Finer Consistency Checks on BPS Black Hole Entropy in String Theory

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

Abhishek Chowdhury

List of Publications arising from the thesis

Journal

1. *Black Hole Bound State Metamorphosis*; Abhishek Chowdhury, Shailesh Lal, Arunabha Saha, Ashoke Sen; *J*HEP 1305 (2013) 020.

2. *Logarithmic Corrections to Twisted Indices from the Quantum Entropy Function*; Abhishek Chowdhury, Rajesh Kumar Gupta, Shailesh Lal, Milind Shyani, Somyadip Thakur; *J*HEP 1411 (2014) 002.

Other publications (not included in thesis)

1. *Phase Structure of Higher Spin Black Holes*; Abhishek Chowdhury, Arunabha Saha; *J*HEP 1502 (2015) 084.

2. *BPS State Counting in $\mathcal{N} = 8$ Supersymmetric String Theory for Pure D-brane Configurations*; Abhishek Chowdhury, Richard S. Garavuso, Swapnamay Mondal, Ashoke Sen; *J*HEP 1410 (2014) 186.

3. *Do All BPS Black Hole Microstates Carry Zero Angular Momentum?*; Abhishek Chowdhury, Richard S. Garavuso, Swapnamay Mondal, Ashoke Sen; *J*HEP 1604 (2016) 082.

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Synopsis

Introduction

String Theory is a serious study for any aspiring theorist. The origin of the whole universe including the space-time itself with its endless bounty of fundamental matter governed by the four forces of nature can be traced back to a singular source, the fluctuations of omnipresent tiny strings floating around in the extra-dimensional space-time. A truly profound statement encouraging a unified outlook ! The concord is potentially illuminating, reinforcing the mysterious symbiosis among several seemingly divergent arenas of theoretical and mathematical physics. One finds prime examples of such deep interplay in the theoretical studies of black holes and in the equivalence of quantum field theories to quantum gravity known colloquially as AdS/CFT correspondence.

Since its discovery in early 20^{th} century black holes have been the muse of generations of physicists. String theory in its attempt to unify the infinitely subtle quantum gravity has benefited largely from the extensive studies on black holes as theoretical laboratories allowing for controlled theoretical predictions. Black holes are space-times with special properties, most notably the existence of Event horizon - a surrounding surface preventing objects from escaping the black hole. Though it has a good classical description, the quantum version behaves as a black body with a finite temperature (Hawking temperature) [1] and hence is governed by the laws of thermodynamics. It emits Hawking radiation and is described by a characteristic set of thermodynamic variables including an entropy proportional to the Area of Event horizon (Bekenstein-Hawking entropy) [2]. One would immediately question the origin of such an entropy and ask, do we have microstates from the statistical point of view ?

It is true that for generic black holes the answer is not known but in the past two decades it has been worked out for special classes of black holes (Extremal Black holes) in string theory [8,9,11,17]. Extremal black holes have zero temperature but they still carry a finite entropy, plus without any Hawking radiation they are usually stable. Often they are invariant under certain supersymmetries and hence the name BPS black holes. A microscopic formulation typically involves fluctuations of fundamental strings over solitonic objects (D-branes). In this picture gravity is weak, allowing us to calculate the degeneracy of states but it is essentially supersymmetry (Witten/Helicity Trace Index) [3,40] that continues the result to strong coupling where the gravitational back-reaction becomes important and the system is described by a single or multiple blackhole(s) [29]. For a wide class of extremal BPS black holes in string theory the macroscopic picture of Bekenstein-Hawking entropy match the degeneracy of BPS states in the microscopic picture.

The primary research presented in this thesis revolves around the central theme of black hole entropy matching, widening its scope and applicability to further reinforce the internal consistency of string theory. Listed below are the major results of the such studies.

- We attempted to give a physical explanation behind a set of ad hoc rules necessary to match the microscopic counting to the entropy of $\frac{1}{4}$ BPS black holes in $\mathcal{N} = 4$ string theory [4].
- We precisely reproduced certain logarithmic terms emerging from the large charge expansion of twisted microscopic indices from respective quantum corrections to $\frac{1}{4}$ BPS & $\frac{1}{8}$ BPS black holes in $\mathcal{N} = 4$ & $\mathcal{N} = 8$ supergravity theories [5].

Following sections contain brief descriptions of the above results.

Black Hole Bound State Metamorphosis

Among the varied microstates of $\mathcal{N} = 4$ supersymmetric string theories an important class are the so–called negative discriminant states carrying charges for which there are no classical supersymmetric single centered black holes but whose microscopic index is nevertheless non-zero [26]. All such states can be accounted for precisely as two–centered black hole configurations, with each centre representing a small half-BPS black hole. To avoid over–counting a crucial assumption is an ad hoc identification of two seemingly different configurations carrying the same total charge. This phenomenon is termed black hole metamorphosis.

We attempted at understanding the physical phenomenon and justify the ad hoc prescription. The existence of such two(multi)–centered black holes depends on the various moduli (like axion–dilaton) of the theory and taking a queue from BPS wall crossing in $\mathcal{N}$=4 SUSY string theory [23], we can draw a codimension one surface called the wall of marginal stability, effectively dividing the moduli space into two and separating the no-solution and two–centered configurations [28]. On the walls of marginal stability the two centers have infinite separation. For two two–centered configurations of the same total charge there is an overlapping region of existence and hence the ad hoc identification. To resolve certain singularities (Enhancon mechanism [36]) in the solution, we have replaced one of the black hole centers with a smooth gauge theory dyon [35]. This reduces the range of moduli for which each solution exists by effectively erecting another codimension one surface, separating the previously overlapping region. Now, at any given point in the moduli space there is no overlap and the total contribution by all existing two–centered configurations to the index adds up to match precisely the microscopic result for the same index [4].

Logarithmic Corrections to Twisted Indices from Quantum Entropy Function

The computation of logarithmic corrections to macroscopic black hole entropy is performed using the formalism of Quantum Entropy Function [31]. This proposal exploits the $AdS_2 \otimes S_2$ factor in the near-horizon geometries of spherically symmetric extremal black holes. The full quantum degeneracy associated with the horizon is a string path integral over all field configurations of fixed charges that asymptote to the attractor geometry of the black hole. The divergences can be regulated in accordance with the AdS/CFT correspondence [6]. The leading saddle point of the path integral is the attractor geometry itself which equals the exponential of Wald entropy [7]. Further, by expanding the massless fields of four-dimensional supergravity upto quadratic fluctuations (1-loop) about this saddle point, the 1-loop exact logarithmic correction to Wald entropy can be extracted and matched successfully to its microscopic index counterpart [54].

If we restrict ourselves to special subspaces of the moduli space which admit discrete symmetry transformations say $\mathbb{Z}_N$, we can average over the generator of the group and define the Twisted Indices (Helicity trace index) [64]. The large charge expansion of the microscopic index for $\frac{1}{4}$ BPS $\mathcal{N} = 4$ & $\frac{1}{8}$ BPS $\mathcal{N} = 8$ SUSY theories results in vanishing log corrections. On the macroscopic side, the original attractor solution is no longer an admissible saddle–point, however the new solution is just a $\mathbb{Z}_N$ orbifold of the original solution carrying (1/N)th the previous Wald entropy. Moreover, the sub-leading $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold saddle points also contribute (1/(NM))th the original Wald entropy.

We attempted to compute the 1–loop exact log corrections about these orbifold saddle points in the macroscopic picture. As a first step, while projecting to states invariant under the $\mathbb{Z}_N$ orbifold, we should account for the fact that various supergravity fields have non trivial transformations under the discrete group. The computation can be tricky, firstly because we are evaluating Heat Kernels [70] over non-compact AdS_2 background, secondly they diverge and finally there are infinitely many zero modes. However, there are some simplifications in defining the Heat Kernels around the $\mathbb{Z}_N$ orbifold of $S_2 \otimes S_2$ and then analytically continuing the results to the orbifold of $AdS_2 \otimes S_2$. Finally we showed that the log corrections vanish not only for the leading orbifold saddle point but also for the subsequent sub-leading orbifold saddle points of the Twisted index [5], in complete agreement with the microscopic results.

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CHAPTER 1

Introduction

"If the doors of perception were cleansed every thing would appear to man as it is, Infinite. For man has closed himself up, till he sees all things thro' narrow chinks of his cavern."

– William Blake, *The Marriage of Heaven and Hell.*

1.1 Why String Theory?

Over many centuries physicists have tried to decipher reality, from the tiny to the enormous, the freezing to the fiery. Their quest? To bring heaven, hell, and everything in between under one set of rules. It's been a long journey, with many dead ends and setbacks. But some are hoping we're nearly there.

Today we physicists subscribe to the idea of nature being governed by four fundamental forces. While electromagnetism and gravity are familiar from everyday existence, strong & weak nuclei forces are only relevant on tiny subatomic scales. Gravity affects the structure of space and time and is much weaker than all the other forces. All these forces are transmitted by messenger particles. The best theories of the 20^{th} & 21^{st} centuries seems to suggest that the universe is composed of tiny particles and they behave according to quantum mechanics. Unlike general relativity, it is a probabilistic theory. The uncertainty principle and superfine discrete energy levels of particles leads to the concept of zero point energy. Seemingly empty space (vacuum) is teeming with energy which causes particles to pop in and out of existence. Quantum field theory extends quantum mechanics from single localized particles to fields that exist everywhere. These fields also represent forces that permeate all of space and time. Although QFT reproduces classical physics easily, naive attempts to compute quantum corrections give infinite answer. Infinity is a physicist's brick wall - it is impossible for real things to have infinite value! As Wilson explained, these infinities arise because the calculations incorrectly assumed QFT applied down to infinitely short distances and renormalization is merely a sensible systematic procedure to discard infinities and extract the finite testable probabilities. The second complication has to do with the complicated machinery involved which allowed only approximate answers through the perturbative approach.

Unfortunately, the problems don't end there. Sadly, the techniques of QFT fail for the fourth fundamental force - gravity. It is said to be non–renormalizable; the high energy theory is worthless for making predictions since it requires an infinite number of constants. For most part, small scale gravity is of no concern; we can safely ignore its tiny impact on the behavior of atoms. But sometimes we would want to understand gravity in tiny regions. The Big Bang theory predicts that the early universe was extremely dense and all forces are on equal footing. Similarly, Black Holes are so massive that they crush space-time to an infinitesimal point. We do need a quantum theory of gravity to explain such scenarios.

The difficulties may be traced back to the mismatch between the fundamental principles that govern general relativity and quantum mechanics. GR tells us that space and time are not absolute but are dynamical objects. On the contrary in QFTs space and time are the fixed background against which fields are defined. Furthermore, our interpretation of quantum mechanics is inseparably entwined with the role we play as observers, but in GR observer makes no difference to the theory. Finally, GR requires a smooth fabric of space but quantum mechanics tells us that even empty space is foaming with quantum fluctuations. Though it seems that the most essential elements of the theories are incompatible, we believe that at high energies all forces will fit into a single theoretical framework, sometimes called a Theory of Everything. Fifty years of progress in QFT has formed the basis of the famous Standard Model of particle physics which gives a common framework for the remaining three forces except gravity. The new theory could attempt to solve the unification problems of GR and quantum mechanics or it may start afresh and establish completely new ideas of reality. String theory is an example of such a theory.

In string theory, the nature of reality changes. We are used to thinking of the world as composed of particles. However, in string theory, particles are no longer fundamental; at very short distances they appear to be tiny vibrating strings. These strings play nicely with both quantum mechanics and relativity and can easily give rise to the particles we observe for both forces and matter. There are open strings which end on spacetime membranes (D-branes) and closed strings which form a loop; they are the source of gravity. Despite its apparent elegance, it requires many elaborate concepts: supersymmetry, extra dimensions etc. but sadly to skeptics it provides no experimental evidence for these exotic demands. Strings are probably incredibly small: Plank scale and thus would continue to elude detection in current and future accelerators. There are many different ways of mushing up the extra dimensions into a tiny space. The current research focuses on Calabi-Yau manifolds as a promising group of compactification. Supersymmetry (SUSY) claims that there's a way to replace fermions (like electron, quark) with bosons (like photon, graviton) such that laws of physics remains the same. Including SUSY makes a big difference to string theory; it removes the impossible negative mass^2 particle (Tachyon) and lay the foundations of a ten dimensional superstring theory. Originally there were five consistent and distinct superstring theories. It would take a revolution to realize that these were all smoothly connected and part of a unified theory now called M–theory. In string theory we witness a rich underlying structure magnified by appealing dualities, unexpectedly linking diverse areas of mathematics and physics. It's true framework is grander than its founders could have possibly imagined.

It might happen that different mathematical theories describe the same physics. We shall call this situation a duality. String theory is full of dualities and they offer us new perspective on reality, improve our ability to compute and more importantly unite disparate areas of physics. Much of the modern research focuses on using these dualities to better understand a broad spectrum of topics. T–duality is perhaps the simplest to appreciate; small extra dimensions are equivalent to large extra dimensions, they produce identical theories. An extension of T–duality is the mirror symmetry which states that Calabi-Yau shapes comes in pairs. Perhaps on a more fundamental footing is S-duality which underpins the success of M–theory. Take two distinct string theories A and B, they each have an adjustable coupling constant. If A has a large coupling constant and B a small one, they predict exactly the same physics. M–theory is not just populated by strings, but also by membranes called M–branes. These are multidimensional surfaces that move through the eleven dimensions of M–theory. For example, a two–dimensional M–brane wrapped around a tiny extra dimension will appear as a one–dimensional string or a D1–brane moving through ten dimensions of the superstring theory. In recent years D–branes have become a central ingredient in modern research. The study of D–branes has shed light on some of the most elusive elements in the universe, black holes. Finally, they played an essential role in the formulating the AdS/CFT correspondence. They are non–perturbative objects allowing physicists to do calculations in regimes where interaction is strong. Historically this was uncharted terrain.

String theory is not a monolithic topic. People differ in interests and aptitudes, and their purposes vary widely. String theory is wielded to tackle quantum gravity and should tackle situations when spacetime become infinitely bent, forming a point called a singularity. In such situations general relativity as we know it breaks down so we can't fathom the physics! A particular example is that of orbifold singularities; it turns out the equations of string theory continue to work nicely at such singularities. Hence strings enable us to move beyond general relativity. More complicated are the conifold singularities where the basic equations of string theory fails but including D–branes we can make the theory consistent again, thereby allowing smooth transitions in spacetime. String theory gives an intuitive picture for the disappearance of infinities. Infinities arise because we assume the particles to be points but instead if we smear them out to extended strings then the infinite answers disappear for distances smaller than the smearing scale. Thankfully it has been confirmed by exact calculations in string perturbation theory. Though string theory provide insights into physics beyond Standard Model, cosmology, strongly interacting particles, mathematics and many more areas of active interests, we shall focus here on AdS/CFT correspondence and black holes.

AdS/CFT correspondence is one of the largest industry of research in string theory. It stand for Anti–de–Sitter space/Conformal Field Theory. It is a deeply surprising example of duality between theories of gravity and quantum field theories. The original example by Maldacena linked two very special theories. The gravitational side involved a particular geometry (5–dim. Anti–de–Sitter space) for a supersymmetric extension of gravity (type IIB supergravity) and on the other side, the QFT (Super Yang–Mills theory) was the unique theory with the largest possible amount of SUSY in four dimensions. There's a specific dictionary that translate between the two theories. This example has by now been extended to gravitational theories in (N+1) dimensions, completely equivalent to non–gravitational QFTs in N dimensions. It is more generally referred to as the gauge–gravity correspondence. The correspondence has a very useful property. When the gravitational theory is hard to solve , the QFT is east to solve , and vice–versa. This opens doors to previously intractable strongly coupled regime of QFT through simple calculations in gravity theories. Moreover AdS/CFT allows for a conceptual reworking of the classical problems in general relativity. Indeed if GR is equivalent to QFT, then neither one of them is deeper than the other. Finally, the correspondence indirectly brings string theory under experimental scrutiny through it's various application in condense matter physics, in particular near phase transitions where the effective theory is a CFT.

Before we move on to the core topic of black holes in string theory, we shall present a few thoughts on string theory's struggle with philosophy and its limitations. Progress in science is often based on a symbiotic allegiance between theory and experiments; a methodology so far missing in string theory. Some would say that the concepts are so deep and rich that they must describe nature, but skeptics would argue that it's only a mathematical framework, not a valid theory in itself. One of the biggest empirical problems string theory faces is under–determinism. M–theory has a very large number - some estimates give 10^{20000} - of low energy solutions, but no idea which one describes our reality. One way to deal with this is to apply the anthropic principle. This states that some properties of our universe has no deep fundamental origin, but take values they do purely because if they didn't, we wouldn't be here! String theory is not without it's limitations, particularly when we don't yet know of any deep underlying principle. Why strings are the fundamental objects of nature, rather than just a convenient concept. We could also ask, what are the fundamental equations of spacetime? A background independent formulation would be genuinely non–perturbative allowing generation of spacetime in the quantum regime. So much remains mysterious to quantum gravity that only time will tell how close we are with string theory. The truth we believe is out there!

1.2 Black Holes in String Theory

A black hole is at once the most simple and the most complex object.

Since its discovery in early 20^{th} century black holes have been the muse of generations of physicists. They are one of the most famous predictions of Einstein's theory of general relativity. Black holes are one of the rare examples in the history of science where a scientific idea has gestured and evolved over several decades into an important conceptual and quantitative tool, almost entirely on the merits of theoretical considerations. That we have journeyed so far with any confidence at all with very little guidance from the experiments indicates the robustness of the basic tenets of physics.

Though black hole is now part of our vocabulary, it's interesting to note that first black hole solution written down over a century ago by Karl Schwarzschild was deemed inadmissible by Einstein himself, for he and others believed that physical theories couldn't tolerate such singularities. The metric was immediately accepted as the correct static description of the gravitational field outside a symmetric mass such as a star like our sun. But the bizarre part was the hidden implication that if the entire mass of the sun is concentrated to a radius below 3 km we would have to face up with a singularity! Slowly, acceptance grew with work of gravitational (stellar) collapse by Chandrasekhar, Oppenheimer and others which sealed the fate of (super)massive stars, doomed to be reincarnated as black holes. Normally, stars balance the gravitational force with the pressure from the nuclear fusion reactions but when it gets old and burns up all its hydrogen and helium into heavier non-fissile elements it either turns into a white dwarf, a neutron star for less than twice the mass of our sun or collapse into a black hole. Astrophysicists now believe that our universe ought to contain many black holes. Cluster of stars collapse and create supermassive black holes at the center of probably every galaxy, with a mass ten to one hundred million times greater than the sun.

Simply put, a black hole is a region of spacetime in which things can fall, or be thrown in, but nothing comes out, not even light. The boundary from which no round–trip tickets are available is called the event horizon. Event horizon and singularity are the two defining features of a black hole. Its simplicity lie in the fact that it is completely specified by its mass, spin and charge. This is a remarkable consequence of the so–called 'No Hair Theorem'. For an astrophysical object like the earth, the gravitational field will not only depend on the mass but the distribution of the mass, on the shapes of valleys and mountains. Not so for the black hole; once the star collapses to form a black hole, the gravitational field around it forgets all details of its formation except for the mass, spin and charge. In this respect, a black hole behaves much like a structure–less elementary particle.

What happens if we add quantum mechanics to the game? Hawking's semiclassical analysis of QFT in a black hole background allows for the energy density of the curvature around the horizon to decay into particle–antiparticle pairs [1]. Once in a while one of the two falls into the black hole horizon, while the other escaped to spatial infinity. Contemporarily, Bekenstein realized that black hole must have entropy [2]. Otherwise, if we throw a bucket of hot water into the black hole then the net entropy of the outside world would seem to decrease, in violation of second law of thermodynamics. These observations paved way for understanding black holes as statistical ensembles and ultimately lead to the formulation of laws of black hole thermodynamics; of particular interest is the statement that entropy is proportional to the area of the event horizon. Entropy gives an account of the number of microscopic states of a system and hence associates an incredibly complex microstructure with the black hole. In this respect, a black hole is very unlike an elementary particle.

Where are the microstates? We may also reflect upon the infamous Information paradox, the failure of the semiclassical setup to establish an unitary evolution of a black hole pure state into thermal radiation. We would like to solve these problems. The solution is in the study of black holes in a quantum gravity theory, that can unify classical GR with quantum mechanics. String theory is a powerful mathematical framework uniquely equipped to address exactly this. String theory in its attempt to unify the infinitely subtle quantum gravity has benefited largely from the extensive studies on black holes as theoretical laboratories allowing for controlled theoretical predictions. In any consistent quantum theory of gravity such as the string theory, the requirement that the thermodynamic entropy must equal the statistical entropy of black hole is an extremely stringent theoretical constraint. Moreover, the constraint is also universal in that it must exist in any phase or compactification of theory that admits a black hole. We shall use this freedom to study special class of phases of string theory with a large amount of unbroken supersymmetry. Since these phases have Bose–Fermi degenerate spectrum of states, they don't describe the observed world. Nevertheless, for the past two decades these extremal or BPS black holes enjoyed a microscopic description within the framework of string theory [8,9,11,17]. Equality between mass and the charges leads to unbroken SUSY, which is ultimately responsible for the disappearance of messy quantum corrections, so that physics near the horizon can be found by relatively simple calculations. A microscopic formulation typically involves fluctuations of fundamental strings over solitonic objects (D-branes). In this picture gravity is weak, allowing us to calculate the degeneracy of states but it is essentially supersymmetry (Witten/Helicity Trace Index) [3,40] that continues the result to strong coupling where the gravitational back-reaction becomes important and the system is described by a single or multiple blackhole(s) [29]. For a wide class of extremal BPS black holes in string theory the macroscopic picture of Bekenstein-Hawking entropy match the degeneracy of BPS states in the microscopic picture.

The primary research presented in this thesis revolves around the central theme of black hole entropy matching, widening its scope and applicability to further reinforce the internal consistency of string theory. We shall mainly focus on the macroscopic or supergravity side of the story and allow for a large charge limit where the black hole description is valid. We have looked at cases where there is a primary agreement between the microscopic and the macroscopic descriptions, but upon closer examination mild disagreements surfaced. Such finer disagreements and their resolutions are more than welcome in string theory as they tend to be sensitive on the 'phase' under consideration and contains a wealth of information about the details of compactification as well as the spectrum of non–perturbative states in the theory. Below are the major results of the such studies.

- We attempted to give a physical explanation behind a set of ad hoc rules necessary to match the microscopic counting to the entropy of $\frac{1}{4}$ BPS black holes in $\mathcal{N} = 4$ string theory [4].
- We precisely reproduced certain logarithmic corrections emerging from the large charge expansion of twisted microscopic indices from respective quantum corrections to $\frac{1}{4}$ BPS & $\frac{1}{8}$ BPS black holes in $\mathcal{N} = 4$ & $\mathcal{N} = 8$ supergravity theories [5].

Since both results are covered in complete detail in the subsequent chapters we shall only provide very brief descriptions below.

1.2.1 Black Hole Bound State Metamorphosis

Among the varied microstates of $\mathcal{N} = 4$ supersymmetric string theories an important class are the so–called negative discriminant states carrying charges for which there are no classical supersymmetric single centered black holes but whose microscopic index is nevertheless non-zero [26]. The result for the degeneracy takes the form of an integration over a three real dimensional subspace (a contour) of the Siegel upper half plane parametrizing genus two Riemann surfaces, and the integrand involves inverse of a certain meromorphic modular form of a subgroup of the Siegel modular group. On the macroscopic side, all such states can be accounted for precisely as two–centered black hole configurations, with

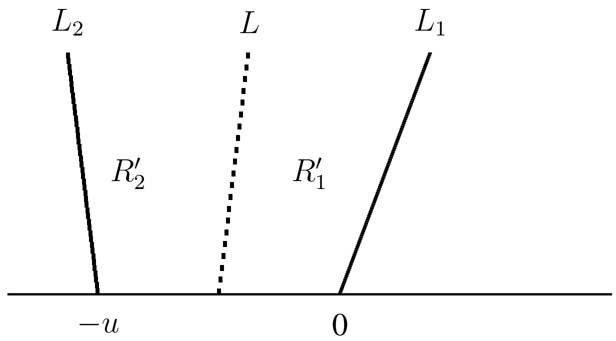

Figure 1.1: Black hole metamorphosis.

each centre representing a small half-BPS black hole. To avoid over–counting a crucial assumption is an ad hoc identification of two seemingly different configurations carrying the same total charge. This phenomenon is termed black hole metamorphosis.

We attempted at understanding the physical phenomenon and justify the ad hoc prescription. The existence of such two(multi)–centered black holes depends on the various moduli (like axion–dilaton) of the theory and taking queue from BPS wall crossing in $\mathcal{N}$=4 SUSY string theory [23], we can draw a codimension one surface called the wall of marginal stability (the lines L_1 & L_2 in Fig. 1.1), effectively dividing the moduli space into two and separating the no-solution and two–centered configurations [28]. On the walls of marginal stability the two centers have infinite separation and the original dyon becomes marginally unstable against decay into a pair of half BPS dyons. Typically the spectrum of the original dyon changes discontinuously as we move through these marginal stability walls in the moduli space. For two two–centered configurations of the same total charge there is an overlapping region of existence, where each configuration exists in the region $R_1' \cup R_2' \equiv R$ in Fig. 1.1 and hence the ad hoc identification. The phenomenon of black hole metamorphosis suggests the existence of a hypothetical line L, shown in Fig. 1.1, such that the first configuration exists only in the region R_1' to the right of L and left of L_1 and the second configuration exists only in the region R_2' to the left of L and the right of L_2. The line L is another codimension one surface erected to address certain singularities

(Enhancon mechanism [36]), which we resolved by replaced one of the black hole centers with a smooth gauge theory dyon [35]. Now, at any given point in the moduli space there is no overlap and the total contribution by all existing two–centered configurations to the index adds up to match precisely the microscopic result for the same index [4].

1.2.2 Logarithmic Corrections to Twisted Indices from Quantum Entropy Function

The computation of logarithmic corrections to macroscopic black hole entropy is performed using the formalism of Quantum Entropy Function [31]. This proposal exploits the $AdS_2 \otimes S_2$ factor in the near-horizon geometries of spherically symmetric extremal black holes. The full quantum degeneracy associated with the horizon is a string path integral over all field configurations of fixed charges that asymptote to the attractor geometry of the black hole. The divergences can be regulated in accordance with the AdS/CFT correspondence [6]. The leading saddle point of the path integral is the attractor geometry itself which equals the exponential of Wald entropy [7]. Further, by expanding the massless fields of four-dimensional supergravity upto quadratic fluctuations (1-loop) about this saddle point, the 1-loop exact logarithmic correction to Wald entropy can be extracted and matched successfully to its microscopic index counterpart [54].

If we restrict ourselves to special subspaces of the moduli space which admit discrete symmetry transformations say $\mathbb{Z}_N$, we can average over the generator of the group and define the Twisted Indices (Helicity trace index) [64]. The large charge expansion of the microscopic index for $\frac{1}{4}$ BPS $\mathcal{N} = 4$ & $\frac{1}{8}$ BPS $\mathcal{N} = 8$ SUSY theories results in vanishing log corrections. On the macroscopic side, the original attractor solution is no longer an admissible saddle–point, however the new solution is just a $\mathbb{Z}_N$ orbifold of the original solution carrying (1/N)th the previous Wald entropy. Moreover, the sub-leading $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold saddle points also contribute (1/(NM))th the original Wald entropy.

We attempted to compute the 1–loop exact log corrections about these orbifold saddle points in the macroscopic picture. As a first step, while projecting to states invariant under the $\mathbb{Z}_N$ orbifold, we should account for the fact that various supergravity fields have non trivial transformations under the discrete group. The computation can be tricky, firstly because we are evaluating Heat Kernels [70] over non-compact AdS_2 background, secondly they diverge and finally there are infinitely many zero modes. However, there are some simplifications in defining the Heat Kernels around the $\mathbb{Z}_N$ orbifold of $S_2 \otimes S_2$ and then analytically continuing the results to the orbifold of $AdS_2 \otimes S_2$. Finally we showed that the log corrections vanish not only for the leading orbifold saddle point but also for the subsequent sub-leading orbifold saddle points of the Twisted index [5], in complete agreement with the microscopic results..

CHAPTER 2

Black Hole Bound State Metamorphosis

2.1 Introduction

Matching of microscopic counting of BPS states to the entropy of supersymmetric black holes is an important problem. Exact microscopic counting of BPS states, including the dependence of the spectrum on the asymptotic moduli, has now been achieved for a wide class of states in $\mathcal{N} = 8$ supersymmetric string theories [8–10] and a wide class of $\mathcal{N} = 4$ supersymmetric string theories [11–25] in four dimensions. An important class of these microscopic states are the so called negative discriminant states – states carrying charges for which there are no classical supersymmetric single centered black holes but whose microscopic index is nevertheless non-zero. In particular such states are abundant in $\mathcal{N} = 4$ supersymmetric string theories [1]. It was shown in [26], following an earlier observation of [24], that all the known negative discriminant states in $\mathcal{N} = 4$ supersymmetric string theories, which appear in the microscopic counting of states, can be accounted for precisely as 2-centered black hole configurations, with each center representing a small

[1] Note that we are not going to discuss positive discriminant states. The matter has been extensively discussed in the literature with full agreement between microscopic and macroscopic counting.

half-BPS black hole [2]. This however required one crucial assumption: certain 2-centered configurations, whose indices can be computed and shown to be the same, had to be treated as identical configurations. This identification was ad hoc, since the configurations which had to be identified appeared to be different configurations carrying the same total charge. Nevertheless [26] gave a precise set of rules for determining when a pair of configurations have to be identified. This phenomenon was called black hole metamorphosis. A similar phenomenon in the context of supersymmetric gauge theories had been discussed earlier in [27]. Note that not all negative discriminant states exhibit metamorphosis but only states where either of the two centers carries charge $Q^2 = -2$ or $P^2 = -2$ or both. In our analysis we have omitted the more complicated case of both $Q^2 = -2$ and $P^2 = -2$.

