EXPLORATIONS IN ENTANGLEMENT OF PURIFICATION AND MULTIPARTITE ENTANGLEMENT

By

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Contents

Synopsis

The quantum states of multiparticle systems contain correlations between the subsystems of the composite system. These correlations are not only of theoretical interest, but also they have been proved to be useful in various quantum information processing and computation tasks. Thus, characterizing and quantifying these correlations are very important for the development of quantum information theory as this can have potential applications in diverse areas of physics.

These correlations can be classified in two different categories called the classical correlations and the quantum correlations. Given a quantum state one tries to capture the total amount of correlation comprising of the classical correlation and quantum correlations. Of the total correlations, the quantum correlations can account for many counter-intuitive features in general [1, 19]. In particular, entanglement is a type of quantum correlation that has been successfully employed to interpret many different phenomena which cannot be accounted for by the laws of classical physics [1]. Not only this, the triumph of entanglement lies in the fact that it can be used as an useful resource for performing various quantum information tasks which cannot be performed by the classical correlations alone. It has also been identified as the basic ingredient for different quantum communication protocols like super-dense coding [2] and quantum teleportation [4]. However, even with such advancement in understanding the entanglement, one is far away from total characterization of entanglement and therefore its utilization in different quantum information tasks. One such difficulty lies in the fact that entanglement is hard to calculate for mixed states with increasing dimensions, and number of parties and for most cases a closed formula for an entanglement measure for mixed states of arbitrary dimensions, parties and ranks is still missing.

Efforts have been made to quantify the total correlations in the form of the quantum mutual information and the entanglement of purification [34]. Though the quantum mutual information [68] is a well explored measure of total correlation, the entanglement of purification is not understood in great detail. In terms of the importances in various quantum information tasks, the regularized entanglement of purification has been shown to give the optimal visible compression rate for mixed states [73]. One remarkable property of it is the monogamy inequality it shares with the quantum advantage of dense coding [41]. Thus the findings on the entanglement of purification gets translated to the quantum advantage of dense coding as well. However, despite its importance many important characteristic properties of entanglement of purification remains unknown owing to the absence of a closed analytical formula for it. In particular finding the exact values, lower bounds for the entanglement of purification and its monogamy and additivity (or non-additivity) properties need to be explored extensively since they have the capability of further enhancing the understanding for the entanglement of purification.

Bipartite entanglement is not the only type of entanglement that makes the arena of quantum physics rich and interesting, but the presence of genuine multipartite entanglement is another fascinating area which baffles quantum physicists even today [1]. The criteria of detection and quantization of genuine multipartite entanglement is an area of vibrant research. One of the measure is generalized geometric measure [52]. With increasing dimensions, number of parties and ranks of the quantum states, the evaluation of the measure of genuine multipartite entanglement becomes extremely arduous. However, with the use of some symmetry [61] this calculation can be simplified for some important classes of states.

In view of the above development, necessity and interest we formulate related questions that we address in the thesis. Thus, we state here the main results obtained in the proposed thesis.

- We study various important aspects of the total correlation captured by the entanglement of purification. In this respect, we have explored some of the most important properties of entanglement of purification which include its exact values, better lower bounds, monogamy and polygamy properties as well as the question of additivity of entanglement of purification. We also discuss its implications on the quantum advantage of dense coding.
- We explore the exact values of the measure of genuine multiparty correlation called the generalized geometric measure by exploiting symmetry properties of the quantum states, for different mixed states of varying dimensions, ranks and number of parties.

In the first part of the thesis, we study the various important aspects of the total correlation captured by the entanglement of purification [96]. For bipartite pure and mixed quantum states, in addition to the quantum mutual information, there is another measure of total correlation, namely, the entanglement of purification. The definition of entanglement of purification is motivated operationally, trying to see how many singlets are required to construct a quantum state with vanishing amount of communication asymptotically [34]. Unlike the quantum mutual information, the characteristics of this measure of total correlation remains unexplored owing to absence of a closed analytical formula and the arduous optimization that needs to be performed in higher dimensional Hilbert space. Thus, bringing out the properties of this measure is a step forward to understanding of this measure, which finds its usefulness in various quantum information tasks. Among various important properties, we study the monogamy, polygamy, exact values, better lower bounds and additivity properties of the entanglement of purification for pure and mixed states [96]. In this paper, we show that, in contrast to the quantum mutual information which is strictly monogamous for any tripartite pure states, the entanglement of purification can be polygamous for the same. This shows that there can be genuinely two types of total correlation across any bipartite cross in a pure tripartite state. Also, we find better lower bounds and actual values of the entanglement of purification for different classes of bipartite, higher dimensional bipartite and tripartite mixed states. In regard to the additivity property, we show that if entanglement of purification is not additive on tensor product states, it is actually subadditive. Using the above results that we obtain, we identify some new classes of states which are additive on tensor products for entanglement of purification. By virtue of the monogamy relation of the entanglement of purification with the quantum advantage of dense coding, we translate the results mentioned as above to the quantum advantage of dense coding. Specifically, we show that for tripartite pure states, the quantum advantage of dense coding is strictly monogamous. Also we find the exact values and upper bounds for the quantum advantage of dense coding and thereafter we show that if it is nonadditive, then it must be superadditive on tensor product states.

The theory of quantum entanglement is one of the most promising field of research in quantum information science. It has found application in numerous quantum information processing tasks such as the quantum teleportation, remote state preparation, superdense coding and the prime factorization to name a few [1]. Proper characterization and manipulation of quantum entanglement holds reserves for implications in quantum technology. The study of entanglement has been many-fold and also along many directions. First, it characterizes the quantum correlation of physical systems. Secondly, it has been used to characterize the total correlation of a quantum system in the form of entanglement of purification [34]. Thirdly, efforts have been made to understand and capture the quantitative aspects of genuine multiparty entanglement, which is different from the bipartite entanglement. In this respect a measure of genuine multipartite entanglement called the generalized geometric measure has been proposed [52], which has been shown to be of importance in detecting quantum phase transitions and other quantum information tasks [52].

