Classical and Quantum Corrections to Entropy of Extremal Black Hole

By

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Certificate

This is to certify that the Ph.D. thesis titled "Classical and Quantum Corrections to Entropy of Extremal Black Hole" submitted by Rajesh Kumar Gupta is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Date: **Professor Ashoke Sen**

Thesis Adviser

Declaration

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

This work was done under guidance of Professor Ashoke Sen, at Harish Chandra Research Institute, Allahabad.

Date: **Rajesh Kumar Gupta**

Ph.D. Candidate

To My Family....

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Synopsis

The low energy limit of string theory is supergravity coupled to various matter fields. These theories have black hole solutions. Black holes are solutions of Einstein's equation of motion with the properties that they are surrounded by a hypothetical surface - known as event horizon- such that no object inside the event horizon can escape the black hole. However in quantum theory black hole behaves as a black body with finite temperature and has entropy. In the Einstein's gravity, this entropy known as the Bekenstein-Hawking entropy S_{BH} is given by the expression

$$S_{BH} = \frac{A}{4G_N} \tag{1}$$

Here A is the area of the event horizon and G_N is Newton's gravitational constant.
One of the success of string theory has been an explanation of the Bekenstein-Hawking entropy of a class of supersymmetric extremal black hole in terms of degeneracy of microscopic quantum states (d_{micro}). These extremal black holes are special class of black holes which have zero temperature and hence do not radiate and are usually stable. Technically extremal black holes are defined to be those whose near horizon geometries have the form $AdS_2 \times K$, where K is some compact space. The initial comparison between S_{BH} and $S_{micro}(= \ln d_{micro})$ was done in the limit of large charges. In the limit of large charges we can work with two derivative action in full string effective action. On the microscopic side we can use the asymptotic formula for d_{micro} for large charges instead of having to compute it exactly. However it is clearly of interest to know if the correspondence between the black hole entropy and the microscopic entropy extends beyond the large charge limit. In order to address this problem we need to open two fronts. First of all we need to compute the degeneracy of states of black holes to greater accuracy so that we can compute corrections to S_{micro}. The other front involves understanding how higher derivative corrections/ string loop corrections affect the black hole entropy. Wald's formalism gives a clear prescription for calculating the effect of tree level higher derivative corrections on the black hole entropy, and for extremal black holes this leads to the entropy function formalism. However in order to implement it we need to know the higher

derivative terms in the action. Inclusion of quantum corrections into the computation of black hole entropy is challenging both conceptually and technically.

My research projects are focussed on the macroscopic side.

For some extremal black holes in string theory the AdS_2 component of the near horizon geometry, together with an internal circle, describes a locally AdS_3 space. More accurately the near horizon geometry of these extremal black holes correspond to that of extremal BTZ black holes. The BTZ solution describes a rotating black hole in three dimensional theory of (super-)gravity with negative cosmological constant. A general three dimensional theory of pure (super-)gravity with arbitrary higher derivative terms without matter fields admits a field redefinition that reduces the action to the standard (super-)gravity action whose gravitational part contains sum of three terms, Einstein-Hilbert term, a cosmological constant term and the Chern-Simons term. We proved that this is true even in presence of matter fields. After field redefinition and consistent truncation (the scalar fields set to constant and the tensor fields set to zero) the action reduces to standard (super-)gravity action. The parameters labelling the truncated action are the cosmological constant and the coefficient of the Chern-Simons term. Of them the coefficient of the Chern-Simons term does not change under the field redefinition but the cosmological constant term is modified and can be calculated explicitly. The corrected central charge in the CFT at the boundary of AdS_3 and hence the corrected entropy of the BTZ black hole is determined in terms of coefficients of Chern-Simons term and the modified cosmological constant.

Since extremal black holes have an AdS_2 factor in their near horizon geometry, one expects that the underlying quantum gravity theory in this background will have a dual description in terms of a conformal quantum mechanics (CQM) living at the boundary of AdS_2. This CQM has only ground states parameterised by the charge of the black hole and there are no excited states. It has been conjectured that the logarithm of the ground state degeneracy of this dual CQM in a fixed charge sector should be taken as the definition of the quantum corrected macroscopic entropy S_{macro} of extremal black holes. Quantitatively this proposal states that quantum degeneracy associated with horizon degrees of freedom of the black hole is given as the finite part of the partition function of string theory on AdS_2. The relation is given as

$$e^{S_{macro}(\vec{q})} = d(\vec{q})_{macro} = \left\langle exp\left[-iq_i \oint d\theta A_\theta^{(i)}\right]\right\rangle_{AdS_2}^{finite} \tag{2}$$

Here $\vec{q}$ are the charges carried by the black hole, $A^{(i)}$ are the gauge field and the path integral is over all fields in string theory with the boundary condition that they asymptote to near horizon geometry of the black hole containing an AdS_2 factor.

We check this proposal in the case of supersymmetric extremal BTZ black hole where we have independent definition of the entropy. The BTZ black hole is identified with states in the dual CFT living at the boundary of the AdS_3. The entropy of this black hole is given as logarithm of degeneracy of the corresponding states in the CFT. In order to compare the proposal with the above definition of entropy, one has to find the relation between CQM dual to AdS_2 and the CFT dual to AdS_3. One finds that the CQM dual to gravity in AdS_2 is described by the chiral half of the (1+1) dimensional CFT dual to gravity in AdS_3. In other words the states of the CQM living on the boundary of AdS_2 are described by the $\bar{L}_0 = 0$ states of the (1+1) dimensional CFT living on the boundary of AdS_3. In fact the degeneracy of the ground states of CQM carrying a given charge q is identified with the degeneracy of the states of the CFT which have $\bar{L}_0 = 0$ and $L_0 - \bar{L}_0 = q$. The later states appear in the definition of entropy of the black hole via AdS_3/CFT_2 correspondence. Hence the two definition of the entropy agree.

In order to compare macroscopic the quantum degeneracy computed from equa.(2) with the microscopic degeneracy, which is known for some of the supersymmetric black holes, one needs to perform the above path integral explicitly. However this is technically a very difficult problem. In the next project, instead of performing the path integral, we simplify this using the supersymmetry of the near horizon geometry. The isometry supergroup of the near horizon geometry has a factor $SU(1,1|2)$. Using supersymmetry and localization techniques we showed that the path integral could receives non-vanishing contribution only from a special class of field configurations which preserve a particular subgroup (we call it H_1) of $SU(1,1|2)$. We identify this subgroup and showed that path integral around such field configuration reduces to integration over H_1 invariant slice passing through the field configuration. This analysis is useful for finding the saddle points, i.e. classical string field configuration which could give non-perturbative corrections to the quantum entropy. We also find out some examples of such field configurations.

List of Publications

1. Duality covariant variables for STU-model in presence of non-holomorphic corrections,
   Authors: S. Banerjee and R. K. Gupta,
   arXiv:0905.2700 [hep-th]

2. Supersymmetry, Localization and Quantum Entropy Function,
   Authors:N. Banerjee, S. Banerjee, R. K. Gupta, I. Mandal and A. Sen,
   arXiv:0905.2686 [hep-th],
   Published in JHEP **1002**, 091 (2010)

3. On the universal hydrodynamics of strongly coupled CFTs with gravity duals,
   Authors:R. K. Gupta and A. Mukhopadhyay,
   arXiv:0810.4851 [hep-th],
   Published in JHEP **0903**, 067 (2009)

4. Ads(3)/CFT(2) to Ads(2)/CFT(1),
   Authors: R.K.Gupta and A.Sen,
   arXiv:0806.0053 [hep-th],
   Published in JHEP **0904**, 034 (2009)

5. Consistent Truncation to Three Dimensional (Super-)gravity,
   Authors:R.K.Gupta and A.Sen,
   arXiv:0710.4177 [hep-th],
   Published in JHEP **0803**, 015 (2008)

Contents

Part I

Introduction

Chapter 1

Introduction

1.1 Black Holes In General Relativity

Black holes are fascinating area of research both in theoretical physics and observational astrophysics. From experimental point of view it poses a challenging observational problem to find black holes in galaxies, however there are evidences of super-massive black holes at the center of most of the galaxies. From theoretical point of view it poses a challenging problem of unifying general relativity with quantum mechanics. Thus black holes are excellent theoretical laboratory for understanding some features of quantum gravity. The goal of the present thesis is to explore few aspects of the quantum theory of the black holes.

General theory of relativity, formulated by Einstein, is a classical theory of gravity according to which the gravity is a manifestation of the curvature of space time. According to Einstein, the gravitational field is not a new field but correspond to deviation of spacetime geometry from that of flat space time. The geometry of a spacetime is described in terms of metric or line element between two points.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

Here $\mu, \nu = 1, .., 4$. The deviation of the spacetime metric from flatness, which accounts for the physical effects usually ascribed to gravity, is determined in terms of Riemann curvature tensor $R_{\mu\nu\rho\sigma}$. The Riemann tensor $R_{\mu\nu\rho\sigma}$ is given in terms of Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ as

$$R^\mu_{\nu\rho\lambda} = \partial_\rho \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\tau\rho} \Gamma^\tau_{\nu\lambda} - \Gamma^\mu_{\tau\lambda} \Gamma^\tau_{\nu\rho} \tag{1.2}$$

The Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are given as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) \tag{1.3}$$

Furthermore the curvature of the spacetime is related to the energy-momentum tensor of the matter in the spacetime via Einstein's equation given as

$$R_{\mu\nu} - \frac{1}{2}(R - 2\Lambda)g_{\mu\nu} = -8\pi T_{\mu\nu} \tag{1.4}$$

Here Λ and $T_{\mu\nu}$ are cosmological constant and energy-momentum tensor of the matter respectively[1]. The Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are given as

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}, \quad R = g^{\mu\nu}R_{\mu\nu} \tag{1.5}$$

The Einstein's equation (1.4) is obtained as the Euler-Lagrange equation of motion from the action

$$S = -\frac{1}{16\pi}\int d^4x\sqrt{-g}(R - 2\Lambda) + S_{matter} \tag{1.6}$$

Here S_{matter} is the matter action.
Classically black holes are solutions of Einstein's equations with special properties. They have a hypothetical surface - known as the event horizon - surrounding them such that no object inside the event horizon can escape the black hole. One of the simplest black hole solution in four dimension is the Schwarzschild solution. The Schwarzschild solution is a static, spherically symmetric solution of vacuum Einstein equation (1.4) (with $T_{\mu\nu} = 0$) which describes the geometry outside the spherically symmetric energy-momentum distribution. The solution for mass M is given as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{1.7}$$

There exist more general black hole solutions which in addition carry charge and angular momentum. In Einstein-Maxwell theory the most general stationary black hole solution is the Kerr-Newmann black hole, which is uniquely characterized by its mass, charge and angular momentum.

1.2 Laws of Black Hole Mechanics

One of the most remarkable results of black hole physics is that one can derive a set of laws, called the laws of black hole mechanics, which have the same structure as the laws of thermodynamics.
Before stating the laws of the black hole mechanics, we first give few definitions [1, 2, 3, 4, 5].

[1] In this report we will put all fundamental constants, G_N, c and $\hbar$ to 1.

Null Hypersurface: Let $S(x)$ be a smooth function of the spacetime coordinates x^μ and consider a family of hypersurfaces $S = constant$. The vector fields normal to the hypersurface are

$$l = f(x) g^{\mu\nu} \partial_\nu S(x) \frac{\partial}{\partial x^\mu} \tag{1.8}$$

where $f(x)$ is some arbitrary non-zero function. If $l^2 = 0$ for a particular hypersurface, $\mathcal{N}$, in the family, then $\mathcal{N}$ is said to be a null hypersurface.
A tangent vector field t on $\mathcal{N}$ is one for which $t \cdot l = 0$. Since for a null hypersurface $\mathcal{N}$, $l \cdot l = 0$, l is itself a tangent vector field i.e.

$$l^\mu = \frac{dx^\mu(\lambda)}{d\lambda} \tag{1.9}$$

for some curve $x^\mu(\lambda)$ in $\mathcal{N}$.
It is simple to prove that the curves $x^\mu(\lambda)$ generated by the vector field l are geodesics. These are called null geodesics and are the generators of the null surface $\mathcal{N}$.
Killing Horizon: A null hypersurface $\mathcal{N}$ is a Killing horizon of a Killing vector field ξ if, on $\mathcal{N}$, ξ is normal to $\mathcal{N}$. Then on $\mathcal{N}$

$$\xi = f(x) l \tag{1.10}$$

for some function $f(x)$ and l is normal to $\mathcal{N}$.
For killing horizon one defines the surface gravity k_s[2] as

$$\xi^\mu D_\mu \xi^\nu = k_s \xi^\nu, \quad on \quad \mathcal{N} \tag{1.11}$$

were D_μ is covariant derivative and $k_s = \xi \cdot \partial \ln |f(x)|$ on $\mathcal{N}$.
Then by simple calculation one can show that on the killing horizon $\mathcal{N}$ the surface gravity k_s is constant on the orbits of the killing vector field ξ generating $\mathcal{N}$.
Event horizons of black holes are null hypersurfaces. In Einstein's theory of gravity all event horizons of stationary black holes are killing horizon. From here onwards we will assume that the event horizon of a stationary asymptotic flat black hole in arbitrary higher derivative theory of gravity is a killing horizon. This is trivially true for static black hole since in this case there exist a time like hypersurface orthonormal killing vector field.
Let us consider the case of non-degenerate killing horizons ($k_s \neq 0$). We choose a coordinate on $\mathcal{N}$ such that

$$\xi = \frac{\partial}{\partial \alpha} \tag{1.12}$$

with group parameter α as one coordinate. Consider the orbit $\alpha = \alpha(\lambda)$ with affine parameter λ generated by ξ. Then

$$\xi|_{orbit} = \frac{\partial \lambda}{\partial \alpha} \frac{\partial}{\partial \lambda} = f l \tag{1.13}$$

[2]In defining surface gravity we choose standard normalisation for ξ at infinity, e.g. for time translation killing field ξ, in asymptotic flat spacetime, we choose normalisation such that $\xi^2 \to -1$ as $r \to \infty$.

Here $f = \frac{\partial\lambda}{\partial\alpha}$ and $l = \frac{\partial}{\partial\lambda}$.
Now we have

$$\frac{\partial \ln |f|}{\partial\alpha} = k_s \tag{1.14}$$

Solving this we get

$$f = \frac{\partial\lambda}{\partial\alpha} = \pm f_0 e^{k_s\alpha} \tag{1.15}$$

Since there is a freedom to shift α by a constant, we can choose $f_0 = k_s$. Hence we get

$$\lambda = \pm e^{k_s\alpha} + constant \tag{1.16}$$

We can choose the constant to be zero. Hence when the group parameter α ranges from $-\infty$ to $+\infty$, λ is either > 0 or < 0. Thus ξ generates two null surface $\mathcal{N}_A$ and $\mathcal{N}_B$ intersecting at $\lambda = 0$ hypersurface Σ corresponding to fixed point of ξ. The killing horizon $\mathcal{N}$ which is union of two null hypersurface $\mathcal{N}_A$ and $\mathcal{N}_B$ is called bifurcate killing horizon and Σ $(\xi|_\Sigma = 0)$ is called bifurcation surface.
As it is evident, in degenerate case $(k_s = 0)$ there are no bifurcate killing horizon and bifurcation surface.

The fact that the entropy S of a stationary black hole with bifurcate killing horizon is 2π times the Noether charge associated with the horizon killing field, normalised so as to have unit surface gravity, was first proved in [6, 7]. The expression for the entropy was derived for a general, classical theory of gravity in d-dimension arising from diffeomorphism invariant Lagrangian with arbitrary matter fields. It is given as

$$S = 2\pi \int_\Sigma \tilde{Q} \tag{1.17}$$

where Σ is the bifurcation surface where killing field $\xi = 0$ and $\tilde{Q}$ is Noether charge associated with killing field $\tilde{\xi} = k_s^{-1}\xi$.
In 4-dimension, for a Lagrangian which does not depend on derivatives of the Riemann tensor, the entropy is given by

$$S = 2\pi \int_\Sigma \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu}\varepsilon_{\rho\sigma}\sqrt{h}dA \tag{1.18}$$

where $\varepsilon^{\mu\nu}$ is binormal[3] of Σ.
As an example, consider a 4-dimensional theory described by Lagrangian $\mathcal{L} = -\frac{1}{16\pi}R$. We consider a static and spherically symmetric metric of the form

$$ds^2 = -e^{2g(r)}dt^2 + e^{2f(r)}(dr^2 + r^2 d\Omega^2) \tag{1.19}$$

[3] In general expression for the Noether charge one of the binormal arises through the relation $D_\mu\xi_\nu = k_s\varepsilon_{\mu\nu}$ on the bifurcation surface Σ.

For this metric the binormal takes the form $\varepsilon_{tr} = -\varepsilon_{rt} = e^{g(r)+f(r)}$. We also have

$$8\pi \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu}\varepsilon_{\rho\sigma} = -\frac{1}{2} g^{\mu[\rho} g^{\sigma]\nu} \varepsilon_{\mu\nu}\varepsilon_{\rho\sigma} = -(\varepsilon_{tr})^2 g^{tt} g^{rr} = 1 \tag{1.20}$$

And the entropy is

$$S = \frac{2\pi}{8\pi} \int_{\Sigma} \sqrt{h} d\Omega = \frac{A}{4} \tag{1.21}$$

where A is the area of the event horizon at $r = 0$.
Now consider a stationary rotating black hole with bifurcate killing horizon. Let the killing field which vanishes on the bifurcation surface be

$$\xi = \xi_t + \Omega \xi_\phi \tag{1.22}$$

where ξ_t and ξ_ϕ are timelike and axial killing field respectively. Then the surface charges associated with these killing field at asymptotic infinity are the mass (M) and the angular momentum (J) of the black hole and their expression in terms of killing field are similar to Komar's formula of mass and angular momentum.
We shall now state the laws of black hole mechanics [8, 9, 10, 3].
Zeroth Law: The zeroth law states that the surface gravity k_s of a stationary black hole is constant over the event horizon,

$$k_s = constant \tag{1.23}$$

First Law: The first law, in general theory of gravity, states[4] that if one considers two infinitesimally close stationary rotating black hole solutions then the change δM of the mass is related to change of entropy, δS and of the angular momentum, δJ via:

$$\delta M = \frac{k_s}{2\pi} \delta S + \Omega \delta J \tag{1.24}$$

Here Ω, defined via (1.22), is the angular velocity at the horizon. Second Law: If $T_{\mu\nu}$ satisfies the weak energy condition and assuming that the cosmic censorship conjecture is true then the total area of the event horizon of black holes in a non-stationary process (e.g. collisions and fusion of black holes) in an asymptotically flat spacetime is a non-decreasing function of time.
The weak energy condition means that $T_{\mu\nu} u^\mu u^\nu \geqslant 0$ at any point in spacetime for any time like vector field u^μ. The cosmic censorship conjecture states that naked singularities can not form from a gravitational collapse in an asymptotically flat spacetime which was non singular on some initial spacelike hypersurface.
The zeroth law of the black hole mechanics resembles the zeroth law of thermodynamics,

[4]In Einstein's theory of gravity it takes the form $\delta M = \frac{k_s}{2\pi} \delta A + \Omega \delta J$, where A is the area of the event horizon.

which says that the temperature is constant in a thermodynamic equilibrium. The first law of black hole mechanics resembles the energy conservation and the second law resembles the thermodynamic law that the entropy in a given non-equilibrium process always increases. These resemblance led one to think that black hole has temperature and thermodynamic entropy which is proportional to surface gravity and area of the event horizon. However the black hole laws are a priori not linked to thermodynamics in any obvious way because they are derived using geometrical properties of event horizons and general covariance. The most obvious reason for not believing in a thermodynamic content is that a classical black hole is just black. It cannot radiate and therefore one should assign temperature zero to it, so that the interpretation of the surface gravity as temperature has no physical content.

This changes dramatically when taking into account quantum effects [11]. One can analyse black holes in the context of quantum field theory in curved backgrounds, where matter is described by quantum field theory while gravity enters as a classical background. In this framework it was discovered that black holes can emit radiation, called Hawking radiation, and the spectrum is Planckian with a temperature, the so-called Hawking temperature, which is indeed proportional to the surface gravity,

$$T_H = \frac{k_s}{2\pi} \tag{1.25}$$

This motivates one to take the laws of the black hole mechanics and the relation between area of the event horizon and the thermodynamic entropy more seriously. In fact one now interprets the laws of black hole mechanics as the laws of black hole thermodynamics. The universal expression for the thermodynamic entropy of the stationary black hole in Einstein's theory of gravity, which we also got in (1.21), is called Bekenstein-Hawking entropy.

$$S_{BH} = \frac{A}{4} \tag{1.26}$$

However in general theory of gravity with arbitrary higher derivative terms, the expression for the entropy deviates from the Bekenstein-Hawking area law. The entropy also deviate from the area law when we take into account the quantum gravity effects.

1.3 Black Holes in String Theory

String theory is a quantum theory of relativistic strings. The low energy limit of string theory gives rise to (super-)gravity coupled to other fields. As a result these theories typically have black hole solutions[5]. At present the string theory is the most prominent candidate for a quantum theory of gravity. Thus it is natural to expect that the string theory is a framework for studying classical and quantum properties of black holes.

[5] For nice review on the black holes in string theory look at [12, 13].

In most physical system the thermodynamic entropy has a statistical description in term of microstates which correspond to same macrostate. Thus one can ask: in string theory can we understand the thermodynamic entropy (1.26) from statistical viewpoint i.e. as logarithm of the number of quantum states associated with the black hole? Although one does not yet have a complete answer to this question, for a special class of black holes in string theory, known as extremal black holes, this question has been answered in the affirmative. These black holes have zero temperature and hence do not Hawking radiate and are usually stable. Technically an extremal black hole is one which has near horizon geometry of the form $AdS_2 \times K$, for some compact space K. Often, but not always, extremal black holes are also invariant under certain number of supersymmetry transformations. In that case they are called BPS black holes. In string theory the BPS black holes can be realised in terms of various configuration of solitonic objects like D-branes, fundamental strings etc. The charges carried by the black hole are realised in terms of quantum numbers, like number of D-brane, Kaluza-Klein momenta, winding number etc, carried by the configuration of solitonic objects. This in turn allows us to calculate the degeneracy of such states at weak coupling where gravitational back reaction of the system can be ignored. Supersymmetry allows us to continue the result to strong coupling where gravitational backreaction becomes important and the system can be described as a black hole. In string theory one finds that for a wide class of extremal BPS black holes we have, in the limit large charges

$$S_{BH}(Q) = S_{micro}(Q) \tag{1.27}$$

where S_{micro} is defined as

$$S_{micro}(Q) = \ln d_{micro}(Q) \tag{1.28}$$

where $d_{micro}(Q)$ is the degeneracy of BPS states in the theory carrying same set of charges Q.

The initial comparison between S_{BH} and S_{micro} was carried out in the limit of large charges. Typically in this limit the horizon size is large so that the curvature and other field strengths at the horizon are small and hence we can calculate the entropy via (1.26) without worrying about the higher derivative corrections to the effective action of string theory. In this limit, S_{micro} also get simplified and often it is given in terms of the degeneracy of a state in $(1+1)$ dimensional CFT with the spatial coordinate compactified on a circle [14, 15]. In this case the BPS black hole with large charges corresponds to state in the CFT with large L_0(or $\bar{L}_0$) eigenvalue h_L(or h_R) and zero $\bar{L}_0$(or L_0) eigenvalue. The degeneracy of such states (e.g. for $h_R = 0$ case) are given in terms of Cardy formula

$$d_{micro} \simeq exp[2\pi\sqrt{\frac{c_L h_L}{6}}] \tag{1.29}$$

where c_L is the left moving central charge.

One of the success of the string theory is that these two completely different computations, -one for $S_{BH}(Q)$ from gravity side and the other for $S_{micro}(Q)$ from CFT side,- give the same answer. Given this success, it is natural to carry out this comparison to finer details. When we move away from the large charge limit, the curvature and other field strengths at the horizon are no longer negligible. In string theory one finds that the low energy effective action contains higher curvature terms. In fact at tree level it contains an Einstein-Hilbert term together with an infinite series of higher curvature terms that are suppressed by powers of α', so that they are subleading at low energy. Thus, string theory deviates from Einstein gravity already at the classical level. On top of these terms the effective action also gets contribution from string loop corrections which involves powers of string coupling g_s. For a large but finite size black hole we expect the effect of these corrections at the horizon will be small but non-zero, giving rise to small modifications of the black hole entropy. On the other hand for finite but large charges the statistical entropy computed from the Cardy formula will also receive corrections which are suppressed by inverse powers of charges. Thus it would be natural to ask: Does the agreement between $S_{BH}(Q)$ and $S_{micro}(Q)$ continue to hold even after taking into account the effects of higher derivative corrections on the black hole side, and deviation from the Cardy formula on the statistical side?

In order to address this problem we need to open two fronts. First of all we need to compute the degeneracy of states of black holes to greater accuracy so that we can compute corrections to S_{micro} given in (1.28). Conceptually this is a straightforward problem since d_{micro} is a well defined number, especially in the case of BPS extremal black holes, since the BPS property gives a clean separation between the spectrum of BPS and non-BPS states. Technically, counting of d_{micro} is a challenging problem, although this has now been achieved for a class of black holes in $N = 4$ supersymmetric string theories [16, 17, 18]. Significant progress has also been made for half BPS black holes in a class of $N = 2$ supersymmetric string theories [19]. The other front involves understanding how higher derivative corrections / string loop corrections affect the black hole entropy. Wald's formalism [6, 7] gives a clear prescription for calculating the effect of tree level higher derivative corrections on the black hole entropy, and for extremal black holes this leads to the entropy function formalism. Thus here there is no conceptual problem, but in order to implement it we need to know the higher derivative terms in the action. Inclusion of quantum corrections into the computation of black hole entropy is more challenging both conceptually and technically.

