Aspects of Supersymmetric Black Holes and Galilean Conformal Algebras

By

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Certificate

This is to certify that the Ph.D. thesis titled “Aspects of Supersymmetric Black Holes and Galilean Conformal Algebras” submitted by Ipsita Mandal is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Date:

Professor Ashoke Sen

Thesis Advisor

Declaration

This thesis is a presentation of my original research work. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

This work was done under guidance of Professor Ashoke Sen, at Harish Chandra Research Institute, Allahabad.

Date: **Ipsita Mandal**

Ph.D. Candidate

To My Family ...

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The exception proves the rule.

Synopsis

The first part of my thesis is concerned with black holes in string theory. Black holes are classical solutions of the equations of motion of general theory of relativity. Each black hole is surrounded by an event horizon that acts as a one way membrane. Nothing, including light, can escape a black hole horizon. Thus classically the horizon of a black hole behaves as a perfect black body at zero temperature.

This picture undergoes a dramatic modification in quantum theory. There a black hole behaves as a thermodynamic system with definite temperature, entropy etc. In particular, the temperature and the Bekenstein-Hawking entropy of a black hole is given by the simple formulæ:

$$T = \frac{\kappa}{2\pi}\,, \qquad S_{BH} = \frac{A}{4\,G_N}\,,$$

where κ is the surface gravity – acceleration due to gravity at the horizon of the black hole (measured by an observer at infinity), A is the area of the event horizon and G_N is the Newton's gravitational constant. We have set $\hbar = c = k_B = 1$.

Now, for ordinary objects, the entropy of a system has a microscopic interpretation. If we fix the macroscopic parameters and count the number of quantum states (dubbed microstates), each of which has the same charge, energy etc., then we can define the microscopic (statistical) entropy as:

$$S_{micro} = \ln\, d_{micro}\,,$$

where d_{micro} is the number of such microstates. This naturally leads to the question whether the entropy of a black hole has a similar statistical interpretation.

In order to investigate the statistical origin of black hole entropy, we need a quantum theory of gravity. Since string theory gives a framework for studying classical and quantum properties of black holes, we shall carry out our investigation in string theory. Now, even though there is a unique string (M)-theory, it can exist in many different stable and metastable phases. However, there are some issues like those involving black hole thermodynamics, which are universal, and hence can be addressed in any phase of string theory. We shall make use of this freedom to study these issues in a special class of phases of string theory with a large amount of unbroken supersymmetry.

One of my research projects focusses on the identification of the hair degrees of freedom for an extremal black hole. Macroscopic entropy of an extremal black hole is expected to be determined completely by its near horizon geometry. Thus two black holes with identical near horizon geometries should have identical macroscopic entropy, and the expected equality between macroscopic and microscopic entropies will then imply that they have identical degeneracies of microstates. An apparent counterexample is provided by the 4D-5D lift relating BMPV black hole to a four dimensional black hole. The two black holes have identical near horizon geometries but different microscopic spectrum. We suggest that this discrepancy can be accounted for by black hole hair, – degrees of freedom living outside the horizon and contributing to the degeneracies. We identify these degrees of freedom for both the four and the five dimensional black holes and show that after their contributions are removed from the microscopic degeneracies of the respective systems, the result for the four and five dimensional black holes match exactly.

The second part of my thesis deals with the Galilean Conformal Algebras (GCA), which correspond to the generators of a non-relativistic conformal symmetry obtained by a parametric contraction of the relativistic conformal group.

In the paper "Supersymmetric Extension of Galilean Conformal Algebras", we extend the analysis to include supersymmetry in four spacetime dimensions. We work at the level of the co-ordinates in superspace to construct the $\mathcal{N} = 1$ Super Galilean conformal algebra. One of the interesting outcomes of the analysis is that one is able to naturally extend the finite algebra to an infinite one. We also comment on the extension of our construction to cases of higher $\mathcal{N}$.

In a subsequent work, "Supersymmetric Extension of GCA in 2d", we derive the infinite dimensional Supersymmetric Galilean Conformal Algebra (SGCA) in the case of two spacetime dimensions by performing group contraction on 2d superconformal algebra. We also obtain the representations of the generators in terms of superspace coordinates. Here we find realisations of the SGCA by considering scaling limits of certain 2d SCFTs which are non-unitary and have their left and right central charges become large in magnitude and opposite in sign. We focus on the Neveu-Schwarz sector of the parent SCFTs and develop the representation theory based on SGCA primaries, Ward identities for their correlation functions and their descendants which are null states.

List of Publications

Research Papers

1. *Supersymmetric Extension of GCA in 2d.*
   Author: Ipsita Mandal
   Ref: arXiv:1003.0209 [hep-th], JHEP 1011:018,2010.

2. *GCA in 2d.*
   Authors: Arjun Bagchi, Rajesh Gopakumar, Ipsita Mandal and Akitsugu Miwa
   Ref: arXiv:0912.1090 [hep-th], JHEP 1008:004,2010.

3. *Supersymmetry, Localization and Quantum Entropy Function.*
   Authors: Nabamita Banerjee, Shamik Banerjee, Rajesh Kumar Gupta, Ipsita Mandal, Ashoke Sen
   Ref:arXiv:0905.2686, JHEP 1002:091,2010

4. *Supersymmetric Extension of Galilean Conformal Algebras.*
   Authors: Arjun Bagchi and Ipsita Mandal
   Ref: arXiv:0905.0580 [hep-th], Phys.Rev.D80:086011,2009.

5. *On Representations and Correlation Functions of Galilean Conformal Algebras.*
   Authors: Arjun Bagchi and Ipsita Mandal
   Ref: hep-th/0802.0544, Phys.Lett.B675:393-397,2009.

6. *Black Hole Hair Removal.*
   Authors: Nabamita Banerjee, Ipsita Mandal and Ashoke Sen
   Ref: arXiv:0901.0359 [hep-th], JHEP 0907:091,2009.

7. *Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions.*
   Authors: Sayantani Bhattacharyya, R. Loganayagam, Ipsita Mandal, Shiraz Minwalla and Ankit Sharma
   Ref: arXiv:0809.4272 [hep-th], JHEP 0812:116,2008.

Review Article

- *Black Hole Microstate Counting and its Macroscopic Counterpart.*
  Authors: Ipsita Mandal and Ashoke Sen
  Ref: arXiv:1008.3801 [hep-th], Class.Quant.Grav.27:214003,2010.

Contents

I Introduction 1

1 Introduction 3
- 1.1 Introduction to black holes . . . 3
- 1.2 Introduction to Galilean Conformal Algebras . . . 7
- 1.3 Plan of the report . . . 8

II Black hole hair removal 11

2 Microstate counting 13
- 2.1 The role of index . . . 13
- 2.2 Microstate counting in heterotic string theory on T^6 . . . 14
- 2.3 Asymptotic expansion . . . 18
- 2.4 Walls of marginal stability . . . 19

3 Black hole hair removal 21
- 3.1 Introducion and summary . . . 21
- 3.2 Analysis of the BMPV black hole entropy . . . 24
- 3.3 Analysis of the $4D$ black hole entropy . . . 31
- 3.4 Appendix . . . 37
  - 3.4.1 Left-moving bosonic modes on the BMPV black hole . . . 37
  - 3.4.2 Left-moving bosonic modes on the $4D$ black hole . . . 40

III Galilean Conformal Algebras 45

4 Galilean Conformal Algebras 47

4.1 Schrödinger symmetry 47
4.2 Contraction of the Relativistic Conformal Group 48
4.3 The Infinite Dimensional Extended GCA 51

5 SGCA in $4d$ 55
5.1 Non-relativistic contraction in superspace 55
5.2 Fermionic generators 56
5.3 Algebra 58
5.4 Infinitely extended SGCA 59
5.5 Generalization to higher $\mathcal{N}$ 61
5.6 Summary 61
5.7 Appendix 62
5.7.1 Spinors: convention and choice 62
5.7.2 Bosonic generators 63
5.7.3 OPE analysis 65

6 SGCA in $2d$ 67
6.1 2d SGCA from group contraction 68
6.1.1 SGCA from SuperVirasoro in 2d 68
6.1.2 Non-relativistic superconformal transformations in the superspace . 70
6.2 Representations of the 2d SGCA 71
6.2.1 Primary states and descendants 71
6.2.2 Transformation laws of superprimary fields 72
6.3 Non-relativistic Ward identities and correlation functions 73
6.3.1 SGCA two point functions 75
6.3.2 SGCA higher point functions 76
6.3.3 SGCA correlation functions from 2d SCFT 77
6.4 SGCA null vectors 78
6.4.1 The intrinsic SGCA analysis 80
6.4.2 SGCA null vectors from 2d SCFT 81
6.4.3 Discussion on SGCA null states not obtained from SCFT null states 84
6.5 Differential equations for SGCA correlators from null states 84
6.6 SGCA fusion rules 86
6.7 Summary 88
6.8 Appendix 88

6.8.1 SGCA Descendants 88
6.8.2 The OPE and SGCA blocks 89

IV Concluding remarks 93

7 Concluding remarks 95

Part I

Introduction

Chapter 1

Introduction

1.1 Introduction to black holes

Black holes are classical solutions of the equations of motion of general theory of relativity. Each black hole is surrounded by an event horizon that acts as a one way membrane. Nothing, including light, can escape a black hole horizon. Thus classically the horizon of a black hole behaves as a perfect black body at zero temperature.

This picture undergoes a dramatic modification in quantum theory [1, 2, 3, 4]. There a black hole behaves as a thermodynamic system with definite temperature, entropy etc. In particular, the temperature and the Bekenstein-Hawking entropy of a black hole is given by the simple formulæ:

$$T = \frac{\kappa}{2\pi}, \qquad S_{BH} = \frac{A}{4\,G_N}, \tag{1.1.1}$$

where κ is the surface gravity – acceleration due to gravity at the horizon of the black hole (measured by an observer at infinity), A is the area of the event horizon and G_N is the Newton's gravitational constant. We have set $\hbar = c = k_B = 1$.

Now, for ordinary objects, the entropy of a system has a microscopic interpretation. If we fix the macroscopic parameters (*e.g.* total electric charge, energy etc.) and count the number of quantum states (dubbed microstates), each of which has the same charge, energy etc., then we can define the microscopic (statistical) entropy as:

$$S_{micro} = \ln d_{micro}\,, \tag{1.1.2}$$

where d_{micro} is the number of such microstates. This naturally leads to the question whether the entropy of a black hole has a similar statistical interpretation. As pointed out by Hawking, answering this question in the affirmative is essential for any consistent theory of quantum gravity as otherwise it leads to violation of the laws of quantum mechanics.

In order to investigate the statistical origin of black hole entropy, we need a quantum theory of gravity. Since string theory gives a framework for studying classical and quantum properties of black holes, we shall carry out our investigation in string theory. Now, even though there is a unique string (M)-theory, it can exist in many different stable and metastable phases. Without knowing precisely which phase of string theory describes the part of the universe we live in, we cannot directly compare string theory to experiments. However, there are some issues like those involving black hole thermodynamics, which are universal, and hence can be addressed in any phase of string theory. We shall make use of this freedom to study these issues in a special class of phases of string theory with a large amount of unbroken supersymmetry. Since these phases have Bose-Fermi degenerate spectrum of states, they do not describe the observed world. Nevertheless they contain black hole solutions and hence can be used to study issues involving black hole thermodynamics.

Many aspects of black hole thermodynamics have been studied in string theory, but we shall focus our attention on one particular aspect: entropy of the black hole in the zero temperature limit (i.e., supersymmetric, extremal black holes). The advantage of studying such a black hole is that it is a stable state of the theory. The general strategy is as follows [5, 6]:

1. Identify a supersymmetric black hole carrying a certain set of electric charges $\{Q_i\}$ and magnetic charges $\{P_i\}$, and calculate its entropy $S_{BH}(Q, P)$ using the Bekenstein-Hawking formula.[1]

2. Identify the supersymmetric quantum states in string theory carrying the same set of charges. These can include not only the fundamental strings but also other objects in string theory which are required for consistency of the theory (*e.g.* D-branes, Kaluza-Klein monopoles). We then calculate the number $d_{micro}(Q, P)$ of these states.

3. Compare $S_{micro} \equiv \ln d_{micro}(Q, P)$ with $S_{BH}(Q, P)$.

For a class of supersymmetric extremal black holes in type IIB string theory on $K3 \times S^1$, Strominger and Vafa [6] computed the Bekenstein-Hawking entropy via (1.1.1) and found agreement with the statistical entropy defined in (1.1.2). This agreement is quite remarkable since it relates a geometric quantity in black hole space-time to a counting problem that does not make any direct reference to black holes. At the same time, one should keep in mind that the Bekenstein-Hawking formula is an approximate formula that holds in classical general theory of relativity. While string theory gives a theory of gravity that reduces to Einstein's theory when gravity is weak, there are

[1] Since we are considering a generic phase of string theory, it may have more that one Maxwell field and hence multiple charges.

corrections.[2] Thus the Bekenstein-Hawking formula for the entropy works well only when gravity at the horizon is weak. Typically this requires the charges to be large. Similarly, the computation of d_{micro} in [6] was also carried out in the limit of large charges, so that instead of having to carry out an exact counting of states, one can use some appropriate asymptotic formula to compute it. Thus the agreement between S_{BH} and S_{micro}, seen in [6], can be regarded as an agreement in the limit of large size.

This leads to the following question: For ordinary systems, thermodynamics provides an accurate description only in the limit of large volume. Is the situation with black holes similar, i.e., do they only capture the information about the system in the limit of large charge and mass? Or, could it be that the relation $A/4G_N = \ln d_{micro}$ is an approximation to an exact result? The goal of the string theorists to argue for the second possibility by giving an exact formula to which the above is an approximation.

In order to address this issue, we have to work on two fronts:

1. Count the number of microstates to greater accuracy.

2. Calculate the black hole entropy to greater accuracy.

We can then compare the two to see if they agree beyond the large charge limit.

Let us review a a summary of the progress on both the fronts:[3]

1. **Progress in microscopic counting:** In a wide class of phases of string theory with 16 or more unbroken supercharges, one now has a complete understanding of the microscopic 'degeneracies' of supersymmetric black holes [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53]. Typically, such theories have multiple Maxwell fields and the black hole is characterized by multiple electric and magnetic charges, collectively denoted by (Q, P). It turns out that for a wide class of charge vectors (all charge vectors in some cases), $d_{micro}(Q, P)$ in these theories can be explicitly computed and can be expressed as Fourier expansion coefficients of some functions with remarkable symmetry properties. This provides us with the 'experimental data' to be explained by a 'theory of black holes', giving a powerful tool for checking the internal consistency of string theory. Needless to say, in the large charge limit, these degeneracies agree with the exponential of the Bekenstein-Hawking entropy of black holes carrying the same set of charges.

2. **Progress in black hole entropy computation:** On the macroscopic side, we would like to ask whether we can find an exact formula for the black hole entropy that can be compared with $\ln d_{micro}(Q, P)$. This will require us to take into account

[2] In string theory, even at classical level, we have higher derivative (α') corrections. This is because strings are not point objects. So even at classical level, there will be corrections to the Bekenstein-Hawking formula. Besides this, there will also be quantum corrections.

[3] A detailed discussion and review on the above issues can be found in [7].

(a) stringy (α') corrections, and

(b) quantum (g_s) corrections.

Quantum entropy function is an approach to finding such a general formula for for the full quantum computation of the black hole entropy from the macroscopic side, using AdS_2/CFT_1 correspondence. This is a proposal for computing quantum corrected entropy in terms of a path integral of string theory in the near horizon geometry [54, 55].

We find that so far string theory has been successful in providing an explanation of the entropy of a certain class of supersymmetric extremal black holes in terms of microscopic degrees of freedom. Initial studies focussed on black holes carrying large charges for which the classical two derivative action, and the associated formula for the entropy due to Bekenstein and Hawking, is sufficient to compute the entropy. This assumption can be relaxed to some extent using Wald's formula for black hole entropy [56, 57, 58, 59] that takes into account higher derivative corrections to the classical action. However a complete expression for the entropy of a black hole receives contribution from higher derivative corrections as well as quantum corrections. In fact, one finds that quantum entropy function formalism is not sensitive to the nature of the solution away from the horizon [54]. Wald's classical formula for the entropy, which coincides with the classical limit of quantum entropy function, also satisfies this criterion.

This simple assumption has a non-trivial consequence: two different black holes with identical near horizon geometries have the same macroscopic entropy. The equality of the macroscopic and the microscopic entropy would then imply that they must have the same microscopic entropy. There is however a counterexample: a rotating black hole in type IIB string theory compactified on $K3 \times S^1$, known as the BMPV black hole [60], placed in a flat transverse space and in Taub-NUT space [61] have identical near horizon geometries [62] but different microscopic degeneracies [8, 9, 10, 12, 17, 28, 63]!

In the paper [64], we suggest that this discrepancy can be accounted for by black hole hair, – degrees of freedom living outside the horizon and contributing to the degeneracies. Whereas an appropriate computation in string theory in the near horizon geometry of the black hole would give the macroscopic entropy associated with the horizon, the full macroscopic entropy also involves contribution from the hair degrees of freedom. For a supersymmetric black hole, the latter can be computed by identifying classical supersymmetry preserving normalizable deformations of the black hole solution with support outside the horizon, and then carrying out geometric quantization on the space of these solutions. We identify a class of such deformations both for the BMPV black hole in flat transverse space and BMPV black hole in Taub-NUT space and found that after removing the contribution from these hair degrees of freedom from the microscopic degeneracy formulæ, one obtains identical result for the two black holes. This can then be identified as the common contribution to the degeneracy coming from the horizon. We will study this in detail in §3.

1.2 Introduction to Galilean Conformal Algebras

The non-relativistic versions of the AdS/CFT conjecture [65] have recently received a lot of attention. The motivation has mainly been studying real-life systems in condensed matter physics via the gauge-gravity duality[4]. It was pointed out in [67] that the Schrödinger symmetry group [68, 69, 70], a non-relativistic version of conformal symmetry, is relevant to the study of cold atoms. A gravity dual possessing these symmetries was then proposed in [71, 72].

Recently, the study of the actual non-relativistic limit of the conjecture was initiated in [73], where the authors proposed to study a non-relativistic conformal symmetry obtained by a parametric contraction of the relativistic conformal group. One of the additional motivations for this is that there might be possibly interesting tractable sectors of the parent conjecture, like the BMN limit [74], which emerge when we look at such a non-relativistic limit. The process of group contraction of the relativistic conformal group $SO(d+1,2)$ in $d+1$ space-time dimensions[75], leads in $d=3$ to a fifteen parameter group (like the parent $SO(4,2)$ group) which contains the ten parameter Galilean subgroup. This Galilean conformal group is to be contrasted with the twelve parameter Schrödinger group (plus central extension) with which it has in common only the non-centrally extended Galilean subgroup. The Galilean conformal group is different from the Schrödinger group in some crucial respects. For instance, the dilatation generator $\tilde{D}$ in the Schrödinger group scales space and time differently $x_i \to \lambda x_i, t \to \lambda^2 t$. Whereas the corresponding generator D in the Galilean Conformal Algebra (GCA) scales space and time in the *same* way $x_i \to \lambda x_i, t \to \lambda t$. Relatedly, the GCA does *not* admit a mass term as a central extension. Thus, in some sense, this symmetry describes "massless" or "gapless" non-relativistic theories, like the parent relativistic group but unlike the Schrödinger group.

One of the most interesting feature of the GCA is its natural extension to an infinite dimensional symmetry algebra, somewhat analogous to the way the finite 2d conformal algebra of $SL(2,C)$ extends to two copies of the Virasoro algebra[5]. It is natural to expect this necessary to be dynamically realized (perhaps partially) in actual systems possesing the finite dimensional Galilean conformal symmetry. This partial realization is actually observed in the non-relativistic Navier-Stokes equations [73].

It has been known (see [77] and references therein) that there is a notion of a

[4] See [66] for a recent review of AdS/Condensed Matter Theory (AdS/CMT) correspondence.

[5] Closely related infinite dimensional algebras have been studied in the context of statistical mechanical systems in [76]. It would be interesting to study the precise connection as well as the potential realisations in statistical mechanics further.

"Galilean isometry" which encompasses the so-called Coriolis group of arbitrary time dependent (but spatially homogeneous) rotations and translations. In this language, the infinite dimensional algebra is that of "Galilean conformal isometries". It contains one copy of a Virasoro together with an $SO(d)$ current algebra (on adding the appropriate central extension). There has been interesting progress in this direction recently in [78].

In a follow-up work [79], we looked at representations and correlation functions of the GCA. Among other things, it was found that the form of the three point function was fixed upto a constant by the requirement of GCA invariance, like in the case of relativistic CFTs. However, in the case of the Schrödinger algebra, the three point function is arbitrary upto a function of a particular combination of variables. This is another indicator of the fact that the GCA is a more natural non-relativistic limit of the parent theory. For other related work on Galilean conformal algebras, see [80, 81].

The analysis in [73] was entirely classical, whereas in [79] (see also [80, 81]) the two and three point correlation functions (of primary fields) were obtained as solutions of the Ward identities for the finite part of the GCA (which arises as the contraction of $SO(d, 2)$). Finally in [82], the quantum mechanical realisation of the GCA in two dimensions was studied in great detail, where 2d GCFTs with nonzero central charges were obtained by considering a somewhat unusual limit of non-unitary 2d CFTs.

A natural and immediate direction of interest is to try and generalize our construction to the supersymmetric case. The algebra was, as emphasised just above, obtained by taking a natural parametric limit of the relativistic conformal algebra. We thus expect that the GCA would be a sub-sector of all relativistic theories. Particularly, in the context of AdS/CFT, it is natural to first try and extend the analysis of the algebra to include supersymmetry before we look to understand the details of the full field theory. We would finally be interested in embedding the GCA in String Theory where we would need to realize supersymmetric configurations.

So first we study the supersymmetric extension of GCA in 4d. Then we consider the $\mathcal{N} = (1, 1)$ supersymmetric extension of GCFTs in 2d, dubbed "SGCFT".

1.3 Plan of the report

We shall now give a brief plan of the chapters to follow:

In chapter 2, we will briefly review some of the known results on the counting of quarter-BPS dyonic black holes in $\mathcal{N} = 4$ supersymmetric string theories. This will lay the necessary groundwork for understanding the next chapter.

In chapter 3, we will describe the work "Black hole hair removal", where we identify the hair degress of freedom both for the BMPV black hole in flat transverse space and BMPV black hole in Taub-NUT space. We show that after subtracting out the contribution from these hair degrees of freedom from the microscopic degeneracy formulæ, one obtains identical result for the two black holes.

In chapter 4, we will review the Galilean Conformal Algebras, describing how to obtain them by the method of group contraction from relativistic conformal algebras. We will also discuss their extension to an infinite dimensional symmetry algebra.

In chapter 5, we will perform a contraction on the $\mathcal{N} = 1$ superconformal algebra in $(3+1)d$ to obtain a supersymmetric extension of the GCA in four spacetime dimensions. We will also see how we can lift the SGCA to an infinite dimensional algebra.

In chapter 6, we will study the $\mathcal{N} = (1,1)$ supersymmetric extension of the GCA in two spacetime dimensions. We will discuss the representation theory, the Ward identities, fusion rules etc. for these SGCA in $2d$.

Lastly, in chapter 7, we end with some concluding remarks.

Part II

Black hole hair removal

Chapter 2

Microstate counting

In this chapter we shall survey the known results on the counting of quarter-BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories.

2.1 The role of index

The counting of microstates is always done in a region of the moduli space where gravity is weak and hence the states do not form a black hole. In order to be able to compare it with the black hole entropy, we must focus on quantities which do not change as we change the coupling from small to large value. So we need an appropriate index which is protected by supersymmetry, and at the same time does not vanish identically when evaluated on the microstates of interest. The relevant index in $D = 4$ turns out to be the helicity trace index [83, 84].

Suppose we have a BPS state that breaks $4n$ supersymmetries. Then there will be $4n$ fermion zero modes (goldstinos) on the world-line of the state. Quantization of these zero modes will produce Bose-Fermi degenerate states. Thus the usual Witten index $Tr(-1)^F$, which measures the difference between the number of bosonic and fermionic states, will receive vanishing contribution from these states. To remedy this situation, we define a new index called the helicity trace index:

$$B_{2n} = \frac{1}{(2n)!}\, Tr\{(-1)^F (2h)^{2n}\} = \frac{1}{(2n)!}\, Tr\{(-1)^{2h} (2h)^{2n}\}\,, \qquad (2.1.1)$$

where h is the third component of the angular momentum in the rest frame. The trace is taken over states carrying a fixed set of charges. For every pair of fermion zero modes, $Tr\{(-1)^F(2h)\}$ gives a non-vanishing result i, leading to a non-zero contribution $(-1)^n$ to B_{2n}. On the other hand, any state that breaks more than $4n$ supersymmetries, will have more then $2n$ pairs of fermion zero modes and will give vanishing contribution to this trace. In particular, non-BPS states will not contribute, and the index will be protected from corrections as we vary the moduli (except at the walls of marginal stability [85, 86, 87, 88, 89], which will be discussed in §2.4).

Quarter-BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories preserve four of the sixteen supersymmetries, and hence break twelve supersymmetries. Thus the relevant helicity trace index is B_6. We shall now describe the microscopic results for B_6 in a class of $\mathcal{N} = 4$ supersymmetric string theories. However, we must keep in mind that, since on the microscopic side we compute an index, on the black hole side also we must compute an index. Otherwise we cannot compare the results of microscopic and macroscopic computations. It is described in [7, 55] how one can use black hole entropy to compute the index B_6 on the black hole side.

2.2 Microstate counting in heterotic string theory on T^6

The simplest example of an $\mathcal{N} = 4$ supersymmetric string theory is heterotic string theory on T^6 (or equivalently type IIA or IIB string theory on $K3 \times T^2$, as they are related by duality transformations). This theory has 28 U(1) gauge fields arising from the Cartan generators of the $E_8 \times E_8$ (or $SO(32)$) gauge group, and the components of the metric and the 2-form field along the six internal directions. Thus a generic charged state is characterized by 28 dimensional electric charge vector Q and 28 dimensional magnetic charge vector P. Under the $O(6, 22;\ \mathbb{Z})$ T-duality symmetry of the theory, the charges Q and P transform as vectors. This allows us to define T-duality invariant bilinears in the charges[1]: Q^2, P^2, $Q \cdot P$.

Our goal is to compute the index $B_6(Q, P)$. The computation is done in the dual frame: type IIB on $K3 \times S^1 \times \widetilde{S}^1$, where S^1 and $\widetilde{S}^1$ represent two circles which are not factored metrically.[2] In this frame, we compute B_6 for a rotating D1-D5-p system[60] in Kaluza-Klein (KK) monopole (or equivalently Taub-NUT) background. More specifically, we take a system containing [10]

1. one KK monopole along $\widetilde{S}^1$;
2. one D5-brane wrapped on $K3 \times S^1$;
3. $(\widetilde{Q}_1 + 1)$ D1-branes wrapped on S^1;
4. $-n$ units of momentum along S^1;

[1] Note that these bilinears are not positive definite as $O(6, 22;\ \mathbb{Z})$-invariant matrices have both positive and negative eigenvalues.

[2] The problem with carrying out this computation in heterotic frame is that there the system will contain $NS5$-branes, and the coupling constant diverges at the core of these branes.

5. J units of momentum along $\widetilde{S}^1$.

The momentum along $\widetilde{S}^1$ appears as an angular momentum at the center of the Taub-NUT space [62]. Thus, macroscopically, the system describes a rotating BMPV black hole [105] at the center of the Taub-NUT space [10]. In the weak coupling limit, the dynamics is given by that of a system of decoupled harmonic oscillators, and an exact computation of B_6 is possible. The result is then expressed in terms of the T-duality invariant bilinears Q^2, P^2, $Q \cdot P$ in the original heterotic frame, using the fact that the system described above has

$$Q^2 = 2n, \qquad P^2 = 2\widetilde{Q}_1, \qquad Q \cdot P = J\,. \tag{2.2.1}$$

If Q^2, P^2 and $Q \cdot P$ were the only T-duality invariants, i.e., if any two dyons with the same Q^2, P^2 and $Q \cdot P$ had been related to each other by a T-duality transformation, then the result for $B_6(Q, P)$ for the specific system described above will give the result for all dyons in the theory. However it turns out that this is not quite correct. Nevertheless, any charge vector satisfying the condition [22]

$$\gcd\{Q_i P_j - Q_j P_i\,, \quad 1 \leq i, j \leq 28\} = 1\,, \tag{2.2.2}$$

can be related to the above system by a T-duality transformation [31]. Thus the formula we quote below is valid only for this special class of charges.

Let us denote by $B_6(\widetilde{Q}_1, n, J)$ the sixth helicity trace associated with the system described above. We define the partition function as:

$$Z(\rho, \sigma, v) = \sum_{\widetilde{Q}_1, n, J} (-1)^J\, B_6(\widetilde{Q}_1, n, J)\, e^{2\pi i(\widetilde{Q}_1 \rho + n\sigma + Jv)}\,. \tag{2.2.3}$$

The computation of Z proceeds as follows. In the weakly coupled type IIB description, the low energy dynamics of the system is described by three weakly interacting pieces:

1. The closed string excitations around the KK monopole.
2. The dynamics of the D1-D5 center of mass coordinate in the KK monopole background.
3. The motion of the D1 branes along $K3$.

The dyon partition function is obtained as the product of the partition functions of these three subsystems [17].[3] The analysis can be simplified by taking the size of S^1 to be

[3] A factor of $(-1)^{J+1}$ in (2.2.3) was missed in [17]. The $(-1)^J$ factor arises because in five dimensions, at the center of the KK monopole, we have $(-1)^F = (-1)^{J+2h}$ instead of $(-1)^{2h}$[13]. An overall factor of -1, which has been absorbed in the definition of B_6 in (2.2.3), arises from the partition function of the quantum mechanics describing the D1-D5-brane motion in the KK monopole background[28]. A detailed derivation of many of the results given in this section has been reviewed in [28].

large compared to other dimensions, so that we can regard each subsystem as a 1+1 dimensional CFT. Since BPS condition forces the modes carrying positive momentum along S^1 (right-moving modes) to be frozen into their ground state, only left-moving modes can be excited. We shall now describe the contribution to Z from each subsystem.

First consider the fields describing the dynamics of KK monopole. These include

1. 3 left-moving and 3 right-moving bosons arising from its motion in the 3 transverse directions;
2. 2 left-moving and 2 right-moving bosons arising from the components of 2-form fields along the harmonic 2-form in Taub-NUT space [120, 121];
3. 19 left-moving and 3 right-moving bosons, arising from the components of the 4-form field along the wedge product of the harmonic 2-form on Taub-NUT and a harmonic 2-form on $K3$;
4. 8 right-moving goldstino fermions associated with the eight supersymmetries which are broken by the KK monopole.

Since the right-moving modes are frozen into their ground state, the contribution to the partition function from the KK-monopole dynamics, after separating out the contribution from fermion zero modes which go into the helicity trace, is equal to that of 24 left-moving bosons [17]:

$$Z_{KK} = e^{-2\pi i\sigma} \prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n\sigma})^{-24} \right\} . \tag{2.2.4}$$

The overall factor of $e^{-2\pi i\sigma}$ is a reflection of the fact that the ground state of the Kaluza-Klein monopole carries a net momentum of 1 along S^1.

The dynamics of the D1-D5 center of mass motion in the KK monopole background is described by a supersymmetric sigma model with Taub-NUT space as the target space. By taking the size of the Taub-NUT space to be large, we can take the oscillator modes to be those of a free field theory, but the zero mode dynamics is described by a supersymmetric quantum mechanics problem. The contribution is found to be [17]

$$Z_{CM} = e^{-2\pi i v} \prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n\sigma})^4 \, (1 - e^{2\pi i n\sigma + 2\pi i v})^{-2} \, (1 - e^{2\pi i n\sigma - 2\pi i v})^{-2} \right\} e^{-2\pi i v} \, (1 - e^{-2\pi i v})^{-2} . \tag{2.2.5}$$

The third component comprises D1-brane motion along $K3$. This can be computed as outlined below [63]:

1. First consider a single D1-brane, wrapped k times along S^1 and carrying fixed momenta along S^1 and $\widetilde{S}^1$. The dynamics of this system is described by a supersymmetric sigma model with target space $K3$. The number of states of this system can

be counted by the standard method of going to the orbifold limit. After removing a trivial degeneracy factor associated with fermion zero mode quantization, the net number of bosonic minus fermionic states, carrying momentum $-l$ along S^1 and j along $\widetilde{S}^1$, is given by $c(4lk-j^2)$, where $c(n)$ is defined as:

$$F(\tau,z) \equiv 8\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}+\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}+\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right], \tag{2.2.6}$$

$$F(\tau,z) = \sum_{j\in\mathbb{Z},n} c(4n-j^2)\, e^{2\pi i n\tau+2\pi i z j}. \tag{2.2.7}$$

Physically, $c(4n-j^2)$ counts the number of BPS states in the supersymmetric sigma model with target space $K3$ with $L_0=n$ and $\mathcal{J}_3=j/2$, where $\mathcal{J}_3$ denotes the third component of the $SU(2)$ R-symmetry current.

2. A generic state contains multiple D1-branes of this type, carrying different amounts of winding along S^1 and different momenta along S^1 and $\widetilde{S}^1$. The total number of states can be determined from the result of step 1 by simple combinatorics.