In the second part of the thesis, we explore the genuine multipartite entanglement captured by the generalized geometric measure (GGM) [97]. The GGM of pure states has already been computed efficiently in several systems for arbitrary number of parties. In this work [97], we define the GGM for mixed states via the convex roof. However, computing genuine multipartite entanglement of an arbitrary multipartite mixed state is in general not an easy task as it usually involves complex optimization. Here we show that exploiting symmetries of different paradigmatic classes mixed states [61, 97], we can compute the generalized geometric measure for such classes of mixed states. The chosen states have different ranks and consist of an arbitrary number of parties. To deal with the obstacle of evaluating the convex roof extension, we use symmetry properties of certain multiparty quantum states and simplify the evaluation of GGM for these classes of mixed states. In particular, we first present the exact value of GGM for certain classes of rank 2 and rank 3 mixed states with arbitrary number of qubits. We then compute the GGM for a specific class of states which is a mixture of Greenberger-Horne-Zeilinger (GHZ) and all the Dicke states, having a variety of ranks. The common property that all these classes possesses is that they remain invariant under the action of same symmetric local unitary operators on each qubit. Moreover, we find the GGM of a class of tripartite states of rank 4 which remains unaltered under different local unitaries on each party. Finally, we show that such symmetry properties can lead to an exact expression of GGM for a class of multiqudit states having varied ranks [97].

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Chapter 1

Introduction

Characterization and quantification of quantum entanglement [1] lies at the heart of quantum information theory, since its early recognition as spooky action at a distance[2] in the Einstein-Podolsky-Rosen article [3]. In quantum world there exists correlations between the subsystems of the multiparticle quantum states, which are of classical and quantum origin. Proper characterization of these correlations is one of the many daunting challenges facing the development of the theory of quantum information and computation. Given a bipartite or multipartite state one tries to characterize the amount of classical correlation, quantum correlation and quantum entanglement contained in the composite quantum system. Different correlations can arise depending on the state preparation procedure, measurements and quantum operations performed on the system. These correlations can account for many counter-intuitive features of the quantum world. In particular, entanglement is a property that has been successfully employed to interpret several physical phenomena which cannot be understood using only the laws of classical physics [1]. It has also been identified as an useful resource for performing different quantum communication protocols like super-dense coding [2], quantum teleportation [4], quantum cryptography [5], remote-state preparation [6, 7] and quantum computational tasks such as the one-way quantum computer [8]. Moreover, entanglement has been shown to be a necessary ingredient in studying quantum state tomography [9], quantum metrology [10], cooperative quantum phenomena in many-body systems like quantum phase transitions [11], etc.

There are myriad challenges facing the growth and development of quantum information science, in particular the theory of the correlations in the composite quantum states. Among many interesting and desirable properties that the correlation measures are expected to follow, a few are particularly interesting and intriguing. This is currently an active field of research and in the past several measures of entanglement with different motivation and interpretations have been proposed over the course of time. In this respect many measures have been defined in mainly two different realms, one is the entanglement- separability paradigm and the other is the information theoretic paradigm. Different correlations have been proposed comprising of the quantum as well as total correlation that belong to these two different paradigm. Of this, the entanglement measures belong to the entanglement -separability paradigm, while the measures like quantum discord and quantum work deficit are quantum correlations measures defined in the information theoretic paradigm. Similarly, in the case of total correlation measure, the entanglement of purification is defined in the entanglement- separability paradigm. In this thesis we focus on two such measures namely the generalized geometric measure and the entanglement of purification respectively. These correlation measures have several important and striking properties, which we discuss briefly as follows.

One fundamental property of quantum correlations in multiparty quantum states is that it can be monogamous [12]. To state this in a qualitative way, if a correlation measure is monogamous, then this says that in a composite quantum state, if two subsystems are more correlated with each other, then they will share a less amount of correlation with the other subsystems with respect to that measure of correlation. In other words, it puts a restriction on the share-ability of correlation between the different parties of a composite quantum state. The monogamy of quantum correlations thus dictates that if the two subsystems are maximally quantum correlated with each other, then they cannot get quantum correlated to any other subsystem at the same time. The measures of classical correlation are never monogamous and therefore are considered to be freely shareable. But, not all measures of quantum correlation satisfy monogamy [13–17]. For example, the square of concurrence and the squashed entanglement satisfy the monogamy inequality [18], whereas the relative entropy of entanglement, the entanglement of formation and other measures do not satisfy monogamy in general. Recently, it has been shown that the monogamous character is not an intrinsic property of other quantum correlation measures. In particular, the quantum discord [19] for tripartite states does not obey monogamy in general [20–22]. However, interestingly, though a quantum correlation measure may not satisfy monogamy, yet the quantum correlation measure raised to a power will certainly obey monogamy [23]. It has been shown that the square of the concurrence, which is a monotonic function of entanglement of formation, is monogamous. Similarly, it has been shown that the square of the quantum discord also satisfies monogamy. The concept of monogamy, not only is important from fundamental point of view, it finds practical importance too. For example, the monogamy of quantum correlations plays a crucial role in the security of quantum cryptography [24].

While the monogamy is an important property to study for various correlation measures, still there remains other desirable properties that the correlation measures are expected to obey from the perspective of being physically meaningful. One such property is the additivity on tensor product of density matrices [25]. The property of additivity on tensor product states dictates that a correlation measure is an additive measure if the value of that measure on the tensor product of density matrices is simply equal to the addition of the values of that correlation measure on the individual density matrices forming the tensor product state. The quantum mutual information is an additive measure of total correlation and the squashed entanglement is another additive measure of quantum correlation [26]. However, all correlation measures are not yet proved to be additive [27]. There are measures of entanglement and capacity of channels that have been proved to be non-additive [28–31]. For example the relative entropy of entanglement is proved to be non-additive [32] and there is strong indication that the bipartite distillable entanglement is also non-additive [27]. Also, the additivity of entanglement of formation still remains an open question, and it is conjectured to obey a strong super-additivity condition [33]. Thus, the question of additivity of the different correlation measures is one of the intriguing and yet to be solved question in the realm of quantum information theory. In the next section we discuss in details on the aspects of total and quantum correlation measures.

1.0.1 Total and quantum correlations

Let us consider a bipartite quantum state ρ consisting of subsystems ρ_A and ρ_B, with marginal states $\rho_A = tr_B(\rho)$ partial trace over part B and $\rho_A = tr_B(\rho)$. A fundamental question in quantum information theory is to classify and quantify the correlations present in the quantum state ρ. For the classification issue, one usually distinguishes between total, quantum, and classical correlations. Landauer showed that the amount of information stored is proportional to the work required to erase the memory of the information content. These ideas were further developed by other researchers into a connection of classical information and thermodynamics. In this regard the quantum mutual information denoted by $I(A:B)$ was proposed as a measure of total correlation. It was motivated operationally by showing that exactly $I(A:B)$ amount of energy was required to remove all correlations from a quantum state ρ_{AB}. Therefore, the quantum mutual information was not motivated from the entanglement separability paradigm, i.e., not in terms of singlets to construct a quantum state, but rather from a thermodynamic or an information-theoretic perspective.