1.4 AdS/CFT Correspondence

According to [20, 21, 22] string theory on AdS_{d+1} times a compact space is dual to a d dimensional conformal field theory (CFT) living on the boundary of AdS_{d+1}. According to this correspondence, in the approximation where the bulk fields are treated classically, the boundary fields parameterizing the boundary conditions of the bulk fields are identified with the source for the dual operators and the classical supergravity partition function

acts as a generating function for the correlation function of the dual operator in the boundary CFT.
In the present thesis we will concentrate on $d = 1$ and 2 case. The $d = 1$ case will be dealt in detail in the next section. For $d = 2$ the AdS/CFT correspondence implies an equivalence between string theory on AdS_3 and CFT_2 living at the boundary of AdS_3. AdS_3 is a homogeneous space of constant negative curvature and it's isometry group is $SL(2, \mathbb{C}) \cong SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. In fact a long time ago Brown and Henneaux [23] made a remarkable observation that the asymptotic symmetry group of AdS_3 is generated by two copies of Virasoro algebra (whose global part is $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$) and therefore the states in the Hilbert space of quantum theory of gravity on AdS_3 must be in the representation of these Virasoro algebra and hence it must be a conformal field theory. They also computed the value of the central charge as $c_L = c_R = \frac{3}{2\sqrt{\Lambda}}$ where Λ is the cosmological constant. The vacuum of this CFT with $L_0 = \bar{L}_0 = 0$ correspond to empty AdS_3 and is invariant under only global part of the Virasoro algebra.
In three dimension the Weyl tensor vanishes and hence Riemann tensor can be expressed in terms of metric, Ricci tensor and Ricci scalar. As a result in case of pure gravity there are no gravitational waves (however it becomes nontrivial in presence of matter fields and higher derivative terms) but there are a black hole solutions, discovered by Bañados, Teitelboim, and Zanelli [24]. BTZ solution describe a rotating black hole solution and often appears as a factor in the near horizon geometry of higher dimensional black holes in string theory. For some extremal black holes in string theory the AdS_2 component of the near horizon geometry together with the an internal circle describes a locally AdS_3 space. Specifically the near horizon geometry of these extremal black hole correspond to that of extremal BTZ black hole. BTZ black hole corresponds to states in the CFT and has non zero L_0 and $\bar{L}_0$ which is determined in terms it's mass and angular momentum. The degeneracy of such states with large L_0 and $\bar{L}_0$ is given in terms of cardy formula and hence one can compute the entropy of such black hole.

1.5 Quantum Entropy of Extremal Black Hole

Wald's formula [6, 7] computes the entropy of a non-extremal black hole in a classical theory of gravity with arbitrary higher derivative terms coupled to arbitrary set of matter fields. It gives entropy in terms of field configurations near the horizon of the black hole. For extremal black holes, defined as the limit of a non-extremal black holes, Wald's formula reduces to a simple algorithm known as the entropy function formalism. We shall review this formalism below. Since the main ingredient of this analysis is the appearance of an AdS_2 factor in the near horizon geometry of extremal black holes, we shall begin by describing the origin of AdS_2 factor.

As an example consider a (3 + 1)-dim. Reissner-Nordstrom black hole. It is given by

$$ds^2 = -(1 - a/\rho)(1 - b/\rho)d\tau^2 + \frac{d\rho^2}{(1 - a/\rho)(1 - b/\rho)} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.30}$$

Here $(\tau, \rho, \theta, \phi)$ are the coordinates of space-time and a and b $(a > b)$ are two parameters labelling the position of the outer and inner horizon of the black hole respectively.
The extremal limit corresponds to $b \to a$. We take this limit keeping the coordinates θ, ϕ and

$$r = 2\left(\rho - \frac{a + b}{2}\right)/(a - b), \qquad t = (a - b)\tau/2a^2 \tag{1.31}$$

fixed.
In this limit the metric takes the form

$$ds^2 = a^2\left(-(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1}\right) + a^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.32}$$

This is the metric of $AdS_2 \times S^2$, with AdS_2 labelled by (r, t) and S^2 labelled by (θ, ϕ). As we mentioned, all known extremal black hole solutions have an AdS_2 factor in their near horizon geometry; furthermore the other near horizon field configurations remain invariant under the $SO(2, 1)$ isometry of AdS_2 . We shall take this as the definition of extremal black hole even in presence of higher derivative terms.
One can give a uniform treatment of all such extremal black holes by regarding the angular directions as part of compact coordinates. Thus we have an effective two dimensional theory of gravity coupled to (infinite number of) other fields obtained by compactifying the fundamental theory on the compact directions. Among them of particular importance are the set of massless fields like abelian gauge fields $A_\mu^{(i)}$ and a set of neutral scalar fields ϕ_s. Let $\mathcal{L}_0$ be the classical Lagrangian density and A be the classical action describing the dynamics of these massless fields

$$A[g_{\mu\nu}, \{A_\mu^{(i)}\}, \{\phi_s\}] = \int d^2x\sqrt{-g}\mathcal{L}_0 \tag{1.33}$$

The most general near horizon geometry consistent with the $SO(2, 1)$ isometries of AdS_2 is given by

$$ds^2 = v\left(-(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1}\right), \quad F_{rt}^{(i)} = e^i, \quad \phi_s = u_s \tag{1.34}$$

where $F_{\mu\nu}^{(i)} = \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)}$ are the gauge field strengths , v, $\{e^i\}$ and $\{u_s\}$ are constants. Note that there are no parameter explicitly labelling the magnetic charges; they are encoded in the component of the gauge field strengths along the compact directions and appear as the parameters labelling the two dimensional theory. Let us denote by $f(\vec{u}, v, \vec{e})$ the Lagrangian density $\sqrt{-g}\mathcal{L}_0$ evaluated for the near horizon geometry (1.34)

$$f(\vec{u}, v, \vec{e}) = \sqrt{-g}\mathcal{L}_0 = v\mathcal{L}_0 \tag{1.35}$$

Then the classical black hole entropy is given by

$$S_{BH} = 2\pi(e^i q_i - f(\vec{u}, v, \vec{e})) \tag{1.36}$$

evaluated at

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v} = 0, \quad \frac{\partial f}{\partial e^i} = q_i \tag{1.37}$$

The function $2\pi(e^i q_i - f(\vec{u}, v, \vec{e}))$ is called the classical entropy function.

We now make an analytic continuation of the solution (1.34) by defining a new coordinate

$$t = -i\theta, \qquad r = \cosh\eta \tag{1.38}$$

In these coordinates the solution (1.34) takes the form

$$ds^2 = v(d\eta^2 + \sinh^2\eta d\theta^2), \quad \phi_s = u_s, \quad F^i_{\theta\eta} = ie^i \sinh\eta \tag{1.39}$$

Also the action (1.33) becomes

$$A = -i\int d\eta d\theta \sqrt{g^E}\mathcal{L}_0 \tag{1.40}$$

In this coordinate the AdS_2 is a disk with boundary at $\eta = \infty$. We can regulate the volume of the AdS_2 by putting an upper cut-off η_{max} on η. The metric in non-singular at the origin $\eta = 0$ if we choose θ to have period 2π.
Integrating the field strength we can get the form of the gauge field (in $A^{(i)}_\eta = 0$ gauge)

$$A^{(i)}_\theta = -ie^i(\cosh\eta - 1) \tag{1.41}$$

The (-1)-factor inside the parenthesis is required in order to make the gauge field non-singular at the origin $\eta = 0$.

We can formally define the partition function of string theory in this background (1.39) as the path integral over all the string fields. In order to properly define this path integral we need to fix the boundary condition on various fields at $\eta = \eta_{max}$. Special care is needed to fix the boundary condition on the gauge fields. In AdS_{d+1} for general d the classical Maxwell equations for a gauge field near the boundary has two independent solutions. One of these represent the constant mode of the asymptotic gauge field and the other one measures the asymptotic electric field or equivalently the charge carried by the solution. Requiring the absence of singularity in the interior of AdS_{d+1} gives a relation between the two coefficients. Thus in defining the path integral over AdS_{d+1} we fix one of the coefficients and allow the other one to fluctuate. For $d \geq 3$ the constant mode of the gauge field is dominant near the boundary; hence it is natural to fix this and allow the mode measuring the charge to be determined dynamically in the classical limit and to fluctuate in the full quantum theory. However for $d = 1$ the mode that measures charge is the dominant one near the boundary; thus it is more natural to think of this as a parameter of the boundary CFT and let the constant mode of the gauge field be determined dynamically. This can be seen for example in (1.41) where the term proportional to $\cosh\eta$ measures the charge and the constant term in the expression for the gauge field is determined in terms of the electric field by requiring the gauge fields to be non-singular at the origin. Thus a more natural definition of the partition function on AdS_2 will be to fix the coefficient of the linear term in $\cosh\eta$ and allow the constant term to fluctuate. In general this is achieved by requiring that[25]

$$\lim_{r\to\infty}\frac{\delta A_{bulk}}{\delta F^{(i)}_{r\theta}} = iq_i, \tag{1.42}$$

Here A_{bulk} is the bulk part of the action iA and $r = \cosh\eta$. In this case under infinitesimal variation of the gauge field component $A^{(i)}_\theta$ we have

$$\delta A_{bulk} = (e.o.m) + iq_i \oint d\theta \delta A^{(i)}_\theta \tag{1.43}$$

In order to cancel this boundary term, we need to add a boundary term $-iq_i \oint d\theta A^{(i)}_\theta$ into the action iA. Similarly boundary conditions on the other fields are fixed in the standard manner, e.g. in the $g_{\eta\eta} = v$, $g_{\theta\eta} = 0$ gauge we freeze the mode of $g_{\theta\theta}$, proportional to $e^{2\eta}$ near the boundary, to $\frac{v}{4}$ and allow the mode independent of η to fluctuate. Appropriate boundary terms must be added to the action so that the variation of the action under arbitrary variation of the various fields, subject to the boundary conditions, vanishes when equations of motion are satisfied. With these suitable boundary conditions we introduce the quantity

$$Z_{AdS_2} = \left\langle \exp\left[-iq_i \oint d\theta\, A^{(i)}_\theta\right]\right\rangle_{AdS_2} \tag{1.44}$$

where $<>_{AdS_2}$ denotes the unnormalised path integral over various fields on AdS_2 with weight factor e^{iA}. Let us evaluate the partition function (1.44) in the classical limit. On-shell the bulk part of the action is given by

$$\begin{aligned} A_{bulk} &= -i\int d\eta d\theta \sqrt{g^E}\mathcal{L}_0 = -2\pi i v \int_0^{\eta_{max}} d\eta \sinh\eta \mathcal{L}_0 \\ &= -2\pi i v(\cosh\eta_{max} - 1)\mathcal{L}_0 = i(\cosh\eta_{max} - 1)(S_{BH} - 2\pi e^i q_i) \end{aligned} \tag{1.45}$$

Here we have used (1.36). This is however not the complete contribution to A ; we can get additional contribution from the boundary terms at $\eta = \eta_{max}$. To determine the form of the boundary contribution, we make a change of coordinates

$$\tilde\eta = \eta_{max} - \eta, \qquad \tilde\theta = \frac{1}{2}e^{\eta_{max}}\theta \tag{1.46}$$

In this coordinate the boundary has period

$$\beta = \pi e^{\eta_{max}} \tag{1.47}$$

Furthermore the metric and the gauge field strengths near the boundary take the form

$$\begin{aligned} ds^2 &= v[d\tilde{\eta}^2 + e^{-2\tilde{\eta}} d\tilde{\theta}^2] + \mathcal{O}(\beta^{-2}) \\ F^i_{\tilde{\eta}\tilde{\theta}} &= ie^i e^{-\tilde{\eta}} + \mathcal{O}(\beta^{-2}) \end{aligned} \tag{1.48}$$

And the Wilson loop at the boundary becomes

$$-iq_i \oint d\theta A^{(i)}_\theta = 2\pi e^i q_i - \beta e^i q_i + \mathcal{O}(\beta^{-1}) \tag{1.49}$$

Now the boundary term in the action is given by some local expression constructed from the metric, the gauge field strength and their derivatives integrated along the boundary. Due to translation symmetry along $\tilde{\theta}$, the integration along the boundary gives a factor of β multiplying the integrand. On the other hand the form of the solution given in (1.48) shows that the integrand is given by a β-independent term plus a contribution of order β^{-2} . Thus up to correction terms of order β^{-1} , the boundary contribution must be proportional to the length β of the boundary circle. Together with (1.45), the complete classical partition function (1.44) is

$$Z_{AdS_2} = \exp[S_{BH} + \beta K(q) + \mathcal{O}(\beta^{-1})] \tag{1.50}$$

Since the term $K(q)$ contains the boundary terms and hence ambiguous but unambiguous and the finite part of the partition function is given by

$$Z^{finite}_{AdS_2} = e^{S_{BH}} \tag{1.51}$$

Thus in the classical limit the finite part of the partition function (1.44) reduces to exponential of Wald entropy.

In full quantum theory we hope to represent the effect of this path integral by a modification of the original Lagrangian density to an appropriate effective Lagrangian density. In flat spacetime the one particle irreducible action is non-local and hence causes an obstruction to express the action as an integral over a local Lagrangian density. However the situation in AdS_2 background is better since the non vanishing curvature of AdS_2 puts a natural infrared cut-off. Thus the quantum Z_{AdS_2} is expected to be given by an expression similar to (1.50) but with quantum corrected entropy (determined in terms of effective Lagrangian density).

We will now give an interpretation of the entropy S_{BH} appearing in (1.50) in terms of an appropriate conformal quantum mechanics living at the boundary of AdS_2. Since Z_{AdS_2} is the partition function of the theory on AdS_2, according to AdS/CFT correspondence[20, 21, 22] one expects this to be the partition function of the dual quantum mechanics living at the boundary $\eta = \eta_{max}$. Thus if H denotes the Hamiltonian generating $\tilde{\theta}$ translation in the dual quantum mechanics living at $\eta = \eta_{max}$ then, according to AdS/CFT correspondence,

$$Z_{AdS_2} = Tr\left(e^{-\beta H}\right) \tag{1.52}$$

In the $\eta_{max} \to \infty$ which corresponds to $\beta \to \infty$ limit, the right hand side of (1.52) will get dominant contribution from the ground state. If $d(q)$ is the degeneracy of the ground states, then in this limit we get

$$Z_{AdS_2} = e^{-\beta E} d(q) \tag{1.53}$$

Here E is the ground state energy.
Comparing with (1.50) and identifying $E = -K(q)$.

$$d(q) = e^{S_{BH}(q)} = \left\langle \exp\left[-iq_i \oint d\theta\, A_\theta^{(i)}\right]\right\rangle_{AdS_2}^{finite}, \tag{1.54}$$

The right hand side of (1.54) is called quantum entropy function.

In case of supersymmetric extremal black hole one expects to make a precise comparison between macroscopic and microscopic entropy. Let $d_{micro}(\vec{q})$ denotes the degeneracy of the BPS microstates carrying total charge $\vec{q}$ in string theory. Then on general grounds one expects the following relation between $d_{micro}(\vec{q})$ and the macroscopic quantities associated with the black hole:

$$d_{micro}(\vec{q}) = d_{macro}(\vec{q}) \tag{1.55}$$

where

$$d_{macro}(\vec{q}) = \sum_n \sum_{\substack{\{\vec{q}_i\},\vec{q}_{hair} \\ \sum_{i=1}^n \vec{q}_i + \vec{q}_{hair} = \vec{q}}} \left\{\prod_{i=1}^n d_{hor}(\vec{q}_i)\right\} d_{hair}(\vec{q}_{hair}; \{\vec{q}_i\}), \tag{1.56}$$

The n-th term on the right hand side of (1.56) represents the contribution to the degeneracy from n-centered black hole configuration and $d_{hair}(\vec{q}_{hair}; \{\vec{q}_i\})$ is the degeneracy of the hair degrees of freedom [26]. $d_{hor}(\vec{q}_i)$ is the degeneracy associated with the horizon degrees of freedom given by (1.54).
One issue which will be not dealt here in detail but need to mention is related to infrared divergence and it's regularisation. In proving that the quantum entropy function (1.54) in classical limit reduces to exponential of Wald entropy we have used, in order to regularise the infrared divergence arises because of infinite volume of AdS_2, a specific infrared cutoff in which we put $\eta = \eta_{max}$. However in [25] it was shown that the right hand side of (1.54) is insensitive to any choice of infrared cutoff in full quantum theory.

1.6 Plan of the report

This thesis is divided into two parts. In the first part we describe the field redefinition and consistent truncation in general three dimensional theory of (super-)gravity and compute the classical correction to BTZ black hole entropy. In the second part we describe the quantum correction to entropy of extremal black hole. The plan of the chapters are as follows:

As we mentioned for some extremal black hole the near horizon geometry correspond to that of extremal BTZ black hole. The entropy of the BTZ black hole are given in terms of central charges. In chapter 2 we describe field redefinition and consistent truncation in three dimensional general higher derivative theory of (super-)gravity coupled to arbitrary set of matter fields. After field redefinition and consistent truncation the action reduces to standard (super-)gravity action which is sum of three terms, Einstein-Hilbert term, a cosmological constant term and the Chern-Simons term. We describe the procedure of computing the parameters of the final truncated theory. We apply these procedure in case of 3-dimensional theory obtained by dimensional reduction of 5-dimensional four derivative theory of $\mathcal{N} = 2$ supergravity. We then compute cosmological constant and the central charges of the boundary CFT which will give higher derivative correction to entropy of the BTZ black hole .

In chapter 3 we describe a check of quantum entropy function proposal. We consider extremal supersymmetric charged BTZ black hole in 3-dimension for which there exist independent definition of entropy based on AdS_3/CFT_2 correspondence. We compare this definition of entropy with that of quantum entropy function proposal. We will show that these two different definition of entropy agrees.

In chapter 4 we tried to find the field configuration which could give non vanishing contribution to quantum entropy function. Using supersymmetry and localization techniques we show that the path integral could receive non-vanishing contribution only from a special class of field configurations which preserve a particular subgroup (we call it H_1) of $SU(1,1|2)$. We identify this subgroup and showed that path integral around such field configuration reduces to integration over H_1 invariant slice passing through the field configuration. This analysis is useful to find the saddle points, i.e. classical string field configuration which could give non-perturbative corrections to the quantum entropy. We also find out some examples of such field configurations.

Part II

Corrections to Black Hole Entropy

Chapter 2

Consistent truncation to three dimensional (super-)gravity

2.1 Introduction

Three dimensional (super-)gravity with negative cosmological constant has played an important role in the study of black holes in string theory. At first sight classical three dimensional theory of gravity described by Einstein-Hilbert action with negative cosmological constant seems to be trivial. The reason is that in three dimension, gravity does not have any physical degree of freedom and hence there are no gravitational waves and any two solution is locally equivalent to AdS_3 [27]. However in three dimension there are black hole solutions, discovered by Bañados, Teitelboim, and Zanelli [24], which make it a much more exciting problem and worth to study. BTZ solution describe a rotating black hole solution and often appears as a factor in the near horizon geometry of higher dimensional black holes in string theory. For some extremal black holes in string theory the AdS_2 component of the near horizon geometry together with an internal circle describes a locally AdS_3 space. More specifically the near horizon geometry of these extremal black holes correspond to that of extremal BTZ black hole [15]. For this reason BTZ solution has provided us with a useful tool for relating black hole entropy to the degeneracy of microstates of the black hole, both in three dimensional theory of gravity and also in string theory [28, 15].
A general BTZ black hole in general higher derivative theory of gravity with scalar curvature $-6/l^2$ is given by the metric

$$ds^2 = -\frac{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}{l^2\rho^2}d\tau^2 + \frac{l^2\rho^2}{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}d\rho^2 + \rho^2(d\phi - \frac{\rho_+\rho_-}{l\rho^2}dt)^2 \tag{2.1}$$

where τ denotes the time coordinate, ρ is the radial variable, ϕ is the azimuthal angle with period 2π. $\rho_\pm(\rho_+ > \rho_-)$ are the parameters labelling the black hole solutions and are related to mass M and angular momentum J of the black hole but the precise relation requires the knowledge of all the higher derivative terms in the Lagrangian. The Wald entropy of the BTZ black hole in this theory is given by

$$S_{BH} = 2\pi\sqrt{\frac{c_L h_L}{6}} + 2\pi\sqrt{\frac{c_R h_R}{6}} \tag{2.2}$$

where $h_{L,R}$[1] are related to M and J by

$$h_L = \frac{lM + J}{2}, \qquad h_R = \frac{lM - J}{2} \tag{2.3}$$

Computation of $c_{L,R}$ requires the knowledge of full Lagrangian and hence it encodes correction to the entropy due to higher derivative terms.
The theories relevant for string theory however are not theories of pure gravity but (super-)gravity coupled to other matter fields containing higher derivative terms. In the absence of other matter fields the higher derivative terms in the action can be removed by field redefinition and the action may be reduced to the standard (super-)gravity action whose gravitational part contains a sum of three terms, -the Einstein-Hilbert term, a cosmological constant term and the Chern-Simons term [29, 27]. An argument based on AdS/CFT correspondence suggests that even when matter fields are present one can carry out a consistent truncation of the theory where only (super-)gravity is present, and action is again that of standard (super-)gravity whose gravitational sector is given by the sum of three terms[30]. The main ingredient of this argument was that in the dual two dimensional (super-)conformal field theory living at the boundary of AdS_3 any correlation function with one matter field and arbitrary number of (super-)stress tensor vanishes, and furthermore the correlation functions of the (super-)stress tensor are determined completely in terms of the central charge and are independent of the matter content of the theory. In this chapter we describe the consistent truncation procedure directly in the bulk theory without any reference to AdS/CFT correspondence. Although our analysis is classical, it can in principle be applied to the full quantum effective action. In our analysis we shall have to assume that the initial action is local, ı.e is given by an integral of a local Lagrangian density that admits a derivative expansion. Since in general the full quantum effective action can contain non-local terms, our analysis will not be directly applicable on these terms. In contrast the argument based on AdS/CFT correspondence works for the full quantum corrected effective action. After consistent truncation and field redefinition that brings the action to the standard form, the parameters labelling the action are the cosmological constant and the coefficient of the Chern-Simons term. Of them the Chern-Simons term does not change under the field redefinition required to bring the action to the standard form but the cosmological constant term is modified. The central charges $c_{L,R}$ are determined in terms of renormalised cosmological constant term and the

[1] h_L and h_R correspond to the eigen value of $L_0 - \frac{c}{24}$ and $\bar{L}_0 - \frac{\bar{c}}{24}$ respectively.

coefficient of the Chern-Simons term. Since the Wald entropy does not change under field redefinition, the central charges $c_{L,R}$ will contain the complete correction to Wald entropy due to higher derivative terms.
The rest of the chapter are organised as follows. In §2.2 and §2.4 we will describe the field redefinition of supergravity fields (metric, gauge fields and gravitino)and consistent truncation (setting scalars to constant and tensor fields in matter multiplet to zero) which reduce the action to standard leading supergravity action. As mentioned above under field redefinition the cosmological constant get renormalised. In §2.3 we describe simple procedure for finding the exact $\Lambda(\phi)$ which at extremum gives exact cosmological constant. In §2.5 we apply these procedure to three dimension (0,4) supergravity obtained by dimensionally reducing on S^2 the five dimensional supergravity with curvature squared term coupled to a set of vector multiplets and determine the exact central charge of the boundary CFT.