The net contribution to the partition function from D1-brane motion along $K3$ is [63]:

$$Z_{D1} = e^{-2\pi i\rho}\prod_{\substack{l,j,k\in Z\\ k>0,l\geq 0}}\left\{1-e^{2\pi i(l\sigma+k\rho+jv)}\right\}^{-c(4lk-j^2)}, \tag{2.2.8}$$

After taking the product of the component partition functions (2.2.4), (2.2.5) and (2.2.8), we get [17]

$$Z = e^{-2\pi i(\rho+\sigma+v)}\prod_{\substack{l,j,k\in Z\\ k\geq 0,l\geq 0,j<0 \text{ for } k=l=0}}\left\{1-e^{2\pi i(l\sigma+k\rho+jv)}\right\}^{-c(4lk-j^2)}, \tag{2.2.9}$$

where we have used the explicit values of $c(u)$ to express the contribution from (2.2.4) and (2.2.5) in terms of $c(n)$. Indeed these two factors give the $k=0$ term in (2.2.9). Eq.(2.2.9) can be expressed as

$$Z(\rho,\sigma,v) = 1/\Phi_{10}(\rho,\sigma,v). \tag{2.2.10}$$

Here Φ_{10} is a well known function, known as the weight 10 Igusa cusp form of $Sp(2,\ \mathbb{Z})$ [108, 109].[4] The formula for Z given above was conjectured in [8].

[4] $Sp(2,\ \mathbb{Z})$ includes the $SL(2,\ \mathbb{Z})$ S-duality group, but it is a much bigger group than the S-duality group of string theory. Thus it is not completely understood why Z has $Sp(2,\ \mathbb{Z})$ symmetry (see [12, 20, 43] for some attempts in this direction). In fact, this property of Z comes out at the very end after combining the results from the individual subsystems. But once we arrive at this final form, these symmetries can be conveniently used to analyse the asymptotic behaviour of Z.

Eq.(2.2.3) can be inverted to express $B_6(\widetilde{Q}_1, n, J)$ as

$$-B_6(\widetilde{Q}_1, n, J) = (-1)^{J+1} \int d\rho d\sigma dv \, e^{-2\pi i(\widetilde{Q}_1\rho + n\sigma + Jv)} \, Z(\rho, \sigma, v) \,. \tag{2.2.11}$$

We shall express this in a more duality invariant notation using (2.2.1):

$$-B_6(Q, P) = (-1)^{Q\cdot P+1} \int d\rho d\sigma dv \, e^{-\pi i(P^2\rho + Q^2\sigma + 2Q\cdot Pv)} \, Z(\rho, \sigma, v) \,. \tag{2.2.12}$$

2.3 Asymptotic expansion

In order to compare (2.2.12) with the black hole entropy, we need to find its behaviour for large Q^2, P^2, $Q.P$. It turns out that this is controlled by the behaviour of Z at its poles, which in turn are at the zeroes of Φ_{10} [8]. The location of the zeroes of Φ_{10} as well as the behaviour of Φ_{10} around these zeroes can be determined using its modular properties. We perform one of the three integrals using the residue theorem, picking up contributions from various poles. The leading contribution comes from the pole at [8]

$$(\rho\sigma - v^2) + v = 0 \,. \tag{2.3.1}$$

After picking up the residue at this pole, we are left with a two dimensional integral:

$$-B_6(Q, P) \simeq \int \frac{d^2\tau}{\tau_2^2} \, e^{F(Q^2, P^2, Q.P, \tau_1, \tau_2)} \,, \tag{2.3.2}$$

where (τ_1, τ_2) parametrize the locus of the zeroes of Φ_{10} at (2.3.1) in the (ρ, σ, v) space and

$$F = \frac{\pi}{2\tau_2}(Q-\tau P)\cdot(Q-\bar{\tau}P) - 24\ln\eta(\tau) - 24\ln\eta(-\bar{\tau}) - 12\ln(2\tau_2) + \ln\left[26 + \frac{\pi}{\tau_2}(Q-\tau P)\cdot(Q-\bar{\tau}P)\right] \tag{2.3.3}$$

We evaluate this integral by the saddle point method. We expand F around its extremum and carry out the integral using perturbation theory. If we consider a limit in which we scale all the charges by some large parameter Λ, then the perturbation expansion around the saddle point generates a series in inverse power of Λ^2, with the leading semi-classical result being of order Λ^2.

Applying the above procedure, first of all we find that, for large charges, $-B_6(Q, P)$ is positive [28] (i.e., $B_6(Q, P)$ is negative). Furthermore [9, 110]:

$$\ln|B_6(Q, P)| = \pi\sqrt{Q^2P^2 - (Q.P)^2} - \phi\left(\frac{Q.P}{P^2}, \frac{\sqrt{Q^2P^2 - (Q.P)^2}}{P^2}\right) + \mathcal{O}\left(\frac{1}{Q^2, P^2, Q.P}\right) , \tag{2.3.4}$$

where

$$\phi(\tau_1, \tau_2) \equiv 12 \ln \tau_2 + 24 \ln \eta(\tau_1 + i\tau_2) + 24 \ln \eta(-\tau_1 + i\tau_2) \,. \tag{2.3.5}$$

The first term, $\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}$, is indeed the Bekenstein-Hawking entropy of the black hole [111, 112]. For an explanation of the macroscopic origin of the other terms, see [7].

2.4 Walls of marginal stability

Our result for the D1-D5-KK monopole system was derived for weakly coupled type IIB string theory. However, as we move around in the moduli space, we may hit walls of marginal stability, at which the quarter-BPS dyon under consideration becomes unstable against decay into a pair of half-BPS dyons. At these walls, the index jumps, and hence we cannot trust our formula on the other side of the wall. It turns out, however, that with the help of S-duality, we can always bring the moduli to a domain where the type IIB theory is in the weakly coupled domain and we can trust our original formula. The net outcome of this analysis is that, in different domains, the index is given by the formula:

$$-B_6(Q, P) = (-1)^{Q.P+1} \int_C d\rho d\sigma dv \, e^{-\pi i(P^2\rho + Q^2\sigma + 2Q.Pv)} / \Phi_{10}(\rho, \sigma, v) \,, \tag{2.4.1}$$

where C denotes the choice of 'contour' that picks a 3 real dimensional subspace of integration in the 3 complex dimensional space:

$$Im(\rho) = M_1, \quad Im(\sigma) = M_2, \quad Im(v) = M_3, \quad 0 \le Re(\rho), Re(\sigma), Re(v) \le 1 \,. \tag{2.4.2}$$

The three real numbers (M_1, M_2, M_3), which specify the choice of the contour C, depend on the domain in the moduli space where we compute the index [21, 22, 25]. For example in the weak coupling limit of type IIB string theory, for the system we have analyzed, we have $M_1, M_2 >> 1$, $1 << |M_3| << M_1, M_2$ and the sign of M_3 is positive or negative depending on whether the angle between S^1 and $\widetilde{S}^1$ is larger or smaller than $\pi/2$ [17, 19]. The jumps in the index, across the walls of marginal stability, are encoded in the residues at the poles in Z that we encounter while deforming the contour corresponding to one domain to the contour corresponding to the other domain. There is a precise correspondence between different walls of marginal stability and different poles of Z. For the decay $(Q, P) \Rightarrow (Q, 0) + (0, P)$, the associated wall is at $v = 0$[17, 18, 19, 21, 22]. This, together with the S-duality invariance of the theory, tells us that for the wall associated with the decay

$$(Q, P) \Rightarrow (\alpha Q + \beta P, \gamma Q + \delta P) + ((1-\alpha)Q - \beta P, -\gamma Q + (1-\delta)P) \,, \tag{2.4.3}$$

the corresponding pole is at

$$\gamma\rho - \beta\sigma + (\alpha - \delta)v = 0 \,. \tag{2.4.4}$$

A precise formula giving (M_1, M_2, M_3) in terms of the moduli and charges can be found in [25]. We should keep in mind, however, that the result is independent of (M_1, M_2, M_3) as long as changing them does not make the contour cross a pole.

On the black hole (macroscopic) side, these jumps correspond to (dis-)appearance of two-centered black holes as we cross walls of marginal stability. There is a precise match between the B_6 index of 2-centered black holes carrying charges given on the right hand side of (2.4.3), and the change in $B_6(Q, P)$ computed from the residues at the poles (2.4.4) [24, 25].

In this context, we would like to mention that the changes in the index across the walls of marginal stability are subleading, as these give corrections which grow as exponentials of single power of the charges. This is related to the fact that only decays of a 1/4-BPS dyon into half-BPS dyons contribute to the wall crossing in an $\mathcal{N} = 4$ supersymmetric string theory [26, 38, 48]. However the contribution from the multi-centered solutions can become significant when we study dyons in $\mathcal{N} = 2$ supersymmetric string theories [89].

Chapter 3

Black hole hair removal

3.1 Introducion and summary

Since the Bekenstein-Hawking entropy of a black hole is proportional to the area of the event horizon of the black hole[1, 2, 3, 90, 91] one expects that the horizon of the black hole contains the key to understanding the black hole microstates. Wald's modification of the Bekenstein-Hawking formula in higher derivative theories of gravity [56, 57, 58, 59] deviates from the area law, but nevertheless expresses the black hole entropy in terms of the horizon data. The situation becomes even better in the extremal limit where an infinite throat separates the horizon from the rest of the black hole space-time and the near horizon configuration can be regarded as a fully consistent solution to the field equations [5, 92, 93, 94, 95]. The classical Wald entropy can be related to the value of the classical Lagrangian density evaluated in this near horizon geometry [96]. This leads one to expect that we should be able to define a macroscopic quantity, computed from quantum string theory in the near horizon geometry, that captures complete information about the microscopic degeneracies of the corresponding black hole. Quantum entropy function is such a proposal relating the microscopic degeneracies of extremal black holes to an appropriate partition function of quantum gravity in the near horizon geometry of the black hole[54, 46] (see also [97, 98]).

Irrespective of any specific proposal, if the postulate that the microscopic degeneracy of an extremal black hole can be related to some computation in the near horizon geometry is correct, then this leads to an immediate consequence: two black holes with identical near horizon geometries will have identical degeneracies of microstates. There are some trivial counterexamples with straightforward resolutions. For example the near horizon geometry of an extremal black hole in flat space-time is independent of the asymptotic values of the moduli fields due to the attractor mechanism[92, 93, 94, 96, 99, 100], but the microscopic degeneracy of states, carrying the same quantum numbers as the black hole, jumps across the walls of marginal stability as we vary the asymptotic moduli[17, 19, 21, 22, 25]. The resolution of this puzzle is provided by the fact that for a given set of charges there are typically many classical solutions. One of these is a single centered black hole solution but the others contain multiple centers[101, 85, 86, 87, 88, 89, 24, 25]. As we cross a wall of marginal stability some of these multi-centered solutions cease to exist and hence cause a jump in the total entropy. This precisely accounts for the jump in the total degeneracy across the walls of marginal stability, thereby showing that the degeneracy of states associated with a single centered black hole remains unchanged as we cross a wall of marginal stability. This suggests a natural modification of the original proposal: *string theory in the near horizon geometry captures information about the microscopic degeneracy of the single centered black holes only.* This is clearly natural from a physical perspective: the near horizon geometry of a given black hole should encode information only about the particular solution which produces the particular near horizon geometry. Multi-centered black holes have multiple horizons with multiple near horizon geometry, and hence the contribution to their degeneracies should involve studying string theory in the near horizon geometry of each of these black holes.[1]

In order to make this modified proposal concrete we must independently define microscopic degeneracy of a single centered black hole. Typically microscopic computation involves studying degeneracies of various brane configurations and cannot distinguish whether a given state would correspond to a single centered or a multi-centered configuration in the limit when the state becomes a black hole. However in asymptotically flat four dimensional space-time there is a simple algorithm for calculating the spectrum of single centered black holes in the microscopic theory; we simply need to set the asymptotic values of the moduli to be equal to their attractor values.[2] In that case all multi-centered black hole solutions disappear and the microstate counting only picks up the contribution from the single centered black holes.

In this thesis, we focus on a different counterexample that cannot be resolved by invoking the existence of multi-centered black holes. This involves the BMPV black hole[60], whose microscopic description involves a D1-D5 system of type IIB string theory on $K3 \times S^1$, carrying momentum along S^1 and equal angular momentum in two planes transverse to the D5-brane. The macroscopic description of this is a five dimensional rotating black hole. By placing this black hole at the center of a Taub-NUT space we get a four dimensional black hole[61]. Since near the origin the Taub-NUT space appears as flat

[1] The near horizon AdS_2 geometry of a black hole can fragment into multiple throats carrying different charges[102, 103, 104]. However for such solutions the charges carried by the fragments are mutually local, i.e. have $(\vec{q}_i \cdot \vec{p}_j - \vec{q}_j \cdot \vec{p}_i) = 0$ where $(\vec{q}_i, \vec{p}_i)$ denote the electric and magnetic charge vectors of the ith throat. Since such configurations do not contribute to the entropy[87, 89], the conclusion that the near horizon geometry of a black hole captures the degeneracies of single centered black holes remains unchanged.

[2] This is sufficient but not necessary; all we need is that the asymptotic values of the moduli should be chosen such that we can continuously deform them to the attractor values without crossing any wall of marginal stability.

space, the near horizon geometries of the four and five dimensional black holes are exactly identical[62, 10]. However the microscopic description of the four dimensional black hole involves D1-D5-brane moving in the background of a Kaluza-Klein monopole and the degeneracies of this system are different from those of just the D1-D5 system[17]. This would seem to contradict the claim that the microscopic degeneracies of single centered black holes are completely encoded in their near horizon geometries.

We suggest the following resolution of this puzzle. Common sense tells us that the near horizon geometry should capture the degeneracies associated with the dynamics of the horizon. If the black hole has no hair, that is no degree of freedom living outside the horizon that could contribute to the degeneracy, then the near horizon geometry would capture the complete information about the microscopic degeneracy of the black hole. However if the black hole solution contains degrees of freedom living outside the horizon then the full degeneracy of the black hole has to be computed by combining the contribution from the horizon with the contribution from the degrees of freedom living outside the horizon, and the combined contribution will then have to be compared with the microscopic degeneracies. Thus two black holes having identical near horizon geometry can have different microscopic degeneracies if they have different sets of degrees of freedom living outside the horizon. We expect that at least for extremal black holes the separation between the contribution from the black hole hair and the contribution from the horizon degrees of freedom can be done rigorously since the horizon is separated from the asymptotic space-time by an infinite throat. Thus two such extremal black holes with identical near horizon geometry will have identical degeneracies of microstates *after we remove the contribution from the degrees of freedom living outside the horizon.*[3]

In the rest of the chapter, we shall identify the degrees of freedom living outside the horizon for both the BMPV black hole and the four dimensional extremal black hole obtained by placing the BMPV black hole in a Taub-NUT geometry, and then show that their microscopic degeneracies agree after we remove the contribution due to the hair. The organisation of the sections will be as follows. In §3.2 we identify the hair degrees of freedom of the five dimensional BMPV black hole, and remove their contribution from the partition function to determine the partition function associated with the horizon degrees of freedom. The result is given in (3.2.17). In §3.3 we repeat the same analysis for the four dimensional black hole obtained by placing the BMPV black hole at the center of Taub-NUT space. The result, given in (3.3.20), is found to agree with (3.2.17). It of course remains a challenge to reproduce these microscopic results from a macroscopic calculation, *e.g.* of the quantum entropy function. In §3.4 we describe explicit construction of the bosonic modes associated with the hair degrees of freedom.

Before concluding this section we would like to add a word of caution. While we

[3]This is similar in spirit to the phenomenon that for a stack of N D3-branes, string theory living in the bulk of the near horizon $AdS_5 \times S^5$ geometry does not capture the $U(1)$ center of mass degrees of freedom of the D3-branes[65].

have identified appropriate hair degrees of freedom for the five and four dimensional black holes after whose removal the result for the partition function of the two black holes agree, we have not proved that these are the only hair degrees of freedom. If there are additional hair degrees of freedom which differ for these two black holes then it could spoil the agreement. On the other hand if there are additional hair degrees of freedom which are common to both black holes then the agreement between the partition functions of the two black holes after hair removal will continue to hold.

3.2 Analysis of the BMPV black hole entropy

We begin with the analysis of microscopic degeneracy of the five dimensional quarter BPS black hole in type IIB string theory on $K3$. The microscopic description involves Q_5 number of D5-branes wrapped on $K3 \times S^1$ and Q_1 number of D1-branes wrapped on S^1 carrying $-n$ units of momentum along S^1 (with $n > 0$) and J units of angular momentum. For simplicity we shall take $Q_5 = 1$ without any loss of generality since the result depends on Q_1 and Q_5 only through the combination $Q_5(Q_1 - Q_5)$. Our convention for angular momentum and supersymmetry generators will be as follows. We denote the $SO(4)$ rotation group of the five dimensional space-time by $SU(2)_L \times SU(2)_R$ and identify the angular momentum J with twice the diagonal generator of $SU(2)_L$. We also denote by h the eigenvalue of the diagonal generator of $SU(2)_R$. Since supersymmetry transformation parameters of type IIB on K3 are chiral spinors in six dimensions, when we regard them as representations of the $SO(1,1) \times SU(2)_L \times SU(2)_R$ subgroup of the Lorentz group, with $SO(1,1)$ acting on the common direction of the D1-brane and the D5-brane, the $SO(1,1)$ quantum numbers will be correlated with the $SU(2)_L \times SU(2)_R$ quantum numbers. We shall now argue that in order that the configuration described above describes a quarter BPS state, we must choose the convention that the left-chiral spinors of $SO(1,1)$ carry $(J = 0, 2h = \pm 1)$ and the right-chiral spinors of $SO(1,1)$ carry $(J = \pm 1, h = 0)$. The argument goes as follows. First of all note that since the D1-D5-brane system carries negative momentum along S^1, it must be allowed to carry left-moving excitations without violating supersymmetry. Thus the left-chiral excitations must be neutral under the unbroken supersymmetries of the system. This in turn implies that these supersymmetry transformation parameters must be left-chiral spinors of $SO(1,1)$, – since left-chiral supersymmetry transformation parameters act on the right-chiral modes and vice versa. We shall now argue that the unbroken supersymmetry transformation parameters must also carry $J = 0$, – this would force us to choose the convention described above. In order that the system can carry macroscopic J charge, a large number of internal modes must carry non-vanishing J charge. Now most of the bosonic degrees of freedom come from the motion of the D1-brane inside the D5-brane, i.e. along the $K3$ direction. This leads to four bosons for each D1-brane describing its position along $K3$. These modes are clearly neutral under the $SO(4)$ rotation along the space transverse to the

D1-D5-brane system, and hence do not carry any J charge. On the other hand for every D1-brane we also have eight fermionic modes, – four carrying $(J \neq 0, h = 0)$ and four carrying $(J = 0, h \neq 0)$.[4] The requirement of unbroken supersymmetry freezes the modes on which supersymmetry acts, i.e. those which form partners of the bosons. Now since we want to excite the modes carrying J charge, we must freeze the ones with $J = 0$. Thus the latter must be acted upon by supersymmetry and paired with the bosons. Since the bosons carry $J = 0$, the supersymmetry transformation parameter must also carry $J = 0$. This establishes the desired result.

We denote by $d_{5D}(n, Q_1, J)$ the helicity trace $-Tr\left((-1)^{2h+J}(2h)^2\right)/2!$ of five dimensional black hole carrying quantum numbers (n, Q_1, J), and define

$$Z_{5D}(\rho, \sigma, v) = \sum_{n,Q_1,J} d_{5D}(n, Q_1, J) \exp\left[2\pi i\{(Q_1 - 1)\,\sigma + (n-1)\,\rho + Jv\}\right] . \tag{3.2.1}$$

The -1 in $(Q_1 - 1)$ reflects the fact that a D5-brane wrapped on $K3$ carries -1 units of D1-brane charge. On the other hand the -1 in $(n-1)$ has been introduced due to the fact that this charge measured at ∞ differs from that measured on the horizon[106, 40, 107] – a Chern-Simons coupling in the action produces -1 unit of this charge from the region between the horizon and infinity. Thus if $-n$ is the total momentum along S^1 carried by the black hole, the charge measured at the horizon will be $-(n-1)$. Explicit computation shows that Z_{5D} defined in (3.2.1) has the form

$$\begin{aligned} Z_{5D}(\rho, \sigma, v) &= e^{-2\pi i\rho - 2\pi i\sigma} \prod_{\substack{k,l,j\in\mathbb{Z}\\ k\geq 1, l\geq 0}} \left(1 - e^{2\pi i(\sigma k + \rho l + vj)}\right)^{-c(4lk - j^2)} \\ &\quad \times \left\{\prod_{l\geq 1} (1 - e^{2\pi i(l\rho + v)})^{-2}\,(1 - e^{2\pi i(l\rho - v)})^{-2}\,(1 - e^{2\pi i l\rho})^4\right\}(-1)\,(e^{\pi i v} - e^{-\pi i v})^2 \\ &\quad + e^{-2\pi i\rho - 2\pi i\sigma} \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k + vj)}\right)^{-c(-j^2)}\,(e^{\pi i v} - e^{-\pi i v})^2 , \end{aligned} \tag{3.2.2}$$

where the coefficients $c(n)$ are defined via the equation

$$8\left[\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2}\right] = \sum_{j,n\in\mathbb{Z}} c(4n - j^2)\, e^{2\pi i n\tau + 2\pi i jz} . \tag{3.2.3}$$

Eq.(3.2.2) requires some explanation. The first line of (3.2.2) denotes the contribution from the relative motion of the D1-D5 system and was computed in [63]. The asymptotic expansion of the degeneracies of this system has been studied recently in [40, 41].

[4] These have opposite relation between the $SO(1,1)$ and $SU(2)_L \times SU(2)_R$ quantum numbers, but we shall not need to use this information here.

The second line represents contribution from the 'center of mass modes' of the D1-D5 system. This contribution can be calculated as follows. Since the D1-D5 system breaks the translation symmetries along the four directions transverse to the brane, the (1+1) dimensional world-volume theory of this system, spanned by the time coordinate and the coordinate along S^1, will contain four goldstone bosons associated with the four broken translation generators. Furthermore since the ground state of the D1-D5 system also breaks eight out of the sixteen supersymmetries of type IIB string theory on K3, we shall have eight goldstino fermions carrying the same quantum numbers as the broken supersymmetry transformation parameters. This leads to four left-moving and four right-moving fermions living on the D1-D5-brane world-volume. In our convention the left-moving fermions carry ($J = 0, 2h = \pm 1$) and the right-moving fermions carry ($J = \pm 1, 2h = 0$). We need to count excitations of this system preserving four supersymmetries, parametrized by left-chiral spinors on the D1-D5-brane world-volume. Since these transformations act on the right-moving fermions and bosons, the BPS condition will freeze all the right-moving excitations except the zero modes. Since the right-moving fermions carry $J = \pm 1$, $h = 0$, quantization of a pair of right chiral zero modes would produce a pair of states with $J = \pm\frac{1}{2}$, $h = 0$. Thus the net contribution of four right chiral zero modes to the trace, containing a factor of $(-1)^J e^{2\pi i v J} = e^{2\pi i J(v+\frac{1}{2})}$, is a factor of $(e^{\pi i(v+\frac{1}{2})} + e^{-\pi i(v+\frac{1}{2})})^2 = -(e^{\pi i v} - e^{-\pi i v})^2$. This accounts for the last two factors in the second line of (3.2.2). The BPS condition does not restrict the left-moving degrees of freedom and the terms in the curly bracket in the second line of (3.2.2) represent contribution from these left-moving excitations. In particular the zero modes of the left-moving fermions, carrying helicities $\pm 1/2$, soak up the factors of $-(2\,h)^2/2!$ in the helicity trace so that if we leave aside these zero modes, contribution to the helicity trace from the rest of the modes involve computing the Witten index $Tr(-1)^F$. Since the left-moving fermions have $J = 0$, their oscillators lead to the last term in the product inside the curly bracket. On the other hand the left-moving bosons, transforming under $(2, 2)$ representation of $SU(2)_L \times SU(2)_R$, carry ± 1 units of J quantum numbers and lead to the first two terms inside the curly bracket. Finally the term in the last line of (3.2.2) removes the contribution of the $n = 0$ term[5] from eq.(3.2.1), i.e. it subtracts the term whose ρ dependence is of the form $e^{-2\pi i\rho}$. The rationale for subtracting this term is that for $n = 0$ the D1-D5 system includes contribution from half-BPS states. Thus it is more natural to consider the partition function of pure quarter BPS states by subtracting the contribution due to the $n = 0$ term.

Now we need to analyze the contribution to the partition function from the degrees of freedom of the black hole living outside the horizon and remove this contribution from (3.2.2) to determine the expected microscopic degeneracies associated with the horizon. We begin by writing down the action and the black hole solution. The relevant part of the action containing the string metric $G_{\mu\nu}$, dilaton Φ and the Ramond-Ramond 3-form

[5]Throughout this chapter we shall denote the additive term proportional to $e^{-2\pi i\rho}$ as the $n = 0$ term.

field strength $F^{(3)} = dC^{(2)}$ takes the form

$$\frac{1}{(2\pi)^7} \int d^{10}x \sqrt{-\det G} \left[e^{-2\Phi} \left(R + 4\, G^{\mu\nu}\, \partial_\mu \Phi \partial_\nu \Phi \right) - \frac{1}{12} F^{(3)}_{MNP} F^{(3)MNP} \right], \tag{3.2.4}$$

in $\alpha' = 1$ unit. For simplicity we shall set the asymptotic values of the moduli to their attractor values for the specific black hole solution we analyze, so that all the moduli fields including the dilaton are constants. The generalization to more general asymptotic values is straightforward. In this case the rotating black hole solution describing Q_5 D5-branes along $K3 \times S^1$, Q_1 D1-branes along S^1, $-n$ units of momentum along S^1 and angular momentum J, takes the form[6]

$$\begin{aligned}
dS^2 &= \left(1 + \frac{r_0}{r}\right)^{-1} \left[-dt^2 + (dx^5)^2 + \frac{r_0}{r}(dt + dx^5)^2 + \frac{\widetilde{J}}{4r}\,(dt + dx^5)\,(dx^4 + \cos\theta\, d\phi) \right] \\
&\quad + \widehat{g}_{mn}(\vec{u})\, du^m du^n + \left(1 + \frac{r_0}{r}\right) ds^2_{flat}\,, \\
ds^2_{flat} &= r\,(dx^4 + \cos\theta d\phi)^2 + \frac{1}{r}\,(dr^2 + r^2 d\theta^2 + r^2\, \sin^2\theta\, d\phi^2)\,, \\
(\theta, \phi, x^4) &\equiv (2\pi - \theta, \phi + \pi, x^4 + \pi) \equiv (\theta, \phi + 2\pi, x^4 + 2\pi) \equiv (\theta, \phi, x^4 + 4\pi)\,, \\
e^{\Phi} &= \lambda\,, \\
F^{(3)} &\equiv \frac{1}{6} F^{(3)}_{MNP} dx^M \wedge dx^N \wedge dx^P \\
&= \frac{r_0}{\lambda} \left(\epsilon_3 + *_6 \epsilon_3 + \frac{1}{r_0} \left(1 + \frac{r_0}{r}\right)^{-1} (dx^5 + dt) \wedge d\zeta \right), \\
\epsilon_3 &\equiv \sin\theta\, dx^4 \wedge d\theta \wedge d\phi\,,
\end{aligned} \tag{3.2.5}$$

where x^5 is the coordinate of the circle S^1 with period $2\,\pi\, R_5$, u^m for $m = 6, ..., 9$ are the coordinates of $K3$, $\widehat{g}_{mn}$ is the metric on $K3$, $(2\pi)^4\, V$ is the volume of $K3$ measured in this metric, λ is the asymptotic value of the string coupling, $*_6$ denotes Hodge dual in the six dimensions spanned by t, x^5, x^4, r, θ and ϕ with the convention $\epsilon^{t54r\theta\phi} = 1$, and

$$r_0 = \frac{\lambda(Q_1 - Q_5)}{4V} = \frac{\lambda Q_5}{4} = \frac{\lambda^2 |n|}{4 R_5^2 V}\,, \tag{3.2.6}$$

$$\widetilde{J} = \frac{J\, \lambda^2}{2\, R_5\, V}\,, \tag{3.2.7}$$

$$\zeta = -\frac{\widetilde{J}}{8r}\, (dx^4 + \cos\theta d\phi)\,. \tag{3.2.8}$$

[6]Conventionally the BMPV black hole as well as the BMPV black hole at the center of Taub-NUT space is expressed as a solution in five dimensional supergravity theory[60, 61, 62]. Here we express them as solutions in a ten dimensional theory so that we can study the excitations which propagate along the internal directions.

Eq.(3.2.6) determines the asymptotic moduli V and λ/R_5^2 in terms of the charges. This corresponds to setting the asymptotic moduli to their attractor values. ds^2_{flat} describes flat euclidean space in the Gibbons-Hawking coordinates. Higher derivative corrections to the entropy of this black hole have been discussed extensively in [113, 114, 115, 106, 116, 40, 107, 117].

Now the black hole solution breaks four translation symmetries and twelve of the sixteen space-time supersymmetries, and hence we expect to have four bosonic zero modes and twelve fermionic zero modes living on the black hole, forming part of the black hole hair.[7] Typically the quantization of the bosonic zero modes do not give rise to additional degeneracies but produces new charge sectors instead, – this was illustrated in [23] in the context of four dimensional black holes. However the quantization of the fermion zero modes does affect the partition function. The (J, h) quantum numbers of the fermion zero modes can be read out by comparison with the microscopic description. Since the four unbroken supersymmetries are labelled by left-chiral spinors on the D1-D5 world-volume, eight of the broken supersymmetries are right-chiral and four of the broken supersymmetries are left-chiral. This leads to eight right-chiral and four left-chiral zero modes. The left-chiral zero modes carrying $(J = 0, h = \pm\frac{1}{2})$ soak up the factors of $-(2\,h)^2/2!$ in the helicity trace, so that for the rest of the degrees of freedom we only need to calculate the Witten index $Tr(-1)^{2h+J}$. On the other hand the right-chiral zero modes carry $(J = \pm 1, h = 0)$ and their contribution to the partition function is given by

$$(e^{\pi i v} - e^{-\pi i v})^4 \,. \tag{3.2.9}$$

This however is not the end of the story. Given a zero mode we can explore whether it is possible to lift it to a full fledged field in (1+1) dimensions spanned by the coordinates (t, x^5). If we can lift them to such fields then the oscillation modes of these fields would produce additional contribution to the partition function of the black hole hair. To this end we note that if the black hole solution had been Lorentz invariant in the (x^5, t) plane, then any broken symmetry would automatically lead to a massless goldstone or goldstino field on the black hole world volume instead of just the zero modes. In particular the bosonic zero modes would lift to scalar fields, left-chiral fermion zero modes would lift to left-moving fermion fields and right-chiral fermion zero modes would lift to right-moving fermion fields. However the black hole solution (3.2.5) does not have $(1 + 1)$ dimensional Lorentz invariance, and hence *a priori* we cannot use results in 1+1 dimensonal quantum field theory to conclude that associated with a broken symmetry we shall have a massless field living on the world-volume of the black hole. Nevertheless we shall now argue that the left-moving modes are not affected by the breaking of Lorentz invariance and continue

[7] Given that black hole solution outside the horizon changes under these translations and supersymmetry transformations, it is clear that these modes are non-vanishing outside the horizon. What is not apparent at this stage is whether they have support entirely outside the horizon. For now we shall proceed by assuming that this is the case, but will study this issue in detail in §3.4.1.

to exist. Our argument will be somewhat heuristic, but we compensate for it by giving a detailed construction of these modes in §3.4.1. First we note that the source of Lorentz non-invariance in (3.2.5) are the $(dt + dx^5)^2$ term and the $(dt + dx^5)\,(dx^4 + \cos\theta\, d\phi)$ terms in the metric. This structure of the metric shows that only the g_{++} and g_{+i} components of the metric violate the Lorentz invariance. Since these lead to g^{--} and g^{-i} components of the metric but no g^{++} or g^{+i} components, we see that the Lorentz violating terms in the equation of motion of various modes around the solution must involve ∂_- derivatives or $_{-\ldots}$ components of fields. In particular the left-moving fields φ for which $\partial_-\varphi = 0$ do not couple to the g^{--} or g^{-i} components of the metric and should continue to describe solutions to linearized equations of motion around the black hole background. Thus we can conclude that the world-volume of the black hole will have four left-moving bosonic fields carrying $(J = \pm 1, 2h = \pm 1)$ and four left-moving fermion fields carrying $(J = 0, 2h = \pm 1)$. Their contribution to the partition function is given by

$$\prod_{l\geq 1}(1 - e^{2\pi i(l\rho+v)})^{-2}\,(1 - e^{2\pi i(l\rho-v)})^{-2}\,(1 - e^{2\pi i l\rho})^4\,. \tag{3.2.10}$$

Multiplying this by the contribution (3.2.9) from the zero modes we get the total contribution to the partition function from the degrees of freedom living outside the horizon

$$Z^{hair}_{5D}(\rho,\sigma,v) = (e^{\pi i v} - e^{-\pi i v})^4 \prod_{l\geq 1}(1 - e^{2\pi i(l\rho+v)})^{-2}\,(1 - e^{2\pi i(l\rho-v)})^{-2}\,(1 - e^{2\pi i l\rho})^4\,. \tag{3.2.11}$$

Let $Z^{hor}_{5D}(\rho,\sigma,v)$ denote the partition function associated with the horizon degrees of freedom of the five dimensional black hole. Naively we have the relation $Z_{5D} = Z^{hor}_{5D} \times Z^{hair}_{5D}$. However we shall now argue that there is an extra additive contribution to Z_{5D}, and the correct relation is

$$Z_{5D} = Z^{hor}_{5D} \times Z^{hair}_{5D} + Z^{extra}_{5D}\,. \tag{3.2.12}$$

The extra contribution Z^{extra}_{5D} comes from starting with a configuration where the black hole does not carry any momentum along S^1, and then exciting its hair degrees of freedom carrying momentum. As can be seen from (3.2.6), the initial configuration is singular in the supergravity approximation. Thus it describes a 'small black hole' in five dimensions,[8] and hence its hair degrees of freedom are different from the ones we analyzed earlier. In particular since the D1-D5 system without momentum breaks only four left-chiral and four right chiral supersymmetries, we have only four right chiral zero modes instead of 8, and hence a factor of $-(e^{\pi i v} - e^{-\pi i v})^2$ will be missing from the hair degrees of freedom. Furthermore since the D1-D5-brane world-volume theory now has full (1+1)

[8]Here, as well as in §3.3, we shall denote by 'small black hole' any object which is singular in the supergravity limit, carrying Q_1, Q_5 and J quantum numbers but no momentum along S^1. Thus it includes small black ring configurations as well[118, 119].

dimensional Lorentz invariance, the right-chiral modes are now lifted to full right-moving fields, However the requirement of unbroken supersymmetry still freezes the right-moving excitations to their ground state. Thus the net contribution from the hair is given by

$$Z^{hair}_{small} = -(e^{\pi iv} - e^{-\pi iv})^2 \prod_{l\geq 1}(1 - e^{2\pi i(l\rho+v)})^{-2}\,(1 - e^{2\pi i(l\rho-v)})^{-2}\,(1 - e^{2\pi il\rho})^4\,. \qquad (3.2.13)$$

Let us denote by Z^{hor}_{small} the contribution from the horizon degrees of freedom of the small black hole. Then Z^{extra}_{5D} will be obtained by taking the product $Z^{hor}_{small} \times Z^{hair}_{small}$ and subtracting the $n = 0$ contribution. On the other hand Z^{hor}_{small} may be determined by identifying the $n = 0$ contribution in $Z^{hor}_{small} \times Z^{hair}_{small}$ with the partition function of the D1-D5 system with no momentum along S^1. The latter is simply the negative of the last term in (3.2.2):

$$-e^{-2\pi i\rho-2\pi i\sigma} \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)} (e^{\pi iv} - e^{-\pi iv})^2\,. \qquad (3.2.14)$$

Dividing (3.2.14) by the ρ independent term in the series expansion of (3.2.13) gives

$$Z^{hor}_{small}(\rho, \sigma, v) = e^{-2\pi i\rho-2\pi i\sigma} \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)}\,. \qquad (3.2.15)$$

Z^{extra}_{5D} is now obtained by multiplying (3.2.15) by (3.2.13) and then subtracting the $n = 0$ term, i.e. the term proportional to $e^{-2\pi i\rho}$ in the series expansion:

$$\begin{aligned} Z^{extra}_{5D}(\rho, \sigma, v) &= -e^{-2\pi i\rho-2\pi i\sigma}\,(e^{\pi iv} - e^{-\pi iv})^2 \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)} \\ &\quad \times \prod_{l\geq 1}(1 - e^{2\pi i(l\rho+v)})^{-2}\,(1 - e^{2\pi i(l\rho-v)})^{-2}\,(1 - e^{2\pi il\rho})^4 \\ &\quad + e^{-2\pi i\rho-2\pi i\sigma}\,(e^{\pi iv} - e^{-\pi iv})^2 \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)} \end{aligned} \qquad (3.2.16)$$

Using (3.2.2), (3.2.11), (3.2.12) and (3.2.16) we now get

$$\begin{aligned} Z^{hor}_{5D}(\rho, \sigma, v) &= (Z_{5D} - Z^{extra}_{5D})/Z^{hair}_{5D} \\ &= -e^{-2\pi i\rho-2\pi i\sigma}\,(e^{\pi iv} - e^{-\pi iv})^{-2} \prod_{\substack{k,l,j\in\mathbb{Z}\\ k\geq 1, l\geq 0}} \left(1 - e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \\ &\quad + e^{-2\pi i\rho-2\pi i\sigma}\,(e^{\pi iv} - e^{-\pi iv})^{-2} \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)}\,. \end{aligned} \qquad (3.2.17)$$

The presence of the $(e^{\pi iv} - e^{-\pi iv})^{-2}$ factor may lead one to believe that Z_{5D}^{hor} has a double pole at $v = 0$ and hence the index extracted from this partition function will suffer from the contour prescription ambiguities discussed in [21, 22, 25]. However using the relation $\sum_j c(4n - j^2) = 24\,\delta_{n,0}$ and the $v \to -v$ symmetry one can show that the sum of the two terms in (3.2.17) has no singularity at $v = 0$. Thus (3.2.17) leads to an unambiguous result for the index of quarter BPS states associated with the horizon degrees of freedom. We also note that since the factor of $-(2\,h)^2/2!$ in the helicity trace is soaked up by the fermion zero modes associated with the hair, the partition function Z_{5D}^{hor} measures the Witten index $Tr(-1)^F = Tr(-1)^{2h+J}$ of the black hole microstates associated with the horizon in a given (n, Q_1, J) sector.