Our main goal in this chapter will be to understand the physical origin of this phenomenon, and justify the ad hoc prescription of [26] for identifying certain apparently different configurations of black holes. What we shall show is that precisely for the class of two centered solutions for which the ad hoc identification rule is to be imposed, one of the black hole centers need to be replaced by a smooth gauge theory dyon to avoid certain singularities in the solution. The effect of this is that the range of moduli for which each solution exists is smaller than the one based on the naive analysis of a two centered black hole solution. Taking into account this effect, we find that at any given point of the moduli space the total index contributed by all the two centered configurations which exist at that point adds up to match precisely the microscopic result for the same index. We shall relegate subtleties arising from dimension reduction and certain computational steps to Appendix A. Although we have carried out our analysis in the context of a specific theory – for heterotic string theory on T^6 – and worked in a region of the moduli space where one of the two centers is light and can be regarded as a test particle in the background produced

[2] Table 1 & 2 in [26]. For example, for net charges $(Q^2, P^2, Q \cdot P) \equiv (2, 2, 3)$, it splits up into $(Q, Q) + (0, P - Q)$ and $(2Q - P, 2Q - P) + ((P - Q), 2(P - Q))$.

by the other center, we expect that our analysis captures the essential physics behind the phenomenon of black hole bound state metamorphosis for more general theories and in generic region of the moduli space.

2.2 Review of black hole metamorphosis

In this section we shall review the phenomenon of black hole bound state metamorphosis discussed in [26]. Although this phenomenon takes place in all $\mathcal{N}=4$ supersymmetric string theories, we shall consider in this chapter the concrete example of heterotic string theory compactified on T^6. At a generic point in the moduli space this theory has 28 U(1) gauge fields and hence a BPS state is characterized by a 28 dimensional electric charge vector Q and a 28 dimensional magnetic charge vector P. We shall denote the combined charge vector as (Q,P). We can associate with these vectors T-duality invariant bilinears Q^2, P^2 and $Q\cdot P$. We consider quarter BPS states carrying charges $(\widehat{Q},\widehat{P})$ satisfying

$$(\widehat{Q}^2\widehat{P}^2-(\widehat{Q}\cdot\widehat{P})^2)<0, \quad \text{and} \quad \gcd\{\widehat{Q}_i\widehat{P}_j-\widehat{Q}_j\widehat{P}_i, \quad 1\leq i,j\leq 28\}=1\,. \qquad (2.2.1)$$

In this case there are no single centered black holes carrying these charges and the only two centered configurations which can contribute to the index carry charges of the form [3] :

$$(aQ,cQ) \quad \text{and} \quad (bP,dP)\,, \qquad (2.2.2)$$

for some vectors Q and P and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\,\mathbb{Z})$, carrying total charge

$$(aQ+bP,cQ+dP)=(\widehat{Q},\widehat{P}) \qquad (2.2.3)$$

[3]Note that as each center is half–BPS in $\mathcal{N}=4$ SUSY theory, the electric and magnetic charges are parallel to each other.

This two centered configuration exists in a certain region of the moduli space of the theory determined by the rules given in [26] [4]. Outside this region the configuration ceases to exist and hence does not contribute to the index. The contribution to the index from this configuration when it exists is given by

$$(-1)^{Q\cdot P+1}|Q\cdot P|\, d_h(Q^2/2)\, d_h(P^2/2)\,, \tag{2.2.4}$$

where

$$\sum_n d_h(n) q^n = q^{-1}\prod_{k=1}^{\infty}\left(1-q^k\right)^{-24}\,. \tag{2.2.5}$$

Physically $d_h(n)$ denotes the index of half BPS states.

The phenomenon of metamorphosis takes place when either P^2 or Q^2 (or both) take the value -2. Let us suppose $P^2 = -2$. In that case the configuration

$$(a'Q', c'Q') \quad \text{and} \quad (b'P', d'P')\,, \qquad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \equiv \begin{pmatrix} a & b-au \\ c & d-cu \end{pmatrix}$$

$$Q' \equiv Q + uP, \quad P' \equiv P, \quad u \equiv Q\cdot P \tag{2.2.6}$$

has the same total charge, satisfies

$$Q'^2 = Q^2, \quad P'^2 = P^2, \quad Q'\cdot P' = -Q\cdot P\,, \tag{2.2.7}$$

[4]For a typical two centered configuration like (2.2.2) the wall of marginal stability is an arc intersecting the real axis of upper half τ–plane at $\frac{b}{d}$ and $\frac{a}{c}$. For $Q\cdot P > 0$, connecting $\frac{b}{d}$ to $\frac{a}{c}$, the two centered configuration is on the left side of the line of marginal stability. For $Q\cdot P < 0$ they are on the right side. The case $c = 0$ implies $a = d = \pm 1$ which in turn creates a line of marginal stability connecting $\pm b$ to $i\,\infty$. Also note that exercising S–duality we can bring (2.2.2) to $(Q, 0)$ and $(0, P)$ and thus the degeneracy being duality covariant is given by (2.2.4).

and hence, according to (2.2.4) gives the same contribution to the index [5]. Note that configuration (2.2.6) is not S-dual to the configuration (2.2.2). Now suppose that the configuration (2.2.2) exists in the region R_1 in the moduli space and the configuration (2.2.6) exists in the region R_2. It turns out that $R_1 \cup R_2$ spans the whole moduli space of the theory. Thus naively one would expect that in the region $R = R_1 \cap R_2$ the total contribution to the index from these two configurations will be given by twice of (2.2.4) whereas outside this region the contribution to the index will be given by (2.2.4). However in order to match the microscopic result we have to assume that in the region R the contribution to the index is given by (2.2.4) while outside this region there is no contribution to the index from these configurations.

The case where $Q^2 = -2$ is related to the above by a duality transformation and need not be discussed separately. In fact with the help of an S-duality transformation by the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ we can map the configurations (2.2.2) and (2.2.6) to

$$(Q, 0) \quad \text{and} \quad (0, P)\,, \quad P^2 = -2, \tag{2.2.8}$$

[5]The reason we should suspect the center(s) with $charge^2 = -2$ is because there is no genuine black hole solution, the metric agrees with that of a black hole only far away from the core. This has been termed enhancon mechanism in [34]. On the other hand a Harvey-Liu dyon has the correct asymptotic behavior and remains smooth all the way to the origin. This is the primary reason why we were able to replace one of the centers with a Harvey-Liu dyon.
With hindsight, it can be shown that the configurations $(Q, 0) + (0, P)\,, (Q + (Q \cdot P)P, 0) + (-(Q \cdot P)P, P)\,\&\,(0, P + (Q \cdot P)Q) + (Q, -(Q \cdot P)Q)$ have overlapping contributions in certain region of the moduli space. But we want more, for example for $Q' \equiv Q + (Q \cdot P)P\,,\ P' \equiv P$ we shall demand (2.2.7). Since, $Q' \cdot P' \equiv (Q \cdot P)(1 + P^2)$, the only choice is $P^2 = -2$. Similarly, for the other configuration we get $Q^2 = -2$. Also note that since we are looking at Heterotic on T^6 each individual center represents a half–BPS state with $\frac{Q^2}{2}\,,\ \frac{P^2}{2} \geq -1$. Also note that for either $Q^2 = -2$ or $P^2 = -2$ the discriminant is negative and hence we don't worry about these issues for positive discriminant states.

and

$$(Q + uP, 0) \quad \text{and} \quad (-uP, P), \quad u \equiv Q \cdot P \,, \tag{2.2.9}$$

with each configuration carrying the same index as (2.2.4). Thus from now on we shall focus on this configuration. In this case Fig. 2.1 shows the regions R_1, R_2 and R in the upper half τ-plane [23] – where $\tau = \tau_1 + i\tau_2$ denotes the asymptotic values of the axion-dilaton modulus of the heterotic string theory on T^6– for fixed asymptotic values of the other fields. The boundaries of R_1, R_2 marked by the thick lines L_1 and L_2 correspond to walls of marginal stability beyond which the configurations (2.2.8) and (2.2.9) cease to exist. The precise slope of these straight lines depend on the details of the charges and the asymptotic values of the other moduli, and will be given in eqs.(2.4.27) and (2.4.38) respectively.

The phenomenon of black hole metamorphosis suggests the existence of a hypothetical line L, shown in Fig. 2.2, such that the configuration (2.2.8) exists only in the region R_1' to the right of L and left of L_1 and the configuration (2.2.9) exists only in the region R_2' to the left of L and the right of L_2. In that case it would explain why the index is given by (2.2.4) in the region $R_1' \cup R_2' = R$ and vanishes outside this region. Our goal will be to understand the physical origin of L [6].

[6] We could justifiably ask whether we would witness the phenomenon of metamorphosis in $\mathcal{N} = 8$ and $\mathcal{N} = 2$ SUSY theories. As $\mathcal{N} = 8$ theories don't have $Q^2 = -2$ or $P^2 = -2$ states we would not encounter metamorphosis. The situation for $\mathcal{N} = 2$ would be very similar to $\mathcal{N} = 4$ and metamorphosis can certainly happen except for the fact that $Q^2 = -2$ or $P^2 = -2$ states would have independent walls beyond which the half–BPS configurations ceases to exist. Unfortunately for $\mathcal{N} = 2$ SUSY theory, till date we don't even know how to extract the precise leading contribution to the entropy from the microscopic side and match it with the entropy from single center supergravity black hole solutions. Though important for negative discriminant states, metamorphosis is a sub–leading issue and hence we leave it till we know better!

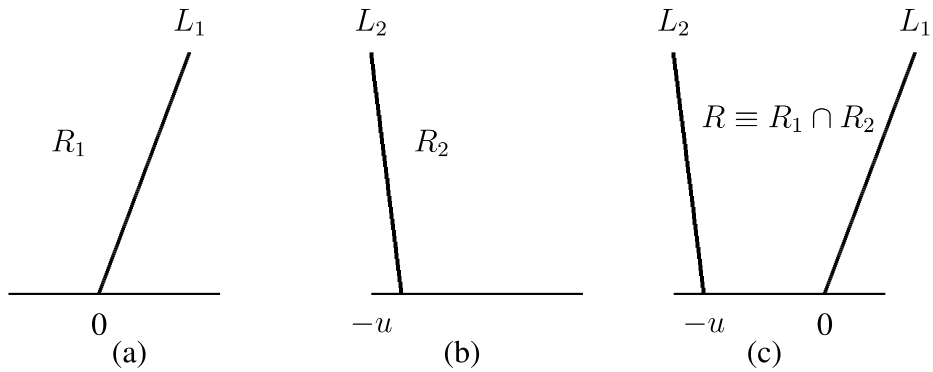

Figure 2.1: Figure illustrating the walls of marginal stability and the region of existence of the configurations described in (2.2.8) and (2.2.9). In Fig. (a) the thick line L_1 labels the wall of marginal stability for the configuration (2.2.8) which exists in the region R_1 to the left of L_1 in the upper half plane. In Fig. (b) the thick line L_2 labels the wall of marginal stability for the configuration (2.2.9) which exists in the region R_2 to the right of L_2 in the upper half plane. Fig. (c) labels the region $R \equiv R_1 \cap R_2$. Thus naively we expect both configurations to exist in the region R and one of the two configurations to exist in the region outside R. However microscopic counting requires that only one of the two configurations exist in the region R and none exist outside this region. In drawing these figures we have implicitly taken u to be positive. If u is negative then each figure has to be reflected about the vertical axis passing through the origin.

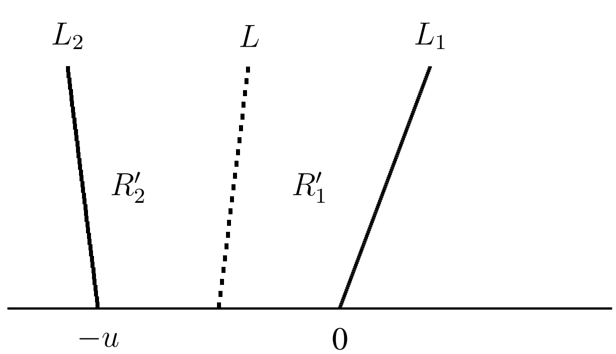

Figure 2.2: The pictorial description of black hole metamorphosis.

2.3 Review of multi-black hole solutions in $N = 2$ supergravity

Although heterotic string theory on T^6 describes an $\mathcal{N} = 4$ supersymmetric string theory, the multi-black hole solutions are best understood in the language of $\mathcal{N} = 2$ supergravity. For this reason in this section we shall review multi-black hole solutions in $\mathcal{N} = 2$ supergravity. The bosonic fields of an $\mathcal{N} = 2$ supergravity coupled to n_v vector multiplet fields are the metric $g_{\mu\nu}$, $n_v + 1$ complex scalars X^I, and $n_v + 1$ gauge fields $\mathcal{A}^I_\mu$ with $0 \le I \le n_v$. The theory has a complex gauge invariance under which all the X^I's scale by an arbitrary complex function $\Lambda(x)$, the metric scales by $|\Lambda|^{-2}$ and the gauge fields remain invariant. The action of the theory is completely fixed by the prepotential F which is a meromorphic, homogeneous function of the X^I's of degree 2. If (q, p) denote the electric and magnetic charge vectors carried by a state with q and p being $n_v + 1$ dimensional vectors, then we define

$$F_I \equiv \partial F/\partial X^I, \quad e^{-K} \equiv i\,(\bar{X}^I F_I - X^I \bar{F}_I), \quad Z(q,p) \equiv (q_I X^I - p^I F_I) e^{K/2}\,. \quad (2.3.1)$$

The gauge fields are normalized so that the action of a test particle carrying electric charges $\hat{q}_I$ and magnetic charges $\hat{p}^I$ takes the form

$$\frac{1}{2}\int (\hat{q}_I \mathcal{A}^I_\mu - \hat{p}^I \mathcal{A}_{I\mu}) dx^\mu, \quad (2.3.2)$$

where $\mathcal{A}^I_\mu$ denotes the usual gauge potential and $\mathcal{A}_{I\mu}$ denotes the dual magnetic potential. Note, as customary in $\mathcal{N} = 2$ supergravity we choose a symplectic inner product w.r.t. the upper and lower index label I.

A general supersymmetric multi-centered black hole solution in such a theory was constructed in [28, 29]. To describe the solution we introduce the functions:

$$H^I = \sum_i \frac{p^I_{(i)}}{|\vec{r}-\vec{r}_i|} - 2\,\mathrm{Im}\left[e^{-i\alpha_\infty} X^I e^{K/2}\right]_\infty , \quad H_I = \sum_i \frac{q_{(i)I}}{|\vec{r}-\vec{r}_i|} - 2\,\mathrm{Im}\left[e^{-i\alpha_\infty} F_I e^{K/2}\right]_\infty , \tag{2.3.3}$$

where $\vec{r}_i$ are the locations of the centers in the three dimensional space, $(q_{(i)}, p_{(i)})$ denote the electric and magnetic charges carried by the i-th center, the subscript $_\infty$ denotes the asymptotic values of the various fields and

$$\alpha_\infty = \mathrm{Arg}\left[Z\left(\sum_i q_{(i)}, \sum_i p_{(i)} \right) \right]_\infty . \tag{2.3.4}$$

Now let

$$S_{BH}(\{q_I\}, \{p^I\}) = \pi\, \Sigma(\{q_I\}, \{p^I\}) , \tag{2.3.5}$$

be the entropy of a single centered black hole solution in this theory with charge (q, p). There is a standard algorithm for computing the function Σ from the knowledge of the function F – it is given by the extremum of $|Z(q,p)|^2$ with respect to the scalar moduli fields. We now define

$$\chi^K(\{q_I\}, \{p^I\}) \equiv -\frac{\partial \Sigma}{\partial q_K}, \quad \chi_K(\{q_I\}, \{p^I\}) \equiv \frac{\partial \Sigma}{\partial p^K} , \tag{2.3.6}$$

$$g^K(\{q_I\}, \{p^I\}) = \chi^K(\{q_I\}, \{p^I\}) + i\, p^K, \quad g_K(\{q_I\}, \{p^I\}) = \chi_K(\{q_I\}, \{p^I\}) + i\, q_K . \tag{2.3.7}$$

Then the solution for the scalar fields, metric and the gauge fields is given by

$$\frac{X^K}{X^0} = \frac{g^K(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})}{g^0(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})}, \quad \frac{F_K}{X^0} = \frac{g_K(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})}{g^0(\{H_I(\vec{r})\}, \{H^I(\vec{r})\})} , \tag{2.3.8}$$

$$ds^2 = e^{2V}(dt + \vec{\omega}\cdot d\vec{x})^2 + e^{-2V} dx^i dx^i , \tag{2.3.9}$$

$$e^{-2V} \equiv \Sigma(\{H_I(\vec{r})\}, \{H^I(\vec{r})\}) , \tag{2.3.10}$$

$$\begin{aligned}\mathcal{A}^K_\mu dx^\mu &= -\Sigma(\{H_I(\vec{r})\},\{H^I(\vec{r})\})^{-1}\chi^K(\{H_I(\vec{r})\},\{H^I(\vec{r})\})(dt+\vec{\omega}\cdot d\vec{x}) \\ &\quad -\sum_i p^K_{(i)}\cos\theta_{(i)} d\phi_{(i)}, \\ \mathcal{A}_{K\mu} dx^\mu &= -\Sigma(\{H_I(\vec{r})\},\{H^I(\vec{r})\})^{-1}\chi_K(\{H_I(\vec{r})\},\{H^I(\vec{r})\})(dt+\vec{\omega}\cdot d\vec{x}) \\ &\quad -\sum_i q_{(i)K}\cos\theta_{(i)} d\phi_{(i)},\end{aligned} \tag{2.3.11}$$

where $(\theta_{(i)}, \phi_{(i)})$ denote the polar and azimuthal angles of the spherical polar coordinate system with origin at $\vec{r}_i$. The general solution for $\vec{\omega}$ exists but we shall not need it, though for single centered solution $\vec{\omega}$ vanishes.

One clarification in necessary here. The combinations X^I/X^0, F_I/X^0 and the gauge fields are invariant under the complex gauge transformation generated by $\Lambda(x)$ and hence it is not necessary to specify in which gauge we have given the solutions. However since the metric is not invariant under this transformation we need to specify the gauge in which the metric is given. (2.3.9) is given in the choice of gauge in which the Einstein-Hilbert term takes the form [29]

$$\frac{1}{16\pi}\int d^4x\,\sqrt{-\det g}\,R\,. \tag{2.3.12}$$

Finally consistency demands that the locations $\vec{r}_i$ be subject to the constraint [7] :

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{q_{(i)I}p^I_{(j)} - q_{(j)I}p^I_{(i)}}{|\vec{r}_i - \vec{r}_j|} = 2\,\mathrm{Im}\,(e^{-i\alpha_\infty} Z_i), \quad Z_i \equiv Z\,\left(q_{(i)}, p_{(i)}\right)\big|_\infty \tag{2.3.13}$$

For a 2-centered solution carrying charges $(q_{(1)}, p_{(1)})$ at $\vec{r}_1$ and $(q_{(2)}, p_{(2)})$ at $\vec{r}_2$, it gives

$$|\vec{r}_1 - \vec{r}_2| = \frac{q_{(2)I}p^I_{(1)} - q_{(1)I}p^I_{(2)}}{2\,\mathrm{Im}(e^{-i\alpha_\infty} Z_2)}, \quad e^{i\alpha_\infty} = \frac{Z_1+Z_2}{|Z_1+Z_2|}\,. \tag{2.3.14}$$

When $|Z(q_{(2)}, p_{(2)})|$ is small and we can ignore the background field produced by the second center in most of the space, then an independent way of arriving at the result

[7] Colloquially known as Denef's rules for composite objects.

(2.3.14) is as follows. Let us consider the background fields produced by a single centered solution carrying charges $(q_{(1)}, p_{(1)})$ placed at the origin. If we now place a test particle carrying charge $(\hat{q}, \hat{p})$ in this background then the action of this test particle takes the form

$$S_t = -\int d\tau \, |Z(\{\hat{q}_I\}, \{\hat{p}^I\})| + \frac{1}{2}\int (\hat{q}_I \mathcal{A}^I_\mu - \hat{p}^I \mathcal{A}_{I\mu}) dx^\mu, \tag{2.3.15}$$

where τ is the proper time (not to be confused with the axion-dilaton moduli), and x^μ denote the coordinates of the test particle. If the test particle is at rest then we have $d\tau = e^V dt$ and hence

$$S_t = \int dt \left[-e^V \, |Z(\{\hat{q}_I\}, \{\hat{p}^I\})| + \frac{1}{2}(\hat{q}_I \mathcal{A}^I_0 - \hat{p}^I \mathcal{A}_{I0}) \right] . \tag{2.3.16}$$

The equilibrium position of the test particle will be at the extremum of the integrand with respect to the spatial coordinates x^1, x^2, x^3. It can be shown that this gives us back (2.3.14) with $(q_{(2)}, p_{(2)})$ replaced by $(\hat{q}, \hat{p})$ if $|Z(q_{(2)}, p_{(2)})|$ is small so that we can treat the second center as a test particle ignoring its backreaction on the geometry [28].

2.4 S-T-U model

In this section we shall analyze a class of 2-centered black hole solutions in heterotic string theory on T^6 and propose a mechanism for black hole metamorphosis. Our analysis will proceed in several steps.

- We shall describe a truncation of heterotic string theory on T^6 which can be mapped to an $\mathcal{N} = 2$ supergravity theory, known as the S-T-U model.
- We then describe the S-T-U model and the maps between the fields in the two descriptions.
- We then consider a two centered configuration in this theory with one center carrying

charge $(0, P)$ with $P^2 = -2$, and take a limit where the other center carrying charge $(Q, 0)$ becomes light and can be regarded as a probe. The technique reviewed in §2.3 then enables us to easily construct the background field associated with the heavy center and find the equilibrium position of the light center.

- We then analyze the solution carefully to find the region of the moduli space where it exists. Although naively it exists in the region R_1 to the left of the line L_1 in Fig. 2.1, we suggest a mechanism by which the region of existence gets truncated to R_1' displayed in Fig. 2.2.

- This analysis also allows us to determine the precise location of the line L in Fig. 2.2.

2.4.1 Truncation of heterotic string theory on T^6

We shall now describe the truncation of heterotic string theory on T^6 that can be mapped to an $\mathcal{N} = 2$ supergravity theory. For this we take T^6 in the form of the product $T^4 \times T^2$ and ignore all excitations of the components of the metric and 2-form fields with one or both legs along T^4 and also all excitations of the ten dimensional gauge fields. This truncated theory will have only four gauge fields corresponding to 4-μ and 5-μ components of the metric and the 2-form gauge fields, with x^4 and x^5 denoting the coordinates along T^2 and x^μ with $0 \le \mu \le 3$ denoting the coordinates along the 3+1 dimensional non-compact space-time. The other relevant fields are the canonical metric $g_{\mu\nu}$, the axion dilaton modulus $S = S_1 + iS_2$, the complex structure modulus $U = U_1 + iU_2$ of T^2 and the complexified Kahler modulus $T = T_1 + iT_2$ of T^2. The four components (Q_1, Q_2, Q_3, Q_4) of the electric charge vector Q correspond respectively to the number of units of momentum along x^5 and x^4 respectively and winding numbers along x^5 and x^4 respectively. On the other hand the components (P_1, P_2, P_3, P_4) of the magnetic charge P denote respectively the heterotic five-brane winding numbers along $T^4 \times x^4$-circle and $T^4 \times x^5$-circle respectively and Kaluza-Klein monopole charges associated with x^5 and x^4 directions respectively.

The bilinears Q^2, P^2, $Q \cdot P$ are given by

$$Q^2 = 2(Q_1Q_3+Q_2Q_4), \quad P^2 = 2(P_1P_3+P_2P_4), \quad Q\cdot P = Q_1P_3+Q_2P_4+Q_3P_1+Q_4P_2\,. \tag{2.4.1}$$

Finally the entropy of a black hole carrying (electric, magnetic) charges (Q, P) is given by

$$S_{BH} = \pi\sqrt{\Sigma}, \quad \Sigma = Q^2P^2 - (Q\cdot P)^2\,. \tag{2.4.2}$$

2.4.2 $\mathcal{N} = 2$ description

This truncated theory can be mapped to an $\mathcal{N} = 2$ supergravity theory coupled to three vector multiplets, with prepotential

$$F = -\frac{X^1X^2X^3}{X^0}\,. \tag{2.4.3}$$

In the notations of Appendix A, the scalar fields S, T and U introduced in §2.4.1 are given by

$$\begin{aligned} S &= \frac{X^1}{X^0} = \Psi + i\,e^{-\Phi^{(4)}}, \\ T &= \frac{X^2}{X^0} = \hat{B}_{45} + i\sqrt{\det\hat{G}} = i\,R\tilde{R}, \\ U &= \frac{X^3}{X^0} = \frac{(\hat{G}_{45} + i\,\hat{G}^{44})}{\sqrt{\det\hat{G}}} = i\,\frac{\tilde{R}}{R}, \end{aligned} \tag{2.4.4}$$

where the expressions involving the $\tilde{R}$ & R (respective radii corresponding to the coordinates 4 & 5) are true only when T & U are completely imaginary. The relations between the gauge fields in the two descriptions can be described by giving the relations between the charges $\{Q_i, P_i\}$ given above with the charges $\{q_I, p^I\}$ in the $\mathcal{N} = 2$ supergravity description. This is as follows [8] (see *e.g.* [31] for a review [9])

$$Q \equiv (Q_1, Q_2, Q_3, Q_4) = (q_0, q_3, -p^1, q_2), \quad P \equiv (P_1, P_2, P_3, P_4) = (q_1, p^2, p^0, p^3)\,. \tag{2.4.5}$$

Eq.(2.4.2) now gives

$$\Sigma(\{q_I\},\{p^I\}) = \left[4(q_2 q_3 - q_0 p^1)(p^0 q_1 + p^2 p^3) - (q_0 p^0 - q_1 p^1 + q_2 p^2 + q_3 p^3)^2\right]^{1/2}\,. \tag{2.4.6}$$

We shall denote the asymptotic values of the various moduli fields as

$$S|_\infty = \zeta \equiv \zeta_1 + i\zeta_2, \quad T|_\infty = \rho \equiv \rho_1 + i\rho_2, \quad U|_\infty = \sigma \equiv \sigma_1 + i\sigma_2\,. \tag{2.4.7}$$

As we shall see in (2.4.21), ζ is related to the modulus τ of §2.2 via the relation $\zeta = -\bar{\tau}$. We also define

$$x^0 \equiv X^0_\infty\,. \tag{2.4.8}$$

From (2.3.1), (2.4.3) and (2.4.7) it follows that

$$F_0 = \frac{X^1 X^2 X^3}{(X^0)^2}, \quad F_1 = -\frac{X^2 X^3}{X^0}, \quad F_2 = -\frac{X^1 X^3}{X^0}, \quad F_3 = -\frac{X^1 X^2}{X^0}\,. \tag{2.4.9}$$

$$\begin{aligned} e^{-K} &= i\,(\bar{X}^1 F_1 - X^1 \bar{F}_1 + \bar{X}^2 F_2 - X^2 \bar{F}_2 + \bar{X}^3 F_3 - X^3 \bar{F}_3 + \bar{X}^0 F_0 - X^0 \bar{F}_0) \\ &= i\, X^0 \bar{X}^0 (-\bar{S}TU + S\bar{T}\bar{U} - \bar{T}SU + T\bar{S}\bar{U} - \bar{U}ST + U\bar{S}\bar{T} \\ &\qquad + STU - \bar{S}\bar{T}\bar{U}) \end{aligned}$$

[8] This differs from the identification made in [30] by the transformation $(Q_1, Q_2, Q_3, Q_4) \to (Q_4, Q_3, Q_2, Q_1)$, $(P_1, P_2, P_3, P_4) \to (P_4, P_3, P_2, P_1)$.

[9] Together with the prepotential (2.4.3), charge redefinitions (2.4.5) and gauge choice $w = 1$, the entropy functional eq. (3.2.5) and entropy eq. (3.2.13) of $\mathcal{N} = 2$ supergravity given in [30] equals respectively eq. (3.1.11) and eq. (3.1.23) of $\mathcal{N} = 4$ supergravity.

$$= \quad 8X^0\bar{X}^0 S_2 T_2 U_2\,. \tag{2.4.10}$$

and from (2.4.8) it follows that

$$e^{-K}\big|_\infty = 8\,x^0\bar{x}^0\zeta_2\sigma_2\rho_2\,. \tag{2.4.11}$$

2.4.3 The two centered solution

In the asymptotic background described above we construct a two centered solution with the first center carrying charge $(0, P)$ and the second center carrying charge $(Q, 0)$, with

$$Q = (a, b, c, d), \quad P = (0, -1, 0, 1)\,. \tag{2.4.12}$$

This gives, from (2.4.1)

$$Q^2 = 2ac + 2bd, \quad P^2 = -2, \quad u \equiv Q \cdot P = b - d\,. \tag{2.4.13}$$

We shall for definiteness take $(b-d) > 0$ so that $u > 0$. Since $P^2 = -2$ this configuration should display the phenomenon of black hole bound state metamorphosis. In particular there must exist a line L in the $\tau = -\bar{\zeta}$ plane such that the bound state ceases to exist to the left of this line. Our goal will be to understand the physical origin of this line L.

