Some Classical and Semi-Classical Aspects of Higher Spin Theories in AdS_3

By

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Enrolment Number : PHYS08200804001

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A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABA NATIONAL INSTITUTE

August 2014

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of publications arising from the thesis

Journal

- "Quantum W-symmetry in AdS_3", Matthias R. Gaberdiel, Rajesh Gopakumar and Arunabha Saha, Journal of High Energy Physiscs,Year 2011,**Volume 2011**, Issue 2, pp 1-22
- "Topologically massive higher spin gravity", Arjun Bagchi, Shailesh lal, Arunabha Saha, Bindusar Sahoo, Journal of High Energy Physiscs,Year 2011,**Volume 2011**, Issue 10,Article 150

E-prints

- "Phase Structure of Higher Spin Black Holes", Abhishek Chowdhury and Arunabha Saha, arXiv:1312.7017 [hep-th]

List of other publications of candidate not related to this thesis

- "One loop partition function for topologically massive higher spin gravity", Arjun Bagchi, Shailesh lal, Arunabha Saha, Bindusar Sahoo, Journal of High Energy Physics,Year 2011,**Volume 2011**,Issue 12,Article 68
- "Black hole bound state metamorphosis", Abhishek Chowdhury, Shailesh Lal, Arunabha Saha, Ashoke Sen, Journal of High Energy Physics,Year 2013, **Volume 2013**,Issue 5, Article 20

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Dedicated to

my grandfather

Acknowledgments

First of all I would like to thank my thesis supervisor Rajesh Gopakumar for giving me the opportunity to work under his guidance and also for the enlightening discussions we had over the course of my PhD. His insights have played a huge role in shaping my thoughts about physics in general.

I would like to thank Mathias Gaberdiel, Arjun Bagchi, Shailesh Lal, Bindusar Sahoo, Abhishek Chowdhury for collaborations on the papers that we wrote. I will like to mention my special gratitude towards Ashoke Sen who gave me the opportunity to work on a beautiful problem on Black Holes in string theory.

Whatever understanding I have of physics till now is largely due to the efforts of very good teachers who taught me. So, a special thanks are due to all my teachers especially Ashoke Sen, Rajesh Gopakumar, Parthasarathi Majumdar, Somnath Bharadwaj, Dwijen Majumdar, Madhab Bannerjee, Shailen Dutta and many others.

I will also like to thank all past and present members of our string theory group at HRI especially Rajesh Gupta, Shamik , Ayan , Sayantani, Binata, Anirban, Anshuman, Bobby, Dileep, Sachi, Sudhakar, Roji, Swapno, Abhishek, Sitender, Mrityunjay, Takimi, Masazumi and many others for creating such a vibrant atmosphere for research and learning through all the discussions at string lunch, the string seminars and in general.

I would also like to thank Arnab Rudra, Justin David, Somyadip Thakur, Abhishek Iyer, Alejandra Castro, Geoffrey Comperè, Gustavo-Lucena Gomez, Andrea Campoleoni, Nacho, Rakib, Eric Perlmutter, Nabil Iqbal, Somdeb Chakraborty, Juan Jottar, Sajidul Haque, Gautam Mandal, Suvankar Dutta, Nabamita Bannerjee, Koushik Ray, Sachin Jain, Diptarka Das and many others with whom I have had the opportunity to have many fruitful and insightful discussions over the years.

These six years at HRI would have been very dull without the special people here. Tarat, Bhuru, Niyogi, Rambha, Hochi, Manoj, Anusree, Vikas, Sanjoy da, JD, Mama, Dibya, Aritra, Rana, Tausif, Landau, Dosh, Bhacha have been special friends. The heated discussions on various relevant and irrelevant topics at our marathon "adda sessions", the cricket matches and the PJs kept me motivated enough to go back to work.

Special thanks are due to my wife Ushoshi. Thanks for being there for me and lifting up my soul whenever I needed it. Thanks for cooking up all those beautiful dishes which instantly recharged me. And thanks for making me understand a lot of "real world physics" through the discussions we had.

My parents and brother have been my pillars of strength throughout my academic life. They have kept a lot of faith in whatever I am doing and it gives me a lot of strength. Thanks a lot for being there for me.

And most probably I would not have gone on this path but for the motivation I received from my grandfather Shri Satyendra Nath Saha who made me understand that unravelling the mysteries of nature is by far one of the best professions that one can be related to.

Contents

Synopsis

AdS/CFT duality is the conjectured duality between a theory of quantum gravity on an Anti-de Sitter(AdS) space-time and a Conformal Field Theory(CFT) living on the asymptotic boundary region of the AdS. By this it is meant that quantities in the gravity side can be calculated by computing some quantities in the CFT and vice versa. This duality has been subject to many tests in the context where it is stated as a duality between a particular superstring theory on AdS space-time and a supersymmetric quantum field theory with a large amount of supersymmetry (which turns out to be a CFT) on the asymptotic boundary of the AdS space. There also most of the focus has been on a particular sub-sector of this string theory called the supergravity limit which is dual to a strongly coupled sector of the field theory. Using this duality quantities in the strongly coupled sector of the field theory can be calculated with much more ease in the supergravity limit of string theory.

So, there are two important aspects of this duality that has to be probed further. One of them being able to move to a sector of string theory different from supergravity. The second being able to test this duality for non-supersymmetric systems. Higher spin theories provide us with the tool to address both these issues simultaneously. In the tensionless limit a particular sub-sector of the full string theory is believed to give rise to the massless higher spin theories. Also, these theories have a spectrum which can in principle be only bosonic and hence we have a good non-supersymmetric theory to test the AdS/CFT duality. A consistent theory of massless higher spin gauge theories in AdS space has been proposed by Vasiliev. In arbitrary dimensions these theories are consistent only in an asymptotic AdS(or de-Sitter) background and their spectrum must include fields with all spin from 2 to ∞ and some scalars.

In my thesis work I have focused on the higher spin theories in AdS_3 . In 3 dimensions there are many simplifications that make the theory much more tractable than in higher dimensions like the spectrum of fields can be consistently truncated at any finite upper value of integral spin. Also, in 3 dimensions for a theory consisting of purely higher spin fields there are no bulk propagating degrees of freedom. The theory is completely defined by the boundary values of its fields. These issues make them computationally much more tractable than other theories. Even after all these simplifications the theory is not void and retains enough complexity to provide us with some intuition into AdS/CFT duality in this context and also in general.

In the first project in this thesis I calculated the first quantum correction to a system consisting of all fields with spin from 2 to ∞ by calculating the leading order correction to the partition function around a classical saddle point which we take to be the thermal AdS. We showed that it is the same as vacuum character of a CFT with an extended symmetry group with a lie algebra called W_∞ . The motivation for this was an earlier established fact that the classical Poisson bracket algebra followed by the generators of the higher spin symmetries which can change the boundary configurations is a W algebra. In this context our work took this matching to the next level where we checked that the spectrum of fluctuations of fields on the AdS side matches with the spectrum of operators corresponding to this sector in the CFT side.

In the second work in this thesis we wrote down the most general action of a parity violating, gauge invariant and 3 derivative action for higher spin theories at the quadratic level. We called this the Topologically Massive Higher Spin Gravity or TMHSG. We found out that a good basis for solutions to the equations of motion at arbitrary point in parameter space consists of three branches called 1) The left moving mode 2) The right moving mode and 3) The massive mode. We showed that the energies of all three of these modes are never non-negative together except at a particular point in parameter space called the chiral point. But we showed that the basis of solution also becomes degenerate at this point as the left moving branch merges with the massive branch. We found out that at this special point there is a new branch of solution called the "log branch" and hence we find a new complete basis of solution at this point. Energy calculation though showed that even at this "chiral point" there are negative energy modes with this new complete basis of solution. So, it looks like that the theory has a genuine linear instability at all points in phase space.

In the third work we studied the phase structure of a higher spin system in AdS_3 with a maximal spin 3 in the canonical approach. There are two types of classical solution in the euclidean system at a finite temperature which has the topology of a torus. This depends on the identification of the contractible cycle of the torus. If the contractible cycle is time like we have a "black hole" like solution and if the contractible cycle is spatial we have a "thermal AdS" type solution. We found out all possible solution of either type and studied their phase structure. We found out 2 types of solutions for contractible spatial cycle and we called them "Thermal AdS" like solution and "Extremal thermal AdS" like solution. Similarly we found out 4 types of solutions for the solution with contractible time cycle. Out of these we discard 2 solutions because they have negative entropy and hence are unphysical. Of the remaining 2 solutions we get a BTZ black hole like solution and an extremal black hole like solution. We found a Hawking-Page like transition between the black hole like and thermal AdS like solutions at a temperature dependent upon chemical potential for spin 3. We find that all the black hole like and thermal AdS like solution do not exist beyond a certain value of temperature for a particular chemical potential and the region of existence of thermal AdS like solution is greater than that of black hole solution. So, we see the presence of a new type of phase transition here where after a particular temperature only the thermal AdS like solution is present. We also studied how by a similarity transformation the field content of this theory can be converted to that of a theory with spectrum containing fields of spin 1, 2 and $\frac{3}{2}$ with a chemical potential for spin $\frac{3}{2}$. Here the black hole solutions exist for all temperatures. Here also, we have 4 branches of solution out of which 2 have negative entropy and hence are discarded. We were able to show that the solutions which were discarded in the earlier case map to the good solutions here and vice versa. Also, we showed that solutions here have a good scaling behaviour at

very high temperatures and hence this may well be the way the phase structure of the system should be studied at very high temperature.

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Chapter 1

Introduction

One of the most prominent tasks of theoretical physics is to understand the properties of the four basic forces (or interactions) of nature. It seems that **Quantum Field Theory** is the right framework to understand (at least perturbatively) three out of these four interactions- 1) **electromagnetic interaction,** 2) **the weak interaction** and 3) **the strong interaction**. Quantum field theory is able to predict to an extremely good precision level the observables of these three interactions. This relies on the fact that they are **renormalisable**. By this what we mean is that short distance singularities of these interactions can be taken care of by suitably redefining a finite set of parameters, at any particular order in accuracy for the parameters of these theories. And the beauty of this is that the number of such necessary redefinitions does not increase for calculations at higher order in accuracy. These three interactions together form what is known as the **"Standard Model of Particle Physics"**.

The fourth interaction -**the gravitational interaction** is very well known at the classical level and observables have been tested against predictions from this theory and found to match to an amazing level of accuracy. The problem starts when we try to quantise it in the conventional manner which has been successful for the other three interactions. If we try to calculate some observable perturbatively , we see that we can get rid of the short distance infinities by redefining finite number of parameters at each order in the interaction parameter which is the Newton's gravitational constant. However, it is impossible to find a finite set of parameters whose redefinitions are sufficient to get rid of the infinities at all orders of accuracy in parameter of the theory.

This is where **String Theory** steps in. Here the basic interacting objects are not point particles, but are one dimensional extended objects called strings. All particles in the universe (observed as well as unobserved) are the modes of excitation of the string and interactions between the strings gives rise to all (observed as well as unobserved) interactions of the universe. Since, there is a well defined minimum length (the string length) so all interactions are by construction safe from infinites due to very short distance interactions.

A particular limit of this theory is the supergravity limit when the string length$\rightarrow 0$ or the string tension$\rightarrow \infty$. This limit of the theory has been studied in quite some detail till now. Another limit which will be related to the primary interest of this thesis is when the string tension$\rightarrow 0$. A particular subsector (called the first Regge trajectory) of string theory in this limit gives rise to fields which behave like massless higher spin fields (spin≥ 2) in Wigner's classification.

It is believed due to many no go theorems [1] that a consistent interacting theory of higher spin fields cannot be constructed on asymptotically flat spacetime backgrounds. One of the remarkable features of AdS spacetimes is the existence of interacting theories of massless particles with spin greater than or equal to two [2]. Typically, the consistency of such theories in AdS spacetimes requires the introduction of an infinite tower of particles with arbitrarily high spin.

These higher spin theories (which include gravity) are thus in some sense intermediate between conventional field theories of gravity involving a finite number of fields on the one hand, and string theories on the other (see [3] for an introduction to these matters). This observation takes on added significance in the context of the AdS/CFT correspondence. Higher spin theories provide an opportunity to understand the correspondence beyond the (super)gravity limit without necessarily having the full string theory under control. In fact, there is the tantalising possibility of a consistent truncation to this subsector within a string theory which might, in itself, be dual to a field theory on the boundary of the AdS spacetime.

For a string theory on AdS spacetime there is actually a dimensionless constant in the bulk given by $\frac{L^2}{\alpha'}$ where 'L' is the AdS radius and α' is the string constant which is related to the square of the string length. For the tensionless limit this quantity needs to be taken to zero. From the AdS/CFT point of view this ratio is related to a coupling of a marginal deformation of the dual CFT side. When the coupling goes to zero we have a free theory on the field theory side. Normally a free field theory has infinite number of conserved currents which are absent when the coupling constant is turned on. Following the AdS/CFT picture all these conserved currents are dual to higher spin gauge fields in the AdS bulk.

Another interesting observation has been made in [11], where it was shown that if in a CFT (in spacetime dimensions ≥ 3) we have a conserved charge with spin ≥ 3 then the CFT will contain conserved charge with all spins ≥ 2 upto ∞. Also it was observed there that the correlation function between the conserved charges correspond to that of a free field theory. From all these observations it is reasonable to assume that the CFT dual to higher spin theories in the bulk is most likely a free CFT. This is in sharp contrast to the CFT dual to the supergravity limit of the string theory which is believed to be dual to a field theory in the strong (t'Hooft) coupling regime.

The argument given above is not valid for CFT_2 (2 dimensional field theories in general) where it is consistent to have interactions and infinite number of conserved charges together. The Minimal model CFTs with extended W-symmetry, which were proposed to be dual to the Vasiliev higher spin system in AdS_3 [64] are examples of field theories in 2 dimensions which have infinite number of conserved charges with correlation function which are non-trivial. These theories have a dimensionless (t'Hooft) coupling constant which takes values between 0 and 1. Hence, even in 2 dimensions the CFTs dual to higher spin theories in the bulk are not strongly coupled. Hence, the higher spin theories are in general dual to CFTs which are not strongly coupled and hence are capable of shedding light on another window of the AdS/CFT duality. So, understanding aspects of higher spin theories in the AdS_3 is a useful exercise to carry out.

So in this thesis keeping in mind the above motivation we explore some aspects of the higher spin theories on the AdS_3 side which may help us gain some insight into the AdS/CFT duality in this context. We start by giving a brief introduction on how higher spin theories are formulated in AdS (in particular the 3 dimensional case). In the sections that follow we first give a brief description of massless higher spin theories in flat and AdS spacetime. Next we give a vierbein like formulation for the higher spin theories. After that we move on to the particular case of AdS_3. Then we discuss higher spin theories in 3 dimensions in the context of AdS_3/CFT_2 correspondence. On the way we give a lightening review of the AdS/CFT conjecture in general.

1.1 Free "Massless" Higher Spin Gravity

The free higher spin fields can be consistently defined on a flat(Minkowski) space. It is only when we try to introduce interactions that we face problems. A higher spin field can be represented by a symmetric rank 's' tensor satisfying the equation [20]

$$\mathcal{F}_{\mu_1\ldots\mu_s} = \Box\phi_{\mu_1\ldots\mu_s} - \partial_{(\mu_1|}\partial^\lambda\phi_{|\mu_2\ldots\mu_s)\lambda} + \partial_{(\mu_1}\partial_{\mu_2}\phi^\lambda_{\mu_3\ldots\mu_s)\lambda} = 0 \tag{1.1.1}$$

where the parentheses indicate symmetrisation without any normalisation factor. The above equation is invariant under the gauge transformation

$$\begin{aligned} \delta\phi_{\mu_1\ldots\mu_s} &= \partial_{(\mu_1}\xi_{\mu_2..\mu_s)} \\ \text{with} \qquad \xi^\lambda_{\mu_1\ldots\mu_{s-3}\lambda} &= 0 \end{aligned} \tag{1.1.2}$$

For spin 2 case the above 2 equations are the linearised Ricci tensor on flat background and diffeomorphism invariance respectively. Over and above this if we put a double traceless constraint given by

$$\phi^{\alpha\beta}_{\mu_1\ldots\mu_{s-4}\alpha\beta} = 0 \tag{1.1.3}$$

to arrive at the correct number of degrees of freedom for the massless spin 's' field. Apart from this the double-tracelessness condition is necessary to have a second order action in 'D' dimensions, which is invariant under the gauge transformation 1.1.2 upto total derivatives given by [20]

$$S = \frac{1}{2}\int d^D x \phi^{\mu_1\ldots\mu_s}\left(\mathcal{F}_{\mu_1\ldots\mu_s} - \frac{1}{2}\eta_{(\mu_1\mu_2}\mathcal{F}^\lambda_{\mu_3\ldots\mu_s)\lambda}\right). \tag{1.1.4}$$

All the contractions above has been done with respect to the background metric. The above procedure can be generalised to AdS space provided we demand a gauge invariance under a transformation of the form

$$\begin{aligned} \delta\phi_{\mu_1\ldots\mu_s} &= \nabla_{(\mu_1}\xi_{\mu_2..\mu_s)} \\ \text{with} \qquad \xi^\lambda_{\mu_1\ldots\mu_{s-3}\lambda} &= 0 \end{aligned} \tag{1.1.5}$$

where, ∇ is the covariant derivative w.r.t. AdS background. This ensures that the fields have the same number of degrees of freedom as massless spin 's' field in flat space [68]. Now, obviously the covariant derivatives don't commute with the commutation relations and other conventions being given in appendix 6.1. The equation of motion invariant under this gauge transformation changes due to this and is given by

$$\hat{\mathcal{F}}_{\mu_1\ldots\mu_s} \equiv \mathcal{F}_{\mu_1\ldots\mu_s} - \frac{1}{l^2}\left\{\left[s^2 + (D-6)s - 2(D-3)\right]\phi_{\mu_1\ldots\mu_s} + 2g_{(\mu_1\mu_2}\phi^{\lambda}_{\mu_3\ldots\mu_s)\lambda}\right\} = 0 \quad (1.1.6)$$

This equation of motion can be obtained from an action (invariant upto total derivatives under 1.1.5, provided the fields are double traceless) given by

$$S = \frac{1}{2}\int d^Dx\sqrt{-g}\phi^{\mu_1\ldots\mu_s}\left(\hat{\mathcal{F}}_{\mu_1\ldots\mu_s} - \frac{1}{2}g_{(\mu_1\mu_2}\hat{\mathcal{F}}^{\lambda}_{\mu_3\ldots\mu_s)\lambda}\right) \quad (1.1.7)$$

1.2 "Vierbein" like formalism

1.2.1 Vierbein formalism for gravity

For normal gravity there is a vierbein like formalism where we use the equivalence principle to define a set of orthogonal coordinate system locally at each point and name them e^a_μ. Here 'a' are the local lorentz indices and μ are the global coordinate indices. The relation between the metric and the vierbeins is given by

$$\eta_{ab}e^a_\mu e^b_\nu = g_{\mu\nu} \quad (1.2.8)$$

This comes from the demand that a very "small" vector $V_\mu = e^a_\mu V_a$ has the length $g_{\mu\nu}V^\mu V^\nu$ in global coordinate system and length $\eta_{ab}V^aV^b$ in local coordinate system and both these lengths are the same and the fact that $V^\mu = e^\mu_a V^a$ where e^μ_a is the inverse of e^a_μ such that $e^\mu_a e^b_\mu = \delta^b_a$. Gravity is obtained by gauging the local lorentz symmetry. The connection necessary to define the covariant derivatives of various fields in this "gauge theory" is called the spin connection. For vectors for example the covariant derivative is defined so that $D_\mu V^a = e^a_\nu D_\mu V^\nu$, which gives

$$\begin{aligned} D_\mu e^a_\nu &= 0 \\ \Rightarrow \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + w^a_{\mu b} e^b_\nu &= 0 \end{aligned} \tag{1.2.9}$$

where $w^a_{\mu b}$ is the spin connection for the vector. From the above equation the spin connection can be solved in terms of vierbeins to get

$$w^{ab}_\mu = \frac{1}{2} e^{\nu a} \left(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu \right) - \frac{1}{2} e^{\nu b} \left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu \right) - \frac{1}{2} e^{\rho a} e^{\sigma b} \left(\partial_\rho e_{\sigma c} - \partial_\sigma e^{\rho c} \right) e^c_\mu \tag{1.2.10}$$

From which it is clear that the spin connection is antisymmetric in its lorentz indices.

1.2.2 Vierbein formalism for higher spin theories

Similarly, the higher spin fields defined above have a "vierbein" like description though it lacks the geometrical insight of the gravity case. The generalised vierbeins and spin connections are $e^{a_1 a_2 \ldots a_{s-1}}_\mu$ and $w^{b, a_1 a_2 \ldots a_{s-1}}_\mu$ respectively, with the a_i indices being symmetric and traceless and the b index being anti-symmetric with all a_i indices. There are similar conditions as above to determine the generalised spin connections in terms of the generalised vierbeins. The spin 's' Fronsdal fields (in the linearised equation of motion) are given by

$$\phi_{\mu_1 \ldots \mu_s} = \bar{e}^{a_1}_{(\mu_1} \ldots \bar{e}^{a_{s-1}}_{\mu_{s-1}} e_{\mu_s) a_1 \ldots a_{s-1}} \tag{1.2.11}$$

where $\bar{e}^{a_1}_{\mu_1}$ are the vierbiens for the AdS background. The above result can only be fixed after we have made a choice about the local lorentz frames and corresponding things for the higher spin vierbeins.

1.3 Higher Spin fields in 3 dimensions

In 3 dimensions the little group of massless fields is $Z_2 \otimes R$ [21]. Removing the continuous spin representations we are only left with 2 types of inequivalent representations of Z_2. So, the usual notion of spin reduces to the distinction between bosons and fermions. Also, in 3 dimensions there are no propagating bulk degrees of freedom for spins ≥ 2. But in presence of cosmological constant fields with different rank 's' have different boundary dynamics though they have no bulk propagating degrees of freedom (for $s > 1$). So, we see it fit to call the rank as the spin of the fields.

1.3.1 Chern Simons formulation of $3d$ HS Gravity

$3d$ gravity with negative cosmological constant can be written as a Chern Simons theory with gauge group $SL(2,R) \times SL(2,R)$. This theory has equations of motion where the field strength is zero and hence does not carry any propagating physical degrees of freedom in the bulk. This property is same for Einstein gravity in 3 dimensions. We define the $SL(2,R)$ potentials by

$$j^a_\mu = w^a_\mu + \frac{1}{l}e^a_\mu, \quad \bar{j}^a_\mu = w^a_\mu - \frac{1}{l}e^a_\mu \tag{1.3.12}$$

where $w^a_\mu = \frac{1}{2}\epsilon^{abc}w^{bc}_\mu$ is the dualised spin connection in $3d$ and 'l' is related to cosmological constant by $\Lambda = -\frac{1}{l^2}$. Similarly, the spin 's' counterpart of this can be defined as

$$t^{a_1\ldots a_{s-1}}_\mu = w^{a_1\ldots a_{s-1}}_\mu + \frac{1}{l}e^{a_1\ldots a_{s-1}}_\mu, \quad \bar{t}^{a_1\ldots a_{s-1}}_\mu = w^{a_1\ldots a_{a-1}}_\mu - \frac{1}{l}e^{a_1\ldots a_{s-1}}_\mu \tag{1.3.13}$$

where $w^{a_1\ldots a_{a-1}}_\mu$ are the dualised spin connections for the spin s case. The gauge fields are then given by

$$\begin{aligned} A &= \left(j^a_\mu J^a + t^{a_1\ldots a_{s-1}}_\mu T^{a_1\ldots a_{s-1}}\right) dx^\mu \\ \bar{A} &= \left(\bar{j}^a_\mu J^a + \bar{t}^{a_1\ldots a_{s-1}}_\mu T^{a_1\ldots a_{s-1}}\right) dx^\mu \end{aligned} \tag{1.3.14}$$

The higher spin generators must transform under the $sl(2,R)$ subalgebra as tensors. So, that the algebra followed by them can be written as

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J^c \\ \left[J_a, T_{b_1 \ldots b_{s-1}}\right] &= \epsilon^{m}_{a(b_1} T_{b_2 \ldots b_{s-1})m}. \end{aligned} \tag{1.3.15}$$

To get the linearised Fronsdal equations from this frame like picture only this much information about the algebra is necessary. We need not know about the commutators between higher spin generators. If the generators form an algebra with a non degenerate bilinear form denoted by 'tr' we can then consider the action for this system to be given by the CS action

$$\begin{aligned} S_{HS} &= S_{CS}[A] - S_{CS}[\bar{A}] \\ S_{CS}[A] &= \frac{k}{4\pi} \int tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \end{aligned} \tag{1.3.16}$$

The above equation reduces to the Einstein Hilbert action with cosmological constant, upto boundary terms when the gauge fields only contain the 'j' potentials. Using the normalisation $tr\,(J_a J_b) = \frac{1}{2}\eta_{ab}$ we get $k = \frac{l}{4G}$ where 'G' is the Newton's Gravitational constant. To check if the Fronsdal equations can be obtained from this we split the gravitational quantities as backgrounds and fluctuations over them and the higher spin quantities as only fluctuations over the gravitational background i.e., $e^a_\mu = \bar{e}^a_\mu + h^a_\mu, w^a_\mu = \bar{w}^a_\mu + v^a_\mu$ etc.. The equations that we get by using 1.3.12,1.3.13,1.3.14,1.3.15 and the equation coming from 1.3.16 given by $F = dA + A \wedge A = 0$ is given by (at the linearised level)

$$
\begin{aligned}
R^{a} \equiv Dh^{a} + \epsilon^{abc}\bar{e}_{b} \wedge v_{c} &= 0 \\
T^{a} \equiv Dv^{a} + \frac{1}{l^{2}}\epsilon^{abc}\bar{e}_{b} \wedge h_{c} &= 0 \\
R^{a_1\ldots a_{s-1}} \equiv Dh^{a_1\ldots a_{s-1}} + \epsilon^{cd(a_1|}\bar{e}_{b} \wedge v_{d}^{|a_2\ldots a_{s-1})} &= 0 \\
T^{a_1\ldots a_{s-1}} \equiv Dv^{a_1\ldots a_{s-1}} + \frac{1}{l^{2}}\epsilon^{cd(a_1|}\bar{e}_{c} \wedge h_{d}^{|a_2\ldots a_{s-1})} &= 0
\end{aligned}
\tag{1.3.17}
$$

where, the form indices have been omitted and the covariant derivative D is given by

$$
Df^{a_1\ldots a_{s-1}} = df^{a_1\ldots a_{s-1}} + \epsilon^{cd(a_1|}\bar{w}_{c} \wedge f_{d}^{|a_2\ldots a_{s-1}}. \tag{1.3.18}
$$

The equations 1.3.17 are left invariant under the following field transformations

$$
\begin{aligned}
\delta h^{a_1\ldots a_{s-1}} &= D\xi^{a_1\ldots a_{s-1}} + \epsilon^{cd(a_1|}\bar{e}_{c}\Lambda_{d}^{|a_2\ldots a_{s-1})} \\
\delta v^{a_1\ldots a_{s-1}} &= D\Lambda^{a_1\ldots a_{s-1}} + \frac{1}{l^{2}}\epsilon^{cd(a_1|}\bar{e}_{c}\xi_{d}^{|a_2\ldots a_{s-1}}
\end{aligned}
\tag{1.3.19}
$$

upto equations of motion of the background fields. The ξ part of the gauge freedom corresponds to "diffeomorphism" like gauge freedom and the Λ part corresponds to a local lorentz frame's choice type freedom.

The 'R' equations of 1.3.17 can be used to get the 'v's' as a function of the 'h's'. Substituting this in the CS action we get a second order quadratic action for

$$
h_{\mu_1,\mu_2\ldots\mu_s} = \bar{e}_{\mu_2}^{a_1}\ldots\bar{e}_{\mu_s}^{a_{s-1}} h_{\mu_1,a_1\ldots a_{s-1}} \tag{1.3.20}
$$

The skewed {s,1} symmetry of the above field can be taken care of to make it a fully symmetric tensor in the μ indices by using up the local lorentz like symmetry due to the Λ parameters mentioned in 1.3.19. Using this and eliminating the background spin connection $\bar{w}^a_\mu$ using the vierbein constraint in 1.2.9 given by

$$\partial_\mu e^a_\nu + \epsilon^a_{bc} w^b_\mu e^c_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda = 0 \tag{1.3.21}$$

we get the Fronsdal action 1.1.7 for spin 's' field given by 1.2.11. Similarly, the ξ part of the gauge transformation in 1.3.19 will correspond to the gauge freedom of Fronsdal like fields in 1.1.2.

Doing all this we have basically transformed the problem of finding consistent interaction between higher spin fields to that of suitably completing the algebra given in 1.3.15. Progress in this direction was made in [22]. They showed that the generators of higher spin algebra $T_{a_1 \ldots a_{s-1}}$ can act as the generators of $sl(N, R)$ algebra. A small evidence in this direction is calculating the number of independent generators in the higher spin theory with spins $\leq$ n. The number of independent generators of a particular spin 's' is the total number of symmetric traceless tensors of rank s-1 in 3 dimensions. The number of independent components of a symmetric tensor of rank 's-1' in 3 dimensions is given by $\frac{(s+1)s}{2}$ and tracelessness imposes $\frac{(s-1)(s-2)}{2}$ constraints. So, the number of generators of spin 's' is given by $2s - 1$. So, the number of independent generators with spin $\leq$ n is given by $\sum_{s=1}^{n-1}(2s-1)$ which is $n^2 - 1$, which is the same as the number of generators of the group $SL(n, R)$. So, the interacting theory of higher spin fields with spin from $s = 2$ to n is given by CS action with gauge group $SL(n, R) \times SL(n, R)$ in AdS_3.

1.3.2 The spin 3 example

For spin 3 case the algebra 1.3.15 can be closed using the Jacobi identity and is given by

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J_c \\ [J_a, T_{bc}] &= \epsilon_{ma(b} T_{c)m} \\ [T_{ab}, T_{bc}] &= \sigma \left(\eta_{a(c} \epsilon_{d)bm} + \eta_{b(c} \epsilon_{d)am} \right) J_m \end{aligned} \tag{1.3.22}$$

where, σ is an undetermined constant. It can be shown that this algebra is isomorphic to the $Sl(3, C)$ whose fundamental representation can be obtained by writing the higher spin generators in terms of the 3 dimensional representation of the $sl(2, C)$ generators as

$$T_{ab} = \sqrt{-\sigma} \left(J_{(a} J_{b)} - \frac{2}{3} J_c J_c \right). \tag{1.3.23}$$

The choice of sign of sigma determines which real non-compact form of $sl(3, C)$ is chosen. for $\sigma \geq 0$ we get the $su(1, 2)$ algebra and for $\sigma \leq 0$ we get the $sl(3, R)$ algebra. In fact for the two types of potentials in 1.3.12 and 1.3.13 we can choose a different real form. But to get a good $l \to \infty$ limit of the algebra , i.e. a good flat space limit of this higher spin theory we need to choose the same real form for both the sectors.

To make the $sl(3)$ nature of the algebra more clear and to get rid of the tracelessness constraint of the T_{ab} generators it is best to go to another basis given by

$$\begin{aligned} & J_0 = \frac{1}{2} (L_1 + L_{-1}), \quad J_1 = \frac{1}{2} (L_1 - L_{-1}), \quad J_2 = L_0 \\ & T_{00} = \frac{1}{4} (W_2 + W_{-2} + 2W_0), \quad T_{01} = \frac{1}{4} (W_2 - W_{-2}) \\ & T_{11} = \frac{1}{4} (W_2 + W_{-2} - 2W_0). \quad T_{02} = \frac{1}{4} (W_1 + W_{-1}) \\ & T_{22} = W_0, \quad T_{12} = \frac{1}{2} (W_1 - W_{-1}) \end{aligned} \tag{1.3.24}$$

where the new generators follow the algebra

$$\begin{aligned}
[L_i, L_j] &= (i-j)L_{i+j} \\
[L_i, W_m] &= (2i-m)W_{i+m} \\
[W_m, W_n] &= \frac{\sigma}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n}
\end{aligned} \tag{1.3.25}$$

1.3.3 The hs$[\lambda]$ algbera: the "sl(∞)" algebra

The Vasiliev's higher spin theories which can have candidate CFT duals have in their spectrum all fields with spin from 2 to ∞ (along with some scalars). So, using the above arguments we need to work with sl(∞) algebra (more specifically the n→ ∞ limit of sl(n) algebra). This algebra contains infinite number of generators corresponding to the infinite number of higher spin fields. Fortunately there exists an algebra which has the requisite infinite number of generators and which has a much more user-friendly definition than 'sl(∞)". This is called the hs[λ] algebra. Its construction will be explained in this subsection. The universal enveloping algebra U$(sl(2))$ is quotiented by ideal generated by $C^{sl} - \frac{1}{4}(\lambda^2-1)\mathbf{1}$ to obtain an algebra given by

$$B[\lambda] = \frac{U(sl(2))}{\langle C^{sl} - \frac{1}{4}(\lambda^2-1)\mathbf{1}\rangle}. \tag{1.3.26}$$

If the $sl(2)$ generators are defined by $J_0, J_\pm$ with the algebra

$$[J_+, J_-] = 2J_0, \quad [J_\pm, J_0] = \pm J_\pm, \tag{1.3.27}$$

then the Casimir is defined by

$$C^{sl} = J_0^2 - \frac{1}{2}(J_+J_- + J_-J_+) \tag{1.3.28}$$

The basis for the algebra consists of one 0 letter word namely the identity denoted by V_0^1 , three 1 letter words $V_1^2 = J^+$, $V_0^2 = J_0$ and $V_{-1}^2 = J_-$. and five 2 letter words and so on. Continuing we get $2s+1$ different $s-1$ letter words given by

$$V_n^s = (-1)^{s-1-n}\frac{(n+s-1)!}{(2s-2)!}\left[J_-, ... \left[J_-, \left[J_-, J_+^{s-1}\right]\right]\right] \tag{1.3.29}$$

where $|n| \leq s-1$. The algebra for the generators is given by 'lone star product'

$$[X, Y] = X \star Y - Y \star X. \tag{1.3.30}$$

We can then define an invariant bilinear trace by

$$tr(X \star Y) = X \star Y|_{J_a=0} \tag{1.3.31}$$

This trace is symmetric and hence the commutators of two $B[\lambda]$ generators will not contain the identity element **1**. The $B[\lambda]$ algebra then decomposes into

$$B[\lambda] = \mathbb{C} \oplus hs[\lambda], \tag{1.3.32}$$

where vector space $\mathbb{C}$ is the vector space generated by the identity element. The rest of the generators V_n^s with $|n| \leq s$ with $s \geq 2$ generates the $hs[\lambda]$ algebra. The V_n^s generators transform in a $2s-1$ dimensional representation with respect to the $sl(2)$ algebra generated by $s=2$ generators given by

$$[V_m^2, V_n^s] = (-n + m(s-1))V_{m+n}^s \tag{1.3.33}$$

so that the fields associated with the V_n^S generators have spacetime spin s and the full spectrum thus consists of fields with spin from 2 to ∞. Another interesting observation is that the bilinear trace defined by 1.3.31 goes to zero for all $s \geq N$ when $\lambda = N$. Also, the algebra shows that for $\lambda = N$ there is an ideal consisting of all generators with $s \geq N$ and hence for integral values of λ the hs[λ] algebra truncates to a finite dimensional algebra given by $sl(N)$ when quotiented by this ideal. Hence hs[λ] algebra contains the sl(N) algebra.