3.3 Analysis of the $4D$ black hole entropy

Now we turn to the degeneracies of four dimensional black holes obtained by placing the five dimensional black hole described above at the center of Taub-NUT space. The corresponding solution is given by[61]

$$\begin{aligned}
dS^2 &= \left(1+\frac{r_0}{r}\right)^{-1}\Big[-dt^2 + (dx^5)^2 + \frac{r_0}{r}(dt+dx^5)^2 \\
&\quad + \frac{\widetilde{J}}{4}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)(dx^4 + \cos\theta\, d\phi)\,(dt+dx^5)\Big] \\
&\quad + \widehat{g}_{mn}\, du^m du^n + \left(1+\frac{r_0}{r}\right) ds_{TN}^2\,, \\
e^{\Phi} &= \lambda\,, \\
F^{(3)} &= \frac{r_0}{\lambda}\left(\epsilon_3 + *_6\epsilon_3 + \frac{1}{r_0}\left(1+\frac{r_0}{r}\right)^{-1}(dx^5+dt)\wedge d\widetilde{\zeta}\right), \qquad (3.3.1)
\end{aligned}$$

where

$$\widetilde{\zeta} = -\frac{\widetilde{J}}{8}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)(dx^4 + \cos\theta d\phi)\,, \qquad (3.3.2)$$

$$ds_{TN}^2 = \left(\frac{4}{R_4^2}+\frac{1}{r}\right)^{-1}(dx^4+\cos\theta d\phi)^2 + \left(\frac{4}{R_4^2}+\frac{1}{r}\right)(dr^2 + r^2 d\theta^2 + r^2\,\sin^2\theta\, d\phi^2)\,. \quad (3.3.3)$$

Here R_4 is a constant labelling the asymptotic radius of the x^4 circle. Note that for $R_4^2 = 4r_0$ the 44, 45 and 55 components of the metric become constant independent of r. Thus $4r_0$ is the attractor value of R_4^2. We shall proceed with the solution for general R_4. Using (3.3.3) we can express the solution given in (3.3.1) as

$$\begin{aligned}
dS^2 &= -e_0^2 + e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + \widehat{g}_{mn}\, du^m\, du^n\,, \\
F^{(3)} &= \frac{r_0}{\lambda\, r^2}\Bigg[\left(1+\frac{r_0}{r}\right)^{-3/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{-1/2}(e_2\wedge e_4\wedge e_5 + e_0\wedge e_1\wedge e_3)
\end{aligned}$$

$$+\frac{\widetilde{J}}{8\,r_0}\left(1+\frac{r_0}{r}\right)^{-2}\left(-e_0\wedge e_2\wedge e_3+e_0\wedge e_4\wedge e_5-e_1\wedge e_2\wedge e_3+e_1\wedge e_4\wedge e_5\right)\Bigg],\tag{3.3.4}$$

where

$$\begin{aligned}
e_0 &= \left(1+\frac{r_0}{r}\right)^{-1}(dt+\widetilde{\zeta}),\\
e_1 &= \left(dx^5+dt-\left(1+\frac{r_0}{r}\right)^{-1}(dt+\widetilde{\zeta})\right),\\
e_2 &= \left(1+\frac{r_0}{r}\right)^{1/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{-1/2}(dx^4+\cos\theta d\phi),\\
e_3 &= \left(1+\frac{r_0}{r}\right)^{1/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{1/2}dr\,,\\
e_4 &= \left(1+\frac{r_0}{r}\right)^{1/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{1/2}r\,d\theta\,,\\
e_5 &= \left(1+\frac{r_0}{r}\right)^{1/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{1/2}r\,\sin\theta\,d\phi\,.
\end{aligned}\tag{3.3.5}$$

Since x^4 has period 4π, the asymptotic circle parametrized by x^4 has finite radius. Thus asymptotically we have four non-compact space-time dimensions. Also since x^4 now represents a compact coordinate, the quantum number J is interpreted as the momentum along x^4 instead of angular momentum. However for small r the solution approaches that given in (3.2.5), and both solutions have identical near horizon geometry. To see this explicitly we take the near horizon limit by first defining new coordinates (ρ,τ,y) via

$$r=r_0\,\beta\rho,\quad t=\tau/\beta,\quad x^5=y-t\tag{3.3.6}$$

and taking the limit $\beta\to 0$. In this limit both (3.2.5) and (3.3.1) take the form[9]

$$\begin{aligned}
dS^2 &= r_0\frac{d\rho^2}{\rho^2}+dy^2+r_0(dx^4+\cos\theta d\phi)^2+\frac{\widetilde{J}}{4r_0}dy(dx^4+\cos\theta d\phi)-2\rho dyd\tau\\
&\quad +r_0\left(d\theta^2+\sin^2\theta d\phi^2\right)+\widehat{g}_{mn}du^m du^n\,,\\
e^{\Phi} &= \lambda\,,\\
F^{(3)} &= \frac{r_0}{\lambda}\left[\epsilon_3+*\epsilon_3+\frac{\widetilde{J}}{8\,r_0^2}\,dy\wedge\left(\frac{1}{\rho}\,d\rho\wedge(d\,x^4+\cos\theta\,d\phi)+\sin\theta\,d\theta\wedge d\phi\right)\right]
\end{aligned}\tag{3.3.7}$$

[9]We could take a more careful limit by beginning with a non-extremal black hole and scaling the non-extremality parameter also by β as reviewed in [54]. However this does not play any role in the present discussion.

Thus we expect that the contribution to the degeneracy from the horizon degrees of freedom will be identical for the four and the five dimensional black holes. In particular the quantum entropy function will give identical results for the two solutions. We shall now try to test this at the microscopic level by computing the degeneracies associated with the four dimensional black hole horizon.

The microscopic degeneracy associated with the four dimensional black hole is different from that of the five dimensional black hole, as it receives additional contribution from the modes living on the Taub-NUT space as well as the modes associated with the motion of the D1-D5-brane in the Taub-NUT space[17]. If we denote by $d_{4D}(n, Q_1, J)$ the sixth helicity trace[10] $-B_6 \equiv -Tr((-1)^{2h+J}(2h)^6)/6!$ for the states of the four dimensional black hole carrying quantum numbers (n, Q_1, J) then the four dimensional partition function defined via

$$Z_{4D}(\rho,\sigma,v) = \sum_{n,Q_1,J} d_{4D}(n-1,Q_1,J)\,\exp\left[2\pi i\{(Q_1-1)\,\sigma+(n-1)\,\rho+Jv\}\right]\,, \quad (3.3.8)$$

has the form[8, 9, 10, 12, 17][11]

$$Z_{4D}(\rho,\sigma,v) = -e^{-2\pi i\rho-2\pi i\sigma-2\pi iv} \prod_{\substack{k,l,j\in\mathbb{Z}\\ k,l\geq 0, j<0\, for\, k=l=0}} \left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)}\,. \quad (3.3.9)$$

Note that we now have $(n-1)$ in the argument of d_{4D} in (3.3.8), matching the coefficient of ρ in the exponent. This reflects the fact that for the four dimensional black holes the charge measured at the horizon agrees with the charge measured by an asymptotic observer. The $e^{-2\pi i\rho}$ factor in (3.3.9) is a reflection of the fact that the ground state of the Taub-NUT space carries -1 unit of momentum along S^1; however this is visible only after taking into account the higher derivative term in the action involving the gravitational Chern-Simons term. Finally we note that there is no need to subtract the $n=0$ contribution from the sum, since in the presence of a Taub-NUT space even the $n=0$ states are quarter BPS. The near horizon geometry of the $n=0$ black hole will however lose the memory of the Taub-NUT background and will have enhanced supersymmetries.

We now need to remove the contribution to Z_{4D} from the degrees of freedom living outside the horizon. We begin by counting the fermionic modes living outside the horizon. First of all, there are 12 broken supersymmetry generators leading to 12 fermion zero modes. They carry $h=\pm\frac{1}{2}$ and soak up the $-(2\,h)^6/6!$ factor from the helicity trace.

[10] h now denotes the third component of the angular momentum in the (3+1) dimensional theory. J represents a U(1) charge in the four dimensional theory and its inclusion in the trace is purely a matter of convenience.

[11] The correct sign of the partition function has been determined in [28]. Note that $d_{4D}(n, Q_1, J)$ used here differ from the index used in [28] by a factor of $(-1)^J$ due to the insertion of $(-1)^J$ in our definition of B_6. However the definition of partition function in [28] has an explicit factor of $(-1)^{J+1}$ inserted.

Thus the effect of removing their contribution is to map the helicity trace index to the Witten index of the remaining system[17, 28]. Had the black hole world-volume theory been Lorentz invariant in the (x^5, t) coordinates, eight of the zero modes would lift to right-moving fermion fields and four of the zero modes would lift to left-moving fermion fields on the black hole world-volume. As in the case of five dimensional black holes, we expect that the breaking of Lorentz invariance does not affect the equations for the left-moving modes and hence we should be able to lift the four left-chiral fermion zero modes into full fledged left-moving fermion fields on the black hole world-volume. These modes produce a contribution to the Witten index of the form

$$\prod_{l=1}^{\infty}(1 - e^{2\pi i l\rho})^4 \,. \tag{3.3.10}$$

Next we turn to the bosonic modes living on the black hole. As before we shall proceed by pretending that the black hole world-volume has Lorentz invariance in the (x^5, t) plane, and then take into account the lack of Lorentz invariance by freezing the right-moving fields. Our arguments will be heuristic, but we give more explicit construction of some of the modes in §3.4.2. The black hole solution given in (3.3.1) admits a normalizable closed 2-form inherited from the normalizable harmonic 2-form of the Taub-NUT space[120, 121]. It is given by

$$\omega = -\frac{r}{4r + R_4^2}\,\sin\theta d\theta \wedge d\phi + \frac{R_4^2}{(4r + R_4^2)^2}\,dr \wedge (dx^4 + \cos\theta d\phi)\,. \tag{3.3.11}$$

Using the metric (3.3.1) one can easily check that this harmonic form is supported outside the near horizon throat geometry. Thus any 2-form field along this harmonic form will give rise to a scalar mode living outside the horizon. From the NSNS and RR 2-form fields of type IIB string theory we get two scalar modes. Furthermore the 4-form field with self-dual field strength, reduced on the 22 internal cycles of K3, generate 3 right chiral and 19 left chiral 2-form fields in type IIB string theory on K3.[12] Picking up the components of these fields along the 2-form ω we get 19 left-moving scalars and 3 right-moving scalars on the black hole world-volume. By the logic given earlier we expect the left-moving modes to survive even after taking into account the breaking of the Lorentz invariance in the $(x^5 - t)$ plane. Besides these there are three goldstone bosons associated with the three broken translational symmetries. After freezing the right-moving modes we get three more left-moving modes on the black hole world-volume. Thus we have altogether 2+19+3=24 left-moving scalars living outside the horizon.[13] Since they do not carry any J quantum number (which now corresponds to momentum along x^4), their contribution

[12]In our convention the right-chiral 2-form fields have self-dual 3-form field strength and the left-chiral 2-form fields have anti-self-dual 3-form field strength in six dimensions.

[13]Explicit form of these deformations can be found in §3.4.2.

to the black hole partition function is given by

$$\prod_{l=1}^{\infty}(1-e^{2\pi i l\rho})^{-24}\,. \tag{3.3.12}$$

We shall now argue that the four dimensonal solution carries four more left-moving bosonic excitations living outside the horizon and carrying J-charge ±1. Explicit construction of these modes have been discussed in §3.4.2. Physically these modes represent the motion of the D1-D5 system relative to the Taub-NUT space. Normally if in a composite system we try to displace one component relative to the other there will be a drastic change in the near horizon geometry and we would not expect such deformations to be described by modes living outside the horizon. However since the Taub-NUT space is non-singular everywhere, the near horizon geometry of a D1-D5-Taub-NUT system is described by that of the D1-D5 system, and hence moving the Taub-NUT space relative to the D1-D5 system should not alter the near horizon geometry. Thus such deformations should be described by modes living outside the horizon. Furthermore since the coordinates labelling the transverse position of the D1-D5 system transform in the vector representation of $SO(4)$, these modes should carry $J=\pm1$. By the standard argument based on the lack of Lorentz invariance in the x^5-t plane, we expect the right-moving modes to be frozen but the left-moving modes should be freely excitable. The contribution from these modes to the partition function is given by

$$\prod_{l=1}^{\infty}\left[\left(1-e^{2\pi i(l\rho+v)}\right)^{-2}\left(1-e^{2\pi i(l\rho-v)}\right)^{-2}\right]\,. \tag{3.3.13}$$

Can there be additional zero modes associated with the motion of the D1-D5-system relative to the Taub-NUT space? The five dimensional black hole world volume in flat transverse space has four left-chiral fermion zero modes with $(J,2h)=(0,\pm1)$ and eight right-chiral fermion zero modes with $(J,2h)=(\pm1,0)$, – all living outside the horizon. By an argument similar to the one in the previous paragraph, we expect them to be approximate zero modes even when we place the five dimensional black hole in the Taub-NUT background. The four left-chiral fermion zero modes form part of the 12 goldstino zero modes of the combined system and have already been counted before. Four of the eight right chiral fermion zero modes must form superpartners of the bosonic zero modes describing the motion of the D1-D5-brane system in transverse space. This gives rise to a factor of $-e^{-2\pi i v}(1-e^{-2\pi i v})^{-2}$ from summing over bound states in the supersymmetric quantum mechanics describing the zero mode dynamics[121, 122, 17, 28]. The other four right-chiral fermion zero modes which are not paired with the bosons under supersymmetry would give a factor of $-(e^{\pi i v}-e^{-\pi i v})^2$ since they carry $J=\pm1$. Thus these two factors cancel exactly and we do not get any additional contribution to the hair from these zero modes.

Combining (3.3.10), (3.3.12) and (3.3.13) we get the net contribution to the four dimensional black hole partition function from the hair:

$$Z_{4D}^{hair}(\rho,\sigma,v) = \prod_{l=1}^{\infty}\left[\left(1-e^{2\pi i l\rho}\right)^{-20}\left(1-e^{2\pi i(l\rho+v)}\right)^{-2}\left(1-e^{2\pi i(l\rho-v)}\right)^{-2}\right]. \quad (3.3.14)$$

Let Z_{4D}^{hor} denote the partition function of the horizon degrees of freedom of the four dimensional black hole. Then naively we have the relation $Z_{4D} = Z_{4D}^{hor} \times Z_{4D}^{hair}$, but as in the case of five dimensional black holes, Z_{4D} receives an extra contribution from the configuration where a small five dimensional black hole carrying no momentum along S^1 is placed at the center of the Taub-NUT space and the momentum along S^1 is carried by the hair degrees of freedom. Denoting the extra contribution by Z_{4D}^{extra} we have

$$Z_{4D} = Z_{4D}^{hor} \times Z_{4D}^{hair} + Z_{4D}^{extra}. \quad (3.3.15)$$

Z_{4D}^{extra} is given by the product of horizon partition function of the small black hole as given in (3.2.15) and the contribution from the hair degrees of freedom. The latter now consists of four bosons and four left- and four right-moving fermions associated with the motion of the small black hole in Taub-NUT space, and eight right-moving fermions, eight right-movimg bosons and twenty four left-moving bosons associated with the fluctuations in Taub-NUT space. Instead of going through a detailed analysis of these modes we simply note that the number and dynamics of these modes is identical to those describing the dynamics of the Taub-NUT space and the overall motion of the D1-D5 system in Taub-NUT space as discussed in [17, 28]. Thus the partition function associated with the hair degrees of freedom can be read out from [17, 28]. In particular the contribution from the degrees of freedom associated with the overall motion of the D1-D5 system can be read out from eq.(5.2.22) of [28] for $N = 1$:[14]

$$-e^{-2\pi i v}\left(1-e^{-2\pi i v}\right)^{-2}\prod_{l\geq 1}(1-e^{2\pi i(l\rho+v)})^{-2}\,(1-e^{2\pi i(l\rho-v)})^{-2}\,(1-e^{2\pi i l\rho})^{4}. \quad (3.3.16)$$

On the other hand the degrees of freedom of the Taub-NUT space contributes

$$\prod_{l\geq 1}(1-e^{2\pi i l\rho})^{-24}. \quad (3.3.17)$$

Taking the product of (3.2.15), (3.3.16) and (3.3.17) gives

$$Z_{4D}^{extra}(\rho,\sigma,v) = -e^{-2\pi i(v+\rho+\sigma)}\left(1-e^{-2\pi i v}\right)^{-2}\prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}}\left(1-e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)}$$

[14] The factor of $-e^{-2\pi i v}(1-e^{-2\pi i v})^{-2}$ arises from the sum over bound states of the quantum mechanics describing the motion of the D1-D5-system in Taub-NUT space. The main difference from the computation of Z_{4D}^{hair} is that when the core of the black hole describing the D1-D5 system carries zero momentum, we have only eight fermion zero modes living on the D1-D5 system instead of twelve. Thus an extra factor of $-(e^{\pi i v}-e^{-\pi i v})^2$ is missing here.

$$\prod_{l=1}^{\infty}\left[\left(1-e^{2\pi il\rho}\right)^{-20}\left(1-e^{2\pi i(l\rho+v)}\right)^{-2}\left(1-e^{2\pi i(l\rho-v)}\right)^{-2}\right]. \quad (3.3.18)$$

Using (3.3.9), (3.3.14), (3.3.15) and (3.3.18), and the relations

$$c(0)=20, \qquad c(-1)=2, \qquad c(u)=0 \quad \text{for } u\leq -2\,, \qquad (3.3.19)$$

we get

$$\begin{aligned} Z_{4D}^{hor}(\rho,\sigma,v) &= (Z_{4D}-Z_{extra})/Z_{4D}^{hair} \\ &= -e^{-2\pi i\rho-2\pi i\sigma}\left(e^{\pi iv}-e^{-\pi iv}\right)^{-2}\prod_{\substack{k,l,j\in\mathbb{Z}\\ k\geq 1,l\geq 0}}\left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \\ &\quad +e^{-2\pi i\rho-2\pi i\sigma}\left(e^{\pi iv}-e^{-\pi iv}\right)^{-2}\prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}}\left(1-e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)}. \end{aligned} \quad (3.3.20)$$

This is identical to Z_{5D}^{hor} given in (3.2.17). We also note that since the $-(2\,h)^6/6!$ term in the trace has been absorbed by the fermion zero modes living outside the horizon, Z_{4D}^{hor} measures the Witten index $Tr(-1)^F$ of the microstates associated with the horizon in a given (n, Q_1, J) sector. The equality of Z_{4D}^{hor} and Z_{5D}^{hor} now shows that the Witten indices associated with the near horizon degrees of freedom of the four and the five dimensional black holes are exactly identical.

It has been shown in [123] that the hair modes describing the transverse oscillations of the five dimensional black hole, and the oscillations of the BMPV black hole relative to the Taub-NUT space for the four dimensional black hole, develop curvature singularities at the future horizon. Thus they should not be included among the hair degrees of freedom. Since they contributed the same amount to the respective partition functions, the agreement between the partition functions of four and five dimensional black holes after hair removal continue to hold.

3.4 Appendix

3.4.1 Left-moving bosonic modes on the BMPV black hole

Since our argument leading to the existence of left-moving modes on the BMPV black hole has been somewhat abstract we shall now explicitly demonstrate the existence of such modes. For simplicity we shall focus on the left-moving bosonic zero modes associated with the transverse oscillations. If we introduce new coordinates

$$\begin{aligned} w^1 &= 2\sqrt{r}\cos\frac{\theta}{2}\cos\frac{x^4+\phi}{2}, \qquad w^2 = 2\sqrt{r}\cos\frac{\theta}{2}\sin\frac{x^4+\phi}{2}, \\ w^3 &= 2\sqrt{r}\sin\frac{\theta}{2}\cos\frac{x^4-\phi}{2}, \qquad w^4 = 2\sqrt{r}\sin\frac{\theta}{2}\sin\frac{x^4-\phi}{2}\,, \end{aligned} \quad (3.4.1)$$

then the solution given in (3.2.5) takes the form

$$\begin{aligned}
dS^2 &= \psi(r)^{-1}\left[dx^+dx^- + (\psi(r)-1)(dx^+)^2\right] + \chi_i(\vec{w})\, dx^+\, dw^i + \widehat{g}_{mn}\, du^m du^n + \psi(r) d\vec{w}^{\,2}, \\
x^\pm &\equiv x^5 \pm t\,, \quad r \equiv \frac{1}{4}\vec{w}^2, \quad \psi(r) \equiv \left(1+\frac{r_0}{r}\right), \quad \chi_i(\vec{w})\, dw^i = \psi(r)^{-1}\,\frac{\widetilde{J}}{4r}\,(dx^4 + \cos\theta\, d\phi) \\
C^{(2)} &= \frac{1}{2}C_{ij}(\vec{w})dw^i \wedge dw^j + C_{+i}(\vec{w})dx^+ \wedge dw^i + C_{+-}(\vec{w})dx^+ \wedge dx^-\,, \qquad (3.4.2)
\end{aligned}$$

where $C^{(2)}$ denotes the RR 2-form field and C_{ij}, C_{+i} and C_{+-} are some fixed functions of $\vec{w}$. We can now use the following algorithm to generate the deformations describing left-moving transverse oscillations of the black hole:

1. We first consider a deformation of the solution generated by the diffeomorphism

$$\begin{aligned}
w^i &\rightarrow w^i + a^i\,(x^+ + c)\, f + (x^+ + c)\,\vec{a}\cdot\vec{w}\, w^i\, g, \\
x^- &\rightarrow x^- - 2\,\vec{a}\cdot\vec{w}\,\psi^2\, f - (x^+ + c)\,\psi\,\left(\vec{a}\cdot\vec{\chi}\, f + \vec{a}\cdot\vec{w}\,\vec{w}\cdot\vec{\chi}\, g\right), \\
x^+ &\rightarrow x^+\,, \qquad (3.4.3)
\end{aligned}$$

   where $\vec{a}$ denotes an arbitrary constant four dimensional vector, $\vec{a}\cdot\vec{w} \equiv a^i w^i$, c is an arbitrary constant and f and g are functions of r satisfying

$$g = \frac{1}{2}\,\psi^{-2}\,(\psi^2 f)'\,. \qquad (3.4.4)$$

   Here $'$ denotes derivative with respect to r. The diffeomorphism has been chosen such that all the terms in $\delta(dS^2)$ to first order in a^i are proportional to $(x^+ + c)$ without any derivative acting on it. By accompanying this diffeomorphism by a suitable gauge transformation of $C^{(2)}$ we can ensure that $\delta C^{(2)}$ also is proportional to $(x^+ + c)$ without any derivative acting on it.

2. We now replace the overall factor of $x^+ + c$ by an arbitrary function $\epsilon(x^+)$ everywhere in the deformed solution. Thus the deformed configuration is proportional to $\epsilon(x^+)$. Furthermore, by construction it is guaranteed to be a solution to the equations of motion for $\epsilon(x^+) = x^+ + c$. This in turn shows that if we substitute the deformed configuration into the equations of motion then the terms proportional to $\epsilon(x^+)$ and $\partial_+\epsilon(x^+)$ must vanish automatically.

3. Our goal is to ensure that the deformed configuration is a solution to the equations of motion to linear order in ϵ for arbitrary function $\epsilon(x^+)$. Since the field equations are second order in derivatives, and terms involving $\epsilon(x^+)$ and $\partial_+\epsilon(x^+)$ are guaranteed to vanish, it only remains to ensure that the terms involving $\partial_+^2\epsilon$ vanish. Such terms

can arise in the ++ component of the metric equation, and the vanishing of the term proportional to $\partial_+^2\epsilon$ can be shown to require[15]

$$G^{ij}\delta G_{ij} = 0\,, \tag{3.4.5}$$

where i, j run over the four transverse spatial coordinates, G_{ij} is the background metric and δG_{ij} denotes the first order deformation of the metric. This imposes one additional constraint on the functions f and g. Once this condition is satisfied we have a set of deformations parametrized by four arbitrary function $a^i\epsilon(x^+)$.[16]

At the end of the second step this procedure gives

$$\begin{aligned}
\delta\left(dS^2\right) &= -\frac{1}{2}\,\epsilon(x^+)\,\psi^{-2}\psi'\,\vec{a}\cdot\vec{w}\,(f+4rg)\,\left(dx^+\,dx^- - (dx^+)^2\right)\\
&\quad+\frac{1}{2}\,\epsilon(x^+)\,\psi'\,\vec{a}\cdot\vec{w}\,(f+4rg)\,\vec{dw}^{\,2} + \epsilon(x^+)\,\psi\,f'\,\vec{a}\cdot\vec{dw}\,\vec{w}\cdot\vec{dw}\\
&\quad+2\,\epsilon(x^+)\,\psi\,g\,\vec{a}\cdot\vec{dw}\,\vec{w}\cdot\vec{dw} + 2\,\epsilon(x^+)\,\psi\,g\,\vec{a}\cdot\vec{w}\,\vec{dw}^{\,2} + \epsilon(x^+)\,\psi\,g'\,\vec{a}\cdot\vec{w}\,\vec{w}\cdot\vec{dw}\,\vec{w}\cdot\vec{dw}\\
&\quad-\epsilon(x^+)\,\psi^{-1}\,dx^+\;d\left(\psi(\vec{a}\cdot\vec{\chi}\,f + \vec{a}\cdot\vec{w}\,\vec{w}\cdot\vec{\chi}\,g)\right)\\
&\quad+\epsilon(x^+)\,\chi_i\,dx^+\,d\left(a^i\,f + \vec{a}\cdot\vec{w}\,w^i\,g\right) + \epsilon(x^+)\,\partial_k\chi^i\,(a^k\,f + \vec{a}\cdot\vec{w}\,w^k\,g)\,dx^+dw^i\,,\\
\delta\,C^{(2)} &= \frac{1}{2}\,\epsilon(x^+)\,(\partial_k\,C_{ij} + \partial_i\,C_{jk} + \partial_j\,C_{ki})\,\left(a^k f + \vec{a}\cdot\vec{w}\,w^k\,g\right)dw^i\wedge dw^j\\
&\quad+\epsilon(x^+)\,\partial_k C_{+-}\,\left(a^k f + \vec{a}\cdot\vec{w}\,w^k\,g\right)dx^+\wedge dx^-\\
&\quad+\epsilon(x^+)\,\partial_k C_{+-}\,dw^k\wedge\,\left(2\,\psi^2\,f\,\vec{a}\cdot\vec{dw} + \vec{a}\cdot\vec{w}\,(\psi^2 f)'\,\vec{w}\cdot\vec{dw}\right)\\
&\quad-\epsilon(x^+)\,\partial_k\,C_{+-}\,\left(\vec{a}\cdot\vec{\chi}\,f + \vec{a}\cdot\vec{w}\,\vec{w}\cdot\vec{\chi}\,g\right)\,\psi\,dw^k\wedge dx^+\\
&\quad+\epsilon(x^+)\,(\partial_l C_{+k} - \partial_k C_{+l})\,(a^l\,f + \vec{a}\cdot\vec{w}\,w^l\,g)\,dx^+\wedge dw^k\,.
\end{aligned} \tag{3.4.6}$$

Substituting this into (3.4.5) gives

$$2\psi'\,(f+4rg) + \psi\,f' + 10\,\psi\,g + 4\,r\,\psi\,g' = 0\,. \tag{3.4.7}$$

Using eq.(3.4.4) we can regard (3.4.7) as a second order linear differential equation for f. Thus it has two independent solutions. It is easy to verify that the general solution to (3.4.4), (3.4.7) is

$$f = (A_0\,r^{-2} + B_0)\,\psi^{-2}, \qquad g = -A_0\,r^{-3}\,\psi^{-2}\,, \tag{3.4.8}$$

where A_0 and B_0 are two arbitrary constants. Requiring that the solution gives a normalizable deformation of the metric and the 2-form field near $r = 0$ we get $A_0 = 0$. Thus we have

$$f = B_0\,\psi^{-2}\,, \qquad g = 0\,. \tag{3.4.9}$$

[15]Note that since the three form field strengths contain at most a single derivative of ϵ, they do not directly contribute any term proportional to $\partial_+^2\epsilon$ in the equations of motion.

[16]This analysis is similar in spirit, although much simpler than, the one carried out in [124].

It is easy to verify that the deformations of the metric and the 2-form field associated with this choice of f is normalizable both at $r = 0$ and at $r = \infty$. Thus we have normalizable deformation of the solution parametrized by four indendent functions $a^i\,\epsilon(x^+)$. This shows the existence of four left-moving modes on the black hole world-volume. Furthermore the contribution to the norm of the deformation from the throat region $r << r_0$ vanishes, showing that these modes are located outside the horizon.

We expect that a similar argument can be used to construct the four left-moving fermionic modes on the black hole world-volume. In this case we shall need to use the broken supersymmetry generators to generate the fermionic deformation of the solution. However we shall not carry out this analysis explicitly.