Now using (2.4.5) we see that in the language of $\mathcal{N} = 2$ supergravity the two centers carry charges $(q_{(1)}, p_{(1)})$ and $(q_{(2)}, p_{(2)})$ where

$$p_{(1)} = (0, 0, -1, 1), \quad q_{(1)} = (0, 0, 0, 0), \qquad p_{(2)} = (0, -c, 0, 0), \quad q_{(2)} = (a, 0, d, b)\,. \tag{2.4.14}$$

We define

$$Z_1 \equiv Z(q_{(1)}, p_{(1)})|_\infty = \left[e^{K/2}(F_2 - F_3)\right]_\infty = \sqrt{\frac{x^0}{\bar{x}^0}}\frac{1}{\sqrt{8\zeta_2\rho_2\sigma_2}}\,\zeta(\rho - \sigma)\,,$$

$$
\begin{aligned}
Z_2 \equiv Z(q_{(2)}, p_{(2)})|_\infty &= \left[e^{K/2}(aX^0 + dX^2 + bX^3 + cF_1)\right]_\infty \\
&= \sqrt{\frac{x^0}{\bar{x}^0}} \frac{1}{\sqrt{8\zeta_2\rho_2\sigma_2}} \left(a + d\rho + b\sigma - c\rho\sigma\right) .
\end{aligned} \tag{2.4.15}
$$

To simplify the analysis we shall work in the limit where ζ_2 is large. In this limit $|Z_2|$ given in (2.4.15) is small showing that the corresponding state is light. Hence we can ignore its effect on the background field and treat this center as a probe. In this limit the background geometry approaches that of a single centered black hole with charge $(q_{(1)}, p_{(1)})$ placed at $\vec{r}_1$, and α_∞ defined in (2.3.4) and the functions H^I and H_I introduced in (2.3.3) take the form

$$
e^{i\alpha_\infty} = \frac{Z_1 + Z_2}{|Z_1 + Z_2|} \simeq \frac{Z_1}{|Z_1|} = \sqrt{\frac{x^0}{\bar{x}^0}} \frac{\zeta}{|\zeta|} \frac{\rho - \sigma}{|\rho - \sigma|} , \tag{2.4.16}
$$

$$
(H^0, H^1, H^2, H^3) \simeq \frac{1}{|\vec{r} - \vec{r}_1|}(0, 0, -1, 1) - \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \mathrm{Im} \left\{ \frac{|\zeta|}{\zeta} \frac{|\rho - \sigma|}{\rho - \sigma} (1, \zeta, \rho, \sigma) \right\} . \tag{2.4.17}
$$

$$
(H_0, H_1, H_2, H_3) \simeq \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}} \mathrm{Im} \left\{ \frac{|\zeta|}{\zeta} \frac{|\sigma - \rho|}{\rho - \sigma} (-\zeta\rho\sigma, \rho\sigma, \zeta\sigma, \zeta\rho) \right\} . \tag{2.4.18}
$$

From this we can construct the solution for the metric, scalars and gauge fields using the prescription reviewed in §2.3. The separation between the two centers is given, according to (2.3.14), by

$$
|\vec{r}_1 - \vec{r}_2| = \frac{b - d}{2} \frac{\sqrt{8\zeta_2\sigma_2\rho_2}}{|\zeta||\sigma - \rho|} \frac{1}{\mathrm{Im}\left[(a + d\rho + b\sigma - c\rho\sigma)/(\zeta(\rho - \sigma))\right]} . \tag{2.4.19}
$$

Before we go on we must mention two subtle points in the relation between the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theory that will be important for our analysis. According to (2.4.15), the total mass of the system is given by

$$
|Z_1 + Z_2| = \frac{1}{\sqrt{8\zeta_2\rho_2\sigma_2}} \sqrt{|A|^2 + |B|^2|\zeta|^2 + 2\,\zeta_1 \mathrm{Re}\,(AB^*) + 2\,\zeta_2\, \mathrm{Im}\,(AB^*)},
$$

$$A \equiv a + d\rho + b\sigma - c\rho\sigma, \quad B \equiv (\rho - \sigma)\,. \tag{2.4.20}$$

Now consider a state carrying total charge (P, P). The BPS mass of this state will be given by setting $a = c = 0$ and $b = -1, d = 1$ in (2.4.20) and its dependence on the axion dilaton modulus ζ will be proportional to $\sqrt{|1+\zeta|^2}/\sqrt{\zeta_2}$. On the other hand in the convention of [23, 26] which we used in presenting the results in §2.2, the dependence of the BPS mass of a particle of charge (P, P) on the axion dilaton modulus is proportional to $\sqrt{|1-\tau|^2}/\sqrt{\tau_2}$. This shows that ζ and τ are related by

$$\zeta = -\bar{\tau}\,. \tag{2.4.21}$$

To discuss the second subtlety, let us return to the general formula (2.4.20). The BPS mass formula in the $\mathcal{N} = 4$ supersymmetric theories (derived in [32, 33] and used *e.g.* in [23] for the analysis of the walls of marginal stability) is given by the same formula as (2.4.20) (after the identification (2.4.21)) except that the coefficient of $\zeta_2 = \tau_2$ under the square root is given by $2|\mathrm{Im}\,(AB^*)|$. Thus the two formulæ agree when $\mathrm{Im}\,(AB^*) > 0$, i.e.

$$(\sigma_2 - \rho_2)(a + d\rho_1 + b\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2) + (\rho_1 - \sigma_1)(d\rho_2 + b\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)) > 0\,. \tag{2.4.22}$$

From now on we shall assume that this condition holds.

2.4.4 The region of existence of the solution

From (2.4.19) we can identify the wall of marginal stability as the curve in the ζ plane on which the right hand side of (2.4.19) diverges. This implies

$$Im\left[(a + d\rho + b\sigma - c\rho\sigma)\bar{\zeta}(\bar{\rho} - \bar{\sigma})\right] = 0 \tag{2.4.23}$$

and gives

$$\frac{\zeta_1}{\zeta_2} = \frac{N}{D}, \quad i.e. \quad \frac{\tau_1}{\tau_2} = -\frac{N}{D}, \tag{2.4.24}$$

where

$$\begin{aligned} N &= -(\sigma_2 - \rho_2)(d\rho_2 + b\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)) \\ &\quad +(\rho_1 - \sigma_1)(a + d\rho_1 + b\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2), \\ D &= (\sigma_2 - \rho_2)(a + d\rho_1 + b\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2) \\ &\quad +(\rho_1 - \sigma_1)(d\rho_2 + b\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)). \end{aligned} \tag{2.4.25}$$

Eq. (2.4.24) marks the location of the line L_1 in Figs. 2.1 and 2.2. In particular the solution exists when the right hand side of (2.4.19) is positive, i.e. for

$$\begin{aligned} \zeta_1 &> \frac{N}{D}\zeta_2 \quad \text{for } (b-d)D > 0, \\ &< \frac{N}{D}\zeta_2 \quad \text{for } (b-d)D < 0. \end{aligned} \tag{2.4.26}$$

Since according to (2.4.22) we have $D > 0$, and we have assumed that $b - d > 0$, this condition translates to

$$\zeta_1 > \frac{N}{D}\zeta_2 \quad \iota.e. \quad \tau_1 < -\frac{N}{D}\tau_2. \tag{2.4.27}$$

Naively one may expect that (2.4.27) is the only condition on τ for the existence of the solution, since as long as (2.4.27) is satisfied, $|\vec{r}_1 - \vec{r}_2|$ given in (2.4.19) remains positive. However upon closer examination one discovers a peculiar property of the solution that can be attributed to the special charge vector carried by the first center. If we take a test particle of charge $(P, 0)$ with $P = (0, -1, 0, 1)$ as in (2.4.12), it maps to $q = (0, 0, 1, -1)$, $p = (0, 0, 0, 0)$ in the $\mathcal{N} = 2$ supergravity variables, and its mass at a point $\vec{r}$ is given by

$$\frac{1}{8\sqrt{S_2(\vec{r})T_2(\vec{r})U_2(\vec{r})}}|T(\vec{r}) - U(\vec{r})|. \tag{2.4.28}$$

Thus it vanishes when $T(\vec{r}) = U(\vec{r})$. Using (2.4.6) and eqs.(2.3.6)-(2.3.8)

$$\begin{aligned}
\chi^2 &= -\frac{1}{2}\frac{1}{\Sigma}\left[4q_3(p^0q_1 + p^2p^3) - 2(q_0p^0 - q_1p^1 + q_2p^2 + q_3p^3)p^2\right] , \\
\chi^3 &= -\frac{1}{2}\frac{1}{\Sigma}\left[4q_2(p^0q_1 + p^2p^3) - 2(q_0p^0 - q_1p^1 + q_2p^2 + q_3p^3)p^3\right] , \quad (2.4.29)
\end{aligned}$$

we see that

$$T(\vec{r}) = U(\vec{r}) \Rightarrow X^2(\vec{r}) = X^3(\vec{r}) \Rightarrow g^2(\vec{r}) = g^3(\vec{r}) \Rightarrow \quad (2.4.30)$$

$$-\tfrac{1}{2}\tfrac{1}{\Sigma}\left[4H_3(H^0H_1 + H^2H^3) - 2(H_0H^0 - H_1H^1 + H_2H^2 + H_3H^3)H^2\right] + i\, H^2$$

$$= -\tfrac{1}{2}\tfrac{1}{\Sigma}\left[4H_2(H^0H_1 + H^2H^3) - 2(H_0H^0 - H_1H^1 + H_2H^2 + H_3H^3)H^3\right] + i\, H^3 ,$$

requires $H_2 = H_3$ and $H^2 = H^3$. Now from (2.4.18) we see that the first condition is satisfied automatically, while eq.(2.4.17) tells us that we have $H^2 = H^3$ when

$$|\vec{r} - \vec{r}_1| = r_e, \quad r_e \equiv \sqrt{8\zeta_2\sigma_2\rho_2}\,\frac{|\zeta|}{\zeta_2|\rho - \sigma|} . \quad (2.4.31)$$

This describes a spherical shell of radius r_e around $\vec{r}_1$ on which an electrically charged test particle carrying charge $(P, 0)$ becomes massless. Physically on this shell the radius of the x^4 direction reaches the self-dual point and hence we have massless non-abelian gauge fields. This in turn shows that at this point the original solution describing the background field produced by the charge $(0, P)$ breaks down and we should not trust the solution for values of $\vec{r}$ for which $|\vec{r} - \vec{r}_1|$ is less than r_e defined in (2.4.31). This has been named the enhancon mechanism in [34]. Indeed, if we ignore this effect and continue to trust the solution for $|\vec{r} - \vec{r}_1| < r_e$, then at some point $\Sigma(\{H_I\}, \{H^I\})$ computed from (2.4.6), (2.4.17) and (2.4.18) vanishes and the solution becomes singular [34]. We shall call r_e the enhancon radius. Thus a two centered solution, obtained by placing in the above background a test charge $(Q, 0)$ at $\vec{r}_2$ is sensible only when we have

$$|\vec{r}_1 - \vec{r}_2| > r_e \,. \tag{2.4.32}$$

Using (2.4.19), (2.4.31) and the positivity of D and $b-d$, this translates to

$$\zeta_1 < \frac{b-d}{2D}\zeta_2|\rho-\sigma|^2 + \frac{N}{D}\zeta_2 \quad \iota.e. \quad \tau_1 > -\frac{b-d}{2D}\tau_2|\rho-\sigma|^2 - \frac{N}{D}\tau_2 \,. \tag{2.4.33}$$

As we shall see in §2.5, the correct description of the solution involves replacing it by a gravitationally dressed smooth BPS dyon obtained by boosting the Harvey-Liu monopole solution [35, 36] in an internal compact direction. As a result the solution begins to differ from that given in §2.4.3 even for $|\vec{r} - \vec{r}_1| > r_e$. However for now we shall take the above bound on ζ_1 seriously and examine its consequences. In this case (2.4.33) gives us the location of the left boundary L of the region R_1' in Fig. 2.2, with the right boundary L_1 of R_1' being given by the wall of marginal stability (2.4.27). In §2.5 we shall see that this in fact is the exact result for the allowed range of ζ_1 in the large ζ_2 limit.

2.4.5 The second two centered solution

Next consider the two centered configuration with charges $(-uP, P)$ and $(Q + uP, 0)$ where $u = Q \cdot P = (b - d)$. In the language of $\mathcal{N} = 2$ supergravity the two centers carry charges $(q_{(1)}, p_{(1)})$ and $(q_{(2)}, p_{(2)})$ where

$$p_{(1)} = (0, 0, -1, 1), \quad q_{(1)} = (0, 0, -u, u), \quad p_{(2)} = (0, -c, 0, 0), \quad q_{(2)} = (a, 0, d+u, b-u) \,. \tag{2.4.34}$$

Again one can argue that in the limit of large ζ_2 the second center carrying only electric charge $(Q + uP, 0)$ is light and hence can be treated as a test particle. Furthermore, for the first center the contribution to the background field from the electric component proportional to uP will be small and hence can be dropped. Thus the problem effectively reduces to studying the test charge $(Q + uP, 0)$ in the background produced by the charge $(0, P)$. Since according to (2.4.12), (2.4.13), $Q + uP$ differs from Q just by the exchange of the quantum numbers b and d, we can derive the various results for this system simply by exchanging b and d in the earlier results. In particular for this system the separation between the two centers is given by

$$|\vec{r}_1 - \vec{r}_2| = \frac{d-b}{2} \frac{\sqrt{8\zeta_2\sigma_2\rho_2}}{|\zeta||\sigma - \rho|} \frac{1}{\mathrm{Im}\left[(a + b\rho + d\sigma - c\rho\sigma)/(\zeta(\rho - \sigma))\right]} . \qquad (2.4.35)$$

The wall of marginal stability, where $|\vec{r}_1 - \vec{r}_2|$ diverges, is at

$$\zeta_1 = \frac{N'}{D'}\zeta_2 \quad \iota.e. \quad \tau_1 = -\frac{N'}{D'}\tau_2 , \qquad (2.4.36)$$

where

$$\begin{aligned} N' &= -(\sigma_2 - \rho_2)(b\rho_2 + d\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)) \\ &\quad +(\rho_1 - \sigma_1)(a + b\rho_1 + d\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2) , \\ D' &= (\sigma_2 - \rho_2)(a + b\rho_1 + d\sigma_1 - c\rho_1\sigma_1 + c\rho_2\sigma_2) = D \\ &\quad +(\rho_1 - \sigma_1)(b\rho_2 + d\sigma_2 - c(\rho_2\sigma_1 + \rho_1\sigma_2)) . \end{aligned} \qquad (2.4.37)$$

Eq. (2.4.36) marks the location of the line L_2 in Figs.2.1 and 2.2. The solution exists for

$$\zeta_1 < \frac{N'}{D}\zeta_2 \quad \iota.e. \quad \tau_1 > -\frac{N'}{D}\tau_2 . \qquad (2.4.38)$$

The enhancon radius remains at the same place as before. The condition that the location of the second center lies outside the enhancon radius can be translated to

$$\zeta_1 > \frac{d-b}{2D}\zeta_2|\rho - \sigma|^2 + \frac{N'}{D}\zeta_2 \quad \iota.e. \quad \tau_1 < \frac{b-d}{2D}\tau_2|\rho - \sigma|^2 - \frac{N'}{D}\tau_2 . \qquad (2.4.39)$$

As before we shall take this to be our estimate for the right boundary of the region R_2' in Fig. 2.2, with the left boundary L_2 of R_2' being given by the constraint (2.4.38).

We now note that

$$\frac{b-d}{2D}|\rho-\sigma|^2 - \frac{N'}{D} = \frac{d-b}{2D}|\rho-\sigma|^2 - \frac{N}{D}\,, \tag{2.4.40}$$

i.e. the right hand sides of (2.4.33) and (2.4.39) match. This in turn shows that the right boundary of R_2' coincides with the left boundary L of R_1', and hence in any region of the moduli space between the two walls of marginal stability L_1 and L_2 in Fig. 2.2, one and only one of the two configurations exists. This is precisely what is required for the black hole metamorphosis hypothesis to hold.

2.4.6 Special case of diagonal T^6

For later use we shall now write down the explicit solutions in the special case

$$\sigma_1 = \rho_1 = 0\,, \tag{2.4.41}$$

corresponding to setting the off-diagonal components of the metric and the 2-form field along T^2 to zero at infinity. Furthermore we shall take the location of the first center at the origin so that

$$\vec{r}_1 = 0, \quad |\vec{r}-\vec{r}_1| = r\,. \tag{2.4.42}$$

Then we can express (2.4.17), (2.4.18) as

$$\begin{aligned}
H^0 &= \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}}\,\mathrm{sign}(\rho_2-\sigma_2)\frac{\zeta_1}{|\zeta|}\,,\\
H^1 &= \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}}\,\mathrm{sign}(\rho_2-\sigma_2)|\zeta|\,,\\
H^2 &= -\frac{1}{r} + \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}}\,\mathrm{sign}(\rho_2-\sigma_2)\frac{\rho_2\zeta_2}{|\zeta|}\,,
\end{aligned}$$

$$
\begin{aligned}
H^3 &= \frac{1}{r} + \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}}\,\mathrm{sign}(\rho_2-\sigma_2)\frac{\sigma_2\zeta_2}{|\zeta|}\,, \\
H_0 &= -\frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}}\,\mathrm{sign}(\rho_2-\sigma_2)\rho_2\sigma_2|\zeta|\,, \\
H_1 &= \frac{2}{\sqrt{8\zeta_2\rho_2\sigma_2}}\,\mathrm{sign}(\rho_2-\sigma_2)\frac{\rho_2\sigma_2\zeta_1}{|\zeta|}\,, \\
H_2 &= 0\,, \\
H_3 &= 0\,.
\end{aligned} \tag{2.4.43}
$$

From (2.4.43) we have $H_1H^1 = -H_0H^0$ and using this with (2.4.6), (2.3.6)-(2.3.11) gives,

$$
\begin{aligned}
\Sigma(\{H_I\},\{H^I\}) &= \left[-4H_0H^1(H^0H_1+H^2H^3)-(H_0H^0-H_1H^1)^2\right]^{\frac{1}{2}} \\
&= [-4H_0H^1H^2H^3]^{1/2}\,. \\
S = \frac{X^1}{X^0} = \frac{g^1}{g^0} &= \frac{-\frac{1}{2\Sigma}[-4H^0H_0H^1+4H_0H^0H^1]+i\,H^1}{-\frac{1}{2\Sigma}\,[-4H^1(H^0H_1+H^2H^3)-4H_0(H^0)^2]+i\,H^0} \\
&= \frac{i\,H^1}{-\frac{1}{2\Sigma}[-4H^1H^2H^3]+i\,H^0} \\
&= \frac{2iH_0H^1}{-\Sigma+2iH_0H^0}\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
T &= \frac{X^2}{X^0} = -\frac{2iH_0H^2}{\Sigma}, \qquad U = \frac{X^3}{X^0} = -\frac{2iH_0H^3}{\Sigma}\,, \\
ds^2 &= -\Sigma^{-1}\,dt^2 + \Sigma\,dx^idx^i\,, \\
\mathcal{A}^0_\mu dx^\mu &= -\frac{2H^1H^2H^3}{\Sigma^2}\,dt\,, \qquad \mathcal{A}^2_\mu dx^\mu = -\frac{2H_0H^0H^2}{\Sigma^2}\,dt + \cos\theta d\phi\,, \\
\mathcal{A}^3_\mu dx^\mu &= -\frac{2H_0H^0H^3}{\Sigma^2}\,dt - \cos\theta d\phi, \qquad \mathcal{A}_{1\mu}dx^\mu = \frac{2H_0H^2H^3}{\Sigma^2}\,dt\,.
\end{aligned} \tag{2.4.44}
$$

Note that we have given the expressions for the electric potentials $\mathcal{A}^0_\mu, \mathcal{A}^2_\mu, \mathcal{A}^3_\mu$ and the magnetic potential $\mathcal{A}_{1\mu}$. This contains full information about all the gauge fields. Eq. (2.4.44) shows that T and U remain purely imaginary for all values of $\vec{r}$ and hence the off diagonal components of the metric and the 2-form field along T^2 continue to vanish at all points.

From eqs.(2.3.16) and (2.4.14) we get the Lagrangian of the test particle carrying charge $(q_{(2)}, p_{(2)})$ in this background to be

$$\begin{aligned}
L_t &= -\Sigma(\{H_I\},\{H^I\})^{-1/2}\frac{1}{\sqrt{8S_2(\vec{r})T_2(\vec{r})U_2(\vec{r})}}|a+d\,T(\vec{r})+b\,U(\vec{r})-c\,T(\vec{r})\,U(\vec{r})| \\
&\quad +\frac{1}{2}(c\,\mathcal{A}_{10}+a\,\mathcal{A}_0^0+d\,\mathcal{A}_0^2+b\,\mathcal{A}_0^3) \\
&= -\frac{1}{4H_0}\left\{1+\frac{H^0H_1}{H^2H^3}\right\}^{1/2}\left[\left(a-c\frac{H_0}{H^1}\right)^2-\frac{H_0}{H^1H^2H^3}(dH^2+bH^3)^2\right]^{1/2} \\
&\quad -\frac{1}{4}\left\{-\frac{a}{H_0}+d\frac{H_1}{H_0H^3}+b\frac{H_1}{H_0H^2}+\frac{c}{H^1}\right\}. \qquad (2.4.45)
\end{aligned}$$

The equilibrium separation (2.4.19) between the two centers can be found by extremizing (2.4.45) with respect to r. The relevant parts of (2.4.45) are terms containing H_2 & H_3. Corresponding result for the second system is obtained by exchanging b and d in (2.4.45).

2.5 Replacing the enhancon by the smooth solution

In this section we shall replace the solution in the S-T-U model described in §2.4.6 by a smooth dyon solution and compute the range of values of ζ_1 for which the solution exists. Since the analysis of this section will be somewhat technical let us first summarize the main result. We shall find that the net effect of smoothening the solution is to replace in the expressions for Σ, S_2, T, U, $\mathcal{A}_0^0$, $\mathcal{A}_0^2$, $\mathcal{A}_0^3$ and $\mathcal{A}_{10}$ given in (2.4.44), the variable r by $\hat{r}$ where

$$\frac{1}{\hat{r}}=\frac{1}{r}-\kappa\coth(\kappa r)+\kappa, \qquad \kappa\equiv\sqrt{\frac{\zeta_2}{8\rho_2\sigma_2}\frac{|\rho_2-\sigma_2|}{|\zeta|}}=\frac{1}{r_e}. \qquad (2.5.1)$$

This does not mean that the new solution is related to the old one by a coordinate transformation since for example the dx^idx^i term in the expression for the metric is still given by

$dr^2 + r^2 d\Omega_2^2$ with $d\Omega_2$ denoting the line element on a unit 2-sphere. Nevertheless it shows that the potential for the test charge in this new background is given by (2.4.45) with r replaced by $\hat{r}$ everywhere. Thus for given values of the asymptotic moduli the equilibrium position of the test charge $(Q, 0)$ or $(Q + uP, 0)$ is given by replacing $|\vec{r}_1 - \vec{r}_2| = |\vec{r}_2|$ by $\hat{r}_2$ in the S-T-U model results (2.4.19) and (2.4.35) respectively, where $\hat{r}_2$ is the value of $\hat{r}$ defined in (2.5.1) for $r = |\vec{r}_2|$.[10] Now since according to (2.5.1) $r = 0$ corresponds to $\hat{r} = r_e$ and $r = \infty$ corresponds to $\hat{r} = \infty$ we see that requiring $0 < |\vec{r}_2| < \infty$ corresponds to $r_e < \hat{r}_2 < \infty$ [11]. This according to the analysis of §2.4 (with r replaced by $\hat{r}$) constraints τ to lie inside the region R_1' of Fig. 2.2 for the first configuration and the region R_2' of Fig. 2.2 for the second configuration. Thus we conclude that the ranges of τ_1 for which the solutions exists remain the same as what we derived in §2.4. However the interpretation of what happens at the left boundary of R_1' and the right boundary of R_2' is slightly different. At the left boundary of R_1', when τ_1 saturates the bound (2.4.33), the second center of the first configuration reaches the center of the smooth dyonic solution. On the other hand at the right boundary of R_2', when τ_1 saturates the bound (2.4.39), the second center of the second configuration reaches the center of the smooth dyonic solution.

We shall now describe how these results arise.

[10] We are again setting $\vec{r}_1 = 0$ i.e. taking the location of the first center as the origin of the coordinate system.

[11] From afar the smooth dyon solution behaves exactly like the black hole solution. Only when we move close enough we realize that the black hole center is plagued by singularities due to the enhancon mechanism whereas the dyon center is throughout smooth.

2.5.1 Harvey-Liu monopole and dyon solutions in the ten dimensional description

We shall consider a truncation of the effective action of ten dimensional heterotic string theory where we keep only a single $SU(2)$ gauge field $\mathcal{V}_\mu^{(a)}$ $(1 \le a \le 3)$ out of $SO(32)$ or $E_8 \times E_8$. This action is given by

$$\begin{aligned} S &= \frac{2\pi}{(2\pi\sqrt{\alpha'})^8} \int d^{10}x \sqrt{-\det G}\, e^{-2\Phi} \Big[R + 4 G^{MN} \partial_M \Phi \partial_N \Phi \\ &\qquad - \frac{1}{12} G^{MM'} G^{NN'} G^{RR'} H_{MNR} H_{M'N'R'} - \frac{\alpha'}{8} \mathcal{W}^{(a)}_{MN} \mathcal{W}^{(a)MN} \Big] , \\ \mathcal{W}^{(a)}_{MN} &\equiv \partial_M \mathcal{V}^{(a)}_N - \partial_N \mathcal{V}^{(a)}_M + \epsilon^{abc} \mathcal{V}^{(b)}_M \mathcal{V}^{(c)}_N , \end{aligned} \tag{2.5.2}$$

$$dH = -\frac{\alpha'}{4} \mathcal{W}^{(a)} \wedge \mathcal{W}^{(a)}, \quad H \equiv \frac{1}{3!} H_{MNP} dx^M \wedge dx^N \wedge dx^P, \quad \mathcal{W}^{(a)} \equiv \frac{1}{2!} \mathcal{W}^{(a)}_{MN} dx^M \wedge dx^N . \tag{2.5.3}$$

Here x^M for $0 \le M \le 9$ are the coordinates labelling the ten dimensional space-time, G_{MN} is the string metric, H is the 3-form field strength and Φ is the dilaton field. We now compactify the theory on T^6 labelled by $x^4, \cdots x^9$ with period $2\pi\sqrt{\alpha'}$ and non-compact coordinates labelled by x^0, x^1, x^2, x^3. In this theory we consider the Harvey-Liu monopole solution [35, 36][12]

$$\begin{aligned} \mathcal{V}^{(a)}_i &= \epsilon_{iak} \frac{x^k}{r^2} (K(C_1 r) - 1), \quad \mathcal{V}^{(a)}_4 = C_2 \frac{x^a}{r^2} H(C_1 r), \quad 1 \le i, k, a \le 3, \quad r \equiv \sqrt{x^k x^k} , \\ &\quad H(x) \equiv x \coth x - 1, \quad K(x) = x / \sinh x , \\ e^{2\Phi} &= C_3^2 + \frac{\alpha'}{4} (C_1^2 - r^{-2} H(C_1 r)^2) , \end{aligned}$$

[12] Strictly speaking, if we take the circles labelled by $x^6, \cdots x^9$ to have self-dual radius, as is the case for the metric given in (2.5.4), we shall get additional massless non-abelian gauge fields. We can avoid this situation by taking the metric along the $x^6, \cdots x^9$ direction to be $K_{mn} dx^m dx^n$ for some constant symmetric matrix K with $\det K = 1$. This does not affect any of the subsequent analysis. Similarly we could also break the rest of the ten dimensional gauge group (SO(32) or $E_8 \times E_8$) by turning on Wilson lines for these gauge fields along the 6-7-8-9 directions without changing any of the subsequent analysis.

$$
\begin{aligned}
ds^2 &= -(dx^0)^2 + e^{2\Phi}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 + C_2^2 (dx^4)^2\right) + C_4^2 (dx^5)^2 \\
&\quad + \sum_{m=6}^{9} dx^m dx^m \,, \\
H_{4ij} &= -2\, C_2\, e^{2\Phi}\, \epsilon_{ijk}\partial_k \Phi \quad 1 \le i,j,k \le 3 \,,
\end{aligned} \tag{2.5.4}
$$

where C_1, C_2, C_3 and C_4 are arbitrary constants, and ϵ_{ijk} is the totally anti-symmetric symbol with $\epsilon_{123} = 1$. Since all the fields in (2.5.4) are invariant under changes in signs of C_1, C_3 and C_4, we can choose

$$
C_1, C_4, C_2 C_3 > 0 \,, \tag{2.5.5}
$$

without any loss of generality. Note that the solution described in (2.5.4) lies outside the truncated theory described in §2.4.1 since we have non-trivial background values of the ten dimensional gauge fields. However we shall see that (the dyonic generalization of) this solution can be mapped to a solution inside the truncated theory by a duality rotation [13].

Physically (2.5.4) represents a gravitationally dressed BPS monopole solution of the $SU(2)$ gauge theory. We can construct from this a dyon solution by making the replacement (see *e.g.* [37]) [14]

$$
x^0 \to \cosh\gamma\, x^0 + C_2 C_3 \sinh\gamma\, x^4, \quad x^4 \to C_2^{-1} C_3^{-1} \sinh\gamma\, x^0 + \cosh\gamma\, x^4 \,, \tag{2.5.6}
$$

[13] I.e. we need to map external gauge fields to gauge fields arising from the dimensional reduction of the metric & the antisymmetric two form along directions 4 & 5.

