i} = R^{-1}_{ij}\frac{D}{Dt_2}V_{(2)j}, \tag{4.11}$$

where

$$R_{ij} = R_{(2)ik}R^{-1}_{(1)kj}, \tag{4.12}$$

as required so that indeed $x_{(2)i} = R_{ij}x_{(1)j}$. Therefore, (4.9) is valid for any two frames, even if both are non-inertial. In particular we will define the covariant velocity $\mathcal{V}^{(rot)}$ as the covariant derivative of the position vector so that

$$\mathcal{V}_i^{(rot)} = \frac{D}{Dt}x_i = \frac{d}{dt}x_i + \Omega_{ij}x_j. \tag{4.13}$$

By construction this transforms covariantly under (4.5), so that

$$\mathcal{V}_i^{(rot)} = R_{ij}^{-1}\mathcal{V}_j^{(rot)'}. \tag{4.14}$$

The above tells us how to modify the acceleration so that we get a covariant vector. We define, $\mathcal{A}^{(rot)}$ the "covariant acceleration" as two covariant time derivatives acting on the position vector as below,

$$\mathcal{A}_i^{(rot)} = \frac{D^2}{Dt^2}x_i = \frac{d^2}{dt^2}x_i + 2\Omega_{ij}v_j + \Omega_{ij}\Omega_{jk}x_k + (\frac{d}{dt}\Omega_{ij})x_j. \tag{4.15}$$

In the non-inertial coordinates in the right hand side of the last expression above the corrections to the usual acceleration are just the Corriolis, centrifugal and Euler forces respectively. [6] By construction, under the transformations (4.5), the covariant acceleration transforms as below,

$$\mathcal{A}_i^{(rot)} = R_{ij}^{-1}\mathcal{A}_i^{(rot)'}, \tag{4.16}$$

where both the primed and unprimed coordinates can be non-inertial.

We also observe that the spatial derivative ∇_i and the symmetric traceless tensor $\sigma_{ij} = \nabla_i v_j + \nabla_j v_i - (2/3)\delta_{ij}(\nabla.\mathbf{v})$ transforms covariantly while divergence of the velocity $\nabla\cdot\mathbf{v}$ transforms invariantly (in the last two cases, of course, we are talking of a velocity field), so that under the transformations (4.5),

$$\begin{aligned}\nabla_i &= R_{ij}^{-1}\nabla_j^{'}, \\ \sigma_{ij} &= R_{ik}^{-1}R_{jl}^{-1}\sigma_{kl}^{'}, \\ \nabla\cdot\mathbf{v} &= \nabla^{'}\cdot\mathbf{v}^{'}.\end{aligned} \tag{4.17}$$

[6] Usually the relation between acceleration in inertial frame and non-inertial frames in the case of three spatial dimensions are written from the "passive" point of view as: $\mathbf{a}^{'} = \mathbf{a} - 2\mathbf{\Omega}\times\mathbf{v} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x}) - (d\mathbf{\Omega}/dt)\times\mathbf{x}$, where the primed coordinates are non-inertial and unprimed ones are inertial. However, one can work out that it is, in fact, equivalent to $\mathbf{a} = \mathbf{a}^{'} + 2\mathbf{\Omega}\times\mathbf{v}^{'} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x}^{'}) + (d\mathbf{\Omega}/dt^{'})\times\mathbf{x}^{'}$. In three spatial dimensions this is just another way of understanding (4.15).

For the last two results above, we have used the fact that $(dR_{ik}/dt)R^{-1}_{kj}$ is antisymmetric in i and j.

To summarize we see that we have two basic operators which transform covariantly, namely the covariant time derivative D/Dt (as defined in (4.7)) and the spatial derivative ∇_i. Further the traceless symmetric tensor ϵ_{ij} transforms covariantly and $\nabla \cdot \mathbf{v}$ transforms invariantly.

4.2.2 Arbitrary time dependent boosts

These transformations are

$$x_i^{'} = x_i + b_i(t), \qquad t^{'} = t. \tag{4.18}$$

We will mathematically interpret the above as the position vector not transforming covariantly. It is easy to see that relative distances, relative velocities and relative accelerations will remain invariant under these transformations. So, one can easily get a invariant acceleration field using the relative acceleration with respect to the absolute acceleration of the frame. Let $\mathcal{B}$ be the absolute acceleration of the non-inertial frame. Then the invariant acceleration field $\mathcal{A}^{(accl)}$ may be defined as,

$$\mathcal{A}_i^{(accl)} = \frac{d}{dt}v_i - \mathcal{A}^{(accl)} = \frac{d}{dt}v_i - \nabla_i(\mathcal{B}.\mathbf{x}). \tag{4.19}$$

This, again, can be proved as before, consider the unprimed coordinates as inertial and primed coordinates as non-inertial in (4.18), then from the passive point of view, the absolute acceleration of the non-inertial frame is $\mathcal{B}_i = d^2b_i/dt^2$. So it is clearly true that

$$\mathcal{A}_i^{(accl)} = \frac{d}{dt}v_i = \frac{d}{dt'}v_i^{'} - \nabla_i^{'}(\mathcal{B}\cdot\mathbf{x}^{'}) = \mathcal{A}_i^{(accl)'}. \tag{4.20}$$

We can repeat the same trick of comparing two non-inertial frames with one inertial frame and then comparing the two non-inertial frames with each other, as described in the previous subsection, to conclude that $\mathcal{A}_i^{(accl)} = \mathcal{A}_i^{(accl)'}$ is valid even if both the primed and unprimed frames are non-inertial. Therefore we conclude that (4.19) indeed defines an invariant acceleration field.

We also observe the operator ∇_i is invariant and so are $\nabla \cdot \mathbf{v}$ and the symmetric traceless tensor ϵ_{ij} under the transformation (4.18).

4.2.3 Spatially correlated time reparametrization

These transformations are

$$\begin{aligned} x_i^{'} &= \frac{df}{dt} x_i, \\ t^{'} &= f(t). \end{aligned} \tag{4.21}$$

The interesting thing about this transformation is that the new frame may be using a different time from absolute time. However, one must ask how can an observer using a frame know that the time being used is different from absolute time? To find that out, let us first note the transformation of the velocity,

$$v_i = v_i^{'} + \frac{\frac{d^2t}{dt'^2}}{\frac{dt}{dt'}} x_i^{'}. \tag{4.22}$$

The divergence of the velocity field transforms as,

$$\nabla \cdot \mathbf{v} = \frac{dt^{'}}{dt} \nabla^{'} \cdot \mathbf{v}^{'} + d \frac{\frac{d^2t}{dt'^2}}{(\frac{dt}{dt'})^2}. \tag{4.23}$$

Combining these one can easily see that one can make an invariant velocity field,

$$\mathcal{V}_i^{(sctr)} = v_i - \frac{\nabla \cdot \mathbf{v}}{d} x_i. \tag{4.24}$$

Firstly let us assume that when the frame is using absolute time the divergence of the velocity field, $\nabla \cdot \mathbf{v}$ vanishes. After a generic transformation as in (4.21), as shown in (4.22), clearly it will no longer be zero. Therefore, if is this is not zero, one knows that the time being used is not using absolute time. Note the divergence of the velocity field remains zero under constant dilatation or shifts, so one can be sure of the use of absolute time only upto a constant dilation or shift. Now, (4.24) shows that *one can construct an invariant velocity field under space correlated time reparametrization, which reduces to the usual velocity field in an inertial frame (where absolute time is used), if and only if, the* *divergence of the velocity field vanishes (or the flow is incompressible) in the inertial frame.* This is precisely why the assumption of incompressible flow is crucial to covariantize the Navier-Stokes' equation under the full GCA.

One can make a covariant acceleration field

$$\mathcal{A}_i^{(sctr)} = \frac{d}{dt}\mathcal{V}_i^{(sctr)}, \tag{4.25}$$

so that

$$\mathcal{A}_i^{(sctr)} = \frac{dt^{'}}{dt}\mathcal{A}_i^{(sctr)'}. \tag{4.26}$$

Finally one notes that the operators ∇_i transforms covariantly and so does the traceless symmetric tensor σ_{ij};

$$\begin{aligned} \nabla_i &= \frac{dt^{'}}{dt}\nabla^{'}_i, \\ \sigma_{ij} &= \frac{dt^{'}}{dt}\sigma_{ij}. \end{aligned} \tag{4.27}$$

4.2.4 Summing all up

We would like to sum up all our results in order to construct a covariant acceleration field which will be covariant under the full GCA. We first observe that any element of the GCA can be written as a succession of a time dependent rotation, a spatially correlated time reparametrization and a time dependent boost (for proof please see appendix B). So without loss of generality, any element of GCA can be written as below:

$$\begin{aligned} x_i^{'} &= \frac{df}{dt}R_{ij}(t)x_j + b_i(t), \\ t^{'} &= f(t). \end{aligned} \tag{4.28}$$

Instead of working out what happens under the full transformation we can, instead, use the following logic. Let us first put $b_i(t)$ to zero so that the position vector transforms covariantly. Then one can define a velocity field which is covariant under the combined action of rotation and spatially correlated time reparametrization.

$$\mathcal{V}_i^{(\mathbf{b}=0)} = v_i + \Omega_{ij}x_j - \frac{\nabla\cdot\mathbf{v}}{d}x_i. \tag{4.29}$$

However, now the angular velocity of the frame Ω_{ij} is defined with the time of the frame, which need not be the absolute time, for instance, in (4.28), if the primed coordinates are non-inertial and the unprimed one is inertial then the angular velocity of the non-inertial frame is $\Omega_{ij} = -(dR_{ik}/dt')R^{-1}_{kj}$. One can easily see the further modification which makes the velocity field covariant as $\nabla \cdot \mathbf{v}$ transforms invariantly under arbitrary rotations. Anyway, using methods pointed out in the previous subsections, one can readily check that when $b_i(t) = 0$, under the transformation (4.28), the covariant velocity field transforms as,

$$\mathcal{V}_i^{(\mathbf{b}=0)} = R^{-1}_{ij}\mathcal{V}_j^{(\mathbf{b}=0)'}. \tag{4.30}$$

If we have a vector V_i, which transforms under (4.28) when $b_i(t) = 0$ as

$$V_i = R^{-1}_{ij}V_j, \tag{4.31}$$

then we define its covariant time derivative as

$$\frac{D}{Dt}V_i = \frac{d}{dt}V_i + \Omega_{ij}V_j. \tag{4.32}$$

Then when $b_i(t) = 0$, under the transformation (4.28) we get

$$\frac{D}{Dt}V_i = \frac{dt'}{dt}R^{-1}_{ij}\frac{D}{Dt'}V'_j. \tag{4.33}$$

The above can be easily proved by our previous trick of comparing two non-inertial frames with an inertial one and then comparing the non-inertial frames with each other so that the above remains valid even when both the primed and unprimed frames are non-inertial. For the sake of convenience of the reader, we will repeat this trick explicitly for our final covariant acceleration field, which we are now in the process of constructing. It is now clear how we should construct a covariant acceleration field when $b_i(t) = 0$. We must make the covariant time derivative act on the covariant velocity field, so that,

$$\mathcal{A}_i^{(\mathbf{b}=0)} = \frac{D}{Dt}\mathcal{V}_i^{(\mathbf{b}=0)} = \frac{d}{dt}(v_i + \Omega_{ij}x_j - \frac{\nabla.\mathbf{v}}{d}x_i) + \Omega_{ij}(v_j + \Omega_{jk}x_k - \frac{\nabla.\mathbf{v}}{d}x_j). \tag{4.34}$$

Therefore, when $b_i(t) = 0$, under the combined transformation (4.28), the covariant acceleration field constructed above transforms as $\mathbf{V}$ in (4.32), so that

$$\mathcal{A}_i^{(\mathbf{b}=0)} = \frac{dt'}{dt}R^{-1}_{ij}\mathcal{A}_j^{(\mathbf{b}=0)'}. \tag{4.35}$$

Again it is clear how we can maintain the above covariance when $b_i(t)$ is not zero. We just take the relative covariant acceleration with respect to $\mathcal{B}$, the acceleration of the frame in the time of the frame (which may not be absolute time). Our final covariant acceleration field, which is covariant with respect to the full GCA is

$$\begin{aligned}\mathcal{A}_i^{(comb)} &= \mathcal{A}_i^{(\mathbf{b}=0)} - \mathcal{B}_i = \mathcal{A}_i^{(\mathbf{b}=0)} - \nabla_i(\mathcal{B}\cdot\mathbf{x}) = \frac{D}{Dt}\mathcal{V}_i^{(\mathbf{b}=0)} - \nabla_i(\mathcal{B}\cdot\mathbf{x}) \\ &= \frac{d}{dt}(v_i + \Omega_{ij}x_j - \frac{\nabla.\mathbf{v}}{d}x_i) + \Omega_{ij}(v_j + \Omega_{jk}x_k - \frac{\nabla.\mathbf{v}}{d}x_j) - \nabla_i(\mathcal{B}\cdot\mathbf{x}).\end{aligned} \tag{4.36}$$

The covariance, under the full GCA is simply

$$\mathcal{A}_i^{(comb)} = \frac{dt'}{dt}R_{ij}^{-1}\mathcal{A}_j^{(comb)'}. \tag{4.37}$$

To check the above, one can go back again to the representation (4.28) of an arbitrary element of GCA. Now let us suppose that the unprimed coordinates are inertial (where the time is absolute time) so that $\Omega_{ij}, \mathcal{B}_i, \nabla.\mathbf{v}$ are all zero in these coordinates. The covariant acceleration field is just the usual acceleration dv/dt in these coordinates. Now one can readily check the validity of (4.37) with the definition (4.36) of the covariant acceleration field with

$$\begin{aligned}\Omega_{ij} &= -(\frac{d}{dt'}R_{ik})R_{kj}^{-1}, \\ \mathcal{B}_i = \frac{D^2}{Dt'^2}b_i(t(t')) &= \frac{d^2}{dt'^2}b_i - 2\Omega_{ij}\frac{d}{dt'}b_j + \Omega_{ij}\Omega_{jk}b_k - (\frac{d}{dt'}\Omega_{ij})b_j.\end{aligned} \tag{4.38}$$

The above relations are familiar in usual Galilean kinematics, except for the use of a general time t' in the non-inertial frame, which may not be the absolute time. Now as before we consider another non-inertial frame $(\mathbf{x}'', t'')$ related to the same inertial frame $(\mathbf{x}, t)$ through the same relation (4.28), but with different parameters $(R'_{ij}(t), f'(t), b'_i(t))$. Then again (4.37) is valid with the definition (4.36) of the covariant acceleration field and with the angular velocities and acceleration of this frame given by (4.38), but (R_{ij}, b_i) replaced by (R'_{ij}, b'_i). As a result

$$\mathcal{A}_i^{(comb)} = \frac{dt'}{dt}R_{ij}^{-1}\mathcal{A}_j^{(comb)'} = \frac{dt''}{dt}R_{ij}'^{-1}\mathcal{A}_j^{(comb)''}. \tag{4.39}$$

The above implies

$$\mathcal{A}_j^{(comb)'} = \frac{dt''}{dt'} R_{ij} R'^{-1}_{jk} \mathcal{A}_k^{(comb)''} = \frac{dt''}{dt'} (R'_{ij} R^{-1}_{jk})^{-1} \mathcal{A}_k^{(comb)''}. \tag{4.40}$$

The last equality above is exactly what is required for the validity of (4.37) between these two non-inertial frames and since by choice they were arbitrary, we have proved that (4.37) is valid for any two coordinates. However, we note that the covariant acceleration field as defined in (4.36) reduces to the usual acceleration field in an inertial frame only if the flow is incompressible in the inertial frame. So, we prove that *it is possible to define a covariant acceleration field as defined in (4.36) which transforms covariantly as in (4.37) under the full GCA if and only if the flow is incompressible (i.e.* $\nabla \cdot \mathbf{v} = 0$*) in an inertial frame (where absolute time is used).*

Finally we note that the operator ∇_i transforms covariantly under the full GCA and so does the traceless symmetric tensor σ_{ij}. Under the transformation (4.28)

$$\nabla_i = \frac{dt'}{dt} R^{-1}_{ij} \nabla'_j, \tag{4.41}$$

$$\sigma_{ij} = \frac{dt'}{dt} R^{-1}_{ik} R^{-1}_{jl} \sigma'_{kl}.$$

4.3 Covariantizing the Navier-Stokes equation

The approach to equilibrium in physical systems is captured usually by three equation, namely, the continuity equation, the Navier-Stokes equation and the equation for evolution of the mean isotropic pressure. Of these three, the Navier-Stokes equation concerned with the approach to mechanical equilibrium is the most fundamental. The continuity equation is valid only if the microscopic interactions conserve particle number. When the flow is incompressible, i.e when the divergence of the velocity field (whose take values corresponding to the local mean particle velocity) vanishes, the pressure actually is not an independent dynamical variable as it does not have an independent equation for its evolution [68].