3.4.2 Left-moving bosonic modes on the $4D$ black hole

In this subsection, we shall give explicit construction of the bosonic zero modes living on the four dimensional black hole. We begin with the left-moving zero modes associated with the harmonic two form ω in the Taub-NUT space given in (3.3.11). For any 2-form field B – either the NSNS or RR sector 2-form field of the ten dimensional type IIB string theory or a four form field with two legs along an internal 2-cycle of $K3$ – we consider a deformation of the form

$$\delta B = \epsilon(x^+)\,\omega\,, \tag{3.4.10}$$

for any function $\epsilon(x^+)$ of $x^+ = x^5 + t$. This gives

$$\begin{aligned} d(\delta\, B) &= \epsilon'(x^+)\, dx^+ \wedge \omega \\ &= -\epsilon'(x^+)\, \frac{1}{r^2\, R_4^2} \left(\frac{1}{r} + \frac{4}{R_4^2}\right)^{-2} \left(1 + \frac{r_0}{r}\right)^{-1} \\ &\qquad (e_0 \wedge e_2 \wedge e_3 + e_0 \wedge e_4 \wedge e_5 + e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5)\,, \end{aligned} \tag{3.4.11}$$

where the 1-forms e_i's have been defined in (3.3.5). $d(\delta B)$ given in (3.4.11) can be shown to be anti-self-dual. Hence $d(\delta B)$ is both closed and co-closed and δB given in (3.4.10) provides a solution to the linearized equations of motion of $B_{\mu\nu}$ around the background (3.3.1). For the 3-form field strength deformation given in (3.4.11) one also finds that there is no contribution to the stress tensor from the interference term between the deformation and the leading order field strength given in (3.3.4). As a result the deformation (3.4.10) also satisfies the metric equation of motion at the linearized level. However in order that (3.4.10) corresponds to a valid configuration in string theory, B must correspond to a left-chiral 2-form (which has anti-self-dual field strength in our convention). Since type IIB on K3 has 2+19=21 left-chiral 2-form fields we get 21 left-moving bosonic modes from this construction. Finally this deformation is normalizable with the metric given in (3.3.1) and the norm is supported outside the throat, i.e. outside the $r << r_0, R_4^2$ region. Thus these modes should be counted as part of the black hole hair.

Next we shall describe the left-moving modes associated with the 3 transverse motion of the black hole. For this we introduce new coordinates (y^1, y^2, y^3) via

$$y^1 = r\,\cos\theta\,\cos\phi, \qquad y^2 = r\,\cos\theta\,\sin\phi, \qquad y^3 = r\,\cos\theta\,. \tag{3.4.12}$$

In this coordinate system the metric given in eq.(3.3.1) takes the form

$$\begin{aligned} dS^2 &= \psi(r)^{-1}\left\{dx^+dx^- + (\psi(r)-1)\,(dx^+)^2\right\} + \frac{\widetilde{J}}{4}\,\chi(r)\,\psi(r)^{-1}\,(dx^4 + A_\alpha(\vec{y})\,dy^\alpha)\,dx^+ \\ &\quad + \psi(r)\,\chi(r)^{-1}\,\left(dx^4 + A_\alpha(\vec{y})\,dy^\alpha\right)^2 + \psi(r)\,\chi(r)\,\vec{dy}^{\,2} + \widehat{g}_{mn}\,du^m du^n\,, \end{aligned} \tag{3.4.13}$$

where

$$\psi(r) = 1 + \frac{r_0}{r}\,, \quad \chi(r) = \frac{1}{r} + \frac{4}{R_4^2}\,, \quad A_\alpha(\vec{y})\,dy^\alpha = \cos\theta\,d\phi\,. \tag{3.4.14}$$

We can now generate an x^+ dependent deformation of this solution by first considering a diffeomorphism

$$\begin{aligned} y^\alpha &\to y^\alpha + (x^+ + c)\,(b^\alpha\,\widetilde{f} + \vec{b}\cdot\vec{y}\,y^\alpha\,\widetilde{g})\,, \\ x^- &\to x^- - 2\,\vec{b}\cdot\vec{y}\,\chi\,\psi^2\,\widetilde{f}\,, \\ x^4 &\to x^4 - (x^+ + c)\,A_\alpha\,(b^\alpha\,\widetilde{f} + \vec{b}\cdot\vec{y}\,y^\alpha\,\widetilde{g})\,, \end{aligned} \tag{3.4.15}$$

and then replacing $(x^+ + c)$ by $\epsilon(x^+)$ in the deformed solution. Here c, b^1, b^2, b^3 are arbitrary parameters, $\vec{b}\cdot\vec{y} \equiv b^\alpha\,y^\alpha$, and $\widetilde{f}$ and $\widetilde{g}$ are functions satisfying

$$\widetilde{g} = \frac{1}{r}\,\psi^{-2}\,\chi^{-1}(\psi^2\,\chi\,\widetilde{f})'\,. \tag{3.4.16}$$

This gives

$$\begin{aligned} \delta(dS^2) &= -\epsilon(x^+)\,\psi^{-2}\,\psi'\,\frac{\vec{b}\cdot\vec{y}}{r}\,(\tilde{f} + r^2\tilde{g})\,\left(dx^+dx^- - (dx^+)^2\right) \\ &\quad + \frac{\widetilde{J}}{4}\,\epsilon(x^+)\,(\psi^{-1}\chi)'\frac{\vec{b}\cdot\vec{y}}{r}\,(\tilde{f} + r^2\tilde{g})\,dx^+\,(dx^4 + \vec{A}\cdot\vec{dy}) \\ &\quad + \frac{\widetilde{J}}{4}\,\epsilon(x^+)\,\psi^{-1}\,\chi\,(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\,(b^\alpha\,\widetilde{f} + \vec{b}\cdot\vec{y}\,y^\alpha\,\widetilde{g})\,dy^\beta\,dx^+ \\ &\quad + \epsilon(x^+)\,(\psi\,\chi^{-1})'\,\frac{\vec{b}\cdot\vec{y}}{r}\,(\tilde{f} + r^2\tilde{g})\,(dx^4 + \vec{A}\cdot\vec{dy})^2 \\ &\quad + 2\,\epsilon(x^+)\,\psi\,\chi^{-1}\,(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\,(b^\alpha\,\widetilde{f} + \vec{b}\cdot\vec{y}\,y^\alpha\,\widetilde{g})\,dy^\beta\,(dx^4 + \vec{A}\cdot\vec{dy}) \\ &\quad + \epsilon(x^+)\,(\psi\,\chi)'\,\frac{\vec{b}\cdot\vec{y}}{r}\,(\tilde{f} + r^2\tilde{g})\,\vec{dy}^2 + 2\,\epsilon(x^+)\,\psi\,\chi\,dy^\alpha\,d\left(b^\alpha\,\widetilde{f} + \vec{b}\cdot\vec{y}\,y^\alpha\,\widetilde{g}\right)\,. \end{aligned} \tag{3.4.17}$$

One can construct the deformation of the 2-form field in a straightforward manner but we shall not do this here.[17] Our construction guarantees that when we substitute the deformation (3.4.17) (and the corresponding deformation of the 2-form field) into the linearized equations of motion in the black hole background, all terms up to first derivative of $\epsilon(x^+)$ vanish. Requiring the coefficient of the $\partial_+^2\epsilon$ term to vanish gives us the equation:

$$\psi^{-1}\,\chi\,(\psi\,\chi^{-1})'(\widetilde{f}+r^2\,\widetilde{g})+3\,\psi^{-1}\,\chi^{-1}\,(\psi\,\chi)'(\widetilde{f}+r^2\,\widetilde{g})+2\,(\widetilde{f}'+4\,r\,\widetilde{g}+r^2\,\widetilde{g}')=0\,. \quad (3.4.18)$$

Using eq.(3.4.16) we can regard (3.4.18) as a second order linear differential equation for $\widetilde{f}$. Thus it has two independent solutions. It is easy to verify that the general solution to (3.4.16), (3.4.18) is

$$\widetilde{f}=(\widetilde{A}_0\,r^{-3}+\widetilde{B}_0)\,\psi^{-2}\,\chi^{-1}, \qquad \widetilde{g}=-3\,\widetilde{A}_0\,r^{-5}\,\psi^{-2}\,\chi^{-1}\,, \quad (3.4.19)$$

where $\widetilde{A}_0$ and $\widetilde{B}_0$ are two arbitrary constants. Requiring that the solution gives a normalizable deformation of the metric and the 2-form field near $r=0$ we get $\widetilde{A}_0=0$. Thus we have

$$\widetilde{f}=\widetilde{B}_0\,\psi^{-2}\,\chi^{-1}, \qquad \widetilde{g}=0\,. \quad (3.4.20)$$

It is easy to verify that the deformations of the metric and the 2-form field associated with this choice of $\widetilde{f}$ is normalizable both at $r=0$ and at $r=\infty$. Thus we have normalizable deformation of the solution parametrized by three indendent functions $b^\alpha\,\epsilon(x^+)$. This shows the existence of three left-moving modes on the black hole world-volume describing the left-moving transverse oscillation modes of the black hole. Furthermore the contribution to the norm of the deformation from the throat region $r << r_0, R_4^2$ vanishes, showing that these modes are located outside the horizon.

Finally we turn to the zero modes describing the motion of the D1-D5 system relative to the Taub-NUT space. We shall not carry out the construction in detail but describe these deformations in the limit $R_4^2 >> r_0$. To leading order in this limit, the deformations associated with these left-moving modes are in fact given by the ones described in (3.4.6). Indeed the arguments of §3.4.1 show that for $r << R_4^2$ when the Taub-NUT metric can be replaced by flat metric, the deformations given in (3.4.6) satisfy the linearized equations of motion. On the other hand since the function f in (3.4.6) approaches a constant for $r >> r_0$, the metric fluctuations fall off as $1/r^2$ and the contribution to the norm of the deformation from this region is small. Thus the deformation given in (3.4.6) is supported in the region $r\sim r_0$, and for $r\sim R_4^2 >> r_0$, where the deviation of the Taub-NUT metric from the flat metric becomes significant, the deformation is close to zero. Thus

[17]For this one needs to accompany the diffeomorphism (3.4.15) by an appropriate gauge transformation of the 2-form field such that every term in the deformation has an explicit factor of $(x^+ + c)$ without any derivative acting on it. We then replace $(x^+ + c)$ by $\epsilon(x^+)$.

we conclude that in the region where the deformation (3.4.6) is supported it remains an approximate solution to the equations of motion.[18]

Our analysis also allows us to determine the J quantum numbers of various deformations. Since in the region $r << R_4^2$ the parameters $\vec{a}$ labelling the deformation in (3.4.6) transform in the vector representation of the $SO(4)$ rotation group acting on the coordinates $\vec{w}$, they carry $J = \pm 1$. This may also be seen by noting that under a translation $x^4 \to x^4 + \beta$, these modes transform with a phase $e^{\pm i\beta/2}$. Since x^4 has period 4π, this shows that these modes carry ± 1 quantum of x^4 momentum. On the other hand the deformations describing the overall transverse motion of the black hole, described by the parameters b^α, are neutral under x^4 translation, and hence has $J = 0$. The different transformation properties of the modes labelled by $\vec{a}$ and $\vec{b}$ help demonstrate that they are distinct deformations of the solution.

[18]While this argument has been somewhat heuristic, we note that even in the microscopic counting the transverse oscillation modes of the D1-D5 system in Taub-NUT space was accounted for by assuming that for large R_4 we can regard the non-zero mode oscillations of the D1-D5 system as free oscillators[17].

Part III

Galilean Conformal Algebras

Chapter 4

Galilean Conformal Algebras

In this chapter, we review the bosonic non-relativistic conformal symmetry. In the next section, we first review the Schrödinger symmetry algebra in order to set the notation, and to contrast it with the Galilean Conformal Algebra obtained by a parametric contraction of the relativistic conformal group [73]. The GCA is studied in great detail in the following sections.

4.1 Schrödinger symmetry

The Schrödinger symmetry group in $(d+1)$ dimensional spacetime (which we will denote as $Sch(d,1)$) has been studied as a non-relativistic analogue of conformal symmetry. It's name arises from being the group of symmetries of the free Schrödinger wave operator in $(d+1)$ dimensions. In other words, it is generated by those transformations that commute with the operator $S = i\partial_t + \frac{1}{2m}\partial_i^2$. However, this symmetry is also believed to be realised in interacting systems, most recently in cold atoms at criticality.

The symmetry group contains the usual Galilean group (denoted as $G(d,1)$) with its central extension.

$$\begin{aligned}
[J_{ij}, J_{rs}] &= so(d) \\
[J_{ij}, B_r] &= -(B_i\delta_{jr} - B_j\delta_{ir}) \\
[J_{ij}, P_r] &= -(P_i\delta_{jr} - P_j\delta_{ir}), \quad [J_{ij}, H] = 0 \\
[B_i, B_j] &= 0, \quad [P_i, P_j] = 0, \quad [B_i, P_j] = m\delta_{ij} \\
[H, P_i] &= 0, \quad [H, B_i] = -P_i.
\end{aligned} \tag{4.1.1}$$

Here J_{ij} $(i, j = 1 \dots d)$ are the usual $SO(d)$ generators of spatial rotations. P_r are the d generators of spatial translations and B_j those of boosts in these directions. Finally H is the generator of time translations. The parameter m is the central extension and has the interpretation as the non-relativistic mass (which also appears in the Schrödinger operator S).

As vector fields on the Galilean spacetime $R^{d,1}$, they have the realisation (in the absence of the central term)

$$\begin{aligned} J_{ij} &= -(x_i\partial_j - x_j\partial_i) \qquad H = -\partial_t \\ P_i &= \partial_i \qquad B_i = t\partial_i \end{aligned} \tag{4.1.2}$$

In addition to these Galilean generators there are *two* more generators which we will denote by $\tilde{K}, \tilde{D}$. $\tilde{D}$ is a dilatation operator, which unlike the relativistic case, scales time and space differently. As a vector field $\tilde{D} = -(2t\partial_t + x_i\partial_i)$ so that

$$x_i \to \lambda x_i, \qquad t \to \lambda^2 t. \tag{4.1.3}$$

$\tilde{K}$ acts something like the time component of special conformal transformations. It has the form $\tilde{K} = -(tx_i\partial_i + t^2\partial_t)$ and generates the finite transformations (parametrised by μ)

$$x_i \to \frac{x_i}{(1+\mu t)}, \qquad t \to \frac{t}{(1+\mu t)}. \tag{4.1.4}$$

These two additional generators have non-zero commutators

$$\begin{aligned} [\tilde{K}, P_i] &= B_i, \quad [\tilde{K}, B_i] = 0, \quad [\tilde{D}, B_i] = -B_i \\ [\tilde{D}, \tilde{K}] &= -2\tilde{K}, \quad [\tilde{K}, H] = -\tilde{D}, \quad [\tilde{D}, H] = 2H. \end{aligned} \tag{4.1.5}$$

The generators $\tilde{K}, \tilde{D}$ are invariant under the spatial rotations J_{ij}. We also see from the last line that $H, \tilde{K}, \tilde{D}$ together form an $SL(2,R)$ algebra. The central extension term of the Galilean algebra is compatible with all the extra commutation relations.

Note that there is no analogue in the Schrödinger algebra of the spatial components K_i of special conformal transformations. Thus we have a smaller group compared to the relativistic conformal group. In $(3+1)$ dimensions, the Schrödinger algebra has twelve generators (ten being those of the Galilean algebra) and the additional central term. In contrast, the relativistic conformal group has fifteen generators. In the next subsection we will discuss how to get a nonrelativistic conformal group through group contraction. In the process of group contraction one does not lose any generators and hence the Galilean Conformal Algebra we find will have the same number of generators as the group $SO(d+1,2)$.

4.2 Contraction of the Relativistic Conformal Group

First let us study a simple example of the Inonu-Wigner Contraction: We know that the $SO(3)$ is the isometry group of S^2 and maps the surface of the sphere to itself. Now the equation for a sphere of radius R (embedded in $\mathcal{R}^3$) in cartesian coordinates is given by $x_1^2 + x_2^2 + x_3^2 = R^2$. The infinitesimal generators of rotation are given by:

$$J_{ij} = x_i\partial_j - x_j\partial_i\,, \tag{4.2.1}$$

with commutation relations

$$[J_{ij}, J_{rs}] = J_{is}\delta_{jr} + J_{jr}\delta_{is} - J_{ir}\delta_{js} - J_{js}\delta_{ir}\,. \tag{4.2.2}$$

Now let us take the limit $R \to \infty$, which physically means we are considering the limit of a two-dimensional plane ($\mathcal{R}^2$). Let us look at the north pole: $(x_1 = 0, x_2 = 0, x_3 = R)$. Next we redefine the generators as

$$\begin{aligned} Y_{12} &= \lim_{R\to\infty} X_{12} = x_1\partial_2 - x_2\partial_1\,, \\ P_i &= \lim_{R\to\infty} \frac{1}{R} X_{i3} = \lim_{R\to\infty} \frac{1}{R}(x_i\partial_3 - x_3\partial_i) \to -\partial_i\,, \end{aligned}$$

which now satisfy the algebra

$$[Y_{12}, P_i] = P_1\delta_{2i} - P_2\delta_{1i}, \quad [P_1, P_2] = 0\,. \tag{4.2.3}$$

This is the algebra of the $ISO(2)$ group. This is what we expected, because at North Pole, with $R \to \infty$, S^2 looks like $\mathcal{R}^2$. We will use this technique to investigate the non-relativistic limit of the conformal algebra.

We know that the Galilean algebra $G(d, 1)$ arises as a contraction of the Poincare algebra $ISO(d, 1)$. Physically this comes from taking the non-relativistic scaling

$$t \to \epsilon^r t \qquad x_i \to \epsilon^{r+1} x_i \tag{4.2.4}$$

with $\epsilon \to 0$.[1] This is equivalent to taking the velocities $v_i \sim \epsilon$ to zero (in units where $c = 1$). We will use (4.2.4) to scale and redefine the generators, as motivated in the example considered at the beginning of the section, so that the leading term scales as unity.

Starting with the expressions for the Poincare generators $(\mu, \nu = 0, 1 \ldots d)$

$$J_{\mu\nu} = -(x_\mu\partial_\nu - x_\nu\partial_\mu) \qquad P_\mu = \partial_\mu, \tag{4.2.5}$$

the above scaling gives us the Galilean vector field generators of (4.1.2)

$$\begin{aligned} J_{ij} &= -(x_i\partial_j - x_j\partial_i) \qquad P_0 = H = -\partial_t \\ P_i &= \partial_i \qquad J_{0i} = B_i = t\partial_i. \end{aligned} \tag{4.2.6}$$

They obey the commutation relations (without central extension) of (4.1.1). This should be contrasted with the Poincare commutators

$$[J_{ij}, J_{rs}] = so(d)\,,$$

[1] We have allowed for a certain freedom of scaling through the parameter r, since we might have other scales in the theory with respect to which we would have to take the above nonrelativistic limit. However, for the process of group contraction, the parameter r will play no role apart from modifying an over all factor which is unimportant. Hence we will simply take $r = 0$.

$$\begin{aligned}
[J_{ij}, B_r] &= -(B_i\delta_{jr} - B_j\delta_{ir})\,, \\
[J_{ij},\, P_r] &= -(P_i\delta_{jr} - P_j\delta_{ir})\,, \quad [J_{ij}, H] = 0 \\
[B_i, B_j] &= -J_{ij}, \quad [P_i, P_j] = 0\,, \quad [B_i, P_j] = \delta_{ij}H \\
[H, P_i] &= 0\,, \quad [H, B_i] = -P_i
\end{aligned} \tag{4.2.7}$$

To obtain the Galilean Conformal Algebra, we simply extend the scaling (4.2.4) to the rest of the generators of the conformal group $SO(d+1, 2)$. Namely to

$$D = -(x \cdot \partial) \qquad K_\mu = -(2x_\mu(x \cdot \partial) - (x \cdot x)\partial_\mu) \tag{4.2.8}$$

where D is the relativistic dilatation generator and K_μ are those of special conformal transformations. The non-relativistic scaling in (4.2.4) now gives (see also [75])

$$\begin{aligned}
D &= -(x_i\partial_i + t\partial_t) \\
K &= K_0 = -(2tx_i\partial_i + t^2\partial_t) \\
K_i &= t^2\partial_i.
\end{aligned} \tag{4.2.9}$$

Note that the dilatation generator $D = -(x_i\partial_i + t\partial_t)$ is the *same* as in the relativistic theory. It scales space and time in the same way $x_i \to \lambda x_i, t \to \lambda t$. Therefore it is different from the dilatation generator $\tilde{D} = -(2t\partial_t + x_i\partial_i)$ of the Schrödinger group. Similarly, the temporal special conformal generator K in (4.2.9) is different from $\tilde{K} = -(tx_i\partial_i + t^2\partial_t)$. Finally, we now have spatial special conformal transformations K_i which were not present in the Schrödinger algebra. Thus the generators of the Galilean Conformal Algebra are $(J_{ij}, P_i, H, B_i, D, K, K_i)$.

Since the usual Galilean algebra $G(d, 1)$ for the generators (J_{ij}, P_i, H, B_i) is a sub-algebra of the GCA, we will not write down their commutators. The other non-trivial commutators of the GCA are [75]

$$\begin{aligned}
[K, K_i] &= 0, \quad [K, B_i] = K_i, \quad [K, P_i] = 2B_i \\
[J_{ij}, K_r] &= -(K_i\delta_{jr} - K_j\delta_{ir}), \quad [J_{ij}, K] = 0, \quad [J_{ij}, D] = 0 \\
[K_i, K_j] &= 0, \quad [K_i, B_j] = 0, \quad [K_i, P_j] = 0, \quad [H, K_i] = -2B_i, \\
[D, K_i] &= -K_i, \quad [D, B_i] = 0, \quad [D, P_i] = P_i, \\
[D, H] &= H, \quad [H, K] = -2D, \quad [D, K] = -K.
\end{aligned} \tag{4.2.10}$$

This can again be contrasted with commutators of the corresponding relativistic generators

$$\begin{aligned}
[K, K_i] &= 0, \quad [K, B_i] = K_i, \quad [K, P_i] = 2B_i \\
[J_{ij}, K_r] &= -(K_i\delta_{jr} - K_j\delta_{ir}), \quad [J_{ij}, K] = 0, \quad [J_{ij}, D] = 0 \\
[K_i, K_j] &= 0, \quad [K_i, B_j] = \delta_{ij}K, \quad [K_i, P_j] = 2J_{ij} + 2\delta_{ij}D \\
[H, K_i] &= -2B_i, \quad [D, K_i] = -K_i, \quad [D, B_i] = 0, \quad [D, P_i] = P_i, \\
[D, H] &= H, \quad [H, K] = -2D, \quad [D, K] = -K.
\end{aligned} \tag{4.2.11}$$

We can also compare the relevant commutators in (4.2.10) with those of (4.1.5) and we notice that they too are different. Thus the Schrödinger algebra and the GCA only share a common Galilean subgroup and are otherwise different. In fact, one can verify using the Jacobi identities for (D, B_i, P_j) that the Galilean central extension in $[B_i, P_j]$ is *not* admissible in the GCA. This is another difference from the Schrödinger algebra, which as mentioned above, does allow for the central extension. Thus in some sense, the GCA is the symmetry of a "massless" (or gapless) nonrelativistic system. We will discuss some possible realisations in the next section. It should be pointed out that the GCA does admit a *different* central extension of the form

$$[K_i, P_j] = N\delta_{ij} \tag{4.2.12}$$

where N commutes with all the other generators of the GCA. The exact interpretation of this term in general is not clear. It will, in fact, turn out to be absent when one considers the infinite dimensional extension of the GCA in the next section.

4.3 The Infinite Dimensional Extended GCA

The most interesting feature of the GCA is that it admits a very natural extension to an infinite dimensional algebra of the Virasoro-Kac-Moody type[2]. To see this we denote

$$\begin{aligned} L^{(-1)} &= H, \qquad L^{(0)} = D, \qquad L^{(+1)} = K, \\ M_i^{(-1)} &= P_i, \qquad M_i^{(0)} = B_i, \qquad M_i^{(+1)} = K_i. \end{aligned} \tag{4.3.1}$$

The finite dimensional GCA which we had in the previous section can now be recast as

$$\begin{aligned} [J_{ij}, L^{(n)}] &= 0, \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)} \\ [J_{ij}, M_k^{(m)}] &= -(M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik}), \qquad [M_i^{(m)}, M_j^{(n)}] = 0, \\ [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)}. \end{aligned} \tag{4.3.2}$$

The indices $m, n = 0, \pm 1$ We have made manifest the $SL(2, R)$ subalgebra with the generators $L^{(0)}, L^{(\pm 1)}$. In fact, we can define the vector fields

$$\begin{aligned} L^{(n)} &= -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t \\ M_i^{(n)} &= t^{n+1}\partial_i \end{aligned} \tag{4.3.3}$$

with $n = 0, \pm 1$. These (together with J_{ij}) are then exactly the vector fields in (4.1.2) and (4.2.9) which generate the GCA (without central extension).

[2]A similar Virasoro extension of the Schrödinger group can be found in ref. [70]. The actual algebra is different from the one described here.

If we now consider the vector fields of (4.3.3) for *arbitrary* integer n, and also define

$$J_a^{(n)} \equiv J_{ij}^{(n)} = -t^n(x_i\partial_j - x_j\partial_i) \tag{4.3.4}$$

then we find that this collection obeys the current algebra

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} \qquad [L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)} \\ [J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \end{aligned} \tag{4.3.5}$$

The index a labels the generators of the spatial rotation group $SO(d)$ and f_{abc} are the corresponding structure constants. We see that the vector fields generate a $SO(d)$ Kac-Moody algebra without any central terms. In addition to the Virasoro and current generators we also have the commuting generators $M_i^{(n)}$ which function like generators of a global symmetry. We can, for instance, consistently set these generators to zero. The presence of these generators therefore do not spoil the ability of the Virasoro-Kac-Moody generators to admit the usual central terms in their commutators.

What is the meaning of this infinite dimensional extension? Do these additional vector fields generate symmetries?

There is a relatively simple interpretation for the generators $M_i^{(n)}, L^{(n)}, J_a^{(n)}$. We know that $P_i = M_i^{(-1)}, B_i = M_i^{(0)}, K_i = M_i^{(1)}$ generate uniform spatial translations, velocity boosts and accelerations respectively. In fact, it is simple to see from (4.3.3) that the $M_i^{(n)}$ generate arbitrary time dependent (but spatially independent) accelerations.

$$x_i \to x_i + b_i(t). \tag{4.3.6}$$

Similarly the $J_{ij}^{(n)}$ in (4.3.4) generate arbitrary time dependent rotations (once again space independent)

$$x_i \to R_{ij}(t)x_j \tag{4.3.7}$$

These two set of generators together generate what is sometimes called the Coriolis group: the biggest group of "isometries" of "flat" Galilean spacetime [77].

Recall that in the absence of gravity Galilean spacetime is characterised by a degenerate metric. The time intervals are much larger than any space-like intervals in the nonrelativistic scaling limit (4.2.4). We thus have an absolute time t and spatial sections with a flat Euclidean metric. We can, in a precise sense, describe the analogue of the isometries in this Galilean spacetime. The Coriolis group by virtue of preserving the spatial slices (at any given time) are the maximal set of isometries. This realisation of the current algebra in our context is a bit like the occurence of a loop group.

The generators $L^{(n)}$ have a more interesting action in acting both on time as well as space. We can read this off from (4.3.3)

$$t \to f(t), \qquad x_i \to \frac{df}{dt}x_i. \tag{4.3.8}$$

Thus it amounts to a reparametrisation of the absolute time t. Under this reparametrisation the spatial coordinates x_i act as vectors (on the worldline t). It seems as if this is some kind of "conformal isometry" of the Galilean spacetime, rescaling coordinates by the arbitrary time dependent factor $\frac{df}{dt}$.

With this interpretation of the infinite extension of the GCA, one might expect that it ought to be partially or fully dynamically realised in physical systems where the finite GCA is (partially or fully) realised. We will see below an example which lends support to this idea. We will also see in Sec. 5 that the bulk geometry which we propose as the dual has the extended GCA among its *asymptotic* isometries. An analogy might be two dimensional conformal invariance where the Virasoro algebra is often a symmetry when the finite conformal symmetry of $SL(2,C)$ is realised. And the (two copies of the) Virasoro generators are reflected in the bulk AdS_3 as asymptotic isometries.

Given that the Galilean limit can be obtained by taking a definite scaling limit within a relativistic theory, we expect to see the GCA (and perhaps its extension) as a symmetry of some subsector within every relativistic conformal field theory. For instance, in the best studied case of $\mathcal{N} = 4$ Yang-Mills theory, we ought to be able to isolate a sector with this symmetry. One clue is the presence of the $SL(2,R)$ symmetry together with the preservation of spatial rotational invariance. One might naively think this should be via some kind of conformal quantum mechanics obtained by considering only the spatially independent modes of the field theory. But this is probably not totally correct for the indirect reasons explained in the next paragraph.

Recently, the nonrelativistic limit of the relativistic conformal hydrodynamics, which describes the small fluctuations from thermal equilibrium, have been studied [125, 126, 127]. One recovers the non-relativistic incompressible Navier-Stokes equation in this limit. The symmetries of this equation were then studied by [126] (see also [127]). One finds that all the generators of the finite GCA are indeed symmetries[3] except for the dilatation operator D [4]. In particular it has the K_i as symmetries. It is not surprising that the choice of a temperature should break the scaling symmetry of D [5]. The interesting point is that the arbitrary accelerations $M_i^{(n)}$ are also actually a symmetry [128] (generating what is sometimes called the Milne group [77]). Thus we have a part of the extended GCA as a symmetry of the non-relativistic Navier-stokes equation which should presumably describe the hydrodynamics in every nonrelativistic field theory. In particular, the closed non-relativistic subsector within every relativistic conformal field theory should have a hydrodynamic description governed by the Navier-Stokes equation. This might seem to suggest that this sector ought to have more than just the degrees of freedom of a conformal quantum mechanics.

[3]For a realisation of the Schrödinger symmetry in the context of the Navier-Stokes equation see [129, 130].

[4]The generator K acts trivially.

[5]However, one can define an action of the $\tilde{D}$ as in (4.1.3) to be a symmetry.

Coming back to the Navier-Stokes equation, if the viscosity is set to zero, one gets the incompressible Euler equations

$$\partial_t v_i(x,t) + v_j \partial_j v_i(x,t) = -\partial_i p(x,t) \tag{4.3.9}$$

In this case one has the entire finite dimensional GCA being a symmetry since D is now also a symmetry. It is the viscous term which breaks the symmetry under equal scaling of space and time. This shows that one can readily realise "gapless" non-relativistic systems in which space and time scale in the same way! [6]

[6] Inonu and Wigner [131] have considered representations of the Galilean group without the mass extension and concluded that a particle interpretation of states of the irreducible representations is subtle. In particular such states are not localisable. Just as in the case of relativistic conformal group it is likely that observables such as the S-matrix are ill-defined.

Chapter 5

SGCA in $4d$

In this chapter, we discuss a systematic construction of the $\mathcal{N} = 1$ supersymmetric extension of the GCA in $(3+1)d$, which we will refer to as SGCA. To that end, we would first look to perform a contraction on the simplest $\mathcal{N} = 1$ case. As in the bosonic case, we would look to implement the contraction at the level of the co-ordinates. We write down the superspace representations of the relativistic algebra and perform the contraction on the ordinary as well as the grassmann co-ordinates. In §5.4, we also lift the SGCA to an infinite dimensional algebra. Lastly, in §5.5, we comment on the generalisation to higher $\mathcal{N}$.

5.1 Non-relativistic contraction in superspace

We have seen that for the bosonic case, the non relativistic limit arises from (4.2.4). Now, we also need to take into account the grassmann co-ordinates θ. We know that square of θ acts like an ordinary co-ordinate. Hence we expect the scaling of θ to go like $\sqrt{\epsilon}$. The various linear combinations of the components of θ that scale differently should scale like $\sqrt{\epsilon}$ and $\frac{1}{\sqrt{\epsilon}}$ respectively. Naive choices of the components would lead to incorrect answers like the vanishing of $\{Q, \overline{Q}\}$. So, one needs to be careful while choosing the appropriate linear combinations of the components of the grassmann variable which would scale in the way mentioned above.

Along with (4.2.4), we choose to scale

$$\theta_+ \to \frac{1}{\sqrt{\epsilon}}\theta_+\,, \quad \theta_- \to \sqrt{\epsilon}\theta_-\,, \tag{5.1.1}$$

where $\theta_\pm$ are projections defined below:

$$\theta_\pm = \frac{1}{2}(1 \pm \gamma^0)\theta = \frac{1}{2}\begin{pmatrix} \mathbf{1}_{2\times 2} & \pm\,\sigma^0 \\ \pm\sigma^0 & \mathbf{1}_{2\times 2} \end{pmatrix}\begin{pmatrix} \theta_\alpha \\ \overline{\theta}^{\dot{\alpha}} \end{pmatrix}\,. \tag{5.1.2}$$

We list the conventions used for the spinor algebra in §5.7.1. To make the process of contraction explicit, let us define variables with the different scaling behaviours as follows:

$$\begin{aligned} \psi_1 &= \frac{1}{2}(\theta_1 - \bar{\theta}_{\dot{2}}), \quad \psi_2 = \frac{1}{2}(\theta_2 + \bar{\theta}_{\dot{1}}) \quad \Rightarrow \psi_{1,2} \to \frac{1}{\sqrt{\epsilon}}\psi_{1,2} \\ \chi_1 &= \frac{1}{2}(\theta_1 + \bar{\theta}_{\dot{2}}), \quad \chi_2 = \frac{1}{2}(\theta_2 - \bar{\theta}_{\dot{1}}) \quad \Rightarrow \chi_{1,2} \to \sqrt{\epsilon}\chi_{1,2} \end{aligned} \tag{5.1.3}$$

Let us briefly comment on the choice of this particular scaling. From the bosonic case, we know that the space part should scale in a way which is different from the time part in the non-relativistic limit. We would have a rotational symmetry present in the spatial part of the non-relativistic quantities of interest. Incorporating this feature in the supersymmetric case, it is necessary to look at spinors of $SO(3)$. The spinors we have described above are indeed spinors of $SO(3)$. Some more details can be found in §5.7.1. There is another hint at what we want if we look at the non-relativistic limit of the Dirac equation. This is precisely the projection that gets rid of the negative energy states and projects onto the Pauli equation in the limit where we take the speed of light to infinity.