[14] Note that (2.5.6) is nothing but a Lorentz boost along the directions x^0 and $\tilde{x}^4 = C_2 C_3 x^4$ with rapidity factor γ. It is a new solution because the new x^4 again has a period of $2\pi\sqrt{\alpha'}$ instead of $2\pi\sqrt{\alpha'}\cosh\gamma$. Most importantly it generates electric fields by boosting magnetic fields.

and taking the new x^4 coordinate defined this way as being periodically identified with period $2\pi\sqrt{\alpha'}$. This gives a solution:

$$
\begin{aligned}
\mathcal{V}_i^{(a)} &= \epsilon_{iak}\frac{x^k}{r^2}(K(C_1 r)-1), \quad \mathcal{V}_4^{(a)} = C_2\,\cosh\gamma\,\frac{x^a}{r^2}H(C_1 r)\,,\\
\mathcal{V}_0^{(a)} &= C_3^{-1}\,\sinh\gamma\,\frac{x^a}{r^2}H(C_1 r)\,,\\
e^{2\Phi} &= C_3^2+\frac{\alpha'}{4}(C_1^2-r^{-2}H(C_1 r)^2)\,,\\
ds^2 &= -(dx^0)^2+e^{2\Phi}\left((dx^1)^2+(dx^2)^2+(dx^3)^2\right)+C_2^2C_3^2(dx^4)^2+C_4^2(dx^5)^2\\
&\quad +\sum_{m=6}^{9}dx^m dx^m+\left(e^{2\Phi}C_3^{-2}-1\right)(\sinh\gamma\,dx^0+C_2C_3\,\cosh\gamma\,dx^4)^2\,,\\
H_{4ij} &= -2\,C_2\,\cosh\gamma\,e^{2\Phi}\,\epsilon_{ijk}\partial_k\Phi\,,\\
H_{0ij} &= -2\,C_3^{-1}\,\sinh\gamma\,e^{2\Phi}\,\epsilon_{ijk}\partial_k\Phi, \quad 1\le i,j,k,a\le 3\,.
\end{aligned}
\tag{2.5.7}
$$

The solutions given above are in the hedgehog gauge. For comparison with the solution in the S-T-U model it will be more appropriate to express the solution in the string gauge (see *e.g.* [36]). We can comb the the hedgehog gauge by a singular gauge transformation U such that

$$
U=\frac{1}{\sqrt{2}}\left(\sqrt{1+n^3}+i\,\frac{n^2\sigma^1-n^1\sigma^2}{\sqrt{1+n^3}}\right)\,, \tag{2.5.8}
$$

where $\vec{n}$ is the normal vector in $\mathbb{R}^3$ and $\sigma^{1,2,3}$ are the Pauli matrices. U is singular at $n^3=-1$ and facilitates the use of

$$
U^\dagger(n^a\sigma^a)U=\sigma^3\,. \tag{2.5.9}
$$

In this gauge the solution takes the form

$$
\begin{aligned}
\mathcal{V}_i^{(3)}dx^i &\simeq \cos\theta d\phi,\\
\mathcal{V}_4^{(3)} &= C_2\,\cosh\gamma\,\frac{1}{r}H(C_1 r)\,,\\
\mathcal{V}_0^{(3)} &= C_3^{-1}\,\sinh\gamma\,\frac{1}{r}H(C_1 r)
\end{aligned}
$$

$$
\begin{aligned}
e^{2\Phi} &= C_3^2 + \frac{\alpha'}{4}(C_1^2 - r^{-2}H(C_1 r)^2) \\
ds^2 &= -(dx^0)^2 + e^{2\Phi}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right) + C_2^2 C_3^2 (dx^4)^2 + C_4^2 (dx^5)^2 \\
&\quad + \sum_{m=6}^{9} dx^i dx^i + \left(e^{2\Phi} C_3^{-2} - 1\right)(\sinh\gamma\, dx^0 + C_2 C_3 \,\cosh\gamma\, dx^4)^2 \\
H_{4ij} &= -2\, C_2\, \cosh\gamma\, e^{2\Phi}\, \epsilon_{ijk}\partial_k \Phi, \\
H_{0ij} &= -2\, C_3^{-1}\, \sinh\gamma\, e^{2\Phi}\, \epsilon_{ijk}\partial_k \Phi, \quad 1 \le i, j, k \le 3\,.
\end{aligned}
\tag{2.5.10}
$$

The $\simeq$ in the first equation describes equality up to terms of order $e^{-C_1 r}$ and also additive constants. From now on we shall work in the $\alpha' = 16$ unit. For reason that will become clear later, we shall choose the constants C_i's and γ such that

$$
G_{44} + 4(\mathcal{V}_4^{(3)})^2 = C_2^2 C_3^2 + 4 C_1^2 C_2^2 \cosh^2\gamma = 1\,. \tag{2.5.11}
$$

2.5.2 Smooth dyon solution in the four dimensional description

We now translate the above solution into a field configuration in an effective four dimensional field theory. For this we dimensionally reduce the theory to four dimensions, keeping a single $U(1)$ gauge field $\mathcal{V}_M^{(3)}$ in ten dimensions, and setting the components of various fields along T^4, labelled by the coordinates $x^6, \cdots x^9$, to their background values given in (2.5.10), and setting $\alpha' = 16$. This leads to an action whose bosonic part is given by:

$$
\begin{aligned}
S &= \frac{1}{32\pi}\int d^4x\, \sqrt{-\det g}\,\Big[R - \frac{1}{2\,S_2^2} g^{\mu\nu}\partial_\mu S \partial_\nu \bar{S} - S_2 F^{(a)}_{\mu\nu}(LML)_{ab}F^{(b)\mu\nu} \\
&\quad + S_1 F^{(a)}_{\mu\nu} L_{ab} \widetilde{F}^{(b)\mu\nu} + \frac{1}{8} g^{\mu\nu}\, Tr(\partial_\mu ML \partial_\nu ML)\Big]\,.
\end{aligned}
\tag{2.5.12}
$$

Here $S = S_1 + iS_2$ is a complex scalar field representing the heterotic axion - dilaton system, $F^{(a)}_{\mu\nu} \equiv \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu$ for $1 \le a \le 5$ are the gauge field strengths associated with five $U(1)$ gauge fields $A^{(a)}_\mu$, $\widetilde{F}_{\mu\nu}$ denotes the dual field strength of $F_{\mu\nu}$, L is the 5×5

matrix

$$L = \begin{pmatrix} 0 & I_2 & \\ I_2 & 0 & \\ & & -1 \end{pmatrix}, \tag{2.5.13}$$

with I_n denoting $n \times n$ identity matrix, and M is a matrix valued scalar field, satisfying

$$MLM^T = L, \qquad M^T = M\,. \tag{2.5.14}$$

The precise relation between the fields appearing here and those in the ten dimensional supergravity was given in [38] and reviewed in Appendix A [15]. We shall use the normalization convention of [39], keeping in mind that $\mathcal{V}_\mu^{(3)}$ is related to the ten dimensional abelian gauge fields $A_\mu^{(10)I}$ used in [39] as $A_\mu^{(10)1} = 2\sqrt{2}\mathcal{V}_\mu^{(3)}$ [16]. In order to facilitate comparison with the fields of the S-T-U model as reviewed in §2.4, where the normalization in front of the Einstein-Hilbert time is given by $1/16\pi$, we shall make a $g_{\mu\nu} \to 2g_{\mu\nu}$ field redefinition, so that the action takes the form:

$$\begin{aligned} S &= \frac{1}{16\pi}\int d^4x\,\sqrt{-\det g}\,\Big[R - \frac{1}{2\,S_2^2}g^{\mu\nu}\partial_\mu S\partial_\nu\bar{S} - \frac{1}{2}S_2 F^{(a)}_{\mu\nu}(LML)_{ab}F^{(b)\mu\nu} \\ &\quad + \frac{1}{2}S_1 F^{(a)}_{\mu\nu}L_{ab}\widetilde{F}^{(b)\mu\nu} + \frac{1}{8}g^{\mu\nu}\,Tr(\partial_\mu ML\partial_\nu ML)\Big]\,. \end{aligned} \tag{2.5.15}$$

If we denote the metric appearing in (2.5.10) by G_{MN} and define

$$A_4 = 2\sqrt{2}\mathcal{V}_4^{(3)}, \tag{2.5.16}$$

then using the results reviewed in Appendix A we find that the four dimensional field

[15] In the convention of [39] that we shall use, S corresponds to the field λ.

[16] We arrive at it by matching the ten dimensional action eq. (1) written in Section 2.1 of [39] and the same action (2.5.2) written in §2.5.1.

configuration corresponding to the background (2.5.10) is given by[17]

$$
\begin{aligned}
M &= \begin{pmatrix}
G_{55}^{-1} & 0 & 0 & 0 & 0 \\
0 & G_{44}^{-1} & 0 & \frac{1}{2}G_{44}^{-1}A_4^2 & G_{44}^{-1}A_4 \\
0 & 0 & G_{55} & 0 & 0 \\
0 & \frac{1}{2}G_{44}^{-1}A_4^2 & 0 & (G_{44}+\frac{1}{2}A_4^2)^2 G_{44}^{-1} & (G_{44}+\frac{1}{2}A_4^2)G_{44}^{-1}A_4 \\
0 & G_{44}^{-1}A_4 & 0 & (G_{44}+\frac{1}{2}A_4^2)G_{44}^{-1}A_4 & 1+G_{44}^{-1}A_4^2
\end{pmatrix} \\
&= \begin{pmatrix}
G_{55}^{-1} & 0 & 0 & 0 & 0 \\
0 & G_{44}^{-1} & 0 & \frac{1}{2}G_{44}^{-1}A_4^2 & G_{44}^{-1}A_4 \\
0 & 0 & G_{55} & 0 & 0 \\
0 & \frac{1}{2}G_{44}^{-1}A_4^2 & 0 & G_{44}^{-1} & G_{44}^{-1}A_4 \\
0 & G_{44}^{-1}A_4 & 0 & G_{44}^{-1}A_4 & 1+G_{44}^{-1}A_4^2
\end{pmatrix},
\end{aligned}
\tag{2.5.18}
$$

where in the last step we have used (2.5.11),

$$S_2 = e^{-2\Phi} C_2 C_4 \sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2 \sinh^2\gamma}\,, \tag{2.5.19}$$

$$S_1 \simeq C_2 C_3 C_4 \, \sinh\gamma \, e^{-2\Phi}\,, \tag{2.5.20}$$

$$\{A_0^{(a)}\} \;=\; -\sqrt{2}\, C_3 \, \sinh\gamma \, \left(e^{2\Phi}\cosh^2\gamma - C_3^2 \sinh^2\gamma\right)^{-1} \times$$

[17] In order to get the expression for S_1 given in (2.5.20), we need to correct the formula for the 4-dimensional 2-form field $B_{\mu\nu}$ given in eq.(3) of [39]. The corrected expression is given by

$$B_{\mu\nu} = B^{(10)}_{\mu\nu} - 4\widehat{B}_{mn} A^{(m)}_\mu A^{(n)}_\nu - 2\left(A^{(m)}_\mu A^{(m+6)}_\nu - A^{(m)}_\nu A^{(m+6)}_\mu\right) - 2\widehat{A}^I_m \left(A^{(I+12)}_\mu A^{(m)}_\nu - A^{(I+12)}_\nu A^{(m)}_\mu\right) \tag{2.5.17}$$

in the notation of [39]. The last term was missed in [39] but is needed to ensure that $B_{\mu\nu}$ transforms correctly under the gauge transformation of $A^{(I+12)}_\mu$.

$$
\begin{pmatrix} 0 \\ -\frac{1}{2\sqrt{2}} C_2^{-1} \cosh\gamma (e^{2\Phi} C_3^{-2} - 1) \\ 0 \\ \sqrt{2}\, C_2\, \cosh\gamma\, r^{-2} H(C_1 r)^2 \\ r^{-1} H(C_1 r) \end{pmatrix},
$$

$$
\{A_i^{(a)} dx^i\} = -\sqrt{2}\, \cos\theta\, d\phi \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.5.21}
$$

$$
\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -\frac{C_2 C_4}{2\sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2 \sinh^2\gamma}} (dx^0)^2 \\ &\quad + \frac{1}{2}\, C_2 C_4 \sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2 \sinh^2\gamma}\; dx^i dx^i \,. \end{aligned} \tag{2.5.22}
$$

We now take the 5×5 matrix

$$
\begin{aligned} W &\equiv \begin{pmatrix} I_2/\sqrt{2} & I_2/\sqrt{2} & \\ I_2/\sqrt{2} & -I_2/\sqrt{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_3 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \begin{pmatrix} I_2/\sqrt{2} & I_2/\sqrt{2} & \\ I_2/\sqrt{2} & -I_2/\sqrt{2} & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \end{aligned} \tag{2.5.23}
$$

satisfying

$$
W^T W = I_5, \qquad W^T L W = L\,, \tag{2.5.24}
$$

and make the field redefinition [18]:

$$M \to W M W^T, \qquad F^{(a)}_{\mu\nu} \to W_{ab} F^{(b)}_{\mu\nu} \,. \tag{2.5.25}$$

The action in the new variables takes the same form as (2.5.12). After this transformation the solution (2.5.18) for M becomes

$$M = \begin{pmatrix} \widetilde{R}^{-2} & 0 & 0 & 0 & 0 \\ 0 & R^{-2} & 0 & 0 & 0 \\ 0 & 0 & \widetilde{R}^{2} & 0 & 0 \\ 0 & 0 & 0 & R^{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.5.26}$$

where

$$\widetilde{R}^2 = G_{55} = C_4^2, \quad R^2 = \frac{1 - \frac{1}{\sqrt{2}} A_4}{1 + \frac{1}{\sqrt{2}} A_4} = \frac{1 - 2\mathcal{V}_4^{(3)}}{1 + 2\mathcal{V}_4^{(3)}} = \frac{1 - 2\, C_2\, \cosh\gamma\, r^{-1} H(C_1 r)}{1 + 2\, C_2\, \cosh\gamma\, r^{-1} H(C_1 r)} \,. \tag{2.5.27}$$

[18] The necessity for the above redefinition lies with the fact that (2.5.21) is not in the correct form and should be brought to the form presented in (2.5.28). It can be achieved in two steps, at first the second matrix in (2.5.23) flips the coordinates 4 & 5 in (2.5.21) and then upon acting with the first matrix in (2.5.23) the the pair of coordinates 1 & 2 mimics 3 & 4 but with a opposite sign. We need the third term in (2.5.23) as it by itself don't satisfy (2.5.24).

The gauge field background takes the form, up to constant shifts (see Appendix A),

$$\{A_0^{(a)}\} \;=\; C_3 C_2^2 \sinh\gamma \,\frac{1}{r}\, H(C_1 r) \begin{pmatrix} 0 \\ \{2C_2 \cosh\gamma\, r^{-1}\, H(C_1 r) - 1\}^{-1} \\ 0 \\ \{2C_2 \cosh\gamma\, r^{-1}\, H(C_1 r) + 1\}^{-1} \\ 0 \end{pmatrix},$$

$$\{A_i^{(a)} dx^i\} \;\simeq\; \cos\theta\, d\phi \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \qquad (2.5.28)$$

The metric and the axion-dilaton fields remain unchanged under this field redefinition.

We now note that for the solution described above the matrix M and the gauge fields are non-trivial only along the first four rows and columns. This corresponds to setting to zero all ten dimensional gauge fields and also setting all components of the metric and 2-form fields with one or both legs along T^4 to trivial values. This is precisely the condition under which the solution can be embedded in the S-T-U model. Rescaling x^i and x^0 as

$$x^i \to \sqrt{\frac{2}{C_2 C_3 C_4}}\, x^i, \quad x^0 \to x^0 \sqrt{\frac{2C_3}{C_2 C_4}}\,, \qquad (2.5.29)$$

and identifying $R\widetilde{R}$ with T_2 and $\widetilde{R}/R$ with U_2 we see that in the variables of the S-T-U model the scalar fields and the metric takes the form:

$$T_1 = 0, \quad U_1 = 0,$$
$$T_2 U_2 = C_4^2, \quad \frac{T_2}{U_2} = \frac{1 - C_2\, \cosh\gamma\, \sqrt{2\, C_2 C_3 C_4}\, r^{-1} H(\sqrt{2}\, C_1 r/\sqrt{C_2 C_3 C_4})}{1 + C_2\, \cosh\gamma\, \sqrt{2\, C_2 C_3 C_4}\, r^{-1} H(\sqrt{2}\, C_1 r/\sqrt{C_2 C_3 C_4})}\,,$$

$$
\begin{aligned}
&S_2 = e^{-2\Phi} C_2 C_4 \sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2\sinh^2\gamma},\\
&S_1 \simeq C_2 C_3 C_4\, \sinh\gamma\, e^{-2\Phi}\,,\\
&g_{\mu\nu}dx^\mu dx^\nu = -e^{2V}\,(dx^0)^2 + e^{-2V}\; dx^i dx^i\,,\\
&\quad e^{2\Phi} = C_3^2 + 4\left(C_1^2 - \frac{C_2C_3C_4}{2\,r^2}\, H(\sqrt{2}\,C_1 r/\sqrt{C_2C_3C_4})^2\right),\\
&e^{2V} \equiv \frac{C_3}{\sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2\sinh^2\gamma}}\,.
\end{aligned}
\tag{2.5.30}
$$

To find the gauge fields in the S-T-U model notation we first note that after the coordinate change (2.5.29) the first four components of gauge fields $A_\mu^{(a)}$ given in (2.5.28) takes the form

$$
\begin{aligned}
\{A_0^{(a)}\} &= C_3^2C_2^2\sinh\gamma\,\frac{1}{r}\,H(\sqrt{2}\,C_1r/\sqrt{C_2C_3C_4})\\
&\quad\begin{pmatrix} 0 \\ \left\{2C_2\cosh\gamma\,\sqrt{\frac{C_2C_3C_4}{2}}\,r^{-1}\,H(\sqrt{2}\,C_1r/\sqrt{C_2C_3C_4}) - 1\right\}^{-1} \\ 0 \\ \left\{2C_2\cosh\gamma\,\sqrt{\frac{C_2C_3C_4}{2}}\,r^{-1}\,H(\sqrt{2}\,C_1r/\sqrt{C_2C_3C_4}) + 1\right\}^{-1}\end{pmatrix},\\
\{A_i^{(a)}dx^i\} &\simeq \cos\theta\,d\phi \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}
\tag{2.5.31}
$$

Now it was shown in [39] that a test charge $(Q, 0)$ couples to this gauge field background through the action

$$
\begin{aligned}
\pm\frac{1}{2}\int dx^\mu A_\mu^{(a)} Q_a &= \pm\frac{1}{2}\int dx^\mu \left[A_\mu^{(1)}\,Q_1 + A_\mu^{(2)}\,Q_2 + A_\mu^{(3)}\,Q_3 + A_\mu^{(4)}\,Q_4\right]\\
&= \pm\frac{1}{2}\int dx^\mu \left[A_\mu^{(1)}\,q_0 + A_\mu^{(2)}\,q_3 - A_\mu^{(3)}\,p^1 + A_\mu^{(4)}\,q_2\right]\,.
\end{aligned}
\tag{2.5.32}
$$

The $\pm$ sign reflects the fact that the analysis of [39] determines the normalization but not the sign of the coupling of the gauge fields to the charges since the bosonic action involving the $U(1)$ gauge fields has an $A_\mu \to -A_\mu$ symmetry. Comparing this with (2.3.2) we get

$$\begin{pmatrix} \mathcal{A}^0_\mu \\ \mathcal{A}^3_\mu \\ \mathcal{A}_{1\mu} \\ \mathcal{A}^2_\mu \end{pmatrix} = \pm \begin{pmatrix} A^{(1)}_\mu \\ A^{(2)}_\mu \\ A^{(3)}_\mu \\ A^{(4)}_\mu \end{pmatrix} . \tag{2.5.33}$$

Eq.(2.5.31) now shows that the magnetic part of the field is given by

$$\mathcal{A}^3_i\, dx^i \simeq \mp \cos\theta d\phi, \quad \mathcal{A}^2_i\, dx^i = \pm \cos\theta dx^i\,. \tag{2.5.34}$$

On the other hand (2.4.44) shows that the expected magnetic field in the S-T-U model, produced by the first center, is given by

$$\mathcal{A}^3_i\, dx^i = -\cos\theta d\phi, \quad \mathcal{A}^2_i\, dx^i = \cos\theta dx^i\,. \tag{2.5.35}$$

Comparing (2.5.34) and (2.5.35) we see that we must use the top sign in (2.5.33). This can now be used to express the electric potentials given in (2.5.31) as

$$\begin{pmatrix} \mathcal{A}^0_0 \\ \mathcal{A}^3_0 \\ \mathcal{A}_{10} \\ \mathcal{A}^2_0 \end{pmatrix} = \begin{pmatrix} A^{(1)}_0 \\ A^{(2)}_0 \\ A^{(3)}_0 \\ A^{(4)}_0 \end{pmatrix} = C_3^2 C_2^2 \sinh\gamma \, \frac{1}{r}\, H\left(\sqrt{2}\, C_1 r/\sqrt{C_2C_3C_4}\right) \times$$
$$\begin{pmatrix} 0 \\ \left\{2C_2 \cosh\gamma \sqrt{\frac{C_2C_3C_4}{2}}\, r^{-1}\, H(\sqrt{2}\, C_1 r/\sqrt{C_2C_3C_4}) - 1\right\}^{-1} \\ 0 \\ \left\{2C_2 \cosh\gamma \sqrt{\frac{C_2C_3C_4}{2}}\, r^{-1}\, H(\sqrt{2}\, C_1 r/\sqrt{C_2C_3C_4}) + 1\right\}^{-1} \end{pmatrix} . \tag{2.5.36}$$

Finally, adding constant terms to the gauge potential, we can bring (2.5.36) to the form:

$$\begin{pmatrix} \mathcal{A}_0^0 \\ \mathcal{A}_0^3 \\ \mathcal{A}_{10} \\ \mathcal{A}_0^2 \end{pmatrix} = C_3^2 C_2^2 \sinh\gamma \,\frac{1}{2C_2\cosh\gamma}\sqrt{\frac{2}{C_2C_3C_4}} \times$$

$$\begin{pmatrix} 0 \\ \left\{2C_2\cosh\gamma\,\sqrt{\frac{C_2C_3C_4}{2}}\,r^{-1}\,H(\sqrt{2}\,C_1 r/\sqrt{C_2C_3C_4}) - 1\right\}^{-1} \\ 0 \\ -\left\{2C_2\cosh\gamma\,\sqrt{\frac{C_2C_3C_4}{2}}\,r^{-1}\,H(\sqrt{2}\,C_1 r/\sqrt{C_2C_3C_4}) + 1\right\}^{-1} \end{pmatrix}. \tag{2.5.37}$$

For example, to match $\mathcal{A}_0^3$ from (2.5.37) to the expression written in (2.5.36) we have to add the constant $(-\sinh\gamma\, C_3^2C_2^2)/(\sqrt{2}C_2\cosh\gamma\sqrt{C_2C_3C_4})$.

Defining

$$\frac{1}{\hat{r}} = \kappa - \frac{1}{r}\,H(\kappa r) = \frac{1}{r} - \kappa\coth(\kappa r) + \kappa, \tag{2.5.38}$$

where

$$\kappa = \frac{\sqrt{2}\,C_1}{\sqrt{C_2C_3C_4}}\,, \tag{2.5.39}$$

we can express (2.5.37), (2.5.30) as

$$\begin{pmatrix} \mathcal{A}_0^0 \\ \mathcal{A}_0^3 \\ \mathcal{A}_{10} \\ \mathcal{A}_0^2 \end{pmatrix} = \frac{1}{2}\,\frac{C_3}{C_2C_4}\,\frac{\sinh\gamma}{\cosh^2\gamma}\begin{pmatrix} 0 \\ \left\{-\frac{1-2C_1C_2\cosh\gamma}{2C_1C_2\cosh\gamma}\kappa - \frac{1}{\hat{r}}\right\}^{-1} \\ 0 \\ \left\{-\frac{1+2C_1C_2\cosh\gamma}{2C_1C_2\cosh\gamma}\kappa + \frac{1}{\hat{r}}\right\}^{-1} \end{pmatrix},$$

$$T_1 = 0, \quad U_1 = 0,$$

$$T_2U_2 = C_4^2, \quad \frac{T_2}{U_2} = \frac{1 - C_2\,\cosh\gamma\,\sqrt{2\,C_2C_3C_4}\,(\kappa - \hat{r}^{-1})}{1 + C_2\,\cosh\gamma\,\sqrt{2\,C_2C_3C_4}\,(\kappa - \hat{r}^{-1})}\,,$$

$$
\begin{aligned}
&S_2 = e^{-2\Phi} C_2 C_4 \sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2 \sinh^2\gamma}\,, \\
&S_1 \simeq C_2 C_3 C_4\, \sinh\gamma\, e^{-2\Phi}\,, \\
&g_{\mu\nu} dx^\mu dx^\nu = -e^{2V}\,(dx^0)^2 + e^{-2V}\, dx^i dx^i\,, \\
&e^{2\Phi} = C_3^2 + 4\left(C_1^2 - \frac{C_2 C_3 C_4}{2}\,(\kappa - \hat{r}^{-1})^2\right), \quad e^{2V} \equiv \frac{C_3}{\sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2\sinh^2\gamma}}\,.
\end{aligned}
\tag{2.5.40}
$$

For large r we have $H(r) \simeq r - 1$ and hence $\hat{r} \simeq r$ up to exponentially suppressed corrections. In that case the field configurations given in (2.5.40) agree with those given in (2.4.44) (up to constant additive terms in the gauge potential) with the choice

$$
\begin{aligned}
&\rho_2 = C_4 \sqrt{\frac{1 - 2C_1 C_2 \cosh\gamma}{1 + 2C_1 C_2 \cosh\gamma}}\,, \quad \sigma_2 = C_4 \sqrt{\frac{1 + 2C_1 C_2 \cosh\gamma}{1 - 2C_1 C_2 \cosh\gamma}}\,, \\
&\zeta_2 = \frac{C_2 C_4}{C_3}, \quad \zeta_1 = \frac{C_2 C_4}{C_3}\,\sinh\gamma\,.
\end{aligned}
\tag{2.5.41}
$$

We can easily check (2.5.41) by taking the $r \to \infty$ limit of the S, T, U fields in (2.5.40). Under this identification, κ given in (2.5.39) becomes (see Appendix A)

$$
\kappa = \sqrt{\frac{\zeta_2}{8\rho_2\sigma_2}}\,\frac{|\rho_2 - \sigma_2|}{|\zeta|} = \frac{1}{r_e}\,.
\tag{2.5.42}
$$

Now note that for finite r the solutions for S_2, T, U, V and $\mathcal{A}_0^I$, $\mathcal{A}_{I0}$ are given by the same expressions as in the case of S-T-U model described in §2.4 with the replacement of r by $\hat{r}$. Since these are the fields which determine the location of the test particle charge (by the extrema of (2.4.45)), we can directly take the results of section 2.4 with r replaced by $\hat{r}$ for determining the location of the test charge. Now from (2.5.38) we see that the condition $r > 0$ corresponds to $\hat{r} > 1/\kappa = r_e$. Thus requiring $|\vec{r}_2|$ to be positive corresponds to requiring $\hat{r}_2$, – the value of $\hat{r}$ corresponding to the vector $\vec{r}_2$ – be larger that r_e. On the other hand for large r we have $r \simeq \hat{r}$. Thus the condition $0 < |\vec{r}_2| < \infty$ translates to $r_e \le \hat{r}_2 < \infty$. Since we can use the results of §2.4 for determining the location of the test charge with r replaced by $\hat{r}$, we see that the condition $r_e \leq \hat{r}_2 < \infty$ translates to requiring τ_1 to lie inside the range given in (2.4.27), (2.4.33) for the first configuration and inside the range given in (2.4.38), (2.4.39) for the second configuration. These two ranges do not overlap, and together they make up the region $R'_1 \cup R'_2$ of the moduli space shown in Fig. 2.2 – precisely in agreement with the microscopic result for the index.

This still leaves open the question as to how the two configurations metamorphose into each other at the boundary L of R'_1 and R'_2. To examine this we apply the inverse of the duality transformation (2.5.23) to map the test electric charges $Q = \begin{pmatrix} a \\ b \\ c \\ d \\ 0 \end{pmatrix}$ and $Q + uP = \begin{pmatrix} a \\ d \\ c \\ b \\ 0 \end{pmatrix}$ to

$$\begin{pmatrix} a \\ (b+d)/2 \\ c \\ (b+d)/2 \\ (b-d)/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a \\ (b+d)/2 \\ c \\ (b+d)/2 \\ (d-b)/\sqrt{2} \end{pmatrix} . \tag{2.5.43}$$

The last entry represents electric charge under the T^3 generator of the SU(2) group. Now at the center of the dyon solution the SU(2) gauge symmetry is restored. Thus at no cost in energy, the test electric charge can undergo an SU(2) rotation of π about the 1-axis flipping the sign of the T^3 charge. This exchanges the quantum number b and d, precisely transforming the test electric charges of the two configurations to each other. Thus we see that the two configurations can transform into each other at the boundary L between R_1' and R_2'. The excess charge $-uP$ is dumped into the background, but we do not detect it in the probe approximation that we are using.

As a parting comment we would like to highlight the fact that whole analysis has been done in the probe approximation where we focused on the large τ_2 region of the moduli space and as a result we could ignore the backreaction of the center with charge $(Q, 0)$ on the background set by the other center $(0, P)$. We believe that the arguments leading up to the line of metamorphosis are fairly general and with more hard work or at least numerically both centers could have been dealt equitably. Also, it should be possible make progress on the relatively complicated case of both $Q^2 = -2$ and $P^2 = -2$. We leave these to future studies.

CHAPTER 3

Logarithmic Corrections to Twisted Indices from the Quantum Entropy Function

3.1 Introduction and Review

In the previous chapter we witnessed one instance of the correct macroscopic interpretation of a collection of BPS states prevalent in the microscopic regime. We have shown that for certain negative discriminant states in four dimensional $\mathcal{N} = 4$ superstring theories, we should replace the two–centered black hole configuration with a BPS configuration consisting of a single–centered black hole and a gauge theory dyon. In the last two decades of matching the microscopic and the macroscopic regimes of black hole entropy, such instances of finer attention are rare. In the current chapter we shall present another such instance where the primary, zeroth order matching of entropy from the macroscopic side is relatively straightforward but not enough. Amongst various higher order corrections we shall particularly focus on the logarithmic corrections to black hole entropy.

