Higher-Spin Theories and the AdS/CFT Correspondence

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Certificate

This is to certify that the Ph. D. thesis titled "Higher-Spin Theories and the AdS/CFT Correspondence" submitted by Shailesh Lal is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Date:

Prof. Rajesh Gopakumar
Thesis Advisor

Declaration

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgment of collaborative research and discussions.

The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

This work was done under guidance of Professor Rajesh Gopakumar, at Harish-Chandra Research Institute, Allahabad.

Date:

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Higher-Spin Theories and the AdS/CFT Correspondence

Thesis Synopsis

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The AdS/CFT Correspondence is a remarkable output of string theory—a theory of quantum gravity—which conjectures that a string theory defined on spacetimes that asymptote to Anti de-Sitter (AdS) spaces are actually completely equivalent to a conformal field theory (CFT) which is defined on the spatial boundary of the AdS. Since a conformal field theory is a usual relativistic Quantum Field Theory, which we understand well, this gives us a tool to explore string theory and Quantum Gravity. However, though physicists have tested this duality and have always found it to be true, many aspects of this duality remain unclear. For example, supersymmetry is ubiquitous in the known instances of this duality. At the same time, the general reasons for expecting a string description of Quantum Field theories are quite independent of supersymmetry. We expect that insight into the mechanics of this duality will give us intuition for more ambitious questions like formulating a string dual for Quantum Chromodynamics, the theory of strong interactions within the nucleus.

This thesis describes out work on the holography of higher-spin theories in AdS. The motivations are two-fold: firstly, in the low-dimensional examples of AdS_3 and AdS_4, these theories are expected to have CFT duals in their own right. These dualities are non-supersymmetric and therefore offer a rare concrete opportunity to explore AdS/CFT in non-supersymmetric settings. Secondly, the major thrust of research in AdS/CFT has been towards using the regime of classical weak gravity (Einstein's General Relativity) as a tool to understand the physics of strongly coupled planar gauge theories, which is typically intractable *via* standard field theory techniques. Alternatively, one might try to probe the opposite corner in the landscape of the duality, that of free planar gauge theories, which correspond to tensionless string theory on AdS spacetimes. This is a natural starting point to get insight about the mechanics of AdS/CFT and gauge-string duality. However, even for supersymmetric settings, this limit of string theory has hitherto proven intractable. Here we may expect that the higher-spin symmetry might play the role of a different large symmetry group that might render the problem tractable. As a subset of the full duality, we also expect that in the tensionless limit, there is a sector of the string theory which consists of an infinite tower of massless, interacting particles, which should be described by a consistent classical theory. It is natural to expect that this theory is a Vasiliev theory. Our work in higher-spin theories is with these (not unrelated) questions in mind.

An important tool in our exploration of higher-spin theories will be the one-loop partition function of the AdS higher-spin theory, which contains the leading quantum corrections to its classical description. We will define this partition function in terms of the heat kernel of the Laplacian acting on symmetric, transverse traceless tensors of arbitrary rank. Evaluating the heat kernel over AdS is *a priori* a hard problem, as it involves determining the eigenvalues and eigenfunctions of the Laplacian in curved spaces over tensors of arbitrary rank. A useful tool to exploit is the fact that AdS is a symmetric space, and these questions have a group theoretic interpretation. A simple example well known from Quantum Mechanics is that the eigenfunctions of the Laplacian acting on the wave function of a particle moving on the two-dimensional sphere are matrix elements of Wigner matrices, and the eigenvalues can be interpreted as quadratic Casimirs of $SU(2)$. We use this fact to determine a closed-form expression the heat kernel of the Laplacian for arbitrary spin fields over quotients of (Euclidean) AdS. This expression is in terms of characters and quadratic Casimirs of unitary representations of the Lorentz group and the special orthogonal group, which have been completely determined in the existing literature.

Also, motivated by the Gaberdiel-Gopakumar conjecture relating higher-spin theories in AdS_3 to $\mathcal{W}_N$ minimal models, we consider a topologically massive deformation of these theories. By this, we mean that we consider a (parity violating) spin-s free field theory which has the linearised higher-spin gauge symmetry, but whose excitations include massive modes. This is nontrivial, because typically adding mass terms to gauge fields destroys the gauge symmetry. In particular, we compute the one-loop partition function of a topologically massive spin-3 field. This provides us with evidence that the $\mathcal{W}$ symmetry survives the chiral limit, which was a question unclear from previous analyses. We also compute the partition function for arbitrary spin fields, which enables us to identify generalisations of the spin-1 excitation seen in the spin-3 field. A spin-s field contains transverse traceless excitations of spin s, $s-1, \ldots, 1$. We expect that these expressions will be an important ingredient in formulating an explicit duality for topologically massive higher-spin theories.

We finally turn to the computation of the partition function of higher-spin theories in AdS_5. We consider the higher-spin theory containing all spins from zero to infinity appearing only once. We find simplifications in the multiparticle partition function. In particular, for thermal AdS_5, we are able to write the answer in a form strongly suggestive of a vacuum character, and also in a form that is reminiscitive of a higher-dimensional MacMahon function, generalising the observation of Gaberdiel, Gopakumar and Saha, who noted that the AdS_3 partition function could be expressed in terms of the two-dimensional MacMahon function. We also turn on chemical potentials for the other AdS_5 Cartans and compute the one-loop partition function. We again obtain a vacuum character-like form for the answer. We take this as evidence that the asymptotic symmetry algebra of higher-spin theories in AdS_5 gets enhanced, in a manner similar to that of AdS_3, to include additional generators that act on multi-particle states of the theory.

List of publications that form the thesis of the candidate

1. *Partition Functions for Higher-Spin theories in AdS.* Rajesh Kumar Gupta, Shailesh Lal.
   JHEP 1207 (2012) 071, [arXiv:1205.1130]
2. *One loop partition function for Topologically Massive Higher Spin Gravity.* Arjun Bagchi, Shailesh Lal, Arunabha Saha, Bindusar Sahoo.
   JHEP 1112 (2011) 068, [arXiv:1107.2063] [arXiv:1107.2063]
3. *The Heat Kernel on AdS.* Rajesh Gopakumar, Rajesh Kumar Gupta, Shailesh Lal.
   JHEP 1111 (2011) 010 [arXiv:1103.3627]

List of papers of the candidate that are not included in the thesis

1. *Topologically Massive Higher Spin Gravity.* Arjun Bagchi, Shailesh Lal, Arunabha Saha, Bindusar Sahoo.
   JHEP 1110 (2011) 150, [arXiv:1107.0915]
2. *Rational Terms in Theories with Matter.* Shailesh Lal, Suvrat Raju.
   JHEP 1008 (2010) 022, [arXiv:1003.5264]
3. *The Next-to-Simplest Quantum Field Theories.* Shailesh Lal, Suvrat Raju.
   Phys. Rev. D81 (2010) 105002, [arXiv:0910.0930]

Contents

Part I

Introduction

Chapter 1

Introduction

Describing how interactions take place in nature has been the single aim of science over the past many centuries. From the efforts of scientists over the past century or so, a paradigm has emerged, containing some key elements which any theory of nature must contain.

First of all, any theory that attempts to describe particles moving near the speed of light must incorporate Einstein's Relativity. If the situation is such that gravitational effects can be ignored (this is true when the particles are very light) then Special Relativity is sufficient, but if gravitational interactions are significant, then we must incorporate General Relativity.

Secondly, a theory that attempts to describe physics at lengths smaller than the size of an atom must incorporate Quantum Mechanics, according to which, particles have a "dual" description as waves smeared out over space. Although this seems bizarre at first sight, it has been tested rigorously by experiments for a hundred years, and has always been confirmed. We now strongly believe that Quantum Mechanics is an essential element of any theory that aims to describe Nature.

The task before us, as physicists, is therefore to invent a framework in which both Relativity and Quantum Mechanics may be incorporated. This is essentially the minimal prerequisite for a unified framework to describe all interactions that occur in nature. For Special Relativity and Quantum Mechanics, this program has been implemented in a mathematical framework known as Quantum Field Theory (QFT). This is the framework in which, for example, the quantum theory of electricity and magnetism is formulated.

At the time of writing, there are several important questions which have been only partially addressed or where only preliminary explorations have been made, largely due to the complexity of the problems. Here are some of them:

1. The interactions within the nucleus of an atom.
2. High temperature superconductivity, and other phenomena such as relativistic fluid flows.
3. A complete theory of Quantum Gravity, which we expect will be crucial to understand concretely many questions about Black Holes and the Big Bang singularity.

String Theory is a theory of quantum gravity, and as such aims to solve 3. In the mid 1990s a remarkable conjecture was proposed by Maldacena [1], provides a novel interrelation between the three questions outlined above. He conjectured that string theory (quantum gravity) defined on asymptotically AdS spaces is completely equivalent to a conformal field theory that lives on the boundary of the AdS. By translating questions about quantum gravity to the quantum field theory, we can address questions about quantum gravity without ever having to directly quantise gravity! Equally interestingly, by using the fact that General Relativity (or Supergravity) is an approximate description of the underlying string theory, one can use this correspondence to shed light on important questions about the hologram QFT. This conjecture is known as the AdS/CFT correspondence (or AdS/CFT).

We shall soon see how higher-spin theories fit into this framework, but let us first review the correspondence itself.

1.1 The AdS/CFT Correspondence: A review

In this section, we shall briefly review some essential elements of the AdS/CFT duality. Firstly, we will recollect some kinematical elements that make such a duality possible. In particular, we shall show how the isometry group of AdS acts as the conformal group on its spatial boundary. A chief element of this dictionary is that the AdS hamiltonian gets identified to the dilatation operator of the CFT. Secondly, we will then recapitulate a string theory argument from which this duality arises. Thirdly, we shall then present some preliminary tests of this duality, which have some non-trivial aspects in their own right.

1.1.1 The Geometry of AdS_{d+1}

We begin with a brief review of the AdS geometry. This has been very extensively reviewed in the literature, and we shall essentially follow [2, 3] here. One way of thinking about AdS space is to embed it in a higher-dimensional pseudo-Euclidean space. Let us consider a $d+2$ dimensional space $\mathbb{M}^{2,d}$ and coordinates $(y^0, \overrightarrow{y}, y^{d+1})$. The invariant length is

$$y^2 = (y^0)^2 + (y^{d+1})^2 - \overrightarrow{y}^2, \tag{1.1.1}$$

which is preserved by the group $SO(2,d)$. Then AdS_{d+1} may be defined as the subspace

$$y^2 = b^2, \tag{1.1.2}$$

where b is a constant. A convenient parametrisation of this space is

$$y^0 = b\,\cosh\rho\cos\tau,\; y^{d+1} = b\,\cosh\rho\sin\tau,\; \overrightarrow{y} = b\,\sinh\rho\overrightarrow{\Omega}, \tag{1.1.3}$$

where $\overrightarrow{\Omega}$ are coordinates on S^{d-1}, $0 \leq \tau < 2\pi$, and $0 \leq \rho$ in order to cover the AdS once. Clearly, $SO(2,d)$ is the symmetry group of AdS, manifest from the embedding (1.1.2) into $\mathbb{M}^{2,d}$.

To access the boundary of AdS, we shall take the ys on the locus (1.1.2) to be very large, via a scaling

$$y^a = R\,u^a, \tag{1.1.4}$$

where R is a real number which we shall take to infinity. In the coordinate system (1.1.3) this corresponds to taking $\rho \simeq 0$. In this limit, the locus (1.1.2) becomes

$$(u^0)^2 + (u^{d+1})^2 = \overrightarrow{u}^2, \tag{1.1.5}$$

which naively looks $d+1$ dimensional. However, we could have scaled by any other large parameter tR instead of R. Therefore, in the u parametrisation of the boundary, there is an equivalence

$$u \mapsto tu, \tag{1.1.6}$$

because of which the boundary is d dimensional. By fixing this arbitrariness in scaling, we can represent the boundary by

$$(u^0)^2 + (u^{d+1})^2 = 1 = \overrightarrow{u}^2. \tag{1.1.7}$$

The boundary is therefore $S^1 \otimes S^{d-1}$. It is further possible to show that the AdS_{d+1} isometry group $SO(2,d)$ acts on the boundary as the conformal group in d dimensions. We refer the reader to [2] for an explicit demonstration of this fact.

In this analysis we have so far omitted an important point. The induced metric on the AdS_{d+1} is given in the 'global' coordinates ρ, τ, Ω by

$$ds^2 = b^2 \left(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega^2\right). \tag{1.1.8}$$

As it stands, since $0 \leq \tau < 2\pi$, the geometry has closed timelike curves. To get around this, we unwrap the τ circle to let $-\infty < \tau < \infty$. The boundary is then $\mathbb{R} \otimes S^{d-1}$. The rest of the statements carry over as before. Especially, the $SO(2)$ that corresponds to translations in the τ coordinate, i.e. the AdS Hamiltonian, gets mapped to dilatations in the boundary conformal group. In particular, there is a dictionary which relates the eigenvalues Δ of operators in the CFT to the masses m of the fields in AdS that they are dual to, which we give some examples of.

- scalars: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2}\right)$,
- vectors: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{(d-2)^2 + 4m^2}\right)$,
- p-forms: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{(d-2p)^2 + 4m^2}\right)$,
- massless spin-2: $\Delta = d$.

We refer the reader to [3] and references therein for a more complete list.

For explicit computations, it is usually more convenient to work in the so-called Poincare patch of the AdS spacetime given by coordinates $(z, t, \vec{x})$

$$\begin{aligned} y^0 &= \frac{z}{2}\left(1 + \frac{1}{z^2}\left(b^2 + \vec{x}\cdot\vec{x} - t^2\right)\right), \quad y^{d+1} = \frac{bt}{z}, \\ y^i &= \frac{bx^i}{z}, \quad i = 1 \ldots d-1, \\ y^d &= \frac{z}{2}\left(1 - \frac{1}{z^2}\left(b^2 - \vec{x}\cdot\vec{x} + t^2\right)\right). \end{aligned} \tag{1.1.9}$$

with the metric

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dx^i dx^i + dz^2\right). \tag{1.1.10}$$

The boundary of the AdS is located at $z = 0$. These coordinates, however, cover only a part of the AdS spacetime.

As a final observation, we note that any geometry which is asymptotically AdS has an induced conformal structure on its boundary. In particular, the asymptotic AdS Killing vectors act as conformal Killing vectors on the boundary. This leads to an important subtlety in identifying the AdS_{d+1} isometry group to the d dimensional conformal group for the special case of $d = 2$. In particular, the AdS isometry group is $SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})$, which is just the global subgroup of the two-dimensional conformal group. How do we interpret the action of the other generators? It turns out that the additional generators act as large gauge transformations on the AdS vacuum, generating multiparticle *boundary graviton* states from the vacuum [7].

1.1.2 Arriving at the AdS_5/CFT$_4$ duality

We have so far demonstrated how the AdS isometry group acts as the group of conformal transformations on the boundary of AdS. This is still very far from arguing how a string theory defined on asymptotically AdS spacetimes may dynamically be the same as a CFT defined on its boundary.

We shall now review the central arguments of [1] from which this conjecture arises. This has been extensively reviewed in the literature, see for example [2–5], and we shall only recapitulate the central argument, closely following [5].

Our starting point will be Type IIB string theory in the supergravity approximation, in which we shall consider the black 3-brane metric for N units of flux. We take N to be large throughout. In the string frame, this metric is given by

$$ds^2 = H^{-\frac{1}{2}}(r)\,\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{1}{2}}(r)\,dx^m dx^m, \tag{1.1.11}$$

where μ, ν range from 0 to 3 and parametrize the directions parallel to the brane, and m runs from 4 to 9 and parametrizes the directions transverse to the brane, and we further have

$$H = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g N \alpha'^2, \quad r^2 = x^m x^m, \tag{1.1.12}$$

where g is the string coupling. The horizon of this geometry is located at $r = 0$. In the near horizon limit, this metric becomes

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2, \tag{1.1.13}$$

which is the $AdS_5 \otimes S^5$ geometry. The curvatures of both the AdS and the sphere are determined by the radius L. When $gN >> 1$, the radius L is large compared to the string length, and supergravity is a good approximation to the string theory.

Parallely, we consider N D3 branes in the IIB string theory in flat space. Perturbation theory about this system is parametrized by gN, every world-sheet boundary on the branes brings a factor of N from the Chan-Paton trace and g from the genus. Clearly, perturbation theory is valid when $gN << 1$.

It turns out that that the D-branes of string theory and the branes of supergravity are complementary descriptions of the same system [6] . When gN is small, string perturbation theory is valid, and the D-brane picture is appropriate, and when gN is large, the supergravity picture should be used.

The AdS/CFT duality arises from considering low-energy limits of both pictures. In the open string desciption, this consists of massless closed and open strings, localised on the brane. The massless closed string excitations are the supergravity multiplet, which decouples in the low-energy limit, and the open string excitations give the field theory defined on the world-volume of the D3 branes, namely $\mathcal{N} = 4$ super-Yang-Mills with gauge group $SU(N)$.[1]

In the supergravity picture, valid when $gN >> 1$, we have massless closed strings away from the brane, which decouple from the near-brane physics due to the infinite red-shift contained in g_{00} as r approaches 0. Near the brane, however, any massive closed string mode will have arbitrarily small energy, again due to the divergent red-shift.

Making the assumption that the low-energy limit commutes with varying g from a small to a large value leads us to conjecture that

$$D = 4, \quad \mathcal{N} = 4 \text{ SU(N) SYM} \equiv \text{IIB String theory on } AdS_5 \times S^5. \tag{1.1.14}$$

Let us consider how the parameters of string theory, the AdS radius L, the Planck length L_P, and the string tension α'^{-1}, relate to the gauge theory parameters, g_{YM} and N.

It turns out (see, for example, [5]) that upto overall proportionality constants,

$$\frac{1}{g^2} \sim \left(\frac{L}{L_P}\right)^8 \sim N^2, \quad \frac{L}{\alpha'^{1/2}} \sim \left(g_{YM}^2 N\right)^{\frac{1}{4}} \equiv \lambda^{\frac{1}{4}}, \tag{1.1.15}$$

where λ is the 't Hooft coupling. From this, it is clear that a planar gauge theory at a large value of λ corresponds to a low-energy (large string tension) classical theory in the bulk *i.e.* supergravity.

[1]Naively, we would get $U(N)$ as the gauge group, but the $U(1)$ corresponds to the motion of the stack of D-branes, and decouples.

We shall shortly consider the 'opposite' high-energy (zero tension) classical bulk theory, which corresponds to $\lambda = 0$ in the boundary CFT.

1.1.3 Computations with the AdS/CFT Correspondence

We have so far presented an argument that string theory on AdS spacetimes is the same theory as a conformal field theory defined on the boundary of the AdS. We now recollect the elements of a precise dictionary between the two theories, i.e. we show how observables of one theory map to observables of the other. This was identified by [11, 12]; and is reviewed, for example, in [3]. We shall follow the treatment in section 3.1.2 of [3]. Essentially, to make the correspondence computable, one posits that the partition functions of the string theory in AdS is equal to that of the CFT on the boundary. In particular, it is proposed that boundary values of AdS fields act as sources for operators in the field theory (this statement needs to be interpreted carefully, as we shall soon see). In particular, that

$$\left\langle e^{\int d^4x \phi_0(x)\mathcal{O}(x)} \right\rangle_{CFT} = \mathcal{Z}_{string}\left[\phi(x,z)|_{z=0} = \phi_0(x) \right]. \tag{1.1.16}$$

In particular, let us consider the simplest case of a massive scalar in AdS_{d+1} sourcing an operator $\mathcal{O}$ in the CFT$_d$. The wave equation for a massive scalar in AdS may be solved in the Poincare coordinates (1.1.10) to obtain the asymptotic behaviour

$$\phi(x,z) \sim A(x) \cdot z^{d-\Delta} + B(x) \cdot z^{\Delta}, \tag{1.1.17}$$

where

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}, \tag{1.1.18}$$

as Δ is typically greater that d. Now in the limit $z \to 0$, the field does not take a definite value, its behaviour is dominated by $z^{d-\Delta}$. We therefore modify (1.1.16) to impose the boundary condition

$$\phi(x,\epsilon) = \epsilon^{d-\Delta}\phi_0(x). \tag{1.1.19}$$

Since ϕ is an AdS scalar, ϕ_0 must have length dimensions of $\Delta - d$, and hence the operator $\mathcal{O}$ must have a conformal dimension Δ. This is precisely the correspondence mentioned previously. As a first application, one can use this prescription to compute, for example, two point functions of the CFT as carried out by Witten in [12]. We will not pursue this course here.

1.1.4 Preliminary Checks on the AdS/CFT Correspondence

We have so far recapitulated the argument for string theory in asymptotically AdS spacetimes to be dual to a conformal field theory on the boundary of the AdS. A very precise instance of this conjecture is the duality between Type IIB string theory and $\mathcal{N} = 4$ super-Yang-Mills with gauge group $SU(N)$. This has a number of appealing features.

1. Firstly, it is a conjecture that a string theory, a theory of quantum gravity, which is prohibitively difficult to solve directly, is actually equivalent to a Yang-Mills theory, which is much easier to quantize. This gives us a framework to explore a quantum theory of gravity without having to directly quantize gravity (or string theory).
2. Secondly, it states – in effect – that a theory of quantum gravity in a given spacetime is equivalent to a theory defined on the boundary of the spacetime. It is natural to expect that the correspondence is holographic in the sense of [8, 9], and one can show [3] that this is indeed the case.
3. Finally, it has long been argued [10] that there is a string description of planar Yang-Mills

theories. This conjecture explicitly realises this expectation. The string description is however in a higher number of spacetime dimensions than the Yang-Mills theory itself.

Thus we see that the AdS/CFT correspondence contains within itself explicit realizations of many profound ideas that have guided physicists for the past many years. A skeptic would perhaps be justified in saying that this seems too good to be true, and it would certainly be required to subject this conjecture to as many tests as possible, pending a complete proof.

We shall now describe a few preliminary checks that can immediately be made. We begin with the AdS_5 example.

1. First, we note that $AdS_5 \otimes S^5$ has the isometry group $SO(2,4) \otimes SO(6)$. This is precisely the group of (bosonic) symmetries of $\mathcal{N} = 4$ super-Yang-Mills in four dimensions. The $SO(2,4)$ is the group of conformal transformations, and the $SO(6)$ is the group of R-symmetries of the theory. This feature plays a key role in formulating a computational prescription for this correspondence [11, 12].
2. Using the symmetry matching as a guideline, it is natural to propose that fields in AdS_5 transforming under a given unitary irreducible representation of $SO(2,4)$ should correspond to gauge invariant operators transforming under the same representation of $SO(2,4)$ as reviewed earlier in the section. Applying this to the case of short representations, we are led to expect that boundary conserved currents should be dual to bulk massless fields. This is indeed the prescription of [11, 12].
3. It will soon become apparent that the window in which classical supergravity is a good approximation to the underlying string theory corresponds to the regime in which the dual gauge theory is a planar, strongly coupled theory. This raises a small puzzle (which is immediately resolved). It is known that supergravity on $AdS_5 \otimes S^5$ has infinite towers of Kaluza Klein modes arising from reduction on the five-sphere [13]. These are massive modes and do not saturate the $SO(2,4)$ unitarity bound, and yet their scaling dimension Δ should be protected from becoming very large in the strongly coupled gauge theory. It turns out that these modes are dual to short representations of the full *superconformal* algebra, *i.e.* BPS states, in the dual field theory [12], and the conformal dimension of the dual states in the QFT is therefore protected.
4. Finally, it is interesting to note that the parameter N is quantized on both sides of the theory, despite the fact that N was taken to be large in 'deriving' the duality. This may be taken to indicate that the duality continues to be true at finite N.

Similar tests have also been carried out for other instances of the AdS/CFT correspondence.

1. In the AdS_3 example, the spectrum matching leads to a non-trivial string exclusion principle from the conformal field theory. The existence of this principle is not manifest from the supergravity in AdS_3, but it may be shown explicitly that the spectrum indeed matches. We refer the reader to [3] for details.
2. In the ABJ(M) dualities [14, 15] in AdS_4, spectrum matching again has very non-trivial consequences. In particular, this requires the introduction of new monopole operators in the dual gauge theory. This was a non-trivial prediction of the AdS/CFT correspondence, which was later confirmed by an explicit construction of these operators. We refer the reader to the review [16] for details of these and related developments.