1.4 Higher Spin theories in the context of AdS$_3$/CFT$_2$ correspondence

First of all we will very briefly discuss some specific aspects of **AdS/CFT correspondence**.

1.4.1 The AdS Geometry

The AdS_{n+1} space can be obtained as an embedding in a pseudo-Euclidean $n+2$ dimensional space with length squared given by

$$y^2 = (y^0)^2 + (y^{n+2})^2 - y_i y^i, \quad i = 1 \quad \text{to} \quad n. \tag{1.4.34}$$

The AdS_{n+1} space being defined as the locus of points with

$$y^2 = b^2 = constant \tag{1.4.35}$$

So, it is clear from the above equation 1.4.35 that the isometry group AdS_{n+1} is $SO(n,2)$. So, some quantum theory on AdS_{n+1} should be an $SO(n,2)$ invariant theory with $\frac{1}{2}(n+1)(n+2)$ generators. One particular coordinate system of use to us is the **global coordinate system**.This is particular parametrisation of the equation 1.4.35 given by

$$y^0 = b\cosh\rho\cos\tau, \quad y^{n+1} = b\cosh\rho\sin\tau, \quad \overrightarrow{y} = b\sinh\rho\overrightarrow{\Omega} \tag{1.4.36}$$

where $\overrightarrow{\Omega}$ represents coordinates on S_{n-1} and and the range of coordinates being $0 \leq \tau < 2\pi$ and $0 \leq \rho < \infty$. The metric in this coordinate system is given by

$$ds^2 = b^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2\right). \tag{1.4.37}$$

With this obviously we have closed time-like curves in the geometry (which are pathological). To get rid of this we get rid of the compactness of the τ coordinates and define its new range as $-\infty < \tau < \infty$.

1.4.2 The Conformal Boundary of AdS

AdS space is not a compact manifold but there is a good way to take the extremum limits of the coordinates and still arrive at a sensible ”boundary“ metric. Lets us make the following redefinitions

$$y^a = R\bar{y}^a, \quad u = R\bar{u}, \quad v = R\bar{v} \tag{1.4.38}$$

and then take the $R \to \infty$ limit so that $y^2 = b^2 \Rightarrow \bar{u}\bar{v} - \overrightarrow{y}^2 = \frac{b^2}{R^2} \to 0$. But tR is as good as R for any $t \in \mathbb{R}$ as the scaling parameter and hence the ”boundary“ of AdS is given by

$$uv - \overrightarrow{y}^2 = o, \quad \text{where,} \quad (u, v, \overrightarrow{y}) \sim t(u, v, \overrightarrow{y}) \tag{1.4.39}$$

We call this boundary conformal boundary as its coordinates are well defined upto a scaling. The topology of this conformal boundary is that of $S^1 \times S^{n-1}$ as can be seen from the equation 1.4.39 and hence it is an n dimensional ”boundary“ of the $n+1$ dimensional manifold.

1.4.3 Arguments leading to the AdS/CFT conjecture

Here we present the arguments given in [52] that led to the famous AdS/CFT conjecture. This has been reviewed at many places [25, 26, 23, 27]. But we will be following closely the discussion given in [28].

Lets us consider a stack of N D3-branes in type IIB string theory. The regime of string perturbation theory is valid when gN$\ll$ 1 and breaks down when gN$\gg$ 1, where $g = e^{\Phi}$ and Φ is the dilaton. Now there are black brane configurations which have the same RR fluxes as the above D-branes and the metric for the black $3-$brane with N units of flux is given in string frame by

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H(r)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(r)}dx^m dx^m, \quad \mu,\nu \in 0,...3, \quad m,n \in 4,...9 \\ H &= 1+\frac{L^4}{r^4}, \quad L^4 = 4\pi g N \alpha'^2, \quad r^2 = x^m x^m \end{aligned} \tag{1.4.40}$$

where α' is the physical string tension. The horizon of this black-brane system is at $r=0$ and the near horizon metric is

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega^2_{S^5}, \tag{1.4.41}$$

so that the near horizon geometry looks like that of $AdS_5 \times S_5$ where L is the radius of curvature for both AdS_5 and S_5. When $gN \gg 1$ then 'L' is very large in string units and low energy supergravity expansion is a good approximation. So, this looks like a regime complementary to the perturbative D-brane description.

Let us look at the low energy regime of both the theories. In the low 'gN' regime this consists of massless open and closed strings. The open strings ending on D3-branes are the usual $U(N)$ adjoint gauge fields and collective coordinates and their fermionic partners while the closed strings give rise to a supergravity multiplet. The open strings remain interacting in the low energy limit as $3+1$ dimensional gauge couplings are dimensionless but massless closed strings have irrelevant interactions and decouple .

In the large 'gN' description we again have closed strings away from the black-brane but there exist states which have smaller energy just because they are present near the horizon where the warp factor g_{00} is small. Therefore, in the large 'gN' limit the low energy regime consists of not only massless string states but also many massive string states which have low energy due to their proximity to the horizon.

Lets also scale the coordinates such that $r \to \zeta^{-1}r$, $x \to \zeta x$ and take the $\zeta \to \infty$ limit. In the black brane picture this is like moving near the horizon where the geometry is

like $AdS_5 \times S_5$. For the D-brane picture this is thought to be a scaling symmetry of the low energy gauge theory (because the closed string phase space collapses to zero and massive open string effects are suppressed by $\frac{\alpha'}{x^2} \sim \zeta^{-2}$).
Now in the final step we assume that the order of taking the low energy limit of the theory and adiabatic continuation in g is irrelevant. Then we have a duality where we have a $U(N)$ gauge theory with scaling symmetry at weak coupling and the full string theory in an $AdS_5 \times S_5$ background in the strong coupling regime.

1.4.4 The statement of duality and matching of parameters

The precise statement of the duality is given by

$$D = 4, \quad \mathcal{N} = 4, \quad \text{SU(N) Yang-Mills} = \text{IIB string theory on} AdS_5 \times S_5 \tag{1.4.42}$$

The parameters on the gauge theory side are g_{YM}^2 and N where the action is normalised to $-\frac{1}{2g_{YM}^2} TrF_{\mu\nu}F^{\mu\nu}$. The parameters on the string theory side are string coupling g and the units of five form flux through S_5 i.e. N. The relation between the couplings being $g = \frac{g_{YM}^2}{4\pi}$. Also, the parameters on the gravity side are expressed best in terms of length scales.

$$\frac{L}{L_{string}} \equiv \frac{L}{\sqrt{\alpha'}} = (4\pi g N)^{\frac{1}{4}} = (g_{YM}^2 N)^{\frac{1}{4}} = \lambda^{\frac{1}{4}} \tag{1.4.43}$$

where, L_{string} is the string length and λ is the t'Hooft coupling. Another length scale of the gravity theory is Planck length in 10 dimensions L_P which is given by

$$\begin{aligned} L_P^8 &= \frac{1}{2}(2\pi)^7 g^2 \alpha'^4 \\ \text{so that} \quad \frac{L}{L_P} &= \frac{1}{\pi^{\frac{5}{8}}}\left(\frac{N}{2}\right)^{\frac{1}{4}} \end{aligned} \tag{1.4.44}$$

A Classical description would require both the ratios 1.4.43 and 1.4.44 to be large, i.e., we need the coupling λ and the number of fields N to be large.

1.4.5 The AdS/CFT dictionary and the guiding principles

Symmetries

The $SO(4,2) \times SO(6)$ isometry of the $AdS_5 \times S_5$ is present as the conformal symmetry group of the gauge theory and the R-symmetry group for the SUSY of the gauge theory. This acts as one of the guiding principles while invoking other examples of the AdS/CFT duality. The isometry group of the background AdS geometry acts as the conformal group of the dual CFT.

In general a gauge symmetry in the bulk should correspond to some global symmetry in the CFT side. Gravity can be thought of as the gauge theory obtained by gauging the underlying isometry group of the background spacetime manifold. So, in this case $SO(4,2)$ is the isometry group of the AdS and hence gravity on it is the gauge theory with gauge group $S)(4,2$ and hence the corresponding field theory dual has this as a global symmetry (the conformal group). Thus, corresponding to every gauge field on the bulk we have a conserved current in the dual field theory side.

Matching of spectrum

Due to operator state correspondence in the gauge theory side (which is a CFT) there is an isomorphism between states and operators. The duality can then be expressed as a one-to-one mapping between particle species in AdS_{D+1} and the single trace chiral primary operators in the CFT.

The statement goes like this : Suppose there is a bulk field ϕ whose boundary behaviour is like z^{Δ} (in Poincare patch coordinates) then the scaled boundary limit of this bulk field is mapped to the CFT operator,

$$\mathcal{O}(x) = C_{\mathcal{O}} \lim_{z \to 0} z^{-\Delta} \phi(x,z) \tag{1.4.45}$$

where $C_{\mathcal{O}}$ is a convention dependent constant. Scale transformation in the bulk takes $\phi(z,x) \to \phi(\zeta z, \zeta x)$ and

$$\mathcal{O} \to C_{\mathcal{O}} \lim_{z\to 0} \left(\frac{z}{\zeta}\right)^{-\Delta} \phi(\zeta x, z) = \zeta^{\Delta}\mathcal{O}(\zeta x) \tag{1.4.46}$$

which is the required scale transformation of an operator of dimension Δ.

Correlators

With the above mentioned dictionary between operators and fields in 1.4.45 the generating functional for correlators in the gauge theory side is given by the partition function in the bulk theory.

$$\langle 0|e^{\int d^D x j(x)\mathcal{O}(x)}|0\rangle = Z_j \to e^{-S_{cl}}. \tag{1.4.47}$$

where, S_{cl} is the on shell action evaluated with that mode of the bulk field which has the leading boundary behaviour. From this it becomes clear that the leading boundary behaviour of the bulk field acts as the source for correlators of the corresponding operators in the boundary and the subleading piece acts as the expectation value of the corresponding operator.

Systems dual to finite temperature CFTs

Thermal systems in the CFT can be studied by euclideanising the spacetime of the CFT. The dual gravity system can also be studied in its euclidean AdS background. The temperature being identified with the inverse of periodicity of the time circle in both cases. If the finite temperature CFT is in thermodynamic equilibrium the dual gravity system should also consist of a thermodynamically stable gravity system which is a black hole. The passing over of a CFT with strong coupling from a confined phase to a deconfined phase as we increase the temperature corresponds in the gravity side to the Hawking-Page transition where the difference in free energy between thermal AdS

background to an AdS black hole background changes sign from negative to positive. The confined phase corresponds to the thermal AdS in the bulk whose free energy is smaller upto Hawking-Page transition and the deconfined phase corresponds to the black hole whose free energy is smaller after the Hawking-Page transition.
With this we end our discussion of the general aspects of the AdS/CFT correspondence. Further details can be found in the reviews that we mentioned.

1.4.6 Back to Higher Spin Theories

The general properties of known AdS/CFT dualities is given below.

- Supersymmetry is all pervading. This makes life simpler obviously but it looks likely that the duality is valid even for non supersymmetric field theories and their corresponding gravity duals. Higher spin theories as envisaged by Vasiliev et al. in [78, 79, 80, 81, 2] are particular type of theories with gravity for which the dual field theory has another set of a large number of symmetries other than supersymmetry in the form of infinite number of conserved currents with higher spin. This large number of symmetries play similar role in simplifying calculations as in the SUSY case.

- The above mentioned theories have CFTs where the number of degrees of freedom scale as $O(N^2)$ where N is some integer parameter of the theory whose $N \to \infty$ limit is studied as the theories are matrix like. It is believed that the CFTs dual to higher spin theories have degrees of freedom which scale like O(N) and hence are vector like. Vector like theories are much simpler to handle than matrix like theories.

The first proposed dualities between a higher spin theory in AdS and a dual CFT was given in [4, 5, 82]. Here it is stated as a duality between Vasiliev's theory in AdS_4 and a O(N) vector model in 3 dimensions, where $N \to \infty$ and correspondingly the spectrum of higher spin fields in the Vasiliev's theory consists of all fields with spin from

2 to ∞ taken once. Lets revisit the equation 1.4.40.There the radius of curvature of the AdS space was given by $L^4 = 4\pi g N \alpha'^2$. For SUGRA limit of the string theory we have very high string tension and hence a very large radius of curvature , so that stringy effects can be neglected. This is the limit where the AdS/CFT conjecture is very well tested. This limit is dual to strongly coupled CFTs , i.e. theories with very high t'Hooft coupling as can be seen from the equation 1.4.43. Massless higher spin theories allow us to probe another window of the AdS/CFT duality as they are thought to be arising from the first Regge trajectory of the 'tension-less' limit of the string theory and hence consists only of massless particles. This hence allows us to probe this conjecture in a limit where the dual field theory is not very strongly coupled ($\lambda \to O(1)$).

Our main focus will be on the AdS_3/CFT_2 duality with higher spin fields in the gravity side. This is particularly interesting because here the bulk theory has no propagating degrees of freedom [1]. This though does not mean that the theory becomes void as the higher spin fields still have non-trivial boundary degrees of freedom due to the presence of a conformal boundary of the AdS_3 space. So, a duality in this case has the computational simplicity to make it a good playing field to better understand the AdS/CFT conjecture. Let us go step by step towards the duality. Since there is a non-trivial conformal boundary of the AdS_3 bulk we have non-trivial boundary degrees of freedom in the bulk even though there are no bulk degrees of freedom. These correspond to the large gauge transformations of the underlying gauged isometry group. These large gauge symmetries of the bulk in this case correspond to the symmetry currents in the CFT_2. The procedure to arrive at a concrete formulation of this statement is to find out the asymptotic symmetry algebra. We will mention that procedure below.

[1]This is true only for the higher spin part of the bulk spectrum. Any additional matter added to make the duality consistent will have bulk degrees of freedom in general.

1.4.7 The Asymptotic symmetry algebra in the bulk side

We first review the procedure for arriving at asymptotic symmetry algebra for the pure gravity case based on [6] and then we review the procedure for finding the asymptotic symmetry algebra for higher spin case following closely [12]. Let us introduce a basis for the $sl(2,R)$ algebra which is the gauge group for the barred (or unbarred) part of the Chern-Simons gauge theory which is equivalent to Einstein-Hilbert Gravity in 3 dimensions with cosmological constant. It consists of the generators $L_0, L_\pm$ with $[L_m, L_n] = (m-n)L_{m+n}$. Also, the solid cylinder (the topology of AdS_3 geometry) on which the Chern-Simons Theory lives is defined by the coordinates (t, ρ, ϕ) where (ρ, ϕ) define the 2d polar coordinate system for the disc part of the cylinder and the time variable along the length of the cylinder is given by t. We can now define the light cone coordinate system by $x^\pm = \frac{t}{l} \pm \phi$ so that the Chern Simons connection takes the form

$$A = A_\rho d\rho + A_+ dx^+ + A_- dx^- \tag{1.4.48}$$

The variation of the action gives rise to a boundary term (as the solid torus has a boundary) which is given by

$$\delta S|_{bdy} = -\frac{k}{4\pi}\int_{R\times S^1} dx^+ dx^- tr\left(A_+\delta A_- - A_-\delta A_+\right) \tag{1.4.49}$$

This needs to be put to zero so that the bulk equation of motion extremises the action. One easy way to do it is to put $A_- = 0$ at the boundary. To characterise the physically inequivalent solution we need to get rid of the gauge redundancies (which vanish at the boundary). This can be partially done by demanding that we have

$$A_\rho = b^{-1}(\rho)\partial_\rho b(\rho) \tag{1.4.50}$$

The equation of motion and boundary condition given above then fixes the other components as

$$A_+ = b^{-1}(\rho)a(x^+)b(\rho), \quad A_- = 0 \tag{1.4.51}$$

A similar analysis for the barred components give

$$\bar{A}_\rho = b(\rho)\partial_\rho b^{-1}(\rho), \quad \bar{A}_+ = 0, \quad \bar{A}_- = b(\rho)\bar{a}(x^-)b^{-1}(\rho) \tag{1.4.52}$$

For the above solution to be asymptotically AdS_3 we need that the $b(\rho)$ to be same for both the barred and unbarred connections. As an illustration we give the connection for the pure AdS_3 case

$$\begin{aligned} A_{AdS} &= b^{-1}\left(L_1 + \frac{1}{4}L_{-1}\right) b dx^+ + b^{-1}\partial_\rho b d\rho \\ \bar{A}_{AdS} &= -b\left(L_{-1} + \frac{1}{4}L_1\right) b^{-1}dx^- + b\partial_\rho b^{-1}d\rho \end{aligned} \tag{1.4.53}$$

with $b = e^{\rho L_0}$.

An Asymptotically AdS_3 space is defined by connections which satisfy the condition

$$(A - A_{AdS})|_{bdy} = \left(\bar{A} - \bar{A}_{AdS}\right)|_{bdy} = \mathcal{O}(1) \tag{1.4.54}$$

This makes sure that the functions $a(x^+)$ and $\bar{a}(x^-)$ that appear in 1.4.51 and 1.4.52 are of the form

$$\begin{aligned} a(\phi) &= L_1 + l^0(\phi)L_0 + l^{-1}(\phi)L_{-1} \\ \bar{a}(\phi) &= L_{-1} + \bar{l}^0(\phi)L_0 + \bar{l}^1(\phi)L_1 \end{aligned} \tag{1.4.55}$$

where for simplicity we are assuming a time independent solution. Whatever gauge freedom (gauge transformations which vanish on the boundary) that is left after the partial gauge fixing of 1.4.50 can be fixed by setting $l^0(\phi) = \bar{l}^0(\phi) = 0$ but the functions $l^{-1}(\phi)$ and its barred counterpart cannot be changed. So, the space of solutions is parametrised by the functions $l^{-1}(\phi)$ and $\bar{l}^{1}(\phi)$ only. The exact process of calculation of the asymptotic symmetry algebra will be demonstrated only for the case of spin 3 now.

The basis algebra for spin 3 case given by $sl(3)$ generators is given in equation 1.3.25. We again fix the gauge partially with the radial gauge choice given in 1.4.50 and following a similar procedure as above the equivalent of 1.4.55 is given by

$$a(\phi) = \sum_{m=-1}^{1} l^m L_m + \sum_{n=-2}^{2} w^n W_n \tag{1.4.56}$$

and a similar thing for the barred connection. The asymptotic AdS condition is still given by 1.4.54 with the pure AdS connection still given by 1.4.53. Again fixing gauge that correspond to gauge transformation which vanish at boundary gives $l^0(\phi) = w^0(\phi) = w^{-1}(\phi) = 0$. So, the space of inequivalent solutions is now parametrised by $l^{-1}(\phi)$ and $w^{-2}(\phi)$ (and their barred counterparts). The most general form of gauge transformation which preserves the form of the connection given in 1.4.50 and 1.4.51 is given by

$$\Gamma(x^+, \rho) = e^{-\rho L_0}\gamma(x^+)e^{\rho L_0} \tag{1.4.57}$$

The infinitesimal form of this gauge transformation is given by

$$\delta a = \gamma' + [a, \gamma] \tag{1.4.58}$$

Let us redefine our generators as V_n^s where $n \leq |s|$ and we get the generators $(L_0, L_{\pm 1})$ for $s = 2$ and the generators $(W_0, W_{\pm 1}, W_{\pm 2})$ for $s = 3$. The gauge transformation can then be parametrised as

$$\gamma(\phi) = \sum_{s=2}^{3} \sum_{n \leq |s|} \gamma_{s,n}(\phi) V_n^s \tag{1.4.59}$$

Demanding that after the above gauge transformation the connection still remains of the form given in 1.4.56 with the condition on the coefficients for asymptotic AdS given above gives rise to the following recursion relations among the gauge transformation parameters

$$\begin{aligned}
\gamma_{2,0} &= \gamma'_{2,1} \\
\gamma_{2,-1} &= \frac{1}{2}\gamma''_{2,1} + \frac{2\pi}{k}\gamma_{2,1}\mathcal{L} + \frac{4\pi}{k}\gamma_{3,2}\mathcal{W} \\
\gamma_{3,1} &= -\gamma'_{3,2} \\
\gamma_{3,0} &= \frac{1}{2}\gamma''_{3,2} + \frac{4\pi}{k}\gamma_{3,2}\mathcal{L} \\
\gamma_{3,-1} &= \frac{1}{6}\gamma'''_{3,2} - \frac{10\pi}{3k}\gamma'_{3,2}\mathcal{L} - \frac{4\pi}{3k}\gamma_{3,2}\mathcal{L}' \\
\gamma_{3,-2} &= \frac{1}{24}\gamma''''_{3,2} + \frac{4\pi}{3k}\gamma''_{3,2}\mathcal{L} + \frac{7\pi}{6k}\gamma'_{3,2}\mathcal{L}' + \frac{\pi}{3k}\gamma_{3,2}\mathcal{L}'' \\
&\quad + \frac{4\pi^2}{k^2}\gamma_{3,2}\mathcal{L}^2 + \frac{\pi}{6k}\gamma_{2,1}\mathcal{W}.
\end{aligned} \tag{1.4.60}$$

where $\mathcal{L}, \mathcal{W}$ are respectively l^{-1}, w^{-2}. The change in the parameters of the connection is given by (assuming that $\epsilon = \gamma_{2,1}$ and $\chi = \gamma_{3,2}$)

$$\begin{aligned}
\delta_\epsilon \mathcal{L} &= \epsilon\mathcal{L}' + 2\epsilon'\mathcal{L} + \frac{k}{4\pi}\epsilon''' \\
\delta_\epsilon \mathcal{W} &= \epsilon\mathcal{W}' + 3\epsilon'\mathcal{W} \\
\delta_\chi \mathcal{L} &= 2\chi\mathcal{W}' + 3\epsilon'\mathcal{W} \\
\delta_\chi \mathcal{W} &= 2\chi\mathcal{L}''' + 9\chi'\mathcal{L}'' + 15\chi''\mathcal{L}' + 10\chi'''\mathcal{L} + \frac{k}{4\pi}\chi^{(5)} + \frac{64}{k}\left(\chi\mathcal{L}\mathcal{L}' + \chi'\mathcal{L}^2\right)
\end{aligned} \tag{1.4.61}$$

We interpret these changes as due to the effect of charges for large gauge transformations and read of the Poisson bracket between the associated currents. The first equation of 1.4.61 shows that $\mathcal{L}$ acts like a stress tensor and from the second equation there we can see that $\mathcal{W}$ acts as a primary of weight 3. The corresponding Poisson brackets are given by

$$\begin{aligned} i\{\mathcal{L}_m, \mathcal{L}_n\} &= (m-n)\mathcal{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \\ i\{\mathcal{L}_m, \mathcal{W}_n\} &= (2m-n)\mathcal{W}_{m+n} \\ i\{\mathcal{W}_m, \mathcal{W}_n\} &= -(m-n)(2m^2+2n^2-mn-8)\mathcal{L}_{m+n} + \frac{96}{c}(m-n)\Lambda^{(4)}_{m+n} \\ &\quad + \frac{c}{12}m(m^2-1)(m^2-4)\delta_{m,-n} \end{aligned} \tag{1.4.62}$$

where $\Lambda^{(4)}_m = \sum_{n\in\mathbb{Z}} \mathcal{L}_n\mathcal{L}_{m-n}$. The above algebra is the algebra for generators of $\mathcal{W}_3^{cl}$ algebra. This is a well defined algebra in the sense that it satisfies the Jacobi identity.
A similar analysis can be done for the case of hs[λ] algebra. The asymptotic symmetry algebra obtained is known as $\mathcal{W}^{cl}_\infty[\lambda]$. This also contains the non-linear terms as described for the spin 3 case in 1.4.62. And hence this algebra is difficult to quantise simply by changing the Poisson bracket to Commutators as we need to define normal ordering of products for the non-linear terms. In the $c \to \infty$ limit all the non-linearities vanish and the generators $\mathcal{W}^{(s)}_n$ with $|n| \leq s-1$ (called the wedge algebra) matches exactly with the $hs[\lambda]$ algebra. From the above analysis it looks plausible that the CFT dual to this higher spin gravity theory should have the $\mathcal{W}$ algebra as the symmetry algebra. Thus the symmetry of the CFT_2 is extended by generators other than the usual Virasoro generators with the new generators behaving as primaries of the underlying Virasoso algebra.
In the next chapter I will talk about my work with Rajesh Gopakumar and Mathias Gaberdiel where we provided the first evidence beyond the classical regime towards the above statement. We will calculate the one-loop partition function around a thermal AdS_3 background of the higher spin theory and find that they match with the Vacuum character of the $\mathcal{W}$ algebra. In the next two chapters I discuss my work on aspects of higher spin gravity theory in AdS_3 not exactly related to their AdS/CFT correspondence in particular.
In chapter 3, I will discuss my work done with Arjun Bagchi, Shailesh Lal and

Bindusar Sahoo on A parity violating version of the higher spin theory in AdS_3. In this work we figure out the action for this theory and calculate the energies of the various modes of the solutions to the Equations of motion and discuss the stability of these modes at a generic point in the space of parameters as well as a particular interesting point called the chiral point.

In chapter 4 will contain a discussion my work with Abhishek Chowdhury on phase structure of asymptotically AdS_3 black holes with higher spin theories. We discuss the stability, physical relevance and thermodynamics of these black holes in the chapter.

Chapter 2

Quantum W-Symmetry of AdS_3

2.1 Introduction

This chapter will be based on the work done in [63]. In this chapter, we will study the massless higher spin theories on AdS_3 discussed in the previous chapter at the quantum level. More specifically, we perform a one loop calculation of the quadratic fluctuations of the fields about a thermal AdS_3 background. This requires a careful accounting of the gauge degrees of freedom of these fields. In particular, we show that the partition function reduces to the ratio of two determinants. For a spin s field these involve Laplacians for transverse traceless modes of helicity $\pm s$ as well as $\pm(s-1)$

$$Z^{(s)} = \left[\det\left(-\Delta + \frac{s(s-3)}{\ell^2}\right)^{TT}_{(s)}\right]^{-\frac{1}{2}} \left[\det\left(-\Delta + \frac{s(s-1)}{\ell^2}\right)^{TT}_{(s-1)}\right]^{\frac{1}{2}} . \qquad (2.1.1)$$

In [15] such determinants were explicitly evaluated, in a thermal background, for arbitrary spin s, using the group theoretic techniques of [16, 17]. By applying the results of [15] we find that the one loop answer factorises neatly into left and right moving pieces

$$Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} , \qquad (2.1.2)$$

where $q = e^{i\tau}$ is the modular parameter of the boundary T^2 of the thermal background. This generalises the expression for the case of pure gravity $(s = 2)$ [18], as explicitly checked in [19]. The expression (2.1.2) is seen to be the contribution to the character for a generator of conformal dimension s. Combining the different fields of spin $s = 3, \ldots, N$, together with the corresponding expression for the spin two case, one obtains indeed the vacuum character of the $\mathcal{W}_N$ algebra.

A straightforward generalisation of an argument of Maloney and Witten [18] can now be made to show that this expression (together with the classical contribution $(q\bar{q})^{-\frac{c}{24}}$) is one loop exact in perturbation theory. This is to be understood in a particular scheme where the Newton constant is suitably renormalised while keeping c fixed.

It is interesting that if we consider the Vasiliev higher spin theory with left and right copies of the $hs(1,1)$ higher spin algebras, then we find a vacuum character of the $\mathcal{W}_\infty$ algebra, which can be written as

$$Z_{hs(1.1)} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2} = \prod_{n=1}^{\infty}|1-q^n|^2 \times \prod_{n=1}^{\infty}\frac{1}{|(1-q^n)^n|^2}\,. \tag{2.1.3}$$

It is interesting that the answer can be naturally expressed in terms of the so-called MacMahon function

$$M(q) = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^n}\,. \tag{2.1.4}$$

The organisation of this chapter is as follows. In the next subsection we review some basic features of the massless higher spin fields. We will find it useful to decompose the fields in terms of transverse traceless modes of various helicities which will enable us to count the physical and gauge degrees of freedom. In Sec.2.2, we lay out the basics for the calculation of the quadratic fluctuations, correctly taking into account the redundancy from the gauge modes. In Sec.2.3 we obtain the one loop answer for the spin three case in a brute force manner. Sec.2.4 shows how the answer for a general spin can be carried out without having to do too much work. Sec.2.5 uses the results of [15] to evaluate the determinants (2.1.1) explicitly in a thermal AdS_3 background to obtain (2.1.2). Sec.2.6 relates these expressions to the vacuum characters of $\mathcal{W}_N$. We also comment on the case of $\mathcal{W}_\infty$ and the relation to the MacMahon function. Sec. 7 contains closing remarks while the appendices describe our conventions and spell out some of the details of the spin three calculation.

2.1.1 Counting Degrees of Freedom

Here we will count the number of independent components of the Higher spin fields in metric like formulation. Recall that a completely symmetric tensor of rank s in three dimensions has $\frac{(s+1)(s+2)}{2}$ independent components. In our case, because of the double trace constraint, many of these components are not actually independent. The constraints are as many in number as those of a symmetric tensor of rank $(s-4)$. Therefore the net number of independent components is given by

$$\frac{(s+1)(s+2)}{2} - \frac{(s-3)(s-2)}{2} = 4s - 2\ . \tag{2.1.5}$$

We now argue that half of these are gauge degrees of freedom. Recall that the gauge parameter is given by a traceless rank $(s-1)$ symmetric tensor $\xi_{\mu_1\mu_2\ldots\mu_{s-1}}$. The number of independent components of $\xi_{(s-1)}$ is therefore

$$\frac{s(s+1)}{2} - \frac{(s-1)(s-2)}{2} = 2s - 1\ . \tag{2.1.6}$$

Therefore the non-gauge components are also $(2s-1)$ in number. Now the number of gauge parameters is equal to the number of equations of motion which are actually constraints and not dynamical in nature. Using up these constraints the number of unconstrained degrees of freedom left are $4s-2-(2s-1) = 2s-1$. So, in a particular gauge we can fix $2s-1$ components of the field with spin s and we are left with no bulk propagating degrees of freedom.

Let us now analyse the representation theoretic content of these different modes. This will give us an important clue to the analysis of the one loop answer. We can decompose the field $\varphi_{(s)}$ in the following way

$$\varphi_{\mu_1\mu_2\dots\mu_s} = \varphi^{\mathrm{TT}}_{\mu_1\mu_2\dots\mu_s} + g_{(\mu_1\mu_2}\tilde{\varphi}_{\mu_3\dots\mu_s)} + \nabla_{(\mu_1}\xi_{\mu_2\dots\mu_s)} \ . \tag{2.1.7}$$

Here $\varphi^{\mathrm{TT}}_{(\mu_1\mu_2\dots\mu_s)}$ is the transverse, traceless piece of $\varphi_{(s)}$ and consists of two independent components carrying helicity $\pm s$. $\tilde{\varphi}_{(\mu_3\dots\mu_s)}$ is the spin $(s-2)$ piece which carries all the trace information of $\varphi_{(s)}$. Finally, $\xi_{(s-1)}$ are the gauge parameters. Note that the double trace constraint on $\varphi_{(s)}$ of eq. (1.1.3) implies that $\tilde{\varphi}_{(s-2)}$ is traceless.

In what follows it will be important for us to make the further decomposition of the gauge field $\xi_{(s-1)}$ into its traceless transverse component ξ^{TT}, as well as

$$\xi_{\mu_1\dots\mu_{s-1}} = \xi^{\mathrm{TT}}_{\mu_1\dots\mu_{s-1}} + \xi^{(\sigma)}_{\mu_1\dots\mu_{s-1}} \ , \tag{2.1.8}$$

where $\xi^{(\sigma)}_{(s-1)}$ is the longitudinal component, that can be written as

$$\xi^{(\sigma)}_{\mu_1\dots\mu_{s-1}} = \nabla_{(\mu_1}\sigma_{\mu_2\dots\mu_{s-1})} - \frac{2}{(2s-3)}\, g_{(\mu_1\mu_2}\nabla^{\lambda}\sigma_{\mu_3\dots\mu_{s-1})\lambda} \ , \tag{2.1.9}$$

with $\sigma_{(s-2)}$ a traceless symmetric tensor. The transverse, traceless component $\xi^{\mathrm{TT}}_{(s-1)}$ carries helicity $\pm(s-1)$.