Indices carry important information about the spectrum of dyons in string theory. In particular, in four dimensional string theories the helicity trace index, defined by [40, 41]

$$B_{2n} = \frac{1}{(2n)!} \mathrm{Tr} \left[(-1)^{2h} (2h)^{2n} \right] , \tag{3.1.1}$$

receives contributions only from those BPS states in the string theory which break less than $4n$ supersymmetries. Here the trace is over all states in the string theory that carry some specified electric and magnetic charges. As mentioned in Chapter 2, this has now been computed exactly for a wide class of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ string theories [8–13, 16–19, 42–44]. In an expansion in large charges it may be shown that this reproduces the correct semiclassical entropy of an extremal black hole carrying the same charges as the dyons. In many cases, higher-derivative and quantum corrections have also been computed on the macroscopic side and the results have been successfully matched with the corresponding corrections computed from the microscopic formula. We refer the reader to the reviews [31, 45–47] covering various aspects of this program for details and a more complete set of references.

3.1.1 Quantum Entropy Function & Log corrections

The computation of the quantum corrections is performed using the formalism of the Quantum Entropy Function [48, 49]. This proposal exploits the fact that the near–horizon geometry of extremal black holes always contains an AdS_2 factor [50, 51]and as such applies AdS/CFT correspondence. In particular, for spherically symmetric black holes in four dimensions, the near–horizon geometry, embedded in 10–dimensional supergravity, contains an $\mathrm{AdS}_2 \otimes \mathrm{S}^2$ factor coupled to background $U(1)$ fluxes and scalar fields. The entire configuration is completely determined by the $SO(2,1) \otimes SO(3)$ isometry of the solution, along with the electric and magnetic charges carried by the black hole. In Euclidean signature, this configuration is given by

$$\begin{aligned} ds^2 &= a^2\left(d\eta^2 + \sinh^2\eta d\theta^2\right) + a^2\left(d\psi^2 + \sin^2\psi d\phi^2\right), \quad 0 \le \eta < \infty,\, 0 \le \theta < 2\pi, \\ F^{(i)}_{\eta\theta} &= e^i \sinh\eta, \quad F^{(i)}_{\psi\phi} = \frac{p_i}{4\pi}\sin\psi, \quad \Phi_w = u_w, \quad 1 \le i \le r, \quad 1 \le w \le s. \end{aligned} \tag{3.1.2}$$

where the background has r U(1) fluxes and s scalar fields, and a is a function of the electric and magnetic charges of the black hole, determined in terms of the (e^i, p_i).

Using this fact it has been argued that the quantum degeneracy $d_{hor}(\vec{q})$ associated with the horizon of an extremal black hole carrying charges $\vec{q} \equiv q_i$ is given by the unnormalized string path integral, with a Wilson line insertion [1], over all field configurations that asymptote to the attractor geometry of the black hole. In particular, [48, 49]

$$d_{hor}(\vec{q}) \equiv \left\langle \exp\left[i \oint q_i d\theta \mathcal{A}^i_\theta\right]\right\rangle^{finite}_{\mathrm{AdS}_2}. \tag{3.1.3}$$

The subscript 'finite' reminds us that the path integral naively contains a volume divergence due to the presence of the AdS_2 factor. Regulating this divergence is carried out in accordance with the AdS/CFT correspondence.

According to the AdS/CFT dictionary, we have

$$\mathcal{Z}_{AdS_2} = \mathcal{Z}_{CFT_1} = \mathrm{Tr}\left(e^{-LH}\right) \underset{L\to\infty}{=} d_0 e^{-LE_0}. \tag{3.1.4}$$

[1] Near the AdS_2 boundary: $A_\eta = 0,\ A_\theta = C_1 + C_2\cosh\eta$. Here C_1 is a chemical potential and a normalizable mode whereas C_2 is the electric charge and a non–normalizable mode. Fluctuations of the non–normalizable modes should be kept fixed at the boundary which in this case amounts to fixed field strength at the boundary and insertion of the Wilson line to properly implement the boundary conditions. Hence, AdS_2 path integral computes entropy in the microcannonical ensemble.

where H is the hamiltonian of the dual boundary CFT_1, L the infinite length of the boundary and (d_0, E_0) are respectively the degeneracy and energy of states in the CFT_1. In the microscopic regime, the CFT_1 is the quantum mechanics describing the infrared limit of the brane system. The ground states form a finite dimensional Hilbert space and they should be identified with the degeneracy associated with the horizon of the black hole. On the other hand the AdS_2 partition function at large L limit takes the form

$$\mathcal{Z}_{AdS_2} \underset{L\to\infty}{=} e^{CL} \times d_{hor} \,. \tag{3.1.5}$$

Where we should think of C as the redefinition of the ground state energy and the finite part as d_{hor} of (3.1.3). Alternately, the full quantum corrected entropy is

$$S_{macro} = \ln d_{hor} = \lim_{L\to\infty} \left(1 - L\frac{d}{dL}\right) \ln \mathcal{Z}_{AdS_2} \,. \tag{3.1.6}$$

The near–horizon AdS_2 space has $SL(2,R)$ isometry and the supersymmetric blackholes we consider are invariant under four supersymmetries. The closure of the symmetry algebra requires many more generators, and leads to the $su(1,1|2)$ algebra. The corresponding symmetry group contains an $SU(2)$ subgroup which can be identified with the spatial rotation group associated with the sphere S^2 and hence the solution carries zero angular momentum. Upon dimensional reduction to AdS_2, the angular momentum behaves as electric charge and following footnote 1, should be fixed for all fluctuations. This implies all allowed fluctuations carry zero angular momentum. After accounting for the BPS multiplet in the helicity trace index (3.1.1), the remaining $(-1)^{2h}$ factor is always 1 on the macroscopic side. Hence, (3.1.3) which computes a degeneracy rather than an index can be compared with the microscopic result [52].

Since its proposal, the conjecture of [48, 49] has been put to a variety of tests. Firstly, the leading saddle-point of the path integral is the attractor configuration (3.1.2) itself, and it may be shown that the value of the path integral (3.1.3) at this saddle-point is the exponential of the Wald entropy associated with the black hole [2]. Further, by expanding the massless fields of four–dimensional supergravity in quadratic fluctuations about this saddle-point, the logarithmic correction to the Wald entropy may be extracted from (3.1.3) and matched with the microscopic answer [53] [3]. This has been successfully carried out for the $\frac{1}{4}$–BPS black holes in $\mathcal{N} = 4$ supergravity and $\frac{1}{8}$–BPS black holes in $\mathcal{N} = 8$ supergravity [54, 55] and for rotating extremal black holes in [56]. The corresponding expressions for $\frac{1}{2}$–BPS black holes in $\mathcal{N} = 2$ supergravity have also now been obtained [57], however in this case the microscopic results are so far not available. Recently, [58] presented a new approach to the computation of logarithmic terms from (3.1.3) which greatly simplifies the intermediate steps encountered in the calculations of [55–57]. We also note here that (3.1.3) has been exactly evaluated for $\mathcal{N} = 4$ and $\mathcal{N} = 8$ string theories using localization in [59–63] and the answer obtained precisely reproduces the microscopic expressions computed from the indices B_n.

[2] Lets focus on the classical limit of (3.1.5), we have

$$\begin{aligned}
\mathcal{Z}_{AdS_2} &= \exp\left[-\,Action - iq_i \oint_{\partial(AdS_2)} d\theta \mathcal{A}^i_\theta\right]_{classical} \qquad L = a\sinh\eta_0 \\
&= \exp\left[-2\pi\left(q_i e_i - \sqrt{\det g_{AdS_2}}\mathcal{L}_{AdS_2}\right)(\cosh\eta_0 - 1)\right] \\
&= \exp\left[-2\pi\left(q_i e_i - \sqrt{\det g_{AdS_2}}\mathcal{L}_{AdS_2}\right) + CL\right] = \exp\left[S_{Wald} + CL\right] . \qquad (3.1.7)
\end{aligned}$$

Throwing away the infinite term we are left with the degeneracy which at zeroth order is exponential of the Wald entropy.

[3] For massive particles propagating in the loops we could write down an effective $1P1$–action and directly apply Wald's analysis [12,16,19]. Integrating out massless modes would however generate a non–local $1P1$–action and hence Wald's analysis is not valid.

3.1.2 Twisted Indices & Log corrections

If we restrict ourselves to special subspaces of the moduli space which admit discrete symmetry transformations generated by an element g and also require that the charges of the dyons be g–invariant, then we may define twisted indices as

$$B^g_{2n} \equiv \frac{1}{(2n)!}\mathrm{Tr}\left[g\,(-1)^{2h}\,(2h)^{2n}\right]. \tag{3.1.8}$$

The group generated by g is taken to be isomorphic to $\mathbb{Z}_N$. These indices were computed in [64, 65], and a proposal for their macroscopic interpretation was also presented in [64]. In particular, [64] considered Type II string theory compactified on $\mathcal{M} \otimes T^2$, where $\mathcal{M}$ could be either T^4 or $K3$, and g was the generator of a geometric $\mathbb{Z}_N$ symmetry that acts on $\mathcal{M}$ and preserves 16 supercharges. The twisted index B^g_6, which receives contributions from dyonic states which preserve 4 supersymmetries all of which are g–invariant, was then computed. It was found that the answer in the large–charge limit takes the form [45]

$$B^g_6\,(Q,P) = e^{\frac{\pi\sqrt{Q^2P^2-(Q\cdot P)^2}}{N}}\left(\mathcal{O}\,(1) + \ldots\right)\,, \tag{3.1.9}$$

where contrary to the e^{Λ^2} scaling of the exponential term in (3.1.9) under the large Λ scaling $Q \to \Lambda Q\,, P \to \Lambda P$, the $\mathcal{O}\,(1)$ term represents functions with arguments like $\frac{Q\cdot P}{P^2}$ which don't scale with Λ, while the terms represented by the $\ldots$ scale with inverse powers of Λ^2. Therefore, if we assign an 'entropy' to the index by taking its logarithm then we find that

$$\ln|B^g_6\,(Q,P)\,| = \frac{S_{BH}}{N} + \mathcal{O}\,(1)\,, \tag{3.1.10}$$

i.e. the logarithmic correction to the entropy vanishes (there is no term which goes like $\ln\Lambda$ under the above scaling). Here

$$S_{BH} = \pi\sqrt{Q^2P^2-(Q\cdot P)^2}, \tag{3.1.11}$$

is the Wald entropy of an extremal black hole carrying electric and magnetic charges (Q, P). This is also the asymptotic expansion arrived at from Type IIB string theory on the CHL orbifold [65]. In this chapter we shall show how this result arises from a macroscopic computation of the kind performed in [54, 55, 66, 67] for the entropy of the black hole.

Before we do so, we briefly review the proposal made in [64] regarding the macroscopic interpretation of the index B_6^g. The key ingredient of the proposal is that B_6^g is indeed captured by a string path integral of the type (3.1.3) in AdS_2. However, the path integral must now be carried out over fields which obey twisted boundary conditions along the θ–circle of the AdS_2. In particular, as θ shifts by 2π the fields must transform by g. This partition function was denoted by Z_g in [64]. When we impose these boundary conditions then the attractor geometry itself is no longer an admissible saddle–point of the path integral as the θ–circle is contractible in the interior of AdS_2, which leads to a singularity. Let us instead consider the following $\mathbb{Z}_N$ orbifold of the attractor geometry (3.1.2), generated by the identification

$$\tilde{g} : (\theta, \phi) \mapsto \left(\theta + \frac{2\pi}{N}, \phi - \frac{2\pi}{N}\right). \tag{3.1.12}$$

Then it may be shown by an appropriate change of coordinates that the resulting field configuration still asymptotes to the full attractor geometry (3.1.2) [4]. Additionally, this orbifold preserves enough supersymmetry that its contribution to the path integral (3.1.3) does not automatically vanish by integration over the fermionic zero modes associated to broken supersymmetries. For these reasons, these field configurations are also admissible

[4]Rescaling $\theta \to \frac{\theta}{N}$ and $r = \cosh\eta \to Nr$, the orbifold metric becomes,

$$ds^2_{AdS_2} = a^2 \left(\left(r^2 - N^{-2}\right) d\theta^2 + \frac{dr^2}{r^2 - N^{-2}} \right), \tag{3.1.13}$$

which asymptotes to the AdS_2 part of the attractor geometry (3.1.2) with $\theta \to \theta + 2\pi$.

saddle–points of the quantum entropy function (3.1.3).[5] Using these inputs, [64] proposed that Z_g would receive contributions from the saddle–point obtained by imposing a $\mathbb{Z}_N$ orbifold generated by the action of $\tilde{g}$ on the attractor geometry, with a combined action of $\tilde{g}$ and g–twisted boundary conditions imposed on the fields. It was further shown that the value of Z_g^{finite} at the saddle–point was given by $e^{\frac{S_{BH}}{N}}$ [6], in agreement with the asymptotic growth of B_6^g from the microscopic side.

3.1.3 Strategy

In this chapter we will show that the correspondence between Z_g and B_6^g exists even at the quantum level. In particular, we will compute the log correction to the 'entropy' given by $\log Z_g$ by expanding about the $\mathbb{Z}_N$ orbifold of the black hole attractor geometry generated by the action of $\tilde{g}$, where we impose g–twisted boundary conditions on the fields. We will find that the answer vanishes, in accordance with the microscopic results. In order to compute log corrections, we shall use the fact that the contributions of the form $\log a$ to

[5]These orbifolds have fixed points at the origin of the AdS$_2$ times the north or south poles of S^2 and *a priori* it is not clear whether or not this is a consistent orbifold of string theory in the presence of background fluxes. If however the 10–dimensional attractor geometry also contains a circle $\mathcal{C}$ which is non–contractible at the origin of AdS$_2$, then one way to avoid this potential pitfall is to accompany the orbifold (3.1.12) by a translation by $\frac{1}{N}$ units along $\mathcal{C}$. The orbifold group then acts freely over the 10–dimensional attractor geometry. If the radius of the circle $\mathcal{C}$ does not scale with the AdS$_2$ and S^2 radii a, the precise details of the shift will not be relevant for us [53]. We do assume tacitly in our analysis that the generator $\tilde{g}$ includes such a shift along the internal directions as well. Such orbifolds have been explicitly defined in the 10–dimensional theory in [52, 68].

Even if we consider a genuine fixed point which would lead to twisted states contributing to the partition function, they shouldn't matter as we are only interested in the log correction. The twisted states would localize around the fixed points in $AdS_2 \times S^2$, completely oblivious to the radius a. Since log correction scales as a, its unchanged.

[6]Naively we expect it to be $\frac{1}{N}e^{S_{BH}}$ but it is $e^{\frac{S_{BH}}{N}} << \frac{1}{N}e^{S_{BH}}$. It signals a very delicate distribution of $\mathcal{Z}_N$ quantum number amongst the $e^{S_{BH}}$ states.

the partition function of a theory defined with a length scale a are completely determined from the one-loop fluctuations about the saddle-point, where we may focus exclusively on massless fields and further neglect higher-derivative terms [53]. Therefore the only fields that can contribute to the log term in $\log Z_g$ are the massless fields about its admissible saddle–points. We shall compute the log correction, focussing on modes which obey appropriate twisted boundary conditions, and find that the answer vanishes. While we do this computation explicitly for $\mathcal{N} = 8$ string theory obtained by compactifying Type II string theory on T^6, this is only for definiteness and we shall see that the results obtained would carry over to the $\mathcal{N} = 4$ case as well.

We now give a brief overview of the computation, emphasizing the overall strategy and the important differences from the analyses previously carried out in [66] and [67]. We will decompose the $\mathcal{N} = 8$ supergravity multiplet into irreducible representations of the $\mathcal{N} = 4$ subalgebra which commutes with g. These are one $\mathcal{N} = 4$ gravity multiplet, four $\mathcal{N} = 4$ gravitini multiplets and six $\mathcal{N} = 4$ vector multiplets, each of which are charged under g as enumerated in Appendix B. Importantly for us the $\mathcal{N} = 4$ gravity multiplet is uncharged under g, and therefore obeys untwisted boundary conditions. Its contribution to the logarithmic term in the large charge expansion of Z_g is therefore identical to that computed in [67]. The contributions of the gravitini and vector multiplets are however different from [67], and are computed in this chapter.

A brief overview of the chapter is as follows. In section 3.2 we compute the heat kernel for scalars, Dirac fermions and 'discrete modes' of the spin–1 and spin–$\frac{3}{2}$ fields on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ with twisted boundary conditions. This is an extension of the analysis of [66] where the heat kernel over orbifold–invariant modes on these spaces was computed. We find that the answer again assembles into a global part, which obeys untwisted boundary conditions, plus conical contributions which are finite in the limit where the heat kernel time t approaches zero. We put these results together to evaluate the contributions of $\mathcal{N} = 4$ vector and gravitino multiplets that obey twisted boundary conditions in section 3.3. We find that the contribution to the log term vanishes for any non-zero value of the twist. These results demonstrate explicitly that the log term in B_6^g vanishes for $\mathcal{N} = 8$ string theory and $\mathcal{N} = 4$ string theory. We then discuss how our results also prove that the log term vanishes even about exponentially suppressed corrections to the leading asymptotic formula for B_6^g and conclude.

3.2 The Heat Kernel for the Laplacian on $\left(\mathbf{AdS}_2 \otimes \mathbf{S}^2\right)/\mathbb{Z}_N$ with Twisted Boundary Conditions

The goal of this chapter is to compute logarithmic corrections to the partition function Z_g defined as the path integral (3.1.3) with g–twisted boundary conditions. These corrections only receive contributions from the one-loop fluctuations of massless fields over the $\mathbb{Z}_N$ orbifold of the attractor geometry generated by $\tilde{g}$. The one–loop partition function about this background is determined in terms of the determinant of the kinetic operator D evaluated over the spectrum of the theory. We shall define this determinant by the means of the heat kernel method [70].

3.2.1 The Heat Kernel

We shall focus on operators of Laplace–type defined over fields on a manifold $\mathcal{M}$ with a length scale a. The eigenvalues of such operators scale as $\frac{1}{a^2}$ and are denoted by $\frac{\kappa_n}{a^2}$ and the corresponding degeneracies are d_n. With these inputs the one–loop partition function for a $d+1$ dimensional theory with overall length scale a takes the form

$$\mathcal{Z}_{\text{1–loop}} = (\det{}'(D))^{-\frac{1}{2}} \cdot \mathcal{Z}_{zero}(a)\,. \tag{3.2.1}$$

The prime indicates the determinant is evaluated on non–zero modes of the operator D . $\mathcal{Z}(a)$ is the zero mode contribution to the partition function and the argument a reminds us that the zero mode contribution also scales non-trivially with a. The determinant of D may be defined via

$$-\ln\det D' = \int_{\epsilon}^{\infty}\frac{dt}{t}\,\mathrm{Tr}'\left(e^{-tD}\right) = \int_{\frac{\epsilon}{a^2}}^{\infty}\frac{d\bar{s}}{\bar{s}}K'\left(\bar{s}\right), \tag{3.2.2}$$

where ϵ is a UV cutoff and $\bar{s} = \frac{t}{a^2}$. We may define the integrated heat kernel (referred from now on as simply 'the heat kernel') as

$$\begin{aligned} K\left(t\right) &= \mathrm{Tr}\left(e^{-tD}\right) = \sum_n d_n e^{-\frac{t}{a^2}\kappa_n} \\ &= \sum_n\sum_{m=1}^{d_n}\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\psi^*_{n,m}(x)\psi_{n,m}(x)e^{-\frac{t}{a^2}\kappa_n} \\ &= \int_{\mathcal{M}} d^{d+1}x\sqrt{g}K(x,x;t)\,. \end{aligned} \tag{3.2.3}$$

Note that the expression is perfectly well defined for compact manifolds like S^2, but is naively divergent for non–compact spaces like hyperboloids. Moreover, for homogeneous spaces like S^2 and AdS_{d+1} the unintegrated heat kernel $K(x,x;t)$ doesn't depend on the spatial coordinates and hence $K(t)$ is proportional to the volume spanned by the spatial coordinates. However, this divergence may be regulated in accordance with the general principles of AdS/CFT correspondence and a sensible answer can be extracted for $K(t)$ [54, 55]. This involves putting a cutoff on the radial coordinate of the global AdS_2 and extracting the order one term in the large radius (η_0) expansion. This procedure also extends nicely to quotients of hyperboloids both with and without fixed points [66,67,76]. We shall denote the orbifold of the space $\mathcal{M}$ by $\mathcal{M}_\beta$, where β is the cone angle $2\pi/N$. Normally, the heat kernel $K(t)$ in (3.2.3) is evaluated over all the eigenfunctions of D, including the zero modes. To obtain the determinant over the non–zero modes one has to subtract out the zero mode contribution

$$\ln \det{}'(D) = -\int_{\epsilon}^{\infty} \frac{dt}{t} \left(K(t) - n_D^0\right) \,, \tag{3.2.4}$$

where n_D^0 is the number of zero modes $\psi_{0,m}$ of the operator D,

$$n_D^0 = \sum_{m=1}^{n_M^0} \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\psi_{0,m}^*(x)\psi_{0,m}(x), \qquad D|\psi_{0,D}\rangle = 0\,. \tag{3.2.5}$$

The heat kernel has a divergent small t expansion of the form

$$K(t) = \frac{1}{(4\pi)^{\frac{d+1}{2}}} \sum_{n=0}^{\infty} \frac{t^n}{t^{\frac{d+1}{2}}} \int_{\mathcal{M}} d^{d+1}x\sqrt{g}a_n(x)\,, \tag{3.2.6}$$

where a few leading coefficients a_n are known explicitly for the Laplacian on both smooth and conical spaces [70]. The contribution from $\ln \det{}' D$ that scales as $\ln a$ comes from the t^0 term in the heat kernel expansion.

$$\ln \det{}' D = \left(\frac{1}{(4\pi)^{\frac{d+1}{2}}} \int_{\mathcal{M}} d^{d+1}x\sqrt{g}\, a_{\frac{d+1}{2}}(x) - n_D^0\right) \ln a + \ldots \tag{3.2.7}$$

where the '$\ldots$' denote terms that are not of the form $\ln a$. From this expression, the term proportional to $\ln a$ in $\ln \mathcal{Z}$ may be extracted. Logarithmic corrections to black hole entropy have been computed from the quantum entropy function in this manner in [54–57, 66, 67].

Before proceeding further, we shall remind the reader that the zero mode contribution needs to be analyzed separately [53–55, 71] when the operator D is only positive semi–definite, i.e. has zero modes. The kinetic operator for which we compute the heat kernel is the one studied in [54, 55, 66, 67]. This has zero modes over spin–2, spin–$\frac{3}{2}$ and spin–1 fields. However, the zero modes of the graviton and gravitino arise only within the $\mathcal{N} = 4$

gravity multiplet [55] which obeys untwisted boundary conditions in the path integral Z_g and have therefore already been accounted in the analysis of [67]. Additionally, it may be shown that the log term for vectors may as well be extracted out by defining the heat kernel over all eigenvalues κ_n, including the zero eigenvalue, and extracting the $\mathcal{O}\left(\bar{s}^0\right)$ term as before [54]. We will therefore ignore the presence of zero modes in our present analysis but the interested reader will find a selective summary in §3.3.3.

We now turn to the main computation of this section, which will provide us with the essential tools we need to compute logarithmic corrections to the partition function $\mathcal{Z}_g$. These are the heat kernels of the Laplacian over scalar fields and of the Dirac operator over spin-$\frac{1}{2}$ fields on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$, where the $\mathbb{Z}_N$ orbifold is generated by $\tilde{g}$. The heat kernel over the fluctuations invariant under the $\tilde{g}$–generated $\mathbb{Z}_N$ orbifold was computed and the log term extracted in [66, 67]. The analysis of this section is entirely analogous, with the only difference being that we now focus on modes which obey twisted boundary conditions under the $\tilde{g}$ orbifold. We find that the essential steps carry over directly from [66, 67] with only minor modifications. Further, as has been shown in [54, 55], the higher–spin fields in the supergravity multiplets may be expanded in a basis obtained by acting on the scalar with the background metric and covariant derivatives and acting on the spin–$\frac{1}{2}$ field with gamma matrices and covariant derivatives. It turns out that the heat kernel over all quadratic fluctuations may be organized into the heat kernel over scalars and spin–$\frac{1}{2}$ fermions with appropriate multiplicities and shifts in eigenvalues [7]. This will also be of

[7] For example, let us consider a $U(1)$ gauge field with euclidean action and gauge fixing term

$$\begin{aligned} \mathcal{S}_A + \mathcal{S}_{gf} &= -\frac{1}{4}\int d^4x\sqrt{\det g}F_{\mu\nu}F^{\mu\nu} + \left(-\frac{1}{2}\right)\int d^4x\sqrt{\det g}\,(D_\mu A^\mu)^2 \\ &= -\frac{1}{2}\int d^4x\sqrt{\det g}A_\mu(\Delta A)^\mu\,, \end{aligned} \tag{3.2.8}$$

where

$$(\Delta A)_\mu \equiv -\Box A_\mu + R_{\mu\nu}A^\nu, \qquad \Box A_\mu \equiv g^{\rho\sigma}D_\rho D_\sigma A_\mu\,. \tag{3.2.9}$$

great utility in our present analysis. Finally, we note that the heat kernel expression (3.2.3) contains both eigenvalues and degeneracies of the kinetic operator D. On manifolds like AdS_2 the notion of degeneracy is subtle and requires a careful definition. It takes the form of the Plancherel measure [72–74]. On quotients of AdS spaces, it turns out to be useful to exploit the fact that harmonic analysis on AdS is related to the sphere by an analytic continuation [72–74]. By exploiting this analytic continuation, one may obtain the heat kernel and degeneracies of the Laplacian on these orbifolded spaces as well [66,67,75,76]. We shall adopt this approach in this chapter as well.

To start with, we will consider the geometry given by

$$ds^2 = a_1^2\left(d\chi^2 + \sin^2\chi d\theta^2\right) + a_2^2\left(d\psi^2 + \sin^2\psi d\phi^2\right), \tag{3.2.12}$$

Here d denotes the exterior derivative operator and δ the operator $-*d*$ where $*$ denotes the Hodge dual operation. then the Laplacian, Δ may be expressed as $\Delta \equiv (d\delta + \delta d)$.

With the choice of harmonic gauge the kinetic operator on $AdS_2 \otimes S^2$ can be expressed as sum of the kinetic operators in S^2 and AdS_2, and a vector in $AdS_2 \otimes S^2$ decomposes into a (vector, scalar) plus a (scalar, vector). Now, suppose that we have a scalar field Φ on AdS_2 or S^2 satisfying

$$\Delta\Phi \equiv \delta d\Phi \equiv -\Box\Phi = \kappa\Phi\,. \tag{3.2.10}$$

Then we can construct two configurations for the gauge field A with the same eigenvalue κ of Δ and the same normalization as Φ:

$$A^{(1)} = \kappa^{-\frac{1}{2}} d\Phi, \qquad A^{(2)} = \kappa^{-\frac{1}{2}} * d\Phi\,. \tag{3.2.11}$$

There are however some corrections to this both in S^2 and AdS_2 due to global issues. On S^2, the constant mode of the scalar don't generate any non–trivial gauge field configuration and hence there contribution should be removed. On the other hand on AdS_2 we have an extra set of discrete modes (3.2.56). Similar decompositions and global issues are present for the metric, p–forms and the gravitino.

which is related via the analytic continuation

$$(a_1, a_2) \mapsto (ia, a)\,, \quad \chi \mapsto i\eta, \tag{3.2.13}$$

to the $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ geometry

$$ds^2 = a^2\left(d\eta^2 + \sinh^2\eta d\theta^2\right) + a^2\left(d\psi^2 + \sin^2\psi d\phi^2\right). \tag{3.2.14}$$

The $\mathbb{Z}_N$ orbifold generated by $\tilde{g}$ acts on both these spaces via

$$\tilde{g}: \quad (\theta, \phi) \mapsto \left(\theta + \frac{2\pi}{N}, \phi - \frac{2\pi}{N}\right). \tag{3.2.15}$$

Following the strategy of [66,67,75,76], we will do the computation on $\left(\mathrm{S}^2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ and analytically continue the result to $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$. We will however need to be mindful of an important subtlety while performing this analytic continuation which arises due to a class of 'discrete modes' of the vector and spin–$\frac{3}{2}$ fields in AdS_2 [72, 73]. These are normalizable eigenfunctions of the Laplacian over AdS_2 which are not related to normalizable eigenfunctions of the Laplacian over S^2. Their contribution is computed separately in Section 3.2.4.

