Counting Microscopic Degeneracy of $N = 4$ Black Holes

By

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Certificate

This is to certify that the Ph.D. thesis titled "Counting Microscopic Degenaracy of $N = 4$ Black Holes" submitted by Shamik Banerjee is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Date:

Ashoke Sen

Professor Ashoke Sen
Thesis Advisor

Declaration

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

This work was done under guidance of Professor Ashoke Sen, at Harish Chandra Research Institute, Allahabad.

Date: **Shamik Banerjee**

Ph.D. Candidate

To My Parents....

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Any truth is better than indefinite doubt.

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Synopsis

Black Holes are classical solutions of the equations of General Theory of Relativity. They have the characteristic feature that the space-time curvature blows up at a point which is called the singularity. It is believed that a black hole produced as a result of gravitational collapse will always be surrounded by a horizon of finite area which allows inflow of matter and radiation but nothing can come out of it. The area of the black hole horizon behaves in many ways like the entropy of a thermodynamic system. For example if two black holes collide and form a single black hole then the area of the horizon of the new black hole will be greater than the sum of the horizon areas of the parent black holes. This, along with many other indications, led Bekenstein to conjecture that black hole carries entropy proportional to its horizon area. Immediately after that Bardeen, Carter and Hawking proposed the four laws of black hole thermodynamics. This proved, to some extent, that a black hole is a thermodynamic system which has entropy and has a finite temperature. But this immediately led to the following puzzle. We know that a body at finite temperature emits radiation. In the case of black hole no radiation can come out of it. Stephen Hawking resolved the puzzle by showing that quantum mechanically black holes can emit and the spectrum matches exactly with that of a black body kept at the same temperature as the black hole. This proved beyond doubt that black holes are thermodynamic objects and so we are allowed to ask all sorts of questions about its "constituents".

The development so far depended only on General Theory of Relativity and Quantum Field theory on Curved Spacetime. But none of them can answer the following question. It is known that the entropy of a thermodynamic system is proportional to the logarithm of the number of micro states accessible to the system. So what are the microstaes of a black hole? According to the General Theory of Relativity if we specify the mass, charge and angular momentum of a black hole then the solution is uniquely specified. So classically the entropy vanishes and we need a theory other than General Theory of Relativity which can give us information about the micro states of a black hole and its nature. String Theory is one such candidate. It is a quantum theory of gravity and it can talk meaningfully and unambiguously about black hole and its micro states.

Superstring theories live in ten dimensions. To reduce the number of large dimensions to four one compactifies the six extra dimensions and obtain in this way a theory, which at low energies, reduces to a field theory containing gravity. Black holes arise as solutions of these low energy field theories. In string theory it is possible to count micro states of many supersymmetric black holes. The entropy obtained in this way matches exactly with the Bekenstein-Hawking entropy.

In this thesis I shall focus my attention on a particular $N = 4$ string theory in four dimensions which is obtained by compactifying Heterotic string theory on T^6. This theory has black hole solutions in four dimensions which preserve 1/4 -th of the original supersymmetry (they are usually called quarter BPS dyonic black holes). They carry both electric and magnetic charges- (Q, P)- under $28U(1)$ gauge fields which are present in the theory. The charge vectors take value in the 28 dimensional Narain lattice which is an even selfdual lattice of signature $(6, 22)$. Q and P are both 28 dimensional vectors.This theory also has a set of massless scalar fields which transform in a specific way under the T and S duality symmetries of the four dimensional theory, along with the charge vectors (Q, P). The microscopic degeneracy of these dyonic black holes was known only for a subset of charge vectors which satisfy a specific irreducibility criterion. Our work extends these results to all charge vectors. The details are as follows:

In our first work, we derived the complete set of T-duality invariants which characterize a pair of charge vectors (Q, P). Using this we could identify the complete set of dyonic black holes to which the previously derived degeneracy formula can be extended. By going near special points in the moduli space of the theory we derived the spectrum of quarter BPS dyons in $N = 4$ supersymmetric gauge theory with simply laced gauge groups. The results are in agreement with those derived from field theory analysis.

In our next work we studied the action of S-duality group on the discrete T-duality invariants and studied its consequence for the dyon degeneracy formula. In particular we found that for dyons with torsion r, the degeneracy formula, expressed as a function of Q^2, P^2 and $Q.P$, is required to be manifestly invariant under only a subgroup of the S-duality group. This subgroup is isomorphic to $\Gamma^0(r)$. Our analysis also showed that for a given torsion r, all other discrete T-duality invariants are characterized by the elements of the coset $\frac{SL(2,Z)}{\Gamma^0(r)}$.

In the third work we proposed a general set of constraints on the partition function of quarter BPS dyons in any $N = 4$ supersymmetric string theory by drawing insight from known examples, and studied the consequences of this proposal. The main ingredients of our analysis are duality symmetries, wall crossing formula and black hole entropy. We used our analysis to constrain the dyon partition function for two hitherto unknown cases - the partition function of dyons of torsion two (i.e. $gcd(Q \wedge P) = 2$) in heterotic string theory on T^6 and the partition function of dyons carrying untwisted sector electric charge in Z_2 CHL model. With the help of these constraints we proposed a candidate for the partition function of dyons of torsion two in heterotic string theory on T^6. This leads to a novel wall crossing formula for decay of quarter BPS dyons into half BPS dyons with non-primitive charge vectors. In an appropriate limit the proposed formula reproduces the known result for the spectrum of torsion two dyons in gauge theory.

The original proposal of Dijkgraaf, Verlinde and Verlinde for the quarter BPS dyon partition function in heterotic string theory on T^6 is known to correctly produce the degeneracy of dyons of torsion 1, i.e. dyons for which $gcd(Q \wedge P) = 1$. In our last work we proposed a generalization of this formula for dyons of arbitrary torsion. Our proposal satisfies the constraints coming from S-duality invariance, wall crossing formula, black hole entropy and the gauge theory limit. Furthermore using our proposal we derive a general wall crossing formula that is valid even when both the decay products are nonprimitive half-BPS dyons.

List of Publications

1. Duality covariant variables for STU-model in presence of non-holomorphic corrections.
   Authors: Shamik Banerjee, Rajesh Kumar Gupta
   Ref: arXiv:0905.2700

2. Supersymmetry, Localization and Quantum Entropy Function
   Authors: Nabamita Banerjee, Shamik Banerjee, Rajesh Kumar Gupta, Ipsita Mandal, Ashoke Sen
   Ref:arXiv:0905.2686, JHEP 1002:091,2010

3. Genus Two Surface and Quarter BPS Dyons: The Contour Prescription
   Authors: Shamik Banerjee, Ashoke Sen, Yogesh K. Srivastava
   Ref: hep-th/0808.1746, JHEP 0903:151,2009

4. Interpreting the M2-brane Action
   Authors: Shamik Banerjee, Ashoke Sen,
   Ref: hep-th/0805.3930, Mod.Phys.Lett.A24:721-724,2009

5. Partition Functions of Torsion > 1 Dyons in Heterotic String Theory on T^6
   Authors:Shamik Banerjee, Ashoke Sen, Yogesh K. Srivastava
   Ref: hep-th/0802.1556, JHEP 0805:098,2008

6. Generalities of Quarter BPS Dyon Partition Function and Dyons of Torsion Two
   Authors: Shamik Banerjee, Ashoke Sen, Yogesh K. Srivastava
   Ref: hep-th/0802.0544, JHEP 0805:101,2008

7. S-duality Action on Discrete T-duality Invariants
   Authors: Shamik Banerjee, Ashoke Sen,
   Ref: hep-th/0801.0149, JHEP 0804:012,2008

8. Duality orbits, dyon spectrum and gauge theory limit of heterotic string theory on T^6
   Authors: Shamik Banerjee, Ashoke Sen,
   Ref: hep-th/0712.0043, JHEP 0803:022,2008

Contents

Chapter 1

Introduction

Einstein proposed the general theory of relativity in 1915. Almost a decade after that Quantum theory was formulated. Since then people have tried to formulate a quantum theory of gravity which will unify the principle of general covariance with the principles of quantum mechanics. String theory is the leading candidate for such a theory.

But what are the effects of quantum gravity ? We do not hope to see quantum gravity effects in our daily life. We do not even hope to see it in the present day high energy particle accelarators. So what is the way out? At present it seems that the only place to look for quantum gravity effects is the sky.

Black Holes are classical solutions of the equations of General Theory of Relativity. They have the characteristic feature that the space-time curvature blows up at a point which is called the singularity. It is almost an experimental fact that at the center of every galaxy there is a black hole. It is believed that a black hole produced as a result of gravitaional collapse will always be surrouned by a horizon of finite area which allows inflow of matter and radiation but nothing can come out of it. Black holes have many things in common with an ordinary thermodynamic system. For example the area of the black hole horizon behaves in many ways like the entropy of a thermodynamic system. For example if two black holes collide and form a single black hole then the area of the horizon of the new black hole will be greater than the sum of the horizon areas of the parent black holes. This, along with many other indications, led Bekenstein to conjecture that black hole carries entropy proportional to its horizon area. Immediately after that Bardeen, Carter and Hawking proposed the four laws of black hole thermodynamics. This proved, to some extent, that a black hole is a thermodynamic system whick has entropy and has a finite temparature. But this immediately led to the following puzzle. We know that a body at finite temparature emits radiation. In the case of black hole no radiation can come out of it. Stephen Hawking resolved the puzzle by showing that quantum mechanically black holes can emit and the spectrum matches exactly with that of a blackbody kept at the same temparature as the black hole. His calculation also fixed the proportionality constant to be 1/4, for a specific choice of units.This proved beyond doubt that black holes are thermodynamic objects and so we are allowed to ask all sorts of questions about its ”constituents”.

The developement so far depended only on General Theory of Relativity and Quantum Field theory on Curved Spacetime. But none of them can answer the following question. It is known that the entropy of a thermodynamic system is proportional to the logarithm of the number of microstates accessible to the system. So what are the microstaes of a black hole? According to the General Theory of Relativity if we specify the mass, charge and angular momentum of a black hole then the solution is uniquely specified. So classically the entropy vanishes and we need a theory other than General Theory of Relativity which can give us information about the microstates of a black hole and its nature. But apparantlyy we also need a new theory of “Quantum Mechanics“ for the following reason. We have already seen that a black hole radiates like a black body. Black body radiation is an incoherent superposition of photons. To be more precise, it is described by a mixed ensemble in quantum mechanics. Now if we prepare matter in a pure quantum state and throw it into a black hole, what will come out at the end of the day is black body radiation. So it seems that in the presence of black holes a pure state can evolve into a mixed state and hence unitarity of quantum mechanical time evolution is violated.

For a class of black holes known as extremal/near extremal black holes, string theory can account for the microstates which are responsible for its entropy. Since string theory is manifestly unitary it seems that the violation of unitaity is an artifact of the semiclassical approach. To make things more concrete we shall describe in some detail how microscopic degeneracy is calculated in string theory. We shall focus our attention on D1-D5 blackholes in five dimensions. Then we shall state the result for quarter BPS black holes in heterotic string theory compactified on T^6, for a specific class of electric and magnetic charge vectors.

1.1 Broad Overview of Black holes in String Theory

Black holes in string theory arise as solutions of the low energy effective field theory containing gravity. This is obtained by compactifying superstring theory to various numbers of spacetime dimensions. They are charged under various gauge fields and not all of them are sourced by the fumdumental string. This makes it necessary to consider various solitonic objects like branes. Among the various types of solitons, D-branes play a fundumental role. It is generally believd that at weak string coupling D-branes are described well by the perturbative open string theory, but at strong coupling the appropriate description is in terms of gravity, which originates in the closed string sector. In many cases the gravity description is in terms of black holes with horizon and this is the key to the understanding of black hole microstates in string theory.

But, with our present ability to perform calculations at strong coupling we may not be able to understand the microscopics of an arbitrary black hole. So we confine our attention to a specific class of supersymmetric black holes. They will correspond to states in the D-brane field theory which preserve some amount of supersymmetry. So by counting states corresponding to a given black hole we can calculate the entropy. Counting of supersymmetric states can be done at weak coupling because of their protected nature. This gurantees that the number of such states (or more appropriately an index) does not change as we vary the coupling from weak to strong.

To make things more concrete, in the next section we give a brief introduction to D-branes and describe the counting of microstates for D1-D5 black holes in five dimensions.

1.2 D Branes

D-branes or more precisely D-p branes are solitonic objects in string theory which couple electrically [1] to Ramond-Ramond $(p + 1)$-form potentials where p is the number of spacelike directions along the brane worldvolume.The allowed values of p depend on the particular superstring theory under consideration- it is odd in type IIB and even in type IIA string theory. The tension or mass per unit volume of a brane goes inversely as the closed string coupling constant. At weak string coupling the quantum theory of branes is postulated to be the theory of open strings which have both the ends on the brane. In the worldsheet conformal field theory of the fundumental open strings,this corresponds to Neumann boudary conditions along $(p+1)$ worldvolume directions and Dirichilet boundary conditions along $(9-p)$ transverse directions of the brane.

One of the most important properties of a D-brane is that it preserves half of the original space-time supersymmetries [1]. This makes it very useful for probing various nonperturbative effects in string theory. The low energy theory describing a single D-p brane in flat spacetime is a U(1) super Yang-Mills theory with 16 supersymmetries in $(p + 1)$ dimensions. If we have a stack of N D-branes then the U(1) is enhanced to U(N) [2]. The U(1) factor of the U(N) gauge group represents the center of mass motion of the branes and the SU(N) can be thought of as describing their relative motion. Various low energy properties of a stack of D-branes can be studied in the effective super Yang-Mills theory [2–5]. For example if we have a certain configuration which from the spacetime point of view preserves only half of the supersymmetries, then they can be thought of as supersymmetric vacua of the D-brane super Yang-Mills theory. Similarly if it preserves quarter of the spacetime supersymmetries then it represents a half-BPS state in the worldvolume field theory. This will play an important role in the microstate counting of black holes.

We can also consider D-branes in a compactified superstring theory. In this case the branes can also wrap various cycles (compact submanifolds without boundary) of the compactification manifold. Our main interest lies in the supersymmetric bound states of branes of various dimensions wrapping various cycles of the compact manifold. To be specific, we shall consider type IIB string theory compactified on $K3 \times S^1$. This theory has D-p branes for odd values of p. We take D-5 branes and D-1 branes and wrap them on $K3 \times S^1$ and S^1 respectively. By T-dualising one can also think of this as D0-D4 system on $K3$ in type IIA string theory [5]. In the following we shall describe in detail the low energy field theory of this system.

Type IIB string theory compactified on $K3 \times S^1$ has 16 unbroken supersymmetries. If we wrap D1-D5 brane system on $K3 \times S^1$ [6] in the manner described above the number of unbroken spacetime supersymmetries are 8. Now we go to a limit where the size of the $K3$ is much smaller than the size of the S^1. In this limit the low energy theory describing the system is expected to be a $(1+1)$ dimensional field theory based on S^1. It can be shown that this theory is actually a $(4,4)$ superconformal sigma model with target space the moduli space of instantons on $K3$. If we want to calculate the degeneracy of some configuration which preserves only 4 out of 16 spacetime supersymmetries, which is precisely the amount of supersymmetries preserved by D1-D5 black hole, we can look for half-BPS states in this two dimensional theory. Since we are looking for supersymmetric states we can just look at the Ramond-Ramond sector of the field theory. In this sector the half-BPS states are precisely the ones in which say the right moving excitations are in their ground state and the left moving excitation is arbitrary.[1] In a unitary conformal field theory the number of states with large conformal weight can be calculated using Cardy formula, without knowing the details of the theory except the central charges of the left movers and right movers. Typically black holes with large charges correspond to states in the conformal field theory with large conformal dimensions. For them the degeneracy can be calculated using Cardy formula and the leading answer matches perfectly with the Bekenstein-Hawking entropy

1.3 Brief Introduction to $E_8 \times E_8$ Heterotic String Theory

In the following we will briefly review the basic facts about heterotic string theory which will be useful for our purpose. Heterotic string has sixteen supersymmetries and $E_8 \times E_8$ gauge symmetry in ten dimensions. Compactification on T^6 does not lead to any supersymmetry breaking. In any compactified string theory there are massless scalar fields whose asymptotic values determine the vacuum. They are called moduli fields. Heterotic string theory also has a set of moduli fields parametrized by 28×28 real symmetric matrix M which takes value in $\frac{O(6,22;R)}{O(6;R)\times O(22;R)}$ and a complex scalar moduli field, τ valued in the upper half plane. Spacetime variation of M represents the fluctuations of the internal geometry and various nongeometric fields on the compactification manifold and τ represents the axion-dilaton of the four dimensional theory. At generic points on the moduli space, there are 28 U(1) gauge fiels. Sixteen of them come from that part of the $E_8 \times E_8$ gauge field which is valued in the Cartan

[1] In a supersymmetric field theory Witten index counts the difference between the number of Bosonic and Fermionic vacua. In this case there is a similar index called elliptic genus which is essentially the Witten index for say the right movers and the usual partition function for the left movers.It does not count the absolute number of states and being an index it is robust against deformation of the asymptotic moduli.

sub-algebra. Six of them come from the rank two antisymmetric tensor field and the rest from the metric. There exist special points on the moduli space where the gauge symmetry is enhanced to a non-abelian group.

The four dimensional theory has black hole solutions which preserve 1/4 -th of the original supersymmetry.They are called quarter BPS dyonic black holes. They are both electrically and magnetically charged under the $U(1)$ gauge fields present in the theory. The electric and magnetic charge vectors, (Q, P), take value in the 28 dimensional Narain lattice which is an even selfdual lattice of signature $(6, 22)$ (a lattice is called even if the length squared of every vector belonging to it is an even integer and self-dual if the dual lattice is isomorphic to the original lattice). They are both 28 dimensional vectors.

An explicit expression can be written down for the matrix M in terms of the internal components of metric, antisymmetric tensor field and gauge fields denoted by G_{ab}, B_{ab} and A^I_a, respectively [71,72]. Here $4 \leq a, b \leq 9$ stand for the internal spacetime indices along the torus direction and $1 \leq I \leq 16$ denote the Lie algebra indices along the cartan direction. We can combine the scalar fields G_{ab}, B_{ab}, and $A^I{}_a$ into an $O(6, 22; R)$ matrix. For this we regard G_{ab} , B_{ab} and A^I_a as 6×6, 6×6, and 6×16 matrices respectively, $C_{ab} = \frac{1}{2}\, A^I{}_a\, A^I{}_b$ as a 6×6 matrix, and define M to be the 28×28 dimensional matrix

$$M = \begin{pmatrix} G^{-1} & G^{-1}(B+C) & G^{-1}\ A \\ (-B+C)\ G^{-1} & (G-B+C)\ G^{-1}(G+B+C) & (G-B+C)\ G^{-1}\ A \\ A^T\ G^{-1} & A\ G^{-1}(G+B+C) & I_{16} +\ A^T\ G^{-1}\ A \end{pmatrix} . \quad (1.3.1)$$

It can be checked that M satisfies

$$MLM^T = L, \qquad M^T = M, \qquad L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix}, \quad (1.3.2)$$

where I_n denotes the $n \times n$ identity matrix. The first equation tells us that M is an $O(6, 22; R)$ matrix. The matrix L has 6 positive eigenvalues $+1$ and 22 negative eigenvalues -1. This is also the metric on the Narain lattice [25,26] and can be used to define the inner product between two charge vectors.

1.3.1 Duality Symmetries

In four dimensions the heterotic string has two types of duality symmetries, T-duality and S-duality [72]. T duality group is $O(6, 22; Z)$ and it acts on the charges and the moduli in the following way,

$$Q \to \Omega Q, \qquad P \to \Omega P \quad (1.3.1)$$

and

$$M \to \Omega M \Omega^T, \qquad \tau \to \tau, \quad (1.3.2)$$

where Ω is an $O(6,22;Z)$ matrix defined such that $\Omega L \Omega^T = L$ and Ω preserves the Narain lattice. Similarly the S-duality acts like

$$Q \to aQ + bP, \qquad P \to cQ + dP \tag{1.3.3}$$

and

$$M \to M, \qquad \tau \to (a\tau + b)/(c\tau + d) \tag{1.3.4}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.3.5}$$

is an $SL_2(Z)$ matrix defined by the property that $ad - bc = 1$ and a, b, c, d are all integers. There are two important points to notice. One is that the T-duality group acts as isometry of the narain lattice and so does not change the lengths and inner products between charge vectors. But S-duality group can change them. And both T and S duality group act on the moduli.

1.4 Degeneracy Of Quarter-BPS Dyonic Black Holes

Heterotic string theory reduces to a field theory containing gravity at low energy. The two derivative action of the theory is uniquely fixed by the requirement of $N = 4$ supersymmetry. This theory has extremal black hole solutions carrying arbitrary electric and magnetic charge vectors (Q, P). The leading order Bekenstein-hawking entropy is given by the expression $\pi\sqrt{|Q^2P^2 - (Q.P)^2|}$. It is known that the black holes for which $Q^2P^2 - (Q.P)^2$ is greater than zero, are supersymmetric. The action can have arbitrary higher derivative term and the resulting change in entropy can be calculated by Entropy Function Formalism.

Now one would like to know if the same thing can be reproduced by calculating the microscopic degeneracy of the black hole state carrying the specified charge vectors. This will give us some insight into the nature of the black hole microstates. In calcualting the degeneracy for generic charge vectors one makes heavy use of the symmetries of the theory. The degeneracy formula should be invariant under the duality symmetries. Naively this fixes the form of the degeneracy formula to some extent. The formula can depend only on the invariant combinations of charge vectors. But there are some fine issues related to this. We have already discussed the fact that the four dimensional vacuum of the theory is determined in terms of the asymptotic values of the moduli fields. So in principle the degeneracy of a state can depend not only on the charges but also on the asymptotic moduli. Generically the duality transformations act on the asymptotic moduli and change the vacuum of the theory.One could construct mixed invariants built out of charges and the moduli, because both of them transform under the duality, and the degeneracy formula could well depend on them. But it turns out that at least for supersymmetric black holes, the dependence of the degeneracy on the asymptotic moduli is rather soft. There are domains in the moduli space in which the degeneracy does not depend on the moduli and it jumps by a known amount once the moduli cross the boundary of the domain (these boundaries are usually called the walls of marginal stability).Therefore the dependence on such mixed invariants can be determined in terms of jumps across the walls of marginal stability. This makes the job somewhat easier. One can calculate the degeneracy for a specific charge vector and then can extend the result to other charge vectors by duality symmetry. This turns out to be crucial for what we are going to do in the next few chapters.

We shall now state the result for the degeneracy of quarter-BPS dyonic black holes carrying charges (Q, P) [7]. Let $d(Q, P)$ stand for the degenaracy in a specific domain of the moduli space. Then,

$$d(Q,P) = (-1)^{Q\cdot P+1} \int_{\mathcal{C}} d\rho d\sigma dv e^{-i\pi(\sigma Q^2+\rho P^2+2vQ\cdot P)} \frac{1}{\Phi_{10}(\rho,\sigma,v)} \tag{1.4.6}$$

where $\mathcal{C}$ is the contour of integration in three complex dimensional space parametrized by (ρ, σ, v) and

$$0 \le \rho_1 \le 1, \quad 0 \le \sigma_1 \le 1, \quad 0 \le v_1 \le 1, \quad \rho_2 = M_1, \quad \sigma_2 = M_2, \quad v_2 = M_3\,, \tag{1.4.7}$$

M_1, M_2 and M_3 are three real constants. Φ_{10} is the unique weight ten Igusa cusp form of $Sp(2, \mathbb{Z})$ and it is invariant under both T and S duality transformations. The effect of the asymptotic moduli is contained in the choice of the contour of integration. We have already seen that the degeneracy in a given domain does not depend on the asymptotic moduli, but it can jump if we cross a wall of marginal stability. There exists a specific moduli dependent choice of contour [22] which beautifully captures both the moduli independence in a given domain and the wall crossing phenomenon. It is given by,

$$\begin{aligned}
\rho_2 &= \Lambda \left\{ \frac{|\tau|^2}{\tau_2} + \frac{Q_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}} \right\}, \\
\sigma_2 &= \Lambda \left\{ \frac{1}{\tau_2} + \frac{P_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}} \right\}, \\
v_2 &= -\Lambda \left\{ \frac{\tau_1}{\tau_2} + \frac{Q_R\cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}} \right\},
\end{aligned} \tag{1.4.8}$$

where,

$$Q_R^2 = Q^T(M+L)Q, \quad P_R^2 = P^T(M+L)P, \quad Q_R\cdot P_R = Q^T(M+L)P\,. \tag{1.4.9}$$

For Λ sufficiently large it gives us the correct contour of integration associated with a specific value of the moduli M and τ. As long as we are varying the moduli inside a given domain this is certainly more than that is necessary. Since the integrand is holomorphic, for given values of the moduli, we can deform the contour and this will not change the integral if we do not cross any pole of the integrand. But the virtue of this formula is that it can correctly capture the wall-crosssing phenomenon. As the moduli cross a wall of marginal stability the contour in the (ρ, σ, v) space crosses a pole of the integrand. This causes the necessary jump in the degenaracy. And moreover the formula is T duality invariant and S duality covariant.

Our next task is to specify the domain of validity of the beforementioned degenaracy formula. We shall follow closely the discussion of [23].

We write the T^6 on which the heterotic string theory is compactified as $T^4 \times \widehat{S}^1 \times S^1$. We consider a state in this theory which carries $\widehat{n}$ unit of momentum and $-\widehat{w}$ unit of winding along $\widehat{S}^1$, n' unit of momentum and $-w'$ unit of winding along S^1, $\widehat{N}$ unit of Kaluza-Klein monopole charge [73, 74] associated with $\widehat{S}^1$, $-\widehat{W}$ unit of NS 5-brane wrapped along $T^4 \times S^1$, N' unit of Kaluza-Klein monopole charge associated with S^1 and W' unit of NS 5-brane wrapped along $T^4 \times \widehat{S}^1$. This describes a four dimensional charge vector of the form

$$Q = \begin{pmatrix} \widehat{n} \\ n' \\ \widehat{w} \\ w' \end{pmatrix}, \qquad P = \begin{pmatrix} \widehat{W} \\ W' \\ \widehat{N} \\ N' \end{pmatrix}. \tag{1.4.10}$$

For this we have,

$$Q^2 = 2(\widehat{n}\widehat{w} + n'w'), \qquad P^2 = 2(\widehat{N}\widehat{W} + N'W'), \qquad P \cdot Q = \widehat{N}\widehat{n} + \widehat{W}\widehat{w} + N'n' + W'w' . \tag{1.4.11}$$

The microscopic calculation of the degeneracy was done for the D1-D5 system moving in the Kaluza-Klein Monopole background [15].The specific type of charge vector for which this formula is valid, is given in the heterotic frame by,

$$Q = \begin{pmatrix} k_3 \\ k_4 \\ k_5 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} l_3 \\ l_4 \\ l_5 \\ 0 \end{pmatrix}, \qquad k_i, l_i \in \mathbb{Z} \quad \text{otherwise}, \quad \text{g.c.d.}(l_3, l_5) = 1 . \tag{1.4.12}$$

But this is not the most general charge vector. We can use various duality transformations to map it to charge vectors of other forms. We have already talked about the T and S duality transformations and we have seen that they also act on the asynptotic moduli. So from the duality invariance of the degeneracy formula we can say that the microscopic degeneracy for two charge vectors realated by the duality transformations are the same provided the asymptotic moduli are also transformed. It turns out that the T duality transformation does not take the asymptotic moduli across the walls of marginal stability. Since the degenaracy does not vary inside a domain bounded by the walls of marginal stability, we can expect that the degeneracy remains unchanged even if the asymptotic moduli are not transformed (in fact this is manifest in the formula for the contour of integration). But on the other hand S duality transformation generically takes the moduli across the wall of marginal stability. So we can extend the result from the original domain where the calculation was done to other domains by S duality transformations.

1.5 Motivation and A Brief Review of The Work Done

We saw in the last section that the degeneracy formula is valid only for a specific class of charge vectors. Moreover it depends only on the T duality invariant combinations Q^2, P^2 and $Q.P$. So it seems that two charge vectors with the same values for these invariants will have the same degenaracy. But this is not quite correct. In fact Q^2, P^2 and $Q.P$ are also invariants of the continious T duality group $O(6, 22; R)$. The discrete T duality group can have more independent invariants than the continious one and as a result two charge vectors having the same values for Q^2, P^2 and $Q.P$, may not be related by a T duality transformation. For example [20][2],

$$r(Q, P) = \text{g.c.d}\{Q_i P_j - Q_j P_i\}, \quad 1 \le i, j \le 28, . \tag{1.5.13}$$

is an invariant of the discrete T duality group but not of the continious one. In [20, 40, 54],it was shown that the degenaracy formula (1.4.6) can at most be valid for charge vectors which satisfy $r(Q, P) = 1$. In general the degeneracy can depend on $Q^2, P^2, Q.P, r(Q, P)$ and other independent invariants of the discrete T duality group.

The motivation behind the collaborative research projects that I had taken up, on which this dissertation is based, is to answer these questions. We describe below, in detail, the results of our investigation.

In our first work [54], we derived the complete set of T-duality invariants which characterize a pair of charge vectors (Q, P). Using this we could identify the complete set of dyonic black holes to which the previously derived degeneracy formula can be extended. By going near special points in the moduli space of the theory we derived the spectrum of quarter BPS dyons in N=4 supersymmetric gauge theory with simply laced gauge groups. The results are in agreement with those derived from field theory analysis.

This work proved that for unit torsion,i.e, for $r(Q, P) = 1$, Q^2, P^2 and $Q.P$ are the only invariants and so the degeneracy formula is valid for all charge vectors with unit torsion.

In our next work [40] we studied the action of S-duality group on the discrete T-duality invariants and studied its consequence for the dyon degeneracy formula. In particular we found that for dyons with torsion r, the degeneracy formula, expressed as a function of Q^2, P^2 and $Q.P$, is required to be manifestly invariant under only a subgroup of the S-duality group. This

[2]This will be called the torsion of the pair (Q, P).

subgroup is isomorphic to $\Gamma^0(r)$. Our analysis also showed that for a given torsion r, all other discrete T-duality invariants are characterized by the elements of the coset $SL(2,Z)/\Gamma^0(r)$.

In the third work [62] we proposed a general set of constraints on the partition function of quarter BPS dyons in any N=4 supersymmetric string theory by drawing insight from known examples, and studied the consequences of this proposal. The main ingredients of our analysis are duality symmetries, wall crossing formula and black hole entropy. We used our analysis to constrain the dyon partition function for two hitherto unknown cases – the partition function of dyons of torsion two (i.e. $gcd(Q \wedge P) = 2$) in heterotic string theory on T^6 and the partition function of dyons carrying untwisted sector electric charge in Z_2 CHL model. With the help of these constraints we proposed a candidate for the partition function of on T^6. This leads to a novel wall crossing formula for decay of quarter BPS dyons into half BPS dyons with non-primitive charge vectors. In an appropriate limit the proposed formula reproduces the known result for the spectrum of torsion two dyons in gauge theory.

In our first work we proved that the original proposal of Dijkgraaf, Verlinde and Verlinde for the quarter BPS dyon partition function in heterotic string theory on T^6 can correctly produce the degeneracy of dyons of torsion 1, i.e. dyons for which $gcd(Q \wedge P) = 1$. In our last work [63] we proposed a generalization of this formula for dyons of arbitrary torsion. Our proposal satisfies the constraints coming from S-duality invariance, wall crossing formula, black hole entropy and the gauge theory limit. Furthermore using our proposal we derive a general wall crossing formula that is valid even when both the decay products are non-primitive half-BPS dyons.

Chapter 2

Duality orbits, dyon spectrum and gauge theory limit of heterotic string theory on T^6

2.1 Introduction

We have a good understanding of the exact spectrum of a class of quarter BPS dyons in a variety of $\mathcal{N} = 4$ supersymmetric string theories [7–23, 38]. In the last chapter we saw one particular example of this, namely heterotic string theory compactified on T^6. Explicit computation of the spectrum was carried out for a special class of charge vectors in a specific region of the moduli space. Using the various duality invariances of the theory we can extend the results to various other charge vectors in various other regions in the moduli space. However in order to do this we need to find out the duality orbits of the charge vectors for which the spectrum has been computed. This is one of the goals of this chapter. Throughout this paper we shall focus on heterotic string theory compactified on T^6.

We have already seen that a duality transformation typically acts on the charges as well as the moduli. Thus using duality invariance we can relate the degeneracy of a given state at one point of the moduli space to that of a different state, carrying different set of charges, at another point of the moduli space. For BPS states however the degeneracy – or more precisely an appropriate index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets for a given set of charges – is invariant under changes in the moduli unless we cross a wall of marginal stability on which the state under consideration becomes marginally unstable. Thus for BPS states, instead of having to describe the spectrum as a function of the continuous moduli parameters we only need to specify it in different domains bounded by walls of marginal stability [19, 22, 23]. It turns out that a T-duality transformation takes a point inside one such domain to another point inside the same domain in a sense described precisely in [19, 23]. Thus once we have calculated the spectrum in one domain for a given charge, T-duality symmetry can be used to find the spectrum in the same domain for all other charges related to the initial charge by a T-duality transformation. For this reason it is important to understand under what condition two different charges are related to each other by a T-duality transformation, i.e. to classify the T-duality orbits. S-duality transformation, on the other hand, takes a point inside one domain to a point in another domain. Thus once we have calculated the spectrum in one domain, S-duality transformation allows us to calculate the spectrum in other domains.

Our results for the T-duality orbit of charges can be summarized as follows. We shall take Q and P to be primitive vectors of the Narain lattice; if not we can express them as integer multiples of primitive vectors and apply our analysis to these primitive vectors, treating the integer factors as additional T-duality invariants. Let Q_i and P_i denote the components of Q and P along some basis of primitive vectors of the lattice Λ and L_{ij} denote the natural metric of signature (6,22) under which the lattice is even and self-dual. Then the complete set of T-duality invariants are as follows. First of all we have the invariants of the continuous T-duality group:

$$Q^2 = Q^T L Q, \qquad P^2 = P^T L P, \qquad Q \cdot P = Q^T L P\,. \tag{2.1.1}$$

Next we have the combination [20, 24]

$$r(Q,P) = \text{g.c.d.}\{Q_i P_j - Q_j P_i, \quad 1 \le i, j \le 28\}\,. \tag{2.1.2}$$

Finally we have

$$u_1(Q,P) = \alpha \cdot P \text{ mod } r(Q,P), \qquad \alpha \in \Lambda, \quad \alpha \cdot Q = 1\,. \tag{2.1.3}$$

$u_1(Q,P)$ can be shown to be independent of the choice of $\alpha \in \Lambda$. One finds first of all that each of the five combinations Q^2, P^2, $Q \cdot P$, $r(Q,P)$ and $u_1(Q,P)$ is invariant under T-duality transformation. Furhermore two pairs (Q,P) and (Q',P') having the same set of invariants can be transformed to each other by a T-duality transformation. Thus a necessary and sufficient condition for two pairs of charge vectors (Q,P) and (Q',P') to be related via a T-duality transformation is that all the five invariants are identical for the two pairs.

The computation of [15] of the spectrum of quarter BPS states in heterotic string theory on T^6 has been carried out for a special class of charge vectors for which $r(Q,P) = 1$, and Q^2, P^2 and $Q \cdot P$ are arbitrary. The invariant $u_1(Q,P)$ is trivially 0 for states with $r(Q,P) = 1$. Let us denote the calculated index by $f(Q^2, P^2, Q \cdot P)$. Then T-duality invariance tells us that for all states with $r(Q,P) = 1$ the index is given by the same function $f(Q^2, P^2, Q \cdot P)$ in the domain of the moduli space in which the original calculation was performed. Since S-duality maps states with $r(Q,P) = 1$ to states with $r(Q,P) = 1$, but maps the original domain to other domains, S-duality invariance allows us to extend the result to all states with $r(Q,P) = 1$ in all domains of the moduli space.

Since at special points in the moduli space of heterotic string theory on T^6 we can get $\mathcal{N} = 4$ supersymmetric gauge theories with simply laced gauge groups [25, 26] in the low energy limit, we can use the dyon spectrum of string theory to extract information about the dyon spectrum of $\mathcal{N} = 4$ supersymmetric gauge theories. For this we need to work near the point in the moduli space where we have enhanced gauge symmetry. Slightly away from this point we have the non-abelian part of the gauge symmetry spontaneously broken at a scale small compared to the string scale, and the spectrum of string theory contains quarter BPS dyons whose masses are of the order of the symmetry breaking scale. These dyons can be identified as dyons in the $\mathcal{N} = 4$ supersymmetric gauge theory. Thus the knowledge of the quarter BPS dyon spectrum in heterotic string theory on T^6 gives us information about the quarter BPS dyon spectrum in all $\mathcal{N} = 4$ supersymmetric gauge theories which can be obtained from the heterotic string theory on T^6. This method has been used in [59] to compute the spectrum of a class of quarter BPS states in $\mathcal{N} = 4$ supersymmetric SU(3) gauge theory.