In order to exhibit the remaining helicitiy components we can now further decompose $\tilde{\varphi}_{(s-2)}$ and $\sigma_{(s-2)}$ into their transverse traceless, as well as their longitudinal spin $(s-3)$ components. The longitudinal pieces that appear in either of these decompositions can then again be decomposed into transverse traceless spin $(s-3)$ components, together with longitudinal components of spin $(s-4)$, *etc*. In this way we can see that both $\tilde{\varphi}_{(s-2)}$ and $\sigma_{(s-2)}$ have helicity components corresponding to all the helicities less or equal to $(s-2)$; this gives rise to $2(s-2)+1 = 2s-3$ components for each of the two fields. In summary we therefore have

(i) a symmetric transverse traceless field $\varphi^{\rm TT}_{(s)}$ of spin s, with helicities $\pm s$ [2 components]

(ii) a symmetric transverse traceless gauge mode $\xi^{\rm TT}_{(s-1)}$ of spin $s-1$, with helicities $\pm(s-1)$ [2 components]

(iii) a symmetric traceless (but not necessarily transverse) field $\tilde{\varphi}_{(s-2)}$ of spin $s-2$, with helicities $0, \pm 1, \pm 2, \ldots, \pm(s-2)$ [$2s-3$ components]

(iv) a symmetric traceless (but not necessarily transverse) gauge field $\sigma_{(s-2)}$ of spin $s-2$, with helicities $0, \pm 1, \pm 2, \ldots, \pm(s-2)$ [$2s-3$ components]

In particular, there are therefore $2s-1$ non-gauge and $2s-1$ gauge components, in agreement with the above counting. Note that there are precisely as many gauge components in $\sigma_{(s-2)}$, as there are components in $\tilde{\varphi}_{(s-2)}$. In fact, if we consider the trace part of (2.1.7), the tracelessness of $\sigma_{(s-2)}$ implies that

$$
\begin{aligned}
\varphi_{\mu_1\mu_2\ldots\mu_{s-2}\lambda}{}^{\lambda} &= (2s-1)\,\tilde{\varphi}_{\mu_1\ldots\mu_{s-2}} + \nabla^{\lambda}\xi^{(\sigma)}_{\mu_1\ldots\mu_{s-2}\lambda} \\
&= (2s-1)\,\tilde{\varphi}_{\mu_1\ldots\mu_{s-2}} + (\mathcal{K}\sigma)_{\mu_1\ldots\mu_{s-2}}\,, \qquad (2.1.10)
\end{aligned}
$$

where $\mathcal{K}$ is a linear second order differential operator. Thus, at least classically, we can gauge away $\tilde{\varphi}_{(s-2)}$ completely [31]. This therefore suggests that in the calculation of the one loop determinant, the $\tilde{\varphi}_{(s-2)}$ and $\sigma_{(s-2)}$ fields will give cancelling contributions. The final answer should therefore only involve the helicity $\pm s$ non-gauge modes of $\varphi^{\rm TT}_{(s)}$, as well as the helicity $\pm(s-1)$ gauge modes of $\xi^{\rm TT}_{(s-1)}$. This is the intuitive explanation of the answer (2.1.1). Below we will see explicitly how this happens from a careful consideration of the quadratic functional integral for the φ-field.

2.2 The general setup

The quadratic fluctuations we are interested in can be computed from the functional integral

$$Z^{(s)} = \frac{1}{\text{Vol(gauge group)}} \int [D\varphi_{(s)}]\, e^{-S[\varphi_{(s)}]} \,. \tag{2.2.11}$$

Here $S[\varphi_{(s)}]$ is the action defined in 1.1.7of a spin-s field in a $D = 3$ dimensional AdS background [20]

In order to evaluate the path integral $Z^{(s)}$ in (2.2.11) it is useful to change variables as

$$[D\varphi_{(s)}] = Z_{\text{gh}}^{(s)}\, [D\varphi_{(s)}^{\text{TT}}]\, [D\tilde{\varphi}_{(s-2)}]\, [D\xi_{(s-1)}] \,, \tag{2.2.12}$$

where we use the same decomposition as in (2.1.7). Here $Z_{\text{gh}}^{(s)}$ denotes the ghost determinant that arises from the change of variables.

The gauge invariance of the action, together with the orthogonality of the first terms of (2.1.7), implies that

$$S[\varphi_{(s)}] = S[\varphi_{(s)}^{\text{TT}}] + S[\tilde{\varphi}_{(s-2)}] \,, \tag{2.2.13}$$

and the first term is simply

$$S[\varphi_{(s)}^{\text{TT}}] = \int d^3x \sqrt{g}\, \varphi^{\text{TT}\,\mu_1 \ldots \mu_s} \left(-\Delta + \frac{s(s-3)}{\ell^2}\right) \varphi^{\text{TT}}_{\mu_1 \ldots \mu_s} \,. \tag{2.2.14}$$

Thus the functional integral over the TT modes is easily evaluated to be

$$Z^{(s)} = Z_{\text{gh}}^{(s)} \left[\det\left(-\Delta + \frac{s(s-3)}{\ell^2}\right)^{\text{TT}}_{(s)}\right]^{-\frac{1}{2}} \int [D\tilde{\varphi}_{(s-2)}]\, e^{-S[\tilde{\varphi}_{(s-2)}]} \,. \tag{2.2.15}$$

The determination of the functional integral requires therefore that we compute $Z_{\text{gh}}^{(s)}$, as well as the quadratic integral over $\tilde{\varphi}_{(s-2)}$. Let us briefly discuss both terms.

2.2.1 The quadratic action of $\tilde{\varphi}_{(s-2)}$

For the component of $\varphi_{(s)}$ proportional to $\tilde{\varphi}_{(s-2)}$, see eq. (2.1.7), the above action simplifies considerably. Indeed, it follows directly from (1.1.7) that

$$S[\tilde{\varphi}_{(s-2)}] = -\frac{s(s-1)(2s-3)}{4} \int d^3x\sqrt{g}\,\tilde{\varphi}^{\mu_1\ldots\mu_{s-2}}\,\widehat{\mathcal{F}}_{\mu_1\ldots\mu_{s-2}\lambda}{}^{\lambda}\,, \tag{2.2.16}$$

where $\widehat{\mathcal{F}}$ is evaluated on $\varphi_{(s)} = g\,\tilde{\varphi}_{(s-2)}$. By an explicit computation one finds that

$$\begin{aligned}\hat{\mathcal{F}}_{\mu_1\ldots\mu_s}(\varphi) &= g_{(\mu_1\mu_2}\left[\Delta\,\tilde{\varphi}_{\mu_3\ldots\mu_s)} - \nabla_{\mu_3}\nabla^{\lambda}\tilde{\varphi}_{\mu_4\ldots\mu_s)\lambda} - \frac{(s^2+s-2)}{\ell^2}\tilde{\varphi}_{\mu_3\ldots\mu_s)}\right] \\ &\quad + \frac{(2s-3)}{2}\nabla_{(\mu_1}\nabla_{\mu_2}\tilde{\varphi}_{\mu_3\ldots\mu_s)}\,.\end{aligned} \tag{2.2.17}$$

Therefore $\hat{\mathcal{F}}_{\mu_3\ldots\mu_s\lambda}{}^{\lambda} = g^{\mu_1\mu_2}\hat{\mathcal{F}}_{\mu_1\ldots\mu_s}$ is given by

$$\begin{aligned}\hat{\mathcal{F}}_{\mu_3\ldots\mu_s\lambda}{}^{\lambda} &= (2s-1)\left[\Delta\tilde{\varphi}_{(\mu_3\ldots\mu_s)} - \nabla_{(\mu_3}\nabla^{\lambda}\tilde{\varphi}_{\mu_4\ldots\mu_s)\lambda} - \tfrac{(s^2+s-2)}{\ell^2}\tilde{\varphi}_{(\mu_3\ldots\mu_s)}\right] \\ &\quad -2g_{(\mu_3\mu_4}\nabla^{\lambda}\nabla^{\nu}\tilde{\varphi}_{\mu_5\ldots\mu_s)\lambda\nu} + \tfrac{(2s-3)}{2}g^{\lambda\nu}\nabla_{(\lambda}\nabla_{\nu}\tilde{\varphi}_{\mu_3\ldots\mu_s)}\,.\end{aligned} \tag{2.2.18}$$

Note that the second but last term in (2.2.18) will not contribute when put into (2.2.16) because of the tracelessness condition on $\tilde{\varphi}$. The last term in (2.2.18) can be evaluated to be

$$\begin{aligned}g^{\mu_1\mu_2}\nabla_{(\mu_1}\nabla_{\mu_2}\tilde{\varphi}_{\mu_3\ldots\mu_s)} &= 2\Delta\tilde{\varphi}_{(\mu_3\ldots\mu_s)} + 2\nabla_{(\mu_3}\nabla^{\lambda}\tilde{\varphi}_{\mu_4\ldots\mu_s)\lambda} + 2\nabla^{\lambda}\nabla_{(\mu_3}\tilde{\varphi}_{\mu_4\ldots\mu_s)\lambda} \\ &= 2\Delta\tilde{\varphi}_{(\mu_3\ldots\mu_s)} + 4\nabla_{(\mu_3}\nabla^{\lambda}\tilde{\varphi}_{\mu_4\ldots\mu_s)} - \tfrac{(s-1)(s-2)}{\ell^2}\tilde{\varphi}_{\mu_3\ldots\mu_s)}\,.\end{aligned} \tag{2.2.19}$$

Plugging (2.2.19) and (2.2.18) into (2.2.17), the quadratic action for $\tilde{\varphi}$ finally becomes

$$S[\tilde{\varphi}_{(s-2)}] = C_s \int d^3x\sqrt{g}\,\tilde{\varphi}^{\mu_3\ldots\mu_s}\,\hat{\mathcal{F}}_{\mu_3\ldots\mu_s\lambda}{}^{\lambda}$$

$$= \quad C_s \int d^3x \sqrt{g}\, \tilde{\varphi}^{\mu_3\ldots\mu_s} \Big[4(s-1) \left(\Delta - \frac{s^2-s+1}{\ell^2} \right) \tilde{\varphi}_{\mu_3\ldots\mu_s} \qquad (2.2.20)$$
$$+ (2s-5) \nabla_{(\mu_3} \nabla^\lambda \tilde{\varphi}_{\mu_4\ldots\mu_s)\lambda} \Big] \,,$$

where C_s is an (unimportant) constant. The path integral over $\tilde{\varphi}_{(s-2)}$ is now straightforward, and can be expressed in terms of the determinant of the differential operator appearing in (2.2.20). Notice, however, that $\tilde{\varphi}_{(s-2)}$ is only traceless, and not transverse. If we want to express this determinant in terms of differential operators acting on transverse traceless operators, more work will be required. This will be sketched for $s = 3$ below in section 2.3.2.

2.2.2 The ghost determinant

For the evaluation of the ghost determinant we shall follow the same strategy as in [76]. This is to say, we write

$$\begin{aligned} 1 \quad &= \quad \int [D\varphi_{(s)}]\, e^{-\langle \varphi_{(s)}, \varphi_{(s)} \rangle} \\ &= \quad Z^{(s)}_{\text{gh}} \int [D\varphi^{\text{TT}}_{(s)}]\, [D\tilde{\varphi}_{(s-2)}]\, [D\xi_{(s-1)}]\, e^{-\langle \varphi_{(s)}, \varphi_{(s)} \rangle} \,, \end{aligned} \qquad (2.2.21)$$

where $\varphi_{(s)} \equiv \varphi_{(s)}(\varphi^{\text{TT}}_{(s)}, \tilde{\varphi}_{(s-2)}, \xi_{(s-1)})$ as in (2.1.7). Next we expand out

$$\begin{aligned} \langle \varphi_{(s)}, \varphi_{(s)} \rangle \quad &= \quad \langle \varphi^{\text{TT}}_{(s)}, \varphi^{\text{TT}}_{(s)} \rangle + \langle g\tilde{\varphi}_{(s-2)}, g\tilde{\varphi}_{(s-2)} \rangle + \langle \nabla\xi_{(s-1)}, \nabla\xi_{(s-1)} \rangle \\ &\quad + \langle g\tilde{\varphi}_{(s-2)}, \nabla\xi_{(s-1)} \rangle + \langle \nabla\xi_{(s-1)}, g\tilde{\varphi}_{(s-2)} \rangle \,. \end{aligned} \qquad (2.2.22)$$

In order to remove the off-diagonal terms of the last line, we rewrite (2.1.7) as

$$\varphi_{\mu_1\mu_2\ldots\mu_s} = \varphi^{\text{TT}}_{\mu_1\mu_2\ldots\mu_s} + g_{(\mu_1\mu_2} \tilde{\varphi}'_{\mu_3\ldots\mu_s)} + \left(\nabla_{(\mu_1} \xi_{\mu_2\ldots\mu_s)} - \tfrac{2}{2s-1} g_{(\mu_1\mu_2} \nabla^\lambda \xi_{\mu_3\ldots\mu_s)\lambda} \right) \,, \qquad (2.2.23)$$

where

$$\tilde{\varphi}'_{\mu_1\ldots\mu_{s-2}} = \tilde{\varphi}_{\mu_1\ldots\mu_{s-2}} + \tfrac{2}{2s-1}\nabla^\lambda \xi_{\mu_1\ldots\mu_{s-2}\lambda} \,. \tag{2.2.24}$$

Then the quadratic term takes the form

$$\begin{aligned} \langle \varphi_{(s)}, \varphi_{(s)} \rangle &= \langle \varphi^{\rm TT}_{(s)}, \varphi^{\rm TT}_{(s)} \rangle + \langle g\tilde{\varphi}'_{(s-2)}, g\tilde{\varphi}'_{(s-2)} \rangle \\ &\quad + \langle (\nabla\xi_{(s-1)} - \tfrac{2}{2s-1} g\nabla\xi), (\nabla\xi_{(s-1)} - \tfrac{2}{2s-1} g\nabla\xi) \rangle \,. \end{aligned} \tag{2.2.25}$$

Both the $\varphi^{\rm TT}_{(s)}$ and the $\tilde{\varphi}'_{(s-2)}$ path integral are now trivial, as is the Jacobian coming from the change of measure in going from $\tilde{\varphi}_{(s-2)}$ to $\tilde{\varphi}'_{(s-2)}$. Thus the ghost determinant simply becomes

$$\frac{1}{Z^{(s)}_{\rm gh}} = \int [D\xi_{(s-1)}]\, e^{-\langle (\nabla\xi_{(s-1)} - \frac{2}{2s-1} g\nabla\xi), (\nabla\xi_{(s-1)} - \frac{2}{2s-1} g\nabla\xi) \rangle} \,. \tag{2.2.26}$$

The exponent can be simplified further by integrating by parts to get

$$\begin{aligned} S_\xi &= \langle (\nabla\xi_{(s-1)} - \tfrac{2}{2s-1} g\nabla\xi), (\nabla\xi_{(s-1)} - \tfrac{2}{2s-1} g\nabla\xi) \rangle \\ &= s \int d^3z \sqrt{g} \Big[\xi_{\mu_1\ldots\mu_{s-1}} \left(-\Delta + \frac{s(s-1)}{\ell^2} \right) \xi^{\mu_1\ldots\mu_{s-1}} \\ &\qquad - \frac{(s-1)(2s-3)}{(2s-1)}\, \xi_{\mu_1\ldots\mu_{s-2}\lambda} \nabla^\lambda \nabla_\nu \xi^{\mu_1\ldots\mu_{s-2}\nu} \Big] \,. \end{aligned} \tag{2.2.27}$$

This is the path integral we have to perform. Before doing the calculation in the general case, it is instructive to analyse the simplest case, $s = 3$, first. The impatient reader is welcome to skip the next section and proceed directly to Sec. 4 where we perform the general analysis of the ghost determinant.

2.3 The Case of Spin Three

As explained in the previous section, see eq. (2.2.15), the calculation of the 1-loop determinant is reduced to determining the ghost determinant (2.2.26) as well as the determinant arising from (2.2.20). We shall first deal with the ghost determinant.

2.3.1 Calculation of the Ghost Determinant

For $s = 3$ the ξ-dependent exponent of (2.2.27) is of the form

$$S_\xi = 3\int d^3x\sqrt{g}\left[\xi_{\nu\rho}\left(-\Delta + \frac{6}{\ell^2}\right)\xi^{\nu\rho} - \frac{6}{5}\,\xi_\nu{}^\rho\nabla_\rho\nabla_\mu\xi^{\mu\nu}\right]. \tag{2.3.28}$$

We shall first do the calculation in a pedestrian manner, following the same methods as in [32]. We shall then explain how our result can be more efficiently obtained. To start with we decompose $\xi^{\mu\nu}$ as

$$\xi^{\mu\nu} = \xi^{\mathrm{TT}\,\mu\nu} + \nabla^\mu\sigma^{\mathrm{T}\nu} + \nabla^\nu\sigma^{\mathrm{T}\mu} + \left(\nabla^\mu\nabla^\nu - \frac{1}{3}g^{\mu\nu}\nabla^2\right)\psi = \xi^{\mathrm{TT}\,\mu\nu} + \nabla^{(\mu}\sigma^{\mathrm{T}\nu)} + \psi^{\mu\nu}\,, \tag{2.3.29}$$

where $\xi^{\mathrm{TT}\,\mu\nu}$ is the transverse and traceless part of $\xi^{\mu\nu}$, while $\sigma^{\mathrm{T}\nu}$ is (traceless) and transverse, *i.e.*

$$\nabla_\nu\sigma^{\mathrm{T}\nu} = 0\,. \tag{2.3.30}$$

Plugging (2.3.29) into (2.3.28) we obtain after a lengthy calculation — some of the details are explained in appendix 6.2 —

$$\begin{aligned} S_\xi = \int d^3x\sqrt{g}\Bigg[& 3\,\xi^{\mathrm{TT}}_{\nu\rho}\left(-\Delta + \frac{6}{\ell^2}\right)\xi^{\mathrm{TT}\,\nu\rho} \\ & + \frac{48}{5}\,\sigma^{\mathrm{T}}_\nu\left(-\Delta + \frac{2}{\ell^2}\right)\left(-\Delta + \frac{7}{\ell^2}\right)\sigma^{\mathrm{T}\nu} \\ & + \frac{18}{5}\,\psi\,(-\Delta)\left(-\Delta + \frac{3}{\ell^2}\right)\left(-\Delta + \frac{8}{\ell^2}\right)\psi\Bigg]. \end{aligned} \tag{2.3.31}$$

The ghost determinant (2.2.26) is then simply

$$Z_{\rm gh}^{(s=3)} = J_1^{-1}\left\{\det\left(-\Delta+\frac{6}{\ell^2}\right)^{\rm TT}_{(2)}\det\left[\left(-\Delta+\frac{2}{\ell^2}\right)\left(-\Delta+\frac{7}{\ell^2}\right)\right]^{\rm T}_{(1)}\right.$$
$$\left.\times\det\left[(-\Delta)\left(-\Delta+\frac{3}{\ell^2}\right)\left(-\Delta+\frac{8}{\ell^2}\right)\right]_{(0)}\right\}^{\frac{1}{2}}, \qquad (2.3.32)$$

where J_1 is the Jacobian from the change of measure in going from ξ to $(\xi^{\rm TT}, \sigma^{\rm T}, \psi)$. This can be calculated from the identity

$$1 = \int D\xi e^{-\langle\xi,\xi\rangle} = \int J_1 D\xi^{\rm TT} D\sigma^{\rm T} d\psi\, e^{-\langle\xi(\xi^{\rm TT},\sigma^{\rm T},\psi),\,\xi(\xi^{\rm TT},\sigma^{\rm T},\psi)\rangle}\,. \qquad (2.3.33)$$

Expanding out the exponential, the terms of interest are

$$\int d^3x\sqrt{g}\ \nabla_{(\mu}\sigma^{\rm T}_{\nu)}\,\nabla^{(\mu}\sigma^{{\rm T}(\nu)} = -2\int \sigma^{\rm T}_\nu\left(\Delta-\frac{2}{\ell^2}\right)\sigma^{{\rm T}\nu}\,. \qquad (2.3.34)$$

and

$$\int d^3x\sqrt{g}\left[\left(\nabla_\mu\nabla_\nu-\frac{1}{3}g_{\mu\nu}\Delta\right)\psi\right]\left[\left(\nabla^\mu\nabla^\nu-\frac{1}{3}g^{\mu\nu}\Delta\right)\psi\right]$$
$$=\frac{2}{3}\int d^3x\sqrt{g}\ \psi\ \left[(-\Delta)\left(-\Delta+\frac{3}{\ell^2}\right)\right]\psi\,. \qquad (2.3.35)$$

Thus the $(-\Delta+\frac{2}{\ell^2})$ term is cancelled from the first line of (2.3.32) and similarly the $(-\Delta)(-\Delta+\frac{3}{\ell^2})$ term from the second line. The complete ghost determinant for $s=3$ therefore equals

$$Z_{\rm gh}^{(s=3)} = \left[\det\left(-\Delta+\frac{6}{\ell^2}\right)^{\rm TT}_{(2)}\det\left(-\Delta+\frac{7}{\ell^2}\right)^{\rm T}_{(1)}\left(-\Delta+\frac{8}{\ell^2}\right)_{(0)}\right]^{\frac{1}{2}}. \qquad (2.3.36)$$

For the following it will be important to observe that this result can be obtained more directly. Indeed, as the above calculation has demonstrated, there are many cancellations between terms arising from S_ξ and the change of measure J_1. Actually, it is not difficult to see how this comes about. Consider for example the vector part. The factor by which the second line of (2.3.31) differs from (2.3.34) is the eigenvalue of the differential operator

$$\left(\mathcal{L}^{(3)}\xi\right)^{\nu\rho} \equiv \left(-\Delta + \frac{6}{\ell^2}\right)\xi^{\nu\rho} - \frac{3}{5}\left(\nabla^\rho\nabla_\mu\xi^{\nu\mu} + \nabla^\nu\nabla_\mu\xi^{\rho\mu}\right) \tag{2.3.37}$$

evaluated on the tensors of the form $\xi^{\nu\rho} = \nabla^{(\nu}\sigma^{\mathrm{T}\rho)}$. This reproduces indeed (2.3.36) since we find

$$\begin{aligned}\left(-\Delta + \frac{6}{\ell^2}\right)\nabla^{(\nu}\sigma^{\mathrm{T}\rho)} - \frac{3}{5}\left(\nabla^\rho\nabla_\mu\nabla^{(\mu}\sigma^{\mathrm{T}\nu)} + \nabla^\nu\nabla_\mu\nabla^{(\mu}\sigma^{\mathrm{T}\rho)}\right)&\\ = \frac{8}{5}\left[\nabla^\nu\left(-\Delta + \frac{7}{\ell^2}\right)\sigma^{\mathrm{T}\rho} + \nabla^\rho\left(-\Delta + \frac{7}{\ell^2}\right)\sigma^{\mathrm{T}\nu}\right]&.\end{aligned} \tag{2.3.38}$$

Similarly, the action of $\mathcal{L}^{(3)}$ on $\psi^{\mu\nu}$ leads to

$$\begin{aligned}\left(-\Delta + \frac{6}{\ell^2}\right)\psi_{\nu\rho} - \frac{6}{10}\left(\nabla_\rho\nabla^\mu\psi_{\mu\nu} + \nabla_\nu\nabla^\mu\psi_{\mu\rho}\right)&\\ = \frac{9}{5}\nabla_\rho\nabla_\nu\left(-\Delta + \frac{8}{\ell^2}\right)\psi - \frac{1}{3}g_{\rho\nu}\,\Delta\left(-\Delta + \frac{12}{\ell^2}\right)\psi&.\end{aligned} \tag{2.3.39}$$

While $\psi^{\mu\nu}$ is not an eigenvector of (2.3.37), the second term proportional to $g_{\rho\nu}$ does not actually matter for the 1-loop calculation since the result is contracted with the traceless tensor $\psi_{\rho\nu}$, see eq. (6.2.6).

2.3.2 The Quadratic Contribution from $\tilde\varphi$

The other piece of the calculation is the quadratic contribution from $\tilde\varphi$, which for $s = 3$ takes the form

$$S[\tilde\varphi_{(1)}] = \frac{9}{2}\int d^3\sqrt{g}\left[8\,\tilde\varphi^\rho\left(-\Delta + \frac{7}{\ell^2}\right)\tilde\varphi_\rho - \tilde\varphi^\rho\,\nabla_\rho\nabla^\lambda\tilde\varphi_\lambda\right]. \tag{2.3.40}$$

In order to express the determinant in terms of those acting on traceless transverse components, we now decompose $\tilde{\varphi}_\rho$ into its transverse and longitudinal component

$$\tilde{\varphi}_\rho = \tilde{\varphi}^{\mathrm{T}}_\rho + \nabla_\rho \chi \,. \tag{2.3.41}$$

By the usual argument the above quadratic action then becomes

$$\begin{aligned} S &= -\frac{9}{2}\int d^3\sqrt{g}\left[8\,\tilde{\varphi}^{\mathrm{T}\,\rho}\left(\Delta - \frac{7}{\ell^2}\right)\tilde{\varphi}^{\mathrm{T}}_\rho \right.\\ &\qquad \left. -8\,\chi\nabla^\rho\left(\Delta - \frac{7}{\ell^2}\right)\nabla_\rho\chi - \chi\,\nabla^\rho\,\nabla_\rho\nabla^\lambda\nabla_\lambda\chi\right] \\ &= \frac{9}{2}\int d^3\sqrt{g}\left[8\,\tilde{\varphi}^{\mathrm{T}\,\rho}\left(-\Delta + \frac{7}{\ell^2}\right)\tilde{\varphi}^{\mathrm{T}}_\rho + 9\chi\left(-\Delta + \frac{8}{\ell^2}\right)(-\Delta)\chi\right] . \end{aligned} \tag{2.3.42}$$

The last $(-\Delta)$ factor is removed by the Jacobian that arises because of the change of variables $\tilde{\varphi} \equiv \tilde{\varphi}(\tilde{\varphi}^{\mathrm{T}}, \chi)$; the relevant term there is simply

$$\int d^3x\sqrt{g}\,(\nabla^\rho\chi)(\nabla_\rho\chi) = \int d^3x\sqrt{g}\,\chi(-\Delta\chi)\,. \tag{2.3.43}$$

The correction term coming from this part of the calculation is therefore of the form

$$Z^{(s=3)}_{\tilde{\varphi}_{(1)}} = \left[\det\left(-\Delta + \frac{7}{\ell^2}\right)^{T}_{(1)} \det\left(-\Delta + \frac{8}{\ell^2}\right)_{(0)}\right]^{-\frac{1}{2}} . \tag{2.3.44}$$

This cancels precisely against two of the factors in $Z^{(s=3)}_{\mathrm{gh}}$ of eq. (2.3.36), as expected from our general considerations above. Combining the different pieces as in eq. (2.2.15), the total 1-loop determinant for $s = 3$ then equals

$$Z^{(s=3)} = \left[\det\,(-\Delta)^{\mathrm{TT}}_{(3)}\right]^{-\frac{1}{2}} \left[\det\left(-\Delta + \frac{6}{\ell^2}\right)^{\mathrm{TT}}_{(2)}\right]^{\frac{1}{2}} . \tag{2.3.45}$$

As expected, only the helicity s and helicity $(s-1)$ terms therefore contribute to this determinant.

2.4 Quadratic Fluctuations for General Spin

The above calculation is fairly technical, and we cannot hope to generalise it directly to higher spin. However, as explained above, we expect that the contributions of $\tilde{\varphi}_{(s-2)}$ and $\sigma_{(s-2)}$ should cancel each other, and it should therefore be possible to organise the calculation in a way in which this becomes manifest. In the following we shall explain how this can be achieved. In particular, we shall explain that most of the ghost determinant will actually just cancel the quadratic contribution from $\tilde{\varphi}_{(s-2)}$.

2.4.1 The ghost determinant

Generalising the definition of $\mathcal{L}^{(3)}$ in (2.3.37) let us define $\mathcal{L}^{(s)}$ to be the differential operator $\mathcal{L}^{(s)}$ appearing in the integral (2.2.27)

$$\left(\mathcal{L}^{(s)}\xi\right)_{\mu_1\ldots\mu_{s-1}} = \left(-\Delta + \frac{s(s-1)}{\ell^2}\right)\xi_{\mu_1\ldots\mu_{s-1}} - \frac{(2s-3)}{(2s-1)}\nabla_{(\mu_1}\nabla^{\lambda}\xi_{\mu_2\ldots\mu_{s-1})\lambda}\,. \qquad (2.4.46)$$

Let us separate ξ into its transverse traceless component as well as $\sigma_{(s-2)}$ as in (2.1.8), *i.e.* $\xi_{(s-1)} = \xi^{\rm TT}_{(s-1)} + \xi^{(\sigma)}_{(s-1)}$ with

$$\xi^{(\sigma)}_{\mu_1\ldots\mu_{s-1}} = \nabla_{(\mu_1}\sigma_{\mu_2\ldots\mu_{s-1})} - \frac{2}{(2s-3)}\, g_{(\mu_1\mu_2}\nabla^{\lambda}\sigma_{\mu_3\ldots\mu_{s-1})\lambda}\,. \qquad (2.4.47)$$

On the transverse traceless components $\xi^{\rm TT}_{(s-1)}$ of $\xi_{(s-1)}$ the second term of $\mathcal{L}^{(s)}$ vanishes, and the operator has a simple form, namely

$$\mathcal{L}^{(s)}\xi^{\rm TT}_{(s-1)} = \left(-\Delta + \frac{s(s-1)}{\ell^2}\right)\xi^{\rm TT}_{(s-1)}\,. \qquad (2.4.48)$$

In order to determine $\mathcal{L}^{(s)}\xi_{(s-1)}$ it therfore remains to calculate $\mathcal{L}^{(s)}\xi^{(\sigma)}_{(s-1)}$ as a differential operator on $\sigma_{(s-2)}$. Since the resulting expression will be contracted with $\xi^{(\sigma)}_{(s-1)}$, we only have to evaluate the operator up to 'trace terms', (*i.e.* terms that are proportional to $g_{\mu_i\mu_j}$ for some indices $i,j\in\{1,\ldots,s-1\}$). The calculation can be broken up into different terms. From the first term of $\mathcal{L}^{(s)}$ we get

$$\left(-\Delta + \frac{s(s-1)}{\ell^2}\right)\nabla_{(\mu_1}\sigma_{\mu_2\ldots\mu_{s-1})} \cong \nabla_{(\mu_1}\left(-\Delta + \frac{(s-1)(s+2)}{\ell^2}\right)\sigma_{\mu_2\ldots\mu_{s-1})}\,, \qquad (2.4.49)$$

where $\cong$ always denotes equality up to trace terms. The action of $(-\Delta + \frac{s(s-1)}{\ell^2})$ on the second term in (2.1.9) only produces a trace term.

This leaves us with evaluating the second term of $\mathcal{L}^{(s)}$. The relevant formulae are

$$\begin{aligned}-\frac{(2s-3)}{(2s-1)}\nabla_{(\mu_1}\nabla^\lambda\nabla_{(\mu_2}\sigma_{\mu_3\ldots\mu_{s-1}\lambda)} \;\cong\; & \frac{(2s-3)}{(2s-1)}\nabla_{(\mu_1}\left(-\Delta + \frac{(s-1)(s-2)}{\ell^2}\right)\sigma_{\mu_2\ldots\mu_{s-1})} \\ & -\frac{(2s-3)}{(2s-1)}\nabla_{(\mu_1}\nabla_{\mu_2}\nabla^\lambda\sigma_{\mu_3\ldots\mu_{s-1})\lambda}\,, \qquad (2.4.50)\end{aligned}$$

as well as

$$-\frac{(2s-3)}{(2s-1)}\nabla_{(\mu_1}\nabla^\lambda\left(-\frac{2}{2s-3}\right)g_{\mu_2\mu_3}\nabla^\nu\sigma_{\mu_4\ldots\mu_{s-1}\lambda)\nu} \cong \frac{2}{(2s-1)}\nabla_{(\mu_1}\nabla_{\mu_2}\nabla^\lambda\sigma_{\mu_3\ldots\mu_{s-1})\lambda}\,. \qquad (2.4.51)$$

Combining (6.3.23), (6.3.24) and (6.3.25) then leads to

$$\begin{aligned}\left(\mathcal{L}^{(s)}\xi^{(\sigma)}\right)_{\mu_1\ldots\mu_{s-1}} \;\cong\; & \frac{1}{(2s-1)}\nabla_{(\mu_1}\Bigg[4(s-1)\left(-\Delta + \frac{s^2-s+1}{\ell^2}\right)\sigma_{\mu_2\ldots\mu_{s-1})} \\ & +(5-2s)\nabla_{\mu_2}\nabla^\lambda\sigma_{\mu_3\ldots\mu_{s-1})\lambda}\Bigg]\,. \qquad (2.4.52)\end{aligned}$$

For the simplest case, $s=3$, we have also worked out the trace piece; this is described in appendix 6.2.1.

Now the important observation is that the differential operator in the square brackets of (2.4.52) acts on $\sigma_{(s-2)}$ in precisely the same way as the differential operator in (2.2.20) acts on $\tilde{\varphi}_{(s-2)}$. Both $\sigma_{(s-2)}$ and $\tilde{\varphi}_{(s-2)}$ are symmetric traceless, but not necessarily transverse tensors of rank $(s-2)$, and thus the eigenvalues of the two operators agree exactly, including multiplicities. As a consequence the contribution to the path integral coming from $\sigma_{(s-2)}$ cancels precisely against that arising from integrating out

$\tilde{\varphi}_{(s-2)}$.

In particular, the only contributions that actually survive are those coming from $\varphi^{\rm TT}_{(s)}$, see eq. (2.2.14), as well as the contribution of $\xi^{\rm TT}_{(s-1)}$ to the ghost determinant, see (2.4.48). Putting these two contributions together gives the full one-loop amplitude for general s in the simple form

$$Z^{(s)} = \left[\det\left(-\Delta + \frac{s(s-3)}{\ell^2}\right)^{\rm TT}_{(s)}\right]^{-\frac{1}{2}} \left[\det\left(-\Delta + \frac{s(s-1)}{\ell^2}\right)^{\rm TT}_{(s-1)}\right]^{\frac{1}{2}} , \qquad (2.4.53)$$

thus proving (2.1.1).