3.2.2 The Heat Kernel for Scalars on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$

In order to compute the heat kernel for the scalar Laplacian on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$, we will first enumerate its spectrum [72]. The eigenvalues of the scalar Laplacian are

$$E_{\lambda,\ell} = \frac{1}{a^2}\left(\lambda^2 + \frac{1}{4} + \ell\,(\ell+1)\right), \tag{3.2.16}$$

and the corresponding eigenfunctions are given by [72]

$$\Phi_{\lambda,\ell,m,n}(\eta,\theta,\psi,\phi) = f_{\lambda,m}(\eta,\theta)\, Y_{\ell,n}(\rho,\phi)\,, \tag{3.2.17}$$

where, omitting normalization factors,

$$f_{\lambda,m}(\eta,\theta) = \left(\sinh^{|m|}\eta\right) {}_2F_1\left(i\lambda + |m| + \frac{1}{2}, -i\lambda + |m| + \frac{1}{2}, |m|+1, -\sinh^2\frac{\eta}{2}\right) e^{im\theta},$$
$$0 < \lambda < \infty, \quad m \in \mathbb{Z}, \tag{3.2.18}$$

and the $Y_{\ell,n}$s are the usual spherical harmonics on S^2. We will impose the projection (3.2.15) generated by $\tilde{g}$ on the modes (3.2.17) as in [66]. Concentrating on the θ and ϕ coordinates, we have

$$\Phi_{\lambda,l,m,n} \simeq e^{i\,m\theta} e^{i\,n\phi}\,. \tag{3.2.19}$$

The modes invariant under this orbifold are those for which $m - n = Np$, where p is an integer. The heat kernel was computed over such modes in [66]. We will look at the more general case for which

$$m - n = Np + q, \quad p \in \mathbb{Z}, \quad 0 \le q \le N-1, \quad q \in \mathbb{Z}. \tag{3.2.20}$$

We will refer to these as q-twisted boundary conditions. Here, q refers to the inherent phase picked up by the fields themselves under the $\mathbb{Z}_N$ twist, i.e.

$$\Phi \to e^{-\frac{2\pi i\,q}{N}}\Phi\,. \tag{3.2.21}$$

However, as mentioned above, we will carry out the computation by imposing the projection (3.2.15) on eigenfunctions of the scalar Laplacian on $S^2 \otimes S^2$, which are given by

$$\Psi_{\tilde{\ell},m,\ell,n}(\chi,\theta,a_1,\rho,\phi,a_2) = Y_{\tilde{\ell},m}(\chi,\theta,a_1)\, Y_{\ell,n}(\rho,\phi,a_2)\,. \tag{3.2.22}$$

The corresponding eigenvalue is given by

$$E_{\tilde{\ell},\ell} = \frac{1}{a_1^2}\tilde{\ell}\left(\tilde{\ell}+1\right) + \frac{1}{a_2^2}\ell\left(\ell+1\right), \tag{3.2.23}$$

which is related to $E_{\lambda\ell}$ by the analytic continuation

$$\tilde{\ell} = i\lambda - \frac{1}{2}, \quad (a_1, a_2) \mapsto (ia, a). \tag{3.2.24}$$

Using the methods of [66], we find that the heat kernel on q–twisted modes on $\left(\mathrm{S}^2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ is given by

$$K_s^q = \sum_{l,\tilde{l}=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-\tilde{l}}^{\tilde{l}} \delta_{m-n-q,Np} e^{-tE_{l,\tilde{l}}}. \tag{3.2.25}$$

We will now use the following representation of the Kronecker delta function

$$\delta_{a-b,Np} = \frac{1}{N}\sum_{s=0}^{N-1} e^{\frac{2\pi i(a-b)s}{N}}. \tag{3.2.26}$$

Then the heat kernel becomes

$$\begin{aligned} K_s^q &= \frac{1}{N}\sum_{s=0}^{N-1}\sum_{\ell,\tilde{\ell}=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{n=-\tilde{\ell}}^{\tilde{\ell}} e^{\frac{2\pi i m s}{N}} e^{\frac{-2\pi i n s}{N}} e^{\frac{-2\pi i q s}{N}} e^{-tE_{\ell\tilde{\ell}}} \\ &= \frac{1}{N}\sum_{s=0}^{N-1}\sum_{\ell,\tilde{\ell}=0}^{\infty} \chi_{\ell,\tilde{\ell}}\left(\frac{\pi s}{N}\right) e^{\frac{-2\pi i q s}{N}} e^{-tE_{\ell\tilde{\ell}}} \\ &= \frac{1}{N}K^s + \frac{1}{N}\sum_{s=1}^{N-1}\sum_{\ell,\tilde{\ell}=0}^{\infty} \chi_{\ell,\tilde{\ell}}\left(\frac{\pi s}{N}\right) e^{\frac{-2\pi i q s}{N}} e^{-tE_{\ell\tilde{\ell}}}, \end{aligned} \tag{3.2.27}$$

where we have separated out the $s = 0$ term K^s, the scalar heat kernel on the full unquotiented $\mathrm{S}^2 \otimes \mathrm{S}^2$ space and the sum from $s = 1$ to $N-1$ represents the contribution from the conical singularities and is expressed in terms of $\chi_{\ell,\tilde{\ell}}$, the $SU(2) \otimes SU(2)$ Weyl character

$$\chi_{\ell,\tilde{\ell}}\left(\frac{\pi s}{N}\right) \equiv \chi_{\ell}\left(\frac{\pi s}{N}\right)\chi_{\tilde{\ell}}\left(\frac{\pi s}{N}\right) \equiv \frac{\sin\frac{(2\ell+1)\pi s}{N}}{\sin\left(\frac{\pi s}{N}\right)}\frac{\sin\frac{\left(2\tilde{\ell}+1\right)\pi s}{N}}{\sin\left(\frac{\pi s}{N}\right)}, \tag{3.2.28}$$

where χ_ℓ and $\chi_{\tilde{\ell}}$ are $SU(2)$ Weyl characters. Formally, K_s is

$$K_s = \left(\sum_{l=0}^{\infty}(2l+1)e^{-t\frac{l(l+1)}{a^2}}\right)\left(\sum_{\tilde{l}=0}^{\infty}(2\tilde{l}+1)e^{-t\frac{\tilde{l}(\tilde{l}+1)}{a^2}}\right). \tag{3.2.29}$$

and the t^0 may be extracted following [54]. Also in (3.2.27) we can do the sum over $\ell, \tilde{\ell}$ to find that

$$K^q = \frac{1}{N}K^s + \frac{1}{N}\sum_{s=1}^{N-1}\frac{1}{4\sin^4\frac{\pi s}{N}}e^{\frac{-2\pi i q s}{N}}e^{-tE_{\ell\tilde{\ell}}}. \tag{3.2.30}$$

For the case $q = 0$ the sum over s could also be evaluated in $t \to 0$ limit and we had found

$$\frac{1}{4N}\sum_{s=1}^{N-1}\frac{1}{\sin^4\frac{\pi s}{N}} = \frac{(N^2-1)(N^2+11)}{180N}. \tag{3.2.31}$$

For general q the sum remains to be done.

We digress here to remind the reader of the discussion in [69, 70] which will be helpful in motivating (3.2.27) and similar expressions encountered in the later sections. Consider the integrated heat kernel over a manifold $\mathcal{M}_\beta$ with conical singularities located at points p_i. Then the heat kernel over $\mathcal{M}_\beta$ may be decomposed into integrals over the small neighborhoods of the singular points and an integral over the rest if the manifold, which is smooth.

$$\int_{\mathcal{M}_\beta} K(x,x;t) = \int_{\mathcal{M}_\beta - \{\cup \epsilon_i\}} K(x,x;t) + \sum_i \int_{\epsilon_i} K(x,x;t)\,. \tag{3.2.32}$$

The integrated heat kernel on the smooth manifold $\mathcal{M}_\beta - \{\cup \epsilon_i\}$ admits an expansion in powers of t where the coefficients are expressible in terms of volume integrals of the local general–coordinate invariant quantities, for example the curvature $\mathcal{R}$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ etc. As S^2 is homogeneous, these invariants are independent of the location on S^2 (or its quotients), and these integrals on such manifolds are just the volume of the manifold times a constant. On $\left(\mathrm{S}^2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ (once the singular points, the north and south poles, have been removed) therefore, the answer is $1/N$ times the answer on $S^2 \otimes S^2$. As there are four conical singularities on $S^2 \otimes S^2$, each cone contributes

$$\frac{1}{4N} \sum_{s=1}^{N-1} \sum_{\ell,\tilde{\ell}=0}^{\infty} \chi_{\ell,\tilde{\ell}} \left(\frac{\pi s}{N}\right) e^{\frac{-2\pi i q s}{N}} e^{-tE_{\ell\tilde{\ell}}} \, . \tag{3.2.33}$$

Coming back to our main focus, the analytic continuation proceeds in the same way as for the untwisted case [66, 67]. Firstly, the heat kernel over the unquotiented $\mathrm{S}^2 \otimes \mathrm{S}^2$ gets continued to the heat kernel over $\mathrm{AdS}_2 \otimes \mathrm{S}^2$. Then the eigenvalue $E_{\tilde{\ell}\ell}$ gets continued to $E_{\lambda\ell}$ via (3.2.24), and the Weyl character $\chi_{\tilde{\ell}}$ gets continued to the Harish–Chandra (global) character for $sl(2, R)$ [77]

$$\chi^b_\lambda\left(\frac{\pi s}{N}\right) = \frac{\cosh\left(\pi - \frac{2\pi s}{N}\right)\lambda}{\cosh\left(\pi\lambda\right)\sin\left(\frac{\pi s}{N}\right)}, \tag{3.2.34}$$

and the conical terms get multiplied by an overall half [66]. The factor of half accounts for the fact that under the $\mathbb{Z}_N$ orbifold (3.2.15), $\mathrm{AdS}_2 \otimes \mathrm{S}^2$ has half the number of fixed points as does $\mathrm{S}^2 \otimes \mathrm{S}^2$. Finally, the sum over $\tilde{\ell}$ gets continued to an integral over λ. We then obtain the heat kernel for the scalar on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ with the q-twisted boundary condition to be

$$K^q_s = \frac{1}{N} K^s + \frac{1}{2N} \sum_{s=1}^{N-1} \sum_{\ell=0}^{\infty} \int_0^\infty d\lambda \chi^b_{\lambda,\ell}\left(\frac{\pi s}{N}\right) e^{-\frac{2\pi i q s}{N}} e^{-tE_{\lambda\ell}}, \tag{3.2.35}$$

where

$$\chi^b_{\lambda,\ell}\left(\frac{\pi s}{N}\right) = \chi^b_\lambda\left(\frac{\pi s}{N}\right)\chi_\ell\left(\frac{\pi s}{N}\right). \tag{3.2.36}$$

By doing the integral over λ and the sum over ℓ as in [67] and Appendix C we find that (3.2.35) reduces to

$$K_s^q = \frac{1}{N} K^s + \frac{1}{2N} \sum_{s=1}^{N-1} \frac{1}{4 \sin^4 \frac{\pi s}{N}} e^{-\frac{2\pi i q s}{N}} + \mathcal{O}\left(t\right). \tag{3.2.37}$$

This is the expression we shall use to compute logarithmic corrections. It contains two terms. The first is the heat kernel of the untwisted scalar evaluated on the unquotiented space $\mathrm{AdS}_2 \otimes \mathrm{S}^2$. The second term is the contribution of the conical singularities. As observed in [67] for the untwisted modes, this term is finite in the limit where t approaches zero. Hence the contribution of this term to the $\mathcal{O}\left(t^0\right)$ term in the heat kernel expansion is independent of the eigenvalue $E_{\lambda\ell}$. This will be of great utility in our further computations. Finally we note that the expressions (3.2.35) and (3.2.37) are divergent due to the infinite volume of AdS_2. However, using the prescription of [48, 49] this divergence may be regulated and a well–defined finite term extracted even on these quotient spaces [66,67].

At this point we shall digress a bit to mention a caveat associated with the analytic continuation of the unquotiented heat kernels. The analytic continuation of the S^2 to AdS_2 should be performed on the unintegrated heat kernel and then multiplied by the appropriate regularized volume to arrive at the integrated heat kernel. Since both S^2 and AdS_2 are homogeneous spaces, we have

$$K^s_{S^2} = \text{Vol.}_{S^2} K^s_1(a), \qquad K^s_{AdS_2} = \text{Vol.}_{AdS_2} K^s_2(a)\,, \tag{3.2.38}$$

where K^s_1 and K^s_2 are the coincident (unintegrated) heat kernels over S^2 and AdS_2 respectively. Both of them have been explicitly computed in [66] but they are also related by analytic continuation. In particular,

$$K^s_1 \;=\; \frac{1}{12\pi a^2} + \frac{1}{4\pi t} + \frac{t}{60\pi a^4} + \mathcal{O}\left(t^2\right)\,,$$

$$K_2^s \;=\; -\frac{1}{12\pi a^2}+\frac{1}{4\pi t}+\frac{t}{60\pi a^4}+\mathcal{O}\left(t^2\right)\,. \tag{3.2.39}$$

The volume of S^2 is $4\pi a^2$ but AdS_2 volume given by

$$V_{AdS_2}=\int_0^\infty\int_0^{2\pi} d\eta d\phi\, a^2\sinh\eta\,, \tag{3.2.40}$$

is clearly divergent. We shall put a cutoff on η at a large value η_0, which gives the regulated volume of AdS_2 to be

$$\begin{aligned} V_{\text{reg}} &= 2\pi a^2(\cosh\eta_0-1)=2\pi a^2\left(\frac{1}{2}e^{\eta_0}-1+\mathcal{O}\left(e^{-\eta_0}\right)\right) \\ &\simeq -2\pi a^2\,. \end{aligned} \tag{3.2.41}$$

Following the discussion in [71], the divergent term e^{η_0} may be expressed in terms of the radius of curvature of the boundary and as such can be cancelled by boundary counterterms. Once this is done, we obtain a well–defined expression for the degeneracy $d^s_{\lambda\ell}$ of the eigenvalue $E_{\lambda\ell}$ in the q–twisted set of modes on $\left(\mathrm{AdS}_2\otimes \mathrm{S}^2\right)/\mathbb{Z}_N$. This is given by

$$d^s_{\lambda\ell}=-\frac{1}{N}\left(\lambda\tanh\pi\lambda\right)\left(2\ell+1\right)+\frac{1}{2N}\sum_{s=1}^{N-1}\chi^b_{\lambda,\ell}\left(\frac{\pi s}{N}\right)e^{-\frac{2\pi i q s}{N}}. \tag{3.2.42}$$

where $-\left(\lambda\tanh\pi\lambda\right)\left(2\ell+1\right)$ is the regularized degeneracy associated with the unquotiented AdS_2 scalar heat kernel [8] .

3.2.3 The Heat Kernel for Fermions on $\left(\mathbf{AdS}_2\otimes\mathbf{S}^2\right)/\mathbb{Z}_N$

We will turn to the heat kernel of the Dirac operator evaluated over Dirac fermions on $\left(\mathrm{AdS}_2\otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ with q–twisted boundary conditions. The computations are entirely similar to those carried out in [66, 67] once the q–twist has been accounted for as we have

[8] As the eigenfunctions (3.2.18) vanishes at $\eta=0$ for $m\neq 0$, only $m=0$ states will contribute to the coincident scalar heat kernel for AdS_2. At $\eta=0$ the $m=0$ state has the value $\sqrt{(\lambda\tanh\pi\lambda)}/\sqrt{2\pi a^2}$.

for the scalar in Section 3.2.2. We briefly recollect facts about the spectrum of the Dirac operator on S^2. Eigenmodes are $\chi^{\pm}_{\ell m}$ and $\eta^{\pm}_{\ell m}$. Again, concentrating on the θ and ϕ coordinates,the χ's are modded by $e^{i\left(m+\frac{1}{2}\right)\phi}$, while the η's are modded by $e^{-i\left(m+\frac{1}{2}\right)\phi}$, where $0 \leq m \leq \ell$ and $0 \leq \ell < \infty$.

It is more convenient to define the eigenfunctions being modded as

$$e^{i\tilde{m}\phi}, \quad \tilde{m} = -\left(\ell + \frac{1}{2}\right), \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, \left(\ell + \frac{1}{2}\right), \tag{3.2.43}$$

and the wavefunctions themselves being $\Psi^I_{\ell\tilde{m}}$, where

$$\Psi^1_{\ell\tilde{m}} = \chi^+_{\ell m}, \quad \tilde{m} > 0, \quad \Psi^1_{\ell\tilde{m}} = \eta^+_{\ell m}, \quad \tilde{m} < 0, \tag{3.2.44}$$

and $\Psi^2 = (\chi^-, \eta^-)$. It is easier to impose the orbifold in this language. Imposing the projection (3.2.15) generated by $\tilde{g}$ on these modes we have

$$\tilde{m} - \tilde{n} = Np, \quad p \in \mathbb{Z}. \tag{3.2.45}$$

We will consider modes which obey the q-twisted boundary conditions

$$\Psi \mapsto e^{-\frac{2\pi i q}{N}} \Psi\,, \tag{3.2.46}$$

and therefore have to impose the condition

$$\tilde{m} - \tilde{n} = Np + q, \quad p \in \mathbb{Z}, \tag{3.2.47}$$

where $0 \leq q \leq N-1$. Then the heat kernel over the q-twisted modes on $\mathrm{S}^2 \otimes \mathrm{S}^2$ is given by [9]

$$K^q = -4 \sum_{\ell,\tilde{\ell}=0}^{\infty} \sum_{\tilde{m}=-\ell+\frac{1}{2}}^{\ell+\frac{1}{2}} \sum_{\tilde{n}=-\tilde{\ell}+\frac{1}{2}}^{\tilde{\ell}+\frac{1}{2}} \delta_{\tilde{m}-(\tilde{n}+q),Np} e^{-tE_{\ell,\tilde{\ell}}} \,. \tag{3.2.48}$$

As for the scalar, we may expand the conical term in a power series in t omitting the $\mathcal{O}(t)$ and higher terms. Using (3.2.26), the representation of the delta function and carrying out the sum over $\tilde{m}$ and $\tilde{n}$, we find that the heat kernel for a Dirac fermion is given by

$$K_f^q = \frac{1}{N} K_f - \frac{4}{N} \sum_{s=1}^{N-1} \sum_{\ell,\tilde{\ell}=0}^{\infty} \chi^f_{\ell+\frac{1}{2},\tilde{\ell}+\frac{1}{2}} \left(\frac{\pi s}{N}\right) e^{-\frac{2\pi i q s}{N}} + \mathcal{O}(t) \,, \tag{3.2.49}$$

which reduces to

$$K_f^q = \frac{1}{N} K_f - \frac{1}{N} \sum_{s=1}^{N-1} \frac{\cos^2\left(\frac{\pi s}{N}\right)}{\sin^4\left(\frac{\pi s}{N}\right)} e^{-\frac{2\pi i q s}{N}} + \mathcal{O}(t) \,, \tag{3.2.50}$$

on summing over ℓ and $\tilde{\ell}$. These expressions may be analytically continued to $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ to obtain

$$K_f^q = \frac{1}{N} K_f - \frac{2}{N} \sum_{s=1}^{N-1} \sum_{\tilde{\ell}=0}^{\infty} \int_0^{\infty} d\lambda \chi^f_{\lambda,\tilde{\ell}+\frac{1}{2}} \left(\frac{\pi s}{N}\right) e^{-\frac{2\pi i q s}{N}} + \mathcal{O}(t) \,, \tag{3.2.51}$$

and

$$K_f^q = \frac{1}{N} K_f - \frac{1}{2N} \sum_{s=1}^{N-1} \frac{\cos^2\left(\frac{\pi s}{N}\right)}{\sin^4\left(\frac{\pi s}{N}\right)} e^{-\frac{2\pi i q s}{N}} + \mathcal{O}(t) \,, \tag{3.2.52}$$

where we have defined

$$\chi^f_{\lambda,\ell+\frac{1}{2}} \left(\frac{\pi s}{N}\right) = \chi^f_{\lambda} \left(\frac{\pi s}{N}\right) \chi_{\ell+\frac{1}{2}} \left(\frac{\pi s}{N}\right) , \tag{3.2.53}$$

[9] For the fermionic heat kernels the $-\frac{1}{2}$ in (3.2.1) is absent and as a consequence we put an extra $-ve$ sign in the definition of the heat kernel.

and χ^f_λ is the Harish–Chandra character for $sl(2,R)$ as given in [77]

$$\chi^f_\lambda\left(\frac{\pi s}{N}\right)=\frac{\sinh\left(\pi-\frac{2\pi s}{N}\right)\lambda}{\sinh\left(\pi\lambda\right)\sin\left(\frac{\pi s}{N}\right)}\,. \tag{3.2.54}$$

We will use (3.2.52) in our computations for the log term in Section 3.3. We shall just mention the final result for the degeneracy of eigenvalues labelled by the quantum numbers λ,ℓ in the q–twisted set of modes on $\left(\mathrm{AdS}_2\otimes\mathrm{S}^2\right)/\mathbb{Z}_N$.

$$d^f_{\lambda\ell}=-\frac{8}{N}\left(\lambda\coth\pi\lambda\right)\left(\ell+1\right)+\frac{2}{N}\sum_{s=1}^{N-1}\chi^f_{\lambda,\ell+\frac{1}{2}}\left(\frac{\pi s}{N}\right)e^{-\frac{2\pi i q s}{N}}\,. \tag{3.2.55}$$

3.2.4 The Heat Kernel over Discrete Modes

Vectors, gravitini and gravitons on the product space $\left(\mathrm{AdS}_2\otimes\mathrm{S}^2\right)/\mathbb{Z}_N$ may be expanded in a basis constructed from the background metric, Gamma matrices and covariant derivatives, allowing us to express the heat kernel of the kinetic operator over supergravity fields in terms of the heat kernel over scalars and spin–$\frac{1}{2}$ fields [54, 55]. However, this analytic continuation fails to capture a set of discrete modes, labelled by a quantum number ℓ, on the AdS space for the spin–1 and higher spin fields [72–74]. The heat kernel over such modes needs to be computed directly on $\left(\mathrm{AdS}_2\otimes\mathrm{S}^2\right)/\mathbb{Z}_N$. The origin of these modes is the following discrete set of modes over AdS_2.

$$f^m_a=\nabla_a\phi^m,\quad \phi^m=\sqrt{\frac{1}{2\pi|m|}}\left[\frac{\sinh\eta}{1+\cosh\eta}\right]^{|m|}e^{im\theta},\ m\in\mathbb{Z}-\{0\}. \tag{3.2.56}$$

They tensor with scalar modes on S^2 to produce discrete modes of the vector field in $\mathrm{AdS}_2\otimes\mathrm{S}^2$ and its $\mathbb{Z}_N$ orbifold. We will calculate the heat kernel contribution of these modes (3.2.56) when they obey the twisted boundary conditions.

$$K^{(v_d,s)}\;=\;\sum_{m\in\mathbb{Z}-\{0\}}\sum_{\ell=0}^{\infty}\sum_{n=-\ell}^{\ell}\int^{\eta_0}\sqrt{g}\,d\eta\, d\theta\, g^{ab}f^{*m}_a f^m_b\,\delta_{m-(n+q),Np}\,e^{-\frac{t}{a^2}\ell(\ell+1)} \tag{3.2.57}$$

$$= \frac{1}{N}\sum_{s=0}^{N-1}\left(\sum_{m\in\mathbb{Z}-\{0\}}\tanh\left(\frac{\eta_0}{2}\right)^{2|m|}e^{im\frac{2\pi s}{N}}\right)\sum_{\ell=1}^{\infty}\chi_\ell\left(\frac{\pi s}{N}\right)e^{-\frac{2\pi iqs}{N}}e^{-\frac{t}{a^2}\ell(\ell+1)}\,.$$

The expression in the brackets involving $\tanh\left(\frac{\eta_0}{2}\right)$ sums up to

$$T(s) = \frac{1}{1-\tanh^2\left(\frac{\eta_0}{2}\right)e^{\frac{2\pi is}{N}}} - 1 + \frac{1}{1-\tanh^2\left(\frac{\eta_0}{2}\right)e^{-\frac{2\pi is}{N}}} - 1\,. \tag{3.2.58}$$

Further,

$$T(0) \simeq \frac{1}{2}e^{\eta_0} - 1 + \mathcal{O}\left(e^{-\eta_0}\right), \quad T(m) \simeq -1 + \mathcal{O}\left(e^{-\eta_0}\right), 1 \leq m \leq N-1\,. \tag{3.2.59}$$

We may then retain the order–one term only to find that

$$K^{(v_d,s)} = -1\left[\frac{1}{N}\sum_{s=0}^{N-1}\sum_{\ell=0}^{\infty}\chi_\ell\left(\frac{\pi s}{N}\right)e^{-\frac{2\pi iqs}{N}}e^{-\frac{t}{a^2}\ell(\ell+1)}\right]. \tag{3.2.60}$$

We find that the degeneracy of an eigenvalue E_ℓ of the Laplacian over vector discrete modes obeying q–twisted boundary conditions is given by[10]

$$d_\ell^{v_d} = -\frac{2\ell+1}{N} - \frac{1}{N}\sum_{s=1}^{N-1}\chi_\ell\left(\frac{\pi s}{N}\right)e^{-\frac{2\pi iqs}{N}}. \tag{3.2.61}$$

Using the methods of [66, 67], the degeneracy over the q–twisted gravitino discrete modes is given by

$$d_\ell^{f_d} = 8\left(\frac{\ell+1}{N}\right) - \frac{4}{N}\sum_{s=1}^{N-1}\frac{\sin\frac{2\pi s(\ell+1)}{N}}{\sin\frac{\pi s}{N}}\cos\frac{\pi s}{N}\,e^{-\frac{2\pi iqs}{N}}. \tag{3.2.62}$$

[10] We point out here that the modes with $\ell = 0$ correspond to vector zero modes of the kinetic operator [54] and hence $d_{\ell=0}^{v_d}$ corresponds to the regularized number of vector zero modes of the kinetic operator. Explicitly evaluating (3.2.61) with $\ell = 0$, so that $\chi_\ell\left(\frac{\pi s}{N}\right) = 1\ \forall s$, we find that $d_{\ell=0}^{v_d}$ vanishes when non-trivial q–twisted boundary conditions are imposed. This is in contrast to the untwisted case, where $d_{\ell=0}^{v_d} = -1$ [66].

Using the degeneracies (3.2.61) and (3.2.62), we can write down corresponding expressions for the heat kernels over these modes, though we do not do so explicitly here.

3.3 Logarithmic Corrections to the Twisted Index

We now turn to the computation of logarithmic corrections to $\mathcal{Z}_g$. We will carry out this computation for Type II string theory on T^6. This compactification preserves 32 supercharges of which 16 commute with g. Also, as we have previously discussed, the only fields which can contribute to the $\log a$ term are the massless fields in $\mathrm{AdS}_2 \otimes \mathrm{S}^2$. These are just the fields of four–dimensional $\mathcal{N} = 8$ supergravity. We will therefore find it useful to organize the spectrum of $\mathcal{N} = 8$ supergravity in terms of representations of the $\mathcal{N} = 4$ subalgebra which commutes with g. All the fields in a single $\mathcal{N} = 4$ multiplet are characterized by a common g-eigenvalue which in turn dictates which twisted modes on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ should the heat kernel be computed over. This information is summarized in Table B.1. In this section we shall compute the contribution of each multiplet in Table B.1 to the log term in $\mathcal{Z}_g$, which requires us to compute the contribution to $\mathcal{Z}_g$ from quadratic fluctuations of massless fields about the $\mathbb{Z}_N$ orbifold generated by the action (3.2.15) of $\tilde{g}$ on the attractor geometry of the black hole. To do so, we shall compute the heat kernel of the kinetic operator derived in [54, 55] about this orbifolded background, imposing g–twisted boundary conditions on the fields as we act on the background with $\tilde{g}$. Therefore, the results of Section 3.2 will be useful for us.

Finally, as in [54, 55, 66, 67], we need to compute the heat kernel over the supergravity fields taking into account their couplings to the background graviphoton fluxes and scalar fields. As shown in [54, 55], the heat kernel over the various quadratic fluctuations can be expressed in terms of the heat kernel over scalars, spin-$\frac{1}{2}$ fermions and discrete modes of higher–spin fields. The coupling to the background fields however changes the eigenvalues of the kinetic operator from those when fields are minimally coupled to background gravity. The new eigenvalues can in principle be computed by rediagonalizing the kinetic operator. However, the flux does not change the degeneracy of the eigenvalue. Hence, to compute the heat kernel over the supergravity fields with our choice of background and boundary conditions, we can use the shifted eigenvalues computed in [54, 55] and the degeneracies computed in Section 3.2. On doing so, we find two more simplifications that are of great benefit. Firstly, as observed in [67], the contribution of the conical terms to the heat kernel is finite in the $t \mapsto 0$ limit. Hence the contribution to the $\mathcal{O}\left(t^0\right)$ term from the conical terms is insensitive to the eigenvalues and can be computed from the degeneracies. Secondly, the other contribution to the $\mathcal{O}\left(t^0\right)$ term in the heat kernel originates from the $\mathcal{O}\left(t^0\right)$ term in the heat kernel computed for the full attractor geometry without imposing any twist on the boundary conditions. This has already been computed in [54, 55]. Using these results, and the g–charges computed in Table B.1, we can now compute the heat kernel over the various supergravity fields and extract the $\mathcal{O}\left(t^0\right)$ term in the heat kernel, which will yield the log term. With these results, we now turn to the main computation of this chapter.

We firstly note that the $\mathcal{N} = 4$ gravity multiplet is g–invariant, and hence its heat kernel should be computed over untwisted modes. It has already been shown in [66] that the contribution of these modes to the log term vanishes. Additionally, the contribution of any g–invariant $\mathcal{N} = 4$ vector multiplet to the log term also vanishes [66]. Therefore we shall concentrate on the gravitino multiplets and the $\mathcal{N} = 4$ vector multiplets which carry a non–trivial g charge, which corresponds to a non–zero twist in the boundary conditions. We find below that the contribution of these multiplets also vanishes for any arbitrary choice of twisting. This is in contrast to the untwisted case where while the contribution of the vector multiplet did vanish, the gravitino multiplet contribution was non-vanishing and was responsible for the non-zero log correction the entropy of $\frac{1}{8}$–BPS black holes in

$\mathcal{N} = 8$ supergravity [67].