Since the result for the quarter BPS dyon spectrum in heterotic string theory on T^6 is known only for the states with $r(Q, P) = 1$, we can use this information to compute the index of only a subset of dyons in $\mathcal{N} = 4$ super Yang-Mills theory with simply laced gauge groups. For this subset of states the result for the index can be stated in a simple manner, – we find that the index is non-zero only for those charges which can be embedded in the root lattice of an SU(3) subalgebra. Thus these states fall within the class of states analyzed in [59] and can be represented as arising from a 3-string junction with the three external strings ending on three parallel D3-branes [27]. This result for general $\mathcal{N} = 4$ supersymmetric gauge theories is in agreement with previous results obtained either by direct analysis in gauge theory [49, 50] or by the analysis of the spectrum of string network on a system of D3-branes [28].

Some related issues have been addressed in [29].

2.2 T-duality orbits of dyon charges in heterotic string theory on T^6

We consider heterotic string theory compactified on T^6. In this case a general dyon is characterized by its electric and magnetic charge vectors (Q, P) where Q and P are 28 dimensional charge vectors taking values in the Narain lattice Λ [25]. We shall express Q and P as linear combinations of a primitive basis of lattice vectors so that the coefficients Q_i and P_i are integers. There is a natural metric L of signature (6,22) on Λ under which the lattice is even and self-dual. The discrete T-duality transformations of the theory take the form

$$Q \to \Omega Q, \qquad P \to \Omega P\,, \tag{2.2.1}$$

where Ω is a 28×28 matrix that preserves the metric L and the Narain lattice Λ

$$\Omega^T L \Omega = L, \qquad \Omega\Lambda = \Lambda\,. \tag{2.2.2}$$

Since Ω must map an arbitrary integer valued vector to another integer valued vector, the elements of Ω must be integers.

We shall assume from the beginning that Q and P are primitive elements of the lattice.[1] Our goal is to find the T-duality invariants which characterize the pair of charge vectors (Q, P). First of all we have the continuous T-duality invariants

$$Q^2 = Q^T L Q, \qquad P^2 = P^T L P, \qquad Q \cdot P = Q^T L P\,. \tag{2.2.3}$$

Besides these we can introduce some additional invariants as follows. Consider the combination [20, 24]

$$r(Q,P) = \text{g.c.d.}\{Q_i P_j - Q_j P_i, \quad 1 \le i,j \le 28\}\,. \tag{2.2.4}$$

We shall first show that $r(Q, P)$ is independent of the choice of basis in which we expand Q and P. For this we note that the component form of Q and P in a different choice of basis will be related to the ones given above by multiplication by a matrix S with integer elements and unit determinant so that the elements of S^{-1} are also integers. Thus in this new basis r will be given by

$$r(SQ, SP) = \gcd\{S_{ik}S_{jl}(Q_k P_l - Q_l P_k), \quad 1 \le i,j \le 28\}\ . \tag{2.2.5}$$

Since S_{ik} are integers, eq.(2.2.5) shows that $r(SQ, SP)$ must be divisible by $r(Q, P)$. Applying the S^{-1} transformation on (SQ, SP), and noting that S^{-1} also has integer elements, we can show that $r(Q, P)$ must be divisible by $r(SQ, SP)$. Thus we have

$$r(Q,P) = r(SQ,SP)\,, \tag{2.2.6}$$

i.e. $r(Q, P)$ is independent of the choice of basis used to describe the vectors (Q, P). As a special case where we restrict S to T-duality transformation matrices Ω, we find

$$r(Q,P) = r(\Omega Q, \Omega P)\,. \tag{2.2.7}$$

Thus $r(Q, P)$ is invariant under a T-duality transformation.

Another set of T-duality invariants may be constructed as follows. Let $\alpha, \beta \in \Lambda$ satisfy

$$\alpha \cdot Q = 1, \qquad \beta \cdot P = 1\,. \tag{2.2.8}$$

Since Q and P are primitive and the lattice is self-dual one can always find such α, β. Then we define

$$u_1(Q,P) = \alpha \cdot P \bmod r(Q,P), \qquad u_2(Q,P) = \beta \cdot Q \bmod r(Q,P)\,. \tag{2.2.9}$$

One can show that [24]

1. u_1 and u_2 are independent of the choice of α, β.

[1] If this is not the case then the gcd a_1 of all the elements of Q and the gcd a_2 of all the elements of P will be separately invariant under discrete T-duality transformation. We can factor these out as $Q = a_1\,\overline{Q}$, $P = a_2\,\overline{P}$ with $a_1, a_2 \in \mathbb{Z}$, $\overline{Q}, \overline{P} \in \Lambda$, and then apply our analysis on the resulting primitive elements $\overline{Q}$ and $\overline{P}$.

2. u_1 and u_2 are T-duality invariants.

3. u_2 is determined uniquely in terms of u_1.

The proof of these statements goes as follows. To prove that u_1 is independent of the choice of α we note that since Q is a primitive vector we can choose a basis of lattice vectors so that the first element of the basis is Q itself. Then in this basis[2]

$$Q = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \qquad P = \begin{pmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ P_{28} \end{pmatrix}, \tag{2.2.10}$$

and we have

$$r(Q, P) = \gcd(P_2, \cdots P_{28}) \,. \tag{2.2.11}$$

Now suppose α_1 and α_2 are two vectors which satisfy $Q \cdot \alpha_1 = Q \cdot \alpha_2 = 1$. Then $(\alpha_1 - \alpha_2) \cdot Q = 0$, and hence we have

$$(\alpha_1 - \alpha_2) \cdot P = (\alpha_1 - \alpha_2) \cdot (P - P_1 Q) = (\alpha_1 - \alpha_2) \cdot \begin{pmatrix} 0 \\ P_2 \\ P_3 \\ \cdot \\ \cdot \\ P_{28} \end{pmatrix} . \tag{2.2.12}$$

Eq.(2.2.11) shows that the right hand side of (2.2.12) is divisible by r. Thus $\alpha_1 \cdot P = \alpha_2 \cdot P$ modulo r. This shows that u_1 defined through (2.2.9) is independent of the choice of α. A similar analysis shows that u_2 defined in (2.2.9) is independent of the choice of β. From now on all equalities involving $u_1(Q, P)$ and $u_2(Q, P)$ will be understood to hold modulo $r(Q, P)$ although we shall not always mention it explicitly.

T-duality invariance of u_1 follows from the fact that if $\alpha \cdot Q = 1$ then $\Omega\alpha \cdot \Omega Q = 1$. Thus

$$u_1(\Omega Q, \Omega P) = \Omega\alpha \cdot \Omega P \bmod r(\Omega Q, \Omega P) = \alpha \cdot P \bmod r(Q, P) = u_1(Q, P) \,. \tag{2.2.13}$$

A similar analysis shows the T-duality invariance of u_2.

To show that u_2 is determined in terms of u_1 and vice versa we first note that for the choice of (Q, P) given in (2.2.10), we have

$$u_1(Q, P) = \alpha \cdot P = \alpha \cdot (P - P_1 Q) + P_1 \, \alpha \cdot Q = P_1 \bmod r(Q, P) \tag{2.2.14}$$

[2]Note that in this basis the metric takes a complicated form, *e.g.* the 11 component of the metric must be equal to Q^2. However all components of the metric are still integers since the inner product between two arbitrary integer valued vectors – representing a pair of elements of the lattice – must be integer.

since $P - P_1 Q$ is divisible by r due to eqs.(2.2.10), (2.2.11), and $\alpha \cdot Q = 1$. On the other hand we have

$$1 = \beta \cdot P = \{\beta \cdot (P - P_1 Q) + P_1 \beta \cdot Q\} = u_1(Q,P) u_2(Q,P) \bmod r(Q,P)\,, \qquad (2.2.15)$$

since $(P - P_1 Q) = 0$ modulo r, $P_1 = u_1$, and $\beta \cdot Q = u_2$. Thus we have

$$u_1(Q,P)\, u_2(Q,P) = 1 \bmod r(Q,P)\,. \qquad (2.2.16)$$

This shows that neither u_1 nor u_2 shares a common factor with r. We shall now show that (2.2.16) also determines u_2 uniquely in terms of u_1. To prove this assume the contrary, that there exists another number v_2 satisfying $u_1\, v_2 = 1 \bmod r(Q,P)$. Then we have

$$u_1(Q,P)\ (u_2(Q,P) - v_2(Q,P)) = 0 \bmod r(Q,P)\,. \qquad (2.2.17)$$

Since u_1 has no common factor with r, this shows that $v_2 = u_2$ modulo r. Hence u_2 is determined in terms of u_1 modulo r.

Thus we have so far identified five separate T-duality invariants characterizing the pair of vectors (Q,P): Q^2, P^2, $Q \cdot P$, $r(Q,P)$ and $u_1(Q,P)$. We shall now show that these are sufficient to characterize a T-duality orbit, i.e. given any two pairs (Q,P) and (Q',P') with the same set of invariants they are related by a T-duality transformation. We begin by defining[3]

$$\widehat{P} = Q^2 P - Q \cdot P\, Q\,, \qquad (2.2.18)$$

and

$$\widetilde{P} = \frac{1}{K}\,\widehat{P}\,, \qquad K \equiv \gcd\{\widehat{P}_1, \cdots \widehat{P}_{28}\}\,. \qquad (2.2.19)$$

By construction $\widetilde{P}$ is a primitive vector of the lattice satisfying

$$Q \cdot \widetilde{P} = 0\,. \qquad (2.2.20)$$

We shall now use the result of [24] that the T-duality orbit of a pair of primitive vectors $(Q,\widetilde{P})$ satisfying $Q \cdot \widetilde{P} = 0$ is characterized completely by the invariants Q^2, $\widetilde{P}^2$, $r(Q,\widetilde{P})$ and $u_1(Q,\widetilde{P})$. Given this, we shall show that the five invariants Q^2, P^2, $Q \cdot P$, $r(Q,P)$ and $u_1(Q,P)$ completely characterize the duality orbits of an arbitrary pair of charge vectors (Q,P). The steps involved in the proof are as follows:

1. We shall first show that the quantities $\widetilde{P}^2$, $r(Q,\widetilde{P})$, $u_1(Q,\widetilde{P})$ and the constant K appearing in (2.2.19) are determined completely in terms of Q^2, P^2, $Q \cdot P$, $r(Q,P)$ and $u_1(Q,P)$

[3]This procedure breaks down for $Q^2 = 0$, but as long as $P^2 \neq 0$ we can carry out our analysis by reversing the roles of Q and P. If both Q^2 and P^2 vanish then our analysis does not apply. However a different proof given in §2.3 applies to this case as well.

via the relaions

$$
\begin{aligned}
&K = r(Q,P)\,\mathrm{gcd}\,\left\{(u_1(Q,P)Q^2 - Q\cdot P)/r(Q,P), Q^2\right\},\\
&r(Q,\widetilde{P}) = Q^2 r(Q,P)/K, \quad u_1(Q,\widetilde{P}) = \frac{1}{K}\,(u_1(Q,P)Q^2 - Q\cdot P) \quad \mathrm{mod}\; r(Q,\widetilde{P})\,,\\
&\widetilde{P}^2 = \frac{1}{K^2}Q^2(Q^2P^2 - (Q\cdot P)^2)\,. \qquad (2.2.21)
\end{aligned}
$$

The last equation follows trivially from the definition of $\widetilde{P}$. To prove the other relations we again use the form of (Q,P) given in (2.2.10). We have

$$
\widehat{P} = Q^2\,P - Q\cdot P\,Q = Q^2(P-P_1Q) - Q\cdot(P-P_1Q)\,Q = r(Q,P)\{Q^2\gamma - Q\cdot\gamma\,Q\}\,, \quad (2.2.22)
$$

where

$$
\gamma = \frac{1}{r(Q,P)}(P-P_1Q) = \frac{1}{r(Q,P)}\begin{pmatrix} 0\\ P_2\\ \cdot\\ \cdot\\ P_{28}\end{pmatrix}\,. \qquad (2.2.23)
$$

γ has integer elements due to (2.2.11). The same equation tells us that

$$
\mathrm{gcd}(\gamma_2, \cdots \gamma_{28}) = 1\,. \qquad (2.2.24)
$$

Expressing (2.2.22) as

$$
\widehat{P} = r(Q,P)\begin{pmatrix} -Q\cdot\gamma\\ Q^2\gamma_2\\ \cdot\\ \cdot\\ Q^2\gamma_{28}\end{pmatrix}\,, \qquad (2.2.25)
$$

and using (2.2.24) we see that K defined in (2.2.19) is given by

$$
K = r(Q,P)\,\mathrm{gcd}(-Q\cdot\gamma, Q^2)\,. \qquad (2.2.26)
$$

Using (2.2.23) and that $P_1 = u_1(Q,P)$ modulo $r(Q,P)$ we may express (2.2.26) as

$$
K = r(Q,P)\,\mathrm{gcd}\,\left\{(u_1(Q,P)Q^2 - Q\cdot P)/r(Q,P), Q^2\right\}\,. \qquad (2.2.27)
$$

This establishes the first equation in (2.2.21). Note that a shift in u_1 by $r(Q,P)$ does not change the value of K. Thus K given in (2.2.27) is independent of which particular representative we use for $u_1(Q,P)$.

To derive an expression for $r(Q,\widetilde{P})$ we note from the form of Q given in (2.2.10), the form of $\widehat{P}$ given in (2.2.25), and (2.2.24) that

$$
r(Q,\widehat{P}) = \mathrm{gcd}\{Q_i\widehat{P}_j - Q_j\widehat{P}_i,\ 1\le i,j\le 28\} = r(Q,P)\,Q^2\,. \qquad (2.2.28)
$$

Since $\widetilde{P} = \widehat{P}/K$ we have

$$r(Q, \widetilde{P}) = Q^2\, r(Q, P)/K\,. \tag{2.2.29}$$

This establishes the second equation in (2.2.21). Finally to calculate $u_1(Q, \widetilde{P})$ we pick the vector α for which $\alpha \cdot Q = 1$, and express $u_1(Q, \widetilde{P})$ as

$$u_1(Q, \widetilde{P}) = \alpha \cdot \widetilde{P} = \frac{1}{K}\left(Q^2 \alpha \cdot P - Q \cdot P\, \alpha \cdot Q\right) = \frac{1}{K}\left(Q^2 u_1(Q, P) - Q \cdot P\right)\,. \tag{2.2.30}$$

This establishes the third equation in (2.2.21). Note that under a shift of $u_1(Q, P)$ by $r(Q, P)$, the expression for $u_1(Q, \widetilde{P})$ given above shifts by $r(Q, \widetilde{P})$. Thus $u_1(Q, \widetilde{P})$ given above is determined unambiguously modulo $r(Q, \widetilde{P})$.

2. Now suppose we have two pairs (Q, P) and (Q', P') with the same set of invariants:

$$Q^2 = Q'^2, \quad P^2 = P'^2, \quad Q \cdot P = Q' \cdot P', \quad r(Q, P) = r(Q', P'), \quad u_1(Q, P) = u_1(Q', P')\,. \tag{2.2.31}$$

Let us define $\widehat{P}'$, K' and $\widetilde{P}'$ as in (2.2.18), (2.2.19) with (Q, P) replaced by (Q', P') so that $Q' \cdot \widetilde{P}' = 0$. Then by eq.(2.2.21), its analog with (Q, P) replaced by (Q', P'), and eq.(2.2.31), we have

$$Q^2 = Q'^2, \quad K' = K, \quad \widetilde{P}^2 = \widetilde{P}'^2, \quad r(Q, \widetilde{P}) = r(Q', \widetilde{P}'), \quad u_1(Q, \widetilde{P}) = u_1(Q', \widetilde{P}')\,. \tag{2.2.32}$$

Thus by the result of [24], $(Q, \widetilde{P})$ and $(Q', \widetilde{P}')$ must be related to each other by a T-duality transformation Ω:

$$Q' = \Omega Q, \qquad \widetilde{P}' = \Omega \widetilde{P}\,. \tag{2.2.33}$$

It follows from this that

$$\widehat{P}' = \Omega \widehat{P}, \quad \longrightarrow \quad P' = \Omega P\,. \tag{2.2.34}$$

Thus (Q, P) and (Q', P') are related by the duality transformation Ω.

This establishes that the T-duality orbits of pairs of charge vectors (Q, P) are completely characterized by the invariants Q^2, P^2, $Q \cdot P$, $r(Q, P)$ and $u_1(Q, P)$. Two pairs of charge vectors, having the same values of all the invariants, can be related to each other by a T-duality transformation.

2.3 An Alternative Proof

In this section we shall give a different proof of the results of the previous section.

We shall begin by giving a physical interpretation of the discrete T-duality invariants $r(Q, P)$ and $u_1(Q, P)$. Let E denote the two dimensional vector space spanned by the vectors Q and P, and $\Lambda' = E \cap \Lambda$ denote the two dimensional lattice containing the points of the

Narain lattice in E. Let (e_1, e_2) denote a pair of primitive basis elements of the lattice Λ'. Since Q is a primitive vector, we can always choose $e_1 = Q$. Then we claim that in this basis

$$Q = e_1, \qquad P = u_1(Q,P)\, e_1 + r(Q,P)\, e_2 \,. \tag{2.3.1}$$

The proof goes as follows. First of all since (e_1, e_2) form a primitive basis of Λ', by a standard result [30] one can show that (e_1, e_2) can be chosen as the first two elements of a primitive basis of the full lattice Λ. In such a basis $Q_1 = 1$, $P_1 = u_1$, $P_2 = r$ and all the other componets of Q and P vanish. Thus we have gcd $\{Q_i P_j - Q_j P_i\} = r$ as required. Furthermore, it is clear from (2.3.1) that if $\alpha \cdot Q = 1$ then $\alpha \cdot P = u_1$ modulo r as required by the definition of u_1. Finally, we see that a different choice of e_2 that preserves the primitivity of the basis (e_1, e_2) is related to the original choice by $e_2 \to e_2 + s\, e_1$ for some integer s. Under such a transformation u_1 defined through (2.3.1) is shifted by a multiple of r. Thus u_1 defined through (2.3.1) is unambiguous modulo r as required. We shall choose e_2 such that u_1 appearing in (2.3.1) lies between 0 and $r - 1$.

Eq.(2.3.1) provides a physical interpretation of u_1 and r in terms of the components of Q and P along a primitive basis of the Narain lattice in the plane spanned by Q and P. As a consequence of (2.3.1) we have

$$\begin{aligned} &e_1^2 = Q^2, \qquad e_2^2 = \left\{P^2 + u_1(Q,P)^2 Q^2 - 2u_1(Q,P) Q\cdot P\right\}/r(Q,P)^2, \\ &e_1 \cdot e_2 = \left\{Q\cdot P - u_1(Q,P) Q^2\right\}/r(Q,P)\,. \end{aligned} \tag{2.3.2}$$

Now take a different pair of charges (Q', P') with the same invariants, *e.g.* satisfying (2.2.31), and define (e_1', e_2') as in (2.3.1) with (Q, P) replaced by (Q', P'). Then as a consequence of (2.2.31) and (2.3.2) we have

$$e_1^2 = (e_1')^2, \qquad e_2^2 = (e_2')^2, \qquad e_1 \cdot e_2 = e_1' \cdot e_2' \,. \tag{2.3.3}$$

Thus the lattices generated by (e_1, e_2) and (e_1', e_2') can be regarded as different primitive embeddings into Λ of an abstract even lattice of rank two with a given metric. We now use the result of [31–33] that an even lattice of signature (m, n) has a unique primitive embedding in an even self-dual lattice Λ of signature (p, q) up to a T-duality transformation if $m + n \le \min(p, q) - 1$. Setting $m + n = 2$ and $(p, q) = (6, 22)$ we see that the required condition is satisfied and hence (e_1, e_2) must be related to (e_1', e_2') by a T-duality transformation:

$$e_1' = \Omega e_1, \qquad e_2' = \Omega e_2 \,. \tag{2.3.4}$$

Eq.(2.3.1) and its analog with $(Q, P) \longrightarrow (Q', P')$, $(e_1, e_2) \longrightarrow (e_1', e_2')$ then tells us that

$$Q' = \Omega Q, \qquad P' = \Omega P \,. \tag{2.3.5}$$

This is the desired result.

One interesting question is: for a given set of values of Q^2, P^2, $Q \cdot P$ and r, what is the maximum number of possible orbits? This is given by the maximum number of allowed values of u_1. Since u_1 and r cannot share a common factor, the number is bounded from above by the number of positive integers below $(r-1)$ with no common factor with r. This in turn is given by

$$r \times \prod_{\text{primes } p,\ p|r} \left(1 - \frac{1}{p}\right) . \tag{2.3.6}$$

2.4 Predictions for gauge theory

At special points in the moduli space heterotic string theory on T^6 has enhanced gauge symmetry. As we move away from this point the gauge symmetry gets spontaneously broken, with the moduli fields describing deformations away from the enhanced symmetry point playing the role of the Higgs field. When the deformation parameter is small the scale of gauge symmetry breaking is small compared to the string scale and the theory contains massive states with mass of the order of the gauge symmetry breaking scale and small compared to the string scale. These states can be identified as the states of the spontaneously broken gauge theory. Thus if we know the spectrum of the string theory, we can determine the spectrum of spontaneously broken gauge theory. In particular the known spectrum of quarter BPS dyons in string theory should give us information about the spectrum of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

The dyon charges in a gauge theory of rank n are labelled by a pair of n-dimensional vectors (q, p) in the root lattice of the gauge algebra. If we choose a set of n simple roots as the basis of the root lattice then the components q_a and p_a will label the coefficients of the simple roots in an expansion of the charge vectors in this basis. When the root lattice is embedded in the Narain lattice the vectors (q, p) correpond to a pair of vectors (Q, P) in the Narain lattice, and the metric L on the Narain lattice, restricted to the root lattice, gives the negative of the Cartan metric. Denoting by $\circ$ the inner product with respect to the Cartan metric, we have

$$q^2 \equiv q \circ q = -Q^2, \qquad p^2 \equiv p \circ p = -P^2, \qquad q \circ p = -Q \cdot P . \tag{2.4.1}$$

Since the Cartan metric is positive definite, we must have $q^2, p^2 \geq 0$, $|q \circ p| \leq (q^2 + p^2)/2$. Furthermore quarter BPS dyons require q and p to be both non-zero and non-parallel. Hence none of the above inequalities can be saturated. This translates to the following conditions on Q, P:

$$Q^2 < 0, \qquad P^2 < 0 \qquad |Q \cdot P| < \left(|Q^2| + |P^2|\right)/2 . \tag{2.4.2}$$

Finally, since the string theory dyon spectrum is known only for charges (Q, P) with $r(Q, P) = 1$ we need to know what this condition translates to on the vectors (q, p). This is done most easily if the Narain lattice admits a primitive embedding of the root lattice, i.e. if we can choose the n simple roots of the root lattice as the first n basis elements of the full 28 dimensional Narain lattice. In that case we can easily identify (q, p) in the root lattice as a pair of charge vectors (Q, P) in the Narain lattice where the first n components of Q (P) are equal to the components of q (p) and the rest of the components of Q, P vanish. Thus we have

$$r(Q, P) = \gcd\{q_i p_j - q_j p_i\} \equiv r_{gauge}(q, p)\,. \tag{2.4.3}$$

The condition $r(Q, P) = 1$ then translates to $r_{gauge}(q, p) = 1$.

Let us now investigate under what condition the Narain lattice does not admit a primitive embedding of the root lattice. Let F be the n-dimensional vector space spanned by the root lattice, and let $\Lambda' = F \cap \Lambda$. Then by a standard result [30] one finds that the root lattice has a primitive embedding in the Narain lattice if Λ' does not contain any element other than the ones in the root lattice. So we need to examine under what condition Λ' can contain elements other than the ones in the root lattice. Now clearly the elements of Λ' must belong to the weight lattice of the algebra. Furthermore, since Narain lattice is even, any element of Λ' will be even. Thus we can classify all possible extra elements of Λ' by examining the Phys. Rev. D **61**, 045003 (2000) [arXiv:hep-th/9907090]. possible even elements of the weight lattice outside the root lattice. For many algebras we have no such element, and hence in those cases the embedding of the root lattice in the Narain lattice is necessarily primitive. Exceptions among the rank ≤ 22 algebras are $so(16)$, $so(32)$, $su(8)$, $su(9)$, $su(16)$ and $su(18)$; for each of these the weight lattice has even elements other than those in the root lattice [34].[4] Hence in these cases Λ' could contain elements other than the ones in the root lattice, preventing the root lattice from having a primitive embedding in the Narain lattice. But since Λ' would have a primitive embedding in the Narain lattice, if we choose a basis for Λ', and define q_i, p_i as the componets of q and p expanded in this basis, then (2.4.3) continues to reproduce the value of $r(Q, P)$.

With this understanding we can now study the implications of the known dyon spectrum in $\mathcal{N} = 4$ supersymmetric string theory. As is well known, for dyons with $r(Q, P) = 1$ the dyon spectrum in different parts of the moduli space can be different. The situation is best described in the axion-dilaton moduli space at fixed values of the other moduli [19]. In particular in the upper half plane labelled by the axion-dilaton field[5] $\tau = a + iS$ the spectrum jumps across walls of marginal stability, which are circles or straight lines passing through rational points on the real axis [19, 22, 38]. These curves do not intersect in the interior of the upper half plane and divide up the upper half plane into different domains, each with three vertices lying either at rational points on the real axis or at ∞. Inside a given domain the index $d(Q, P)$ that counts the number of bosonic supermultiplets minus the number of fermionic supermultiplets remains constant, but as we move from one domain to another the index changes. We shall first consider the domain bounded by a straight line passing through 0, a straight line passing

[4] Both for $so(16)$ and $su(9)$, inclusion of the extra even elements of the weight lattice makes the lattice $F \cap \Lambda$ into the root lattice of e_8. Thus for such embeddings we are actually counting the dyon spectrum of an E_8 gauge theory rather than $SO(16)$ or $SU(9)$ gauge theory. On the other hand for $su(8)$ the extra even elements of the weight lattice makes $F \cap \Lambda$ into the root lattice of e_7. Thus in this case we get an E_7 gauge theory.

[5] From the point of view of the gauge theory the axion-dilaton moduli correspond to the theta parameter and the inverse square of the coupling constant.

through 1 and a circle passing through 0 and 1, – the domain called $\mathcal{R}$ in [19, 23]. This has vertices at 0, 1 and ∞. In this domain the only non-zero values of $d(Q,P)$ for $Q^2 < 0$, $P^2 < 0$ are obtained at $Q^2 = P^2 = -2$. For $Q^2 = P^2 = -2$ the result for $d(Q,P)$ is [15, 19, 59]

$$d(Q,P) = \begin{cases} 0 & \text{for } Q\cdot P \geq 0 \\ j(-1)^{j-1} & \text{for } Q\cdot P = -j,\ j>0 \end{cases} . \tag{2.4.4}$$

The condition (2.4.2) on $Q \cdot P$ now shows that for (Q,P) describing the elements of the root lattice, non-vanishing index exists only for $Q^2 = P^2 = -2$, $Q\cdot P = -1$. Translated to a condition on the charge vectors in the gauge theory this gives[6]

$$d_{gauge}(q,p) = \begin{cases} 1 & \text{for } q^2 = p^2 = 2,\ q\circ p = 1,\ r_{gauge}(q,p) = 1 \\ 0 & \text{for other } (q,p) \text{ with } r_{gauge}(q,p) = 1 \end{cases} . \tag{2.4.5}$$

This condition in turn implies that q and $-p$ can be regarded as the simple roots of an $su(3)$ subalgebra of the full gauge algebra, with the Cartan metric of su(3) being equal to the restriction of the Cartan metric of the full algebra. Thus we learn that in the domain $\mathcal{R}$ the only dyons with $r_{gauge}(q,p) = 1$ and non-vanishing index are the ones which can be regarded as $SU(3)$ dyons for some level one $su(3)$ subalgebra of the gauge algebra, with q and $-p$ identified with the simple roots α and β of the $su(3)$ algebra.

The index in other domains can be found using the S-duality invariance of the theory. An S-duality transformation of the form $\tau \to (a\tau+b)/(c\tau+d)$ maps the domain $\mathcal{R}$ to another domain with vertices

$$\frac{a}{c}, \quad \frac{b}{d}, \quad \frac{a+b}{c+d} . \tag{2.4.6}$$

Under the same S-duality transformation the charge vector $(q,p) = (\alpha, -\beta)$ gets mapped to

$$(q,p) = (a\alpha - b\beta, c\alpha - d\beta) . \tag{2.4.7}$$

It can be easily seen that $r(Q,P)$ remains invariant under an $SL(2, \mathbb{Z})$ transformation:

$$r(Q,P) = r(aQ + bP, cQ + dP), \quad a,b,c,d \in \mathbb{Z}, \quad ad - bc = 1 . \tag{2.4.8}$$

Thus we conclude that in the domain (2.4.6), the index of gauge theory dyons with $r_{gauge}(q,p) = 1$ is given by

$$d_{gauge}(q,p) = \begin{cases} 1 & \text{for } (q,p) = (a\alpha - b\beta, c\alpha - d\beta), \\ 0 & \text{otherwise} \end{cases} , \tag{2.4.9}$$

with (α, β) labelling the simple roots of some level one su(3) subalgebra of the full gauge algebra.

[6] It is easy to show that for states with $Q^2 = P^2 = -2$, $Q\cdot P = \pm 1$ the condition $r(Q,P) = 1$ is satisfied automatically. Thus we do not need to state this as a separate condition.

This general result agrees with the known results for quarter BPS dyons in gauge theories [27, 28, 49, 50, 69].[7] In particular in the representation of $SU(N)$ dyons as string network with ends on a set of parallel D3-branes, this is a reflection of the fact that networks with three external strings ending on three D3-barnes are the only quarter BPS configurations at a generic point in the moduli space [28].

[7]In codimension ≥ 1 subspaces of the moduli space the dyon spectrum, computed in some approximation, has a rich structure [28, 49, 50, 69]. However the index associated with these dyons vanish and these results are not in contradiction with the spectrum of string theory.

Chapter 3

S-duality Action on Discrete T-duality Invariants

In the last chapter we found out the complete set of T-duality invariants which uniquely characterse a pair of charge vectors (Q, P) modulo T duality transformations. These include the invariants of the continuous T-duality group $O(6, 22; \mathbb{R})$

$$Q^2, \quad P^2, \quad Q \cdot P\,, \tag{3.0.1}$$

together with a set of invariants of the discrete T-duality group $O(6, 22; \mathbb{Z})$. These were defined as follows. We shall assume that the dyon is primitive so that (Q, P) cannot be written as an integer multiple of (Q_0, P_0) with $Q_0, P_0 \in \Lambda$, but we shall not assume that Q and P themselves are primitive. Now consider the intersection of the two dimensional vector space spanned by (Q, P) with the Narain lattice Λ. The result is a two dimensional lattice Λ_0. Let (e_1, e_2) be a pair of basis elements whose integer linear combinations generate this lattice. We can always choose (e_1, e_2) such that in this basis

$$\begin{aligned} Q = r_1 e_1, \qquad P = r_2(u_1 e_1 + r_3 e_2), \qquad r_1, r_2, r_3, u_1 \in \mathbb{Z}^+, \\ \gcd(r_1, r_2) = 1, \quad \gcd(u_1, r_3) = 1, \quad 1 \le u_1 \le r_3\,. \end{aligned} \tag{3.0.2}$$

We saw that besides Q^2, P^2 and $Q \cdot P$, the integers r_1, r_2, r_3 and u_1 are T-duality invariants. Furthermore it was found that this is the complete set of T-duality invariants. Thus a pair of charge vectors (Q, P) can be transformed into another pair (Q', P') via a T-duality transformation if and only if all the invariants agree for these two pairs.

In this chapter our first goal is to study some aspects of the action of the S-duality transformation

$$Q \to Q' = aQ + bP, \quad P \to P' = cQ + dP, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1\,, \tag{3.0.3}$$

on the invariants r_1, r_2, r_3 and u_1. Substituting (3.0.2) into (3.0.3), and expressing the resulting (Q', P') as $(r_1' e_1', r_2'(u_1' e_1' + r_3' e_2'))$ for some primitive basis (e_1', e_2') of Λ_0 we can determine

(r'_1, r'_2, r'_3, u'_1). Since the resuting expressions are somewhat complicated and not very illuminating we shall not describe them here. Instead we shall focus on some salient features of the transformation laws of (r_1, r_2, r_3, u_1). We first note that the torsion $r(Q, P)$ associated with a pair of charges (Q, P), defined as [20, 24]

$$r(Q, P) = Q_1 P_2 - Q_2 P_1 \,, \tag{3.0.4}$$

with Q_i, P_i being the components of Q and P along e_i, is invariant under the S-duality transformation (3.0.3). Furthermore, for the charge vectors (Q, P) given in (3.0.2) we have

$$r(Q, P) = r_1 r_2 r_3 \,. \tag{3.0.5}$$

We shall now show that one can always find an S-duality transformation that brings the T-duality invariants (r_1, r_2, r_3, u_1) to $(r_1 r_2 r_3, 1, 1, 1)$ together with an appropriate transformation on Q^2, P^2 and $Q{\cdot}P$ induced by (3.0.3). For this we note that under the S-duality transformation (3.0.3), (Q, P) given in (3.0.2) transforms to

$$Q' = \{ar_1 + br_2(u_1 + kr_3)\}e_1 + br_2 r_3(e_2 - ke_1), \quad P' = \{cr_1 + dr_2(u_1 + kr_3)\}e_1 + dr_2 r_3(e_2 - ke_1)\,, \tag{3.0.6}$$

where k is an arbitrary integer. We shall choose

$$k = \prod_i p_i \,, \tag{3.0.7}$$

where $\{p_i\}$ represent the collection of primes which are factors of r_1 but not of u_1. Now we know from (3.0.2) that $\gcd(r_1, r_2) = 1$. On the other hand it follows from a result derived in appendix E of [29] that for the choice of k given in (3.0.7) we have $\gcd(r_1, u_1 + kr_3) = 1$. Thus if we choose

$$b = r_1, \quad a = -r_2(u_1 + kr_3)\,, \tag{3.0.8}$$

we have $\gcd(a, b) = 1$ and hence we can always find c, d satisfying $ad - bc = 1$. For this particular choice of $SL(2, \mathbb{Z})$ transformation we have

$$Q' = r_1 r_2 r_3(e_2 - ke_1), \qquad P' = -e_1 + dr_2 r_3(e_2 - ke_1)\,. \tag{3.0.9}$$

We now define

$$e'_1 = (e_2 - ke_1), \qquad e'_2 = -e_1 + (dr_2 r_3 - 1)(e_2 - ke_1)\,. \tag{3.0.10}$$

Since the matrix relating (e'_1, e'_2) to (e_1, e_2) has unit determinant, (e'_1, e'_2) is a primitive basis of the lattice Λ_0. In this basis (Q', P') can be expressed as

$$Q' = r_1 r_2 r_3 e'_1, \qquad P' = e'_1 + e'_2\,. \tag{3.0.11}$$

Comparing this with (3.0.2) we see that for the new charge vector (Q', P') we have

$$r'_1 = r_1 r_2 r_3, \quad r'_2 = 1, \quad r'_3 = 1, \quad u'_1 = 1\,. \tag{3.0.12}$$

This proves the desired result.

Next we shall study the subgroup of S-duality transformations which takes a configuration with $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ to another configuration with $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$. The initial configuration has

$$Q = re_1, \qquad P = e_1 + e_2 \,. \tag{3.0.13}$$

An S-duality transformation (3.0.3) takes this to

$$Q' = are_1 + b(e_1 + e_2)\,, \qquad P' = cre_1 + d(e_1 + e_2)\,. \tag{3.0.14}$$

In order that Q' is r times a primitive vector, we must demand

$$b = 0 \text{ mod } r \,. \tag{3.0.15}$$

Expressing b as $b_0 r$ with $b_0 \in \mathbb{Z}$ we get

$$Q' = re'_1, \qquad P' = e'_1 + e'_2\,, \tag{3.0.16}$$

where

$$e'_1 = (a + b_0)e_1 + b_0 e_2, \qquad e'_2 = (cr + d - a - b_0)e_1 + (d - b_0)e_2 \,. \tag{3.0.17}$$

Since the determinant of the matrix relating (e'_1, e'_2) to (e_1, e_2) is given by

$$(a + b_0)(d - b_0) - b_0(cr + d - a - b_0) = ad - bc = 1\,, \tag{3.0.18}$$

we conclude that (e'_1, e'_2) is a primitive basis of Λ_0. Comparison with (3.0.2) now shows that (Q', P') has $r'_1 = r$, $r'_2 = r'_3 = u'_1 = 1$ as required. Thus the only condition on the $SL(2, \mathbb{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for preserving the $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ condition is that it must have $b = 0$ mod r, i.e. it must be an element of $\Gamma^0(r)$.