As mentioned in the Introduction, we will dissect the Navier-Stokes equation into the kinematic term, the pressure term and the viscous term, and establish the covariance of each of these terms under GCA.

4.3.1 The kinematic term

The kinematic term, in an inertial frame, is simply $d\mathbf{v}/dt$, the acceleration field. Now the total time derivative d/dt acting on any field is simply the Euler operator $\mathcal{D} = \partial/\partial_t + \mathbf{v}.\nabla$ acting on the field. Therefore the covariant form of the kinematic term, under the full GCA is just the covariant acceleration field (4.36) where, we may replace d/dt with $\mathcal{D}$

$$(\mathcal{D}v)_i^{(comb)} = \mathcal{D}(v_i+\Omega_{ij}(t)x_j - \frac{\nabla.\mathbf{v}}{d}x_i)+\Omega_{ij}(t)(v_j+\Omega_{jk}(t)x_k - \frac{\nabla.\mathbf{v}}{d}x_j) - \nabla_i(\mathcal{B}(t)\cdot\mathbf{x}). \quad (4.42)$$

Above we have made explicit that the angular velocity and acceleration of the frame is time dependent only. As we have proved in the previous section, the kinematic term transforms as (4.37) under the full GCA, so under the transformation (4.28), the covariant acceleration field transforms as

$$(\mathcal{D}v)_i^{(comb)} = \frac{dt'}{dt} R_{ij}^{-1} (\mathcal{D}v)_i^{(comb)'}. \quad (4.43)$$

Note the covariant kinematic term (4.42) becomes the usual kinematic term in an inertial frame, where absolute time is also used, only when the flow is incompressible in any inertial frame. So, it is crucial that the flow, is indeed, incompressible, in an inertial frame. *The kinematic term can be made GCA covariant only if the flow is incompressible in an inertial frame so that it reduces to just the Euler derivative acting on the velocity field in an inertial frame.*

We also note that since the centrifugal force is a conservative force, one may also write the centrifugal term like a derivative of the potential term as has been done in the case of the term involving the acceleration of the frame, but it will obscure the covariance of the kinematic term, which could be easily constructed from the logic given in the previous section. Also, written in the form (4.42), we readily see that the acceleration of the frame mimics the effect of an uniform gravitational field. It is reminiscent of the relativistic case where to achieve Weyl covariance we also promote ordinary derivatives to covariant derivatives which also conforms with the equivalence principle.

4.3.2 The pressure term

The pressure term in a non-inertial frame is just $-(\nabla_i p)/\rho$. We will see that the pressure term and even the viscous term requires no modification and by themselves transform covariantly under the full GCA.

The pressure term is

$$-\frac{\nabla_i p}{\rho}. \tag{4.44}$$

We make a natural assumption that the density transforms homogeneously under GCA, so that

$$\rho(\mathbf{x}, t) = (\frac{dt^{'}}{dt})^a \rho^{'}(\mathbf{x}^{'}, t^{'}), \tag{4.45}$$

where a, is an undetermined constant. Therefore the pressure term should remain covariant if the pressure p transforms in exactly the same manner as the density ρ, so that

$$p(\mathbf{x}, t) = (\frac{dt^{'}}{dt})^a p^{'}(\mathbf{x}^{'}, t^{'}). \tag{4.46}$$

Finally one gets,

$$-\frac{\nabla_i p}{\rho} = -\frac{dt^{'}}{dt} R^{-1}_{ij} \frac{\nabla^{'}_i p^{'}}{\rho^{'}}, \tag{4.47}$$

as claimed.

4.3.3 The viscous term

The viscous term in non-inertial frame is:

$$-\frac{\nabla_i(\eta\sigma_{ij})}{\rho} = -\frac{\nabla_i \left(\eta \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}(\nabla \cdot \mathbf{v})\right)\right)}{\rho}. \tag{4.48}$$

We will see that this term is covariant by itself under the full GCA without any modification. We have already seen in (4.41) that ∇_i and the traceless symmetric tensor σ_{ij} both transform covariantly. We have already seen how the density field should transform in (4.45). So clearly, the viscous term transforms like the kinematic term provided

$$\eta(\mathbf{x}, t) = (\frac{dt^{'}}{dt})^{a-1} \eta^{'}(\mathbf{x}^{'}, t^{'}). \tag{4.49}$$

With the above rule for transformation of the viscosity we get as desired.

$$-\frac{\nabla_i(\eta\sigma_{ij})}{\rho} = -\frac{dt^{'}}{dt} R^{-1}_{jl} \frac{\nabla^{'}_k(\eta^{'}\sigma^{'}_{kl})}{\rho^{'}}. \tag{4.50}$$

4.3.4 Summing all up

The full covariant form of the Navier-Stokes equation is:

$$(\mathcal{D}v)_i^{(comb)} = -\frac{\nabla_i p}{\rho} - \frac{\nabla_j\left(\eta\left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}(\nabla\cdot\mathbf{v})\right)\right)}{\rho}, \tag{4.51}$$

or,

$$\mathcal{D}(v_i + \Omega_{ij}(t)x_j - \frac{\nabla.\mathbf{v}}{d}x_i) + \Omega_{ij}(t)(v_j + \Omega_{jk}(t)x_k - \frac{\nabla.\mathbf{v}}{d}x_j) - \nabla_i(\mathcal{B}(t).\mathbf{x}) = -\frac{\nabla_i p}{\rho} - \frac{\nabla_j\left(\eta\left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}(\nabla.\mathbf{v})\right)\right)}{\rho}. \tag{4.52}$$

Besides, the density, pressure and viscosity transforms as follows,

$$\begin{aligned} \rho(\mathbf{x},t) &= (\frac{dt'}{dt})^a\rho'(\mathbf{x}',t'), \\ p(\mathbf{x},t) &= (\frac{dt'}{dt})^a p'(\mathbf{x}',t'), \\ \eta(\mathbf{x},t) &= (\frac{dt'}{dt})^{a-1}\eta'(\mathbf{x}',t'). \end{aligned} \tag{4.53}$$

We will now investigate some interesting consequences of the above transformations. Let us first consider the case when all chemical potentials are zero as in a gas of phonons in a metal. Then both the density and pressure are functions of temperature, which must transform appropriately under GCA to reproduce (4.45) and (4.46). The speed of sound c_s in the comoving frame (i.e in the local inertial frame comoving with the local velocity v of the flow) is given by $c_s^2 = dp/d\rho$. Since the pressure and density transform identically under GCA, we find that c_s is invariant under GCA.

In a typical Galilean invariant theory this is not surprising, as for instance, for monoatomic ideal gases, with molecular weight m, $c_s = \sqrt{(5kBT/3m)}$. The temperature field being Galilean invariant, Galilean invariance of c_s is automatic. The problem is that a GCA invariant microscopic theory (as argued in [57]) cannot have any mass parameter. Here, the temperature T does transform non-trivially under GCA, so c_s must either be a fundamental constant like the speed of light or be given in terms of the microscopic parameters of the theory. The situation is the same in a relativistic conformal system where the speed of sound is $c/\sqrt{3}$, where c is the speed of light. In a typical non-relativistic theory there is no fundamental speed. However, there is a novel possibility, when the number of spatial dimensions is two. We have seen that, in this case, the GCA admits a central charge, Θ, which has the dimension of $(1/speed)^2$ and also being a central charge, this is invariant under GCA. So, in this case, we have a natural origin for a fundamental speed, which is $1/\sqrt{|\Theta|}$. In other dimensions, c_s must be given in terms of microscopic parameters, for instance it can be the ratio of a microscopic length parameter and a microscopic time parameter. We will have more to say about this possibility later. In any case, for a system without chemical potentials, cs must be a constant. However, if we have chemical potentials too, cs need not be so and the analysis above is insufficient to make any conclusion in this case.

4.4 The influence of the continuity equation

We will see here that the constant a, which governs the transformation of density and pressure under the full GCA can be fixed uniquely by the continuity equation. The continuity equation is

$$\mathcal{D}\rho + \rho(\nabla \cdot \mathbf{v}) = 0. \tag{4.54}$$

Let us study how this equation transforms under the full GCA (say as represented in (4.28). We assume, as we did in the previous section that the density field transforms homogeneously, so that

$$\rho(\mathbf{x}, t) = (\frac{dt^{'}}{dt})^{a} \rho^{'}(\mathbf{x}^{'}, t^{'}). \tag{4.55}$$

With this assumption, we readily see that

$$\mathcal{D}\rho + \rho(\nabla.\mathbf{v}) = (\frac{dt^{'}}{dt})^{a+1}(\mathcal{D}^{'}\rho^{'} + \rho^{'}(\nabla^{'}.\mathbf{v}^{'})) + \rho^{'}(\frac{dt^{'}}{dt})^{a-1}(\frac{d^2t^{'}}{dt^2})(a-d). \tag{4.56}$$

So clearly we have covariance for the continuity equation only if $a = d$. So *the continuity equation, if valid, predicts the transformation of the density under GCA.*

We will see what consequences we now have for the Navier-Stokes' equation. If the pressure term has to be covariant under GCA and transform exactly like the kinematic term, we require that the pressure transforms in the same way as the density, so

$$p(\mathbf{x},t) = (\frac{dt^{'}}{dt})^{d} p^{'}(\mathbf{x}^{'},t^{'}). \tag{4.57}$$

We immediately see that the pressure transforming in the same way as the density again makes the speed of sound c_s a constant, when all chemical potentials vanish.

We now turn to the viscous term. Again we easily see that to achieve GCA covariance of the viscous term, we require that the viscosity transforms under GCA as below,

$$\eta(\mathbf{x},t) = (\frac{dt^{'}}{dt})^{d-1} \eta^{'}(\mathbf{x}^{'},t^{'}). \tag{4.58}$$

Finally, we note that if there is no particle number conservation the continuity equation written in the form (4.54) should not hold. In this case the RHS must be non-vanishing owing to say, particle absorption or emission. However, we will still have the same conclusions as it will be natural to demand that the LHS of this modified equation, which will be the same as before, must be covariant under GCA on its own.

4.5 GCA covariance and the viscosity

The covariance of the Navier-Stokes equation and the continuity equation under the full GCA requires that the viscosity should transform in a certain specified manner as given by (4.58). Now, the viscosity can transform only through its dependence on the thermodynamic variables which are pressure and density. Here, as before, we will assume the absence of chemical potentials. We note that p/ρ does not transform under GCA as both the pressure and density transform exactly the same way. So the only way, in which we can achieve the required transformation of the viscosity under the full GCA is that it depends on the pressure and density in the following manner,

$$\eta = A(\frac{p}{\rho})^{x} p^{\frac{d-1}{d}}, \tag{4.59}$$

where A is a *dimensionful* microscopic parameter. The dimension of A turns out to be:

$$[A] = M^{\frac{1}{d}} (\frac{L}{T})^{-\frac{d-2}{d}-2x}. \tag{4.60}$$

In the equation above, A is a (dimensionful) parameter and not a field, so it does not transform under GCA. It is a parameter because it is independent of the thermodynamic quantities like the pressure and density and of course it is independent of the velocity field as well. So, A must be given by some microscopic parameters and fundamental constants like the Planck's constant h. However, as argued in [57], no microscopic theory which is GCA invariant, can contain any mass parameter, so the mass dimension of A can come only through the Planck's constant h. Without any loss of generality, we may also assume that we have a length scale l_f in the theory, which by definition is a parameter in the theory and unlike the thermal wavelength this has no dependence on the temperature or any other thermodynamic variable by definition. Since generically we do not have any fundamental speed like the speed of light in a non-relativistic theory, we need an independent microscopic time scale t_f also, which is again by definition independent of thermodynamic variables, to soak the time dimension of A. We need an independent time scale in the microscopic theory, because unless there is a fundamental speed or a fundamental quantity with dimension of speed, we cannot form a time scale out of a length scale. Finally, without loss of generality, we can say that A should take the form below

$$A \approx h^{\frac{1}{d}} l_f^{-1-2x} t_f^{\frac{d-1}{d}+2x}. \tag{4.61}$$

It is clear from the above equation that we cannot make the dependence of A on the microscopic length scale l_f and the microscopic time scale t_f vanish simultaneously. Therefore, we conclude that we can explain the required transformation of the viscosity under the full GCA only if we have a microscopic length scale or a microscopic time scale or both in our theory. We also note that even when $d = 2$, in which case the central Θ allows to define a "fundamental speed," given by $\sqrt{1/|\Theta|}$, it is impossible to soak the dimension of A with the Planck's constant and Θ alone. So it is impossible to do without introducing a microscopic length scale or microscopic time scale or both.

The conclusion, therefore, is that in a GCA invariant theory, either the viscosity is zero or it contains a microscopic length parameter or a microscopic time parameter or both. This is indeed contrary to the case of a relativistic conformal field theory where we cannot have any intrinsic length parameter or time parameter and any quantity can have a dimension only through the Planck's constant and the speed of light. At this moment, we do not know any GCA invariant microscopic theory so we can be open to the possibility that such theories can contain intrinsic length or time parameters or both. If this is not possible, then the viscosity should vanish. Of course, as in the case with our analysis of c_s , our conclusions may change if we introduce chemical potentials.

One may however, ponder if it is possible that GCA could be a symmetry of the theory only in the presence of non-zero chemical potentials so that the above considerations for the case of vanishing chemical potentials can be avoided. In our opinion, this point of view is rather unnatural, because the symmetry of a theory is usually a fundamental property of the theory and though its manifestation might be modified, it can neither appear or disappear at specific values of thermodynamic intensive variables like temperature or chemical potentials. An easy example which supports this point is the usual relativistic conformal symmetry of $\mathcal{N} = 4$ SYM theory, in which case in presence of a finite temperature we still have conformal symmetry, however the thermodynamic variables also transform under conformal transformations. In the Discussion section, we will point out possible significances of the analysis done here in the case of vanishing chemical potentials for gauge/gravity duality realization of GCA.

4.6 Possible GCA covariant corrections to the Navier-Stokes equation

The Navier-Stokes equation, being a phenomenological equation, is susceptible to higher derivative corrections, which could be, in principle, calculated from kinetic theory. We will see that GCA is powerful in constraining these corrections, quite like in the case of hydrodynamics covariant under the relativistic conformal group. So, this will give us further evidence, that GCA indeed is a credible physical symmetry, that is a symmetry which can constrain phenomenological laws (in absence of known GCA invariant microscopic theories). [7]

Usually, for instance, if calculated from the kinetic theory of gases, the corrections to the Navier-Stokes involve corrections to the dissipative part of the stress tensor τ_{ij}, which at the first-order in derivatives is just $\eta\sigma_{ij}$. The next-order corrections to the Navier-Stokes equation are contained in the two derivative corrections, $\tau^{(2)}_{ij}$, to the dissipative stress tensor, so that $\tau_{ij} = \eta\sigma_{ij} + \tau^{(2)}_{ij}$ and the corrected Navier-Stokes' equation in the inertial frame, now takes the form,

$$\mathcal{D}v_i = -\frac{\nabla_i p}{\rho} - \nabla_i(\tau_{ij}) = -\frac{\nabla_i p}{\rho} - \nabla_i(\eta\sigma_{ij} + \tau^{(2)}_{ij}). \tag{4.62}$$

Now, we would demand that like σ_{ij}, $\tau^{(2)}_{ij}$ contains spatial derivatives only as is indeed that case if these corrections are calculated from kinetic theory. Also, we will assume, that these corrections involve derivatives of the velocity only.