In addition, the correspondence has been successfully tested for the planar theory very extensively at arbitrary values of the 't-Hooft coupling using integrability (see the review [17]) and finally, these dualities have also been put to very non-trivial tests at the non-planar level using localisation [18]. These highly non-trivial tests are very convincing evidence for the validity of this duality.

1.2 The Higher-Spin CFT Correspondence

Despite the fact that no particle with spin greater than one has ever been experimentally detected, and gravitational waves – which would have helicity 2 – have yet to be seen, Higher-Spin theories have been explored with a great deal of interest for the past many years. This has notably resulted in the construction, by Vasiliev and collaborators, of classical theories in AdS involving an infinite number of massless particles of arbitrary spin. See [19] for a review of Vasiliev theories in general dimensions.

There are a variety of reasons for this interest. First of all, there is *a priori* no reason why there should be particles of spins only upto two. The Poincare group admits unitary irreducible representations of arbitrary (half-integer) spin. It is therefore interesting to explore if there could be consistent interacting theories with particles of spin higher than two. Also, the existence of conserved charges of spins higher than those of the Poincare group generators is explicitly forbidden by the Coleman-Mandula theorem, and its supersymmetric extension, the Haag-Lopuszanski-Sohnius theorem [20]. Again, it would be interesting to see how this theorem might be evaded, and this is another motivation to study higher-spin theories[2]. Also, from the form of equations of motion of Vasiliev's higher-spin theories, there is an expectation that these will be in some way a controllable toy model for String Field Theory.

In this thesis, however, we shall focus on the special relevance of higher-spin theories to the AdS/CFT Correspondence as the chief motivation for their exploration. The holography of higher-spin theories has been a topic of some sustained exploration [24]–[36]. This interest has essentially been fuelled by two motivations.

Firstly, as we shall shortly review, it is reasonable to expect that the twist-two sector of a (free, planar) CFT$_d$—such as $\mathcal{N} = 4$ SYM—would be described in the AdS_{d+1} bulk by a consistent classical theory with an infinite tower of arbitrary-spin particles. It is natural to expect that this bulk theory would be a Vasiliev theory. See [37] for details. Since a free CFT is in some sense a natural starting point for organising the boundary theory into a string theory (see [38–40] for a systematic approach) probing these questions would be helpful in order to learn about the general mechanics of gauge-string duality. Equally intriguingly, there is the additional possibility that even when the boundary theory is not free, higher-spin symmetries are still present in a higgsed phase [41]. This, if true, would afford us insight into the larger set of gauge invariances of string theory.

Secondly, in the low-dimensional examples of AdS_3 and AdS_4, these theories are expected to have CFT duals in their own right [24, 29, 31, 34, 35]. Typically, the best known and studied examples of AdS/CFT are supersymmetric [1], because supersymmetry allows us the freedom to reliably compute and extrapolate results on both sides of the duality. However, there are good reasons to believe that the general phenomena of gauge-string duality and holography are not tied to supersymmetry. A natural allied question is to ask what role *does* supersymmetry play in AdS/CFT. In this regard, the holography of higher-spin theories is interesting because many of these dualities are non-supersymmetric [24, 29, 32] and offer us an opportunity to concretely explore AdS/CFT away from supersymmetry.

The above statements require some qualification for the case of AdS_3/CFT$_2$ dualities like [29]. In particular, the CFTs dual to higher-spin theories in AdS_3 are not free, and in fact can even be strongly coupled [42]. This is not surprising; the Coleman-Mandula theorem breaks down in two dimensions.

[2] In three dimensional CFTs, the recent Maldacena-Zhiboedov theorem [21] corresponds to the Coleman-Mandula theorem. In particular, the existence of a single higher-spin conserved current can be shown to imply the existence of infinitely many such conserved currents, and the correlation functions of the currents of the CFT equal that of a free theory. The corresponding case, for arbitrary dimensions is interestingly more involved, and there is the possibility of having additional structures in conserved currents, not arising from free field theory [22, 23]

1.2.1 The Higher-Spin window of the AdS/CFT Correspondence

To get a sense of how higher-spin theories can be related to the AdS/CFT duality let us consider the AdS_5/CFT_4 duality, and the dictionary 1.1.15, which is the context in which the connection of higher-spin theories to AdS/CFT originally arose [37, 43]. Similar statements would hold in all known examples of the AdS/CFT duality.

We consider the window where $g_{YM} = 0$ and $N \to \infty$. This would correspond in the bulk to classical tensionless string theory (or equivalently, classical string theory formulated on a zero radius AdS). In the boundary, this is a free, planar gauge theory with all matter in the adjoint representation (from supersymmetry requirements) and is in some sense, a natural starting point for organising a gauge theory into its string dual. See [38–40] for a general approach to this problem.

While the string theory in this limit is notoriously difficult to deal with – even with supersymmetry – we can use the boundary theory to develop an expectation of what the bulk will be like. This is essentially the approach adopted in [37, 43].

We begin by observing that a free theory contains an infinite number of conserved currents of arbitrary spin. These have been constructed systematically in d dimensions, and we give the final form only.

$$J_{\mu_1\ldots\mu_s} = \sum_{k=0}^{s} \frac{(-1)^k}{k!\left(k+\frac{d-4}{2}\right)!(s-k)!\left(s-k+\frac{d-4}{2}\right)!}\partial_{(\mu_1}\ldots\partial_{\mu_k}\phi^*(x)\cdot\partial_{\mu_{k+1}}\ldots\partial_{\mu_s)}\phi(x) - traces, \tag{1.2.20}$$

where the dot product is the invariant inner product defined over the representation of the gauge group that the fields ϕ transform in. For example, it would be a trace in the adjoint representation. It can further be shown that these operators form a closed subsector of the free conformal field theory, in the sense that the OPE of these operators closes among themselves. We therefore expect that there is a subsector of the tensionless string theory in the bulk which is described by a consistent, classical, interacting theory of an infinite number of particles of spins ranging from zero to infinity. As mentioned previously, such theories are independently known of in AdS spacetimes and it seems natural to try and connect the conserved current subsector of free Yang-Mills theories to these higher-spin theories.

1.3 Free Higher-Spin Theories in AdS

We will begin with obtaining the quadratic action for free massless[3] higher-spin fields when expanded about the AdS vacuum. By a spin-s field, we mean a symmetric rank-s tensor, subject to constraints which we shall come to in a moment. It turns out that for (Anti-) de Sitter spacetimes, one can determine this action by taking the action for a massless spin-s field and minimally coupling it to background gravity. This has been worked out in [44, 45]. We refer the reader to [28, 49] for a modern review, and [46–48] for recent progress. Free higher-spin theory in flat space is also reviewed in [50].

The main ingredient for the analyses carried out in this thesis is a quadratic action for a symmetric rank-s tensor field, invariant under the gauge transformation

$$\phi_{\mu_1\ldots\mu_s} \mapsto \phi_{\mu_1\ldots\mu_s} + \nabla_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}. \tag{1.3.21}$$

The brackets imply symmetrisation over the enclosed indices. This is achieved by adding the minimal number of terms necessary to get a completely symmetrised object. There is no over-all weight multiplied. For a consistent description, it turns out that the fields ϕ necessarily satisfy a

[3] The notion of masslessness for a tensor field is slightly ambiguous on a curved manifold such as AdS, by 'massless', we mean that the action has a gauge freedom, which we specify in (1.3.21).

double-tracelessness constraint which is trivial for spins 3 and lower.

$$\phi_{\mu_1\ldots\mu_{s-4}\nu\rho}{}^{\nu\rho} = 0. \tag{1.3.22}$$

The gauge transformation parameter ξ then satisfies a tracelessness constraint

$$\xi_{\mu_1\ldots\mu_{s-3}\nu}{}^{\nu} = 0. \tag{1.3.23}$$

Note that this constraint is non-trivial—and is imposed—even for the spin-3, which has no double tracelessess constraint. With these conditions, a quadratic action for the spin-s field in AdS_D, invariant under the gauge transformation (1.3.21) may be written down.

$$S\left[\phi_{(s)}\right] = \int d^D x\sqrt{g}\phi^{\mu_1\ldots\mu_s}\left(\hat{\mathcal{F}}_{\mu_1,\ldots,\mu_s} - \frac{1}{2}g_{(\mu_1\mu_2}\hat{\mathcal{F}}_{\mu_3\ldots\mu_s)\lambda}{}^{\lambda}\right), \tag{1.3.24}$$

where

$$\hat{\mathcal{F}}_{\mu_1\ldots\mu_s} = \mathcal{F}_{\mu_1\ldots\mu_s} - \frac{s^2+(D-6)\,s-2\,(D-3)}{\ell^2}\phi_{\mu_1\ldots\mu_s} - \frac{2}{\ell^2}g_{(\mu_1\mu_2}\phi_{\mu_3\ldots\mu_s)\lambda}{}^{\lambda}, \tag{1.3.25}$$

and

$$\mathcal{F}_{\mu_1\ldots\mu_s} = \Delta\phi_{\mu_1\ldots\mu_s} - \nabla_{(\mu_1}\nabla^{\lambda}\phi_{\mu_2\ldots\mu_s)\lambda} + \frac{1}{2}\nabla_{(\mu_1}\nabla_{\mu_2}\phi_{\mu_3\ldots\mu_s)\lambda}{}^{\lambda}. \tag{1.3.26}$$

From this action, it already seems plausible that the result of evaluating the path integral in the one-loop approximation will be expressible in terms of the determinant of the Laplacian evaluated over spin-s fields. It will soon become apparent that this is indeed the case. We shall therefore focus on evaluating precisely these determinants in the next chapter.

Part II

Partition Functions for Higher-Spin Theories in AdS.

Chapter 2

The Heat Kernel on AdS

2.1 The Heat Kernel Method

As mentioned in the previous chapter, a complete theory of quantum gravity is a main outstanding goal of physics. However, important information may be extracted about the quantum theory by performing a loop expansion about a classical vacuum. In particular, it turns out that the leading quantum properties may be extracted purely from classical data. This is clearly important from the point of view of quantum gravity as this behaviour is potentially tractable even when we don't know the underlying quantum theory. We refer the reader to [51] for more details. To see how this works, consider a path integral for a field $\phi(x)$.

$$\mathcal{Z}\left[\Phi\right] = \int \mathcal{D}\Phi e^{-\frac{i}{\hbar}S[\Phi]}. \tag{2.1.1}$$

We will evaluate this in the limit $\hbar \to 0$ by the method of steepest descents to incorporate the leading quantum corrections. In this limit, the path integral is dominated by classical configurations, which solve

$$\frac{\delta}{\delta\Phi}S\left[\Phi\right]|_{\Phi=\Phi_{cl}} = 0 \tag{2.1.2}$$

Let us decompose

$$\Phi = \Phi_{cl} + \phi, \tag{2.1.3}$$

where ϕ denotes small (quantum) fluctuations about the classical configuration Φ_{cl}. In an expansion about Φ_{cl}, we may heuristically write

$$S\left[\Phi_{cl}+\phi\right] \simeq S\left[\Phi_{cl}\right] + \frac{\delta^2}{\delta\Phi^2}S\left[\Phi\right]|_{\Phi=\Phi_{cl}}\phi^2. \tag{2.1.4}$$

Or more correctly,

$$S = S_{cl} + \langle\phi, D\phi\rangle \text{ where } \quad \langle\phi_1,\phi_2\rangle = \int d^n x\sqrt{g}\phi_1(x)\phi_2(x). \tag{2.1.5}$$

The linear term in the variation vanishes because Φ_{cl} obeys the classical equation of motion and hence extremises the action. We have omitted higher-order terms in this expansion which correspond to higher loop corrections to $\mathcal{Z}$. The one-loop path integral is therefore a Gaussian integral and may be easily evaluated.

$$\mathcal{Z}_{1-\ell}\left[\phi\right] = \det{}^{-\frac{1}{2}}\left(D\right). \tag{2.1.6}$$

The operator D should at least be a self-adjoint operator, we will also assume that it is elliptic (so that its eigenvalues are of definite sign). There are additional technical assumptions for which we refer the reader to [51]. In this chapter, we will consider the case that D is the Laplacian and ϕ is an arbitrary spin field moving on an arbitrary-dimensional AdS background and explicitly evaluate the determinant above by relating it to the trace of the exponential of the Laplacian.

As a final remark, if Φ is a gauge field then there is a gauge redundancy which has to be fixed while evaluating the path integral. Typically, this involves the introduction of ghost fields, which yield determinants of their own. See for example, [52] for the corresponding Yang-Mills example. We will also explicitly evaluate these by these heat kernel methods.

2.2 The Heat Kernel for the Laplacian

Given the normalised eigenfunctions $\psi^{(S)}_{n,a}(x)$ of the Laplacian $\Delta_{(S)}$ for a spin-S field on a manifold $\mathcal{M}_{d+1}$, and the spectrum of eigenvalues $E^{(S)}_n$, we can define the heat kernel between two points x and y as

$$K^{(S)}_{ab}(x,y;t) = \langle y,b|e^{t\Delta_{(S)}}|x,a\rangle = \sum_n \psi^{(S)}_{n,a}(x)\,\psi^{(S)}_{n,b}(y)^*\,e^{tE^{(S)}_n}, \qquad (2.2.7)$$

where a and b are local Lorentz indices for the field. We can trace over the spin and spacetime labels to define the traced heat kernel as

$$K^{(S)}(t) \equiv \mathrm{Tr}e^{t\Delta_{(S)}} = \int_{\mathcal{M}} \sqrt{g}d^{d+1}x \sum_a K^{(S)}_{aa}(x,x;t)\,. \qquad (2.2.8)$$

The one-loop partition function is related to the trace of the heat kernel through

$$ln\, Z^{(S)} = \ln\, det\left(-\Delta_{(S)}\right) = \mathrm{Tr}\ln\left(-\Delta_{(S)}\right) = -\int_0^\infty \frac{dt}{t}\mathrm{Tr}e^{t\Delta_{(S)}}. \qquad (2.2.9)$$

For a general manifold, the computation of the heat kernel is a formidable task, even for the scalar Laplacian. Typically one has asymptotic results. However, for symmetric spaces such as spheres and hyperbolic spaces (Euclidean AdS) there are many simplifications. This is because these spaces can be realised as cosets G/H and one can therefore use the powerful methods of harmonic analysis on group manifolds. In [53] these techniques were used to explicitly compute the heat kernel on thermal AdS_3 for fields of arbitrary spin. Though [53] exploited the group theory, it also used at several places the particular fact that S^3 (from which one continued to AdS_3) itself is the group manifold $SU(2)$. In fact, many properties of $SU(2)$ were used in intermediate steps and it was not completely obvious how these generalise to higher dimensional spheres or AdS spacetimes.

In this chapter, we will generalise the methods of [53] to compute the heat kernel for the Laplacian for arbitrary spin tensor fields on thermal AdS spacetimes. This therefore includes the cases of AdS_4, AdS_5 and AdS_7 which play a central role in the AdS/CFT correspondence. Since we are primarily interested in evaluating the one-loop partition function for a spin-S particle, we shall concentrate on the traced heat kernel though we will see that the techniques are sufficiently general. One may, for instance, also adapt these techniques to evaluate other objects of interest such as the (bulk to bulk) propagator.

We shall mostly focus on $\mathcal{M} = AdS_{2n+1}/\Gamma$ where Γ is the thermal quotient. For practical reasons we will obtain the answer for thermal AdS by analytic continuation of the answer for an appropriate "thermal quotient" of the N-sphere. As explained in [53] there are good reasons to believe that this analytic continuation of the harmonic analysis works for odd dimensional spheres and hyperboloids (Euclidean AdS). For even dimensional hyperboloids there are some additional discrete representations other than the continuous ones one obtains by a straightforward analytic continuation. However, this additional set of representations does not contribute for Laplacians over a wide class of tensor fields, which in particular include the symmetric transverse traceless (STT) tensors. We provide necessary details in Section 2.8.

We shall begin with a brief review of harmonic analysis on homogeneous spaces, which is the mainstay of these computations.

2.3 The Heat Kernel on Homogeneous Spaces

The heat kernel of the spin-S Laplacian[1] may be evaluated over the spacetime manifold $\mathcal{M}$ by solving the appropriate heat equation. Alternatively, one may attempt a direct evaluation by constructing the eigenvalues and eigenfunctions of the spin-S Laplacian and carrying out the sum over n that appears in (2.2.7). Both these methods quickly become forbidding when applied to an arbitrary spin-S field. However, if $\mathcal{M}$ is a homogeneous space G/H, then the use of group-theoretic techniques greatly simplifies the evaluation. The main simplifications arise from the following facts which we will review below and then heavily utilise:

1. The eigenvalues $E_n^{(S)}$ of the Laplacian $\Delta_{(S)}$ are determined in terms of the quadratic Casimirs of the symmetry group G and the isotropy subgroup H. There is thus a large degeneracy of eigenvalues.
2. The eigenfunctions $\psi_{n,a}^{(S)}(x)$ are matrix elements of unitary representation matrices of G.
3. This enables one to carry out the sum over degenerate eigenstates using the group multiplication properties of the matrix elements. Thus a large part of the sum in (2.2.7) can be explicitly carried out.

We begin with a brief recollection of some basic facts about harmonic analysis on coset spaces. This will collate the necessary tools with which we evaluate (2.2.7) and (2.2.8), and will set up our notation. The interested reader is referred to [54–56] for introduction and details and to [53] for explicit examples of these constructions. Given compact Lie groups G and H, where H is a subgroup of G, the coset space G/H is constructed through the right action of H on elements of G

$$G/H = \{gH\}. \tag{2.3.10}$$

(We will also need to consider left cosets, $\Gamma\backslash G$, where Γ will act on elements of G from the left.)

We recall that G is the principal bundle over G/H with fibre isomorphic to H. Let π be the projection map from G to G/H, *i.e.*

$$\pi\left(g\right) = gH \quad \forall g \in G. \tag{2.3.11}$$

Then a section $\sigma\left(x\right)$ in the principal bundle is a map

$$\sigma : G/H \mapsto G, \quad \text{such that } \pi\circ\sigma = e, \tag{2.3.12}$$

where e is the identity element in G, and x are coordinates in G/H. A class of sections which will be useful later is of the form

$$\sigma\left(gH\right) = g_\circ, \tag{2.3.13}$$

where $g_\circ$ is an element of the coset gH, which is chosen by some well-defined prescription. The so called 'thermal section' that we choose in Section 2.4.1 is precisely of this form. Let us label representations of G by R and representations of H by S. We will sometimes refer to representations

[1] By spin-S we refer to the representation under which the field transforms under tangent space rotations. In the case of S^{2n+1} or (euclidean) AdS_{2n+1}, this will be a representation of $SO(2n+1)$. The Laplacian is that of a tensor field transforming in this representation. In the case of spheres and hyberboloids we will consider the Laplacian with the Christoffel connection.

S of H as spin-S representations.[2] The vector space which carries the representation R is called $\mathcal{V}_R$, and has dimension d_R, while the corresponding vector space for the representation S is $\mathcal{V}_S$ of dimension d_S.

Eigenfunctions of the spin-S Laplacian are then given by the matrix elements

$$\psi_a^{(S)I}(x) = \mathcal{U}^{(R)}\left(\sigma(x)^{-1}\right)_a^{\ \ I}, \tag{2.3.14}$$

where S is the unitary irreducible representation of H under which our field transforms, and R is any representation of G that contains S when restricted to H. a is an index in the subspace $\mathcal{V}_S$ of $\mathcal{V}_R$, while I is an index in the full vector space $\mathcal{V}_R$. Generally, a given representation S can appear more than once in R. However, we shall be interested in the coset spaces $SO(N+1)/SO(N)$ and $SO(N,1)/SO(N)$, for which a representation S appears at most once [57, 58]. We have therefore dropped a degeneracy factor associated with the index a, which appears in the more complete formulae given in [58].

The corresponding eigenvalues are given by

$$-E_{R,I}^{(S)} = C_2(R) - C_2(S). \tag{2.3.15}$$

The index n for the eigenvalues of the spin-S Laplacian that appeared in (2.2.7) is therefore a pair of labels, viz. (R,I),[3] where the eigenfunctions that have the same label R but a different I are necessarily degenerate. We will therefore drop the subscript I for $E^{(S)}$.

The expression (2.2.7) for the heat kernel then reduces to

$$K_{ab}^{(S)}(x,y;t) = \sum_{R,I} a_R^{(S)} \psi_{(R,I),a}^{(S)}(x)\, \psi_{(R,I),b}^{(S)}(y)^*\, e^{tE_R^{(S)}}, \tag{2.3.16}$$

where $a_R^{(S)} = \frac{d_R}{d_S}\frac{1}{V_{G/H}}$ is a normalisation constant (see Appendix A). This can be further simplified by putting in the expression (2.3.14) for the eigenfunctions.

$$\begin{aligned} K_{ab}^{(S)}(x,y;t) &= \sum_R \sum_{I=1}^{d_R} a_R^{(S)} \mathcal{U}^{(R)}\left(\sigma(x)^{-1}\right)_a^{\ \ I} \left[\mathcal{U}^{(R)}\left(\sigma(y)^{-1}\right)_b^{\ \ I}\right]^* e^{tE_R^{(S)}} \\ &= \sum_R a_R^{(S)} \mathcal{U}^{(R)}\left(\sigma(x)^{-1}\sigma(y)\right)_a^{\ \ b} e^{tE_R^{(S)}}, \end{aligned} \tag{2.3.17}$$

where we have used the fact that the $\mathcal{U}^{(R)}$ furnish a unitary representation of G. As an aside, we note that this matrix representation of the group composition law is the generalisation of the addition theorem for spherical harmonics on S^2 to arbitrary homogeneous vector bundles on coset spaces.

To establish notation for later use, we define the heat kernel with traced spin indices

$$K^{(S)}(x,y;t) \equiv \sum_{a=1}^{d_S} K_{aa}^{(S)}(x,y;t) = \sum_R a_R^{(S)} \mathrm{Tr}_S\left(\mathcal{U}^{(R)}\left(\sigma(x)^{-1}\sigma(y)\right)\right) e^{tE_R^{(S)}}, \tag{2.3.18}$$

where the symbol

$$\mathrm{Tr}_S(\mathcal{U}) \equiv \sum_{a=1}^{d_S} \langle a, S|\mathcal{U}|a, S\rangle, \tag{2.3.19}$$

[2] In the case of sphere and hyperboloids, H is isomorphic to the group of tangent space rotations for the manifold G/H. See footnote 1.

[3] Note that a labels the components of the eigenfunction and is not a part of the index n.

and can be thought of as a trace over the subspace $\mathcal{V}_S$ of $\mathcal{V}_R$. Note that this restricted trace is invariant under a unitary change of basis of $\mathcal{V}_S$ and *not* invariant under the most general unitary change of basis in $\mathcal{V}_R$.

2.3.1 The Heat Kernel on S^{2n+1}

As a prelude to evaluating the traced heat kernel on the "thermal quotient" of the odd-dimensional sphere, let us evaluate (2.3.17) for the case without any quotient. That is, we focus first on $S^{2n+1} \simeq SO(2n+2)/SO(2n+1)$. We will describe the eigenfunctions (2.3.14) and define the sum over R explicitly. This will be useful when we analytically continue our results to the corresponding hyberbolic space. We begin by recalling some facts from the representation theory of special orthogonal groups.

Unitary irreducible representations of $SO(2n+2)$ are characterised by a highest weight, which can be expressed in the orthogonal basis as the array

$$R = (m_1, m_2, \ldots, m_n, m_{n+1}), \quad m_1 \geq m_2 \geq \ldots \geq m_n \geq |m_{n+1}| \geq 0 \tag{2.3.20}$$

where the $m_1 \ldots m_{n+1}$ are all (half-)integers. Similarly, unitary irreducible representations of $SO\,(2n+1)$ are characterised by the array

$$S = (s_1, s_2, \ldots, s_n), \quad s_1 \geq s_2 \geq \ldots \geq s_n \geq 0, \tag{2.3.21}$$

where the $s_1 \ldots s_n$ are all (half-)integers.