Using this we can now determine the subgroup of $SL(2, \mathbb{Z})$ that takes a pair of charge vectors (Q, P) with invariants (r_1, r_2, r_3, u_1) to another pair of charge vectors with the same invariants. For this we note that any $SL(2, \mathbb{Z})$ transformation matrix $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b given in (3.0.8) takes the set (r_1, r_2, r_3, u_1) to the set $(r_1 r_2 r_3, 1, 1, 1)$. Since the latter set is preserved by the $\Gamma^0(r)$ subgroup of $SL(2, \mathbb{Z})$, the original set must be preserved by the subgroup $g_0^{-1}\Gamma^0(r)g_0$. This is isomorphic to the group $\Gamma^0(r)$.

To see an example of this consider the case

$$r_1 = r_2 = 1, \quad r_3 = 2, \quad u_1 = 1\,. \tag{3.0.19}$$

In this case the $SL(2, \mathbb{Z})$ transformation $g_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ takes a configuration given in (3.0.19) to a configuration with $r_1 = 2$, $r_2 = r_3 = u_1 = 1$. Thus the $SL(2, \mathbb{Z})$ transformations which take a configuration with $(r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1)$ to a configuration with the same discrete invariants will be of the form:

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 2b_0 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-c & a-c-d+2b_0 \\ c & c+d \end{pmatrix} . \tag{3.0.20}$$

Since the condition $ad - 2b_0c = 1$ requires a and d to be odd, we have

$$a' + b' \in 2\,\mathbb{Z} + 1, \qquad c' + d' \in 2\,\mathbb{Z} + 1 . \tag{3.0.21}$$

Conversely given any $SL(2, \mathbb{Z})$ matrix $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ satisfying (3.0.21), it can be written as g_0 conjugate of the $\Gamma^0(2)$ matrix $\begin{pmatrix} a'+c' & -a'-c'+b'+d' \\ c' & -c'+d' \end{pmatrix}$. Thus (3.0.21) characterizes the subgroup of S-duality group which preserves the condition (3.0.19).

The results derived so far make it clear that for a given torsion r the discrete T-duality invariants are in one to one correspondence with the elements of the coset $SL(2, \mathbb{Z})/\Gamma^0(r)$. The representative element for a given set of invariants (r_1, r_2, r_3, u_1) is the element $g_0^{-1} \in SL(2, \mathbb{Z})$ that takes a configuration with $(r_1r_2r_3, 1, 1, 1)$ to a configuration with discrete invariants (r_1, r_2, r_3, u_1). Multiplying g_0^{-1} by a $\Gamma^0(r)$ element from the right does not change the final values (r_1, r_2, r_3, u_1) of the discrete invariants since a $\Gamma^0(r)$ transformation does not change the discrete T-duality invariants of the initial configuration.

We shall now examine the consequences of these results for the formula expressing the degeneracy $d(Q, P)$ – or more precisely an appropriate index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets for a given set of charges[1] – of quarter BPS dyons as a function of (Q, P). We note first of all that besides depending on (Q, P), the degeneracy can also depend on the asymptotic values of the moduli fields, collectively denoted as ϕ. We expect the dependence on ϕ to be mild, in the sense that the degeneracy formula should be ϕ independent within a given domain bounded by walls of marginal stability. It follows from the analysis of [36, 37] that the decays relevant for the walls of marginal stability are of the form

$$(Q, P) \to (\alpha Q + \beta P, \gamma Q + \delta P) + ((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P) , \tag{3.0.22}$$

where α, β, γ, δ are not necessarily integers, but must be such that $\alpha Q + \beta P$ and $\gamma Q + \delta P$ belong to the Narain lattice Λ. If we denote by $m(Q, P; \phi)$ the BPS mass of a dyon of charge (Q, P) then the wall of marginal stability associated with the set $(\alpha, \beta, \gamma, \delta)$ is given by the solution to the equation

$$m(Q, P; \phi) = m(\alpha Q + \beta P, \gamma Q + \delta P; \phi) + m((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P; \phi) . \tag{3.0.23}$$

[1]Up to a normalization this is equal to the helicity trace $B_6 = Tr(-1)^{2h} h^6$ over all states carrying charge quantum numbers (Q, P). Here h denotes the helicity of the state.

For appropriate choice of $(\alpha, \beta, \gamma, \delta)$ this describes a codimension one subspace of the moduli space labelled by ϕ. Since the BPS mass formula is invariant under a T-duality transformation $Q \to \Omega Q,\ P \to \Omega P,\ \phi \to \phi_\Omega$:

$$m(\Omega Q, \Omega P; \phi_\Omega) = m(Q, P; \phi) \qquad \Omega \in O(6, 22;\ \mathbb{Z}), \tag{3.0.24}$$

eq.(3.0.23) may be written as

$$m(\Omega Q, \Omega P; \phi_\Omega) = m(\alpha\Omega Q+\beta\Omega P, \gamma\Omega Q+\delta\Omega P; \phi_\Omega)+m((1-\alpha)\Omega Q-\beta\Omega P, -\gamma\Omega Q+(1-\delta)\Omega P; \phi_\Omega)\,. \tag{3.0.25}$$

This is identical to eq.(3.0.23) with (Q, P, ϕ) replaced by $(\Omega Q, \Omega P, \phi_\Omega)$. This shows that under a T-duality transformation on charges and moduli, the wall of marginal stability associated with the set $(\alpha, \beta, \gamma, \delta)$ gets mapped to the wall of marginal stability associated with the same $(\alpha, \beta, \gamma, \delta)$. Thus if we consider a domain bounded by the walls of marginal stability associated with the sets $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ for $1 \le i \le n$ – collectively denoted by a set of discrete variables $\vec{c}$ – then under a simultaneous T-duality transformation on the charges and the moduli this domain gets mapped to a domain labelled by the same vector $\vec{c}$. The precise shape of the domain of course changes since the locations of the walls in the moduli space depends not only on $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ for $1 \le i \le n$ but also on the charges (Q, P) which transform to $(\Omega Q, \Omega P)$.

We now use the fact that the dyon degeneracy formula must be invariant under a simultaneous T-duality transformation on the charges and the moduli, and also the fact that the dependence of $d(Q, P; \phi)$ on the moduli ϕ comes only through the domain in which ϕ lies, i.e. the vector $\vec{c}$. Since $\vec{c}$ remains unchanged under a T-duality transformation, we have

$$d(Q, P; \vec{c}) = d(\Omega Q, \Omega P; \vec{c})\,, \qquad \Omega \in O(6, 22;\ \mathbb{Z})\,. \tag{3.0.26}$$

This shows that $d(Q, P; \vec{c})$ must depend only on (Q, P) via the T-duality invariants:

$$d(Q, P; \vec{c}) = f(Q^2, P^2, Q \cdot P, r_1, r_2, r_3, u_1; \vec{c})\,, \tag{3.0.27}$$

for some function f.

Let us now study the effect of S-duality transformation on this formula. Typically an S-duality transformation will act on the charges and hence on all the T-duality invariants and also on the vector $\vec{c}$ labelling the domain bounded by the walls of marginal stability [19,20,22]. Indeed, as is clear from the condition (3.0.23), under an S-duality transformation of the form (3.0.3), the wall associated with the parameters $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ gets mapped to the wall associated with

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\,. \tag{3.0.28}$$

Thus S-duality invariance of the degeneracy formula now gives

$$f(Q^2, P^2, Q \cdot P, r_1, r_2, r_3, u_1; \vec{c}) = f(Q'^2, P'^2, Q' \cdot P', r_1', r_2', r_3', u_1'; \vec{c}\,')\,, \tag{3.0.29}$$

where $\vec{c}\,'$ stands for the collection of the sets $\{\alpha_i', \beta_i', \gamma_i', \delta_i'\}$ computed according to (3.0.28). We now use the result that there exists a special class of S-duality transformations under which

$$(r_1', r_2', r_3', u_1') = (r_1 r_2 r_3, 1, 1, 1)\,. \tag{3.0.30}$$

Using this S-duality transformation we get

$$f(Q^2, P^2, Q\cdot P, r_1, r_2, r_3, u_1; \vec{c}) = f(Q'^2, P'^2, Q'\cdot P', r_1 r_2 r_3, 1, 1, 1; \vec{c}\,')\,. \tag{3.0.31}$$

Thus the complete information about the spectrum of quarter BPS dyons is contained in the set of functions

$$g(Q^2, P^2, Q\cdot P, r; \vec{c}) \equiv f(Q^2, P^2, Q\cdot P, r, 1, 1, 1; \vec{c})\,. \tag{3.0.32}$$

We shall focus our attention on this function during the rest of our analysis. Using the fact that $\Gamma^0(r)$ transformations leave the set $(r_1 = r, r_2 = 1, r_3 = 1, u_1 = 1)$ fixed, we see that

$$g(Q^2, P^2, Q\cdot P, r; \vec{c}) = g(Q'^2, P'^2, Q'\cdot P', r; \vec{c}\,') \quad \text{for} \begin{pmatrix} Q' \\ P' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(r)\,. \tag{3.0.33}$$

In other words, the function $g(Q^2, P^2, Q\cdot P, r; \vec{c})$ is expected to have manifest invariance under the $\Gamma^0(r)$ subgroup of S-duality transformations. So far our discussion has been independent of any specific formula for the function $g(Q^2, P^2, Q\cdot P, r; \vec{c})$. For $r = 1$ dyons an explicit formula for the function g has been found in a wide class of $\mathcal{N} = 4$ supersymmetric theories [7–23, 38]. In all the known examples the function g is obtained as a contour integral of the inverse of an appropriate modular form of a subgroup of $Sp(2, \mathbb{Z})$. In particular for heterotic string theory on T^6 the modular form is the well known Igusa cusp form of weight 10 of the full $Sp(2, \mathbb{Z})$ group, with the S-duality group $SL(2, \mathbb{Z})$ embedded in $Sp(2, \mathbb{Z})$ in a specific manner. Furthermore the dependence on the domain labelled by $\vec{c}$ is encoded fully in the choice of the integration contour and not in the integrand. If a similar formula exists for $g(Q^2, P^2, Q\cdot P, r; \vec{c})$ for $r > 1$, then our analysis would suggest that the integrand should involve a modular form of a subgroup of $Sp(2, \mathbb{Z})$ that contains $\Gamma^0(r)$ in the same way that the full $Sp(2, \mathbb{Z})$ contains $SL(2, \mathbb{Z})$. It remains to be seen if this constraint together with other physical constraints reviewed in [23] can fix the form of the integrand.

Chapter 4

Generalities of Quarter BPS Dyon Partition Function

Our goal in this chapter is to draw insight from the known results to postulate the general structure of dyon partition function for any class of quarter BPS dyons in any $\mathcal{N} = 4$ supersymmetric string theory.

The results of our analysis can be summarized as follows.

1. **Definition of the partition function:** Let (Q, P) denote the electric and magnetic charges carried by a dyon, and Q^2, P^2 and $Q \cdot P$ be the T-duality invariant quadratic forms constructed from these charges.[1] In order to define the dyon partition function we first need to identify a suitable infinite subset $\mathcal{B}$ of dyons in the theory with the property that if we have two pairs of charges $(Q, P) \in \mathcal{B}$ and $(Q', P') \in \mathcal{B}$ with $Q^2 = Q'^2$, $P^2 = P'^2$ and $Q \cdot P = Q' \cdot P'$, then they must be related by a T-duality transformation. Furthermore given a pair of charge vectors $(Q, P) \in \mathcal{B}$, all other pairs of charge vectors related to it by T-duality should be elements of the set $\mathcal{B}$. We shall generate such a set $\mathcal{B}$ by beginning with a family $\mathcal{A}$ of charge vectors (Q, P) labelled by three integers such that Q^2, P^2 and $Q \cdot P$ are independent linear functions of these three integers, and then define $\mathcal{B}$ to be the set of all (Q, P) which are in the T-duality orbit of the set $\mathcal{A}$. We denote by $d(Q, P)$ the degeneracy, – or more precisely an index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets – of quarter BPS dyons of charge (Q, P). Since $d(Q, P)$ should be invariant under a T-duality transformation, for $(Q, P) \in \mathcal{B}$ it should depend on (Q, P) only via the T-duality invariant combinations

[1]Irrespective of what description we are using, we shall denote by S-duality transformation the symmetry that acts on the complex scalar belonging to the gravity multiplet. In heterotic string compactification this would correspond to the axion-dilaton modulus. On the other hand T-duality will denote the symmetry that acts on the matter multiplet scalars. In the heterotic description these scalars arise from the components of the metric, anti-symmetric tensor fields and gauge fields along the compact directions.

Q^2, P^2 and $Q \cdot P$:

$$d(Q,P) = f(Q^2, P^2, Q\cdot P)\,. \tag{4.0.1}$$

Note that for (4.0.1) to hold it is necessary to choose $\mathcal{B}$ in the way we have described. In particular if $\mathcal{B}$ had contained two elements with same Q^2, P^2 and $Q \cdot P$ but not related by a T-duality transformation, then $d(Q,P)$ can be different for these two elements and (4.0.1) will not hold. Even when $\mathcal{B}$ is chosen according to the prescription given above, eq.(4.0.1) cannot be strictly correct since $d(Q,P)$ could depend on the asymptotic moduli besides the charges and one can construct more general T-duality invariants using these moduli and the charges. Indeed, even though the index is not expected to change under a continuous change in the moduli, it could jump across the walls of marginal stability giving $d(Q,P)$ a dependence on the moduli. The reason that we can still write eq.(4.0.1) is that it is possible to label the different domains bounded by the walls of marginal stability by a set of discrete parameters $\vec{c}$ such that T-duality transformation does not change the parameters $\vec{c}$ [19, 23, 40]. Physically, if a domain is bounded by n walls of marginal stability, with the ith wall associated with the decay $(Q,P) \to (\alpha_i Q + \beta_i P, \gamma_i Q + \delta_i P) + ((1-\alpha_i)Q - \beta_i P, -\gamma_i Q + (1-\delta_i)P)$, then $\vec{c}$ is the collection of the numbers $\{(\alpha_i, \beta_i, \gamma_i, \delta_i);\ 1 \le i \le n\}$. Due to T-duality invariance of $\vec{c}$, $d(Q,P)$ inside a given domain labelled by $\vec{c}$ will be invariant under a T-duality transformation on the charges only and will have the form (4.0.1). For different $\vec{c}$ the function f will be different, i.e. f has a hidden $\vec{c}$ dependence. We now define the dyon partition function associated with the set $\mathcal{B}$ to be[2]

$$\frac{1}{\widehat{\Phi}(\check\rho, \check\sigma, \check v)} \equiv \sum_{Q^2, P^2, Q\cdot P} (-1)^{Q\cdot P+1} f(Q^2, P^2, Q\cdot P) e^{i\pi(\check\sigma Q^2 + \check\rho P^2 + 2\check v Q\cdot P)}\,, \tag{4.0.2}$$

where the sum runs over all the distinct triplets $(Q^2, P^2, Q\cdot P)$ which are present in the set $\mathcal{B}$. This relation can be inverted as

$$f(Q^2, P^2, Q\cdot P) \propto (-1)^{Q\cdot P+1} \int_{\mathcal{C}} d\check\rho d\check\sigma d\check v\, e^{-i\pi(\check\sigma Q^2 + \check\rho P^2 + 2\check v Q\cdot P)}\, \frac{1}{\widehat{\Phi}(\check\rho, \check\sigma, \check v)}\,, \tag{4.0.3}$$

where $\mathcal{C}$ denotes an appropriate three dimensional subspace of the complex $(\check\rho, \check\sigma, \check v)$ space. Along the 'contour' $\mathcal{C}$ the imaginary parts of $\check\rho$, $\check\sigma$ and $\check v$ are fixed at values where the sum in (4.0.2) converges, and the real parts of $\check\rho$, $\check\sigma$ and $\check v$ vary over an appropriate unit cell determined by the quantization laws of Q^2, P^2 and $Q\cdot P$ inside the set $\mathcal{B}$.

In all known cases the function f in different domains $\vec{c}$ is given by (4.0.3) with identical integrand, but the integration contour $\mathcal{C}$ depends on the choice of $\vec{c}$. Put another way, the same function $1/\widehat{\Phi}(\check\rho, \check\sigma, \check v)$ admits different Fourier expansion in different regions in the complex $(\check\rho, \check\sigma, \check v)$ space, since a Fourier expansion that is convergent in one region

[2]For $\mathcal{N}=4$ supersymmetric $\mathbb{Z}_N$ orbifolds reviewed in [23] the function $\widehat{\Phi}$ is related to the function $\widetilde{\Phi}$ of [23] by the relation $\widehat{\Phi}(\check\rho, \check\sigma, \check v) = \widetilde{\Phi}(\widetilde\rho, \widetilde\sigma, \widetilde v)$ with $(\widetilde\rho, \widetilde\sigma, \widetilde v) = (\check\sigma/N, N\check\rho, \check v)$.

may not be convergent in another region. The coefficients of expansion in these different regions in the complex $(\check\rho, \check\sigma, \check v)$ plane may then be regarded as the index $f(Q^2, P^2, Q\cdot P)$ in different domains in the asymptotic moduli space labelled by $\vec c$. We shall assume that *this result holds for all sets of dyons in all $\mathcal{N}=4$ string theories.*

2. **Consequences of S-duality symmetry:** We now consider the effect of an S-duality transformation on the set $\mathcal{B}$. A generic S-duality transformation will take an element of $\mathcal{B}$ to outside $\mathcal{B}$, – we denote by H the subgroup of the S-duality group that leaves $\mathcal{B}$ invariant. This is the subgroup relevant for constraining the dyon partition function associated with the set $\mathcal{B}$. Since a generic element of H takes us from one domain bounded by walls of marginal stability to another such domain, it relates the function f for one choice of $\vec c$ to the function f for another choice of $\vec c$. However since we have assumed that the dyon partition function $1/\widehat\Phi$ is independent of the domain label $\vec c$, we can use invariance under H to constrain the form of $\widehat\Phi$. In particular one finds that an S-duality symmetry of the form $(Q,P)\to(aQ+bP, cQ+dP)$ with $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in H$ gives the following constraint on $\widehat\Phi$:

$$\widehat\Phi(\check\rho,\check\sigma,\check v) = \widehat\Phi(d^2\check\rho + b^2\check\sigma + 2bd\check v, c^2\check\rho + a^2\check\sigma + 2ac\check v, cd\check\rho + ab\check\sigma + (ad+bc)\check v)\,. \tag{4.0.4}$$

Defining

$$\check\Omega = \begin{pmatrix} \check\rho & \check v\\ \check v & \check\sigma\end{pmatrix}, \tag{4.0.5}$$

we can express (4.0.4) as

$$\widehat\Phi((A\check\Omega + B)(C\check\Omega + D)^{-1}) = (\det(C\check\Omega + D))^k\,\widehat\Phi(\Omega)\,, \tag{4.0.6}$$

where

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \begin{pmatrix} d & b & 0 & 0\\ c & a & 0 & 0\\ 0 & 0 & a & -c\\ 0 & 0 & -b & d\end{pmatrix}, \tag{4.0.7}$$

and k is as yet undermined since $\det(C\Omega + D) = 1$.

Besides this symmetry, quantization of Q^2, P^2 and $Q\cdot P$ within the set $\mathcal{B}$ also gives rise to some translational symmetries of $\widehat\Phi$ of the form $\widehat\Phi(\check\rho,\check\sigma,\check v) = \widehat\Phi(\check\rho + a_1, \check\sigma + a_2, \check v + a_3)$ with a_1, a_2, a_3 taking values in an appropriate set. These can also be expressed as (4.0.6) with

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \begin{pmatrix} 1 & 0 & a_1 & a_3\\ 0 & 1 & a_3 & a_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{4.0.8}$$

3. **Wall crossing formula:** Given that the indices in different domains in the moduli space are given by different choices of the 3-dimensional integration contour in the $(\check{\rho}, \check{\sigma}, \check{v})$ space, the jump in the index as we cross a wall of marginal stability must be given by the residue of the integrand at the pole(s) encountered while deforming one contour to another. The walls across which the index jumps are the ones associated with decays into a pair of half-BPS states.[3] We can label the decay products as [19]

$$(Q, P) \rightarrow (a_0 d_0 Q - a_0 b_0 P, c_0 d_0 Q - c_0 b_0 P) + (-b_0 c_0 Q + a_0 b_0 P, -c_0 d_0 Q + a_0 d_0 P)\,, \quad (4.0.9)$$

where a_0, b_0, c_0 and d_0 are normalized so that $a_0 d_0 - b_0 c_0 = 1$. In a generic situation a_0, b_0, c_0 and d_0 are not necessarily integers but are constrained by the fact that the final charges satisfy the charge quantization laws. In all known examples there is a specific correlation between a wall corresponding to a given decay and the location of the pole of the integrand that the contour crosses as we cross the wall in the moduli space. The location of the pole associated with the decay (4.0.9) is given by:

$$\check{\rho} c_0 d_0 + \check{\sigma} a_0 b_0 + \check{v}(a_0 d_0 + b_0 c_0) = 0\,. \quad (4.0.10)$$

We shall assume that *this formula continues to hold in all cases.* This then relates the jump in the index across a given wall of marginal stability to the residue of the partition function at a specific pole. An explicit choice of moduli dependent contour that satisfies this requirement can be found by generalizing the result of Cheng and Verlinde [22] to generic quarter BPS dyons in generic $\mathcal{N} = 4$ supersymmetric string theories:

$$\begin{aligned}
\Im(\check{\rho}) &= \Lambda\left(\frac{|\tau|^2}{\tau_2} + \frac{Q_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}}\right), \\
\Im(\check{\sigma}) &= \Lambda\left(\frac{1}{\tau_2} + \frac{P_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}}\right), \\
\Im(\check{v}) &= -\Lambda\left(\frac{\tau_1}{\tau_2} + \frac{Q_R \cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R \cdot P_R)^2}}\right), \quad (4.0.11)
\end{aligned}$$

[3] For a certain class of dyons kinematics allows decay into a pair of quarter BPS states or a half BPS and a quarter BPS states on a codimension 1 subspace of the moduli space. These correspond to decays of the form $(Q, P) \rightarrow (\alpha Q + \beta P, \gamma Q + \delta P) + ((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P)$ with some of the $\alpha, \beta, \gamma, \delta$ fractional so that we can have $0 < (\alpha\delta - \beta\gamma) < 1$ and $0 \leq ((1-\alpha)(1-\delta) - \beta\gamma < 1$ [36,37]. However a naive counting of the number of fermion zero modes on a half BPS - quarter BPS and quarter BPS - quarter BPS combination suggests that there are additional fermion zero modes besides the ones associated with the broken supersymmetry generators. This makes the index associated with such a configuration vanish. Although a rigorous analysis of this system is lacking at present, we shall proceed with the assumption that the result is valid so that such decays do not change the index. Otherwise the dyon partition function will have additional poles associated with the jump in the index across these additional walls of marginal stability. We wish to thank F. Denef for a discussion on this point.

where Λ is a large positive number, $\Im(z)$ denotes the imaginary part of z,

$$Q_R^2 = Q^T(M+L)Q, \quad P_R^2 = P^T(M+L)P, \quad Q_R\cdot P_R = Q^T(M+L)P\,, \tag{4.0.12}$$

$\tau \equiv \tau_1 + i\tau_2$ denotes the asymptotic value of the axion-dilaton moduli which belong to the gravity multiplet and M is the asymptotic value of the symmetric matrix valued moduli field of the matter multiplet satisfying $MLM^T = L$. The choice (4.0.11) of course is not unique since we can deform the contour without changing the result for the index as long as we do not cross a pole of the partition function.

Independent of the above analysis, the change in the index across a wall of marginal stability can be computed using the wall crossing formula [19, 41–46]. This tells us that as we cross a wall of marginal stability associated with the decay $(Q,P) \to (Q_1,P_1) + (Q_2,P_2)$, the index jumps by an amount[4]

$$(-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}\,(Q_1\cdot P_2 - Q_2\cdot P_1)\, d_h(Q_1,P_1) d_h(Q_2,P_2) \tag{4.0.13}$$

up to a sign, where $d_h(Q,P)$ denotes the index of half-BPS states carrying charge (Q,P). For the decay described in (4.0.9) the relevant half-BPS indices are of the form $d_h(a_0M_0, c_0M_0)$ and $d_h(b_0N_0, d_0N_0)$ where $M_0 \equiv d_0Q - b_0P$ and $N_0 \equiv -c_0Q + a_0P$. T-duality invariance implies that – modulo some subtleties discussed below eqs.(4.3.11) – the dependence of $d_h(a_0M_0, c_0M_0)$ and $d_h(b_0N_0, d_0N_0)$ on M_0 and N_0 must come via the combinations M_0^2 and N_0^2 respectively. We now define

$$\phi_e(\tau; a_0, c_0) \equiv \sum_{M_0^2} e^{\pi i\tau M_0^2} d_h(a_0M_0, c_0M_0), \quad \phi_m(\tau; b_0, d_0) \equiv \sum_{N_0^2} e^{\pi i\tau N_0^2} d_h(b_0N_0, d_0N_0)\,, \tag{4.0.14}$$

where the sums are over the sets of (M_0^2, N_0^2) values which arise in the possible decays of the dyons in the set $\mathcal{B}$ via (4.0.9). Then (4.0.13) agrees with the residue of the partition function at the pole (4.0.10) if we assume that $\widehat{\Phi}$ has a double zero at (4.0.10) where it behaves as

$$\begin{aligned} \widehat{\Phi}(\check\rho,\check\sigma,\check v) \;\;\propto\;\; & \check v'^2\,\phi_e(\check\sigma'; a_0, c_0)\,\phi_m(\check\rho'; b_0, d_0)\,, \qquad \check v' \equiv \check\rho c_0 d_0 + \check\sigma a_0 b_0 + \check v(a_0d_0 + b_0c_0), \\ & \check\sigma' \equiv c_0^2\check\rho + a_0^2\check\sigma + 2a_0c_0\check v, \quad \check\rho' \equiv d_0^2\check\rho + b_0^2\check\sigma + 2b_0d_0\check v\,. \end{aligned} \tag{4.0.15}$$

Since for any given system the allowed values of (a_0, b_0, c_0, d_0) can be found from charge quantization laws, (4.0.15) gives us information about the locations of the zeroes on $\widehat{\Phi}$ and its behaviour at these zeroes in terms of the spectrum of half-BPS states in the theory.

[4]Eq.(4.0.13) holds only if the dyons (Q_1,P_1) and (Q_2,P_2) are primitive. As will be discussed later, this formula gets modified for non-primitive decay.

4. **Additional modular symmetries:** Often the partition functions associated with $d_h(Q,P)$ have modular properties, *e.g.* the function $\phi_m(\tau; a_0, c_0)$ could transform as a modular form under $\tau \to (\alpha\tau+\beta)/(\gamma\tau+\delta)$ and $\phi_e(\tau; b_0, d_0)$ could transform as a modular form under $\tau \to (p\tau+q)/(r\tau+s)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ belonging to certain subgroups of $SL(2, \mathbb{Z})$. Some of these may be accidental symmetries, but some could be consequences of exact symmetries of the full partition function $\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v})^{-1}$. Using (4.0.15) one finds that those which can be lifted to exact symmetries of $\widehat{\Phi}$ can be represented as symplectic transformations of the form (4.0.6) with $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ given by

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix} \tag{4.0.16}$$

and

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix} \tag{4.0.17}$$

respectively. These represent additional symmetries of $\widehat{\Phi}$ besides the ones associated with S-duality invariance and charge quantization laws. Furthermore the constant k appearing in (4.0.6) is given by the weight of ϕ_e and ϕ_m minus 2.

It is these additional symmetries which make the symmetry group of $\widehat{\Phi}$ a non-trivial subgroup of $Sp(2, \mathbb{Z})$. The S-duality transformations (4.0.7) and the translation symmetries (4.0.8) are both associated with $Sp(2, \mathbb{Z})$ matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $C = 0$. In contrast the trnsformations (4.0.16), (4.0.17) typically have $C \neq 0$.

Since we do not *a priori* know which part of the modular symmetries of ϕ_e and ϕ_m survive as symmetries of $\widehat{\Phi}$, this does not give a foolproof method for identifying symmetries of $\widehat{\Phi}$. However often by combining information from the behaviour of $\widehat{\Phi}$ around different zeroes one can make a clever guess.

5. **Black hole entropy:** Additional constraints may be found be requiring that in the limit of large charges the index reproduces correctly the black hole entropy.[5] In particular, by

[5]Here we are implicitly assuming that when the effect of interactions are taken into account, only index worth of states remain as BPS states so that the black hole entropy can be compared to the logarithm of the index.

requiring that we reproduce the black hole entropy $\pi\sqrt{Q^2P^2-(Q\cdot P)^2}$ that arises in the supergravity approximation one finds that $\widehat{\Phi}$ is required to have a zero at [7,8,12,15,23]

$$\check\rho\check\sigma-\check v^2+\check v=0\,. \tag{4.0.18}$$

In order to find the behaviour of $\widehat{\Phi}$ near this zero one needs to calculate the first non-leading correction to the black hole entropy and compare this with the first non-leading correction to the formula for the index. In general the former requires the knowledge of the complete set of four derivative terms in the effective action, but in all known examples one can reproduce the answer for the index just by taking into account the effect of the Gauss-Bonnet term in the action. If we assume that this continues to hold in general then by matching the first non-leading corrections on both sides one can relate the behaviour of $\widehat{\Phi}$ near (4.0.18) to the coefficient of the Gauss-Bonnet term. The result is

$$\widehat{\Phi}(\check\rho,\check\sigma,\check v)\propto(2v-\rho-\sigma)^k\left\{v^2\,g(\rho)\,g(\sigma)+\mathcal{O}(v^4)\right\}, \tag{4.0.19}$$

where

$$\rho=\frac{\check\rho\check\sigma-\check v^2}{\check\sigma},\quad \sigma=\frac{\check\rho\check\sigma-(\check v-1)^2}{\check\sigma},\quad v=\frac{\check\rho\check\sigma-\check v^2+\check v}{\check\sigma}, \tag{4.0.20}$$

and $g(\tau)$ is a modular form of weight $k+2$ of the S-duality group, related to the Gauss-Bonnet term

$$\int d^4x\,\sqrt{-\det g}\,\phi(a,S)\,\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2\right\}, \tag{4.0.21}$$

via the relation

$$\phi(a,S)=-\frac{1}{64\pi^2}\left((k+2)\ln S+\ln g(a+iS)+\ln g(-a+iS)\right)+\text{constant}\,. \tag{4.0.22}$$

Here $\tau=a+iS$ is the axion-dilaton modulus.

In §4.5 we apply the considerations described above to several examples. These include known examples involving unit torsion dyons in heteroric string theory on T^6 and CHL orbifolds and also some unknown cases like dyons of torsion 2 in heterotic string theory on T^6 (i.e. dyons for which gcd($Q\wedge P$)=2 [20]) and dyons carrying untwisted sector charges in $\mathbb{Z}_2$ CHL orbifold [47, 48]. In the latter cases we determine the constraints imposed by the S-duality invariance and wall crossing formulæ and also try to use the known modular properties of half-BPS states to guess the symmetry group of the quarter BPS dyon partition function.

In §4.6 we propose a formula for the dyon partitions function of torsion two dyons in heterotic string theory on T^6. The formula for the partition function when Q and P are both primitive but $(Q\pm P)$ are twice primitive vectors is

$$\frac{1}{\widehat{\Phi}(\check\rho,\check\sigma,\check v)}=\frac{1}{8}\left[\frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v)}+\frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma+\frac12,\check v)}+\frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac14,\check v+\frac14)}\right.$$

$$+\frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{3}{4},\check v+\frac{1}{4})}+\frac{1}{\Phi_{10}(\check\rho+\frac{1}{2},\check\sigma+\frac{1}{2},\check v+\frac{1}{2})}$$
$$+\frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v+\frac{1}{2})}+\frac{1}{\Phi_{10}(\check\rho+\frac{3}{4},\check\sigma+\frac{3}{4},\check v+\frac{3}{4})}+\frac{1}{\Phi_{10}(\check\rho+\frac{1}{4},\check\sigma+\frac{1}{4},\check v+\frac{3}{4})}\Bigg]$$
$$+\frac{2}{\Phi_{10}(\check\rho+\check\sigma+2\check v,\check\rho+\check\sigma-2\check v,\check\sigma-\check\rho)} \qquad (4.0.23)$$

where Φ_{10} is the weight 10 Igusa cusp form of $Sp(2,\ \mathbb{Z})$ describing the inverse partition function of torsion one dyons. The sum of the first eight terms on the right hand side of (4.0.23) coincides with the partition function of unit torsion dyons subject to the constraints that $Q^2+P^2\pm 2Q\cdot P$ are multiples of 8; the last term is a new addition. We show that (4.0.23) satisfies all the required consistency conditions. First of all it has the required S-duality invariance. It also satisfies the wall crossing formulæ at all the walls of marginal stability at which the original dyon decays into a pair of primitive dyons. It satisfies the constraint (4.0.19) coming from the requirement that the statistical entropy and the black hole entropy agrees up to the first non-leading order in inverse powers of charges. Furthermore by taking an appropriate limit of this formula we can reproduce the known results for torsion two dyons in gauge theories [49, 50, 68, 69].

In the case of torsion two dyons with Q, P both primitive, the vectors $Q\pm P$ are not primitive, but $(Q\pm P)/2$ are primitive vectors [40]. As a result for the decay into

$$(Q_1,P_1)=(Q-P,0), \qquad (Q_2,P_2)=(P,P)\,, \qquad (4.0.24)$$

the charge vector (Q_1,P_1) is not primitive. Computing the jump in the index from (4.0.23) we find that in this case the change in the index across this wall of marginal stability is given by

$$\Delta d(Q,P)=(-1)^{Q_1\cdot P_2-Q_2\cdot P_1+1}(Q_1\cdot P_2-Q_2\cdot P_1)\left\{d_h\left(Q_1,P_1\right)+d_h\left(\frac{1}{2}Q_1,\frac{1}{2}P_1\right)\right\}d_h\left(Q_2,P_2\right)\,. \qquad (4.0.25)$$

This differs from the formula (4.0.13). A similar modification of the wall crossing formula for decays into non-primitive states in $\mathcal{N}=2$ supersymmetric string theories has been suggested in [46].

There are two more classes of dyons of torsion two, – one where Q is primitive and P is twice a primitive vector and the other where P is primitive and Q is twice a primitive vector. The partition functions for these dyons can be recovered from the one given above by S-duality transformations $(Q,P)\to(Q,P-Q)$ and $(Q,P)\to(Q-P,P)$ respectively [40]. This amount to making replacements $(\check\rho,\check\sigma,\check v)\to(\check\rho,\check\sigma+\check\rho+2\check v,\check v+\check\rho)$ and $(\check\rho,\check\sigma,\check v)\to(\check\rho+\check\sigma+2\check v,\check\sigma,\check v+\check\sigma)$ respectively in eq.(4.0.23).

Although we have presented most of our analysis as a way of extracting information about the partition function of quarter BPS states from known spectrum of half-BPS states, we could also use it in the reverse direction. In the final section §4.7 we provide some examples in the context of $\mathbb{Z}_N$ orbifold models where the knowledge of the quarter BPS partition function can be used to compute the spectrum of a certain class of half-BPS states.

4.1 The dyon partition function

Let us consider a particular $\mathcal{N} = 4$ supersymmetric string theory in four dimensions with a total of r U(1) gauge fields including the six graviphotons. The electric and magnetic charges in this theory are represented by r dimensional vectors Q and P, and there is a T-duality invariant metric L of signature $(6, r-6)$ that can be used to define the inner product of the charges. Let us consider an (infinite) set $\mathcal{B}$ of dyon charge vectors (Q, P) with the property that if two different members of the set have the same values of $Q^2 \equiv Q^T LQ$, $P^2 \equiv P^T LP$ and $Q \cdot P \equiv Q^T LP$ then there must exist a T-duality transformation that relates the two members. In other words if there are T-duality invariants other than Q^2, P^2 and $Q \cdot P$ then for all members of the set $\mathcal{B}$ with a given set of values of $(Q^2, P^2, Q \cdot P)$ these other T-duality invariants must have the same values. We shall generate such a set $\mathcal{B}$ by beginning with a family $\mathcal{A}$ of charge vectors (Q, P) labelled by three integers such that the triplet $(Q^2, P^2, Q \cdot P)$ are independent linear functions of these three integers, and then define $\mathcal{B}$ to be the set of all (Q, P) which are in the T-duality orbit of the set $\mathcal{A}$. Such a set $\mathcal{B}$ automatically satisfies the restriction mentioned above since given two elements of $\mathcal{B}$ with the same values of $(Q^2, P^2, Q \cdot P)$, each will be related by a T-duality transformation to the unique element of $\mathcal{A}$ with these values of $(Q^2, P^2, Q \cdot P)$. An example of such a set $\mathcal{A}$ can be found in eqs.(4.5.1.3), (4.5.1.4).