2.5 One loop Determinants and Holomorphic Factorisation

Given the explicit formula for $Z^{(s)}$, it is now straightforward to calculate the one loop determinant on thermal AdS$_3$. As was explained in detail in [15], the relevant determinant is of the form

$$-\log\det\left(-\Delta + \frac{m_s^2}{\ell^2}\right)^{\rm TT}_{(s)} = \int_0^\infty \frac{dt}{t}\, K^{(s)}(\tau,\bar{\tau};t)\, e^{-m_s^2 t} , \qquad (2.5.54)$$

where $K^{(s)}$ is the spin s heat kernel

$$K^{(s)}(\tau,\bar{\tau};t) = \sum_{m=1}^{\infty} \frac{\tau_2}{\sqrt{4\pi t}|\sin\frac{m\tau}{2}|^2} \cos(sm\tau_1) e^{-\frac{m^2\tau_2^2}{4t}} e^{-(s+1)t} . \qquad (2.5.55)$$

Here $q = e^{i\tau}$, with $\tau = \tau_1 + i\tau_2$ the complex structure modulus of the T^2 boundary of thermal AdS$_3$. Note that for the case at hand we have for the helicitiy s component $m_s^2 = s(s-3)$, and hence the total t-exponent is $s(s-3)+(s+1) = (s-1)^2$, while for the helicity $(s-1)$ component $m_{s-1}^2 = s(s-1)$ and we get $s(s-1)+s = s^2$. Performing the t-integral with the help of the identity

$$\frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-\frac{\alpha^2}{4t} - \beta^2 t} = \frac{1}{\alpha} e^{-\alpha\beta} \ , \tag{2.5.56}$$

we therefore obtain

$$\begin{aligned} -\log\det\left(-\Delta + \frac{s(s-3)}{\ell^2}\right)^{\mathrm{TT}}_{(s)} &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cos(sm\tau_1)}{|\sin\frac{m\tau}{2}|^2} e^{-m\tau_2(s-1)} \\ &= \sum_{m=1}^{\infty} \frac{2}{m} \frac{1}{|1-q^m|^2} \left(q^{ms} + \bar{q}^{ms}\right) \ , \end{aligned} \tag{2.5.57}$$

as well as

$$\begin{aligned} -\log\det\left(-\Delta + \frac{s(s-1)}{\ell^2}\right)^{\mathrm{TT}}_{(s-1)} &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cos((s-1)m\tau_1)}{|\sin\frac{m\tau}{2}|^2} e^{-m\tau_2 s} \\ &= \sum_{m=1}^{\infty} \frac{2}{m} \frac{q^m \bar{q}^m}{|1-q^m|^2} \left(q^{m(s-1)} + \bar{q}^{m(s-1)}\right) \end{aligned} \tag{2.5.58}$$

Hence we find for

$$\begin{aligned} -\log Z^{(s)} &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{|1-q^m|^2} \Big[(q^{ms} + \bar{q}^{ms}) - q^m \bar{q}^m \left(q^{m(s-1)} + \bar{q}^{m(s-1)}\right) \Big] \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{|1-q^m|^2} \Big[q^{ms}(1-\bar{q}^m) + \bar{q}^{ms}(1-q^m) \Big] \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{q^{ms}}{1-q^m} + \frac{\bar{q}^{ms}}{1-\bar{q}^m} \right] . \end{aligned} \tag{2.5.59}$$

Thus the result is the sum of a purely holomorphic, and a purely anti-holomorphic expression. Expanding the denominator by means of a geometric series and noting that the sum over m just gives the series expansion of the logarithm we hence obtain

$$Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \ , \tag{2.5.60}$$

thus proving (2.1.2).

2.6 $\mathcal{W}_N$, $\mathcal{W}_\infty$ and the MacMahon Function

As is explained in [12] it is consistent to consider higher spin gauge theories with only finitely many spin fields. More specficially, the construction of [12] in terms of a Chern-Simons action based on $SL(N) \times SL(N)$ leads to a theory that has, in addition to the graviton of spin $s = 2$, a family of fields of spin $s = 3, \ldots, N$. The quadratic part of its action is just the sum of the actions $S[\varphi_{(s)}]$ with $s = 2, \ldots, N$. The above calculation therefore implies that the corresponding one-loop determinant equals

$$Z_{SL(N)} = \prod_{s=2}^{N} \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} = \chi_0(\mathcal{W}_N) \times \bar{\chi}_0(\mathcal{W}_N) \,, \tag{2.6.61}$$

where $\chi_0(\mathcal{W}_N)$ is the vacuum character of the $\mathcal{W}_N$ algebra

$$\chi_0(\mathcal{W}_N) = \prod_{s=2}^{N} \prod_{n=s}^{\infty} \frac{1}{(1-q^n)} \,, \tag{2.6.62}$$

see *e.g.* Sec. 6.3.2 of [33]. Indeed, by the usual Poincare-Birkhoff-Witt theorem (see for example [**?**]), a basis for the vacuum representation of $\mathcal{W}_N$ is given by

$$W^{(N)}_{-n_1^{(N)}} \cdots W^{(N)}_{-n_{l_N}^{(N)}} \, W^{(N-1)}_{-n_1^{(N-1)}} \cdots W^{(N-1)}_{-n_{l_{N-1}}^{(N-1)}} \cdots W^{(2)}_{-n_1^{(2)}} \cdots W^{(2)}_{-n_{l_2}^{(2)}} \, \Omega \,, \tag{2.6.63}$$

where $W_n^{(K)}$ are the modes of the field of conformal dimension K, and

$$n_1^{(K)} \geq n_2^{(K)} \geq \cdots \geq n_{l_K}^{(K)} \geq K \,. \tag{2.6.64}$$

Here we have used that $W_n^{(K)}\Omega = 0$ for $n \geq -K+1$ — this is the reason for the lower bound in (2.6.64) — but we have assumed that there are no other null vectors in the vacuum representation; this will be the case for generic central charge c. We have furthermore denoted the Virasoro modes by $W_n^{(2)} \equiv L_n$. It is then easy to see that (2.6.62) is just the counting formula for the basis (2.6.63). Thus our one loop calculation produces the partition function of the $\mathcal{W}_N$ algebra, as suggested by the analysis of [12].

Maloney and Witten [18] have argued that the corresponding answer in the case of pure (super-)gravity should be one loop exact. Essentially, the argument is that for the representation of the (super-)Virasoro symmetry algebra of the theory corresponding to the vacuum character, the energy levels cannot be corrected. One may, in principle, have other states contributing to the partition function. However, we know semi-classically ($c \to \infty$) that there are no propagating gravity states in the bulk. Therefore any additional states that might contribute to the partition function must have energies going to infinity in the semi-classical limit, such as black hole states or other geometries. But these would correspond to non-perturbative corrections from the point of view of the bulk path integral computation.

All the ingredients of this argument are also present in our case of massless higher spin theories. We do not have any propagating states in the bulk, and the only semi-classical physical states are the generalised Brown-Henneaux excitations of the vacuum. These are boundary states and their energies are governed by the $\mathcal{W}_N$ algebra as argued above. Thus any additional contributions would be non-perturbative, and it follows that the above one loop answer is perturbatively exact.

In [10] a classical Brown-Henneaux analysis was also performed for the Blencowe theory based on (two copies of) the infinite dimensional Vasiliev higher spin algebra $hs(1,1)$ [35, 22]. This theory possesses one spin field for each spin $s = 2, 3, \ldots$, and thus the one-loop partition function becomes the $N \to \infty$ limit of $Z_{SL(N)}$, *i.e.*

$$Z_{hs(1,1)} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} = |M(q)|^2 \times \prod_{n=1}^{\infty} |1-q^n|^2 \,, \tag{2.6.65}$$

where $M(q)$ is the MacMahon function (2.1.4). Note, in particular, that the MacMahon function is essentially the $\mathcal{W}_\infty$ vacuum character. This connection appears not to be widely known.[1]

[1]This form of the character for $\mathcal{W}_\infty$ (or rather $\mathcal{W}_{1+\infty}$) appears, for instance, in [36, 37], and the connection also features in the appendix of [38]. We thank B. Szendroi for bringing these references to

2.7 Concluding Remarks for this chapter

We have seen that a computation of the leading quantum effects for higher spin theories on AdS$_3$ can be carried out explicitly. Our result suggests strongly that the quantum Hilbert space can be organised in terms of the vacuum representation of the $\mathcal{W}_N$ algebra. This also leads to the conclusion that this answer is perturbatively exact. Thus we have control over the quantum theory, at least to all orders in the power series expansion in Newton's constant. It would be very interesting to understand whether the full non-perturbative quantum theory is well defined. In the case of pure gravity it was argued in [18] that, under some reasonable looking assumptions, pure gravity on AdS$_3$ does not exist non-perturbatively. It would be very interesting to revisit this question for the higher spin theories considered here. In particular, one may hope that the situation could be different for the $hs(1,1)$ theory with $\mathcal{W}_\infty$ symmetry. A positive answer would probably give some impetus to investigations of these symmetries in higher dimensional AdS spacetimes.[2]

In this context we find the appearance of the MacMahon function as the $\mathcal{W}_\infty$ vacuum character very significant. The MacMahon function first appeared in string theory in the non-perturbative investigation of topological strings [41, 42]. It was further interpreted in terms of a quantum stringy Calabi-Yau geometry in [43, 44]. Perhaps, we should now interpret the ubiquitous appearance of the MacMahon function in the context of topological strings in terms of a hidden $\mathcal{W}_\infty$ symmetry. It is also rather suggestive that the MacMahon function (together with the η-function prefactor of (2.6.65)) precisely accounts for the contribution of the supergravity modes to the elliptic genus of M-theory on AdS$_3 \times S^2 \times X_6$ [45, 46]. This might provide a concrete link between their appearance in topological strings and in AdS$_3$.

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our attention.

[2] $\mathcal{W}_N$ and $\mathcal{W}_\infty$ algebras have also appeared as spacetime symmetries of non-critical string theories, see *e.g.* [39, 40]. It would be interesting to explore the connection, if any, to the above realisations.

2.7.1 Further works in this direction

This work led to the Gopakumar-Gaberdiel conjecture about gravity systems dual to minimal model CFTs[64]. They conjectured that coset CFTs of the form

$$\frac{SU(N)_k \otimes SU(N)_1}{SU(N)_{k+1}} \tag{2.7.66}$$

called the W$_N$ are dual to Vasiliev's system of higher spin theories with gauge algebra hs[λ]$\oplus$hs[λ] in the t'Hooft limit where $\lambda = \frac{N}{N+k}$. In order to match with the spectrum in the bulk to that of the dual CFT the higher spin theories is coupled to two complex scalar fields. Further checks of this duality in the form of calculation of full partiion function was done in [85]. Checks in the form of matching correlation fucntions was done in [70, 71, 72].

At a technical level, it might be interesting to redo the analysis of the quadratic fluctuations within the Chern-Simons formulation of the higher spin theories [35]. It will have a two fold advantage. First of all this will be useful to compute the one loop partition function for more complicated backgrounds like conical surplus discussed in [86, 91]. The second thing is that this will shed more light into the dictionary between the Chern-Simons formulation of higher spin theories and the metric like formulation. It should also be mentioned that the considerations of this chapter have been straightforwardly extended to the supersymmetric case in [47] .

Chapter 3

Topologically Massive Higher Spin Gravity

3.1 Introduction

This chapter will be based on the work done in [89]. After the computation of the one-loop partition function in the previous chapter we move to the study of some classical aspects of Fronsdal like formulation of higher spin theories. In this chapter we will study a parity violating generalisation of higher spin theories studied in the last 2 chapters which are a generalisation of the Topologically Massive Gravity (TMG) in 3 spacetime dimensions.

It is well known that the main difference of three dimensional gravity with higher dimensional gravity arises from the fact that there are no local degrees of freedom for gravity in 3d. There are no gravitational waves and curvature is concentrated at the locations of matter. For topologically trivial spacetimes, there are no gravitational degrees of freedom at all.

To make the dynamics of three dimensional gravity more like gravity in higher dimensions, one needs to restore local degrees of freedom. In 3d, there is the unique opportunity of adding a gravitational Chern-Simons term to the action which now becomes

$$S_3 = S_{EH} + S_{CS} \tag{3.1.1}$$

$$\text{where} \quad S_{EH} = \int d^3x\sqrt{-g}(R - 2\Lambda) \tag{3.1.2}$$

$$\text{and} \quad S_{CS} = \frac{1}{2\mu}\int d^3x \epsilon^{\mu\nu\rho}\left(\Gamma^{\sigma}_{\mu\lambda}\partial_{\nu}\Gamma^{\lambda}_{\rho\sigma} + \frac{2}{3}\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\theta}\Gamma^{\theta}_{\rho\sigma}\right) \tag{3.1.3}$$

The linearlised equations of motion of this theory are those of a massive scalar field. The existence of this massive excitation can also be traced to the effective interaction of static external sources where one finds a Yukawa attraction with interaction energies as expected for a massive scalar graviton. The theory is called topologically massive gravity [50, 51]

Topologically massive gravity theories in three dimensions with a negative cosmological constant ($\Lambda = -1/\ell^2$) have been recently extensively studied in the context of AdS/CFT [52]. Without the Chern-Simons term, 3d gravity in AdS space has the additional feature of having black hole solutions [53]. Now with the topological term, we have both black holes and propagating gravitons. For a generic value of the coefficient of the gravitational Chern-Simons term, the theory has been shown to be inconsistent: either the black hole or the gravitational waves have negative energy. It was conjectured in [54] that the theory becomes sensible at a special point where $\mu\ell = 1$. The authors claimed that the dual boundary CFT became a chiral CFT with one of the central charges vanishing ($c_L = 0$). This claim, however, was soon hotly contested [55] and in following works [56], topologically massive gravity at the chiral point was shown to be more generally dual to a logarithmic CFT. The energies of these logarithmic solutions were calculated and it was shown that these carried negative energy at the chiral point indicating an instability and the breakdown of the Chiral gravity conjecture. A more complete analysis based on techniques of holographic renormalisation showed that this claim was indeed justified [58]. It was discussed that the original chiral gravity conjecture might also hold in a limited sense when one can truncate the LCFT to a chiral CFT provided certain three-point functions vanish [1]. Similar claims were also made in [61].

Motivated by these features of topologically massive gravity, a natural question to ask is what happens when these higher-spin theories are similarly deformed by the addition of a Chern-Simons term. In this chapter we study these issues by considering the effect of parity violating, three-derivative terms added to the quadratic action of spin-3 Fronsdal fields in AdS_3. These are the spin-3 analogues of the linearisation of the gravitational Chern-Simons term described in (3.1.3), and we shall continue to refer to them as “Chern-Simons" terms.

The outline of the chapter is as follows: we start out in Sec.3.2 by giving details about the construction of the curved space analogue of the action for massive gravity coupled to higher spin modes in [66]. The equations of motion are derived from there. After relating the coefficient of the spin-three “Chern-Simons" term to the spin-two term in Sec.3.3 by looking at the frame-like formulation, we enter a detailed analysis of the equations of motion in Sec.3.4.

Here in Sec.3.4, following a strategy similar to the spin-two case, the equations are rewritten in terms of three commuting differential operators. At the chiral point, two of these operators become identical indicating an inadequacy of the basis of solutions and thereby necessitating the existence of a logarithmic solution. The solutions of the equations of motion are given explicitly. Unlike the spin-2 counterpart, the trace of the spin-3 cannot be generically set to zero and will be responsible for giving rise to non-trivial solutions in the bulk which carry a trace, in addition to the traceless mode. Construction of the logarithmic solutions corresponding to both the trace and traceless mode are given. Energies for all the solutions are computed. Away from the chiral point, the massive traceless mode carries negative energy, making this a generalisation of the spin-two example. The novelty of the analysis here is the existence of the trace mode. The massive trace mode carries positive energy away from the chiral point

[1]The existence of such a truncation only shows that a set of operators of the LCFT form a closed sub-sector, not that this sub-sector has a dual of its own [58].

and is not a gauge artefact. At the chiral point, both the traceless and the trace mode have zero energy. The logarithmic partner of the trace mode at the chiral point carries positive energy whereas the logarithmic partner of the traceless mode has negative energy indicating an instability similar to the case of the spin-two example. The massless branch solutions, and hence massive branch solutions at the chiral point, can be gauged away by appropriate choice of residual gauge transformation. This along with the fact that left branch and massive branch solutions carry zero energy at the chiral point suggests that these can be regarded as being gauge equivalent to vacuum. But the logarithmic branch solutions are not pure gauge and the negative energy for the logarithmic partner of the traceless mode is a genuine instability in the bulk, similar to the spin-2 example. Apart from all this we will see a peculiar “resonant" behaviour for the trace modes at $\mu\ell = \frac{1}{2}$, which needs some understanding from the CFT perspective.

In Sec.3.5, we make several comments on the nature of the asymptotic symmetry with the gravitational Chern-Simons term. At the chiral limit, we argue that the natural symmetry algebra to look at is a contraction of the W_3 algebra which essentially reduces to the Virasoro algebra. We comment on other possible realisations at this limit. We end in Sec.3.6 with discussions and comments

3.2 Spin-3 fields in AdS$_3$ with a Chern Simons term

We define the tensor G_{MNP} by

$$G_{MNP} = \mathcal{F}_{MNP} - \frac{1}{2}\eta_{(MN}\mathcal{F}_{P)A}{}^{A}. \tag{3.2.4}$$

It was shown in [66] that the most general action with up to three derivatives and parity violating terms could be written as

$$S[\phi] = \frac{1}{2}\int d^3x\phi^{MNP}G_{MNP} + \frac{1}{2\mu'}\int d^3x\phi^{MNP}\epsilon_{QR(M}\partial^{Q}G^{R}{}_{NP)} \tag{3.2.5}$$

The two terms appearing in this action are each invariant under the gauge transformation

$$\phi_{MNP} \mapsto \phi_{MNP} + \partial_{(M}\xi_{NP)}, \tag{3.2.6}$$

where ξ is a traceless symmetric rank two tensor. The first term is just the usual Fronsdal action for massless spin-3 fields given in 1.1.7, while the second term is the linearised Chern-Simons term.

In this chapter, we will study the covariantisation of this action to AdS_3. To do so, we minimally couple the background gravity to the spin-3 fluctuation by promoting all partial derivatives to covariant derivatives, and demanding invariance under the gauge transformations

$$\phi_{MNP} \mapsto \phi_{MNP} + \nabla_{(M}\xi_{NP)}, \tag{3.2.7}$$

where ∇ is the covariant derivative defined using the background AdS_3 connection. To construct the AdS generalisation of (3.2.5), it is helpful to recollect what happens in the case where there is no topological term, *i.e.* the covariantisation of the Fronsdal action. As reviewed in chapter 1, the Fronsdal operator 1.1.6 (defined now with covariant derivatives instead of partial derivates) is no longer invariant under the gauge transformation (3.2.7), what is invariant (for the spin-3 field in AdS_3) is the combination [68]

$$\tilde{\mathcal{F}}_{MNP} = \mathcal{F}_{MNP} - \frac{2}{\ell^2} g_{(MN}\phi_{P)A}{}^{A}, \tag{3.2.8}$$

and if we now define

$$G_{MNP} = \tilde{\mathcal{F}}_{MNP} - \frac{1}{2} g_{(MN}\tilde{\mathcal{F}}_{P)A}{}^{A}, \tag{3.2.9}$$

the gauge invariant Fronsdal action is given by [68]

$$S[\phi] = \frac{1}{2}\int d^3x \sqrt{-g}\,\phi^{MNP} G_{MNP}. \tag{3.2.10}$$

It turns out that the case with the Chern-Simons terms is essentially similar. The gauge invariant action is given by

$$S[\phi] = \frac{1}{2}\int d^3x\sqrt{-g}\phi^{MNP}G_{MNP} + \frac{1}{2\mu'}\int d^3x\sqrt{-g}\phi^{MNP}\varepsilon_{QR(M}\nabla^Q G^R{}_{NP)}, \tag{3.2.11}$$

where G_{MNP} is now defined through (3.2.9), and

$$\varepsilon^{MNP} \equiv \frac{1}{\sqrt{-g}}\epsilon^{MNP}. \tag{3.2.12}$$

We remind the reader that ε^{MNP} is a tensor and all indices are raised and lowered by the background metric. We can write the above action more compactly by defining

$$\hat{\mathcal{F}}_{MNP} = \tilde{\mathcal{F}}_{MNP} + \frac{1}{\mu'}\varepsilon_{QR(M}\nabla^Q\tilde{\mathcal{F}}^R{}_{NP)}, \tag{3.2.13}$$

in terms of which the action becomes

$$S[\phi] = \frac{1}{2}\int d^3x\sqrt{-g}\phi^{MNP}\left(\hat{\mathcal{F}}_{MNP} - \frac{1}{2}g_{(MN}\hat{\mathcal{F}}_{P)}\right). \tag{3.2.14}$$

One may further show that this action gives rise to the equations of motion

$$\mathcal{D}^{(M)}\tilde{\mathcal{F}}_{MNP} \equiv \hat{\mathcal{F}}_{MNP} = \tilde{\mathcal{F}}_{MNP} + \frac{1}{\mu'}\varepsilon_{QR(M}\nabla^Q\tilde{\mathcal{F}}^R_{NP)} = 0. \tag{3.2.15}$$

Alternatively, one could have started with constructing the most general parity violating, three derivative equations of motion for ϕ_{MNP} in flat space in three dimensions consistent with the gauge invariance (3.2.6), and attempted a covariantisation to AdS. We had initially followed this procedure and obtained identical results. In the above equations, however, the coefficient μ' is arbitrary and is not fixed by the gauge invariant structure. In the next section, we will look at the relation of our action with the $SL(3,R) \times SL(3,R)$ Chern-Simons formulation of spin-3 gravity [12] with unequal levels and obtain the relation of μ' with the coefficient of gravitational Chern-Simons term μ, given in terms of the left and right levels a_L and a_R as,

$$\frac{a_L - a_R}{2} = \frac{1}{\mu}. \tag{3.2.16}$$

3.3 Relation with Chern-Simons formulation of high spin gravity and fixing the normalisation

It has been observed in [10, 12] that higher spin gravity in three dimensions can have a Chern-Simons formulation. The levels of the Chern-Simons action in [10, 12], were taken to be equal and hence it produced only the higher-spin extension of Einstein gravity. Since it is known that if we take unequal levels of the Chern-Simons action in pure gravity and impose the torsion constraints, we get parity violating Chern-Simons term and the action becomes that of a topologically massive gravity. We should also be able to do the same for spin-3 massive gravity by taking unequal levels of the Chern-Simons terms. After taking unequal levels for the $SL(3,R) \times SL(3,R)$ Chern-Simons action in [12], and imposing the torsion constraints, we arrive at the following action

$$\begin{aligned} S &= \frac{1}{8\pi G}\int e^a \wedge \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c - 2\sigma\epsilon_{abc}\omega^{bd} \wedge \omega_d^{\ c}\right) \\ &\quad - 2\sigma e^{ab} \wedge \left(d\omega_{ab} + 2\epsilon_{cda}\omega^c \wedge \omega_b^{\ d}\right) + \frac{1}{6l^2}\epsilon_{abc}\left(e^a \wedge e^b \wedge e^c - 12\sigma e^a \wedge e^{bd} \wedge e_d^{\ c}\right) \\ &\quad + \frac{1}{\mu}\int \omega^a \wedge d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c - 2\sigma\omega^{ab} \wedge d\omega_{ab} - 4\sigma\epsilon_{abc}\omega^a \wedge \omega^b_{\ d} \wedge \omega^{dc} \end{aligned} \tag{3.3.17}$$

Subject to the torsion constraint

$$\begin{aligned} &de^a + \epsilon^{abc}\omega_b \wedge e_c - 4\sigma\epsilon^{abc}e_{bd} \wedge \omega^d_{\ c} = 0, \\ &de^{ab} + \epsilon^{cd(a}\omega_c \wedge e_d^{\ b)} + \epsilon^{cd(a}e_c \wedge \omega_d^{\ b)} = 0. \end{aligned} \tag{3.3.18}$$

This is the full non-linear action for spin-3 massive gravity. But since we are interested in linearised equations of motion, we can expand this action around AdS_3 background

$$\bar{e}^a = e^a_{AdS}, \quad \bar{e}^{ab} = 0. \tag{3.3.19}$$

And then take linearised fluctuations $h_M^{\ a}$ and $h_M^{\ ab}$ around this background. And finally we should be able to write everything in terms of the physical Fronsdal fields $\tilde{h}_{MN}$ and ϕ_{MNP}, defined as

$$\begin{aligned} \tilde{h}_{MN} &= \frac{1}{2}\bar{e}_{(M}^{\ a} h_{N)a}, \\ \phi_{MNP} &= \frac{1}{3}\bar{e}^a_{(M}\bar{e}_N^{\ b} h_{P)ab}. \end{aligned} \tag{3.3.20}$$

The above action (3.3.17) is, however, given in terms of the frame fields

$$\begin{aligned} h_{MN} &= \bar{e}_M^{\ a} h_{Na}, \\ h_{MNP} &= \bar{e}^a_M \bar{e}_N^{\ b} h_{Pab}. \end{aligned} \tag{3.3.21}$$

The frame fields has an additional Λ gauge symmetry [12] which can be gauge fixed to write down the entire action in terms of the physical Fronsdal fields (3.3.20).

If one is able to successfully implement the programme, one should arrive at the action (3.2.14), since the structure is completely determined by gauge invariance. Since we already have the action, we will bypass the complete programme and just use the Chern-Simons formulation to fix the normalisation of the coefficient μ'. For that it is sufficient to find the coefficient of some simple terms. Hence, we use the action (3.3.17), to find the coefficients of $\phi^{MNP}\nabla^2\phi_{MNP}$ and $\phi^{MNP}\epsilon_{QRM}\nabla^Q\nabla^2\phi_{NP}^{\ \ R}$. These coefficients can be found after a simple exercise and the quadratic action is

$$S = \frac{1}{2}\int\sqrt{-g}\left(\phi^{MNP}\nabla^2\phi_{MNP} + \frac{1}{2\mu}\phi^{MNP}\epsilon_{QRM}\nabla^Q\nabla^2\phi_{NP}^{\ \ R} + \cdots\right). \tag{3.3.22}$$

Here we have used $\frac{1}{16\pi G} = 1$ and $2\sigma = -1$. Comparing the coefficients of the above terms to the coefficient of similar terms in (3.2.14), we see that, μ and μ' are related by

$$\mu' = 6\mu. \tag{3.3.23}$$

3.4 Analysis of the linearised equations of motion

3.4.1 Solving the linearised equations of motion

In this section, we will analyse the linearised equations of motion (3.2.15). We wish to cast this equation in a form $\mathcal{D}^{(M)}\mathcal{D}^{(L)}\mathcal{D}^{(R)}\phi_{MNP} = 0$ for three commuting differential operators $\mathcal{D}^{(M)}$, $\mathcal{D}^{(L)}$ and $\mathcal{D}^{(R)}$. $\mathcal{D}^{(M)}$ is defined in (3.2.15). So we have to put $\tilde{\mathcal{F}}_{MNP}$ (3.2.8) into the form $\mathcal{D}^{(L)}\mathcal{D}^{(R)}\phi_{MNP}$. Note that generically this cannot be done. One has to do a suitable field redefinition and use a suitable gauge condition to be able to do it. After a careful analysis, one finds that there is a unique field redefinition and gauge condition which solves the above purpose. They are

$$\begin{aligned} \phi_{MNP} &= \tilde{\phi}_{MNP} - \frac{1}{9} g_{(MN}\tilde{\phi}_{P)}, \\ \nabla^Q \tilde{\phi}_{QMN} &= \frac{1}{2}\nabla_{(M}\tilde{\phi}_{N)}. \end{aligned} \tag{3.4.24}$$

Using this field redefinition and gauge condition, we get

$$\tilde{\mathcal{F}}_{MNP} = \nabla^2\tilde{\phi}_{MNP} - \frac{1}{6}\nabla_{(M}\nabla_N\tilde{\phi}_{P)} - \frac{8}{9l^2}g_{(MN}\tilde{\phi}_{P)} - \frac{1}{9}\nabla^2\tilde{\phi}_{(M}g_{NP)} + \frac{1}{9}g_{(MN}\nabla_{P)}\nabla^Q\tilde{\phi}_Q. \tag{3.4.25}$$

One can further see that this $\tilde{\mathcal{F}}_{MNP}$ can be cast into the desired form as

$$\tilde{\mathcal{F}}_{MNP} = -\frac{4}{\ell^2}\mathcal{D}^{(R)}\mathcal{D}^{(L)}\tilde{\phi}_{MNP}, \tag{3.4.26}$$

where $\mathcal{D}^{(R)}$ and $\mathcal{D}^{(L)}$ are defined as

$$\begin{aligned}\mathcal{D}^{(L)}\tilde{\phi}_{MNP} &= \tilde{\phi}_{MNP} + \frac{\ell}{6}\varepsilon^{QR}{}_{(M|}\nabla_Q\tilde{\phi}_{R|NP)},\\ \mathcal{D}^{(R)}\phi_{MNP} &= \tilde{\phi}_{MNP} - \frac{\ell}{6}\varepsilon^{QR}{}_{(M|}\nabla_Q\tilde{\phi}_{R|NP)}.\end{aligned} \tag{3.4.27}$$

Now, putting this together with (3.2.15), our equations of motion become

$$\mathcal{D}^{(M)}\mathcal{D}^{(L)}\mathcal{D}^{(R)}\tilde{\phi}_{MNP} = 0. \tag{3.4.28}$$

One can also check that $\mathcal{D}^{(M)}$, $\mathcal{D}^{(L)}$ and $\mathcal{D}^{(R)}$ are three sets of mutually commuting operators. The superscripts (M), (L) and (R) stand for massive, left moving and right moving branches, respectively. Taking trace of the equation (3.4.28) and contracting it with ∇^M, one finds that

$$\nabla^M\tilde{\phi}_M = 0 \tag{3.4.29}$$

However, we see that we do not get any tracelessness constraint from the equation of motion and we will soon see that the trace will be responsible for giving rise to some non-trivial solutions to the equation of motion.