2.2 Field redefinition of the bosonic fields

In this section we shall describe how the bosonic part of a (super-)gravity action coupled to matter fields and containing higher derivative terms can be brought into the form of a standard supergravity action via field redefinition and consistent truncation. We begin with a three dimensional general coordinate invariant theory of gravity coupled to an arbitrary set of matter fields. We denote by $g_{\mu\nu}$ the metric, by ϕ the set of all the scalar fields, by Σ the set of all other tensor fields, by $R_{\mu\nu}$ the Ricci tensor associated with the metric $g_{\mu\nu}$ and by R the scalar curvature. At the level of two derivative terms, the action takes the form:

$$S_0 + S_{matter} \,, \tag{2.1}$$

where

$$S_0 = \int d^3x \sqrt{-g}\,(R + \Lambda_0(\phi)), \tag{2.2}$$

and S_{matter} denotes the kinetic term for the matter fields. $-\Lambda_0(\phi)$ represents the scalar field potential. We have already carried out an appropriate redefinition of the metric to remove a possible ϕ dependent function multiplying R in the Einstein-Hilbert term. If $\Lambda_0(\phi)$ has an extremum at $\phi = \phi_0$ then this theory has a solution where ϕ is set equal to ϕ_0, all other tensor fields are set to zero, and the metric is given by that of an AdS_3 space of size $l_0 = \sqrt{2/\Lambda_0(\phi_0)}$ for $\Lambda_0(\phi_0) > 0$ and a dS_3 space of size $\bar{l}_0 = \sqrt{-2/\Lambda_0(\phi_0)}$ for $\Lambda_0(\phi_0) < 0$. In this case $\Lambda_0(\phi_0)$ corresponds to the *negative* of the cosmological constant. We shall now consider the effect of adding higher derivative terms. For this we shall assume that these terms are small compared to the leading term, in the sense that the length parameter l_s that controls these terms is small compared to the length scale l_0

over which the leading order solution varies[2]. We shall also assume that we can associate with each higher derivative term in the Lagrangian density an index n that counts how many powers of l_s accompanies this term compared to the leading term. For example if the three dimensional theory is obtained via a dimensional reduction of type IIB string theory on $K3 \times S^1 \times S^2 \times AdS_3$ with $K3$ and S^1 having size of the order of string scale and S^2 and AdS_3 having large size, then α' corrections as well as corrections coming from integrating out the heavy modes associated with $K3 \times S^1$ compactification will have index $n > 0$, whereas all the terms associated with compactification of supergravity on $S^2 \times AdS_3$ - including the ones involving massive Kaluza-Klein modes - will have index 0. An efficient way to keep track of the derivative expansion is to introduce a derivative counting parameter λ and accompany a term of index n by a factor of λ^n. We shall carry out our analysis in a power series expansion in λ even though at the end we shall set $\lambda = 1$.
Since in three dimension the Riemann tensor $R_{\mu\nu\rho\sigma}$ can be expressed in terms of the Ricci tensor, all the higher derivative terms can be expressed in terms of the Ricci tensor, its covariant derivatives and covariant derivatives of the matter fields. We shall now reorganize these terms as follows. We first note that under $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$,

$$S_0 \rightarrow S_0 - \int d^3x\sqrt{-g}P^{\mu\nu}\delta g_{\mu\nu} + \mathcal{O}(\delta g^2)\,, \tag{2.3}$$

where

$$P_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}(R + \Lambda_0(\phi))g_{\mu\nu}. \tag{2.4}$$

Defining

$$P \equiv P^\mu_\mu = -\frac{1}{2}R - \frac{3}{2}\Lambda_0(\phi) \tag{2.5}$$

(2.4) can be rewritten as

$$R_{\mu\nu} = P_{\mu\nu} - (P + \Lambda_0(\phi))g_{\mu\nu}\,. \tag{2.6}$$

We now eliminate the variables $R_{\mu\nu}, R$ and their covariant derivatives in higher derivative terms by $P_{\mu\nu}, P$ and their covariant derivatives.
In this convention the most general action takes the form:[3]

$$S = S_0 + \lambda\, S_{cs} + \tilde{S}_{matter} + \lambda^n\, S_n\,. \tag{2.7}$$

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[2]Often the three dimensional theory is obtained from dimensional reduction of a higher dimensional theory on a compact space of size of order l_0. In this case if we integrate out the Kaluza-Klein modes we shall generate higher derivative terms which are not suppressed by powers of l_s. To avoid this situation we include all the Kaluza-Klein modes in the set Σ without integrating them out.

[3]During the process of replacing $R_{\mu\nu}$ by the right hand side of (2.6) we may generate some terms of the form $\int d^3x\sqrt{-g}f(\phi)$. Since these cannot be absorbed into $\tilde{S}_{matter}$ or S_n, we need to absorb them into the scalar field potential $\Lambda_0(\phi)$ appearing inside S_0. Thus $\Lambda_0(\phi)$ needs to be determined in a self-consistent manner. To any order in power series expansion in λ this can be done using an iterative procedure.

S_0 is given in (2.2). $\lambda\, S_{cs}$ is the gravitational Chern-Simons term

$$S_{cs} = K \int d^3x \Omega^{(3)}(\Gamma), \qquad \Omega^{(3)}(\Gamma) \equiv \epsilon^{\mu\nu\rho} \left[\frac{1}{2} \Gamma^{\tau}_{\mu\sigma} \partial_\nu \Gamma^{\sigma}_{\rho\tau} + \frac{1}{3} \Gamma^{\tau}_{\mu\sigma} \Gamma^{\sigma}_{\nu\kappa} \Gamma^{\kappa}_{\rho\tau} \right] \tag{2.8}$$

where K is a constant and $\Gamma^{\mu}_{\nu\rho}$ denotes the Christoffel symbol. Note that we have included a factor of λ in S_{cs} since in string theory the gravitational Chern-Simons term typically arises from α' corrections. $\tilde{S}_{matter}$ denotes the matter terms (including the standard kinetic terms) which are *quadratic and higher order in* Σ*, derivatives of* Σ *and derivatives of* ϕ. $\lambda^n S_n$ denotes all other terms, i.e. manifestly general coordinate invariant terms up to linear order in Σ, $\partial_\mu \phi$ and their derivatives, but not terms of the form $\int d^3x \sqrt{-g} R f(\phi)$ since they can be included in S_0. Most general higher derivative terms in the action will have the form given in (2.7) with $n = 1$ but for later use we have allowed for the fact that the higher derivative terms which cannot be included in S_0, $\tilde{S}_{matter}$ or λS_{cs} may actually begin their expansion at order λ^n. It is easy to see that S_n must contain at least one power of $P_{\mu\nu}$, since the $P_{\mu\nu}$ independent terms which do not involve Σ, $\partial_\mu\phi$ or their derivatives can be absorbed into $\Lambda_0(\phi)$ and $P_{\mu\nu}$ independent terms which are linear in Σ, $\partial_\mu\phi$ or their derivatives either vanish or become quadratic in Σ, $\partial_\mu\phi$ or their derivatives after integration by parts and hence may be included in $\tilde{S}_{matter}$. [4] Thus S_n has the form

$$S_n = \int d^3x \sqrt{-g}\, P^{\mu\nu} K_{\mu\nu}(\phi, \Sigma, \nabla_\rho, g_{\rho\sigma}, P_{\rho\sigma}, \lambda)\,. \tag{2.9}$$

where $K_{\mu\nu}$ is some combination of matter fields, $P_{\mu\nu}$ and their covariant derivatives, and can contain non-negative powers of λ. Now consider a redefinition of the metric of the form

$$g_{\mu\nu} \to g_{\mu\nu} + \lambda^n K_{\mu\nu} \tag{2.10}$$

Under this

$$S_0 \to S_0 - \lambda^n \int d^3x \sqrt{-g}\, P^{\mu\nu} K_{\mu\nu} + O(\lambda^{2n}) = S_0 - \lambda^n S_n + O(\lambda^{2n})\,, \tag{2.11}$$

$$S_{cs} \to S_{cs} + O(\lambda^{n+1})\,, \tag{2.12}$$

and

$$\lambda^n\, S_n \to \lambda^n\, S_n + O(\lambda^{2n})\,. \tag{2.13}$$

Thus

$$S_0 + \lambda\, S_{cs} + \lambda^n S_n \to S_0 + \lambda\, S_{cs} + O(\lambda^{n+1})\,. \tag{2.14}$$

[4]In case where a symmetric rank 2 tensor $A_{\mu\nu}$ has a coupling proportional to $\sqrt{-g}\, f_1(\phi)\, g^{\mu\nu} A_{\mu\nu}$ or an antisymmetric rank three tensor $C_{\mu\nu\rho}$ has a coupling proportional to $f_2(\phi)\, \epsilon^{\mu\nu\rho} C_{\mu\nu\rho}$. We express $A_{\mu\nu}$ as $A g_{\mu\nu} + A'_{\mu\nu}$ with $A = g^{\mu\nu} A_{\mu\nu}/3$, and $A'_{\mu\nu}$ a traceless symmetric matrix, and $C_{\mu\nu\rho}$ as $C(\sqrt{-g})\epsilon_{\mu\nu\rho}$ with $C = (\sqrt{-g})^{-1} \epsilon^{\mu\nu\rho} C_{\mu\nu\rho}/6$, and treating A and C as scalar fields. In this case these terms can be included in the scalar field potential $\Lambda_0(\phi)$ appearing in S_0.

Furthermore $\widetilde{S}_{matter}$ remains quadratic in Σ, $\partial_\mu\phi$ or their derivatives under this field redefinition. The order λ^{n+1} term on the right hand side of (2.14) can now be regrouped into a term of the form $\sqrt{-g}\,f(\phi)$ that can be absorbed into a redefinition of $\Lambda_0(\phi)$, a term quadratic in Σ and $\partial\phi$ that can be absorbed into $\widetilde{S}_{matter}(\phi)$ and a term containing at least one power in $P_{\mu\nu}$. Thus the resulting action may be expressed as:

$$S = S_0' + \lambda\, S_{cs} + \widetilde{S}'_{matter} + \lambda^{n+1}\, S_{n+1}, \tag{2.15}$$

where

$$S_0' = \int d^3x\sqrt{-g}\,(R + \Lambda_0'(\phi)), \tag{2.16}$$

$\tilde{S}'_{matter}$ contains terms which are quadratic and higher order in Σ and derivatives of ϕ, Σ and

$$S_{n+1} = \int d^3x\sqrt{-g}P^{\mu\nu}K'_{\mu\nu}(\phi, \Sigma, \nabla_\rho, g_{\rho\sigma}, P_{\rho\sigma}, \lambda) \tag{2.17}$$

for some $K'_{\mu\nu}$. Thus the new action has the same form as our starting action with n replaced by $n+1$. Repeating this process we can ensure that to any fixed order in an expansion in λ, the action can be brought to the form:

$$S = \int d^3x\sqrt{-g}(R + \Lambda(\phi)) + \lambda S_{cs} + \widetilde{S}_{matter}\,, \tag{2.18}$$

for some choice of $\Lambda(\phi)$ and $\tilde{S}_{matter}$.

Now suppose $\Lambda(\phi)$ has an extremum at $\phi = \phi_0$. Introducing new fields $\xi = \phi - \phi_0$ we may express the action as

$$S = \int d^3x\sqrt{-g}(R + \Lambda(\phi_0)) + \lambda S_{cs} + \cdots\,, \tag{2.19}$$

where $\cdots$ contain terms which are at least quadratic in ξ, Σ and their covariant derivatives. We can now carry out a consistent truncation of the theory by setting $\xi = 0$, $\Sigma = 0$. This leaves us with a purely gravitational action with Einstein-Hilbert term, cosmological constant term and Chern-Simons term.

If the theory contains a 2-form field B with gauge invariance $B \to B + d\Lambda$ then we can consider a slightly more general truncation where instead of setting B to zero we set it to have a constant field strength $C\sqrt{-g}\,\epsilon_{\mu\nu\rho}$ for some constant C. Let $\widetilde{B}$ denote the fluctuation around this fixed background. Since $C\sqrt{-g}\,\epsilon_{\mu\nu\rho}$ is a general coordinate invariant tensor, and since the Lagrangian density depends on B only through the combination $(dB)_{\mu\nu\rho} = C\sqrt{-g}\,\epsilon_{\mu\nu\rho} + (d\widetilde{B})_{\mu\nu\rho}$, it depends on $(d\widetilde{B})_{\mu\nu\rho}$ in a manifestly general coordinate invariant fashion. We can then proceed with our analysis as before, including $\widetilde{B}$ in the list of tensor fields Σ.

If instead of considering a theory of gravity we consider (extended) supergravity theories, then the theory contains additional fields. In particular the additional bosonic fields in the theory are gauge fields with Chern-Simons terms. We showed that higher derivative terms involving higher powers of gauge fields can also be removed by field redefinition. This follows from the fact that under $A_\mu \to A_\mu + \delta A_\mu$ the gauge Chern-Simons term changes by a term proportional to $\epsilon^{\mu\nu\rho} Tr\,(F_{\mu\nu}\delta A_\rho)$. If we assume that all other terms in the action depend on the gauge field only through $F_{\mu\nu}$ and not explicitly A_μ i.e. there are no other charged fields on the theory.[5] Then a term of the form $\lambda^n \int \sqrt{-g}\, Tr\,(F_{\mu\nu}L^{\mu\nu})$ in the action can be removed (up to order λ^{2n} terms) by a shift of A_μ proportional to $\sqrt{-g}\epsilon_{\mu\nu\rho}L^{\nu\rho}$. Once this has been done, one can then carry out the field redefinition of the metric and the scalar fields as described earlier, and obtain a consistent truncation to a theory of metric and gauge fields with gauge Chern-Simons terms, Einstein-Hilbert term, cosmological constant term and gravitational Chern-Simons term. Supersymmetry then relates the coefficient of the gauge and gravitational Chern-Simons terms to the cosmological constant term.

2.3 Algorithm for determining $\Lambda(\phi)$

Of the various parameters labelling the final theory the coefficients of the Chern-Simons terms are easy to determine since they do not get renormalized from their initial values. On the other hand the cosmological constant term does get renormalized during the field redefinition. In this section we shall outline a simple procedure for finding the exact $\Lambda(\phi)$ appearing in (2.18) without having to carry out all the steps described in the last section. The cosmological constant of the final truncated theory can then be found by determining the value of $\Lambda(\phi)$ at its extremum.
Suppose our initial action, including all higher derivative terms, has the form

$$S = \int d^3x \sqrt{-g}\mathcal{L} + \lambda S_{cs}. \tag{2.20}$$

In anticipation of the fact that the final truncation involves setting the scalars ϕ to constants and other tensor fields Σ to 0, let us consider a theory of pure gravity obtained by setting Σ to 0 and ϕ to some constant values in (2.20). Thus ϕ can now be regarded as a set of external parameters labelling the action. We now consider a background

$$\begin{aligned} &ds^2 = -l^2(1+r^2)dt^2 + l^2(1+r^2)^{-1}dr^2 + l^2r^2d\varphi^2\,, \\ &\phi = \text{constant}, \qquad \Sigma = 0\,, \end{aligned} \tag{2.21}$$

[5]Even if these charged fields present, they can be set to zero in a consistent truncation scheme provided the gauge symmetry is not spontaneously broken.In the latter case the would be Goldstone boson associated with the symmetry breaking would mix with the gauge field via a two point coupling and we cannot have a consistent truncation to pure supergravity.

representing an AdS_3 space of size l.

Now we define

$$F(l,\phi) = l^3 \mathcal{L} \tag{2.22}$$

evaluated in the background (2.21), then the metric satisfies its equation of motion if l is chosen to be at the extremum l_{ext} of F. Since the variation of Chern-Simons term automatically vanishes for the AdS_3 metric (2.21), it will not contribute in the equation of motion.
Now we consider the truncated action (2.18) after setting ϕ to a constant and Σ to 0

$$S = \int d^3x \sqrt{-g}(R + \Lambda(\phi)) + \lambda S_{cs} \,. \tag{2.23}$$

If we evaluate $\sqrt{-g}\,(R + \Lambda(\phi))$ for the AdS_3 background (2.21), we get a new function $r\,H(l,\phi)$ with

$$H(l,\phi) = l^3 \left[-\frac{6}{l^2} + \Lambda(\phi) \right] \,. \tag{2.24}$$

Since we have carried out a field redefinition of the metric but not of Σ or ϕ, $F(l,\phi)$ and $H(l,\phi)$ are related by a redefinition of the parameter l for any fixed ϕ and hence the value of these function at the extremum must be same. The extremum of H occurs at

$$\tilde{l}_{ext} = \sqrt{\frac{2}{\Lambda(\phi)}}, \qquad H(\tilde{l}_{ext},\phi) = -\sqrt{\frac{32}{\Lambda(\phi)}}, \tag{2.25}$$

we get, by setting the right hand side of (2.25) to $F(l_{ext},\phi)$,

$$\Lambda(\phi) = \frac{32}{F(l_{ext},\phi)^2} \tag{2.26}$$

provided $F(l_{ext},\phi)$ is negative. This determines $\Lambda(\phi)$.
It may so happen that $F(l,\phi)$ defined in (2.22) has an extremum at an imaginary value of l and hence $F(l,\phi)$ is imaginary at the extremum. This will give a negative $\Lambda(\phi)$ and hence a positive cosmological constant. A better way to analyze this case is to consider a de Sitter metric of the form

$$ds^2 = -\bar{l}^2(1-r^2)dt^2 + \bar{l}^2\,(1-r^2)^{-1}\,dr^2 + \bar{l}^2 r^2\,d\varphi^2 \tag{2.27}$$

instead of the anti-de Sitter metric given in (2.21), and define

$$\bar{F}(\bar{l},\phi) = \bar{l}^3 \mathcal{L}, \tag{2.28}$$

evaluated in this background with ϕ set to constants and Σ set to zero. On the other hand (2.24) is now replaced by

$$\bar{H}(\bar{l},\phi) = \bar{l}^3 \left[\frac{6}{\bar{l}^2} + \Lambda(\phi) \right] \,. \tag{2.29}$$

and the value of $\bar{H}(\bar{l}, \phi)$ at the extremum with respect to $\bar{l}$ is given by $\sqrt{-32/\Lambda(\phi)}$. Equating this to the value of $\bar{F}$ at its extremum we get:

$$\Lambda(\phi) = -\frac{32}{\bar{F}(\bar{l}_{ext}, \phi)^2} \tag{2.30}$$

provided $\bar{F}(\bar{l}_{ext}, \phi)$ is positive.
Finally we note that there is always a possibility that neither $F(l, \phi)$ nor $\bar{F}(\bar{l}, \phi)$ has an extremum for real values of l or $\bar{l}$, or even if such extrema exist, the resulting function $\Lambda(\phi)$ does not have an extremum as a function of ϕ. In this case the theory under consideration does not admit an AdS_3 or dS_3 solution and we cannot carry out the consistent truncation following the procedure described above.

2.4 Field redefinition of gravitino

In §2.2 we have described how via a field redefinition the bosonic part of the supergravity action can be brought into the standard form. One would expect that once the bosonic part of the action has been shown to coincide with that of the supergravity action, supersymmetry will fix the fermionic part of the action uniquely (up to a possible field redefinition involving the fermions) to be that of the standard supergravity action. In this section we shall briefly discuss how such a result might be proven.
We begin with an action where the purely bosonic part has already been brought into the standard form using the field redefinition described in §2.2. At the onset we shall assume that supersymmetry is unbroken at the extremum ϕ_0 of $\Lambda(\phi)$; otherwise we expect the gravitino to mix with the Goldstino and hence the matter and the gravity multiplet will no longer be decoupled. This in turn requires $\Lambda(\phi_0)$ to be positive since we do not have unbroken supersymmetry in de Sitter space. If the theory has altogether $\mathcal{N}$ supersymmetries then there are $\mathcal{N}$ gravitino fields ψ^i_μ with $1 \leq i \leq \mathcal{N}$. In the supergravity action of [31]the gravitino action has the form:

$$S_0^\psi = -\int d^3x\, \epsilon^{\mu\nu\rho} \bar{\psi}^i_\mu \mathcal{D}_\nu \psi^i_\rho\,, \tag{2.31}$$

where

$$\mathcal{D}_\mu \psi^i_\nu = \partial_\mu \psi^i_\nu + \frac{1}{8}\omega_{ab\mu}[\gamma^a, \gamma^b]\psi^i_\nu \pm \sqrt{\frac{\Lambda(\phi_0)}{32}} e_{a\mu}\gamma^a \psi^i_\nu + A^a_\mu (T^a)_{ij}\psi^j_\nu\,, \tag{2.32}$$

$\omega_{\mu ab}$ being the spin connection, $e_{a\mu}$ the vielbeins, A^a_μ the gauge fields and T^a are the generators of the representation of the gauge group in which the gravitinos transform. The + (−) sign correspond to the gravitinos associated with left (right) supersymmetries. Under a general variation of the gravitino fields

$$\delta S_0^\psi = -\int d^3x\, \epsilon^{\mu\nu\rho} \left[\delta\bar{\psi}^i_\mu \mathcal{D}_\nu \psi^i_\rho + h.c.\right] \tag{2.33}$$

leading to the gravitino equation of motion

$$\mathcal{D}_\nu \psi^i_\rho - \mathcal{D}_\rho \psi^i_\nu = 0\,. \tag{2.34}$$

The supersymmetry transformation law of the gravitino fields takes the form

$$\delta_s \psi^i_\mu = \mathcal{D}_\mu\, \epsilon^i\,, \tag{2.35}$$

where ϵ^i are the supersymmetry transformation parameters.
We shall now examine the possibility of adding higher derivative terms in the action and also possibly in the supersymmetry transformation laws. We will show that *it is not possible to add higher derivative terms in the action involving gravitino consistent with supersymmetry (2.35) (possibly modifying supersymmetry transformation by higher derivative terms) apart from the terms which are proportional to lowest order supergravity equation of motion and hence can be removed by field redefinition.*
Let us denote by η the set of all the bosonic and fermionic fields coming from the matter sector with the scalars measured relative to ϕ_0 (i.e. the set η contains the shifted fields ξ introduced above (2.19)). We are interested in higher derivative terms which are linear in η or derivative of η [6] and do not vanish identically by lower order supergravity equation of motion as any term that is proportional to the equation of motion of the metric, the gauge fields or the gravitinos derived from the leading supergravity action can be absorbed into a redefinition of these fields at the cost of generating higher order terms. We call such terms as dangerous terms since, if present, they will prevent us from consistently truncating the theory to the one described by the standard supergravity action. Thus the dangerous terms may be expressed as general coordinate invariant and local Lorentz invariant combinations of the gravitino fields, their symmetrized covariant derivatives and the metric. We shall organise these terms according to the power of the derivative counting parameter λ that they carry. Let us suppose that the first dangerous higher derivative terms in the Lagrangian density appear at order λ^k. We consider all the order λ^k dangerous terms and organise them by their rank, – defined as the total power of ψ_μ and $\bar\psi_\mu$ contained in that term. We begin with the terms of lowest rank, – call it m_0. m_0 cannot vanish since we have already argued earlier that all the dangerous terms without the gravitino field can be removed by field redefinition. (For this we need to include in the set Σ of §2.3 all the matter fermions as well.) For non-zero m_0 the lowest order supersymmetry variation of the gravitino described in (2.35) has the effect of producing a term of rank (m_0-1), constructed out of the gravitino fields, their symmetrized covariant derivatives, the metric, and covariant derivatives of the supersymmetry transformation parameter. In order for supersymmetry to be preserved, such terms need to be cancelled by some other terms. The terms arising from the supersymmetry variation of the bosons in the original rank m_0 term are of rank $\geq m_0$ and hence cannot cancel the rank (m_0-1) term. Thus there are two possibilities: 1) the rank (m_0-1) terms arising from the

[6]Any term quadratic or higher order in η in consistent truncation scheme can be set zero.

variation of the gravitino cancel among themselves after we integrate by parts and move all the derivatives from ϵ, $\bar{\epsilon}$ to the fields, possibly after modifying the supersymmetry transformation laws of the supergravity fields, and 2) we can try to cancel these terms against terms coming from supersymmetry variation of the bosons in a term of rank $(m_0 - 2)$. Of these the first possibility would mean that the dangerous terms are invariant under the transformation (2.35) of the gravitino alone up to terms which vanish by lowest order supergravity equations of motion. [7] To see if this is possible we first focus on the terms with maximum number of derivatives where all the covariant derivatives have been replaced by ordinary derivatives in the order λ^k, rank m_0 term in the action. The net supersymmetry variation of these terms under the supersymmetry transformation law (2.35) must vanish after using the lowest order gravitino equations of motion (2.34) with $\mathcal{D}_\mu$ replaced by ∂_μ in (2.35) and (2.34), since this is the term in $\delta_s S$ with maximum number of derivatives at this order. In this case the gravitino satisfying its lowest order equations of motion has the form $\psi^i_\mu = \partial_\mu \chi^i$, $\bar{\psi}^i_\mu = \partial_\mu \bar{\chi}^i$ for some χ^i, $\bar{\chi}^i$. Let us evaluate the order λ^k, rank m_0 term in the action in this background. By assumption the result is not identically zero, – otherwise we could have removed these terms from the action by a field redefinition of the gravitino field. Now for $\psi^i_\mu = \partial_\mu \chi^i$, $\bar{\psi}^i_\mu = \partial_\mu \bar{\chi}^i$ the gauge transformation laws of the gravitino field take the form $\chi^i \to \chi^i + \epsilon^i$, $\bar{\chi}^i \to \bar{\chi}^i + \bar{\epsilon}^i$. Invariance under supersymmetry transformation then tells us that the term under consideration is invariant under $\chi^i \to \chi^i + \epsilon^i$ for an arbitrary function ϵ^i. In other words the term is independent of χ^i. Thus it must vanish since it vanishes when we set all the χ^i and $\bar{\chi}^i$ to zero. This contradicts our original assertion that the term does not vanish identically. This leads us to the conclusion that the original order λ^k, rank m_0 term in the action, with covariant derivatives replaced by ordinary derivatives, must have been such that after suitable integration by parts and commutation of the derivative operators it vanishes when the gravitino satisfies its lowest order equation of motion.

How does the conclusion change when the ordinary derivatives are replaced by covariant derivatives? Since we know that the term can be manipulated and shown to vanish when covariant derivatives are replaced by ordinary derivatives, we can carry out the same manipulation. The only possible extra terms which could arise must be proportional to the commutators $[D_\mu, D_\nu]$ since the covariant derivatives can be manipulated in the same manner as the ordinary derivatives except for their commutators. However these commutators can be reduced to terms with lower number of derivatives using the lowest order metric and gauge field equations of motion. We can now repeat our analysis on these left-over terms with lower number of derivatives and show that they must be further reducible to terms with lower number of derivatives. Repeating this procedure we can show that a term that is invariant under the lowest order supersymmetry transformation

[7] The terms proportional to the lowest order equations of motion of the supergravity fields can be cancelled by modifying the supersymmetry transformation laws of the supergravity fields, since the additional variation of the lowest order supergravity action under the modified supersymmetry transformation laws will be a linear combination of the lowest order equations of motion of these fields.

of the gravitino alone, must vanish as a consequence of lowest order supergravity field equations, and hence can be removed by a field redefinition.
We now turn to the second possibility. This requires the action to contain higher derivative terms of order λ^k and rank $(m_0 - 2)$. Since by assumption the action does not contain any dangerous term of rank $(m_0 - 2)$ to order λ^k, the only possibility is to try to generate these terms from the supersymmetry variation of a non-dangerous term of rank $(m_0 - 2)$. In order to rule out this possibility we need to make one assumption: *as a consequence of unbroken supersymmetry the matter sector fields transform to terms which contain at least a single power of the matter sector field,* i.e. we have $\delta_s \eta \sim \mathcal{O}(\eta)$.[8]In this case terms quadratic and higher order in η transform to terms quadratic and higher order in η and cannot cancel terms which are at most linear in η. This rules out the last possibility. Thus we see that it is not possible to add higher derivative dangerous terms in the action in a manner consistent with supersymmetry.