3.3.1 The Heat Kernel for the $\mathcal{N} = 4$ Vector Multiplet

We will now put the results of Section 3.2 together, using the arguments presented above, to prove the first of our main results : the log correction in $\mathcal{Z}_g$ receives vanishing contribution from any $\mathcal{N} = 4$ vector multiplet with q–twisted boundary conditions. As in [54, 66], the heat kernel for any $\mathcal{N} = 4$ vector multiplet receives contributions from two Dirac fermions, 6 real scalars and one gauge field, along with two scalar ghosts. We will focus on the contribution of the conical terms to the $\mathcal{O}(t^0)$ term in the heat kernel. We denote this contribution by $K^c(t;0)$. Firstly the contribution from the two Dirac fermions is given by

$$K_c^F(t;0) = -\frac{1}{N}\sum_{s=1}^{N-1}\frac{\cos^2\left(\frac{\pi s}{N}\right)}{\sin^4\left(\frac{\pi s}{N}\right)}e^{-\frac{2\pi iqs}{N}}. \qquad (3.3.1)$$

We now turn to the contribution from the integer–spin fields. These are the 6 real scalars, the gauge field and two scalar ghosts. Two of the scalars mix with the gauge field due to the graviphoton flux [54] and we have

$$K^B = 4K^s + K^{(v+2s)} - 2K^s, \qquad (3.3.2)$$

where K^s is the scalar heat kernel along $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ with q–twisted boundary conditions, and $K^{(v+2s)}$ is the heat kernel of the mixed vector–scalar fields due to the background graviphoton flux. As we have previously argued, to extract the t^0 term from the fixed–point contribution to the heat kernel, we don't have to take into account the coupling of the gauge field to the scalars *via* the graviphoton flux and can just add the various contributions piecewise. We therefore find that (3.3.2) reduces to

$$K_c^B = 6K_c^s + K_c^v - 2K_c^s = 4K_c^s + K_c^v. \qquad (3.3.3)$$

K_c^s can be read off from (3.2.37), but we need to compute K_c^v. As shown in [54], the heat kernel K^v of a vector field over $\mathrm{AdS}_2 \otimes \mathrm{S}^2$ may be decomposed into $K^{(v,s)}$, which is the heat kernel of a vector field along AdS_2 times the heat kernel of a scalar along S^2 and $K^{(s,v)}$, the heat kernel of a vector field along S^2 times the heat kernel of a scalar along AdS_2. Further, the modes of the vector field along AdS_2 and S^2 may be further decomposed into longitudinal and transverse modes. There is an additional discrete mode contribution from the vector field on AdS_2. These statements carry over to the case of the $\mathbb{Z}_N$ orbifolds with twisted boundary conditions as well. K^v therefore receives the following contributions.

$$K^v = K^{(v_T+v_L+v_d,s)} + K^{(s,v_T+v_L)}. \tag{3.3.4}$$

Now the modes of longitudinal and transverse vector fields along AdS_2 and S^2 are in one–to–one correspondence with the modes of the scalar with the only subtlety being that along S^2 the $\ell = 0$ mode of the scalar does not give rise to a non-trivial gauge field [54] (see footnote 7). We therefore have

$$K^{(v_T,s)} = K^{(v_L,s)} = K^s, \quad K^{(s,v_T)} = K^{(s,v_L)} = K^s - K^{(s,\ell=0)}, \tag{3.3.5}$$

where, as we have mentioned previously, K^s is the scalar heat kernel along $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ with q–twisted boundary conditions, and $K^{(s,\ell=0)}$ is again the scalar heat kernel along $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$, however we only sum over the modes with $\ell = 0$ along the S^2 direction. We therefore find that the contribution of the conical terms(3.3.4) reduces to

$$K_c^v\,(t;0) = 4K_c^s\,(t;0) + K_c^{(v_d,s)}\,(t;0) - 2K_c^{(s,\ell=0)}\,(t;0)\,. \tag{3.3.6}$$

Further, using (3.2.42), we may show that [11]

$$K_c^{(s,\ell=0)}(t;0) = \frac{1}{4N}\sum_{s=1}^{N-1}\frac{1}{\sin^2\left(\frac{\pi s}{N}\right)}e^{-\frac{2\pi i q s}{N}}, \tag{3.3.7}$$

and that $K_c^s(t;0)$ is given by

$$K_c^s(t;0) = \frac{1}{2N}\sum_{s=1}^{N-1}\frac{1}{4\sin^4\frac{\pi s}{N}}e^{-\frac{2\pi i q s}{N}}. \tag{3.3.8}$$

Finally, using (3.2.61), $K_c^{(v_d,s)}(t;0)$ is given by

$$K_c^{(v_d,s)}(t;0) = -\frac{1}{2N}\sum_{s=1}^{N-1}\frac{1}{\sin^2\left(\frac{\pi s}{N}\right)}e^{-\frac{2\pi i q s}{N}}. \tag{3.3.9}$$

Using (3.3.3) and (3.3.6), and then putting (3.3.7), (3.3.8) and (3.3.9) together, we find that the total integer–spin contribution is given by

$$K_c^B(t;0) = \frac{1}{N}\sum_{s=1}^{N-1}e^{-\frac{2\pi i q s}{N}}\left(\frac{1-\sin^2\left(\frac{\pi s}{N}\right)}{\sin^4\left(\frac{\pi s}{N}\right)}\right). \tag{3.3.10}$$

Then the total contribution of the conical terms from bosons and fermions is obtained by adding (3.3.1) and (3.3.10) to obtain

$$K^c(t;0) = K_c^B(t;0) + K_c^F(t;0) = \frac{1}{N}\sum_{s=1}^{N-1}e^{-\frac{2\pi i q s}{N}}\left(\frac{1-\sin^2\left(\frac{\pi s}{N}\right)-\cos^2\left(\frac{\pi s}{N}\right)}{\sin^4\left(\frac{\pi s}{N}\right)}\right) = 0. \tag{3.3.11}$$

This vanishes for arbitrary values of q. Now, using the arguments at the beginning of the section, the heat kernel for the $\mathcal{N}=4$ vector multiplet about the $\tilde{g}$–generated $\mathbb{Z}_N$ orbifold

[11] An easier approach is to read off the result directly from (3.2.37). In (3.2.37) for $S^2\otimes S^2$, each sum over 0 to ∞ for l and $\tilde{l}$ gives $\chi_{l\text{ or }\tilde{l}}\left(\frac{\pi s}{N}\right) = \frac{1}{2\sin^2\frac{\pi s}{N}}$. Remembering $\chi_{\tilde{l}=0}\left(\frac{\pi s}{N}\right) = 1$ and analytically continuing to $AdS_2\otimes S^2$ we get the desired result.

of the attractor geometry is given, on imposing q–twisted boundary conditions, by

$$K^q = \frac{1}{N} K + K^c (t; 0) + \mathcal{O}(t) \,, \tag{3.3.12}$$

where K is the heat kernel on the unquotiented near–horizon geometry. We therefore have, for the t^0 term in the heat kernel expansion,

$$K^q (t; 0) = \frac{1}{N} K (t; 0) + K^c (t; 0) \ . \tag{3.3.13}$$

We have shown in 3.3.11 that $K^c (t; 0)$ equals zero. In addition, it was shown in [54] that $K (t; 0)$ also vanishes. This implies that $K^q (t; 0)$ also vanishes, which proves that the contribution to the log term from the vector multiplet vanishes even for q–twisted boundary conditions[12].

3.3.2 The Heat Kernel for the $\mathcal{N} = 4$ Gravitino Multiplets

We now compute the contribution of the $\mathcal{N} = 4$ gravitino multiplets to the log term in $\mathcal{Z}_g$ for $\mathcal{N} = 8$ string theory. From Table B.1, we see that the $\mathcal{N} = 4$ gravitino multiplets obey q–twisted boundary conditions. There are four such multiplets, where the highest-weight field is a Majorana spin–$\frac{3}{2}$ fermion, which we organize into two multiplets where the highest–weight field is a Dirac spin–$\frac{3}{2}$ fermion. One multiplet obeys twisted boundary conditions with $q = +1$, and the other with $q = -1$. Further, since we are considering quadratic fluctuations, the background flux in the attractor geometry does not cause gravitino multiplets with different g–charge, and hence different q–twist, to mix with each

[12] We emphasize here that though the contribution of an $\mathcal{N} = 4$ vector multiplet vanishes for both twisted and untwisted boundary conditions, the origin of the result is different in both cases. For the untwisted case, the zero and non–zero modes of the kinetic operator give non–vanishing contributions to the log term which cancel against each other [66], while for the twisted case these contributions are individually zero as shown in this section and in footnote 10 of this chapter.

other. We will therefore focus on the contribution of the log term from one q–twisted multiplet where the highest–weight field is a Dirac spin–$\frac{3}{2}$ fermion.

Now we shall compute the contribution of the conical terms to the t^0 term in the heat kernel expansion for this multiplet. Firstly, we focus on the integer–spin fields. There are 8 gauge fields and 16 real scalars. Further, gauge fixing introduces two ghost scalars for every gauge field. Hence the contribution of the integer–spin fields to the $\mathcal{O}\left(t^0\right)$ term from the conical terms in the heat kernel is

$$K_c^B\left(t;0\right) = 8K_c^v\left(t;0\right) + 16K_c^s\left(t;0\right) - 16K_c^s\left(t;0\right) = 8K_c^v\left(t;0\right), \qquad (3.3.14)$$

which therefore implies that

$$K_c^B\left(t;0\right) = \frac{4}{N}\sum_{s=1}^{N-1}\frac{1}{\sin^4\frac{\pi s}{N}}e^{-\frac{2\pi iqs}{N}} - \frac{8}{N}\sum_{s=1}^{N-1}\frac{1}{\sin^2\frac{\pi s}{N}}e^{-\frac{2\pi iqs}{N}}. \qquad (3.3.15)$$

We have used (3.3.6) with (3.3.7), (3.3.8) and (3.3.9) to arrive at this expression. We now turn to the contribution of the half–integer spin fields. We will focus on the contribution of one Dirac gravitino multiplet, which contains one Dirac gravitino and 7 Dirac spin-$\frac{1}{2}$ fields. The degrees of freedom reorganize themselves into in 4 Dirac fermions with $\ell \geq 0$, 6 Dirac fermions with only $\ell = 0$ modes along the S^2, 7 Dirac fermions with only $\ell \geq 1$ modes along the S^2, one discrete Dirac fermion, and 3 ghost Dirac fermions [55, 67] [13]. We can then show that

$$K_c^F\left(t;0\right) = 8K_c^f\left(t;0\right) - K_c^{(f,\ell=0)}\left(t;0\right) + K_c^{f_d}\left(t;0\right), \qquad (3.3.16)$$

[13] $(4-3=1)$ Dirac fermions + ((7 Dirac fermions with $l \leq 1$) + (7 $l=0$ modes) = 7 Dirac fermion) + $(-7+6=-1,\ l=0$ modes).

where K^{f} is the heat kernel for the Dirac fermion, $K^{(f,\ell=0)}$ is the heat kernel for the Dirac fermion with only $\ell = 0$ modes along the S^2 and K^{f_d} is the heat kernel over one discrete Dirac fermion. Now

$$K_c^{f}\left(t;0\right) = -\frac{1}{2N}\sum_{s=1}^{N-1}\frac{\cos^2\frac{\pi s}{N}}{\sin^4\frac{\pi s}{N}}e^{-\frac{2\pi i q s}{N}}, \tag{3.3.17}$$

and

$$K_c^{(f,\ell=0)}\left(t;0\right) = -\frac{2}{N}\sum_{s=1}^{N-1}\frac{\cos^2\frac{\pi s}{N}}{\sin^2\frac{\pi s}{N}}e^{-\frac{2\pi i q s}{N}}. \tag{3.3.18}$$

Further, using (3.2.62), we find that the discrete mode contribution from the conical terms is given by

$$K_c^{f_d}\left(t;0\right) = +\frac{2}{N}\sum_{s=1}^{N-1}\frac{\cos^2\frac{\pi s}{N}}{\sin^2\frac{\pi s}{N}}e^{-\frac{2\pi i q s}{N}}. \tag{3.3.19}$$

We finally obtain that the full half–integer spin contribution is given by

$$K_c^{F}\left(t;0\right) = -\frac{4}{N}\sum_{s=1}^{N-1}\frac{1}{\sin^4\frac{\pi s}{N}}e^{-\frac{2\pi i q s}{N}} + \frac{8}{N}\sum_{s=1}^{N-1}\frac{1}{\sin^4\frac{\pi s}{N}}e^{-\frac{2\pi i q s}{N}} - \frac{4}{N}\sum_{s=1}^{N-1}e^{-\frac{2\pi i q s}{N}}. \tag{3.3.20}$$

Adding (3.3.15) and (3.3.20), we find that the conical contribution to the t^0 term in the heat kernel for a given value of q is

$$K_c\left(t;0\right) = -\frac{4}{N}\sum_{s=1}^{N-1}e^{-\frac{2\pi i q s}{N}} = +\frac{4}{N}, \tag{3.3.21}$$

which is independent of q. Then the contribution of the g–twisted $\mathcal{N}=4$ gravitino multiplets to the log term in $\mathcal{Z}_g$ is given by

$$K^{g}\left(t;0\right) = \frac{1}{N}K\left(t;0\right) + 2K_c\left(t;0\right), \tag{3.3.22}$$

where $K\left(t;0\right)$ is the coefficient of the t^0 term in the heat kernel expansion of the gravitino multiplets about the unquotiented near–horizon geometry. This was computed to be -8

in [55]. We therefore find that $K^g\,(t;0)$ is given by

$$K^g\,(t;0) = -\frac{8}{N} + \frac{8}{N} = 0. \qquad (3.3.23)$$

Hence, the contribution of the $\mathcal{N} = 4$ gravitini multiplets to the logarithmic term in $\mathcal{Z}_g$ also vanishes.

3.3.3 The Zero Mode Analysis

We will now take into account the presence of zero modes of the kinetic operator for $\mathcal{N} = 8$ supergravity fields expanded about the black hole near horizon geometry. The final result, as mentioned above, is that the zero mode analysis of [67] goes through unchanged, but since the zero mode analysis is an important part of the computation, we shall present the setup in some detail and then quote the results. The following general result [54, 55], see also [71], will be useful for us [14].

To start with, let A_μ be the vector field on $AdS_2 \otimes S^2$ and $g_{\mu\nu}$ be the background metric of the form $a^2 g^{(0)}_{\mu\nu}$ where a is the radius of both S^2 and AdS_2 and $g^{(0)}_{\mu\nu}$ is independent of a. We normalize the path integral over A_μ such that

$$\begin{aligned} &\int [DA_\mu] \exp\left[-\int d^4x\sqrt{\det g}\, g^{\mu\nu} A_\mu A_\nu\right] = 1, \\ &\int [DA_\mu] \exp\left[-a^2\int d^4x\sqrt{\det g^{(0)}}\, g^{(0)\mu\nu} A_\mu A_\nu\right] = 1\,. \end{aligned} \qquad (3.3.24)$$

From above we infer that $[DA_\mu]$ actually corresponds to the integration with measure $\Pi_{\mu,x} d(aA_\mu(x))$. Zero modes for the gauge fields are associated with deformations produced by non–normalizable gauge transformation parameters: $\delta A_\mu \propto \partial_\mu \Lambda(x)$ for some

[14] Unless we subtract the zero modes from the heat kernel the integration over $\tilde{s}$ is divergent. A constant term in the heat kernel will never produce a factor of $\ln a^2$ as they arise from terms which remain constant in the range $a^{-2} << \tilde{s} << 1$ and fall off for $\tilde{s} >> 1$.

$\Lambda(x)$ with a–independent integration range. The Jacobian picks up a factor of a for each zero mode and for n_0 such modes we shall get a overall factor of a^{n_0}. Note that n_0 is infinite but AdS_2 volume regularization gives a finite result [15]. Remember that the eigenvalues in the heat kernel scales as a^{-2} and upon integration the non–zero modes produces a factor proportional to a. Including the zero mode contribution to the heat kernel is equivalent to counting the same factor of a from integration over the zero modes as well. Thus, when we remove from the determinant the contribution of the zero modes, we remove a factor of a for each zero mode. Then for the vector zero modes as long as we keep n_0 finite there is no need to explicitly subtract and then add the contribution of the zero modes to the heat kernel [16] . As we shall see this is not true for the gravity or the gravitino zero modes.

The effect of integrating over the zero mode fluctuations $h_{\mu\nu}$ of the metric can be found

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[15] For example, let f_a^m denote the set of normalized discrete zero mode wave functions of the gauge fields on AdS_2 given in (3.2.56), where index m denotes modes invariant under twisting. For the case of untwisted modes i.e. $q = 0$, the the total number of zero modes would be

$$\begin{aligned} n_0 &= \sum_{p\in\mathbb{Z}-\{0\}} \int d\theta d\eta \sqrt{\det g_{AdS_2}} g^{ab}_{AdS_2} f_a^{m*} f_b^m \qquad \text{where} \quad m = Np \\ &= \sum_{p\in\mathbb{Z}-\{0\}} \int_{AdS_2}^{\eta_0} d\theta d\eta \sqrt{\det g_{AdS_2}} g^{ab}_{AdS_2} f_a^{Np*} f_b^{Np} = \sum_{p\in\mathbb{Z}-\{0\}} \left(\tanh\frac{\eta_0}{2}\right)^{2N|p|} \\ &= 2\frac{\left(\tanh\frac{\eta_0}{2}\right)^{2N}}{1-\left(\tanh\frac{\eta_0}{2}\right)^{2N}} \simeq \frac{1}{2N} e^{\eta_0} - 1 + \mathcal{O}\left(t\right)\left(e^{-\eta_0}\right) \,, \end{aligned} \tag{3.3.25}$$

in the large η_0 limit. We drop the divergent factor in AdS radial coordinate and keep the order one term as the number of zero modes. Hence for the quotient space $AdS_2/\mathbb{Z}_N$, the number of zero modes is given by

$$n^0_{AdS_2/\mathbb{Z}_N} = -1\,. \tag{3.3.26}$$

The case for non–zero q is discussed in footnote 10.

[16] This is almost true except that the dimensional reduction of the metric on S^2 produces a massless $SU(2)$ gauge field with zero modes on AdS_2. They scale as metric zero modes scale i.e. a^2 rather than a.

by replacing (3.3.24) with

$$\int [Dh_{\mu\nu}] \exp\left[-\int d^4x \sqrt{\det g} g^{\mu\nu} g^{\rho\sigma} h_{\mu\rho} h_{\nu\sigma}\right] = 1,$$
$$\int [Dh_{\mu\nu}] \exp\left[-\int d^4x \sqrt{\det g^{(0)}} g^{(0)\mu\nu} g^{(0)\rho\sigma} h_{\mu\rho} h_{\nu\sigma}\right] = 1\,. \qquad (3.3.27)$$

The zero modes are associated with diffeomorphisms with non–renormalizable parameters: $h_{\mu\nu} \propto D_\mu \xi_\nu + D_\nu \xi_\mu$, with the diffeomorphism parameter ξ^μ having a independent integration range. Lowering the index of ξ^μ gives a factor of a^2, leading to a factor of a^2 for each zero mode through the Jacobian. Note removal of zero modes from the heat kernel still takes away a factor of a. Similarly, for gravitino zero modes [17] we have

$$\int [D\psi_\mu]\,[D\bar{\psi}_\mu]\; exp\left[-\int d^4x \sqrt{\det g} g^{\mu\nu} \bar{\psi}_\mu \psi_\nu\right] = 1\,,$$
$$\int [D\psi_\mu]\,[D\bar{\psi}_\mu]\; exp\left[-a^2 \int d^4x \sqrt{\det g^{(0)}} g^{(0)\mu\nu} \bar{\psi}_\mu \psi_\nu\right] = 1\,. \qquad (3.3.28)$$

indicating that $a\psi_\mu$ and $a\bar{\psi}_\mu$ are the correctly normalized integration variables. The fermionic zero modes are associated with asymptotic supersymmetry transformations, with the anti–commutator of a pair of supersymmetry transformations generating a diffeomorphism with parameter $\bar{\epsilon}\Gamma^\mu\epsilon$. Since $\Gamma^\mu \sim a^{-1}$, and $\bar{\epsilon}\Gamma^\mu\epsilon$ is a independent, the correct normalization for ϵ is $\epsilon = a^{\frac{1}{2}}\epsilon_0$, where ϵ_0 has a a independent integration range. Thus a ψ_μ zero mode is equivalent to $a\psi_\mu \sim a^{\frac{3}{2}}\epsilon_0$, producing a factor of $a^{-\frac{3}{2}}$. Since the kinetic operator for fermion is of order a^{-1}, removing a fermion zero mode takes away a factor of $a^{-\frac{1}{2}}$. The net effect is to add back three times the amount we remove from the heat kernel.

To summarize, consider a theory with a length scale a and fields ϕ_i such that the kinetic operator for quadratic fluctuations about a given background has $n^0_{\phi_i} \geq 0$ number of zero

[17] Naively, integration over fermion zero modes should vanish but in supersymmetric theories the zero is cancelled by the infinities coming from integration over bosonic zero modes of the metric.

modes. Further, let the zero mode contribution to the path integral scale with a as

$$\mathcal{Z} \simeq a^{n^0_{\phi_i}\beta_{\phi_i}} \mathcal{Z}_0, \tag{3.3.29}$$

where $\mathcal{Z}_0$ does not scale with a, and the numbers β_{ϕ_i} have been explicitly determined as above for the vector, gravitino and metric zero modes. In particular

$$\beta_v = 1, \quad \beta_{\frac{3}{2}} = 3, \quad \beta_g = 2. \tag{3.3.30}$$

In that case, the log term for the partition function is given by

$$\ln \mathcal{Z}_{\log} = \left(K\left(0;t\right) + \sum_{\phi_i} n^0_{\phi_i}\left(\beta_{\phi_i} - 1\right) \right) \ln a, \tag{3.3.31}$$

where $K\left(0;t\right)$ is the coefficient of the t^0 term in the heat kernel expansion of the kinetic operator over of all fields ϕ_i, evaluated on both zero and non–zero modes. Further, for the $\mathcal{N} = 8$ kinetic operator, all the zero modes of the spin–$\frac{3}{2}$ and spin–2 fields are contained in the $\mathcal{N} = 4$ gravity multiplet [55]. This is quantized with untwisted boundary conditions (very similar to footnote 15) and its contribution has already been evaluated on the orbifold space in [67], where it was determined that

$$n_{\frac{3}{2}} = 2, \quad n_g = -2. \tag{3.3.32}$$

3.3.4 Logarithmic Corrections to the Twisted Index

Now we are in a position to put together the above results to show that the logarithmic corrections to the partition function $\mathcal{Z}_g$ vanish for the $\mathcal{N} = 8$ theory. To do so, we will need the coefficients $K\left(0;t\right)$ from the $\mathcal{N} = 4$ vector, gravitini and gravity multiplets, as well as the corresponding zero mode contributions. It has already been proven in [66] that an untwisted $\mathcal{N} = 4$ vector multiplet has a vanishing contribution to the log term about our background. Further, we have seen in Section 3.3.1 that $K(0;t)$ for the $\mathcal{N}=4$ vector multiplet with twisted boundary conditions vanishes, and in (3.3.23) that $K(0;t)$ for the $\mathcal{N}=4$ gravitini multiplets with twisted boundary conditions also vanishes. Hence, the only non–vanishing contributions to $\ln(\mathcal{Z}_g)_{\text{log}}$ come from the $\mathcal{N}=4$ gravity multiplet, which obeys untwisted boundary conditions. For this multiplet (see Eq. 5.46 of [67])

$$K(0;t) = -2. \tag{3.3.33}$$

Putting these results in (3.3.31) with (3.3.32), we find that

$$\ln(\mathcal{Z}_g)_{\text{log}} = 0, \tag{3.3.34}$$

which completes the proof that the logarithmic term in $\mathcal{Z}_g$ vanishes, in accordance with the microscopic results for B_6^g for $\mathcal{N}=8$ string theory.

3.4 Conclusions

In this chapter we exploited the heat kernel techniques developed in [66] to compute the logarithmic terms in the large charge expansion of the twisted index B_6^g in $\mathcal{N}=8$ string theory. These vanish, matching perfectly with the microscopic computation. Further, the result may be extended to the $\mathcal{N}=4$ case as follows. Firstly, since g commutes with all 16 supercharges in this case, we continue to classify fields into multiplets of the four–dimensional $\mathcal{N}=4$ supersymmetry algebra. Secondly, we need to focus only on the massless supergravity fields over the near–horizon geometry as only these can contribute to the log term. Finally, the g action on the various $\mathcal{N}=4$ multiplets can be found out using techniques similar to the ones employed in the $\mathcal{N}=8$ case. Since g acts geometrically on the compact directions, the $\mathcal{N}=4$ gravity multiplet still does not transform, and its contribution to the log term vanishes as per the analysis of [67]. The $\mathcal{N}=4$ vector multiplets would carry non–trivial g–charges, corresponding to non–trivial q–twists for these fields in the path integral $\mathcal{Z}_g$. We have already seen that the contribution to the log term from $\mathcal{N} = 4$ vector multiplets vanishes for arbitrary twists q. Therefore, the log term vanishes even for $\mathcal{N} = 4$ string theory.

As a final observation, we note that the microscopic expression for B_6^g contains exponentially suppressed corrections of the form

$$B^g_{6,p}\left(Q,P\right) \simeq e^{\frac{\pi\sqrt{Q^2P^2-(Q\cdot P)^2}}{NM}}\left(\mathcal{O}\left(1\right)+\ldots\right), \quad M \in \mathbb{Z}_+,\, M \geq 2. \tag{3.4.1}$$

Using the arguments of [66] for the untwisted index we find that the logarithmic correction vanishes about these saddle–points as well. Following through the arguments of [64], a natural candidate for the macroscopic origin of these corrections corresponds to a saddle–point of $\mathcal{Z}_g$ obtained by taking a $\mathbb{Z}_{NM}$ orbifold of the attractor geometry, where again g–twisted boundary conditions should be imposed on the fields in the path integral. From the analysis presented in this chapter, it follows that the log corrections to $\mathcal{Z}_g$ vanish about these saddle–points as well, which matches with the expectation from the microscopic side.

APPENDIX A

Dimensional Reduction & Computational details

A.1 10D to 4D map

Following [38, 39] we begin with a $\mathcal{N}=1$ supergravity theory coupled to $\mathcal{N}=1$ super Yang-Mills theory in ten dimensions and dimensionally reduce it to four dimensions. Though we start with a super Yang-Mills theory, at generic points in the moduli space only abelian gauge fields gives rise to massless fields in four dimensions and hence we will restrict to one U(1) gauge field. The ten dimensional action is given by

$$\frac{1}{32\pi}\int d^{10}x\sqrt{-G^{(10)}}e^{\Phi^{(10)}}\Big(R^{(10)}+G^{(10)MN}\partial_M\Phi^{(10)}\partial_N\Phi^{(10)}-\frac{1}{12}H^{(10)}_{MNP}H^{(10)MNP} -\frac{1}{4}F^{(10)I}_{MN}F^{(10)IMN}\Big)\ , \tag{A.1.1}$$

where $G^{(10)}_{MN}$, $B^{(10)}_{MN}$, $A^{(10)I}_{M}$, and $\Phi^{(10)}$ are the ten dimensional metric, anti-symmetric two form field, U(1) gauge field and the scalar dilaton respectively ($0\leq M\leq 9,\ I=1$), and

$$\begin{aligned} F^{(10)I}_{MN} &= \partial_M A^{(10)I}_N-\partial_N A^{(10)I}_M\,. \\ H^{(10)}_{MNP} &= \left(\partial_M B^{(10)}_{NP}-\frac{1}{2}A^{(10)I}_M F^{(10)I}_{NP}\right)+\text{cyclic permute. in } M,\ N,\ P\,. \end{aligned} \tag{A.1.2}$$

For dimensional reduction it is convenient to introduce the four dimensional fields $\hat{G}_{mn}$, $\hat{B}_{mn}$, $\hat{A}^I_m$, $A^{(a)}_\mu$, $G_{\mu\nu}$, $B_{\mu\nu}$. We will denote the corresponding ten dimensional fields by a superscript (10), for example $G^{(10)}_{\mu\nu}$. As we are only concerned with the coordinates 4 & 5, the range of indices are $(1 \leq m \leq 2,\, 0 \leq \mu \leq 3,\, 1 \leq a \leq 5)$. The map from the ten dimensional fields to four dimensional fields are given below. The main idea is to make apparent the four dimensional gauge transformations as derived from the ten dimensional gauge transformations .

$$\begin{aligned}
\hat{G}_{mn} &= G^{(10)}_{m+3,n+3}, \qquad \hat{B}_{mn} = B^{(10)}_{m+3,n+3}, \qquad \hat{A}^I_m = A^{(10)I}_{m+3}, \\
A^{(m)}_\mu &= \frac{1}{2}\hat{G}^{mn} G^{(10)}_{n+3,\mu}, \qquad A^{I+4}_\mu = -\left(\frac{1}{2}A^{(10)I}_\mu - \hat{A}^I_n A^{(n)}_\mu\right), \\
A^{(m+2)}_\mu &= \frac{1}{2}B^{(10)}_{(m+3)\mu} - \hat{B}_{mn}A^{(n)}_\mu + \frac{1}{2}\hat{A}^I_m A^{(I+4)}_\mu, \\
G_{\mu\nu} &= G^{(10)}_{\mu\nu} - G^{(10)}_{(m+3)\mu} G^{(10)}_{(n+3)\nu} \hat{G}^{mn} \\
B_{\mu\nu} &= B^{(10)}_{\mu\nu} - 4\hat{B}_{mn}A^{(m)}_\mu A^{(n)}_\nu - 2\left(A^{(m)}_\mu A^{(m+2)}_\nu - A^{(m)}_\nu A^{(m+2)}_\nu\right) \\
&\quad -2\hat{A}^I_m\left(A^{(I+4)}_\mu A^{(m)}_\nu - A^{(I+4)}_\nu A^{(m)}_\mu\right), \\
\Phi^{(4)} &= \Phi^{(10)} - \frac{1}{2}\ln\det\hat{G}\,.
\end{aligned} \tag{A.1.3}$$

Here $\hat{G}^{mn}$ denotes the inverse of the matrix $\hat{G}_{mn}$. We now combine the scalar fields $\hat{G}_{mn}$, $\hat{B}_{mn}$, and $\hat{A}^I_m$ into $O(2,3)$ matrix valued scalar field M. For this we regard $\hat{G}_{mn}$, $\hat{B}_{mn}$, and $\hat{A}^I_m$ as 2×2, 2×2, and 2×1 matrices respectively, and $\hat{C}_{mn} = \frac{1}{2}\hat{A}^I_m\hat{A}^I_n$ as a 2×2 matrix, and define M to be the 5×5 dimensional matrix

$$M = \begin{pmatrix} \hat{G}^{-1} & \hat{G}^{-1}(\hat{B}+\hat{C}) & \hat{G}^{-1}\hat{A} \\ (-\hat{B}+\hat{C})\hat{G}^{-1} & (\hat{G}-\hat{B}+\hat{C})\hat{G}^{-1}(\hat{G}+\hat{B}+\hat{C}) & (\hat{G}-\hat{B}+\hat{C})\hat{G}^{-1}\hat{A} \\ \hat{A}^T\hat{G}^{-1} & \hat{A}\hat{G}^{-1}(\hat{G}+\hat{B}+\hat{C}) & 1+\hat{A}^T\hat{G}^{-1}\hat{A} \end{pmatrix}. \tag{A.1.4}$$

Note that throughout Chapter 2 of the thesis we have relabelled coordinate 4 as coordinate 5 and vice versa. For example, following (A.1.3) we should have $\hat{G}_{11} = G^{(10)}_{44}$ and $\hat{G}_{22} =$

$G_{55}^{(10)}$ but instead we have $\hat{G}_{22} = G_{44}^{(10)}$ and $\hat{G}_{11} = G_{55}^{(10)}$.