Let us first look at terms in $\tau^{(2)}_{ij}$ which have the structure of $(\nabla u)^2$. For that, we need to find if there is any other tensor with structure (∇u) which transforms like σ_{ij}. One can easily see that there is only one more such tensor, which we denote as ω_{ij} and is defined as

$$\omega_{ij} = \frac{1}{2}(\nabla_i u_j - \nabla_j u_i - 2\Omega_{ij}(t)). \tag{4.63}$$

Once again by invoking the trick of comparing one inertial frame with two non-inertial frames and then comparing the two non-inertial frames with each other one can readily prove that ω_{ij} transforms under full GCA like σ_{ij}. Therefore $\tau^{(2)}_{ij}$ involve the following combinations $\lambda_1\sigma_{ik}\sigma_{kj} + \lambda_2(\sigma_{ik}\omega_{kj} + \omega_{ik}\sigma_{kj}) + \lambda_3\omega_{ik}\omega_{kj}$, where the three λ's are arbitrary transport coefficients like the shear viscosity η. For the covariance of the corrected Navier-Stokes we now require them to transform as below,

$$\lambda_i(\mathbf{x}, t) = (\frac{dt'}{dt})^{a-2}\lambda'_i(\mathbf{x}', t'), \tag{4.64}$$

where $i = 1, 2, 3$ and a is defined through the transformation of the density as given in (4.45). We can proceed to find the dependence of the λ's on the thermodynamic variables exactly as we have done for the shear viscosity η, however we will not repeat it here.

[7] The author would like to thank Rajesh Gopakumar for pointing out this significance of the constraints imposed by GCA on the correction to the Navier-Stokes' equation.

Now let us look for possible corrections to $\tau_{ij}^{(2)}$ which contains the structure $(\nabla^2 u)$. Now since $\mathbf{v}.\nabla$ does not transform covariantly, we cannot try combinations like $(\mathbf{v}.\nabla)\sigma_{ij}$. Moreover, though the Laplacian, $\Box$, transforms covariantly, we cannot use it on any polynomial of the velocity like $u_i u_j$, as it is not covariant. It is not, thus hard to see, that there is only one possible covariant term which contains a $(\nabla^2 u)$ term and it is $\nabla_k(\sigma_{ij}\mathcal{V}_k^{(\mathbf{b=0})})$, where $\mathcal{V}_k^{(\mathbf{b=0})}$ is as defined in (4.29). We can still get a covariant term, though $\mathcal{V}_k^{(\mathbf{b=0})}$ is covariant only in absence of boosts, because the full covariant velocity field will differ from this by a purely time-dependent quantity, so it doesn't make any difference when we apply the spatial derivative. We note that, in an inertial frame, however, this new term is just $(\mathbf{v}.\nabla)\sigma_{ij}$. We will denote the coefficient corresponding to this term as λ_0.

Therefore, the most general form of $\tau_{ij}^{(2)}$ is:

$$\tau_{ij}^{(2)} = \lambda_0 \nabla_k(\sigma_{ij}\mathcal{V}_k^{(\mathbf{b=0})}) + \lambda_1 \epsilon_{ik}\epsilon_{kj} + \lambda_2(\epsilon_{ik}\omega_{kj} + \omega_{ik}\epsilon_{kj}) + \lambda_3 \omega_{ik}\omega_{kj}, \tag{4.65}$$

with all λ's having appropriate dependence on thermodynamic variables so that it transforms as in (4.64).

Similarly, we can proceed to constrain higher order corrections of the Navier-Stokes' equation containing more than three derivatives. We observe that our four possible GCA covariant corrections, have analogues in the relativistic conformal case, as all the four possible corrections in flat space-time [37], reduce in the non-relativistic limit to our four terms in an inertial frame when the flow is incompressible. This is intriguing because the covariant forms in the two cases are very different in content. It will be interesting to see if this correspondence also exist at higher orders. There can be another term in our case involving the curvature of the spatial metric as in the relativistic case (the relativistic term involves contractions of the Reimann tensor), but since we have throughout restricted ourselves to the flat spatial metric, this possibility lies outside the scope of our present investigation.

4.7 Discussion

We have shown that the macroscopic Navier-Stokes equation for incompressible flows has covariance under full GCA. So we can conclude that GCA can be realized as a symmetry of a phenomenological law like the Navier-Stokes equation only if we covariantize the usual form of the laws which holds in inertial frames, however not any arbitrary law with mere Galilean covariance can be covariantized. In the case of the Navier-Stokes equation we have needed that the flow is incompressible. We have also seen that the higher derivative corrections to the Navier-Stokes equation can be constrained by requiring GCA covariance.

Our analysis also leads us to conclude that when all chemical potentials vanish, c_s, which denotes the speed of sound in a comoving frame, is a constant. Further, we have seen that in the absence of chemical potentials, the viscosity should either vanish or in the microscopic theory we must have a length scale or a time scale or both.

We would now like to discuss the possible implications of the above analysis for gauge/gravity duality realization of GCA. The presence of both length and time scales in the GCA invariant microscopic theory firstly tallies with the fact that we need to introduce objects like absolute angular velocity and absolute acceleration of the non-inertial frame which brings in dimensions of both length and time into play. This is in contrast with the case of covariantizing under relativistic conformal group where we need not bring in any additional dimensionful parameter. This observation possibly indicates that we need to first deform the action of the relativistic parent theory like $\mathcal{N} = 4$ SYM by *non-marginal* operators such that a deformed $SO(d, 2)$ relativistic conformal group is the symmetry of the theory and then take the contraction which takes $SO(d, 2)$ relativistic conformal group to GCA so that we get a sensible dynamical limit [8]. The deformation parameters of the symmetry being dimensionful, should bring in the required microscopic length scales and time scales in the final GCA invariant theory obtained via the contraction. Further, the deformation parameters will also transform non-trivially under GCA so that the covariantizing will bring in new structures. In fact, if we take the contraction without

[8]A related example could be the omega-deformation [73] of $\mathcal{N} = 2$ SYM theories under which the deformed theory retains the BRST supersymmetry though this supersymmetry itself gets deformed by combining with other supersymmetries.

deformation for the classical $N = 4$ SYM theory, one may readily check that we get a non-dynamical equations of motion for all the fields [9]. This supports our point of view. In the future, we would like to find out the appropriate operators which could give rise to the deformations such that the contraction produces a sensible dynamical theory.

Finally, we mention, that it would be an interesting challenge to construct gravitational duals for GCA covariant hydrodynamic flows. Aside from finding the dynamics of gravity in the bulk, we see now, we also need to find a suitable bulk interpretation of the absolute angular velocity and the absolute acceleration of the boundary coordinate system, as they are surely needed in the covariant formulation of the hydrodynamics of the boundary theory. Some earlier work in [62] could be useful in this direction.

[9] The author thanks Rajesh Gopakumar for valuable discussions regarding these points.

Part IV

Conclusion

Chapter 5

Concluding remarks

We have obtained some important structural information on nonequilibrium states and also some new hints on the origin of irreversibility through our investigations. Firstly, we have argued that generic quantum field theories have purely hydrodynamic states which can be completely characterized by hydrodynamic variables alone even far away from equilibrium and their dynamics can also be determined completely by the equations of fluid mechanics which can be obtained in a systematic derivative expansion. The dimensionless parameter for the derivative expansion is the ratio of typical spatio-temporal scale of variation of hydrodynamic variables and the mean free path at final equilibrium.

We have also argued that there are also a more general class of states, the conservative states, which could be completely characterized by the energy-momentum tensor and the dynamics of these states can also be determined completely by a closed set equations of motion for the energy-momentum tensor. The purely hydrodynamic states are special instances of the hydrodynamic states. If the equations of higher derivative fluid mechanics governing these purely hydrodynamic states are known up to certain orders, we can systematically construct the equations of motion of the energy-momentum tensor determining the evolution of conservative states phenomenologically up to appropriate orders in two expansion parameters and unknown values of nonhydrodynamic coefficients like the relaxation time. These equations of motion include the conservation of energy and momentum but also an independent equation of motion for the shear-stress tensor which we need to expand in the two expansion parameters, one of which is the derivative expansion parameter, while the other is the amplitude expansion parameter which is the ratio of the typical value of the nonhydrodynamic shear-stress tensor and the hydrostatic pressure at final equilibrium.

Moreover any arbitrary state which thermalizes can be approximated by an appropriate conservative state at sufficiently long time, which in turn can be approximated by an appropriate purely hydrodynamic state at a sufficiently longer time. The approach to equilibrium is sufficiently fast at strong coupling, for higher order effects like the hydrodynamic transport coefficients at second order in the derivative expansion to be experimentally observable. Therefore the equations of motion of energy-momentum tensor determining the evolution of conservative states can be used for modeling generic irreversible phenomena like hydrodynamics, relaxation and decoherence. It is in fact a generalization of the equations of fluid mechanics to capture phenomenology of generic irreversible phenomena particularly at strong coupling.

Earlier results also suggest that purely hydrodynamic states have an entropy current with positive definite divergence, at least at strong coupling in conformal gauge theories with gravity duals, just like in the case of the Boltzmann equation. The tubewise black-brane solutions, which by our logic should constitute the normal or purely hydrodynamic solutions at large rank of the gauge group and strong 't Hooft coupling, indeed demonstrate the existence of a family of entropy currents [30]. However, the structure of this entropy current is very different. Unlike for the normal solutions of the Boltzmann equation, or in the Israel-Stewart-Muller formalism, these exact entropy currents at strong coupling are not of the form su^{μ}, where s could be interpreted as the nonequilibrium entropy density.

Our analysis also suggests that such entropy currents do not exist for a generic conservative state. In fact for spatially homogeneous conservative states, which are purely nonhydrodynamic, irreversibility occurs only when we follow the envelope of an appropriate global function and a Lyapunov function probably does not exist.

It would be a real challenge to find a general principle for construction for purely hydrodynamic states and the more general class of conservative states in generic quantum field theories, which would generalize Gibbs' distribution for equilibrium states. In fact, this is the only way one can give microscopic meaning to the hydrodynamic and nonhydrodynamic transport coefficients which appear in the phenomenological equations of motion of energy-momentum tensor determining evolution of conservative states and cannot be defined by linear response theory.

We will now mention some of the developments on which we would like to focus in the immediate future. The first could be in the realm of early-time dynamics, especially in the understanding of decoherence. This should be convenient because we can understand a lot by just considering the higher order corrections to the homogeneous nonhydrodynamic configurations which solve (3.19). We have already observed the possibility of an oscillatory approach to equilibrium here. To uncover the physics, we need to compare with homogeneous conservative solutions in quantum kinetic theories which can capture the physics of decoherence. We can test whether the same dynamics of the energy-momentum tensor in conservative solutions of quantum kinetic theories which captures decoherence, also gives rise to horizon formation in the bulk.

A second important issue would be a better understanding of whether the hydrodynamic limit of nonequilibrium dynamics always leads to generation of an entropy current generically. We hope to get a better understanding of the physics of this entropy current by investigating the existence and form of the entropy currents in the normal solutions of untruncated BBGKY heirarchy which, as mentioned before, are solutions of exact microscopic dynamics. Construction of conservative states and investigation of entropy currents for these states, will also confirm whether entropy currents exist strictly in the hydrodynamic limit and is replaced by an oscillatory approach to equilibrium for states away from this limit.

Thirdly, we have given a framework for general universal nonequilibrium behavior in strongly coupled gauge theories with gravity duals. It would be interesting to see how much of this framework may apply to physics of quark-gluon plasma at the RHIC. We can expect some qualitative similarities because the QCD coupling evolves very slowly at the scales corresponding to the temperature of the quark-gluon plasma at RHIC and so the evolution of the plasma within certain time window should be well described by Weyl covariant equations. Moreover, at finite temperature supersymmetry is broken, hence it is expected that there should be more similarities between $\mathcal{N} = 4$ SYM theory at strong coupling and the quark gluon plasma of QCD at finite temperature than the phenomena in these two theories at zero temperature. Some investigations indeed show such similarities indeed hold to some degree beyond the good matching of η/s, even at the quantitative level [74].

Finally, we need to understand better the origin of universality in the dynamics at strong 't Hooft coupling and large rank of the gauge group in the class of conformal gauge theories with gravity duals studied here. This should give us better insights into existence of other universality classes where the dynamics of an entire class of states become the same for field theories belonging to such classes at critical values of the couplings and parameters.

Chapter 6

Appendices

6.1 Appendix A : Proof of the power series solution for AdS_5 asymptotics

Here we will prove that any asymptotically AdS_5 solution of Einstein's equation with a negative cosmological constant, in the Fefferman-Graham coordinates, has a solution for $g_{\mu\nu}$ which is a power series in the radial coordinate when the boundary metric is flat. Though not explicitly mentioned in most of what follows, it should be kept in mind that here we are specifically investigating five-dimensional solutions with a flat boundary metric. At the end, we will mention if our proof can be generalized to other cases.

To simplify the proof we first rearrange the tensor and the scalar components of Einstein's equation (2.5) while keeping the vector components of Einstein's equation unchanged. The old scalar equation is added with an appropriate linear combination of the trace of the old tensor equation so that now it does not contain any term which has second derivative of $g_{\mu\nu}$ with respect to the radial coordinate ρ. Since the vector equation also does not contain any term with second derivative of $g_{\mu\nu}$ with respect to the radial coordinate we can now think of the vector and scalar components as a set of five constraint equations. We also change the tensor components of Einstein's equation by appropriately replacing $Tr(g^{-1}g')$ using the new scalar equation. We do this so that now the tensor equation by itself is sufficient to determine all the ρ^n coefficients of $g_{\mu\nu}$. The old tensor equation had the feature that to determine $g_{(8)\mu\nu}$, the coefficient of ρ^8 in $g_{\mu\nu}$, we had to use the scalar equation as well, but now this can be fully determined using the tensor equation alone. So our equations are now

$$\begin{aligned}&\frac{1}{2}g'' - \frac{3}{2\rho}g' - \frac{1}{2}g'g^{-1}g' + \frac{1}{4}Tr(g^{-1}g')g' - Ric(g)\\&+ g[\frac{1}{6}R(g) + \frac{1}{24}Tr(g^{-1}g'g^{-1}g') - \frac{1}{24}(Tr(g^{-1}g'))^2] = 0,\end{aligned} \tag{6.1}$$

$$\nabla_\mu Tr(g^{-1}g') - \nabla^\nu g'_{\mu\nu} = 0, \tag{6.2}$$

$$R(g) + \frac{3}{\rho}Tr(g^{-1}g') + \frac{1}{4}Tr(g^{-1}g'g^{-1}g') - \frac{1}{4}[Tr(g^{-1}g')]^2 = 0. \tag{6.3}$$

It is not difficult to see that we can use a power series ansatz to solve the tensor equation as at the n-th order. At the n-th order the only terms which can contain $g_{(n)_{\mu\nu}}$ or $Tr(g_{(n)})\eta_{\mu\nu}$ are $g''_{\mu\nu}$, $g'_{\mu\nu}$ and $Tr(g^{-1}g')g_{\mu\nu}$. Now since the tensor equation contains no term with $Tr(g^{-1}g')g_{\mu\nu}$, at the n-th order,for $n > 4$, the tensor equation gives us $n(n-4)g_{(n)\mu\nu}/2 = f(t_{\mu\nu})$, where $f(t_{\mu\nu})$ is a polynomial in $t_{\mu\nu}$ and its various derivatives with respect to the boundary coordinates only. Hence, for $n > 4$, we can always solve $g_{(n)\mu\nu}$ using the tensor equation alone.