2.5 Dimensional reduction of five dimensional supergravity

In this section we shall consider five dimensional supergravity with curvature squared term coupled to a set of vector multiplets[32] and dimensionally reduce this theory on S^2 in the presence of background magnetic flux through S^2 to get a three dimensional (0,4) supergravity with curvature squared term, coupled to a set of matter fields. We then apply the procedure of §2.2 and §2.3 to truncate this to a pure supergravity theory with gravitational Chern-Simons term, but no other higher derivative terms.
We shall concentrate our attention on the part of the action involving the bosonic fields only. In the three dimensional theory this involves the metric and an SU(2) gauge field that arises during the dimensional reduction of the five dimensional theory on S^2. As we have seen at the end of §2.2, reducing the gauge field action to pure Chern-Simons term is relatively simple; hence we shall focus on the part of the action involving the metric. For this we can restrict the fields to the SU(2) invariant sector from the beginning. Since the SU(2) R-symmetry of the three dimensional supergravity can be identified with the rotational symmetry of the compact S^2, this allows us to carry out the dimensional reduction by restricting the field configurations to rotationally invariant form.
The five dimensional $\mathcal{N} = 2$ supergravity has a Weyl multiplet, a set of vector multiplets and a compensator hypermultiplet. After gauge fixing to Poincare supergravity, the bosonic fields of the theory include the metric g_{ab}, the two-form auxiliary field v_{ab}, a scalar auxiliary field D, a certain number (n_V) of one-form gauge fields A^I_a with $1 \leq I \leq n_V$, and an equal number of scalars M^I. Here $a, b, ..$ are five dimensional coordinate labels

[8]This is of course true at the lowest order in λ but we shall assume that this property continues to hold even after including possible higher derivative corrections to the supersymmetry transformation laws.

and run from 0 to 4. We shall denote by $F^I = dA^I$ the field strength associated with the gauge field A^I.

The action for bosonic fields including curvature squared terms can be written as

$$S = \frac{1}{4\pi^2}\int d^5x\sqrt{-g^{(5)}}[\mathcal{L}_0 + \mathcal{L}_1] \tag{2.36}$$

where $\mathcal{L}_0$ is the Lagrangian at two derivative order and $\mathcal{L}_1$ denotes the supersymmetric completion of the curvature squared terms. The explicit forms of $\mathcal{L}_0$ and $\mathcal{L}_1$ are[32, 33]

$$\begin{aligned}\mathcal{L}_0 \;=\;& -2\left(\frac{1}{4}D - \frac{3}{8}R - \frac{1}{2}v^2\right) + N\left(\frac{1}{2}D + \frac{1}{4}R + 3v^2\right) + 2N_I v^{ab} F^I_{ab} \\ & + N_{IJ}\left(\frac{1}{4}F^I_{ab}F^{Jab} + \frac{1}{2}\partial_a M^I \partial^a M^J\right) + \frac{1}{24}e^{-1}c_{IJK}A^I_a F^J_{bc}F^K_{de}\epsilon^{abcde}\end{aligned} \tag{2.37}$$

$$\begin{aligned}\mathcal{L}_1 \;=\;& \frac{c_{2I}}{24}\Big[\frac{1}{16}e^{-1}\epsilon_{abcde}A^{Ia}C^{bcfg}C^{de}_{fg} + \frac{1}{8}M^I C^{abcd}C_{abcd} + \frac{1}{12}M^I D^2 + \frac{1}{6}F^{Iab}v_{ab}D \\ & -\frac{1}{3}M^I C_{abcd}v^{ab}v^{cd} - \frac{1}{2}F^{Iab}C_{abcd}v^{cd} + \frac{4}{3}M^I\nabla^a v^{bc}\nabla_a v_{bc} + \frac{4}{3}M^I\nabla^a v^{bc}\nabla_b v_{ca} \\ & +\frac{8}{3}M^I\left(v_{ab}\nabla^b\nabla_c v^{ac} + \frac{2}{3}v^{ac}v_{cb}R^b_a + \frac{1}{12}v^{ab}v_{ab}R\right) - \frac{2}{3}e^{-1}M^I\epsilon_{abcde}v^{ab}v^{cd}\nabla_f v^{ef} \\ & +\frac{2}{3}e^{-1}F^{Iab}\epsilon_{abcde}v^{cf}\nabla_f v^{de} + e^{-1}F^{Iab}\epsilon_{abcde}v^c_f\nabla^d v^{ef} - \frac{4}{3}F^{Iab}v_{ac}v^{cd}v_{db} \\ & -\frac{1}{3}F^{Iab}v_{ab}v^2 + 4M^I v_{ab}v^{bc}v_{cd}v^{da} - M^I(v_{ab}v^{ab})^2\Big]\end{aligned} \tag{2.38}$$

where c_{IJK} and c_{2I} are parameters of the theory, $e \equiv \sqrt{-g}$, and

$$N = \frac{1}{6}c_{IJK}M^I M^J M^K \tag{2.39}$$

$$N_I = \frac{1}{2}c_{IJK}M^J M^K \tag{2.40}$$

$$N_{IJ} = c_{IJK}M^K\,, \tag{2.41}$$

and C_{abcd} is the Weyl tensor defined as

$$C^{ab}_{cd} = R^{ab}_{cd} + \frac{1}{6}R\delta^{[a}_{[c}\delta^{b]}_{d]} - \frac{4}{3}\delta^{[a}_{[c}R^{b]}_{d]}\,. \tag{2.42}$$

The parameters c_{2I} appear in the coefficients of the higher derivative terms; thus we can keep track of the derivative expansion by simply counting the power of c_{2I} appearing in the various terms.

We now carry out the dimensional reduction on S^2 and focus on the sector invariant under the $SO(3)$ isometry group of S^2. This can be done using the following ansatz for the five dimensional fields

$$
\begin{aligned}
ds^2 &= g^{(3)}_{\mu\nu}(x)dx^\mu dx^\nu + \chi^2(x)d\Omega^2, \qquad 0 \le \mu,\nu \le 2 \\
& \quad v_{\theta\phi} = V(x)\sin\theta \\
& \quad F^I_{\theta\phi} = \frac{p^I}{2}\sin\theta, \qquad F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu \,,
\end{aligned} \tag{2.43}
$$

with the mixed components of F^I_{ab} and v_{ab} set to zero. Here x^μ denote the three dimensional coordinates. All the scalar fields can be arbitrary functions of x but are independent of the coordinates (θ,ϕ) of S^2. For the metric given in (2.43) the non-vanishing components of the Riemann tensor are

$$
\begin{aligned}
& R_{\mu\nu\sigma\rho} = R^{(3)}_{\mu\nu\sigma\rho}, \quad R_{i\mu j\nu} = -\chi^{-1}\, g_{ij}\, \nabla_\mu\nabla_\nu\chi, \quad R_{ijkl} = \chi^{-2}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)\left(1 - g^{(3)\mu\nu}\partial_\mu\chi\partial_\nu\chi\right)\,, \\
& \qquad 0 \le \mu,\nu \le 2, \qquad i,j = \theta,\phi\,.
\end{aligned} \tag{2.44}
$$

Here $R^{(3)}_{\mu\nu\rho\sigma}$ is the Riemann tensor and ∇_μ is the covariant derivative computed using the three dimensional metric $g^{(3)}_{\mu\nu}$. Using these relations we get the dimensionally reduced action to be

$$
\begin{aligned}
S &= -\frac{c_2\cdot p}{96\pi}\int d^3x\Omega^{(3)}(\Gamma) \\
&\quad + \int d^3x\sqrt{-g^{(3)}}\frac{\chi^2}{\pi}\Big(\frac{3}{4} + \frac{1}{4}N + \frac{c_2\cdot M}{288}\frac{1}{\chi^2} + \frac{c_2\cdot M}{72}\frac{V^2}{\chi^4} - \frac{c_2\cdot p}{288}\frac{V}{\chi^4}\Big)R^{(3)} \\
&\quad + \int d^3x\sqrt{-g^{(3)}}\frac{\chi^2}{\pi}U(\chi, M^I, V, p^I, D) \\
&\quad + \int d^3x\sqrt{-g^{(3)}}\frac{\chi^2}{\pi}\frac{c_2\cdot M}{192}\Big(\frac{8}{3}R^{(3)}_{\mu\nu}R^{(3)\mu\nu} - \frac{5}{6}R^{(3)2} + \frac{16}{3\chi}R^{(3)}_{\mu\nu}\nabla^\mu\nabla^\nu\chi - \frac{4}{3\chi}R^{(3)}\nabla^2\chi\Big) \\
&\quad + \int d^3x\sqrt{-g^{(3)}}\widehat{\mathcal{L}}(\chi, v_{\mu\nu}, M^I, F^I_{\mu\nu}, R^{(3)}_{\mu\nu})
\end{aligned} \tag{2.45}
$$

where

$$
\begin{aligned}
U(\chi, M^I, V, p^I, D) &= \frac{2}{\chi^2}\Big(\frac{3}{4} + \frac{1}{4}N\Big) - 2\Big(\frac{1}{4}D - \frac{V^2}{\chi^4}\Big) + N\Big(\frac{1}{2}D + \frac{6V^2}{\chi^4}\Big) \\
&\quad + \frac{2(N\cdot p)V}{\chi^4} + \frac{N_{IJ}p^Ip^J}{8\chi^4} + \frac{c_2\cdot M}{96\chi^4} + \frac{c_2\cdot M}{288}D^2 + \frac{c_2\cdot p}{144}\frac{VD}{\chi^4} \\
&\quad - \frac{5}{36}(c_2\cdot M)\frac{V^2}{\chi^6} - \frac{c_2\cdot p}{48}\frac{V}{\chi^6} + \frac{c_2\cdot p}{36}\frac{V^3}{\chi^8} + \frac{c_2\cdot M}{6}\frac{V^4}{\chi^8}
\end{aligned} \tag{2.46}
$$

and $\widehat{\mathcal{L}}(\chi, v_{\mu\nu}, M^I, F^I_{\mu\nu}, R^{(3)}_{\mu\nu})$ denotes terms which are at least quadratic in $\nabla_\mu\chi, v_{\mu\nu}, \nabla_\mu M^I$ and $F^I_{\mu\nu}$. In eq.(2.45) all covariant derivatives are computed using the three dimensional

metric $g^{(3)}_{\mu\nu}$.
We first need to redefine our metric in such a manner that the coefficient of $R^{(3)}$ in the second line of the action (2.45) can be absorbed into the metric. One does this by redefining the metric as

$$\widetilde{g}_{\mu\nu} = \psi^{-2}\, g^{(3)}_{\mu\nu} \tag{2.47}$$

with

$$\psi^{-1} = \frac{\chi^2}{\pi}\Big(\frac{3}{4} + \frac{1}{4}N + \frac{c_2\cdot M}{288}\frac{1}{\chi^2} + \frac{c_2\cdot M}{72}\frac{V^2}{\chi^4} - \frac{c_2\cdot p}{288}\frac{V}{\chi^4}\Big) \tag{2.48}$$

Substitution of this will produce quadratic and higher derivative terms of scalar fields in the action.
Further following the general procedure given in §2.3 we define

$$\begin{aligned} P_{\mu\nu} &= \widetilde{R}_{\mu\nu} - \frac{1}{2}\widetilde{g}_{\mu\nu}[\widetilde{R} + \Lambda_0(\phi)] \\ P &= -\frac{1}{2}\widetilde{R} - \frac{3}{2}\Lambda_0(\phi)\,, \end{aligned} \tag{2.49}$$

where for shorthand notation we have denoted $(\chi, M^I, V, p^I, D) \equiv \phi$ and $\Lambda_0(\phi)$ is a function to be determined later. Now we rewrite the action as

$$\begin{aligned} S &= -\frac{c_2\cdot p}{96\pi}\int d^3x\Omega^{(3)}(\widetilde{\Gamma}) + \int d^3x\sqrt{-\widetilde{g}}\Big[\widetilde{R} + Z(\phi)\Big] + \int d^3x\sqrt{-\widetilde{g}}P_{\mu\nu}K^{\mu\nu} \\ &\quad + \int d^3x\sqrt{-\widetilde{g}}\frac{\chi^2}{\psi\pi}\frac{c_2\cdot M}{384}\Lambda_0^2(\phi) \\ &\quad + \int d^3x\sqrt{-\widetilde{g}}\,\widetilde{\mathcal{L}} \end{aligned} \tag{2.50}$$

where

$$\begin{aligned} K_{\mu\nu} &= \frac{\chi^2}{\psi\pi}\frac{c_2\cdot M}{192}\Big[\frac{8}{3}P_{\mu\nu} - \frac{2}{3}\widetilde{g}_{\mu\nu}P + \frac{2}{3}\widetilde{g}_{\mu\nu}\Lambda_0(\phi) - \frac{16}{3\psi}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\psi \\ &\quad + \frac{8}{3\psi}\widetilde{g}_{\mu\nu}\widetilde{\nabla}^2\psi + \frac{16}{3\chi}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\chi - \frac{8}{3\chi}\widetilde{g}_{\mu\nu}\widetilde{\nabla}^2\chi\Big]\,, \end{aligned} \tag{2.51}$$

Here$\widetilde{\mathcal{L}}$ denotes terms quadratic and higher order in the derivatives of the scalar fields and other tensor fields.
We now choose $\Lambda_0(\phi)$ to be the solution to the equation

$$\Lambda_0(\phi) = Z(\phi) + \frac{\chi^2}{\psi\pi}\frac{c_2\cdot M}{384}\Lambda_0(\phi)^2\,, \tag{2.52}$$

so that the action (2.50) may be expressed as

$$\begin{aligned} S &= -\frac{c_2\cdot p}{96\pi}\int d^3x\Omega^{(3)}(\widetilde{\Gamma}) + \int d^3x\sqrt{-\widetilde{g}}\Big[\widetilde{R} + \Lambda_0(\phi)\Big] + \int d^3x\sqrt{-\widetilde{g}}P_{\mu\nu}K^{\mu\nu} \\ &\quad + \int d^3x\sqrt{-\widetilde{g}}\,\widetilde{\mathcal{L}}\,. \end{aligned} \tag{2.53}$$

In this case, as we mentioned earlier, the required field redefinition which will remove the four derivative terms from the action (2.53) is

$$\widetilde{g}_{\mu\nu} \to \widetilde{g}_{\mu\nu} + K_{\mu\nu}\,. \tag{2.54}$$

To this order the scalar field potential $-\Lambda(\phi)$ is given by

$$\Lambda(\phi) = \Lambda_0(\phi) = Z(\phi) + \frac{\chi^2}{\psi\pi}\frac{c_2\cdot M}{384}Z^2(\phi) + \mathcal{O}(c_2^2)\,. \tag{2.55}$$

This process can now be repeated to remove the six and higher derivative terms from the action, but we shall not go through the details of the analysis. Our interest is in finding the exact expression for $\Lambda(\phi)$ since this is what controls the final truncated action. We have already described the algorithm for finding $\Lambda(\phi)$ in §2.3. The first step is to compute $F(l,\phi)$ for the action (2.50) by evaluating the Lagrangian density (without the Chern-Simons term) in the AdS_3 background (2.21) with constant scalar fields and vanishing tensor fields. We get

$$F(l,\phi) = -6l + l^3 Z(\phi) + 2a\frac{1}{l} \tag{2.56}$$

where

$$a = \frac{\chi^2}{\psi\pi}\frac{c_2\cdot M}{192}\,. \tag{2.57}$$

The extremum of $F(l,\phi)$ with respect to l occurs at[9]

$$l_{ext}^2 = \frac{1}{Z(\phi)} + \frac{1}{Z(\phi)}\sqrt{1+\frac{2a}{3}Z(\phi)}\,. \tag{2.58}$$

Hence $\Lambda(\phi)$ is given by

$$\Lambda(\phi) = \frac{32}{F(l_{ext},\phi)^2} = \frac{32Z(\phi)}{W(\phi)}\left(2a\frac{Z(\phi)}{W(\phi)} + W(\phi) - 6\right)^{-2}, \qquad W(\phi) \equiv 1+\sqrt{1+\frac{2a}{3}Z(\phi)}\,. \tag{2.59}$$

Before we proceed we note that to order c_{2I} terms, i.e. order a term, eq.(2.59) reduces to

$$\Lambda(\phi) = Z(\phi) + \frac{1}{2}\,a\,Z(\phi)^2 + \mathcal{O}(a^2)\,. \tag{2.60}$$

This agrees with the result (2.55) of the explicit calculation to this order. We now return to the full expression (2.59) for $\Lambda(\phi)$. $\Lambda(\phi)$ has an extremum at[33, 34]

$$\chi = \frac{pb}{2}$$

[9]There is, in principle, another extremum at $l_{ext}^2 = (Z(\phi))^{-1}\left(1-\sqrt{1+2aZ(\phi)/3}\right)$. This could in principle describe a de Sitter solution. However since for this solution $|l_{ext}| \sim a$, the radius is small and there is no systematic derivative expansion.

$$\begin{aligned} M^I &= \frac{p^I}{pb} \\ V &= -\frac{3pb}{8} \\ D &= \frac{12}{p^2b^2} \end{aligned} \tag{2.61}$$

where

$$p^3 \equiv \frac{1}{6} c_{IJK} p^I p^J p^K, \qquad b^3 = 1 + \frac{c_2 \cdot p}{12 p^3} \tag{2.62}$$

The value of $\Lambda(\phi)$ at it's extremum is given by

$$\Lambda(\phi_0) = \frac{32\pi^2}{p^6} \Big[1 + \frac{c_2 \cdot p}{8p^3} \Big]^{-2} \tag{2.63}$$

Thus the final truncated theory, obtained by setting ϕ to its value at the extremum and other matter fields to zero, is given by

$$S = \int d^3x \sqrt{-\widetilde{g}} \, (\widetilde{R} + \Lambda(\phi_0)) - \frac{c_2 \cdot p}{96\pi} \int d^3x \Omega^{(3)}(\widetilde{\Gamma}) \, . \tag{2.64}$$

From this one can compute the central charges of the conformal field theory living on boundary of AdS using standard formulae[30]. The result is

$$\begin{aligned} c_L &= 24\pi \Big(\sqrt{\frac{2}{\Lambda(\phi_0)}} - \frac{c_2 \cdot p}{96\pi} \Big) = 6p^3 + \frac{1}{2} c_2 \cdot p \\ c_R &= 24\pi \Big(\sqrt{\frac{2}{\Lambda(\phi_0)}} + \frac{c_2 \cdot p}{96\pi} \Big) = 6p^3 + c_2 \cdot p \end{aligned} \tag{2.65}$$

These results agree with the predictions of [35, 36] from the requirement of (0,4) supersymmetry, as well as the explicit calculations of [34, 33, 37] from the computation of the black hole entropy.

Chapter 3

A test of quantum entropy function

3.1 Introduction

Wald's formula for black hole entropy when applied to extremal black holes leads to classical entropy function formalism. However Wald's formula is a classical and can not be used for full 1PI effective action which in general contain no-local terms. Recently Sen made a proposal [38, 39] for computing quantum corrected entropy of single centered extremal black hole. In §1.5 we had given a brief introduction of this quantum entropy function. Quantum entropy function is a proposal which relates the entropy associated with horizon degree of freedom of extremal black hole to the degeneracy of the ground states of the CQM living at the boundary of AdS_2. This proposal suggested that the latter should be taken as the definition of the entropy of extremal black holes in the full quantum theory. In this chapter we would like to test this proposal in case of supersymmetric extremal BTZ black hole of mass M and angular momentum J.
For BTZ black hole there exist an independent definition of entropy of the black hole. Since the extremal BTZ black hole is a black hole solution in three dimensional theory of gravity with negative cosmological constant, by AdS/CFT correspondence it corresponds to states in the dual CFT with $\bar{L}_0$ eigen value $h_R = 0$ and L_0 eigen value $h_L = Ml$. In order that the state preserves supersymmetry it must belong to Ramond sector of the anti-holomorphic part of the superconformal algebra of the CFT, so that the condition $\bar{L}_0 = 0$ forces the state to be in the ground state of the Ramond sector[40]. The entropy of the black hole is then given by the degeneracy of this state. Hence in this case one would like to compare these two definition of the entropy of the black hole. In particular in order to compare these two definition one needs a relationship between the CQM and the CFT. In this chapter we derive this relationship and show that these two independent definitions of entropy agree.

3.2 AdS_3/CFT_2 to AdS_2/CFT_1

The general BTZ black hole solution in an AdS_3 space with scalar curvature $-6/l^2$ is given by

$$ds_3^2 = -\frac{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}{l^2\rho^2}d\tau^2 + \frac{l^2\rho^2}{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}d\rho^2 + \rho^2\left(dy - \frac{\rho_+\rho_-}{l\rho^2}d\tau\right)^2, \quad (3.1)$$

where τ denotes the time coordinate, ρ is the radial variable, y is the azimuthal angle with period 2π and $\rho_\pm$ are parameters labelling the black hole solution satisfying $\rho_+ > \rho_-$. M and J are determined in terms of $\rho_\pm$, but the precise relation requires the knowledge of higher derivative terms. Nevertheless the extremal limit always corresponds to $\rho_+ \to \rho_-$. Following [38] we take this limit by first defining new variables λ, t, r, ϕ and R through

$$\rho_+ - \rho_- = 2\lambda, \quad \rho - \rho_+ = \lambda(r-1), \quad \tau = l^2\, t/(4\lambda), \quad y = \phi + \frac{l}{4\lambda}\left(1-\frac{2\lambda}{\rho_+}\right)t, \quad \rho^+ = \frac{lR}{2}, \quad (3.2)$$

and then taking $\lambda \to 0$ with t, r, ϕ and R fixed. In this limit the metric (3.1) takes the form

$$ds_3^2 = \frac{l^2}{4}\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + R^2\left(d\phi + \frac{1}{R}(r-1)dt\right)^2\right]. \quad (3.3)$$

The metric (3.3) is locally AdS_3. Thus by the standard rules of AdS/CFT correspondence any quantum theory of gravity in the background (3.3) has a dual (1+1) dimensional conformal field theory. Since locally this AdS_3 space is the same as the one in which we embed the BTZ black hole, we expect that as a local field theory the (1+1) dimensional CFT living on the boundary of the near horizon geometry of the BTZ black hole must be identical to that living on the boundary of the AdS_3 in which the full BTZ black hole solution is embedded. The conformal structure of the two dimensional space in which the theory lives will however be quite different for the theory dual to AdS_3 and the one dual to the near horizon geometry of the black hole.

Now via a dimensional reduction we can also regard the three dimensional metric (3.3) as a two dimensional field configuration. For this we introduce a two dimensional metric ds_2^2, a scalar field χ and a gauge field a_μ via the relation:

$$ds_3^2 = ds_2^2 + \chi\,(d\phi + a_\mu dx^\mu)^2\,, \quad (3.4)$$

where $\{x^\mu\}$ for $\mu = 0, 1$ represent the two dimensional coordinates (t, r). From the two dimensional viewpoint, the background (3.3) takes the form

$$ds_2^2 = \frac{l^2}{4}\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right], \quad \chi = \frac{l^2\,R^2}{4}, \quad a_\mu dx^\mu = \frac{1}{R}(r-1)dt\,. \quad (3.5)$$

$$e \equiv F_{rt} = 1/R\,. \quad (3.6)$$

This describes an AdS_2 space-time with background scalar and electric field. Then via the rules of AdS/CFT correspondence the theory is dual to a CQM living on the boundary of AdS_2. In particular we can relate the partition function of the quantum gravity theory on AdS_2 to the partition function of the CQM living on the boundary of AdS_2[38].
Since (3.3) and (3.5) describe the same background, the quantum theories dual to them must also be identical. Consequently the CQM living on the boundary of (3.5) and the (1+1) dimensional CFT living on the boundary of (3.3) are also different descriptions of the same quantum theory. Our goal will be to exploit this equivalence to learn about the CQM living on the boundary of AdS_2.
First we consider the two dimensional viewpoint. The metric is that of AdS_2, and the boundary is located at $r = r_0$. The induced metric, scalar and gauge field on the boundary are

$$ds_B^2 = -\frac{l^2}{4}(r_0^2 - 1)dt^2, \qquad \chi_B = \frac{l^2 R^2}{4}, \qquad a_t|_B = \frac{1}{R}(r_0 - 1)\,. \tag{3.7}$$

We shall denote by H_t the total Hamiltonian of the CQM living on the boundary of AdS_2 including the effect of the background gauge fields and by Q the conserved charge in the CQM conjugate to the gauge field a_μ in the bulk.
We now turn to the three dimensional viewpoint. The dual (1+1) dimensional CFT lives on the two dimensional boundary labelled by (t, ϕ) with induced metric

$$ds_B^2 = \frac{l^2}{4}\left[-(r_0^2 - 1)dt^2 + R^2\left(d\phi + \frac{1}{R}(r_0 - 1)dt\right)^2\right]. \tag{3.8}$$

We now introduce new coordinates

$$\widetilde{t} = R^{-1}\sqrt{r_0^2 - 1}\,t, \quad \widetilde{\phi} = \phi + \frac{1}{R}(r_0 - 1)t\,, \tag{3.9}$$

so that the metric (3.8) becomes

$$ds_B^2 = \frac{l^2 R^2}{4}[-d\widetilde{t}^2 + d\widetilde{\phi}^2]\,. \tag{3.10}$$

Thus up to the overall scale factor the metric is the standard Minkowski metric, and the space coordinate $\widetilde{\phi}$ is compact with period 2π. This gives a standard 1+1 dimensional CFT on a cylinder, and the generators $i\partial_{\widetilde{t}}$ and $-i\partial_{\widetilde{\phi}}$ are identified as

$$i\partial_{\widetilde{t}} = L_0 + \bar{L}_0, \qquad -i\partial_{\widetilde{\phi}} = L_0 - \bar{L}_0\,. \tag{3.11}$$

In order that in the extremal limit we get a supersymmetric black hole, we impose Ramond boundary condition along $\widetilde{\phi}$ on the anti-holomorphic part of the superconformal algebra.