Then the quadratic Casimirs for the unitary irreducible representations for an orthogonal group of rank $n+1$ can be expressed as (see e.g.[57, 59]).

$$C_2\,(m_1, \ldots, m_{n+1}) = m^2 + 2r \cdot m. \tag{2.3.22}$$

Here the dot product is the usual euclidean one, and the Weyl vector r is given by

$$r_i = \left\{ \begin{array}{ll} n - i + 1 & \text{if } \mathrm{G} = \mathrm{SO}\,(2\mathrm{n}+2)\,, \\ \left(n + \frac{1}{2}\right) - i & \text{if } \mathrm{G} = \mathrm{SO}\,(2\mathrm{n}+1)\,, \end{array} \right\} \tag{2.3.23}$$

where i runs from 1 to $n+1$.

Let us now consider the expression (2.3.17) for the spin-S Laplacian on S^{2n+1}. The eigenvalues $E_R^{(S)}$ are given by (2.3.15) and (2.3.22) and may be written down compactly as

$$-\,E_R^{(S)} = m^2 + 2r_{SO(2n+2)} \cdot m - s^2 - 2r_{SO(2n+1)} \cdot s. \tag{2.3.24}$$

The corresponding eigenfunctions are given by (2.3.14) where we have to specify which are the representations R of $SO\,(2n+2)$ that contain a given representation S of $SO\,(2n+1)$. This is determined by the branching rules, which for our case state that a representation R given by (2.3.20) contains the representation S if

$$m_1 \geq s_1 \geq m_2 \geq s_2 \geq \ldots \geq m_n \geq s_n \geq |m_{n+1}|, \quad (m_i - s_i) \in \mathbb{Z}. \tag{2.3.25}$$

Using these branching rules, one can show that the expression that appears on the right of (2.3.24) is indeed positive definite, so that the eigenvalue itself is negative definite as per our conventions.

These rules further simplify if we restrict ourselves to symmetric transverse traceless (STT) representations of H. These tensors of rank s correspond to the highest weight $(s, 0, \ldots, 0)$. In this case, some of these inequalities get saturated, and one obtains the branching rule

$$m_1 \geq s = m_2 \geq 0, \tag{2.3.26}$$

with all other m_i, s_i zero, [4] and the equality follows from requiring that R contain S in the maximal possible way. Essentially, this is equivalent to the transversality condition. The sum over R that appears in (2.3.17) is now a sum over the admissible values of m_1 in the above inequality. Thus if we restrict ourselves to evaluating the heat kernel for STT tensors, then we will be left with a sum over m_1 only.

The expression (2.3.24) for the eigenvalue also simplifies in this case. With the benefit of hindsight, we will write this in a form that is suitable for analytic continuation to AdS

$$-E_R^{(S)} = (m_1 + n)^2 - s - n^2. \tag{2.3.27}$$

Using these tools, one can write down a formal expression for the heat kernel for a spin-S particle between an arbitrary pair of points x and y on S^{2n+1} using (2.3.17). This is given by

$$K_{ab}^S(x, y; t) = \sum_{m_i} \frac{n!}{2\pi^{n+1}} \frac{d_R}{d_S} \mathcal{U}^{(R)} \left(\sigma(x)^{-1} \sigma(y)\right)_a^{\ b} e^{tE_R^{(S)}}, \tag{2.3.28}$$

where we have simply expanded out the sum over R into a sum over the permissible values of m_i determined by (2.3.25), and inserted the expression for the volume of the $(2n+1)$-sphere. Expressions for the dimensions d_R and d_S are well known (see for example[57, 59]). We list them here for the reader's convenience.

$$d_R = \prod_{i<j=1}^{n+1} \frac{l_i^2 - l_j^2}{\mu_i^2 - \mu_j^2}, \quad d_S = \prod_{i<j=1}^{n} \frac{\tilde{l}_i^2 - \tilde{l}_j^2}{\tilde{\mu}_i^2 - \tilde{\mu}_j^2} \prod_{i=1}^{n} \frac{\tilde{l}_i}{\tilde{\mu}_i}, \tag{2.3.29}$$

where $l_i = m_i + (n+1) - i$, $\mu_i = (n+1) - i$, $\tilde{l}_i = s_i + n - i + \frac{1}{2}$, and $\tilde{\mu}_i = n - i + \frac{1}{2}$. As explained above, this expression further simplifies for the STT tensors, and we obtain

$$K_{ab}^S(x, y; t) = \sum_{m_1} \frac{n!}{2\pi^{n+1}} \frac{d_{(m_1,s)}}{d_s} \mathcal{U}^{(m_1,s)} \left(\sigma(x)^{-1} \sigma(y)\right)_a^{\ b} e^{tE_R^{(S)}}, \tag{2.3.30}$$

where the labels (m_1, s) and s that appear on the RHS are shorthand for $R = (m_1, s, 0 \ldots, 0)$ and $S = (s, 0 \ldots, 0)$ respectively. This expression should be compared with the equation (3.9) obtained in [53] for the case of 3 dimensions. The traced heat kernel for the STT tensors is then given by

$$K^S(x, y; t) = \sum_{m_1} \frac{n!}{2\pi^{n+1}} \frac{d_{(m_1,s)}}{d_s} \mathrm{Tr}_S \left(\mathcal{U}^{(m_1,s)} \left(\sigma(x)^{-1} \sigma(y)\right)\right) e^{tE_R^{(S)}}. \tag{2.3.31}$$

As mentioned earlier, we can in principle use these formulae to construct explicit expressions for the heat kernel between two points *à la* [53], and thus for the bulk to bulk propagator. This would, however also require using explicit matrix elements of $SO(2n+2)$ representations, and we shall not pursue this direction further here.

2.4 The Heat Kernel on Quotients of Symmetric Spaces

We consider the heat kernel on the quotient spaces $\Gamma \backslash G/H$ where Γ is a discrete group which can be embedded in G. Though it is not essential to our analysis, we will assume that Γ is of finite order and is generated by a single element. This is indeed true for the quotients (2.4.34) we consider here. In particular, for the "thermal quotient" on the N-sphere, Γ is isomorphic to $\mathbb{Z}_N$. To evaluate the heat kernel (2.2.7) on this space, a choice of section that is *compatible* with the quotienting by Γ is useful. By this we mean that if $\gamma \in \Gamma$ acts on points $x = gH \in G/H$ by

[4] For the special case of $n = 1$, i.e. S^3, the branching rule is $m_1 \geq s = |m_2| \geq 0$.

$\gamma : gH \mapsto \gamma \cdot gH$, then a section $\sigma(x)$ is said to be compatible with the quotienting Γ iff

$$\sigma(\gamma(x)) = \gamma \cdot \sigma(x) \,. \tag{2.4.32}$$

The utility of this choice of section will become clear when we explicitly evaluate the traced heat kernel (2.2.8) for such geometries.

2.4.1 The Thermal Quotient of S^5

As an example, we consider the thermal quotient of S^5. This will serve as a useful prototype to keep in mind. We shall also see that we can extrapolate the analysis to the general odd-dimensional sphere. To begin with, let us express the thermal quotient in terms of "triple-polar" coordinates on S^5, which are complex numbers (z_1, z_2, z_3) such that

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1. \tag{2.4.33}$$

We consider the quotient

$$\gamma : \{\phi_i\} \mapsto \{\phi_i + \alpha_i\}. \tag{2.4.34}$$

Here ϕ_1, ϕ_2, ϕ_3 are the phases of the z's and $n_i \alpha_i = 2\pi$ for some $n_i \in \mathbb{Z}$ and not all n_is are simultaneously zero.[5] However, to embed Γ in $SO(6)$, it is more useful to decompose these complex numbers into 6 real coordinates that embed the S^5 into $\mathbb{R}^6$,

$$\begin{aligned} x_1 &= \cos\theta\, \cos\phi_1 & x_2 &= \cos\theta\, \sin\phi_1, \\ x_3 &= \sin\theta\, \cos\psi\, \cos\phi_2 & x_4 &= \sin\theta\, \cos\psi\, \sin\phi_2, \\ x_5 &= \sin\theta\, \sin\psi\, \cos\phi_3 & x_6 &= \sin\theta\, \sin\psi\, \sin\phi_3. \end{aligned} \tag{2.4.35}$$

Now, we construct a coset representative in $SO(6)$ for this point x with coordinates as in (2.4.35). To do so, we start with the point $(1, 0, 0, 0, 0, 0)$ in R^6, and construct a matrix $g(x)$ that rotates this point, the north pole, to the generic point x. By construction, $g(x) \in SO(6)$, and there is a one-to-one correspondence between the points x on S^5 and matrices $g(x)$, upto a right multiplication by an element of the $SO(5)$ which leaves the north pole invariant. Such a representative matrix $g(x)$ can be taken to be

$$g(x) = e^{i\phi_1 Q_{12}} e^{i\phi_2 Q_{34}} e^{i\phi_3 Q_{56}} e^{i\psi Q_{35}} e^{i\theta Q_{13}}, \tag{2.4.36}$$

where Q's are the generators of $SO(6)$. This is clearly an instance of a section in G over G/H.

The action of the thermal quotient (2.4.34) on the coset representative $g(x)$ is

$$\gamma : g(x) \mapsto g(\gamma(x)) = e^{i\alpha_1 Q_{12}} e^{i\alpha_2 Q_{34}} e^{i\alpha_3 Q_{56}} \cdot x = \gamma \cdot g(x) \,, \tag{2.4.37}$$

where the composition '$\cdot$' is the usual matrix multiplication.[6] This section has the property (2.4.32) that we demand from a thermal section. Hence, we choose the thermal section to be

$$\sigma_{th}(x) = g(x) \,. \tag{2.4.38}$$

This analysis can be repeated for any odd-dimensional sphere to find the same expression for the thermal section. Essentially the only difference is that for a $(2n+1)$-dimensional sphere, we need to consider $(n+1)$ complex numbers z_i and proceed exactly as above. We find that the thermal section can always be chosen to be (2.4.38).

[5]Note that this is a more general identification than the thermal quotient we will need, where one can take $\alpha_i = 0\, (\forall i \neq 1)$

[6]This gives the embedding of Γ in $SO(6)$.

2.4.2 The Method of Images

Since the heat kernel obeys a linear differential equation- the heat equation- we can use the method of images to construct the heat kernel on $\Gamma\backslash G/H$ from that on G/H (see, for example [60]). The relation between the two heat kernels is

$$K_{\Gamma}^{(S)}(x, y; t) = \sum_{\gamma\in\Gamma} K^{(S)}(x, \gamma(y); t), \tag{2.4.39}$$

where $K_{\Gamma}^{(S)}$ is the heat kernel between two points x and y on $\Gamma\backslash G/H$, $K^{(S)}$ is the heat kernel on G/H and the spin indices have been suppressed. We shall use this relation to determine the traced heat kernel on the thermal quotient of S^{2n+1}.

2.4.3 The Traced Heat Kernel on thermal S^{2n+1}

We use the formalism developed above to evaluate the traced heat kernel for the thermal quotient of the odd-dimensional sphere. Using the method of images, the quantity of interest is

$$K_{\Gamma}^{(S)}(t) = \sum_{k\in\mathbb{Z}_N} \int_{\Gamma\backslash G/H} d\mu(x) \sum_a K_{aa}^{(S)}\left(x, \gamma^k(x); t\right), \tag{2.4.40}$$

where $d\mu(x)$ is the measure on $\Gamma\backslash G/H$ obtained from the Haar measure on G, and x labels points in $\Gamma\backslash G/H$. We have also used the fact that $\Gamma \simeq \mathbb{Z}_N$, and since the sum over m is a finite sum, the integral has also been taken through the sum. Further, since

$$\sum_a K_{aa}^{(S)}\left(\gamma x, \gamma^k(\gamma x); t\right) = \sum_a K_{aa}^{(S)}\left(x, \gamma^k(x); t\right), \tag{2.4.41}$$

the integral over $\Gamma\backslash G/H$ can be traded in for the integral over G/H. We therefore multiply by an overall volume factor, and evaluate

$$\int_{G/H} d\mu(x) \sum_a K_{aa}^{(S)}\left(x, \gamma^k(x); t\right), \tag{2.4.42}$$

where $d\mu(x)$ is the left invariant measure on G/H obtained from the Haar measure on G, and x now labels points in the full coset space G/H. Putting the expression (2.3.18) into this, and choosing the section (2.4.38) we obtain

$$\int_{G/H} d\mu(x) \sum_a K_{aa}^{(S)}\left(x, \gamma^k(x); t\right) = \int_{G/H} d\mu(x) \sum_R a_R^{(S)} \mathrm{Tr}_S \left(g_x^{-1}\gamma^k g_x\right)^{(R)} e^{tE_R^{(S)}}, \tag{2.4.43}$$

where $\left(g_x^{-1}\gamma^k g_x\right)^{(R)}$ is an abbreviation for $\mathcal{U}^{(R)}\left(g(x)^{-1}\gamma^k g(x)\right)$. As this expression stands, the trace is only over some subspace $\mathcal{V}_S \subset \mathcal{V}_R$ so the cyclic property of the trace cannot be used to annihilate the g_x^{-1} with the g_x. To proceed further, we move the integral into the summation to obtain

$$\int_{G/H} d\mu(x) \sum_a K_{aa}^{(S)}\left(x, \gamma^k(x); t\right) = \sum_R a_R^{(S)} \int_{G/H} d\mu(x)\, \mathrm{Tr}_S \left(g_x^{-1}\gamma^k g_x\right)^{(R)} e^{tE_R^{(S)}}. \tag{2.4.44}$$

Since G and H are compact, we may use the property that [61]

$$\int_G dg f(g) = \int_{G/H} d\mu(\tilde{x}) \left[\int_H dh f(\tilde{x}h)\right], \tag{2.4.45}$$

where dg is the Haar measure on G, $d\mu$ and dh are the invariant measures on G/H and H respectively. $\tilde{x}$ is an arbitrary choice of coset representatives that we make to label points in G/H. In what follows we shall choose the coset representative to be g_x. Let us consider the function

$$f(g) = \mathrm{Tr}_S \left(g^{-1}\gamma^k g\right)^{(R)}. \tag{2.4.46}$$

This function has the property that $f(g_x h) = f(g_x)$. Putting this in (2.4.45), we see that the integral over H becomes trivial, and we get

$$\int_{G/H} d\mu(x)\, \mathrm{Tr}_S \left(g_x^{-1}\gamma^k g_x\right)^{(R)} = \frac{1}{V_H}\int_G dg \mathrm{Tr}_S \left(g^{-1}\gamma^k g\right)^{(R)}. \tag{2.4.47}$$

Now, as in [53], we note that the integral $I_G = \int_G dg\left(g^{-1}\gamma^k g\right)$ commutes with all group elements $\tilde{g} \in G$ because

$$I_G \cdot \tilde{g} = \int_G dg \left(g^{-1}\gamma^k g\right)\cdot \tilde{g} = \tilde{g}\cdot \int_G d(g\tilde{g})\left((g\tilde{g})^{-1}\gamma^k (g\tilde{g})\right) = \tilde{g}\cdot I_G, \tag{2.4.48}$$

where we have used the right invariance of the measure, viz. $dg = d(g\tilde{g})$. We therefore have, from Schur's lemma, that

$$I_G = \int_G dg\left(g^{-1}\gamma^k g\right) \propto \mathbb{I}, \tag{2.4.49}$$

from which we obtain that

$$\int_G dg \mathrm{Tr}_S \left(g^{-1}\gamma^k g\right)^{(R)} = \frac{d_S}{d_R}\int_G dg \mathrm{Tr}_R \left(g^{-1}\gamma^k g\right)^{(R)} = \frac{d_S}{d_R} V_G \mathrm{Tr}_R\left(\gamma^k\right). \tag{2.4.50}$$

The quotient γ is just the exponential of the Cartan generators of $SO(2n+2)$ (see, for example (2.4.37) for the S^5). This trace, therefore is just the $SO(2n+2)$ character χ_R in the representation R. Putting this result in (2.4.47), we find that

$$\int_{G/H} d\mu(x)\, \mathrm{Tr}_S \left(g_x^{-1}\gamma^k g_x\right)^{(R)} = V_{G/H}\frac{d_S}{d_R}\chi_R\left(\gamma^k\right), \tag{2.4.51}$$

where we have normalised volumes so that $V_G = V_{G/H}V_H$. Therefore,

$$\sum_{k\in\mathbb{Z}_N}\int_{G/H} d\mu(x)\, K^{(S)}\left(x, \gamma^k(x); t\right) = \sum_{k\in\mathbb{Z}_N}\sum_R \chi_R\left(\gamma^k\right) e^{tE_R^{(S)}}, \tag{2.4.52}$$

where we have inserted the value of the normalisation constant $a_R^{(S)}$ from (A.0.3). Now for the thermal quotient, we have γ such that

$$\alpha_1 \neq 0,\ \alpha_i = 0, \quad \forall\, i = 2, \ldots, n+1. \tag{2.4.53}$$

The volume factor for this quotient is just $\frac{\alpha_1}{2\pi}$. This gives us the traced heat kernel on the thermal S^{2n+1}

$$K_\Gamma^{(S)}(t) = \frac{\alpha_1}{2\pi}\sum_{k\in\mathbb{Z}_N}\sum_R \chi_R\left(\gamma^k\right) e^{tE_R^{(S)}}. \tag{2.4.54}$$

We find that the answer assembles naturally into a sum of characters, in various representations of G, of elements of the quotient group Γ. Here the representations R of $SO(2n+2)$ are those which contain S when restricted to $SO(2n+1)$. The reader should compare this expression to the equation (4.20) obtained in [53].

2.5 The Heat Kernel on AdS_{2n+1}

We have so far obtained an expression for the heat kernel on a compact symmetric space (2.3.17) and have extended our analysis to its left quotients. In particular, we have shown how the traced heat kernel on (a class of) quotients of S^{2n+1} assembles into a sum over characters of the orbifold group Γ. We now extend our analysis to the hyperbolic space $\mathbb{H}_{2n+1}$. Following the analysis of [53], we will use the fact that the N-dimensional sphere admits an analytic continuation to the corresponding euclidean AdS geometry. We now give an account of how one can exploit this fact to determine the heat kernel on AdS_{2n+1} and its quotients.

2.5.1 Preliminaries

Euclidean AdS is the N dimensional hyperbolic space $\mathbb{H}_N^+$ which is the coset space

$$\mathbb{H}_N \simeq SO(N,1)/SO(N), \tag{2.5.55}$$

where the quotienting is done, as in the sphere, by the right action. As was done for the three-dimensional case, we will view $SO(N,1)$ as an analytic continuation of $SO(N+1)$.

For explicit expressions, we shall employ the generalisation of the triple-polar coordinates that we introduced in (2.4.35) to the general S^{2n+1}. In these coordinates, the S^{2n+1} metric is

$$d\theta^2 + cos^2\theta d\phi_1^2 + sin^2\theta\, d\Omega_{2n-1}^2. \tag{2.5.56}$$

We perform the analytic continuation

$$\theta \mapsto -i\rho, \quad \phi_1 \mapsto it, \tag{2.5.57}$$

where ρ and t take values in $\mathbb{R}$, to obtain

$$ds^2 = -\left(d\rho^2 + cosh^2\rho dt^2 + sinh^2\rho\, d\Omega_{2n-1}^2\right). \tag{2.5.58}$$

The reader will recognize this as the metric on global AdS_{2n+1}, upto a sign.

Now, to construct eigenfunctions on $\mathbb{H}_N$, we need to write down a section in $SO(N,1)$. The Lie algebra of $SO(N,1)$ is an analytic continuation of $SO(N+1)$ where we choose a particular axis- say '1'- as the time direction and perform the analytic continuation $Q_{1j} \to iQ_{1j}$ to obtain the $so(N,1)$ algebra from the $so(N+1)$ algebra. This is equivalent to the analytic continuation of the coordinates described above. Therefore, the section in $SO(N,1)$ can be obtained from that in $SO(N+1)$ by analytically continuing the coordinates via (2.5.57).

2.5.2 Harmonic Analysis on $\mathbb{H}_{2n+1}$

We have recollected basic results from harmonic analysis on coset spaces in Section 2.3, which we have exploited for compact groups G and H. In fact, all the basic ingredients that we have employed in our analysis can be carried over to the case of non-compact groups as well. The eigenvalues of the spin-S Laplacian are still given by (2.3.15), and the eigenfunctions are still (2.3.14), *i.e.* they are determined by matrix elements of unitary representations of G. These unitary representations are now infinite dimensional, given that G is non-compact. However, for $SO(N,1)$, these representations have been classified [58, 62], and we shall use these results to determine the traced heat kernel on AdS_{2n+1}.

The only unitary representations of $SO(N,1)$ that are relevant to us are those that contain unitary representations of $SO(N)$. For odd-dimensional hyperboloids, where $N = 2n+1$, these are just the so-called principal series representations of $SO(2n+1,1)$ which are labelled by the array

$$R = (i\lambda, m_2, m_3, \cdots, m_{n+1}), \quad \lambda \in \mathbb{R},\, m_2 \geq m_3 \geq \cdots \geq m_n \geq |m_{n+1}|, \tag{2.5.59}$$

where the $m_2, \cdots, m_n$ and $|m_{n+1}|$ are non-negative (half-)integers. We shall usually denote the array $(m_2, m_3, \ldots, m_{n+1})$ by $\vec{m}$. We also note that the principal series representations $(i\lambda, \vec{m})$ that contain a representation S of $SO(2n+1)$ are determined by the branching rules [58, 63]

$$s_1 \geq m_2 \geq s_2 \geq \ldots \geq m_n \geq s_n \geq |m_{n+1}|, \tag{2.5.60}$$

which, for the STT tensors just reduce to $m_2 = s$ with all other m_is and s_is set to zero (*cf.* 2.3.26), except for the case of $n = 1$, where the branching rule is $|m_2| = s$. Comparing (2.5.59) to (2.3.20) suggests that the appropriate analytic continuation is

$$m_1 \mapsto i\lambda - n, \quad \lambda \in \mathbb{R}_+, \tag{2.5.61}$$

which is indeed the analytic continuation used by [64]. Let us consider how the eigenvalues $E_R^{(S)}$ transform under this analytic continuation. It turns out that the eigenvalues (2.3.24) get continued to

$$E^{(S)}_{R,AdS_{2n+1}} = -\left(\lambda^2 + \zeta\right), \quad \zeta \equiv C_2(S) - C_2(\vec{m}) + n^2, \tag{2.5.62}$$

and that the eigenvalue for the STT tensors (2.3.27) gets continued to

$$E^{(S)}_{R,AdS_{2n+1}} = -\left(\lambda^2 + s + n^2\right). \tag{2.5.63}$$

The eigenvalues on AdS have an extra minus sign apart from what is obtained by the analytic continuation because the metric S^{2n+1} under the analytic continuation goes to minus of the metric on AdS_{2n+1}. This analytic continuation preserves the corresponding energy eigenvalue as a negative definite real number, which it must, because the Laplacian on Euclidean AdS is an elliptic operator, and its eigenvalues must be of definite sign.

2.5.3 The coincident Heat Kernel on AdS_{2n+1}

In computing the heat kernel over AdS_{2n+1} by analytically continuing from S^{2n+1}, the sum over m_1 that entered in (2.3.28), (2.3.30) and (2.3.31) is now continued to an integral over λ. In general this integral over λ is hard to perform, but it simplifies significantly in the coincident limit and is evaluated below for this case. The traced heat kernel for STT tensors has previously been obtained directly in this limit by [64] and this calculation therefore serves as a check of the prescription (2.5.57) of analytic continuation that we have employed. We will also see that the normalisation constant a_R^S that appeared for the S^{2n+1} gets continued to μ_R^S, which is essentially the measure for this integral. This is a brief summary of the calculation, the reader will find more details in Appendix B.