On the other hand, the theory of quantum entanglement in the entanglement-separability paradigm also aims at quantifying and characterizing uniquely quantum correlations. This is done by analyzing how entangled quantum states can be processed and transformed by quantum operations. A very important role in the theory is played by the class of Local Operations and Classical Communication (LOCC), since quantum entanglement is non-increasing under these operations. By considering this class of operations one is able to clearly distinguish between the quantum entanglement and the classical correlations that are present in the quantum state. This theory is quite successful, and this has led the authors in [34] pose the question whether one can similarly construct a theory of purely classical correlations in quantum states and their behavior under local or nonlocal processing. However, there the authors note that such an effort cannot succeed in general since merely local actions can convert quantum entanglement into classical correlations. Namely, Alice and Bob who possess an entangled state $|\Psi\rangle = \sum_i \sqrt{\lambda_i}|a\rangle_i \otimes |b\rangle_i$ with Schmidt coefficients $_i$ can, by local measurements, obtain a joint probability distribution with mutual information equal to $H(\lambda)$. Thus, from the above observation, it does not seem possible to separate the classical correlations from the entanglement in an operational way similar to that of quantum entanglement. However, note that it may be possible to separate quantum and classical correlations in a nonoperational way, see for example Ref. [19]. However, an operational approach to the quantification of quantum and classical correlations was recently formulated in Ref. [35].

In the approach to correlation as described above, the authors in [34] propose to treat quantum entanglement and classical correlation in a unified framework, namely they express both correlations in units of entanglement. This theory of all correlations may have potential applications outside quantum information theory as well. Researchers have started to look at entanglement properties of many-particle systems for example at quantum phase transitions [11]. Instead of considering the entanglement of formation in these studies, one may also investigate such properties using the total correlation measure.

In an effort as stated in the above paragraph, the authors, in their paper [34], introduce such a measure, called the entanglement of purification. It is important to note that the entanglement of purification is not an entanglement measure, but a measure of correlations expressed in terms of the entanglement of a pure state. Many times it has been seen that in quantum information theory that questions in the asymptotic approximate regime are easier to answer than the exact non-asymptotic questions. Thus, in this vein the authors in [34] ask how to create a bipartite quantum state ρ in the asymptotic regime, allowing approximation, from an initial supply of EPR-pairs by means of local operations and asymptotically vanishing communication. This latter class of operations were denoted as LOq (Local Operations with o(n) communication in the asymptotic regime) versus the class LO for strictly Local Operations. After this they define this formation cost E_{LOq} as follows:

$$E_{LOq}(\rho) = lim_{\epsilon\rightarrow 0}\{\frac{m}{n}|L_{LOq}, D(L_{LOq}(|\Psi\rangle\langle\Psi|^{\otimes m}), \rho^{\otimes n}) \leq \epsilon\}. \tag{1.1}$$

Here $|\Psi\rangle$ is the singlet state in $H_2 \otimes H_2$ and L_{LOq} is a local super-operator using $o(n)$ quantum communication. D is the Bures distance $D(\rho, \rho) = 2\sqrt{1 - F(\rho', \rho)}$ and the square-root-fidelity is defined as $F(\rho, \rho) = Tr(\sqrt{\rho^{\frac{1}{2}}\rho\rho^{\frac{1}{2}}})$ [36]. One observes that by allowing asymptotically vanishing communication, one has preserved the interconvertibility result for pure states [37]. This is due to the fact that both the process of entanglement dilution as well as entanglement concentration can be achieved with no more than asymptotically vanishing amount of communication [38]. Thus, it is clear that the cost $E_{LOq}(\rho)$ of creating the state ρ is defined analogously to the entanglement cost $E_c(\rho)$ [39, 40], with the restriction that Alice and Bob can only do a negligible amount of communication. It is immediately clear that $E_{LOq}(\rho)$ will in general be larger than $E_c(\rho)$. In particular, for a separable density matrix $E_c(\rho) = 0$ whereas that for any correlated density matrix, i.e. not of the form $\rho_{AB} = \rho_A \otimes \rho_B$, $E_{LOq}(\rho) > 0$. The entanglement cost E_c was found to be equal to $E_c(\rho) = lim_{n\rightarrow\infty}\frac{E_f(\rho^{\otimes n})}{n}$, where $E_f(\rho)$ is the entanglement of formation [39]. Similarly, the authors in Ref.[34] find an expression for $E_{LOq} = lim_{n\rightarrow\infty}\frac{E_p(\rho^{\otimes n})}{n} = E_p^{\infty}(\rho)$, where $E_p(\rho)$ is a new quantity, the entanglement of purification of ρ.

Now, we discuss briefly a few points and results in existing literature on the measure of total correlation captured by the entanglement of purification.

Firstly, it should be emphasized that entanglement of purification [34] is not a measure of entanglement, but a measure of total correlation defined in units of pure state entanglement. As clear from the discussion in the previous paragraph, this definition of entanglement of purification was motivated operationally, trying to see if quantum states could be constructed from EPR pairs, i.e., the Einstein-Podolsky-Rosen pairs, with vanishing amount of communication asymptotically. It is based on the entanglement-separability paradigm, trying to capture the classical and quantum correlations in an unified way. It was shown to be satisfying the properties of a genuine measure of total correlation. Also, a monogamy relation between the entanglement of purification and the quantum advantage of dense coding was given by Horodecki et al. [41]. However, the conditions for the monogamy or polygamy nature of entanglement of purification have not been found yet. The investigation in the monogamy or polygamy nature of entanglement of purification is motivated from the fact that the mutual information, a measure of total correlation is strictly monogamous for any tripartite pure states [21]. Therefore, if the entanglement of purification is a measure of total correlation can it be strictly monogamous for all tripartite pure states? We find that the entanglement of purification of a tripartite pure state ρ_{ABC} across $A : BC$ partition is never less than its sum for the reduced density matrices ρ_{AB} and ρ_{AC}, and is mostly polygamous. This observation calls for further investigation in understanding the nature of correlation captured by the entanglement of purification. At first, we prove that similar to the mutual information, the entanglement of purification does not increase upon discarding ancilla. Thereafter, we explore the monogamy, polygamy and the additivity properties of the entanglement of purification for pure as well as mixed tripartite states. Furthermore, we find analytically the lower bound and actual value of the entanglement of purification for different classes of mixed states. We also present some conditions for the monogamy of entanglement of purification in terms of monogamy of entanglement of formation and other entropic inequalities. We use these properties of entanglement of purification to explore the monogamy and additivity properties of the quantum advantage of dense coding.

Next, we shift our focus to the measure of genuine multipartite entanglement. It is an area of vibrant research, in which many important properties and features of the mutipartite entanglement remains to be revealed and understood. It is interesting to mention here the work in [42], which generalizes the entanglement entropy as the bipartite entanglement measure to the α-entanglement entropy as a multipartite entanglement measure. Thereafter, the multipartite entanglement measures using tools from lattice theory have also been calculated in [42].