In relating this (1+1) dimensional CFT to the CQM living on the boundary of AdS_2, we must identify the total Hamiltonian H_t of the CQM as the generator of t-translation in the CFT. On the other hand the charge Q of the CQM can be identified as the generator of ϕ translation. This gives

$$\begin{aligned} H_t &= i\partial_t = iR^{-1}\sqrt{r_0^2-1}\frac{\partial}{\partial\widetilde{t}} + i\frac{r_0-1}{R}\frac{\partial}{\partial\widetilde{\phi}} = 2\,R^{-1}r_0\bar{L}_0 + R^{-1}(L_0-\bar{L}_0) + \mathcal{O}(r_0^{-1})\,, \\ Q &= -i\partial_\phi = -i\partial_{\widetilde{\phi}} = L_0 - \bar{L}_0\,. \end{aligned} \tag{3.12}$$

Thus in the $r_0 \to \infty$ limit, the only states with finite H_t eigenvalues are those with minimal value of $\bar{L}_0$. Since we have Ramond boundary condition, the minimal value of $\bar{L}_0$ is 0. In other words the states of the CQM living on the boundary of AdS_2 are described by the $\bar{L}_0 = 0$ states of the 1+1 dimensional CFT living on the boundary of AdS_3 [41, 42, 43]. In other words the CQM living at the boundary of AdS_2 is described by the chiral half (1+1) dimensional CFT living at the boundary of AdS_3. In particular *the ground state degeneracy $d(q)$ of the CQM, carrying a given charge q, can be identified as the degeneracy of the states of the CFT which are in the ground state of the Ramond sector in the anti-holomorphic sector and carries $(L_0-\bar{L}_0)$ eigenvalue q.* The former is the quantity that appears in the definition of the entropy via AdS_2/CFT_1 correspondence[38] whereas the latter appears in the definition of the entropy of the extremal BTZ black hole via AdS_3/CFT_2 correspondence. Thus we see that the two definitions of entropy agree.
Using the identification of the CQM as a specific compactification of the CFT we can compute the partition function of the theory. For this we make the Euclidean continuation $t \to -iu$. Regularity of the metric (3.3) (or (3.5)) at the horizon $r = 1$ requires u to be a periodic coordinate with period 2π. Under the replacement $t \to -iu$ the boundary metric (3.8) takes the form

$$ds_B^2 = \frac{l^2}{4}\left[(r_0^2-1)du^2 + R^2\left(d\phi - \frac{i}{R}(r_0-1)du\right)^2\right] = \frac{l^2R^2}{4}[\tau_2^2du^2 + (d\phi+\tau_1 du)^2]\,, \tag{3.13}$$

where

$$\tau_1 = -\frac{i}{R}(r_0-1), \quad \tau_2 = \frac{\sqrt{r_0^2-1}}{R}\,. \tag{3.14}$$

Since u and ϕ both have period 2π, the partition function of the CFT with this background metric will be given by

$$Z = Tr\left[e^{2\pi i(\tau_1+i\tau_2)L_0 - 2\pi i(\tau_1-i\tau_2)\bar{L}_0}\right] = Tr\left[e^{-4\pi r_0 R^{-1}\bar{L}_0 - 2\pi R^{-1}(L_0-\bar{L}_0)} + \mathcal{O}(r_0^{-1})\right]\,. \tag{3.15}$$

This agree with the partition function of the CQM which is given by $Tr(e^{-2\pi H_t})$ with H_t given in (3.12). Eq.(3.15) again shows that in the $r_0 \to \infty$ limit only the $\bar{L}_0 = 0$ states contribute to the trace. We also see that in this limit the contribution to the partition function from states with a given charge $Q = q$ is given by

$$d(q)\,e^{-2\pi eq}\,, \tag{3.16}$$

where q is the $L_0 - \bar{L}_0$ eigenvalue, $e = 1/R$ is the near horizon electric field, and $d(q)$ is the degeneracy of the states with charge q.
Till now we have considered only neutral BTZ black hole. However one can repeat the same analysis for charged BTZ black hole. Let us now suppose that the three dimensional theory has additional $U(1)$ gauge fields $A_M^{(i)}$ with Chern-Simons action of the form

$$\frac{1}{2}\int d^3x\, \epsilon^{MNP}\, C_{ij}\, A_M^{(i)} F_{NP}^{(j)}, \qquad F_{NP}^{(i)} \equiv \partial_N A_P^{(i)} - \partial_P A_N^{(i)}\,, \tag{3.17}$$

where M, N, P run over the three coordinates of AdS_3 and C_{ij} are constants. Then we can construct charged black hole solutions by superimposing on the original BTZ solution (3.1) constant gauge fields:

$$A_M^{(i)} dx^M = w_i \left[dy - \frac{1}{l}\frac{\rho_-}{\rho_+}\, d\tau \right]. \tag{3.18}$$

Here w_i are constants. The term proportional to $d\tau$ has been chosen so as to make the gauge fields non-singular at the horizon. Even though the gauge field strength vanishes, the background (3.18) induces a charge on the black hole since the latter, being proportional to $\delta S/\delta F_{\rho t}^{(i)}$ (in the classical limit), is given by $C_{ij} A_y^{(j)}$ up to a constant of proportionality. Taking the near horizon limit as in (3.2) we arrived at the background

$$ds_3^2 = \frac{l^2}{4}\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + R^2\left(d\phi + \frac{1}{R}(r-1)dt \right)^2 \right], \qquad A_M^{(i)} dx^M = w_i d\phi\,. \tag{3.19}$$

In order to make contact with the two dimensional viewpoint we define two dimensional gauge fields $a_\mu^{(i)}$ and scalar fields $\chi^{(i)}$ via the relations:

$$A_M^{(i)} dx^M = \chi^{(i)}(d\phi + a_\mu dx^\mu) + a_\mu^{(i)}\, dx^\mu\,, \tag{3.20}$$

where a_μ has been defined in (3.4). For the background (3.19) we have $a_\mu dx^\mu = \frac{1}{R}(r-1)dt$, and hence[15]

$$\chi^{(i)} = w_i, \quad a_\mu^{(i)} dx^\mu = e^{(i)}(r-1)dt, \quad e^{(i)} \equiv -\frac{w_i}{R}\,. \tag{3.21}$$

$e^{(i)}$ is the near horizon electric field associated with the two dimensional gauge fields $a_\mu^{(i)}$.

We shall now compute the partition function of the CQM living on the boundary of AdS_2 in the presence of these background gauge fields. This is equivalent to computing the partition function of the CFT living on the boundary of the space-time given in (3.19). According to AdS/CFT correspondence [20, 22], the presence of the gauge fields in the bulk will induce a current in the boundary CFT. Let $(J_{(i)}^\phi, J_{(i)}^t)$ be the currents in the CFT dual to the gauge fields $A_M^{(i)}$ in the bulk. Typically in AdS_3/CFT_2 correspondence the currents dual to gauge fields are either holomorphic or anti-holomorphic depending on the sign of the Chern-Simons term in the bulk theory[44]. We shall assume for simplicity that all our gauge fields are dual to holomorphic currents. This gives a relation between $J^{\phi}_{(i)}$ and $J^{t}_{(i)}$. To determine this relation we note from (3.13) that in the Euclidean theory the holomorphic coordinate z is given by $\phi + \tau_1 u + i\tau_2 u$. Using the relation $u = it$ and the values of τ_1, τ_2 given in (3.14) we get

$$z = \phi - \frac{1}{R}t + \mathcal{O}(r_0^{-1})\,. \tag{3.22}$$

Requiring holomorphicity gives $J^{z}_{(i)} = 0$ since by virtue of current conservation $\partial_z J^{z}_{(i)} = 0$, $J^{z}_{(i)}$ would have described an anti-holomorphic current. Thus we have

$$J^{\phi}_{(i)} - \frac{1}{R}J^{t}_{(i)} = 0\,. \tag{3.23}$$

In the presence of the gauge fields in the bulk the partition function in the CFT will have an insertion of

$$\exp\left[iw_i \int dt d\phi \sqrt{-\det g} J^{\phi}_{(i)}\right]\,, \tag{3.24}$$

Substituting (3.23) in the above equation and using the definition of the charge $Q_{(i)}$,

$$Q_{(i)} = \int d\phi \sqrt{-\det g} J^{t}_{(i)}\,, \tag{3.25}$$

we can express (3.24) as

$$\exp\left[iw_i \int dt\, Q_{(i)}/R\right] = \exp(2\pi\, w_i\, Q_{(i)}/R) = \exp(-2\pi\, e^{(i)} Q_{(i)})\,, \tag{3.26}$$

where in the last step we have used (3.21). Inserting this into (3.15) and using $e = 1/R$ we get

$$Z = Tr\left[e^{-4\pi r_0 R^{-1}\bar{L}_0 - 2\pi \sum_I e^I Q_I}\right]\,, \tag{3.27}$$

where the index I now runs over all the two dimensional gauge fields, – the one coming from the dimensional reduction of the three dimensional metric as well as the ones coming from the three dimensional gauge fields. From (3.27) we see that in the $r_0 \to \infty$ limit we are still restricted to the $\bar{L}_0 = 0$ states. The contribution from the sector with charge $\vec{q}$ is given by

$$d(\vec{q})\, e^{-2\pi \sum_I q_I e^I}\,, \tag{3.28}$$

Here $d(\vec{q})$ denotes the degeneracy of $\bar{L}_0 = 0$ states in the CFT carrying charge $\vec{q}$. It can also be interpreted as the degeneracy of the lowest energy states in the CQM carrying charge $\vec{q}$.

One issue that we have not completely resolved is the following. From (3.10) we see that in the $(\widetilde{t}, \widetilde{\phi})$ coordinate system the conformal factor in front of the metric remains finite as $r_0 \to \infty$, suggesting that we have a finite ultraviolet cut-off. In particular the size of the $\widetilde{\phi}$ circle is of the order of the cut-off. We do not have a direct understanding of the role of this cut-off in the CFT. However studying the effect of this cut-off in the bulk gives us some insight. First of all note that in conventional AdS_3, it is more natural to define the partition function by summing over states of all charges with a fixed value of the chemical potential. However in AdS_2 the modes representing fluctuation of the total charge represent non-normalizable deformations and hence it is more natural to define the partition function by summing over a fixed charge sector[38]. Thus it would seem that the effect of the finite ultraviolet cut-off in the CFT must be to restrict the Hilbert space of a given CFT to a fixed charge sector. There are also other effects of this finite cut-off in the bulk when we embed the BTZ black hole in a supersymmetric theory with additional moduli scalars and vector fields. When we view the extremal BTZ black hole from the point of view of the asymptotically AdS_3 space-time by setting $\rho_+ = \rho_-$ in (3.1) then the ultraviolet cut-off is small compared to the size of the y circle since the latter approaches ∞ as $\rho \to \infty$, but such asymptotic space-time could admit other multi-centered black hole solutions[45]. On the other hand when we view the same extremal black hole from the point of view of its near horizon geometry as in (3.3), then the size of the ϕ circle becomes comparable to the ultra-violet cut-off, but this space-time geometry no longer admits the other multi-centered black hole solutions in AdS_2 since the values of the various scalar fields are fixed at their attractor values.[1] Thus it would seem that the ultraviolet cut-off weeds out the contribution due to the multi-centered black hole configurations of the type discussed in [45] from the CFT spectrum. In support of this speculation we would like to note that for large R the size of the ϕ circle is large compared to the ultra-violet cut-off and hence effect of the cut-off is expected to be small. This is precisely the region in which the entropy of a single centered black hole gives the dominant contribution to the entropy[45].

Even though it is more natural to work in a fixed charge sector of AdS_2, one can get some insight into the OSV conjecture if one does sum over the contribution from different charge sectors. After summing over charges the full partition function is given by

$$Z(\vec{e}) = \sum_{\vec{q}} d(\vec{q})\, e^{-2\pi \vec{e}\cdot\vec{q}}. \tag{3.29}$$

For large charges the dominant contribution to this sum comes from $\vec{q}$ satisfying $\partial \ln d(\vec{q})/\partial q_I = 2\pi e^I$, in agreement with the classical relation between the electric field and the charge. The right hand side of (3.29) has the flavor of the black hole partition function defined in [47]. On the other hand, using AdS/CFT correspondence, the left hand side can be

[1] Possible exceptions are multi-centered black holes with mutually local charges, i.e. charges satisfying $(\vec{q}_i \cdot \vec{p}_j - \vec{q}_j \cdot \vec{p}_i) = 0$ where $(\vec{q}_i, \vec{p}_i)$ denote the electric and magnetic charge vectors of the ith black hole. But they do not contribute to the degeneracy[46].

expressed as a functional integral over the fields in the bulk theory.[2] Now, as was shown in [38], after ignoring terms linear in r_0 in the exponent – which must cancel among themselves – the classical result for the partition function in the $r_0 \to \infty$ limit is given by

$$Z = e^{-2\pi f}\,, \tag{3.30}$$

where f is the classical Lagrangian density evaluated in the near horizon geometry. One might expect that the effect of quantum corrections would be to replace the classical Lagrangian density by some effective Lagrangian density. As we shall now review, if we assume that the effective Lagrangian density that contributes to the partition function is governed only by the F-type terms, i.e. terms which can be encoded in the prepotential $\mathcal{F}$[55], then Z takes the form predicted in the original OSV conjecture.

In $\mathcal{N} = 2$ supergravity theories in four dimensions the information about the 'F-type terms' can be encoded in a function $\mathcal{F}(\{X^I\}, \widehat{A})$ – known as the prepotential – of a set of complex variables X^I which are in one to one correspondence with the gauge fields and an auxiliary complex variable $\widehat{A}$ related to the square of the graviphoton field strength[55, 56]. Supersymmetry demands that $\mathcal{F}$ is a homogeneous function of degree two in its arguments:

$$\mathcal{F}(\{\lambda X^I\}, \lambda^2\widehat{A}) = \lambda^2\mathcal{F}(\{X^I\}, \widehat{A})\,. \tag{3.31}$$

For a given choice of electric field one finds that the extremum of the effective Lagrangian density computed with the F-term effective action occurs at the attractor point where[57, 58, 59, 60, 61, 62, 63]

$$\widehat{A} = -4w^2, \quad 4(\bar{w}^{-1}\bar{X}^I + w^{-1}X^I) = e^I, \quad 4(\bar{w}^{-1}\bar{X}^I - w^{-1}X^I) = -ip^I\,. \tag{3.32}$$

Here w is an arbitrary complex parameter and p^I are the magnetic charges carried by the black hole. These magnetic charges have not appeared explicitly in our discussion so far because from the point of view of the near horizon geometry they represent fluxes through compact two cycles and appear as parameters labelling the two (or three) dimensional field theory describing the near horizon dynamics. The value of the effective Lagrangian density at the extremum (3.32) is given by[60]

$$f = 16\,i\,(w^{-2}\mathcal{F} - \bar{w}^{-2}\bar{\mathcal{F}})\,. \tag{3.33}$$

[2]Note that we have switched back from the three dimensional viewpoint to the two dimensional viewpoint. The black hole partition function has been analyzed using AdS/CFT correspondence earlier (see *e.g.* [48, 49]). Also various other approaches to relating the entropy function formalism to Euclidean action formalism and / or OSV conjecture can be found in [50, 51, 52]. The advantage of our approach lies in the fact that since we apply AdS/CFT correspondence on the near horizon geometry, the chemical potentials dual to the charges are directly related to the near horizon electric field, and hence, via the attractor mechanism, to other near horizon field configuration. Furthermore the path integral needs to be performed only over the near horizon geometry where we have enhanced supersymmetry and hence stronger non-renormalization properties. The approach closest to ours is the one given in [53]; we shall comment on it later. A different approach to deriving the OSV conjecture using AdS/CFT correspondence can be found in [54].

Note that (3.32) determines X^I in terms of the unknown parameter w. However due to the scaling symmetry (3.31), f given in (3.33) is independent of w. Using this scaling symmetry we can choose

$$w = -8\,i\,, \tag{3.34}$$

and rewrite (3.32), (3.33) as

$$\widehat{A} = 256, \quad X^I = -i(e^I + ip^I)\,, \tag{3.35}$$

$$f = -\frac{i}{4}(\mathcal{F}(\{X^I\}, 256) - \overline{\mathcal{F}(\{X^I\}, 256)})\,. \tag{3.36}$$

Thus we have

$$Z(\vec{e}) = e^{-\pi\, Im\, \mathcal{F}(\{p^I - ie^I\}, 256)}\,. \tag{3.37}$$

This is precisely the original OSV conjecture[47].

It has however been suggested in subsequent papers that agreement with statistical entropy requires modifying this formula by including additional measure factors on the right hand side of (3.37)[64, 65, 46]. A careful analysis of the path integral keeping track of the holomorphic anomaly[66, 67] may be able to reproduce these corrections. Some of these corrections are in fact necessary for restoring the duality invariance of the final result for the entropy[65]. However we will not discuss these issues in the present chapter.

Chapter 4

Localization and quantum entropy function

4.1 Introduction

Quantum entropy function, as described in §1.5, proposed a definite relation between the degeneracy $d_{hor}(q)$ associated with the black hole horizon and the partition function of string theory on the near horizon geometry. This relation takes the form

$$d_{hor} = \left\langle \exp\left[-iq_i \oint d\theta\, A^{(i)}_\theta\right]\right\rangle^{finite}_{AdS_2}, \tag{4.1}$$

where $\langle\ \rangle_{AdS_2}$ denotes the unnormalized path integral over all the fields in string theory. In this path integral the string fields required to satisfy the boundary condition that asymptotically the field configuration approaches the near horizon configuration of the black hole. Here $\{A^{(i)}\}$ denote the set of all U(1) gauge fields, q_i is the i-th electric charge carried by the black hole [1] and $\oint d\theta\, A^{(i)}_\theta$ denotes the integral of the i-th gauge field along the boundary of AdS_2 which is labelled by the coordinate θ.
In four space-time dimensions supersymmetry requires the black holes to be spherically symmetric, and as a consequence the near horizon geometry has an $AdS_2 \times S^2$ factor. For 1/8 BPS black holes in $\mathcal{N} = 8$ supersymmetric theories, 1/4 BPS black holes in $\mathcal{N} = 4$ supersymmetric theories and 1/2 BPS black holes in $\mathcal{N} = 2$ supersymmetric theories, the $SL(2, \mathbb{R}) \times SO(3)$ isometry of the near horizon geometry gets enhanced to the $SU(1, 1|2)$ supergroup. Using enhanced supersymmetry and arguments of localization we will show in this chapter that the path integral (4.1) could receive non-vanishing contribution only from a special class of field configurations which are invariant under a particular subgroup of the supergroup $SU(1, 1|2)$. Intuitively the localization of an integral can be understood

[1] In particular q_i also include the angular momentum of the black hole.

in terms of arguments similar to [68] as follows.
Consider an arbitrary quantum field theory, with some function space $\mathcal{M}$ over which one wishes to integrate.

$$Z = \int_{\mathcal{M}} e^{-\mathcal{S}} \tag{4.2}$$

where $\mathcal{S}$ is the action of the theory.

Let F be a supergroup of global symmetries of the theory generated by fermionic generator Q and a compact U(1) bosonic generator X which satisfy the algebra

$$Q^2 = X. \tag{4.3}$$

To begin with let us suppose that F acts freely on $\mathcal{M}$ i.e. there are no points in $\mathcal{M}$ which are invariant under the elements of the supergroup F. Then one has a fibration $\mathcal{M} \to \mathcal{M}/F$, and by first integrating over the fibers of this fibration, one can reduce the integral over $\mathcal{M}$ to an integral over $\mathcal{M}/F$. The integral over the fibers will give a factor which is the volume of the supergroup F.

$$Z = \int_{\mathcal{M}} e^{-\mathcal{S}} = vol(F) \int_{\mathcal{M}/F} e^{-\mathcal{S}} \tag{4.4}$$

However the volume of the supergroup F is zero. The integral over the parameter x of the compact generator will give a finite factor and the fermionic θ integral will make the integral zero.

$$vol(F) = \int dx d\theta = 0 \tag{4.5}$$

Hence (4.4) suggests that if the supergroup F acts freely on $\mathcal{M}$, the partition function vanishes.
In general, F does not act freely, but has a fixed point locus $\mathcal{M}_0$. Let $\mathcal{C}$ be an F -invariant neighborhood of $\mathcal{M}_0$ and $\mathcal{M}'$ its complement. Then the path integral restricted to $\mathcal{M}'$ vanishes, by the above reasoning. So the entire contribution to the path integral comes from the integral over $\mathcal{C}$. Since $\mathcal{C}$ can be an arbitrary small neighborhood and the integral is independent of the choice of $\mathcal{C}$, the result is a localization formula expressing the path integral as an integral on $\mathcal{M}_0$. Identifying $\mathcal{M}_0$ in our case is the main subject of this chapter.
The rest of the chapter is organised as follows. In §4.2 we describe the algebra underlying the $SU(1,1|2)$ group and also the reality condition on the various generators required to represent the symmetries of the Euclidean near horizon geometry. In §4.3 we use localization techniques developed in [69, 70] to argue that the path integral receives contribution only from a special class of string field configurations invariant under a special subgroup H_1 of the $SU(1,1|2)$ group. In §4.4 we use the results of §4.3 to show that integration over the bosonic and fermion zero modes, generated by an infinite dimensional group of asymptotic symmetries, actually gives a finite result to the path integral. In §4.5 we give

some examples of H_1-invariant saddle points which contribute to the path integral. In §4.6 we discuss possible application of our result to further simplify the analysis of quantum entropy function and also a possible application to computing the expectation values of circular 't Hooft - Wilson loop operators in superconformal gauge theories following [71]. In appendix A we analyze the killing spinors in the near horizon geometry of a specific class of quarter BPS black holes in type IIB string theory compactified on $K3 \times T^2$ and show that they indeed generate the $su(1,1|2)$ algebra described in §4.2.

4.2 Symmetries of Euclidean $AdS_2 \times S^2$

Global supersymmetry of $AdS_2 \times S^2$ is described by $su(1,1|2)$ super Lie algebra. It is the global part of $\mathcal{N} = 4$ superconformal algebra in (1+1) dimension. We begin by writing down the global part of the $\mathcal{N} = 4$ superconformal algebra. Its non-vanishing commutators are

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} \\
&[J^3, J^\pm] = \pm J^\pm\,, \qquad [J^+, J^-] = 2\,J^3 \\
&[L_n, G_r^{\alpha\pm}] = \left(\frac{n}{2} - r\right) G^{\alpha\pm}_{r+n} \\
&[J^3, G_r^{\alpha\pm}] = \pm\,\frac{1}{2}\,G_r^{\alpha\pm}\,, \quad [J^\pm, G_r^{\alpha\mp}] = G_r^{\alpha\pm} \\
&\{G_r^{+\alpha}, G_s^{-\beta}\} = 2\,\epsilon^{\alpha\beta}\,L_{r+s} - 2\,(r-s)\,(\epsilon\sigma^i)_{\beta\alpha}\,J^i \\
&\epsilon^{+-} = -\epsilon^{-+} = 1, \quad \epsilon^{++} = \epsilon^{--} = 0\,, \qquad m,n = 0,\pm1, \quad r,s = \pm\frac{1}{2}, \quad \alpha,\beta = \pm
\end{aligned}
\tag{4.6}
$$

The right superscript of G_r denotes the transformation properties under the SU(2) current algebra whose zero modes are denoted by ($J^3, J^\pm = J^1 \pm iJ^2$). There is also an $SU(2)$ group acting on the left superscript. This describes an outer automorphism of the supersymmetry algebra but is not in general a symmetry of the theory.
In the above, the action of the Virasoro generators on the coordinate u labelling the upper half plane (UHP) is of the form

$$L_n = -u^{n+1}\partial_u - \bar{u}^{n+1}\partial_{\bar{u}}\,. \tag{4.7}$$

However while describing symmetries of the Euclidean $AdS_2 \times S^2$, which is isomorphic to $UHP \times S^2$ it is more natural to use the Virasoro generators

$$L_n = -\left(i\,u^{n+1}\partial_u + i\,\bar{u}^{n+1}\partial_{\bar{u}}\right)\,, \tag{4.8}$$

so that the elements of SL(2,$\mathbb{R}$) can be labelled as $\exp(is_nL_n)$ with real parameters s_n, just as $\exp(it_iJ^i)$ labels an element of the $SU(2)$ group for real t_i. The corresponding algebra is obtained from (4.6) by scaling the Virasoro generators by i. For later convenience we shall also multiply $G_r^{+\alpha}$ by e^{is_0} and $G_r^{-\alpha}$ by $i\,e^{-is_0}$ for some arbitrary fixed phase e^{is_0}. This gives[2]

$$\begin{aligned}
&[L_m, L_n] = i(m-n)L_{m+n} \\
&[J^3, J^\pm] = \pm J^\pm\,, \qquad [J^+, J^-] = 2\,J^3 \\
&[L_n, G_r^{\alpha\pm}] = i\left(\frac{n}{2} - r\right) G_{r+n}^{\alpha\pm} \\
&[J^3, G_r^{\alpha\pm}] = \pm\frac{1}{2}\,G_r^{\alpha\pm}\,, \quad [J^\pm, G_r^{\alpha\mp}] = G_r^{\alpha\pm} \\
&\{G_r^{+\alpha}, G_s^{-\beta}\} = 2\,\epsilon^{\alpha\beta}\,L_{r+s} - 2\,i\,(r-s)\,(\epsilon\sigma^i)_{\beta\alpha}\,J^i \\
&\epsilon^{+-} = -\epsilon^{-+} = 1, \quad \epsilon^{++} = \epsilon^{--} = 0\,, \qquad m,n = 0,\pm 1, \quad r,s = \pm\frac{1}{2}, \quad \alpha,\beta = \pm
\end{aligned} \tag{4.9}$$

However it is convenient to represent AdS_2 as a unit disk labelled by a coordinate w related to u via:

$$w = \frac{1 + i\,u}{1 - i\,u}\,. \tag{4.10}$$

In the w coordinate system

$$L_n = \frac{i}{2}\,\left[i^n\,(1+w)^{1-n}\,(1-w)^{1+n}\,\partial_w + c.c.\right]\,. \tag{4.11}$$

On the other hand the action of the J^i's on the stereographic coordinate z of the sphere S^2 takes the form

$$J^3 = (z\partial_z - \bar{z}\partial_{\bar{z}}), \qquad J^+ = z^2\partial_z + \partial_{\bar{z}}, \qquad J^- = -\bar{z}^2\partial_{\bar{z}} - \partial_z\,. \tag{4.12}$$

It is easy to see that the $AdS_2 \times S^2$ metric

$$ds^2 = 4\,v\,\frac{dw\,d\bar{w}}{(1-\bar{w}w)^2} + 4\,u\,\frac{dz\,d\bar{z}}{(1+\bar{z}z)^2}\,, \tag{4.13}$$

where u and v are constants, is invariant under these transformations. Making the coordinate transformations

$$w = \tanh\frac{\eta}{2}\,e^{i\theta}, \qquad z = \tan\frac{\psi}{2}e^{i\phi}\,, \tag{4.14}$$

we can express the metric (4.13) as

$$ds^2 = v(d\eta^2 + \sinh^2\eta\,d\theta^2) + u(d\psi^2 + \sin^2\psi\,d\phi^2)\,. \tag{4.15}$$

[2]Note that while computing the commutators we regard the action of the generators as active transformation.