Our object of interest is the index $d(Q, P)$, measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets of quarter BPS dyons carrying charges $(Q, P) \in \mathcal{B}$. Typically the index, besides depending on (Q, P), also depends of the domain in which the asymptotic moduli lie. These domains are bounded by walls of marginal stability associated with decays of the form $(Q, P) \to (\alpha Q + \beta P, \gamma Q + \delta P) + ((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P)$ for appropriate values of $(\alpha, \beta, \gamma, \delta)$ associated with the quantization conditions [36,37,51]. For fixed values of the other moduli these walls describe circles or straight lines in the axion-dilaton moduli space labelled by the complex parameter τ [19, 36]. We denote by $\vec{c}$ the collection of $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ bordering a particular domain in the moduli space; inside any such domain the index remains unchanged. It has been shown in [19] that the parameters $\vec{c}$ labelling a domain remain invariant under a simultaneous T-duality transformation on the charges and the moduli. Since $d(Q, P)$ must be invariant under simultaneous T-duality transformation on the charges and the moduli, we can conclude that for a given $\vec{c}$ the index $d(Q, P)$ for $(Q, P) \in \mathcal{B}$ will be a function only of the T-duality invariants $(Q^2, P^2, Q \cdot P)$. We shall express this as $f(Q^2, P^2, Q \cdot P, \vec{c})$.

Let us now introduce the partition function

$$\frac{1}{\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v})} \equiv \sum_{Q^2, P^2, Q\cdot P} (-1)^{Q\cdot P+1} f(Q^2, P^2, Q \cdot P; \vec{c}_0) e^{i\pi(\check{\sigma} Q^2 + \check{\rho} P^2 + 2\check{v} Q\cdot P)} . \tag{4.1.1}$$

where $\vec{c}_0$ denotes some specific domain in the moduli space bounded by a set of walls of marginal stability. The sum runs over allowed values of Q^2, P^2 and $Q \cdot P$ for the dyons belonging to the set $\mathcal{B}$. The factor of $(-1)^{Q\cdot P+1}$ has been included for convenience. $\widehat{\Phi}$ so defined is expected to be a periodic function of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$, with the periods depending on the quantization condition on P^2, Q^2 and $Q \cdot P$. Let the periods be T_1, T_2 and T_3 respectively – these represent inverses of the quanta of $P^2/2$, $Q^2/2$ and $Q \cdot P$ belonging to the set $\mathcal{B}$. The sum given in (4.1.1) is typically not convergent for real values of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$. However often it may be made convergent by treating $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ as complex variables and working in appropriate domain in the complex plane. We shall assume that this can be done. We may now invert (4.1.1) as

$$\begin{aligned} f(Q^2, P^2, Q \cdot P; \vec{c}_0) &= \frac{(-1)^{Q\cdot P+1}}{T_1 T_2 T_3} \int_{iM_1-T_1/2}^{iM_1+T_1/2} d\check{\rho} \int_{iM_2-T_2/2}^{iM_2+T_2/2} d\check{\sigma} \int_{iM_3-T_3/2}^{iM_3+T_3/2} d\check{v} \\ & \qquad e^{-i\pi(\check{\sigma}Q^2+\check{\rho}P^2+2\check{v}Q\cdot P)} \frac{1}{\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v})}, \end{aligned} \tag{4.1.2}$$

provided the imaginary parts M_1, M_2 and M_3 of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ are fixed in a region where the original sum (4.1.1) is convergent.

During the above discussion we have implicitly assumed that the quantization laws of Q^2, P^2 and $Q \cdot P$ are uncorrelated so that $\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$ is separately invariant under $\check{\rho} \to \check{\rho} + T_1$, $\check{\sigma} \to \check{\sigma} + T_2$ and $\check{v} \to \check{v} + T_3$. In general we can have more complicated periods which involve simultaneous shifts of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$. In this case the integration in (4.1.2) needs to be carried out over an appropriate unit cell in the $(\Re(\check{\rho}), \Re(\check{\sigma}), \Re(\check{v}))$ space and the factor of $T_1 T_2 T_3$ in the denominator will be replaced by the volume of the unit cell.

4.2 Consequences of S-duality symmetry

Let us now assume that the theory has S-duality symmetries of the form

$$Q \to Q'' = aQ + bP, \qquad P \to P'' = cQ + dP\,, \tag{4.2.1}$$

for appropriate choice of (a, b, c, d). Under this transformation

$$Q''^2 = a^2Q^2 + b^2P^2 + 2abQ\cdot P, \quad P''^2 = c^2Q^2 + d^2P^2 + 2cdQ\cdot P, \quad Q''\cdot P'' = acQ^2 + bdP^2 + (ad+bc)Q\cdot P\,. \tag{4.2.2}$$

A generic S-duality transformation acting on an arbitrary element of $\mathcal{B}$ will give rise to (Q'', P'') outside the set $\mathcal{B}$ for which the index formula is given by the function f. We shall restrict ourselves to a subset of S-duality transformations which takes an element of the set $\mathcal{B}$ to another element of the set $\mathcal{B}$. For such transformations, the S-duality invariance of the theory tells us that

$$f(Q''^2, P''^2, Q''\cdot P'', \vec{c}_0'') = f(Q^2, P^2, Q\cdot P, \vec{c}_0)\,, \tag{4.2.3}$$

where $\vec{c}_0''$ denotes the collection $\{(\alpha_i'', \beta_i'', \gamma_i'', \delta_i'')\}$ of domain walls related to the set $\{(\alpha_i, \beta_i, \gamma_i, \delta_i)\}$ associated with $\vec{c}_0$ by the relation [19, 40]

$$\begin{pmatrix} \alpha_i'' & \beta_i'' \\ \gamma_i'' & \delta_i'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}. \tag{4.2.4}$$

Physically the domain corresponding to $\vec{c}_0''$ represents the image of the one corresponding to $\vec{c}_0$ under simultaneous S-duality transformation on the charges and the moduli. Making a change of variables

$$\check{\rho} = d^2\check{\rho}'' + b^2\check{\sigma}'' + 2bd\check{v}'', \quad \check{\sigma} = c^2\check{\rho}'' + a^2\check{\sigma}'' + 2ac\check{v}'', \quad \check{v} = cd\check{\rho}'' + ab\check{\sigma}'' + (ad+bc)\check{v}'', \tag{4.2.5}$$

in (4.1.2) and using the fact that

$$(-1)^{Q\cdot P} = (-1)^{Q''\cdot P''}, \quad \check{\sigma}Q^2 + \check{\rho}P^2 + 2\check{v}Q\cdot P = \check{\sigma}''Q''^2 + \check{\rho}''P''^2 + 2\check{v}''Q''\cdot P'',$$
$$d\check{\rho}\wedge d\check{\sigma}\wedge d\check{v} = d\check{\rho}''\wedge d\check{\sigma}''\wedge d\check{v}'', \tag{4.2.6}$$

under an S-duality transformation, we can express (4.2.3) as

$$f(Q''^2, P''^2, Q''\cdot P'', \vec{c}_0'') = \frac{(-1)^{Q''\cdot P''+1}}{T_1T_2T_3}\int_{\mathcal{C}} d\check{\rho}''d\check{\sigma}''d\check{v}''\, e^{-i\pi(\check{\sigma}''Q''^2+\check{\rho}''P''^2+2\check{v}''Q''\cdot P'')}\,\frac{1}{\widehat{\Phi}(\check{\rho},\check{\sigma},\check{v})}, \tag{4.2.7}$$

where $\mathcal{C}$ is the image of the original region of integration (4.1.2) in the complex $(\check{\rho}'', \check{\sigma}'', \check{v}'')$ plane:

$$\Im(\check{\rho}'') = a^2M_1 + b^2M_2 - 2abM_3, \quad \Im(\check{\sigma}'') = c^2M_1 + d^2M_2 - 2cdM_3,$$
$$\Im(\check{v}'') = -acM_1 - bdM_2 + (ad+bc)M_3\,. \tag{4.2.8}$$

We would like to get some constraint on the function $\widehat{\Phi}$ by comparing (4.1.2) with (4.2.7). For this we note that we can replace (Q, P) by (Q'', P'') and $(\check{\rho}, \check{\sigma}, \check{v})$ by $(\check{\rho}'', \check{\sigma}'', \check{v}'')$ everywhere in (4.1.2) since they are dummy variables. This gives

$$f(Q''^2, P''^2, Q''\cdot P''; \vec{c}_0) = \frac{(-1)^{Q''\cdot P''+1}}{T_1T_2T_3}\int_{iM_1-T_1/2}^{iM_1+T_1/2} d\check{\rho}''\int_{iM_2-T_2/2}^{iM_2+T_2/2} d\check{\sigma}''\int_{iM_3-T_3/2}^{iM_3+T_3/2} d\check{v}''\; e^{-i\pi(\check{\sigma}''Q''^2+\check{\rho}''P''^2+2\check{v}''Q''\cdot P'')}\,\frac{1}{\widehat{\Phi}(\check{\rho}'',\check{\sigma}'',\check{v}'')}\,. \tag{4.2.9}$$

Since in general $\vec{c}_0$ and $\vec{c}_0''$ describe different domains, we cannot compare (4.2.7) and (4.2.9) to constrain the form of $\widehat{\Phi}$ without any further input.[6] However the dyon spectrum in a variety of $\mathcal{N}=4$ supersymmetric string theories displays the feature that the spectrum in two different domains $\vec{c}_0''$ and $\vec{c}_0$ are both given as integrals with the same integrand, but for $\vec{c}_0''$ the integration over $(\check{\rho}'', \check{\sigma}'', \check{v}'')$ is carried out over a different subspace than the one given in (4.2.9). In particular if $\vec{c}_0''$ is related to $\vec{c}_0$ by an S-duality transformation then this subspace is given by the integration region $\mathcal{C}$ given in (4.2.8). We shall assume that this feature continues to hold in the general situation. In that case the effect of replacinglabels. $\vec{c}_0$ by $\vec{c}_0''$ in eq.(4.2.9) is to replace the integration contour by $\mathcal{C}$ on the right hand side:

$$f(Q''^2, P''^2, Q''\cdot P''; \vec{c}_0'') = \frac{(-1)^{Q''\cdot P''+1}}{T_1T_2T_3}\int_{\mathcal{C}} d\check{\rho}''d\check{\sigma}''d\check{v}''\, e^{-i\pi(\check{\sigma}''Q''^2+\check{\rho}''P''^2+2\check{v}''Q''\cdot P'')}\,\frac{1}{\widehat{\Phi}(\check{\rho}'',\check{\sigma}'',\check{v}'')}\,. \tag{4.2.10}$$

[6]The only exceptions are those S-duality transformations which leave the domain $\vec{c}_0$ unchanged [19].

Comparing (4.2.10) and (4.2.7) we get

$$\widehat{\Phi}(\check\rho, \check\sigma, \check v) = \widehat{\Phi}(\check\rho'', \check\sigma'', \check v'') \,. \tag{4.2.11}$$

For future reference we shall rewrite the transformation laws (4.2.5) in a suggestive form. We define

$$\check\Omega = \begin{pmatrix} \check\rho & \check v \\ \check v & \check\sigma \end{pmatrix} \,. \tag{4.2.12}$$

Then the transformations (4.2.5) may be written as

$$\check\Omega = (A\check\Omega'' + B)(C\check\Omega'' + D)^{-1} \,, \tag{4.2.13}$$

where A, B, C and D are 2×2 matrices, given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix} \,. \tag{4.2.14}$$

Eq.(4.2.11) now gives (after replacing the dummy variable $\check\Omega''$ by $\check\Omega$ on both sides),

$$\widehat{\Phi}((A\check\Omega + B)(C\check\Omega + D)^{-1}) = \det(C\check\Omega + D)^k \widehat{\Phi}(\check\Omega) \,, \tag{4.2.15}$$

for A, B, C, D given in (4.2.14). Here k is an arbitrary number. Since $\det(C\Omega + D) = 1$, we cannot yet ascertain the value of k.

To this we can also append the translational symmetries of $\widehat{\Phi}$:

$$\widehat{\Phi}(\check\rho, \check\sigma, \check v) = \widehat{\Phi}(\check\rho + a_1, \check\sigma + a_2, \check v + a_3) \,, \tag{4.2.16}$$

where a_i's are integer multiples of the T_i's. It is convenient, although not necessary, to work with appropriately rescaled Q and/or P so that the T_i's and hence the a_i's are integers. This symmetry can also be rewritten as (4.2.15) with the choice

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_1 & a_3 \\ 0 & 1 & a_3 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,. \tag{4.2.17}$$

Again since $\det(C\check\Omega + D) = 1$ the choice of k is arbitrary.

The alert reader would have noticed that although we have expressed the consequences of S-duality invariance and charge quantization conditions as symmetries of the function $\widehat{\Phi}$ under a symplectic transformation, the symplectic transformations arising this way are trivial, – for all the transformations arising this way the matrix C vanishes and hence the transformations act linearly on the variables $\check{\rho}$, $\check{\sigma}$ and $\check{v}$. In order to show that the function $\widehat{\Phi}$ has non-trivial modular properties we need to find symmetries of $\widehat{\Phi}$ which have non-vanishing C. This will also determine the weight of $\widehat{\Phi}$ under the modular transformation. To get a hint about any possible additional symmetries of $\widehat{\Phi}$ we need to make use of the wall crossing formula for the dyon spectrum of $\mathcal{N} = 4$ supersymmetric string theories. This will be the subject of discussion in §4.3.

4.3 Constraints from wall crossing

As has already been discussed, the index associated with the quarter BPS dyon spectrum in $\mathcal{N} = 4$ supersymmetric string theories can undergo discontinuous jumps across walls of marginal stability. *A priori* the formula for the dyon spectrum in different domains labelled by the vector $\vec{c}$ could be completely different. However the study of dyon spectrum in a variety of $\mathcal{N} = 4$ supersymmetric string theories shows that in different domains the index continues to be given by an expression similar to (4.1.2), the only difference being that the choice of the 3 real dimensional subspace (contour) over which we carry out the integration in the complex $(\check{\rho}, \check{\sigma}, \check{v})$ plane is different in different domains. As a result the difference between the indices in two different domains is given by the sum of residues of the integrand at the poles we encounter while deforming the contour associated with one domain to the contour associated with another domain. As a special example of this we can consider the decay $(Q, P) \to (Q, 0) + (0, P)$. In all known examples change in the index across this wall of marginal stability is accounted for by the residue of a double pole of the integrand at $\check{v} = 0$, i.e. as we cross this particular wall of marginal stability in the moduli space, the integration contour crosses the pole at $\check{v} = 0$. Since the change in the index as we cross a given wall can be found using the wall crossing formula [19,41–46], this provides information on the residue of the integrand at the $\check{v} = 0$ pole.

There are many other possible decays of a quarter BPS state into a pair of half BPS states. All such decays may be parametrized as [19]

$$(Q, P) \to (a_0d_0Q - a_0b_0P, c_0d_0Q - c_0b_0P) + (-b_0c_0Q + a_0b_0P, -c_0d_0Q + a_0d_0P)\,, \quad a_0d_0 - b_0c_0 = 1\,. \tag{4.3.1}$$

a_0, b_0, c_0, d_0 are not necessarily all integers, but must be such that the charges carried by the decay products belong to the charge lattice. One can try to use the wall crossing formulæ associated with these decays to further constrain the form of $\widehat{\Phi}$. For unit torsion states in heterotic string theory on T^6, a_0, b_0, c_0 and d_0 are integers and the decay given in (4.3.1) is related to the decay $(Q, P) \to (Q, 0) + (0, P)$ via an S-duality transformation $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. Thus the change in the index across the wall is controlled by the residue of the partition function at a new pole that is related to the $\check{v} = 0$ pole by the S-duality transformation (4.2.5). This gives the location of the pole to be at

$$\check{\rho}c_0d_0 + \check{\sigma}a_0b_0 + \check{v}(a_0d_0 + b_0c_0) = 0\,. \tag{4.3.2}$$

As long as $\widehat{\Phi}$ is manifestly S-duality invariant, i.e. satisfies (4.2.14), (4.2.15), the residues at these poles will automatically satisfy the wall crossing formula. Thus they do not provide any new information. However in a generic situation new walls may appear, labelled by fractional values of a_0, b_0, c_0, d_0. Also the S-duality group is smaller. As a result not all the walls can be related to each other by S-duality transformation. It is tempting to speculate that the jump across any wall of marginal stability associated with the decay (4.3.1) is described by the residue of the partition function at the pole at (4.3.2). We shall proceed with this assumption – this will be one of our key postulates.[7]

Before we proceed we shall show that this postulate is internally consistent, i.e. it is possible to choose $\mathcal{C}$ at different points in the moduli space consistent with this postulate. For this we generalize the contour prescription of [22], assuming that it holds for all $\mathcal{N} = 4$ supersymmetric string theories. Let $\tau = \tau_1 + i\tau_2$ be the axion-dilaton moduli, M be the usual $r \times r$ symmetric matrix valued moduli satisfying $MLM^T = L$, and

$$Q_R^2 \equiv Q^T(M+L)Q, \quad P_R^2 \equiv P^T(M+L)P, \quad Q_R \cdot P_R \equiv Q^T(M+L)P\,. \tag{4.3.3}$$

Then at the point (τ, M) in the space of asymptotic moduli we choose $\mathcal{C}$ to be

$$\begin{aligned}
\Im(\check\rho) &= \Lambda\left(\frac{|\tau|^2}{\tau_2} + \frac{Q_R^2}{\sqrt{Q_R^2P_R^2 - (Q_R\cdot P_R)^2}}\right), \\
\Im(\check\sigma) &= \Lambda\left(\frac{1}{\tau_2} + \frac{P_R^2}{\sqrt{Q_R^2P_R^2 - (Q_R\cdot P_R)^2}}\right), \\
\Im(\check v) &= -\Lambda\left(\frac{\tau_1}{\tau_2} + \frac{Q_R\cdot P_R}{\sqrt{Q_R^2P_R^2 - (Q_R\cdot P_R)^2}}\right),
\end{aligned} \tag{4.3.4}$$

where Λ is a large positive number. Then on $\mathcal{C}$

$$\begin{aligned}
&\Im(c_0d_0\check\rho + a_0b_0\check\sigma + (a_0d_0+b_0c_0)\check v) \\
&= \frac{c_0d_0}{\tau_2}\Lambda\left\{\left(\tau_2 + \frac{E}{2c_0d_0}\right)^2 + \left(\tau_1 - \frac{a_0d_0+b_0c_0}{2c_0d_0}\right)^2 - \left(1 + \frac{E^2}{4c_0^2d_0^2}\right)\right\},
\end{aligned} \tag{4.3.5}$$

where

$$E = \frac{c_0d_0Q_R^2 + a_0b_0P_R^2 - (a_0d_0+b_0c_0)Q_R\cdot P_R}{\sqrt{Q_R^2P_R^2 - (Q_R\cdot P_R)^2}}\,. \tag{4.3.6}$$

[7] Of course the translation symmetries (4.2.16) allow us to shift a pole at (4.3.2) to other equivalent locations. Our postulate asserts that the contribution comes from poles which can be brought to (4.3.2) using the translation symmetries (4.2.16). In that case we can choose the unit cell over which we carry out the integration in (4.1.2) in such a way that only the pole at (4.3.2) contributes to the jump in the index across the wall at (4.3.1). A possible exception to this will be discussed in the paragraphs above eq.(4.3.12) where we address some subtle issues.

As shown in [19], the right hand side of (4.3.5) vanishes on the wall of marginal stability associated with the decay given in (4.3.1). Thus it follows from (4.3.5) that as we cross this wall of marginal stability, the contour (4.3.4) crosses the pole at (4.3.2) in accordance with our postulate.

This postulate allows us to identify the possible poles of the partition function besides those related to the $\check{v} = 0$ pole by the S-duality transformation (4.2.5), – they occur at (4.3.2) for those values of a_0, b_0, c_0 and d_0 for which the decay (4.3.1) is consistent with the charge quantization laws. One can also get information about the residues at these poles since they are given by the jumps in the index. This jump can be expressed using the wall crossing formula [19, 41–46] that tells us that as we cross a wall of marginal stability associated with the decay $(Q, P) \to (Q_1, P_1) + (Q_2, P_2)$ the index jumps by an amount

$$(-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\,(Q_1 \cdot P_2 - Q_2 \cdot P_1)\,d_h(Q_1, P_1)d_h(Q_2, P_2) \tag{4.3.7}$$

up to a sign, where $d_h(Q, P)$ denotes the index measuring the number of bosonic minus the number of fermionic half BPS supermultiplets carrying charges (Q, P). Thus this relates the residues at the poles of the integrand to the indices of half BPS states.

We shall now study the consequence of (4.3.7) on the residue at the pole (4.3.2). First let us consider the special case associated with the decay $(Q, P) \to (Q, 0) + (0, P)$. In this case the jump in the index is given by

$$(-1)^{Q\cdot P+1}\,Q \cdot P\,d_e(Q)\,d_m(P)\,, \tag{4.3.8}$$

where $d_e(Q) = d_h(Q, 0)$ is the index of purely electrically charged states and $d_m(P) = d_h(0, P)$ is the index of purely magnetically charged state. This jump is to be accounted for by the residue of a pole of the integrand at $\check{v} = 0$. The result (4.3.8) is reproduced if near $\check{v} = 0$, $\widehat{\Phi}$ behaves as

$$\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v})^{-1} \propto \{\phi_m(\check{\rho})^{-1}\,\phi_e(\check{\sigma})^{-1}\check{v}^{-2} + \mathcal{O}(\check{v}^0)\}\,, \tag{4.3.9}$$

where $1/\phi_m(\check{\rho})$ and $1/\phi_e(\check{\sigma})$ denote respectively the partition functions of purely magnetic and purely electric states:

$$d_m(P) = \frac{1}{T_1}\int_{iM_1-T_1/2}^{iM_1+T_1/2} d\check{\rho}\, e^{-i\pi P^2\check{\rho}}\frac{1}{\phi_m(\check{\rho})}\,, \qquad d_e(Q) = \frac{1}{T_2}\int_{iM_2-T_2/2}^{iM_2+T_2/2} d\check{\sigma}\, e^{-i\pi Q^2\check{\sigma}}\frac{1}{\phi_e(\check{\sigma})}\,. \tag{4.3.10}$$

Substituting (4.3.10) into the integrand in (4.1.2) and picking up the residue from the pole at $\check{v} = 0$ we get the change in the index to be

$$(-1)^{Q\cdot P+1}\,Q \cdot P\,d_e(Q)\,d_m(P)\,, \tag{4.3.11}$$

in agreement with (4.3.8), provided we choose the constant of proportionality in (4.3.9) appropriately. Note that the $Q \cdot P$ factor comes from the $\check{v}$ derivative of the exponential factor in (4.1.2) arising due to the double pole of $\widehat{\Phi}^{-1}$ at $\check{v} = 0$.

In writing (4.3.9), (4.3.10) we have implicitly assumed that the allowed values of Q^2 and P^2 inside the set $\mathcal{B}$ are independent of each other, i.e. the possible values that Q^2 can take for a given P^2 is independent of P^2 and vice versa. If this is not so then instead of having a single product the right hand side of (4.3.9) will contain a sum of products. For example if in the set $\mathcal{B}$, $Q^2/2$ and $P^2/2$ are correlated so that $Q^2/2$ is odd (even) when $P^2/2$ is odd (even) then the coeffcient of $\check{v}^{-2}$ in the expression for $\widehat{\Phi}^{-1}$ will contain two terms, – the product of the partition function with odd $Q^2/2$ electric states with that of odd $P^2/2$ magnetic states and the product of the partition function of even $Q^2/2$ electric states with that of even $P^2/2$ magnetic states.

There is one more assumption that has gone into writing (4.3.9), (4.3.10). We have assumed that given two pairs of charge vectors (Q, P) and $(\widehat{Q}, \widehat{P})$ in $\mathcal{B}$, if $Q^2 = \widehat{Q}^2$ then Q and $\widehat{Q}$ are related by a T-duality transformation. Otherwise $d_e(Q)$ will not be a function of Q^2 and one cannot define an electric partition function via eq.(4.3.10). A similar restriction applies to the magnetic charges as well. Now since the set $\mathcal{B}$ has been chosen such that if the triplets $(Q^2, P^2, Q \cdot P)$ are identical for two charge vectors then they must be related by T-duality transformation, if two different Q's with same Q^2 are not related by T-duality then they must come from triplets with different values of P^2 and/or $Q \cdot P$. In other words the different T-duality orbits for a given Q^2 must be correlated with P^2 and/or $Q \cdot P$. If the correlation is with P^2 then we follow the procedure described in the previous paragraph, *e.g.* if one set of Q's arise from even $P^2/2$ and another set of Q's arise from odd $P^2/2$, we define two separate electric partition function for these two different sets of Q's and identify the coefficient of $\check{v}^{-2}$ in the partition function $\widehat{\Phi}^{-1}$ as a sum of terms. If on the other hand the correlation is with $Q \cdot P$ then the procedure is more complicated. We first project onto different $Q \cdot P$ sectors by adding to $\widehat{\Phi}^{-1}$ other terms obtained by appropriate shifts of $\check{v}$, so that the subset of states which contribute to the new partition function now has a unique Q for a given Q^2 up to T-duality transformations. The singularities of this new partition functions near $\check{v} = 0$ will now be described by equation of the type (4.3.9), (4.3.10). For example if one set of Q's come from odd $Q \cdot P$ and the second set of Q's come from even $Q \cdot P$, then we can consider the quarter BPS partition functions $\frac{1}{2}\{\widehat{\Phi}^{-1}(\check{\rho}, \check{\sigma}, \check{v}) \pm \widehat{\Phi}^{-1}(\check{\rho}, \check{\sigma}, \check{v} + \frac{1}{2})\}$. These pick up even $Q \cdot P$ and odd $Q \cdot P$ states respectively, and hence the contribution to these partition functions will come from charge vectors (Q, P) with the property that for a given Q^2, there will be a unique Q up to a T-duality transformation. Thus the behaviour of these combinations will now be controlled by equations of the type given in (4.3.9), (4.3.10). Conversely, for the original set $\mathcal{B}$ the jump in the index associated with the decay $(Q, P) \to (Q, 0) + (0, P)$ is now controlled by the zeroes of $\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$ at $\check{v} = 0$ and also at $\check{v} = 1/2$. Similar considerations apply when the same P^2 in the set $\mathcal{B}$ comes from more than one P's which are not related by T-duality.

Often both the subtleties mentioned above can be avoided by a judicious choice of the set $\mathcal{B}$. In fact in all the explicit examples we shall study in §4.5, we shall be able to avoid these subtleties.

We now return to the general case associated with the decay described in (4.3.1). Since here

$$(Q_1, P_1) = (a_0 d_0 Q - a_0 b_0 P, c_0 d_0 Q - c_0 b_0 P), \qquad (Q_2, P_2) = (-b_0 c_0 Q + a_0 b_0 P, -c_0 d_0 Q + a_0 d_0 P)\,, \tag{4.3.12}$$

we have

$$Q_1 \cdot P_2 - Q_2 \cdot P_1 = -Q^2 c_0 d_0 - P^2 a_0 b_0 + Q \cdot P (a_0 d_0 + b_0 c_0)\,. \tag{4.3.13}$$

Let us now make a change of variables

$$\check\rho' = d_0^2 \check\rho + b_0^2 \check\sigma + 2 b_0 d_0 \check v, \qquad \check\sigma' = c_0^2 \check\rho + a_0^2 \check\sigma + 2 a_0 c_0 \check v, \qquad \check v' = c_0 d_0 \check\rho + a_0 b_0 \check\sigma + (a_0 d_0 + b_0 c_0) \check v\,, \tag{4.3.14}$$

and define

$$Q' = d_0 Q - b_0 P, \qquad P' = -c_0 Q + a_0 P\,. \tag{4.3.15}$$

Under this change of variables

$$d\check\rho \wedge d\check\sigma \wedge d\check v = d\check\rho' \wedge d\check\sigma' \wedge d\check v'\,, \tag{4.3.16}$$

$$Q_1 \cdot P_2 - Q_2 \cdot P_1 = Q' \cdot P'\,, \tag{4.3.17}$$

$$(Q_1, P_1) = (a_0 Q', c_0 Q'), \qquad (Q_2, P_2) = (b_0 P', d_0 P')\,, \tag{4.3.18}$$

and

$$\frac{1}{2}\check\rho P^2 + \frac{1}{2}\check\sigma Q^2 + \check v Q \cdot P = \frac{1}{2}\check\rho' P'^2 + \frac{1}{2}\check\sigma' Q'^2 + \check v' Q' \cdot P'\,. \tag{4.3.19}$$

Thus the jump in the index given in (4.3.7) can be expressed as

$$(-1)^{Q'\cdot P'+1}\, Q' \cdot P'\, d_h(a_0 Q', c_0 Q') d_h(b_0 P', d_0 P')\,. \tag{4.3.20}$$

Furthermore in these variables the pole at (4.3.2) is at $\check v' = 0$. Thus we can identify (4.3.20) with the residue of the integrand from $\check v' = 0$. Using (4.3.16), (4.3.19) the latter may be expressed as

$$(-1)^{Q\cdot P+1} \int d\check\rho' d\check\sigma' d\check v' e^{i\pi(\check\rho' P'^2 + \check\sigma' Q'^2 + 2\check v' Q'\cdot P')} \frac{1}{\widehat\Phi(\check\rho, \check\sigma, \check v)}\,, \tag{4.3.21}$$

where the integration contour is around $\check v' = 0$. We now note that this result can be reproduced if we assume that near the pole (4.3.2) the partition function behaves as[8]

$$\widehat\Phi(\check\rho, \check\sigma, \check v)^{-1} \propto \{\phi_e(\check\sigma'; a_0, c_0)^{-1} \phi_m(\check\rho'; b_0, d_0)^{-1} \check v'^{-2} + \mathcal{O}(\check v'^0)\}\,, \tag{4.3.22}$$

where $1/\phi_{e,m}(\tau; k, l)$ denote the partition functions of half BPS dyons in the set $\mathcal{B}$ such that

$$\begin{aligned} d_h(a_0 Q', c_0 Q') &= \frac{1}{T} \int_{iM-T/2}^{iM+T/2} d\tau\, e^{-i\pi Q'^2 \tau} \frac{1}{\phi_e(\tau; a_0, c_0)}, \\ d_h(b_0 P', d_0 P') &= \frac{1}{T'} \int_{iM-T'/2}^{iM+T'/2} d\tau\, e^{-i\pi P'^2 \tau} \frac{1}{\phi_m(\tau; b_0, d_0)}\,. \end{aligned} \tag{4.3.23}$$

[8]This formula suffers from the same type of subtleties described below eq.(4.3.11) with (Q, P) replaced by (Q', P') and $(\check\rho, \check\sigma, \check v)$ replaced by $(\check\rho', \check\sigma', \check v')$.

The integration over τ run parallel to the real axis over unit period with the imaginary part fixed at some large positive value M. Substituting (4.3.22) into (4.3.21) and picking up the residue from the pole at $\check{v}' = 0$ we get the change in the index to be

$$(-1)^{Q'\cdot P'+1}\, Q'\cdot P'\, d_h(a_0Q', c_0Q')d_h(b_0P', d_0P')\,, \tag{4.3.24}$$

in agreement with (4.3.20).

To summarize, (4.3.2) gives us the locations of the zeroes of $\widehat{\Phi}$, whereas eq.(4.3.9) and more generally (4.3.22) give us information about the behaviour of $\widehat{\Phi}$ near this zero. We shall now show that these results suggest additional symmetries of $\widehat{\Phi}$ of the type described in (4.2.15). Typically in any theory the partition functions of half BPS states have modular properties. Let us for definiteness consider the decay $(Q,P) \to (Q,0) + (0,P)$. In this case the functions $\phi_m(\check{\rho})$ and $\phi_e(\check{\sigma})$ transform as modular forms of a subgroup of $SL(2,\ \mathbb{Z})$ since they arise from quantization of a fundamental string or a dual magnetic string. These relations take the form

$$\phi_m((\alpha\check{\rho}+\beta)(\gamma\check{\rho}+\delta)^{-1}) = (\gamma\check{\rho}+\delta)^{k+2}\phi_m(\check{\rho}), \qquad \phi_e((p\check{\sigma}+q)(r\check{\sigma}+s)^{-1}) = (r\check{\sigma}+s)^{k+2}\phi_e(\check{\sigma})\,, \tag{4.3.25}$$

where k is an integer specific to the theory under study, and $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ belong to appropriate subgroups of $SL(2,\ \mathbb{Z})$. Given that ϕ_m and ϕ_e have these symmetries, we conclude from (4.3.9) that near $\check{v}=0$, $\widehat{\Phi}$ also has some additional symmetries. Even though there is no guarantee that these will be symmetries of the full quarter BPS partition function, one could hope that some part of these do lift to symmetries of the partition function and hence of $\widehat{\Phi}$. Those which do can be represented by symplectic transformations of the type (4.2.15) with

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \begin{pmatrix}\alpha & 0 & \beta & 0\\ 0 & 1 & 0 & 0\\ \gamma & 0 & \delta & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}A & B\\ C & D\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & p & 0 & q\\ 0 & 0 & 1 & 0\\ 0 & r & 0 & s\end{pmatrix}. \tag{4.3.26}$$

The first transformation generates

$$\check{\rho} \to \frac{\alpha\check{\rho}+\beta}{\gamma\check{\rho}+\delta}, \quad \check{\sigma} \to \check{\sigma} - \frac{\gamma\check{v}^2}{\gamma\check{\rho}+\delta}, \quad \check{v} \to \frac{\check{v}}{\gamma\check{\rho}+\delta}\,, \tag{4.3.27}$$

while the second transformation generates

$$\check{\rho} \to \check{\rho} - \frac{r\check{v}^2}{r\check{\sigma}+s}, \quad \check{\sigma} \to \frac{p\check{\sigma}+q}{r\check{\sigma}+s}, \quad \check{v} \to \frac{\check{v}}{r\check{\sigma}+s}\,. \tag{4.3.28}$$

Both transformations leave the $\check{v}=0$ surface invariant. Furthermore applying these transformations on (4.2.15) and using (4.3.9) near $\check{v}=0$ we generate the transformation laws (4.3.25).

The symplectic transformations given in (4.3.26), if present, give us the additional symmetries required to have $\widehat{\Phi}$ transform as a modular form under a non-trivial subgroup of $Sp(2, \mathbb{Z})$. We can use this to determine the subgroup of $Sp(2, \mathbb{Z})$ under which we expect $\widehat{\Phi}$ to transform as a modular form and also the weight k of the modular form. However since we do not know *a priori* which part of the symmetry groups of ϕ_e and ϕ_m will lift to the symmetries of $\widehat{\Phi}$, this is not a fool proof method. Nevertheless these can serve as guidelines for making an educated guess.

The behaviour of $\widehat{\Phi}$ near the other zeroes given in (4.3.2) could provide us with additional information. If the zero of $\widehat{\Phi}$ at (4.3.2) is related to the one at $\check{v} = 0$ by an S-duality transformation then this information is not new. Since S-duality transformation acts by multiplying the matrix $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ associated with a wall from the left [19], this means that if $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ itself is an S-duality transformation then we do not get a new information. To this we must also add the information that multiplying $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ from the right by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for any λ or by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ does not change the wall [19]. However in many cases even after imposing these equivalence relations one finds inequivalent walls.[9] In such cases the associated zero of $\widehat{\Phi}$ cannot be related to the zero at $\check{v} = 0$ by an S-duality transformation, and we get new information.[10] Let $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ be the matrix associated with such a decay. If the corresponding partition functions $\phi_m(\tau; b_0, d_0)$ and $\phi_e(\tau; a_0, c_0)$ have modular groups containing matrices of the form $\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$ and $\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}$ respectively, then they may be regarded as symplectic transformations generated by the matrices

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 & 0 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_1 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix} \tag{4.3.29}$$

and

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_1 & 0 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & r_1 & 0 & s_1 \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix} \tag{4.3.30}$$

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[9] For example in $\mathbb{Z}_6$ CHL model with S-duality group $\Gamma_1(6)$ the wall corresponding to the matrix $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ is not equivalent to the wall corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We shall discuss this example in some detail in §4.7.

[10] Typically the number of such additional zeroes is a finite number, providing us with a finite set of additional information.

respectively, acting on the original variables $(\check{\rho}, \check{\sigma}, \check{v})$. Again we could hope that a part of this symmetry is a symmetry of $\widehat{\Phi}$.

We shall illustrate these by several examples in §4.5.

4.4 Black hole entropy

Another set of constraints may be derived by requiring that the formula for the index of quarter BPS states match the entropy of the black hole carrying the same charges in the limit when the charges are large. The consequences of this constraint have been analyzed in detail in the past [7, 8, 12, 15] and reviewed in [23]. Hence our discussion will be limited to a review of the salient features.