However, in this thesis, we present results on the generalized geometric measure, which is a measure of genuine multipartite entanglement. It is a generalization of the geometric measure of entanglement , which is defined as a measure of bipartite entanglement, to the multipartite scenario. A pure state is said to be genuinely multiparty entangled if it is not product in any bipartition. It measures the shortest geometric distance from the closest bi-separable states to quantify the genuine multiparty entanglement. In the case of systems composed of $m > 2$ subsystems the classification of entangled states is richer than in the bipartite case. Indeed, in multipartite entanglement apart from fully separable states and fully entangled states, there also exists the notion of partially separable states. Quantification of multipartite entanglement is also essential for characterization of successful preparations of quantum states in the multiparty domains, in the laboratories [43]. The notion of entanglement is rather well understood in the bipartite regime, especially for pure states [44–48]. While several entanglement measures can be computed for bipartite pure states, the situation for mixed states is difficult, and there are only a few entanglement measures which can be computed efficiently. The logarithmic negativity [47] can be obtained for arbitrary bipartite states, while the entanglement of formation [45, 46] can be computed for all two-qubit states. The situation becomes complicated even for the pure states when the number of parties increases. However, there have been significant advances in recent times to quantify multipartite entanglement of pure quantum states in arbitrary dimensions [1]. They are broadly classified in two categories, distance-based measures [49–52] and monogamy-based ones [44, 53–55]. On the other hand, quantifying entanglement for arbitrary multiparty mixed states is still an arduous task [56]. Recently, experiments by using photon polarization [57] and ions [58] have been reported in which multiparty states of the order of ten parties have been created successfully. Such physical implementations demand a general tool to compute multiparty entanglement measures for arbitrary mixed states. Recently there have also been notable advancements in this direction [**?**]. Moreover, when an entanglement measure can only be evaluated for pure states, the entanglement-assisted study of cooperative phenomena becomes restricted to only a system which is at zero temperature.

1.1 Summary of thesis

In this thesis, we have explored the entanglement of purification and the generalized geometric measure. In this section, we give a summary of the results we have obtained for these two measures.

In the first part, we find that contrary to the monogamy nature of the mutual information for tripartite pure states, the entanglement of purification can be polygamous for such states. This shows that even though the mutual information and the entanglement of purification are supposed to capture total correlation, the nature of these correlations can be completely opposite at least for tripartite systems. In case of pure and mixed states, the monogamy of entanglement of purification is related to the monogamy of entanglement of formation. Also, we have found a necessary condition for monogamy of entanglement of purification for a special class of mixed states, in terms of the interaction information or the polygamy of the quantum mutual information. A new lower bound of the entanglement of purification has been given for the tripartite mixed states and higher dimensional bipartite systems. Using the formula for the lower bound we have been able to find the exact values of entanglement of purification for some classes of states. Furthermore, in this thesis, we have also shown that if entanglement of purification is not additive, it has to be a sub-additive quantity. Using these results we have also shown that the quantum advantage of dense coding is strictly monogamous for all tripartite pure states and it is super-additive on tensor products. We have also identified some of the quantum states with no quantum advantage of dense coding. We have brought forward these important aspects of the measure of total correlation as well as that of the quantum advantage of dense coding to the forefront. These will help us understand better the nature of total and quantum correlations of composite quantum states. This calls for more explorations and a deeper understanding of the total correlation present in a composite mixed state.

In the second part of the thesis, we address the question of computing the generalized geometric measure (GGM) for mixed states. The GGM of pure states has already been computed efficiently in several systems for an arbitrary number of parties [59]. We define the GGM for mixed states via the convex roof construction. To deal with the obstacle of evaluating the convex roof extension, we use symmetry properties of certain multiparty quantum states and simplify the evaluation of GGMs for these classes of mixed states, as prescribed in Refs. [60–62] (cf. [63]). Exploiting such symmetries, we are able to compute the GGM of different paradigmatic classes of mixed states having different ranks. In particular, we first present the exact value of the GGM for certain classes of rank 2 and rank 3 mixed states with an arbitrary number of qubits. We then compute the GGM for a specific class of states which is a mixture of Greenberger-Horne-Zeilinger [64] and all the Dicke states [65], having a variety of ranks. The common property that all these classes possess is that they remain invariant under the action of the same symmetric local unitary operators on each qubit. Moreover, we find the GGM of a class of tripartite states of rank 4 which remains unaltered under different local unitaries on each party. Finally, we show that such symmetry properties can lead to an exact expression of the GGM for a class of multiqudit states having varied ranks.

Chapter 2

Background Theory

In the present thesis, we have analyzed and derived results on the measures of total correlation and genuine multipartite entanglement. In this analysis, we have used the various properties of the Von-Neumann entropy, the properties of the quantum states and the properties of total and quantum correlations and lastly symmetry properties of the quantum states. Specifically, the study of quantum correlations requires the use of the concepts of Hilbert space, quantum states, quantum measurements, quantum operations in terms of the trace preserving completely positive maps, the measures of entanglement and its properties etc. Therefore, in the next sections, we have given a brief necessary outline of these background theories in the sections that follow hereafter.

2.1 Quantum formalism

2.1.1 Hilbert space

Hilbert space formalism is widely used and very helpful in the mathematical formulation of quantum mechanics. Hilbert spaces were named after David Hilbert, who studied them in the context of integral equations. The elements of an abstract Hilbert space are also called vectors, which are usually sequences of complex numbers or complex functions. In quantum mechanics for example, a physical system is described by a complex Hilbert space which contains the wavefunctions. The wavefunctions represent the possible states of the quantum system. We now state the formal definition of Hilbert space as follows

Definition

An inner product space is a vector space V over $K(= R \; or \; C)$ together with a map $\langle \dot{,} \dot{} \rangle : VXV \to K$ satisfying (for $x, y, z \in V$ and $\lambda \in K$):

- $\langle x + y|z \rangle = \langle x|z \rangle + \langle y|z \rangle$
- $\langle y|x \rangle = \langle x|y \rangle^{\dagger}$
- $\langle \lambda x|y \rangle = \lambda^{*} \langle x|y \rangle$
- $\langle x|x \rangle \geq 0$
- $\langle x|x \rangle = 0 => x = 0$

An inner product on V gives rise to a norm $||x|| = |\langle x|x \rangle|$. If the inner product space is complete in this norm, or in other words if it is a Banach space with this norm, then we call it a Hilbert space. Another way to define it is that a Hilbert space is a Banach space where the norm arises from inner product. All finite-dimensional inner product spaces are Hilbert spaces. However, the infinite-dimensional Hilbert space is also extremely important in applications in mathematical formulations of quantum mechanics. Now, with this definition stated as above, we now move on to the definition and properties of the quantum states.