We now define

$$\widehat{L}_0 = \frac{1}{2}\,(L_1 + L_{-1})\,, \quad \widehat{L}_\pm = L_0 \pm \frac{i}{2}(L_1 - L_{-1})\,, \quad \widehat{G}^{\alpha\beta}_\pm = G^{\alpha\beta}_{1/2} \mp i\, G^{\alpha\beta}_{-1/2}\,. \tag{4.16}$$

From eqs.(4.11), (4.16) we see that the action of $\widehat{L}_0$, $\widehat{L}_\pm$ on the w-plane is given by

$$\widehat{L}_0 = (w\partial_w - \bar{w}\partial_{\bar{w}}), \quad \widehat{L}_+ = -i(w^2\partial_w - \partial_{\bar{w}}), \quad \widehat{L}_- = i(\partial_w - \bar{w}^2\partial_{\bar{w}})\,. \tag{4.17}$$

This shows that $\widehat{L}_0$ has the interpretation of the generator of rotation about the origin in the w-plane. In terms of these new generators the non-vanishing (anti-)commutators of the $su(1,1|2)$ algebra take the form

$$\begin{aligned}
&[\widehat{L}_0, \widehat{L}_\pm] = \pm\,\widehat{L}_\pm\,, \qquad [\widehat{L}_+, \widehat{L}_-] = -2\,\widehat{L}_0\,,\\
&[J^3, J^\pm] = \pm J^\pm\,, \qquad [J^+, J^-] = 2\,J^3,\\
&[\widehat{L}_0, \widehat{G}^{\alpha\beta}_\pm] = \pm\,\frac{1}{2}\,\widehat{G}^{\alpha\beta}_\pm\,, \quad [\widehat{L}_\pm, \widehat{G}^{\alpha\beta}_\mp] = -i\,\widehat{G}^{\alpha\beta}_\pm,\\
&[J^3, \widehat{G}^{\alpha\pm}_\beta] = \pm\,\frac{1}{2}\,\widehat{G}^{\alpha\pm}_\beta\,, \quad [J^\pm, \widehat{G}^{\alpha\mp}_\beta] = \widehat{G}^{\alpha\pm}_\beta,\\
&\{\widehat{G}^{+\alpha}_\pm, \widehat{G}^{-\beta}_\mp\} = 4\,\epsilon^{\alpha\beta}\,\widehat{L}_0 \pm 4\,(\epsilon\sigma^i)_{\beta\alpha}\,J^i\,, \qquad \{\widehat{G}^{+\alpha}_\pm, \widehat{G}^{-\beta}_\pm\} = \mp\,4\,i\,\epsilon^{\alpha\beta}\,\widehat{L}_\pm,
\end{aligned} \tag{4.18}$$

Note that an element of the form $\exp\left[i(\xi^0\widehat{L}_0 + \xi^+\widehat{L}_+ + \xi^-\widehat{L}_- + \eta_3 J^3 + \eta_+ J^+ + \eta_- J^-)\right]$ will be an element of the $SL(2,\mathbb{R}) \times SU(2)$ group if we have

$$(\xi^0)^* = \xi^0, \quad (\xi^\pm)^* = \xi^\mp, \quad (\eta_3)^* = \eta_3, \quad (\eta_\pm)^* = \eta_\mp\,. \tag{4.19}$$

We shall call these the reality conditions on the bosonic generators. We shall now impose a similar reality condition on the fermionic generators, i.e. specify the condition on the complex grassman parameters $\theta^\gamma_{\alpha\beta}$ under which $\exp\left[i\theta^\gamma_{\alpha\beta}G^{\alpha\beta}_\gamma\right]$ describes an element of the $SU(1,1|2)$ group. Any such rule must be compatible with the requirement that if $\exp(iT_1)$ and $\exp(iT_2)$ are two elements of the $SU(1,1|2)$ group, then $\exp([T_1,T_2])$ must also be an element of this group. The following constraint on $\theta^\gamma_{\alpha\beta}$ is compatible with this rule:[3]

$$\left(\theta^\gamma_{\alpha\beta}\right)^* = \epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\,\theta^{-\gamma}_{\alpha'\beta'}\,. \tag{4.20}$$

Equivalently we can say that

$$\exp\left[i\theta\left(\widehat{G}^{\alpha\beta}_\gamma + \epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\widehat{G}^{\alpha'\beta'}_{-\gamma}\right)\right] \quad \text{and} \quad \exp\left[\theta\left(\widehat{G}^{\alpha\beta}_\gamma - \epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\widehat{G}^{\alpha'\beta'}_{-\gamma}\right)\right]\,, \tag{4.21}$$

[3]We need to remember that if θ_1 and θ_2 are real grassman parameters then $(\theta_1\theta_2)^* = \theta_2\theta_1 = -\theta_1\theta_2$.

are elements of $SU(1,1|2)$ for real θ. We shall proceed with this choice. If we now define

$$\begin{aligned} Q_1 &= \widehat{G}_+^{++} + \widehat{G}_-^{--}, \qquad Q_2 = -i\left(\widehat{G}_+^{++} - \widehat{G}_-^{--}\right), \\ Q_3 &= -i\left(\widehat{G}_+^{-+} + \widehat{G}_-^{+-}\right), \qquad Q_4 = \widehat{G}_+^{-+} - \widehat{G}_-^{+-}, \\ \widetilde{Q}_1 &= \widehat{G}_-^{++} + \widehat{G}_+^{--}, \qquad \widetilde{Q}_2 = -i\left(\widehat{G}_-^{++} - \widehat{G}_+^{--}\right), \\ \widetilde{Q}_3 &= -i\left(\widehat{G}_-^{-+} + \widehat{G}_+^{+-}\right), \qquad \widetilde{Q}_4 = \widehat{G}_-^{-+} - \widehat{G}_+^{+-}, \end{aligned} \tag{4.22}$$

then $\exp(i\theta Q_i)$ and $\exp(i\theta\widetilde{Q}_i)$ are elements of $SU(1,1|2)$ for real θ. In that case we have

$$\{Q_i, Q_j\} = 8\,\delta_{ij}\,(\widehat{L}_0 - J^3), \quad \{\widetilde{Q}_i, \widetilde{Q}_j\} = 8\,\delta_{ij}\,(\widehat{L}_0 + J^3), \quad [\widehat{L}_0 - J^3, Q_i] = 0, \quad [\widehat{L}_0 + J^3, \widetilde{Q}_i] = 0\,. \tag{4.23}$$

Besides this, $\{Q_i, \widetilde{Q}_j\}$ are given by linear combinations of $J^\pm$ and $\widehat{L}_\pm$, $[\widehat{L}_0 - J^3, \widetilde{Q}_i]$ are given by linear combinations of $\widetilde{Q}_i$, $[\widehat{L}_0 + J^3, Q_i]$ are given by linear combinations of Q_i, $[J^\pm, Q_i]$ and $[\widehat{L}_\pm, Q_i]$ are given by linear combinations of $\widetilde{Q}_i$ and $[J^\pm, \widetilde{Q}_i]$, $[\widehat{L}_\pm, \widetilde{Q}_i]$ are given by linear combinations of Q_i. Precise form of these relations can be determined from (4.18) and (4.22), but we shall not write them down explicitly.
Given the reality condition on the various generators, we can label an element of $SU(1,1|2)$ as

$$\begin{aligned} g(\xi, \bar{\xi}, \eta, \bar{\eta}, \sigma, \widetilde{\sigma}, \{\theta_{\alpha\beta}\}, \{\chi_i\}) \;=\; & \exp\left[i\left\{\bar{\xi}\widehat{L}_+ + \xi\widehat{L}_- + \bar{\eta}J^+ + \eta J^- + \theta_{\alpha+}\widehat{G}_-^{\alpha+} + \theta_{\alpha-}\widehat{G}_+^{\alpha-}\right\}\right] \\ & \times \exp\left[i\sigma\,(\widehat{L}_0 + J^3)\right] \times \exp\left[i\left\{\sum_{k=1}^{4}\chi_k Q_k + \widetilde{\sigma}(\widehat{L}_0 - J^3)\right\}\right], \end{aligned} \tag{4.24}$$

where ξ, η are complex bosonic parameters, σ, $\widetilde{\sigma}$ are real bosonic parameters, χ_i are real grassman parameters and $\theta_{\alpha\beta}$ are complex grassman parameters satisfying the reality condition

$$(\theta_{\alpha\beta})^* = \epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\theta_{\alpha'\beta'}\,. \tag{4.25}$$

Let us also denote by H_0 the subgroup of $SU(1,1|2)$ generated by

$$\widehat{L}_0 - J^3, \quad Q_1, \quad Q_2, \quad Q_3, \quad Q_4\,. \tag{4.26}$$

The non-vanishing (anti-)commutators of H_0 are

$$\left[\widehat{L}_0 - J^3, Q_i\right] = 0, \qquad \{Q_i, Q_j\} = 8\,\delta_{ij}\,(\widehat{L}_0 - J^3)\,. \tag{4.27}$$

Then in (4.24) the parameters $\widetilde{\sigma}$ and $\{\chi_i\}$ parametrize an element of H_0 and the parameters σ, ξ, η and $\theta_{\alpha\beta}$ parametrize the coset G/H_0.

Finally another subgroup of $SU(1,1|2)$ (and H_0) that will play an important role in our analysis is the subgroup H_1 generated by Q_1 and $(\widehat{L}_0 - J^3)$.

4.3 Localization

In computing the quantum entropy function, – the partition function of string theory on the near horizon geometry of the black hole – we need to integrate over all string field configurations. In order to carry out the path integral, which involves integration over infinite number of modes, it will be useful to fix the order in which we carry out the integration. We shall adopt the following definition of the path integral: first we shall integrate over the orbits of the subgroup H_1 generated by Q_1 and $(\widehat{L}_0 - J^3)$, then over the orbits generated by the others Q_i's belonging to the subgroup H_0 and then carry out the integration over the remaining variables in some order. As we shall see this definition will allow us to arrive at simple results on which configurations could contribute to the path integral. Our approach follows closely that of [69]. Throughout this analysis we shall implicitly assume that the theory admits a formalism in which at least the H_1 subalgebra of the $su(1,1|2)$ algebra, generated by Q_1 and $(\widehat{L}_0 - J^3)$, is realized off-shell. It may be possible to achieve this by generalizing the trick used in [72] for $\mathcal{N} = 4$ supersymmetric gauge theories. Finally we shall ignore the various issues related to gauge fixing. For supersymmetric gauge theories in four dimensions gauge fixing introduces various subtleties in the proof of localization[72]. However eventually these can be overcome, and we shall assume that similar results will hold for supergravity as well. Formally the division of the path integral into orbits of H_1 and directions transverse to these orbits can be done by manipulating the integral using Fadeev-Popov method.[4] By expressing an element of H_1 as

$$h = \exp(i\alpha Q_1 + i\beta(\widehat{L}_0 - J^3)) \tag{4.28}$$

we can express the path integral as

$$\left[\int dh\right]\left[\int e^{-A}\left(\prod_a \delta(F^a)\right)\,\mathrm{sdet}\frac{\delta F^a_{\vec{\tau}}}{\partial \tau^b}\bigg|_{\vec{\tau}=0}\right], \tag{4.29}$$

where $\int dh$ denotes integration over the group H_1 with Haar measure, A is the Euclidean action,[5] F^a are a pair of 'gauge fixing functionals' of the field configuration, $\vec{\tau}$ denote

[4] Unlike in the case of a gauge symmetry here we do not divide the path integral by the volume of the 'gauge group' H_1. The rest of the manipulation proceeds exactly as in the case of a gauge theory.

[5] We are including in A the bulk and the boundary contributions to the action including the $i\oint \vec{q}\cdot\vec{A}$ term that is necessary to make the path integral well defined [39, 25]. We shall also be implicitly assuming that the boundary terms have been chosen so that all the supersymmetries of the bulk theory are preserved.

collectively the parameters (α, β) labelling the elements of the group H_1 and $F^a_{\vec{\tau}}$ is the transform of F^a by the group element corresponding to the parameters $\vec{\tau}$. We now note that the integration over H_1 has a bosonic direction β which parameterizes a compact $U(1)$ group and hence gives a finite result, and a fermionic direction α. By the standard rules of integration over grassman parameters the fermionic integral gives a zero, making the whole integral vanish.

This argument breaks down around a configuration Φ which is invariant under a subgroup of H_1, since the matrix $(\delta F^a_{\vec{\tau}}/\delta\tau^b)$ in (4.29) becomes degenerate at this point. In this case we proceed as follows. First of all we note that a subgroup of H_1 can either be the whole of H_1 or the $U(1)$ group generated by $(\widehat{L}_0 - J^3)$. However if Φ is invariant only under $(\widehat{L}_0 - J^3)$, then the zero eigenvector of the matrix $\delta F^a_{\vec{\tau}}/\partial\tau^b\big|_{\vec{\tau}=0}$ is along the bosonic direction corresponding to the $U(1)$ transformation. This makes the sdet factor in (4.29) vanish on the configuration Φ but does not generate any divergence in the integrand. Hence our earlier argument can still be applied to show that the $\int dh$ factor makes the integral vanish. Thus the configuration Φ must be invariant under both Q_1 and $(\widehat{L}_0 - J^3)$. This allows us to choose the coordinates of the configuration space, measuring fluctuations around the configuration Φ, as follows. First by Fourier decomposing these fluctuations in the $(\theta - \phi)$ coordinates we can choose them to be eigenvectors of $(\widehat{L}_0 - J^3)$ with definite eigenvalues $m \in \mathcal{Z}$. For example for a scalar field a deformation of the form $e^{im(\theta-\phi)/2} f(\theta + \phi, r, \psi)$ for any arbitrary function f will have this property. Let us parametrize the set of all such bosonic fluctuations by coordinates $z^s_{(m)}$. The complex conjugate deformation, labelled by $z^{s*}_{(m)}$ will have $(\widehat{L}_0 - J^3)$ eigenvalue $-m$. To avoid double counting we shall denote the fluctuations with positive m by $z^s_{(m)}$ and fluctuations with negative m by $z^{s*}_{(m)}$. As s runs over different values, the parameters $z^s_{(m)}$ produce the complete set of bosonic deformations with $(\widehat{L}_0 - J^3)$ eigenvalue m. Now for $m \neq 0$, the action of the generator Q_1 on such a bosonic deformation cannot vanish since $Q_1^2 = 4(\widehat{L}_0 - J^3)$ acting on the fluctuation does not vanish. Instead this will generate a particular fermionic deformation with $(\widehat{L}_0 - J^3)$ eigenvalue m. Let us denote the parameter associated with the fermionic deformation by $\zeta^s_{(m)}$. Finally we shall call the $m = 0$ bosonic and fermionic modes collectively as $\vec{y}$. Since the original configuration Φ is the origin of the coordinate system, all the coordinates vanish at Φ. We can now write[6]

$$Q_1\, z^s_{(m)} = \zeta^s_{(m)}, \quad Q_1\, \zeta^s_{(m)} = 4m\, z^s_{(m)}\,, \tag{4.30}$$

where the second equation follows from the fact that $(Q_1)^2\, z^s_{(m)} = 4\, m\, z^s_{(m)}$. Using the reality of the operator $(i\epsilon Q_1)$ and the rules for complex conjugation of grassman variables described in footnote 3, the complex conjugate relations of (4.30) can be expressed in the

[6]Our convention for defining the action of Q_1 on the parameters will be as follows. Take a general field configuration labelled by $(\{z^s_{(m)}\}, \{\zeta^s_{(m)}\}, \vec{y})$ and act on it by the transformation $(1 + i\epsilon Q_1)$. The new configuration can be associated with a new set of values of the various parameters. We call the parameters associated with the new configuration as $(\{z^s_{(m)} + i\epsilon Q_1\, z^s_{(m)}\}, \{\zeta^s_{(m)} + i\epsilon Q_1\, \zeta^s_{(m)}\}, \vec{y} + i\epsilon Q_1\, \vec{y})$.

form[7]

$$Q_1\, z^{s*}_{(m)} = \zeta^{s*}_{(m)}, \quad Q_1\, \zeta^{s*}_{(m)} = -4m\, z^{s*}_{(m)}\,. \tag{4.31}$$

$\zeta^s_{(m)}$ for different values of s give the complete set of fermionic deformations with $(\widehat{L}_0 - J^3)$ eigenvalue m and $\zeta^{s*}_{(m)}$ for different values of s give the complete set of fermionic deformations with $(\widehat{L}_0 - J^3)$ eigenvalue $-m$. To see this let us assume the contrary, i.e. that there is a fermionic coordinate $\chi_{(m)}$ carrying $\widehat{L}_0 - J^3$ eigenvalue m that is linearly independent of the $\zeta^s_{(m)}$'s (up to quadratic and higher powers of the other coordinates). Since the origin is Q_1 invariant, $Q_1\chi_{(m)}$ must vanish at the origin. On the other hand if $Q_1\chi_{(m)}$ is bilinear in the coordinates $(\{z^s_{(m)}\}, \{z^{s*}_{(m)}\}, \{\zeta^s_{(m)}\}, \{\zeta^{s*}_{(m)}\}, \chi_{(m)}, \vec{y})$ then it will be impossible to satisfy the $Q_1^2\chi_{(m)} = 4m\chi_{(m)}$ condition since the action of Q_1 on each of the coordinates produces a term linear and higher order in these coordinates. Thus $Q_1\chi_{(m)}$ must be a linear combinations of the complete set of bosonic coordinates $\{z^s_{(m)}\}$ carrying $\widehat{L}_0 - J^3$ eigenvalue m up to additional higher order terms in the coordinates. Applying Q_1 on either side we see that $\chi_{(m)}$ must be a linear combination of the coordinates $\zeta^s_{(m)}$ up to additional higher order terms, in contrary to our original assumption that $\chi_{(m)}$ is linearly independent of the other $\zeta^s_{(m)}$'s.

The coordinates $(\{z^s_{(m)}\}, \{z^{s*}_{(m)}\}, \{\zeta^s_{(m)}\}, \{\zeta^{s*}_{(m)}\})$ will in particular include the deformations generated by the elements of $SU(1,1|2)$ outside the subgroup generated by the Q_i's and $(\widehat{L}_0 \pm J^3)$, since such deformations will carry non-zero $\widehat{L}_0 - J^3$ charge. If for example we use the parameterization given in (4.24) for an element of $SU(1,1|2)$, then the parameters ξ, $\bar{\eta}$ and $\theta_{\alpha+}$ will carry $(\widehat{L}_0 - J^3)$ eigenvalue $+1$, and their complex conjugate parameters will carry $(\widehat{L}_0 - J^3)$ eigenvalue -1.

Now the path integral over the various fields can be regarded as integral over the parameters $z^s_{(m)}$, $z^{s*}_{(m)}$, $\zeta^s_{(m)}$ and $\zeta^{s*}_{(m)}$ for different values of s and $m \neq 0$ together with integration over the variables $\vec{y}$. Thus we have an integral

$$I = \int d\vec{y} \prod_{m>0,s} dz^s_{(m)}\, dz^{s*}_{(m)}\, d\zeta^s_{(m)}\, d\zeta^{s*}_{(m)}\, \mathcal{J}\, e^{-A}\,. \tag{4.32}$$

where $\mathcal{J}$ represents any measure factor which might arise from changing the integration variables to $(\vec{y}, \vec{z}, \vec{z}^*, \vec{\zeta}, \vec{\zeta}^*)$. We now deform this to another integral

$$I(t) = \int d\vec{y} \prod_{m>0} dz^s_{(m)}\, dz^{s*}_{(m)}\, d\zeta^s_{(m)}\, d\zeta^{s*}_{(m)}\, \mathcal{J}\, e^{-A - t\, Q_1\, F}\,, \tag{4.33}$$

[7] To see this we can write $z^s_{(m)} = z^s_{(m)R} + iz^s_{(m)I}$, $\zeta^s_{(m)} = \zeta^s_{(m)R} + i\zeta^s_{(m)I}$ with real $z^s_{(m)R}$, $z^s_{(m)I}$, $\zeta^s_{(m)R}$ and $\zeta^s_{(m)I}$, and then compare the real and imaginary parts of (4.30) after multiplying both sides by $i\theta$, keeping in mind that the operator $i\theta Q$ for real grassman parameter θ takes a real variable to a real variable, and also that given two real grassman variables θ_1, θ_2, $\theta_1\theta_2$ is imaginary. Eq.(4.31) follows from this immediately.

where t is a positive real parameter and

$$F = \sum_{m>0} \sum_{s} z^{s*}_{(m)} \, \zeta^{s}_{(m)} \, . \tag{4.34}$$

This gives

$$Q_1 \, F = \sum_{m>0} \sum_{s} \left[4m \, z^{s*}_{(m)} \, z^{s}_{(m)} + \zeta^{s*}_{(m)} \, \zeta^{s}_{(m)} \right] \, . \tag{4.35}$$

Furthermore, since by construction F is invariant under $(\widehat{L}_0 - J^3)$, we have

$$Q_1^2 F = 0 \, . \tag{4.36}$$

This equation, together with the supersymmetry invariance of the action ($Q_1 A = 0$) can be used to get

$$\begin{aligned} \partial_t I(t) &= \int d\vec{y} \prod_{m>0} dz^{s}_{(m)} \, dz^{s*}_{(m)} \, d\zeta^{s}_{(m)} \, d\zeta^{s*}_{(m)} \, \mathcal{J} \, (-Q_1 F) \, e^{-A - t \, Q_1 \, F} \\ &= - \int d\vec{y} \prod_{m>0} dz^{s}_{(m)} \, dz^{s*}_{(m)} \, d\zeta^{s}_{(m)} \, d\zeta^{s*}_{(m)} \, \mathcal{J} \, Q_1 \left(F \, e^{-A - t \, Q_1 \, F} \right) = 0 \, , \end{aligned} \tag{4.37}$$

where in the last step we have used Q_1 invariance of the path integral measure. Thus $I(t)$ is independent of t, and has the same value in the limits $t \to 0$ and $t \to \infty$. Noting that in the $t \to 0$ limit $I(t)$ reduces to I, and using (4.33), (4.35) we get

$$I = \lim_{t \to \infty} \int d\vec{y} \prod_{m>0} dz^{s}_{(m)} \, dz^{s*}_{(m)} \, d\zeta^{s}_{(m)} \, d\zeta^{s*}_{(m)} \, \mathcal{J} \, e^{-A - t \sum_{m>0} \sum_s \left[4 \, m \, z^{s*}_{(m)} \, z^{s}_{(m)} + \zeta^{s*}_{(m)} \, \zeta^{s}_{(m)} \right]} \, . \tag{4.38}$$

In the $t \to \infty$ limit the $z^{s}_{(m)}$ and $\zeta^{s}_{(m)}$ dependent terms inside the action A are subleading. Thus up to an overall t independent normalization constant,[8] the $e^{-t \sum_{m>0} \sum_s \left[4 \, m \, z^{s*}_{(m)} \, z^{s}_{(m)} + \zeta^{s*}_{(m)} \, \zeta^{s}_{(m)} \right]}$ term in the $t \to \infty$ limit is equivalent to inserting in the path integral a factor of

$$\prod_{m>0} \prod_{s} \delta \left(z^{s}_{(m)} \right) \delta \left(z^{s*}_{(m)} \right) \delta (\zeta^{s*}_{(m)}) \delta (\zeta^{s}_{(m)}) \, . \tag{4.39}$$

This shows that the path integral is localized in the subspace of $(\widehat{L}_0 - J^3)$ invariant deformations parameterized by the coordinates $\vec{y}$. In particular it restricts integration

[8]This normalization constant can of course be absorbed into a redefinition of the measure $\mathcal{J}$. Alternatively, we could define $\zeta^{s}_{(m)}$ with a different normalization so that eqs.(4.30) take the form $Q_1 \, z^{s}_{(m)} = \alpha_m \zeta^{s}_{(m)}$, $Q_1 \, \zeta^{s}_{(m)} = 4m \, \alpha_m^{-1} \, z^{s}_{(m)}$ for some constant α_m. By adjusting α_m we could ensure that the replacement of the t dependent exponential factor by (4.39) does not require any additional normalization.

over the orbits of $SU(1,1|2)$, generated by the action of (4.24) on any $(\widehat{L}_0 - J^3)$ invariant configuration, to the subspace

$$\xi = 0, \quad \eta = 0, \quad \theta_{\alpha\beta} = 0\,. \tag{4.40}$$

More generally, since $\widehat{L}_0$ and J^3 generate translations along θ and ϕ directions of $AdS_2 \times S^2$ respectively, restriction to $\widehat{L}_0 - J^3$ invariant subspace amounts to restricting the path integral over field configurations which depend on θ and ϕ only through the combination $(\theta + \phi)$.