In the approximation where we keep the supergravity part of the action containing only the two derivative terms, the black hole entropy is given by

$$\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}\,. \tag{4.4.1}$$

In all known cases this result is reproduced by the asymptotic behaviour of (4.1.2) for large charges. Furthermore the leading asymptotic behaviour comes from the residue of the partition function at the pole at [7, 8, 12, 15, 23]

$$\check{\rho}\check{\sigma} - \check{v}^2 + \check{v} = 0\,, \tag{4.4.2}$$

up to translations of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ by their periods. We shall assume that this result continues to hold in the general case. Thus $\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v})$ must have a zero at (4.4.2). In order to find the behaviour of $\widehat{\Phi}$ near this zero one needs to know the first non-leading correction to the leading formula (4.4.1) for the black hole entropy. *A priori* these corrections depend on the complete set of four derivative terms in the quantum effective action of the theory and are difficult to calculate. However in all known examples one finds that the entropy calculated just by including the Gauss-Bonnet term in the effective action reproduces correctly the first non-leading correction to the statistical entropy. If we assume that this result continues to hold for a general theory then we can use this to determine the behaviour of $\widehat{\Phi}$ near (4.4.2) in terms of the coefficient of the Gauss-Bonnet term in the effective action.

Since this procedure has been extensively studied in [7, 8, 12, 15] and reviewed in [23], we shall only quote the result. Typically the Gauss Bonnet term in the Lagrangian has the form

$$\int d^4x\,\sqrt{-\det g}\,\phi(a,S)\,\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right\}\,, \tag{4.4.3}$$

where $\tau = a + iS$ is the axion-dilaton modulus and the function $\phi(a, S)$ has the form

$$\phi(a,S) = -\frac{1}{64\pi^2}\,\left((k+2)\ln S + \ln g(a+iS) + \ln g(-a+iS)\right) + \text{constant}\,. \tag{4.4.4}$$

Here k is the same integer that appeared in (4.2.15) and $g(\tau)$ transforms as a modular form of weight $k+2$ under the S-duality group. In a given theory $g(\tau)$ can be calculated in string perturbation theory [52, 53]. To the first non-leading order in the inverse power of charges, the effect of this term is to change the black hole entropy to [23]

$$S_{BH} = \pi\sqrt{Q^2P^2-(Q\cdot P)^2} + 64\,\pi^2\,\phi\left(\frac{Q\cdot P}{P^2}, \frac{\sqrt{Q^2P^2-(Q\cdot P)^2}}{P^2}\right) + \cdots \tag{4.4.5}$$

The analysis of [7, 8, 12, 15, 23] shows that this behaviour can be reproduced if we assume that near the zero at (4.4.2)

$$\widehat{\Phi}(\check\rho,\check\sigma,\check v) \propto (2v-\rho-\sigma)^k \left\{v^2\, g(\rho)\, g(\sigma) + \mathcal{O}(v^4)\right\}, \tag{4.4.6}$$

where

$$\rho = \frac{\check\rho\check\sigma-\check v^2}{\check\sigma}, \quad \sigma = \frac{\check\rho\check\sigma-(\check v-1)^2}{\check\sigma}, \quad v = \frac{\check\rho\check\sigma-\check v^2+\check v}{\check\sigma}. \tag{4.4.7}$$

If we assume that eq.(4.4.6) holds in general, then it gives us information about the behaviour of $\widehat{\Phi}(\check\rho,\check\sigma,\check v)$ near the zero at (4.4.2). On the other hand if we can determine $\widehat{\Phi}$ from other considerations then the validity of (4.4.6) would provide further evidence for the postulate that in $\mathcal{N}=4$ supersymmetric string theories the Gauss-Bonnet term gives the complete correction to black hole entropy to first non-leading order.

4.5 Examples

In this section we shall describe several applications of the general procedure described in §4.1. Some of them will involve known cases and will provide a test for our procedure, while others will be new examples where we shall derive a set of constraints on certain dyon partition functions which have not yet been computed from first principles.

4.5.1 Dyons with unit torsion in heterotic string theory on T^6

We consider a dyon of charge (Q,P) in the heterotic string theory on T^6. Q and P take values in the Narain lattice Λ [25, 26]. Let S^1 and $\widetilde{S}^1$ be two circles of T^6, each labelled by a coordinate with period 2π and let us denote by $n', \widetilde{n}$ the momenta along S^1 and $\widetilde{S}^1$, by $-w', -\widetilde{w}$ the fundamental string winding numbers along S^1 and $\widetilde{S}^1$, by $N', \widetilde{N}$ the Kaluza-Klein monopole charges associated with S^1 and $\widetilde{S}^1$, and by $-W', -\widetilde{W}$ the H-monopole charges associated with S^1 and $\widetilde{S}^1$ [23]. Then in the four dimensional subspace consisting of charge vectors

$$Q = \begin{pmatrix}\widetilde{n}\\ n'\\ \widetilde{w}\\ w'\end{pmatrix}, \qquad P = \begin{pmatrix}\widetilde{W}\\ W'\\ \widetilde{N}\\ N'\end{pmatrix}, \tag{4.5.1.1}$$

the metric L takes the form

$$L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{4.5.1.2}$$

where I_2 denotes 2×2 identity matrix. In this subspace we consider a three parameter family of charge vectors (Q, P) with

$$Q = \begin{pmatrix} 0 \\ m \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}. \tag{4.5.1.3}$$

This has

$$Q^2 = -2m, \quad P^2 = 2K, \quad Q \cdot P = -J. \tag{4.5.1.4}$$

We shall identify this set of charge vectors as the set $\mathcal{A}$. As required, Q^2, P^2 and $Q \cdot P$ are independent linear functions of m, K and J so that for a pair of distinct values of (m, K, J) we get a pair of distinct values of $(Q^2, P^2, Q \cdot P)$. All the charge vectors in this family have unit torsion, i.e. if we express the charges as linear combinations $\sum Q_i e_i$ and $\sum P_i e_i$ of primitive basis elements e_i of the lattice Λ, then the torsion

$$r(Q, P) \equiv \gcd\{Q_i P_j - Q_j P_i\}, \tag{4.5.1.5}$$

is equal to 1. In this case it is known that Q^2, P^2 and $Q \cdot P$ are the complete set of T-duality invariants [54], i.e. beginning with a pair (Q, P) with unit torsion we can reach any other pair with unit torsion and same values of Q^2, P^2 and $Q \cdot P$ via a T-duality transformation. Since the set $\mathcal{A}$ contains all integer triplets $(Q^2/2, P^2/2, Q \cdot P)$ we conclude that the set $\mathcal{B}$ is the set of all (Q, P) with unit torsion. The corresponding partition function is known [7] – it is the inverse of the weight ten Igusa cusp form Φ_{10} of the full $Sp(2, \mathbb{Z})$ group.

We shall now examine how Φ_{10} satisfies the various constraints derived in the previous sections. First of all note that since S-duality transformation does not change the torsion r, the full $SL(2, \mathbb{Z})$ group is a symmetry of this set. Furthermore in this set $Q^2/2$, $P^2/2$ and $Q \cdot P$ are all quantized in integer units. Thus the partition function is invariant under translation of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ by arbitrary integer units. These correspond to symplectic transformations of the form

$$\begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & a_1 & a_3 \\ 0 & 1 & a_3 & a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad a_1, a_2, a_3 \in \mathbb{Z}. \tag{4.5.1.6}$$

Clearly each of these transformations belong to $Sp(2, \mathbb{Z})$ and is a symmetry of Φ_{10}.

Next we turn to the constraints from the wall crossing formula. In this case all the walls are related by S-duality transformation to the wall corresponding to the decay $(Q,P) \to (Q,0)+(0,P)$. So it is sufficient to study the consequences of the wall crossing formula at this wall. Clearly Q^2 and P^2 given in (4.5.1.4) are uncorrelated. Furthermore in heterotic string theory on T^6 all Q's with a given Q^2 are related by T-duality transformation [35]. The same is true for P. Thus the subtleties mentioned below eq.(4.3.11) are absent, and the behaviour of $\widehat{\Phi}(\check\rho,\check\sigma,\check v)$ near $\check v = 0$ is expected to be given by (4.3.9). In this case both the electric and the magnetic half-BPS partition functions are given by $\eta(\tau)^{-24}$ where η denotes the Dedekind function. Thus we have, as a consequence of the wall crossing formula,

$$\widehat{\Phi}(\check\rho,\check\sigma,\check v) \propto \{\check v^2\,(\eta(\check\rho))^{24}\,(\eta(\check\sigma))^{24} + \mathcal{O}(\check v^4)\}\,. \tag{4.5.1.7}$$

$\eta(\tau)^{24}$ transforms as a modular form of weight 12 under an $SL(2,\ \mathbb{Z})$ transformation. From eqs.(4.3.26) it follows that these $SL(2,\ \mathbb{Z})$ transformations may be regarded as the following symplectic transformations of $\check\rho$, $\check\sigma$, $\check v$

$$\begin{pmatrix}\alpha & 0 & \beta & 0\\ 0 & 1 & 0 & 0\\ \gamma & 0 & \delta & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & p & 0 & q\\ 0 & 0 & 1 & 0\\ 0 & r & 0 & s\end{pmatrix}$$
$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL(2,\ \mathbb{Z}), \quad \begin{pmatrix}p & q\\ r & s\end{pmatrix} \in SL(2,\ \mathbb{Z})\,. \tag{4.5.1.8}$$

Furthermore $\widehat{\Phi}$ should have weight $12-2=10$.

Let us now compare these with the known properties of Φ_{10}. $\Phi_{10}(\check\rho,\check\sigma,\check v)$ is indeed known to have the factorization property (4.5.1.7). Furthermore since Φ_{10} transforms as a modular form of weight 10 under the full $Sp(2,\ \mathbb{Z})$ group, and since (4.5.1.8) are $Sp(2,\ \mathbb{Z})$ matrices, they represent symmetries of Φ_{10}. Thus we see that in this case the full set of symmetries of ϕ_m and ϕ_e lift to symmetries of $\widehat{\Phi}$. It is worth noting that the matrices given in (4.5.1.6) and (4.5.1.8) generate the full $Sp(2,\ \mathbb{Z})$ group. Thus in this case by assuming that the full modular groups of ϕ_e and ϕ_m lift to symmetries of the partition function we could determine the symmetries of the partition function.

Finally let us consider the constraints coming from the knowledge of black hole entropy. In this case the function $g(\tau)$ appearing in (4.4.4) is given by $\eta(\tau)^{24}$. Thus (4.4.6) takes the fom

$$\widehat{\Phi}(\check\rho,\check\sigma,\check v) \propto (2v-\rho-\sigma)^{10}\,\{v^2\,\eta(\rho)^{24}\,\eta(\sigma)^{24} + \mathcal{O}(v^4)\}\,, \tag{4.5.1.9}$$

where $(\check\rho,\check\sigma,\check v)$ and (ρ,σ,v) are related via eq.(4.4.7). The Siegel modular form $\Phi_{10}(\check\rho,\check\sigma,\check v)$ satisfies these properties. In fact since (4.4.7) represents an $Sp(2,\ \mathbb{Z})$ transformation, the property (4.5.1.9) of $\widehat{\Phi}(\check\rho,\check\sigma,\check v)$ follows from the factorization property (4.5.1.7). This however will not be the case in a more generic situation.

4.5.2 Dyons with unit torsion and even $Q^2/2$ in heterotic on T^6

We now consider again heterotic string theory on T^6, but choose the set $\mathcal{A}$ to be collection of (Q, P) of the form:

$$Q = \begin{pmatrix} 0 \\ 2m \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}. \tag{4.5.2.1}$$

This has

$$Q^2 = -4m, \quad P^2 = 2K, \quad Q \cdot P = -J. \tag{4.5.2.2}$$

We note that all the charge vectors have $Q^2/2$ even. Since Q^2 is T-duality invariant, any other charge vector which can be obtained from this one by a T-duality transformation has $Q^2/2$ even. Thus the set $\mathcal{B}$ now consists of charge vectors which have even $Q^2/2$ and arbitrary integer values of $P^2/2$ and $Q \cdot P$. Since this set $\mathcal{B}$ is a subset of charges for which the spectrum was analyzed in §4.5.1 we do not expect to derive any new results. Nevertheless we have chosen this example as this will serve as a useful guide to our analysis in later sections.

We first note that the quantization conditions of Q^2, P^2 and $Q \cdot P$ imply the following periods of the partition function:

$$(\check\rho, \check\sigma, \check v) \to (\check\rho + a_1, \check\sigma + a_2, \check v + a_3), \quad a_1 \in \mathbb{Z}, \quad a_2 \in \frac{1}{2}\mathbb{Z}, \quad a_3 \in \mathbb{Z}. \tag{4.5.2.3}$$

The period along $\check\sigma$ is not an integer. We can remedy this by defining

$$Q_s = Q/2, \qquad \check\sigma_s = 4\check\sigma, \qquad \check v_s = 2\check v. \tag{4.5.2.4}$$

so that $Q_s^2/2$ and $Q_s \cdot P$ are now quantized in half integer units. The periods $(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3)$ of the variables $(\check\rho, \check\sigma_s, \check v_s)$ conjugate to $(P_s^2/2, Q_s^2/2, Q_s \cdot P_s)$ are now integers, given by,

$$\widetilde{a}_1 \in \mathbb{Z}, \quad \widetilde{a}_2 \in 2\,\mathbb{Z}, \quad \widetilde{a}_3 \in 2\,\mathbb{Z}. \tag{4.5.2.5}$$

The dyon partition function in this case can be easily calculated from the one for §4.5.1 by taking into account the evenness of $Q^2/2$. This amounts to adding to the original partition function another term where $\check\sigma$ is shifted by 1/2. Thus we have

$$\frac{1}{\widehat{\Phi}(\check\rho, \check\sigma, \check v)} = \frac{1}{2}\left(\frac{1}{\Phi_{10}(\check\rho, \check\sigma, \check v)} + \frac{1}{\Phi_{10}(\check\rho, \check\sigma + \frac{1}{2}, \check v)}\right), \tag{4.5.2.6}$$

or, in terms of the rescaled variables,

$$\frac{1}{\widehat{\Phi}(\check\rho, \check\sigma, \check v)} = \frac{1}{2}\left(\frac{1}{\Phi_{10}(\check\rho, \frac{1}{4}\check\sigma_s, \frac{1}{2}\check v_s)} + \frac{1}{\Phi_{10}(\check\rho, \frac{1}{4}\check\sigma_s + \frac{1}{2}, \frac{1}{2}\check v_s)}\right). \tag{4.5.2.7}$$

Let us determine the symmetries of this partition function. For this it will be useful to work in terms of the original unscaled variables $(\check\rho, \check\sigma, \check v)$ and at the end go back to the rescaled variables. The first term on the right hand side of (4.5.2.6) has the usual $Sp(2, \mathbb{Z})$ symmetries acting on the variables $(\check\rho, \check\sigma, \check v)$. However not all of these are symmetries of the second term. Given an $Sp(2, \mathbb{Z})$ matrix $\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$, it is a symmetry of the second term provided its action on $(\check\rho, \check\sigma, \check v)$ can be regarded as an $Sp(2, \mathbb{Z})$ action $\begin{pmatrix} a_1' & b_1' & c_1' & d_1' \\ a_2' & b_2' & c_2' & d_2' \\ a_3' & b_3' & c_3' & d_3' \\ a_4' & b_4' & c_4' & d_4' \end{pmatrix}$ on $(\check\rho, \check\sigma + \frac{1}{2}, \check v)$ followed by a translation on $\check\sigma$ by 1/2. Since a translation of $\check\sigma$ by 1/2 can be regarded as a symplectic transformation with the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the above condition takes the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} = \begin{pmatrix} a_1' & b_1' & c_1' & d_1' \\ a_2' & b_2' & c_2' & d_2' \\ a_3' & b_3' & c_3' & d_3' \\ a_4' & b_4' & c_4' & d_4' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (4.5.2.8)$$

This gives

$$\begin{pmatrix} a_1' & b_1' & c_1' & d_1' \\ a_2' & b_2' & c_2' & d_2' \\ a_3' & b_3' & c_3' & d_3' \\ a_4' & b_4' & c_4' & d_4' \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 - \frac{1}{2}b_1 \\ a_2 + \frac{1}{2}a_4 & b_2 + \frac{1}{2}b_4 & c_2 + \frac{1}{2}c_4 & d_2 + \frac{1}{2}(d_4 - b_2) - \frac{1}{4}b_4 \\ a_3 & b_3 & c_3 & d_3 - \frac{1}{2}b_3 \\ a_4 & b_4 & c_4 & d_4 - \frac{1}{2}b_4 \end{pmatrix} . \qquad (4.5.2.9)$$

The coefficients a_i, b_i, c_i and d_i are integers. Requiring that there exist integer a_i', b_i', c_i' and d_i' satisfying the above constraints we get further conditions on a_i, b_i, c_i and d_i. These take the following form:

$$a_4, b_4, c_4, b_1, b_3 \in 2\,\mathbb{Z}, \quad b_4 - 2(d_4 - b_2) \in 4\,\mathbb{Z}\,. \qquad (4.5.2.10)$$

On the other hand the requirement that the original matrix $\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$ is symplectic, together with the first set of conditions given in (4.5.2.10), can be used to show that b_2 and d_4 are both odd. As a result $(b_2 - d_4)$ is even, and hence b_4 must be a multiple of 4 in order to satisfy (4.5.2.10). Thus we have

$$a_4 = 2\widehat{a}_4, \quad b_4 = 4\widehat{b}_4, \quad c_4 = 2\widehat{c}_4, \quad b_1 = 2\widehat{b}_1, \quad b_3 = 2\widehat{b}_3, \quad \widehat{a}_4, \widehat{b}_4, \widehat{c}_4, \widehat{b}_1, \widehat{b}_3 \in \mathbb{Z}\,. \qquad (4.5.2.11)$$

This determines the subgroup of $Sp(2, \mathbb{Z})$ which leaves the individual terms in (4.5.2.6) invariant. To this we must add the additional element corresponding to $\check{\sigma} \to \check{\sigma} + \frac{1}{2}$ which exchanges the two terms in (4.5.2.6). This corresponds to the symplectic transformation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.5.2.12}$$

The full symmetry group is then generated by the matrices:

$$\begin{pmatrix} a_1 & 2\widehat{b}_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & 2\widehat{b}_3 & c_3 & d_3 \\ 2\widehat{a}_4 & 4\widehat{b}_4 & 2\widehat{c}_4 & d_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.5.2.13}$$

We can easily determine how these transformations act on the rescaled variables $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$. This is done with the help of conjugation by the symplectic matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \tag{4.5.2.14}$$

relating $(\check{\rho}, \check{\sigma}, \check{v})$ to $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$. This converts the generators given in (4.5.2.13) to

$$\begin{pmatrix} a_1 & \widehat{b}_1 & c_1 & 2d_1 \\ 2a_2 & b_2 & 2c_2 & 4d_2 \\ a_3 & \widehat{b}_3 & c_3 & 2d_3 \\ \widehat{a}_4 & \widehat{b}_4 & \widehat{c}_4 & d_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.5.2.15}$$

We now note that all the matrices appearing in (4.5.2.15) have the form

$$\begin{pmatrix} * & * & * & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix} \quad \text{mod } 2 , \tag{4.5.2.16}$$

with $*$ denoting an arbitrary integer subject to the condition that (4.5.2.16) describes a symplectic matrix. Furthetmore the set of matrices (4.5.2.16) are closed under matrix multiplication. Thus the group generated by the matrices (4.5.2.15) is contained in the group $\check{G}$ consisting of $Sp(2, \mathbb{Z})$ matrices of the form (4.5.2.16). It is in fact easy to show that the group generated by the matrices (4.5.2.15) is the whole of $\check{G}$, i.e. any element of $\check{G}$ given in (4.5.2.16) can be written as a product of the elements given in (4.5.2.15).

We shall now set aside this result for a while and study the implications of S-duality symmetry and the wall crossing formula on the partition function. The eventual goal is to test the conclusions drawn from the general arguments along the lines of §4.2 and §4.3 against the known results for $\widehat{\Phi}$ given above. It follows from (4.2.2) and (4.5.2.2) that in order that an S-duality transformation generated by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ takes an arbitrary element of the set $\mathcal{B}$ to another element of the set $\mathcal{B}$ we must have b even. Thus S-duality transformations which preserve the set $\mathcal{B}$ take the form:

$$Q \to Q'' = aQ + bP, \quad P \to P'' = cQ + dP, \quad a, c, d \in \mathbb{Z}, \quad b \in 2\,\mathbb{Z}, \quad ad - bc = 1\,. \quad (4.5.2.17)$$

On the original variables $(\check{\rho}, \check{\sigma}, \check{v})$ the associated transformation can be represented by the symplectic matrix (4.2.14). After conjugation by the matrix (4.5.2.14) we get the symplectic matrix acting on the rescaled variables $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$:

$$\begin{pmatrix} d & \widetilde{b} & 0 & 0 \\ \widetilde{c} & a & 0 & 0 \\ 0 & 0 & a & -\widetilde{c} \\ 0 & 0 & -\widetilde{b} & d \end{pmatrix}, \quad a, \widetilde{b} \equiv b/2, d \in \mathbb{Z}, \quad \widetilde{c} \equiv 2c \in 2\,\mathbb{Z}\,. \quad (4.5.2.18)$$

This clearly has the form given in (4.5.2.16). Also the periodicities along the $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$ directions, as given in (4.5.2.5), are represented by the symplectic transformation

$$\begin{pmatrix} 1 & 0 & \widetilde{a}_1 & \widetilde{a}_3 \\ 0 & 1 & \widetilde{a}_3 & \widetilde{a}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{a}_1 \in \mathbb{Z}, \quad \widetilde{a}_2, \widetilde{a}_3 \in 2\,\mathbb{Z}\,. \quad (4.5.2.19)$$

These also are of the form given in (4.5.2.16).

Next we turn to the information obtained from the wall crossing relations. Consider first the wall associated with decay $(Q,P) \to (Q,0) + (0,P)$, – this controls the behaviour of $\widehat{\Phi}$ near $\check{v} = 0$ via eq.(4.3.9). Since $Q^2 = -4m$ and $P^2 = 2K$ can vary independently inside the set $\mathcal{A}$, and since any two charge vectors of the same norm can be related by a T-duality transformation [35], there is no subtlety of the type described below (4.3.11). The inverse of the magnetic partition function ϕ_m entering (4.3.9) is the same as the one that appeared in (4.5.1.7):

$$\phi_m(\check{\rho}) = (\eta(\check{\rho}))^{24}\,. \quad (4.5.2.20)$$

The electric partition function gets modified from the corresponding expression given in (4.5.1.7) due to the fact that we are only including even $Q^2/2$ states. As a result the partition function now becomes $\frac{1}{2}\left(\eta(\check{\sigma})^{-24} + \eta\left(\check{\sigma} + \frac{1}{2}\right)^{-24}\right)$. Replacing $\check{\sigma}$ by $\check{\sigma}_s/4$ we get

$$\phi_e(\check{\sigma})^{-1} = \frac{1}{2}\left(\eta\left(\frac{\check{\sigma}_s}{4}\right)\right)^{-24} + \frac{1}{2}\left(\eta\left(\frac{\check{\sigma}_s}{4} + \frac{1}{2}\right)\right)^{-24}\,. \quad (4.5.2.21)$$

This leads to the following behaviour of $\widehat{\Phi}$ near $\check{v}_s = 0$:

$$\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto \left[\check{v}_s^2 \, \eta(\check{\rho})^{24} \left\{ \left(\eta\left(\frac{\check{\sigma}_s}{4}\right)\right)^{-24} + \left(\eta\left(\frac{\check{\sigma}_s}{4} + \frac{1}{2}\right)\right)^{-24} \right\}^{-1} + \mathcal{O}(\check{v}_s^4) \right] \qquad (4.5.2.22)$$

$\widehat{\Phi}$ given in (4.5.2.7) can be shown to satisfy this property.

$\phi_m(\check{\rho})$ given in (4.5.2.20) transforms as a modular form of weight 12 under

$$\check{\rho} \to \frac{\alpha\check{\rho} + \beta}{\gamma\check{\rho} + \delta}, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z}) . \qquad (4.5.2.23)$$

On the other hand $\phi_e(\check{\sigma})$ given in (4.5.2.21) can be shown to transform as a modular form of weight 12 under

$$\check{\sigma}_s \to \frac{p\check{\sigma}_s + q}{r\check{\sigma}_s + s}, \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma^0(2) , \qquad (4.5.2.24)$$

i.e. $SL(2, \mathbb{Z})$ matrices with q even. (4.5.2.23) and (4.5.2.24) can be represented as symplectic transformations of $(\check{\rho}, \check{\sigma}_s, \check{v}_s)$ generated by the $Sp(2, \mathbb{Z})$ matrices

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix}, \quad q \in 2\,\mathbb{Z}, \quad \alpha, \beta, \gamma, \delta, p, r, s \in \mathbb{Z} . \qquad (4.5.2.25)$$

We now note that these transformations fall in the class given in (4.5.2.16). Thus in this case the modular symmetries of the half-BPS partition function associated with pole at $\check{v} = 0$ are lifted to symmetries of the full partition function.

In this case there is one additional wall which is not related to the wall considered above by the $\Gamma^0(2)$ S-duality transformation (4.5.2.17) acting on the original variables. This corresponds to the decay $(Q, P) \to (Q - P, 0) + (P, P)$. Comparing this with (4.3.1) we see that here

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \qquad (4.5.2.26)$$

Following (4.3.14), (4.3.15) and the relationship (4.5.2.4) between the original variables and the rescaled variables we have

$$\check{\rho}' = \check{\rho} + \frac{1}{4}\check{\sigma}_s + \check{v}_s, \quad \check{\sigma}' = \frac{1}{4}\check{\sigma}_s, \quad \check{v}' = \frac{1}{2}\check{v}_s + \frac{1}{4}\check{\sigma}_s , \qquad (4.5.2.27)$$

$$Q' = Q - P, \quad P' = P . \qquad (4.5.2.28)$$

Thus the pole of the partition function is at $\check{v}_s + \frac{1}{2}\check{\sigma}_s = 0$. Furthermore since from the relations (4.5.2.2) we see that the allowed values of $(Q - P)^2/2 = J + K - 2m$ and $P^2/2 = K$ are uncorrelated and can take arbitrary integer values, it follows from (4.3.22) that at this zero $\widehat{\Phi}$ goes as

$$\widehat{\Phi}(\check{\rho},\check{\sigma}_s,\check{v}_s) \propto (2\check{v}_s+\check{\sigma}_s)^2\,\phi_m\left(\check{\rho}+\frac{1}{4}\check{\sigma}_s+\check{v}_s;1,1\right)\,\phi_e\left(\frac{1}{4}\check{\sigma}_s;1,0\right)+\mathcal{O}\left((2\check{v}_s+\check{\sigma}_s)^4\right)\,. \quad (4.5.2.29)$$

$\phi_m(\tau;1,1)$ denotes the partition function of half-BPS states carrying charges (P,P), with τ being conjugate to the variable $P'^2/2 = P^2/2$. Thus we have $\phi_m(\tau;1,1) = (\eta(\tau))^{24}$. On the other hand $(\phi_e(\tau;1,0))^{-1}$ is the partition function of half BPS states carrying charges $(Q',0) = (Q-P,0)$ with τ being conjugate to $Q'^2/2 = (Q-P)^2/2$. Since $(Q-P)^2/2 = (-2m+K+J)$ can take arbitrary integer values, the corresponding partition function is also given by $\eta(\tau)^{-24}$. Thus we have near $(\check{\sigma}_s+2\check{v}_s)=0$

$$\widehat{\Phi}(\check{\rho},\check{\sigma}_s,\check{v}_s) \propto \left\{(2\check{v}_s+\check{\sigma}_s)^2\eta\left(\check{\rho}+\frac{1}{4}\check{\sigma}_s+\check{v}_s\right)^{24}\eta\left(\frac{\check{\sigma}_s}{4}\right)^{24}+\mathcal{O}((2\check{v}_s+\check{\sigma}_s)^4)\right\} \quad (4.5.2.30)$$

$\widehat{\Phi}$ given in (4.5.2.7) can be shown to satisfy this property.

$\phi_m(\tau;1,1)$ transforms as a modular form of weight 12 under $\tau\to(\alpha_1\tau+\beta_1)/(\gamma_1\tau+\delta_1)$ with $\alpha_1,\beta_1,\gamma_1,\delta_1\in\mathbb{Z}$, $\alpha_1\delta_1-\beta_1\gamma_1=1$. On the other hand $\phi_e(\tau;1,0)$ transforms as a modular form of weight 12 under $\tau\to(p_1\tau+q_1)/(r_1\tau+s_1)$ with $p_1,q_1,r_1,s_1\in\mathbb{Z}$, $p_1s_1-q_1r_1=1$. Using (4.3.29), (4.3.30) and (4.5.2.14) we see that the the action of these transformations on the variables $(\check{\rho},\check{\sigma}_s,\check{v}_s)$ may be represented by the symplectic matrices

$$\begin{pmatrix}\alpha_1 & (\alpha_1-1)/2 & \beta_1 & 0\\ 0 & 1 & 0 & 0\\ \gamma_1 & \gamma_1/2 & \delta_1 & 0\\ \gamma_1/2 & \gamma_1/4 & (\delta_1-1)/2 & 1\end{pmatrix},\quad \begin{pmatrix}1 & (1-p_1)/2 & q_1 & -2q_1\\ 0 & p_1 & -2q_1 & 4q_1\\ 0 & 0 & 1 & 0\\ 0 & r_1/4 & (1-s_1)/2 & s_1\end{pmatrix}. \quad (4.5.2.31)$$

By comparing with the matrices given in (4.5.2.16) we see however that the transformations (4.5.2.31) generates symmetries of the full partition function only after we impose the additional constraints

$$r_1,\gamma_1\in 4\,\mathbb{Z}\,. \quad (4.5.2.32)$$

Thus here we encounter a case where only a subset of the symmetries of the partition function near a pole is lifted to a full symmetry of the partition function. By examining the details carefully one discovers that in this case the pole comes from the first term in (4.5.2.7). Whereas this term displays the full symmetry given in (4.5.2.31), requiring that the other term also transforms covariantly under this symmetry generates the additional restrictions given in (4.5.2.32).

Finally we turn to the constraint from black hole entropy. As in §4.5.1, in this case we have $g(\tau)=\eta(\tau)^{24}$ in (4.4.4). Thus (4.4.6) takes the form

$$\widehat{\Phi}(\check{\rho},\check{\sigma},\check{v}) \propto (2v-\rho-\sigma)^{10}\left\{v^2\,\eta(\rho)^{24}\,\eta(\sigma)^{24}+\mathcal{O}(v^4)\right\}, \quad (4.5.2.33)$$

where $(\check\rho, \check\sigma, \check v)$ and (ρ, σ, v) are related via (4.4.7). $\widehat\Phi(\check\rho, \check\sigma, \check v)$ given in (4.5.2.6) can be shown to satisfy this property. In fact the relevent pole of $\widehat\Phi^{-1}$ comes from the first term on the right hand side of (4.5.2.6). The location of the zeroes of Φ_{10} are given in (4.6.8), and it follows from this that the second term does not have a pole at $v = 0$.

4.5.3 Dyons of torsion 2 in heterotic string theory on T^6

We consider again heterotic string theory on T^6 and take the set $\mathcal{A}$ to consist of charge vectors of the form

$$Q = \begin{pmatrix} 1 \\ 2m+1 \\ 1 \\ 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 2K+1 \\ 2J+1 \\ 1 \\ -1 \end{pmatrix}, \qquad m, K, J \in \mathbb{Z}\,. \tag{4.5.3.1}$$

This has

$$Q^2 = 4(m+1), \quad P^2 = 4(K-J), \quad Q\cdot P = 2(K+J-m+1)\,. \tag{4.5.3.2}$$

Furthermore gcd$\{Q_iP_j - Q_jP_i\}$=2. Thus we have a family of charge vectors with torsion 2. It was shown in [40,54] that for $r = 2$ there are three T-duality orbits for given $(Q^2, P^2, Q\cdot P)$ – in the first Q is twice a primitive lattice vector, in the second P is twice a primitive lattice vector and in the third both Q and P are primitive but $Q\pm P$ are twice primitive lattice vectors. The dyon charges given in (4.5.3.1) are clearly of the third kind. In the notation of [40] the discrete T-duality invariants of these charges are $(r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1)$. Note that as we vary m, J and K, $Q^2/2$ and $P^2/2$ take all possible even values and $Q\cdot P$ takes all possible values subject to the restriction that $Q \pm P$ are twice primitive lattice vectors. The latter condition requires $Q\cdot P$ to be even and $Q\cdot P - \frac{1}{2}Q^2 - \frac{1}{2}P^2$ to be a multiple of four. It now follows from the result of [40,54] that the T-duality orbit $\mathcal{B}$ of the set $\mathcal{A}$ consists of all the pairs (Q, P) with $(r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1)$ and even values of $Q^2/2$, $P^2/2$.

Since $Q^2/2$, $P^2/2$ and $Q\cdot P$ are all even and $Q^2 + P^2 + 2Q\cdot P$ is a multiple of 8, it is natural to introduce new charge vectors and variables

$$Q_s \equiv Q/2, \quad P_s \equiv P/2, \quad \check\rho_s \equiv 4\check\rho, \quad \check\sigma_s \equiv 4\check\sigma, \quad \check v_s \equiv 4\check v\,, \tag{4.5.3.3}$$

so that we have

$$\frac{1}{2}Q_s^2 = \frac{1}{2}(m+1), \quad \frac{1}{2}P_s^2 = \frac{1}{2}(K-J), \quad Q_s\cdot P_s = \frac{1}{2}(K+J-m+1)\,, \tag{4.5.3.4}$$

quantized in half integer units subject to the constraint that

$$\frac{1}{2}Q_s^2 + \frac{1}{2}P_s^2 + Q_s\cdot P_s = K+1\,, \tag{4.5.3.5}$$

is an integer. Since $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ are conjugate to $(P_s^2/2, Q_s^2/2, Q_s \cdot P_s)$, the partition function (4.1.1) will be periodic under

$$(\check{\rho}_s, \check{\sigma}_s, \check{v}_s) \to (\check{\rho}_s + 2, \check{\sigma}_s, \check{v}_s),\ (\check{\rho}_s, \check{\sigma}_s + 2, \check{v}_s),\ (\check{\rho}_s, \check{\sigma}_s, \check{v}_s + 2),\ (\check{\rho}_s + 1, \check{\sigma}_s + 1, \check{v}_s + 1)\,. \quad (4.5.3.6)$$

The group generated by these transformations can be collectively represented by symplectic matrices of the form

$$\begin{pmatrix} 1 & 0 & \widetilde{a}_1 & \widetilde{a}_3 \\ 0 & 1 & \widetilde{a}_3 & \widetilde{a}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3 \in \mathbb{Z}, \quad \widetilde{a}_1 + \widetilde{a}_2, \widetilde{a}_2 + \widetilde{a}_3, \widetilde{a}_1 + \widetilde{a}_3 \in 2\,\mathbb{Z}\,, \qquad (4.5.3.7)$$

acting on the variables $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$. For future reference we note that the change of variables from $(\check{\rho}, \check{\sigma}, \check{v})$ to $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ can be regarded as a symplectic transformation of the form

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. \qquad (4.5.3.8)$$

We now need to determine the subgroup of the S-duality group that leaves the set $\mathcal{B}$ invariant. If we did not have the restriction that $Q^2/2$ and $P^2/2$ are even, then this subgroup would consist of $SL(2, \mathbb{Z})$ matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ subject to the restriction $a + b \in 2\,\mathbb{Z} + 1$ and $c + d \in 2\,\mathbb{Z} + 1$ [40], – these conditions guarantee that the new charge vectors (Q'', P'') are each primitive and hence have the same set of discrete T-duality invariants ($r_1 = 1, r_2 = 1, r_3 = 2, u_1 = 1$). We shall now argue that the same subgroup also leaves the set $\mathcal{B}$ invariant. For this we need to note that if we begin with a (Q, P) for which $Q^2/2$, $P^2/2$ and $Q \cdot P$ are all even then their S-duality transforms given in (4.2.2) will automatically have the same properties. Thus requiring the transformed pair (Q'', P'') to have even $Q''^2/2$ and $P''^2/2$, as is required for (Q'', P'') to belong to the set $\mathcal{B}$, does not put any additional restriction on the S-duality transformations. Since both Q and P are scaled by the same amount to get the rescaled charges Q_s and P_s, the S-duality group action on (Q_s, P_s) is identical to that on (Q, P) and hence its action on $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ is identical to that on $(\check{\rho}, \check{\sigma}, \check{v})$. Using (4.2.14) we see that the representations of these symmetries as symplectic matrices are given by

$$\begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad a + c \in 2\,\mathbb{Z} + 1, \quad b + d \in 2\,\mathbb{Z} + 1\,, \qquad (4.5.3.9)$$

acting on the variables $(\check{\rho}, \check{\sigma}, \check{v})$ and also on $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$.