On using (2.3.29) for the special case of $R = (m, s, 0, \ldots, 0)$, one can show that a_R^S gets continued via (2.5.57) to μ_R^S, where

$$\mu_R^S = \frac{1}{d_s} \frac{\left[\lambda^2 + \left(s + \frac{N-3}{2}\right)^2\right] \prod_{j=0}^{\frac{N-5}{2}} \left(\lambda^2 + j^2\right)}{2^{N-1}\pi^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} \frac{(2s+N-3)(s+N-4)!}{s!\,(N-3)!}, \tag{2.5.64}$$

where $N = 2n+1$. A little algebra reveals this as the combination

$$\mu_R^S = \frac{C_N g(s)}{d_S} \frac{\mu(\lambda)}{\Omega_{N-1}}, \tag{2.5.65}$$

in the notation of [64], (see their expressions 2.7 to 2.13 and 2.108 or our Appendix B). We have omitted the overall sign $(-1)^{\frac{N-1}{2}}$ in writing the above. The quantity $\mu(\lambda)$ is known as the

Plancherel measure. We now consider the expression (2.3.31) on S^{2n+1}, which in the coincident limit, reduces to

$$K^S(x,x;t) = \sum_{m_1} a_R^S d_S e^{tE_R^{(S)}}. \qquad (2.5.66)$$

where $a_R^S d_S = \frac{n!}{2\pi^{n+1}} d_{(m_1,s)}$. We analytically continue this expression via our prescription (2.5.57) to the coincident heat kernel on AdS_{2n+1}.

$$K^S(x,x;t) = \int d\lambda\, \mu_R^S(\lambda)\, d_S e^{tE_R^{(S)}} = \frac{C_N}{\Omega_{N-1}} g(s) \int d\lambda\, \mu(\lambda)\, e^{-t\left(\lambda^2+s+n^2\right)}, \qquad (2.5.67)$$

which is precisely the expression obtained by [64].

2.6 The Heat Kernel on Thermal AdS_{2n+1}

2.6.1 The Thermal Quotient of AdS

We are now in a position to calculate the traced heat kernel of an arbitrary tensor particle on thermal AdS_{2n+1}. This space is the hyperbolic space $\mathbb{H}_{2n+1}$ with a specific $\mathbb{Z}$ identification (in the generalised polar coordinates)

$$t \sim t + \beta, \quad \beta = i\alpha_1 \qquad (2.6.68)$$

which is just the analytic continuation by (2.5.57) of the identification (2.4.53) on the sphere. Since t is the global time coordinate, β is to be interpreted as the inverse temperature.

2.6.2 The Heat Kernel

In section 2.5 we have discussed how the heat kernel on $\mathbb{H}_{2n+1}$ can be calculated by analytically continuing the harmonic analysis on S^{2n+1} to $\mathbb{H}_{2n+1}$. As discussed in Sec. 6.2 of [53], we expect to be able to continue the expressions for the heat kernel on the thermal sphere to thermal AdS, with the difference that now $\Gamma \simeq \mathbb{Z}$, rather than $\mathbb{Z}_N$. Also, as noted in [53], essentially the only difference that arises for the traced heat kernel is that the character of $SO(2n+2)$ that appears in (2.4.54) is now replaced by the Harish-Chandra (or global) character for the non-compact group $SO(2n+1,1)$.

With these inputs, the traced heat kernel on thermal AdS_{2n+1} is given by

$$K^{(S)}(\gamma,t) = \frac{\beta}{2\pi} \sum_{k\in\mathbb{Z}} \sum_{\vec{m}} \int_0^\infty d\lambda\, \chi_{\lambda,\vec{m}}\left(\gamma^k\right) e^{tE_R^{(S)}}, \qquad (2.6.69)$$

where $\chi_{\lambda,\vec{m}}$ is the Harish-Chandra character in the principal series of $SO(2n+1,1)$, which has been evaluated [65] to be

$$\chi_{\lambda,\vec{m}}(\beta,\phi_1,\phi_2,\ldots,\phi_n) = \frac{e^{-i\beta\lambda}\chi_{\vec{m}}^{SO(2n)}(\phi_1,\phi_2,\ldots,\phi_n) + e^{i\beta\lambda}\chi_{\check{\vec{m}}}^{SO(2n)}(\phi_1,\phi_2,\ldots,\phi_n)}{e^{-n\beta}\prod_{i=1}^n |e^\beta - e^{i\phi_i}|^2}, \qquad (2.6.70)$$

for the group element

$$\gamma = e^{i\beta Q_{12}} e^{i\phi_1 Q_{23}} \ldots e^{i\phi_n Q_{2n+1,2n+2}}, \qquad (2.6.71)$$

where $\check{\vec{m}}$ is the conjugated representation, with the highest weight $(m_2,\ldots,-m_{n+1})$, and $\chi_{\vec{m}}^{SO(2n)}$ is the character in the representation $\vec{m}$ of $SO(2n)$. The sum over $\vec{m}$ that appears in (2.6.69) is the sum over permissible values of m as determined by the branching rules (2.5.60). We also recall that '1' is the time-like direction.

For the thermal quotient, we have $\beta \neq 0$ and $\phi_i = 0\ \forall\, i$. The character of $SO(2n)$ that appears in the character formula (2.6.70) above is then just the dimension of the corresponding representation.

Using the fact that the dimensions of this representation is equal to that of its conjugate, we have for the character

$$\chi_{\lambda,\vec{m}}(\beta,\phi_1,\phi_2,\ldots,\phi_n) = \frac{cos\,(\beta\lambda)}{2^{2n-1}\sinh^{2n}\frac{\beta}{2}} d_{\vec{m}}. \tag{2.6.72}$$

We therefore obtain, for the traced heat kernel AdS_{2n+1} (2.6.69),

$$K^{(S)}(\beta,t) = \frac{\beta}{2^{2n}\pi}\sum_{k\in\mathbb{Z}}\sum_{\vec{m}} d_{\vec{m}}\int_0^\infty d\lambda\,\frac{cos\,(k\beta\lambda)}{\sinh^{2n}\frac{k\beta}{2}} e^{-t(\lambda^2+\zeta)}. \tag{2.6.73}$$

The integral over λ is a Gaussian integral, which we can evaluate to obtain

$$K^{(S)}(\beta,t) = \frac{\beta}{2^{2n}\sqrt{\pi t}}\sum_{k\in\mathbb{Z}_+}\sum_{\vec{m}} d_{\vec{m}}\frac{1}{\sinh^{2n}\frac{k\beta}{2}} e^{-\frac{k^2\beta^2}{4t}-t\zeta}, \tag{2.6.74}$$

where we have dropped the term with $k=0$, which diverges. This divergence arises due to the infinite volume of AdS, over which the coincident heat kernel on the full AdS_{2n+1} is integrated. It can be reabsorbed into a redefinition of parameters of the gravity theory under study and is independent of β and is therefore not of interest to us.

This expression further simplifies for the case of the STT tensors. The branching rules determine that $\vec{m} = (s,0,\ldots,0)$ and therefore the sum over $\vec{m}$ gets frozen out and we obtain

$$K^{(S)}(\beta,t) = \frac{\beta}{2^{2n}\sqrt{\pi t}}\sum_{k\in\mathbb{Z}_+}\frac{d_{\vec{m}}}{\sinh^{2n}\frac{k\beta}{2}} e^{-\frac{k^2\beta^2}{4t}-t(s+n^2)}. \tag{2.6.75}$$

The reader may compare this expression to the equation (3.9) obtained in [53] for the AdS_3 case (where one would have to specialise to $\tau_1 = 0$, $\tau_2 = \beta$, and $d_{\vec{m}} \equiv 1$, and further include a factor of 2 that appears because the branching rule for AdS_3 leads us to sum over $m_2 = \pm s$ rather than $m_2 = s$ for $s > 0$).

2.7 The one-loop Partition Function

As a consequence of the above, we can calculate the one-loop determinant of a spin-S particle on AdS_5. To do so, we need the result that

$$\int_0^\infty \frac{dt}{t^{\frac{3}{2}}} e^{-\frac{\alpha^2}{4t}-\beta^2 t} = \frac{2\sqrt{\pi}}{\alpha} e^{-\alpha\beta}. \tag{2.7.76}$$

Then the one-loop determinants can be deduced from the heat kernel by using

$$-\,log\,det\left(-\Delta_{(S)} + m_S^2\right) = \int_0^\infty \frac{dt}{t} K^{(S)}(\beta,t)\, e^{-m_S^2 t}, \tag{2.7.77}$$

which we can simplify to obtain

$$-\,log\,det\left(-\Delta_{(S)} + m_S^2\right) = \sum_{k\in\mathbb{Z}_+}\sum_{\vec{m}} d_{\vec{m}}\frac{2}{e^{-nk\beta}\left(e^{k\beta}-1\right)^{2n}}\frac{1}{k} e^{-k\beta\sqrt{\zeta+m_S^2}}. \tag{2.7.78}$$

This expression further simplifies for the case of STT tensors and one obtains

$$-\,log\,det\left(-\Delta_{(S)} + m_S^2\right) = \sum_{k\in\mathbb{Z}_+} d_{\vec{m}}\frac{2}{e^{-nk\beta}\left(e^{k\beta}-1\right)^{2n}}\frac{1}{k} e^{-k\beta\sqrt{s+n^2+m_S^2}}. \tag{2.7.79}$$

2.7.1 The scalar on AdS_5

Let us evaluate the above expression for scalars in AdS_5, where $s = 0$. In units where the AdS radius is set to one, $\sqrt{m_S^2 + 4} = \Delta - 2$, where Δ is the conformal dimension of the scalar. We therefore have

$$- \, log\, det\left(-\Delta_{(S)} + m_S^2\right) = \sum_{k \in \mathbb{Z}_+} \frac{2}{k\left(1 - e^{-k\beta}\right)^4} e^{-k\beta\Delta} \tag{2.7.80}$$

We can evaluate the sum to find that the one-loop determinant is given by

$$- \, log\, det\left(-\Delta_{(S)} + m_S^2\right) = -2 \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} log\left(1 - e^{-\beta(\Delta+n)}\right). \tag{2.7.81}$$

Now since $logZ_{(S)} = -\frac{1}{2} log\, det\left(-\Delta_{(S)} + m_S^2\right)$, we have, for the one-loop partition function of a scalar,

$$logZ_{(S)} = -\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} log\left(1 - e^{-\beta(\Delta+n)}\right). \tag{2.7.82}$$

This expression matches exactly with that which is obtained with the method of [66] (see also, earlier work by [67, 68]).

2.8 An extension to Even Dimensions

We have so far considered the case of the odd-dimensional hyperboloids. This is mainly because we have obtained the heat kernel answer by means of an analytic continuation from the sphere which essentially captures the 'principal series' contribution to the heat kernel. For the odd-dimensional case there is no other contribution. The case of even-dimensional hyperboloids is however a bit more subtle. There can in principle be a contribution from the discrete series also. However, as we outline below, this series does not contribute for a wide class of tensor fields. In particular, this includes the case of the STT tensors which have been of special interest to us[7]. The answer for the heat kernel in such cases is again captured by the usual analytic continuation, as was explicitly shown in [64].

We will now briefly sketch how the computation of the traced heat kernel would proceed for the even-dimensional hyperboloids. We begin by observing that the expression (2.4.54) is valid for cosets of compact Lie groups G and H, which therefore includes the case of the even-dimensional spheres also. That a thermal section on such spheres may be defined is apparent via the geometric construction outlined in the main text[8]. The expression for the heat kernel on $S^{2n} \simeq SO\left(2n+1\right)/SO\left(2n\right)$, $n \geq 2$ is then a sum of characters of the 'thermal' quotient group Γ embedded in $SO\left(2n+1\right)$.

The hyperboloid AdS_{2n} is the quotient space $SO\left(2n,1\right)/SO\left(2n\right)$. The principal series of unitary irreducible representations of $SO\left(2n,1\right)$ are labelled by an array

$$R = \left(i\lambda, m_2, m_3, \cdots, m_n\right), \quad \lambda \in \mathbb{R},\, m_2 \geq m_3 \geq \cdots \geq m_n, \tag{2.8.83}$$

where the $m_2, \cdots, m_n$ are non-negative (half-)integers. These contain a representation S of $SO\left(2n\right)$ if [58, 63]

$$s_1 \geq m_2 \geq s_2 \geq \ldots \geq m_n \geq |s_n|. \tag{2.8.84}$$

The analytic continuation from the unitary irreducible representations of $SO\left(2n+1\right)$ to the prin-

[7]These remarks are true for higher (than two)-dimensional hyperboloids. There is an additional discrete series contribution in AdS_2 *even* for the STT tensors. See [64] for details.

[8]As an example, we see that setting $\phi_3 = 0$ in (2.4.35) gives us a parametrisation of S^4. The construction of the thermal section is then exactly analogous.

cipal series of $SO\,(2n,1)$ may be deduced as in Section 2.5.2. The answer is

$$m_1 \mapsto i\lambda - \frac{2n-1}{2}, \quad \lambda \in \mathbb{R}_+, \tag{2.8.85}$$

which is precisely the continuation obtained in [64]. There is in addition an additional discrete series of representations which is not captured by this analytic continuation. However, this contains the representation S only if $s_n \geq \frac{1}{2}$, see [58] for details. Therefore, this additional series never contributes for the STT tensors (for which s_n equals zero). The naive analytic continuation is therefore sufficient to give the full heat kernel answer.

The methods outlined in Sections 2.5 and 2.6 may therefore be extended to even dimensional hyperboloids as well. The expressions for the global characters of $SO\,(2n,1)$ are well known [65]. We therefore have all the ingredients needed to compute the heat kernel on thermal AdS_{2n}.

For example, in this manner the one-loop partition function for a scalar on AdS_4 may be calculated. We find that

$$logZ_{(S)} = \sum_{k\in\mathbb{Z}_+} \frac{1}{k\left(1-e^{-k\beta}\right)^3} e^{-k\beta\Delta}, \tag{2.8.86}$$

where Δ is determined in terms of the mass of the scalar via

$$\Delta = \sqrt{m^2 + \frac{9}{4}} + \frac{3}{2}. \tag{2.8.87}$$

This matches, for instance, with the expressions obtained via the Hamiltonian analysis done in [69].

2.9 Conclusions

We have computed the principal ingredients that go into the calculation of one loop effects on odd dimensional thermal AdS spacetimes. In the following two chapters, we shall apply these results to evaluate the one-loop partition function for higher-spin theories in AdS spacetimes.

Chapter 3

The Partition Function for Higher-Spin Theories in AdS

3.1 Introduction

Studies of the asymptotic symmetries of higher-spin theories in AdS_3 have recently led to remarkable progress in their holography, including the formulation of an explicit duality between Vasiliev theories in AdS_3 and $\mathcal{W}_N$ minimal models [29]. It was shown in [27, 28] that the asymptotic symmetry algebra of (a class of) higher-spin theories comprises of two copies of a $\mathcal{W}$ algebra. This was checked in [70] by computing the one-loop partition function of the theory, to obtain a vacuum character of left- and right-moving $\mathcal{W}$ algebras. The results of [70] were in fact a key ingredient in formulating the duality of [29]. The analysis carried out in [70] drew heavily on the heat kernel methods of [53]. Essentially, the message to take away from these computations is that the second-quantised partition function contains information about additional symmetries which act on *multiparticle* states of the theory.

While a Brown-Henneaux like analysis for Vasiliev theories in dimensions higher than three is prohibitive, the one-loop partition function may be computed readily. The essential ingredients are again the quadratic action for higher-spin fields, which is well known (see [28] for a recent review), and the determinants of the Laplacian, which have been computed in [71].

In this chapter, we shall carry out precisely such a computation for a Vasiliev theory in general-dimensional AdS spacetimes. In particular, we shall evaluate this partition function on a thermal quotient of AdS. As mentioned above, for the one-loop partition function of a theory the details of the interactions are not important. The only information that enters is the quadratic action and the spectrum. We shall choose the spectrum of all spins $s = 0, 1, 2, \ldots \infty$ appearing once [1]. We shall obtain a form for the partition function which is suggestive of a vacuum character of a symmetry group. We also express the partition function in a form (3.2.17) which is closely related to the d-dimensional MacMahon function.

A brief overview of this chapter is as follows. In section 3.2 we compute the partition function of a massless spin-s field in AdS_D by the heat kernel method.[2] We show that the degrees of freedom that contribute to the one-loop partition function are encoded in symmetric, transverse, traceless (STT) tensors of rank s, and $s-1$. These determinants have been evaluated in [71] We obtain an answer consistent with the the partition function of the spin-s conserved current in CFT$_{D-1}$. As we outline later (see the discussion just above section 3.2.1), while this analysis goes through for general dimensional AdS spacetimes, though the expressions for AdS_5 are perhaps the nicest, and we concentrate mostly on this case here.

For evaluating these determinants to arrive at concrete expressions for the partition function, we begin with the relatively simpler case of thermal AdS_5, *i.e.* turning off all other chemical

[1] As is well known, there is a also minimal Vasiliev theory with only even spins. We consider the theory containing all spins in its spectrum. On the boundary CFT, this corresponds to considering conserved currents built out of complex scalars. Also see [72, 73] for representations of the higher-spin algebra in AdS_5.

[2] We evaluate these determinants explicitly for odd-dimensional AdS spacetimes. For a very general class of tensors, which includes STT tensors, the analysis for even dimensions greater than two is entirely analogous and we do not repeat it here. See [71] and references therein.

potentials, in section 3.2.1. We observe simplifications due to which the partition function of the theory arranges itself into a form reminisctive of the MacMahon function, and also write down a nested, vacuum character like form for this partition function. Expressions for the partition function of the higher-spin theories in AdS_4 and AdS_7 are available in Appendix C.

In section 3.3 we generalise to the case of non-zero chemical potentials for the $SO(4)$ Cartans of the AdS_5 isometry group. We find that the essential simplifications in section 3.2.1 remain, due to which it is still possible to write the answer in terms of a (shifted) two-dimensional MacMahon function. We also obtain a nested form for the answer, suggestive of a vacuum character interpretation. We then discuss our results in section 3.4. Some calculational details are available in the Appendices.

3.2 The Partition Function for a Higher-Spin Field in AdS

We begin with computing the partition function of a massless spin-s field in an arbitrary dimensional (Euclidean) AdS spacetime. Our starting point will be the functional integral

$$Z^{(s)} = \frac{1}{\text{Vol(gauge group)}} \int \left[D\phi_{(s)}\right] e^{-S[\phi_{(s)}]}, \tag{3.2.1}$$

which we shall compute in the one-loop approximation. Hence, only the spectrum of the theory and the quadratic terms in the action are of relevance to us. The quadratic action for higher-spin fields has already been worked out in an arbitrary dimensional AdS spacetime and is mentioned in (1.3.26).

We now evaluate the functional integral (4.3.5) with the action (1.3.24) using the method adopted in [70, 74, 75] which we now summarise. Since this action has a gauge invariance associated with it, we need to gauge fix. Typically this involves the introduction of Faddeev-Popov ghosts. In this case however, there is a natural choice of integration variables for the functional integral. We use the decomposition

$$\phi_{\mu_1\ldots\mu_s} = \phi^{TT}_{\mu_1\ldots\mu_s} + g_{(\mu_1\mu_2}\tilde{\phi}_{\mu_3\ldots\mu_s)} + \nabla_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}, \tag{3.2.2}$$

where $\tilde{\phi}$ and ξ are respectively symmetric traceless rank $s-2$ and $s-1$ tensors. We note that the last term is just the gauge transformation of the field ϕ under which the action 1.3.24 is invariant. This is the choice of integration variables employed in [74] for the spin-2 field, and subsequently generalised in [70] to higher-spin fields in AdS_3. The measure for the functional integral changes under this change of variables.

$$[D\phi_{(s)}] = \mathcal{Z}^{(s)}_{gh}[D\phi^{TT}_{(s)}][D\tilde{\phi}_{(s-2)}]\left[D\xi_{(s-1)}\right], \tag{3.2.3}$$

where $\mathcal{Z}^{(s)}_{gh}$ is the ghost determinant that arises from making the change of variables. Now, using the gauge invariance of the action, we have

$$S\left[\phi_{(s)}\right] = S\left[\phi^{TT}_{(s)} + g\tilde{\phi}_{(s-2)}\right]. \tag{3.2.4}$$

One may further show that

$$S\left[\phi^{TT} + g\tilde{\phi}\right] = S\left[\phi^{TT}\right] + S\left[g\tilde{\phi}\right], \tag{3.2.5}$$

where

$$S\left[\phi^{TT}\right] = \phi^{\mu_1\ldots\mu_s}_{TT}\left[\Box - \frac{s^2 + (D-6)\,s - 2\,(D-3)}{\ell^2}\right]\phi^{TT}_{\mu_1\ldots\mu_s}. \tag{3.2.6}$$

Therefore, the contribution to the one-loop partition function is given by

$$\mathcal{Z}^{(s)} = \mathcal{Z}^{(s)}_{gh}\left[\det\left(-\Box + \frac{s^2+(D-6)\,s-2\,(D-3)}{\ell^2}\right)_{(s)}\right]^{-\frac{1}{2}} \int\left[D\tilde{\phi}_{(s-2)}\right]e^{-S[\tilde{\phi}_{(s-2)}]}, \qquad (3.2.7)$$

where the subscript (s) on the second term reminds us that the determinant should be evaluated over rank s STT tensors. This expression should be compared to Equation (2.8) of [70]. Also, the ghost determinant may be evaluated using the identity

$$1 = \int\left[\mathcal{D}\phi_{(s)}\right]e^{-\langle\phi_{(s)},\phi_{(s)}\rangle} = \mathcal{Z}_{gh}\int\left[\mathcal{D}\phi_{(s)}\right]\left[\mathcal{D}\tilde{\phi}_{(s-2)}\right]\left[\mathcal{D}\xi_{(s-1)}\right]e^{-\langle\phi_{(s)},\phi_{(s)}\rangle}, \qquad (3.2.8)$$

as in [70, 74]. The evaluation is entirely analogous to the procedure adopted in [70], and we merely mention the final result. The partition function $\mathcal{Z}^{(s)}$ is determined to be a ratio of functional determinants evaluated over STT tensor fields. In particular,

$$\mathcal{Z}^{(s)} = \frac{\left[\det\left(-\Box - \frac{(s-1)(3-D-s)}{\ell^2}\right)_{(s-1)}\right]^{\frac{1}{2}}}{\left[\det\left(-\Box + \frac{s^2+(D-6)s-2(D-3)}{\ell^2}\right)_{(s)}\right]^{\frac{1}{2}}}. \qquad (3.2.9)$$

The numerator is a determinant evaluated over rank $s-1$ STT tensor fields. These determinants were evaluated explicitly in [71], and reviewed in the previous chapter, for Laplacians over quotients of AdS. In particular, for the thermal quotient of AdS_{2n+1}, we find that

$$\log \mathcal{Z}^{(s)} = \sum_{m=1}^{\infty}\frac{e^{-m\beta(s+2n-2)}}{\left(1-e^{-m\beta}\right)^{2n}}\left(d_s - d_{s-1}e^{-m\beta}\right), \qquad (3.2.10)$$

where we adopt the notation d_s for the dimension of the $(s,0,\ldots,0)$ representation of $SO\,(2n)$. We would like to exponentiate this expression to determine the partition function $\mathcal{Z}^{(s)}$. The following identity is useful for this purpose.

$$\sum_{m=1}^{\infty}\frac{q^{-m\Delta}}{m\left(1-q^m\right)^{2n}} = \sum_{m=1}^{\infty}\binom{2n+m-2}{2n-1}\log\frac{1}{1-q^{(\Delta+m-1)}}, \qquad (3.2.11)$$

where we have defined $q = e^{-\beta}$. Using (3.2.11), we find that

$$\mathcal{Z}^{(s)} = \prod_{m=1}^{\infty}\left(\frac{\left(1-q^{(s+m+2n-2)}\right)^{d_{s-1}}}{\left(1-q^{(s+m+2n-3)}\right)^{d_s}}\right)^{\binom{m+2n-2}{2n-1}}. \qquad (3.2.12)$$

This is the contribution of a single spin-s field to the partition function of the theory in an odd dimensional AdS spacetime. The case of even dimensions is similar, see Appendix C. Now, it will be apparent to the reader that many of the subsequent steps that we shall now undertake will follow in arbitrary dimensions as well. However, the resulting expressions are perhaps the nicest for the case of AdS_5 and we shall focus on this case from now onwards. We refer the reader to Appendix C for the expressions for the blind partition function in AdS_4 and AdS_7.