2.1.2 Quantum states

In quantum physics, quantum state refers to the state of a quantum system. According to the quantum theory, the quantum state encodes all the information about the physical system. A quantum state provides a recipe to calculate the probability distributions [66] for the possible values of each observable, i.e., for the outcome of each possible measurement on the system, corresponding to the observable whose value is to be measured. Knowledge of the quantum state together with the rules for the system's evolution in time given by the Schrodinger's equation exhausts all that can be predicted about the system's behavior.

From physical considerations, there exists specific rules about the proper mathematical expression representing a quantum state. The density matrix formalism is a very useful and effective tool in proper and complete mathematical description of quantum states living in Hilbert space. Specifically, if we represent a quantum state by a density matrix ρ, then the following conditions must be satisfied

- Hermiticity: $\rho = \rho^{\dagger}$,
- Normalization: $Tr(\rho) = 1$,
- Positivity: $\rho \geq 0$.

The quantum states can be divided into two main classes namely the *pure states* and the *mixed states*. Mathematically, a pure quantum state can be represented by a ray or a vector in a Hilbert space over the field of complex numbers. The ray or a vector is a set of nonzero vectors differing by just a complex scalar factor such that any of them can be chosen as a state vector to represent the ray and thus the state. A unit vector is picked to represent a pure quantum state, but usually its phase factor can be chosen freely. However, relative phase factors are important when state vectors with relative phase differences are added together to form a superposition which is again a valid quantum state. In terms of the density matrix representation, pure states are represented by density matrices which additionally satisfy the condition $Tr(\rho^n) = 1$ for any positive integer n.

A mixed quantum state corresponds to a probabilistic mixture of pure states. In other words, in terms of density matrices, a mixed state ρ can be represented as $\rho = \sum_i p_i \sigma_i$, where $0 \leq p_i \leq 1$, $\sum_i p_i = 1$ and σ_i represent the pure states. However, different distributions of pure states can give equivalent (i.e., physically indistinguishable) mixed states. Mixed states are effectively described by the density matrices. For mixed states, the density matrices satisfy the condition $Tr(\rho^n) < 1$ for any positive integer n.

The above formalism of density matrices are very effective in representing the states of quantum systems. However, not only do we need the representation of quantum states, but also to describe the dynamic behavior of quantum states such as the transformations due to quantum measurements, or as a result of interaction with the environment, we briefly discuss the formalism used for representing the quantum operations in the next paragraph.

2.1.3 Quantum Operations

In quantum mechanics, a quantum operation is a mathematical formalism used to describe a broad class of transformations that a quantum mechanical system can undergo. This tool is highly powerful since it can address a wide range of physical processes and transformations that a quantum system can undergo. The quantum operation formalism describes not only unitary time evolution, but also the effects of measurement on quantum systems and interactions with an environment. The quantum operations therefore has different classes, of which the Trace-preserving completely positive maps form a very important and powerful formalism for representing almost all kind of quantum operations. The TPCP maps have far reaching applications and usefulness in the field of quantum information theory and quantum computation and form an indispensable part of the field of quantum information and computation. In this thesis, we have used some properties of these maps, therefore we discuss some important points of these TPCP maps in the sections that follow.

2.1.3.1 TPCP Maps

Trace-preserving completely positive maps or TPCP Maps in short arise naturally in quantum information theory where one wishes to restrict attention to a quantum system that should be considered a subsystem of a larger system or the environment with which it interacts. In such situations, if the system of interest is described by a Hilbert space $\mathcal{H}_{\mathcal{A}}$ and the larger system by a tensor product $H = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$.

The result of an interaction with the larger system also called the environment is described by a map $\Lambda : B(\mathcal{H}_{\mathcal{A}}) \otimes \mathcal{B}(\mathcal{H}_{\mathcal{B}})$ that takes one density matrix ρ to another density matrix $\Lambda(\rho)$. In the general scenario and in an axiomatic approach to quantum operation, this can be defined as a map Λ from the set of density operators of the input space Q_A to the set of density operators for the output space Q_B with three properties which are stated as follows:

- $Tr(\Lambda(\rho))$ is the probability that the process represented by Λ occurs, when ρ is the initial state. Therefore, we have $0 \leq Tr(\Lambda(\rho)) \leq 1$ for any state ρ. This property is for mathematical convenience.

- Λ is a convex-linear map on the set of density operators , which implies the following equation
$$\Lambda(\sum_i p_i \rho_i) = \sum_i p_i \Lambda(\rho_i). \tag{2.1}$$
This property comes due to the requirement on quantum operations from the point of view of physical considerations.

- Λ is a completely positive map. This means that if Λ maps density operators of the system Q_A to density operators of system Q_B, then $\Lambda(A)$ must be positive for any positive operator A. Also, we need an additional extra condition. The extra condition arises from the fact that if we introduce an extra system R of arbitrary dimensionality, it must be true that $(I \otimes \Lambda)(A)$ is positive for any positive operator A on the combined system RQ_A, where I denotes an identity map on system R. This property also stems from a very important physical requirement, that not only $\Lambda(\rho)$ must be a valid density matrix (upto proper normalization factor) as long as ρ is valid as a density operator, but also if $\rho = \rho_{RQ}$ is the density matrix of a joint system RQ, is Λ acts only on Q, then $\Lambda(\rho_{RQ})$ must also be a valid density matrix (upto proper normalization factor) representing the joint system.

  In general the above three are sufficient in defining a quantum operation. Though the above formalism is powerful to describe quantum operations in general, yet there exists a more convenient representation of these maps that prove very useful in quantum information theory and computation.

2.1.3.2 Kraus Operators

The Kraus representation or the operator sum representation is very useful and effective tool to represent quantum operations. The main ingredient is the Kraus theorem. The Kraus theorem characterizes maps that represent quantum operations between density operators. It is stated as follows:

Theorem:

Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces of dimension n and m respectively, and Λ be a quantum operation taking the density matrices acting on $\mathcal{H}$ to those acting on $\mathcal{G}$. Then there are matrices $\{B_i\}_{1\leq i\leq nm}$ mapping $\mathcal{G}$ to $\mathcal{H}$, such that $\Lambda(\rho) = \sum_i B_i^\dagger \rho B_i$. Conversely, any map Λ of this form is a quantum operation provided by $\{B_i\}_{1\leq i\leq nm}$ which satisfy $\sum_i B_i B_i^\dagger \leq 1$..