Next we turn to the constraints from the wall crossing formula. We begin with the wall associated with the decay $(Q,P) \to (Q,0)+(0,P)$, – this controls the behaviour of $\widehat{\Phi}$ at $\check{v}=0$. The analysis is straightforward. We note that both electric and magnetic partition functions involve summing over all possible even $Q^2/2$ and $P^2/2$ values. An analysis similar to the one leading to (4.5.2.21) give

$$\phi_e(\check{\sigma})^{-1} = \frac{1}{2}\left\{\eta\left(\frac{\check{\sigma}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\sigma}_s+2}{4}\right)^{-24}\right\}, \tag{4.5.3.10}$$

and

$$\phi_m(\check{\rho})^{-1} = \frac{1}{2}\left\{\eta\left(\frac{\check{\rho}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\rho}_s+2}{4}\right)^{-24}\right\}. \tag{4.5.3.11}$$

Thus we have

$$\widehat{\Phi}(\check{\rho},\check{\sigma},\check{v}) \sim \left[\check{v}_s^2\left\{\eta\left(\frac{\check{\sigma}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\sigma}_s+2}{4}\right)^{-24}\right\}^{-1}\left\{\eta\left(\frac{\check{\rho}_s}{4}\right)^{-24} + \eta\left(\frac{\check{\rho}_s+2}{4}\right)^{-24}\right\}^{-1} + \mathcal{O}(\check{v}_s^4)\right], \tag{4.5.3.12}$$

near $\check{v}=0$. One can easily verify that the functions $\phi_e(\check{\sigma})$ and $\phi_m(\check{\rho})$ transform as modular forms of weight 12 under the transformation $\check{\sigma}_s \to (p\check{\sigma}_s+q)/(r\check{\sigma}_s+s)$ and $\check{\rho}_s \to (\alpha\check{\rho}_s+\beta)/(\gamma\check{\rho}_s+\delta)$ with $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma^0(2)$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^0(2)$. These can be regarded as symplectic transformations of the form

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix},$$
$$\alpha\delta-\beta\gamma=1, \quad ps-qr=1, \quad p,r,s,\alpha,\gamma,\delta \in \mathbb{Z}, \quad q,\beta \in 2\,\mathbb{Z}, \tag{4.5.3.13}$$

acting on $(\check{\rho}_s,\check{\sigma}_s,\check{v}_s)$.

Next we consider the wall associated with the decay $(Q,P) \to ((Q-P)/2,(P-Q)/2) + ((Q+P)/2,(Q+P)/2)$. From (4.3.1) we see that the associated matrix can be taken to be

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{4.5.3.14}$$

According to (4.3.2) this controls the behaviour of $\widehat{\Phi}(\check{\rho},\check{\sigma},\check{v})$ near its zero at

$$\check{\rho}-\check{\sigma}=0\,. \tag{4.5.3.15}$$

Following the procedure outlined in eqs.(4.3.14)-(4.3.22) we can find the coefficient of $(\check{\rho}-\check{\sigma})^2$ in the expression for $\widehat{\Phi}$. One can see from (4.5.3.1) that in this case $\frac{1}{2}((Q+P)/2)^2$ and

$\frac{1}{2}((Q-P)/2)^2$ can take all possible independent integer values $(K+1)$ and $(m-J)$ respectively. We find from (4.3.23) that the inverses of the relevant half-BPS partition functions are:

$$\phi_e(\tau; a_0, c_0) = \eta(2\tau)^{24}, \quad \phi_m(\tau; b_0, d_0) = \eta(2\tau)^{24}. \tag{4.5.3.16}$$

The factor of 2 in the argument of η is due to the fact that $Q'^2/2 = (d_0Q - b_0P)^2/2 = (Q-P)^2/4$ and $P'^2/2 = (-c_0Q + a_0P)^2/2 = (Q+P)^2/4$ entering in (4.3.23) are twice the usual integer normalized combinations $\frac{1}{8}(Q \pm P)^2$. This gives, from (4.3.14), (4.3.22) and (4.5.3.3)

$$\widehat{\Phi}(\check\rho, \check\sigma, \check v) \sim \left\{(\check\rho_s - \check\sigma_s)^2\, \eta((\check\rho_s + \check\sigma_s - 2\check v_s)/4)^{24}\, \eta((\check\rho_s + \check\sigma_s + 2\check v_s)/4)^{24} + \mathcal{O}((\check\rho_s - \check\sigma_s)^4)\right\}, \tag{4.5.3.17}$$

near $\check\rho_s \simeq \check\sigma_s$. Since $\eta(2\tau)$ transforms covariantly under $\tau \to (\alpha\tau + \frac{1}{2}\beta)/(2\gamma\tau + \delta)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$, both ϕ_e and ϕ_m have full $SL(2, \mathbb{Z})$ symmetry. Using (4.3.29), (4.3.30) and (4.5.3.8) to represent them as symplectic transformations on the variables $(\check\rho_s, \check\sigma_s, \check v_s)$ we get the following two sets of symplectic matrices:

$$\frac{1}{2}\begin{pmatrix} \alpha_1+1 & \alpha_1-1 & 2\beta_1 & 2\beta_1 \\ \alpha_1-1 & \alpha_1+1 & 2\beta_1 & 2\beta_1 \\ \gamma_1/2 & \gamma_1/2 & \delta_1+1 & \delta_1-1 \\ \gamma_1/2 & \gamma_1/2 & \delta_1-1 & \delta_1+1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} p_1+1 & -p_1+1 & 2q_1 & -2q_1 \\ -p_1+1 & p_1+1 & -2q_1 & 2q_1 \\ r_1/2 & -r_1/2 & s_1+1 & -s_1+1 \\ -r_1/2 & r_1/2 & -s_1+1 & s_1+1 \end{pmatrix},$$
$$\alpha_1, \beta_1, \gamma_1, \delta_1, p_1, q_1, r_1, s_1 \in \mathbb{Z}, \quad \alpha_1\delta_1 - \beta_1\gamma_1 = p_1s_1 - q_1r_1 = 1. \tag{4.5.3.18}$$

Next we turn to the wall corresponding to the decay $(Q,P) \to (Q-P, 0) + (P,P)$. This corresponds to the choice

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{4.5.3.19}$$

and is associated with the zero of $\widehat{\Phi}$ at

$$\check\sigma + \check v = 0. \tag{4.5.3.20}$$

Since $(Q-P)^2/8 = m-J$ and $P^2/4 = K-J$ can take independent integer values, we should be able to use (4.3.9), (4.3.10). The behaviour of $\widehat{\Phi}$ near this zero is however somewhat ambiguous since one of the decay products – the state carrying charge $(Q-P, 0)$ – is not a primitive dyon. As a result the index associated with this state is ambiguous.[11] Nevertheless if we go ahead

[11]For half-BPS states in $\mathcal{N} = 2$ supersymmetric theories a modification of the wall crossing formula for such non-primitive decays has been suggested in [46]. It is not clear *a priori* how to modify it for the decays of quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories. In §4.6 we shall propose a formula for the partition function of the states being studied in this section and examine it to find what the modification should be.

and assume the naive index that follows from tree level spectrum of elementary string states, we get the following factorization behaviour of $\widehat{\Phi}$:

$$\widehat{\Phi}(\check{\rho},\check{\sigma},\check{v}) \overset{?}{\sim} \left\{(\check{\sigma}_s+\check{v}_s)^2\phi_e\left(\frac{\check{\sigma}_s}{4};1,0\right)\phi_m\left(\frac{\check{\rho}_s+\check{\sigma}_s+2\check{v}_s}{4};1,1\right)+\mathcal{O}((\check{\sigma}_s+\check{v}_s)^4)\right\} \quad \text{for } \check{v}\simeq-\check{\sigma}\,, \tag{4.5.3.21}$$

with

$$\begin{aligned}\phi_e\left(\frac{\tau}{4};1,0\right) &= \frac{1}{4}\left\{\eta(\tau/4)^{-24}+\eta((\tau+1)/4)^{-24}+\eta((\tau+2)/4)^{-24}+\eta((\tau+3)/4)^{-24}\right\}^{-1},\\ \phi_m\left(\frac{\tau}{4};1,1\right) &= \frac{1}{2}\left\{\eta(\tau/4)^{-24}+\eta(\tau+2)/4)^{-24}\right\}^{-1}.\end{aligned} \tag{4.5.3.22}$$

$\phi_e(\tau/4)$ has duality symmetries of the form $\tau\to(p_2\tau+q_2)/(r_2\tau+s_2)$ with $\begin{pmatrix}p_2 & q_2\\ r_2 & s_2\end{pmatrix}\in\Gamma_0(2)$. On the other hand $\phi_m(\tau/4)$ has duality symmetries of the form $\tau\to(\alpha_2\tau+\beta_2)/(\gamma_2\tau+\delta_2)$ with $\begin{pmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{pmatrix}\in\Gamma^0(2)$. Using (4.3.29), (4.3.30) and (4.5.3.8) we find that the modular properties in this factorized limit correspond to the following symplectic transformations acting on $(\check{\rho}_s,\check{\sigma}_s,\check{v}_s)$

$$\begin{pmatrix}\alpha_2 & \alpha_2-1 & \beta_2 & 0\\ 0 & 1 & 0 & 0\\ \gamma_2 & \gamma_2 & \delta_2 & 0\\ \gamma_2 & \gamma_2 & \delta_2-1 & 1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}1 & 1-p_2 & q_2 & -q_2\\ 0 & p_2 & -q_2 & q_2\\ 0 & 0 & 1 & 0\\ 0 & r_2 & 1-s_2 & s_2\end{pmatrix},$$
$$\alpha_2\delta_2-\beta_2\gamma_2=1=p_2s_2-r_2q_2,\quad \alpha_2,\gamma_2,\delta_2,p_2,q_2,s_2\in\mathbb{Z},\quad \beta_2,r_2\in 2\,\mathbb{Z}\,. \tag{4.5.3.23}$$

We can now try to see if all the symplectic transformation matrices (4.5.3.7), (4.5.3.9), (4.5.3.13), (4.5.3.18) and (4.5.3.23), representing possible symmetries of $\widehat{\Phi}$, fit into some subgroup of $Sp(2,\mathbb{Z})$ defined by some congruence condition. As it stands there does not seem to be a simple congruence subgroup of $Sp(2,\mathbb{Z})$ that fits all the matrices since some of these matrices do not even have integer entries. However if we restrict γ and r in (4.5.3.13) to be even, i.e. assume that only a $\Gamma(2)\times\Gamma(2)$ subgroup of the symmetry group $\Gamma^0(2)\times\Gamma^0(2)$ of the $\check{v}\to 0$ limit survives as a symmetry of the full partition function, and restrict γ_1 and r_1 in (4.5.3.18) to be multiples of 4, i.e. assume that only a $\Gamma_0(4)\times\Gamma_0(4)$ subgroup of the $\check{\rho}_s\to\check{\sigma}_s$ limit survives as a symmetry of the full partition function, then there is a simple congruence subgroup of $Sp(2,\mathbb{Z})$ into which all the matrices fit:

$$\begin{pmatrix}1+u & u & v & v\\ u & 1+u & v & v\\ w & w & 1+u & u\\ w & w & u & 1+u\end{pmatrix} \mod 2,\quad u,v,w=0,1\,. \tag{4.5.3.24}$$

We speculate that this could be the symmetry group of the dyon partition function under consideration.

Finally we turn to the constraint from black hole entropy. As in §4.5.1, in this case we have $g(\tau) = \eta(\tau)^{24}$ in (4.4.4). Thus (4.4.6) takes the form

$$\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto (2v - \rho - \sigma)^{10} \left\{ v^2 \, \eta(\rho)^{24} \, \eta(\sigma)^{24} + \mathcal{O}(v^4) \right\}, \tag{4.5.3.25}$$

where (ρ, σ, v) and $(\check{\rho}, \check{\sigma}, \check{v})$ are related via (4.4.7).

Before concluding this section we would like to note that we can easily extend the analysis of this section to the complementary subset of torsion 2 dyons with Q, P primitive and $Q^2/2$ and $P^2/2$ odd. For this we consider six dimensional electric and magnetic charge vectors with metric

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & I_2 & 0 \end{pmatrix}, \tag{4.5.3.26}$$

and take the set $\mathcal{A}$ to be the collection of charge vectors (Q, P) with

$$Q = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2m+1 \\ 1 \\ 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 \\ 1 \\ 2K+1 \\ 2J+1 \\ 1 \\ -1 \end{pmatrix}, \qquad m, K, J \in \mathbb{Z}. \tag{4.5.3.27}$$

This has

$$\frac{Q^2}{2} = 2m + 1, \quad \frac{P^2}{2} = 2(K - J) + 1, \quad Q \cdot P = 2(K + J - m + 1). \tag{4.5.3.28}$$

Thus we have $Q^2/2$ and $P^2/2$ odd and $Q \cdot P$ even. Furthermore we still have the constraint that $Q^2 + P^2 + 2Q \cdot P$ is a multiple of 8. Thus with $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ defined as in (4.5.3.3), the partition function is antiperiodic under $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s) \to (\check{\rho}_s + 2, \check{v}_s, \check{\sigma}_s), (\check{\rho}_s, \check{\sigma}_s + 2, \check{v}_s)$ and periodic under $(\check{\rho}_s, d_0\check{\sigma}_s, \check{v}_s) \to (\check{\rho}_s, \check{\sigma}_s, \check{v}_s + 2), (\check{\rho}_s + 1, \check{\sigma}_s + 1, \check{v}_s + 1)$. We can now repeat the analysis of this section for this set of dyons. The results are more or less identical except for some relative minus signs between the terms in the curly brackets in eqs.(4.5.3.10)-(4.5.3.12) and the second equation in (4.5.3.22).

4.5.4 Dyons in $\mathbb{Z}_2$ CHL orbifold with twisted sector electric charge

We now consider a $\mathbb{Z}_2$ CHL orbifold defined as follows [47, 48]. We begin with $E_8 \times E_8$ heterotic string theory on $T^4 \times S^1 \times \widetilde{S}^1$ with S^1 and $\widetilde{S}^1$ labelled by coordinates with period 4π and 2π respectively, and take a quotient of the theory by a $\mathbb{Z}_2$ symmetry that involves 2π shift along S^1 together with an exchange of the two E_8 factors. In the four dimensional subspace of charges given in (4.5.1.1), now the momentum n' along S^1 is quantized in units of 1/2 whereas the Kaluza-Klein monopole charge N' along S^1 is quantized in units of 2 [15]. We shall take the set $\mathcal{A}$ to be consisting of charge vectors of the form

$$Q = \begin{pmatrix} 0 \\ m/2 \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \qquad m, K, J \in \mathbb{Z}\,. \tag{4.5.4.1}$$

For this state we have

$$Q^2 = -m, \quad P^2 = 2K, \quad Q \cdot P = -J\,. \tag{4.5.4.2}$$

As usual we denote by $\mathcal{B}$ the set of all (Q, P) which are related to the ones given in (4.5.4.2) by a T-duality transformation. Since $Q^2/2$, $P^2/2$ and $Q \cdot P$ are quantized in units of 1/2, 1 and 1 respectively, $\widehat{\Phi}$ satisfies the periodicity conditions (4.2.16) with

$$a_1 \in \mathbb{Z}, \quad a_2 \in 2\,\mathbb{Z}, \quad a_3 \in \mathbb{Z}\,. \tag{4.5.4.3}$$

Comparison of (4.5.4.1) and (4.5.1.1) shows that the winding charge $-w'$ along S^1 is 1 for this state. Thus it represents a twisted sector state.

Our next task is to determine the subgroup of the S-duality group that leaves the set $\mathcal{B}$ invariant. In this case the full S-duality group is $\Gamma_0(2)$, generated by matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, d \in \mathbb{Z}$, $c \in 2\,\mathbb{Z}$, $ad - bc = 1$. It was shown in [23] that the set $\mathcal{B}$ is closed under the full S-duality group. Thus the full S-duality group must be a symmetry of the partition function.

We now turn to the constraints from the wall crossing formula. Consider first the wall associated with the decay $(Q, P) \to (Q, 0) + (0, P)$, – this in fact is the only case we need to analyze since all the walls are related to this one by S-duality transformation [19]. First of all note from (4.5.4.1), (4.5.4.2) that for a given $Q^2 = -m$ the charge vector $Q \in \mathcal{A}$ is fixed uniquely. Thus the index of half-BPS states with charge $(Q, 0)$ can be regarded as a function of Q^2. On the other hand for a given $P^2 = 2K$ there is a family of $P \in \mathcal{A}$ labelled by J, but these can be transformed to the vector corresponding to $J = 0$ by the T-duality transformation matrix [23] $\begin{pmatrix} 1 & 0 & 0 & J \\ 0 & 1 & -J & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Thus the index of the charge vector $(0, P)$ can also be expressed as a function of P^2. Finally we see from (4.5.4.2) that the allowed values of Q^2 and P^2 are uncorrelated. Thus we can use eqs.(4.3.9), (4.3.10) to extract the behaviour of $\widehat{\Phi}$ near $\check{v} = 0$. The electric partition function can be calculated by examining the spectrum of twisted sector states in the heterotic string theory [55–58]. On the other hand the magnetic partition function can be calculated by examing the spectrum of D1-D5 system in a dual type

IIB description of the theory [23]. The results are

$$\phi_e(\check\sigma) = \eta(\check\sigma)^8\eta(\check\sigma/2)^8\,, \qquad \phi_m(\check\rho) = \eta(\check\rho)^8\eta(2\check\rho)^8\,. \tag{4.5.4.4}$$

Eq.(4.3.9) then gives, near $\check v = 0$,

$$\widehat\Phi(\check\rho,\check\sigma,\check v) \propto \left\{\check v^2\eta(\check\sigma)^8\eta(\check\sigma/2)^8\eta(\check\rho)^8\eta(2\check\rho)^8 + \mathcal{O}(\check v^4)\right\}\,. \tag{4.5.4.5}$$

$\phi_e(\check\sigma)$ and $\phi_m(\check\rho)$ transform as modular forms of weight 8 under

$$\check\sigma \to \frac{p\check\sigma+q}{r\check\sigma+s}, \quad p,r,s\in\mathbb{Z}, \quad q\in 2\,\mathbb{Z}, \quad ps-qr=1\,, \tag{4.5.4.6}$$

and

$$\check\rho \to \frac{\alpha\check\rho+\beta}{\gamma\check\rho+\delta}, \quad \alpha,\beta,\delta\in\mathbb{Z}, \quad \gamma\in 2\,\mathbb{Z}, \quad \alpha\delta-\beta\gamma=1\,. \tag{4.5.4.7}$$

The corresponding groups are $\Gamma^0(2)$ and $\Gamma_0(2)$ respectively. Thus from (4.2.14), (4.2.17), (4.3.26) we see that if (4.5.4.6) and (4.5.4.7) lift to symmetries of the full partition function then the partition function transforms as a modular form of weight 6 under the $Sp(2,\mathbb{Z})$ transformations of the form

$$\begin{pmatrix} d & b & 0 & 0\\ c & a & 0 & 0\\ 0 & 0 & a & -c\\ 0 & 0 & -b & d\end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & a_1 & a_3\\ 0 & 1 & a_3 & a_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 & \beta & 0\\ 0 & 1 & 0 & 0\\ \gamma & 0 & \delta & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & p & 0 & q\\ 0 & 0 & 1 & 0\\ 0 & r & 0 & s\end{pmatrix}, \tag{4.5.4.8}$$

with

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(2), \quad \begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(2), \quad \begin{pmatrix} p & q\\ r & s\end{pmatrix}\in\Gamma^0(2), \quad a_1,a_3\in\mathbb{Z}, \quad a_2\in 2\,\mathbb{Z}\,. \tag{4.5.4.9}$$

All the $Sp(2,\mathbb{Z})$ matrices in (4.5.4.8) subject to the constraints (4.5.4.9) have the form

$$\begin{pmatrix} 1 & * & * & *\\ 0 & 1 & * & 0\\ 0 & 0 & 1 & 0\\ 0 & * & * & 1\end{pmatrix} \quad \text{mod } 2\,. \tag{4.5.4.10}$$

Furthermore the set of matrices (4.5.4.10) are closed under matrix multiplication. Thus the group generated by the set of $Sp(2,\mathbb{Z})$ matrices (4.5.4.8) subject to the condition (4.5.4.9) is contained in the group $\check G$ of $Sp(2,\mathbb{Z})$ matrices (4.5.4.10).

All the symmetries listed in (4.5.4.8) are indeed symmetries of the dyon partition function of this model proposed in [12] and proved in [15]. Furthermore near $\check v = 0$ the partition function is known to have the factorization property given in (4.5.4.5) [19,22,38]. One question that one can ask is: do the matrices given in (4.5.4.8) generate the full symmetry group of the partition function (which is known in this case)? It turns out that the answer is no. This group does not include the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \tag{4.5.4.11}$$

since this is not of the form given in (4.5.4.10). This generates the transformation

$$\check\rho \to \frac{\check\rho}{(1-\check v)^2 - \check\sigma\check\rho}, \quad \check\sigma \to \frac{\check\sigma}{(1-\check v)^2 - \check\sigma\check\rho}, \quad \check v \to \frac{\check\rho\check\sigma + \check v(1-\check v)}{(1-\check v)^2 - \check\sigma\check\rho}, \tag{4.5.4.12}$$

and is known to be a symmetry of the partition function.[12]

Finally we turn to the constraints from black hole entropy. In this case the function $g(\tau)$ is given by [53, 55]:

$$g(\tau) = \eta(\tau)^8\,\eta(2\tau)^8\,. \tag{4.5.4.13}$$

Thus (4.4.6) takes the fom

$$\widehat\Phi(\check\rho,\check\sigma,\check v) \propto (2v-\rho-\sigma)^6\,\{v^2\,\eta(\rho)^8\eta(2\rho)^8\,\eta(\sigma)^8\eta(2\sigma)^8 + \mathcal{O}(v^4)\}\,, \tag{4.5.4.14}$$

where (ρ,σ,v) and $(\check\rho,\check\sigma,\check v)$ are related via (4.4.7). The dyon partition function of the $\mathbb{Z}_2$ CHL model is known to satisfy this property. In fact historically this is the property that was used to guess the form of the partition function [12].

This analysis can be easily generalized to the dyons of $\mathbb{Z}_N$ CHL orbifolds carrying twisted sector electric charges.

4.5.5 Dyons in $\mathbb{Z}_2$ CHL model with untwisted sector electric charge

We again consider the $\mathbb{Z}_2$ CHL model introduced in §4.5.4, but now take the set $\mathcal{A}$ to consist of dyons with charge vectors

$$Q = \begin{pmatrix} 0 \\ (2m+1)/2 \\ 0 \\ -2 \end{pmatrix}, \qquad P = \begin{pmatrix} 2K+1 \\ J \\ 1 \\ 0 \end{pmatrix}, \qquad m, K, J \in \mathbb{Z}\,. \tag{4.5.5.1}$$

Since $w' = -2$ for this state, it represents an untwisted sector state. For this state we have

$$Q^2 = -2(2m+1), \quad P^2 = 2(2K+1), \quad Q\cdot P = -2J\,. \tag{4.5.5.2}$$

[12] This is the symmetry refered to as $g_3(1,0)$ in [12] in a different representation.

Note that Q and P are both primitive. Since $Q \cdot P$ is quantized in units of 2, we shall define

$$Q_s = \frac{Q}{\sqrt{2}}, \quad P_s = \frac{P}{\sqrt{2}}, \quad \check{\rho}_s = 2\check{\rho}, \quad \check{\sigma}_s = 2\check{\sigma}, \quad \check{v}_s = 2\check{v}\,. \tag{4.5.5.3}$$

Thus we have

$$Q_s^2 = -(2m+1)\,, \quad P_s^2 = 2K+1, \quad Q_s \cdot P_s = -J\,. \tag{4.5.5.4}$$

Since $Q_s^2/2$ is quantized in units of $1/2$, we expect the partition function to have $\check{\sigma}_s$ period 2. However except for an overall additive factor of $1/2$, $Q_s^2/2$ is actually quantized in integer units. Thus the partition function has the additional property that it is odd under $\check{\sigma}_s \to \check{\sigma}_s + 1$. Similarly since P_s^2 is an odd integer, the partition function picks up a minus sign under $\check{\rho}_s \to \check{\rho}_s + 1$. We shall call these symmetries of $\widehat{\Phi}$. Finally since $Q_s \cdot P_s$ is quantized in integer units, the period in the $\check{v}_s$ direction is also unity. The corresponding symplectic transformations acting on $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$ are of the form

$$\begin{pmatrix} 1 & 0 & \widetilde{a}_1 & \widetilde{a}_3 \\ 0 & 1 & \widetilde{a}_3 & \widetilde{a}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3 \in \mathbb{Z}\,. \tag{4.5.5.5}$$

Under this transformation the partition function picks up a multiplier factor of $(-1)^{\widetilde{a}_1 + \widetilde{a}_2}$.

Our next task is to determine the subgroup of the S-duality group $\Gamma_0(2)$ that leaves the set $\mathcal{B}$ – defined as the T-duality orbit of $\mathcal{A}$ – invariant. For this let us apply the S-duality transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ on the charge vector (4.5.5.1). This gives

$$Q' = aQ + bP = \begin{pmatrix} b(2K+1) \\ (2m+1)a/2 + bJ \\ b \\ -2a \end{pmatrix}, \qquad P' = cQ + dP = \begin{pmatrix} d(2K+1) \\ (2m+1)c/2 + dJ \\ d \\ -2c \end{pmatrix}. \tag{4.5.5.6}$$

We need to choose a, b, c, d such that (4.5.5.6) is inside the set $\mathcal{B}$, i.e. it can be brought to the form (4.5.5.1) after a T-duality transformation. The T-duality transformations acting within this four dimensional subspace are generated by matrices of the form [23]:

$$\begin{pmatrix} n_1 & -m_1 & & \\ -l_1 & k_1 & & \\ & & k_1 & l_1 \\ & & m_1 & n_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k_2 & & & -l_2 \\ & k_2 & l_2 & \\ & m_2 & n_2 & \\ -m_2 & & & n_2 \end{pmatrix}, \quad \begin{pmatrix} k_i & l_i \\ m_i & n_i \end{pmatrix} \in \Gamma_0(2)\,. \tag{4.5.5.7}$$

Now suppose b in (4.5.5.6) is even. Then we can apply a T-duality transformation on the charge vector given in (4.5.5.6) with the matrix

$$\begin{pmatrix} 1 & & & l_0 \\ & 1 & -l_0 & \\ & 0 & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} d & -2c & & \\ -b/2 & a & & \\ & & a & b/2 \\ & & 2c & d \end{pmatrix}, \qquad l_0 \equiv \frac{1}{2} bd(2K+1) - \frac{c}{2}\{(2m+1)a + 2bJ\}\,. \tag{4.5.5.8}$$

It is straightfoward to verify that this brings (4.5.5.6) back to the set $\mathcal{A}$ consisting of pairs of charge vectors of the form given in (4.5.5.1). This shows that a sufficient condition for (4.5.5.6) to lie in the set $\mathcal{B}$ is to have b even, i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. Using (4.2.2) we can also see that this condition is necessary since acting on a pair (Q,P) with $Q^2/2$ odd, $P^2/2$ odd and $Q\cdot P$ even, an S-duality transformation produces a (Q',P') with odd $Q'^2/2$ only if b is even.

Thus we identify the subgroup $\Gamma(2)$ of the S-duality group $\Gamma_0(2)$ as the symmetry of the set $\mathcal{B}$. The overall scaling of Q and P does not change the symmetry group. Thus the quarter BPS dyon partition function associated with the set $\mathcal{B}$ must be invariant under the $\Gamma(2)$ S-duality symmetry. This in turn corresponds to symplectic transformations of the form

$$\begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}, \qquad ad - bc = 1, \quad a, d \in \mathbb{Z}, \quad b, c \in 2\,\mathbb{Z}\,, \tag{4.5.5.9}$$

acting on $(\check\rho, \check\sigma, \check v)$ and also on $(\check\rho_s, \check\sigma_s, \check v_s)$.

Next we turn to the analysis of the constraints from wall crossing. First consider the wall corresponding to the decay $(Q,P) \to (Q,0) + (0,P)$, and examine whether there are subtleties of kind mentioned below eq.(4.3.11) in applying eqs.(4.3.9), (4.3.10). For this we note that here $Q^2 = -2(2m+1)$ and $P^2 = 2(2K+1)$ are uncorrelated. For a given $Q^2 = -2(2m+1)$ there is a unique charge vector in the list given in (4.5.5.1). On the other hand even though for a given $P^2 = 2(2K+1)$ there is an infinite family of P labelled by J, they are all related by the T-duality transformation matrix $\begin{pmatrix} 1 & 0 & 0 & -J \\ 0 & 1 & J & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ to the vector $\begin{pmatrix} 2K+1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Thus there are no subtleties of the kind mentioned below (4.3.11) and we have

$$\widehat\Phi(\check\rho, \check\sigma, \check v) \propto \left\{ \check v^2\, \phi_m(\check\rho) \phi_e(\check\sigma) + \mathcal{O}(\check v^4) \right\} \quad \text{for } \check v \simeq 0\,. \tag{4.5.5.10}$$

The magnetic partition function is obtained from (4.5.4.4) by projection to odd values of $P^2/2$ followed by $\check\rho \to \check\rho_s/2$ replacement. This gives

$$\phi_m(\check\rho)^{-1} = \frac{1}{2} \left\{ \eta(\check\rho_s/2)^{-8} \eta(\check\rho_s)^{-8} - \eta((\check\rho_s+1)/2)^{-8} \eta(\check\rho_s)^{-8} \right\}\,. \tag{4.5.5.11}$$

On the other hand the electric partition function can be calculated by analyzing the untwisted sector BPS spectrum of the fundamental heterotic string [55–58]. After taking into account the fact that we are computing the partition function of odd $Q^2/2$ states only, and the $\check{\sigma} \to \check{\sigma}_s/2$ replacement, the result is

$$\begin{aligned} \phi_e^{-1}(\check{\sigma}) &= \frac{1}{2}\left(\psi_e(\check{\sigma}_s) - \psi_e\left(\check{\sigma}_s + 1\right)\right), \\ \psi_e(\check{\sigma}_s) &= 8\,\eta(\check{\sigma}_s/2)^{-24}\left[\frac{1}{2}\left(\vartheta_2(\check{\sigma}_s)^8 + \vartheta_3(\check{\sigma}_s)^8 + \vartheta_4(\check{\sigma}_s)^8\right) - \vartheta_3(\check{\sigma}_s/2)^4\vartheta_4(\check{\sigma}_s/2)^4\right]. \end{aligned} \tag{4.5.5.12}$$

In (4.5.5.12) ψ_e describes the partition function before projecting on to the odd $Q^2/2$ sector [56].

$\phi_m(\check{\rho}_s)$ given in (4.5.5.11) transforms as a modular form of weight 8 under $\check{\rho}_s \to (\alpha\check{\rho}_s + \beta)/(\gamma\check{\rho}_s + \delta)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2)$ with a multiplier $(-1)^\beta$. On the other hand $\phi_e(\check{\sigma})$ given in (4.5.5.12) can be shown to transform as a modular form of weight 8 under $\check{\sigma}_s \to (p\check{\sigma}_s+q)/(r\check{\sigma}_s+s)$ for $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(2)$, with a multiplier $(-1)^q$. These duality symmetries correspond to the symplectic transformations

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & r & 0 & s \end{pmatrix},$$

$$\alpha\delta - \beta\gamma = 1, \quad ps - qr = 1, \quad \alpha, \beta, \delta, p, q, s \in \mathbb{Z}, \quad \gamma, r \in 2\,\mathbb{Z}, \tag{4.5.5.13}$$

acting on $(\check{\rho}_s, \check{\sigma}_s, \check{v}_s)$.

Since for the set $\mathcal{B}$ the S-duality group is $\Gamma(2)$, in this case there is another wall of marginal stability, associated with $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which cannot be related to the previous wall by an S-duality transformation. This corresponds to the decay $(Q, P) \to (Q - P, 0) + (P, P)$ and controls the behaviour of $\widehat{\Phi}$ near $\check{v} + \check{\sigma} = 0$. As usual we first need to determine if there are any subtleties of the type mentioned below eq.(4.3.11). Eq.(4.5.5.1) shows that for a given $(Q-P)^2 = 4(K+J-m)$ there is an infinite family of $(Q-P) = \begin{pmatrix} -(2K+1) \\ (2m-2J+1)/2 \\ -1 \\ -2 \end{pmatrix}$ labelled by $2K$. However all of these can be related by T-duality transformation $\begin{pmatrix} 1 & 0 & 0 & K \\ 0 & 1 & -K & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

to the vector $\begin{pmatrix} -1 \\ (2m-2J-2K+1)/2 \\ -1 \\ -2 \end{pmatrix}$ which is determined completely in terms of $(Q-P)^2$. We have already seen earlier that all choices of P for a given P^2 are also related by T-duality transformations. Finally we note that in this case $(Q-P)^2/2$ and $P^2/2$ can take independent even and odd integer values respectively. It then follows that there are no subtleties of the kind mentioned below (4.3.11). After evaluating ϕ_e and ϕ_m by standard procedure we find that near $\check{v}+\check{\sigma}=0$ $\widehat{\Phi}$ behaves as

$$\widehat{\Phi}(\check{\rho},\check{\sigma},\check{v}) \propto \left[\check{v}_s'^2 \left\{\eta(\check{\rho}_s'/2)^{-8} - \eta((\check{\rho}_s'+1)/2)^{-8}\right\}^{-1} \eta(\check{\rho}_s')^8 \{\psi_e(\check{\sigma}_s') + \psi_e(\check{\sigma}_s'+1)\}^{-1} + \mathcal{O}(\check{v}_s'^4)\right], \tag{4.5.5.14}$$

where

$$\check{v}_s' = \check{v}_s + \check{\sigma}_s, \quad \check{\sigma}_s' = \check{\sigma}_s, \quad \check{\rho}_s' = \check{\rho}_s + \check{\sigma}_s + 2\check{v}_s\,. \tag{4.5.5.15}$$

This has duality symmetry $\check{\rho}_s' \to (\alpha_1\check{\rho}_s'+\beta_1)/(\gamma_1\check{\rho}_s'+\delta_1)$ and $\check{\sigma}_s' \to (p_1\check{\sigma}_s'+q_1)/(r_1\check{\sigma}_s'+s_1)$ for $\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \in \Gamma_0(2)$ and $\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \in \Gamma_0(2)$ with a multiplier $(-1)^{\beta_1}$. We can express them as symplectic transformations acting on $(\check{\rho}_s,\check{\sigma}_s,\check{v}_s)$ using (4.3.29), (4.3.30). This gives

$$\begin{pmatrix} \alpha_1 & \alpha_1-1 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_1 & \gamma_1 & \delta_1 & 0 \\ \gamma_1 & \gamma_1 & \delta_1-1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 1-p_1 & q_1 & -q_1 \\ 0 & p_1 & -q_1 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & r_1 & 1-s_1 & s_1 \end{pmatrix},$$

$$\alpha_1\delta_1 - \beta_1\gamma_1 = 1 = p_1 s_1 - r_1 q_1, \quad \alpha_1,\beta_1,\delta_1,p_1,q_1,s_1 \in \mathbb{Z}, \quad \gamma_1, r_1 \in 2\,\mathbb{Z}\,. \tag{4.5.5.16}$$

As usual, we would like to know if there is a natural subgroup of $Sp(2,\,\mathbb{Z})$ defined by some congruence relation into which all the $Sp(2,\,\mathbb{Z})$ matrices (4.5.5.5), (4.5.5.9), (4.5.5.13) and (4.5.5.16) fit. There is indeed such a subgroup defined as the collection of matrices of the form

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{mod } 2\,. \tag{4.5.5.17}$$

It is natural to speculate that (4.5.5.17) is the symmetry group of the partition function under consideration.

Finally we turn to the constraint from black hole entropy. Since we are considering $\mathbb{Z}_2$ CHL orbifold, the function $g(\tau)$ appearing in the coefficient of the Gauss-Bonnet term in the effective action is the same as the one in §4.5.4:

$$g(\tau) = \eta(\tau)^8\,\eta(2\tau)^8\,. \tag{4.5.5.18}$$

Thus (4.4.6) takes the fom

$$\widehat{\Phi}(\check\rho,\check\sigma,\check v) \propto (2v-\rho-\sigma)^6 \left\{v^2\,\eta(\rho)^8\eta(2\rho)^8\,\eta(\sigma)^8\eta(2\sigma)^8 + \mathcal{O}(v^4)\right\}, \tag{4.5.5.19}$$

where (ρ,σ,v) and $(\check\rho,\check\sigma,\check v)$ are related via (4.4.7).