3.2.1 The Blind Partition Function in AdS_5

While we shall finally work with nonzero chemical potentials for the $SO(4)$ Cartans of AdS_5 as well, it is useful, for intuition, to start with the case where they are zero, *i.e.* thermal AdS_5. This will also yield some curious forms for the partition function. We have, on setting $n = 2$ in (3.2.12),

$$\mathcal{Z}^{(s)} = \prod_{m=1}^{\infty} \left(\frac{\left(1 - q^{(s+m+2)}\right)^{s^2}}{\left(1 - q^{(s+m+1)}\right)^{(s+1)^2}} \right)^{\binom{m+2}{3}} . \tag{3.2.13}$$

This is the contribution of a single spin-s field to the partition function of the theory. To determine the full partition function, we need to specify the spectrum of the theory. We choose the spectrum $s = 0, 1, 2, \ldots, \infty$, with each spin appearing only once. The full partition function of the theory is then

$$\mathcal{Z} = \prod_{s=0}^{\infty} \mathcal{Z}^{(s)}. \tag{3.2.14}$$

Note that there are cancellations between the numerator of $\mathcal{Z}^{(s)}$ and the denominator of $\mathcal{Z}^{(s+1)}$, for $s \geq 1$ because they are both of the form $\left(1 - q^{(s+m+2)}\right)^{\alpha}$. We finally obtain

$$\mathcal{Z} = \prod_{s=0}^{\infty} \prod_{m=1}^{\infty} \frac{1}{(1 - q^{s+m+2})^{4(s+1)\binom{m+2}{3}}} \cdot \prod_{m=1}^{\infty} \frac{1}{(1 - q^{m+1})^{\binom{m+2}{3}}}. \tag{3.2.15}$$

We can assemble the first product into a single product over $k \equiv s + n$. This is given by

$$\prod_{s=0}^{\infty} \prod_{m=1}^{\infty} \frac{1}{(1 - q^{s+m+2})^{4(s+1)\binom{m+2}{3}}} = \prod_{k=1}^{\infty} \frac{1}{(1 - q^{k+2})^{4\binom{k+4}{5}}}. \tag{3.2.16}$$

We then obtain

$$\mathcal{Z} = \prod_{k=1}^{\infty} \frac{1}{(1 - q^{k+2})^{4\binom{k+4}{5}}} \cdot \prod_{m=1}^{\infty} \frac{1}{(1 - q^{m+1})^{\binom{m+2}{3}}}. \tag{3.2.17}$$

This form of the answer is closely related to the d-dimensional MacMahon function (see [76] for a recent review)

$$M_d(q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^{\binom{n+d-2}{d-1}}}. \tag{3.2.18}$$

We remind the reader that in [70] a form for the partition function of a higher-spin theory in AdS_3 was obtained in terms of the (two-dimensional) MacMahon function and bore an interpretation as the vacuum character of $\mathcal{W}_N$, the asymptotic symmetry algebra of the theory [27, 28]. We also observe the existence of a 'nested product' form for this answer.

$$\mathcal{Z} = \prod_{s_1=3}^{\infty} \prod_{s_2=s_1}^{\infty} \prod_{s_3=s_2}^{\infty} \prod_{s_4=s_3}^{\infty} \prod_{s_5=s_4}^{\infty} \prod_{n=s_5}^{\infty} \frac{1}{(1 - q^n)^4}. \tag{3.2.19}$$

The reader may verify that this expression does indeed reduce to (3.2.17). We also remind the reader that an analogous expression for AdS_3 was interpreted in [70] as the vacuum character of the $\mathcal{W}_\infty$ algebra. It would be very interesting to see if our five-dimensional answer also admits a similar interpretation.

3.2.2 Additional Chemical Potentials in AdS_5

From now onwards we turn on chemical potentials $(\beta, \alpha_1, \alpha_2)$ for AdS_5, where β is the inverse temperature and α_1, α_2 are chemical potentials for the $SO(4)$ Cartans of the AdS_5 isometry group $SO(4,2)$. To explicitly evaluate (3.2.9), we will need the character given in equation 5.3 of [71], where we specialise to $n=2$, *i.e.* AdS_5, and focus on the case of STT tensors.

$$\chi_{(\lambda,s)}\left(\beta,\alpha_1,\alpha_2\right)=\frac{e^{-i\beta\lambda}\chi^{SO(4)}_{(s,0)}\left(\phi_1,\phi_2\right)+e^{i\beta\lambda}\chi^{SO(4)}_{(s,0)}\left(\phi_1,\phi_2\right)}{e^{-2\beta}|e^{\beta}-e^{i\phi_1}|^2|e^{\beta}-e^{i\phi_2}|^2}. \tag{3.2.20}$$

We can evaluate these characters to obtain

$$\chi_{(\lambda,s)}\left(\beta,\alpha_1,\alpha_2\right)=\frac{2\cos\left(\beta\lambda\right)}{e^{-2\beta}|e^{\beta}-e^{i\phi_1}|^2|e^{\beta}-e^{i\phi_2}|^2}\frac{\sin\left(\left(s+1\right)\alpha_1\right)}{\sin\left(\alpha_1\right)}\frac{\sin\left(\left(s+1\right)\alpha_2\right)}{\sin\left(\alpha_2\right)}, \tag{3.2.21}$$

where the ϕs and αs are related by

$$\phi_1=\alpha_1+\alpha_2,\quad \phi_2=\alpha_1-\alpha_2. \tag{3.2.22}$$

The partition function of a massless spin-s particle may then be computed to obtain

$$\log\mathcal{Z}^{(s)}=\sum_{m=1}^{\infty}\frac{1}{m}\frac{e^{-m\beta(s+2)}}{|1-e^{-m(\beta-i\phi_1)}|^2|1-e^{-m(\beta-i\phi_2)}|^2}\left[\chi^{SO(4)}_{(s,0)}-\chi^{SO(4)}_{(s-1,0)}e^{-m\beta}\right], \tag{3.2.23}$$

where the $SO(4)$ characters are evaluated over the angles $(m\phi_1,m\phi_2)$. This is precisely the answer that would be obtained via a Hamiltonian computation, as carried out in [69] using the results of [77, 78].

3.3 The Refined Partition Function in AdS_5

Having acquired some intuition for the kind of simplifications we may expect by looking at the blind partition function, we shall now now turn to the case of non-zero chemical potentials α_1 and α_2 and exponentiate the expression (3.2.23). This will enable us to write down another nested form for the partition function. It would be again useful to introduce notation

$$e^{-\beta}=q,\quad e^{i\alpha_1}=p,\quad e^{i\alpha_2}=r. \tag{3.3.24}$$

We finally obtain

$$\mathcal{Z}^{(s)}=\prod_{m_i=0}^{\infty}\frac{\prod_{k,l=0}^{s-1}\left(1-q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s-1}r^{m_1+m_4+s-1}\bar{p}^{m_2+m_4+2k}\bar{r}^{m_2+m_3+2l}\right)}{\prod_{k,l=0}^{s}\left(1-q^{m_1+m_2+m_3+m_4+s+2}p^{m_1+m_3+s}r^{m_1+m_4+s}\bar{p}^{m_2+m_4+2k}\bar{r}^{m_2+m_3+2l}\right)}, \tag{3.3.25}$$

where the product over m_i collectively denotes a product over $m_1,\ldots,m_4$. It turns out that even for the refined case, the ratio of the numerator of $\mathcal{Z}^{(s)}$ and the denominator of $\mathcal{Z}^{(s+1)}$ is again simple. This leads to the following form for the one-loop partition function of the theory.

$$\mathcal{Z}=\prod_{s=0}^{\infty}\prod_{m_i=0}\frac{1}{\hat{\prod}\left(1-q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s-1}r^{m_1+m_4+s-1}\bar{p}^{m_2+m_4+2k}\bar{r}^{m_2+m_3+2l}\right)}, \tag{3.3.26}$$

where the hatted product in the denominator runs over the following values of k,l,

$$\begin{gathered}k=0,\,l=0;\quad k=0,\,l=s+1;\quad k=s+1,\,l=0\quad k=s+1,\,l=s+1\\ k=0,\,l=1\ldots s;\quad l=0,\,k=1\ldots s;\quad k=s+1,\,l=1\ldots s;\quad l=s+1,\,k=1\ldots s,\end{gathered}$$

and we have used the fact that $(p\bar{p})^2(r\bar{r})^2=1$. It is useful to define the following combinations

$$q_1=pr,\quad q_2=\bar{p}\bar{r},\quad q_3=p\bar{r},\quad q_4=\bar{p}r. \tag{3.3.27}$$

The expression (3.3.26) may be simplified using the procedure outlined in Appendix D. We finally find that the partition function may be written in terms of the product of four nested product expressions.

$$\mathcal{Z} \equiv \mathcal{Z}_{(0)} \cdot \mathcal{Z}_{(1)} \cdot \mathcal{Z}_{(2)} \cdot \mathcal{Z}_{(3)} \cdot \mathcal{Z}_{(4)}, \tag{3.3.28}$$

where $\mathcal{Z}_{(0)}$ is the partition function of the scalar, which may be computed to obtain

$$\mathcal{Z}_{(0)} = \prod_{m_{1,2,3,4}} \frac{1}{\left(1 - q^2 \left(q q_1\right)^{m_1} \left(q q_2\right)^{m_2} \left(q q_3\right)^{m_3} \left(q q_4\right)^{m_4}\right)}, \tag{3.3.29}$$

and

$$\mathcal{Z}_{(1)} = \prod_{k=0}^{\infty} \prod_{m_{2,3,4}}^{\infty} \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \frac{1}{\left(1 - \left(q q_1\right)^n q^{k+m_2+m_3+m_4+2} q_1^k q_2^{m_2+k} q_3^{m_3+k} q_4^{m_4}\right)}. \tag{3.3.30}$$

This has been explicitly evaluated in appendix D. The other $\mathcal{Z}_{(i)}$s have similar expressions which we now enumerate.

$$\mathcal{Z}_{(2)} = \prod_{k=0}^{\infty} \prod_{m_{1,2,4}}^{\infty} \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \frac{1}{\left(1 - \left(q q_3\right)^n q^{k+m_1+m_2+m_4+2} q_1^{m_1} q_2^{m_2+k} q_3^k q_4^{m_4+k}\right)}, \tag{3.3.31}$$

$$\mathcal{Z}_{(3)} = \prod_{k=0}^{\infty} \prod_{m_{1,2,3}}^{\infty} \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \frac{1}{\left(1 - \left(q q_4\right)^n q^{k+m_1+m_2+m_4+2} q_1^{m_1+k} q_2^{m_2} q_3^{m_3+k} q_4^k\right)}, \tag{3.3.32}$$

$$\mathcal{Z}_{(4)} = \prod_{k=0}^{\infty} \prod_{m_{1,3,4}}^{\infty} \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \frac{1}{\left(1 - \left(q q_2\right)^n q^{k+m_1+m_3+m_4+2} q_1^{m_1+k} q_2^k q_3^{m_3} q_4^{m_4+k}\right)}. \tag{3.3.33}$$

3.4 Discussion

In this chapter, we applied the heat kernel results of [71] to compute the partition function of a higher-spin theory in AdS_5. To get a better feeling for the possible content of the answer it is useful to recollect elements of the corresponding story for pure gravity and higher-spin gravity in AdS_3. This discussion is qualitative, and we refer the reader to the original papers for more details. As we remarked previously, the asymptotic symmetry algebra of pure gravity on AdS_3 comprises of two copies of the Virasoro algebra [7] and for higher-spin gravity comprises of two copies of the $\mathcal{W}$ algebra [27, 28], while the gauge symmetry is $SL(2,R) \times SL(2,R)$ and $SL(N,R) \times SL(N,R)$ or $hs(1,1)$ respectively. One manifestation of the asymptotic symmetry algebra is in the one-loop partition function of the theories expanded about their AdS vacua. For the higher-spin theory with spectrum of spins $s = 2, \ldots, \infty$, the partition function is [70]

$$\mathcal{Z} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} = \chi_0\left(\mathcal{W}_\infty\right) \times \bar{\chi}_0\left(\mathcal{W}_\infty\right). \tag{3.4.34}$$

This is (two copies of) the character evaluated over the Verma module built out of the AdS_3 vacuum as the lowest weight state and the raising operators $\mathcal{W}^{(s)}_{-m}$, $s = 2, \ldots, \infty$, acting on it. This is the sense in which the second-quantised partition function is supposed to encode information about multiparticle symmetries of the theory. The corresponding expressions for pure gravity [53, 60, 79] are perhaps more familiar to the reader. The partition function is then a character of the Virasoro algebra, again with the AdS_3 vacuum as the lowest weight state and the generators L_{-m} acting as raising operators, for $m \geq 2$. These generators, when acting on the AdS_3 vacuum generate large gauge transformations to create multiparticle states—boundary gravitons—over the AdS_3 vacuum.

We obtained a form for the answer that is suggestive of a vacuum character of a larger symmetry group than the higher-spin symmetry. In particular, following the reasoning suggested above, it leads us to speculate that there are additional generators in the symmetry algebra that act on multiparticle states built out of twist-two operators, which sit in the vacuum representation of the symmetry algebra. It would be very interesting to verify if this is indeed the case. A natural candidate symmetry to relate these results to is the Yangian symmetry of $\mathcal{N} = 4$ SYM. However, we do not know of an apparent connection, especially since the Yangian algebra does not close over the set of gauge-invariant operators.[3] Alternatively, this might be related to the additional symmetries of free Yang-Mills noted in [80]. There is additionally the intriguing possibility that these might be connected to multiparticle symmetries already noted in the literature for higher-spin theories [81–83].[4]

Additionally enhanced symmetries for higher-spin theories have also been observed in [84–86] .[5] It is also possible that our results are related to these symmetries.

[3] We thank N. Beisert for this remark.

[4] We thank M. Vasiliev for this observation.

[5] We thank D. Sorokin for bringing this to our attention.

Chapter 4

The Partition Function for Topologically Massive Higher-Spin Theory

4.1 Three-Dimensional Gravity and its Topologically Massive deformation

Possibly the most viable testing ground for theories of quantum gravity has been three dimensional gravity in Anti de-Sitter space. Pure Anti de-Sitter gravity in three dimensions, as is well known, has no local propagating degrees of freedom. At the same time, it has non-trivial solutions, including Black Holes [89]. This opens up the possibility of using three-dimensional gravity as a toy model for studying various interesting phenomena such as the black hole information paradox. At the same time, three-dimensional gravity is a particularly promising arena to probe other interesting questions about quantum gravity, such as the roles of time, topology and topology change in qauntum gravity. We refer the reader to [90] for a review of these aspects of three dimensional quantum gravity.

Motivated by the AdS/CFT conjecture, and the above considerations, there have been attempts to find the dual of the 3d AdS theory in terms of a conformal field theory. The field theory dual to the bulk Einstein theory in AdS_3 was initially conjectured to be an extremal CFT [91]. But a partition function computation by taking into account contributions from classical geometries and also including quantum corrections [79] showed that the expected holomorphic factorisation does not hold and there were other studies [92] which indicated that this conjecture was incorrect.

Motivated by [91], the authors of [93] looked at Topologically Massive Gravity (TMG) in AdS_3, which consists of Einstein gravity, modified by the addition of a gravitational Chern-Simons term [87, 88]

$$S_3 = S_{EH} + S_{CS} \tag{4.1.1}$$

$$\text{where} \qquad S_{EH} = \int d^3x\sqrt{-g}(R - 2\Lambda), \tag{4.1.2}$$

$$\text{and} \qquad S_{CS} = \frac{1}{2\mu}\int d^3x \epsilon^{\mu\nu\rho}\left(\Gamma^{\sigma}_{\mu\lambda}\partial_\nu\Gamma^{\lambda}_{\rho\sigma} + \frac{2}{3}\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\theta}\Gamma^{\theta}_{\rho\sigma}\right). \tag{4.1.3}$$

This theory already has the appealing feature that in contrast to usual Einstein gravity in AdS_3, it has local propagating gravitational degrees of freedom, thus making it more like the higher-dimensional theories.[1] However, a closer look reveals some disquieting features. In particular, it

[1]The theory is of course, parity violating, in contrast to Einstein gravity in higher dimensions.

turns out that at a generic value of μ, the theory has negative energy gravitons. [2] However, it turns out that at the 'chiral' point $\mu\ell = 1$ the negative energy gravitons (which are purely left-moving) have zero energy, and are thus pure gauge. Also, at this point, the corresponding central charge $c_{\pm} = \frac{3l}{2G}(1 \mp \frac{1}{\mu\ell})$ [94, 95] vanishes. This therefore gives rise to a holomorphically factorised (trivially so, since its purely right-moving) partition function. It was therefore conjectured in [93] that the boundary theory dual to TMG in AdS_3 is a right-moving chiral CFT.

This conjecture recieved intense scrutiny (see, for example, the works [96]), and in [97, 98] it was shown that TMG at the chiral point was more generally dual to a Logarithmic Conformal Field Theory (LCFT) and that there were additional normalisable solutions to TMG which carried negative energy at the chiral point, thus invalidating the earlier conjecture. A more complete analysis in terms of holographic renormalisation techniques [99] was carried out and the results supported the latter claim of the duality to LCFT. One of the more robust checks of this conjecture was the recent computation of the one-loop partition function [74]. It was conclusively shown that there is no holomorphic factorisation of the one-loop partition function at the chiral point. The structure of the gravity calculation also matched with expectations from LCFT. The authors [74] found an exact matching with a part of the answer, *viz.* the single-particle excitations and offered substantial numerical evidence of the matching of the full partition functions.

4.2 Higher-Spin Theories in AdS_3

Much as in the case of gravity, three-dimensional higher-spin theories are a useful playground for studying dynamics of higher-spin fields in AdS spacetimes. In contrast to the higher-dimensional cases, in three dimensions there can be a truncation to fields with spin less than and equal to N for any N.[3] It has been argued in [27, 28] on the basis of a Brown-Henneaux analysis that classically, these theories have an extended classical $\mathcal{W}_N$ asymptotic symmetry algebra (see also the recent work [100]). A one-loop computation in [70], using the techniques developed in [53], showed that this symmetry is indeed perturbatively realised at the quantum level as well. This was an important ingredient in formulating the duality [29] between higher-spin theories and $\mathcal{W}_N$ minimal models in the large-N limit. There was also a subsequent work [101], in which the authors provided a bound on the amount of higher spin gauge symmetry present. This was regarded as a gravitational exclusion principle, where quantum gravitational effects place an upper bound on the number of light states in the theory. Different tests of this duality have been performed successfully subsequent to its proposal, see [102].

The higher spin theories described above, like Einstein AdS gravity in three dimensions, do not have any propagating degrees of freedom in the 'gravitational' sector.[4] It is natural thus to ask if one can generalise Topologically Massive Gravity to theories of higher spin. A construction of this theory, called Topologically Massive Higher Spin Gravity (TMHSG) was initated in [103, 104]. As a first step to this end, the quadratic action for a spin-3 field was studied in the linearised approximation about AdS. The results of this analysis were strongly suggestive of the interpretation that the spin-3 theory at the so-called chiral point is dual to a logarithmic CFT. Specifically, the space of solutions developed an extra logarithmic branch at the chiral point.

In this chapter, we shall describe a test of this conjecture at the (leading) quantum level. In particular, we shall compute the one-loop partition function of the spin-3 theory on a thermal quotient of AdS_3 and show that it does not factorize holomorphically at the chiral point. Our analysis is along the lines of [74], and our results may be viewed as a higher-spin generalisation

[2]We adopt sign conventions such that black holes have positive energy.

[3]It is also notable that these simpler theories with a finite number of spins are consistent truncations of the more non-trivial Vasiliev theories in AdS_3.

[4]The Vasiliev theories to which the $\mathcal{W}_N$ minimal models are believed to be dual contain propagating scalars. In contrast to higher-dimensions, however, the scalars fall into separate representations of the higher-spin algebra than the spins two and higher.

of theirs. We shall also compute the one-loop partition function for a spin-N generalisation of the action proposed in [103] and show that this property continues to hold. We interpret these results as an indication that the dual CFT at the chiral point is not chiral, and that the results are consistent with the expectation of a high spin extension of a dual logarithmic CFT.

In particular, we find the contribution to the one loop partition function at the chiral point, from the spin-3 fields to be

$$Z^{(3)}_{TMHSG} = \prod_{n=3}^{\infty} \frac{1}{|1-q^n|^2} \prod_{m=3}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{(1-q^m \bar{q}^{\bar{m}})} \prod_{k=4}^{\infty} \prod_{\bar{k}=3}^{\infty} \frac{1}{(1-q^k \bar{q}^{\bar{k}})}, \tag{4.2.4}$$

where $q \equiv e^{i\tau}$, and τ is the modular parameter on the boundary torus of thermal AdS_3. The first term is the holomorphic contribution determined from a study of the massless theory in [70], the other terms are non-holomorphic, and new. The middle term is the contribution from the transverse traceless spin-3 determinant, while the last term is the transverse spin-1 contribution coming from the trace of the spin-3 field.

The case of general spins yields analogous results. The relevant excitations are transverse traceless spin-s, spin-$s-1$ (coming from the ghost determinant), transverse traceless spin-$s-2$ ones (which is the trace of the spin-s field), transverse traceless spin-$s-3$ (coming from the longitudinal component of trace), transverse traceless spin-$s-4$ (coming from the longitudinal component of the longitudinal component of the trace) and so on upto transverse traceless spin-1, of which the spin-s, $s-2$, $s-3$, $\cdots$, 1 contribute non-holomorphically to the one-loop partition function. There are no other relevant excitations coming from the spin-s analysis because of the double-tracelessness condition reviewed, for example, in [28]. In section 4.6, we conclude with a brief interpretation of our results. We will also do a classical analysis for arbitrary spins in appendix E and show that the contributions to the partition function also appear in the classical spectrum.