We have now got to show that the power series we have so obtained as a solution to the tensor equation is consistent with the vector and scalar constraints. We will do this by the method of induction iterating over the various coefficients of ρ^n in $g_{\mu\nu}$, order by order in n. We will first establish the following fact that the ρ-derivative of the vector and scalar constraints vanish when the tensor equation along with the vector and scalar constraints are satisfied. This just articulates the intuition that once the initial data consisting of $g_{\mu\nu}$ and $g'_{\mu\nu}$ satisfy the vector and scalar constraints on hypersurface with a fixed value of the radial coordinate ρ, the dynamical evolution in ρ should be such that the constraints should be automatically satisfied for any other hypersurface. To show this we will need the following:

$$\Gamma^{\mu}_{\nu\sigma}{}' = \frac{1}{2}g^{\mu\alpha}(\nabla_\nu g'_{\alpha\sigma} + \nabla_\sigma g'_{\alpha\nu} - \nabla_\alpha g'_{\nu\sigma}), \tag{6.4}$$

$$R^{\mu}_{\nu\alpha\beta}{}' = \frac{1}{2}g^{\mu\gamma}[\nabla_\alpha\nabla_\nu g'_{\gamma\beta} - \nabla_\alpha\nabla_\gamma g'_{\nu\beta} - \nabla_\beta\nabla_\nu g'_{\gamma\alpha} + \nabla_\beta\nabla_\gamma g'_{\nu\alpha}].$$

One can use the tensor (6.1) and scalar (6.3) equations to write

$$\begin{aligned} R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R =& \frac{1}{2}g^{\mu\alpha}g''_{\alpha\nu} - \frac{3}{2\rho}g^{\mu\alpha}g'_{\alpha\nu} - \frac{1}{2}g^{\mu\alpha}g'_{\alpha\beta}g^{\beta\gamma}g'_{\gamma\nu} + \frac{1}{4}Tr(g^{-1}g')g^{\mu\alpha}g'_{\alpha\nu} + \frac{5}{4\rho}Tr(g^{-1}g')\delta^\mu_\nu \\ &- \frac{1}{4}\delta^\mu_\nu[Tr(g^{-1}g'') - Tr(g^{-1}g'g^{-1}g') + \frac{1}{2}(Tr(g^{-1}g'))^2]. \end{aligned} \tag{6.5}$$

When all the equations (6.1), (6.2) and (6.3) are satisfied, the ρ-derivative of the vector constraint can also be written as:

$$\begin{aligned} (\nabla_\mu Tr(g^{-1}g') - \nabla^\nu g'_{\mu\nu})' =& \partial_\mu[Tr(g^{-1}g'' - \frac{3}{4}g^{-1}g'g^{-1}g') + \frac{1}{4}(Tr(g^{-1}g'))^2] \\ &- \nabla_\nu(g^{\alpha\nu}g''_{\mu\alpha} - g^{\alpha\beta}g'_{\beta\gamma}g^{\gamma\nu}g'_{\alpha\mu} + \frac{1}{2}g^{\nu\alpha}g'_{\alpha\mu}Tr(g^{-1}g')). \end{aligned} \tag{6.6}$$

Comparing the right hand sides of (6.5) and (6.6) using all the equations of motion again, we see that

$$(\nabla_\mu Tr(g^{-1}g') - \nabla^\nu g'_{\mu\nu})' = \nabla_\nu(R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R). \tag{6.7}$$

So the Bianchi identity implies that the ρ-derivative of the vector equation should vanish when all the equations of motion are satisfied. We will now get to the scalar equation.

When the vector equation of motion (6.2) is satisfied we get

$$R_{\mu\nu}' = -\frac{1}{2}R_{\alpha\mu}(g^{\alpha\beta}g'_{\beta\nu}) + \frac{1}{2}R^\gamma_{\nu\alpha\mu}(g^{\alpha\beta}g'_{\beta\gamma}) + \frac{1}{2}\nabla_\mu\nabla_\nu Tr(g^{-1}g') - \frac{1}{2}\nabla^2 g'_{\mu\nu}. \tag{6.8}$$

This implies that when the vector equation of motion is satisfied, we have

$$R' = -g^{\mu\nu}g'_{\nu\sigma}g^{\sigma\alpha}R_{\mu\alpha}. \tag{6.9}$$

On the other hand the vanishing of the ρ-derivative of the scalar constraint (6.3) ought to give us

$$\begin{aligned} R' =& -\frac{1}{2}Tr(g^{-1}g'g^{-1}g'') + \frac{3}{2\rho}Tr(g^{-1}g'g^{-1}g') \\ &+ \frac{1}{2}Tr(g^{-1}g'g^{-1}g'g^{-1}g') - \frac{1}{4}Tr(g^{-1}g')Tr(g^{-1}g'g^{-1}g') + \frac{1}{2\rho}[Tr(g^{-1}g')]^2. \end{aligned} \tag{6.10}$$

Now using the tensor and scalar equations of motion, we can see that the right hand sides of (6.9) and (6.10) are the same, or in other words the ρ-derivative of the scalar constraint indeed vanishes when all the equations of motion are satisfied. So we have established that the ρ-derivatives of the all the five constraints vanish when all the equations of motion are satisfied, or to state compactly

$$(6.1), (6.2), (6.3) \Rightarrow (6.2)', (6.3)'. \tag{6.11}$$

To prove that the power series solution of the tensor equation is consistent with the constraints, we will use the above at $\rho = 0$. To obtain a condition for $g_{(n)\mu\nu}$ (the coefficient of ρ^n in $g_{\mu\nu}$) from the tensor equation we need to differentiate it n-2 times with respect to ρ and then set $\rho = 0$. Similarly to obtain a condition for $g_{(n)\mu\nu}$ from the vector and scalar constraints we need to differentiate each of them n-1 times with respect to ρ and then set $\rho = 0$.

The vector and scalar constraints imply that $g_{(2)\mu\nu}$ should vanish while the tensor equation identically vanishes at this order. The tensor equation for $g_{(4)\mu\nu}$ which we have appropriately renamed $t_{\mu\nu}$, also identically vanishes while the vector constraint gives us the conservation equation $\partial^\mu t_{\mu\nu} = 0$ and the scalar constraint gives the tracelessness condition $Tr(t) = 0$. We can start our induction from here, since the three equations are all consistent with each other up to this order

Let us suppose, by the induction hypothesis that the solution for $g_{(n-1)\mu\nu}$ obtained from the tensor equation is consistent with the vector and scalar constraints. We now denote the m-th ρ-derivative as $m^{'}$. So, by induction hypothesis, the three equations $(n-3)'(6.1)(\rho = 0)$, $(n-2)'(6.2)(\rho = 0)$ and $(n-2)'(6.3)(\rho = 0)$ are consistent with each other. Now we iterate by determining $g_{(n)\mu\nu}$ from the tensor equation, or in other words we solve

$$(n-2)'(6.1)(\rho = 0). \tag{6.12}$$

But by induction hypothesis we can assume $(n-2)'(6.3)(\rho = 0)$ and $(n-2)'(6.2)(\rho = 0)$ are consistent with the tensor equation. Now our result (6.11) for a general fixed ρ hypersurface implies that

$$(n-2)'(6.1), (n-2)'(6.2), (n-2)'(6.3) \Rightarrow (n-1)'(6.2), (n-1)'(6.3). \tag{6.13}$$

We can apply the above at $\rho = 0$ [1] to iterate and say that if the solution for $g_{(n-1)\mu\nu}$ from the tensor equation is consistent with the constraints so would the solution for $g_{(n)\mu\nu}$ from the tensor equation be. This completes the proof by induction that the power series solution of the tensor equation is consistent with the constraints.

Let us see if our proof can be generalized to other cases, in particularly for all dimensions if the boundary metric is flat. The only change in the equation of motion happens to be the coefficient of $g'_{\mu\nu}$ in the tensor equation. Let us, for example, take the case when the number of boundary coordinates is six. We can check by hand that all $g_{(n)\mu\nu}$ vanish for all n such that $0 < n < 6$ and $g_{(6)\mu\nu}$ cannot be determined from the tensor equation for an exactly similar reason as for $g_{(4)\mu\nu}$ when the number of boundary coordinates was four, namely the tensor equation identically vanishes. The vector and scalar constraints imply conservation and tracelessness of $g_{(6)\mu\nu}$ implying that it should be identified with the stress tensor (and indeed it has been shown in [9] that this agrees with with the Balasubramanian-Krauss stress tensor). We can begin our induction, from here as before and hence our proof generalizes. So, the general problem in applying the induction is to show that the equations of motion are consistent with the power series ansatz at $g_{(d)\mu\nu}$. We have not been able to prove it generally but we have checked it up to $d = 6$. The same problem appears when we try to apply induction to prove the validity of the power series solution when the number of boundary coordinates is odd, but the boundary metric is arbitrary. Before we apply induction, we need to prove that the power series works at $g_{(d)\mu\nu}$, (in fact this is harder to show, because when the boundary metric is not flat $g_{(n)\mu\nu}$'s do not vanish for $0 < n < d$). However, Fefferman and Graham have proved the validity of the power series solution by a different method for an arbitrary boundary metric when the number of boundary coordinates is odd.

[1] At $\rho = 0$ the statement (6.13) has a non-trivial content strictly for $n > 2$, because of the slight technicality that what we really need to use to find a condition for $g_{(n)\mu\nu}$ is that we need to differentiate (ρ(6.3)) not really (6.3) n-1 times. So at $\rho = 0$, this result is trivial for the scalar constraint when $n = 2$ and we do not need to use the result (6.13), but since the first step of induction starts from $n = 4$, it is safe to use this in the iteration procedure.

6.2 Appendix B: On fixing η/s by calculating curvature invariants

We have already done the regularity analysis of our first order solution in Fefferman Graham coordinates by translating to Eddington-Finkelstein coordinates where the regularity or irregularity becomes manifest. However, one may ask if the regularity analysis can be done also by calculating some curvature invariants. We will see that indeed at the first order, this analysis can also be done by calculating an appropriate curvature invariant, but we will argue that there may not be a finite number of curvature invariants which can be reliably used to fix all the coefficients in the hydrodynamic stress tensor at higher orders in the derivative expansion.

Before we do that, we want to point out that though the metric in Fefferman-Graham coordinates and in Eddington-Finkelstein coordinates could be made coordinate equivalent up to any given order in the derivative expansion for an arbitrary hydrodynamic stress tensor, the curvature invariants calculated from the two metrics will typically never be the same! Let us examine why this should happen at the first order itself. Any typical curvature invariant, like the Ricci scalar R itself, will show a divergence only when we expand it to second order in derivatives of the boundary coordinates. In this case, this should be so, because the metric in either coordinate system is a solution of the equations of motion up to first order in derivatives of boundary coordinates. However, the second order piece in R calculated from the metric in either coordinate system will not be the same, because the two metrics are related by a coordinate transformation only up to first order in derivatives. In fact we will explicitly demonstrate that R itself can be used to fix the value of $\eta/4\pi s$ in the Eddington-Finkelstein metric at first order but not in the Fefferman-Graham metric at first order. So the procedures of using curvature invariants to fix the coefficients in the hydrodynamic stress tensor in the two coordinate systems are indeed very different!

Another crucial aspect should be kept in mind because this also features in comparing curvature invariants calculated from the metrics in the two coordinate systems. Fundamentally, solving Einstein's equations in either of the two coordinate systems involves a trade-off between manifest regularity and manifest asymptotic boundary condition. The solution in Eddington-Finkelstein coordinate system at the zeroth order and also at the the first order for the right value of $\eta/4\pi s$ are manifestly regular so any curvature invariant calculated at the horizon will be regular to all orders as well. However, the solution preserves the asymptotic AdS boundary condition only up to first order in derivatives as it can be translated to Fefferman-Graham coordinate system only up to that order. The solution in Fefferman-Graham coordinate system at first order, of course preserves boundary condition to all orders, but even for the right choices of $\eta/4\pi s$ it is not regular to all orders. In other words, for the right choice of $\eta/4\pi s$ all order divergences should vanish when we calculate curvature invariants from the metric in Eddington-Finkelstein coordinate system, but in case of the solution in Fefferman-Graham coordinates at first order, at most the leading divergence at the second order vanishes for the right choice of $\eta/4\pi s$. In fact, for certain curvature invariants even that do not happen. Of course, eventually if we add a right second order correction to the Fefferman-Graham metric, all divergences in the curvature invariants at the second order should vanish, but still divergences at higher orders will remain and so on. We will illustrate the first order case with examples below.

To compute curvature invariants it is useful to first choose a velocity and temperature profile. As mentioned before, the vector constraint in Einstein's equations of motion demand that the velocity-temperature profile should be a solution of the relativistic Euler equation

$$\frac{\partial_\mu b}{b} = (u.\partial)u_\mu - u_\mu \frac{\partial.u}{3}. \tag{6.14}$$

We call our boundary coordinates (t, x, y, z) and we select the following static velocity profile which is a relativistic version of laminar flow

$$u^\mu = \frac{1}{\sqrt{1-a^2y^2}}(1, ay, 0, 0), \tag{6.15}$$

where a is a constant of dimension 1/length. The advantages of using this velocity profile are twofold, namely,

- The relativistic Euler equation gives us that temperature, hence b, should be a constant.

- It is easy to employ the derivative expansion by using the following trick. We note that the only non-trivial derivatives of the boundary coordinates are the y-derivatives. Any y-derivative of the velocities will bring in an extra a which is unpaired with a y so that it picks up the right dimension. Hence to do the derivative expansion we may first set $y = p/a$ and simply do a Taylor expansion in a about $a = 0$. The correct dimensionless parameter of the derivative expansion, of course will be ab.

We can use the above velocity-temperature profile in the first order solution in any coordinate system. Though away from the boundary the boundary coordinates (or, in other words, the field-theoretic coordinates) in a given coordinate system will mix with all the coordinates in another coordinate system, at the boundary they will always align with other. This is, how solutions in two different coordinate systems come to share the same boundary stress tensor and also the same conservation equation, which in this case, is the relativistic Euler equation.

If we use the above velocity-temperature profile to calculate R in the Eddington-Finkelstein coordinate system we will find that

$$\begin{aligned} R = -20 + a^2 \frac{1}{8(1-a^2y^2)^2 b^4 r^6} \\ \left(\frac{(\gamma-1)(9+b^2r^2(3\gamma-2\pi)+16b^5r^5-2\pi b^6r^6))}{(br-1)(1+br+b^2r^2)} + \right. \\ (\gamma-1)(\gamma+1-8b^3r^3)b^2r^2 Log(br-1) + O(1)) + O(a^3). \end{aligned} \tag{6.16}$$

At the zeroth order in a, R should of course be -20 and at first order in a, R should of course vanish because our metric is a solution of equations of motion up to first order. At order a^2, we indeed expect some divergence at the horizon, which is at $r = 1/b$, because the metric is explicitly not regular there unless $\gamma = \eta/4\pi s = 1$. We see that when $\eta/4\pi s = \gamma = 1$ all divergences go away. This feature replicates also at higher orders in a. [2] On the other hand, if we calculate R from the Fefferman-Graham metric at first order,

[2] We would like to thank Sayantani Bhattacharya for confirming that this indeed happens for arbitrary velocity and temperature profiles.

we get

$$R = -20 + a^2[\frac{128b^{10}\rho^8(12b^4\gamma^2 + 4b^2\rho^2 + 3\gamma^2\rho^4)}{(1-a^2y^2)^2(4b^4-\rho^4)^2(4b^4+\rho^4)^3} + \frac{16b^6\gamma^2}{(1-a^2y^2)^2(4b^4+\rho^4)^2}Log(\frac{4b^4-\rho^4}{4b^4+\rho^4})] + O(a^3). \tag{6.17}$$

At order a^2, we see that there is a leading inverse power two divergence for any value of γ and a subleading log divergence except when $\gamma = 0$. So this is useless to figure out the right value of γ. Of course this will certainly be useful to fix certain coefficients of the hydrodynamic stress tensor at second order, because these divergences should go away for any right second order correction to the Fefferman-Graham metric.

It turns out, however, that, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ can be used to fix the value of γ in the Fefferman-Graham metric. We get

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4(1280b^{16} + 1280b^{12}\rho^4 + 2784b^8\rho^8 + 80b^4\rho^{12} + 5\rho^{16})}{(4b^4+\rho^4)^4} - a^2[\frac{2(-1+\gamma^2)b^6}{(1-a^2y^2)^2(\rho-\sqrt{2}b)^4} + O(\frac{1}{(\rho-\sqrt{2}b)^2}) + O(Log(\sqrt{2}b-\rho)) + O(1)]. \tag{6.18}$$

We see that the zeroth order piece is always finite and independent of γ and at order a (for some reason we do not understand) the scalar vanishes. However, at order a^2, we find that when γ is 1 or -1 the leading divergence at $\rho = \sqrt{2}b$ goes away, though, the subleading divergences remain and as before, they should disappear when we add any right second order contribution to the Fefferman-Graham metric. We are also not sure, if by computing $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ itself we can fix the values of all the coefficients in the hydrodynamic stress tensor at second order. To fix all the coefficients of the second order hydrodynamic stress tensor, one may have to look for another appropriate curvature invariant.

It is certainly, worth exploring, if the "hydrodynamic" Fefferman-Graham solutions are "'special" enough so that computing a finite number of curvature invariants will suffice to determine regularity, hence in fixing all the coefficients in the hydrodynamic stress tensor to all orders. We will leave this for a future work. Nevertheless, our procedure of fixing the coefficients in the hydrodynamic stress tensor by translating to Eddington-Finkelstein coordinate system works for all orders in the derivative expansion.

6.3 Appendix C: Proof of existence of conservative solutions of the Boltzmann equation

We will now present the details of our proof for the existence and uniqueness of conservative solutions of the Boltzmann equation. In order to keep the proof reasonably self-contained, we give further details on the Boltzmann equation and how one can obtain the hydrodynamic equations seen earlier. We follow the notation of [55, 43] mostly for this part of the discussion. This will be followed by the proof in full detail.