The matrices $\{B_i\}_{1\leq i\leq nm}$ are called Kraus operators. The Stinespring factorization theorem extends the above result to arbitrary separable Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$. However, the representation of quantum operations by Kraus operators is not a unique representation. The Kraus operators are unique only upto an unitary equivalence which is stated as below

Unitary equivalence:

Kraus matrices are not uniquely determined by the quantum operation Λ in general. The following theorem states that all systems of Kraus matrices which represent the same quantum operation are related by a unitary operation.

Theorem:

Let Λ be a (not necessarily trace preserving) quantum operation on a finite-dimensional Hilbert space $\mathcal{H}$ with two representing sequences of Kraus matrices $\{B_i\}_{1\leq i\leq nm}$ and$\{C_i\}_{1\leq i\leq nm}$. Then there is a unitary operator matrix U, such that $C_i = \sum_j u_{ij}B_j$. In the infinite-dimensional case, this generalizes to a relationship between two minimal Stinespring representations. Also, according to the Stinespring's theorem that all quantum operations can be implemented via unitary evolution after coupling a suitable ancilla or the environment to the original system.

Apart from the above properties and theorem, there is another theorem on the Kraus operators concerning the maximum number of Kraus operators one needs to represent a quantum operation. This theorem is stated as below:

Theorem:

All quantum operations Λ on a system of Hilbert space of dimension d can be generated by an operator sum representation or the Kraus operator representation containing at most d^2 elements,

$$\Lambda(\rho) = \sum_{k=1}^{M} E_k\rho E_k^\dagger, \tag{2.2}$$

where, $1 \leq M \leq d^2$ and $\sum E_k^\dagger E_k = I$.

All of the above theorems and features of the Kraus representation makes quantum operations formalism more convenient to handle and gives rise to various interesting features.

2.1.4 von-Neumann entropy

The von-Neumann entropy is one of the most important and widely used physical quantity in the quantum information theory and computation. We define the von-Neumann entropy as follows.

Given the density matrix ρ, von Neumann defined the entropy as $S(\rho) = -\mathrm{tr}(\rho \ln \rho)$, which is a proper extension of the Gibbs entropy (up to a factor kB) and the Shannon entropy to the quantum domain. To calculate $S(\rho)$ it is very convenient to compute the eigen-decomposition of ρ as $\rho = \sum_j \eta_j |j\rangle\langle j|$. The von Neumann entropy corresponding to ρ is then given by $S(\rho) = -\sum_j \eta_j \ln \eta_j$. The von-Neumann entropy has some interesting properties, which prove crucial in proving many important features of the different correlation measures in quantum information and computation. These properties are as follows:

- $S(\rho)$ is zero if and only if ρ represents a pure state.
- $S(\rho)$ is maximal and equal to lnN for a maximally mixed state, N being the dimension of the Hilbert space.
- $S(\rho)$ is invariant under any kind of basis transformation to represent ρ, that is, $S(\rho) = S(U\rho U^\dagger)$, with U being a unitary transformation.
- $S(\rho)$ is concave, that is, given a set of positive numbers λ_i which sum to unity ($\Sigma_i \lambda_i = 1$) and density operators ρ_i, we have $S\bigg(\sum_{i=1}^k \lambda_i \, \rho_i \bigg) \geq \sum_{i=1}^k \lambda_i \, S(\rho_i)$.
- $S(\rho)$ is additive for independent systems. In other words, this means that given two density matrices ρ_A ,ρ_B describing independent systems A and B, we have $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$.
- $S(\rho)$ is strongly subadditive for any three systems A, B, and C. This implies that $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$. This directly implies that $S(\rho)$ is subadditive which is nothing but the equation $S(\rho_{AC}) \leq S(\rho_A) + S(\rho_C)$. Below, the concept of subadditivity is discussed, followed by its generalization to strong subadditivity.

The above properties are extremely useful and are widely used. We will repeatedly take help of these properties to prove our results on entanglement of purification and the generalized geometric measure in this thesis.

Sub-additivity

If ρ_A, ρ_B are the reduced density matrices of a general state ρ_{AB}, then one has the general inequality $|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$. The right hand inequality is known as sub-additivity. The two inequalities together are also popularly known as the triangle inequality. While in Shannon's theory the entropy of a composite system can never be lower than the entropy of any of its parts, i.e., the conditional entropy of any system cannot be negative, however, in quantum theory this is not the case, i.e., it is possible that $S(\rho_{AB}) = 0$, while $S(\rho_A) = S(\rho_B) > 0$ [67], or to simply put it $S(A|B) = S(AB) - S(B) < 0$, i.e., conditional von-Neumann entropy can be negative.

Strong subadditivity

The von Neumann entropy is also strongly sub-additive. Given three Hilbert spaces, $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$,, strong sub-additivity inequality states that $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$. By using the proof technique that establishes the left side of the triangle inequality above, one can show that the strong subadditivity inequality is equivalent to the inequality $S(\rho_A) + S(\rho_C) \leq S(\rho_{AB}) + S(\rho_{BC})$ when ρ_{AB}, etc. are the reduced density matrices of a density matrix ρ_{ABC}. If we apply ordinary subadditivity to the left side of this inequality, and consider all permutations of A, B, C, we obtain the triangle inequality for ρ_{ABC}, then each of the three numbers $S(\rho_{AB})$, $S(\rho_{BC})$, $S(\rho_{AC})$ is less than or equal to the sum of the other two.

The above two inequalities are two most important and widely used inequalities that are frequently used in exploring various properties of quantum correlation and total correlations. Likewise, we have used these inequalities to prove a few of our results on entanglement of purification.

2.1.5 Some important properties of total and quantum correlations

2.1.5.1 Monogamy/ Polygamy

Monogamy is a property of a multiparticle quantum state that can be studied with respect to a particular correlation measure. It is an important property that tells us about the nature of the correlation at our disposal, in particular, whether it is freely shareable or not. Classical correlations [68] are always polygamous, whereas certain quantum correlation measures satisfy this property and some others do not [14, 15, 20, 26]. For example, the quantum discord is not in general a monogamous quantity for even some cases of the pure tripartite states, whereas the total correlation given by the quantum mutual information is strictly monogamous for all tripartite pure states. Therefore, the monogamy or polygamy nature of the total correlation measure that supposedly contains some amount of quantum and classical correlation is an important question to consider. Now, according to the definition of monogamy, it is a property which does not allow the free sharing of correlation between the subparts of a composite system. Mathematically, if a correlation measure $Q(\rho)$ satisfies

$$Q(A:BC) \geq Q(A:B) + Q(A:C) \tag{2.3}$$

for any tripartite state ρ_{ABC}, then the correlation measure is called monogamous, otherwise it is called polygamous. This definition can be extended to the case of n parties as well. A correlation measure Q is said to be n partite monogamous if the following inequality is satisfied

$$Q(A_1 : A_2..A_n) \geq Q(A_1 : A_2) + Q(A_1 : A_3) + ..Q(A_1 : A_n)$$

and otherwise it is called n partite polygamous.