4.3 The basic set up for $s = 3$

In this section we will compute the one loop partition function for spin 3 TMHSG, and in the process build up a mechanism to generalise our calculations to arbitrary spin in the subsequent section. Following the method adopted in [70], we shall compute the one-loop partition function in the Euclideanised version of theory *via* the path integral

$$Z^{(s)} = \frac{1}{\text{Vol(gauge group)}} \int \left[D\phi_{(s)}\right] e^{-S[\phi_{(s)}]}. \tag{4.3.5}$$

In the one-loop approximation, only the quadratic part of the action $S\left[\phi_{(s)}\right]$ is relevant. This has been worked out for TMHSG, for the case of $s = 3$, in [103]. In the Euclidean signature it takes the form [5]

$$S = \frac{1}{2} \int d^3x \, \sqrt{g} \phi^{MNP} \left[\hat{\mathcal{F}}_{MNP} - \frac{1}{2} \hat{\mathcal{F}}_{(M} g_{NP)}\right], \tag{4.3.6}$$

where

$$\hat{\mathcal{F}}_{MNP} = \mathcal{D}^{(M)} \mathcal{F}_{MNP} \equiv \mathcal{F}_{MNP} + \frac{i}{6\mu} \varepsilon_{QR(M} \nabla^Q \mathcal{F}^R{}_{NP)}, \tag{4.3.7}$$

and

$$\mathcal{F}_{MNP} = \Delta\phi_{MNP} - \nabla_{(M|} \nabla^Q \phi_{|NP)Q} + \frac{1}{2} \nabla_{(M} \nabla_N \phi_{P)} - \frac{2}{\ell^2} g_{(MN} \phi_{P)}. \tag{4.3.8}$$

As always, the brackets "( )" denote the sum of the minimum number of terms necessary to achieve complete symmetrisation in the enclosed indices without any normalisation factor. Let

[5] This is the most general parity violating, three derivative action for the free spin-3 field. It can also be obtained from linearising the parity violating, non-linear theory of [104]

us also define the operation of $\mathcal{D}^{(M)}$ on the trace of the spin-3 field by taking the trace of the expression in (4.3.7)

$$\mathcal{D}^{(M)}\phi_M = \phi_M + \frac{i}{6\mu}\varepsilon_{QRM}\nabla^Q\phi^R \tag{4.3.9}$$

We now decompose the fluctuations ϕ_{MNP} into transverse traceless $\phi^{(TT)}$, trace $\tilde{\phi}_M$ and longitudinal parts $\nabla_{(M}\xi_{NP)}$ as[6]

$$\phi_{MNP} = \phi^{(TT)}_{MNP} + \tilde{\phi}_{(M}g_{NP)} + \nabla_{(M}\xi_{NP)}. \tag{4.3.10}$$

Following [70], we use gauge invariance, and orthogonality of the first two terms in (4.3.10) to decompose the action (4.3.6) as

$$S[\phi_{MNP}] = S[\phi^{(TT)}_{MNP}] + S[\tilde{\phi}_M], \tag{4.3.11}$$

where

$$S[\phi^{(TT)}_{MNP}] = -\frac{1}{2}\int d^3x\sqrt{g}\phi^{(TT)MNP}\left[-\mathcal{D}^{(M)}\Delta\right]\phi^{(TT)}_{MNP}, \tag{4.3.12}$$

and

$$S[\tilde{\phi}_M] = \frac{9}{4}\int d^3x\ \sqrt{g}\left[8\phi^M\mathcal{D}^{(M)}\left(-\Delta + \frac{7}{\ell^2}\right)\phi_M - \phi^M\mathcal{D}^{(M)}\nabla_M\nabla^Q\phi_Q\right]. \tag{4.3.13}$$

Now one can further decompose $\tilde{\phi}_M$ into its transverse and longitudinal parts as

$$\tilde{\phi}_M = \tilde{\phi}^{(T)}_M + \nabla_M\chi. \tag{4.3.14}$$

The action for ϕ_M then becomes

$$S[\tilde{\phi}_M] = \frac{9}{4}\int d^3x\ \sqrt{g}\left[8\tilde{\phi}^{(T)M}\mathcal{D}^{(M)}\left(-\Delta + \frac{7}{\ell^2}\right)\tilde{\phi}^{(T)}_M + 9\chi\left(-\Delta + \frac{8}{\ell^2}\right)(-\Delta)\chi\right]. \tag{4.3.15}$$

We see that the χ part of the action is same as that obtained for the massless theory in [70] and will subsequently cancel with the relevant term coming from the ghost determinant, which arises from making the change of variables (4.3.10) in the path integral (4.3.5). The ghost determinant– being independent of the structure of the action– is essentially be the same as that obtained [70], see their expression (2.14). This turns out to imply that although that the trace contribution does not cancel with the ghost determinant, the contribution coming from its longitudinal part does. This is in accordance with the observation in [103] that the trace in TMHSG is not pure gauge unlike the massless theory. It is however still true even in the topologically massive theory that $\nabla^M\phi_M$ is pure gauge and hence the longitudinal contribution from the trace ϕ_M does cancel with the ghost determinant.

The spin-3 contribution to the one loop partition function for TMHSG is given by

$$\begin{aligned} Z^{(3)}_{TMGHS} = Z^{(3)}_{gh} \quad &\times \quad (\det[(-\mathcal{D}_{(M)}\Delta)]^{TT}_{(3)})^{-\frac{1}{2}} \times (\det[\mathcal{D}_{(M)}(-\Delta + \frac{7}{\ell^2})]^{T}_{(1)})^{-\frac{1}{2}} \\ &\times \quad (\det[-\Delta(-\Delta + \frac{8}{\ell^2})]_{(0)})^{-\frac{1}{2}}, \end{aligned} \tag{4.3.16}$$

where $Z^{(3)}_{gh}$ is the ghost determinant arising as a Jacobian factor corresponding to the change of variables (4.3.10). The ghost determinant has been obtained in [70], see their expression (3.9), and is given by

$$Z^{(3)}_{gh} = [\det(-\Delta + \frac{6}{\ell^2})^{TT}_{(2)} \times \det(-\Delta + \frac{7}{\ell^2})^{T}_{(1)} \times \det(-\Delta + \frac{8}{\ell^2})_{(0)}]^{\frac{1}{2}}. \tag{4.3.17}$$

[6] Note that this decomposition is not orthogonal, in the sense that the trace part contains longitudinal terms, and the longitudinal part is not traceless. See [70].

Therefore, the spin-3 contribution to the one loop partition function is given by

$$\begin{aligned} Z^{(3)}_{TMGHS} &= [\det[-\mathcal{D}_{(M)}\Delta]^{TT}_{(3)}]^{-\frac{1}{2}}[\det[\mathcal{D}_{(M)}]^{T}_{(1)}]^{-\frac{1}{2}}[\det[-\Delta+\frac{6}{\ell^2})]^{TT}_{(2)}]^{\frac{1}{2}} \\ &\equiv Z^{(3)}_{massless}Z^{(3)}_{M}, \end{aligned} \quad (4.3.18)$$

where $Z^{(3)}_{massless}$ is the massless spin-3 partition function obtained in [70], which is

$$Z^{(3)}_{massless} = \prod_{s=3}^{\infty}\frac{1}{|1-q^n|^2}. \quad (4.3.19)$$

It remains to determine $Z^{(3)}_{M}$. In order to do so, we shall follow the method of [74], and first calculate the absolute value $|Z^{(3)}_{M}|$. Following [74], we define

$$|Z^{(3)}_{M}| = \left[\det(\mathcal{D}_{(M)}\bar{\mathcal{D}}_{(M)})^{TT}_{(3)}\right]^{-\frac{1}{4}} \times \left[\det(\mathcal{D}_{(M)}\bar{\mathcal{D}}_{(M)})^{T}_{(1)}\right]^{-\frac{1}{4}}. \quad (4.3.20)$$

The subscript (3) and (1) signifies that the operators are acting on transverse traceless spin-3 and transverse spin-1 respectively. Using (4.3.7) and (4.3.9), we can show that

$$\begin{aligned} \mathcal{D}_{(M)}\bar{\mathcal{D}}_{(M)}\phi^{TT}_{MNP} &= -\frac{1}{4\mu^2}(\Delta+\frac{4}{\ell^2}-4\mu^2)\phi^{TT}_{MNP} \\ \mathcal{D}_{(M)}\bar{\mathcal{D}}_{(M)}\phi^{T}_{P} &= -\frac{1}{36\mu^2}[-\Delta+(36\mu^2-\frac{2}{\ell^2})]\phi^{T}_{P}. \end{aligned} \quad (4.3.21)$$

Therefore,

$$|Z^{(3)}_{M}| = \left[\det\left(-\Delta+4(\mu^2-\frac{1}{\ell^2})\right)_{(3)}\right]^{-\frac{1}{4}} \times \left[\det\left(-\Delta+(36\mu^2-\frac{2}{\ell^2})\right)_{(1)}\right]^{-\frac{1}{4}}. \quad (4.3.22)$$

4.4 One-loop determinants for spin-3

We shall evaluate the one-loop determinants utilising the machinery developed in [53], according to which the relevant determinant takes the following form

$$-\log\det(-\Delta+\frac{m_s^2}{\ell^2})^{TT}_{(s)} = \int_0^\infty \frac{dt}{t}K^{(s)}(\tau,\bar{\tau};t)e^{-m^2_{(s)}t}, \quad (4.4.23)$$

where $K^{(s)}$ is the spin-s heat kernel given by

$$K^{(s)}(\tau,\bar{\tau};t) = \sum_{m=1}^{\infty}\frac{\tau_2}{\sqrt{4\pi t}|sin\frac{m\tau}{2}|^2}cos(sm\tau_1)e^{-\frac{m^2\tau_2^2}{4t}}e^{-(s+1)t}. \quad (4.4.24)$$

We therefore obtain

$$\begin{aligned} -\log\det\left(-\Delta+\frac{4(\mu^2l^2-1)}{\ell^2}\right)^{TT}_{(3)} &= \sum_{m=1}^{\infty}\frac{1}{m}\frac{cos(3m\tau_1)}{|sin\frac{m\tau}{2}|^2}e^{-2\mu\ell m\tau_2} \\ &= \sum_{m=1}^{\infty}\frac{2}{m}\frac{q^{3m}+\bar{q}^{3m}}{(1-q^m)(1-\bar{q}^m)}(q\bar{q})^{m(\mu\ell-1)}, \end{aligned} \quad (4.4.25)$$

and similarly,

$$-\log\det\left[-\Delta+\frac{36\mu^2l^2-2}{\ell^2}\right]^T_{(1)}=\sum_{m=1}^{\infty}\frac{2}{m}\frac{q^m+\bar q^m}{(1-q^m)(1-\bar q^m)}(q\bar q)^{3\mu\ell m}. \qquad (4.4.26)$$

We have used the notations and conventions of [53] throughout. We remind the reader that $q\equiv e^{i\tau}$, where $\tau=\tau_1+i\tau_2$ is the complex structure modulus on the boundary torus of thermal AdS_3. We can now put together all the contributions into the expression for the one-loop partition function

$$\begin{aligned}\log|Z_M^{(3)}| &= -\frac{1}{4}\log\det\left(-\Delta+\frac{4(\mu^2l^2-1)}{\ell^2}\right)^{TT}_{(3)}-\frac{1}{4}\log\det\left[-\Delta+\frac{36\mu^2l^2-2}{\ell^2}\right]^T_{(1)}\\ &= \sum_{m=1}^{\infty}\frac{1}{2m}\frac{q^{3m}+\bar q^{3m}}{(1-q^m)(1-\bar q^m)}(q\bar q)^{m(\mu\ell-1)}+\sum_{m=1}^{\infty}\frac{1}{2m}\frac{q^m+\bar q^m}{(1-q^m)(1-\bar q^m)}(q\bar q)^{3\mu\ell m}.\end{aligned} \qquad (4.4.27)$$

Following [74], we infer from this that at the chiral point we have (after rewriting $\log|Z_M^{(3)}|=\frac{1}{2}\log Z_M^{(3)}+\frac{1}{2}\log\bar Z_M^{(3)}$)

$$\log Z_M^{(3)}=\sum_{m=1}^{\infty}\frac{1}{m}\frac{\left(q^{3m}+(q\bar q)^{3m}q^m\right)}{(1-q^m)(1-\bar q^m)}. \qquad (4.4.28)$$

Let us now note that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{3n}}{(1-q^n)(1-\bar q^n)} &= -\sum_{m=3}^{\infty}\sum_{\bar m=0}^{\infty}\log\left(1-q^m\bar q^{\bar m}\right)\\ \sum_{n=1}^{\infty}\frac{1}{n}\frac{(q\bar q)^{3n}q^n}{(1-q^n)(1-\bar q^n)} &= -\sum_{m=4}^{\infty}\sum_{\bar m=3}^{\infty}\log\left(1-q^m\bar q^{\bar m}\right),\end{aligned} \qquad (4.4.29)$$

and hence the spin-3 contribution to the full partition function at the ciral point, factorizes as (after including the massless contribution from [70])

$$Z^{(3)}_{TMHSG}=\prod_{n=3}^{\infty}\frac{1}{|1-q^n|^2}\prod_{m=3}^{\infty}\prod_{\bar m=0}^{\infty}\frac{1}{(1-q^m\bar q^{\bar m})}\prod_{k=4}^{\infty}\prod_{\bar k=3}^{\infty}\frac{1}{(1-q^k\bar q^{\bar k})}. \qquad (4.4.30)$$

This is the expression (4.2.4) that we had mentioned in the introduction. As noted there, the partition function does not factorize holomorphically. A chiral CFT would not give rise to such a partition function. We therefore take this as a one-loop clue that the dual CFT to TMHSG is indeed not chiral, but logarithmic. We remind the reader that an analogous result was found in [74] for the case of topologically massive gravity, from which the authors were able to conclude that the dual CFT was indeed logarithmic.

4.5 Generalisation to arbitrary spin

In this section, we will consider a natural generalisation of our spin-3 action (4.3.6) to an arbitrary spin-s field. We will perform the analysis of Sections 4.3 and 4.4 and show that holomorphic factorisation does not occur. We take the action[7]

$$S=\frac{1}{2}\int d^3x\,\sqrt{g}\phi^{M_1M_2\cdots M_s}\left[\hat{\mathcal{F}}_{M_1M_2\cdots M_s}-\frac{1}{2}\hat{\mathcal{F}}^P{}_{P(M_1\cdots M_{s-2}}g_{M_{s-1}M_s)}\right], \qquad (4.5.31)$$

[7] See [105] for a derivation

where

$$\hat{\mathcal{F}}_{M_1\cdots M_s} = \mathcal{D}^{(M)}\mathcal{F}_{M_1M_2\cdots M_s} \equiv \mathcal{F}_{M_1\cdots M_s} + \frac{i}{s(s-1)\mu}\varepsilon_{QR(M_1}\nabla^Q\mathcal{F}^R{}_{M_2\cdots M_s)}, \tag{4.5.32}$$

and

$$\begin{aligned}\mathcal{F}_{M_1M_2M_3.....M_s} \equiv\ & \Delta\phi_{M_1M_2.....M_s} - \nabla_{(M_1|}\nabla^Q\phi_{|M_2.....M_s)Q} + \frac{1}{2}\nabla_{(M_1}\nabla_{M_2}\phi_{M_3M_4....M_s)P}{}^P \\ & -\frac{1}{\ell^2}\left\{\left[s^2-3s\right]\phi_{M_1M_2....M_s} + 2g_{(M_1M_2}\phi_{M_3M_4...M_s)P}{}^P\right\}.\end{aligned} \tag{4.5.33}$$

Now we would like to consider the possibility that the higher-spin theory with spins upto N is also dual to a high spin extension of LCFT. One would then expect that along the lines of what we have seen for spin-3, the one-loop partition function again would not factorize holomorphically. We will now do a one-loop analysis using the action (4.5.31) with this goal in mind. An essential ingredient for this analysis – as in the case of spin 3 – is the action of $\mathcal{D}^{(M)}$ on symmetric transverse traceless (STT) tensors of rank s and below. Using the definition of $\mathcal{D}^{(M)}$ in (4.5.32), one can show that

$$\begin{aligned}\bar{\mathcal{D}}^{(M)}\mathcal{D}^{(M)}\phi^{(TT)}_{M_1\cdots M_s} &= \frac{1}{(s-1)^2\mu^2}\left[-\Delta+\frac{m_s^2}{\ell^2}\right]\phi^{(TT),}_{M_1\cdots M_s} \\ \bar{\mathcal{D}}^{(M)}\mathcal{D}^{(M)}\phi^{(s-2,TT)}_{M_3\cdots M_s} &= \frac{(s-2)^2}{(s-1)^2s^2\mu^2}\left[-\Delta+\frac{m_{s-2}^2}{\ell^2}\right]\phi^{(s-2,TT)}_{M_1\cdots M_s}, \\ \bar{\mathcal{D}}^{(M)}\mathcal{D}^{(M)}\chi^{(s-m,TT)}_{M_{s-m+1}\cdots M_s} &= \frac{(s-m)^2}{(s-1)^2s^2\mu^2}\left[-\Delta+\frac{m_{s-m}^2}{\ell^2}\right]\chi^{(s-m,TT)}_{M_{s-m+1}\cdots M_s},\end{aligned} \tag{4.5.34}$$

where

$$\begin{aligned}m_s^2 &= \mu^2\ell^2(s-1)^2-(s+1) \\ m_{s-m}^2 &= \mu^2\ell^2\frac{s^2(s-1)^2}{(s-m)^2}-(s-m+1), \quad s>m\geq 2.\end{aligned} \tag{4.5.35}$$

Here $\phi^{(s-2)}$ and $\chi^{(s-n)}$ are STT tensors of rank $s-2$ and $s-n$ respectively, where $n \geq 3$.

Following the same steps as for spin-3 in the previous section, the one loop partition function can be factorized as

$$Z^{(s)}_{TMHSG} = Z^{(s)}_{(massless)}Z^{(s)}_M, \tag{4.5.36}$$

The contributions to $Z^{(s)}_M$ come from the determinants of $\mathcal{D}^{(M)}$ acting on $\phi^{(TT)}$, $\phi^{(s-2,TT)}$ and $\chi^{(s-m,TT)}$ (for $s>m\geq 3$). This is apparent from (4.5.34), which suggests that $\mathcal{D}^{(M)}$ acts non trivially on the STT modes of spin-1 and higher. The action of $\mathcal{D}^{(M)}$ stops at $\chi^{(1)}$ as it acts trivially on scalars, being just the identity map. After a careful analysis, one sees that

$$\log|Z^{(s)}_M| = -\frac{1}{4}\log\det\left(-\Delta+\frac{m_s^2}{\ell^2}\right)^{(TT)}_{(s)} - \frac{1}{4}\sum_{m=2}^{s-1}\log\det\left(-\Delta+\frac{m_{s-m}^2}{\ell^2}\right)^{(TT)}_{(s-m)}. \tag{4.5.37}$$

We find, using the spin-s heat kernel (4.4.24) and the expression (4.4.23) for the determinant,

$$
\begin{aligned}
-\log\det\left(-\Delta+\frac{m_s^2}{\ell^2}\right)^{(TT)}_{(s)} &= \sum_{m=1}^{\infty}\frac{2}{m}(q\bar{q})^{\left[m\frac{(s-1)}{2}(\mu\ell-1)\right]}\frac{q^{ms}+\bar{q}^{ms}}{|1-q^m|^2},\\
-\log\det\left(-\Delta+\frac{m_{s-m}^2}{\ell^2}\right)^{(TT)}_{(s-m)} &= \sum_{n=1}^{\infty}\frac{2}{n}(q\bar{q})^{[k(s,m,\mu\ell)n]}\frac{q^{n(s-m)}+\bar{q}^{n(s-m)}}{|1-q^n|^2},
\end{aligned}
\tag{4.5.38}
$$

Where,

$$
k(s,m,\mu\ell)=\frac{s(s-1)\mu\ell-(s-m)(s-m-1)}{2(s-m)}. \tag{4.5.39}
$$

At the chiral point $\mu\ell=1$, it becomes

$$
k(s,m,1)\equiv k(s,m)=\frac{s(s-1)-(s-m)(s-m-1)}{2(s-m)}=\frac{m(2s-m-1)}{2(s-m)}, \tag{4.5.40}
$$

and hence at the chiral point the partition function $Z_M^{(s)}$ becomes

$$
\log Z_M^{(s)}=\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(q^{sn}+\sum_{m=2}^{s-1}(q\bar{q})^{k(s,m)n}q^{(s-m)n}\right)}{(1-q^n)(1-\bar{q}^n)}. \tag{4.5.41}
$$

After using the identities

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{sn}}{(1-q^n)(1-\bar{q}^n)} &= -\sum_{m=s}^{\infty}\sum_{\bar{m}=0}^{\infty}\log\left(1-q^m\bar{q}^{\bar{m}}\right),\\
\sum_{n=1}^{\infty}\frac{1}{n}\frac{(q\bar{q})^{k(s,m)n}q^{(s-m)n}}{(1-q^n)(1-\bar{q}^n)} &= -\sum_{p=r(s,m)}^{\infty}\sum_{\bar{p}=k(s,m)}^{\infty}\log\left(1-q^p\bar{q}^{\bar{p}}\right)
\end{aligned}
\tag{4.5.42}
$$

where,

$$
r(s,m)=k(s,m)+s-m=\frac{s(s-1)+(s-m)(s-m+1)}{2(s-m)}, \tag{4.5.43}
$$

the contribution to the full partition function from spin-s field at the chiral point, becomes

$$
Z^{(s)}_{TMHSG}=\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2}\prod_{m=s}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{(1-q^m\bar{q}^{\bar{m}})}\prod_{t=2}^{s-1}\prod_{p=r(s,t)}^{\infty}\prod_{\bar{p}=k(s,t)}^{\infty}\frac{1}{(1-q^p\bar{q}^{\bar{p}})}. \tag{4.5.44}
$$

Hence, the full partition function at the chiral point, in a theory with fields of spin $s=3,\cdots N$ in addition to the spin 2 graviton, becomes

$$
\begin{aligned}
Z_{TMHSG} &= \prod_{s=2}^{N}\left[\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2}\prod_{m=s}^{\infty}\prod_{\bar{m}=0}^{\infty}\frac{1}{(1-q^m\bar{q}^{\bar{m}})}\right]\times\left[\prod_{s=3}^{N}\prod_{t=2}^{s-1}\prod_{p=r(s,t)}^{\infty}\prod_{\bar{p}=k(s,t)}^{\infty}\frac{1}{(1-q^p\bar{q}^{\bar{p}})}\right],\\
\text{Where}\qquad & k(s,m)=\frac{s(s-1)-(s-m+1)(s-m-1)}{2(s-m)},\quad r(s,m)=k(s,m)+s-m
\end{aligned}
\tag{4.5.45}
$$

We will see in appendix E that $r(s,m)$ and $k(s,m)$ appear respectively as left and right weights of classical left moving primary solution (or massive primary at the chiral point). In particular, we will also see that, $r(s,m,\mu\ell)\equiv k(s,m,\mu\ell)+s-m$, and $k(s,m,\mu\ell)$ will be the weights of massive primary at a generic point. Thus we also see that for generic spins, our one loop computations and classical computations are also mutually consistent.

Note that the first square bracket contribution starts from $s = 2$ whereas the second square bracket contribution starts from $s = 3$. This is a novel feature of topologically massive higher spin theory and technically this comes from the fact the trace is not a pure gauge. As the spin increases, the factors contributing to the partition function also increases. This, as discussed before, comes from the fact that longitudinal component of trace, longitudinal components of the longitudinal components of the trace and so on are not pure gauge and starts contributing till one hits a scalar, which is a pure gauge and the contribution terminates.

One might be worried about the fact that $k(s, m)$ and $r(s, m)$ in (4.5.45) are not integers for generic spin, and this might not be consistent with the periodicity in τ. However one might note that $r(s, m) - k(s, m) = s - m$, which is an integer and hence the periodicity in τ is not affected because of the following identity

$$q^{r(s,m)}\bar{q}^{k(s,m)} = (q\bar{q})^{k(s,m)}q^{s-m}. \tag{4.5.46}$$

4.6 Conclusions

In this chapter, we computed the one loop partition function for topologically massive higher spin gravity (TMHSG) for spin-3 and later generalised it to arbitrary spin. We find that the one loop partition function does not factorize holomorphically giving strong evidence that the dual theory is a high spin extension of LCFT. This was also anticipated in the classical calculation for spin-3 TMHSG in [103] as extra logarithmic modes emerged at the chiral point. Additionally there is the presence of a non trivial spin one contribution (in case of spin 3 TMHSG) in the one loop partition function. This points to the fact that the trace modes cannot be gauged away, both of which results are in accordance with the analysis of classical solutions in [103].

Part III

Conclusions

Chapter 5

Conclusions and Outlook

In this thesis, we have studied aspects of the holography of higher-spin theories. We were motivated both by the central role that higher-spin theories seem to have in unravelling the tensionless limit of string theory as well as the novel holographic dualities that the three-dimensional higher-spin theories admit.

We firstly constructed as a main tool, the heat kernel for the Laplacian on arbitrary-spin fields on AdS spacetimes in chapter 2. We subsequently used this to gain insight on the multiparticle symmetries of higher-spin theory in chapters 3 and 4. We provided evidence in chapter 3 that the higher-spin theories in higher dimensions, such as AdS_5 have additional symmetries that act on the multiparticle states of the theory, much like the enhanced symmetries in AdS_3, which are present even for pure gravity. Finally in chapter 4 we computed the partition function of topologically massive higher-spin theories. We found that the partition function does not factorise holomorphically. The overall structure of the answer leads us to expect the theory to be dual to a logarithmic CFT with extended $\mathcal{W}$ symmetries.

Finally, we believe that the work in this thesis is only the beginning of a very important line of exploration. Firstly, it is important to precisely elucidate the nature of the enhanced symmetries conjectured in chapter 3. This would have important implications for the AdS/CFT duality. Secondly, we would like to completely carry out the program we have initiated for topologically massive higher-spin theories to construct a duality between higher-spin theories in AdS_3 and logarithmic CFTs with $\mathcal{W}$ symmetry. There are many additional problems to be solved to meet this overall goal, not the least of which is constructing such CFTs. However, such a duality would have important condensed matter applications and is a significant overall goal of our program.