6.3.1 C.1 A short description of the Boltzmann equation

The Boltzmann equation for the one-particle phase space distribution $f(\mathbf{x}, \xi)$ for a gas of nonrelativistic monoatomic molecules of unit mass interacting through a central force is

$$(\frac{\partial}{\partial t} + \xi \cdot \frac{\partial}{\partial \mathbf{x}})f(\mathbf{x}, \xi) = J(f, f)(\mathbf{x}, \xi) \quad , \tag{6.19}$$

where

$$J(f, g) = \int \Big(f(\mathbf{x}, \xi^{'})g(\mathbf{x}, \xi^{*'}) - f(\mathbf{x}, \xi)g(\mathbf{x}, \xi^{*})\Big) B(\theta, V)d\xi^{*}d\epsilon d\theta \quad , \tag{6.20}$$

is the collision integral. (ξ, ξ^{*}) are the velocities of the molecules before a binary collision and $(\xi^{'}, \xi^{*'})$ are their corresponding velocities after the collision. The angular coordinates (θ, ϵ) are the coordinates related to the collision, and $\mathbf{V} = \xi - \xi^{*}$ is the relative velocity with magnitude V. We assume that the collision takes place due to a central force acting between the molecules.

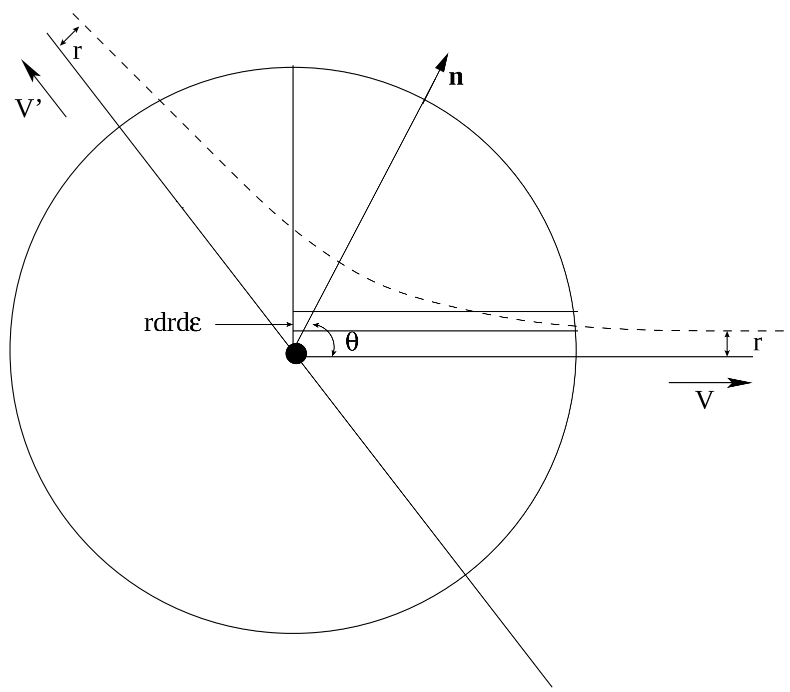

Fig. 1 : The collision coordinates

Figure 1 illustrates the coordinates (θ, ϵ) used for describing the collision. The black dot in the center of the figure refers to the first molecule–the target molecule. The dotted line indicates the trajectory of the second molecule which we call the bullet molecule, with respect to the target molecule. The target molecule is placed at the center where its trajectory comes closest to that of the bullet molecule. We have drawn a sphere around the target molecule and **n** is the unit vector in the direction of the point of closest approach of the bullet molecule. The beginning of the trajectory asymptotes in the direction opposite $\mathbf{V}$ and the end of the trajectory asymptotes in the direction opposite $\mathbf{V}'$, which is the relative velocity $\xi' - \xi^{*'}$ after the collision. The co-ordinates (r, ϵ) are polar co-ordinates in the plane orthogonal to the plane containing the trajectory of the bullet molecule and the target molecule as shown in the figure. The radial coordinate r is just the impact parameter as shown in the figure. The angular coordinate θ is the angle between $\mathbf{n}$ and the initial relative velocity $\mathbf{V}$. Thus the unit vector $\mathbf{n}$ is determined by the angular coordinates θ and ϵ.

Solving Newton's second law for the given central force, we can determine r as a function of θ and V, i.e. if the force is known we know $r(\theta, V)$. The collision kernel $B(\theta, V)$ is defined as

$$B(\theta, V) = Vr\frac{\partial r(\theta, V)}{\partial \theta} \quad . \tag{6.21}$$

Finally the velocities of the target and bullet molecule are related to the initial velocities of the target and bullet molecule kinematically through

$$\begin{aligned} \xi_i^{'} &= \xi_i - n_i(\mathbf{n} \cdot \mathbf{V}) \quad , \\ \xi_i^{*'} &= \xi_i^* + n_i(\mathbf{n} \cdot \mathbf{V}) \quad , \end{aligned} \tag{6.22}$$

so that $\mathbf{V}^{'} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$.

This completes our description of the Boltzmann equation. When the molecules interact via an attractive or repulsive central force which is proportional to the fifth inverse power of the distance ρ between the molecules, we say the system is a gas of Maxwellian molecules. The simplification for Maxwellian molecules is that r is independent of θ which can be seen from the fact that the trajectories of both the target and the bullet molecules lie on the circumference of a circle in the center of mass frame. As a consequence, B is also independent of θ.

To proceed further we need to develop some notation. Let $\phi(\xi)$ be a function of ξ. We will call it a collision invariant if

$$\Phi(\xi, \xi^*, \xi^{'}, \xi^{*'}) \equiv \phi(\xi) + \phi(\xi^*) - \phi(\xi^{'}) - \phi(\xi^{*'}) = 0 \quad . \tag{6.23}$$

Clearly there are five collision invariants - $(1, \xi_i, \xi^2)$ - which we will collectively denote as ψ_α.

Let us also define, for convenience of notation,

$$\mathcal{J}(f,g) = \frac{J(f,g) + J(g,f)}{2} \quad . \tag{6.24}$$

Notation: We will use the following notation in the rest of this section. Let $A^{(m)}$ and $B^{(n)}$ be two tensors of rank m and n respectively, completely symmetric in all their indices. Then,

- $A^{(m)}B^{(n)}$ will denote the symmetric product of the tensors so that it is completely symmetric in all its $m+n$ indices.
- $A^{(m)}B_i^{(n)}$ will denote a tensor of rank $(m+n)$ where all indices except the i-th index in $B^{(n)}$ have been completely symmetrized.
- We will use ν as in A_ν to denote all the m indices in A
- If A_{ij} and B_{kl} are symmetric second rank tensors, then $(A_{ij}B_{kl} + + + ++)$ will denote the combination of all the six terms required to make the sum symmetric in its indices i, j, k and l.

The above notations will hold even when A or B is a tensorial operator containing spatial derivatives.

The hydrodynamic equations can be derived from the Boltzmann equation as follows. Using symmetry one can easily prove that

$$\int \phi(\xi)\mathcal{J}(f,g)d\xi = \frac{1}{4}\int \Phi(\xi,\xi^*,\xi^{'},\xi^{*'})\mathcal{J}(f,g)d\xi \quad . \tag{6.25}$$

Using (6.23) it is clear that if $\phi(\xi)$ is a collision invariant, that is $\phi(\xi) = \psi_\alpha(\xi)$, then

$$\int \psi_\alpha(\xi)\mathcal{J}(f,g)d\xi = 0 \quad . \tag{6.26}$$

A special case of the preceding result gives

$$\int \psi_\alpha(\xi)J(f,f)d\xi = 0 \quad . \tag{6.27}$$

The Boltzmann equation [on multiplying by $\psi_\alpha(\xi)$ and integrating] implies

$$\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x_i}\left(\int \xi_i \psi_\alpha f d\xi\right) = 0 \quad , \tag{6.28}$$

where ρ_α are the locally conserved quantities defined by

$$\rho_\alpha = \int \psi_\alpha f d\xi \quad . \tag{6.29}$$

These equations are equivalent to the hydrodynamic equations (3.2) once we make the identifications $[\rho_0 = \rho,\ \rho_i = \rho u_i\ (i = 1, 2, 3),\ \rho_4 = (3p/2)]$.

The next few velocity moments, needed for later reference, are

$$\begin{aligned} p_{ij} &= \int (c_i c_j - RT\delta_{ij}) f d\xi \quad , \\ S_{ijk} &= \int c_i c_j c_k f d\xi \quad , \\ Q_{ijkl} &= \int c_i c_j c_k c_l f d\xi \quad , \end{aligned} \tag{6.30}$$

where $c_i = \xi_i - u_i$.

6.3.2 C.2 The moment equations

Multiplying both sides of the Boltzmann equation by higher polynomials of ξ and integrating over ξ, we find the equations satisfied by the moments $f^{(n)}$'s for $n \geq 2$ to be

$$\frac{\partial f^{(n)}}{\partial t} + \frac{\partial}{\partial x_i}\left(u_i f^{(n)} + f_i^{(n+1)}\right) + \frac{\partial \mathbf{u}}{\partial x_i} f_i^{(n)} - \frac{1}{\rho} f^{(n-1)} \frac{\partial f_i^{(2)}}{\partial x_i} = J^{(n)} \quad , \tag{6.31}$$

where

$$J^{(n)} = \int \mathbf{c}^n B(f' f_1' - f f_1) d\theta d\epsilon d\xi d\xi_1 \quad , \tag{6.32}$$

is the n-th velocity moment of the collision kernel.

It can be shown that

$$J_\mu^{(n)} = \sum_{p,q=0;p\geq q}^{\infty} B_{\mu\nu\rho}^{(n,p,q)}(\rho, T) f_\nu^{(p)} f_\rho^{(q)} \quad , \tag{6.33}$$

with a particular simplification for $B^{(2,2,0)}_{ijkl}$, which can be written as

$$B^{(2,2,0)}_{ijkl}(\rho, T) = B^{(2)}(\rho, T)\delta_{ik}\delta_{jl} \quad . \tag{6.34}$$

For Maxwellian molecules, there is yet another remarkable simplification that $B^{(n,p,q)}$'s are nonzero only if $p+q = n$. This happens essentially because the collision kernel $B(\theta, V)$ in (6.20) is independent of θ in this case (for more details please see [55]).

We will also denote $f^{(4)}_{ijkl}$ as Q_{ijkl} and its explicit form will be useful.

6.3.3 C.3 Formal Proof of Existence of Conservative Solutions

We now outline the proof that demonstrates existence of conservative solutions for the Boltzmann equation. The one-particle phase space distribution f will be functionally determined by the hydrodynamic variables and the shear-stress tensor (and their spatial derivatives). It must be emphasized that we proceed *exactly* along the same lines as used by Enskog in proving the existence of the normal (or purely hydrodynamic solutions) of the Boltzmann equation.

The proof for the existence of normal solutions of the Boltzmann equation [40, 41, 42, 43] (first given in Enskog's thesis) rests on the following theorem due to Hilbert [54, 43].
Theorem: *Consider the following linear integral equation for g:*

$$J(f_0, g) + J(g, f_0) = \mathcal{K} \quad , \tag{6.35}$$

where $J(f_0, g)$ is defined through (6.20) and f_0 is a locally Maxwellian distribution. This equation has a solution if and only if the source term $\mathcal{K}$ is orthogonal to the collision invariants ψ_α so that:

$$\int \psi_\alpha \mathcal{K} d\xi = 0 \quad , \tag{6.36}$$

provided the potential $U(\rho)$ satisfies the condition that $|U(\rho)| \geq \mathcal{O}(\rho^{-n+1})$ as $\rho \to 0$ for $n \geq 5$. [That is, when the distance (ρ) between molecules vanishes, the absolute value of the potential should grow faster than $(1/\rho)^4$.] Further the solution is unique up to an additive linear combination of the ψ_α's.

This theorem will be important in proving the existence of conservative solutions too, wherein we have to actually solve for the functional dependence on the hydrodynamic variables and the shear-stress tensor. For any conservative solution, we will just need to specify the initial data for the hydrodynamic variables and the shear-stress tensor. The only requirement will be that these initial data are analytic, because the functional dependence of f on the hydrodynamic variables and the shear-stress tensor will involve spatial derivatives of all orders. Clearly all normal solutions are conservative solutions, but not vice versa.

The method of proof can be briefly outlined thus. We will extract a *purely nonhydrodynamic* part from the shear-stress tensor p_{ij}, and denote it as $p_{ij}^{(nh)}$. This $p_{ij}^{(nh)}$ will satisfy a simpler equation of motion which schematically reads $(\partial p^{(nh)}/\partial t) = \sum_{n=1}^{\infty} c_n (p^{(nh)})^n$, involving just a single time derivative [although the initial data for $p_{ij}^{(nh)}$ can have any (analytic) spatial dependence]. The full shear-stress tensor p_{ij} can be solved as a functional of the hydrodynamic variables and the $p_{ij}^{(nh)}$. One can functionally invert this to reinstate p_{ij} as the independent variable in place of $p_{ij}^{(nh)}$ and also determine the equation for p_{ij}. In the process we will see that there is an interesting class of nontrivial homogeneous conservative solutions, where all the hydrodynamic variables are constants over space and time, while the shear-stress tensor is exactly $p_{ij}^{(nh)}$, which is just a function of time. This class of solutions is thus *purely nonhydrodynamic*, representing equilibration in velocity space.

The proof begins by writing the Boltzmann equation abstractly as

$$\mathcal{D} = J(f, f) \quad , \tag{6.37}$$

where

$$\mathcal{D} = \frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial \mathbf{x}} \quad , \tag{6.38}$$

and $J(f, f)$ is as defined through (6.20).

For a conservative solution, f is a functional of the nonhydrodynamic shear-stress tensor, $p_{ij}^{(nh)}(\mathbf{x}, t)$ and the five hydrodynamic variables, namely $u_i(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$ and $T(\mathbf{x}, t)$.

We expand f in two formal expansion parameters ϵ and δ such that

$$f = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \epsilon^n \delta^m f_{(m,n)} \quad . \tag{6.39}$$

The physical meanings of the expansion parameters will soon be made precise. For the moment, if the reader so pleases, she can think of ϵ as a hydrodynamic and δ as a nonhydrodynamic expansion parameter. Following Enskog, we will also expand the time derivative in powers of ϵ and δ as :

$$\frac{\partial}{\partial t} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \epsilon^n \delta^m \frac{\partial^{(n,m)}}{\partial t} \quad . \tag{6.40}$$

The above expansion of the time derivative might seem a little strange, but it will be necessary for us precisely for the same reason it was necessary for Enskog - the solutions of the equations of motion of hydrodynamic variables and $p_{ij}^{(nh)}$ cannot be expanded analytically in ϵ and δ, though their equations of motion could be through the subdivision of the partial time derivative. The proof will actually rely on the subdivision of the equations of motion just as in Enskog's purely hydrodynamic normal solutions and will not require the solutions to have analytic expansions [3].

This automatically results in a similar expansion for $\mathcal{D}$, such that

- For $n \geq 1$ and for all m

$$\mathcal{D}^{(n,m)} \equiv \sum_{k=1}^{n}\sum_{l=0}^{m} \frac{\partial^{(k,l)} f_{(n-k,m-l)}}{\partial t} + \xi \cdot \frac{\partial f_{(n-1,m)}}{\partial \mathbf{x}} \quad . \quad (n \geq 1; m = 0,1,2,...) \tag{6.41}$$

- For $n = m = 0$,

$$\mathcal{D}^{(0,0)} = 0 \quad . \tag{6.42}$$

[3]There is an analog of this in the fluid/gravity correspondence too. The existence of solutions in gravity dual to hydrodynamic configurations in the boundary which could be analytically expanded depends on the derivative expansion of the hydrodynamic equations and is independent of the fact that the solutions of the hydrodynamic equations themselves have no analytic expansion in the derivative expansion parameter.

With the assumption that f is a functional of the hydrodynamic variables and $p_{ij}^{(nh)}$, the time derivative acts on f schematically as

$$\begin{aligned}\frac{\partial f}{\partial t} &= \sum_{k=0}^{\infty}\frac{\partial f}{\partial(\nabla^k\rho)}\frac{\partial(\nabla^k\rho)}{\partial t}+\sum_{k=0}^{\infty}\frac{\partial f}{\partial(\nabla^k u_i)}\frac{\partial(\nabla^k u_i)}{\partial t}\\ &\quad+\sum_{k=0}^{\infty}\frac{\partial f}{\partial(\nabla^k T)}\frac{\partial(\nabla^k T)}{\partial t}+\sum_{k=0}^{\infty}\frac{\partial f}{\partial(\nabla^k p_{ij}^{(nh)})}\frac{\partial(\nabla^k p_{ij}^{(nh)})}{\partial t}\quad . \end{aligned} \tag{6.43}$$

Above, ∇^k schematically denotes k-th order spatial derivatives. Any time derivative acting on a hydrodynamic variable can be replaced by a functional of the hydrodynamic variables and the nonhydrodynamic shear-stress tensor by using the hydrodynamic equations of motion. These functional forms have a systematic derivative expansion in terms of the number of spatial derivatives present and contain only spatial derivatives and no time derivatives. So the expansion of the time derivative in ϵ is actually a derivative expansion, where the expansion parameter ϵ is the ratio of the typical length scale of spatial variation of f and the mean-free path. This naturally "explains" (6.41).