5.2 Fermionic generators

We would implement the above described scaling on the fermionic generators[1]. We would look at particular combinations of the generators and perform the contraction in a way similar to the bosonic case.

We look at the supersymmetry generators first. They are

$$Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} + \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\,, \quad \bar{Q}_{\dot{\alpha}} = -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\,. \tag{5.2.1}$$

where $\alpha, \dot{\alpha} = 1, 2$.

More explicitly:

$$Q_1 = i\frac{\partial}{\partial\theta_2} - \bar{\theta}_{\dot{2}}\partial_t + \bar{\theta}_{\dot{2}}\partial_3 - \bar{\theta}_{\dot{1}}\partial_1 + i\bar{\theta}_{\dot{1}}\partial_2\,, \quad Q_2 = -i\frac{\partial}{\partial\theta_1} + \bar{\theta}_{\dot{1}}\partial_t + \bar{\theta}_{\dot{1}}\partial_3 + \bar{\theta}_{\dot{2}}\partial_1 + i\bar{\theta}_{\dot{2}}\partial_2\,,$$
$$\bar{Q}_{\dot{2}} = i\frac{\partial}{\partial\bar{\theta}_{\dot{1}}} - \theta_1\partial_t - \theta_1\partial_3 - \theta_2\partial_1 + i\theta_2\partial_2\,, \quad \bar{Q}_{\dot{1}} = -i\frac{\partial}{\partial\bar{\theta}_{\dot{2}}} + \theta_2\partial_t - \theta_2\partial_3 - \theta_1\partial_1 + i\theta_1\partial_2\,.$$

In order to perform the contraction, we would need to take linear combinations of the above equations as shown below :

$$\begin{aligned} \tilde{\mathcal{Q}}^+_1 &= Q_1 - \bar{Q}_{\dot{2}}\,, \quad \tilde{\mathcal{Q}}^+_2 = Q_2 + \bar{Q}_{\dot{1}}\,, \\ \tilde{\mathcal{Q}}^-_1 &= Q_1 + \bar{Q}_{\dot{2}}\,, \quad \tilde{\mathcal{Q}}^-_2 = Q_2 - \bar{Q}_{\dot{1}}\,. \end{aligned} \tag{5.2.2}$$

[1]Here we would be keeping track of factors of $\pm i$ which we had not taken into account in the purely bosonic subalgebra.

Expressing them in the variables ψ_a and χ_a defined before, we find

$$\begin{aligned}\tilde{\mathcal{Q}}_a^+ &= i\epsilon^{ab}\partial_{\chi_b} + 2\psi_a\partial_t + 2\chi_b\sigma^j_{ab}\partial_j\,,\\ \tilde{\mathcal{Q}}_a^- &= i\epsilon^{ab}\partial_{\psi_b} - 2\chi_a\partial_t - 2\psi_b\sigma^j_{ab}\partial_j\,,\end{aligned}$$

where $a = 1, 2$.

We now perform the contraction by scaling t, x^i, ψ_a and χ_a and choosing redefined generators in the way below :

$$\begin{aligned}\mathcal{Q}_a^+ &= \lim_{\epsilon\to 0}\epsilon^{1/2}\tilde{\mathcal{Q}}_a^+ = i\epsilon^{ab}\partial_{\chi_b} + 2\psi_a\partial_t + 2\chi_b\sigma^j_{ab}\partial_j\,,\\ \mathcal{Q}_a^- &= \lim_{\epsilon\to 0}\epsilon^{3/2}\tilde{\mathcal{Q}}_a^- = -2\psi_b\sigma^j_{ab}\partial_j\,.\end{aligned} \tag{5.2.3}$$

The anticommutators of the algebra involving $\mathcal{Q}^-_{1,2}$ are all zero. The non-zero anticommutators are given by

$$\{\mathcal{Q}_1^+,\mathcal{Q}_2^+\} = -4i\partial_3,\quad \{\mathcal{Q}_1^+,\mathcal{Q}_1^+\} = 4i(\partial_1 - i\partial_2),\quad \{\mathcal{Q}_2^+,\mathcal{Q}_2^+\} = -4i(\partial_1 + i\partial_2). \tag{5.2.4}$$

Now we turn our attention to the other fermionic generators of the relativistic superconformal group,- the super-conformal transformations S.

The S-supersymmetry generators are

$$\begin{aligned}S_\alpha &= -i\epsilon^{\dot\beta\dot\gamma}(\sigma_\mu)_{\alpha\dot\gamma}x^\mu_{(+)}\theta^\beta\sigma^\nu_{\beta\dot\beta}\partial_\nu + 2i(\theta\theta)\partial_\alpha + \epsilon^{\dot\beta\dot\gamma}(\sigma_\mu)_{\alpha\dot\gamma}x^\mu_{(-)}\overline{\partial}_{\dot\beta}\,,\\ \overline{S}_{\dot\alpha} &= -i\epsilon^{\beta\gamma}(\sigma_\mu)_{\gamma\dot\alpha}x^\mu_{(-)}\overline{\theta}^{\dot\beta}\sigma^\nu_{\beta\dot\beta}\partial_\nu - 2i(\overline{\theta\theta})\overline{\partial}_{\dot\alpha} + \epsilon^{\beta\gamma}(\sigma_\mu)_{\gamma\dot\alpha}x^\mu_{(+)}\partial_\beta\,,\end{aligned}$$

where we have defined $x^\mu_{(\pm)} = x^\mu \pm i\theta\sigma^\mu\overline{\theta}$.

As in the case of the Q-generators, we take linear combinations of the above equations as follows:

$$\begin{aligned}\tilde{S}_1^+ &= S_1 - \overline{S}_{\dot 2}\,,\quad \tilde{S}_2^+ = S_2 + \overline{S}_{\dot 1}\,,\\ \tilde{S}_1^- &= S_1 + \overline{S}_{\dot 2}\,,\quad \tilde{S}_2^- = S_2 - \overline{S}_{\dot 1}\,.\end{aligned} \tag{5.2.5}$$

We now perform the contraction by scaling t, x^i, ψ_a and χ_a and choosing redefined generators in the way below :

$$\begin{aligned}\mathcal{S}_a^+ = \lim_{\epsilon\to 0}\epsilon^{1/2}\tilde{S}_a^+ &= (2it\chi_b - 8\chi_1\chi_2\psi_b)\sigma^j_{ab}\partial_j + 2i\sigma^j_{ab}\sigma^k_{bc}\psi_c x^j\partial_k + \psi_a(2it - 8\epsilon^{bc}\psi_b\chi_c)\partial_t\\ &\quad -2i\psi_a\psi_b\partial_{\psi_b} + 4i\psi_a\chi_b\partial_{\chi_b} - 6i\epsilon^{ab}\epsilon^{cd}\psi_c\chi_d\partial_{\chi_b} - t\epsilon^{ab}\partial_{\chi_b}\,,\\ \mathcal{S}_a^- = \lim_{\epsilon\to 0}\epsilon^{3/2}\tilde{S}_a^- &= 2it\psi_b\sigma^j_{ab}\partial_j + 8\psi_1\psi_2\chi_b\sigma^j_{ab}\partial_j - 6i\psi_1\psi_2\epsilon^{ab}\partial_{\chi_b}\,,\end{aligned} \tag{5.2.6}$$

where $a = 1, 2$.

Again, the only non-zero anticommutators involve the $\mathcal{S}_a^+$ generators as shown below:

$$\{\mathcal{S}_1^+, \mathcal{S}_2^+\} = 4\mathcal{K}_3, \quad \{\mathcal{S}_1^+, \mathcal{S}_1^+\} = -4(\mathcal{K}_1 - i\mathcal{K}_2), \quad \{\mathcal{S}_2^+, \mathcal{S}_2^+\} = 4(\mathcal{K}_1 + i\mathcal{K}_2). \tag{5.2.7}$$

We should mention that the bosonic generators now also have fermionic pieces. The details can be found in §5.7.2. The algebra of the bosonic generators, as expected, remains the same with these additional pieces. Along with all the usual bosonic generators, there are also the extra R-symmetry generators which rotate the fermionic generators. For the $\mathcal{N} = 1$ case at hand, this is just a single generator, representing the $U(1)$ R-symmetry. (Again, more details are provided in §5.7.2.)

5.3 Algebra

We list the algebra here but omitting the purely bosonic subalgebra. One should also note that the commutator of $\mathcal{A}$ with any bosonic generator is zero.

The non-zero anticommutators of the fermionic generators are given by

$$\begin{aligned}
\{\mathcal{Q}_1^+, \mathcal{Q}_1^+\} &= -4(P_1 - iP_2),\ \{\mathcal{Q}_2^+, \mathcal{Q}_2^+\} = 4(P_1 + iP_2),\ \{\mathcal{Q}_1^+, \mathcal{Q}_2^+\} = 4P_3, \\
\{\mathcal{S}_1^+, \mathcal{S}_1^+\} &= -4(\mathcal{K}_1 - i\mathcal{K}_2),\ \{\mathcal{S}_2^+, \mathcal{S}_2^+\} = 4(\mathcal{K}_1 + i\mathcal{K}_2),\ \{\mathcal{S}_1^+, \mathcal{S}_2^+\} = 4\mathcal{K}_3, \\
\{\mathcal{S}_1^+, \mathcal{Q}_1^+\} &= 4i(B_1 - iB_2),\ \{\mathcal{S}_1^+, \mathcal{Q}_2^+\} = -4iB_3 - 12\mathcal{A}, \\
\{\mathcal{S}_2^+, \mathcal{Q}_1^+\} &= -4iB_3 + 12\mathcal{A},\ \{\mathcal{S}_2^+, \mathcal{Q}_2^+\} = -4i(B_1 + iB_2).
\end{aligned} \tag{5.3.1}$$

The commutators of $\mathcal{Q}_a^\pm, \mathcal{S}_a^\pm$ with $P_i, \mathcal{K}_i, B_i$ are as follows:

$$\begin{aligned}
[P_i, \mathcal{Q}_a^\pm] &= 0,\ \ [P_i, \mathcal{S}_a^+] = -\sigma^i_{ab}\mathcal{Q}_b^-,\ \ [P_i, \mathcal{S}_a^-] = 0, \\
[\mathcal{K}_i, \mathcal{Q}_a^+] &= -\sigma^i_{ab}\mathcal{S}_b^-,\ \ [\mathcal{K}_i, \mathcal{Q}_a^-] = 0,\ \ [\mathcal{K}_i, \mathcal{S}_a^\pm] = 0, \\
[B_i, \mathcal{Q}_a^+] &= \frac{i}{2}\sigma^i_{ab}\mathcal{Q}_b^-,\ \ [B_i, \mathcal{Q}_a^-] = 0, \\
[B_i, \mathcal{S}_a^+] &= \frac{i}{2}\sigma^i_{ab}\mathcal{S}_b^-,\ \ [B_i, \mathcal{S}_a^-] = 0.
\end{aligned} \tag{5.3.2}$$

The commutators of $\mathcal{Q}_a^\pm, \mathcal{S}_a^\pm$ with the angular momentum generators J_i are given by

$$[J_i, \mathcal{Q}_a^\pm] = -\frac{1}{2}\sigma^i_{ab}\mathcal{Q}_b^\pm, \quad [J_i, \mathcal{S}_a^\pm] = -\frac{1}{2}\sigma^i_{ab}\mathcal{S}_b^\pm, \tag{5.3.3}$$

which tell us that the fermionic generators transform as spinors of $SO(3)$, the three-dimensional rotation group.

Finally, the commutators of $\mathcal{Q}_a^\pm, \mathcal{S}_a^\pm$ with $H, \mathcal{K}, D, \mathcal{A}$ are as follows:

$$\begin{aligned}
[H, \mathcal{Q}_a^\pm] &= 0,\ \ [H, \mathcal{S}_a^+] = \mathcal{Q}_a^+,\ \ [H, \mathcal{S}_a^-] = -\mathcal{Q}_a^-, \\
[\mathcal{K}, \mathcal{Q}_a^+] &= -\mathcal{S}_a^+,\ \ [\mathcal{K}, \mathcal{Q}_a^-] = \mathcal{S}_a^-,\ \ [\mathcal{K}, \mathcal{S}_a^\pm] = 0,
\end{aligned}$$

$$\begin{aligned}
[D, \mathcal{Q}_a^{\pm}] &= -\frac{i}{2}\mathcal{Q}_a^{\pm}, \;\; [D, \mathcal{S}_a^{\pm}] = \frac{i}{2}\mathcal{S}_a^{\pm}, \\
[\mathcal{A}, \mathcal{Q}_a^{+}] &= -\frac{1}{2}\mathcal{Q}_a^{-}, \;\; [\mathcal{A}, \mathcal{Q}_a^{-}] = 0, \\
[\mathcal{A}, \mathcal{S}_a^{+}] &= \frac{1}{2}\mathcal{S}_a^{-}, \;\; [\mathcal{A}, \mathcal{S}_a^{-}] = 0.
\end{aligned} \tag{5.3.4}$$

5.4 Infinitely extended SGCA

We have seen in §4.3 that the bosonic GCA admitted an infinite dimensional extension. This was one of the very interesting aspects of the GCA. As we have systematically employed an analogous non-relativistic limit on the supersymmetric version of the bosonic conformal algebra, we would expect that similar infinite dimensional extensions are valid even in this case. To that end, we would now re-write the finite dimensional contracted algebra in a suggestive form.
We define:

$$G_{-1/2} = \begin{pmatrix} G^{+a}_{-1/2} \\ G^{-a}_{-1/2} \end{pmatrix} = \begin{pmatrix} i\mathcal{Q}_a^{+} \\ -i\mathcal{Q}_a^{-} \end{pmatrix}, \quad G_{1/2} = \begin{pmatrix} G^{+a}_{1/2} \\ G^{-a}_{1/2} \end{pmatrix} = \begin{pmatrix} \mathcal{S}_a^{+} \\ \mathcal{S}_a^{-} \end{pmatrix}. \tag{5.4.1}$$

Remembering the definitions of L_n, M_n^i, J_{ij} from §4.2, we can re-write the finite dimensional superconformal algebra in the following way: [2]

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}, \;\; [M_n^i, M_m^j] = 0, \;\; [L_m, M_n^i] = (m-n)M_{m+n}^i, \\
[L_n, G_r^{\pm a}] &= (\frac{n}{2} - r)G_{n+r}^{\pm a}, \;\; [M_n^i, G_r^{+a}] = (r - \frac{n}{2})\sigma^i_{ab}G^{-b}_{n+r}, \;\; [M_n^i, G_r^{-a}] = 0, \\
\{G_r^{+a}, G_s^{+b}\} &= 4i(\sigma^i\epsilon)_{ab}M^i_{r+s} - 12if_1(r,s)\epsilon^{ab}\mathcal{A}_{r+s}, \;\; \{G_r^{-a}, G_s^{\pm b}\} = 0,
\end{aligned} \tag{5.4.2}$$

for $n = 0, \pm 1$ and $r = \pm\frac{1}{2}$.

At the level of the algebra, we can continue (5.4.2) for all integral values of n, m and all half integral values of r, s. The commutators of the supercharges of the finite algebra with the L_n, M_n^i for arbitrary n generate the higher supercharges. We see that to fit with the Jacobi identity we need to promote $\mathcal{A}$ to have a Virasoro index. Let us try and derive a consistent infinite lift for $\mathcal{A}$ so that the above algebra closes. The first thing to note is that the symmetry of the total fermionic anticommutator and the antisymmetry of ϵ^{ab} forces the function $f_1(r,s)$ to be antisymmetric in r, s. Let us take the simplest function and try and build a consistent infinite dimensional extension of the contracted algebra, which is

$$f_1(r,s) = r - s\,. \tag{5.4.3}$$

[2]Here we return to a convention devoid of i for the purely bosonic subalgebra to compare with the results in the infinite bosonic algebra. More explicitly, the H, B_i, K, K_i, J_{ij} used here can be obtained from those defined in §5.7.2 and used in §5.3, by multiplying with $(-i)$, whereas the D, P_i used here can be obtained by multiplying with i.

To generate the rest of the $\mathcal{A}_n$ algebra, let us look at the Jacobi identity involving $G_r^{+a}, G_s^{+b}, \mathcal{A}_n$:

$$\begin{aligned} & [\{G_r^{+a}, G_s^{+b}\}, \mathcal{A}_n] - \{[G_s^{+b}, \mathcal{A}_n], G_r^{+a}\} + \{[\mathcal{A}_n, G_r^{+a}], G_s^{+b}\} = 0\,, \\ \Rightarrow\ & 4i(\sigma^i\epsilon)_{ab}[M^i_{r+s}, \mathcal{A}] - 12i\epsilon^{ab}(r-s)[\mathcal{A}_{r+s}, \mathcal{A}_n] = 0\,, \end{aligned} \tag{5.4.4}$$

where we have used $[\mathcal{A}_n, G_r^+] = f_2(n,r)G^-_{n+r}$.

We find that

$$[M^i_m, \mathcal{A}_n] = 0, \quad [\mathcal{A}_m, \mathcal{A}_n] = 0 \tag{5.4.5}$$

is a consistent choice. The first commutator can be further motivated by the fact that the vector index on the RHS must come from $M^i_{n'}$ and not $\sigma^i_{ab}\mathcal{A}_{n'}$ as $\mathcal{A}_{n'}$ does not have any a or b index to contract with the indices coming from σ^i_{ab}. Then the RHS would at most have factors of M^i_{m+n}. Any natural definition of $[\mathcal{A}_m, \mathcal{A}_n]$ would not generate $M^i_{n'}$ on the RHS. And hence, to satisfy (5.4.4), both must be given by (5.4.5).

We can now look at the Jacobi identity of $L_m, L_n, \mathcal{A}_{m'}$ to arrive at a choice for $[L_n, \mathcal{A}_m]$ which fits with the finite algebra. The simplest choice is

$$[L_n, \mathcal{A}_m] = -m\mathcal{A}_{m+n}\,. \tag{5.4.6}$$

The rest of the algebra of $\mathcal{A}_n$, and indeed the full contracted finite case, also follows from various Jacobi identities. We list a simple consistent choice for the infinite lift of the rest of the algebra:

$$[\mathcal{A}_n, G_r^{+a}] = \frac{1}{2}G^{-a}_{n+r}, \quad [\mathcal{A}_n, G_r^{-a}] = 0, \quad [J^i, G_r^{\pm a}] = \frac{i}{2}\sigma^i_{ab}G_r^{\pm b}, \quad [J^i, \mathcal{A}_n] = 0. \tag{5.4.7}$$

Together with (5.4.2), (5.4.5), and (5.4.6), (5.4.7) constitutes the infinite dimensional extension of the non-relativistic superconformal algebra[3].

The interesting thing to note here is that the supercharge anticommutators generate the M^i_n and not the L_n as would be the usual expectation from the bosonic algebra. The reason for this counter-intuitive behaviour can be traced back to the fact that we chose to scale the fermionic generators in a way which meant that the $SL(2,R)$ part always dropped out of the fermionic anti-commutators. The above algebra looks structurally like the usual superconformal algebra in 2 dimensions. This is an infinitely extended $\mathcal{N}=1$ Super Galilean conformal algebra.

There might be a possible way to extract the usual supeconformal algebra in 2 dimensions by choosing to contract the fermionic generators in a different way. One has to make sure that the $SL(2,R)$ part is the one that remains after scaling and not the

[3]Here we cannot give the rotation generators J^i's an infinite lift as discussed in §5.7.3. The OPE analysis of §5.7.3 also suggests that this simple choice is the unique infinite dimensional extension consistent with the finite part for this particular contraction.

part with the vector indices. This would be a different non-relativistic limit of the parent super-conformal algebra and hence an inequivalent Super-Galilean conformal algebra. It might be interesting to explore this in more detail. The classification of the possible supersymmetric GCAs is also something worth pondering about.

5.5 Generalization to higher $\mathcal{N}$

The generalization to extended SUSY is immediate. For $\mathcal{N} > 1$ superconformal algebras, the difference with the $\mathcal{N} = 1$ case is that the number of fermionic generators increases. (With each of the Q and S, a label 'p' will now be attached.) This, in turn, implies that the R-symmetry will be enhanced for these algebras. We can use constructions very similar to what we have described above to arrive at the non-relativistic extended superconformal algebras. The linear combinations used earlier would just need to be augmented by the extra internal index 'p' and the R-symmetry generator $\mathcal{A}$ gets promoted to $\mathcal{A}_{pq}$ (for example, $SO(6)$ symmetry for $\mathcal{N} = 4$).

There is a point to note here. Unlike in the case of extended supersymmetry algebras, extended superconformal algebras don't allow for central extensions in the fermionic generators. This is clear if one looks at the Jacobi identities. For example, if we want to put a central term in the $\{\overline{Q}, \overline{Q}\}$ anticommutator, then we can check using the Jacobi identity of $\{P, S, \overline{Q}\}$ that this cannot be consistent. So, in the extended superconformal algebras, we would not have to worry about additional central terms coming from the relativistic algebra when we are looking to perform the non-relativistic contraction. The process of contraction and the contracted algebra are just the same as in the $\mathcal{N} = 1$ with extra indices fitted wherever required.

The infinite dimensional lift can also be implemented along the lines mentioned in §5.4. Except for extra indices, the rest of the construction remains the same.

5.6 Summary

In this chapter, we have systematically derived a non-relativistic limit for superconformal algebras. We used a simple representation of the $\mathcal{N} = 1$ superconformal algebra and took a limit on that by demanding that the linear combinations of the spinors we look at would be spinors of $SO(3)$. We also found a way to embed this finite dimensional algebra in an infinitely extended algebra, along the lines similar to the bosonic construction. The suprise there was that instead of the appearance of the expected infinite $\mathcal{N} = 1$ superconformal algebra in $d = 2$, we found a close cousin of that algebra. The extension to extended superconformal algebras was immediate. These are supersymmetric versions of the bosonic Galilean Conformal Algebra studied in [73, 79] and reviewed in §4. It is interesting that one has a simple framework in which all types of superconformal algebras can be handled in the non-relativistic limit. We also mentioned the fact that there might be other contractions of the superconformal algebra, which realise different types of super-extended GCAs. In this context, we would like to mention that in [132], the authors looked at similar contractions of the relativistic super-conformal group $PSU(2,2|4)$ while looking at non-relativistic limits of AdS/CFT correspondence from the world-sheet point of view. The linear combination of spinors used there is motivated by kappa-symmetry of the relativistic string action and seem to be different to our limit[4].

5.7 Appendix

5.7.1 Spinors: convention and choice

In this subsection, we provide some details of first the conventions and then why we choose to scale the spinors in the way we have done while performing group contraction.

Convention

For the calculations in this chapter, we have followed the notation of Wess-Bagger.

$$\begin{aligned}
\sigma^\mu &= (\sigma^0, \sigma^i), \quad \overline{\sigma}^\mu = (\sigma^0, -\sigma^i) \\
\sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
(\sigma^{\mu\nu})_\alpha{}^\beta &= -\frac{1}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)_\alpha{}^\beta, \quad (\tilde{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta} \\
\epsilon^{12} &= -\epsilon^{21} = \epsilon^{\dot1\dot2} = -\epsilon^{\dot2\dot1} = 1 \\
\epsilon_{12} &= -\epsilon_{21} = \epsilon_{\dot1\dot2} = -\epsilon_{\dot2\dot1} = -1
\end{aligned}$$

We have used the a, b indices running over $1, 2$, when we have dealt with the components of the non-relativistic $SO(3)$ spinors, and it does not matter whether they have been written as subscripts or superscripts. More explicitly,

$$\epsilon^{ab} = \epsilon_{ab}\,, \quad (\sigma^i)_{ab} = (\sigma^i)^{ab} = (\sigma^i)^a{}_b = (\sigma^i)_a{}^b\,. \tag{5.7.1}$$

Choice of spinors

In the non-relativistic limit there is a sharp distinction between time and space which was not present in the relativistic theory. So, as we have already stressed, when we look at the symmetries of quantities of interest in a non-relativistic setting, we would expect

[4] Ref. [133] seems more in this direction.

a rotational invariance in the spatial part only. In $d = 3 + 1$, when we consider fermions, we would thus choose to work with spinors of $SO(3)$.

From the commutators of the $\mathcal{Q}$'s with the J_{ij} generators, it is clear that under spatial rotations, they have the transformation properties of the spinors of $SO(3)$.

More explicitly, let us look at how the combinations mentioned in the paper form the two components of an $SU(2)$ or an $SO(3)$ spinor.
Under $SO(3)$ transformations:

$$\delta\theta_\alpha = (\sigma_{ij})_\alpha{}^\beta \theta_\beta w^{ij}, \quad \delta\overline{\theta}_{\dot\beta} = -\overline{\theta}_{\dot\alpha}(\tilde\sigma_{ij})^{\dot\alpha}{}_{\dot\beta} w^{ij}. \tag{5.7.2}$$

Now,

$$\sigma_{ij} = \frac{i}{2}\epsilon_{ijk}\sigma^k \Rightarrow w^{ij}\sigma_{ij} = \begin{pmatrix} c & a-ib \\ a+ib & -c \end{pmatrix}. \tag{5.7.3}$$

So, we get

$$\begin{array}{rcl} \delta\theta_1 &=& c\theta_1 + (a-ib)\theta_2, \quad \delta\theta_2 = (a+ib)\theta_1 - c\theta_2 \\ \delta\overline{\theta}_{\dot 1} &=& -(a+ib)\overline{\theta}_{\dot 2} - c\overline{\theta}_{\dot 1}, \quad \delta\overline{\theta}_{\dot 2} = c\overline{\theta}_{\dot 2} - (a-ib)\overline{\theta}_{\dot 1}. \end{array} \tag{5.7.4}$$

Hence,

$$\begin{array}{l} \delta(\theta_1 - \overline{\theta}_{\dot 2}) = c(\theta_1 - \overline{\theta}_{\dot 2}) + (a-ib)(\theta_2 + \overline{\theta}_{\dot 1}), \\ \delta(\theta_2 + \overline{\theta}_{\dot 1}) = (a+ib)(\theta_1 - \overline{\theta}_{\dot 2}) - c(\theta_2 + \overline{\theta}_{\dot 1}). \end{array} \tag{5.7.5}$$

Or more compactly,

$$\theta_+ = \begin{pmatrix} \theta_1 - \overline{\theta}_{\dot 2} \\ \theta_2 + \overline{\theta}_{\dot 1} \end{pmatrix} \Rightarrow \delta\theta_+ = (\hat{n}.\vec{\sigma})\theta_+. \tag{5.7.6}$$

This shows that the linear combinations that we work with are indeed $SO(3)$ spinors.

5.7.2 Bosonic generators

As already mentioned in §5.2, the bosonic generators in the supersymmetric case will get contributions from the fermionic coordinates. In this subsection, we list the various bosonic generators that have extra pieces and note how the generators need to be rescaled in the non-relativistic limit. As expected, the scaling behaviour is the same as in the case of the purely bosonic algebra.

We first derive explicit expressions for the angular momentum and boost generators, which are obtained from the J_{ij} and the J_{0i} components respectively, of the Lorentz generators $J_{\mu\nu}$ given by

$$J_{\mu\nu} = i(x_\nu\partial_\mu - x_\mu\partial_\nu + \sigma^{\alpha\beta}_{\mu\nu}\theta_\alpha\partial_\beta - \tilde\sigma^{\dot\alpha\dot\beta}_{\mu\nu}\overline{\theta}_{\dot\alpha}\overline{\theta}_{\dot\beta}). \tag{5.7.7}$$

We now perform contraction by scaling x^i, t, ψ_a and χ_a. The J_{ij} generators are invariant under this scaling and are explicitly given by

$$J_{ij} = i(x_j\partial_i - x_i\partial_j) + \frac{1}{2}\epsilon_{ijk}\left\{\psi_a(\sigma^k)^T_{ab}\partial_{\psi_b} + \chi_a(\sigma^k)^T_{ab}\partial_{\chi_b}\right\}. \quad (5.7.8)$$

For notational convenience, we define

$$J_i = \frac{1}{2}\epsilon_{ijk}J_{jk}. \quad (5.7.9)$$

For the J_{0i} generators, however, we have to choose redefined operators as shown below:

$$B_i = \lim_{\epsilon\to 0}\epsilon J_{0i} = it\partial_i + \frac{i}{2}\psi_a(\sigma^i)^T_{ab}\partial_{\chi_b}. \quad (5.7.10)$$

The dilatation generator is given by the expression

$$\begin{aligned} D &= ix^\mu\partial_\mu + \frac{i}{2}(\theta^\alpha\partial_\alpha + \overline{\theta}^{\dot{\alpha}}\overline{\partial}_{\dot{\alpha}}) \\ &= it\partial_t + ix^j\partial_j + \frac{i}{2}(\psi_a\partial_{\psi_a} + \chi_a\partial_{\chi_a}). \end{aligned} \quad (5.7.11)$$

We can see clearly that D does not scale when we perform contraction by scaling x^i, t, ψ_a and χ_a. Hence we need not redefine it.

The remaining bosonic generators are given by the temporal and spatial special conformal transformation generators, obtained from the K_0 and the K_i components, respectively, of the relativistic special conformal transformation generator K_μ given by

$$\begin{aligned} K_\mu &= -4i\left\{x_\mu x^\nu + (\theta\sigma_\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})\right\}\partial_\nu + 2i\left\{x^\nu x_\nu + 2(\theta\theta)(\overline{\theta\theta})\right\}\partial_\mu \\ &\quad +4i\epsilon^{\beta\gamma}(\sigma^\nu\overline{\sigma}_\mu)_\gamma{}^\alpha\theta_\alpha x_\nu\partial_\beta. \end{aligned} \quad (5.7.12)$$

We perform contraction by scaling x^i, t, ψ_a and χ_a. For these generators we have to choose redefined operators as shown below :

$$\begin{aligned} \mathcal{K}_i &= \lim_{\epsilon\to 0}\epsilon K_i = i(t^2 + 16\psi_1\psi_2\chi_1\chi_2)\partial_i + it\psi_a(\sigma^i)^T_{ab}\partial_{\chi_b} + 4\psi_1\psi_2\chi_a(\sigma^i)^T_{ab}\partial_{\chi_b}, \\ \mathcal{K} &= \lim_{\epsilon\to 0}K_0 = -i(t^2 - 16\psi_1\psi_2\chi_1\chi_2)\partial_t - 2itx^j\partial_j - it(\psi_a\partial_{\psi_a} + \chi_a\partial_{\chi_a}) \\ &\quad +4\psi_1\psi_2\chi_a\partial_{\psi_a} - 4\chi_1\chi_2\psi_a\partial_{\chi_a} - ix^j\psi_a(\sigma^j)^T_{ab}\partial_{\chi_b}. \end{aligned} \quad (5.7.13)$$

The R-symmetry generator is given by the expression

$$A = \frac{1}{2}(\theta^\alpha\partial_\alpha - \overline{\theta}^{\dot{\alpha}}\overline{\partial}_{\dot{\alpha}})$$

$$= \frac{1}{2}(\psi_a \partial_{\chi_a} + \chi_a \partial_{\psi_a}) \,. \tag{5.7.14}$$

Performing the contraction by scaling x^i, t, ψ_a and χ_a, we find that we need to redefine it as

$$\mathcal{A} = \lim_{\epsilon \to 0} \epsilon A = \frac{1}{2}\psi_a \partial_{\chi_a} \,. \tag{5.7.15}$$

Just for the sake of completeness, we give here the expressions used in Sec(3.3) for the Hamiltonian H and the momentum generators P_i, which do not have any piece contributed by the fermionic coordinates. They are:

$$H = -i\partial_t \,, \quad P_i = -i\partial_i \,. \tag{5.7.16}$$

5.7.3 OPE analysis

We consider the holomorphic currents $T(z), M^i(z), \mathcal{A}(z), G^{\pm a}(z)$, where z denotes the two-dimensional complex plane. Now we regard the generators $L_n, M^i_n, \mathcal{A}_n, G^{\pm a}_r$ of our infinite-dimensional algebra as the Laurent coefficients or mode operators of the Laurent expansion of the above holomorphic currents respectively. Then the (anti)commutators of the mode operators can be found by the standard Operator Product Expansion (OPE) and contour integral method.

If we could give J^i also an infinite lift, it would have to be promoted to a holomorphic current like the others. However, for the choice $[J^i_n, \mathcal{A}_m] = 0$, the Jacobi identity for $J^i_n, G^{+a}_r, G^{+b}_s$ is not satisfied for $n \neq 0$, which tells us that J^i cannot be given an infinite lift in this framework.

However, let us adopt an OPE analysis to see if we could have modified our infinite algebra, assuming J^i can be promoted to a holomorphic current $J^i(z)$ in this modified framework. We will then find all the (anti)commutators and fix the infinite algebra by requiring that all the Jacobi identities should be satisfied, and the finite part should coincide with (5.4.2) and have the commutators of $\mathcal{A}_0$ with $L_0, L_{\pm 1}, M^i_0, M^i_{\pm 1}, J^i_0$ equal to zero.