We can further localize the $\vec{y}$ integral onto Q_1-invariant subspace. Intuitively this can be understood by noting that unless $\vec{y}$ is invariant under Q_1, the orbit of Q_1 through a point $\vec{y}$ will give a vanishing contribution to the integral[73, 70]. Thus the contribution to the integral must come from the Q_1 invariant subspace of the $(\widehat{L}_0 - J^3) = 0$ subspace. Formally this can be established as follows. Let $(\{\vec{w}^\alpha\}, \{\zeta^a\})$ denote the bosonic and fermionic components of $\vec{y}$. Then we can write

$$Q_1\, \zeta^a = f^a(\vec{w}, \vec{\zeta})\,, \tag{4.41}$$

for some functions f^a. We now insert into the path integral a term

$$\exp\left[-tQ_1 \sum_a \zeta^a\, f^a(\vec{w}, \vec{\zeta})\right] = \exp\left[-t\,\sum_a f^a(\vec{w}, \vec{\zeta})\, f^a(\vec{w}, \vec{\zeta})\right]\,. \tag{4.42}$$

Nilpotence of Q_1 and Q_1 invariance of the original action can be used to argue that the path integral is independent of t. Restriction of the path integral to the purely bosonic subspace $\zeta^a = 0$ now has a factor $\exp\{-t\sum_a f^a(\vec{w}, \vec{0})\, f^a(\vec{w}, \vec{0})\}$. Thus in the $t \to \infty$ limit the path integral is restricted to the subspace $f^a(\vec{w}, \vec{0}) = 0$ in the $\vec{\zeta} = 0$ sector. This is precisely the Q_1 invariant subspace of purely bosonic configurations.

This establishes that *in order to get a non-vanishing contribution from integration around a saddle point Φ it must be invariant under the group H_1 generated by Q_1 and $\widehat{L}_0$*. Furthermore after taking into account appropriate measure factors we can express the path integral as integration over an H_1 invariant slice passing through Φ.

4.4 Integrating Over the Orbit of the Superconformal Current Algebra

String theory on $AdS_2 \times S^2$ space, describing the near horizon geometry of a BPS black hole, has an infinite group of asymptotic symmetries besides the global $SU(1,1|2)$ transformations which leave the $AdS_2 \times S^2$ background invariant. These more general transformations do not leave the $AdS_2 \times S^2$ background invariant but preserve the asymptotic condition on the various fields. Hence they can be used to generate new solutions from a given solution. As was shown in [25], the Euclidean action of the theory remains unchanged under these transformations even after taking into account the effect of the infrared cut-off. Thus they represent zero modes. In a non-supersymmetric theory where only bosonic zero modes are present, integration over these zero modes will generate an infinite factor in the partition function. Hence integration over these directions must be restricted by declaring the corresponding transformations as gauge transformations. However as was pointed out in [25], in a supersymmetric theory there is a possibility of cancellation between the bosonic and fermionic zero mode integrals yielding a finite result. We shall now demonstrate that this is indeed what happens.

The generators of the extended superconformal algebra may be labelled as $\widetilde{L}_n$, $\widetilde{J}^i_n$ and $\widetilde{G}^{\alpha\beta}_r$ with $n \in \mathbb{Z}$, $r \in \mathbb{Z}+\frac{1}{2}$, $1 \leq i \leq 3$ and $\alpha,\beta = \pm$. The generators of $su(1,1|2)$ discussed in §4.2 are special cases of these generators with the identification

$$\widehat{L}_0 = \widetilde{L}_0, \quad \widehat{L}_\pm = \widetilde{L}_{\mp 1}, \quad J^i = \widetilde{J}^i_0, \quad \widehat{G}^{\alpha\beta}_\pm = \widetilde{G}^{\alpha\beta}_{\mp\frac{1}{2}}\,. \tag{4.43}$$

For our analysis we shall not need the full superconformal current algebra, but only the commutators of the various generators with $\widehat{L}_0$ and J^3. They are given by

$$\begin{aligned}
&[\widehat{L}_0, \widetilde{L}_n] = -n\,\widetilde{L}_n, \quad [\widehat{L}_0, \widetilde{J}^i_n] = -n\,\widetilde{J}^i_n, \quad [\widehat{L}_0, \widetilde{G}^{\alpha\beta}_r] = -r\,\widetilde{G}^{\alpha\beta}_r,\\
&[J^3, \widetilde{L}_n] = [J^3, \widetilde{J}^3_n] = 0, \quad [J^3, \widetilde{J}^\pm_n] = \pm\widetilde{J}^\pm_n, \quad [J^3, \widetilde{G}^{\alpha\beta}_r] = \frac{1}{2}\,\beta\,\widetilde{G}^{\alpha\beta}_r,\\
&\qquad \widetilde{J}^\pm_n \equiv \widetilde{J}^1_n \pm i\widetilde{J}^2_n\,.
\end{aligned} \tag{4.44}$$

This gives

$$\begin{aligned}
&[\widehat{L}_0 - J^3, \widetilde{L}_n] = -n\,\widetilde{L}_n, \quad [\widehat{L}_0 - J^3, \widetilde{J}^3_n] = -n\,\widetilde{J}^3_n, \quad [\widehat{L}_0 - J^3, \widetilde{J}^\pm_n] = (-n \mp 1)\,\widetilde{J}^\pm_n,\\
&[\widehat{L}_0 - J^3, \widetilde{G}^{\alpha\beta}_r] = \left(-\frac{1}{2}\beta - r\right)\widetilde{G}^{\alpha\beta}_r\,.
\end{aligned} \tag{4.45}$$

Consider now an H_1-invariant saddle point and analyze the contribution from the zero modes generated by the action of the superconformal algebra. First note that most of the modes generated by the superconformal algebra carry non-zero eigenvalues under $\widehat{L}_0 - J^3$. They are part of the deformations labelled by $z^s_{(m)}$ and $\zeta^s_{(m)}$ in §4.3 and are eliminated by the localization procedure described in §4.3. Thus we only need to worry about deformations generated by $\widehat{L}_0 - J^3$ invariant generators. Of these several are part of the global symmetry group $SU(1,1|2)$ and have already been taken into account in the analysis of §4.3. From (4.45) we see that the only $\widehat{L}_0 - J^3$ invariant generators which are not part of $SU(1,1|2)$ are $\widetilde{J}^+_{-1}$ and $\widetilde{J}^-_1$. Since together with $\widetilde{J}^3_0 = J^3$ they generate an $SU(2)$ group, the integration over these zero modes will give us a finite factor proportional to the volume of $SU(2)$. This shows that around an H_1-invariant saddle point, integration over the fermionic and bosonic zero modes generated by the full superconformal current algebra gives a finite result.

4.5 Examples of H_1-invariant Saddle Points

In this section we shall review the construction of a class of saddle points from orbifolds of the near horizon geometry of the black hole[74, 25, 75] and verify their H_1-invariance. We shall focus on type IIB string theory on $K3$ – the theory discussed in appendix A – and consider six dimensional geometries whose asymptotic form coincide with that of $S^1 \times \widetilde{S}^1 \times AdS_2 \times S^2$ with background 3-form fluxes.[9] The simplest example of H_1-invariant saddle point is $S^1 \times \widetilde{S}^1 \times AdS_2 \times S^2$ with background fluxes and is given by

$$\begin{aligned} ds^2 &= v \left(d\eta^2 + \sinh^2\eta d\theta^2\right) + u \left(d\psi^2 + \sin^2\psi d\phi^2\right) + \frac{R^2}{\tau_2}\left|dx^4 + \tau dx^5\right|^2 , \\ G^I &\equiv \frac{1}{3!} G^I_{MNP} dx^M \wedge dx^N \wedge dx^P \\ &= \frac{1}{8\pi^2}\left[Q_I \sin\psi\, dx^5 \wedge d\psi \wedge d\phi + P_I \sin\psi\, dx^4 \wedge d\psi \wedge d\phi + \text{dual}\right] , \\ {V_I}^i &= \text{constant}, \quad {V_I}^r = \text{constant} . \end{aligned} \tag{4.46}$$

Here 'dual' denotes the dual 3-form required to make G^I satisfy the self-duality constraint given in (A-2), v, u, R are real constants and $\tau = \tau_1 + i\tau_2$ is a complex constant. (η, t) label an AdS_2 space, (ψ, ϕ) label a 2-sphere and x^4, x^5 label coordinates along $\widetilde{S}^1$ and S^1 respectively, each taken to have period 2π. Q_I and P_I denote the fluxes through the 3-cycles $S^1 \times S^2$ and $\widetilde{S}^1 \times S^2$ respectively, and are related to the integer charges carried by the black hole whose near horizon geometry is described by (A-8).
This background is not only H_1-invariant but is invariant under the full $SU(1,1|2)$ symmetry group. The classical contribution to the quantum entropy function from this saddle point is given by $\exp(S_{wald})$ where S_{wald} denotes the classical Wald entropy[6].

We shall now construct other H_1 invariant saddle points with the same asymptotic behaviour as (A-8) by taking orbifold of the above background by some discrete $\mathbb{Z}_s$ group. Since H_1 is generated by Q_1 and Q_1^2, in order to preserve H_1 the $\mathbb{Z}_s$ action must commute with Q_1. Typically the generator of the $\mathbb{Z}_s$ transformation will involve an element of $SU(1,1|2)$ together with an internal symmetry transformation that commutes with $SU(1,1|2)$. Now one can see from the algebra (4.18) that the only bosonic generator of $su(1,1|2)$ that commutes with Q_1 is $(\widehat{L}_0 - J^3)$. Thus the part of the orbifold group generator that belongs to $SU(1,1|2)$ must be an element of the $U(1)$ subgroup generated by $(\widehat{L}_0 - J^3)$. However since $(\widehat{L}_0 - J^3)$ commutes with the $H_0 \times U(1)$ subgroup of $SU(1,1|2)$ generated by $Q_1, \cdots Q_4$ and $(\widehat{L}_0 \pm J^3)$, we see that any such saddle point will automatically also be invariant under this bigger subgroup of $SU(1,1|2)$. We shall now give some specific examples of such orbifolds.

[9] Note S^1 and $\widetilde{S}^1$ are not factored metrically, i.e. we allow the metric to have components which mix S^1 and $\widetilde{S}^1$ coordinates.

It was shown in [76, 77] that with the help of a duality transformation we can bring the charge vector to the form

$$(Q,P) = (\ell Q_0, P_0)\,, \tag{4.47}$$

for some integer ℓ, representing a duality invariant combination of the charges[78]. Here (Q_0, P_0) are primitive vectors of the charge lattice, satisfying

$$\gcd\left(\{Q_{0I}P_{0J} - Q_{0J}P_{0I}\}\right) = 1\,. \tag{4.48}$$

We now consider an orbifold of the background (A-8) by the $\mathbb{Z}_s$ transformation[25]

$$(\theta, \phi, x^5) \to \left(\theta + \frac{2\pi}{s}, \phi - \frac{2\pi}{s}, x^5 + \frac{2\pi k}{s}\right), \quad k, s \in \mathbb{Z}, \quad \gcd(s,k) = 1\,. \tag{4.49}$$

Since the circle parametrized by x^5 is non-contractible, this is a freely acting orbifold. At the origin $\eta = 0$ of the AdS_2 space we have a non-contractible 3-cycle spanned by (x^5, ψ, ϕ), with the identification $(x^5, \psi, \phi) = (x^5 + 2\pi k/s, \psi, \phi - 2\pi/s)$. As a result of this identification the total flux of G^I through this cycle is equal to $Q_I/s = (l/s)Q_{0I}$. Since the flux quantization constraints require the fluxes through this new 3-cycle to be integers, we see that this orbifold is an allowed configuration in string theory only when ℓ/s is an integer.

Since $(\widehat{L}_0 - J^3)$ shifts θ and ϕ in opposite directions, the $\mathbb{Z}_s$ transformation described in (4.49) is generated by $(\widehat{L}_0 - J^3)$ together with a shift along x^5. Since all the generators of $SU(1,1|2)$ are invariant under a shift along x^5, we see that the subgroup of $SU(1,1|2)$ that commutes with $(\widehat{L}_0 - J^3)$ will be a symmetry of this orbifold. This is precisely the group H_0 together with the $U(1)$ subgroup generated by $(\widehat{L}_0 + J^3)$.

It was shown in [25] that the orbifold described above has the correct asymptotic behaviour. For this we rename the coordinates $(\eta, \theta, \phi, x^5)$ appearing in (A-8) as $(\widetilde{\eta}, \widetilde{\theta}, \widetilde{\phi}, \widetilde{x}^5)$ and express the new configuration as

$$\begin{aligned}
ds^2 &= v\left(d\widetilde{\eta}^2 + \sinh^2\widetilde{\eta}\, d\widetilde{\theta}^2\right) + u(d\psi^2 + \sin^2\psi\, d\widetilde{\phi}^2) + \frac{R^2}{\tau_2}\left|dx^4 + \tau d\widetilde{x}^5\right|^2, \\
G^I &= \frac{1}{8\pi^2}\left[Q_I \sin\psi\, d\widetilde{x}^5 \wedge d\psi \wedge d\widetilde{\phi} + P_I \sin\psi\, dx^4 \wedge d\psi \wedge d\widetilde{\phi} + \text{dual}\right], \\
(\widetilde{\theta}, \widetilde{\phi}, \widetilde{x}^5) &\equiv \left(\widetilde{\theta} + \frac{2\pi}{s}, \widetilde{\phi} - \frac{2\pi}{s}, \widetilde{x}^5 + \frac{2\pi k}{s}\right) \equiv (\widetilde{\theta} + 2\pi, \widetilde{\phi}, \widetilde{x}^5) \\
&\equiv (\widetilde{\theta}, \widetilde{\phi} + 2\pi, \widetilde{x}^5) \equiv (\widetilde{\theta}, \widetilde{\phi}, \widetilde{x}^5 + 2\pi)\,.
\end{aligned} \tag{4.50}$$

We now make the coordinate transformation:

$$\theta = s\widetilde{\theta}, \quad \phi = \widetilde{\phi} + (1-s)\widetilde{\theta}, \quad x^5 = \widetilde{x}^5 - k\widetilde{\theta}, \quad \eta = \widetilde{\eta} - \ln s\,. \tag{4.51}$$

In these coordinates the background (4.50) takes the form

$$\begin{aligned}
ds^2 &= v\left(d\eta^2+\sinh^2\eta\left(1+\frac{(1-s^{-2})e^{-\eta}}{2\sinh\eta}\right)^2 d\theta^2\right)+u(d\psi^2+\sin^2\psi(d\phi+d\theta-s^{-1}d\theta)^2)\\
&\quad +\frac{R^2}{\tau_2}\left|dx^4+\tau(dx^5+ks^{-1}d\theta)\right|^2,\\
G^I &= \frac{1}{8\pi^2}\big[Q_I\,\sin\psi\,(dx^5+ks^{-1}d\theta)\wedge d\psi\wedge(d\phi+d\theta-s^{-1}d\theta)\\
&\quad +P_I\,\sin\psi\,dx^4\wedge d\psi\wedge(d\phi+d\theta-s^{-1}d\theta)+\text{dual}\big],\\
&\quad (\theta,\phi,x^5)\equiv\left(\theta+2\pi,\phi,x^5\right)\equiv\left(\theta,\phi+2\pi,x^5\right)\equiv\left(\theta,\phi,x^5+2\pi\right). \qquad (4.52)
\end{aligned}$$

Since the asymptotic region lies at large η, we see that this has the same asymptotic behaviour as the $S^1\times\widetilde{S}^1\times AdS_2\times S^2$ background described in (A-8). Note the presence of the $d\theta-s^{-1}d\theta$ terms added to $d\phi$ and $ks^{-1}d\theta$ terms added to dx^5. From the point of view of the two dimensional theory living on AdS_2 these represent constant values of the gauge fields arising from the 5θ and $\phi\theta$ components of the metric. Thus (4.52) is an allowed configuration over which the path integral should be performed. The classical contribution to the quantum entropy function from this saddle point is given by $\exp(S_{wald}/s)$[25]. These match with the asymptotic behaviour of specific extra terms in the microscopic formula which appear when the integer ℓ introduced in (4.47) is larger than 1.

Finally we can consider another class of orbifolds for which k appearing in (4.49) vanishes, or more generally, has a common factor with s. The orbifold group still commutes with H_0 and hence we expect H_0 to be a symmetry of this orbifold. However in this case the orbifold action has fixed points and we no longer have a freely acting orbifold. Let us consider the $k=0$ case for definiteness[74]. The points $(\eta=0;\psi=0,\pi)$ are fixed points of this orbifold group, and the 3-cycles spanned by (x^5,ψ,ϕ) and (x^4,ψ,ϕ) at $\eta=0$ now pass through these fixed points. The fluxes through these three cycles from regions outside the fixed points are given by Q_I/s and P_I/s respectively. However flux quantization rule does not put any constraints on the charge vectors Q_I and P_I. Instead it requires that there must be additional flux at the fixed points which make the total flux through these 3-cycles satisfy the correct quantization rules.[10] As was argued in [74], the contribution to the partition function from these saddle points is given by $\exp(S_{wald}/s)$ if we ignore the contribution from the fixed points. Furthermore the contribution from the fixed points add at most constants of order unity to S_{wald}/s whereas S_{wald} grows quadratically with the charges carried by the black hole. Thus for large charges the contribution from the fixed points to the exponent is subleading.

In a dual description of these theories in M-theory the near horizon geometry of these black holes can have an extra circle that combines with the AdS_2 to give a locally AdS_3 space. In this case one can get freely acting $\mathbb{Z}_s$ orbifolds by accompanying the

[10] Such fluxes have been considered before in [79] in a different context.

orbifold action by a translation along this extra circle[75], without imposing any additional arithmetic condition on the charges of the type $\ell/s \in \mathbb{Z}$. This could provide a possible way to analyze the orbifolds with fixed points in the type IIB description.

4.6 Discussion

In this chapter we have used the localization procedure to classify the saddle points which will contribute to string theory path integral over the near horizon geometry of extremal BPS black holes. This path integral is required for the computation of quantum entropy function, which appears in the macroscopic computation of the entropy of extremal black holes via AdS_2/CFT_1 correspondence[39].
We hope that the same localization techniques will also simplify the computation of the path integral around each of the saddle points, *e.g.* by reducing the path integral over the fields to a finite dimensional integral. In particular for quarter BPS black holes in type IIB string theory on $K3 \times T^2$ if the contribution to the path integral from some of the saddle points can be expressed as finite dimensional integrals, they can then be compared with the corresponding microscopic results derived in [16, 80, 81, 82, 83, 84, 85], providing us with a precision test of the AdS_2/CFT_1 correspondence. The formulation of string theory on $AdS_2 \times S^2$ described in [86] could also be a useful tool in this venture.
Finally we note that [71] expressed the expectation value of circular 't Hooft - Wilson loop operators in an $\mathcal{N} = 4$ supersymmetric gauge theory as a path integral over the field theory on $AdS_2 \times S^2$ background. Except for the replacement of the string theory by $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, this path integral is identical to what appears in the definition of the quantum entropy function. Thus we expect that any method (like the one in the present paper) developed for the study of quantum entropy function is likely to be useful for the study of the 't Hooft - Wilson loop operators in $\mathcal{N} = 4$ super-Yang-Mills theory. Similarly any method developed for computing 't Hooft - Wilson loop operators in $\mathcal{N} = 4$ super-Yang Mills theory (like the one developed in [71]) may be useful for the computation of quantum entropy function in string theory. It will also be useful to explore whether the correspondence between the 't Hooft - Wilson loop and the quantum entropy function is just a mathematical coincidence or whether there is some deeper physical reason behind it.

Chapter 5

Conclusion

We have studied the correction to the entropy of extremal black hole in string theory. In the first part we discussed the higher derivative correction to the entropy of the BTZ black hole. In string theory one finds that the low energy effective action contains higher curvature terms. In fact at tree level it contains an Einstein-Hilbert term together with an infinite series of higher curvature terms that are suppressed by powers of α', so that they are subleading at low energy. We described field redefinition and consistent truncation in three dimensional general higher derivative theory of (super-)gravity coupled to arbitrary set of matter fields. After field redefinition and consistent truncation the action reduces to standard (super-)gravity action which is sum of three terms, Einstein-Hilbert term, a cosmological constant term and the Chern-Simons term. The effect of higher derivative corrections are encoded in the correction of the central charges. Thses will give classical correction to the entropy of extremal black hole in string theory whose near horizon geometry corresponds to that of extremal BTZ black hole.
In the second part we described the quantum entropy of the extremal black hole. The quantum degeneracy associated with horizon degrees of freedom of the extremal black hole is given as the finite part of the partition function of string theory on AdS_2. According to this proposal the macroscopic entropy in full quantum theory is equal to logarithm of degeneracy of the ground states of the CQM living on the boundary of AdS_2. We first check this proposal in case of extremal BTZ black hole where there exist an independent definition of entropy via AdS_3/CFT_2 correspondence. We found that both the definition of entropy agrees. We also simplify this path integral using the supersymmetry of the near horizon geometry. The isometry supergroup of the near horizon geometry has a factor $SU(1,1|2)$. Using supersymmetry and localization techniques we showed that the path integral could receives non-vanishing contribution only from a special class of field configurations which preserve a particular subgroup of $SU(1,1|2)$.
The next thing one would like to do is to compare this proposal with the known microscopic results. However in order to do this, one needs to know the full spacetime string effective action and perform the path integral explicitly.

Appendix A

Killing Spinors in Six Dimensional Supergravity on $S^1 \times \widetilde{S}^1 \times AdS_2 \times S^2$

In this appendix we shall analyze the Killing spinors in six dimensional $\mathcal{N} = 4$ chiral supergravity compactified on $S^1 \times \widetilde{S}^1 \times AdS_2 \times S^2$. This theory is dual to M-theory on $K3 \times T^3 \times AdS_2 \times S^2$, for which the Killing spinor equations have been analyzed in [53]. Thus we could try to recover our answer by dualizing the results of [53]. We shall however analyze the Killing spinor equations directly in the six dimensional chiral supergravity in the presence of arbitrary background fluxes. This will make the duality covariance of the equations manifest.

We begin with the six dimensional supergravity theory obtained by dimensional reduction of type IIB supergravity on $K3$[87, 88]. We shall follow the conventions of [89]. The bosonic fields in the theory are the metric g_{MN}, matrix valued scalar fields ${V_I}^i$, ${V_I}^r$ ($1 \le i \le 5$, $6 \le r \le 26$) satisfying

$$V L V^T = L, \qquad L = diag(+^5, -^{21})\,, \tag{A-1}$$

and 2-form fields B^I_{MN} ($1 \le I \le 26$) with field strengths $G^I = dB^I$ satisfying the following self duality constraint:

$$H^i_{MNP} = \frac{1}{3!}\, e_{MNPQRS}\, H^{iQRS}, \qquad H^r_{MNP} = -\frac{1}{3!}\, e_{MNPQRS}\, H^{rQRS}\,, \tag{A-2}$$

where

$$H^i_{MNP} = G^I_{MNP} {V_I}^i, \qquad H^r_{MNP} = G^I_{MNP} {V_I}^r\,. \tag{A-3}$$

e_{MNPQRS} is a six form defined via

$$e^{MNPQRS} = |\det g|^{-1/2}\, \epsilon^{MNPQRS}\,, \tag{A-4}$$

ϵ being the totally antisymmetric symbol. We shall label the time coordinate by t and the space-coordinates by $(x^4, x^5, \eta, \psi, \phi)$ and choose the convention

$$\epsilon^{t45\eta\psi\phi} = 1\,. \tag{A-5}$$

Indices of e are raised and lowered by the metric g_{MN}. Not all components of V describe physical degrees of freedom since there is an identification

$$V \equiv V\,O\,, \tag{A-6}$$

where O is an $SO(5)\times SO(21)$ matrix acting on the first five and the last twenty one indices respectively.