We would like to return to these questions in the future.

Part IV

Appendices

Appendix A

Normalising the Heat Kernel on G/H

We determine the normalisation factor $a_R^{(S)}$ that arises in the expressions (2.3.16) onwards. This is fixed by demanding that the integrated traced heat kernel obey

$$\int_{G/H} d\mu\left(x\right) K^{(S)}\left(x, x; t\right) = \sum_R d_R e^{tE_R^{(S)}}. \tag{A.0.1}$$

Using the expression (2.3.17) for the heat kernel on G/H, we have

$$\begin{aligned}\int_{G/H} d\mu\left(x\right) K^{(S)}\left(x, x; t\right) &= \sum_R \int_{G/H} d\mu\left(x\right) a_R^{(S)} \mathcal{U}^{(R)}\left(\sigma\left(x\right)^{-1}\sigma\left(x\right)\right)_a{}^a e^{tE_R^{(S)}} \\ &= \sum_R a_R^{(S)} V_{G/H} d_S e^{tE_R^{(S)}}.\end{aligned} \tag{A.0.2}$$

On comparing (A.0.1) and (A.0.2), we obtain the required relation

$$a_R^{(S)} = \frac{d_R}{V_{G/H} d_S}, \tag{A.0.3}$$

which we use in the main text from (2.3.16) onwards.

Appendix B

The Plancherel Measure for STT tensors on H_N

We show how the normalisation constant a_R^S gets analytically continued to μ_R^S, the measure for the λ integration that appears in the AdS heat kernel. Let us consider the expression for $d_{(m,s)}$ which we obtain from (2.3.29).

$$d_{m_1,s} = \prod_{j=2}^{n+1} \frac{l_1^2 - l_j^2}{\mu_1^2 - \mu_j^2} \prod_{j=3}^{n+1} \frac{l_2^2 - l_j^2}{\mu_2^2 - \mu_j^2}. \tag{B.0.1}$$

The first product in the numerator gets analytically continued via (2.5.57) to

$$\prod_{j=2}^{n+1} \left(l_1^2 - l_j^2\right) \mapsto (-1)^n \left[\lambda^2 + (s+n-1)^2\right] \prod_{j=0}^{n-2} \left(\lambda^2 + j^2\right), \tag{B.0.2}$$

while the second product evaluates to

$$\prod_{j=2}^{n+1} \left(l_2^2 - l_j^2\right) = \frac{(s+n-1)\,(s+2n-3)!}{s!}. \tag{B.0.3}$$

The denominator $\prod_{j=2}^{n+1} \left(\mu_1^2 - \mu_j^2\right) \prod_{j=3}^{n+1} \left(\mu_1^2 - \mu_j^2\right)$ evaluates to

$$\prod_{j=2}^{n+1} \left(\mu_1^2 - \mu_j^2\right) \prod_{j=3}^{n+1} \left(\mu_1^2 - \mu_j^2\right) = (2n-2)! \frac{2^{2n-2}}{\sqrt{\pi}} n! \Gamma\left(n + \frac{1}{2}\right). \tag{B.0.4}$$

The dimension $d_{(m,s)}$ then gets continued to

$$(-1)^{\frac{N-1}{2}} \frac{\left[\lambda^2 + \left(s + \frac{N-3}{2}\right)^2\right] \prod_{j=0}^{\frac{N-5}{2}} \left(\lambda^2 + j^2\right)}{\frac{2^{N-2}}{\sqrt{\pi}} \Gamma\left(\frac{N}{2}\right) \left(\frac{N-1}{2}\right)!} \frac{(2s+N-3)\,(s+N-4)!}{s!\,(N-3)!}, \tag{B.0.5}$$

where we have changed variables from n to $N = 2n+1$. We now use the fact that for odd N

$$V_{S^N} = \frac{(N+1)\,\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+3}{2}\right)} = \frac{2\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)}, = \frac{2\pi^{\frac{N+1}{2}}}{\left(\frac{N-1}{2}\right)!} \tag{B.0.6}$$

and hence the combination $\frac{d_{(m,s)}}{V_{G/H}}$ gets mapped to

$$(-1)^{\frac{N-1}{2}} \frac{\left[\lambda^2 + \left(s + \frac{N-3}{2}\right)^2\right] \prod_{j=0}^{\frac{N-5}{2}} \left(\lambda^2 + j^2\right)}{2^{N-1} \pi^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)} \frac{(2s+N-3)\,(s+N-4)!}{s!\,(N-3)!}. \tag{B.0.7}$$

Using the expressions from [64] quoted in the main text, *i.e.*

$$\Omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \quad c_N = \frac{2^{N-2}}{\pi}, \quad g(s) = \frac{(2s+N-3)(s+N-4)!}{(N-3)!s!} \tag{B.0.8}$$

and

$$\mu(\lambda) = \frac{\pi[\lambda^2 + (s+\frac{N-3}{2})^2]\prod_{j=0}^{\frac{N-5}{2}}(\lambda^2+j^2)}{\left[2^{N-2}\Gamma\left(\frac{N}{2}\right)\right]^2}, \tag{B.0.9}$$

we see that the normalisation constant a_R^S gets mapped to

$$\mu_R^S = \frac{C_N g\left(s\right)}{d_S}\frac{\mu\left(\lambda\right)}{\Omega_{N-1}}, \tag{B.0.10}$$

where we have omitted the overall sign that appears for some values of N as an artefact of the analytic continuation, since the measure is always positive definite. The coincident heat kernel is therefore

$$K^S\left(x,x,t\right) = \int d\lambda\, \mu_R^S\left(\lambda\right) d_S e^{tE_R^{(S)}} = \frac{C_N}{\Omega_{N-1}} g\left(s\right) \int d\lambda\, \mu\left(\lambda\right) e^{-t\left(\lambda^2+s+n^2\right)}. \tag{B.0.11}$$

Now, the coincident heat kernel may also be written down using 2.7 of [64] (in their notation) as

$$K^S\left(x,x,t\right) = \sum_u \int_0^\infty d\lambda\, \hat{h}^{\lambda u *}\cdot \hat{h}^{\lambda u}\left(x\right) e^{-t\left(\lambda^2+s+n^2\right)}. \tag{B.0.12}$$

On choosing x to be the origin (which we can do for arbitrary x), and using their expression 2.10, we conclude that

$$K^S\left(x,x,t\right) = \frac{C_N}{\Omega_{N-1}} g\left(s\right) \int d\lambda\, \mu\left(\lambda\right) e^{-t\left(\lambda^2+s+n^2\right)}, \tag{B.0.13}$$

which is precisely the expression we have obtained via analytic continuation.

Appendix C

Blind Partition functions for AdS_4 and AdS_7

We shall begin by using (3.2.12) to obtain the blind partition function of the Vasiliev theory in AdS_7. Observations parallel to those in Section 3.2.1 lead us to conclude that the expressions for the partition function are again simple. We need to evaluate the product

$$\prod_{s=1}^{\infty}\prod_{m=1}^{\infty}\frac{1}{(1-q^{s+m+4})^{\binom{m+4}{5}(d_{s+1}-d_{s-1})}}, \tag{C.0.1}$$

where d_s denotes the $(s,0,0)$ representation of $SO(6)$. Computing these dimensions (see [71] and references therein), we find that the partition function of the Vasiliev theory may be expressed as the product

$$\mathcal{Z}=\mathcal{Z}_{(0)}\cdot\prod_{s=0}^{\infty}\prod_{m=1}^{\infty}\frac{1}{(1-q^{s+m+4})^{\binom{m+4}{5}(d_{s+1}-d_{s-1})}}, \tag{C.0.2}$$

where $\mathcal{Z}_{(0)}$ is the partition function of the scalar, given by

$$\mathcal{Z}_{(0)}=\prod_{m=1}^{\infty}\frac{1}{(1-q^{m+3})^{\binom{m+4}{5}}}. \tag{C.0.3}$$

We finally obtain

$$\mathcal{Z}=\prod_{m=1}^{\infty}\frac{1}{(1-q^{m+3})^{\binom{m+4}{5}}}\cdot\frac{1}{(1-q^{m+4})^{4\binom{m+8}{9}+\binom{m+6}{7}+\binom{m+5}{6}}}, \tag{C.0.4}$$

which is again related to the d-dimensional MacMahon function (3.2.18).

We now turn to the case of AdS_4. The entire heat kernel analysis of this paper and [71] may be applied to even dimensional hyperboloids. We merely mention the final result for AdS_4. We find that

$$\mathcal{Z}=\frac{1}{(1-q)(1-q^2)^3}\prod_{m=2}^{\infty}\frac{1}{(1-q^m)^{\binom{m+1}{2}}(1-q^{m+1})^{3\binom{m+1}{2}+4\binom{m+2}{3}}}. \tag{C.0.5}$$

This is again related to the d-dimensional MacMahon function (3.2.18).

Appendix D

Computing the Refined Partition Function

We outline some of the main steps involved in reducing (3.3.26) to the nested form (3.3.28). We begin by expanding out $\hat{\prod}$ that appears in (3.3.26) into two products A and B. We will evaluate

$$A = \prod_{m_i=0}^{\infty}\prod_{s=0}^{\infty}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s+1}r^{m_1+m_4+s+1}\bar{p}^{m_2+m_4}\bar{r}^{m_2+m_3}\right)$$
$$\prod_{m_i=0}^{\infty}\prod_{s=0}^{\infty}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s+1}r^{m_1+m_4+s+1}\bar{p}^{m_2+m_4}\bar{r}^{m_2+m_3+2(s+1)}\right)$$
$$\prod_{m_i=0}^{\infty}\prod_{s=0}^{\infty}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s+1}r^{m_1+m_4+s+1}\bar{p}^{m_2+m_4+2(s+1)}\bar{r}^{m_2+m_3}\right)$$
$$\prod_{m_i=0}^{\infty}\prod_{s=0}^{\infty}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s+1}r^{m_1+m_4+s+1}\bar{p}^{m_2+m_4+2(s+1)}\bar{r}^{m_2+m_3+2(s+1)}\right), \tag{D.0.1}$$

and

$$B = \prod_{m_i=0}^{\infty}\prod_{s=1}^{\infty}\prod_{k=1}^{s}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s+1}r^{m_1+m_4+s+1}\bar{p}^{m_2+m_4}\bar{r}^{m_2+m_3+2k}\right)$$
$$\prod_{m_i=0}^{\infty}\prod_{s=1}^{\infty}\prod_{k=1}^{s}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s+1}r^{m_1+m_4+s+1}\bar{p}^{m_2+m_4+2k}\bar{r}^{m_2+m_3+2(s+1)}\right)$$
$$\prod_{m_i=0}^{\infty}\prod_{s=1}^{\infty}\prod_{k=1}^{s}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s+1}r^{m_1+m_4+s+1}\bar{p}^{m_2+m_4+2k}\bar{r}^{m_2+m_3}\right)$$
$$\prod_{m_i=0}^{\infty}\prod_{s=1}^{\infty}\prod_{k=1}^{s}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}p^{m_1+m_3+s+1}r^{m_1+m_4+s+1}\bar{p}^{m_2+m_4+2(s+1)}\bar{r}^{m_2+m_3+2k}\right). \tag{D.0.2}$$

The full partition function of the theory is the inverse of the product of A and B. We will now simplify these expressions.

$$A = \prod_{m_i,s}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}q_1^{m_1+s+1}q_2^{m_2}q_3^{m_3}q_4^{m_4}\right)\left(1 - q^{m_1+m_2+m_3+m_4+s+3}q_1^{m_1}q_2^{m_2}q_3^{m_3+s+1}q_4^{m_4}\right)$$
$$\prod_{m_i,s}\left(1 - q^{m_1+m_2+m_3+m_4+s+3}q_1^{m_1}q_2^{m_2}q_3^{m_3}q_4^{m_4+s+1}\right)\left(1 - q^{m_1+m_2+m_3+m_4+s+3}q_1^{m_1}q_2^{m_2+s+1}q_3^{m_3}q_4^{m_4}\right), \tag{D.0.3}$$

where we have used the fact that $p\bar{p} = 1 = r\bar{r}$. In each product, there is a pairing of one of m_i and s. We use this to write

$$\begin{aligned} A = \prod_{m_i=0}^{\infty} \left(1 - q^{m+s+3} q_1^{m_1+1} q_2^{m_2} q_3^{m_3} q_4^{m_4}\right)^{m_1+1} \prod_{m_i=0}^{\infty} \left(1 - q^{m+s+3} q_1^{m_1} q_2^{m_2} q_3^{m_3+1} q_4^{m_4}\right)^{m_3+1} \\ \prod_{m_i=0}^{\infty} \left(1 - q^{m+s+3} q_1^{m_1} q_2^{m_2} q_3^{m_3} q_4^{m_4+1}\right)^{m_4+1} \prod_{m_i=0}^{\infty} \left(1 - q^{m+s+3} q_1^{m_1} q_2^{m_2+1} q_3^{m_3} q_4^{m_4}\right)^{m_2+1}, \end{aligned} \tag{D.0.4}$$

where we have defined $m \equiv \sum_i m_i$. Now each product is of the form

$$\prod_{n=1}^{\infty} (1 - \alpha q^n)^n = \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} (1 - \alpha q^n), \tag{D.0.5}$$

where α is independent of n. This gives a (two-dimensional) MacMahon function form for this sector of the partition function, which we can arrange into a nested form. For example, the first product reduces to

$$\prod_{m_2,m_3,m_4=0}^{\infty} \left(\prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \left(1 - (qq_1)^n q^{m_2+m_3+d+2} q_2^{m_2} q_3^{m_3} q_4^{m_4}\right)\right). \tag{D.0.6}$$

There are similar expressions for the other three products, which arise by permuting q_i's above. We will now look at the B term, and obtain its nested product representation. Written in terms of the q_is, the first product that enters in B is

$$\begin{aligned} &\prod_{m_i=0}^{\infty} \prod_{s=1}^{\infty} \prod_{k=1}^{s} \left(1 - (qq_1)^{(m_1+s+1)} q^{m_2+m_3+d+2} q_2^{m_2+k} q_3^{m_3+k} q_4^{m_4}\right) \\ &\quad = \prod_{m_i=0}^{\infty} \prod_{k=1}^{\infty} \prod_{s=k}^{\infty} \left(1 - (qq_1)^{(m_1+s+1)} q^{m_2+m_3+d+2} q_2^{m_2+k} q_3^{m_3+k} q_4^{m_4}\right), \end{aligned} \tag{D.0.7}$$

where we have changed the products over k and s. This enumerates the same combinations (k, s) as the previous product. Again, redefining $a + s$ to a, and $s - k$ to s, we get

$$\begin{aligned} &\prod_{m_i=0}^{\infty} \prod_{k=1}^{\infty} \prod_{s=0}^{s} \left(1 - (qq_1)^{(m_1+s+1)} q^{k+m_2+m_3+m_4+2} q_1^k q_2^{m_2+k} q_3^{m_3+k} q_4^{m_4}\right) \\ &\quad = \prod_{k=1}^{\infty} \prod_{m_i=0}^{\infty} \left(1 - (qq_1)^{(m_1+1)} q^{k+m_2+m_3+m_4+2} q_1^k q_2^{m_2+k} q_3^{m_3+k} q_4^{m_4}\right)^{m_1+1}, \end{aligned} \tag{D.0.8}$$

from which we can write the nested form

$$\prod_{k=1}^{\infty} \prod_{m_{2,3,4}}^{\infty} \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \left(1 - (qq_1)^n q^{k+m_2+m_3+m_4+2} q_1^k q_2^{m_2+k} q_3^{m_3+k} q_4^{m_4}\right) \tag{D.0.9}$$

Note that (D.0.6) is just the $k = 0$ term in this product. We can write the contribution of these two terms together as

$$\mathcal{Z}_{(1)}^{-1} = \prod_{k=0}^{\infty} \prod_{m_{2,3,4}}^{\infty} \left(\prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \left(1 - (qq_1)^n q^{k+m_2+m_3+m_4+2} q_1^k q_2^{m_2+k} q_3^{m_3+k} q_4^{m_4}\right)\right). \tag{D.0.10}$$

The other three contributions to the partition function may be similarly written down by combining the second, third and fourth terms respectively from A and B, we obtain the nested product forms

$$\mathcal{Z}_{(2)}^{-1} = \prod_{k=0}^{\infty} \prod_{m_{1,2,4}}^{\infty} \left(\prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \left(1 - (qq_3)^n \, q^{k+m_1+m_2+m_4+2} q_1^{m_1} q_2^{m_2+k} q_3^k q_4^{m_4+k} \right) \right), \tag{D.0.11}$$

$$\mathcal{Z}_{(3)}^{-1} = \prod_{k=0}^{\infty} \prod_{m_{1,2,3}}^{\infty} \left(\prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \left(1 - (qq_4)^n \, q^{k+m_1+m_2+m_4+2} q_1^{m_1+k} q_2^{m_2} q_3^{m_3+k} q_4^k \right) \right), \tag{D.0.12}$$

$$\mathcal{Z}_{(4)}^{-1} = \prod_{k=0}^{\infty} \prod_{m_{1,3,4}}^{\infty} \left(\prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \left(1 - (qq_2)^n \, q^{k+m_1+m_3+m_4+2} q_1^{m_1+k} q_2^k q_3^{m_3} q_4^{m_4+k} \right) \right). \tag{D.0.13}$$

Putting these expressions together yields the expression for $\mathcal{Z}$ in (3.3.28).

Appendix E

Classical analysis for generic spins

In this appendix, we will study the classical analysis for generic spins and show that $k(s, m, \mu\ell)$ (4.5.39) and $r(s, m, \mu\ell) \equiv k(s, m, \mu\ell) + s - m$ will appear as weights (right and left respectively) of massive primaries in the linearised spectrum of generic spins.

The classical equation of motion for a generic spin is

$$\hat{\mathcal{F}}_{M_1\cdots M_s} \equiv \mathcal{D}^{(M)}\mathcal{F}_{M_1M_2\cdots M_s} \equiv \mathcal{F}_{M_1\cdots M_s} + \frac{1}{s(s-1)\mu}\varepsilon_{QR(M_1}\nabla^Q\mathcal{F}^R{}_{M_2\cdots M_s)} = 0, \qquad \text{(E.0.1)}$$

where

$$\begin{aligned} \mathcal{F}_{M_1M_2M_3.....M_s} \quad \equiv \quad & \Delta\phi_{M_1M_2.....M_s} - \nabla_{(M_1|}\nabla^Q\phi_{|M_2.....M_s)Q} + \frac{1}{2}\nabla_{(M_1}\nabla_{M_2}\phi_{M_3M_4....M_s)P}{}^P \\ & -\frac{1}{\ell^2}\left\{\left[s^2-3s\right]\phi_{M_1M_2....M_s} + 2g_{(M_1M_2}\phi_{M_3M_4...M_s)P}{}^P\right\}. \end{aligned} \qquad \text{(E.0.2)}$$

Using the field redefinition and the gauge condition defined below,

$$\begin{aligned} \phi_{M_1\cdots M_s} \quad &= \quad \tilde{\phi}_{M_1\cdots M_s} - \frac{1}{s^2}g_{(M_1M_2}\tilde{\phi}^{(s-2)}_{M_3\cdots M_s)}, \\ \nabla^M\tilde{\phi}_{MM_2\cdots M_s} \quad &= \quad \frac{1}{2}\nabla_{(M_2}\tilde{\phi}^{(s-2)}_{M_3\cdots M_s)}, \end{aligned} \qquad \text{(E.0.3)}$$

we can write

$$\mathcal{F}_{M_1M_2M_3.....M_s} = \mathcal{D}^{(L)}\mathcal{D}^{(R)}\tilde{\phi}_{M_1\cdots M_s}, \qquad \text{(E.0.4)}$$

where

$$\begin{aligned} \mathcal{D}^{(L)}\tilde{\phi}_{M_1\cdots M_s} \quad &\equiv \quad \tilde{\phi}_{M_1\cdots M_s} + \frac{\ell}{s(s-1)}\varepsilon_{QR(M_1}\nabla^Q\tilde{\phi}^R{}_{M_2\cdots M_s)}, \\ \mathcal{D}^{(R)}\tilde{\phi}_{M_1\cdots M_s} \quad &\equiv \quad \tilde{\phi}_{M_1\cdots M_s} - \frac{\ell}{s(s-1)}\varepsilon_{QR(M_1}\nabla^Q\tilde{\phi}^R{}_{M_2\cdots M_s)}. \end{aligned} \qquad \text{(E.0.5)}$$

One can now check that the equations of motion implies that

$$\begin{aligned} & g^{M_1M_2}\nabla^{M_3}\cdots\nabla^{M_s}\hat{\mathcal{F}}_{M_1\cdots M_s} = 0 \\ \Longrightarrow \quad & g^{M_1M_2}\nabla^{M_3}\cdots\nabla^{M_s}\mathcal{F}_{M_1\cdots M_s} = 0 \\ \Longrightarrow \quad & \nabla^{M_3}\cdots\nabla^{M_s}\tilde{\phi}^{(s-2)}_{M_3\cdots M_s} = 0. \end{aligned} \qquad \text{(E.0.6)}$$

Now let us write down the massive branch equation $\mathcal{D}^{(M)}\tilde{\phi} = 0$, by operating on it from the left by $\bar{\mathcal{D}}^{(M)}$ which is the same as $\mathcal{D}^{(M)}$ with μ replaced by $-\mu$. By replacing $\mu\ell = \pm 1$ on the solutions obtained here, we can recover the left and right branch solutions. The equations are

$$\begin{aligned}
\Delta\tilde{\phi}_{M_1\cdots M_s} - \frac{m_s^2}{\ell^2}\tilde{\phi}_{M_1\cdots M_s} &= \frac{(2s-3)}{2s^2}\nabla_{(M_1}\nabla_{M_2}\tilde{\phi}^{(s-2)}_{M_3\cdots M_s)} + \frac{1}{\ell^2 s^2}(s^2-s+2)g_{(M_1M_2}\tilde{\phi}^{(s-2)}_{M_3\cdots M_s)} \\
&\quad + \frac{1}{s^2}g_{(M_1M_2}\Delta\tilde{\phi}^{(s-2)}_{M_3\cdots M_s)} - \frac{1}{s^2}g_{(M_1M_2}\nabla_{M_3}\chi^{(s-3)}_{M_4\cdots M_s)}, \\
\Delta\tilde{\phi}^{(s-2)}_{M_3\cdots M_s} - \frac{m_{s-2}^2}{\ell^2}\tilde{\phi}^{(s-2)}_{M_3\cdots M_s} &= \frac{(2s-5)}{(s-2)^2}\nabla_{(M_3}\chi^{(s-3)}_{M_4\cdots M_s)} - \frac{2}{(s-2)^2}g_{(M_3M_4}\chi^{(s-4)}_{M_5\cdots M_s)}, \quad s>2 \\
\Delta\chi^{(s-3)}_{M_4\cdots M_s} - \frac{m_{s-3}^2}{\ell^2}\chi^{(s-3)}_{M_4\cdots M_s} &= \frac{(2s-7)}{(s-3)^2}\nabla_{(M_4}\chi^{(s-4)}_{M_5\cdots M_s)} - \frac{2}{(s-3)^2}g_{(M_4M_5}\chi^{(s-5)}_{M_6\cdots M_s)}, \quad s>3 \\
&\vdots \\
\Delta\chi^{(2)}_{MN} - \frac{m_2^2}{\ell^2}\chi^{(2)}_{MN} &= \frac{3}{4}\nabla_{(M}\chi^{(1)}_{N)}, \\
\Delta\chi^{(1)}_{M} - \frac{m_1^2}{\ell^2}\chi^{(1)}_{M} &= 0,
\end{aligned} \tag{E.0.7}$$

where

$$\chi^{(s-m)} = \nabla^{M_3}\cdots\nabla^{M_m}\tilde{\phi}^{(s-2)}_{M_3\cdots M_s}, \tag{E.0.8}$$

and m_s and m_{s-m} are given in (4.5.35). The procedure to solve the above equations is same as the spin-3 analysis done in [103]. We need to start with the last equation of (E.0.7), the solution of which would be given by $h-\bar{h}=\pm 1$, where h and $\bar{h}$ are the left and right weights. We will put the solution of $\chi^{(1)}$ into the equation of $\chi^{(2)}$ and decompose $\chi^{(2)}$ into two parts, one carrying the weights of $\chi^{(1)}$ and the other would be transverse, the solution of which would be given by $h-\bar{h}=\pm 2$. Following the steps similarly we will decompose $\chi^{(3)}$ into three parts, one carrying the weight of $\chi^{(1)}$, another carrying the weights of transverse $\chi^{(2)}$ and a part which is transverse on its own carrying weight $h-\bar{h}=\pm 3$. In this way we will finally obtain the full solution. We are not interested in obtaining the explicit form of the final solution (although there is no technical obstacle in doing so). We will, however, be interested in obtaining the weights of the different modes present in the final solution. For that it is sufficient (as per the argument above) to analyse the transverse traceless parts of the equations (E.0.7). For that let us write the Laplacian acting on traceless rank-s tensor as

$$\Delta\phi_{M_1\cdots M_s} = \left[-\frac{2}{\ell^2}\left(L^2+\bar{L}^2\right) - \frac{s(s+1)}{\ell^2}\right]\phi_{M_1\cdots M_s} \tag{E.0.9}$$

Where, L^2 and $\bar{L}^2$ are the left and right $SL(2,R)$ casimirs of the isometry group of AdS_3 (see eq 4.11 and 4.12 of [103]). The eigenvalues of L^2 and $\bar{L}^2$ are given by $h(1-h)$ and $\bar{h}(1-\bar{h})$, where h and $\bar{h}$ are the left and right weights of the solution. We will also use the fact that for a rank-s, transverse traceless primary, $h-\bar{h}=\pm s$. Thus the different modes of equation (E.0.7), will carry weights $h-\bar{h}=\pm s$, $h-\bar{h}=\pm(s-2)$, $h-\bar{h}=\pm(s-3)$, $\cdots$, $h-\bar{h}=\pm 1$. By using (E.0.7) and (E.0.9), we obtain the weights of the different modes as

$$\begin{aligned}
h-\bar{h}=+s: &\qquad h = \frac{s^2(\mu\ell+1)-s(\mu\ell-1)}{2s}, \quad \bar{h} = \frac{s(s-1)(\mu\ell-1)}{2s}, \\
h-\bar{h}=-s: &\qquad h = \frac{s(s-1)(\mu\ell-1)}{2s}, \quad \bar{h} = \frac{s^2(\mu\ell+1)-s(\mu\ell-1)}{2s}, \\
h-\bar{h}=+(s-m): &\qquad h = \frac{(s-m+1)(s-m)+s(s-1)\mu\ell}{2(s-m)} = r(s,m,\mu\ell), \\
&\qquad \bar{h} = \frac{s(s-1)\mu\ell-(s-m-1)(s-m)}{2(s-m)} = k(s,m,\mu\ell), \quad (s-1)\geq m\geq 2, \\
h-\bar{h}=-(s-m): &\qquad h = k(s,m,\mu\ell), \quad \bar{h} = r(s,m,\mu\ell), \quad (s-1)\geq m\geq 2
\end{aligned} \tag{E.0.10}$$

The modes with negative $h - \bar{h}$ will not belong to the massive branch (they appear because we had the operation of $\bar{\mathcal{D}}^{(M)}$ on the original equation of motion). But these modes will become the right branch solution by taking $\mu\ell = 1$. The modes with positive $h - \bar{h}$ will belong to the massive branch solution. For $\mu\ell = 1$, they will become the left branch solutions. These are the weights that also appear in the one loop partition function at the chiral point (4.5.45).