On the other hand, it will be seen that the time derivative of the nonhydrodynamic shear-stress tensor can be replaced, using its equation of motion, by an infinite series of polynomials of the nonhydrodynamic shear-stress tensor. Thus the expansion of the time derivative in δ as in (6.42); but we expand the solution of the equation of motion as an amplitude expansion with the expansion parameter δ identified as the ratio of the typical amplitude of the nonhydrodynamic shear-stress tensor with the pressure in final equilibrium. For the moment, these are just claims, to be borne out by an appropriate definition of the expansion of f and the time derivative.

C.3.1 Subdivisions in terms of ϵ and δ

We outline here the expansion of the various quantities in the Boltzmann equation and the full Boltzmann equation itself in terms of the two expansion parameters ϵ and δ and thereby arrive at various constraints that must be satisfied by these expansions. Our proof eventually will involve recursion while expanding the full Boltzmann equation in these expansion parameters.

1. In close analogy with Enskog's original subdivision of f, we impose some further properties on the subdivision of f.

   - First we require, as in the case of normal solutions of Enskog and Chapman, that the hydrodynamic variables are unexpanded in ϵ and δ and therefore are exactly the same as in the zeroth-order solution $f_{(0,0)}$, which will turn out to be locally Maxwellian. This is required because solutions of the hydrodynamic equations cannot be expanded analytically in these expansion parameters though the hydrodynamic equations themselves could be, as mentioned above. Therefore we should have

   $$\int \psi_\alpha f_{(n,m)} d\xi = 0; \qquad (n+m \geq 1; \alpha = 0,1,2,3,4) \tag{6.44}$$

   where ψ_α are the collision invariants $(1, \xi_i, \xi^2)$. It follows that

   $$\rho_\alpha = \int \psi_\alpha f_{(0,0)} d\xi \quad . \tag{6.45}$$

   ρ_α are the locally conserved quantities defined through (6.29). We may recall that these are just some combinations of the hydrodynamic variables.

   - We also require that the purely nonhydrodynamic part of the shear-stress tensor, $p_{ij}^{(nh)}$ has no expansion in ϵ and δ, analogous to the hydrodynamic variables. Being purely nonhydrodynamic, it determines $f_{(0,m)}$ for all m, i.e. the part of f which is zeroth order in ϵ, but contains all orders of δ in the conservative solutions. Since it vanishes at equilibrium, it is of first order in δ and is given exactly by $f_{(0,1)}$. More explicitly, for $m \geq 2$ and $n = 0$, we should have

   $$\int (c_i c_j - RT\delta_{ij}) f_{(0,m)} d\xi = 0 \quad (m \geq 2) \quad , \tag{6.46}$$

   so that

   $$p_{ij}^{(nh)} = \int (c_i c_j - RT\delta_{ij}) f_{(0,1)} d\xi \quad . \tag{6.47}$$

2. The subdivision of the time derivative is defined next. Following Enskog, we impose

on the time-derivative the condition that

$$\begin{aligned} \frac{\partial^{(0,m)}\rho^\alpha}{\partial t} &= 0 \quad , \\ \int \mathcal{D}^{(n,m)}\psi_\alpha d\xi &= 0; \quad (n \geq 1, m = 0, 1, 2, ...) \quad . \end{aligned} \tag{6.48}$$

Using (6.44), the second equation above can be simplified to

$$\frac{\partial^{(n,m)}\rho^\alpha}{\partial t} + \frac{\partial}{\partial x_i}\left(\int \xi_i \psi_\alpha f_{(n-1,m)} d\xi\right) = 0 \quad (n \geq 1, m = 0, 1, 2, ..) \quad . \tag{6.49}$$

Since the ρ^α are a redefinition of the hydrodynamic variables, this above condition amounts to expanding the hydrodynamic equations in a particular way. From this expansion we know how each subdivision of the time derivative acts on the (unexpanded) hydrodynamic variables. It is clear from (6.43) that if we now specify how the subdivisions of the time derivative act on $p_{ij}^{(nh)}$, we have defined the time derivative. Indeed, we have to solve for the action of the time-derivative because specifying this amounts to proving the existence of conservative solutions [4].

3. The next thing is to note that the full shear-stress tensor p_{ij} (just like any other higher moment) has an expansion in both ϵ and δ. If we denote $\delta p_{ij} = p_{ij} - p_{ij}^{(nh)}$, then for $n \geq 1$

$$\delta p_{ij}^{(n,m)} = \int (c_i c_j - RT\delta_{ij}) \left(f_{(n,m)} - f_{(0,1)}\right) d\xi \quad , \tag{6.50}$$

need not vanish. The expansion of $\delta p_{ij}^{(n,m)}$ as a functional of the hydrodynamic variables and $p_{ij}^{(nh)}$ in ϵ is the derivative expansion, with the power of ϵ essentially counting the number of spatial derivatives (which act both on hydrodynamic variables and the nonhydrodynamic shear-stress tensor). The expansion in δ is the "amplitude" expansion in terms of $p_{ij}^{(nh)}$, which we may recall is first order in δ.

4. On the basis of the above subdivisions one can now expand both sides of (6.37) and equate the terms of the same order on both sides. This enables us to write down the following set of equations that $J(f, f)$ must satisfy for different values of (n, m).

[4]Even for the hydrodynamic variables, the action of the subdivisions of the time derivative on them could have been treated as unknowns. But we have chosen the logically equivalent path of declaring them beforehand from our experience with the case of normal solutions.

- For $n = m = 0$, substituting (6.41), (6.42) and (6.39) in (6.37) we get

$$J(f_{(0,0)}, f_{(0,0)}) = 0 \quad , \tag{6.51}$$

so that $f_{(0,0)}$ has to be a locally Maxwellian distribution.

- Using the above fact, for $n = 0$ and $m \geq 1$, we get

$$\begin{aligned} J(f_{(0,0)}, f_{(0,m)}) + J(f_{(0,m)}, f_{(0,0)}) - \frac{\partial^{(0,0)}}{\partial t} f_{(0,m)} \\ = \sum_{l=1}^{m} \frac{\partial^{(0,l)}}{\partial t} f_{(0,m-l)} - S_{(0,m)}; \quad (m \geq 1) \quad . \end{aligned} \tag{6.52}$$

- Finally, for $n \geq 1$ and for all m

$$\begin{aligned} J(f_{(0,0)}, f_{(n,m)}) + J(f_{(n,m)}, f_{(0,0)}) = \\ \sum_{k=1}^{n} \sum_{l=0}^{m} \frac{\partial^{(k,l)} f_{(n-k,m-l)}}{\partial t} + \xi \cdot \frac{\partial f_{(n-1,m)}}{\partial \mathbf{x}} - S_{(n,m)}; \quad (n \geq 1). \end{aligned} \tag{6.53}$$

- The $S_{(n,m)}$ are given by, for $(n + m \geq 2)$

$$\begin{aligned} S_{(n,m)} = \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} J(f_{(k,l)}, f_{(n-k,m-k)}) \\ + \sum_{k=1}^{n-1} J(f_{(k,0)}, f_{(n-k,m)}) + \sum_{l=1}^{m-1} J(f_{(0,l)}, f_{(n,m-k)}) \\ + \sum_{k=1}^{n-1} J(f_{(k,m)}, f_{(n-k,0)}) + \sum_{l=1}^{m-1} J(f_{(n,l)}, f_{(0,m-k)}) \\ + J(f_{(n,0)}, f_{(0,m)}) + J(f_{(0,m)}, f_{(n,0)}); \quad (n + m) \geq 2 \end{aligned} \tag{6.54}$$

and

$$S_{(0,1)} = S_{(1,0)} = 0 \quad . \tag{6.55}$$

C.3.2 A recursive proof

With all of the above, we will now prove the existence and uniqueness of conservative solutions recursively. Recall that the key idea in this proof is to understand how the time derivative operator $\frac{\partial}{\partial t}$ acts on the hydrodynamic variables and the nonhydrodynamic part of the shear-stress tensor, $p_{ij}^{(nh)}$. We already know the action of this operator on the hydrodynamic variables from Eqs.(6.48) and (6.49). Now we will solve for the action of this operator on $p_{ij}^{(nh)}$. The action of the time derivative, when expanded in ϵ and δ, can be understood by analyzing the subdivisions of the Boltzmann equation given by Eqs. (6.51), (6.52) and (6.53).

1. It is clear from (6.51) that at the zeroth-order in m and n, $f_{(0,0)}$ is a locally Maxwellian distribution which is uniquely fixed by the choice of the five hydrodynamic variables (6.45) and hence can uniquely be specified as

$$f_{(0,0)} = \frac{\rho}{(2\pi RT)^{\frac{3}{2}}} \exp\left(-\frac{\mathbf{c}^2}{2RT}\right) \quad . \tag{6.56}$$

2. Next let us consider (6.52). The usual trick here is to rewrite $f_{(0,m)}$ as $f_{(0,0)}h_{(0,m)}$. The advantage is that since $f_{(0,0)}$ contains hydrodynamic variables only,

$$\frac{\partial^{(0,m)}}{\partial t} f_{(0,0)} = 0 \quad . \tag{6.57}$$

Therefore (6.52) can be rewritten as

$$\begin{aligned} J(f_{(0,0)}, f_{(0,0)}h_{(0,m)}) + J(f_{(0,0)}h_{(0,m)}, f_{(0,0)}) - f_{(0,0)}\frac{\partial^{(0,0)}}{\partial t}h_{(0,m)} \\ = f_{(0,0)}\sum_{l=1}^{m}\frac{\partial^{(0,l)}}{\partial t}h_{(0,m-l)} - \sum_{l=1}^{m-1} J(f_{(0,0)}h_{(0,l)}, f_{(0,0)}h_{(0,m-l)}); \qquad (m \geq 2). \end{aligned} \tag{6.58}$$

Now we analyze (6.58) for $m = 1$ and $m = 2$.

- **m=1**:

For $m = 1$, (6.58) reduces to

$$\begin{aligned} J(f_{(0,0)}, f_{(0,0)}h_{(0,1)}) + J(f_{(0,0)}h_{(0,1)}, f_{(0,0)}) \\ = f_{(0,0)}\frac{\partial^{(0,0)}}{\partial t}h_{(0,1)} \quad , \end{aligned} \tag{6.59}$$

while it follows from (6.47) that

$$h_{(0,1)} = \frac{1}{2!}\frac{p_{ij}^{(nh)}(\mathbf{x},t)}{pRT}(c_i c_j - RT\delta_{ij}) \quad . \tag{6.60}$$

These two equations imply that

$$\frac{\partial^{(0,0)}}{\partial t} p_{ij}^{(nh)} = B^{(2)}(\rho, T) p_{ij} \quad , \tag{6.61}$$

where $B^{(2)}$ has been defined in (6.34) [5].

- **m=2**:

At the second order, (6.58) implies

$$\begin{aligned} & J(f_{(0,0)}, f_{(0,0)}h_{(0,2)}) + J(f_{(0,0)}h_{(0,2)}, f_{(0,0)}) \\ = f_{(0,0)}\frac{\partial^{(0,0)}}{\partial t}h_{(0,2)} &+ f_{(0,0)}\frac{\partial^{(0,1)}}{\partial t}h_{(0,1)} - J(f_{(0,0)}h_{(0,1)}, f_{(0,0)}h_{(0,1)}) \quad . \end{aligned} \tag{6.62}$$

We then need to solve for two things, $h_{(0,2)}$ and the operator $\left(\partial^{(0,1)}/\partial t\right)$. To do this we first write $h_{(0,2)}$ as

$$\begin{aligned} h_{(0,2)} &= \frac{1}{3!}\frac{S_{ijk}^{(0,2)}}{p(RT)^2}\Big(c_i c_j c_k - RT(c_i\delta_{jk} + +)\Big) \\ &+ \frac{1}{4!(RT)^2}\left[\frac{Q_{ijkl}^{(0,2)}}{p(RT)} - \left(\frac{p_{ij}^{(nh)}}{p}\delta_{kl} + + + + +\right) - (\delta_{ij}\delta_{kl} + +)\right] \\ &\times\Big[c_i c_j c_k c_l - RT(c_i c_j \delta_{kl} + + + + +) + (RT)^2(\delta_{ij}\delta_{kl} + +)\Big] \quad . \end{aligned} \tag{6.63}$$

The idea behind guessing this form is to expand $h_{(0,2)}$ in two higher order Hermite polynomials of **c**'s and re expressing the Hermite coefficients through the ordinary moments. This method of expansion is due to Grad [55]. For the moment we can just take it as the most general possible form of $h_{(0,2)}$,

[5] Note that (6.61) is consistent with the amplitude expansion of the time-derivative in δ. The time-derivative expansion in δ should consistently start at the zeroth order as both sides of the equation contain one $p_{ij}^{(nh)}$.

since if higher Hermite polynomials are included here, the coefficients would have vanished. It also turns out that $S^{(0,2)}_{ijk}$ vanishes. Similarly all the other higher odd moments vanish, *so far as their purely nonhydrodynamic parts (or expansion in* m *for* $n = 0$*) is concerned.* Obviously this does not mean that these higher odd moments have no dependence on $p^{(nh)}_{ij}$. For $n > 0$ there is indeed a nonvanishing expansion in m for these moments. We can now compare the coefficients of Hermite polynomials on both sides of our Eq. (6.62). For Maxwellian molecules (thus determining the form of J) we have

$$\begin{aligned}\frac{\partial^{(0,1)}}{\partial t} p^{(nh)}_{ij} &= B^{(2,2,2)}_{ijklmn}(\rho, T) p^{(nh)}_{kl} p^{(nh)}_{mn} + B^{(2,4,0)}_{ijklmn}(\rho, T) Q^{(0,2)}_{klmn} \quad , \qquad (6.64)\\ \frac{\partial^{(0,0)}}{\partial t} Q^{(0,2)}_{ijkl} &= B^{(4,4,0)}_{ijklmnpq}(\rho, T) Q^{(0,2)}_{mnpq} + B^{(4,2,2)}_{ijklmnpq}(\rho, T) p^{(nh)}_{mn} p^{(nh)}_{pq} \quad .\end{aligned}$$

Since we know the action of $(\partial^{(0,0)}/\partial t)$ on $p^{(nh)}_{ij}$ and the hydrodynamic variables, we can solve for $Q^{(0,2)}_{ijkl}$ as a functional of $p^{(nh)}_{ij}$ and the hydrodynamic variables; the solution turns out to be

$$Q^{(0,2)}_{klmn} = X_{klmnpqrs} p^{(nh)}_{pq} p^{(nh)}_{rs} \quad , \qquad (6.65)$$

where $X_{klmnpqrs}$ satisfies the equation [6]

$$2B^{(2)} X_{klmnpqrs} = B^{(4,4,0)}_{klmnijtu} X_{ijtupqrs} + B^{(4,2,2)}_{klmnpqrs} \quad . \qquad (6.66)$$

This in turn provides the solution for the operator $\left(\partial^{(0,1)}/\partial t\right)$:

$$\frac{\partial^{(0,1)}}{\partial t} p^{(nh)}_{ij} = B^{(2,2,2)}_{ijklmn} p^{(nh)}_{kl} p^{(nh)}_{mn} + B^{(2,4,0)}_{ijklmn} X_{klmnpqrs} p^{(nh)}_{pq} p^{(nh)}_{rs} \quad . \qquad (6.67)$$

[6] The solution for $X_{(klmn)(pqrs)}$ regarded as an 81×81 matrix is $(2B^{(2)}\delta_{(klmn)(ijtu)} - B^{(4,4,0)}_{(klmn)(ijtu)})^{-1} B^{(4,2,2)}_{(ijtu)(pqrs)}$ where $\delta_{(klmn)(ijtu)}$ is defined as a 81×81 matrix whose entries are 1 if $k = i$, $l = j$, $m = t$, $n = u$ and zero otherwise. It is quite evident that when $(2B^{(2)}\delta_{(klmn)(ijtu)} - B^{(4,4,0)}_{(klmn)(ijtu)})$ fails to be invertible, there is a singularity in our solution and in fact this may happen when ρ and T takes appropriate values. Such singularities also appeared in Born and Green's normal solutions of BBGKY heirarchy and was interpreted as describing local nucleation of the solid phase. In our case too, the singularities of the conservative solutions may signal local condensation of the liquid phase.