4.6 A proposal for the partition function of dyons of torsion two

In this section we shall consider the set of dyons described in §4.5.3, carrying charge vectors (Q,P) with torsion 2, Q, P primitive and $Q^2/2$, $P^2/2$ even, and propose a form of the partition function that satisfies all the constraints derived in §4.5.3. The proposed form of the partition function is

$$\begin{aligned}
\frac{1}{\widehat{\Phi}(\check\rho,\check\sigma,\check v)} = & \frac{1}{16}\Bigg[\frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v)} + \frac{1}{\Phi_{10}(\check\rho,\check\sigma+\frac12,\check v)} + \frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma,\check v)} \\
& + \frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma+\frac12,\check v)} + \frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac14,\check v+\frac14)} \\
& + \frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac34,\check v+\frac14)} + \frac{1}{\Phi_{10}(\check\rho+\frac34,\check\sigma+\frac14,\check v+\frac14)} \\
& + \frac{1}{\Phi_{10}(\check\rho+\frac34,\check\sigma+\frac34,\check v+\frac14)} + \frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma+\frac12,\check v+\frac12)} \\
& + \frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma,\check v+\frac12)} + \frac{1}{\Phi_{10}(\check\rho,\check\sigma+\frac12,\check v+\frac12)} + \frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v+\frac12)} \\
& + \frac{1}{\Phi_{10}(\check\rho+\frac34,\check\sigma+\frac34,\check v+\frac34)} + \frac{1}{\Phi_{10}(\check\rho+\frac34,\check\sigma+\frac14,\check v+\frac34)} \\
& + \frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac34,\check v+\frac34)} + \frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac14,\check v+\frac34)}\Bigg] \\
& + \Bigg[\frac{1}{\Phi_{10}(\check\rho+\check\sigma+2\check v,\check\rho+\check\sigma-2\check v,\check\sigma-\check\rho)} \\
& + \frac{1}{\Phi_{10}(\check\rho+\check\sigma+2\check v+\frac12,\check\rho+\check\sigma-2\check v+\frac12,\check\sigma-\check\rho+\frac12)}\Bigg].
\end{aligned} \tag{4.6.1}$$

The index $d(Q,P)$ is computed from this partition function using the formula

$$\begin{aligned}
d(Q,P) = & \frac14 (-1)^{Q\cdot P+1} \int_{\mathcal{C}} d\check\rho_s d\check\sigma_s d\check v_s\, e^{-i\frac{\pi}{4}(\check\sigma_s Q^2+\check\rho_s P^2+2\check v_s Q\cdot P)}\, \frac{1}{\widehat{\Phi}(\check\rho,\check\sigma,\check v)}, \\
& (\check\rho_s,\check\sigma_s,\check v_s) \equiv (4\check\rho, 4\check\sigma, 4\check v),
\end{aligned} \tag{4.6.2}$$

where the contour $\mathcal{C}$ is defined by fixing the imaginary parts of $\check\rho_s$, $\check\sigma_s$, $\check v_s$ to appropriate values depending on the domain of the moduli space in which we want to compute the index, and the real parts span the unit cell defined by the periodicity condition

$$(\check\rho_s, \check\sigma_s, \check v_s) \to (\check\rho_s + 2, \check\sigma_s, \check v_s),\ (\check\rho_s, \check\sigma_s + 2, \check v_s),\ (\check\rho_s, \check\sigma_s, \check v_s + 2),\ (\check\rho_s + 1, \check\sigma_s + 1, \check v_s + 1)\,. \quad (4.6.3)$$

The overall multiplicative factor of 1/4 in (4.6.2) accounts for the fact that the unit cell defined by eqs.(4.6.3) has volume 4. The factor of 1/4 in the exponent in (4.6.2) accounts for the replacement of $(\check\rho, \check\sigma, \check v)$ by $(\check\rho_s/4, \check\sigma_s/4, \check v_s/4)$ in (4.1.2). Note that the sixteen terms inside the first square bracket in (4.6.1) together gives the partition function of dyons of unit torsion subject to the constraint that $Q^2/2$, $P^2/2$, $Q\cdot P$ are even and $Q^2 + P^2 - 2Q\cdot P$ is a multiple of 8. The second term is new and reflects the effect of considering states with torsion two.

We shall now check that this formula satisfies all the constraints derived in §4.5.3. We begin with the S-duality transformations. The first term inside the first square bracket $(\Phi_{10}(\check\rho, \check\sigma, \check v))^{-1}$ is S-duality invariant since S-duality transformation can be regarded as an $Sp(2, \mathbb{Z})$ transformation on $(\check\rho, \check\sigma, \check v)$. The other terms inside the first square bracket have the form $(\Phi_{10}(\check\rho + b_1, \check\sigma + b_2, \check v + b_3))^{-1}$ for appropriate choices of (b_1, b_2, b_3). Since these shifts may be represented as symplectic transformations of $(\check\rho, \check\sigma, \check v)$, S-duality transformation (4.5.3.9) will change this term to $(\Phi_{10}(\check\rho + b_1'', \check\sigma + b_2'', \check v + b_3''))^{-1}$ with

$$\begin{pmatrix} 1 & 0 & b_1'' & b_3'' \\ 0 & 1 & b_3'' & b_2'' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & b_1 & b_3 \\ 0 & 1 & b_3 & b_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix},$$

$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad a + c \in 2\,\mathbb{Z} + 1, \quad b + d \in 2\,\mathbb{Z} + 1\,. \quad (4.6.4)$$

One finds that under such a transformation the sixteen triplets (b_1, b_2, b_3) appearing in the sixteen terms inside the first square bracket get permuted up to integer shifts which are symmetries of Φ_{10}. This proves the S-duality invariance of the first 16 terms.

To test the S-duality invariance of the second term we define

$$\check\rho' = (\check\rho + \check\sigma + 2\check v), \quad \check\sigma' = (\check\rho + \check\sigma - 2\check v), \quad \check v' = (\check\sigma - \check\rho)\,. \quad (4.6.5)$$

It is easy to verify that the effect of S-duality transformation (4.5.3.9) on the $(\check\rho', \check\sigma', \check v')$ variables is represented by the symplectic matrix

$$\begin{pmatrix} (a+b+c+d)/2 & (a+b-c-d)/2 & 0 & 0 \\ (a-b+c-d)/2 & (a-b-c+d)/2 & 0 & 0 \\ 0 & 0 & (a-b-c+d)/2 & -(a-b+c-d)/2 \\ 0 & 0 & -(a+b-c-d)/2 & (a+b+c+d)/2 \end{pmatrix}. \quad (4.6.6)$$

Given the conditions (4.6.4) on a, b, c, d, this is an $Sp(2, \mathbb{Z})$ transformation. Thus the first term inside the second square bracket in (4.6.1), given by $(\Phi_{10}(\check\rho', \check\sigma', \check v'))^{-1}$, is manifestly S-duality invariant. The second term involves a shift of $(\check\rho', \check\sigma', \check v')$ by $(1/2, 1/2, 1/2)$. One can easily check that this commutes with the symplectic transformation (4.6.6) up to integer shifts in $(\check\rho', \check\sigma', \check v')$. Thus the second term is also S-duality invariant.

We now turn to the wall crossing formulæ. First consider the wall associated with the decay $(Q,P) \to (Q,0) + (0,P)$. The jump in the index across this wall is controlled by the residue of the pole at $\check v = 0$. In order to evaluate this residue it will be most convenient to choose the unit cell over which the integration (4.6.2) is performed to be $-1 \le \Re(\check\rho_s) < 1$, $-1 \le \Re(\check\sigma_s) < 1$ and $-\frac{1}{2} \le \Re(\check v_s) < \frac{1}{2}$ so that the image of the pole at $\check v_s = 0$ under $(\check\rho_s, \check\sigma_s, \check v_s) \to (\check\rho_s \pm 1, \check\sigma_s \pm 1, \check v_s \pm 1)$ is outside the unit cell, – otherwise we would need to include the contribution from this pole as well. Using (4.6.2) we see that the change in the index across this wall is given by

$$\Delta d(Q,P) = \frac{1}{4}\,(-1)^{Q\cdot P+1} \int_{iM_1-1}^{iM_1+1} d\check\rho_s \int_{iM_2-1}^{iM_2+1} d\check\sigma_s \oint d\check v_s\, e^{-i\frac{\pi}{4}(\check\sigma_s Q^2+\check\rho_s P^2+2\check v_s Q\cdot P)} \frac{1}{\widehat\Phi(\check\rho,\check\sigma,\check v)}\,, \tag{4.6.7}$$

where $\oint$ denotes the contour around $\check v_s = 0$ and M_1, M_2 are large positive numbers. Now the poles in (4.6.1) can be found from the known locations of the zeroes in $\Phi_{10}(x,y,z)$:

$$\begin{aligned} & n_2(xy - z^2) + jz + n_1 y - m_1 x + m_2 = 0 \\ & m_1, n_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z}+1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,. \end{aligned} \tag{4.6.8}$$

Using this we find that the poles in (4.6.1) at $\check v_s = 0$ can come from the first four terms inside the first square bracket. There is no pole at $\check v_s = 0$ from the terms in the second square bracket. The residue at the pole can be calculated by using the fact that

$$\Phi_{10}(x,y,z) \simeq -4\pi^2\, z^2\, \eta(x)^{24}\eta(y)^{24} + \mathcal{O}(z^4)\,, \tag{4.6.9}$$

near $z = 0$. This gives

$$\widehat\Phi(\check\rho,\check\sigma,\check v) = -4\pi^2\, \check v_s^2 \left\{\eta\left(\frac{\check\sigma_s}{4}\right)^{-24} + \eta\left(\frac{\check\sigma_s+2}{4}\right)^{-24}\right\}^{-1} \left\{\eta\left(\frac{\check\rho_s}{4}\right)^{-24} + \eta\left(\frac{\check\rho_s+2}{4}\right)^{-24}\right\}^{-1} + \mathcal{O}(\check v_s^4)\,. \tag{4.6.10}$$

Substituting this into (4.6.7) and using the convention that the $\check v_s$ contour encloses the pole clockwise, we get

$$\begin{aligned} \Delta d(Q,P) \;=\; & \frac{1}{16}\,(-1)^{Q\cdot P+1}\, Q\cdot P \int_{iM_1-1}^{iM_1+1} d\check\rho_s\, \left\{\eta(\check\rho_s/4)^{-24} + \eta((\check\rho_s+2)/4)^{-24}\right\} e^{-i\pi\check\rho_s P^2/4} \\ & \int_{iM_2-1}^{iM_2+1} d\check\sigma_s\, \left\{\eta(\check\sigma_s/4)^{-24} + \eta((\check\sigma_s+2)/4)^{-24}\right\} e^{-i\pi\check\sigma_s Q^2/4}\,. \end{aligned} \tag{4.6.11}$$

We now want to compare this with the general wall crossing formula given in (4.3.7). Here the relevant half-BPS partition functions are to be computed with Q^2 and P^2 restricted to be even. This gives

$$\begin{aligned} d_h(Q,0) &= \int_{iM-1/4}^{iM+1/4} d\tau \left\{ \eta(\tau)^{-24} + \eta\left(\tau+\frac{1}{2}\right)^{-24} \right\} e^{-i\pi\tau Q^2} \\ &= \frac{1}{4}\int_{4iM-1}^{4iM+1} d\check\sigma_s \left\{\eta(\check\sigma_s/4)^{-24} + \eta((\check\sigma_s+2)/4)^{-24}\right\} e^{-i\pi\check\sigma_s Q^2/4}, \\ d_h(0,P) &= \int_{iM-1/4}^{iM+1/4} d\tau \left\{ \eta(\tau)^{-24} + \eta\left(\tau+\frac{1}{2}\right)^{-24} \right\} e^{-i\pi\tau P^2} \\ &= \frac{1}{4}\int_{4iM-1}^{4iM+1} d\check\rho_s \left\{\eta(\check\rho_s/4)^{-24} + \eta((\check\rho_s+2)/4)^{-24}\right\} e^{-i\pi\check\rho_s P^2/4}, \end{aligned} \tag{4.6.12}$$

for some large positive number M. Using this we can rewrite (4.6.11) as

$$\Delta d(Q,P) = (-1)^{Q\cdot P+1}\, Q\cdot P\, d_h(Q,0)\, d_h(0,P)\,, \tag{4.6.13}$$

in agreement with the wall crossing formula.

Next we consider the wall associated with the decay $(Q,P) \to ((Q-P)/2,(P-Q)/2) + ((Q+P)/2,(Q+P)/2)$. The associated pole of the partition function is at $\check\rho = \check\sigma$. With the help of (4.6.9) we find that this pole arises from the first term inside the second square bracket in (4.6.1), and near this pole

$$\widehat\Phi(\check\rho,\check\sigma,\check v) = -\frac{\pi^2}{4}(\check\sigma_s-\check\rho_s)^2\,\eta((\check\rho_s+\check\sigma_s+2\check v_s)/4)\,\eta((\check\rho_s+\check\sigma_s-2\check v_s)/4) + \mathcal{O}\left((\check\sigma_s-\check\rho_s)^4\right)\,. \tag{4.6.14}$$

In order to compute the change in the index as we cross this wall, we change variables to

$$\check\rho_s' = (\check\rho_s+\check\sigma_s+2\check v_s)/2, \quad \check\sigma_s' = (\check\rho_s+\check\sigma_s-2\check v_s)/2, \quad \check v_s' = (\check\sigma_s-\check\rho_s)/2\,. \tag{4.6.15}$$

The periodicity properties (4.6.3) on $(\check\rho_s,\check\sigma_s,\check v_s)$ take the form

$$(\check\rho_s',\check\sigma_s',\check v_s') \to (\check\rho_s'+2,\check\sigma_s',\check v_s'),\ (\check\rho_s',\check\sigma_s'+2,\check v_s'),\ (\check\rho_s',\check\sigma_s',\check v_s'+2),\ (\check\rho_s'+1,\check\sigma_s'+1,\check v_s'+1)\,. \tag{4.6.16}$$

We choose the unit cell in the $(\Re(\check\rho_s'),\Re(\check\sigma_s'),\Re(\check v_s'))$ to be $-1\le\Re(\check\rho_s')<1$, $-1\le\Re(\check\sigma_s')<1$ and $-\frac{1}{2}\le\Re(\check v_s')<\frac{1}{2}$. Since the jacobian of the transformation associated with (4.6.15) is unity, the change in the index across the wall is given by an expression analogous to (4.6.7)

$$\Delta d(Q,P) = \frac{1}{4}(-1)^{Q\cdot P+1}\int_{iM_1'-1}^{iM_1'+1} d\check\rho_s' \int_{iM_2'-1}^{iM_2'+1} d\check\sigma_s' \oint d\check v_s'\, e^{-i\frac{\pi}{4}(\check\sigma_s' Q'^2+\check\rho_s' P'^2+2\check v_s' Q'\cdot P')}\,\frac{1}{\widehat\Phi(\check\rho,\check\sigma,\check v)}\,, \tag{4.6.17}$$

where

$$Q' = \frac{Q-P}{\sqrt2}, \qquad P' = \frac{Q+P}{\sqrt2}\,. \tag{4.6.18}$$

Substituting (4.6.14) into (4.6.17) we get

$$\begin{aligned}\Delta d(Q,P) &= \frac{1}{4}\,(-1)^{Q\cdot P+1}\,Q'\cdot P'\int_{iM_1'-1}^{iM_1'+1} d\check\rho_s'\,\eta(\check\rho_s'/2)^{-24}\,e^{-i\pi\check\rho_s' P'^2/4} \\ &\qquad \int_{iM_2'-1}^{iM_2'+1} d\check\sigma_s'\,\eta(\check\sigma_s'/2)^{-24}\,e^{-i\pi\check\sigma_s' Q'^2/4}\,. \end{aligned} \tag{4.6.19}$$

On the other hand now the indices of the half-BPS decay products carrying charges

$$(Q_1,P_1)=((Q-P)/2,(P-Q)/2), \qquad (Q_2,P_2)=((Q+P)/2,(Q+P)/2)\,, \tag{4.6.20}$$

are given by

$$d_h(Q_1,P_1)=\int_{iM-1/2}^{iM+1/2} d\tau(\eta(\tau))^{-24}e^{-i\pi\tau((Q-P)/2)^2}=\frac{1}{2}\int_{2iM-1}^{2iM+1} d\check\sigma_s'\,\eta(\check\sigma_s'/2)^{-24}\,e^{-i\pi\check\sigma_s' Q'^2/4}\,, \tag{4.6.21}$$

and

$$d_h(Q_2,P_2)=\int_{iM-1/2}^{iM+1/2} d\tau(\eta(\tau))^{-24}e^{-i\pi\tau((Q+P)/2)^2}=\frac{1}{2}\int_{2iM-1}^{2iM+1} d\check\rho_s'\,\eta(\check\rho_s'/2)^{-24}\,e^{-i\pi\check\rho_s' P'^2/4}\,. \tag{4.6.22}$$

Using these results and the identities

$$Q_1\cdot P_2-Q_2\cdot P_1=Q'\cdot P', \quad (-1)^{Q_1\cdot P_2-Q_2\cdot P_1}=(-1)^{(Q-P)^2/2-P^2+Q\cdot P}=(-1)^{Q\cdot P}\,, \tag{4.6.23}$$

we can express (4.6.19) as

$$\Delta d(Q,P)=(-1)^{Q_1\cdot P_2-Q_2\cdot P_1+1}\,(Q_1\cdot P_2-Q_2\cdot P_1)\,d_h(Q_1,P_1)\,d_h(Q_2,P_2)\,, \tag{4.6.24}$$

in agreement with the wall crossing formula.

Next consider the decay $(Q,P)\to(Q-P,0)+(P,P)$. This is controlled by the pole at $\check\sigma+\check v=0$. To analyze this contribution we define

$$Q'=Q-P, \quad P'=P \tag{4.6.25}$$

and

$$\check\rho_s'=\check\rho_s+\check\sigma_s+2\check v_s, \quad \check\sigma_s'=\check\sigma_s, \quad \check v_s'=\check v_s+\check\sigma_s\,, \tag{4.6.26}$$

so that $\check\rho_s P^2+\check\sigma_s Q^2+2\check v_s Q\cdot P=\check\rho_s' P'^2+\check\sigma_s' Q'^2+2\check v_s' Q'\cdot P'$. In terms of these variables the periods are

$$(\check\rho_s',\check\sigma_s',\check v_s')\to(\check\rho_s'+2,\check\sigma_s',\check v_s'),\ (\check\rho_s',\check\sigma_s'+1,\check v_s'),\ (\check\rho_s',\check\sigma_s',\check v_s'+2)\,. \tag{4.6.27}$$

The behaviour of $\widehat{\Phi}$ near $\check{v}'_s = 0$ can be found by the usual procedure and result is

$$\begin{aligned}\widehat{\Phi}(\check{\rho},\check{\sigma},\check{v})^{-1} &= -\frac{1}{4\pi^2\check{v}_s'^2}\left\{\eta\left(\frac{\check{\rho}'_s}{4}\right)^{-24}+\eta\left(\frac{\check{\rho}'_s+2}{4}\right)^{-24}\right\}\\ &\quad\times\left\{\eta\left(\frac{\check{\sigma}'_s}{4}\right)^{-24}+\eta\left(\frac{\check{\sigma}'_s+1}{4}\right)^{-24}+\eta\left(\frac{\check{\sigma}'_s+2}{4}\right)^{-24}+\eta\left(\frac{\check{\sigma}'_s+3}{4}\right)^{-24}\right\}\\ &\quad-\frac{1}{\pi^2\check{v}_s'^2}\left\{\eta\left(\frac{\check{\rho}'_s}{4}\right)^{-24}+\eta\left(\frac{\check{\rho}'_s+2}{4}\right)^{-24}\right\}\eta(\check{\sigma}'_s)^{-24}+\mathcal{O}\left(\check{v}_s'^0\right) \qquad (4.6.28)\end{aligned}$$

Note that the first set of terms represent correctly the factorization behaviour given in (4.5.3.21), but the second set of terms are extra. Thus the wall crossing formula gets modified for the decay into non-primitive states. Using (4.6.28) we can compute the jump in the index across the wall

$$\begin{aligned}\Delta d(Q,P) &= \frac{1}{16}(-1)^{Q\cdot P+1}\,Q'\cdot P'\int_{iM_1'-1}^{iM_1'+1}d\check{\rho}'_s\left\{\eta\left(\frac{\check{\rho}'_s}{4}\right)^{-24}+\eta\left(\frac{\check{\rho}'_s+2}{4}\right)^{-24}\right\}e^{-i\pi\check{\rho}'_sP'^2/4}\\ &\int_{iM_2'-1/2}^{iM_2'+1/2}d\check{\sigma}'_s\left\{\eta\left(\frac{\check{\sigma}'_s}{4}\right)^{-24}+\eta\left(\frac{\check{\sigma}'_s+1}{4}\right)^{-24}\right.\\ &\qquad\left.+\eta\left(\frac{\check{\sigma}'_s+2}{4}\right)^{-24}+\eta\left(\frac{\check{\sigma}'_s+3}{4}\right)^{-24}\right\}e^{-i\pi\check{\sigma}'_sQ'^2/4}\\ &+\frac{1}{4}(-1)^{Q\cdot P+1}\,Q'\cdot P'\int_{iM_1'-1}^{iM_1'+1}d\check{\rho}'_s\left\{\eta\left(\frac{\check{\rho}'_s}{4}\right)^{-24}+\eta\left(\frac{\check{\rho}'_s+2}{4}\right)^{-24}\right\}e^{-i\pi\check{\rho}'_sP'^2/4}\\ &\quad\times\int_{iM_2'-1/2}^{iM_2'+1/2}d\check{\sigma}'_s\,\eta(\check{\sigma}'_s)^{-24}\,e^{-i\pi\check{\sigma}'_sQ'^2/4}\,. \qquad (4.6.29)\end{aligned}$$

Defining

$$(Q_1,P_1)=(Q-P,0),\quad (Q_2,P_2)=(P,P)\,, \qquad (4.6.30)$$

we can express (4.6.29) as

$$\Delta d(Q,P)=(-1)^{Q_1\cdot P_2-Q_2\cdot P_1+1}(Q_1\cdot P_2-Q_2\cdot P_1)\left\{d_h\left(Q_1,P_1\right)+d_h\left(\frac{1}{2}Q_1,\frac{1}{2}P_1\right)\right\}d_h\left(Q_2,P_2\right)\,. \qquad (4.6.31)$$

The second term is extra compared to (4.3.7); it represents the effect of non-primitivity of the final state dyons.

Finally let us turn to the analysis of the black hole entropy. For this we need to identify the zeroes of $\widehat{\Phi}$ at $\check{\rho}\check{\sigma}-\check{v}^2+\check{v}=0$ and show that $\widehat{\Phi}$ has the behaviour given in (4.5.3.25) near this pole. This is easily done using (4.6.1) and the locations of the zeroes of Φ_{10} given in

(4.6.8). One finds that the only term that has a zero at the desired location is the first term inside the first square bracket in (4.6.1). Furthermore this term is proportional to the dyon partition function $1/\Phi_{10}(\check\rho,\check\sigma,\check v)$ of the unit torsion states discussed in §4.5.1. Thus this term clearly will have the desired factorization property given in (4.5.3.25).

Our proposal for the dyon partition function can be easily generalized to the torsion 2, primitive Q, P and odd $Q^2/2$, $P^2/2$ dyons discussed at the end of §4.5.3. This requires changing the signs of appropriate terms in (4.6.1) so that the partition function is odd under $\check\rho\to\check\rho+\frac{1}{2}$ and also under $\check\sigma\to\check\sigma+\frac{1}{2}$. The result is

$$\begin{aligned}
\frac{1}{\widehat{\Phi}(\check\rho,\check\sigma,\check v)} &= \frac{1}{16}\Bigg[\frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v)}-\frac{1}{\Phi_{10}(\check\rho,\check\sigma+\frac12,\check v)}-\frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma,\check v)}\\
&+\frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma+\frac12,\check v)}+\frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac14,\check v+\frac14)}\\
&-\frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac34,\check v+\frac14)}-\frac{1}{\Phi_{10}(\check\rho+\frac34,\check\sigma+\frac14,\check v+\frac14)}\\
&+\frac{1}{\Phi_{10}(\check\rho+\frac34,\check\sigma+\frac34,\check v+\frac14)}+\frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma+\frac12,\check v+\frac12)}\\
&-\frac{1}{\Phi_{10}(\check\rho+\frac12,\check\sigma,\check v+\frac12)}-\frac{1}{\Phi_{10}(\check\rho,\check\sigma+\frac12,\check v+\frac12)}+\frac{1}{\Phi_{10}(\check\rho,\check\sigma,\check v+\frac12)}\\
&+\frac{1}{\Phi_{10}(\check\rho+\frac34,\check\sigma+\frac34,\check v+\frac34)}-\frac{1}{\Phi_{10}(\check\rho+\frac34,\check\sigma+\frac14,\check v+\frac34)}\\
&-\frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac34,\check v+\frac34)}+\frac{1}{\Phi_{10}(\check\rho+\frac14,\check\sigma+\frac14,\check v+\frac34)}\Bigg]\\
&+\Bigg[\frac{1}{\Phi_{10}(\check\rho+\check\sigma+2\check v,\check\rho+\check\sigma-2\check v,\check\sigma-\check\rho)}\\
&-\frac{1}{\Phi_{10}(\check\rho+\check\sigma+2\check v+\frac12,\check\rho+\check\sigma-2\check v+\frac12,\check\sigma-\check\rho+\frac12)}\Bigg]\,.
\end{aligned} \tag{4.6.32}$$

This together with (4.6.1) exhausts all the dyons of torsion two with Q, P primitive since there are no dyons of this type with $Q^2/2$ even, $P^2/2$ odd or vice versa. To see this we note that since $(Q\pm P)$ are $2\times$ primitive vectors, $(Q\pm P)^2/2$ must be multiples of four. Taking the sum and difference we find that $(Q^2+P^2)/2$ and $Q\cdot P$ must be even.

Since (4.6.1) and (4.6.32) contains information about all the torsion two dyons with primitive (Q,P), the full partition function for such dyons in obtained by taking the sum of these two functions. This gives the result quoted in (4.0.23).

Given this result on torsion two dyons in string theory we can go to appropriate gauge theory limit to extract information about torsion two dyons in gauge theories as in [54,59]. For simplicity we shall consider $SU(3)$ gauge theories. If we denote by α_1 and α_2 the two simple roots of $SU(3)$, then, since the metric L reduces to the negative of the Cartan metric of the gauge group, we have

$$\alpha_1^2 = -2, \qquad \alpha_2^2 = -2, \qquad \alpha_1 \cdot \alpha_2 = 1\,. \tag{4.6.33}$$

Let us now consider a dyon of charge vectors (Q, P) with

$$Q = \alpha_1 - \alpha_2, \qquad P = \alpha_1 + \alpha_2\,. \tag{4.6.34}$$

This has torsion 2. Furthermore both Q and P are primitive. Thus this falls in the class of dyons analyzed in this section. In fact, since

$$\frac{Q^2}{2} = -3, \qquad \frac{P^2}{2} = -1, \qquad Q \cdot P = 0\,, \tag{4.6.35}$$

the index of these quarter BPS dyons in gauge theory must be contained in (4.6.32). We shall first show that the 16 terms inside the first square bracket in (4.6.32) do not contribute to the index of the dyons described in (4.6.34) in any domain in the moduli space. For this we note that the index computed from these terms is identical to the index of dyons of torsion 1 with appropriate constraints on Q^2, P^2 and $Q \cdot P$. In absence of these constraints the index is known to reproduce the index of unit torsion gauge theory dyons correctly [59], – these are dyons of charge (α_1, α_2) or ones related to it by $SL(2, \mathbb{Z})$ S-duality transformation:

$$(Q, P) = (a\alpha_1 + b\alpha_2, c\alpha_1 + d\alpha_2), \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1\,. \tag{4.6.36}$$

Such dyons will always have $Q^2P^2 - (Q \cdot P)^2 = 3$, and hence can never give a state of the form given in (4.6.34) which has $Q^2P^2 - (Q \cdot P)^2 = 12$. This in turn shows that the 16 terms inside the first square bracket in (4.6.32) can never give a non-vanishing contribution to the index of a gauge theory state with charge vector given in (4.6.34).

Thus the only possible contribution to the index of the dyons with charges $(\alpha_1 - \alpha_2, \alpha_1 + \alpha_2)$ can come from the two terms inside the second square bracket in (4.6.32). In fact when $Q^2/2$ and $P^2/2$ are odd then both terms give equal contribution; so we can just calculate the contribution from the first term and multiply it by a factor of 2. Equivalently we could use (4.0.23) where only the first term is present with a factor of 2. Defining

$$\check\rho' = \check\rho + \check\sigma + 2\check v, \quad \check\sigma' = \check\rho + \check\sigma - 2\check v, \quad \check v' = \check\sigma - \check\rho\,, \tag{4.6.37}$$

we can identify the relevant term in $1/\widehat\Phi(\check\rho, \check\sigma, \check v)$ as $2/\Phi_{10}(\check\rho', \check\sigma', \check v')$. As usual the contribution of this term to the index depends on the choice of the integration contour, which in turn is determined by domain in the moduli space in which we want to compute the index. Equivalently we can say that in different domains we need to use different Fourier series expansion of $1/\widehat\Phi$. Now the index of a charge vector of the type given in (4.6.35) will come from a term in the expansion of $1/\widehat\Phi$ of the form

$$e^{-2i\pi(\check\rho + 3\check\sigma)}\,. \tag{4.6.38}$$

Using (4.6.37) this takes the form

$$e^{-2i\pi(\check\rho' + \check\sigma' + \check v')}\,. \tag{4.6.39}$$

Thus in whichever domain the Fourier expansion of $1/\widehat{\Phi}$ contains a term of the form (4.6.39) we have a non-vanishing index for the dyon in (4.6.35) with the index being equal to $(-1)^{Q\cdot P+1}$ times the coefficient of this term. Since here $Q \cdot P = 0$, the index is -2 times the coefficient of (4.6.39) in the Fourier expansion of $1/\Phi_{10}(\check\rho', \check\sigma', \check v')$. Now from the analysis of partition function of torsion one dyons (see *e.g.* [59]) we know that this expansion indeed has a term of the form (4.6.39) for one class of choices of contour; these are the contours for which

$$\Im(\check\rho'), \Im(\check\sigma') >> -\Im(\check v') >> 0\,. \tag{4.6.40}$$

Using (4.3.4), or equivalently by an $SL(2, \mathbb{R})$ transformation of the results of [59], one can figure out the domain in the moduli space in which this choice of contour is the correct one. It turns out to be the domain bounded by the walls associated with the decays of (Q, P) into

$$((Q-P)/2, (P-Q)/2) + ((Q+P)/2, (Q+P)/2), \quad (-P, P) + (P+Q, 0), \quad (Q-P, 0) + (P, P)\,. \tag{4.6.41}$$

In this domain $1/\Phi_{10}$ has to be first expanded in powers of $e^{2\pi i\check\rho'}$ and $e^{2\pi i\check\sigma'}$ and then each coefficient needs to be expanded in powers of $e^{-2\pi i\check v'}$. (4.6.39) is the leading term in this expansion and its coefficient is 1. As a result the index of the dyons is -2. This agrees with the results of [49, 50, 68, 69] where it was shown that in an appropriate domain in the moduli space dyons of torsion r has index $(-1)^{r-1}r$, since these dyonic states are obtained by tensoring the basic supermultiplet with a state of spin $(r-1)/2$.

Using a string junction picture [27, 28] ref. [68] also showed that the dyon considered above exists in a domain of the moduli space bounded by three walls of marginal stability, – one asscociated with the decay into $(\alpha_1, \alpha_1) + (-\alpha_2, \alpha_2)$, the second associated with the decay into $(2\alpha_1, 0) + (-\alpha_1 - \alpha_2, \alpha_1 + \alpha_2)$ and the third one associated with the decay into $(-2\alpha_2, 0) + (\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)$. These are precisely the walls listed in (4.6.41). Since the gauge theory dyons cease to exist outside these walls, the index computed in gauge theory jumps by 2 across these three walls of marginal stability. Does this agree with the prediction of the proposed dyon partition function in string theory? We can calculate the change in the index associated with these decays using the standard formula (4.3.7) for decay into primitive dyons and the modified formula (4.6.31) for decay into non-primitive dyons since the proposed partition function satisfies these relations. We find a jump in the index equal to 2 across each of these walls as predicted by the gauge theory results.

4.7 Reverse applications

In our analysis so far we have used the information on half BPS partition function to extract information about quarter BPS partition function. However we can turn this around. If the quarter BPS partition function is known then we can use it to extract information about the half-BPS partition function by first identifying an appropriate wall on which one of the decay products is the half BPS state under consideration and then studying the behaviour of the quarter BPS partition function near the pole that controls the jump in the index at this particular wall.

As an example we can consider the $\mathbb{Z}_2$ CHL model of §4.5.4. The decay $(Q,P) \to (Q,0)+(0,P)$ is controlled by the behaviour of $\widehat{\Phi}$ near $\check{v}=0$. Thus if we did not know the spectrum of magnetically charged half BPS states in this theory, we could study the behaviour of $\widehat{\Phi}$ near $\check{v}=0$ to get this information. In this case since all the other walls are related to this one by S-duality transformation, this is the only independent information we can get. However for more complicated models there can be more information.

To illustrate this we shall consider the example of the $\mathbb{Z}_6$ CHL model [60,61] mentioned in footnote 9. Our set $\mathcal{A}$ consists of charge vectors of the form

$$Q = \begin{pmatrix} 0 \\ m/6 \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \qquad m, K, J \in \mathbb{Z}, \tag{4.7.1}$$

as in (4.5.4.1). We now consider the decay associated with the matrix

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}. \tag{4.7.2}$$

From (4.3.1) we see that this corresponds to the decay

$$(Q,P) \to (M_0, 2M_0) + (N_0, 3N_0), \quad M_0 \equiv 3Q - P, \quad N_0 \equiv -2Q + P. \tag{4.7.3}$$

The charge vectors M_0 and N_0 are not related to Q or P by a T-duality transformation since they correspond to charges that are triple and double twisted respectively. Furthermore the dyon charges $(M_0, 2M_0)$ and $(N_0, 3N_0)$ cannot be related by S-duality group $\Gamma_1(6)$ to either a purely electric or a purely magnetic state whose index is known. On the other hand the partition function of quarter BPS states of the type given in (4.7.3) is known [17,23]. Thus the latter can be used to extract information about the partition function of these half BPS states.

From (4.3.2) it follows that the relevant zero of $\widehat{\Phi}$ we need to examine is at

$$6\check{\rho} + \check{\sigma} + 5\check{v} = 0. \tag{4.7.4}$$

The zeroes of $\widehat{\Phi}$ have been classified in [15,23]. For a generic $\mathbb{Z}_N$ model $\widehat{\Phi}$ has double zeroes at

$$\begin{gathered} n_2(\check{\sigma}\check{\rho} - \check{v}^2) + j\check{v} + n_1\check{\sigma} - \check{\rho} m_1 + m_2 = 0 \\ m_1 \in N\,\mathbb{Z}, \quad n_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z}+1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}. \end{gathered} \tag{4.7.5}$$

For the $N = 6$ model, taking

$$m_1 = -6, \quad n_1 = 1, \quad m_2 = n_2 = 0, \quad j = 5\,, \tag{4.7.6}$$

we see that $\widehat{\Phi}$ indeed has a zero at (4.7.4). Thus by examining the known expression for $\widehat{\Phi}$ near this zero we can determine the half-BPS partition functions of interest. This can be done in a straightforward manner following the general procedure described in [15, 23].

We note in passing that in an arbitrary $\mathbb{Z}_N$ model with the set $\mathcal{A}$ chosen as

$$Q = \begin{pmatrix} 0 \\ m/N \\ 0 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} K \\ J \\ 1 \\ 0 \end{pmatrix}, \quad m, K, J \in \mathbb{Z}\,, \tag{4.7.7}$$

the walls of marginal stability are controlled by matrices $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ subject to the conditions [19]

$$a_0 d_0 - b_0 c_0 = 1, \quad a_0, b_0, c_0, d_0 \in \mathbb{Z}, \quad c_0 d_0 \in N\,\mathbb{Z}\,. \tag{4.7.8}$$

According to our hypothesis this decay will be controlled by a double zero of $\widehat{\Phi}$ at

$$\check{\rho} c_0 d_0 + \check{\sigma} a_0 b_0 + \check{v}(a_0 d_0 + b_0 c_0) = 0\,. \tag{4.7.9}$$

This corresponds to the choice

$$m_1 = -c_0 d_0, \quad n_1 = a_0 b_0, \quad m_2 = n_2 = 0, \quad j = a_0 d_0 + b_0 c_0\,, \tag{4.7.10}$$

in eq.(4.7.5). We now see that the m_i's, n_i's and j given in (4.7.10) satisfies all the constraints mentioned in (4.7.5) as a consequence of (4.7.8). Thus our proposal that the decay associated with the matrix $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ is always controlled by the zero at (4.7.9) is at least consistent with the locations of the zeroes of $\widehat{\Phi}$ for $\mathbb{Z}_N$ orbifold models.