2.1.5.2 Additivity

Quantum information theory has its share of very challenging mathematical problems. These problems often can be formulated in simple terms, however finding a solution to these problems prove to be quite arduous. One group of these kind of open problems concerns the additivity properties of various quantities characterizing quantum channels, entanglement of formation and the entanglement of purification.

Coming to its definition, if a correlation measure $E(\rho_{AB})$ is additive, then it should follow the equation as given below

$$E(\rho_{AB} \otimes \sigma_{CD}) = E(\rho_{AB}) + E(\sigma_{CD}). \tag{2.4}$$

This means that the correlation measure of a density matrix in product form is just the sum of the correlation measure of the constitutent density matrices. If the additivity holds, this also implies that the regularized value of the measure is equal to its value for the single copy. This is given by the equation

$$E(\rho_{AB}) = E^{\infty}(\rho_{AB}) = Lim_{n\to\infty}\frac{E(\rho_{AB}^{\otimes n})}{n} \tag{2.5}$$

Specifically, it has been shown that four important questions of additivity in quantum information theory are completely equivalent to each other. Namely, the conjectures of additivity of the minimum output entropy of a quantum channel, additivity of the Holevo expression for the classical capacity of a quantum channel, additivity of the entanglement of formation, and strong super-additivity of the entanglement of formation, are either all true or all false. However, the additivity of entanglement of purification is not yet shown to be equivalent to these four additivity questions. Instead, it has been suspected to be a non-additive quantity. The question of additivity in quantum information theory has various important implications, in particular, according to the additivity conjecture, the classical capacity or the maximal purity of outputs cannot be increased by using entangled inputs of the channel.

2.1.6 Total correlations

We consider multiparticle quantum system with each subsystem defined on a finite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ be the set of all linear operators acting on $\mathcal{H}$ and $D(\mathcal{H})$ be the set of all density operators ρ with $\rho \geq 0$ and $Tr(\rho) = 1$. The composite state $\rho_{ABC} \in D(\mathcal{H}_{ABC})$ is a general state that may contain classical and quantum correlations including entanglement. The von-Neumann entropy for a density operator ρ_A is defined as $S(A) = -Tr(\rho_A \log_2 \rho_A)$, where $\rho_A = Tr_{BC}(\rho_{ABC})$. In this section we discuss two important measures of total correlation in the bipartite scenario, namely, the quantum mutual information and the entanglement of purification. The measures of total correlation try to capture quantitatively the total correlations comprising of the classical as well as the quantum correlations in a bipartite state $\rho_{AB} = Tr_C(\rho_{ABC})$.

2.1.6.1 Quantum mutual information

The quantum mutual information is a measure of total correlation in a quantum system. It is a generalization of the classical mutual information. The quantum mutual information is obtained by just replacing the Shannon entropy by the von-Neumann entropy for the respective terms in the expression for the classical mutual information. Thus, for a bipartite quantum state, the quantum mutual information of the state ρ_{AB} is defined as

$$I(A:B) = S(A) + S(B) - S(AB) \tag{2.6}$$

.

Quantum mutual information satisfies some natural properties, all of which, a total correlation measure is expected to satisfy. They are as follows

- It never increases upon discarding of quantum systems, i.e., $I(A:BC) \geq I(A:B)$.
- Secondly, the quantum mutual information is additive on tensor product of density matrices, which is $I(AC:BD) = I(A:B) + I(C:D)$ for $\rho_{AB} \otimes \sigma_{CD}$.
- Apart from these, the monogamy properties of the mutual information have been studied in Ref.[21]. There, it was shown that a necessary and sufficient condition for the monogamy of quantum mutual information can be stated in terms of the interaction information [20, 21]. Specifically, it can be shown that for any pure tripartite state $|\Psi\rangle_{ABC}$, we have

  $$I(A:B) + I(A:C) = I(A:BC),$$

  which implies that the quantum mutual information is strictly monogamous for a pure tripartite state. The necessary and sufficient criteria for quantum mutual information to be monogamous for mixed tripartite state is that the interaction information should be positive [21].

In classical information theory, interaction information of state ρ_{ABC} is defined as

$$\tilde{I}(\rho_{ABC}) = H(AB) + H(BC) + H(AC) - H(A) - H(B) - H(C) - H(ABC), \tag{2.7}$$

where $H(AB)$ denote the Shannon entropies [69]. Replacing the Shannon entropies by the von-Neumann entropies we obtain the quantum generalization of the interaction information. The quantum interaction information is therefore nothing but

$$\tilde{I}(\rho_{ABC}) = S(AB) + S(BC) + S(AC) - S(A) - S(B) - S(C) - S(ABC), \quad (2.8)$$

where $S(AB)$ denote the von-Neumann entropy of the density matrix ρ_{AB}. Interaction information is a measure of the effect of the presence of a third party C on the amount of correlations shared by the other two parties as it is given by the difference between the information shared between the parties A and B when C is present and when C is not present. Quantum interaction information can be positive as well as negative. It is invariant under the action of local unitaries and non-increasing under the action of unilocal measurements [20]. It has been used to provide necessary and sufficient conditions for the monogamy of quantum discord in Ref.[20]. Quantum mutual information is an important measure of correlation and finds application in a large number of settings primarily in studying the channel capacities [70, 71]. Also, an operational interpretation has been given of the quantum mutual information in Ref.[72]. There, it was interpreted as the total amount of randomness or noise needed to erase the correlations in a bipartite quantum state completely.

2.1.6.2 Entanglement of purification

The entanglement of purification is a measure of total correlation along a bipartition in a quantum state, [34]defined using the notion of the entanglement separability paradigm. Interestingly, in this approach the authors in [34] have treated both the quantum entanglement and the classical correlation in a unified framework, by defining a measure of total correlation namely the entanglement of purification in units of pure state entanglement. By their definition, the entanglement of purification is expressed as the entanglement of the purified version of the mixed state as follows.Suppose we have a mixed state ρ_{AB}, and we purify it to a pure state $|\Psi\rangle_{ABA'B'}$. Then, the entanglement of purification is defined as

$$E_p(A:B) = \min_{A'B'} E_f(AA':BB'), \quad (2.9)$$

where $E_p(A:B)$ denotes the entanglement of purification of the state ρ_{AB} across $A:B$ partition, and $E_f(AA':BB')$ is the entanglement of formation across the bipartition $AA':BB'$ of the pure state $|\Psi\rangle_{ABA'B'}$, obtained from ρ_{AB} by any standard purification procedure such as $|\Psi_s\rangle_{AA':BB'} = \sum_i \sqrt{\lambda_i}|\Psi_i\rangle_{AB} \otimes |0\rangle_{A'}|i\rangle_{B'}$. Here, the λ_i are the Schmidt coefficients and $|\Psi_i\rangle$ are the corresponding Schmidt vectors in $\mathcal{H}_{\mathcal{AB}}$.