The equation above shows the action of the operator on $p_{ij}^{(nh)}$; we already know how it acts on the hydrodynamic variables. This implies that we have solved for this operator at this order. Note that the solution for the operator corroborates the intuitive understanding that this operator is the next order in amplitude expansion. Another important point is that the solution of the operator is not independent of the solution for $Q_{ijkl}^{(0,2)}$ and is just given by the logic of our expansion once $Q_{ijkl}^{(0,2)}$ has been solved as a functional of $p_{ij}^{(nh)}$. This feature is the same for all the higher terms in the expansion of the time derivative operator as well.

For non-Maxwellian molecules things are a bit complicated because the equation for Q_{ijkl} in (6.64) also contains a term linear in $p_{ij}^{(nh)}$, so that now

$$\begin{aligned} \frac{\partial^{(0,1)}}{\partial t} p_{ij}^{(nh)} &= \delta B^{(2)} p_{ij}^{(nh)} + B_{ijklmn}^{(2,2,2)} p_{kl}^{(nh)} p_{mn}^{(nh)} \\ &\quad + B_{ijklmn}^{(2,4,0)} X_{klmnpqrs} p_{pq}^{(nh)} p_{rs}^{(nh)} \quad . \end{aligned} \tag{6.68}$$

However this feature also appears in the usual derivative expansion (the expansion in ϵ) of the time-derivative. Despite appearance, $\delta B^{(2)} p_{ij}^{(nh)}$ is a small quantity as $(\delta B^{(2)}/B^{(2)})$ is a pure number which is smaller than unity (for a proof of this and also for the statement of convergence of such corrections in the context of normal solutions, please see [41, 42]). This result can be translated here, as the normal solutions are just special cases of our conservative solutions and at a sufficiently late time our solutions will be just appropriate normal solutions [7].

This is indeed remarkable considering that we have no parametric suppression here. Formally however, aside from the convergence problem, there is no obstruction because δ is just a formal parameter and is only intuitively connected

[7] We note that we can simply borrow Burnett's results here because all the hydrodynamic transport coefficients are not independent of the nonhydrodynamic parameters like the relaxation time $1/B^{(2)}$, for instance viscosity η is at leading order $(p/B^{(2)})$. Since p has no expansion, convergence of the viscosity implies convergence of $B^{(2)}$ as any generic "conservative" solution will be approximated by an appropriate normal solution at sufficiently late times.

to the amplitude expansion.

- **Higher m**:

  We can proceed in the same way to the next order in m when n is zero. At every stage we have to deal with $f_{(0,m)}$ which we may write as $f_{(0,0)}h_{(0,m)}$ and further expand $h_{(0,m)}$ in a series containing up to m-th order Hermite polynomial in c's. We have to solve for the coefficients of these Hermite polynomials, which depend on x only, and this leads to the definition of the m-th subdivision of the time derivative operator in the δ expansion when the ϵ expansion is at the zeroth order. The equation for evolution of $p_{ij}^{(nh)}(\mathbf{x},t)$ thus finally involves only a single time derivative which we have expanded in δ. This is highly nonlinear, involving an infinite series of $p_{ij}^{(nh)}$. The presence of just a single time derivative in the equation of motion for $p_{ij}^{(nh)}(\mathbf{x},t)$ makes it essentially an ordinary differential equation in one variable and so for any initial data existence and uniqueness of solution is guaranteed.

  We note that we can consistently truncate our solution at $n = 0$ so that there is no expansion of f in ϵ, provided all the hydrodynamic variables are constants over both space and time and $p_{ij}^{(nh)}$ is constant over space but a function of time. This gives us the simplest class of conservative solutions which is homogeneous in space; the Boltzmann equation becomes equivalent to an ordinary differential equation involving a single time derivative for p_{ij}. Physically this solution corresponds to the most general conservative solution which is homogeneous in space, but generically far away from equilibrium in the velocity space (so that the velocity distribution is far from being Maxwellian).

3. The next task is to prove the existence of solutions for the recursive series of equations in (6.53). To see if solutions will exist we need to employ Hilbert's theorem. $S_{(n,m)}$ contains either pairs of the form $J(f_{(p,q)}, f_{(r,s)}) + J(f_{(p,q)}, f_{(r,s)})$ or just $J(f_{(l,l)}, f_{(l,l)})$. So when the collision invariants are integrated with $S_{(n,m)}$, as in $\int \psi_\alpha S_{(n,m)} d\xi$, the integrals vanish as a consequence of (6.26). Therefore, the exis-

tence of the solution to $f_{(n,m)}$ follows from Hilbert's theorem as a consequence of (6.48). The solution is unique because the condition (6.44) fixes the arbitrariness of the dependence of $f_{(n,m)}$ on the collision invariants ψ_α. The details for $n \geq 1$, are thus, exactly the same as in the case of normal solutions. The action of $(\partial^{(n,m)}/\partial t)$ on $p_{ij}^{(nh)}$ is also determined as soon as the functional dependence of δp_{ij} and the relevant higher moments on $p_{ij}^{(nh)}$ and the hydrodynamic variables are determined.

The explicit calculations become extremely complex even when, say $n = 2, m = 0$ or $n = 1, m = 1$. We give some explicit results for the first few terms in the expansion for δp_{ij} as

$$
\begin{aligned}
\delta p_{ij}^{(1,0)} &= \frac{p}{B^{(2)}}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} - \frac{2}{3}\delta_{mn}\frac{\partial u_r}{\partial x_r}\right) \quad , \\
\delta p_{ij}^{(1,1)} &= \frac{1}{B^{(2)}}\left(\frac{\partial}{\partial x_r}(u_r p_{ij}^{(nh)}) + \frac{\partial u_j}{\partial x_r}p_{ir}^{(nh)} + \frac{\partial u_i}{\partial x_r}p_{jr}^{(nh)} - \frac{2}{3}\delta_{ij}p_{rs}^{(nh)}\frac{\partial u_r}{\partial x_s}\right) \\
&\quad - \frac{2pB_{ijklmn}^{(2,2,2)}}{(B^{(2)})^2}p_{kl}^{(nh)}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} - \frac{2}{3}\delta_{mn}\frac{\partial u_r}{\partial x_r}\right) \quad .
\end{aligned}
\tag{6.69}
$$

It is clear that the terms in the expansion involve spatial derivatives of both the hydrodynamic variables and $p_{ij}^{(nh)}$. From the expression for $\delta p_{ij}^{(1,0)}$ one can determine the shear viscosity η which is of course the same as in the purely hydrodynamic normal solutions, so that

$$
\eta \approx \frac{p}{B^{(2)}}(\rho, T) \quad . \tag{6.70}
$$

We also give some terms in the expansion for the heat flow vector

$$
\begin{aligned}
S_i^{(1,0)} &= \frac{15pR}{2B^{(2)}}\frac{\partial T}{\partial x_i} \quad , \\
S_i^{(1,1)} &= \frac{3}{2B^{(2)}}\left(2RT\frac{\partial p_{ir}^{(nh)}}{\partial x_r} + 7Rp_{ir}^{(nh)}\frac{\partial T}{\partial x_r} - \frac{2p_{ir}^{(nh)}}{\rho}\frac{\partial p}{\partial x_r}\right) \quad ,
\end{aligned}
\tag{6.71}
$$

It is clear that the heat conductivity χ is also the same as in purely hydrodynamic normal solutions so that

$$
\chi \approx \frac{15R}{2}\frac{p}{B^{(2)}}(\rho, T) \approx \frac{15R}{2}\eta \quad . \tag{6.72}
$$

Corrections to the above relation appear in the higher order for non-Maxwellian molecules but again these are the same as in the case of normal solutions.

This completes our proof for the existence of conservative solutions for the non-relativistic Boltzmann equation. As mentioned before, we can now reinstate p_{ij} as the independent variable. Our independent variables satisfy the following equations of motion

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}(\rho u_r) &= 0 \quad , \\
\frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho}\frac{\partial (p\delta_{ir} + p_{ir})}{\partial x_r} &= 0 \quad , \\
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r}(u_r p) + \frac{2}{3}(p\delta_{ir} + p_{ir})\frac{\partial u_i}{\partial x_r} + \frac{1}{3}\frac{\partial S_r}{\partial x_r} &= 0 \quad , \\
\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{\partial S_{ijr}}{\partial x_r} - \frac{1}{3}\delta_{ij}\frac{\partial S_r}{\partial x_r} & \\
+\frac{\partial u_j}{\partial x_r} p_{ir} + \frac{\partial u_i}{\partial x_r} p_{jr} - \frac{2}{3}\delta_{ij} p_{rs}\frac{\partial u_r}{\partial x_s} & \\
+p(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_r}{\partial x_r}) &= \sum_{p,q=0,p\geq q;(p,q)\neq(2,0))}^{\infty} B^{(2,p,q)}_{ij\nu\rho}(\rho,T) f^{(p)}_{\nu} f^{(q)}_{\rho} \\
&\quad + B^{(2)}(\rho,T) p_{ij} \quad .
\end{aligned} \tag{6.73}$$

The first three equations are just the hydrodynamic equations, while the equation for p_{ij} can be obtained from (6.31).

The crucial point of this proof is that we have now solved for all higher moments $f^{(n)}_\nu$'s for $n \geq 3$ (which includes, of course, S_{ijk} and thus S_i) as functionals of our ten variables (ρ, u_i, p, p_{ij}) with $T = p/(R\rho)$. Any solution of these ten equations of motion can be uniquely lifted to a full solution of the Boltzmann equation as all the higher moments are dependent on these ten variables through a unique functional form. Also, some solutions for p_{ij} in the last of our system of equations are purely hydrodynamic and these constitute the normal solutions [8].

[8] This can be readily seen as follows. If we assume that p_{ij} is functionally dependent on the hydrodynamic variables, from the equation for its evolution, it is clear that at the first order in the derivative expansion $p_{ij} = \eta(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_r}{\partial x_r})$, where $\eta = \frac{p}{B^{(2)}}$. Now we can substitute this in place of p_{ij} in

6.4 Appendix D: A Simple Mathematical Interpretation of the GCA

Mathematically, the infinite dimensional GCA can be motivated as follows: *Consider two particles with velocities* $\mathbf{v_1}$ *and* $\mathbf{v_2}$ *respectively at the same point in space* $\mathbf{x}$ *and at the same time* t*. Then the infinite dimensional GCA is the largest possible group of space-time transformations under which the relative velocity* $(\mathbf{v_1} - \mathbf{v_2})$ *transforms covariantly (as a vector under rotation) while its norm remains invariant.*. We will now prove this statement.

Let us consider an arbitrary space-time transformation from $(\mathbf{x}, t)$ to $(\mathbf{x}', t')$. Let us denote

$$M_{ij} = \frac{\partial x_i'}{\partial x_j}, N_i = \frac{\partial x_i'}{\partial t}, P_i = \frac{\partial t'}{\partial x_j}, Q = \frac{\partial t'}{\partial t}. \tag{6.74}$$

Then the following holds,

$$\begin{aligned} dx_i' &= M_{ij} dx_j + N_i dt, \\ dt' &= P_i dx_i + Q dt. \end{aligned} \tag{6.75}$$

So, we have

$$v_i' = \frac{M_{ij} v_j + N_i}{P_k v^k + Q}. \tag{6.76}$$

The relative velocity of two particles at the same point in space at a given time transforms as below,

$$v_{(1)i}' - v_{(2)i}' = \frac{(M_{ij} v_{(1)j} P_k v_{(2)k} - M_{ij} v_{(2)j} P_k v_{(1)k}) + Q(M_{ij} v_{(1)j} - M_{ij} v_{(2)j}) + N_i (P_k v_{(2)k} - P_k v_{(1)k})}{(P_l v_{(1)l} + Q)(P_m v_{(2)m} + Q)}. \tag{6.77}$$

For this transformation to be covariant, we require $P_k = 0$, in which case

$$v_{(1)i}' - v_{(2)i}' = \frac{M_{ij} v_{(1)j} - M_{ij} v_{(2)j}}{Q}. \tag{6.78}$$

If we also require the norm to remain the same, we should have,

$$\frac{M_{ij}}{Q} = R_{ij}, \tag{6.79}$$

the equation of evolution for p_{ij} to get the second order correction and so on. In the substitution, the time derivative acting on the hydrodynamic variables can be replaced by spatial derivatives by using the hydrodynamic equations of motion.

where, R_{ij} is a rotation matrix. Now, $P_i = (\partial t^{'}/\partial x_i) = 0$ implies that

$$t^{'} = f(t), Q = \frac{df(t)}{dt}. \tag{6.80}$$

Then we have

$$M_{ij} = \frac{\partial x_i^{'}}{\partial x_j} = QR_{ij}(\mathbf{x}, t) = \frac{df(t)}{dt} R_{ij}(\mathbf{x}, t). \tag{6.81}$$

The integrability condition requires that

$$\frac{\partial M_{ij}}{\partial x_k} = \frac{\partial M_{ik}}{\partial x_j}, \tag{6.82}$$

which in turn implies that

$$\frac{\partial R_{ij}(\mathbf{x}, t)}{\partial x_k} = \frac{\partial R_{ik}(\mathbf{x}, t)}{\partial x_j}. \tag{6.83}$$

The above condition at a fixed value of i, the implies that the curl of a vector vanishing so that we must have

$$R_{ij}(\mathbf{x}, t) = \frac{\partial V_i(\mathbf{x}, t)}{\partial x_j}. \tag{6.84}$$

A rotation matrix satisfies the property that $R_{ij}^{-1} = R_{ji}$, so we should have

$$\frac{\partial V_i}{\partial x_j} \frac{\partial V_k}{\partial x_j} = \delta_{ik}. \tag{6.85}$$

The solution to the above system of equations is
$V_i = \overline{R}_{ij}(t) x_j +$ a function of time,
so, we have $R_{ij} = \overline{R}_{ij}(t)$. To sum up, $(\partial x_i^{'}/\partial x_j) = QM_{ij} = (df(t)/dt) R_{ij}(t)$, therefore

$$x_i^{'} = \frac{df(t)}{dt} R_{ij}(t) x_j + b_i(t). \tag{6.86}$$

The above together with (6.80) belongs to our group of spacetime transformations denoted by GCA.

It is also easy to check that any transformation belonging to the GCA makes the relative velocity of two particles at a given point in space at a given time transform covariantly while preserving its norm. So we have proved, that the largest group of spacetime transformations under which the relative velocity of two particles at the same point in space at a given time transforms covariantly while its norm is preserved, is the GCA. This mathematical result can have physical applications in constructing local interactions of particles in a GCA-invariant microscopic theory.

6.5 Appendix E: $G = MLR$

Here, we will prove that any arbitrary element (G) of GCA, can be written uniquely as a succession of a time dependent rotation (R), a spatially correlated time reparametrization (L) and a time dependent boost (M).

Let us denote the space-time coordinates $(\mathbf{x}, t)$ together as X. Let G be an arbitrary element of the GCA and let two coordinates X and $X^{'}$ be related so that $X^{'} = G.X$, i.e. $X^{'}$ is the result of action of G on X.

However, we now note that there is a *unique* time-dependent boost M such that $M.X$ and $X^{'}$ will *will share the same origin of spatial coordinates at all times.* Let us denote $M^{-1}.X^{'}$ as $X^{''}$. So, by construction $X^{''}$ and X share the same origin of spatial coordinates *at all times.*

Now, if two space-time coordinates share the same origin of spatial coordinates at all times, it is also easy to see, that there is a *unique* spatially correlated time reparametrization L which relate their times. Therefore, there is a *unique* L such that $X^{'''} = L^{-1}.X^{''}$ and X share the same time.

By construction, we see that $X^{'''}$ and X share the same time and the same origin of spatial coordinates. Therefore, they must be related by a *unique* time-dependent rotation R, so that $X = R^{-1}.X^{'''}$.

Summing all up, $X = R^{-1}.X^{'''} = R^{-1}L^{-1}X^{''} = R^{-1}L^{-1}M^{-1}X^{'}$. But we assumed $X = GX^{'}$, so $G = MLR$, with M, L and R being unique because they were unique in each stage of our argument above. So, we have proved that any arbitrary element (G) of GCA, can be written as a succession of a time dependent rotation (R), a spatially correlated time reparametrization (L) and a time dependent boost (M).