Chapter 5

Partition Functions of Torsion > 1 Dyons

Since the original proposal of Dijkgraaf, Verlinde and Verlinde [7] for quarter BPS dyon spectrum in heterotic string theory compactified on T^6, there has been extensive study of dyon spectrum in a variety of $\mathcal{N} = 4$ supersymmetric string theories [8–23, 38] and also in $\mathcal{N} = 8$ and $\mathcal{N} = 2$ supersymmetric string theories [39, 70]. However it has been realised for some time that even in heterotic string theory on T^6 the proposal of [7] gives the correct dyon spectrum only for a subset of dyons, – those with unit torsion, i.e. for which the electric and magnetic charge vectors Q and P satisfy gcd($Q \wedge P$)=1 [20, 40, 54]. In the previous chapter we wrote down a general set of constraints which must be satisfied by the partition function of quarter BPS dyons in any $\mathcal{N} = 4$ supersymmetric string theory and used these constraints to propose a candidate for the dyon partition function for torsion two dyons in heterotic string theory on T^6 [62]. In the present chapter we extend our analysis to dyons of arbitrary torsion and propose a form of the partition function of such dyons.

Proposal for the partition function: We consider the set $\mathcal{B}$ of all dyons of charge vectors (Q, P) in heterotic string theory on T^6, with Q being r times a primitive vector, P a primitive vector and Q/r and P admitting a primitive embedding in the Narain lattice [25, 26], i.e. all lattice vectors lying in the plane of Q and P can be expressed as integer linear combinations of Q/r and P. These dyons have torsion r, i.e. $\gcd(Q \wedge P) = r$. It was shown in [40, 54] that given any pair (Q, P) of this type with the same values of Q^2, P^2 and $Q \cdot P$, they are related by T-duality transformation. We denote by $d(Q, P)$ the index measuring the number of bosonic supermultiplets minus the number of fermionic supermultiplets of quarter BPS dyons carrying charges (Q, P) – up to a normalization this can be identified with the helicity supertrace B_6 introduced in [53]. T-duality invariance of the theory tells us that $d(Q, P)$ must be a function of the T-duality invariants, and hence has the form $f(Q^2, P^2, Q \cdot P)$. Then the dyon partition function $1/\widehat{\Phi}(\check\rho,\check\sigma,\check v)$ is defined as

$$\frac{1}{\widehat{\Phi}(\check\rho,\check\sigma,\check v)} = \sum_{Q^2,P^2,Q\cdot P} (-1)^{Q\cdot P+1} f(Q^2,P^2,Q\cdot P)\, e^{i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)}\,. \tag{5.0.1}$$

The sum in (5.0.1) runs over all possible values of Q^2, P^2 and $Q\cdot P$ in the set $\mathcal{B}$. This in particular requires

$$Q^2/2 \in r^2\,\mathbb{Z}, \qquad P^2/2 \in \mathbb{Z}, \qquad Q\cdot P \in r\,\mathbb{Z}\,. \tag{5.0.2}$$

The imaginary parts of $(\check\rho,\check\sigma,\check v)$ in (5.0.1) need to be adjusted to lie in a region where the sum is convergent. Although the index f and hence the partition function $1/\widehat{\Phi}$ so defined could depend on the domain of the asymptotic moduli space of the theory in which we are computing the partition function [19, 23, 40, 62], in all known examples the dependence of $\widehat{\Phi}$ on the domain is found to come through the region of the complex $(\check\rho,\check\sigma,\check v)$ plane in which the sum is convergent. Thus (5.0.1) computed in different domains in the asymptotic moduli space of the theory describes the same analytic function $\widehat{\Phi}$ in different domains in the complex $(\check\rho,\check\sigma,\check v)$ plane. We shall assume that the same feature holds for the partition function under consideration.

Since the quantization laws of Q^2, P^2 and $Q\cdot P$ imply that $\widehat{\Phi}(\check\rho,\check\sigma,\check v)$ is periodic under independent shifts of $\check\rho$, $\check\sigma$ and $\check v$ by 1, $1/r^2$ and $1/r$ respectively, eq.(5.0.1) can be inverted as

$$\begin{aligned} d(Q,P) &= (-1)^{Q\cdot P+1} r^3 \int_{iM_1-1/2}^{iM_1+1/2} d\check\rho \int_{iM_2-1/(2r^2)}^{iM_2+1/(2r^2)} d\check\sigma \int_{iM_3-1/(2r)}^{iM_3+1/(2r)} d\check v \\ &\qquad e^{-i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)}\, \frac{1}{\widehat{\Phi}(\check\rho,\check\sigma,\check v)}\,, \end{aligned} \tag{5.0.3}$$

provided the imaginary parts M_1, M_2 and M_3 of $\check\rho$, $\check\sigma$ and $\check v$ are fixed in a region where the original sum (5.0.1) is convergent.

Our proposal for $\widehat{\Phi}(\check\rho,\check\sigma,\check v)$ is

$$\widehat{\Phi}(\check\rho,\check\sigma,\check v)^{-1} = \sum_{\substack{s\,\in\,\mathbb{Z},\, s|r \\ \bar s\,\equiv\, r/s}} g(s)\,\frac{1}{\bar s^3} \sum_{k=0}^{\bar s^2-1}\sum_{l=0}^{\bar s-1} \Phi_{10}\left(\check\rho, s^2\check\sigma+\frac{k}{\bar s^2}, s\check v+\frac{l}{\bar s}\right)^{-1}\,, \tag{5.0.4}$$

where

$$g(s) = s\,, \tag{5.0.5}$$

and $\Phi_{10}(\check\rho,\check\sigma,\check v)$ is the weight 10 Igusa cusp form of $Sp(2,\mathbb{Z})$. The sum over k and l in (5.0.4) makes $\widehat{\Phi}$ periodic under $\check\sigma\to\check\sigma+(1/r^2)$ and $\check v\to\check v+(1/r)$ as required. Even though the function $g(s)$ has a simple form given in (5.0.5), we shall carry out our analysis keeping $g(s)$ arbitrary so that we can illustrate at the end how we fix the form of $g(s)$ from the known wall crossing formula for decay into a pair of primitive half-BPS dyons. In particular we shall show that across a wall of marginal stability associated with the decay of the original quarter BPS dyon into a pair of half BPS states carrying charges (Q_1, P_1) and (Q_2, P_2) with (Q_1, P_1) being N_1 times a primitive lattice vector and (Q_2, P_2) being N_2 times a primitive lattice vector, the index jumps by an amount

$$\Delta d(Q,P)) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\,(Q_1\cdot P_2 - Q_2\cdot P_1)\,\left\{\sum_{L_1|N_1} d_h\left(\frac{Q_1}{L_1},\frac{P_1}{L_1}\right)\right\}\left\{\sum_{L_2|N_2} d_h\left(\frac{Q_2}{L_2},\frac{P_2}{L_2}\right)\right\} \tag{5.0.6}$$

for the choice $g(s) = s$ in (5.0.4). Here $d_h(q,p)$ denotes the index of half BPS states carrying charges (q,p). When $N_1 = N_2 = 1$ both the decay products are primitive and (5.0.6) reduces to the standard wall crossing formula [19, 41–46, 64, 65].

Substituting (5.0.4) into (5.0.3), extending the ranges of $\check\sigma$ and $\check v$ integral with the help of the sums over k and l, and using the periodicity of Φ_{10} under integer shifts of its arguments we can get a simpler expression for the index:

$$\begin{aligned} d(Q,P) &= (-1)^{Q\cdot P+1}\sum_{s|r} g(s)\, s^3 \int_{iM_1-1/2}^{iM_1+1/2} d\check\rho \int_{iM_2-1/(2s^2)}^{iM_2+1/(2s^2)} d\check\sigma \int_{iM_3-1/(2s)}^{iM_3+1/(2s)} d\check v \\ &\quad e^{-i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)}\,\Phi_{10}\left(\check\rho, s^2\check\sigma, s\check v\right)^{-1}. \end{aligned} \tag{5.0.7}$$

The set of dyons considered above contains only a subset of dyons of torsion r. This subset is known to be invariant under a $\Gamma^0(r)$ subgroup of the S-duality group [40]. This requires $\widehat\Phi$ to be invariant under the transformation [62]

$$\widehat\Phi(\check\rho',\check\sigma',\check v') = \widehat\Phi(\check\rho,\check\sigma,\check v) \qquad \text{for} \qquad \begin{pmatrix}\check\rho' & \check v'\\ \check v' & \check\sigma'\end{pmatrix} = \begin{pmatrix} d & b\\ c & a\end{pmatrix}\begin{pmatrix}\check\rho & \check v\\ \check v & \check\sigma\end{pmatrix}\begin{pmatrix} d & c\\ b & a\end{pmatrix},$$
$$a,c,d\in \mathbb{Z}, \quad b\in r\,\mathbb{Z}, \quad ad-bc=1\,. \tag{5.0.8}$$

On the other hand a general S-duality transformation matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ outside $\Gamma^0(r)$ will take us to dyons of torsion r outside the set $\mathcal{B}$ [40]. Thus with the help of these S-duality transformations on $\widehat\Phi$ we can determine the partition function for other torsion r dyons lying outside the set $\mathcal{B}$ considered above. In particular if we consider the set of dyons carrying charges (Q', P') related to (Q,P) via an S-duality transformation

$$(Q,P) = (aQ'+bP', cQ'+dP'), \quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\,\mathbb{Z})\,, \tag{5.0.9}$$

and denote by $1/\widehat\Phi'$ the partition function of these dyons, then $\widehat\Phi'$ is related to $\widehat\Phi$ via the relation

$$\widehat\Phi'(\check\rho',\check\sigma',\check v') = \widehat\Phi(\check\rho,\check\sigma,\check v) \quad \text{for} \quad \begin{pmatrix}\check\rho' & \check v'\\ \check v' & \check\sigma'\end{pmatrix} = \begin{pmatrix} d & b\\ c & a\end{pmatrix}\begin{pmatrix}\check\rho & \check v\\ \check v & \check\sigma\end{pmatrix}\begin{pmatrix} d & c\\ b & a\end{pmatrix}. \tag{5.0.10}$$

This allows us to determine the partition function of all other sets of torsion r dyons from the partition function given in (5.0.4). In particular for $r=2$, choosing $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we recover the dyon partition function proposed in [62].

We shall now show that the proposed partition function (5.0.4) satisfies various consistency tests described in [62].

S-duality invariance: We shall first verify the required S-duality invariance of the partition function described in eq.(5.0.8). Using $Sp(2, \mathbb{Z})$ invariance of $\Phi_{10}(x,y,z)$, and that b is a multiple of r for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(r)$ one can show that

$$\Phi_{10}\left(\check\rho', s^2\check\sigma' + \frac{k}{\bar s^2}, s\check v' + \frac{l}{\bar s}\right) = \Phi_{10}\left(\check\rho, s^2\check\sigma + \frac{k'}{\bar s^2}, s\check v + \frac{l'}{\bar s}\right), \tag{5.0.11}$$

where

$$k' = kd^2 - 2cdlr \in \mathbb{Z}, \qquad l' = (ad+bc)l - bdk/r \in \mathbb{Z}. \tag{5.0.12}$$

Thus

$$\sum_{k=0}^{\bar s^2-1}\sum_{l=0}^{\bar s-1} \Phi_{10}\left(\check\rho', s^2\check\sigma' + \frac{k}{\bar s^2}, s\check v' + \frac{l}{\bar s}\right)^{-1} = \sum_{k'=0}^{\bar s^2-1}\sum_{l'=0}^{\bar s-1} \Phi_{10}\left(\check\rho, s^2\check\sigma + \frac{k'}{\bar s^2}, s\check v + \frac{l'}{\bar s}\right)^{-1}, \tag{5.0.13}$$

and we have the required relation (5.0.8).

Wall crossing formula: We shall now verify that (5.0.7) is consistent with the wall crossing formula. As in [62] we shall only consider the decay into a pair of half-BPS dyons [19,36,37,51]

$$(Q,P) \to (Q_1,P_1) + (Q_2,P_2), \tag{5.0.14}$$

$$(Q_1,P_1) = (\alpha Q + \beta P, \gamma Q + \delta P), \qquad (Q_2,P_2) = (\delta Q - \beta P, -\gamma Q + \alpha P), \tag{5.0.15}$$

$$\alpha\delta = \beta\gamma, \qquad \alpha + \delta = 1. \tag{5.0.16}$$

Since any lattice vector lying in the plane of Q and P can be expressed as a linear combination of Q/r and P with integer coefficients, we must have $\beta,\delta,\alpha \in \mathbb{Z}$, $\gamma \in \mathbb{Z}/r$. Thus we can write $\gamma = \gamma'/K$, where $K \in \mathbb{Z}$, $K|r$ and $\gcd(\gamma',K)$=1. The condition $\alpha\delta = \beta\gamma$ together with the integrality of α,β,δ now tells us that β must be of the form $K\beta'$ with $\beta' \in \mathbb{Z}$. Thus we have

$$\beta = K\,\beta', \quad \gamma = \frac{\gamma'}{K}, \quad K,\alpha,\delta,\beta',\gamma' \in \mathbb{Z}, \quad K|r, \quad \gcd(\gamma',K) = 1. \tag{5.0.17}$$

Using eqs.(5.0.16), (5.0.17) we have

$$\alpha + \delta = 1, \qquad \alpha\delta = \beta'\gamma', \qquad \alpha,\beta',\gamma',\delta \in \mathbb{Z}. \tag{5.0.18}$$

The analysis of [19] now shows that we can find a', b', c', d' such that

$$\alpha = a'd', \quad \beta' = -a'b', \quad \gamma' = c'd', \quad \delta = -b'c', \quad a',b',c',d' \in \mathbb{Z}, \quad a'd' - b'c' = 1. \tag{5.0.19}$$

As a consequence of (5.0.19) and the relation $\gcd(\gamma', K) = 1$ we have

$$\gcd(a', b') = \gcd(a', c') = \gcd(c', d') = \gcd(b', d') = 1, \quad \gcd(c', K) = \gcd(d', K) = 1\,. \tag{5.0.20}$$

Using eqs.(5.0.17)-(5.0.19) we can now express (5.0.15) as

$$(Q_1, P_1) = (a'K, c')(d'\bar{K}Q/r - b'P), \qquad (Q_2, P_2) = (b'K, d')(-c'\bar{K}Q/r + a'P), \qquad \bar{K} \equiv r/K\,. \tag{5.0.21}$$

Since according to (5.0.20), $\gcd(a'K, c')$=1, $\gcd(b'K, d')$=1, and since any lattice vector lying in the Q-P plane can be expressed as integer linear combinations of Q/r and P, it follows from (5.0.21) that (Q_1, P_1) can be regarded as N_1 times a primitive vector and (Q_2, P_2) can be regarded as N_2 times a primitive vector where

$$N_1 = \gcd(d'\bar{K}, b') = \gcd(\bar{K}, b'), \qquad N_2 = \gcd(c'\bar{K}, a') = \gcd(\bar{K}, a')\,. \tag{5.0.22}$$

In the last steps we have again made use of (5.0.20). It follows from (5.0.20) and (5.0.22) that

$$\gcd(N_1, N_2) = 1, \quad N_1 N_2 | \bar{K}\,. \tag{5.0.23}$$

We shall now use the formula (5.0.7) for the index in different regions of the moduli space and calculate the change in the index as we cross the wall of marginal stability associated with the decay (5.0.14). For this we need to know how to choose the imaginary parts of $\check{\rho}$, $\check{\sigma}$ and $\check{v}$ along the integration contour for the two domains lying on the two sides of this wall of marginal stability. A prescription for choosing this contour was postulated in [62] according to which as we cross the wall of marginal stability associated with the decay (5.0.14), the integration contour crosses a pole of the partition function at

$$\check{\rho}\gamma - \check{\sigma}\beta + \check{v}(\alpha - \delta) = 0\,. \tag{5.0.24}$$

Thus the change in the index can be calculated by evaluating the residue of the partition function at this pole. We shall now examine for which values of s the $\Phi_{10}(\check{\rho}, s^2\check{\sigma}, s\check{v})^{-1}$ term in the expansion (5.0.7) has a pole at (5.0.24). The poles of $\Phi_{10}(\check{\rho}, s^2\check{\sigma}, s\check{v})^{-1}$ are known to be at

$$\begin{aligned} & n_2\, s^2\,(\check{\rho}\check{\sigma} - \check{v}^2) + n_1 s^2 \check{\sigma} - m_1 \check{\rho} + m_2 + j s \check{v} = 0 \\ & m_1, n_1, m_2, n_2 \in \mathbb{Z}, \quad j \in 2\,\mathbb{Z} + 1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,. \end{aligned} \tag{5.0.25}$$

Comparing (5.0.25) and (5.0.24) we see that we must have

$$m_2 = n_2 = 0, \qquad j = \frac{\lambda}{s}(\alpha - \delta), \qquad n_1 = -\frac{\lambda}{s^2}\beta, \qquad m_1 = -\lambda\gamma\,, \tag{5.0.26}$$

for some λ. The last condition in (5.0.25), together with eqs.(5.0.16) now gives

$$\lambda = s\,. \tag{5.0.27}$$

Thus we have from (5.0.17), (5.0.26)

$$j = \alpha - \delta, \qquad m_1 = -s\gamma = -\gamma' s/K, \qquad n_1 = -\beta/s = -K\beta'/s\,. \tag{5.0.28}$$

Since $\gcd(\gamma', K) = 1$, the second equation in (5.0.28) shows that s must be a multiple of K:

$$s = K\,\tilde{s}, \qquad \tilde{s} \in \mathbb{Z}\,. \tag{5.0.29}$$

Substituting this into the last equation in (5.0.28) and using (5.0.19) we see that

$$n_1 = \frac{a'b'}{\tilde{s}} \quad \Rightarrow \quad \frac{a'b'}{\tilde{s}} \in \mathbb{Z}\,. \tag{5.0.30}$$

Since $\gcd(a', b')$=1, we must have a unique decomposition

$$\tilde{s} = L_1 L_2, \qquad L_1 | b', \qquad L_2 | a'\,. \tag{5.0.31}$$

On the other hand since s divides r, it follows from (5.0.29) that $\tilde{s}$ must divide $r/K = \bar{K}$. Thus

$$L_1 | \bar{K}, \qquad L_2 | \bar{K}\,. \tag{5.0.32}$$

It now follows from (5.0.22) that

$$L_1 | N_1, \qquad L_2 | N_2\,. \tag{5.0.33}$$

Conversely, given any pair (L_1, L_2) satisfying (5.0.33), it follows from (5.0.22) that L_1, L_2 will satisfy (5.0.31), (5.0.32). This allows us to find integers m_1, n_1, j satisfying (5.0.26) via eqs.(5.0.27)-(5.0.31).

This shows that the poles of $\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v})^{-1}$ at (5.0.24) can come from the $s = KL_1L_2$ terms in (5.0.4) for $L_1|N_1$ and $L_2|N_2$. Our next task is to find the residues at these poles to compute the change in the index as we cross this wall. For this we define

$$\bar{a} = a'/L_2, \quad \bar{\mathrm{D}} = d'L_2, \quad \bar{b} = b'/L_1, \quad \bar{c} = c'L_1, \quad s_0 = KL_1L_2\,. \tag{5.0.34}$$

It follows from (5.0.31) that $\bar{a}, \bar{b}, \bar{c}, \bar{\mathrm{D}}$ are all integers. In terms of these variables the location of the pole given in (5.0.24) can be expressed as

$$s_0^{-1}\,\bar{c}\bar{\mathrm{D}}\check{\rho} + \bar{a}\bar{b}s_0\check{\sigma} + (\bar{a}\bar{\mathrm{D}} + \bar{b}\bar{c})\check{v} = 0\,. \tag{5.0.35}$$

We now define

$$\check{\rho}' = \bar{\mathrm{D}}^2\check{\rho} + \bar{b}^2 s_0^2\check{\sigma} + 2\bar{b}\bar{\mathrm{D}}s_0\check{v}, \quad \check{\sigma}' = \bar{c}^2\check{\rho} + \bar{a}^2 s_0^2\check{\sigma} + 2\bar{a}\bar{c}s_0\check{v}, \quad \check{v}' = \bar{c}\bar{\mathrm{D}}\check{\rho} + \bar{a}\bar{b}s_0^2\check{\sigma} + (\bar{a}\bar{\mathrm{D}} + \bar{b}\bar{c})s_0\check{v}\,. \tag{5.0.36}$$

The change of variables from $(\check{\rho}, s_0^2\check{\sigma}, s_0\check{v})$ to $(\check{\rho}', \check{\sigma}', \check{v}')$ is an $Sp(2, \mathbb{Z})$ transformation. Thus we have

$$\Phi_{10}(\check{\rho}, s_0^2\check{\sigma}, s_0\check{v}) = \Phi_{10}(\check{\rho}', \check{\sigma}', \check{v}')\,. \tag{5.0.37}$$

In the primed variables the desired pole at (5.0.35) is at $\check{v}'=0$. We also have

$$\check{\rho}P^2+\check{\sigma}Q^2+2\check{v}Q\cdot P=\check{\rho}'P'^2+\check{\sigma}'Q'^2+2\check{v}'Q'\cdot P'\,, \tag{5.0.38}$$

where

$$Q'=\bar{\mathrm{D}}\,Q/s_0-\bar{b}P,\qquad P'=-\bar{c}Q/s_0+\bar{a}P\,. \tag{5.0.39}$$

Finally we have

$$d\check{\rho}'d\check{\sigma}'d\check{v}'=s_0^3 d\check{\rho}d\check{v}d\check{\sigma}\,. \tag{5.0.40}$$

This is consistent with the fact that $\Phi_{10}(\check{\rho}',\check{\sigma}',\check{v}')$ is invariant under integer shifts in $\check{\rho}'$, $\check{\sigma}'$ and $\check{v}'$ so that in the primed variables the volume of the unit cell is 1, while in the unprimed variables the volume of the unit cell is $1/s_0^3$. We can now express the change in the index from the $s=s_0$ term in (5.0.7) as

$$\begin{aligned}(\Delta d(Q,P))_{s_0} \;=\;& (-1)^{Q\cdot P+1}\,g(s_0)\int_{iM_1'-1/2}^{iM_1'+1/2}d\check{\rho}'\int_{iM_2'-1/2}^{iM_2'+1/2}d\check{\sigma}'\oint d\check{v}'\\ & e^{-i\pi(\check{\sigma}'Q'^2+\check{\rho}'P'^2+2\check{v}'Q'\cdot P')}\,\Phi_{10}\left(\check{\rho}',\check{\sigma}',\check{v}'\right)^{-1}\,,\end{aligned} \tag{5.0.41}$$

where the $\check{v}'$ contour is around the origin, – as in [62] we shall use the convention that the contour is in the clockwise direction. Using the fact that

$$\Phi_{10}\left(\check{\rho}',\check{\sigma}',\check{v}'\right)=-4\pi^2\,\check{v}'^2\,\eta(\check{\rho}')^{24}\,\eta(\check{\sigma}')^{24}+\mathcal{O}\left(\check{v}'^4\right)\,, \tag{5.0.42}$$

near $\check{v}'=0$, we get

$$\begin{aligned}(\Delta d(Q,P))_{s_0} \;=\;& (-1)^{Q\cdot P+1}\,g(s_0)\,Q'\cdot P'\int_{iM_1'-1/2}^{iM_1'+1/2}d\check{\rho}'e^{-i\pi\check{\rho}'P'^2}\,\eta(\check{\rho}')^{-24}\\ & \int_{iM_2'-1/2}^{iM_2'+1/2}d\check{\sigma}'e^{-i\pi\check{\sigma}'Q'^2}\,\eta(\check{\sigma}')^{-24}\\ =\;& (-1)^{Q\cdot P+1}\,g(s_0)\,Q'\cdot P'\,d_h(Q',0)\,d_h(P',0)\,,\end{aligned} \tag{5.0.43}$$

where $d_h(q,0)$ denotes the index measuring the number of bosonic half BPS supermultiplets minus the number of fermionic half BPS supermultiplets carrying charge $(q,0)$.

We shall now express the right hand side of (5.0.43) in terms of the charges (Q_1,P_1) and (Q_2,P_2) of the decay products. First of all it is easy to see that

$$(-1)^{Q\cdot P}=(-1)^{Q_1\cdot P_2-Q_2\cdot P_1}\,. \tag{5.0.44}$$

Furthermore it follows from (5.0.21), (5.0.34) and (5.0.39) that

$$(Q_1,P_1)=L_1\,\left(a'KQ',c'Q'\right),\qquad (Q_2,P_2)=L_2\,\left(b'KP',d'P'\right)\,, \tag{5.0.45}$$

$$Q_1\cdot P_2-Q_2\cdot P_1=s_0\,Q'\cdot P'\,. \tag{5.0.46}$$

Now according to (5.0.20) the pair of integers $(a'K, c')$ are relatively prime, and the pair of integers $(b'K, d')$ are also relatively prime. Thus using S-duality invariance we can write

$$\begin{aligned} d_h\left(\frac{Q_1}{L_1}, \frac{P_1}{L_1}\right) &= d_h\left(a'KQ', c'Q'\right) = d_h(Q', 0), \\ d_h\left(\frac{Q_2}{L_2}, \frac{P_2}{L_2}\right) &= d_h\left(b'KP', d'P'\right) = d_h(P', 0). \end{aligned} \tag{5.0.47}$$

Using (5.0.44), (5.0.46) and (5.0.47) we can now express (5.0.43) as

$$(\Delta d(Q,P))_{s_0} = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\, g(s_0)\, \frac{1}{s_0}\, (Q_1\cdot P_2 - Q_2\cdot P_1)\, d_h\left(\frac{Q_1}{L_1}, \frac{P_1}{L_1}\right) d_h\left(\frac{Q_2}{L_2}, \frac{P_2}{L_2}\right). \tag{5.0.48}$$

For a given decay K is fixed but L_1 and L_2 can vary over all the factors of N_1 and N_2. Thus the total change in the index is obtained by summing over all possible values of s_0 of the form KL_1L_2. Thus gives

$$\begin{aligned} \Delta d(Q,P)) &= (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\, (Q_1\cdot P_2 - Q_2\cdot P_1) \\ & \sum_{L_1|N_1}\sum_{L_2|N_2} g(KL_1L_2)\, \frac{1}{KL_1L_2}\, d_h\left(\frac{Q_1}{L_1}, \frac{P_1}{L_1}\right) d_h\left(\frac{Q_2}{L_2}, \frac{P_2}{L_2}\right). \end{aligned} \tag{5.0.49}$$

We can now fix the form of the function $g(s)$ by considering a decay where the decay products are primitive, i.e. $N_1 = N_2 = 1$. In this case we have $L_1 = L_2 = 1$, and (5.0.49) takes the form[1]

$$\Delta d(Q,P)) = (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1+1}\, (Q_1\cdot P_2 - Q_2\cdot P_1)\, g(K)\, \frac{1}{K}\, d_h(Q_1, P_1)\, d_h(Q_2, P_2). \tag{5.0.50}$$

In order that this agrees with the standard wall crossing formula for primitive decay [19, 41–46, 64] we must have $g(K) = K$. Since this result should hold for all $K|r$ we see that we must set $g(s) = s$ as given in (5.0.5). Using this we can simplify the wall crossing formula (5.0.49) for generic non-primitive decays to the form given in (5.0.6).

Black hole entropy: In order to reproduce the leading contribution to the black hole entropy in the limit of large charges, the partition function must have a pole at [7]

$$\check\rho\check\sigma - \check v^2 + \check v = 0. \tag{5.0.51}$$

[1] In this argument we have implicitly assumed that for a given K, it is possible to find integers a', b', c', d' satisfying $a'd' - b'c' = 1$, $\gcd(c', K) = \gcd(d', K) = \gcd(a', \bar K) = \gcd(b', \bar K) = 1$, so that (5.0.20) holds and we have $N_1 = N_2 = 1$ according to (5.0.22). If either K or $\bar K$ is odd then this assumption holds with the choice $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. If K and $\bar K$ are both even then we cannot find a', b', c' and d' satisfying all the requirements since in order to satisfy $a'd' - b'c' = 1$ at least one of them must be even. However in this case if we choose $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ then we satisfy (5.0.20) and have $N_1 = 1$ and $N_2 = 2$ according to (5.0.22). We can now demand that the wall crossing formula in this case agrees with the one derived in [62] for decays where one of the decay products is twice a primitive vector. This gives $g(K) = K$ even for K, $\bar K$ both even.

Furthermore, in order to reproduce the black hole entropy to first non-leading order, the inverse of the partition function near this pole must behave as [8, 62]

$$\widehat{\Phi}(\check{\rho}, \check{\sigma}, \check{v}) \propto (2v - \rho - \sigma)^{10} \{v^2 \, \eta(\rho)^{24} \, \eta(\sigma)^{24} + \mathcal{O}(v^4)\}\,, \tag{5.0.52}$$

where

$$\rho = \frac{\check{\rho}\check{\sigma} - \check{v}^2}{\check{\sigma}}, \quad \sigma = \frac{\check{\rho}\check{\sigma} - (\check{v} - 1)^2}{\check{\sigma}}, \quad v = \frac{\check{\rho}\check{\sigma} - \check{v}^2 + \check{v}}{\check{\sigma}}\,. \tag{5.0.53}$$

We shall now examine the poles of $\widehat{\Phi}$ given in (5.0.4) to see if it satisfies the above relations. For this we recall eq.(5.0.25) giving the pole of $\Phi_{10}(\check{\rho}, s^2\check{\sigma}, s\check{v})^{-1}$. Comparing (5.0.25) with (5.0.51) we see that in order to get a pole at (5.0.51) we must choose in (5.0.25)

$$n_2 = \lambda/s^2, \qquad j = \lambda/s, \qquad n_1 = m_1 = m_2 = 0\,, \tag{5.0.54}$$

for some λ. The requirement $m_1 n_1 + m_2 n_2 + \frac{1}{4} j^2 = \frac{1}{4}$ then gives

$$\lambda = s, \qquad n_2 = \frac{1}{s}\,. \tag{5.0.55}$$

Since n_2 must be an integer this gives $s = 1$. Thus only the $s = 1$ term in (5.0.4) has a pole at (5.0.51). It follows from the known behaviour of Φ_{10} near its zeroes that $\widehat{\Phi}$ defined in (5.0.4) satisfies the requirement (5.0.52).

Gauge theory limit: Finally we shall show that in special regions of the Narain moduli space where the low lying states in string theory describe a non-abelian gauge theory, the proposed dyon spectrum of string theory reproduces the known results in gauge theory. Since the T-duality invariant metric L in the Narain moduli space descends to the negative of the Cartan metric in gauge theories, and since the Cartan metric is positive definite, all the gauge theory dyons have the property

$$Q^2 < 0, \qquad P^2 < 0, \qquad Q^2 P^2 > (Q \cdot P)^2\,. \tag{5.0.56}$$

Thus in order to identify dyons which could be interpreted as gauge theory dyons in an appropriate limit we must focus on charge vectors satisfying (5.0.56).

In order to identify such dyons we need to expand the partition function $\widehat{\Phi}^{-1}$ in powers of $e^{2\pi i\check{\rho}}$, $e^{2\pi i\check{\sigma}}$ and $e^{2\pi i\check{v}}$ and pick up the appropriate terms in the expansion. For this we need to identify a region in the $(\check{\rho}, \check{\sigma}, \check{v})$ space (or equivalently in the asymptotic moduli space) where we carry out the expansion, since in different regions we have different expansion. We shall consider the region

$$\Im(\check{\rho}), \Im(\check{\sigma}) >> -\Im(\check{v}) >> 1\,, \tag{5.0.57}$$

where $\Im(z)$ denotes the imaginary part of z. The results in other relevant regions will be related to the ones in this region by S-duality transformation. In the region (5.0.57) the only term in $\Phi_{10}\left(\check{\rho}, s^2\check{\sigma}+\frac{k}{s^2}, s\check{v}+\frac{l}{s}\right)^{-1}$ which has powers of $e^{2\pi i\check{\rho}}$, $e^{2\pi i\check{\sigma}}$ and $e^{2\pi i\check{v}}$ compatible with the requirement (5.0.56) is

$$e^{-2\pi i\check{\rho}-2\pi i(s^2\check{\sigma}+\frac{k}{s^2})-2\pi i(s\check{v}+\frac{l}{s})}\,. \tag{5.0.58}$$

This is in fact the leading term in the expansion in the limit (5.0.57). Substituting this into (5.0.4) and performing the sum over k and l we see that only the $s=r$ term in the sum survives and the result is

$$r\, e^{-2\pi i(\check{\rho}+r^2\check{\sigma}+r\check{v})}\,. \tag{5.0.59}$$

This corresponds to dyons with

$$Q^2/2=-r^2,\quad P^2/2=-1,\quad Q\cdot P=-r\,, \tag{5.0.60}$$

with an index of $(-1)^{r+1}\,r$.

We can also determine the walls of marginal stability which border the domain in which these dyons exist. This requires determining the region in the $(\Im(\check{\rho}),\Im(\check{\sigma}),\Im(\check{v}))$ space in which the expansion (5.0.59) of $\Phi_{10}(\check{\rho},r^2\check{\sigma},r\check{v})^{-1}$ is valid since then we can determine the associated walls of marginal stability using (5.0.24). For this we shall utilize the known results for $r=1$; in this case the walls of marginal stability bordering the domain in which (5.0.59) is valid correspond to the decays into $(Q,0)+(0,P)$, $(Q-P,0)+(P,P)$ and $(0,P-Q)+(Q,Q)$ respectively [59]. Using (5.0.24) we now see that validity of the expansion of $\Phi_{10}(\check{\rho},\check{\sigma},\check{v})^{-1}$ given by (5.0.59) with $r=1$ is bounded by the following surfaces in the $(\Im(\check{\rho}),\Im(\check{\sigma}),\Im(\check{v}))$ space:

$$\Im(\check{v})=0\,,\qquad \Im(\check{v}+\check{\sigma})=0,\qquad \Im(\check{v}+\check{\rho})=0\,. \tag{5.0.61}$$

We can now simply scale $\check{\sigma}$ by r^2 and $\check{v}$ by r to determine the region of validity of the expansion (5.0.59) for $\Phi_{10}(\check{\rho},r^2\check{\sigma},r\check{v})$:

$$\Im(\check{v})=0\,,\qquad \Im(\check{v}+r\check{\sigma})=0,\qquad \Im(r\check{v}+\check{\rho})=0\,. \tag{5.0.62}$$

Comparing these with (5.0.24) we now see that the corresponding walls of marginal stability are associated with the decays into

$$(Q,0)+(0,P),\qquad (Q-rP,0)+(rP,P),\qquad \left(0,P-\frac{1}{r}Q\right)+\left(Q,\frac{1}{r}Q\right)\,. \tag{5.0.63}$$

Let us now compare these results with dyons in $\mathcal{N}=4$ supersymmetric $SU(3)$ gauge theory. If we denote by α_1 and α_2 a pair of simple roots of $SU(3)$ with $\alpha_1^2=\alpha_2^2=-2$ and $\alpha_1\cdot\alpha_2=1$, then the analysis of [49, 50, 66, 68, 69] shows that the gauge theory contains dyons of charge

$$(Q,P)=(r\alpha_1,-\alpha_2)\,, \tag{5.0.64}$$

with index $(-1)^{r+1}\,r$. These are precisely the dyons of the type given in (5.0.60). Furthermore using string junction picture [27, 28], ref. [68] also determined the walls of marginal stability bordering the domain in which these dyons exist. These also coincide with (5.0.63).

The spectrum in gauge theory contains other dyons of torsion r related to the ones described above by S-duality transformation. Since our construction is manifestly S-duality invariant, the results for these dyons can also be reproduced from the general formula given in (5.0.4).

Gauge theory also contains other dyons which are not related to the ones described here by S-duality [49, 50]. These typically require higher gauge groups and has additional fermionic zero modes besides the ones required by broken supersymmetry. Quantization of these additional zero modes gives rise to additional bose-fermi degeneracy, and as a result the index being computed here vanishes for these dyons. This is also apparent from the fact that these dyons typically exist only in a subspace of the full moduli space; as a result when we move away from this subspace the various states combine and become non-BPS. Some aspects of these dyons have been discussed recently in [66, 67].

Chapter 6

Concluding Remarks

Complete understanding of the microstates of an arbitrary black hole is one the motivations behind a quantum theory of grvity. String theory is the leading candidate for a quantum gravity theory and there is much progress in understanding the microscopics of supersymmetric black holes.

In this thesis our aim was to calculate the microscopic degeneracy of Black holes, in $N = 4$ superstring theory, which preserve $\frac{1}{4}$-th of the supersymmetries. The formula was known [7] for black holes which carry charges (Q, P) subject to the condition, $g.c.d(Q \wedge P) = 1$ [20, 40, 54]. We extended the result to black holes carrying arbitrary charge vectors. Although we could not provide a first principle derivation of the result, it was done after some time in the reference [76].

There are some aspects of these formulae which require better understanding. We have already seen the appearance of the modular group of the genus two surface, namely $Sp(2, \mathbb{Z})$. The partition function of the theory is given by a modular form of this group or one of its subgroups. But the appearance of $Sp(2, \mathbb{Z})$ is not expected because it is not a symmetry group of the theory. The S-duality group is only a subgroup of this. The first step towards explaining this has been taken in [10, 18]. The contour prescription which we discussed in the introduction can also be derived [75] from the M-theory lift described in the last two papers. It will be very interesting to see how much more can be learned from this picture.

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