Let us now try to solve for the massive branch. We can obtain the left moving and right moving solution from this by putting $\mu\ell = 1$ and $\mu\ell = -1$ respectively. The massive branch equation is

$$\mathcal{D}^{(M)}\tilde{\phi}_{MNP} = 0, \tag{3.4.30}$$

where $\mathcal{D}^{(M)}$ is defined in (3.2.15). Let $\tilde{\mathcal{D}}^{(M)}$ be the same as $\mathcal{D}^{(M)}$ with $\mu \to -\mu$. By acting on (3.4.30) with $\tilde{\mathcal{D}}^{(M)}$, we get

$$\begin{aligned}\nabla^2\tilde{\phi}_{MNP} &- \left(4\mu^2 - \frac{4}{\ell^2}\right)\tilde{\phi}_{MNP}\\ &= \frac{1}{6}\nabla_{(M}\nabla_N\tilde{\phi}_{P)} + \frac{8}{9\ell^2}g_{(MN}\tilde{\phi}_{P)} + \frac{1}{9}\nabla^2\tilde{\phi}_{(M}g_{NP)}.\end{aligned} \tag{3.4.31}$$

The equations for the massless branch is the same as above with $\mu \to \frac{1}{\ell}$. Taking the trace of the above equation, we get

$$\left(\nabla^2 - 36\mu^2 + \frac{2}{\ell^2}\right)\tilde{\phi}_M = 0. \qquad (3.4.32)$$

We will solve the equations in AdS_3 background with the metric

$$ds^2 = \ell^2\left(-\cosh^2\rho d\tau^2 + \sinh^2\rho d\phi^2 + d\rho^2\right). \qquad (3.4.33)$$

The metric has the isometry group $SL(2,R)_L \times SL(2,R)_R$. The $SL(2,R)_L$ isometry generators are [54]

$$\begin{aligned} L_0 &= i\partial_u, \\ L_{-1} &= ie^{-iu}\left[\frac{\cosh 2\rho}{\sinh 2\rho}\partial_u - \frac{1}{\sinh 2\rho}\partial_v + \frac{i}{2}\partial_\rho\right], \\ L_1 &= ie^{iu}\left[\frac{\cosh 2\rho}{\sinh 2\rho}\partial_u - \frac{1}{\sinh 2\rho}\partial_v - \frac{i}{2}\partial_\rho\right], \end{aligned} \qquad (3.4.34)$$

where $u \equiv \tau + \phi$ and $v \equiv \tau - \phi$. The $SL(2,R)_R$ generators $(\bar{L}_0, \bar{L}_{\pm 1})$ are given by the above expressions with $u \to v$. The quadratic Casimirs are

$$\begin{aligned} L^2 &= \frac{1}{2}\left(L_1 L_{-1} + L_{-1}L_1\right) - L_0^2, \\ \bar{L}^2 &= \frac{1}{2}\left(\bar{L}_1 \bar{L}_{-1} + \bar{L}_{-1}\bar{L}_1\right) - \bar{L}_0^2. \end{aligned} \qquad (3.4.35)$$

The Laplacian acting on tensors of various ranks can be written in terms of $SL(2,R)$ Casmirs as

$$\begin{aligned} \nabla^2 h &= -\frac{2}{\ell^2}\left(L^2 + \bar{L}^2\right)h, \\ \nabla^2 h_M &= -\frac{2}{\ell^2}\left(L^2 + \bar{L}^2\right)h_M - \frac{2}{\ell^2}h_M, \\ \nabla^2 h_{MN} &= -\frac{2}{\ell^2}\left(L^2 + \bar{L}^2\right)h_{MN} - \frac{6}{\ell^2}h_{MN} + \frac{2}{\ell^2}h g_{MN}, \\ \nabla^2 h_{MNP} &= -\frac{2}{\ell^2}\left(L^2 + \bar{L}^2\right)h_{MNP} - \frac{12}{\ell^2}h_{MNP} + \frac{2}{\ell^2}h_{(M}g_{NP)}. \end{aligned} \qquad (3.4.36)$$

Now we are in a position to solve the equations of motion. We will first solve for the trace (3.4.32), put it back into the full equation (3.4.31) and obtain the solution to the full equation which carries this trace. Using (3.4.36), we can solve for the trace and classify it in terms of $SL(2,R)$ primaries and descendants. Using (3.4.36), we can write (3.4.32) as

$$\left[-2\left(L^2+\bar{L}^2\right)-36\mu^2\ell^2\right]\tilde{\phi}_M=0. \tag{3.4.37}$$

Let us specialise to "primary" states with weights $(h,\bar{h})$, i.e

$$\begin{aligned}&L_0\tilde{\phi}_M=h\tilde{\phi}_M,\quad \bar{L}_0\tilde{\phi}_M=\bar{h}\tilde{\phi}_M,\\&L_1\tilde{\phi}_M=0,\quad \bar{L}_1\tilde{\phi}_M=0.\end{aligned} \tag{3.4.38}$$

From the explicit form of the generators (3.4.45), one can see that (u,v) dependence of $\tilde{\phi}_M$ is

$$\tilde{\phi}_M=e^{-ihu-i\bar{h}v}\psi_M(\rho), \tag{3.4.39}$$

The primary conditions (second line of (3.4.38)) are satisfied for $h-\bar{h}=0,\pm1$, but the only solutions compatible with the condition $\nabla^M\tilde{\phi}_M=0$ are

$$\begin{aligned}&h-\bar{h}=1,\quad \psi_v=0,\quad \psi_\rho=\frac{2i}{\sinh(2\rho)}f(\rho),\quad \psi_u=f(\rho),\\ \text{or}\quad &h-\bar{h}=-1,\quad \psi_u=0,\quad \psi_\rho=\frac{2i}{\sinh(2\rho)}f(\rho),\quad \psi_v=f(\rho),\end{aligned} \tag{3.4.40}$$

where $f(\rho)$ satisfies [2]

$$\begin{aligned}&\partial_\rho f(\rho)+\left[\frac{(h+\bar{h})\sinh^2(\rho)-\cosh^2(\rho)}{\sinh\rho\cosh\rho}\right]f(\rho)=0\\ \Longrightarrow\ &f(\rho)=\frac{1}{\ell^2}\left(\cosh\rho\right)^{-(h+\bar{h})}\sinh(\rho).\end{aligned} \tag{3.4.41}$$

[2]We have put an overall factor of $\frac{1}{\ell^2}$ in the solution to $f(\rho)$. This is because (for dimensional consistency) we want to obtain the solution to $\tilde{\phi}_{MNP}$ which are dimensionless so that at the end of the day we can multiply appropriate powers of ℓ to the solution to match it with its canonical dimension. And since we want the full solution to be dimensionless, the trace has to be multiplied by the factor of $\frac{1}{\ell^2}$

The first line of (3.4.40) is the solution to our original equation of motion (3.4.30), whereas the second line is the solution to the original equation of motion with $\mu \to -\mu$. The second line will therefore not belong to the massive branch, but by putting $\mu\ell = 1$ in the second line we will get the right branch solution and by putting $\mu\ell = 1$ in the first line, we will get the left branch solution. Putting (3.4.40) in (3.4.32), we get

$$\begin{aligned} & h = 1 \pm 3\mu\ell, \quad \bar{h} = \pm 3\mu\ell, \\ \texttt{or} \quad & h = \pm 3\mu\ell \quad \bar{h} = 1 \pm 3\mu\ell. \end{aligned} \tag{3.4.42}$$

It is easy to see that $f(\rho)$ in (3.4.41) will blow up at $\rho \to \infty$ if $h + \bar{h} < 1$. Since $\mu\ell \geq 1$, this rules out the lower sign in (3.4.42). To summarise, the different branch solution will carry the following weights.

$$\begin{aligned} \texttt{Massive:} \quad & h = 1 + 3\mu\ell \quad & \bar{h} = 3\mu\ell, \\ \texttt{Left:} \quad & h = 4 \quad & \bar{h} = 3, \\ \texttt{Right:} \quad & h = 3 \quad & \bar{h} = 4. \end{aligned} \tag{3.4.43}$$

We can successively apply L_{-1} and $\bar{L}_{-1}$ on the primary solutions obtained above and obtain the descendant solutions. After obtaining the solution for the trace, let us try to obtain the solution to the full equation (3.4.31). Using (3.4.36), we can write (3.4.31) as

$$\frac{1}{\ell^2}\left[-2\left(L^2 + \bar{L}^2\right) - 8 - 4\mu^2\ell^2\right]\tilde{\phi}_{MNP} = \frac{1}{6}\nabla_{(M}\nabla_N\tilde{\phi}_{P)} - \frac{4}{3\ell^2}\left(1 - 3\mu^2\ell^2\right)\tilde{\phi}_{(M}g_{NP)}. \tag{3.4.44}$$

We have to put the solution obtained for the trace in the RHS of the above equation and obtain the solution to the full equation. If we take the primary (or descendant) trace solutions (3.4.39,3.4.40,3.4.41) in the RHS of (3.4.44), then one can show that $\tilde{\phi}_{MNP}$, should also be a primary (or descendant) solution. This is because of the following identity (which we prove in appendix 6.3)

$$L_\xi \nabla_{(M}\nabla_N \tilde{\phi}_{P)} = \nabla_{(M}\nabla_N L_\xi \tilde{\phi}_{P)}, \tag{3.4.45}$$

where L_ξ is an isometry generator.

Since the trace carries weights $(h, \bar{h})$ given by (3.4.42), we can break the full $\tilde{\phi}_{MNP}$ as

$$\tilde{\phi}_{MNP} = \chi_{MNP} + \Sigma_{MNP}, \tag{3.4.46}$$

where all the parts of $\tilde{\phi}_{MNP}$ which carry the weights $(h, \bar{h})$ are put into χ_{MNP} and the rest in Σ_{MNP}. They satisfy the equations

$$\begin{aligned} \frac{1}{\ell^2}\left[-2\left(L^2+\bar{L}^2\right)-8-4\mu^2\ell^2\right]\chi_{MNP} &= \frac{1}{6}\nabla_{(M}\nabla_N\tilde{\phi}_{P)} - \frac{4}{3\ell^2}\left(1-3\mu^2\ell^2\right)\tilde{\phi}_{(M}g_{NP)}, \\ \frac{1}{\ell^2}\left[-2\left(L^2+\bar{L}^2\right)-8-4\mu^2\ell^2\right]\Sigma_{MNP} &= 0. \end{aligned} \tag{3.4.47}$$

Since the RHS of (3.4.44) carries the weights (3.4.42), hence it should be equated with a part of LHS which carries the same weights and hence the equation is decomposed in the above way. The first of the equation in (3.4.47) becomes (by using the weights (3.4.42))

$$\frac{8}{\ell^2}\left(4\mu^2\ell^2-1\right)\chi_{MNP} = \frac{1}{6}\nabla_{(M}\nabla_N\tilde{\phi}_{P)} - \frac{4}{3\ell^2}\left(1-3\mu^2\ell^2\right)\tilde{\phi}_{(M}g_{NP)}. \tag{3.4.48}$$

The solution to χ_{MNP} is therefore

$$\chi_{MNP} = \frac{\ell^2}{8\left(4\mu^2\ell^2-1\right)}\left[\frac{1}{6}\nabla_{(M}\nabla_N\tilde{\phi}_{P)} - \frac{4}{3\ell^2}\left(1-3\mu^2\ell^2\right)\tilde{\phi}_{(M}g_{NP)}\right]. \tag{3.4.49}$$

We see that the solution has a divergence at $\mu\ell = \frac{1}{2}$. This is not something unusual since we are solving the equation with a source (RHS of (3.4.44)) of specific weights $(h, \bar{h})$. This divergent behaviour is analogous to the resonance in forced oscillations.

From (3.4.49), we notice that

$$g^{NP}\chi_{MNP} = \tilde{\phi}_M \quad \nabla^M\chi_{MNP} = \frac{1}{2}\nabla_{(N}\tilde{\phi}_{P)}. \tag{3.4.50}$$

Using (3.4.50) in the decomposition (3.4.46) and in the gauge condition (3.4.24), we get

$$g^{NP}\Sigma_{MNP} = 0, \quad \nabla^M\Sigma_{MNP} = 0. \tag{3.4.51}$$

Let us now solve the equation of motion for Σ_{MNP} (the second line of (3.4.47)) subject to the tracelessness and gauge condition (3.4.51) [3]. We specialise to "primary" states with weights $(h, \bar{h})$, i.e

$$\begin{aligned} &L_0\Sigma_{MNP} = h\Sigma_{MNP}, \quad \bar{L}_0\Sigma_{MNP} = \bar{h}\Sigma_{MNP} \\ &L_1\Sigma_{MNP} = 0, \quad \bar{L}_1\Sigma_{MNP} = 0. \end{aligned} \tag{3.4.52}$$

From the explicit form of the generators, one can see that the (u, v) dependence of Σ_{MNP} is

$$\Sigma_{MNP} = e^{-ihu-i\bar{h}v}\sigma_{MNP}(\rho), \tag{3.4.53}$$

The primary conditions are solved for $h - \bar{h} = 0, \pm1, \pm2, \pm3$. But the only solutions compatible with the gauge conditions and tracelessness condition (3.4.51) are

$$\begin{aligned} h - \bar{h} &= 3, \\ \sigma_{MNv} &= 0 \\ \sigma_{\rho uu} = \frac{if(\rho)}{\cosh\rho\sinh\rho} \quad \sigma_{uuu} = f(\rho) \quad \sigma_{\rho\rho\rho} &= \frac{-if(\rho)}{\cosh^3(\rho)\sinh^3(\rho)} \quad \sigma_{u\rho\rho} = \frac{-f(\rho)}{\cosh^2(\rho)\sinh^2(\rho)}, \end{aligned} \tag{3.4.54}$$

[3] This solution is similar to the one obtained in [65].

and

$$\begin{aligned} h-\bar{h} &= -3, \\ \sigma_{MNu} &= 0 \\ \sigma_{\rho vv} = \frac{if(\rho)}{\cosh\rho\sinh\rho} \quad \sigma_{vvv} = f(\rho) \quad \sigma_{\rho\rho\rho} &= \frac{-if(\rho)}{\cosh^3(\rho)\sinh^3(\rho)} \quad \sigma_{v\rho\rho} = \frac{-f(\rho)}{\cosh^2(\rho)\sinh^2(\rho)}, \end{aligned} \tag{3.4.55}$$

where $f(\rho)$ satisfies

$$\begin{aligned} &\partial_\rho f(\rho) + \left[\frac{(h+\bar{h})\sinh^2(\rho) - 3\cosh^2(\rho)}{\sinh\rho\cosh\rho}\right] f(\rho) = 0 \\ &\implies f(\rho) = (\cosh\rho)^{-(h+\bar{h})}\sinh^3(\rho). \end{aligned} \tag{3.4.56}$$

Now putting the above into the second line of (3.4.47), we get

$$\begin{aligned} & h = 2 \pm \mu\ell \quad \bar{h} = -1 \pm \mu\ell \\ \texttt{or} \quad & h = -1 \pm \mu\ell \quad \bar{h} = 2 \pm \mu\ell \end{aligned} \tag{3.4.57}$$

The solution with $h-\bar{h} = 3$ belongs to the original massive branch whereas $h-\bar{h} = -3$ belongs to the massive branch with $\mu \to -\mu$. The left branch is obtained by putting $\mu\ell = 1$ in the $h - \bar{h} = 3$ solution and right branch is obtained by putting $\mu\ell = 1$ in the $h - \bar{h} = -3$ solution. It is also easy to check that $f(\rho)$ in (3.4.56) diverges at $\rho \to \infty$ unless $h + \bar{h} \geq 3$. This rules out the lower sign in (3.4.57). To summarise we obtain the following solution

$$\begin{aligned} \texttt{Massive:} \quad & h = 2 + \mu\ell \quad \bar{h} = -1 + \mu\ell \\ \texttt{Left:} \quad & h = 3 \quad \bar{h} = 0 \\ \texttt{Right:} \quad & h = 0 \quad \bar{h} = 3 \end{aligned} \tag{3.4.58}$$

We can successively apply L_{-1} and $\bar{L}_{-1}$ on the primary solutions obtained above to obtain the descendant solutions. At the chiral point $\mu\ell = 1$, the massive and left branch solutions coincide and and hence the basis of solutions become insufficient to describe the dynamics. However following the construction of [56], one sees that a new logarithmic mode emerges (which is annihilated by $\mathcal{D}^{(L)2}$ and not by $\mathcal{D}^{(L)}$). We now turn to this point.

3.4.2 Logarithmic modes at the chiral point

Let us denote the massive branch, left branch and right branch solutions with superscripts M, L and R respectively. At the chiral point $\mu\ell = 1$, the massive branch and left branch coincides and hence the basis of solutions become insufficient to describe the dynamics. However following the construction of [56], one sees that a new logarithmic mode emerges (which is annihilated by $\mathcal{D}^{(L)2}$ and not by $\mathcal{D}^{(L)}$). The logarithmic mode is obtained as

$$\Phi^{(new)} = \lim_{\mu\ell\to 1} \frac{\Phi^{(M)}(\mu\ell) - \Phi^{(L)}}{\mu\ell - 1} = \frac{d\Phi^{(M)}(\epsilon)}{d\epsilon}|_{\epsilon=0}, \tag{3.4.59}$$

where $\epsilon \equiv \mu\ell - 1$. We have schematically used Φ to denote any mode which has a decomposition into massless and massive branches and have suppressed any possible spacetime indices. It can be easily seen that since $\Phi^{(M)}$ and $\Phi^{(L)}$ are annihilated by $\mathcal{D}^{(M)}$ and $\mathcal{D}^{(L)}$ respectively, the term inside the limit is annihilated by $\mathcal{D}^{(M)}\mathcal{D}^{(L)}$ but not by $\mathcal{D}^{(M)}$ or $\mathcal{D}^{(L)}$ separately. After taking the limit, therefore the mode is annihilated by $\mathcal{D}^{(L)2}$ but not by $\mathcal{D}^{(L)}$. Now let us find out the logarithmic partner of the mode χ_{MNP} in (3.4.49). Expressing $\mu\ell$ in terms of ϵ and then taking the derivative wrt ϵ, we get

$$\begin{aligned} \hat{\chi}_{MNP} &\equiv \frac{d\chi_{MNP}(\epsilon)}{d\epsilon}|_{\epsilon=0} \\ &= -\frac{\ell^2}{9}\left[\frac{1}{6}\nabla_{(M}\nabla_N\tilde{\phi}^{(L)}_{P)} - \frac{1}{3\ell^2}\tilde{\phi}^{(L)}_{(M}g_{NP)}\right] + \frac{\ell^2}{24}\left[\frac{1}{6}\nabla_{(M}\nabla_N\hat{\phi}_{P)} + \frac{8}{3\ell^2}\hat{\phi}_{(M}g_{NP)}\right], \end{aligned} \tag{3.4.60}$$

where $\tilde{\phi}_M^{(L)}$ is the trace of the left branch solution and $\hat{\phi}_M \equiv \frac{d\tilde{\phi}_M^{(M)}(\epsilon)}{d\epsilon}|_{\epsilon=0}$. It can be easily seen from the definition of $\hat{\phi}_M$ that

$$\hat{\phi}_M = \left[-3i(u+v) - 6\log\cosh\rho\right]\tilde{\phi}_M^{(L)}, \tag{3.4.61}$$

and hence

$$\begin{aligned} & L_0\hat{\phi}_M = 3\tilde{\phi}_M^{(L)} + 4\hat{\phi}_M \quad \bar{L}_0\hat{\phi}_M = 3\tilde{\phi}_M^{(L)} + 3\hat{\phi}_M \quad L_1\hat{\phi}_M = \bar{L}_1\hat{\phi}_M = 0 \\ \implies \quad & L^2\hat{\phi}_M = -21\tilde{\phi}_M^{(L)} - 12\hat{\phi}_M \quad \bar{L}^2\hat{\phi}_M = -15\tilde{\phi}_M^{(L)} - 6\hat{\phi}_M \\ \implies \quad & \left(\nabla^2 - \frac{34}{\ell^2}\right)\hat{\phi}_M = \left[-\frac{2}{\ell^2}\left(L^2 + \bar{L}^2\right) - \frac{36}{\ell^2}\right]\hat{\phi}_M = \frac{72}{\ell^2}\tilde{\phi}_M^{(L)}. \end{aligned} \tag{3.4.62}$$

Using the above set of equations and taking the trace of (3.4.60), we get, as expected, that $\hat{\phi}_M$ is the trace of $\hat{\chi}_{MNP}$. We also see that $\hat{\chi}_{MNP}$ satisfies

$$L_0\hat{\chi}_{MNP} = 3\chi_{MNP}^{(L)} + 4\hat{\chi}_{MNP}, \quad \bar{L}_0\hat{\chi}_{MNP} = 3\chi_{MNP}^{(L)} + 3\hat{\chi}_{MNP}, \quad L_1\hat{\chi}_{MNP} = \bar{L}_1\hat{\chi}_{MNP} = 0. \tag{3.4.63}$$

We have thus obtained the logarithmic partner of the mode $\chi_{MNP}^{(L)}$ at the chiral point. Using the same trick we can also obtain the logarithmic partner of the mode $\Sigma_{MNP}^{(L)}$ and we get[4]

$$\hat{\Sigma}_{MNP} \equiv \frac{d\Sigma_{MNP}^{(M)}(\epsilon)}{d\epsilon}|_{\epsilon=0} = \left[-i(u+v) - 2\log\cosh\rho\right]\Sigma_{MNP}^{(L)}, \tag{3.4.64}$$

and hence $\hat{\Sigma}_{MNP}$ satisfies

$$L_0\hat{\Sigma}_{MNP} = \Sigma_{MNP}^{(L)} + 3\hat{\Sigma}_{MNP} \quad \bar{L}_0\hat{\Sigma}_{MNP} = \Sigma_{MNP}^{(L)} \quad L_1\hat{\Sigma}_{MNP} = \bar{L}_1\hat{\Sigma}_{MNP} = 0. \tag{3.4.65}$$

We have so far obtained traceless as well as traceful solutions to the equation of motion (3.2.15). We also obtained their logarithmic partners at the chiral point. We label

[4] This is the same as the logarithmic mode obtained in [65].

the massive, left and right branch χ modes (3.4.49) as (M_χ), (L_χ) and (R_χ) respectively. We also label the logarithmic solution to the χ mode (3.4.60) as $(\log_\chi)$. Similarly we label the massive, left, right and logarithmic Σ modes (3.4.54, 3.4.55, 3.4.56, 3.4.58, 3.4.64) as (M_Σ), (L_Σ), (R_Σ) and $(\log_\Sigma)$ respectively. We will now obtain the energies of all the above modes.

3.4.3 Energy of the fluctuations

After imposing the field redefinition and gauge condition (3.4.24), we obtain the action (3.2.14) (up to total derivatives) as,

$$\begin{aligned} S = \frac{1}{2}\int \sqrt{-g} \quad & \left[-\nabla_Q \tilde{\phi}^{MNP} \nabla^Q \tilde{\phi}_{MNP} - \frac{1}{2\mu} \varepsilon_{QRM} \nabla^Q \tilde{\phi}^{MNP} \nabla^2 \tilde{\phi}^R_{\ NP} \right. \\ & + \frac{19}{9l^2} \left(\tilde{\phi}^M \tilde{\phi}_M + \frac{1}{6\mu} \varepsilon_{QRM} \tilde{\phi}^M \nabla^Q \tilde{\phi}^R \right) \\ & \left. + \frac{17}{18} \left(\nabla^Q \tilde{\phi}^M \nabla_Q \tilde{\phi}_M + \frac{1}{6\mu} \varepsilon_{QRM} \nabla^Q \tilde{\phi}^M \nabla^2 \tilde{\phi}^R \right) \right] \end{aligned} \tag{3.4.66}$$

The momentum conjugate to $\tilde{\phi}_{MNP}$ is

$$\begin{aligned} \Pi^{(1)MNP} \equiv & \frac{\delta S}{\delta \dot{\tilde{\phi}}_{MNP}} \\ = & \frac{\sqrt{-g}}{2} \left[-\nabla^0 \left(2\tilde{\phi}_{MNP} + \frac{1}{6\mu} \varepsilon^{QR(M} \nabla_Q \tilde{\phi}_R^{\ NP)} \right) \right. \\ & + \frac{17}{18 \times 3} \nabla^0 \left(2\tilde{\phi}^{(M} g^{NP)} + \frac{1}{6\mu} \varepsilon^{QR(M} \nabla_Q \tilde{\phi}_R g^{NP)} \right) \\ & \left. - \frac{1}{6\mu} \varepsilon^{0R(M} \nabla^2 \tilde{\phi}_R^{\ NP)} - \frac{19}{9 \times 18} \frac{1}{\mu \ell^2} \varepsilon^{0R(M} \tilde{\phi}_R g^{NP)} + \frac{17}{18 \times 18\mu} \varepsilon^{0R(M} \nabla^2 \tilde{\phi}_R g^{NP)} \right]. \end{aligned} \tag{3.4.67}$$

Since we have three time derivatives, we should also implement the Ostrogradsky method (following [54]), and introduce $K_{MNP} \equiv \nabla_0 \tilde{\phi}_{MNP}$ as a canonical variable and find the momentum conjugate to that which is,

$$\begin{aligned}
\Pi^{(2)MNP} &\equiv \frac{\delta S}{\delta \dot{K}_{MNP}} \\
&= \frac{\sqrt{-g}}{2}\left[\frac{1}{6\mu} g^{00}\varepsilon^{QR(M}\nabla_Q \tilde{\phi}_R^{\ NP)} - \frac{17}{18\times 18\mu} g^{00}\varepsilon^{QR(M}\nabla_Q \tilde{\phi}_R g^{NP)}\right] \qquad (3.4.68)
\end{aligned}$$

The above expressions are the most generic expressions for the conjugate momenta and can be applied on any modes. The conjugate momenta for the different modes are listed in appendix 6.4. In oder to obtain the energy we must put the expressions for the conjugate momenta in the Hamiltonian

$$\begin{aligned}
H &= \int d^2x \left(\dot{\tilde{\phi}}_{MNP}\Pi^{(1)MNP} + \dot{K}_{MNP}\Pi^{(2)MNP} - \mathcal{L}\right) \\
&= \int d^2x \left(\dot{\tilde{\phi}}_{MNP}\Pi^{(1)MNP} - K_{MNP}\dot{\Pi}^{(2)MNP} - \mathcal{L}\right) + \frac{d}{d\tau}\int d^2x\, K_{MNP}\Pi^{(2)MNP} \\
&\equiv E^0 + E^1, \qquad (3.4.69)
\end{aligned}$$

where the integral is over ϕ and ρ and $\mathcal{L}$ is the Lagrangian density. We have defined the first integral in the second line of (3.4.69) as E^0 and second integral as E^1. Also note that $\mathcal{L} = 0$ on the solutions. Now we can put the conjugate momenta obtained in appendix (6.4) and the real part of the solutions obtained in the previous sections to get the energy expressions for different modes. One can see by explicitly putting the solutions in the above integrals that E^1 for all the non-logarithmic modes vanishes but logarithmic modes get non-trivial contribution from E^1. Putting the real part of the logarithmic solutions and expressions for the conjugate momenta for the logarithmic modes in Mathematica, we get [5]

$$
\begin{aligned}
E^1_{(\log_\chi)} &= \frac{d}{d\tau}\int d^2x\,\frac{\sqrt{-g}}{2}\left[-\nabla^0\hat{\chi}_{MNP}\left(\hat{\chi}^{MNP}+\chi^{(L)MNP}\right)+\frac{17}{18}\nabla^0\hat{\chi}_M\left(\chi^M+\chi^M\right)\right],\\
&= \frac{79\pi}{280\ell^5}\\
E^1_{(\log_\Sigma)} &= \frac{d}{d\tau}\int d^2x\,\frac{-\sqrt{-g}}{2}\left[\nabla^0\hat{\Sigma}_{MNP}\left(\hat{\Sigma}^{MNP}+\Sigma^{(L)MNP}\right)\right]\\
&= -\frac{4\pi}{15\ell^5}.
\end{aligned}
\tag{3.4.70}
$$

We can now put the expressions for the real part of the solutions obtained in the previous sections and conjugate momenta in appendix (6.4), to get the expressions for E^0 for different modes. For the non logarithmic χ modes we get,

$$
\begin{aligned}
E^0_{(M_\chi)} &= -\frac{3}{\mu}\left(3\mu^2-\frac{1}{\ell^2}\right)\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\,\dot{\chi}^{(M)}_{MNP}\chi^{(M)\;NP}_R\\
&\quad+\frac{1}{6\mu}\left(17\mu^2-\frac{5}{\ell^2}\right)\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\,\dot{\chi}^{(M)}_M\chi^{(M)}_R\\
E^0_{(L_\chi)} &= \left(-1+\frac{1}{\mu\ell}\right)\int d^2x\,\sqrt{-g}\,\left[\dot{\chi}^{(L)}_{MNP}\nabla^0\chi^{(L)MNP}-\frac{17}{18}\dot{\chi}^{(L)}_M\nabla^0\chi^{(L)M}\right]\\
&\quad-\frac{6}{\mu\ell^2}\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\,\dot{\chi}^{(L)}_{MNP}\chi^{(L)\;NP}_R+\frac{2}{\mu\ell^2}\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\,\dot{\chi}^{(L)}_M\chi^{(L)}_R\\
E^0_{(R_\chi)} &= \left(-1-\frac{1}{\mu\ell}\right)\int d^2x\,\sqrt{-g}\,\left[\dot{\chi}^{(R)}_{MNP}\nabla^0\chi^{(R)MNP}-\frac{17}{18}\dot{\chi}^{(R)}_M\nabla^0\chi^{(R)M}\right]\\
&\quad-\frac{6}{\mu\ell^2}\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\dot{\chi}^{(R)}_{MNP}\chi^{(R)\;NP}_R+\frac{2}{\mu\ell^2}\int\sqrt{-g}\varepsilon^{0RM}\dot{\chi}^{(R)}_M\chi^{(R)}_R
\end{aligned}
\tag{3.4.71}
$$

[5] All the expressions of energy that we will obtain will have the dimension of $\frac{1}{\ell^5}$. This is due to our choice of units $\frac{1}{16\pi G}=1$ and using dimensionless solutions of $\tilde{\phi}_{MNP}$. If we re-instate the factor of $\frac{1}{16\pi G}=1$ and multiply the solutions of $\tilde{\phi}_{MNP}$ with appropriate powers of ℓ matching their canonical dimensions, we will get the correct dimensions of energy. However this will not change any of the qualitative features of the discussion

For the non logarithmic Σ modes, we get

$$\begin{aligned}
E^0_{(M_\Sigma)} &= \frac{1}{\mu}\left(\mu^2 - \frac{1}{\ell^2}\right)\int d^2x\,\sqrt{-g}\,\varepsilon^{R0M}\,\dot{\Sigma}^{(M)}_{MNP}\Sigma^{(M)NP}_{R} \\
E^0_{(L_\Sigma)} &= \left(-1 + \frac{1}{\mu\ell}\right)\int d^2x\,\sqrt{-g}\,\dot{\Sigma}^{(L)}_{MNP}\nabla^0\Sigma^{(L)MNP} \\
E^0_{(R_\Sigma)} &= \left(-1 - \frac{1}{\mu\ell}\right)\int d^2x\,\sqrt{-g}\,\dot{\Sigma}^{(R)}_{MNP}\nabla^0\Sigma^{(R)MNP}
\end{aligned} \tag{3.4.72}$$

For the logarithmic modes (trace as well as traceless), we get

$$\begin{aligned}
E^0_{(\log_\chi)} &= \int d^2x\,\frac{\sqrt{-g}}{2}\left[\dot{\hat{\chi}}_{MNP}\nabla^0\chi^{(L)MNP} + \dot{\chi}^{(L)}_{MNP}\nabla^0\hat{\chi}^{MNP} - \frac{17}{18}\left(\dot{\hat{\chi}}_M\nabla^0\chi^{(L)M} + \hat{\chi}^{(L)}_M\nabla^0\hat{\chi}^M\right)\right] \\
&\quad -\frac{6}{\ell}\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\,\dot{\hat{\chi}}_{MNP}\hat{\chi}^{NP}_R + \frac{2}{\ell}\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\dot{\hat{\chi}}_M\hat{\chi}_R \\
&\quad -\frac{18}{\ell}\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\dot{\hat{\chi}}_{MNP}\chi^{(L)\,NP}_R + \frac{17}{3l}\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\,\dot{\hat{\chi}}_M\chi^{(L)}_R \\
E^0_{(\log_\Sigma)} &= \int d^2x\,\frac{\sqrt{-g}}{2}\left(\dot{\hat{\Sigma}}_{MNP}\nabla^0\Sigma^{(L)MNP} + \dot{\Sigma}^{(L)}_{MNP}\nabla^0\hat{\Sigma}^{MNP}\right) \\
&\quad -\frac{2}{\ell}\int d^2x\,\sqrt{-g}\,\varepsilon^{0RM}\,\dot{\hat{\Sigma}}_{MNP}\Sigma^{(L)NP}_R
\end{aligned} \tag{3.4.73}$$

All the integrands above are t and ϕ independent. From the above expressions, one can easily see that for M_Σ, L_Σ and R_Σ, the expression is quite simple, being given by single integrals, and by putting the solutions in the integrals, one find that they are negative. Hence one finds that $E^0_{R_\Sigma}$ is always positive, $E^0_{L_\Sigma}$ is positive for $\mu\ell > 1$ and $E^0_{M_\Sigma}$ is positive for $\mu\ell < 1$. And since E^1 vanishes for non-logarithmic modes, we find, in agreement with [65], that the qualitative feature for the non-logarithmic Σ modes is the same as that of the spin-2 case [54]. The energy expressions for the left and right χ modes are obtained after putting the solutions in Mathematica as

$$\begin{aligned}
E_{(L_\chi)} &= E^0_{(L_\chi)} = \frac{\pi}{3\mu\ell^6}\,(1-\mu\ell)\,, \\
E_{(R_\chi)} &= E^0_{(R_\chi)} = \frac{\pi}{3\mu\ell^6}\,(1+\mu\ell)\,.
\end{aligned} \tag{3.4.74}$$

Thus we see that even for the χ modes the energy of the right branch is always positive and the energy of the left branch is positive for $\mu\ell < 1$ and is zero for $\mu\ell = 1$. Although a direct analytic expression for E_{M_χ} is not possible, but using Mathematica it can be seen that it is zero for $\mu\ell = 1$, positive for $\mu\ell > 1$ and negative for $\mu\ell < 1$. We mention some of the numerical results for E_{M_χ} obtained using Mathematica.

$$\begin{aligned}
\mu\ell &= \frac{1}{3}: \quad E_{(M_\chi)} = E^0_{(M_\chi)} = -\frac{16\pi}{45\ell^5}. \\
\mu\ell &= 1: \quad E_{(M_\chi)} = E^0_{(M_\chi)} = 0, \\
\mu\ell &= 2: \quad E_{(M_\chi)} = E^0_{(M_\chi)} = \frac{\pi}{40\ell^5}, \\
\mu\ell &= 3: \quad E_{(M_\chi)} = E^0_{(M_\chi)} = \frac{16\pi}{315\ell^5}.
\end{aligned} \tag{3.4.75}$$

The energies E^0 for the logarithmic branch solutions are obtained (after putting the solutions in Mathematica) as:

$$\begin{aligned}
E^0_{(\log_\chi)} &= \frac{859\pi}{504\ell^5}, \\
E^0_{(\log_\Sigma)} &= -\frac{132\pi}{25\ell^5}.
\end{aligned} \tag{3.4.76}$$

This, along with (3.4.70), shows that the $\log_\chi$ modes has positive energy and the $\log_\Sigma$ modes has negative energy.

3.4.4 Residual gauge transformation

In this section, we will show that the massless branch solutions and massive branch solution at the chiral point (both the trace as well as traceless modes) can be removed by an appropriate choice of residual gauge transformation. But since the residual gauge parameters does not vanish at the boundary, the modes can be regarded as gauge equivalent to the vacuum only if they have vanishing energy. Hence, as per the calculations of the energies above, we will see that massive and left moving solution at the chiral point (both the trace as well as traceless mode) can be regarded as gauge equivalent to vacuum. The gauge transformation in terms of the variable $\tilde{\phi}_{MNP}$ (3.4.24) is

$$\begin{aligned} \delta\tilde{\phi}_{MNP} &= \nabla_{(M}\xi_{NP)} + \frac{1}{2}\nabla_Q \xi^Q_{(M} g_{NP)}, \\ \delta\tilde{\phi}_M &= \frac{9}{2}\nabla_N \xi^N_{\ M}. \end{aligned} \tag{3.4.77}$$

We need to find the residual gauge transformation obeying the gauge condition (3.4.24) and the auxiliary condition (3.4.29) implied by the equation of motion. We find that the residual gauge transformation satisfying these properties is

$$\begin{aligned} &\nabla^2 \xi_{MN} - \frac{6}{\ell^2}\xi_{MN} = \frac{3}{4}\nabla_{(M}\nabla_Q \xi^Q_{\ N)}, \\ &\nabla_M \nabla_N \xi^{MN} = 0. \end{aligned} \tag{3.4.78}$$

One can use the above equation to deduce the following equation for $\nabla_M \xi^M_{\ N}$

$$\nabla^2\left(\nabla_M \xi^M_{\ N}\right) - \frac{34}{\ell^2}\nabla_M \xi^M_{\ N} = 0. \tag{3.4.79}$$

We thus see that $\nabla_M \xi^M_{\ N}$ satisfies the same equation as $\tilde{\phi}_M$ (3.4.32) at the chiral point $\mu\ell = 1$, obeying the same condition (3.4.29). Thus one can choose the residual gauge transformation to remove the trace of the massless branch solution and of the massive branch solution at the chiral point which subsequently gauge away the appropriate χ modes.