Bibliography

[1] C. Fefferman and C. Robin Graham, "Conformal Invariants" in *Elie Cartan et les Mathématiques d'aujourd'hui* (Astérisque, 1985)95.

[2] C.R. Graham, "Volume and area renormalizations for conformally compact Einstein metrics", [arxiv:math.DG/9909042]

[3] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Commun. Math. Phys. **104**, 207 (1986).

[4] R. Penrose and W. Rindler, "SPINORS AND SPACE-TIME. VOL. 2: SPINOR AND TWISTOR METHODS IN SPACE-TIME, *Cambridge, Uk: Univ. Pr. (1986) 501p*, chapter 9

[5] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[6] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[7] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[8] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087].

[9] S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS/CFT Commun. Math. Phys. **217**, 595 (2001) [arXiv:hep-th/0002230].

[10] M. Bianchi, D. Z. Freedman and K. Skenderis, JHEP **0108**, 041 (2001) [arXiv:hep-th/0105276].

[11] M. Bianchi, D. Z. Freedman and K. Skenderis, "Holographic Renormalization," Nucl. Phys. B **631**, 159 (2002) [arXiv:hep-th/0112119].

[12] V. Balasubramanian and P. Kraus, "A stress tensor for anti-de Sitter gravity," Commun. Math. Phys. **208**, 413 (1999) [arXiv:hep-th/9902121].

[13] K. Skenderis, "Lecture notes on holographic renormalization," Class. Quant. Grav. **19**, 5849 (2002) [arXiv:hep-th/0209067].

[14] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7**, 2333 (1973)

[15] J. M. Bardeen, B. Carter and S. W. Hawking, "The four laws of black hole mechanics," Commun Math. Phys. **31**, 161 (1973).

[16] D. T. Son and A. O. Starinets, "Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications," JHEP **0209**, 042 (2002) [arXiv:hep-th/0205051].

[17] G. Policastro, D. T. Son and A. O. Starinets, "From AdS/CFT correspondence to hydrodynamics," JHEP **0209**, 043 (2002) [arXiv:hep-th/0205052].

[18] P. Kovtun, D. T. Son and A. O. Starinets, "Viscosity in strongly interacting quantum field theories from black hole physics," Phys. Rev. Lett. **94**, 111601 (2005) [arXiv:hep-th/0405231]

[19] T. Schaefer and D. Teaney, "Nearly Perfect Fluidity: From Cold Atomic Gases to Hot Quark Gluon Plasmas," arXiv:0904.3107 [hep-ph].

[20] P. K. Kovtun and A. O. Starinets, "Quasinormal modes and holography," Phys. Rev. D **72**, 086009 (2005) [arXiv:hep-th/0506184].

[21] N. Iqbal and H. Liu, "Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm," Phys. Rev. D **79**, 025023 (2009) [arXiv:0809.3808 [hep-th]].

[22] A. Buchel, R. C. Myers, M. F. Paulos and A. Sinha, "Universal holographic hydrodynamics at finite coupling," Phys. Lett. B **669**, 364 (2008) [arXiv:0808.1837 [hep-th]];

[23] I. Kanitscheider and K. Skenderis, "Universal hydrodynamics of non-conformal branes," [arXiv:0901.1487 [hep-th]].

[24] A. Sen, "Universality of the tachyon potential," JHEP **9912**, 027 (1999) [arXiv:hep-th/9911116].

[25] R. A. Janik and R. B. Peschanski, "Asymptotic perfect fluid dynamics as a consequence of AdS/CFT," Phys. Rev. D **73**, 045013 (2006) [arXiv:hep-th/0512162];

M. P. Heller and R. A. Janik, "Viscous hydrodynamics relaxation time from AdS/CFT," Phys. Rev. D **76**, 025027 (2007) [arXiv:hep-th/0703243].

[26] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," JHEP **0802**, 045 (2008) [arXiv:0712.2456 [hep-th]].

[27] M. Rangamani, "Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence," arXiv:0905.4352 [hep-th].

[28] G. Beuf, M. P. Heller, R. A. Janik and R. Peschanski, "Boost-invariant early time dynamics from AdS/CFT," JHEP **0910**, 043 (2009) [arXiv:0906.4423 [hep-th]].

[29] R. A. Janik, "The dynamics of quark-gluon plasma and AdS/CFT," arXiv:1003.3291 [hep-th].

[30] R. Loganayagam, "Entropy Current in Conformal Hydrodynamics," JHEP **0805**, 087 (2008) [arXiv:0801.3701 [hep-th]];

S. Bhattacharyya *et al.*, "Local Fluid Dynamical Entropy from Gravity," JHEP **0806**, 055 (2008) [arXiv:0803.2526 [hep-th]].

[31] R. G. Cai, "Gauss-Bonnet black holes in AdS spaces," Phys. Rev. D **65**, 084014 (2002) [arXiv:hep-th/0109133].

[32] S. Dutta, "Higher Derivative Corrections to Locally Black Brane Metrics," JHEP **0805**, 082 (2008) [arXiv:0804.2453 [hep-th]].

[33] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, "Large rotating AdS black holes from fluid mechanics," JHEP **0809**, 054 (2008) [arXiv:0708.1770 [hep-th]].

[34] S. Kinoshita, S. Mukohyama, S. Nakamura and K. y. Oda, "A Holographic Dual of Bjorken Flow," arXiv:0807.3797 [hep-th].

duals,"

[35] P. Arnold, G. D. Moore and L. G. Yaffe, "Transport coefficients in high temperature gauge theories: (I) Leading-log results," JHEP **0011**, 001 (2000) [arXiv:hep-ph/0010177]; P. Arnold, G. D. Moore and L. G. Yaffe, "Transport coefficients in high temperature gauge theories. II: Beyond leading log," JHEP **0305**, 051 (2003) [arXiv:hep-ph/0302165].

[36] R. Baier, P. Romatschke and U. A. Wiedemann, "Dissipative hydrodynamics and heavy ion collisions," Phys. Rev. C **73**, 064903 (2006) [arXiv:hep-ph/0602249]; M. A. York and G. D. Moore, "Second order hydrodynamic coefficients from kinetic theory," arXiv:0811.0729 [hep-ph].

[37] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, "Relativistic viscous hydrodynamics, conformal invariance, and holography," JHEP **0804**, 100 (2008) [arXiv:0712.2451 [hep-th]].

[38] J. C. Maxwell, *On the dynamical theory of gases*, Philosophical Transactions of the Rpyal Society of London, **157** (1867), 49-88.

[39] L. Boltzmann, *Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen*, Sitzungsberichte Akad. Wiss. Vienna, part II, **6** (1872), 275-370.

[40] D. Enskog, Dissertation, Uppsala (1917): "Arkiv. Mat., Ast. och. Fys" **16**, 1, (1921).

[41] D. Burnett, *Proc. Lond. Math. Soc.* **s2-39**, 385, 1935.

[42] S. Chapman and T. Cowling, "The Mathematical Theory of Non-Uniform Gases", (Cambridge University Press, Cambridge, England), Chapters 7, 8, 10, 15 and 17.

[43] C. Cercignani, "The Boltzmann Equation and its Applications", Springer-Verlag, New York, 1988, Chapters 2, 4 and 5.

[44] J. M. Stewart, Ph. D. dissertation, University of Cambridge, 1969.

[45] E. A. Uehling and G. E. Uhlenbeck, *Phys. Rev. 43*, 552 (1933).

[46] G. Rupak and T. Schäfer, Phys Rev A **76**, 053607 (2007) [arxiv: 0707.1520 [cond-mat.other]].

[47] I. Müller, "Speeds of propagation in classical and relativistic extended thermodynamics," living Reviews in Relativity (http://relativity.livingreviews.org/Articles/lrr-1999-1/).

[48] J Yvon, "Theorie Statistique des Fluides et l'Equation d'Etat," Actes scientifique et industrie, No 203, Paris: Hermann (1935), N. N. Bogoliubov, "Kinetic Equations," Journal of Physics USSR **10** (3), 265-274, 1946 ; J. G. Kirkwood, *J. Cem. Phys.* **14**, 180, 1946; **15**, 72, 1947 ; M. Born and H. S. Green, "A General Kinetic Theory of Liquids I. The Molecular Distribution Functions," Proc. Roy. Soc. A **188**, 10, 1946,**189**, 103, 1947.

[49] M. Born and H. S. Green, *Proc. Roy. Soc.* A, **190**, 455, 1947.

[50] U. W. Heinz, "Early collective expansion: Relativistic hydrodynamics and the transport properties of QCD matter," arXiv:0901.4355 [nucl-th].

[51] K. Skenderis and M. Taylor, "Kaluza-Klein holography," JHEP **0605**, 057 (2006) [arXiv:hep-th/0603016].

[52] I. Müller, *Z. Physik* **198**, 310 (1967); W. Israel, "Nonstationary irreversible thermodynamics: A causal relativistic theory",Annals of Physics *100* 310 (1976); J. M. Stewart, "On transient relativistic thermodynamics and kinetic theory" Proc. R. Soc. Lond. A **357** 59 (1977); W. Israel and J. M. Stewart, "Transient Relativistic Thermodynamics and Kinetic Theory," Annals of Physics **118**, 341 (1979).

[53] S. Kinoshita, S. Mukohyama, S. Nakamura and K. y. Oda, "A Holographic Dual of Bjorken Flow," Prog. Theor. Phys. **121**, 121 (2009) [arXiv:0807.3797 [hep-th]].

[54] R. Courant and D. Hilbert, *Methoden der Math. Physik* , **1** (2nd ed.), pp. 99 and 129.

[55] H. Grad, *On the Kinetic Theory of rarefied gases*, Comm. Pure Appl. Math.,**2** (1949), 331-407.

[56] J. Lukierski, P. C. Stichel and W. J. Zakrzewski, "Exotic Galilean conformal symmetry and its dynamical realisations," Phys. Lett. A **357**, 1 (2006) [arXiv:hep-th/0511259].

[57] A. Bagchi and R. Gopakumar, "Galilean Conformal Algebras and AdS/CFT," arXiv:0902.1385 [hep-th].

[58] M. Alishahiha, A. Davody and A. Vahedi, "On AdS/CFT of Galilean Conformal Field Theories," arXiv:0903.3953 [hep-th];

A. Bagchi and I. Mandal, 'On Representations and Correlation Functions of Galilean Conformal Algebras," arXiv:0903.4524 [hep-th];

A. Davody, "Weyl Anomaly in NonRelativistic CFTs," Phys. Lett. B **685**, 341 (2010) [arXiv:0909.3705 [hep-th]].

[59] D. Martelli and Y. Tachikawa, "Comments on Galilean conformal field theories and their geometric realization," arXiv:0903.5184 [hep-th].

[60] C. Duval, P. A. Horvathy, "Non-relativistic conformal symmetries and Newton-Cartan structures," arxiv:0904.0531 [math-ph].

[61] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, “GCA in 2d,” arXiv:0912.1090 [hep-th];

I. Mandal, “Supersymmetric Extension of GCA in 2d,” arXiv:1003.0209 [hep-th].

[62] Carlos Leiva and Mikhail S. Plyushchay, ”Conformal symmetry of relativistic and nonrelativistic systems and Ads / CFT correspondence,” Ann.Phys.307:372-391,2003 [hep-th/0301244].

[63] K. Hotta, T. Kubota and T. Nishinaka, “Galilean Conformal Algebra in Two Dimensions and Cosmological Topologically Massive Gravity,” arXiv:1003.1203 [hep-th].

[64] J. Lukierski, P. C. Stichel and W. J. Zakrzewski, “Galilean-invariant (2+1)-dimensional models with a Chern-Simons-like term and D = 2 noncommutative geometry,” Annals Phys. **260**, 224 (1997) [arXiv:hep-th/9612017].

[65] Peter A. Horvathy and Mikhail S. Plyushchay, ”Nonrelativistic anyons, exotic Galilean symmetry and noncommutative plane,” JHEP 0206:033,2002 [hep-th/0201228];

Peter A. Horvathy and Mikhail S. Plyushchay; ”Anyon wave equations and the noncommutative plane,” Phys.Lett.B595:547-555,2004 [hep-th/0404137];

M.S. Plyushchay, ”Relativistic particle with torsion, Majorana equation and fractional spin,” Phys.Lett.B262:71-78,1991; M.S. Plyushchay, ”The Model of relativistic particle with torsion,” Nucl.Phys.B362:54-72,1991.

[66] D. T. Son, “Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry,” Phys. Rev. D **78**, 046003 (2008) [arXiv:0804.3972 [hep-th]];

K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,” Phys. Rev. Lett. **101**, 061601 (2008) [arXiv:0804.4053 [hep-th]];

C. P. Herzog, M. Rangamani and S. F. Ross, ”Heating up Galilean holography,” JHEP 0811, 080 (2008) [arXiv:0807.1099 [hep-th]];

J. Maldacena, D. Martelli and Y. Tachikawa, “Comments on string theory backgrounds with non-relativistic conformal symmetry,” JHEP **0810**, 072 (2008) [arXiv:0807.1100 [hep-th]];

A. Adams, K. Balasubramanian and J. McGreevy, “Hot Spacetimes for Cold Atoms,” JHEP **0811**, 059 (2008) [arXiv:0807.1111 [hep-th]];

A. Volovich and C. Wen, “Correlation Functions in Non-Relativistic Holography,” JHEP **0905**, 087 (2009) [arXiv:0903.2455 [hep-th]];

M. Taylor, “Non-relativistic holography,” arXiv:0812.0530 [hep-th].

[67] H. P. Kunzle, “Covariant Newtonian Limit of Lorentz Space-Times,” General Relativity and Gravitation, Vol. 7, No. 5 (1976), pp.445-457

[68] Landau and Lifshitz, “Fluid Mechanics”, Pergamon Press (1959), Chapter 2

[69] M. Rangamani, S. F. Ross, D. T. Son and E. G. Thompson, “Conformal non-relativistic hydrodynamics from gravity,” JHEP **0901**, 075 (2009) [arXiv:0811.2049 [hep-th]];

C. Eling, I. Fouxon and Y. Oz, ”The Incompressible Navier-Stokes Equations From Membrane Dynamics,” arXiv:0905.3638 [hep-th]

[70] S. Bhattacharyya, S. Minwalla and S. R. Wadia, “The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity,” arXiv:0810.1545 [hep-th];

I. Fouxon and Y. Oz, “CFT Hydrodynamics: Symmetries, Exact Solutions and Gravity,” JHEP **0903**, 120 (2009) [arXiv:0812.1266 [hep-th]];

P. A. Horvthy, P. M. Zhang, ”Non-relativistic conformal symmetries in fluid mechanics,” [arXiv:0906.3594[physics.flu-dyn]]

[71] M. Hassaine and P. A. Horvathy, ”Field-dependent symmetries of a non-relativistic fluid model,” Annals Phys. 282, 218 (2000) [arXiv:math-ph/9904022];

M. Hassaine and P. A. Horvathy, ”Symmetries of fluid dynamics with polytropic exponent,” Phys. Lett. A 279, 215 (2001) [arXiv:hep-th/0009092].

[72] Francisco Correa, Vit Jakubsky, Mikhail S. Plyushchay, "Aharonov-Bohm effect on AdS(2) and nonlinear supersymmetry of reflectionless Poschl-Teller system," Annals Phys.324:1078-1094,2009,. [arXiv:0809.2854];

Pedro D. Alvarez, Jose L. Cortes, Peter A. Horvathy, Mikhail S. Plyushchay, "Super-extended noncommutative Landau problem and conformal symmetry," JHEP 0903:034,2009,. [arXiv:0901.1021];

M. Sakaguchi, "Super Galilean conformal algebra in AdS/CFT," arXiv:0905.0188 [hep-th];

A. Bagchi and I. Mandal, "Supersymmetric Extension of Galilean Conformal Algebras," arXiv:0905.0580 [hep-th].

J. A. de Azcarraga, J. Lukierski, "Galilean Superconformal Symmetries" arxiv:0905.0141[math-ph]

[73] N. A. Nekrasov, "Seiberg-Witten Prepotential From Instanton Counting," Adv. Theor. Math. Phys. **7**, 831 (2004) [arXiv:hep-th/0206161].

[74] D. Bak, A. Karch and L. G. Yaffe, "Debye screening in strongly coupled N=4 supersymmetric Yang-Mills plasma," JHEP **0708**, 049 (2007) [arXiv:0705.0994 [hep-th]].