The OPEs are given by

$$\begin{aligned}
T(z_1)T(z_2) &\sim \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial_{z_2}T(z_2)}{(z_1-z_2)} \,, \\
T(z_1)M^i(z_2) &\sim \frac{a_1 J^i(z_2)}{(z_1-z_2)^3} + \frac{a_2 \partial_{z_2} J^i(z_2)}{(z_1-z_2)^2} + \frac{2M^i(z_2)}{(z_1-z_2)^2} + \frac{\partial_{z_2}M^i(z_2)}{(z_1-z_2)} \,, \\
T(z_1)J^i(z_2) &\sim \frac{J^i(z_2)}{(z_1-z_2)^2} + \frac{\partial_{z_2}J^i(z_2)}{(z_1-z_2)} + \frac{a_3 M^i(z_2)}{(z_1-z_2)} \,,
\end{aligned}$$

$$
\begin{aligned}
T(z_1)\mathcal{A}(z_2) &\sim \frac{a_4}{(z_1-z_2)^3}+\frac{\mathcal{A}(z_2)}{(z_1-z_2)^2}+\frac{\partial_{z_2}\mathcal{A}(z_2)}{(z_1-z_2)}+\frac{a_5 T(z_2)}{(z_1-z_2)}\,,\\
T(z_1)G^{\pm a}(z_2) &\sim \frac{\frac{3}{2}G^{\pm a}(z_2)}{(z_1-z_2)^2}+\frac{\partial_{z_2}G^{\pm a}(z_2)}{(z_1-z_2)}\,,\\
M^i(z_1)G^{+a}(z_2) &\sim -\frac{\frac{3}{2}\sigma^i_{ab}G^{+b}(z_2)}{(z_1-z_2)^2}-\frac{\partial_{z_2}\sigma^i_{ab}G^{+b}(z_2)}{(z_1-z_2)}\,,\\
M^i(z_1)G^{-a}(z_2) &\quad,\quad M^i(z_1)M^i(z_2),\ \ \mathcal{A}(z_1)\mathcal{A}(z_2)\ \sim \text{non-singular}\,,\\
G^{+a}(z_1)G^{+b}(z_2) &\sim -\frac{2\times 12i\epsilon^{ab}\mathcal{A}(z_2)}{(z_1-z_2)^2}+\frac{4i(\sigma^i\epsilon)_{ab}M^i(z_2)-12i\epsilon^{ab}\partial_{z_2}\mathcal{A}(z_2)}{(z_1-z_2)}\,,\\
M^i(z_1)\mathcal{A}(z_2) &\sim \text{non-singular}\,,\ \ J^i(z_1)\mathcal{A}(z_2)\sim\frac{a_6}{(z_1-z_2)}J^i(z_2)\,,\\
\mathcal{A}(z_1)G^{+a}(z_2) &\sim \frac{G^{-a}(z_2)}{2(z_1-z_2)}+a_7\frac{G^{+a}(z_2)}{(z_1-z_2)}\,,\ \ \mathcal{A}(z_1)G^{-a}(z_2)\sim a_8\frac{G^{-a}(z_2)}{(z_1-z_2)}+a_9\frac{G^{+a}(z_2)}{(z_1-z_2)}\,,
\end{aligned}
$$

where $a_l = 0$ $(1 \le l \le 9)$ for our algebra of §5.3.

Now let us see if we can allow for non-zero values of the a_l's. One can check that we have allowed for all possible terms in the OPEs when one considers the weights of the various holomorphic fields and the index structure involving i and a.

With the above consideration, we have the following modified commutators:

$$
\begin{aligned}
[L_m, M^i_n] &= (m-n)M^i_{m+n}+\frac{a_1}{2}m(m+1)J^i_{m+n}-a_2(m+1)(m+n+1)J^i_{m+n}\,,\\
[L_m, J^i_n] &= -nJ^i_{m+n}+a_3M^i_{m+n}\,,\\
[L_m, \mathcal{A}_n] &= -n\mathcal{A}_{m+n}+\frac{a_4}{2}m(m+1)\delta_{m+n,0}+a_5L_{m+n}\,,\\
[J^i_n, \mathcal{A}_m] &= a_6J^i_{m+n}\,,\ \ [\mathcal{A}_n, G^{+a}_r]=\frac{1}{2}G^{-a}_{n+r}+a_7G^{+a}_{n+r}\,,\ \ [\mathcal{A}_n, G^{-a}_r]=a_8G^{-a}_{n+r}+a_9G^{+a}_{n+r}\,.
\end{aligned}
$$

But now one can easily check that consistency with the finite part of the algebra sets $a_1, a_2, a_3, a_6, a_7, a_8, a_9$ to zero, while the Jacobi identity for $\{L_m, L_n, \mathcal{A}_p\}$ sets a_4 and a_5 to zero. The above analysis suggests that we cannot give the J^i's an infinite lift.

Chapter 6

SGCA in $2d$

In the present chapter, we study the $\mathcal{N} = (1,1)$ supersymmetric extension of GCFTs in 2d, dubbed "SGCFT". "Two spacetime dimensions" is special because the relativistic conformal algebra there is infinite dimensional. Now the ineteresting question to ask is whether there exists a map between the relativistic and non-relativistic infinite algebras. We will see that the answer is in the affirmative and one can obtain the non-relativistic infinite algebra from the parent (infinite dimensional) relativistic algebra by the usual group contraction. Hence the infinite GCA, which was first written by observation, can be derived as a simple limit of the algebra of 2D CFTs. Most of the algebraic structures of the 2d CFTs can be extended to their supersymmetric extensions, and the associated representation theory can also be developed along similar lines. The superconformal symmetries are also relevant for the superstring theory and the Tricritical Ising Model. We refer the reader to [134]-[140] (and references therein) for an extensive study of the 2d superconformal theories.

As in [82], the families of 2d SCFTs we will need to consider are rather unusual in that their left and right central charges, c and $\bar{c}$, are scaled (as we take the non-relativistic limit) such that their magnitudes go to infinity but are opposite in sign. The parent theories are thus necessarily non-unitary and, not unsurprisingly, this non-unitariness is inherited by the daughter GCFTs. Since non-unitary 2d CFTs arise in a number of contexts in statistical mechanics (e.g. the Yang-Lee model with central charge $c = -22/5$) as well as string theory, one might expect that the 2d GCFTs realised here would also be interesting objects to study.

In this chapter, we focus our attention on the Neveu-Schwarz (NS) sector. Our study of 2d SGCAs in this paper proceeds along two parallel lines. The first line of development is as described above and consists of taking carefully the non-relativistic scaling limit of the parent 2d SCFT. We find that this limit, while unusual, appears to give sensible answers. Specifically, we will study in this way, the representation theory (including null vectors), the Ward identities, fusion rules, and finally the equations for correlation functions following from the existence of level $\frac{3}{2}$ null states. In all these cases we find that a non-trivial scaling limit of the 2d SCFTs exists. This is not *a priori* obvious, since the limit involves keeping terms both of $\mathcal{O}(\frac{1}{\epsilon})$ and of $\mathcal{O}(1)$ (where ϵ is the scaling parameter which is taken to zero). The second line of development obtains many of these same results by carrying out an autonomous analysis of the SGCA, i.e., independent of the above limiting procedure. It is also an important consistency check of our investigation that these two strands of development agree.

6.1 2d SGCA from group contraction

In this section, we derive the supersymmetric extension of the GCA in 2d, by performing group contraction on the 2d superconformal algebra studied in [134]-[140].

6.1.1 SGCA from SuperVirasoro in 2d

The finite dimensional subalgebra of the GCA, which consists of taking $n = 0, \pm 1$ for the $L^{(n)}, M_i^{(n)}$ together with $J_a^{(0)}$, is obtained by considering the non-relativistic contraction of the usual (finite dimensional) global conformal algebra $SO(d,2)$ (in $d > 2$ spacetime dimensions).

However, in two spacetime dimensions, as is well known, the situation is special. The relativistic conformal algebra is infinite dimensional and consists of two copies of the Virasoro algebra. In [82], GCA with central charges was realised by taking a special limit of a non-unitary relativistic 2d CFT.

Here we take a similar limit on the 2d relativistic superconformal algebra, which is also infinite dimensional and consists of two copies of the SuperVirasoro algebra.

The two copies of the SuperVirasoro algebra are given by:

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} + \frac{c}{8}m(m^2-1)\delta_{m+n,0}\,, \\
[\mathcal{L}_m, \mathcal{G}_r] &= (\frac{1}{2}m - r)\mathcal{G}_{m+r} \\
\{\mathcal{G}_r, \mathcal{G}_s\} &= 2\mathcal{L}_{r+s} + \frac{c}{2}(r^2 - \frac{1}{4})\delta_{r+s,0}\,, \\
[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] &= (m-n)\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{8}m(m^2-1)\delta_{m+n,0}\,, \\
[\bar{\mathcal{L}}_m, \bar{\mathcal{G}}_r] &= (\frac{1}{2}m - r)\bar{\mathcal{G}}_{m+r}\,, \\
\{\bar{\mathcal{G}}_r, \bar{\mathcal{G}}_s\} &= 2\bar{\mathcal{L}}_{r+s} + \frac{\bar{c}}{2}(r^2 - \frac{1}{4})\delta_{r+s,0}\,,
\end{aligned} \tag{6.1.1}$$

where $m, n \in \mathcal{Z}$ and *either* $r, s \in \mathcal{Z}$ [Ramond case] *or* $r, s \in \mathcal{Z} + \frac{1}{2}$ [Neveu-Schwarz case].

We now perform group contraction with the new generators defined as:

$$\begin{aligned} L_n &= \lim_{\epsilon\to 0}(\bar{\mathcal{L}}_n + \mathcal{L}_n)\,, \quad M_n = \lim_{\epsilon\to 0}\epsilon\,(\bar{\mathcal{L}}_n - \mathcal{L}_n)\,, \\ G_n &= \lim_{\epsilon\to 0}(\bar{\mathcal{G}}_n + \mathcal{G}_n)\,, \quad H_n = \lim_{\epsilon\to 0}\epsilon\,(\bar{\mathcal{G}}_n - \mathcal{G}_n)\,, \end{aligned} \tag{6.1.2}$$

where for the bosonic part we have followed [82], and for the fermionic part we have chosen a limit so as to get all the bosonic generators as anticommutators of the fermionic ones (here we have followed the scaling used in [133]).

The above generators define the SGCA and obey the algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + C_1 m(m^2-1)\delta_{m+n,0}\,, \\ [L_m, M_n] &= (m-n)M_{m+n} + C_2 m(m^2-1)\delta_{m+n,0}\,, \\ [M_m, M_n] &= 0\,, \\ \{G_r, G_s\} &= 2L_{r+s} + 4C_1(r^2 - \frac{1}{4})\delta_{r+s,0}\,, \\ \{H_r, H_s\} &= 0\,, \\ \{G_r, H_s\} &= 2M_{r+s} + 4C_2(r^2 - \frac{1}{4})\delta_{r+s,0}\,, \\ [L_m, G_r] &= (\frac{1}{2}m - r)G_{m+r}\,, \quad [L_m, H_r] = (\frac{1}{2}m - r)H_{m+r}\,, \\ [M_m, G_r] &= (\frac{1}{2}m - r)H_{m+r}\,, \quad [M_m, H_r] = 0\,, \end{aligned} \tag{6.1.3}$$

where the central charges are given by:

$$C_1 = \lim_{\epsilon\to 0}\frac{\bar{c}+c}{8}\,, \quad C_2 = \lim_{\epsilon\to 0}\epsilon\,\frac{\bar{c}-c}{8}\,. \tag{6.1.4}$$

Thus, for a non-zero C_2 in the limit $\epsilon \to 0$, we see that we need $\bar{c} - c \propto \mathcal{O}(\frac{1}{\epsilon})$. At the same time, requiring C_1 to be finite, we find that $c + \bar{c}$ should be $\mathcal{O}(1)$. As in [82], we will make the slightly stronger assumption that $\bar{c} - c = \mathcal{O}(\frac{1}{\epsilon}) + \mathcal{O}(\epsilon)$.) Actually this is motivated by the fact that $\bar{\mathcal{L}}_n - \mathcal{L}_n$ and $\bar{\mathcal{G}}_r - \mathcal{G}_r$ have vanishing $\mathcal{O}(1)$ pieces, when we write their transformation-actions on supercoordinates and take the appropriate scalings (see (6.1.7) and (6.1.9)). Thus (6.1.4) can hold only if c and $\bar{c}$ are large and opposite in sign (in the limit $\epsilon \to 0$). This immediately implies that the original 2d SCFT, on which we take the non-relativistic limit, cannot be unitary. This is of course not a problem, since there are many statistical mechanical models which are described at a fixed point by non-unitary CFTs.

6.1.2 Non-relativistic superconformal transformations in the superspace

In the superspace formalism, for $\mathcal{N} = (1,1)$ supersymmetry, we introduce the fermionic coordinates $\theta, \bar{\theta}$ for the holomorphic and the antiholomorphic sectors respectively[1]. A superfield is a function defined on superspace, and can be expanded as a power series in $\theta, \bar{\theta}$:

$$\Phi(\mathcal{Z}, \bar{\mathcal{Z}}) = \phi(z,\bar{z}) + \theta\psi(z,\bar{z}) + \bar{\theta}\bar{\psi}(z,\bar{z}) + \theta\bar{\theta}F(z,\bar{z}), \tag{6.1.5}$$

where

$$\mathcal{Z} \equiv (z,\theta), \quad \bar{\mathcal{Z}} \equiv (\bar{z},\bar{\theta}). \tag{6.1.6}$$

The superfields correspond to irreducible representations of the Neveu-Schwarz algebra. The irreducible representations of the Ramond algebra correspond to conformal fields distinct from the superfields, which are in fact non-local (i.e., double-valued) with respect to the fermionic parts of the superfields. These are called *spin fields* and they intertwine the two sectors (see, e.g., [136]).

As in conformal transformations, in superconformal transformations too the unbarred and the barred parts are independent. In superspace, the superconformal transformations corresponding to the holomorphic sector are given by:

$$\begin{aligned} (z\,,\,\theta) &\to \delta'\mathcal{L}_n(z - \delta'\,z^{n+1}\,,\,\theta - \delta'\,\frac{n+1}{2}z^n\,\theta) \\ (z\,,\,\theta) &\to \eta\,\mathcal{G}_r(z + \eta\,\theta\,z^{r+\frac{1}{2}}\,,\,\theta - \eta\,z^{r+\frac{1}{2}})\,, \end{aligned} \tag{6.1.7}$$

where η is an anticommuting parameter. Similarly, one can write down transformations for the antiholomorphic sector.

In terms of spacetime coordinates, $z = t + x$, $\bar{z} = t - x$. Analogously, we take linear combinations of $\theta, \bar{\theta}$ and define the new anticommuting variables:

$$\alpha = \frac{\theta + \bar{\theta}}{2}, \quad \beta = \frac{\theta - \bar{\theta}}{2}. \tag{6.1.8}$$

The non-relativistic contraction corresponding to (6.1.2) consists of taking the scalings:

$$t \to t\,, \quad x \to \epsilon x\,, \quad \alpha \to \alpha\,, \quad \beta \to \epsilon\beta\,, \tag{6.1.9}$$

which immediately gives the coordinates in the non-relativistic superspace transforming as:

$$\begin{aligned} \delta_{\delta' L_n}\{t,x,\alpha,\beta\} &= -\delta'\,\{t^{n+1}\,,\,(n+1)\,t^n\,x\,,\,\frac{1}{2}(n+1)\,t^n\,\alpha\,,\,\frac{1}{2}(n+1)(\,t^n\,\beta + n\,t^{n-1}\,x\,\alpha)\}\,, \\ \delta_{\delta' M_n}\{t,x,\alpha,\beta\} &= \delta'\,\{0\,,\,t^{n+1}\,,\,0\,,\,\frac{1}{2}(n+1)\,t^n\,\alpha\}\,, \end{aligned}$$

[1] More details can be found in [137].

$$\delta_{\eta G_r}\{t,x,\alpha,\beta\} = \eta\,\{t^{r+\frac{1}{2}}\,\alpha\,,\,t^{r+\frac{1}{2}}\,\beta + (r+\frac{1}{2})\,t^{r-\frac{1}{2}}\,x\,\alpha\,,-t^{r+\frac{1}{2}}\,,-(r+\frac{1}{2})\,t^{r-\frac{1}{2}}\,x\}\,,$$
$$\delta_{\eta H_r}\{t,x,\alpha,\beta\} = \eta\,\{0\,,-t^{r+\frac{1}{2}}\,\alpha\,,0\,,\,t^{r+\frac{1}{2}}\}\,. \tag{6.1.10}$$

6.2 Representations of the 2d SGCA

We now turn to the representations of the 2d SGCA. In all our subsequent discussions, we consider the NS sector and hence $r,s\in\mathcal{Z}+\frac{1}{2}$ in all formulae and equations that follow. We will be guided in this by the representation theory of the SuperVirasoro algebra.

6.2.1 Primary states and descendants

We will construct the representations by considering states having definite scaling dimensions:

$$L_0|\Delta\rangle = \Delta|\Delta\rangle\,. \tag{6.2.1}$$

Using the commutation relations (6.1.3), we obtain

$$L_0L_n|\Delta\rangle = (\Delta-n)L_n|\Delta\rangle, \quad L_0M_n|\Delta\rangle = (\Delta-n)M_n|\Delta\rangle. \tag{6.2.2}$$

Then the L_n, M_n with $n>0$ lower the value of the scaling dimension, while those with $n<0$ raise it. If we demand that the dimensions of the states be bounded from below, then we are led to defining primary states in the theory with the properties:

$$L_n|\Delta\rangle_p = 0\,, \quad M_n|\Delta\rangle_p = 0\,, \quad G_r|\Delta\rangle_p = 0\,, \quad H_r|\Delta\rangle_p = 0 \quad (\text{for all } n>0 \text{ and } r>0)\,. \tag{6.2.3}$$

Since the conditions (6.2.3) are compatible with M_0 in the sense

$$L_nM_0|\Delta\rangle_p = 0\,, \quad M_nM_0|\Delta\rangle_p = 0\,, \tag{6.2.4}$$

and also since L_0 and M_0 commute, we may introduce an additional label, which we will call "rapidity" ξ:

$$M_0|\Delta,\xi\rangle_p = \xi|\Delta,\xi\rangle_p\,. \tag{6.2.5}$$

Starting with a primary state $|\Delta,\xi\rangle_p$, one can build up a tower of operators by the action of $L_{-n}\,,\,M_{-n}\,,\,G_{-r}\,,\,H_{-r}$ with $n,r>0$. These will be called the SGCA descendants of the primary. The primary state together with its SGCA descendants form a representation of SGCA. As in the SuperVirasoro case, we have to be careful about the presence of null states. We will look at these in some detail later in Sec. 6.4.

The above construction is quite analogous to that of the relativistic 2d SCFT. In fact, from the viewpoint of the limit (6.1.2), we see that the two labels Δ and ξ are related to the conformal weights in the 2d SCFT as

$$\Delta = \lim_{\epsilon \to 0}(h + \bar{h}) \,, \qquad \xi = \lim_{\epsilon \to 0} \epsilon(\bar{h} - h) \,, \tag{6.2.6}$$

where h and $\bar{h}$ are the eigenvalues of $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$, respectively. We will proceed to assume that such a scaling limit (as $\epsilon \to 0$) of the 2d SCFT exists. In particular, we will assume that the operator-state correspondence in the 2d SCFT gives a similar correspondence between the states and the operators in the SGCA[2]:

$$\mathcal{O}(t, x) \leftrightarrow \mathcal{O}(0, 0)\,|0\rangle \,, \tag{6.2.7}$$

where $|0\rangle$ would be the vacuum state which is invariant under the generators $L_0, L_{\pm 1}$, $M_0, M_{\pm 1}$. Indeed in the rest of the paper, we will offer several pieces of evidence that the scaling limit gives a consistent quantum mechanical system.

6.2.2 Transformation laws of superprimary fields

We consider the transformation laws of SGCA primary superfields arising from the transformation laws of primary superfields in 2d SCFT, which are given by (following [137]):

$$\begin{aligned}
[\,\mathcal{L}_n, \Phi(z, \bar{z}, \theta, \bar{\theta})\,] &= [\, z^{n+1}\partial_z + \frac{1}{2}(n+1)z^n\theta\partial_\theta + h(n+1)z^n\,]\Phi(z, \bar{z}, \theta, \bar{\theta}) \,, \\
[\,\eta\,\mathcal{G}_r, \Phi(z, \bar{z}, \theta, \bar{\theta})\,] &= \eta\,[\, z^{r+\frac{1}{2}}(\partial_\theta - \theta\partial_z) - 2h(r + \frac{1}{2})z^{r-\frac{1}{2}}\,\theta\,]\Phi(z, \bar{z}, \theta, \bar{\theta}) \,;
\end{aligned} \tag{6.2.8}$$

the transformations corresponding to $\bar{\mathcal{L}}_n$, $\bar{\mathcal{G}}_r$ are given by replacing $z \to \bar{z}$, $\theta \to \bar{\theta}$ and $h \to \bar{h}$. We should note here that $(h, \bar{h})$ corresponds to the conformal weights of the lowest component ϕ of the superfield Φ in (6.1.5).

Motivated from the relation (6.1.2), we may define the transformations generated by L_n, M_n, G_n, H_n as:

$$\begin{aligned}
[L_n, \Phi] &= \lim_{\epsilon \to 0} [\,\bar{\mathcal{L}}_n + \mathcal{L}_n \,,\, \Phi] \,, \qquad & [M_n, \Phi] &= \lim_{\epsilon \to 0} \epsilon\,[\,\bar{\mathcal{L}}_n - \mathcal{L}_n \,,\, \Phi] \,, \\
[G_r, \Phi] &= \lim_{\epsilon \to 0} [\,\bar{\mathcal{G}}_r + \mathcal{G}_r \,,\, \Phi] \,, \qquad & [H_r, \Phi] &= \lim_{\epsilon \to 0} \epsilon\,[\,\bar{\mathcal{G}}_r - \mathcal{G}_r \,,\, \Phi] \,,
\end{aligned} \tag{6.2.9}$$

where the superfield Φ is now a function of $\{t, x, \alpha, \beta\}$ and is expanded as:

$$\Phi(t, x, \alpha, \beta) = \phi_1(t, x) + \alpha\,\psi_1(t, x) + \beta\,\psi_2(t, x) + \alpha\beta\,\phi_2(t, x) \,. \tag{6.2.10}$$

[2]We emphasize that this is an assumption we are making (without any justification). Our approach here is to go ahead with this assumption and examine whether this leads to interesting structures and whether the various algebraic considerations lead to a consistent picture.

Then by taking the limits on the superspace coordinates, we obtain:

$$\begin{aligned}
[L_n\,,\Phi] &= [\,t^{n+1}\,\partial_t \,+\, (n+1)\,t^n\,x\,\partial_x \,+\, (n+1)\,(\Delta\,t^n - n\,\xi\,t^{n-1}\,x) \\
&\qquad + \frac{1}{2}(n+1)\,\{\,t^n\,(\alpha\,\partial_\alpha \,+\, \beta\,\partial_\beta) \,+\, n\,t^{n-1}\,x\,\alpha\,\partial_\beta\}\,]\,\Phi\,, \\
[M_n\,,\Phi] &= [\,-t^{n+1}\,\partial_x + \,(n+1)\,\xi\,t^n - \frac{1}{2}(n+1)\,t^n\,\alpha\,\partial_\beta]\,\Phi\,, \\
[\eta_G\,G_r\,,\Phi] &= \eta_G\,[\,t^{r+\frac{1}{2}}\,(-\alpha\partial_t - \beta\partial_x + \,\partial_\alpha) + (\,r + \frac{1}{2}\,)\,t^{r-\frac{1}{2}}\,x\,(-\alpha\partial_x + \partial_\beta) \\
&\qquad + 2\,(\,r + \frac{1}{2}\,)\,t^{r-\frac{1}{2}}\,(\xi\,\beta - \Delta\,\alpha) + \,2\,(\,r^2 - \frac{1}{4}\,)\,\xi\,t^{r-\frac{3}{2}}\,x\,\alpha\,]\Phi\,, \\
[\eta_H\,H_r\,,\Phi] &= \eta_H\,[\,t^{r+\frac{1}{2}}\,(\alpha\partial_x - \partial_\beta) - 2\,(\,r + \frac{1}{2}\,)\,\xi\,t^{r-\frac{1}{2}}\,\alpha\,]\,\Phi\,.
\end{aligned} \tag{6.2.11}$$

where η_G, η_H are anticommuting parameters. Note that the part of the transformation laws independent of Δ and ξ , involving superspace derivatives, encodes the change due to superspace coordinate dependence of Φ, and is in perfect agreement with (6.1.10).

Introducing the vacuum state $|0\rangle$ satisfying

$$\begin{aligned}
L_n|0\rangle = 0\,, \quad M_n|0\rangle = 0\,, \quad (\text{for } n \geq -1) \\
G_r|0\rangle = 0\,, \quad H_r|0\rangle = 0\,, \quad (\text{for } r \geq -\tfrac{1}{2})\,,
\end{aligned} \tag{6.2.12}$$

one immediately finds from (6.2.11) that

$$\begin{aligned}
G_{\frac{1}{2}}|\phi_1\rangle &= 0\,, & G_{-\frac{1}{2}}|\phi_1\rangle &= |\psi_1\rangle\,, \\
H_{\frac{1}{2}}|\phi_1\rangle &= 0\,, & H_{-\frac{1}{2}}|\phi_1\rangle &= -|\psi_2\rangle\,, \\
G_{-\frac{1}{2}}G_{-\frac{1}{2}}|\phi_1\rangle &= L_{-1}|\phi_1\rangle\,, & H_{-\frac{1}{2}}H_{-\frac{1}{2}}|\phi_1\rangle &= 0\,, \\
H_{-\frac{1}{2}}G_{-\frac{1}{2}}|\phi_1\rangle &= M_{-1}|\phi_1\rangle - |\phi_2\rangle\,, & G_{-\frac{1}{2}}H_{-\frac{1}{2}}|\phi_1\rangle &= M_{-1}|\phi_1\rangle + |\phi_2\rangle\,,
\end{aligned} \tag{6.2.13}$$

where the state $|\phi_1\rangle = \phi_1(0,0)\,|0\rangle$ satisfies the conditions (6.2.3) for a primary state.

6.3 Non-relativistic Ward identities and correlation functions

In [137], the two and three point functions for the 2d SCFT were found using the superspace formalism. Here we take the appropriate limits of the those correlation functions to get the SGCA correlation functions and check that these obey the Ward identities coming from the global part comprising $\{L_0,\, L_{\pm1},\, M_0,\, M_{\pm1},\, G_{\pm\frac{1}{2}},\, H_{\pm\frac{1}{2}}\}$. One can solve the differential equations coming from the Ward identities to find the correlation functions directly using (6.2.11). However, the calculation becomes cumbersome because here one cannot use the nice property of the independence of holomorphic and antiholomorphic sectors of the SCFT. We solve the differential equations for the two point functions directly with the SGCA operators, whereas, for the three point function, we find the expression only by taking the limit of the SCFT answer.

For the sake of completeness, we state here the differential equations that an n-point function,

$$G^{(n)}_{2\mathrm{d\,SGCA}}(\{t_i,x_i,\alpha_i,\beta_i\}) = \langle \Phi_1(t_1,x_1,\alpha_1,\beta_1)\,\Phi_2(t_2,x_2,\alpha_2,\beta_2)\cdots\Phi_n(t_n,x_n,\alpha_n,\beta_n)\rangle\,, \tag{6.3.1}$$

should satisfy:

$$\begin{aligned}
&\Big[\sum_{i=1}^{n}\partial_{t_i}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\partial_{x_i}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\{t_i\partial_{t_i}+x_i\partial_{x_i}+\Delta_i+\frac{1}{2}(\alpha_i\partial_{\alpha_i}+\beta_i\partial_{\beta_i})\}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\{-t_i\partial_{x_i}+\xi_i-\frac{1}{2}\alpha_i\partial_{\beta_i}\}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\{t_i^2\partial_{t_i}+2t_ix_i\partial_{x_i}+2(\Delta_it_i-\xi_ix_i)+t_i(\alpha_i\partial_{\alpha_i}+\beta_i\partial_{\beta_i})+x_i\alpha_i\partial_{\beta_i}\}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\{-t_i^2\partial_{x_i}+2\xi_it_i-t_i\alpha_i\partial_{\beta_i}\}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\{-\alpha_i\partial_{t_i}-\beta_i\partial_{x_i}+\partial_{\alpha_i}\}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\{\alpha_i\partial_{x_i}-\partial_{\beta_i}\}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\{t_i(-\alpha_i\partial_{t_i}-\beta_i\partial_{x_i}+\partial_{\alpha_i})+x_i(-\alpha_i\partial_{x_i}+\partial_{\beta_i})+2(\xi_i\beta_i-\Delta_i\alpha_i)\}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,,\\
&\Big[\sum_{i=1}^{n}\{t_i(\alpha_i\partial_{x_i}-\partial_{\beta_i})-2\xi_i\alpha_i\}\Big]G^{(n)}_{2\mathrm{d\,SGCA}}=0\,.
\end{aligned} \tag{6.3.2}$$

The above constraints follow from invariance under the generators L_{-1}, M_{-1}, L_0, M_0, L_1, M_1, $G_{-\frac{1}{2}}$, $H_{-\frac{1}{2}}$, $G_{\frac{1}{2}}$ and $H_{\frac{1}{2}}$ respectively.

6.3.1 SGCA two point functions

We derive the two point functions between all components of two superfields

$$\Phi_i(t_i, x_i, \alpha_i, \beta_i) = \phi_{i1}(t_i, x_i) + \alpha_i\, \psi_{i1}(t_i, x_i) + \beta_i\, \psi_{i2}(t_i, x_i) + \alpha_i\beta_i\, \phi_{i2}(t_i, x_i)\,, \qquad (6.3.3)$$

with $i = 1, 2$. Here the lowest component fields ϕ_{i1} are the primary fields (see the definition (6.2.3)) and are labelled by the eigenvalues (Δ_i, ξ_i).

Here we consider the transformation rules for each component by comparing the coefficients of $\alpha^m\, \beta^n$ (where $m, n = 0, 1$) on both sides of (6.2.11).

One immediately finds that the field $\phi_1(t, x)$ in (6.2.11) has the same transformation properties under the bosonic SGCA generators as the primary fields of GCA (see eq. (4.5) and eq. (4.6) of [82]). Hence the ϕ_{i1} two point function will have the same form as derived in [79], i.e.,

$$\langle \phi_{11}(t_1, x_1)\phi_{21}(t_2, x_2)\rangle = C_{12}\, \delta_{\Delta_1,\Delta_2}\, \delta_{\xi_1,xi_2}\, t_{12}^{-2\Delta_1}\, \exp(\frac{2\xi_1 x_{12}}{t_{12}})\,, \qquad (6.3.4)$$

where

$$t_{ij} = t_i - t_j\,, \quad x_{ij} = x_i - x_j\,, \qquad (6.3.5)$$

and C_{12} is an arbitrary constant. We can take $C_{12} = 1$ by choosing the normalization of the operators.

Starting from this expression, we apply the constraints coming from the fermionic generators $G_{\pm\frac{1}{2}}, H_{\pm\frac{1}{2}}$ of the global part of the SGCA to obtain all other two point functions of the superfield components, as indicated below.

Using the fact that the two point function should be a function of products of the fermionic coordinates which are Grassmann even, we immediately infer:

$$\langle \phi_{1a}\, \psi_{2b}\rangle = 0\,, \quad \langle \psi_{1a}\, \phi_{2b}\rangle = 0 \qquad (\text{where } a, b = 1, 2)\,. \qquad (6.3.6)$$

Evaluating the trivial constraint $\delta_{G_{-\frac{1}{2}}}\langle \phi_{11}\, \psi_{21}\rangle = 0$, one gets the expression:

$$\langle \psi_{11}\, \psi_{21}\rangle = \partial_{t_{12}}\langle \phi_{11}\, \phi_{21}\rangle = -\,\frac{2}{t_{12}}\,(\,\Delta_1 + \frac{\xi_1 x_{12}}{t_{12}}\,)\langle \phi_{11}\, \phi_{21}\rangle\,. \qquad (6.3.7)$$

The trivial constraint $\delta_{H_{-\frac{1}{2}}}\langle \phi_{11}\, \psi_{22}\rangle = 0$ gives:

$$\langle \psi_{12}\, \psi_{22}\rangle = 0 \qquad (6.3.8)$$

The trivial constraints $\delta_{G_{-\frac{1}{2}}}\langle \phi_{11}\, \psi_{22}\rangle = 0$ and $\delta_{G_{\frac{1}{2}}}\langle \psi_{12}\, \phi_{21}\rangle = 0$, on using (6.3.8), give the results:

$$\langle \phi_{12}\, \phi_{21}\rangle = 0\,, \qquad (6.3.9)$$

$$\langle \psi_{11}\, \psi_{22}\rangle = \partial_{x_{12}}\langle \phi_{11}\, \phi_{21}\rangle = \frac{2\xi_1}{t_{12}}\langle \phi_{11}\, \phi_{21}\rangle\,. \qquad (6.3.10)$$

Using $\delta_{G_{-\frac{1}{2}}}\langle\psi_{12}\,\phi_{21}\rangle = 0$, we get:

$$\langle\psi_{12}\,\psi_{21}\rangle = \partial_{x_{12}}\langle\phi_{11}\,\phi_{21}\rangle = \frac{2\xi_1}{t_{12}}\langle\phi_{11}\,\phi_{21}\rangle\,. \tag{6.3.11}$$

Using $\delta_{G_{\frac{1}{2}}}\langle\psi_{12}\,\phi_{21}\rangle = 0$ along with (6.3.10) and (6.3.11), we get:

$$\langle\phi_{12}\,\phi_{21}\rangle = 0\,. \tag{6.3.12}$$

Lastly, $\delta_{G_{\frac{1}{2}}}\langle\psi_{11}\,\phi_{22}\rangle = 0$, on using (6.3.9), (6.3.10) and (6.3.11), gives:

$$\langle\phi_{12}\,\phi_{22}\rangle = \frac{4\xi_1^2}{t_{12}^2}\langle\phi_{11}\,\phi_{21}\rangle\,. \tag{6.3.13}$$

Hence we find that all non-vanishing two point functions of the components of the two superfields are determined in terms of the two point function of their lowest components.