The canonical metric $g_{\mu\nu}$ is related to the string metric $G_{\mu\nu}$ through the relation

$$g_{\mu\nu} = e^{-\Phi^{(4)}} G_{\mu\nu} \,. \tag{A.1.5}$$

The field strength $H^{\mu\nu\rho}$ corresponding to the antisymmetric two form $B_{\mu\nu}$ is dual to a scalar field Ψ through the relation

$$H^{\mu\nu\rho} = -\left(\sqrt{-g}\right)^{-1} e^{2\Phi^{(4)}} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \Psi \,. \tag{A.1.6}$$

The axion-dilaton modulus is

$$S = S_1 + S_2 = \Psi + i\, e^{-\Phi^{(4)}} \,. \tag{A.1.7}$$

A.2 A rough guide to computations

The computations in §2.5 are fairly algebraic and we will supplement them with few more steps here in this section. This is by no means comprehensive. Also note that in the final expressions of §2.5 we have omitted the superscript (10) from all ten dimensional fields. In the paragraphs below we shall mention the equation number from Chapter 2 followed by a few steps.

In arriving at (2.5.18) we have used

$$\hat{G}^{-1} = \begin{pmatrix} G_{55}^{-1} & 0 \\ 0 & G_{44}^{-1} \end{pmatrix}, \qquad \hat{C} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}(A_4)^2 \end{pmatrix}, \qquad \hat{B} = 0 \,,$$

$$\hat{G}^{-1}(\hat{B}+\hat{C}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}G_{44}^{-1}(A_4)^2 \end{pmatrix}, \qquad \hat{G}^{-1}\hat{A} = \begin{pmatrix} 0 \\ G_{44}^{-1}, \end{pmatrix}$$

$$(\hat{G}-\hat{B}+\hat{C})\hat{G}^{-1}(\hat{G}+\hat{B}+\hat{C}) = \begin{pmatrix} G_{55} & 0 \\ 0 & \left(G_{44}+\frac{1}{2}(A_4)^2\right)^2 G_{44}^{-1} \end{pmatrix}, \quad \text{(A.2.8)}$$

where

$$\begin{aligned} \hat{G}_{11} &= C_4^2, \qquad \hat{G}_{12} = G_{54}^{(10)} = G_{45}^{(10)} = \hat{G}_{21}\,, \\ \hat{G}_{22} &= G_{44}^{(10)} = -C_2^2 C_3^2 \sinh^2\gamma + C_2^2 e^{2\Phi}\cosh^2\gamma\,, \\ G_{40}^{(10)} &= C_2 C_3\left(e^{2\Phi}C_3^{-2}-1\right)\sinh\gamma\cosh\gamma\,, \\ G_{00}^{(10)} &= -\cosh^2\gamma + C_3^{-2}e^{2\Phi}\sinh^2\gamma\,. \end{aligned} \quad \text{(A.2.9)}$$

To obtain (2.5.19), first we should note that the ten dimensional dilaton (A.1.1) is twice the dilaton associated with the Harvey-Liu monopole (2.5.4).

$$\begin{aligned} S_2 &= e^{-\Phi^{(4)}} = e^{-2\Phi}\sqrt{\det\hat{G}} \\ &= e^{-2\Phi}C_2C_4\sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2\sinh^2\gamma}\,. \end{aligned} \quad \text{(A.2.10)}$$

To obtain (2.5.22) we have used

$$\begin{aligned} G_{00} &= G_{00}^{(10)} - (G_{40}^{(10)})^2\hat{G}^{44} = \frac{G_{44}G_{00}^{(10)} - (G_{40}^{(10)})^2}{\hat{G}^{44}} \\ &= \frac{1}{\hat{G}^{44}}\left[\left(C_2^2C_3^2 + \left(e^{2\Phi}C_3^{-2}-1\right)C_2^2C_3^2\cosh^2\gamma\right)\left(-1+\left(e^{2\Phi}C_3^{-2}-1\right)\sinh^2\gamma\right)\right. \\ &\quad \left. -\left(e^{2\Phi}C_3^{-2}-1\right)\sinh^2\gamma\cosh^2\gamma C_2^2C_3^2\right] \\ &= -\frac{e^{2\Phi}}{\left(e^{2\Phi}\cosh^2\gamma - C_3^2\sinh^2\gamma\right)}\,, \\ g_{00} &= e^{-\phi^{(4)}}G_{00} = -\frac{C_2C_4}{\sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2\sinh^2\gamma}}\,. \end{aligned} \quad \text{(A.2.11)}$$

Recall that the $g_{\mu\nu}$ here is twice the $g_{\mu\nu}$ of the S-T-U model. Hence, the g_{00} for the S-T-U model is

$$g_{00} = -\frac{C_2 C_4}{2\sqrt{e^{2\Phi}\cosh^2\gamma - C_3^2 \sinh^2\gamma}}\,. \tag{A.2.12}$$

To obtain (2.5.21), first we have to remember that all fields are independent of the coordinates 4 & 5. Secondly, we have to list the values of ten dimensional $B^{(10)}_{MN}$ fields to evaluate the four dimensional gauge fields. Lastly, in $r,\ \theta,\ \phi$ coordinates

$$\epsilon_{r\theta\phi} = r^2 \sin\theta\, \hat{\epsilon}_{r\theta\phi} \qquad \text{where} \quad \hat{\epsilon}_{r\theta\phi} = 1\,. \tag{A.2.13}$$

We also have

$$\hat{A}^1_1 = A^{(10)1}_5 = 0, \qquad \hat{A}^1_2 = A^{(10)}_4 = 2\sqrt{2}C_2\cosh\gamma\,\frac{1}{r}H(C_1 r)\,. \tag{A.2.14}$$

Choose partial gauge fixing $B^{(10)}_{05} = B^{(10)}_{i\,5} = 0$, we have from Harvey–Liu solution,

$$\begin{aligned}
H^{(10)}_{045} &= \cancel{\partial_0 B^{(10)}_{45}} - \frac{1}{2}A^{(10)}\cancel{F^{(10)}_{45}} + \cancel{\partial_5 B^{(10)}_{04}} - \frac{1}{2}\cancel{A^{(10)}_5}F^{(10)}_{04} + \cancel{\partial_4 B^{(10)}_{50}} - \frac{1}{2}A^{(10)}_4\cancel{F^{(10)}_{50}} \\
&= 0\,, \\
H^{(10)}_{r45} &= \partial_r B^{(10)}_{45} - \frac{1}{2}\cancel{A^{(10)}_r F^{(10)}_{45}} - \frac{1}{2}\cancel{A^{(10)}_5}F^{(10)}_{r4} - \frac{1}{2}A^{(10)}_4\cancel{F^{(10)}_{5r}} \\
&= 0\,.
\end{aligned} \tag{A.2.15}$$

Similar zero results for θ & ϕ coordinates implies $\hat{B}_{12} = B^{(10)}_{54} = 0$. Again from $H^{(10)}_{0\,i\,4} = 0$ we get $B^{(10)}_{40} = 0$ and from $H^{(10)}_{4r\theta} = 0$ we get $B^{(10)}_{\theta 4} = 0$. From (2.5.10) in the large r limit, we get

$$\begin{aligned}
H^{(10)}_{4\theta\phi} &= -C_2\cosh\gamma\, r^2\sin\theta\,\partial_r e^{2\phi} \\
&= 8\, C_2\cosh\gamma\, r H(C_1 r)\partial_r\left(\frac{H(C_1 r)}{r}\right)\sin\theta
\end{aligned}$$

$$
\begin{aligned}
&= \quad 8\, C_2 \cosh\gamma\, r H(C_1 r) \partial_r \left(C_1 \coth(C_1 r) - \frac{1}{r} \right) \\
&\underset{r\to\infty}{=} \quad 8\, C_2 \cosh\gamma\, \frac{H(C_1 r)}{r} \sin\theta \,. \qquad \text{(A.2.16)}
\end{aligned}
$$

Writing $H^{(10)}_{4\theta\phi}$ in terms of $B^{(10)}_{MN}$ and then matching with the large r behavior of (A.2.16) we get

$$
\begin{aligned}
H^{(10)}_{4\theta\phi} \quad &= \quad \cancel{\partial_4 B^{(10)}_{286}} - \frac{1}{2} A^{(10)}_4 F^{(10)}_{\theta\phi} + \cancel{\partial_\phi B^{(10)}_{4\theta}} - \frac{1}{2} A^{(10)}_\phi \cancel{F^{10}_{4\theta}} + \partial_\theta B^{(10)}_{\phi 4} - \frac{1}{2} \cancel{A^{(10)}_\theta} F^{(10)}_{\phi 4} \\
&= \quad 4\, C_2 \cosh\gamma\, \sin\theta\, \frac{1}{r} H(C_1 r) + \partial_\theta B^{(10)}_{\phi 4} \,, \\
B^{(10)}_{\phi 4} \quad &\underset{r\to\infty}{=} \quad -4\, C_2 \cosh\gamma\, \cos\theta\, \frac{1}{r} H(C_1 r) \,. \qquad \text{(A.2.17)}
\end{aligned}
$$

Finally, coming back to (2.5.21) we have

$$
\begin{aligned}
A^{(1)}_0 \quad &= \quad \frac{1}{2} \hat{G}^{55} \cancel{G^{(10)}_{50}} = 0 \,, \\
A^{(2)}_0 \quad &= \quad \frac{1}{2} \hat{G}^{44} G^{(10)}_{40} = \frac{C_3 \left(e^{2\Phi} C_3^{-2} - 1 \right) \sinh\gamma\, \cosh\gamma}{2 C_2 \left(e^{2\Phi} \cosh^2\gamma - C_3^2 \sinh^2\gamma \right)} \,, \\
A^{(5)}_0 \quad &= \quad - \left(\frac{1}{2} A^{(10)}_0 - \cancel{A^{(10)}_5} A^{(1)}_0 - A^{(10)}_4 A^{(2)}_0 \right) \\
&= \quad - \frac{\sqrt{2}\, r^{-1} H(C_1 r)}{\left(e^{2\Phi} \cosh^2\gamma \dots \right)} \left[\frac{1}{2} C_3^{-1} \sinh\gamma \left(e^{2\Phi} \cosh^2\gamma - C_3^2 \sinh^2\gamma \right) \right. \\
&\qquad \left. - \frac{1}{2} \cosh\gamma\, C_3 \left(e^{2\Phi} C_3^{-2} - 1 \right) \sinh\gamma\, \cosh\gamma \right] \\
&= \quad - \frac{\sqrt{2}\, r^{-1} H(C_1 r)}{\left(e^{2\Phi} \cosh^2\gamma \dots \right)} [C_3] \,, \\
A^{(4)}_i \quad &= \quad \frac{1}{2} B^{(10)}_{4\,i} + \frac{1}{2} A^{(10)}_4 A^{(5)}_i \,, \\
A^{(4)}_\phi \quad &= \quad \frac{1}{2} B^{(10)}_{4\phi} - 2 C_2 \cosh\gamma\, \frac{1}{r} H(C_1 r) \cos\theta \\
&\underset{r\to\infty}{=} \quad 0 \,, \\
A^{(5)}_i \quad &= \quad - \left(\frac{1}{2} A^{(10)}_i - \cancel{A^{(10)}_5} A^{(1)}_0 - A^{(10)}_4 \cancel{A^{(2)}_i} \right) \underset{r\to\infty}{=} -\sqrt{2} \cos\theta\, d\phi \,. \qquad \text{(A.2.18)}
\end{aligned}
$$

To obtain (2.5.20) we begin with the four dimensional $H_{\mu\nu\rho}$

$$
\begin{aligned}
H_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} + 2A^{(a)}_\mu L_{ab} F^{(b)}_{\nu\rho} + \text{c.p.} \\
&= \partial_\mu B_{\nu\rho} + 2\cancel{A^{(1)}_\mu} F^{(3)}_{\nu\rho} + 2A^{(2)}_\mu F^{(4)}_{\nu\rho} + 2\cancel{A^{(3)}_\mu} F^{(1)}_{\nu\rho} + 2A^{(4)}_\mu F^{(2)}_{\nu\rho} - 2A^{(5)}_\mu F^{(5)}_{\nu\rho} + \text{c.p.} \\
&= \partial_\mu \Big[B^{(10)}_{\mu\nu} - 2\left(\cancel{A^{(1)}_\mu A^{(3)}_\nu} - \cancel{A^{(1)}_\nu A^{(3)}_\nu}\right) - 2\cancel{A^{(10)}_5}\left(A^{(5)}_\mu A^{(1)}_\nu - A^{(5)}_\nu A^{(1)}_\mu\right) \\
&\quad -2A^{(10)}_4 \left(A^{(5)}_\mu A^{(2)}_\nu - A^{(5)}_\nu A^{(2)}_\mu\right)\Big] + 2A^{(2)}_\mu F^{(4)}_{\nu\rho} + 2A^{(4)}_\mu F^{(2)}_{\nu\rho} - 2A^{(5)}_\mu F^{(5)}_{\nu\rho} + \text{c.p.} \\
&= H^{(10)}_{\mu\nu\rho} + \frac{1}{2}A^{(10)}_\mu F^{(10)}_{\nu\rho} - 2\partial_\mu \left[A^{(10)}_4 \left(A^{(5)}_\mu A^{(2)}_\nu - A^{(5)}_\nu A^{(2)}_\mu\right)\right] + 2A^{(2)}_\mu F^{(4)}_{\nu\rho} \\
&\quad +2A^{(4)}_\mu F^{(2)}_{\nu\rho} - 2A^{(5)}_\mu F^{(5)}_{\nu\rho} + \text{c.p.}\,, \\
H_{0\theta\phi} &= H^{(10)}_{0\theta\phi} + 2A^{(10)}_4 A^{(2)}_0 F^{(10)}_{\theta\phi} \\
&= -2C_3^{-1}\sinh\gamma\, r^2 \sin\theta\, e^{2\Phi}\partial_r\Phi - \frac{8C_3\cosh^2\gamma\, \sinh\gamma\, r^{-1} H(C_1 r)\left(e^{2\Phi}C_3^{-2} - 1\right)}{\left(e^{2\Phi}\cosh^2\gamma - C_3^2\sinh^2\gamma\right)} \\
&\underset{r\to\infty}{=} \frac{8C_3\sinh\gamma\, \sin\theta\, r^{-1} H(C_1 r)}{\left(e^{2\Phi}\cosh^2\gamma\ldots\right)}\left[C_3^{-2}\left(e^{2\Phi}\cosh^2\gamma\ldots\right) - \cosh^2\gamma\left(e^{2\Phi}C_3^{-2} - 1\right)\right] \\
&\underset{r\to\infty}{=} \frac{8C_3\sinh\gamma\, r^2\sin\theta}{\left(e^{2\Phi}\cosh^2\gamma\ldots\right)}\frac{H(C_1 r)}{r^3} \underset{r\to\infty}{=} -\frac{C_3\sinh\gamma\, r^2\sin\theta}{\left(e^{2\Phi}\cosh^2\gamma\ldots\right)}\partial_r\left(\frac{1}{e^{-2\Phi}}\right) \\
&\underset{r\to\infty}{=} \frac{C_3\sinh\gamma\, r^2\sin\theta}{\left(e^{2\Phi}\cosh^2\gamma\ldots\right)} e^{4\Phi}\partial_r\left(e^{-2\Phi}\right)\,. \qquad \text{(A.2.19)}
\end{aligned}
$$

From (A.1.6) and matching with (A.2.19) we get

$$
\begin{aligned}
H_{0\theta\phi} &= -C_2C_4\, r^2\sin\theta\, e^{2\Phi^{(4)}}\partial_r S_1 = -\frac{C_2C_4\, r^2\sin\theta\, e^{4\Phi}\partial_r S_1}{C_2^2C_4^2\left(e^{2\Phi}\cosh^2\gamma - C_3^2\sinh^2\gamma\right)}\,, \\
S_1 &\underset{r\to\infty}{=} -C_2C_3C_4\sinh\gamma\, e^{-2\Phi}\,. \qquad \text{(A.2.20)}
\end{aligned}
$$

Following the discussion around (2.4.21) we note that the S_1 associated with the S-T-U model is minus (A.2.20).

The next step is to enforce the field redefinitions (2.5.23) primarily on the gauge fields.

In obtaining (2.5.28) we encounter the following redefinitions of $\{A_0^{(a)}\}$ and $\{A_i^{(a)}\,dx^i\}$:

$$
\{A_0^{(a)}\} \;\to\; \begin{pmatrix} 0 \\ \frac{1}{2}A_0^{(2)}+\frac{1}{2}A_0^{(4)}+\frac{1}{\sqrt{2}}A_0^{(5)} \\ 0 \\ \frac{1}{2}A_0^{(2)}+\frac{1}{2}A_0^{(4)}-\frac{1}{\sqrt{2}}A_0^{(5)} \\ \frac{1}{\sqrt{2}}A_0^{(2)}-\frac{1}{\sqrt{2}}A_0^{(4)} \end{pmatrix},
$$

$$
\{A_i^{(a)}\,dx^i\} \;\to\; \begin{pmatrix} 0 \\ \frac{1}{2}A_i^{(4)}+\frac{1}{\sqrt{2}}A_0^{(5)} \\ 0 \\ \frac{1}{2}A_i^{(4)}-\frac{1}{\sqrt{2}}A_0^{(5)} \\ -\frac{1}{\sqrt{2}}A_0^{(4)} \end{pmatrix}. \tag{A.2.21}
$$

Let us focus on one example term $\frac{1}{\sqrt{2}}A_0^{(2)}-\frac{1}{\sqrt{2}}A_0^{(4)}$,

$$
\begin{aligned}
\frac{1}{\sqrt{2}}A_0^{(2)}-\frac{1}{\sqrt{2}}A_0^{(4)} &= \left(-\frac{1}{2\sqrt{2}\left(e^{2\Phi}\cosh^2\gamma-C_3^{-2}\sinh^2\gamma\right)}C_2^{-1}C_3\sinh\gamma\,\cosh\gamma\right) \\
&\quad \left[\left(e^{2\Phi}C_3^{-2}-1\right)+C_2^2\left(\frac{H(C_1r)}{r}\right)^2\right] \\
&= (\dots)\left[e^{2\Phi}C_3^{-2}-1+c_2^2C_3^2+4C_2^2C_1^2-C_2^2e^{2\Phi}\right] \\
&= (\dots)4C_3^{-2}C_1^2C_2^2\left[e^{2\Phi}\cosh^2\gamma-C_3^{-2}\sinh^2\gamma\right] \\
&= -\sqrt{2}C_1^2C_2C_3^{-1}\sinh\gamma\,\cosh\gamma=\text{const.}\simeq 0\,.
\end{aligned} \tag{A.2.22}
$$

We can obtain (2.5.42) by noting that from (2.4.44) and (2.5.40) we get,

$$
\frac{T_2}{U_2}=\frac{H^2}{H^3}=\frac{\frac{1}{r}+\frac{2}{\sqrt{8\xi_2\rho_2\sigma_2}}\frac{\rho_2\xi_2}{|\xi|}}{\frac{1}{r}+\frac{2}{\sqrt{8\xi_2\rho_2\sigma_2}}\frac{\sigma_2\xi_2}{|\xi|}}=\frac{\frac{-1}{C_2\cosh\gamma\sqrt{2C_2C_3C_4}}+\kappa-\frac{1}{\hat{r}}}{\frac{-1}{C_2\cosh\gamma\sqrt{2C_2C_3C_4}}-\kappa-\frac{1}{\hat{r}}}\,. \tag{A.2.23}
$$

which for $r \to \infty$ implies

$$\kappa = \sqrt{\frac{\zeta_2}{8\rho_2\sigma_2}}\frac{\text{sign}(\rho_2 - \sigma_2)(\rho_2 - \sigma_2)}{|\zeta|} = \sqrt{\frac{\zeta_2}{8\rho_2\sigma_2}}\frac{|\rho_2 - \sigma_2|}{|\zeta|}\,. \tag{A.2.24}$$

APPENDIX B

The g–Charges for the $\mathcal{N}=8$ Supergravity Fields

In this appendix we shall review how the g–twist acts on the fields of four–dimensional $\mathcal{N}=8$ supergravity. As g commutes with the $\mathcal{N}=4$ subalgebra of the full $\mathcal{N}=8$ algebra, we expect that the $\mathcal{N}=8$ gravity multiplet will decompose into $\mathcal{N}=4$ multiplets, each of which carry some charge under g. We shall obtain these charges by working with Type IIB supergravity compactified on $T^4 \otimes T^2$ and studying the action of g on the supergravity fields, which are the graviton h_{MN}, the two–form B_{MN}, the three–form flux C_{MNP} and two 16–component Majorana–Weyl spinors.[1] This action of the g-twist on the Type IIB supergravity fields compactified on $T^4 \otimes T^2$ can be realized in an appropriate complex coordinate system(z^1, z^2) on T^4 and (z^3) on T^2 as [31].

$$dz^1 \to e^{\frac{2\pi i}{N}} dz^1\,, \qquad dz^2 \to e^{-\frac{2\pi i}{N}} dz^2\,, \qquad d\bar{z}^1 \to e^{-\frac{2\pi i}{N}} d\bar{z}^1\,, \qquad d\bar{z}^2 \to e^{\frac{2\pi i}{N}} d\bar{z}^2\,,$$
$$dz^3 \to dz^3\,, \qquad d\bar{z}^3 \to d\bar{z}^3\,. \tag{B.0.1}$$

[1]The indices M, N take values $0, \ldots, 9$, while μ, ν will take values $0, \ldots, 3$ which label the non–compact directions. The indices m, n will take values $4, \ldots, 9$ and label the compact directions.

Multiplet	Number of Multiplets	g–Eigenvalue
Gravity	1	1
Gravitino	2	$e^{-\frac{2\pi i}{N}}$
Gravitino	2	$e^{\frac{2\pi i}{N}}$
Vector	4	1
Vector	1	$e^{-\frac{4\pi i}{N}}$
Vector	1	$e^{\frac{4\pi i}{N}}$

Table B.1: The g–charges of the $\mathcal{N} = 4$ multiplets. It is natural to expect the gravity multiplet to remain invariant since the 4D spacetime metric $h_{\mu\nu}$ is a spacetime field and is unaffected by coordinate transformations on the internal directions.

These transformations can be thought of as individual rotations along the two cycles of T^4. The g–action on the ten dimensional fields is realized as a field transformation under the different representations of the Lorentz group. In the four dimensional theory obtained on compactification, the g–action may be thought of as an internal symmetry.

The compactification of the $\mathcal{N} = 2$ supergravity fields on $T^4 \otimes T^2$ gives one $\mathcal{N} = 8$ gravity multiplet in four dimensions. This contains one graviton $h_{\mu\nu}$, 8 spin–$\frac{3}{2}$ Majorana fields, 28 spin–1 fields, 56 spin–$\frac{1}{2}$ Majorana fields and 70 real scalars. The spin–2 field $h_{\mu\nu}$ is just the spacetime metric. The spin–1 fields come from $G_{\mu m}$, $B_{\mu m}$, $C_{mn\mu}$ and A_μ. The scalars come from G_{mn}, B_{mn}, A_m, C_{mnp}, dualizing the components $C_{m\mu\nu}$ of the three–form field, and the axion and the dilation. The origin of the 8 spin–$\frac{3}{2}$ fields and 48 spin–$\frac{1}{2}$ fields lie in the spin–$\frac{3}{2}$ ψ^α_μ and spin–$\frac{1}{2}$ φ^α_m multiplets obtained on compactification of the two 16 component Majorana–Weyl spinors over T^6. Eight of the remaining spin–$\frac{1}{2}$ fields come from the compactification of the two ten-dimensional $\psi^\alpha_{[10]}$ spinors.

The g–twist commutes with 16 of the 32 supersymmetries. Hence we split the $\mathcal{N} = 8$ gravity multiplet into one $\mathcal{N} = 4$ gravity multiplet, four gravitino, and six vector multiplets. As g commutes with the $\mathcal{N} = 4$ subalgebra, all the members of a given $\mathcal{N} = 4$ multiplet carry the same g–charge. The g-charge of every field has been found to conform with the g-charge of the membership multiplet. The final results of this computation have been summarized in Table B.1.

APPENDIX C

Useful Summation Formulae

In this appendix we shall list various formulae useful when summing over the conical terms in the Heat Kernels. Let us denote the $SU(2)$ Weyl character as χ_l and the $sl(2,R)$ Harish-Chandra (global) character as χ_λ, then

$$\chi_l\left(\frac{\pi s}{N}\right) = \frac{\sin\left[\frac{\pi s(1+2l)}{N}\right]}{\sin\left[\frac{\pi s}{N}\right]}, \qquad \chi_{l+\frac{1}{2}}\left(\frac{\pi s}{N}\right) = \frac{\sin\left[\frac{2\pi s(1+l)}{N}\right]}{\sin\left[\frac{\pi s}{N}\right]},$$
$$\chi^b_\lambda\left(\frac{\pi s}{N}\right) = \frac{\cosh\left(\pi - \frac{2\pi s}{N}\lambda\right)}{\cosh(\pi\lambda)\sin\left(\frac{\pi s}{N}\right)}, \qquad \chi^f_\lambda\left(\frac{\pi s}{N}\right) = \frac{\sinh\left(\pi - \frac{2\pi s}{N}\lambda\right)}{\sinh(\pi\lambda)\sin\left(\frac{\pi s}{N}\right)}, \tag{C.0.1}$$

where b & f respectively denote bosons and fermions.

In the bosonic computation, the following formulae will be useful,

$$\sum_{l=0}^{\infty}\chi_l\left(\frac{\pi s}{N}\right) = \frac{1}{2\sin^2\left[\frac{\pi s}{N}\right]}, \qquad \sum_{l=1}^{\infty}\chi_l\left(\frac{\pi s}{N}\right) = \frac{\cos\left[\frac{2\pi s}{N}\right]}{2\sin^2\left[\frac{\pi s}{N}\right]},$$
$$\sum_{l=0}^{\infty}\int_0^\infty d\lambda\chi^b_\lambda\left(\frac{\pi s}{N}\right)\chi_l\left(\frac{\pi s}{N}\right) = \frac{1}{4\sin^4\left[\frac{\pi s}{N}\right]},$$
$$\sum_{l=1}^{\infty}\int_0^\infty d\lambda\chi^b_\lambda\left(\frac{\pi s}{N}\right)\chi_l\left(\frac{\pi s}{N}\right) = \frac{\cos\left[\frac{2\pi s}{N}\right]}{4\sin^4\left[\frac{\pi s}{N}\right]}. \tag{C.0.2}$$

In the fermionic computation, the following formulae will be useful,

$$\sum_{l=0}^{\infty}\chi_{l+\frac{1}{2}}\left(\frac{\pi s}{N}\right)=\frac{\cos\left[\frac{\pi s}{N}\right]}{2\sin^2\left[\frac{\pi s}{N}\right]},\qquad \sum_{l=1}^{\infty}\chi_{l+\frac{1}{2}}\left(\frac{\pi s}{N}\right)=\frac{\cos\left[\frac{3\pi s}{N}\right]}{2\sin^2\left[\frac{\pi s}{N}\right]},$$
$$\sum_{l=0}^{\infty}\int_0^{\infty}d\lambda\chi_\lambda^f\left(\frac{\pi s}{N}\right)\chi_{l+\frac{1}{2}}\left(\frac{\pi s}{N}\right)=\frac{\cos^2\left[\frac{\pi s}{N}\right]}{4\sin^4\left[\frac{\pi s}{N}\right]},$$
$$\sum_{l=1}^{\infty}\int_0^{\infty}d\lambda\chi_\lambda^f\left(\frac{\pi s}{N}\right)\chi_{l+\frac{1}{2}}\left(\frac{\pi s}{N}\right)=\frac{\cos\left[\frac{\pi s}{N}\right]\cos\left[\frac{3\pi s}{N}\right]}{4\sin^4\left[\frac{\pi s}{N}\right]}. \tag{C.0.3}$$

Also, the following summation formulae will be useful

$$\sum_{s=1}^{N-1}\frac{1}{\sin^2\left[\frac{\pi s}{N}\right]}=\frac{N^2-1}{3},\qquad \sum_{s=1}^{N-1}\sin^2\left[\frac{\pi s}{N}\right]=\frac{N}{2},\qquad \sum_{s=0}^{N-1}\cos^2\left[\frac{\pi s}{N}\right]=\frac{N}{2},$$
$$\sum_{s=1}^{N-1}\frac{1}{\sin^4\left[\frac{\pi s}{N}\right]}=\frac{N^4+10N^2-11}{45},\qquad \sum_{s=1}^{N-1}\frac{\cos\left[\frac{2\pi s}{N}\right]}{\sin^2\left[\frac{\pi s}{N}\right]}=\frac{N^2-6N+5}{3},$$
$$\sum_{s=1}^{N-1}\frac{\cos\left[\frac{4\pi s}{N}\right]}{\sin^2\left[\frac{\pi s}{N}\right]}=\frac{N^2-12N+23}{3},\qquad \sum_{s=1}^{N-1}\frac{\cos\left[\frac{2\pi s}{N}\right]}{\sin^4\left[\frac{\pi s}{N}\right]}=\frac{N^4-20N^2+19}{45},$$
$$\sum_{s=1}^{N-1}\frac{\cos\left[\frac{4\pi s}{N}\right]}{\sin^4\left[\frac{\pi s}{N}\right]}=\frac{N^4-110N^2+360N-251}{45}. \tag{C.0.4}$$

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