In the sector where the bosonic fields are taken to be space-time independent constants, the equations of motion take the form

$$\begin{aligned} R_{MN} = H^i_{MPQ}\,H^{iPQ}_N + H^r_{MPQ}\,H^{rPQ}_N \\ H^i_{MNP}H^{rMNP} = 0\,, \end{aligned} \tag{A-7}$$

where R_{MN} is the Ricci tensor defined in the sign convention in which on the sphere the Ricci scalar $g^{MN}R_{MN}$ is positive. We now look for a solution in this theory of the form

$$\begin{aligned} ds^2 &= v\,\left(d\eta^2 - \sinh^2\eta dt^2\right) + u\,\left(d\psi^2 + \sin^2\psi d\phi^2\right) + \frac{R^2}{\tau_2}\left|dx^4 + \tau dx^5\right|^2\,, \\ G^I &\equiv \frac{1}{3!}G^I_{MNP}dx^M\wedge dx^N\wedge dx^P \\ &= \frac{1}{8\pi^2}\left[Q_I\,\sin\psi\,dx^5\wedge d\psi\wedge d\phi + P_I\,\sin\psi\,dx^4\wedge d\psi\wedge d\phi + \text{dual}\right]\,, \\ {V_I}^i &= \text{constant},\quad {V_I}^r = \text{constant}\,. \end{aligned} \tag{A-8}$$

Here 'dual' denotes the dual 3-form required to make G^I satisfy the self-duality constraint given in (A-2), v, u, R are real constants and $\tau = \tau_1 + i\tau_2$ is a complex constant. (η, t) label an AdS_2 space, (ψ, ϕ) label a 2-sphere and x^4, x^5 label coordinates along $\widetilde{S}^1$ and S^1 respectively, each taken to have period 2π. Q_I and P_I denote the fluxes through the 3-cycles $S^1\times S^2$ and $\widetilde{S}^1\times S^2$ respectively, and are related to the integer charges carried by the black hole whose near horizon geometry is described by (A-8). In order to solve (A-7) we note that given any charge vectors (Q,P) satisfying

$$Q^2 > 0,\quad P^2 > 0,\quad Q^2P^2 > (Q\cdot P)^2\,, \tag{A-9}$$

where

$$Q^2 = Q^TLQ,\quad P^2 = P^TLP,\quad Q\cdot P = Q^TLP\,, \tag{A-10}$$

we can always find a matrix S satisfying $SLS^T = L$ such that

$$Q = SQ_0,\quad P = SP_0\,, \tag{A-11}$$

where

$$Q_0 = \begin{pmatrix} Q\cdot P/\sqrt{P^2} \\ \\ \sqrt{Q^2P^2-(Q\cdot P)^2}/\sqrt{P^2} \\ \\ 0 \\ \cdot \\ \cdot \end{pmatrix}, \qquad P_0 = \begin{pmatrix} \sqrt{P^2} \\ \\ 0 \\ \\ 0 \\ \cdot \\ \cdot \end{pmatrix}. \tag{A-12}$$

In that case eqs.(A-7) is solved by (A-8) for the choice

$$\begin{aligned} &V = (S^T)^{-1}, \quad \tau_1 = Q\cdot P/P^2, \quad \tau_2 = \sqrt{Q^2P^2-(Q\cdot P)^2}/P^2, \\ &v = u = \frac{1}{16\pi^4 R^2}\sqrt{Q^2P^2-(Q\cdot P)^2}. \end{aligned} \tag{A-13}$$

Using eq.(A-3) this gives

$$H^i = \frac{1}{8\pi^2}\left[Q_0^i \sin\psi\, dx^5\wedge d\psi\wedge d\phi + P_0^i \sin\psi\, dx^4\wedge d\psi\wedge d\phi + \text{dual}\right], \quad H^r = 0\,. \tag{A-14}$$

Note that R is arbitrary. Furthermore S defined through (A-11) is ambiguous up to an $SO(3,21)$ transformation from the right acting on the last 24 elements. Thus V given in (A-13) is determined only up to an $SO(3,21)$ multiplication from the right. Due to the identification (A-6) only an $SO(3,21)/SO(3)\times SO(21)$ family of these describe physically inequivalent configurations. These parameters which are left undetermined by the equations of motion describe flat directions of the entropy function.

The fermion fields in this theory consist of a set of gravitini ψ_M and a set of spin 1/2 fermions χ^r. χ^r transforms as **21** of $SO(21)$, **4** of $SO(5)$ and a right chiral spinor of $SO(5,1)$ where $SO(5,1)$ denotes the tangent space Lorentz group, $SO(21)$ is the internal symmetry group acting on the index r, and $SO(5)$ is the internal symmetry group acting on the index i. In what follows we shall suppress all the $SO(5)\times SO(5,1)$ spinor indices. For each M, ψ_M transforms as **4** of $SO(5)$ and a left-chiral spinor of $SO(5,1)$. Finally the supersymmetry transformation parameter ϵ transforms as a **4** of $SO(5)$ and a left chiral spinor of $SO(5,1)$. Let us denote the vielbeins by $e_M^{\ A}$ with A labelling an $SO(5,1)$ tangent space index, the SO(5,1) gamma matrices by $\widetilde{\Gamma}^A$ and the SO(5) gamma matrices by $\widehat{\Gamma}^i$. We shall also use the symbol Γ^M to denote the $SO(5,1)$ gamma matrices in the coordinate basis, i.e. we have

$$\widetilde{\Gamma}^A = e_M^{\ A}\,\Gamma^M\,. \tag{A-15}$$

Then the $SO(5,1)$ chirality conditions on various spinors may be described as

$$\left(\Gamma^{QRS} - \frac{1}{3!}e^{MNPQRS}\Gamma_{MNP}\right)\chi^r = 0,$$

$$\left(\Gamma^{QRS}+\frac{1}{3!}e^{MNPQRS}\Gamma_{MNP}\right)\psi_K=0,$$
$$\left(\Gamma^{QRS}+\frac{1}{3!}e^{MNPQRS}\Gamma_{MNP}\right)\epsilon=0, \tag{A-16}$$

where

$$\Gamma^{M_1\cdots M_k}=\frac{1}{k!}\left(\Gamma^{M_1}\cdots\Gamma^{M_k}+\text{permutations with sign}\right). \tag{A-17}$$

Besides this all the spinors ψ_M, χ^r and ϵ satisfy the symplectic Majorana condition, *e.g.* we have

$$\bar\epsilon=\epsilon^T\,C\,\Omega\,, \tag{A-18}$$

where Ω is the SO(5) charge conjugation matrix acting on the $SO(5)$ spinor index and C is the SO(5,1) charge conjugation matrix acting on the SO(5,1) spinor index. The supersymmetry transformation laws of various fields take the form

$$\begin{aligned}
&\delta e_M^{\ A}=\bar\epsilon\widetilde\Gamma^A\psi_M\\
&\delta\psi_M=D_M\epsilon-\frac{1}{4}H^i_{MNP}\Gamma^{NP}\widehat\Gamma^i\epsilon,\qquad D_M\epsilon\equiv\partial_M\epsilon+\frac{1}{4}\omega_M^{AB}\widetilde\Gamma_{AB}\epsilon-\frac{1}{4}\,Q_M^{ij}\,\widehat\Gamma^{ij}\epsilon\,,\\
&\quad\omega_M^{AB}\equiv-g^{NP}e_N^{\ B}\partial_M e_P^{\ A}+e_N^{\ A}e_P^{\ B}g^{PQ}\Gamma^N_{QM},\qquad \Gamma^M_{NP}\equiv\frac{1}{2}\,g^{MR}\left(\partial_N g_{PR}+\partial_P g_{NR}-\partial_R g_{NP}\right),\\
&\delta B^I_{MN}=-V^{Ii}\bar\epsilon\Gamma_{[M}\widehat\Gamma^i\psi_{N]}+\frac{1}{2}V^{Ir}\bar\epsilon\Gamma_{MN}\chi^r,\\
&\delta\chi^r=\frac{1}{\sqrt2}\Gamma^M P_M^{ir}\widehat\Gamma^i\epsilon+\frac{1}{12}\Gamma^{MNP}H^r_{MNP}\epsilon,\\
&\delta V_I^{\ i}=\bar\epsilon\widehat\Gamma^i\chi^r V_I^{\ r}\,,\\
&\delta V_I^{\ r}=\bar\epsilon\,\widehat\Gamma^i\chi^r V_I^{\ i}\,,
\end{aligned} \tag{A-19}$$

where the index I is raised and lowered by the matrix L and

$$P_M^{\ ir}=\frac{1}{\sqrt2}\,\partial_M V_I^{\ i}(V^{-1})_r^{\ I}\,,\qquad Q_M^{\ ij}=\partial_M V_I^{\ i}(V^{-1})_j^{\ I}\,. \tag{A-20}$$

Thus the Killing spinor equations, obtained by setting the variation of χ^r and ψ_M to zero, are given by

$$D_M\epsilon-\frac{1}{4}H^i_{MNP}\Gamma^{NP}\widehat\Gamma^i\epsilon=0,$$
$$\frac{1}{\sqrt2}\Gamma^M P_M^{ir}\widehat\Gamma^i\epsilon+\frac{1}{12}\Gamma^{MNP}H^r_{MNP}\epsilon=0\,. \tag{A-21}$$

We shall try to solve these equations in the background (A-8), (A-13). The analysis simplifies if we note that in this background

$$P_M^{\ ir}=0,\qquad Q_M^{ij}=0\,,\qquad H^r_{MNP}=0\,. \tag{A-22}$$

Thus the second set of equations in (A-21) are satisfied automatically. The first set of equations can be split into two sets by taking $M = (4,5)$ and $M = (\eta, t, \psi, \phi)$:

$$
\begin{aligned}
& H^i_{a\mu\nu}\Gamma^{\mu\nu}\widehat{\Gamma}^i\epsilon = 0\,, \quad a = 4,5, \quad \mu,\nu = \eta, t, \psi, \phi\,, \\
& D_\mu\epsilon + \frac{1}{2}H^i_{a\mu\nu}\Gamma^{a\nu}\widehat{\Gamma}^i\epsilon = 0\,. \qquad \text{(A-23)}
\end{aligned}
$$

Since we shall eventually be interested in finding the Killing spinors in the euclidean theory, we shall now make a euclidean continuation of the theory. This is done by making the replacement

$$
t \to -i\theta\,, \qquad \text{(A-24)}
$$

and replacing (A-4), (A-5) by

$$
e^{MNPQRS} = i|\det g|^{-1/2}\,\epsilon^{MNPQRS}\,, \qquad \epsilon^{\theta 45\eta\psi\phi} = 1\,. \qquad \text{(A-25)}
$$

This will guarantee that a solution obtained by euclidean rotation of a Minkowski solution will satisfy the self-duality conditions (A-2) with e_{MNPQRS} defined via (A-25). Furthermore the chirality projection rules (A-16), the supersymmetry transformation rules (A-19) and the killing spinor equations (A-21) all remain unchanged as long as we use the new definition (A-25). Finally since the **4** representation of $SO(6)$ is different from its conjugate representation $\bar{\mathbf{4}}$, we can no longer impose the symplectic Majorana condition on the spinors. However we shall now take (A-18) as the definition of $\bar\epsilon$ appearing in the supersymmetry transformation laws. Equivalently, we could first replace $\bar\epsilon$ in the supersymmetry transformation laws in terms of ϵ using (A-18), and then make the Euclidean continuation. The charge conjugation matrices C and Ω have to be chosen so that $\epsilon_1^T\, C\,\Omega\,\widetilde{\Gamma}^A\,\epsilon_2$ and $\epsilon_1^T\, C\,\Omega\,\widehat{\Gamma}^i\,\epsilon_2$ transform as $SO(6)$ vectors and $SO(5)$ vectors respectively for arbitrary ϵ_1 and ϵ_2.

Under the euclidean continuation the solution given in (A-8) takes the form:

$$
\begin{aligned}
ds^2 &= v\left(d\eta^2 + \sinh^2\eta\, d\theta^2\right) + u(d\psi^2 + \sin^2\psi\, d\phi^2) + \frac{R^2}{\tau_2}\left|dx^4 + \tau dx^5\right|^2\,, \\
G^I &= \frac{1}{8\pi^2}\left[Q_I\,\sin\psi\, dx^5\wedge d\psi\wedge d\phi + P_I\,\sin\psi\, dx^4\wedge d\psi\wedge d\phi + \text{dual}\right]\,, \\
{V_I}^i &= \text{constant}, \quad {V_I}^r = \text{constant}\,, \\
H^i &= \frac{1}{8\pi^2}\left[Q_0^i\,\sin\psi\, dx^5\wedge d\psi\wedge d\phi + P_0^i\,\sin\psi\, dx^4\wedge d\psi\wedge d\phi + \text{dual}\right]\,, \quad H^r = 0\,,
\end{aligned}
\qquad \text{(A-26)}
$$

with the various parameters determined from (A-13). The equations (A-23) take the form

$$
\begin{aligned}
& H^i_{a\mu\nu}\Gamma^{\mu\nu}\widehat{\Gamma}^i\epsilon = 0\,, \quad a = 4,5, \quad \mu,\nu = \eta, \theta, \psi, \phi\,, \\
& D_\mu\epsilon + \frac{1}{2}H^i_{a\mu\nu}\Gamma^{a\nu}\widehat{\Gamma}^i\epsilon = 0\,. \qquad \text{(A-27)}
\end{aligned}
$$

Using the self-duality constraints (A-2), the chirality constraints (A-16), the explicit form of the solutions given in (A-13), and (A-12), the first set of equations in (A-27) takes the simple form

$$\widehat{\Gamma}^1\epsilon = \Gamma_{45}\,(\det g^{(45)})^{-1/2}\,\widehat{\Gamma}^2\epsilon\,, \qquad \text{(A-28)}$$

where $g^{(45)}$ denotes the metric on $S^1\times\widetilde{S}^1$. We shall now use (A-28) to simplify the second set of equations in (A-27). For this we need to choose the vielbeins $e_M^{\ A}$ consistent with the background (A-26). We define $e^A \equiv e_M^{\ A} dx^M$ and take

$$\begin{aligned} &e^0 = \sqrt{v}\sinh\eta\, d\theta, \quad e^1 = \sqrt{v}\, d\eta, \quad e^2 = \sqrt{u}\sin\psi\, d\phi, \quad e^3 = \sqrt{u} d\psi, \\ &e^4 = \frac{R}{\sqrt{\tau_2}}\,(dx^4 + \tau_1 dx^5), \quad e^5 = R\,\sqrt{\tau_2}\,dx^5\,. \end{aligned} \qquad \text{(A-29)}$$

We also denote by x^m for $m=2,3$ the coordinates (ϕ,ψ) along S^2 and by x^α for $\alpha=0,1$ the coordinates (θ,η) along AdS^2. In that case the second set of equations in (A-27) are given by

$$\begin{aligned} &D_m\epsilon - \frac{1}{2}\sqrt{u}\varepsilon^{S^2}_{mn}\Gamma^n\,\widetilde{\Gamma}^4\widehat{\Gamma}^1\epsilon = 0\,, \\ &D_\alpha\epsilon + \frac{i}{2}\,\sqrt{v}\,\varepsilon^{AdS_2}_{\alpha\beta}\,\Gamma^\beta\,\widetilde{\Gamma}^4\widehat{\Gamma}^2\epsilon = 0\,, \end{aligned} \qquad \text{(A-30)}$$

where $\widetilde{\Gamma}^A$ have been defined in (A-15), and

$$\varepsilon^{S^2}_{mn}dx^m\wedge dx^n = \sin\psi\, d\psi\wedge d\phi\,, \qquad \varepsilon^{AdS_2}_{\alpha\beta}dx^\alpha\wedge dx^\beta = \sinh\eta\, d\eta\wedge d\theta\,. \qquad \text{(A-31)}$$

We can analyze these equations by choosing the following representation of the gamma matrices:

$$\begin{aligned} &\widetilde{\Gamma}^0 = \sigma_1\otimes I\otimes I\otimes I\otimes I, \quad \widetilde{\Gamma}^1 = \sigma_2\otimes I\otimes I\otimes I\otimes I, \quad \widetilde{\Gamma}^2 = \sigma_3\otimes\sigma_1\otimes I\otimes I\otimes I \\ &\widetilde{\Gamma}^3 = \sigma_3\otimes\sigma_2\otimes I\otimes I\otimes I, \quad \widetilde{\Gamma}^4 = \sigma_3\otimes\sigma_3\otimes\sigma_1\otimes I\otimes I, \quad \widetilde{\Gamma}^5 = \sigma_3\otimes\sigma_3\otimes\sigma_2\otimes I\otimes I \\ &\widehat{\Gamma}^1 = I\otimes I\otimes I\otimes\sigma_1\otimes I, \quad \widehat{\Gamma}^2 = I\otimes I\otimes I\otimes\sigma_2\otimes I, \quad \widehat{\Gamma}^3 = I\otimes I\otimes I\otimes\sigma_3\otimes\sigma_1 \\ &\widehat{\Gamma}^4 = I\otimes I\otimes I\otimes\sigma_3\otimes\sigma_2, \quad \widehat{\Gamma}^5 = I\otimes I\otimes I\otimes\sigma_3\otimes\sigma_3\,, \end{aligned} \qquad \text{(A-32)}$$

where the σ_i are Pauli matrices and I is the 2×2 identity matrix. In this basis the $SO(6)$ charge conjugation matrix C and the $SO(5)$ charge conjugation matrix Ω have the form:

$$C = i\sigma_1\otimes\sigma_2\otimes\sigma_1\otimes I\otimes I, \quad \Omega = I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2, \qquad \text{(A-33)}$$

so that C and Ω satisfy respectively the conditions for $SO(6)$ and $SO(5)$ invariance[1]

$$(C\widetilde{\Gamma}^A)^T = -C\widetilde{\Gamma}^A, \qquad (\Omega\widehat{\Gamma}^i)^T = -\Omega\widehat{\Gamma}^i\,, \qquad \text{(A-34)}$$

[1]Note that (A-34) does not fix the overall phases of C and Ω. We have chosen them according to our convenience.

for all A and i. We now note that the chirality condition (A-16) and the Killing spinor condition (A-28) leads to the constraints:

$$(\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes I \otimes I)\,\epsilon = \epsilon, \qquad (I \otimes I \otimes \sigma_3 \otimes \sigma_3 \otimes I)\,\epsilon = -\epsilon\,. \tag{A-35}$$

Due to these constraints we can parameterize ϵ by eight complex parameters $(\{A_i\},\{B_i\})$:

$$\begin{aligned}\epsilon \;=\;& \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}A_1\\B_1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}A_2\\B_2\end{pmatrix}\\ &+\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}A_3\\B_3\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}A_4\\B_4\end{pmatrix}.\end{aligned}$$

Further simplification occurs due to the fact that eqs.(A-30) do not mix the A_i's with B_i's and in fact remain invariant under the replacement $A_i \leftrightarrow B_i$. Thus we need to solve the Killing spinor equations in the four dimensional subspace parameterized by the A_i's (or B_i's). We get eight solutions $\zeta_\gamma^{\alpha\beta}$ $(\alpha,\beta,\gamma=\pm)$. We shall first write down the solutions for $\zeta_\gamma^{+\beta}$. All of these solutions have $B_i=0$ and the A_i's given by:

$$\begin{aligned}
\zeta_+^{++}:\ \begin{pmatrix}A_1\\A_2\\A_3\\A_4\end{pmatrix} &= 2\,v^{1/4}\,e^{i(\theta+\phi)/2}\begin{pmatrix}\sin\frac{\psi}{2}\sinh\frac{\eta}{2}\\ -\sin\frac{\psi}{2}\cosh\frac{\eta}{2}\\ -\cos\frac{\psi}{2}\sinh\frac{\eta}{2}\\ \cos\frac{\psi}{2}\cosh\frac{\eta}{2}\end{pmatrix},\\
\zeta_+^{+-}:\ \begin{pmatrix}A_1\\A_2\\A_3\\A_4\end{pmatrix} &= 2\,v^{1/4}\,e^{i(\theta-\phi)/2}\begin{pmatrix}-\cos\frac{\psi}{2}\sinh\frac{\eta}{2}\\ \cos\frac{\psi}{2}\cosh\frac{\eta}{2}\\ -\sin\frac{\psi}{2}\sinh\frac{\eta}{2}\\ \sin\frac{\psi}{2}\cosh\frac{\eta}{2}\end{pmatrix},\\
\zeta_-^{++}:\ \begin{pmatrix}A_1\\A_2\\A_3\\A_4\end{pmatrix} &= 2\,v^{1/4}\,e^{-i(\theta-\phi)/2}\begin{pmatrix}-\sin\frac{\psi}{2}\cosh\frac{\eta}{2}\\ \sin\frac{\psi}{2}\sinh\frac{\eta}{2}\\ \cos\frac{\psi}{2}\cosh\frac{\eta}{2}\\ -\cos\frac{\psi}{2}\sinh\frac{\eta}{2}\end{pmatrix},\\
\zeta_-^{+-}:\ \begin{pmatrix}A_1\\A_2\\A_3\\A_4\end{pmatrix} &= 2\,v^{1/4}\,e^{-i(\theta+\phi)/2}\begin{pmatrix}\cos\frac{\psi}{2}\cosh\frac{\eta}{2}\\ -\cos\frac{\psi}{2}\sinh\frac{\eta}{2}\\ \sin\frac{\psi}{2}\cosh\frac{\eta}{2}\\ -\sin\frac{\psi}{2}\sinh\frac{\eta}{2}\end{pmatrix}.
\end{aligned}\tag{A-36}$$

The solutions for $\zeta_\gamma^{-\beta}$ are obtained by replacing the A_i's by B_i's and vice versa. The normalization factor $2\,v^{1/4}$ has been included for convenience.

To check the regularity of the Killing spinors at the origin $\eta=0$ and / or $\psi=0,\pi$, we need to express the $AdS_2\times S^2$ metric in the (z,w) coordinates as in (4.13) and choose the vielbeins as

$$\hat e^0 = \frac{2\sqrt{v}}{1-\bar w w}\,dw_I,\quad \hat e^1 = \frac{2\sqrt{v}}{1-\bar w w}\,dw_R,\quad \hat e^2 = \frac{2\sqrt{u}}{1+\bar z z}\,dz_I,\quad \hat e^3 = \frac{2\sqrt{u}}{1+\bar z z}\,dz_R, \tag{A-37}$$

$$w_R + i w_I \equiv w, \quad z_R + i z_I \equiv z\,. \tag{A-38}$$

Since these vielbeins are regular at $w = 0$ and / or $z = 0$, the Killing spinors will be regular at these points if they are free from any singularity in this frame. Now using (A-29) we get

$$\begin{aligned}
\hat{e}^0 &= \cos\theta\, e^0 + \sin\theta\, e^1, \quad \hat{e}^1 = -\sin\theta\, e^0 + \cos\theta\, e^1, \\
\hat{e}^2 &= \cos\phi\, e^2 + \sin\phi\, e^3, \quad \hat{e}^3 = -\sin\phi\, e^2 + \cos\phi\, e^3\,.
\end{aligned} \tag{A-39}$$

The $\hat{e}^A$ are related to e^A's by a rotation by θ in the 0-1 plane and a rotation by ϕ in the 2-3 plane in the tangent space. Since from (A-32) we see that on the spinors rotations in the 0-1 plane and 2-3 plane are generated by $\frac{1}{2}\sigma_3\otimes I\otimes I\otimes I\otimes I$ and $\frac{1}{2}I\otimes\sigma_3\otimes I\otimes I\otimes I$ respectively, the rotation (A-39) is represented by the matrix

$$\begin{pmatrix} e^{i\theta/2} & \\ & e^{-i\theta/2}\end{pmatrix}\otimes\begin{pmatrix} e^{i\phi/2} & \\ & e^{-i\phi/2}\end{pmatrix}\otimes I\otimes I\otimes I\,. \tag{A-40}$$

Applying this on (A-36) and using (A-36) we get the Killing spinors in the new frame:

$$\begin{aligned}
&\hat{\zeta}^{++}_{+}: \begin{pmatrix} A_1\\ A_2\\ A_3\\ A_4\end{pmatrix} = N\begin{pmatrix} zw\\ -z\\ -w\\ 1\end{pmatrix}, \qquad \hat{\zeta}^{+-}_{+}: \begin{pmatrix} A_1\\ A_2\\ A_3\\ A_4\end{pmatrix} = N\begin{pmatrix} -w\\ 1\\ -\bar{z}w\\ \bar{z}\end{pmatrix}, \\
&\hat{\zeta}^{++}_{-}: \begin{pmatrix} A_1\\ A_2\\ A_3\\ A_4\end{pmatrix} = N\begin{pmatrix} -z\\ z\bar{w}\\ 1\\ -\bar{w}\end{pmatrix}, \qquad \hat{\zeta}^{+-}_{-}: \begin{pmatrix} A_1\\ A_2\\ A_3\\ A_4\end{pmatrix} = N\begin{pmatrix} 1\\ -\bar{w}\\ \bar{z}\\ -\bar{z}\bar{w}\end{pmatrix}, \\
&N \equiv \frac{2\, v^{1/4}}{\sqrt{(1+\bar{z}z)(1-\bar{w}w)}}\,.
\end{aligned} \tag{A-41}$$

Similar expressions are obtained for $\zeta^{-\alpha}_{\beta}$ by replacing the A_i's by B_i's. Eq.(A-41) shows that all the Killing spinors are regular at $z = 0$ and / or $w = 0$.

If $\theta^{\gamma}_{\alpha\beta}$ denotes a grassman parameter labelling the supersymmetry transformations, then the supersymmetry transformation by the spinor parameter $\epsilon = \theta^{\gamma}_{\alpha\beta}\zeta^{\alpha\beta}_{\gamma}$ can be identified as the action of $i\theta^{\gamma}_{\alpha\beta}\widehat{G}^{\alpha\beta}_{\gamma}$ on various fields. Using the known supersymmetry transformation rules for various fields given in (A-19) and the definition (A-18) of $\bar{\epsilon}$ one finds

$$\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2} = \epsilon_2^T\, C\,\Omega\,\Gamma^M\,\epsilon_1\,\partial_M\,, \tag{A-42}$$

up to possible gauge transformations of the type given in (A-6). Using this we can verify that commutator of these supersymmetry generators with themselves and the other symmetries follow the $su(1,1|2)$ algebra given in (4.18).

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