6.3.2 SGCA higher point functions

Using the fact that the lowest components ϕ_{i1}'s obey the same transformation rules as the GCA primaries under the bosonic generators of the SGCA, we conclude that all correlation functions involving these fields have the same form as one gets in the GCA case. In particular, the result derived for three point function in [79] is applicable here for $\langle\phi_{11}\phi_{21}\phi_{31}\rangle$. For the four point function of the ϕ_{i1}'s, we can apply the same analysis as discussed in [82], where one of the ϕ_{i1}'s have a descendant null state at some level[3]. Then, as in the case of the two point function, the fermionic generators of the global part will relate the n-point function $\langle\phi_{i1}\,\phi_{i+11}\,\cdots\,\phi_{i+n1}\rangle$ to the n-point functions involving arbitrary component fields of the relevant superfields $\{\Phi_i, \Phi_{i+1}, \cdots, \Phi_{i+n}\}$.

We remind the reader that the above property follows from the fact that, in 2d CFTs and GCAs, the descendant field correlators can be derived from the primary field correlators. Here the component fields $\psi_{i1}, \psi_{i2}, \phi_{i2}$ are descendants of the primary ϕ_{i1}, as shown in (6.2.13). The global part of the SGCA, which closes by itself and hence forms a subgroup, allows us to group these four fields into the superfield Φ_i (supermultiplet), which is nothing but an irreducible representation of the global subalgebra.

[3]Note that here we can have half-integer level null states. In particular, we show in Sec. 5 that the first non-trivial null state is obtained at level $\frac{3}{2}$ and one can derive the four point function with a primary having such a descendant null state.

6.3.3 SGCA correlation functions from 2d SCFT

We now show that the above expressions for the SGCA two point functions can also be obtained by taking an appropriate scaling limit of the 2d SCFT answers. This limit requires scaling the quantum numbers of the operators as (6.2.6), along with the non-relativistic limit (6.1.9) for the coordinates.

Let us first study the scaling limit of the two point correlator of two superfields (see [137]) given by the expression

$$G^{(2)}_{\rm 2d\,SCFT} = \langle\, \Phi_1(\mathcal{Z}_1, \bar{\mathcal{Z}}_1)\, \Phi_2(\mathcal{Z}_2, \bar{\mathcal{Z}}_2)\,\rangle = \delta_{h_1,h_2}\,\delta_{\bar{h}_1,\bar{h}_2}\tilde{z}_{12}^{-2h_1}\,\tilde{\bar{z}}_{12}^{-2\bar{h}_1}\,, \tag{6.3.14}$$

where

$$\begin{aligned} z_{ij} &= z_i - z_j\,, \quad \bar{z}_{ij} = \bar{z}_i - \bar{z}_j\,, \\ \tilde{z}_{ij} &= z_{ij} - \theta_i\,\theta_j\,, \quad \tilde{\bar{z}}_{ij} = \bar{z}_{ij} - \bar{\theta}_i\,\bar{\theta}_j\,. \end{aligned} \tag{6.3.15}$$

On scaling the above expression according to (6.1.9) and taking the limit using (6.2.6), it reduces to:

$$\begin{aligned} G^{(2)}_{\rm 2d\,SGCA} &= \lim_{\epsilon\to 0}\delta_{h_1,h_2}\,\delta_{\bar{h}_1,\bar{h}_2}\,\{t_{12}-\alpha_1\alpha_2+\epsilon\,(x_{12}-\alpha_1\beta_2+\alpha_2\beta_1)+\epsilon^2\,\beta_1\beta_2\}^{-2h_1} \\ &\qquad\qquad \times\{t_{12}-\alpha_1\alpha_2+\epsilon\,(x_{12}-\alpha_1\beta_2+\alpha_2\beta_1)+\epsilon^2\,\beta_1\beta_2\}^{-2\bar{h}_1} \\ &= \lim_{\epsilon\to 0}\delta_{h_1,h_2}\,\delta_{\bar{h}_1,\bar{h}_2}\,(t_{12}-\alpha_1\alpha_2)^{-2(h_1+\bar{h}_1)} \\ &\qquad\qquad \times\exp\{-2(h_1-\bar{h}_1)(\epsilon\,\frac{(x_{12}-\alpha_1\beta_2+\alpha_2\beta_1)}{(t_{12}-\alpha_1\alpha_2)}+\mathcal{O}(\epsilon^2)\,)\} \\ &= \delta_{\Delta_1,\Delta_2}\,\delta_{\xi_1,\xi_2}\,\tilde{t}_{12}^{-2\Delta_1}\exp\Big(\frac{2\xi_1\tilde{x}_{12}}{\tilde{t}_{12}}\Big)\,, \end{aligned} \tag{6.3.16}$$

where

$$\tilde{t}_{ij} = t_{ij} - \alpha_i\alpha_j\,, \quad \tilde{x}_{ij} = x_{ij} - \alpha_i\beta_j + \alpha_j\beta_i\,. \tag{6.3.17}$$

Now expanding the LHS $G^{(2)}_{\rm 2d\,SGCA} \equiv \langle\Phi_1\Phi_2\rangle$ using (6.2.10), and comparing the coefficients of $\alpha_1^k\beta_1^l\alpha_2^m\beta_2^n$ (for $k,l,m,n=0,1$) on both sides of (6.3.16), we get the values of all possible two point functions of the component fields. One can check that these answers exactly match with those obtained in (6.3.9)-(6.3.13). Also, working in superfield formalism, one can check that (6.3.16) satisfies the constraints coming from the global part of the SGCA using directly (6.3.2) (i.e., without considering the transformations of the component fields separately).

Another interesting point to note is the following: Transforming the non-relativistic superspace coordinates $\{t_1,\,x_1,\,\alpha_1,\,\beta_1\}$ (using (6.1.10)) successively by $t_2\,L_{-1}$, $-x_2\,M_{-1}$, $\alpha_2\,G_{-\frac{1}{2}}$ and $-\beta_2\,H_{-\frac{1}{2}}$, we move to the point in the superspace labelled by $\{\tilde{t}_{12}\,,\,\tilde{x}_{12}\,,\,\alpha_1-$

$\alpha_2\,,\, \beta_1 - \beta_2\}$. The vacuum being invariant under these global transformations, one can easily see that the two-point function should be a function of these combinations of the six coordinates $\{t_i,\, x_i,\, \alpha_i,\, \beta_i\}$.

A similar analysis yields the three point function of the SGCA from the relativistic three point function. The relativistic three point function is written as:

$$\begin{aligned} G^{(3)}_{\mathrm{2d\,CFT}} &= \langle\, \Phi_1(\mathcal{Z}_1, \bar{\mathcal{Z}}_1)\, \Phi_2(\mathcal{Z}_2, \bar{\mathcal{Z}}_2)\, \Phi_3(\mathcal{Z}_3, \bar{\mathcal{Z}}_3)\,\rangle \\ &= [\, \tilde{z}_{12}^{\,h_3-h_1-h_2}\;\; \tilde{z}_{23}^{\,h_1-h_2-h_3}\;\; \tilde{z}_{31}^{\,h_2-h_3-h_1} \times \text{(antiholomorphic)}\,] \\ &\quad \times [\, C_{123} + \frac{\tilde{C}_{123}}{|\tilde{z}_{12}\, \tilde{z}_{23}\, \tilde{z}_{31}|} \{(\theta_1\, \tilde{z}_{23} + \theta_2\, \tilde{z}_{31} + \theta_3\, \tilde{z}_{12} + \theta_1\, \theta_2\, \theta_3) \times \text{(antiholomorphic)}\}\,]\,. \end{aligned} \tag{6.3.18}$$

One should note that there are two arbitrary constants $C_{123}\,,\, \tilde{C}_{123}$ in $G^{(3)}_{\mathrm{2d\,CFT}}$.

Taking the non-relativistic limit, we obtain the SGCA three point function as:

$$\begin{aligned} G^{(3)}_{\mathrm{2d\,SGCA}} &= C_{123}\; \tilde{t}_{12}^{\,\Delta_3-\Delta_1-\Delta_2}\;\; \tilde{t}_{23}^{\,\Delta_1-\Delta_2-\Delta_3}\;\; \tilde{t}_{31}^{\,\Delta_2-\Delta_3-\Delta_1} \\ &\qquad \times \exp\{\frac{(\xi_1+\xi_2-\xi_3)\, \tilde{x}_{12}}{\tilde{t}_{12}} + \frac{(\xi_2+\xi_3-\xi_1)\, \tilde{x}_{23}}{\tilde{t}_{23}} + \frac{(\xi_1+\xi_3-\xi_2)\, \tilde{x}_{31}}{\tilde{t}_{31}}\}\,. \end{aligned} \tag{6.3.19}$$

Again, one can check that (6.3.19) satisfies the differential equations (6.3.2). Comparing the coefficients of the parts involving no fermionic coordinates $\{\alpha_i, \beta_i\}$ on both sides, we find that $\langle \phi_{11}\phi_{21}\phi_{31}\rangle$ is exactly what was derived in [79], and this is what one should get following the discussion in Sec. 4.2.

Here we note that the contribution from the part multiplying $\tilde{C}_{123}$ in (6.3.18) is zero in the non-relativistic limit. However, it may so happen that $\tilde{C}_{123}$ scales in a manner so as to give a finite contribution in combination with the $\mathcal{O}(\epsilon)$ terms. This cannot be ascertained just from the relativistic answer. We need to examine whether the second part survives in the non-relativistic limit by verifying whether it is possible to satisfy (6.3.2) by keeping the $\mathcal{O}(\epsilon)$ terms. Examining the three point functions of the various component fields with the extra terms, we find that (6.3.2) is not satisfied. Hence we conclude that $G^{(3)}_{\mathrm{2d\,SGCA}}$ is completely specified by (6.3.19).

6.4 SGCA null vectors

Just as in the representation of the SuperVirasoro algebra, we will find that there are null states in the SGCA tower built on a primary $|\Delta, \xi\rangle$ for special values of (Δ, ξ). These are states which are orthogonal to all states in the tower including itself.

We can find the null states at a given level by writing the most general state at that level as a combination of the $L_{-m}, M_{-n}, G_{-\frac{r}{2}}, H_{-\frac{s}{2}}$'s and their products (for $m, n, r, s > 0$)

acting on the SGCA primary, and then imposing the condition that all the positive modes $L_m, M_n, G_{\frac{r}{2}}, H_{\frac{s}{2}}$ (with $m,n,r,s>0$) annihilate this state. This will give conditions that fix the relative coefficients in the linear combination as well as give a relation between Δ, ξ and the central charges C_1, C_2. This procedure will give us null states which are primaries and descendants at the same time. These are called "singular vectors".

In this context, we would like to mention that for $C_2 = 0$, since the vacuum state satisfies (6.2.12), all states of the form $M_{-n}|0\rangle$ and $H_{-s}|0\rangle$ (for $n,s>0$) are null states, as their correlation functions with other primaries and secondaries will vanish. Similarly, for $C_1 = C_2 = 0$, $L_{-m}|0\rangle$ and $G_{-r}|0\rangle$ (for $m,r>0$) will also be null states.[4] Hence, for these special cases, the correlation functions will satisfy much stronger constraints as stated below:

(a) For $C_2 = 0$, the correlators are invariant under the generators M_{-n} and H_{-s}, resulting in the equations:

$$\Big[\sum_{i=1}^{k}\Big\{-\frac{1}{t_i^{n-1}}\,\partial_{x_i} - \frac{(n-1)\,\xi_i}{t_i^n} + \frac{(n-1)\,\alpha_i}{2\,t_i^n}\,\partial_{\beta_i}\Big\}\Big]\,G^{(k)}_{\text{2d SGCA}} = 0\,, \tag{6.4.1}$$

$$\Big[\sum_{i=1}^{k}\Big\{\frac{1}{t_i^{s-\frac{1}{2}}}\,(\alpha_i\partial_{x_i} - \partial_{\beta_i}) + 2\,(s-\frac{1}{2})\,\frac{\xi_i\,\alpha_i}{t_i^{s+\frac{1}{2}}}\Big\}\Big]\,G^{(k)}_{\text{2d SGCA}} = 0\,. \tag{6.4.2}$$

Acting on the two point function $G^{(2)}_{\text{2d SGCA}}$, these constraints give the condition $\xi = 0$, which removes the spatial and β dependence of the correlators[5].

(b) For $C_1 = C_2 = 0$, the correlators are invariant under the generators L_{-m}, G_{-r}, M_{-n} and H_{-s}, resulting in the equations:

$$\begin{aligned}\Big[\sum_{i=1}^{k}\Big\{&\frac{1}{t_i^{m-1}}\,\partial_{t_i} - \frac{(m-1)\,x_i}{t_i^m}\,\partial_{x_i} - \frac{m-1}{t_i^m}\Big(\Delta_i + \frac{m\,\xi_i\,x_i}{t_i}\Big)\\ &-\frac{m-1}{2\,t_i^m}\Big(\alpha_i\,\partial_{\alpha_i} + \beta_i\,\partial_{\beta_i} - \frac{m\,x_i\,\alpha_i}{t_i}\,\partial_{\beta_i}\Big)\Big\}\Big]\,G^{(k)}_{\text{2d SGCA}} = 0\,,\end{aligned} \tag{6.4.3}$$

$$\begin{aligned}\Big[\sum_{i=1}^{k}\Big\{&\frac{1}{t_i^{r-\frac{1}{2}}}\,(-\alpha_i\partial_{t_i} - \beta_i\partial_{x_i} + \partial_{\alpha_i}) - (r-\frac{1}{2})\,\frac{x_i}{t_i^{r+\frac{1}{2}}}\,(-\alpha_i\partial_{x_i} + \partial_{\beta_i})\\ &-2\,(r-\frac{1}{2})\,\frac{1}{t_i^{r+\frac{1}{2}}}\,(\xi_i\,\beta_i - \Delta_i\,\alpha_i) + 2\,(r^2-\frac{1}{4})\,\frac{\xi_i\,x_i\,\alpha_i}{t_i^{r+\frac{3}{2}}}\Big\}\Big]\,G^{(k)}_{\text{2d SGCA}} = 0\,,\end{aligned} \tag{6.4.4}$$

---

[4] Note that these null states are not highest-weight states, and hence not singular vectors. We thank the referee for emphasizing this point.

[5] This follows from the fact that all x and β dependence arises in combination with the ξ dependence so as to survive the non-relativistic limit.

in addition to (6.4.1) and (6.4.2). Acting on the two point function $G^{(2)}_{2d\,\mathrm{SGCA}}$, these constraints give the condition $\xi = \Delta = 0$, which simply means that there is no primary in the theory except the vacuum state.

Hence, these sectors are quite trivial, and in all discussions that follow, we will assume that at least $C_2 \neq 0$.

6.4.1 The intrinsic SGCA analysis

At level $\frac{1}{2}$, we can consider a general state $(a\,G_{-\frac{1}{2}} + b\,H_{-\frac{1}{2}})\,|\Delta,\xi\rangle$. One can check that we get two linearly independent null states: $G_{-\frac{1}{2}}|\Delta = 0, \xi = 0\rangle$ and $H_{-\frac{1}{2}}|\Delta, \xi = 0\rangle$.

At level one, we have the general state $(a\,L_{-1} + b\,M_{-1} + c\,G_{-\frac{1}{2}}H_{-\frac{1}{2}})\,|\Delta,\xi\rangle$ (note that this is the most general linear combination of the lowering operators at this level, remembering the relation $\{G_{-\frac{1}{2}}, H_{-\frac{1}{2}}\} = 2M_{-1}$). It is easy to check that one has three linearly independent null states given by $L_{-1}|\Delta = 0, \xi = 0\rangle$, $M_{-1}|\Delta, \xi = 0\rangle$ and $G_{-\frac{1}{2}}H_{-\frac{1}{2}}|\Delta, \xi = 0\rangle$.

At level $\frac{3}{2}$, things are a little more non-trivial. Let us consider the most general level $\frac{3}{2}$ state of the form

$$|\chi\rangle = (a\,G_{-\frac{3}{2}} + b\,L_{-1}G_{-\frac{1}{2}} + c\,M_{-1}G_{-\frac{1}{2}} + d\,H_{-\frac{3}{2}} + e\,L_{-1}H_{-\frac{1}{2}} + f\,M_{-1}H_{-\frac{1}{2}})\,|\Delta,\xi\rangle\,. \quad (6.4.5)$$

We now impose the conditions that $G_{\frac{1}{2},\frac{3}{2}}, H_{\frac{1}{2},\frac{3}{2}}, L_1, M_1$ annihilate this state[6], using (6.1.3). This gives us the following set of conditions:

$$\begin{aligned}
&\xi\,[\,2a + (1+2\Delta)b + 2\xi e\,] = 0\,,\\
&\Delta\,[\,2a + (1+2\Delta)b + 2\xi e\,] + \xi\,[\,(1+2\Delta)c + 2d + e + 2\xi f\,] = 0\,,\\
&\xi^2\,b = 0\,,\\
&(2\Delta+1)\xi b + 2\xi(a+\xi c) = 0\,,\\
&(\Delta+4C_1)a + 2\Delta b + (\xi+4C_2)d + 2(c+e)\xi = 0\,,\\
&(\xi+4C_2)a + 2\xi b = 0\,,\\
&\xi\,[\,2a + (1+2\Delta)b + 2\xi c\,] = 0\,,\\
&\Delta\,[\,2a + (1+2\Delta)b + 2\xi c\,] + \xi\,[\,c + 2d + (1+2\Delta)e + 2\xi f\,] = 0
\end{aligned} \quad (6.4.6)$$

We will now separately consider the two cases where $\xi \neq 0$ and $\xi = 0$.

For the case $\xi \neq 0$, we get the conditions: $b = 0$, $c = e = -\frac{a}{\xi}$, and $f = \frac{(\Delta+1)a}{\xi^2} - \frac{d}{\xi}$. Now we have two further options: either $a = 0$ or $a \neq 0$.

For $a = 0$, to get a non-trivial solution, we must have $\xi = -4C_2$, and the null state is of the form:

$$|\chi^{(1)}\rangle = (H_{-\frac{3}{2}} - \frac{1}{\xi}M_{-1}H_{-\frac{1}{2}})\,|\Delta,\xi\rangle\,. \quad (6.4.7)$$

[6] This is sufficient as the annihilation condition for all the other higher level positive modes are then automatically satisfied.

For $a \neq 0$, we are led to the following consistency conditions: $\xi = -4C_2$ and $\Delta = 4(1-C_1)$. In this case, both a and d can be arbitrary and all other coefficients are determined in terms of these. However, by taking a suitable linear combination with $|\chi^{(1)}\rangle$, we can choose $d = 0$, and then we get another null state of the form:

$$|\chi^{(2)}\rangle = [\, G_{-\frac{3}{2}} - \frac{1}{\xi} M_{-1} G_{-\frac{1}{2}} - \frac{1}{\xi} L_{-1} H_{-\frac{1}{2}} + \frac{(\Delta+1)}{\xi^2} M_{-1} H_{-\frac{1}{2}} \,]\, |\Delta, \xi\rangle \,. \qquad (6.4.8)$$

For the case $\xi = 0$, since $C_2 \neq 0$, we must have $a = 0$, $\Delta(2\Delta+1)b = 0$, $d = -\frac{\Delta b}{2C_2}$, and c, e, f are undetermined. For $\Delta \neq -\frac{1}{2}, 0$ in general, we therefore get three null states: $G_{-\frac{1}{2}} M_{-1} |\Delta, \xi = 0\rangle$, $L_{-1} H_{-\frac{1}{2}} |\Delta, \xi = 0\rangle$ and $M_{-1} H_{-\frac{1}{2}} |\Delta, \xi = 0\rangle$. For $\Delta = 0$, we also obtain $d = 0$, and in this case b is also undetermined. Hence, by taking suitable linear combinations with the three null states for $b = 0$, we get a new null state of the form $L_{-1} H_{-\frac{1}{2}} |\Delta = 0, \xi = 0\rangle$. For $\Delta = -\frac{1}{2}$, we also have $d = \frac{b}{4C_2}$, and again b is also undetermined. Taking appropriate linear combinations with the three states for $b = 0$, we get a new null state of the form:

$$|\chi^{(3)}\rangle = (L_{-1} G_{-\frac{1}{2}} + \frac{1}{4C_2} H_{-\frac{3}{2}})\, |\Delta = -\frac{1}{2}, \xi = 0\rangle \,. \qquad (6.4.9)$$

Crucially, we note that all the above null states for $\xi = 0$, except $|\chi^{(3)}\rangle$, are descendants of the level $\frac{1}{2}$ and level 1 null states.

6.4.2 SGCA null vectors from 2d SCFT

If we want to examine the SGCA null states at a general level, we would have to perform an analysis similar to that in the SuperVirasoro representation theory. A cornerstone of this analysis is the Kac determinant which gives the values of the weights of the SuperVirasoro Primaries $h\,(\bar{h})$ for which the matrix of inner products at a given level has a zero eigenvalue. For the NS algebra, this determinant is given by (found by Kac [139]):

$$\det M_{(l)} = const. \prod (h - h_{p,q}(c))^{P_{NS}(l - \frac{pq}{2})} \,, \qquad (6.4.10)$$

where the product runs over positive integers p, q with $\frac{pq}{2} \leq l$ and $|p-q|$ even. Here $P_{NS}(k)$ is the number of states, arising from a ground state, at level k:

$$\sum_{k=0}^{\infty} \frac{1 + t^{k-\frac{1}{2}}}{1 - t^k} \,. \qquad (6.4.11)$$

The functions $h_{p,q}(c)$ can be expressed in a variety of ways. One convenient representation is:

$$h_{p,q}(c) = h_0 + \frac{1}{4}(p\,\alpha_+ + q\,\alpha_-)^2\,, \tag{6.4.12}$$

$$h_0 = \frac{1}{16}(c-1)\,, \tag{6.4.13}$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{9-c}}{4}\,. \tag{6.4.14}$$

One can write a similar expression for the antiholomorphic sector. The values $h_{p,q}$ are the ones for which we have zeroes of the determinant and hence null vectors (and their descendants).

One could presumably generalise our analysis for SGCA null vectors at level $\frac{3}{2}$ and directly obtain the SGCA determinant at a general level. This would give us a relation for Δ and ξ in terms of C_1, C_2 for which there are null states, generalising the results obtained at level $\frac{3}{2}$. However, instead of a direct analysis, here we will simply take the non-relativistic limit of the Kac formula and see whether one obtains sensible expressions for the Δ and ξ on the SGCA side.

In taking the non-relativistic limit, C_2 is chosen to be positive. Therefore (from (6.1.4)) we need to take $c \ll -1$ and $\bar{c} \gg 1$ as $\epsilon \to 0$. We then find

$$h_{p,q} = \frac{C_2}{4\epsilon}\,(p^2-1) + \frac{1}{16}[-1+5p^2-4pq-4C_1(p^2-1)\,] + \mathcal{O}(\epsilon)\,, \tag{6.4.15}$$

$$\bar{h}_{p',q'} = -\frac{C_2}{4\epsilon}\,(p'^2-1) + \frac{1}{16}[-1+5p'^2-4p'q'-4C_1(p'^2-1)\,] + \mathcal{O}(\epsilon)\,. \tag{6.4.16}$$

Using (6.2.6) and taking $p = p'$ [7]

$$\Delta_{p(q,q')} = \lim_{\epsilon\to 0}(h_{p,q} + \bar{h}_{p,q'}) = -\frac{1}{2}C_1\,(p^2-1) + \frac{1}{8}\,[\,5p^2 - 2p(q+q') - 1\,]\,, \tag{6.4.17}$$

$$\xi_{p(q,q')} = -\lim_{\epsilon\to 0}\epsilon\,(h_{p,q} - \bar{h}_{p,q'}) = -\frac{1}{2}C_2\,(p^2-1)\,. \tag{6.4.18}$$

However, we would like to caution the reader that this non-relativistic limit of the Kac formula does not give us all the null states of the SGCA (see the following subsection).

In the following discussion, we will focus on the null vectors at level $\frac{3}{2}$. The null vector at level $\frac{3}{2}$ in a SuperVirasoro tower is given by (see [137])

$$|\chi_L\rangle = (\,\mathcal{G}_{-\frac{3}{2}} + \eta\,\mathcal{L}_{-1}\mathcal{G}_{-\frac{1}{2}}\,)\,|h\rangle \otimes |\bar{h}\rangle\,, \tag{6.4.19}$$

with

$$\eta = -\frac{2}{2h+1}\,, \tag{6.4.20}$$

[7]Requiring that Δ should not have a $\frac{1}{\epsilon}$ piece immediately implies that $p = p'$.

$$h = \frac{1}{4}\Big\{3 - c \pm \sqrt{(1-c)(9-c)}\Big\}\,, \tag{6.4.21}$$

where the positive and negative signs before the square root correspond to the primaries of conformal weights $h_{3,1}$ and $h_{1,3}$, respectively (see (6.4.12)). One has a similar null state for the antiholomorphic SuperVirasoro obtained by replacing $\mathcal{L}_n \to \bar{\mathcal{L}}_n$, $\mathcal{G}_r \to \bar{\mathcal{G}}_r$, $h \to \bar{h}$ and $c \to \bar{c}$.

For $h = h_{3,1}$ and $\bar{h} = \bar{h}_{3,1}$, we get

$$\xi = -4C_2\,, \qquad \Delta = 4(1 - C_1)\,. \tag{6.4.22}$$

These are precisely the relations we obtained in the previous section if we require the existence of both the SGCA null states $|\chi^{(1)}\rangle$, $|\chi^{(2)}\rangle$ at level $\frac{3}{2}$.

These states themselves can be obtained by taking the non-relativistic limit on appropriate combinations of the relativistic null vectors $|\chi_L\rangle$ and its antiholomorphic counterpart $|\chi_R\rangle$. Consider

$$|\chi^{(1)}\rangle = \lim_{\epsilon\to 0} \epsilon\,(-|\chi_L\rangle + |\chi_R\rangle\,)\,, \qquad |\chi^{(2)}\rangle = \lim_{\epsilon\to 0}(\,|\chi_L\rangle + |\chi_R\rangle\,). \tag{6.4.23}$$

From the expressions (6.2.6), we obtain $\eta = \frac{2\epsilon}{\xi}(1 + \frac{(\Delta+1)\,\epsilon}{\xi})$ and $\bar{\eta} = -\frac{2\epsilon}{\xi}(1 - \frac{(\Delta+1)\,\epsilon}{\xi})$ upto terms of order ϵ^2 . Substituting this into (6.4.23) and using the relations (6.1.2), we obtain

$$\begin{aligned} |\chi^{(1)}\rangle &= (H_{-\frac{3}{2}} - \frac{1}{\xi}M_{-1}H_{-\frac{1}{2}})\,|\Delta,\xi\rangle\,,\\ |\chi^{(2)}\rangle &= \Big\{\,G_{-\frac{3}{2}} - \frac{1}{\xi}M_{-1}G_{-\frac{1}{2}} - \frac{1}{\xi}L_{-1}H_{-\frac{1}{2}} + \frac{(\Delta+1)}{\xi^2}M_{-1}H_{-\frac{1}{2}}\,\Big\}\,|\Delta,\xi\rangle\,, \end{aligned} \tag{6.4.24}$$

which are exactly what we found from the intrinsic SGCA analysis in (6.4.7) and (6.4.8).

For the case $h = h_{1,3}$ and $\bar{h} = \bar{h}_{1,3}$, we find that $\Delta_{1(3,3)} = -1$ and $\xi_{1(3,3)} = 0$. This is also easily seen to correspond to the null states constructed in §6.4.1 for $\xi = 0$ and $\Delta \neq -\frac{1}{2}\,, 0$ (which we have seen are descendants of level one null states).

A point to observe here is that the expansion of relativistic null state expressions (such as (6.4.19)) in powers of ϵ gives us non-relativistic null states when we consider only the coefficients of the first two lowest powers of ϵ.[8] Also, one should consistently expand h and $\bar{h}$ only upto $\mathcal{O}(1)$ (and not beyond) while considering any such expression, because of the definition of the non-relativistic generators in (6.1.2).

[8]In fact this is true for any expression/result of the relativistic theory, from which we want to extract the corresponding non-relativistic analogue. This directly follows from the fact that we have obtained the non-relativistic algebra by retaining only the $\mathcal{O}(\frac{1}{\epsilon})$ and $\mathcal{O}(1)$ terms of the relativistic algebra.

6.4.3 Discussion on SGCA null states not obtained from SCFT null states

We would like to point out that though we find the limiting process gives answers consistent with the intrinsic SGCA analysis, working purely within SGCA, we get some null states which are not obtained in the SCFT case. These extra null states are not initally null in SCFT, but become null in the non-relativistic scaling limit. We list such null states obtained at level $\frac{3}{2}$:

(i) $|\chi^{(1)}\rangle$ in (6.4.7) has $\xi = -4C_2$ but no restriction on Δ. On the other hand, $|\chi^{(1)}\rangle$ obtained in (6.4.24) from SCFT null states, has $\Delta = 4(1 - C_1)$ in addition to $\xi = -4C_2$. This clearly shows that we have more null states for $\xi = -4C_2$ from the intrinsic SGCA analysis.

(ii) $|\chi^{(3)}\rangle$ in (6.4.9) descends from a state

$$\Big\{\bar{\mathcal{G}}_{-3/2} - \mathcal{G}_{-\frac{3}{2}} + \frac{1}{2}(\bar{c}-c)\,(\,\bar{\mathcal{L}}_{-1}\bar{\mathcal{G}}_{-\frac{1}{2}} + \mathcal{L}_{-1}\mathcal{G}_{-\frac{1}{2}} + \bar{\mathcal{L}}_{-1}\mathcal{G}_{-\frac{1}{2}} + \mathcal{L}_{-1}\bar{\mathcal{G}}_{-1/2}\,)\Big\}|h = -\frac{1}{4}, \bar{h} = -\frac{1}{4}\rangle \tag{6.4.25}$$

on the SCFT side, which is not null. This is because, while analysing null state conditions, we never take linear combinations of descendants having different $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$ eigenvalues. On the other hand, descendant states in SGCA are eigenstates of L_0, but not necessarily of M_0. Hence we get a valid null state from the above state in SCFT, after the limiting process.

Hence we conclude that within the SGCA framework, we get more constraints arising from the differential equations involving the extra null states, over and above those resulting from SCFT. This means we get new fusion rules involving the primaries corresponding to these null states.[9]

6.5 Differential equations for SGCA correlators from null states

The presence of the null states gives additional relations between correlation functions which is at the heart of the solvability of relativistic (rational) (super)conformal field theories. To obtain these relations one starts with the differential operator realisations $\hat{\mathcal{L}}_{-n}$ and $\hat{\mathcal{G}}_{-r}$ of $\mathcal{L}_{-n}$ and $\mathcal{G}_{-r}$ respectively (with $n, r > 0$). Thus one has

$$\begin{aligned}
\langle\Phi_k(\mathcal{Z}_k, \bar{\mathcal{Z}}_k)\,\cdots\,\Phi_2(\mathcal{Z}_2, \bar{\mathcal{Z}}_2)\{\mathcal{L}_{-n}\,\Phi_1(0,0)\}\rangle &= \hat{\mathcal{L}}_{-n}\langle\Phi_k(\mathcal{Z}_k, \bar{\mathcal{Z}}_k)\,\cdots\,\Phi_2(\mathcal{Z}_2, \bar{\mathcal{Z}}_2)\Phi_1(0,0)\rangle,\\
\langle\Phi_k(\mathcal{Z}_k, \bar{\mathcal{Z}}_k)\,\cdots\,\Phi_2(\mathcal{Z}_2, \bar{\mathcal{Z}}_2)\{\mathcal{G}_{-r}\,\Phi_1(0,0)\}\rangle &= \hat{\mathcal{G}}_{-r}\langle\Phi_k(\mathcal{Z}_k, \bar{\mathcal{Z}}_k)\,\cdots\,\Phi_2(\mathcal{Z}_2, \bar{\mathcal{Z}}_2)\Phi_1(0,0)\rangle,
\end{aligned}$$

[9] We discuss these issues a bit more elaborately in the concluding remarks, where we also mention the future directions we would like to follow to get a better understanding.

where

$$\hat{\mathcal{L}}_{-n} = \sum_{i=2}^{k} \left\{ \frac{(n-1)h_i}{z_i^n} + \frac{n-1}{2}\, \frac{\theta_i}{z_i^n}\, \partial_{\theta_i} - \frac{1}{z_i^{n-1}}\, \partial_{z_i} \right\} , \tag{6.5.1}$$

$$\hat{\mathcal{G}}_{-r} = \sum_{i=2}^{k} \text{sign}_i \left\{ \frac{(2r-1)h_i}{z_i^{r+\frac{1}{2}}} + \frac{1}{z_i^{r-\frac{1}{2}}}\, (\, \partial_{\theta_i} - \theta_i\, \partial_{z_i}\,) \right\} , \tag{6.5.2}$$

where sign_i is $+1$ and -1 for bosonic and fermionic superfields respectively. One can write analogous expressions for the antiholomorphic sector.

For the SGCA also we can construct such operators. Firstly, we derive the expressions entirely from the SGCA side.