For the traceless Σ modes, the residual gauge transformation should obey the equations

$$\begin{aligned} \nabla^2 \xi_{MN} - \frac{6}{\ell^2}\xi_{MN} &= 0, \\ \nabla_M \xi^M_N &= 0. \end{aligned} \tag{3.4.80}$$

We can once again see from (3.4.80) that for the residual gauge transformation parameter for the Σ mode satisfying the above equation (3.4.80), $\nabla_{(M}\xi_{NP)}$ satisfies

$$\begin{aligned} \nabla^2\nabla_{(M}\xi_{NP)} &= 0, \\ \nabla^M\nabla_{(M}\xi_{NP)} &= 0. \end{aligned} \tag{3.4.81}$$

These equations are the same as the massless Σ equations of motion and massive equations of motion at the chiral point (3.4.47) and Σ gauge condition (3.4.51) and hence one can appropriately choose the parameters to gauge away the massless branch solution for Σ_{MNP} and massive branch solution for Σ_{MNP} at the chiral point.

To summarise, we find that both the massless χ and Σ modes and their respective massive modes at the chiral point can be gauged away by an appropriate choice of residual gauge transformation parameters. Since the gauge transformation parameters do not vanish at the boundary, the modes can however be treated as gauge equivalent to vacuum only if they have vanishing energy. Hence, as per the energy calculations in the previous section, the left branch solution and massive branch solution at the chiral point can be regarded as gauge equivalent to vacuum. Since the logarithmic modes do not satisfy the same equations as their left moving partners, they cannot be regarded as pure gauge and are therefore physical propagating modes in the bulk. Thus the logarithmic traceless modes indicate a genuine instability in the bulk since they carry negative energy.

3.5 Asymptotic Symmetries and the Chiral Point

In our analysis of three dimensional gravity with spin three fields, we have seen that while solving the equations of motion for the linearised spin three, we find that there is a point where the basis for the solution becomes insufficient to describe it. This is the indication of the development of a logarithmic branch to the solution. This happens at a point where $\mu\ell = 1$. This is the same point where the spin-two excitations develop a logarithmic branch and the central charge of the left moving Virasoro algebra vanishes.

Topological Massive Gravity at the chiral point was conjectured to be dual to a logarithmic conformal field theory with $c = 0$. In our bulk analysis above, we have provided indications that a similar picture emerges when one includes the spin-three fields. To further our understanding of the symmetries of the boundary theory, let us look at the asymptotic symmetry structures.

3.5.1 The $c = 0$ confusion

The asymptotic symmetry analysis for the theory with spin three fields in AdS (without the parity violating gravitational C-S term) was performed recently in [10, 12]. The asymptotic symmetry algebra that was obtained was the classical W_3 algebra.

$$\begin{aligned}
[\, L_m, L_n \,] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
[\, L_m, V_n \,] &= (2m-n)W_{m+n} \\
[\, W_m, W_n \,] &= \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0} + \frac{16}{5c}(m-n)\Lambda_{m+n} \\
&+ (m-n)\Big(\frac{1}{15}(m+n+2)(m+n+3) - \frac{1}{6}(m+2)(n+2)\Big)L_{m+n},
\end{aligned} \tag{3.5.82}$$

where

$$\Lambda_m = \sum_{n=-\infty}^{+\infty} L_{m-n}\, L_n \,. \tag{3.5.83}$$

sums quadratic nonlinear terms. Here the central charge for both the Virasoro and the pure W_3 is given by the Brown-Henneaux central term $c = \frac{3\ell}{2G}$ for AdS.

When one adds the parity violating gravitational C-S term, in the case of the usual AdS_3 without any higher spin terms, one ends up with corrected central terms where the left-right symmetry is broken, viz. $c_\pm = \frac{3\ell}{2G}(1 \mp \frac{1}{\mu\ell})$. The "chiral-point" corresponds to $\mu\ell = 1$ where $c_+ = 0$.

The shift of the central terms, which is the effect of gravitational anomalies on the boundary stress tensor [69, 73], does not change with the addition of the spin three fields. Thus the asymptotic symmetry algebra for the bulk theory with the Chern-Simons terms added is two copies of W_3 algebra, now with differing central charges.

Now, when we look at the chiral point of the W_3 algebra, we see a potential problem. The non-linear term (3.5.83) in (3.5.82) has a coefficient which is inversely proportional to the central term and hence in the chiral limit would blow up.

3.5.2 The solution

We propose a simple solution to the above problem. The blowing up of an algebra in a particular limit is indicative of the fact that one should look at an Inönü-Wigner contraction of the algebra at that point. To achieve this, let us rescale the generators as follows:

$$\mathcal{L}_n = L_n, \quad Y_n = \sqrt{c} W_n. \tag{3.5.84}$$

The rescaled W3 algebra now looks like

$$\begin{aligned}
\left[\, \mathcal{L}_m, \mathcal{L}_n \,\right] &= (m-n)\mathcal{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}, \qquad (3.5.85)\\
\left[\, \mathcal{L}_m, Y_n \,\right] &= (2m-n) Y_{m+n},\\
\left[\, Y_m, Y_n \,\right] &= \frac{c^2}{360} m(m^2-1)(m^2-4)\delta_{m+n,0} + \frac{16}{5}(m-n)\Lambda_{m+n}\\
&+ c(m-n)\Big(\frac{1}{15}(m+n+2)(m+n+3) - \frac{1}{6}(m+2)(n+2)\Big)\mathcal{L}_{m+n}.
\end{aligned}$$

Now, at the chiral point, the algebra would be the contracted version of the W3 algebra.

$$\begin{aligned}
\left[\, \mathcal{L}_m, \mathcal{L}_n \,\right] &= (m-n)\mathcal{L}_{m+n}, \quad \left[\, \mathcal{L}_m, Y_n \,\right] = (2m-n) Y_{m+n}, \qquad (3.5.86)\\
\left[\, Y_m, Y_n \,\right] &= \frac{16}{5}(m-n)\Lambda_{m+n}.
\end{aligned}$$

The Y and Λ actually generate an ideal and so one must set them to zero in any irreducible representation of the W_3 algebra. So the classical W_3 in the chiral limit essentially reduces to the Virasoro algebra.

What we are advocating here is the classical analogue of what happens for the quantum W_3 for $c = -22/5$ [34]. Let us remind the reader of the quantum version of the W_3 algebra is. The quantum effects enter into the regularisation of the quadratic non-linear term (3.5.83). This shifts the overall quadratic coefficient of the quadratic term from $\frac{16}{5c} \to \frac{16}{5c+22}$ in (3.5.82). As is obvious, $c = -22/5$ represents a blowing up of the quantum W_3 algebra and [?] prescribes a similar procedure to what we have outlined above.

The logarithmic degeneracy at the chiral point that we would go on to construct, in this light would be related to a left moving LCFT with $c = 0$, very similar to the original construction of the spin-two example.

3.5.3 Comments on other possible solutions

The above procedure is certainly a correct one, but one might think that this is not the most general procedure that can be followed at the chiral point. Let us comment on a couple of other possible solutions.

One way to argue that $c = 0$ is not a problem in this context is to say that in this limit one should actually be looking at the quantum version of the W_3, instead of the classical algebra. Then the shifting of the non-linear term described above would mean that the algebra is perfectly fine in the chiral limit. When c is small, and the curvature of space-time is large, it may be more sensible to look at the quantum algebra. The question obviously would be how an asymptotic symmetry analysis would see the change from classical to quantum and this is far from obvious. That this feature does not have any analogue in the well-studied spin-two example makes this an attractive avenue of further exploration.

Another possible solution is to say that nothing is wrong at $c = 0$. Λ is actually a null field and the c=0 singularity is cancelled by Λ become null. Let us take the quantum counterpart $c = -22/5$. Let us suppose that Λ is a null field. We can work the commutation relations and see for example,

$$\left[\, L_m, \Lambda_n \,\right] = (3m - n)\Lambda_{m+n} + \frac{22 + 5c}{16}[m(m^2 - 1)L_{m+n}]. \tag{3.5.87}$$

So we see that indeed at $c = -22/5$, this commutator closes to Λ. This is consistent with the fact that Λ is a null field. We can similarly work out the consequences for W_n. The obstacle in this path is trying to figure out how to carry out an essentially quantum mechanical analysis in a classical algebra. We leave these issues for future work.

3.6 Concluding remarks of this chapter

In this chapter, we reviewed the the linearised action for spin-3 Fronsdal fields with a Chern- Simons term in flat space [66] and generalised it to AdS space. The structure of the action is uniquely fixed by gauge invariance. Its relation to the $SL(3, R) \times SL(3, R)$ Chern-Simons action [12, 10] with unequal levels was explored and the normalisation of the gauge invariant action found earlier is fixed. We then looked at the equations of motion which was decomposed it into left, right and massive branch.

The trace cannot be set to zero unlike the spin-2 case [54]. The trace gives rise to non-trivial solutions to the equations of motion which has no counterpart in the spin-2 case. The trace solution has a “resonant" behaviour at $\mu\ell = \frac{1}{2}$. The massive branch trace mode carries positive energy for $\mu\ell > 1$ and negative energy for $\mu\ell < 1$ and zero energy for $\mu\ell = 1$. The left branch solution carries positive energy for $\mu\ell < 1$ and negative energy for $\mu\ell > 1$ and zero energy for $\mu\ell = 1$. Apart from the “trace" solutions we also have the usual traceless mode. However the traceless mode has energy behaviour which is opposite to that of the trace mode (and similar to the spin-2 counterpart [54]) i.e massive traceless mode carries positive energy for $\mu\ell < 1$ and negative energy for $\mu\ell > 1$ and zero energy for $\mu\ell = 1$ and the left branch traceless solution carries positive energy for $\mu\ell > 1$ and negative energy for $\mu\ell < 1$ and zero energy for $\mu\ell = 1$. The right branch solution carries positive energy for both the trace and traceless mode.

At the chiral point the massive and left branch solution coincide and develop a new logarithmic branch both for the trace and traceless modes. The logarithmic solution for the trace mode carries positive energy whereas the logarithmic solution for the traceless mode carries negative energy. The left branch and massive branch solution at the chiral point are pure gauge and have vanishing energy and hence can be treated as gauge equivalent to the vacuum. But the logarithmic modes are not pure gauge and are therefore physical propagating modes in the bulk. And since the logarithmic solution for the traceless mode carries negative energy, it indicates an instability in the bulk at the chiral point. It is therefore tempting to conjecture that higher spin massive gravity constructed here at the chiral point is dual to a higher spin extension of $LCFT_2$. But there are some conceptual issues which should be dealt with before making this conjecture which are:

1. Variational principle is well defined for the new logarithmic solutions:
   The logarithmic solutions are the non trivial solutions to spin-3 massive gravity at the chiral point that grows linearly in time and linearly in ρ asymptotically. It is found to have finite time-independent negative energy. But before it can be accepted as a valid classical solution one must check that the variational principle is well defined, i.e. the boundary terms vanish on-shell for the logarithmic solutions. Similar questions for the spin-2 counterpart was asked with an affirmative answer in [56]. A similar check needs to be done for both of our logarithmic solutions and as a by product obtain the boundary currents dual to the logarithmic modes.

2. Consistent boundary conditions for the logarithmic modes:

We should be able to find consistent set of boundary conditions which encompasses the new logarithmic solutions i.e. there are consistent set of boundary conditions for which the generator of the asymptotic symmetry group is finite. Similar questions for the spin-2 case was asked with an affirmative answer in [57]. A similar analysis needs to be performed for our logarithmic branch solutions.

3. Correlation function calculation:

   We should be able to compute correlation function in the gravity side. This should put us in a position to compare them with boundary correlators expected from a higher spin extension of LCFT. Similar questions were addressed in [58, 59] for the spin-2 case. The comparison in that case was however with correlators in LCFT which is well known in the literature. To our knowledge there is no higher spin extension of LCFT in the literature so far[6]. The correlation function calculations should open up interesting questions to be answered about the higher spin extension of LCFT. Some progress in this direction has already been made in [60] where using methods of holographic renormalisation the 2-point functions were calculated from the bulk side and also consistency conditions were put on certain 3-point functions so that they match with the desired asymptotic symmetry algebra.

Another check for the TMHSG/LCFT duality is the calculation of one loop partition function in the thermal AdS background which was done in [90], generalising the results of [63, 76]. It was shown there that the partition function does not factorise holomorphically (even at the chiral point) which is what it should be for a theory with dual LCFT description.

Apart from all the above issues, the boundary CFT needs to be understood better. For example, there is the peculiar “resonant" behaviour found for the trace modes at $\mu\ell = \frac{1}{2}$ which should show up even in the CFT. Apart from that we find a positive

[6] See however some very recent work [75].

energy propagating mode in the bulk at the chiral point, which is the logarithmic solution corresponding to the trace mode. This has no counterpart in the spin-2 example and one would like to understand what this means from the CFT perspective.

Chapter 4

Phase Structure of Higher Spin Black Holes

4.1 Introduction

This chapter is based on the work done in [93]. After studying some aspects of Fronsdal like formulation of higher spin fields in AdS_3 in the previous chapters we move on to study a particular aspect of the Chern-Simons formulation of higher spin theories. In this chapter we study some thermodynamical aspects of "Black-Holes" in the context of HS theories in AdS_3. We start by giving a brief introduction to the black holes in AdS_3 with higher spin charges. This is by no means meant to be complete. For further details the reader can look up the references mentioned as we go along.

4.1.1 Higher Spin Black Holes in 3 dimensions

In 3 dimensions, the topology of Euclidean space-time with asymptotic AdS geometry is that of a solid torus. The contractible cycle is either spatial or temporal depending on whether we are in a thermal AdS background or a black hole background. In black holes the non-contractible cycle being spatial points towards the existence of a "horizon". For Euclidean black holes the temperature is defined by assuming that the time cycle is periodic. The periodicity is such that at the horizon there are no conical singularities i.e. the horizon is smooth. This periodicity in time cycle is related to the inverse of the temperature of the black hole.

In higher spin theories the concept of a metric is blurred by the fact that there are higher spin gauge transformations under which the metric is not invariant. Hence, the normal procedure of identifying black hole geometries to metrics with horizons doesn't work. In [94], a procedure to identify the higher spin black hole geometry in AdS_3, in the Chern-Simons formulation was given. There the black hole geometry was identified with those configurations where the connection is smooth in the interior of the torus geometry with a temporal contractible cycle. This is equivalent to demanding a trivial holonomy for the connection along the temporal cycle (i.e. falls in the centre of the gauge group). This ensures that when the contractible cycle is shrunk to zero, the connection comes back to itself after moving around the cycle once. But this does not ensure that the corresponding metric will look like that of an ordinary black hole. In [95] a gauge transformation was found in which the metric obtained resembled that of a conventional black hole. It was also shown that the RG flow by an irrelevant deformation triggered by a chemical potential corresponding to a spin 3 operator takes us from the principle embedding of $SL(2, R)$ to the diagonal embedding of $SL(2, R)$ in $SL(3, R)$. Also to get a a higher spin black black hole a chemical potential corresponding to the independent charges had to be added so that the system is stable thermodynamically. So, a black hole solution with higher spin charges necessarily causes the system to flow from one fixed point to another. In [96] the partition function for the black hole solution was obtained as a series expansion in spin 3 chemical potential with $hs[\lambda] \times hs[\lambda]$ algebra (this gives the higher spin symmetry algebra when the spin is not truncated to any finite value) and matched with the known CFT results for free bosonic ($\lambda = 1$) and free fermionic case ($\lambda = 0$). This answer also matches the one for general λ obtained from CFT calculations in [104]. A review of these aspects of black holes in higher spin theories can be found in [99].The $\lambda \to \infty$

limit for partition function was studied in [97], where an exact expression for partition function and spin 4 charge was obtained for any temperature and spin 3 chemical potential. Analysis of a HS black holes in presence of spin 4 chemical potential was done in [98].

A different approach to study the thermodynamics of these black holes was carried out in [100, 101, 103, 102]. A good variational principle was obtained by adding proper boundary terms to Chern Simons theories on manifolds with boundaries. The free energy was obtained from the on-shell action and an expression for entropy was obtained from that. This expression for entropy was different from that obtained in [94]. It was also shown that the stress energy tensor obtained from the variational principle mixes the holomorphic and antiholomorphic components of the connection. This formalism for obtaining the thermodynamics variables is referred to as the "canonical formalism" in the literature. The CFT calculations done in [104] seem to match with the "holomorphic formalism" given in [94], but the canonical approach seems to be much more physically plausible. In [105] a possible solution to this discrepancy was suggested,where they changed the bulk to boundary dictionary in a way suited to the addition of chemical potential which deforms the theory.

In [107], the process of adding chemical potential was unified for the full family of solutions obtained by modular transformation from the conical defect solution. The black holes that we talked about is only one member of the family. It was shown that the same boundary terms need to be added to the action to get a good variational principle for all members of the family. The definitions for all thermodynamic quantities for any arbitrary member of the family were obtained there.

4.1.2 Phase structure of higher spin black holes in AdS$_3$

The phase structure of spin 3 black holes in AdS_3 was studied in [109] using the holomorphic variables. In the principle embedding of $SL(2,R)$ in $SL(3,R)$ (with spectrum consisting of fields with spin 2 and 3) , they found 4 solutions to the equations corresponding to a trivial holonomy along the time circle. They relaxed the condition that the spin 3 charge has to approach 0 as the corresponding chemical potential is taken to 0. It was shown there that of the 4 branches one is unphysical as its entropy is negative. Of the remaining three branches one is the BTZ branch (here spin 3 charge goes to zero as chemical potential goes to 0), one is the extremal branch (having a non-zero charge configuration at zero temperature) and a third branch. The negative specific heat of the extremal branch makes it an unstable branch. A more analytical treatment of phase structure was done in [108] for spin 3 and $\tilde{4}$ black holes.

The phase diagram given there shows that the BTZ and extremal branch exist only in the low temperature regime, after which the thermodynamic quantities for this two branches do not remain real. The third branch that is present has real thermodynamic variables at all temperatures. It is shown there that in the low temperature regime only the BTZ branch has the expected scaling behavior while the third branch does not have the correct scaling behaviour at high temperature. By scaling behaviour they meant that charges and other thermodynamic quantities have the right power law behaviour with temperature, e.g. spin 3 charge $\propto T^3$ etc. From the free energy perspective the BTZ branch dominates over all other branches (i.e. the BTZ branch has the lowest free energy). In the region of existence of the BTZ and extremal branch, the extremal branch dominates over the third branch and at high enough temperature only the third branch survives. The free energy of the unphysical 4th branch is greatest among all the branches.

They then argued that the correct thermodynamics at high temperature is given by the diagonal embedding. The diagonal embedding is obtained as the end point of the RG flow initiated by the addition of chemical potential corresponding to the spin 3 charge in the principle embedding. The spectrum in the diagonal embedding has a pair of fields with spin $\frac{3}{2}$, a pair with spin 1 and a spin 2 field [62] . It also has a chemical potential corresponding to the spin $\frac{3}{2}$. The scaling behavior of the thermodynamic quantities was found to be correct at high temperature for this embedding. Near the zero chemical potential (for spin $\frac{3}{2}$) limit, i.e. near the end point of the RG flow, the holonomy equations have 2 real solutions. Of them the one with the lower free energy was conjectured to be the "third branch"(from the principal embedding) at the end of the RG flow. Since this branch has the correct scaling behaviour(w.r.t. the diagonal embedding) it was argued that beyond the point where BTZ and extremal branches of the principle embedding cease to exist the third branch takes over and it is actually the black hole solution in the diagonal embedding.

In summary they showed that the principle embedding is the correct IR picture valid at low temperature regime and diagonal embedding is the correct UV picture valid in the high temperature regime.

4.1.3 Our Work

In this chapter we study the phase structure of $SL(3, R) \times SL(3, R)$ higher spin system in the canonical formalism. We will first study the principle embedding. We will use the definition of thermodynamic quantities for conical surplus solution (which go to the thermal AdS branch when chemical potential and spin 3 charges are taken to zero) given in [107]. The conical surplus has a contractible spatial cycle. So, we demand that the holonomy of connection along this cycle be trivial. Using this condition we are able to get the undeformed spin 2 and 3 charges in terms of temperature and chemical potential for spin 3 charge. We will use this to study the phase structure of the conical surplus. From the phase diagram we will see that there are 2 branches of solutions with real values for undeformed spin 2 and spin 3 charges for a given temperature and chemical potential. One of the branch reduces to the thermal AdS branch (with zero spin 3 charge) when the chemical potential (μ) is taken to zero. The other one is a new branch which like the extremal black holes has a non-trivial charge configuration even when $\mu \to 0$. This we call the "extremal thermal AdS" branch. This extremal branch has a lower free energy for all values of μ and T. But this is unphysical as its energy is unbunded from below.

We then move to studying the phase structure of black hole in this embedding. We again solve for the chages in terms of chemical potential (μ) and temperature (T), but now with the time cycle contractible. Here we will get 4 branches of solutions. We will find that two of these branches have negative entropy and hence are unphysical. Among the other two branches, one is the BTZ branch (which reduces to BTZ black hole when chemical potential $\mu \to 0$) and the other is the extremal branch (having a non-trivial charge configuration at $T = 0$). The extremal branch has negative specific heat and hence is unstable. The BTZ branch is stable, has lower free energy and hence is the dominant of the two good solutions. Given a chemical potential both the black hole and the thermal AdS solutions exist till a certain temperature, which is different for the black hole and thermal AdS. Crossing the temperature leads to complex values of the thermodynamic quantities and hence the solutions are no longer trust-able.

We then undertake a study of the phase structure for the conical surplus and black hole together. Between the 3 solutions- the BTZ black hole, the extremal black hole and the thermal AdS branch we will try to see which branch has the minimum free energy for a given chemical potential and temperature. We will find that for a particular chemical potential, at very low temperature the thermal AdS has the lowest free energy and then as we gradually increase the temperature the BTZ branch starts dominating over the thermal AdS. This is the analogue of Hawking-page transition. After a particular temperature the extremal black hole also dominates over the thermal AdS though it is sub-dominant to the BTZ branch. Still increasing the temperature further the black holes cease to exist and the thermal AdS is the only solution which survives.

Next part of our study will involve studying black holes in the diagonal embedding of $SL(2,R)$ in $SL(3,R)$. There is a consistent truncation where the spin $\frac{3}{2}$ fields are put to zero [112]. But here we don't want to do this. The reason being that we want to use the fact that this diagonal embedding is actually the UV limit of the flow initiated in principal embedding by the spin 3 chemical potential. We want to study the full theory obtained from this procedure and there all the mentioned fields are present. First of all we will be able to give a map between the parameters of the theory at UV and IR fixed points. Also, here we obtain 4 solutions to the holonomy equations and by similar arguments as above two of them are unphysical. Of the other two branches the one with the lower free energy is throughout stable. We will also show that the good solution near IR fixed point actually maps to the bad solution near the UV fixed point and vice versa. We give a plausible reasoning for this mapping between the good and bad branches.

4.1.4 Organization of the rest of the chapter

In section 4.2 we give a brief review of the geometry of higher spin theories and their thermodynamics. In the section 4.3 we give the analysis for the thermodynamics of conical surplus, black hole and Hawking-Page transition for principle embedding. In section 4.4 we give a similar description for black hole in the diagonal embedding. Lastly we give a summary of our results in section 4.5.

4.2 Review of higher spin geometry in AdS$_3$ and thermodynamics

Let us briefly elaborate on the 'Canonical formalism' for BTZ Black Holes in higher spin scenarios. We will mostly follow the conventions given in [101, 107]. In $2+1$ dimensions higher spin theories coupled to gravity with negative cosmological constant can be written as a Chern-Simons theory with gauge group $G \simeq SL(N,R) \times SL(N,R)$ [35]. For $N = 2$ it reduces to ordinary gravity but for $N \geq 3$ depending on possible embeddings of the $SL(2,R)$ subgroup into $SL(N,R)$ it generates a spectrum of fields with different spins. We are mostly interested in an Euclidean Chern-Simons theory on a three-dimensional manifold M with the topology $S^1 \times D$ where the S^1 factor is associated with the compactifed time direction and $\partial D \simeq S^1$. It is customary to introduce coordinates $(\rho, z, \bar{z})$ on M, where ρ is the radial coordinate and $\rho \to \infty$ is the boundary with the topology of a torus where the $z,\ \bar{z}$ coordinates are identified as $z(\bar{z}) \simeq z(\bar{z}) + 2\pi \simeq z(\bar{z}) + 2\pi\tau(\bar{\tau})$. For Chern-Simons theory the field strength is zero, so the connection is pure gauge. We will be working in a gauge where the connections have a radial dependence given by

$$A = b^{-1}db + b^{-1}ab \qquad \bar{A} = bdb^{-1} + b\bar{a}b^{-1}$$

with $b = b(\rho) = e^{\rho L_0}$ and $a, \bar{a}$ being functions of boundary $z,\ \bar{z}$ coordinates only.
The holonomies associated with the identification along the temporal direction are

$$Hol_{\tau,\bar{\tau}}(A) = b^{-1}e^{h}b \qquad Hol_{\tau,\bar{\tau}}(\bar{A}) = be^{\bar{h}}b^{-1} \tag{4.2.1}$$

where the matrices h and $\bar{h}$ are

$$h = 2\pi(\tau a_z + \bar{\tau}a_{\bar{z}}) \qquad \bar{h} = 2\pi(\tau\bar{a}_z + \bar{\tau}\bar{a}_{\bar{z}}) \tag{4.2.2}$$

Triviality of the holonomy forces it to be an element of the center of the gauge group and a particularly interesting choice which corresponds to the choice for uncharged BTZ black hole gives

$$Tr[h \cdot h] = -8\pi^2 \qquad Tr[h \cdot h \cdot h] = 0 \tag{4.2.3}$$

A different choice of the center element is synonymous to a scaling of τ and hence is not very important for us as we focus on a particluar member of the centre of the group and are not interested in a comparative study between various members.

With this setup in mind the Euclidean action is

$$I^{(E)} = I^{(E)}_{CS} + I^{(E)}_{Bdy}$$

where

$$I_{CS}^{(E)} = CS[A] - CS[\bar{A}], \quad CS[A] = \frac{ik_{cs}}{4\pi} \int_M Tr[A \wedge dA + \frac{2}{3} A \wedge A \wedge A]$$

For a good variational principal on the manifold we need to add some boundary terms to the above action. To get a variation of action of the form $\delta I \sim Q_i \delta\mu_i$ (for grand canonical ensemble) we need to add a boundary term of the form

$$I_{Bdy}^{(E)} = -\frac{k_{cs}}{2\pi} \int_{\partial M} d^2z \; Tr\left[(a_z - 2L_1)a_{\bar{z}}\right] - \frac{k_{cs}}{2\pi} \int_{\partial M} d^2z \; Tr\left[(\bar{a}_{\bar{z}} - 2L_{-1})a_z\right]$$

We will be interested in an asymptotically AdS boundary which will give rise to the W_N algebra as the asymptotic symmetry algebra in the absence of any chemical potential. This is satisfied by the connections written in the Drinfeld-Sokolov form

$$a = (L_1 + Q)\, dz - (M + \ldots)\, d\bar{z} \quad (4.2.4)$$

$$\bar{a} = \left(L_{-1} - \bar{Q}\right) d\bar{z} + \left(\bar{M} + \ldots\right) dz \quad (4.2.5)$$

with $[L_{-1}, Q] = [L_1, M] = 0$ (and similarly for $\bar{Q},\ \bar{M}$). We adopt a convention that the highest (lowest) weights in a_z ($\bar{a}_{\bar{z}}$) are linear in the charges, and the highest (lowest) weights in $\bar{a}_z$ ($a_{\bar{z}}$) are linear in the chemical potentials corresponding to charges other than spin 2. The convention for definition of chemical potential that we use is given by

$$Tr\left[(a_z - L_1)(\bar{\tau} - \tau)a_{\bar{z}}\right] = \sum_{i=3}^{N} \mu_i Q_i \quad (4.2.6)$$

$$Tr\left[(-\bar{a}_{\bar{z}} + L_{-1})(\bar{\tau} - \tau)\bar{a}_z\right] = \sum_{i=3}^{N} \bar{\mu}_i \bar{Q}_i \quad (4.2.7)$$

Varying $I^{(E)}$ on-shell we arrive at

$$\begin{aligned}\delta I_{os}^{(E)} = -\ln Z = &-2\pi i k_{cs} \int_{\partial M} \frac{d^2 z}{4\pi^2 Im(\tau)}\, Tr\Big[(a_z - L_1)\delta\left((\bar{\tau}-\tau)a_{\bar{z}}\right) + \left(\frac{a_z^2}{2} + a_z a_{\bar{z}} - \frac{\bar{a}_z^2}{2}\right)\delta\tau \\ &- (-\bar{a}_{\bar{z}} + L_{-1})\delta\left((\bar{\tau}-\tau)\bar{a}_z\right) - \left(\frac{\bar{a}_{\bar{z}}^2}{2} + \bar{a}_{\bar{z}}\bar{a}_z - \frac{a_{\bar{z}}^2}{2}\right)\delta\bar{\tau}\Big] \\ = &-2\pi i k_{cs} \int_{\partial M} \frac{d^2 z}{4\pi^2 Im(\tau)} \left(T\delta\tau - \bar{T}\delta\bar{\tau} + \sum_{i=3}^{N}\left(Q_i\delta\mu_i - \bar{Q}_i\delta\bar{\mu}_i\right)\right)\end{aligned} \tag{4.2.8}$$

So, the added boundary terms are the correct one as we get the desired variation of the action on-shell.

The black hole geometry that we discussed can be obtained by a $SL(2,Z)$ modular transformation acting on a conical surplus geometry and vice versa. This property was used to show in [107] that the variational principle for either geometry (or for that matter any geometry obtained by a $SL(2,Z)$ transformation on the conical surplus geometry) goes through correctly if we use the boundary terms given above. This in principle means that we have the same definition of stress tensor for all the members of the '$SL(2,Z)$' family and is given by

$$\mathcal{T} = Tr[\frac{a_z^2}{2} + a_z a_{\bar{z}} - \frac{\bar{a}_z^2}{2}], \quad \bar{\mathcal{T}} = Tr[\frac{\bar{a}^2}{2} + \bar{a}_{\bar{z}}\bar{a}_z - \frac{a_{\bar{z}}^2}{2}] \tag{4.2.9}$$

The on-shell action evaluated for a member gives the free energy for that particular member. Please note that in arriving at equation (4.2.8) we have to go through an intermediate coordinate transformation pushing the τ dependence of the periodicity $z \simeq z + 2\pi\tau$ to the integrand. This is necessary to make sure that the variation does not affect the limits of integration. However, this procedure of making periodicities of the coordinates constant is dependent upon the member of the '$SL(2,Z)$' family of interest. The free energy for any arbitrary member was evaluated in [107] and is given by

$$-\beta F = -I_{on-shell} = \pi i k_{cs} Tr\Big[\big(h_A h_B - \bar{h}_A \bar{h}_B\big) - 2i(a_z - 2L_1)a_{\bar{z}} - 2i(\bar{a}_{\bar{z}} - 2L_{-1})\bar{a}_z\Big] \quad (4.2.10)$$

where h_A and h_B are respectively the holonomy along the contractible and non-contractible cycles.

Performing a Legendre transform of the free energy (i.e. from a function of chemical potentials/sources to function of charges) we arrive at an expression for the entropy. The expression for entropy of the black hole solution turns out to be

$$S = -2\pi i k_{cs} Tr\Big[(a_z + a_{\bar{z}})(\tau a_z + \bar{\tau} a_{\bar{z}}) - (\bar{a}_z + \bar{a}_{\bar{z}})(\tau \bar{a}_z + \bar{\tau}\bar{a}_{\bar{z}})\Big] \quad (4.2.11)$$

The entropy for a conical surplus turns out to be zero as expected of a geometry without any "horizons".

All of the above statements can very easily be generalized to non-principal embedding. The things that will be different are the value of the label 'k' associated with different $SL(2,R)$ embedding in $SL(3,R)$ and their definition of charges and chemical potentials. The value of k is related to k_{cs} by

$$k_{cs} = \frac{k}{2Tr\left[\Lambda^0\Lambda^0\right]}, \quad (4.2.12)$$

where k_{cs} is the label associated with the $SL(3,R)$ CS theory. The central charge of the theory for a particular embedding is given by $c = 6k$. $\Lambda_{-1}, \Lambda_0, \Lambda_1$ are the generators giving rise to the $sl(2,R)$ sub-algebra in the particular embedding.

4.3 The principal embedding for SL(3,R)

We give the conventions for connections and the thermodynamic quantities that we use in this chapter. The connection that we use here is based on [94, 107, 101]. We will confine our connections in the radial gauge and use the conventions for generators of $SL(3,R)$ given in[1] [99]

$$\begin{aligned} a &= (L_1 - 2\pi\mathcal{L}L_{-1} - \frac{\pi}{2}\mathcal{W}W_{-2})dz + \frac{mT}{2}(W_2 + 4\pi\mathcal{W}L_{-1} - 4\pi\mathcal{L}W_0 + 4\pi^2\mathcal{L}^2W_{-2})d\bar{z}, \\ \bar{a} &= (L_{-1} - 2\pi\bar{\mathcal{L}}L_1 - \frac{\pi}{2}\bar{\mathcal{W}}W_2)d\bar{z} + \frac{\bar{m}T}{2}(W_{-2} + 4\pi\bar{\mathcal{W}}L_{-1} - 4\pi\bar{\mathcal{L}}W_0 + 4\pi^2\bar{\mathcal{L}}^2W_2)dz. \end{aligned} \tag{4.3.13}$$

We are interested in studying only non-rotating solutions, hence we require $g_{zz} = g_{\bar{z}\bar{z}}$ for the metric which when converted to the language of connections in radial gauge becomes

$$Tr[a_{\bar{z}}a_{\bar{z}} - 2a_{\bar{z}}\bar{a}_{\bar{z}} + \bar{a}_{\bar{z}}\bar{a}_{\bar{z}}] = Tr[a_z a_z - 2a_z\bar{a}_z + \bar{a}_z\bar{a}_z]. \tag{4.3.14}$$

With our convention this is satisfied if $\bar{m} = -m$, $\bar{\mathcal{W}} = -\mathcal{W}$, $\bar{\mathcal{L}} = \mathcal{L}$

These connections automatically satisfy the equations of motion $[a_z, a_{\bar{z}}] = 0$. With our conventions the equation (4.2.6) becomes

$$Tr[(a_z - L_1)a_{\bar{z}}(\bar{\tau} - \tau)] = 4im\pi\mathcal{W}. \tag{4.3.15}$$

So, demanding that $\mathcal{W}$,which is the measure of spin 3 charge in our conventions be real, the chemical potential μ_3 is imaginary, whose measure is given by *'im'*.