Bibliography

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200]. 3, 5, 9

[2] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS / CFT," Int. J. Mod. Phys. A **14** (1999) 3597 [hep-th/9902131]. 4, 5

[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323** (2000) 183 [hep-th/9905111]. 4, 5, 7, 8

[4] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS / CFT correspondence," hep-th/0201253. 5

[5] J. Polchinski, "Introduction to Gauge/Gravity Duality," arXiv:1010.6134 [hep-th]. 5, 6

[6] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," Phys. Rev. Lett. **75** (1995) 4724 [hep-th/9510017]. 6

[7] J. D. Brown, M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," Commun. Math. Phys. **104**, 207-226 (1986). 5, 36

[8] G. 't Hooft, "Dimensional reduction in quantum gravity," gr-qc/9310026. 7

[9] L. Susskind, "The World as a hologram," J. Math. Phys. **36** (1995) 6377 [hep-th/9409089]. 7

[10] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B **72** (1974) 461. 7

[11] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428** (1998) 105 [hep-th/9802109]. 7, 8

[12] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150]. 7, 8

[13] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, "The Mass Spectrum of Chiral N=2 D=10 Supergravity on S**5," Phys. Rev. D **32** (1985) 389. 8

[14] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," JHEP **0810** (2008) 091 [arXiv:0806.1218 [hep-th]]. 8

[15] O. Aharony, O. Bergman and D. L. Jafferis, "Fractional M2-branes," JHEP **0811** (2008) 043 [arXiv:0807.4924 [hep-th]]. 8

[16] J. Bagger, N. Lambert, S. Mukhi and C. Papageorgakis, "Multiple Membranes in M-theory," arXiv:1203.3546 [hep-th]. 8

[17] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov and R. A. Janik *et al.*, "Review of AdS/CFT Integrability: An Overview," Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982 [hep-th]]. 8

[18] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824 [hep-th]]. 8

[19] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, “Nonlinear higher spin theories in various dimensions,” hep-th/0503128. 9

[20] J. Wess and J. Bagger, “Supersymmetry and supergravity,” Princeton, USA: Univ. Pr. (1992) 259 p 9

[21] J. Maldacena and A. Zhiboedov, “Constraining Conformal Field Theories with A Higher Spin Symmetry,” arXiv:1112.1016 [hep-th]. 9

[22] Y. S. Stanev, “Correlation Functions of Conserved Currents in Four Dimensional Conformal Field Theory,” arXiv:1206.5639 [hep-th]. 9

[23] A. Zhiboedov, “A note on three-point functions of conserved currents,” arXiv:1206.6370 [hep-th]. 9

[24] I. R. Klebanov and A. M. Polyakov, “AdS dual of the critical O(N) vector model,” Phys. Lett. B **550** (2002) 213 [hep-th/0210114]. 9

[25] S. Giombi and X. Yin, “Higher Spin Gauge Theory and Holography: The Three-Point Functions,” JHEP **1009** (2010) 115 [arXiv:0912.3462 [hep-th]].

[26] S. Giombi and X. Yin, “Higher Spins in AdS and Twistorial Holography,” JHEP **1104** (2011) 086 [arXiv:1004.3736 [hep-th]].

[27] M. Henneaux, S. -J. Rey, “Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity,” JHEP **1012**, 007 (2010). [arXiv:1008.4579 [hep-th]]. 31, 34, 36, 40

[28] A. Campoleoni, S. Fredenhagen, S. Pfenninger, S. Theisen, “Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields,” JHEP **1011**, 007 (2010). [arXiv:1008.4744 [hep-th]]. 10, 31, 34, 36, 40, 41

[29] M. R. Gaberdiel, R. Gopakumar, “An AdS_3 Dual for Minimal Model CFTs,” Phys. Rev. **D83**, 066007 (2011). [arXiv:1011.2986 [hep-th]]. 9, 31, 40

[30] A. Castro, A. Lepage-Jutier and A. Maloney, “Higher Spin Theories in AdS(3) and a Gravitational Exclusion Principle,” JHEP **1101**, 142 (2011) [arXiv:1012.0598 [hep-th]]. 40

[31] S. Minwalla, P. Narayan, T. Sharma, V. Umesh and X. Yin, “Supersymmetric States in Large N Chern-Simons-Matter Theories,” JHEP **1202** (2012) 022 [arXiv:1104.0680 [hep-th]]. 9

[32] C. -M. Chang and X. Yin, “Higher Spin Gravity with Matter in AdS_3 and Its CFT Dual,” arXiv:1106.2580 [hep-th]. 9

[33] A. Bagchi, S. Lal, A. Saha and B. Sahoo, “Topologically Massive Higher Spin Gravity,” JHEP **1110** (2011) 150 [arXiv:1107.0915 [hep-th]]. 40, 41, 42, 47, 66

[34] O. Aharony, G. Gur-Ari and R. Yacoby, “d=3 Bosonic Vector Models Coupled to Chern-Simons Gauge Theories,” JHEP **1203** (2012) 037 [arXiv:1110.4382 [hep-th]]. 9

[35] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia and X. Yin, “Chern-Simons Theory with Vector Fermion Matter,” arXiv:1110.4386 [hep-th]. 9

[36] M. A. Vasiliev, “Holography, Unfolding and Higher-Spin Theory,” arXiv:1203.5554 [hep-th].

[37] P. Haggi-Mani and B. Sundborg, "Free large N supersymmetric Yang-Mills theory as a string theory," JHEP **0004** (2000) 031 [hep-th/0002189]. 9
9, 10

[38] R. Gopakumar, "From free fields to AdS," Phys. Rev. D **70** (2004) 025009 [hep-th/0308184]. 9, 10

[39] R. Gopakumar, "From free fields to AdS. 2.," Phys. Rev. D **70** (2004) 025010 [hep-th/0402063]. 9, 10

[40] R. Gopakumar, "From free fields to AdS: III," Phys. Rev. D **72** (2005) 066008 [hep-th/0504229]. 9, 10

[41] N. Beisert, M. Bianchi, J. F. Morales and H. Samtleben, "Higher spin symmetry and N=4 SYM," JHEP **0407** (2004) 058 [hep-th/0405057]. 9

[42] R. Gopakumar, A. Hashimoto, I. R. Klebanov, S. Sachdev and K. Schoutens, "Strange Metals in One Spatial Dimension," arXiv:1206.4719 [hep-th]. 9

[43] E. Witten, in JHS/60: Conf. in Honor of John Schwarzs 60th Birthday, California Institute of Technology, Pasadena, CA, USA, Nov. 3 4, 2001; http://theory.caltech.edu/jhs60/witten/1.html 10

[44] C. Fronsdal, "Massless Fields with Integer Spin," Phys. Rev. D **18** (1978) 3624. 10

[45] C. Fronsdal, "Singletons and Massless, Integral Spin Fields on de Sitter Space (Elementary Particles in a Curved Space. 7," Phys. Rev. D **20** (1979) 848. 10

[46] I. L. Buchbinder, A. Pashnev and M. Tsulaia, "Lagrangian formulation of the massless higher integer spin fields in the AdS background," Phys. Lett. B **523** (2001) 338 [hep-th/0109067]. 10

[47] A. Sagnotti and M. Tsulaia, "On higher spins and the tensionless limit of string theory," Nucl. Phys. B **682** (2004) 83 [hep-th/0311257]. 10

[48] A. Campoleoni and D. Francia, "Maxwell-like Lagrangians for higher spins," arXiv:1206.5877 [hep-th]. 10

[49] N. Bouatta, G. Compere and A. Sagnotti, "An Introduction to free higher-spin fields," hep-th/0409068. 10

[50] D. Sorokin, "Introduction to the classical theory of higher spins," AIP Conf. Proc. **767**, 172 (2005) [hep-th/0405069]. 10

[51] D. V. Vassilevich, "Heat kernel expansion: User's manual," Phys. Rept. **388** (2003) 279 [hep-th/0306138]. 15, 16

[52] M. E. Peskin and D. V. Schroeder, Reading, USA: Addison-Wesley (1995) 842 p

[53] J. R. David, M. R. Gaberdiel and R. Gopakumar, "The Heat Kernel on AdS_3 and its Applications," JHEP **1004** (2010) 125 [arXiv:0911.5085 [hep-th]]. 16

[54] A. Salam, J. A. Strathdee, "On Kaluza-Klein Theory," Annals Phys. **141** (1982) 316-352. 16, 17, 20, 23, 24, 26, 27, 31, 36, 40, 43, 44

[55] R. Camporesi, "Harmonic analysis and propagators on homogeneous spaces," Phys. Rept. **196** (1990) 1. 17

[56] R. Camporesi, A. Higuchi, "On the Eigen functions of the Dirac operator on spheres and real hyperbolic spaces," J. Geom. Phys. **20** (1996) 1-18. [gr-qc/9505009] 17

[57] A. O. Barut and R. Raczka, "Theory Of Group Representations And Applications," *Singapore, Singapore: World Scientific (1986).* 17

[58] R. Camporesi, A. Higuchi, "The Plancherel measure for p-forms in real hyperbolic spaces," J. Geom. Phys. **15** (1994) 57-94. 18, 19, 20

[59] W. Fulton, J. Harris, "Representation Theory", *Springer-Verlag, (1991).* 18, 24, 25, 28, 29

[60] S. Giombi, A. Maloney, X. Yin, "One-loop Partition Functions of 3D Gravity," JHEP **0808** (2008) 007. [arXiv:0804.1773 [hep-th]]. 19, 20

[61] Sigurdur Helgason, "Groups and geometric analysis: integral geometry, invariant differential operator and spherical function", *American Mathematical Society.* 22, 36

[62] Takeshi Hirai "On irreducible representations of the Lorentz group of n-th order", Proc. Japan Acad. Volume 38, Number 6 (1962), 258-262. 22

[63] U. Ottoson "A classification of unitary irreducible representations of $SO_0(N,1)$", Commun. Math. Phys. 8 (1968) 228-244 24

[64] R. Camporesi and A. Higuchi, "Spectral functions and zeta functions in hyperbolic spaces," J. Math. Phys. **35** (1994) 4217. 25, 28

[65] Takeshi Hirai, "The characters of irreducible representations of the Lorentz group of n-th order" Proc. Japan Acad. Volume 41, Number 7 (1965), 526-531 25, 26, 28, 29, 58

[66] F. Denef, S. A. Hartnoll and S. Sachdev, "Black hole determinants and quasinormal modes," Class. Quant. Grav. **27** (2010) 125001 [arXiv:0908.2657 [hep-th]]. 26, 29

[67] T. Hartman, L. Rastelli, "Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT," JHEP **0801** (2008) 019. [hep-th/0602106]. 28

[68] D. E. Diaz, H. Dorn, "Partition functions and double-trace deformations in AdS/CFT," JHEP **0705** (2007) 046. [hep-th/0702163 [HEP-TH]]. 28

[69] G. W. Gibbons, M. J. Perry, C. N. Pope, "Partition functions, the Bekenstein bound and temperature inversion in anti-de Sitter space and its conformal boundary," Phys. Rev. **D74** (2006) 084009. [hep-th/0606186]. 28
29, 35

[70] M. R. Gaberdiel, R. Gopakumar and A. Saha, "Quantum W-symmetry in AdS_3," JHEP **1102**, 004 (2011) [arXiv:1009.6087 [hep-th]]. 31, 32, 33, 34, 36, 40, 41, 42, 43, 44

[71] R. Gopakumar, R. K. Gupta and S. Lal, "The Heat Kernel on AdS," JHEP **1111** (2011) 010 [arXiv:1103.3627 [hep-th]]. 31, 33, 34, 36, 59

[72] E. Sezgin and P. Sundell, "Doubletons and 5-D higher spin gauge theory," JHEP **0109** (2001) 036 [hep-th/0105001]. 31

[73] E. Sezgin and P. Sundell, "Towards massless higher spin extension of D=5, N=8 gauged supergravity," JHEP **0109** (2001) 025 [hep-th/0107186]. 31

[74] M. R. Gaberdiel, D. Grumiller and D. Vassilevich, "Graviton 1-loop partition function for 3-dimensional massive gravity," JHEP **1011**, 094 (2010) [arXiv:1007.5189 [hep-th]]. 32, 33, 40, 43, 44

[75] A. Bagchi, S. Lal, A. Saha and B. Sahoo, "One loop partition function for Topologically Massive Higher Spin Gravity," JHEP **1112** (2011) 068 [arXiv:1107.2063 [hep-th]]. 32

[76] S. Balakrishnan, S. Govindarajan and N. S. Prabhakar, "On the asymptotics of higher-dimensional partitions," J. Phys. A A **45** (2012) 055001 [arXiv:1105.6231 [cond-mat.stat-mech]]. 34

[77] F. A. Dolan, "Character formulae and partition functions in higher dimensional conformal field theory," J. Math. Phys. **47** (2006) 062303 [hep-th/0508031]. 35

[78] A. Barabanschikov, L. Grant, L. L. Huang and S. Raju, "The Spectrum of Yang Mills on a sphere," JHEP **0601** (2006) 160 [hep-th/0501063]. 35

[79] A. Maloney and E. Witten, "Quantum Gravity Partition Functions in Three Dimensions," JHEP **1002**, 029 (2010) [arXiv:0712.0155 [hep-th]]. 36, 39

[80] Y. Kimura and S. Ramgoolam, "Enhanced symmetries of gauge theory and resolving the spectrum of local operators," Phys. Rev. D **78** (2008) 126003 [arXiv:0807.3696 [hep-th]]. 37

[81] M. A. Vasiliev, "Conformal higher spin symmetries of 4-d massless supermultiplets and osp(L,2M) invariant equations in generalized (super)space," Phys. Rev. D **66** (2002) 066006 [hep-th/0106149]. 37

[82] O. A. Gelfond and M. A. Vasiliev, "Higher rank conformal fields in the Sp(2M) symmetric generalized space-time," Theor. Math. Phys. **145** (2005) 1400 [Teor. Mat. Fiz. **145** (2005) 35] [hep-th/0304020]. 37

[83] O. A. Gelfond and M. A. Vasiliev, "Unfolded Equations for Current Interactions of 4d Massless Fields as a Free System in Mixed Dimensions," arXiv:1012.3143 [hep-th]. 37

[84] I. A. Bandos, J. Lukierski and D. P. Sorokin, "Superparticle models with tensorial central charges," Phys. Rev. D **61** (2000) 045002 [hep-th/9904109]. 37

[85] M. Plyushchay, D. Sorokin and M. Tsulaia, "Higher spins from tensorial charges and OSp(N—2n) symmetry," JHEP **0304** (2003) 013 [hep-th/0301067]. 37

[86] I. Bandos, P. Pasti, D. Sorokin and M. Tonin, "Superfield theories in tensorial superspaces and the dynamics of higher spin fields," JHEP **0411** (2004) 023 [hep-th/0407180]. 37

[87] S. Deser, R. Jackiw and S. Templeton, "Topologically Massive Gauge Theories," Annals Phys. **140**, 372 (1982) [Erratum-ibid. **185**, 406 (1988)] [Annals Phys. **185**, 406 (1988)] [Annals Phys. **281**, 409 (2000)]. 39

[88] S. Deser, R. Jackiw and S. Templeton, "Three-Dimensional Massive Gauge Theories," Phys. Rev. Lett. **48**, 975 (1982). 39

[89] M. Banados, C. Teitelboim and J. Zanelli, "The Black hole in three-dimensional space-time," Phys. Rev. Lett. **69** (1992) 1849 [hep-th/9204099]. 39

[90] Steven Carlip, "Quantum Gravity in 2+1 Dimensions: The Case of a Closed Universe," Living Rev. Relativity 8, (2005), 1. URL (cited on 21.08.2012): http://www.livingreviews.org/lrr-2005-1 39

[91] E. Witten, "Three-Dimensional Gravity Revisited," arXiv:0706.3359 [hep-th]. 39

[92] M. R. Gaberdiel, "Constraints on extremal self-dual CFTs," JHEP **0711**, 087 (2007) [arXiv:0707.4073 [hep-th]]. 39
M. R. Gaberdiel and C. A. Keller, "Modular differential equations and null vectors," JHEP **0809**, 079 (2008) [arXiv:0804.0489 [hep-th]].

[93] W. Li, W. Song and A. Strominger, "Chiral Gravity in Three Dimensions," JHEP **0804**, 082 (2008) [arXiv:0801.4566 [hep-th]]. 39, 40

[94] S. N. Solodukhin, "Holography with gravitational Chern-Simons," Phys. Rev. D **74**, 024015 (2006) [arXiv:hep-th/0509148]. 40

[95] P. Kraus and F. Larsen, "Holographic gravitational anomalies," JHEP **0601**, 022 (2006) [arXiv:hep-th/0508218]. 40

[96] S. Carlip, S. Deser, A. Waldron, D. K. Wise, "Cosmological Topologically Massive Gravitons and Photons," Class. Quant. Grav. **26**, 075008 (2009). [arXiv:0803.3998 [hep-th]]; 40
W. Li, W. Song, A. Strominger, "Comment on 'Cosmological Topological Massive Gravitons and Photons'," [arXiv:0805.3101 [hep-th]].
D. Grumiller, R. Jackiw, N. Johansson, "Canonical analysis of cosmological topologically massive gravity at the chiral point," [arXiv:0806.4185 [hep-th]].
S. Carlip, S. Deser, A. Waldron, D. K. Wise, "Topologically Massive AdS Gravity," Phys. Lett. **B666**, 272-276 (2008). [arXiv:0807.0486 [hep-th]].
G. Giribet, M. Kleban, M. Porrati, "Topologically Massive Gravity at the Chiral Point is Not Chiral," JHEP **0810**, 045 (2008). [arXiv:0807.4703 [hep-th]].
A. Strominger, "A Simple Proof of the Chiral Gravity Conjecture," [arXiv:0808.0506 [hep-th]].

[97] D. Grumiller, N. Johansson, "Instability in cosmological topologically massive gravity at the chiral point," JHEP **0807**, 134 (2008). [arXiv:0805.2610 [hep-th]]. 40
D. Grumiller and N. Johansson, "Consistent boundary conditions for cosmological topologically massive gravity at the chiral point," Int. J. Mod. Phys. D **17**, 2367 (2009) [arXiv:0808.2575 [hep-th]].

[98] D. Grumiller and I. Sachs, "AdS (3) / LCFT (2) —¿ Correlators in Cosmological Topologically Massive Gravity," JHEP **1003**, 012 (2010) [arXiv:0910.5241 [hep-th]]. 40

[99] K. Skenderis, M. Taylor, B. C. van Rees, "Topologically Massive Gravity and the AdS/CFT Correspondence," JHEP **0909**, 045 (2009). [arXiv:0906.4926 [hep-th]]. 40
K. Skenderis, M. Taylor, B. C. van Rees, "AdS boundary conditions and the Topologically Massive Gravity/CFT correspondence," [arXiv:0909.5617 [hep-th]].

[100] A. Campoleoni, S. Fredenhagen and S. Pfenninger, "Asymptotic W-symmetries in three-dimensional higher-spin gauge theories," arXiv:1107.0290 [hep-th]. 40

[101] A. Castro, A. Lepage-Jutier and A. Maloney, "Higher Spin Theories in AdS(3) and a Gravitational Exclusion Principle," JHEP **1101**, 142 (2011) [arXiv:1012.0598 [hep-th]]. 40

[102] M. R. Gaberdiel, T. Hartman, "Symmetries of Holographic Minimal Models," JHEP **1105**, 031 (2011). [arXiv:1101.2910 [hep-th]]. 40
C. Ahn, "The Large N 't Hooft Limit of Coset Minimal Models," [arXiv:1106.0351 [hep-th]].
M. R. Gaberdiel, R. Gopakumar, T. Hartman, S. Raju, "Partition Functions of Holographic Minimal Models," [arXiv:1106.1897 [hep-th]].
M. R. Gaberdiel and C. Vollenweider, "Minimal Model Holography for SO(2N)," arXiv:1106.2634 [hep-th].
C. M. Chang and X. Yin, "Higher Spin Gravity with Matter in AdS3 and Its CFT Dual," arXiv:1106.2580 [hep-th].

[103] A. Bagchi, S. Lal, A. Saha and B. Sahoo, "Topologically Massive Higher Spin Gravity," arXiv:1107.0915 [hep-th]. 40, 41, 42, 47, 66

[104] B. Chen, J. Long and J. b. Wu, "Spin-3 Topological Massive Gravity," arXiv:1106.5141 [hep-th]. 40, 41

[105] B. Chen and J. Long, "High Spin Topologically Massive Gravity," JHEP **1112** (2011) 114 [arXiv:1110.5113 [hep-th]]. 44