Let us assume that we have a null state at a level l, which is a descendant of (the lowest component of) the primary superfield $\Phi_1(t_1, x_1, \alpha_1, \beta_1)$, represented as $f(\{L_{-n}, M_{-m}, G_{-r}, H_{-s}\})\, \Phi$ where f is the appropriate linear combination of the products of the SGCA generators (with $n, m, r, s > 0$ and the level adding up to l) such that the null state conditions are satisfied. Since the null states are orthogonal to all states, we have the condition:

$$\langle 0|\, \Phi_k(t_k, x_k, \alpha_k, \beta_k)\, \cdots\, \Phi_2(t_2, x_2, \alpha_2, \beta_2)\, [\, f(\{L_{-n}, M_{-m}, G_{-r}, H_{-s}\})\, \Phi_1(0,0,0,0)\,]\, |0\rangle = 0\,. \tag{6.5.3}$$

Using (6.2.11) and the fact that L_{-n} , M_{-m} , G_{-r} , H_{-s} annihilate $\langle 0|$, we commute f past all the Φ_i's and obtain the expression:

$$f(\{\hat{L}_{-n}, \hat{M}_{-m}, \hat{G}_{-r}, \hat{H}_{-s}\})\, \langle 0|\, \Phi_k(t_k, x_k, \alpha_k, \beta_k)\, \cdots\, \Phi_2(t_2, x_2, \alpha_2, \beta_2)\, \Phi_1(0,0,0,0)\, |0\rangle = 0\,, \tag{6.5.4}$$

where the differential operators acting on the correlation function are given by:

$$\begin{aligned}
\hat{L}_{-n} &= -\sum_{i=2}^{k} \Big\{ \frac{1}{t_i^{n-1}}\, \partial_{t_i} - \frac{(n-1)\, x_i}{t_i^n}\, \partial_{x_i} - \frac{n-1}{t_i^n} \Big(\Delta_i + \frac{n\, \xi_i\, x_i}{t_i} \Big) \\
&\qquad - \frac{n-1}{2\, t_i^n} \Big(\alpha_i\, \partial_{\alpha_i} + \beta_i\, \partial_{\beta_i} - \frac{n\, x_i\, \alpha_i}{t_i}\, \partial_{\beta_i} \Big) \Big\} , \\
\hat{M}_{-m} &= -\sum_{i=2}^{k} \Big\{ -\frac{1}{t_i^{m-1}}\, \partial_{x_i} - \frac{(m-1)\, \xi_i}{t_i^m} + \frac{(m-1)\, \alpha_i}{2\, t_i^m}\, \partial_{\beta_i} \Big\} , \\
\hat{G}_{-r} &= -\sum_{i=2}^{k} \text{sign}_i \Big\{ \frac{1}{t_i^{r-\frac{1}{2}}} (-\alpha_i \partial_{t_i} - \beta_i \partial_{x_i} + \partial_{\alpha_i}) - (r - \frac{1}{2})\, \frac{x_i}{t_i^{r+\frac{1}{2}}} (-\alpha_i \partial_{x_i} + \partial_{\beta_i}) \\
&\qquad -2\, (r - \frac{1}{2})\, \frac{1}{t_i^{r+\frac{1}{2}}} (\xi_i\, \beta_i - \Delta_i\, \alpha_i) + 2\, (r^2 - \frac{1}{4})\, \frac{\xi_i\, x_i\, \alpha_i}{t_i^{r+\frac{3}{2}}} \Big\} , \\
\hat{H}_{-s} &= -\sum_{i=2}^{k} \text{sign}_i \Big\{ \frac{1}{t_i^{s-\frac{1}{2}}} (\alpha_i \partial_{x_i} - \partial_{\beta_i}) + 2\, (s - \frac{1}{2})\, \frac{\xi_i\, \alpha_i}{t_i^{s+\frac{1}{2}}} \Big\} ,
\end{aligned} \tag{6.5.5}$$

[10]Note that $\Phi_1(0,0,0,0)\, |0\rangle = \phi_{11}(0,0)\, |0\rangle$.

where once again we note that the factor sign_i is necessary to account for the minus sign when commuting f through a fermionic superfield[11].

It follows directly from (6.1.2) that expanding the operators $\hat{\mathcal{L}}_{-n}$ and $\hat{\mathcal{G}}_{-r}$ as

$$\begin{aligned}\hat{\mathcal{L}}_{-n} &= \epsilon^{-1}\hat{\mathcal{L}}^{(-1)}_{-n} + \hat{\mathcal{L}}^{(0)}_{-n} + \mathcal{O}(\epsilon)\,,\\ \hat{\mathcal{G}}_{-r} &= \epsilon^{-1}\hat{\mathcal{G}}^{(-1)}_{-r} + \hat{\mathcal{G}}^{(0)}_{-r} + \mathcal{O}(\epsilon)\,,\end{aligned}$$

(and similarly for the antiholomorphic part), we get expressions for the differential operators $\hat{M}_{-n}$, $\hat{L}_{-n}$, $\hat{H}_{-r}$ and $\hat{G}_{-r}$ which match exactly with (6.5.5).

Therefore, correlation functions involving an SGCA descendant of a primary field are given in terms of the correlators of the primaries by the action of the corresponding differential operators $\hat{M}_{-n}$, $\hat{L}_{-n}$, $\hat{H}_{-r}$ and $\hat{G}_{-r}$.

Now we will study the consequences of having null states at level $\frac{3}{2}$. We will consider the two null states $|\chi^{(1)}\rangle$, $|\chi^{(2)}\rangle$ of Sec. 5.1, or rather correlators involving the corresponding fields $\chi^{(1,2)}(t_1, x_1)$. Setting the null state and thus its correlators to zero gives rise to differential equations for the correlators involving the primary superfield $\Phi_{\Delta_1,\xi_1}(t_1, x_1, \alpha_1, \beta_1)$[12] with other fields. Using the forms (6.4.7) and (6.4.8), we find that the differential equations take the form

$$(\hat{H}_{-\frac{3}{2}} - \frac{1}{\xi}\hat{M}_{-1}\hat{H}_{-\frac{1}{2}})\,\langle\, \Phi_k(t_k, x_k, \alpha_k, \beta_k)\ \cdots\ \Phi_2(t_2, x_2, \alpha_2, \beta_2)\, \Phi_1(0, 0, 0, 0)\,\rangle = 0\,, \tag{6.5.6}$$

$$\begin{aligned}&\Big[\,\hat{G}_{-\frac{3}{2}} - \frac{1}{\xi_1}\hat{M}_{-1}\hat{G}_{-\frac{1}{2}} - \frac{1}{\xi_1}\hat{L}_{-1}\hat{H}_{-\frac{1}{2}}\\ &\quad + \frac{(\Delta_1 + 1)}{\xi_1^2}\,\hat{M}_{-1}\hat{H}_{-\frac{1}{2}}\,\Big]\,\langle\, \Phi_k(t_k, x_k, \alpha_k, \beta_k)\ \cdots\ \Phi_2(t_2, x_2, \alpha_2, \beta_2)\, \Phi_1(0, 0, 0, 0)\,\rangle = 0\,,\end{aligned} \tag{6.5.7}$$

with $\hat{M}_{-n}$, $\hat{L}_{-n}$, $\hat{H}_{-r}$ and $\hat{G}_{-r}$ as given in (6.5.5).

6.6 SGCA fusion rules

Analogous to the relativistic case (see [135] and [138]), we can derive "*Fusion rules*",

$$[\Phi_1] \times [\Phi_2] \simeq \sum_f [\Phi_f]\ , \tag{6.6.1}$$

for the SGCA superconformal families, that determine which families $[\Phi_f]$ have their primaries and descendants occurring in an OPE of any two members of the families $[\Phi_1]$ and $[\Phi_2]$. Here we have denoted a family $[\Phi_i]$ by the corresponding primary superfield Φ_i.

[11] However, the reader should note that, though not stated explicitly, we have assumed correlation functions of bosonic superfields everywhere in this paper.

[12] Note that $\chi^{(1,2)}(t_1, x_1)$ is the descendant of the lowest component of Φ_{Δ_1,ξ_1}.

We illustrate how the fusion rules can be obtained for the families $[\Phi_{\Delta_1,\xi_1}]$ and $[\Phi_{\Delta_2,\xi_2}]$, where both fields are members of the the non-relativistic limit of the Kac table as specified by (6.4.15) and (6.4.16). As mentioned in footnote 7, we need to take $p = p'$. The resulting (Δ, ξ) are thus labelled by a triple $\{p(q,q')\}$. In particular, we will consider below the case of $\Delta_1 = \Delta_{3(1,1)}$ and $\xi_1 = \xi_{3(1,1)}$.

The fusion rules are derived from applying the condition that Φ_{Δ_1,ξ_1} has a null descendant at level $\frac{3}{2}$. For (Δ_2, ξ_2), we will consider a general member $\Phi_{p(q,q')}$.[13] Thus we have from (6.4.22), (6.4.17) and (6.4.18):

$$\begin{aligned} \Delta_1 &= \Delta_{3(1,1)} = 4(1-C_1)\,, \quad \xi_1 = \xi_{3(1,1)} = -4C_2\,; && (6.6.2)\\ \Delta_2 &= \Delta_{p(q,q')} = -\frac{1}{2}C_1\,(p^2-1) + \frac{1}{8}\,[\,5p^2 - 2p(q+q') - 1\,]\,, && (6.6.3)\\ \xi_2 &= \xi_{p(q,q')} = -\frac{1}{2}C_2\,(p^2-1)\,. && (6.6.4) \end{aligned}$$

We need to consider the conditions (6.5.6) and (6.5.7) for the case of the three point function. With

$$G^{(3)}_{\text{2d SGCA}}(\{t_i, x_i, \alpha_i, \beta_i\}) = \langle\, \Phi_{\Delta_3,\xi_3}(t_3, x_3, \alpha_1, \beta_1)\, \Phi_{\Delta_2,\xi_2}(t_2, x_2, \alpha_2, \beta_2)\, \Phi_{\Delta_1,\xi_1}(0,0,0,0)\,\rangle\,, \qquad (6.6.5)$$

these give the constraints:

$$\Big[-\sum_{i=2}^{3}\Big\{\frac{1}{t_i}(\alpha_i\partial_{x_i} - \partial_{\beta_i}) + \frac{2\xi_i}{t_i^2}\,\alpha_i\Big\} + \frac{1}{\xi_1}\sum_{i=2}^{3}\partial_{x_i}\sum_{j=2}^{3}(\alpha_j\partial_{x_j} - \partial_{\beta_j})\Big]\, G^{(3)}_{\text{2d SGCA}} = 0\,, \quad (6.6.6)$$

$$\begin{aligned} &\Big[-\sum_{i=2}^{3}\Big\{\frac{1}{t_i}(-\alpha_i\partial_{t_i} - \beta_i\partial_{x_i} + \partial_{\alpha_i}) - \frac{\xi_i}{t_i^2}\,(-\alpha_i\partial_{x_i} + \partial_{\beta_i}) - \frac{2}{t_i^2}\,(\xi_i\beta_i - \Delta_i\alpha_i) + \frac{2\xi_i}{t_i^3}x_i\alpha_i\Big\}\\ &+ \frac{1}{\xi_1}\sum_{i=2}^{3}\partial_{x_i}\sum_{j=2}^{3}(-\alpha_i\partial_{t_i} - \beta_i\partial_{x_i} + \partial_{\alpha_i}) - \frac{1}{\xi_1}\sum_{i=2}^{3}\partial_{t_i}\sum_{j=2}^{3}(\alpha_i\partial_{x_j} - \partial_{\beta_j})\\ &- \frac{\Delta_1+1}{\xi_1^2}\sum_{i=2}^{3}\partial_{x_i}\sum_{j=2}^{3}(\alpha_j\partial_{x_j} - \partial_{\beta_j})\Big]\, G^{(3)}_{\text{2d SGCA}} = 0\,, \end{aligned}$$

respectively. Now by using (6.3.19), these translate into

$$\begin{aligned} &\xi_1\,(\xi_2+\xi_3) - (\xi_2-\xi_3)^2 = 0\,,\\ &(\Delta_2+\Delta_3-1)\,\xi_1^2 - 2\,(\Delta_2-\Delta_3)\,(\xi_2-\xi_3)\,\xi_1 + (\Delta_1+1)\,(\xi_2-\xi_3)^2 = 0\,. \end{aligned}$$

[13] Here we assume that $\Phi_{p(q,q')}$ has no extra null descendant other than those obtained from the non-relativistic limit of the $(h_{p,q}, \bar{h}_{p,q'})$ null states. While this is seen to be true for the level $\frac{3}{2}$, one needs to construct a formalism to verify this for any arbitrary level in the Kac table.

Solving the above equations, we get two simple sets of solutions:

$$\xi_3 = -\frac{1}{2}C_2\left\{(p\pm 2)^2-1\right\}, \; \Delta_3 = -\frac{1}{2}C_1\left\{(p\pm 2)^2-1\right\}+\frac{1}{8}\left\{5\,(p\pm 2)^2-2(p\pm 2)\,(q+q')-1\right\}. \tag{6.6.7}$$

Comparing with (6.6.3) and (6.6.4), we see that

$$\Delta_3 = \Delta_{p\,\pm\,2\,(q,q')}\,, \qquad \xi_3 = \xi_{p\,\pm\,2\,(q,q')}\,, \tag{6.6.8}$$

which is exactly what the relativistic fusion rules imply, namely

$$[\Phi_{3(1,1)}]\times[\Phi_{p(q,q')}] = [\Phi_{p+2\,(q,q')}] + [\Phi_{p-2\,(q,q')}]\,. \tag{6.6.9}$$

Thus once again we see evidence for the consistency of the SGCA limit of the 2d SCFT. However, we would like to remind the reader that in the SCFT case, two independent fusion rules (dubbed "even" and "odd") arise (for each of the holomorphic and antiholomorphic sectors), as shown in [138], and their composition gives the full fusion rule. This is due to the presence of two independent constants for the SCFT three point function in each sector. But we have seen in (6.3.19) that when we multiply the results for the two sectors and take the limit, the contributions coming from the Grassmann odd terms of the corresponding sectors do not survive. So in the context of SGCA, only the *even* fusion rules of SCFT are relevant.

6.7 Summary

This concludes our present study of the supersymmetric extension of the GCA in two dimensions. We found that 2d SGCFTs, with non-zero central charges C_1 and C_2, can be readily obtained by considering a somewhat unusual limit of a non-unitary 2d SCFT. While the resulting Hilbert space of the SGCFT is again non-unitary, the theory seems to be otherwise well-defined. We found that many of the structures are parallel to those in the SuperVirasoro algebra and indeed arise from them when we realise the SGCA by means of the scaling limit. But in most cases we could also obtain many of the same results autonomously from the definition of the SGCA itself, showing that these are features of any realisation of this symmetry.

6.8 Appendix

6.8.1 SGCA Descendants

By means of the differential operators $\hat{M}_{-m}$, $\hat{L}_{-n}$, $\hat{H}_{-r}$ and $\hat{G}_{-s}$ (with $m,n,r,s>0$) in (6.5.5), we may express the correlation function including a general SGCA descendant in terms of the correlation function of the corresponding primary superfield $\Phi_{\Delta\xi}(t,x,\alpha,\beta)$. We have in fact already used this in Sec. 6 as (6.5.6) and (6.5.7), for the simple cases of these descendants corresponding to null states. The general expression can be written as

$$\begin{aligned}&\langle\, \Phi_k(t_k,x_k,\alpha_k,\beta_k)\,\cdots\,\Phi_2(t_2,x_2,\alpha_2,\beta_2)\,\Phi_1^{\{\vec{l},\vec{q},\vec{u},\vec{v}\}}(0,0,0,0)\,\rangle\\ &= \hat{L}_{-l_i}\cdots\hat{L}_{-l_1}\,\hat{M}_{-q_j}\cdots\hat{M}_{-q_1}\,\hat{G}_{-u_{i'}}\cdots\hat{G}_{-u_1}\\ &\qquad \hat{H}_{-v_{j'}}\cdots\hat{H}_{-v_1}\,\langle\, \Phi_k(t_k,x_k,\alpha_k,\beta_k)\,\cdots\,\Phi_2(t_2,x_2,\alpha_2,\beta_2)\,\Phi_1(0,0,0,0)\,\rangle\,,\end{aligned}$$

$$\begin{aligned}\text{for}\qquad &\Phi_1^{\{\vec{l},\vec{q},\vec{u},\vec{v}\}}(0,0,0,0)\,|0\rangle\\ &= L_{-l_i}\cdots L_{-l_1}\,M_{-q_j}\cdots M_{-q_1}\,G_{-u_{i'}}\cdots G_{-u_1}\,H_{-v_{j'}}\cdots H_{-v_1}\,\Phi_1(0,0,0,0)\,|0\rangle\,,\end{aligned}$$

where

$$\begin{aligned}&\vec{l}=(l_1,l_2,\cdots,l_i)\,,\quad \vec{q}=(q_1,q_2,\cdots,q_j)\,,\\ &\vec{u}=(u_1,u_2,\cdots,u_{i'})\quad\text{and}\quad \vec{v}=(v_1,v_2,\cdots,v_{j'})\end{aligned}$$

are sequences of positive integers such that $l_1\le l_2\cdots\le l_i$ and similarly for the q, u and v's. Also note that $\Phi_1^{\{0,0,0,0\}}(t_1,x_1,\alpha_1,\beta_1)$ denotes the primary $\Phi_1(t_1,x_1,\alpha_1,\beta_1)$ itself.

6.8.2 The OPE and SGCA blocks

Just as in the relativistic case, the OPE of two SGCA primary superfields can be expressed in terms of the SGCA primary superfields and their descendants as

$$\Phi_1(t,x,\alpha,\beta)\,\Phi_2(0,0,0,0)=\sum_p\sum_{\{\vec{l},\vec{q},\vec{u},\vec{v}\}}C_{12}^{p\{\vec{l},\vec{q},\vec{u},\vec{v}\}}(t,x,\alpha,\beta)\,\Phi_p^{\{\vec{l},\vec{q},\vec{u},\vec{v}\}}(0,0,0,0)\,. \qquad (6.8.1)$$

We should mention that, unlike in the case of a 2d SCFT, such an expansion is not analytic (see (6.8.6) below), as was also true for GCA in [82]. The form of the two and three point functions clearly exhibit essential singularities. Nevertheless we will go ahead with the expansion assuming it makes sense in individual segments such as $x,t>0$. One can find the first few coefficients $C_{12}^{p\{\vec{k},\vec{q},\vec{u},\vec{v}\}}(t,x,\alpha,\beta)$ by considering the three point function of the primary superfields $\langle\Phi_3\Phi_1\Phi_2\rangle$. In such a situation one can replace $\Phi_1\Phi_2$ in the three point function with the RHS of (6.8.1), and obtain

$$\begin{aligned}&\langle\, \Phi_3(t',x',\alpha',\beta')\,\Phi_1(t,x,\alpha,\beta)\,\Phi_2(0,0,0,0)\,\rangle\\ &=\sum_{p,\{\vec{l},\vec{q},\vec{u},\vec{v}\}}C_{12}^{p\{\vec{l},\vec{q},\vec{u},\vec{v}\}}(t,x,\alpha,\beta)\,\langle\, \Phi_3(t',x',\alpha',\beta')\,\Phi_p^{\{\vec{l},\vec{q},\vec{u},\vec{v}\}}(0,0,0,0)\,\rangle\,. \qquad (6.8.2)\end{aligned}$$

We can find $C_{12}^{p\{0,0,0,0\}}$, $C_{12}^{p\{0,0,1,0\}}$, $C_{12}^{p\{0,0,0,1\}}$, $C_{12}^{p\{1,0,0,0\}}$, $C_{12}^{p\{0,1,0,0\}}$ and $C_{12}^{p\{0,0,1,1\}}$ by expanding the LHS of (6.8.2) in powers of the parameter $\frac{t}{t'}$ with $\frac{x'}{t'}$, $\frac{x}{t}$, $\frac{\alpha'\alpha}{t'}$, $\frac{\alpha'\beta}{t'}$ and $\frac{\alpha\beta'}{t'}$ as coefficients, and comparing the $\{t', x', \alpha', \beta'\}$-dependence of both the sides. To make the final formulae simple, we concentrate on the case with $\Delta_1 = \Delta_2 = \Delta$ and $\xi_1 = \xi_2 = \xi$.

The expansion of the LHS is given as:

$$\begin{aligned}
&\langle\, \Phi_3(t',x',\alpha',\beta')\, \Phi_1(t,x,\alpha,\beta)\, \Phi_2(0,0,0,0)\,\rangle \\
=\;& C_{312}\,(t'-t-\alpha'\alpha)^{-\Delta_3}\, t^{\Delta_3-2\Delta}\,(-t')^{-\Delta_3} \exp\Big\{\xi_3\,\frac{x'-x-\alpha'\beta+\alpha\beta'}{t'-t-\alpha'\alpha} + (2\xi-\xi_3)\frac{x}{t} + \xi_3\frac{x'}{t'}\Big\} \\
=\;& C'_{312}\, t'^{-2\Delta_3}\, e^{2\xi_3\frac{x'}{t'}} \cdot t^{\Delta_3-2\Delta}\, e^{(2\xi-\xi_3)\frac{x}{t}} \Big[1 + \Delta_3\frac{\alpha'\alpha}{t'} + \xi_3\,(\frac{\alpha\beta'}{t'} - \frac{\alpha'\beta}{t'} + \frac{\alpha'\alpha}{t'}\frac{x'}{t'}) \\
&+ \Big\{\Delta_3 + \xi_3\,(\frac{x'}{t'} - \frac{x}{t}) + \Delta_3\frac{\alpha'\alpha}{t'} + \xi_3\,(\frac{\alpha\beta'}{t'} - \frac{\alpha'\beta}{t'} + 2\,\frac{\alpha'\alpha}{t'}\frac{x'}{t'} - \frac{\alpha'\alpha}{t'}\frac{x}{t})\Big\}\,\frac{t}{t'} \\
&+ \Big\{\Delta_3\frac{\alpha'\alpha}{t'} + \xi_3\,(\frac{\alpha\beta'}{t'} - \frac{\alpha'\beta}{t'} + \frac{\alpha'\alpha}{t'}\frac{x'}{t'})\Big\}\Big\{\Delta_3 + \xi_3\,(\frac{x'}{t'} - \frac{x}{t}) + \xi_3\,(\frac{\alpha\beta'}{t'} - \frac{\alpha'\beta}{t'})\Big\}\,\frac{t}{t'} + \mathcal{O}((t/t')
\end{aligned} \tag{6.}$$

where $C'_{312} = (-1)^{\Delta_3}\, C_{312}$.

The RHS is given by

$$\begin{aligned}
&\sum_{p,\{\vec{l},\vec{q},\vec{u},\vec{v}\}} C_{12}^{p\{\vec{l},\vec{q},\vec{u},\vec{v}\}}(t,x,\alpha,\beta)\ \langle\, \Phi_3(t',x',\alpha',\beta')\, \Phi_p^{\{\vec{l},\vec{q},\vec{u},\vec{v}\}}(0,0,0,0)\,\rangle \\
=\;& \Big[\, C_{12}^{3\{0,0,0,0\}}(t,x,\alpha,\beta) \,+\, C_{12}^{3\{0,0,1,0\}}(t,x,\alpha,\beta)\,\hat{G}_{-\frac{1}{2}} \,+\, C_{12}^{3\{0,0,0,1\}}(t,x,\alpha,\beta)\,\hat{H}_{-\frac{1}{2}} \\
&+ C_{12}^{3\{1,0,0,0\}}(t,x,\alpha,\beta)\,\hat{L}_{-1} \,+\, C_{12}^{3\{0,1,0,0\}}(t,x,\alpha,\beta)\,\hat{M}_{-1} \\
&+ C_{12}^{3\{0,0,1,1\}}(t,x,\alpha,\beta)\,\hat{G}_{-\frac{1}{2}}\,\hat{H}_{-\frac{1}{2}} + \ldots \Big]\, t'^{-2\Delta_3}\, e^{2\xi_3\frac{x'}{t'}} \\
=\;& t'^{-2\Delta_3}\, e^{2\xi_3\frac{x'}{t'}} \Big[\, C_{12}^{3\{0,0,0,0\}} \,+\, 2\,C_{12}^{3\{0,0,1,0\}}(-\Delta_3\alpha' - \xi_3\frac{x'\alpha'}{t'} + \xi_3\beta')\,\frac{1}{t'} - 2\xi_3\, C_{12}^{3\{0,0,0,1\}}\,\frac{\alpha'}{t'} \\
&+ 2\,C_{12}^{3\{1,0,0,0\}}(\Delta_3 + \xi_3\frac{x'}{t'})\frac{1}{t'} +\, 2\xi_3\, C_{12}^{3\{0,1,0,0\}}\,\frac{1}{t'} + 2\,\xi_3\, C_{12}^{3\{0,0,1,1\}}(1 + 2\xi_3\frac{\alpha'\beta'}{t'})\,\frac{1}{t'} + \cdots \Big]\,.
\end{aligned} \tag{6.8.4}$$

One can easily read off the coefficients by comparing (6.8.3) and (6.8.4)[14]:

$$\begin{aligned}
C_{12}^{3\{0,0,0,0\}} &= C'_{312}\, t^{\Delta_3-2\Delta}\, e^{(2\xi-\xi_3)\frac{x}{t}}\,, \\
C_{12}^{3\{0,0,1,0\}} &= \frac{1}{2}\,C'_{312}\, t^{\Delta_3-2\Delta}\, e^{(2\xi-\xi_3)\frac{x}{t}}\,\alpha\,, \\
C_{12}^{3\{0,0,0,1\}} &= -\frac{1}{2}C'_{312}\, t^{\Delta_3-2\Delta}\, e^{(2\xi-\xi_3)\frac{x}{t}}\,\beta\,, \\
C_{12}^{3\{1,0,0,0\}} &= \frac{1}{2}\,C'_{312}\, t^{\Delta_3-2\Delta+1}\, e^{(2\xi-\xi_3)\frac{x}{t}}\,,
\end{aligned} \tag{6.8.5}$$

[14]The reader should note that we have compared the full functional dependence on the coordinates $\{t, x, \alpha, \beta, t', x', \alpha', \beta'\}$ on both sides, though the LHS has been shown upto a certain order in $\frac{t}{t'}$ (which is just a convenient trick to extract out the expression for the $C_{12}^{p\{\vec{l},\vec{q},\vec{u},\vec{v}\}}$'s).

$$
\begin{aligned}
C_{12}^{3\{0,1,0,0\}} &= -\frac{1}{2}\, C'_{312}\, x\, t^{\,\Delta_3 - 2\Delta}\, \mathrm{e}^{(2\xi - \xi_3)\frac{x}{t}}\,, \\
C_{12}^{3\{0,0,1,1\}} &= 0\,.
\end{aligned}
$$

So in this case, the SGCA OPE is

$$
\begin{aligned}
& \Phi_1(t,x,\alpha,\beta)\,\Phi_2(0,0,0,0) \\
= \;& \sum_p C'_{p12}\, t^{\Delta_p - 2\Delta}\, \mathrm{e}^{(2\xi - \xi_p)\frac{x}{t}} \Big(\Phi_p(0,0,0,0) + \frac{\alpha}{2}\, \Phi_p^{\{0,0,1,0\}}(0,0,0,0) - \frac{\beta}{2}\, \Phi_p^{\{0,0,0,1\}}(0,0,0,0) \\
& \qquad + \frac{t}{2}\, \Phi_p^{\{1,0,0,0\}}(0,0,0,0) - \frac{x}{2}\, \Phi_p^{\{0,1,0,0\}}(0,0,0,0) + \ldots \Big)\,.
\end{aligned}
\tag{6.8.6}
$$

Part IV

Concluding remarks

Chapter 7

Concluding remarks

In the first part of this thesis, we studied "Black hole hair removal". We noted that BMPV black holes in flat transverse space and in Taub-NUT space have identical near horizon geometries but different microscopic degeneracies. We showed that this difference can be accounted for by different contribution to the degeneracies of these black holes from hair modes, – degrees of freedom living outside the horizon. After removing the contribution due to the hair degrees of freedom from the microscopic partition function, the partition functions of the two black holes agree. We constructed the bosonic hair modes, but only by a linear analysis.

Ref. [123] filled some of the gaps in the analysis outlined in §3. These are of three types:

1. We identified the bosonic deformations of the black hole solution by working with the linearized equations of motion. Ref. [123] extended them to the solutions to full non-linear equations of motion.

2. We gave a general argument for the existence of a certain set of fermionic deformations but did not construct them explicitly. Ref. [123] constructed these fermionic modes by solving the equations of motion of the fermions around the BMPV black hole background.

3. We did not study supersymmetry properties of the deformations explicitly. Ref. [123] demonstrated that the deformations preserve the same number of supersymmetries as the original BMPV black hole background.

The authors also found that one set of deformations for each black hole have mild curvature singularities in the future horizon, when viewed as ten-dimensional geometries [141, 142]. Hence one must remove these modes from the counting of the hair degrees of freedom. However, fortunately they give identical contributions to the partition functions for the two black holes. Consequently, even after removing their contribution from the hair partition function, one continues to get agreement between the partition functions of the two black holes after hair removal.

In the remaining part of the thesis, we concentrated on supersymmetric extention of the Galilean Conformal Algebras in 4 and 2 dimensions respectively. For $d = 4$, we could find an infinite lift of the SGCA, though the relativistic superconformal algebra has finite number of generators. However, we saw that d=2 is special $\rightarrow$ because the relativistic superconformal algebra is infinite dimensional. The usual group contraction of the parent relativistic algebra then provided a map between the relativistic and non-relativistic infinite algebras. Our study of $2d$ SGCFTs proceeds along two parallel lines: One involved carrying out an autonomous analysis of the SGCA, and the other consisted of taking carefully the non-relativistic scaling limit of the parent $2d$ SCFTs.

For the $4d$ case, there exists a large literature on the supersymmetric extensions of the Schrödinger algebra; some of the recent works include [143]–[150]. Among these, [145, 146] rely on clever re-writings of the relativistic algebras to get at the super-schrodinger algebra which exists as a subgroup inside the relativistic one. In our construction of SGCA, the supercharge anticommutators generate the M_n^i and not the L_n as would be the usual expectation from the bosonic algebra. The reason is: we chose to scale the fermionic generators in a way which meant that the $SL(2, R)$ part always dropped out of the fermionic anti-commutators. There are other possible ways [133, 151] to obtain SGCA by choosing to contract the fermionic generators such that the $SL(2, R)$ part is the one that remains after scaling and not the part with the vector indices. This is a different non-relativistic limit of the parent super-conformal algebra and hence an inequivalent Super-Galilean conformal algebra.

There are numerous avenues left to explore in the context of SGCA in $4d$. Construction of representations and correlation functions for the supersymmetric case, along the lines of [79] is an immediate step. Now that we understand the non-relativistic scaling in a super-conformal setting, we are better equipped to deal with the main subject of interest, viz. $\mathcal{N} = 4$ SYM. The primary objective of the supersymmetric extension of the GCA that we have looked at in this paper is to build a platform from which we can understand the symmetries of the $\mathcal{N} = 4$ SYM. Once we understand how to take this systematic non-relativistic limit of $\mathcal{N} = 4$ SYM, we might be able to isolate some interesting and tractable sector of the theory, somewhat analogous to the plane wave sector in the BMN limit.

The bulk dual of the GCA proposed in [73] was a novel Newton-Cartan like limit of AdS_{d+2} with a base AdS_2 and R^d fibres. The GCA emerges as the asymptotic isometry of this Newton-Cartan structure. A better understanding of the boundary theory is also a step in understanding the bulk. Taking the non-relativistic limit on the $\mathcal{N} = 4$ SYM would be a first step in trying to understand the novel bulk-boundary dictionary in this case. Now that we understand how to extend the GCA to SGCA, using the contraction on the $\mathcal{N} = 4$ SYM should become clearer.

There are numerous avenues to explore in the study of non-relativistic 2d theories, whose algebra can be obtained by group contraction of the well-studied relativistic theories. One of the immediate things one needs to understand better are the extra constraints arising purely within the non-relativistic sector (as we have explained in §6.4.3), whose analogues do not not exist in the parent relativistic theory. Our present understanding of the above issue is the following: This means that what was an irreducible representation ("irrep") of the (Super)Virasoro algebra (i.e. modulo the original null states) is no longer an irrep after taking the limit. This in itself is not surprising or unusual. The irreps of the original group need not to go over into irreps of the contracted group. It is therefore not too surprising that if we further choose to restrict to the irreps formed by modding out by the additional null states, there would be additional relations. The physical translation of these statements is that, if we choose to set the additional null states to zero, then the correlation functions have some further selection properties. Another way to put it is that the correlation functions of operators, which lie in the smaller vector space, may satisfy additional relations which would not be true of the full vector space. This is because we are choosing to work with a subclass of operators (states) which can close amongst themselves consistently rather than the full set of operators (states). The main thing to check is that the additional conditions are not incompatible. For all the specific cases we have dealt with in the present work, we have not found any inconsistency. Similar checks must be done for the higher levels, but at the moment we do not have a general way.

It is clear from the above discussion that it is not obvious to conclude that the (S)GCA arises as a limit of the (S)CFT without further analysis. We would like to stress that, in the present work as well as in [82], we have not established in any strong way the existence of our limit of the (S)CFT. We have just performed a series of consistency checks. But one can look at the possibility whether one can construct a consistent (S)GCA where the extra relations do not play a role. The fusion rules found in GCA and SGCA indicate that one can truncate to the states in the usual Kac table. In other words, can the primaries (with the special values of ξ and Δ corresponding to the extra null states) appear in the RHS of fusion rules of the other null state primaries? If they do not appear, then we think that we can consider a truncation where these kinds of null states do not have to be considered. We can then consider the family of primaries which have only the values in the non-relativistic limit of the usual Kac table and the OPEs will close in this sector. We have found this to be true for the lowest level(s) where we get non-trivial null states. However, we have not proven this for states at any arbitrary level and we would like to explore whether it is possible to give a general proof that the fusion rules in the non-relativistic theory always give other members of the original Kac table (and not anything else).

We would like to emphasize that we have not been able to provide any strong evidence of the presence of (S)GCA in possible field theories. This will also require proving our assumption of the state-operator correspondence.

Here we have focussed on the Neveu-Schwarz sector of the $\mathcal{N} = (1, 1)$ supersymmetric extension of GCFTs in 2d. One can try to work on the Ramond sector and find the analogous results there, where one cannot use the superfield formalism. Also, one can try to find out the consequences when we increase the number of supersymmetries. All these studies can be easily done along the framework presented in this work.

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