4.3.1 The Conical Surplus Solution

Here as stated above the contractible cycle is spatial and hence we demand that the holonomy of connection defined as e^{ih} where $h = 2\pi(a_z + a_{\bar{z}})$ to be trivial along the

[1]Here we redefine our variables to absorb the k appearing in the connections given in [101]

contractible cycle. It follows the same holonomy equation as that given in (4.2.3). This choice of center is the same as that for thermal AdS.

The boundary terms that we use (given by the equation above the Drienfeld-Sokolov connection in (4.2.4)) is suited for a study in grand canonical ensemble where, the chemical potentials are the parameters of the theory.

The first among the two holonomy equations in (4.2.3) can be used to get $\mathcal{W}$ in terms of $\mathcal{L}$

$$\mathcal{W}_{CS} = \frac{1}{12m\pi T} + \frac{2\mathcal{L}}{3mT} + \frac{16}{9}m\pi\mathcal{L}^2T \tag{4.3.16}$$

and using the second equation we get an equation for $\mathcal{L}$ in terms of m and T given by

$$-\frac{1}{mT}-\frac{8\pi\mathcal{L}}{mT}+\frac{mT}{3}-\frac{8}{3}m\pi\mathcal{L}T+64m\pi^2\mathcal{L}^2T+\frac{128}{9}m^3\pi^2\mathcal{L}^2T^3-\frac{512}{3}m^3\pi^3\mathcal{L}^3T^3+\frac{4096}{27}m^5\pi^4\mathcal{L}^4T^5 = 0. \tag{4.3.17}$$

From equation (4.2.9) the stress energy tensor is given by

$$\mathcal{T}_{CS} = 8\pi\mathcal{L} - 12m\pi\mathcal{W}T - \frac{64}{3}m^2\pi^2\mathcal{L}^2T^2, \tag{4.3.18}$$

and the spin 3 charge which was $\mathcal{W}$ in absence of chemical potential is left unchanged in presence of chemical potential. The free energy given in equation (4.2.10) in this case becomes

$$F_{CS} = 16\pi\mathcal{L} - 8m\pi\mathcal{W}T - \frac{128}{3}m^2\pi^2\mathcal{L}^2T^2. \tag{4.3.19}$$

Now there are 4 solutions to (4.3.17) out of which only 2 turn out to be real solutions. Using this we can get the solutions for $\mathcal{T}_{CS}$ and F_{CS} in terms of m and T. From equation (4.3.17) we see that all relevant quantities are functions of $\mu_c = mT$ and hence we plot them in terms μ_c in figure (4.1).

From the figure (4.1) we see that the blue branch is the branch that goes to ther-

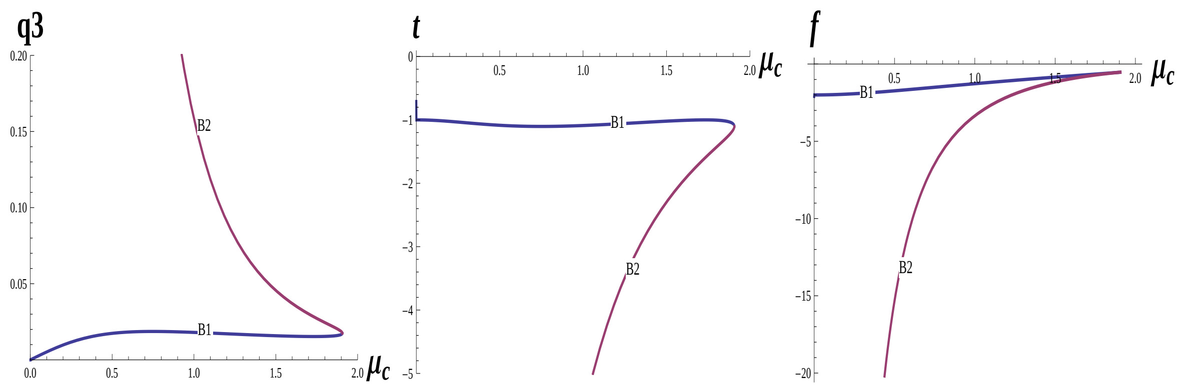

Figure 4.1: Here the phase diagram of conical surplus solution is given. The horizontal axes in all the figures is the parameter μ_c. The vertical axes are respectively the spin 3 charge $\mathcal{W}_{CS}$, stress tensor $\mathcal{T}_{CS}$ and free energy F_{CS}.

mal AdS (without charges) when $\mu_c \to 0$. The other branch in red is a special branch where as $\mu_c \to 0$ we have $\frac{\mathcal{W}^{\frac{2}{3}}}{\mathcal{L}} = -\frac{1}{6(2\pi)^{\frac{2}{3}}}$. This special branch starts from an "extremal point" analogous to black holes discussed in [94] and [109]. Let us call it, the "extremal branch". This is a bit of a misnomer as for thermal AdS in any gauge there is no concept of horizon. The two branches merge at the value of the parameter $\mu_c = \frac{3}{4}\sqrt{3+2\sqrt{3}}$, and after that the conical surplus solution ceases to exist. The "extremal branch" has the lowest value for free energy at all points in the parameter space where it exists among all solutions. But from the energy plots we see that its energy is unbounded from below for $mT \to 0$. So, this is not a physically acceptable solution. So, this branch will not destroy the overall phase structure as this is an unphysical branch.

4.3.2 The black hole solution

The black hole solution is obtained by demanding that the time circle is contractible and holonomy defined in equation (4.2.1) satisfy the equations in (4.2.3). The holonomy equations in this case are

$$\begin{aligned} 2-\frac{32m^2\mathcal{L}^2}{3}-\frac{4\mathcal{L}}{\pi T^2}-\frac{6m\mathcal{W}}{\pi T} &= 0, \\ -\frac{128}{9}m^3\mathcal{L}^3+\frac{6m^3\mathcal{W}^2}{\pi}+\frac{3\mathcal{W}}{2\pi^2T^3}+\frac{16m\mathcal{L}^2}{\pi T^2}+\frac{12m^2\mathcal{L}\mathcal{W}}{\pi T} &= 0. \end{aligned} \tag{4.3.20}$$

Using the same procedure as after (4.3.16) we get the final holonomy equation for the black hole as

$$\frac{4m\mathcal{L}}{3}-\frac{64}{3}m^3\mathcal{L}^3-\frac{\mathcal{L}}{m\pi^2T^4}+\frac{1}{2m\pi T^2}+\frac{8m\mathcal{L}^2}{\pi T^2}+\frac{2}{3}m\pi T^2-\frac{64}{9}m^3\pi\mathcal{L}^2T^2+\frac{512}{27}m^5\pi\mathcal{L}^4T^2=0. \tag{4.3.21}$$

The free energy of equation (4.2.10) in this case is given by

$$F_{BH}=-16\pi\mathcal{L}-8m\pi\mathcal{W}T+\frac{128}{3}m^2\pi^2\mathcal{L}^2T^2. \tag{4.3.22}$$

The entropy defined in equation (4.2.11) in our case becomes

$$S=\frac{32\pi\mathcal{L}}{T}-\frac{256}{3}m^2\pi^2\mathcal{L}^2T. \tag{4.3.23}$$

We see that equation (4.3.21) is an equation for $l=\frac{\mathcal{L}}{T^2}$ in terms of $\mu_b=mT^2$. So, μ_b is a good variable to study the phase structure for the black hole[2] The phase diagram for spin 3 black hole is given in figure (4.2). We denote the 4 branches of solutions with the following color code- branch-1-Blue,branch-2-Red,branch-3-Orange and branch-4-green.

[2]The variables which will be used to study the phase structure in terms of μ_b are

$$t=\frac{\mathcal{T}}{T^2},\quad w=\frac{\mathcal{W}}{T^3},\quad f=\frac{F}{T^2},\quad s=\frac{S}{T}. \tag{4.3.24}$$

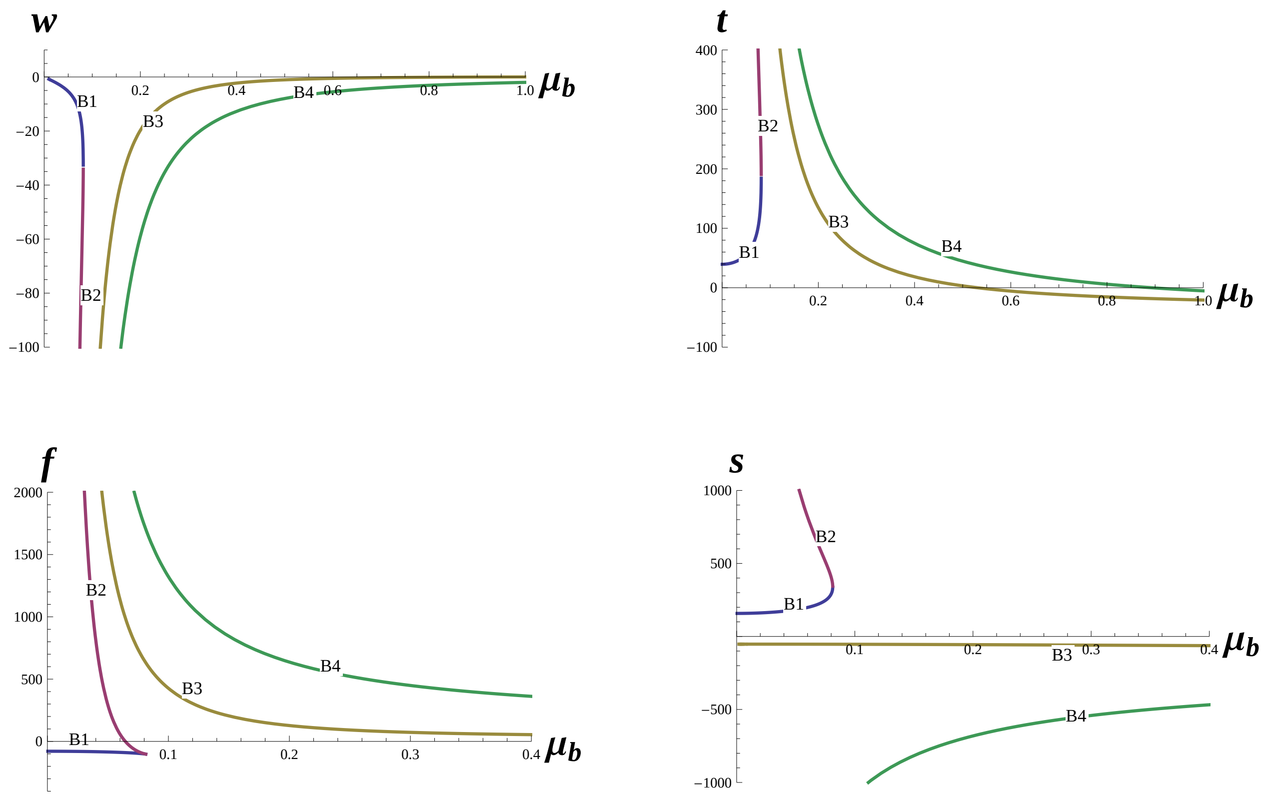

Figure 4.2: This figure gives the phase structure for spin 3 black hole. The horizontal axis is μ_b and the vertical axis on the upper panel are respectively the spin 3 charge $\mathcal{W}_{BH}$ and stress tensor $\mathcal{T}_{BH}$ and in the lower panel are free energy F_{BH} and entropy S respectively

From the plots we see that branches 3 and 4 are unphysical with negative entropy. Branches 1 and 2 merge at the point $\mu_b = \frac{3\sqrt{-3+2\sqrt{3}}}{8\pi}$. Beyond this point the black hole solutions cease to exist. For branch 2 the stress tensor decreases with $\mu_d = mT^2$, i.e. it decreases with T^2 if we keep chemical potential m fixed, so this branch has negative specific heat and hence is unstable. So, the branches 1 and 2 correspond respectively to the large (stable) and small (unstable) black hole solutions in AdS space [110, 109]. For branch 2, in the limit $\mu \to 0$ we get $\frac{w^{\frac{2}{3}}}{t} = \frac{1}{6}(\frac{-1}{2\pi})^{\frac{2}{3}}$, so the branch 2 evolves from the extremal point having a non trivial configuration at $T = 0$. The branches 3 and 4 also evolve from the extremal point, but they evolve to unphysical branches. From the free energy plot we see that the BTZ black hole branch is the dominant solution in the temperature regime where it exists.

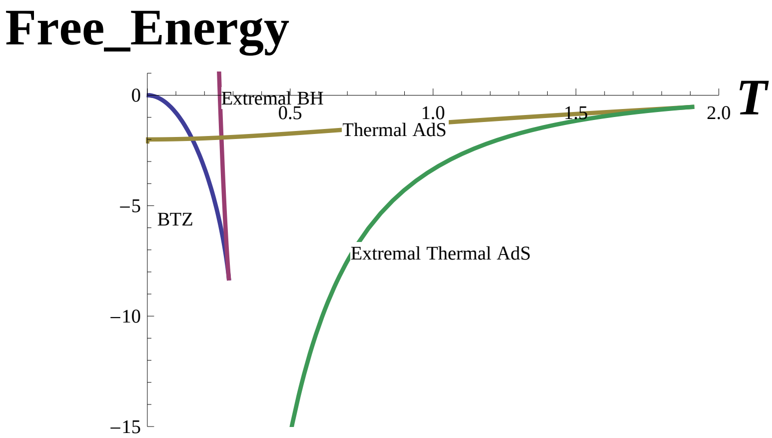

Figure 4.3: Comparison between free energy of black hole and conical surplus at m=1. The blue branch is the BTZ branch of black hole and red branch is extremal branch. The brown branch is the conical surplus branch which goes to pure AdS in absence of chemical potential and the green branch is the new "extremal branch" of conical surplus

4.3.3 The "Hawking-Page" Transition

We will now study a phase transition first studied for Einstein-Hilbert gravity with negative cosmological constant on AdS_4 in [110]. There it was shown that in asymptotically AdS space, out of the two phases $1)$ a gas of gravitons and $2)$ a black hole, the former dominates at low temperature and after a particular temperature the black hole solution becomes more dominant. The dominant phase was obtained by identifying which solution had the lowest free energy for a particular temperature. Both pure AdS (gas of gravitons) and black hole were put at the same temperature by keeping the identification of the time circle at the same value. The free energy was calculated by calculating the on shell action in Euclidean signature with proper boundary terms added. For the AdS_3 case, the thermal AdS and BTZ black hole configurations are related by a modular transformation $\tau_{BTZ} = -\frac{1}{\tau_{AdS}}$. At the point of Hawking-Page transition i.e. $\tau_{BTZ} = \tau_{AdS}$, $T = \frac{1}{2\pi}$ (putting the AdS radius to unity). We are studying the phase structure in a grand canonical ensemble and we will try to find out the regions in parameter space where this phase transition takes place.

At $m = 0$ the temperature at which transition takes place is $T = \frac{1}{2\pi}$. Let us introduce a chemical potential for spin 3 and see how the temperature deviates from this point. For the moment we will study the branches which go to BTZ black holes and thermal AdS in the limit $m \to 0$ i.e. the branch 1 in both cases. We will assume the following form for the transition temperature after the introduction of a non zero chemical potential m.

$$T = \frac{1}{2\pi} + \#_1 m + \#_2 m^2 + \#_3 m^3 + \#_4 m^4 + ... \quad . \tag{4.3.25}$$

We will find the difference between the free energy of black hole given in equation (4.3.22) and that of thermal AdS like solution given in equation (4.3.19), both at the same temperature and chemical potential. We then find the temperature where this difference is zero which will give the various coefficients in (4.3.25) order by order. Upon doing this we arrive at the following temperature where the transition takes place to $O(m^6)$

$$T_{HP} = \frac{1}{2\pi} - \frac{1}{12\pi^3}m^2 + \frac{7}{144\pi^5}m^4 - \frac{71}{1728\pi^7}m^6 + ... \quad . \tag{4.3.26}$$

For a chemical potential given by $m = 1$ [3] we plot the free energies of both the black hole and the conical surplus in figure (4.3). The color coding is explained in the caption there. We see that the "thermal AdS branch" dominates over the black hole for low temperature and the BTZ branch black hole solutions take over at higher temperatures. The unstable (extremal) black hole always is the sub-dominant contribution to the free energy compared to BTZ branch. The extremal black hole branch also starts dominating over the thermal AdS as we increase the temperature further. Beyond the

[3] This is for the purpose of illustration only as this helps in bringing out all the features nicely in a single diagram. Since introduction of chemical potential violates the boundary falloff conditions we want $m << 1$ if we want the theory to be studied for high enough temperature as the deformation is by a term of the form $mTW_2e^{2\rho}$

temperature of existence of the black hole the thermal AdS like solutions are the only solutions available. This "phase transition" can be explained by a physical argument based on the fact that all even spin fields are self-attractive and all odd spin fields are self-repulsive [4]. So, at very low temperature when there are very few excitations the thermal AdS is the dominating solution. As we increase the temperature the number of excitation of both the spin 2 and 3 fields increase but the attractive nature of spin 2 field dominates and formation of a black hole is more favourable. Further increasing the temperature causes the number of excitations to increase further and the repulsive nature of spin 3 dominating over the attractive nature of spin 2 and makes it unfavourable to form a black hole. The only issue of concern is that the "extremal" conical surplus solution seems to dominate over all solutions in the low temperature regime. We will try to shed some more light on this issue later.

We numerically give the region of dominance of the black hole and thermal AdS like solutions as well as the region of existence of the solutions in figure (4.4). We see that the temperature where the "Hawking-Page" transition takes place is lower for higher values of chemical potential.

From the figure (4.4) we see that at any temperature, for high enough chemical potential the black hole solution ceases to exist and only thermal AdS like solutions are present. The lower plot in figure (4.4) puts this in perspective where we plot the region of existence of the black hole and thermal AdS like solutions . The region of existence of the thermal AdS like solutions is much larger (the full coloured region)than the black hole (region bounded by the axes and the blue line boundary).

In all this we have to be careful of the fact that introducing a chemical potential corresponds to breaking the asymptotic AdS boundary conditions. The asymptotic AdS falloff conditions which gives rise to the Virasoro symmetry algebra is $A - A_{AdS} = O(1)$, but by introducing a chemical potential this gets broken down to $A - A_{AdS} = mTe^{2\rho}$ [5].

[4] This was brought to our notice by Arnab Rudra and the physical argument rose from a discussion with him.

[5] We have reintroduced the radial dependence by $A = b^{-1}db + b^{-1}ab$

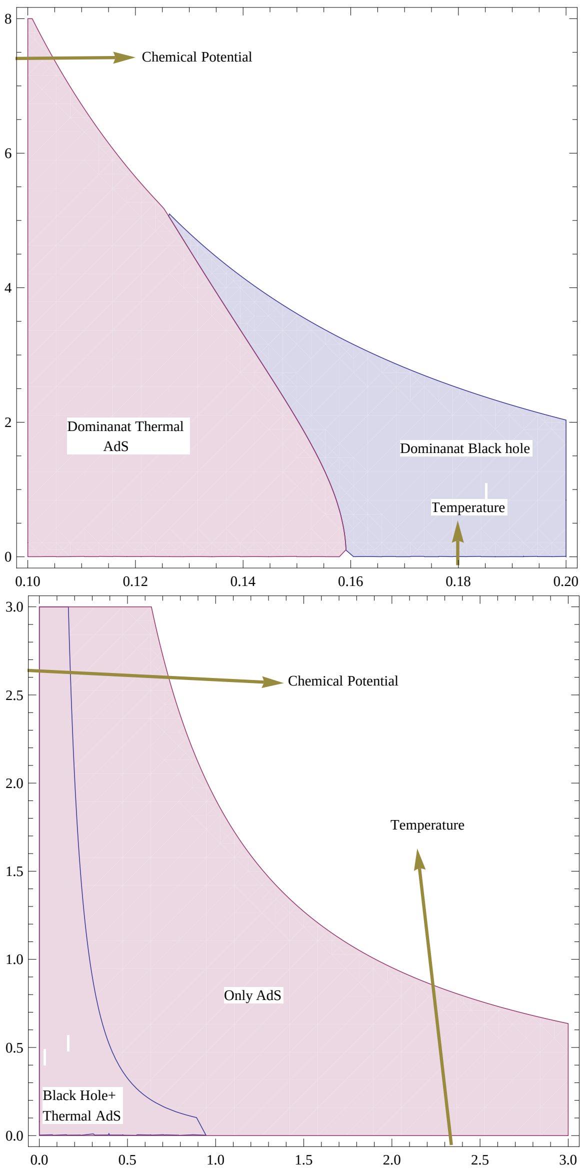

Figure 4.4: In these figures the x axis represents temperature(T) and y axis the chemical potential(m). For the upper figure pink region is where the conical surplus dominates and the blue region indicates where black hole dominates. The boundary between this two regions represents the temperature where the "Hawking-Page" transition takes place for a particular chemical potential. The lower figure represents the region of existence of conical surplus and black holes solutions. The black hole solutions exist in the region bound by the axes and the blue line boundary and the conical surplus solution exists in the full coloured region

So,the definition of charges that we are using are not valid if we move too far away from the fixed point. Since, we want to study the property of the system for high enough temperatures we have to confine ourselves to very small values of chemical potential. Also introduction of this deformation induces a RG flow which takes us to another non trivial fixed point in the UV with a completely different spectrum, to be studied next. Hence for large values of m the the parameters of the UV fixed point may be the correct parameters to use.

Across the point of transition we see that not only does the stress energy tensor changes sign which was expected, but the spin 3 charge also changes sign. This can be inferred from the fact that in the allowed regime for conical surplus the spin 3 charge is always positive which can be seen from figure (4.1), and that for black hole it is always negative as can be seen in figure (4.2).

4.4 The diagonal embedding for SL(3)

The definition of $SL(2,R)$ sub-algebra generators in diagonal embedding in terms of generators of principle embedding is given in [109] and [95] by $\frac{1}{2}L_0$, $\pm\frac{1}{4}W_{\pm 2}$. The spectrum here consists of fields of spin 2, spin $\frac{3}{2}$ and spin 1. The generators for spin $\frac{3}{2}$ multiplet in the bulk are given by (W_1, L_{-1}) and (W_{-1}, L_1) and that for spin 1 is W_0. The on-shell connection for this theory is given by

$$\begin{aligned} a &= \left(\frac{1}{4}W_2 + \mathcal{G}L_{-1} + \mathcal{J}W_0 + \mathcal{J}^2 W_{-2}\right)dz + \frac{\lambda T_d}{2}\left(L_1 + 2\mathcal{J}L_{-1} - \frac{\mathcal{G}}{2}W_{-2}\right)d\bar{z} \\ \bar{a} &= -\frac{\bar{\lambda}T_d}{2}\left(L_{-1} + 2\bar{\mathcal{J}}L_1 - \frac{1}{2}\bar{\mathcal{G}}W_2\right)dz - \left(\frac{1}{4}W_{-2} + \bar{\mathcal{G}}L_1 + \bar{\mathcal{J}}W_0 + \bar{\mathcal{J}}^2 W_2\right)d\bar{z} \end{aligned} \quad (4.4.27)$$

The non rotating condition (4.3.14) applied here gives

$$\bar{\mathcal{G}} = -\mathcal{G}, \quad \bar{\mathcal{J}} = \mathcal{J}, \quad \bar{\lambda} = -\lambda \quad (4.4.28)$$

Though this embedding looks like an independent theory by itself. But in [95] it was shown that after adding a deformation with chemical potential corresponding to spin $\frac{3}{2}$ (λ above), this theory becomes the correct UV behaviour of a theory whose behaviour near IR fixed point is given by the principle embedding studied earlier. If we reintroduce the radial dependence in (4.4.27) the leading term comes from $\frac{1}{4}W_2$. So, the way to go to the UV theory from the IR side is to change the coefficient of W_2 in $\bar{z}$ component of connection in equation (4.3.13) from $\frac{mT}{2}$ to $\frac{1}{4}$ by a similarity transformation (also found out in [109])

$$\begin{aligned} a_z^{UV} &= e^{xL_0} a_{\bar{z}}^{IR} e^{-xL_0} \quad , \quad a_{\bar{z}}^{UV} = e^{xL_0} a_z^{IR} e^{-xL_0}, \\ \bar{a}_z^{UV} &= e^{-xL_0} \bar{a}_{\bar{z}}^{IR} e^{xL_0} \quad , \quad \bar{a}_{\bar{z}}^{UV} = e^{-xL_0} \bar{a}_z^{IR} e^{xL_0} \quad \text{where} \quad x = \ln(\sqrt{2mT}), \end{aligned} \tag{4.4.29}$$

where a^{UV} is the connection given in equation (4.4.27) and a^{IR} is the one given in equation (4.3.13). We see from the map given in (4.4.29) that the holomorphic and anti holomorphic components change into each other in going from the IR to UV picture. Demanding that equation (4.4.29) holds we get a relation between parameters of the theories near the UV and IR fixed points like in [109] given by

$$\mathcal{G} = 2\sqrt{2}\pi\mathcal{W}(mT)^{\frac{3}{2}}, \quad \mathcal{J} = -2\pi\mathcal{L}mT, \quad \lambda T_d = \frac{\sqrt{2}}{\sqrt{mT}} \tag{4.4.30}$$

The holonomy equation calculated here as in the case of principle embedding is given by

$$-\frac{8\mathcal{J}^3}{3\pi^3 T_d^3} + \frac{2\mathcal{J}}{3\pi T_d} - \frac{64\mathcal{J}^4}{27\pi^3 T_d^{52}} + \frac{32\mathcal{J}^2}{9\pi T_d^{32}} - \frac{4\pi}{3{T_d}^2} - \frac{\mathcal{J}^{22}}{\pi^3 T_d} - \frac{{T_d}^2}{4\pi} - \frac{\mathcal{J}{T_d}^4}{8\pi^3} = 0 \tag{4.4.31}$$

The value of spin $\frac{3}{2}$ charge is obtained in terms of the spin 1 field using the holonomy condition and is given by

$$\mathcal{G}_{diag} = -\frac{16\mathcal{J}^2}{9T_d} + \frac{4\pi^2 T_d}{3} + \frac{2\mathcal{J}T_d}{3}. \tag{4.4.32}$$

The holonomy equation should also evolve along the RG flow from IR to UV, i.e. the holonomy equation (4.4.31) should reduce to (4.3.21), under the transformation of variables given in (4.4.30) . This happens if over and above the above transformation we assume that the definition of temperature on both limits is the same i.e, $T_d = T$ and the chemical potentials are related by $= \frac{\sqrt{2}}{T\sqrt{mT}}$.

The definition of the thermodynamic quantities in terms of connection are the same as they were for principal embedding given in [101, 107]. Here their definition in terms of the parameters of diagonal embedding are given by

$$\begin{aligned} \mathcal{T}_{diag} &= \frac{16\mathcal{J}^2}{3} - 3\mathcal{G}T_d + 2\mathcal{J}T_d^{22} \\ F_{diag} &= -\frac{32\mathcal{J}^2}{3} + 4\mathcal{G}T_d - 4\mathcal{J}T_d^{22} \\ S_{diag} &= \frac{64\mathcal{J}^2}{3T_d} + 8\mathcal{J}T_d{}^2 \end{aligned} \tag{4.4.33}$$

The equation (4.4.31) is an equation for $\frac{\mathcal{J}}{T_d}$ in $l = \sqrt{T_d}$. So, The correct parameter for drawing phase diagram is l and the quantities which are a function of l only, are

$$g = \frac{\mathcal{G}}{T_d^{\frac{3}{2}}}, \quad j = \frac{\mathcal{J}}{T_d}, \quad t = \frac{\mathcal{T}}{T_d^2}, \quad s = \frac{S_{CS}}{T_d}, \quad f = \frac{F_{CS}}{T_d^2} \tag{4.4.34}$$

In the phase diagram for black holes given in figure (4.5) the 4 branches of solution are color coded as branch 1-Blue, branch 2-Red, branch 3-Orange and branch 4-Green.

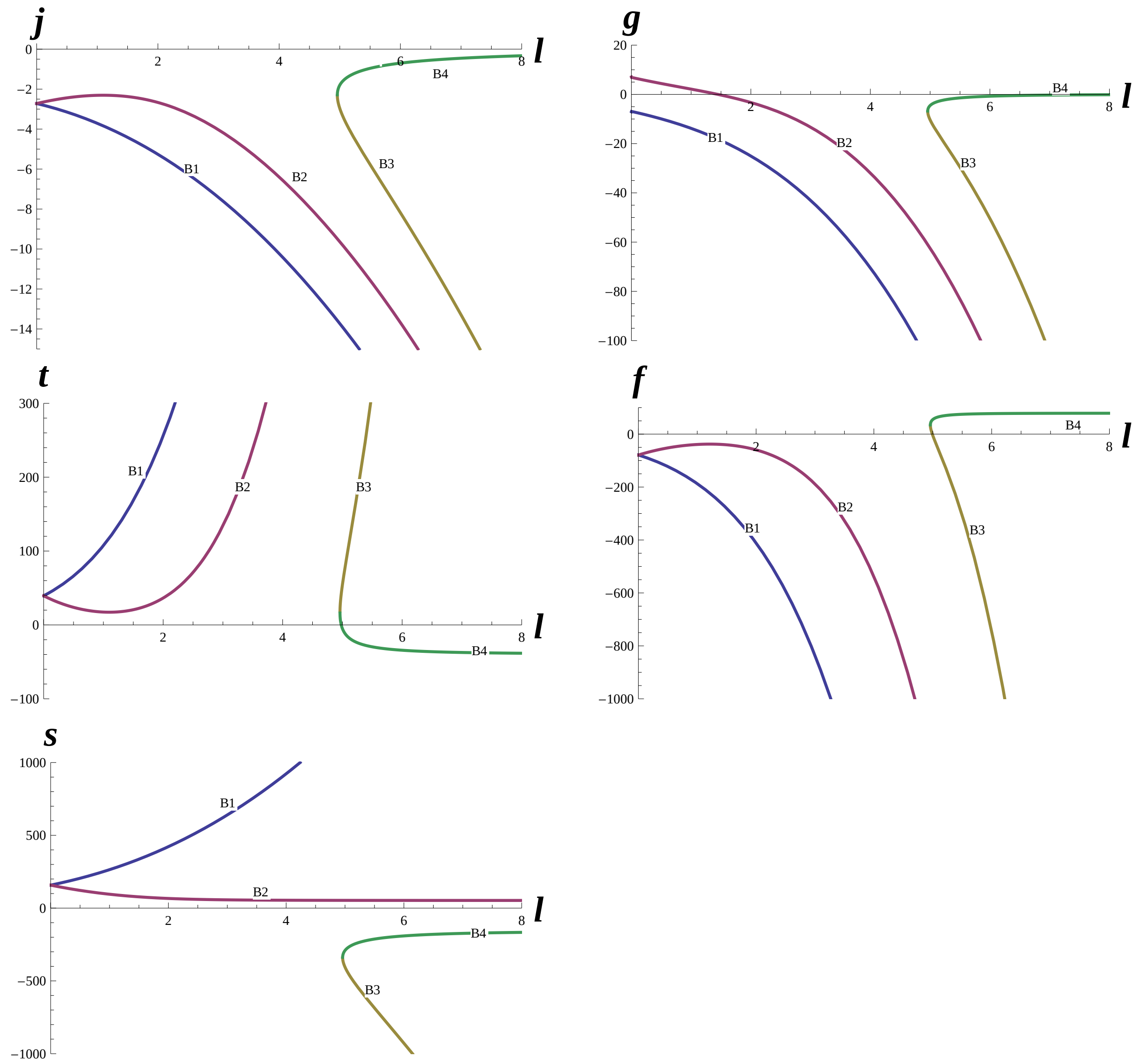

Figure 4.5: Phase structure for spin 3 black hole in diagonal embedding. The horizontal axis is l parameter that we used.

From the phase diagram we see that branches 3 and 4 are unphysical because they have negative entropy. If we assume that the chemical potential is fixed at some value then these are plots with respect to square root of temperature. So, if somewhere the gradient of stress tensor is negative then in those region it decreases with temperature and hence the system has negative specific heat. The 2nd branch has a region of negative specific heat for lower temperature but at higher temperatures it is stable for a given chemical potential. The branch 1 is the dominant solution when we look at the free energy plot. Another interesting thing that we notice is that for branch 1 and 2 as $\lambda \to 0$ we have

$$\mathcal{G} \to -\frac{2\sqrt{2}\pi^{\frac{3}{2}}T_d^{\frac{3}{2}}}{3^{\frac{3}{4}}}, \quad \mathcal{J} \to -\frac{1}{2}\sqrt{3}\pi T_d. \tag{4.4.35}$$

So, both the spin $\frac{3}{2}$ and spin 1 charges are non-zero even when the chemical potential corresponding to that charge is zero, i.e. even when the theory is undeformed. This was also obtained in [109]. This is different than the principle embedding case where spin 3 charge goes to zero when the chemical potential goes to zero for the dominant branch. This stems from the fact that $\lambda \to 0$ limit corresponds to $m \to \infty$ limit and hence it is not exactly an undeformed theory that we are studying but a theory which has been deformed in IR.

We see that this embedding has a valid high temperature behaviour i.e. as $T_d \to \infty$ we have

$$\mathcal{G} \to -\frac{-\lambda^3 T_d^3}{2} \to \frac{l^3 T_d^{\frac{3}{2}}}{2}, \quad \mathcal{J} \to -\frac{2\lambda^2 T_d^2}{8} \to -\frac{2l^2 T_d}{8}. \tag{4.4.36}$$

We see that in the high temperature limit the charges have the correct scaling behaviour in terms of the only dimensionful parameter (T) [6] and they are real. So, to study the high temperature behaviour the diagonal embedding is the correct theory to use.

In equation (4.4.30) we have the mapping between the parameters of the UV and IR theory. Now if we substitute solutions of branch $1, 2, 3, 4$ of the IR theory in this

[6] l is dimensionless as in terms of l all thermodynamic parameters have correct scaling behaviour with temperature as seen from equation (4.4.34)

map, it respectively matches with branches $4, 3, 2, 1$ of the UV theory. So, this suggests that along the flow the good solutions in one end go to the bad solutions in the other and vice versa. This is easy to see if we plot branches of $-2\pi\mathcal{L}mT$ and the corresponding branches of $\mathcal{J}$ with its parameters and T_d replaced by m and T using equation (4.4.30), the two plots merge with the mentioned identifications of the branches at the two ends. This is expected since from the expression of entropy given in (4.2.11) we see that the sign of the expressions changes if we replace the z-component of connection by $\bar{z}$ component. Also, since we expect the RG flow from IR to UV to happen when m goes from 0 to ∞, we see that initially the $\bar{z}$ component of the connection acts like a perturbation near IR fixed point as can be seen from equation (4.3.13), but near UV fixed point due to $m \to \infty$ $\bar{z}$ component is dominant part. So, the sign of entropy of the branches changes between the UV and IR fixed points and hence the good and the bad solutions get swapped. This is the reason for the bad branches in IR being able to explain the high temperature behaviour of the theory in the UV.

4.5 Concluding Remarks of this chapter

At first glance our analysis may look very similar to [109]. But we differ on many aspects of formulation, convention and results. We differ from [109] in the following respect

- We added a particular set of boundary terms in our bulk action so that the variation of the full action is like $\delta I \sim \mathcal{T}\delta\tau + Q_i \delta m^i$ which ensures that our on shell partition function is of the form $Z = e^{\tau\mathcal{T} + m^i Q_i}$. But the in [109] they added a boundary term which made sure that the variation of the on shell action is of the form $\delta I \sim \mathcal{L}\delta\tau + Q_i\delta(Tm^i)$, so that in that case the on shell partition function is like $Z = e^{\tau\mathcal{L} + \tau m^i Q_i}$, with the convention for connection being that of (4.3.13). Due to this difference in convention our physical quantities are not finite in the $m \to 0$ or $T \to 0$, whereas they are finite for the conventions used in [109]. For a batter comparison of our results with that of [109] one needs to add a boundary term to our action so that its variation becomes $\delta I \sim \mathcal{T}\delta\tau + Q_i\delta(m^i T)$.

- In the study of black hole phase structure in principle embedding in [109] they get one unphysical branch and 3 physical branch. Of the remaining 3 branches they have one unstable branch with negative specific heat. Of the remaining 2 stable branches they have one branch with the expected scaling behavior and one with wrong scaling behaviour with temperature for quantities having a CFT description. In our case we get 2 unphysical and 2 physical branch. Our BTZ branch has the correct scaling behavior $(e.g. Stress\ Tensor \propto T^2)$ whereas our extremal branch apparently does not have one $(\propto \frac{1}{m^2T^2})$. But using the correct dimensionless thermodynamical variable that we use to plot our phase plots, the scaling of the extremal branch goes like $\propto \frac{T^2}{\mu^2}$ and hence, it too has the correct scaling behaviour. So, both our branches can have a consistent CFT description.

- In [109] they argued that the third real and physical branch which had the wrong scaling behaviour with temperature at high temperatures infact had the correct scaling behavior when looked at from the diagonal embedding. They argued this by stating that in the diagonal embedding, of the two real branches near the fixed point, the one with the lower free energy must map to the lower free energy branch among the two surviving branches from the principal embedding [7]. Whereas in our case we explicitly show that the good branches in the principal embedding map to the bad branches in diagonal embedding and vice-versa by giving an explicit one-to-one mapping between the branches at the two ends. We also show that in terms of the temperature and the dimensionless parameter used the thermodynamic quantities have the correct scaling behaviour at high temperature in the diagonal embedding.

- In addition to this we studied the thermodynamics of the thermal AdS like so-

[7] There this was the branch with positive entropy for principal embedding

lutions also in the principal embedding and in the process we were able to show that a " Hawking-Page" like transition takes place in the low temperature regime. Also, after a certain temperature when the black hole solutions in the principal embedding cease to exist the thermal AdS like solution again takes over as for a particular chemical potential its regime of existence extends to a higher temperature than that of the black hole. As we have stated earlier this was due to the self repulsive nature of spin 3 fields due to which at high enough temperatures black hole formation is prevented.

Another point to notice is that we did our calculations in the canonical formalism which is different from the holomorphic formalism used in [109]. But the canonical formalism is in apparent disagreement with the CFT calculations of [104]. But in the canonical formalism the thermodynamic quantities are obtained much more naturally so we think that there has to be a way to see if it matches with the CFT calculation in the highest spin going to ∞ limit, where the CFT calculations have been done. A possible solution for this was suggested in [105].

In the the recent work [106] where it has been proposed that for higher spin theories the correct way to add chemical potential preserving the Brown-Henneaux fall off conditions necessary for definition of charges that we are using is to add them along the time component of the connection rather than the antiholomorphic component. In the light of this our analysis should be redone to see if some extra features emerge other than that we already have here. The ideal situation would be to derive the the asymptotic charges in the presence of a chemical potential exactly.
Another recent development in the study of higher spin black holes is the concept of generalised black holes in 3 dimensions [117]. There it claimed that the HS black holes of [94] are not the black holes of principal embedding spectrum but rather that of a special case of diagonal embedding spectrum and they point this out as the reason for the discrepancy between holomorphic and canonical formulation of HS black holes. They constructed the most general black holes with principal as well as diagonal embedding

black holes and they claim that the entropy computation matches for the holomorphic and canonical formulation for these black holes. It will be interesting to pursue this line of thought further and come up with possible CFT duals to these systems which can reproduce the entropy computed from the bulk. Also, a phase structure analysis of these generalised black holes can be performed following the procedure of this chapter.

Chapter 5

Concluding Remarks and Future Directions

In this thesis I have presented the work done for my PhD on some aspects of massless higher spin theories in AdS_3. These general motivation for this works stems from need to examine the massless higher spin theories as a consistent sub-sector to study the AdS/CFT conjecture.

In the first work I have described how we matched the one-loop partition function of the massless higher spin theories in an AdS_3 background with the vacuum character of the W-symmetry algebra. This acts as the first "quantum-mechanical" check of the duality between the higher spin theories and the CFT with W-symmetry.

The second work involved the formulation of Topologically Massive Higher Spin Gravity a.k.a. TMHSG. Here we constructed the action (quadratic in fields) and gave basic evidence towards the statement that the CFT dual to this theory is a logarithmic CFT. Also, we showed that the theory suffers from having a genuine instability at the linear equation of motion level at all points in the parameter space.

In the third and final work we studied the phase structure of the higher spin black holes in canonical formulation i.e. a formulation based on finding the thermodynamic quantities based on the variation of the action. We found out how the "Hawking-Page" transition changes with the introduction of the spin 3 chemical potential. We also, gave connection between the variables that explain the low and high temperature regimes of the phase structure correctly.

There are many further aspects of work done in this thesis that can be pursued further. Some of them being

- Being able to compute the one loop amplitude of the higher spin theories in the Chern-Simons formulation. This will give help us in finding the one-loop partition function with background values for higher spin fields. A case in point is the conical surplus background which has non-trivial background values for higher spin fields.

- Formulation of TMHSG in the Chern-Simons like formulation is another interesting aspect that needs to be pursued further. Initial work in this direction has been done in [92] where it was proposed that Chern Simons action with different levels for the barred and unbarred connection gives rise to TMHSG provided we impose the additional constraint that the field strength for both types of connections are same (off-shell). It has been shown that the linearised equations of motion obtained in terms of the Fronsdal like fields matches with what we have found out. This needs to be checked further that this indeed gives the TMHSG as this will help in identifying the CFT dual.

- Classical solutions of TMHSG were found out in [118]. But these solutions were generalisations of solutions of parity conserving case when the levels are assumed to be different. Finding solutions which are present only for the parity violating case will be really interesting. For this also a robust Chern-Simons like formulation will be useful as in that case we will be able to use holonomy to differentiate various classical solutions.

- Studying how the Hawking-Page transition temperature deviates from the pure gravity case by introduction of chemical potential corresponding to various charges will be an interesting task to accomplish. Going further being able to do our analysis will help us to understand the possible conflict between our result and the result obtained in [119, 120] where it was shown that due to the growth of number of light states the Hawking-Page transition is smoothened out for hs[λ] case, though there no chemical potential was introduced.

These and several other issues need to studied better in future works. With this I conclude this thesis.

Chapter 6

Appendix

6.1 Conventions

In our conventions the commutator of two covariant derivatives, evaluated on a totally symmetric rank s contravariant tensor, is equal to

$$[\nabla_\mu, \nabla_\nu]\,\xi^{\rho_1\ldots\rho_s} = \sum_{j=1}^{s} R^{\rho_j}{}_{\delta\mu\nu}\,\xi^{\rho_1\ldots\widehat{\rho}_j\ldots\rho_s\delta}\ , \tag{6.1.1}$$

where the notation $\widehat{\rho}_j$ means that ρ_j is excluded. The Riemann curvature tensor for AdS_3 is of the form

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\ . \tag{6.1.2}$$

The Ricci tensor is then

$$R^{\mu}{}_{\nu\mu\sigma} = -\frac{2}{\ell^2}\,g_{\nu\sigma}\ . \tag{6.1.3}$$

We shall also use the conventions of [12] that by an index $(\mu_1 \ldots \mu_s)$ we mean the symmetrised expression without any combinatorial factor, but with the understanding that terms that are obviously symmetric will not be repeated. So for example, the tensor $\nabla_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}$ equals

$$\nabla_{(\mu_1}\xi_{\mu_2\ldots\mu_s)} = \sum_{j=1}^{s} \nabla_{\mu_j}\xi_{\mu_1\ldots\widehat{\mu}_j\ldots\mu_s} \,, \tag{6.1.4}$$

if $\xi_{(s-1)}$ is a symmetric tensor, *etc*. By Δ we always mean the Laplace operator

$$\Delta = \nabla^{\lambda}\nabla_{\lambda} \,. \tag{6.1.5}$$

Because of (6.1.1), the explicit action depends on the spin s of the field on which Δ acts.

6.2 The calculation for $s = 3$

In this appendix we give some of the details of the calculation of section **??**. First we explain how to obtain our explicit formula (2.3.31) for the ξ-dependent exponent (2.3.28) of (2.2.27). To start with we plug (2.3.29) into (2.3.28) to obtain

$$\begin{aligned} S_\xi \;\; = \;\; 3\int d^3x\sqrt{g}\,\Bigg[& \xi^{\mathrm{TT}}_{\nu\rho}\left(-\Delta+\frac{6}{\ell^2}\right)\xi^{\mathrm{TT}\,\nu\rho} \\ & +\nabla_{(\nu}\sigma^{\mathrm{T}}_{\rho)}\left(-\Delta+\frac{6}{\ell^2}\right)\nabla^{(\nu}\sigma^{\mathrm{T}\rho)} - \frac{6}{5}\nabla_{(\nu}\sigma^{\mathrm{T}}_{\rho)}\nabla^{\rho}\nabla_{\mu}\nabla^{(\mu}\sigma^{\mathrm{T}\nu)} \\ & +\psi_{\nu\rho}\left(-\Delta+\frac{6}{\ell^2}\right)\psi^{\nu\rho} - \frac{6}{5}\psi_{\nu\rho}\nabla^{\rho}\nabla_{\mu}\psi^{\mu\nu}\Bigg] \,. \end{aligned} \tag{6.2.6}$$

Note that there are no cross-terms between ξ^{TT}, σ^{T} and $\psi^{\mu\nu}$, simply because any potential index contractions lead to vanishing results on account of the tracelessness and transversality of ξ^{TT} and $\sigma^{\mathrm{T}\nu}$. We want to simplify the expressions in the second and third line.

First we consider the $\sigma^{\mathrm{T}\nu}$ terms. To this end we observe that

$$\left(-\Delta+\frac{6}{\ell^2}\right)\nabla_{(\mu}\sigma^{\mathrm{T}}_{\nu)} = \nabla_{(\mu}\left(-\Delta+\frac{10}{\ell^2}\right)\sigma^{\mathrm{T}}_{\nu)} \,, \tag{6.2.7}$$

as one checks explicitly. It then follows that the first term of the second line of (6.2.6) leads to

$$\begin{aligned} S_\xi^{[\sigma,1]} &= 3\int d^3x\sqrt{g}\left[\nabla_{(\nu}\sigma^{\rm T}_{\rho)}\right]\left[\left(-\Delta+\frac{6}{\ell^2}\right)\nabla^{(\nu}\sigma^{{\rm T}\rho)}\right] \\ &= 6\int d^3x\sqrt{g}\,\sigma^{\rm T}_\nu\left(-\Delta+\frac{2}{\ell^2}\right)\left(-\Delta+\frac{10}{\ell^2}\right)\sigma^{{\rm T}\nu}\,. \end{aligned} \tag{6.2.8}$$

In order to evaluate the second term of the second line we now calculate

$$\nabla^\mu\,\nabla_{(\mu}\sigma^{\rm T}_{\nu)} = \Delta\sigma^{\rm T}_\nu - \frac{2}{\ell^2}\sigma^{\rm T}_\nu\,, \tag{6.2.9}$$

where we have used the transversality of $\sigma^{{\rm T}\mu}$, (2.3.30). Using integration by parts we therefore get

$$S_\xi^{[\sigma,2]} = \frac{18}{5}\int d^3x\sqrt{g}\;\sigma^{\rm T}_\nu\left(-\Delta+\frac{2}{\ell^2}\right)\left(-\Delta+\frac{2}{\ell^2}\right)\sigma^{{\rm T}\nu}\,. \tag{6.2.10}$$

Putting the two calculations together we thus arrive at the result

$$S_\xi^{[\sigma,1]} + S_\xi^{[\sigma,2]} = \frac{48}{5}\int d^3x\sqrt{g}\;\sigma^{\rm T}_\nu\left(-\Delta+\frac{2}{\ell^2}\right)\left(-\Delta+\frac{7}{\ell^2}\right)\sigma^{{\rm T}\nu}\,, \tag{6.2.11}$$

which is the second line of (2.3.31).

Next we deal with the $\psi^{\mu\nu}$ terms. The analogue of (6.2.7) is now

$$\left(-\Delta+\frac{6}{\ell^2}\right)\psi_{\mu\nu} = \left(\nabla_\mu\nabla_\nu - \frac{1}{3}g_{\mu\nu}\Delta\right)\left(-\Delta+\frac{12}{\ell^2}\right)\psi\,. \tag{6.2.12}$$

The first term of the third line then leads to

$$\begin{aligned} S_\xi^{[\psi,1]} &= 3\int d^3x\sqrt{g}\left[\left(\nabla_\nu\nabla_\rho-\frac{1}{3}g_{\mu\rho}\Delta\right)\psi\right]\cdot\left[\left(-\Delta+\frac{6}{\ell^2}\right)\left(\nabla^\nu\nabla^\rho-\frac{1}{3}g^{\mu\rho}\Delta\right)\psi\right] \\ &= 2\int d^3x\sqrt{g}\;\psi\,(-\Delta)\left(-\Delta+\frac{3}{\ell^2}\right)\left(-\Delta+\frac{12}{\ell^2}\right)\psi\,. \end{aligned} \tag{6.2.13}$$

For the second term we calculate

$$\nabla_\rho \nabla^\mu \psi_{\mu\nu} = \frac{2}{3} \nabla_\rho \nabla_\nu \left(\Delta - \frac{3}{\ell^2}\right) \psi \,, \tag{6.2.14}$$

thus leading to

$$\begin{aligned} S_\xi^{[\chi,2]} &= -\frac{18}{5} \int d^3x \sqrt{g} \left[\left(\nabla^\nu \nabla^\rho - \frac{1}{3} g^{\nu\rho} \Delta \right) \psi \right] \cdot \left[(\nabla_\rho \nabla^\mu) \left(\nabla_\mu \nabla_\nu - \frac{1}{3} g_{\mu\nu} \Delta \right) \psi \right] \\ &= \frac{8}{5} \int d^3x \sqrt{g}\; \psi\, (-\Delta) \left(-\Delta + \frac{3}{\ell^2}\right) \left(-\Delta + \frac{3}{\ell^2}\right) \psi \,. \end{aligned} \tag{6.2.15}$$

Putting the two calculations together we therefore get

$$S_\xi^{[\chi,1]} + S_\xi^{[\chi,2]} = \frac{18}{5} \int d^3x \sqrt{g}\; \psi\, (-\Delta) \left(-\Delta + \frac{3}{\ell^2}\right) \left(-\Delta + \frac{8}{\ell^2}\right) \psi \,, \tag{6.2.16}$$

which is the third line of (2.3.31).

6.2.1 The full action of $\mathcal{L}^{(3)}$

In this appendix we work out the full action of $\mathcal{L}^{(3)}$, including the trace piece. Actually, in order to do this calculation efficiently, it is convenient to modify $\mathcal{L}^{(3)}$ by a trace piece so that it maps traceless tensors to traceless tensors. The resulting operator is

$$\left(\hat{\mathcal{L}}^{(3)}\xi\right)^{\nu\rho} = \left(-\Delta + \frac{6}{\ell^2}\right) \xi^{\nu\rho} - \frac{3}{5} \left(\nabla^\nu \nabla_\mu \xi^{\mu\rho} + \nabla^\rho \nabla_\mu \xi^{\mu\nu}\right) + \frac{2}{5} g^{\nu\rho} \nabla_\alpha \nabla_\beta \, \xi^{\alpha\beta} \,. \tag{6.2.17}$$

Next we consider the action of $\hat{\mathcal{L}}^{(3)}$ on the traceless tensor

$$\xi^{(\sigma)\,\mu\nu} \equiv \nabla^\mu \sigma^\nu + \nabla^\nu \sigma^\mu - \frac{2}{3} g^{\mu\nu} \, \nabla_\alpha \sigma^\alpha \,. \tag{6.2.18}$$

After a lengthy calculation one finds

$$\left(\hat{\mathcal{L}}^{(3)}\xi^{(\sigma)}\right)^{\nu\rho} = \frac{1}{5}\left[8\,\nabla^{(\nu}\left(-\Delta + \frac{7}{\ell^2}\right)\sigma^{\rho)} - \nabla^{(\nu}\nabla^{\rho)}\left(\nabla_\lambda\sigma^\lambda\right) - 6\,g^{\nu\rho}\,\nabla_\lambda\left(-\Delta + \frac{6}{\ell^2}\right)\sigma^\lambda\right]. \tag{6.2.19}$$

The first two terms obviously agree with (2.4.52) for $s = 3$. To understand how to obain (6.2.19) let us look at the various terms of $\hat{\mathcal{L}}^{(3)}$ separately: from the first term of $\hat{\mathcal{L}}^{(3)}$ one gets

$$\left(-\Delta + \frac{6}{\ell^2}\right)\xi^{(\sigma)\,\nu\rho} = \nabla^{(\nu}\left(-\Delta + \frac{10}{\ell^2}\right)\sigma^{\rho)} + \frac{2}{3}\,g^{\nu\rho}\,\nabla_\lambda\Delta\sigma^\lambda - \frac{20}{3\,\ell^2}\,g^{\nu\rho}\left(\nabla_\lambda\sigma^\lambda\right). \tag{6.2.20}$$

One easily checks that the right-hand side is indeed traceless, as must be. We group the remaining terms into two traceless parts, namely

$$\begin{aligned} &-\frac{3}{5}\left(\nabla^\nu\nabla_\mu\nabla^{(\mu}\sigma^{\rho)} + \nabla^\rho\nabla_\mu\nabla^{(\mu}\sigma^{\rho)}\right) + \frac{2}{5}g^{\nu\rho}\,\nabla_\lambda\nabla_\nu\nabla^{(\lambda}\sigma^{\nu)} \\ &\quad = \frac{3}{5}\nabla^{(\nu}\left(-\Delta + \frac{2}{\ell^2}\right)\sigma^{\rho)} - \frac{3}{5}\nabla^{(\nu}\nabla^{\rho)}(\nabla_\lambda\sigma^\lambda) + \frac{4}{5}g^{\nu\rho}\,\nabla_\lambda\Delta\,\sigma^\lambda\,, \end{aligned} \tag{6.2.21}$$

and

$$\begin{aligned} &-\frac{3}{5}\Big(\nabla^\nu\nabla_\mu(-\tfrac{2}{3})g^{\mu\rho}(\nabla_\lambda\sigma^\lambda) + \nabla^\rho\nabla_\mu(-\tfrac{2}{3})g^{\mu\nu}(\nabla_\lambda\sigma^\lambda)\Big) + \frac{2}{5}g^{\nu\rho}\,\nabla_\lambda\nabla_\tau(-\tfrac{2}{3})g^{\lambda\tau}(\nabla_\alpha\sigma^\alpha) \\ &\quad = \frac{2}{5}\nabla^{(\nu}\nabla^{\rho)}\left(\nabla_\lambda\sigma^\lambda\right) - \frac{4}{15}g^{\nu\rho}\,\nabla_\lambda\left(\Delta + \frac{2}{\ell^2}\right)\sigma^\lambda\,. \end{aligned} \tag{6.2.22}$$

One then easily checks that the sum of (6.2.20), (6.2.21) and (6.2.22) gives indeed the right hand-side of (6.2.19).

6.3 Taking the isometry generator across symmetrised covariant derivatives

In this appendix we give the proof of the statement that the isometry generator can be taken across symmetrised covariant derivatives. Let the isometry generator be

$$L_\xi = \xi^M \partial_M, \tag{6.3.23}$$

where ξ_M satisfies

$$\nabla_{(M}\xi_{N)} = 0. \tag{6.3.24}$$

This generator acts on tensors of rank (r, s) as

$$\begin{aligned} L_\xi T^{M_1 M_2 \ldots\ldots M^r}_{N_1 N_2 \ldots N_s} &= \xi^M \partial_M T^{M_1 M_2 \ldots\ldots M^r}_{N_1 N_2 \ldots N_s} - \partial_Q \xi^{M_1} T^{Q M_2 \ldots\ldots M^r}_{N_1 N_2 \ldots N_s} - \partial_Q \xi^{M_2} T^{M_1 Q \ldots\ldots M^r}_{N_1 N_2 \ldots N_s} \cdots \\ &- \partial_Q \xi^{M_r} T^{M_1 M_2 \ldots\ldots Q}_{N_1 N_2 \ldots N_s} + \partial_{N_1} \xi^Q T^{M_1 M_2 \ldots\ldots M^r}_{Q N_2 \ldots N_s} \cdots + \partial_{N_s} \xi^Q T^{M_1 M_2 \ldots\ldots M^r}_{N_1 N_2 \ldots Q} \\ = \quad & \xi^M \nabla_M T^{M_1 M_2 \ldots\ldots M^r}_{N_1 N_2 \ldots N_s} - \nabla_Q \xi^{M_1} T^{Q M_2 \ldots\ldots M^r}_{N_1 N_2 \ldots N_s} - \nabla_Q \xi^{M_2} T^{M_1 Q \ldots\ldots M^r}_{N_1 N_2 \ldots N_s} \cdots \\ &- \nabla_Q \xi^{M_r} T^{M_1 M_2 \ldots\ldots Q}_{N_1 N_2 \ldots N_s} + \nabla_{N_1} \xi^Q T^{M_1 M_2 \ldots\ldots M^r}_{Q N_2 \ldots N_s} \cdots + \nabla_{N_s} \xi^Q T^{M_1 M_2 \ldots\ldots M^r}_{N_1 N_2 \ldots Q}. \end{aligned} \tag{6.3.25}$$

In the last equality we have added and subtracted Christoffel connections to write the partial derivatives as covariant derivatives. Now let us apply (6.3.25) to a tensor of rank 1 and its covariant derivative

$$\begin{aligned} L_\xi \phi_N &= \xi^M \nabla_M \phi_N + \left(\nabla_N \xi^M\right) \phi_M \\ L_\xi \left(\nabla_P \phi_N\right) &= \xi^M \nabla_M \nabla_P \phi_N + \left(\nabla_N \xi^M\right) \nabla_P \phi_M + \left(\nabla_P \xi^M\right) \nabla_M \phi_N. \end{aligned} \tag{6.3.26}$$

Taking a covariant derivative of the first expression in (6.3.26) and subtracting it from the second, we obtain after some algebra

$$\nabla_P L_\xi \phi_N - L_\xi \left(\nabla_P \phi_N\right) = \frac{1}{\ell^2} \xi_{[N} \phi_{P]} - \phi_M \nabla^M \nabla_P \xi_N. \tag{6.3.27}$$

Therefore symmetrising the indices we get

$$\nabla_{(P}L_\xi\phi_{N)} - L_\xi\left(\nabla_{(P}\phi_{N)}\right) = 0. \tag{6.3.28}$$

Now let us define $T_{PN} \equiv \nabla_{(P}\phi_{N)}$. Performing the same analysis as before we obtain

$$\nabla_M\left(L_\xi T_{PN}\right) - L_\xi\left(\nabla_M T_{PN}\right) = \frac{1}{\ell^2}\left[\xi_{[P}T_{M]N} + \xi_{[N}T_{M]P}\right] - T_{PQ}\nabla^Q\nabla_M\xi_N - T_{NQ}\nabla^Q\nabla_M\xi_P. \tag{6.3.29}$$

And hence once again symmetrising the indices we get

$$\nabla_{(M}\left(L_\xi T_{PN)}\right) - L_\xi\left(\nabla_{(M}T_{PN)}\right) = 0. \tag{6.3.30}$$

Combining this with (6.3.28), we get

$$\nabla_{(M}\nabla_N L_\xi\phi_{P)} - L_\xi\left(\nabla_{(M}\nabla_N\phi_{P)}\right) = 0. \tag{6.3.31}$$

This is what we wanted to prove.

6.4 Conjugate momenta of different modes

In this appendix, we list all the conjugate momenta of the different modes that we obtained from the equation of motion. The conjugate momenta of the first kind are

$$\begin{aligned}
\Pi^{(1)MNP}_{(M_\chi)} &= \frac{\sqrt{-g}}{2}\Bigg[-\nabla^0\chi^{(M)MNP}+\frac{17}{18\times 3}\nabla^0\chi^{(M)(M}g^{NP)} \\
&\quad -\frac{2}{\mu}\left(3\mu^2-\frac{1}{\ell^2}\right)\varepsilon^{0R(M}\chi_R^{(M)\ NP)}+\frac{1}{9\mu}\left(17\mu^2-\frac{5}{\ell^2}\right)\varepsilon^{0R(M}\chi_R^{(M)}g^{NP)}\Bigg], \\
\Pi^{(1)MNP}_{(L_\chi)} &= \frac{\sqrt{-g}}{2}\Bigg[-\left(2-\frac{1}{\mu\ell}\right)\nabla^0\chi^{(L)MNP}+\frac{17}{18\times 3}\left(2-\frac{1}{\mu\ell}\right)\nabla^0\chi^{(L)(M}g^{NP)} \\
&\quad -\frac{4}{\mu\ell^2}\varepsilon^{0R(M}\chi_R^{(L)\ NP)}+\frac{4}{3\mu\ell^2}\varepsilon^{0R(M}\chi_R^{(L)}g^{NP)}\Bigg], \\
\Pi^{(1)MNP}_{(R_\chi)} &= \frac{\sqrt{-g}}{2}\Bigg[-\left(2+\frac{1}{\mu\ell}\right)\nabla^0\chi^{(R)MNP}+\frac{17}{18\times 3}\left(2+\frac{1}{\mu\ell}\right)\nabla^0\chi^{(R)(M}g^{NP)} \\
&\quad -\frac{4}{\mu\ell^2}\varepsilon^{0R(M}\chi_R^{(R)\ NP)}+\frac{4}{3\mu\ell^2}\varepsilon^{0R(M}\chi_R^{(R)}g^{NP)}\Bigg], \\
\Pi^{(1)MNP}_{(\log_\chi)} &= \frac{\sqrt{-g}}{2}\Bigg[-\nabla^0\left[\hat{\chi}^{MNP}-\chi^{(L)MNP}\right]+\frac{17}{18\times 3}\nabla^0\left[\hat{\chi}^{(M}g^{NP)}-\chi^{(L)(M}g^{NP)}\right] \\
&\quad -\frac{4}{\ell}\varepsilon^{0R(M}\hat{\chi}_R^{\ NP)}+\frac{4}{3l}\varepsilon^{0R(M}\hat{\chi}_R g^{NP)}-\frac{12}{\ell}\varepsilon^{0R(M}\chi_R^{(L)\ NP)}+\frac{34}{9l}\varepsilon^{0R(M}\chi_R^{(L)}g^{NP)}\Bigg].
\end{aligned} \tag{6.4.32}$$

And

$$\begin{aligned}
\Pi^{(1)MNP}_{(M_\Sigma)} &= \frac{\sqrt{-g}}{2}\left[-\nabla^0\Sigma^{(M)MNP}-\frac{2}{3\mu}\left(\mu^2-\frac{1}{\ell^2}\right)\varepsilon^{0R(M}\Sigma_R^{(M)NP)}\right], \\
\Pi^{(1)MNP}_{(L_\Sigma)} &= -\frac{\sqrt{-g}}{2}\left(2-\frac{1}{\mu\ell}\right)\nabla^0\Sigma^{(L)MNP}, \\
\Pi^{(1)MNP}_{(R_\Sigma)} &= -\frac{\sqrt{-g}}{2}\left(2+\frac{1}{\mu\ell}\right)\nabla^0\Sigma^{(R)MNP}, \\
\Pi^{(1)MNP}_{(\log_\Sigma)} &= \frac{\sqrt{-g}}{2}\left[-\nabla^0\left(\hat{\Sigma}^{MNP}-\Sigma^{(L)MNP}\right)-\frac{4}{3l}\varepsilon^{0R(M}\Sigma_R^{(L)NP)}\right].
\end{aligned} \tag{6.4.33}$$

And the conjugate momenta of the second kind are

$$\begin{aligned}
\Pi^{(2)MNP}_{(M_\chi)} &= \frac{\sqrt{-g}}{2}\left[-g^{00}\chi^{(M)MNP} + \frac{17}{18\times 3}g^{00}\chi^{(M)(M}g^{NP)}\right],\\
\Pi^{(2)MNP}_{(L_\chi)} &= \frac{\sqrt{-g}}{2}\left[-\frac{1}{\mu\ell}g^{00}\chi^{(L)MNP} + \frac{17}{18\times 3\mu\ell}g^{00}\chi^{(L)(M}g^{NP)}\right],\\
\Pi^{(2)MNP}_{(R_\chi)} &= \frac{\sqrt{-g}}{2}\left[\frac{1}{\mu\ell}g^{00}\chi^{(R)MNP} - \frac{17}{18\times 3\mu\ell}g^{00}\chi^{(R)(M}g^{NP)}\right],\\
\Pi^{(2)MNP}_{(\log_\chi)} &= \frac{\sqrt{-g}}{2}\left[-g^{00}\left[\hat{\chi}^{MNP} + \chi^{(L)MNP}\right] + \frac{17}{18\times 3}g^{00}\left[\hat{\chi}^{(M}g^{NP)} + \chi^{(L)(M}g^{NP)}\right]\right].
\end{aligned} \tag{6.4.34}$$

And

$$\begin{aligned}
\Pi^{(2)MNP}_{(M_\Sigma)} &= -\frac{\sqrt{-g}}{2}g^{00}\Sigma^{(M)MNP},\\
\Pi^{(2)MNP}_{(L_\Sigma)} &= -\frac{\sqrt{-g}}{2\mu\ell}g^{00}\Sigma^{(L)MNP},\\
\Pi^{(2)MNP}_{(R_\Sigma)} &= \frac{\sqrt{-g}}{2\mu\ell}g^{00}\Sigma^{(R)MNP},\\
\Pi^{(2)MNP}_{(\log_\Sigma)} &= -\frac{\sqrt{-g}}{2}g^{00}\left[\hat{\Sigma}^{MNP} + \Sigma^{(L)MNP}\right].
\end{aligned} \tag{6.4.35}$$

The labels L, M, R and $\log$ labels labelling the left, massive, right and logarithmic modes respectively are kept inside "( )" braces and hence should not be confused with the spacetime indices MNP. The following relations have been used

$$\begin{aligned}
&\mathcal{D}^{(L)}(\hat{\chi},\hat{\Sigma})_{MNP} \equiv (\hat{\chi},\hat{\Sigma})_{MNP} + \frac{\ell}{6}\varepsilon_{QR(M}\nabla^{Q}(\hat{\chi},\hat{\Sigma})^{R}_{\ NP)} = -(\chi,\Sigma)^{(L)}_{MNP},\\
&\mathcal{D}^{(M)}(\chi,\Sigma)^{(M)}_{MNP} = \mathcal{D}^{(L)}(\chi,\Sigma)^{(L)}_{MNP} = \mathcal{D}^{(R)}(\chi,\Sigma)^{(R)}_{MNP} = 0,\\
&\nabla^2\hat{\chi}_{MNP} = \frac{72}{\ell^2}\chi^{(L)}_{MNP} + \frac{24}{\ell^2}\hat{\chi}_{MNP} + \frac{2}{\ell^2}\hat{\chi}_{(M}g_{NP)},\\
&\nabla^2\chi^{(L,R)}_{MNP} = \frac{24}{\ell^2}\chi^{(L,R)}_{MNP} + \frac{2}{\ell^2}\chi^{(L,R)}_{(M}g_{NP)}, \quad \nabla^2\chi^{(M)}_{MNP} = 12\left(3\mu^2 - \frac{1}{\ell^2}\right)\chi^{(M)}_{MNP} + \frac{2}{\ell^2}\chi^{(M)}_{(M}g_{NP)},\\
&\nabla^2\Sigma^{(L,R)}_{MNP} = 0, \quad \nabla^2\Sigma^{(M)}_{MNP} = \left(4\mu^2 - \frac{4}{\ell^2}\right)\Sigma^{(M)}_{MNP},\\
&\nabla^2\hat{\Sigma}_{MNP} = \frac{8}{\ell^2}\Sigma^{(L)}_{MNP}.
\end{aligned} \tag{6.4.36}$$

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