The above expression can be reformulated in terms of the trace preserving completely positive (TPCP) maps, since every quantum operation can be written in terms of the TPCP maps. Following Ref.[34], we get $E_p(A:B)$ of ρ_{AB} as the following minimum over unitary matrices as

$$E_p(A:B) = \min_{U_{A'B'}} E_f(AA':BB'), \tag{2.10}$$

where $E_f(AA':BB')$ is the entanglement of formation across the $AA':BB'$ partition of the pure state $(I_{AB} \otimes U_{A'B'})(|\Psi_s\rangle\langle\Psi_s|)(I_{AB} \otimes U_{A'B'})^\dagger$ obtained from ρ_{AB} by a standard purification procedure and then acting unitary matrices over the ancilla part. This is nothing but the entropy as follows

$$\min_{U_{A'B'}} S(Tr_{AA'}((I_{AB} \otimes U_{A'B'})(|\Psi_s\rangle\langle\Psi_s|)(I_{AB} \otimes U_{A'B'})^\dagger)). \tag{2.11}$$

Now by tracing out the AA' part from the pure state as well as the unitary operator, one obtains the following equivalent form of entanglement of purification in terms of the TPCP map

$$\begin{aligned} E_p(A:B) = \min_{\Lambda_{B'}} S((I_B \otimes \Lambda_{B'})(\mu_{BB'}(\rho_{AB}))); \\ \Lambda_{B'}(\nu) = Tr_{A'}(U_{A'B'}(\nu_{B'} \otimes |0\rangle\langle 0|_{A'})U^\dagger_{A'B'}); \\ \mu_{BB'}(\rho_{AB}) = Tr_{AA'}(|\Psi\rangle\langle\Psi|), \end{aligned} \tag{2.12}$$

where $\Lambda_{B'}$ is a TPCP map. The above form is derived in Ref.[34]. Therefore, the minimization over unitary matrices in Eq.(2.11) is now represented as a minimization over all TPCP maps $\Lambda_{B'}$, since a TPCP map is equivalently represented as an unitary transformation on the larger system followed by tracing over the ancilla. It was shown that the above optimization can be successfully performed in a Hilbert space of a limited dimension $d_{A'} = d_{AB}$ and $d_{B'} = d^2_{AB}$, due to the result by Terhal *et al.* [34]. For pure states, the entanglement of purification is equal to the entanglement of formation and for a mixed state ρ_{AB}, one has $E_p(A:B) \geq E_f(A:B)$. Alongside, the authors have introduced the regularised entanglement of purification $E_p^\infty(A:B)$. It was shown that the asymptotic cost of preparing n copies of ρ_{AB} from singlets using only local operations and an asymptotically vanishing amount of quantum or classical communication is equal to the regularised entanglement of purification. This implies that the regularised entanglement of purification is actually the entanglement cost (with LO_q) of the quantum states ρ on $\mathcal{H}_d \otimes \mathcal{H}_d$ [34], i.e., $E_{LO_q}(A:B) = E_p^{\infty}(A:B)$. Later, from an operational point of view it was shown that if it is additive on tensor product states then $E_P^{\infty}(A:B)$ is actually the optimal visible compression rate for mixed states [73]. Other operational interpretations have been explored for this quantity. In particular, the regularized entanglement of purification was shown to be equal to the entanglement assisted noisy channel capacity [74]. On another note it was shown that the regularized entanglement of purification $E_{LO_q}(A:B)$ gives the communication cost of simulating a channel without the presence of prior entanglement [75]. However, the entanglement of purification is mostly an unexplored quantity since it is a difficult quantity to calculate analytically owing to the optimization needed to be done in a larger Hilbert space. But, using the monogamy property of entanglement, the authors in Ref.[18] have found the entanglement of purification for a class of bipartite states supported in symmetric or antisymmetric subspaces analytically to be $S(A)$. However, one of the unanswered question regarding the entanglement of purification is the property of additivity. It is still not known whether the entanglement of purification is additive on tensor product states or not. But, some progress has been made in this direction by, where entanglement of purification has been proved to be non-additive within a certain numerical tolerance [76]. The entanglement of purification has been related to some other information theoretic quantities as well. It has also been shown that the entanglement of purification is related to the partial quantum information, through its monogamy relation with the quantum advantage of dense coding [41].

2.1.7 Quantum advantage of dense coding

In quantum information theory, superdense coding is a technique used to send two bits of classical information using only one qubit. It is the inverse of quantum teleportation, which sends one qubit with two classical bits. Both superdense coding and quantum teleportation require, and use up, entanglement between the sender and receiver in the form of Bell pairs. In other words, superdense coding or simply the quantum dense coding is a method of utilizing shared quantum entanglement to increase the rate at which information can be sent through a noiseless quantum channel. Sending a single qubit noiselessly between two parties gives a maximum rate of communication of one bit per qubit (given by the HSW Theorem). If the sender's qubit is maximally entangled with a qubit in the receiver's possession, then dense coding increases the maximum rate to two bits per qubit. We discuss briefly the dense coding for qubits method as below.

If the sender (Alice) and receiver (Bob) share a maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ then Alice may encode two bits of information into the shared state by using one of four unitary operations corresponding to the different two bit strings. The operations consist of the identity (doing nothing), a bit flip σ_X (where $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$), a phase flip σ_Z (where $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow -|1\rangle$), or a combination of both giev effectively by σ_Y. After encoding, Alice and Bob share one of the states

$$
\begin{aligned}
&|00\rangle \rightarrow (I \otimes I)|\Phi^+\rangle = |\Phi^+\rangle, \\
&|01\rangle \rightarrow (\sigma_X \otimes I)|\Phi^+\rangle = \sqrt{\tfrac{1}{2}}(|1\rangle|0\rangle + |0\rangle|1\rangle) = |\Psi_+\rangle, \\
&|10\rangle \rightarrow (\sigma_Y \otimes I)|\Phi^+\rangle = i\sqrt{\tfrac{1}{2}}(|1\rangle|0\rangle - |0\rangle|1\rangle) = |\Psi_-\rangle, \\
&|11\rangle \rightarrow (\sigma_Z \otimes I)|\Phi^+\rangle = \sqrt{\tfrac{1}{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle) = |\Phi_-\rangle.
\end{aligned}
$$

These resultant shared states are orthogonal, and if Alice sends her state to Bob, he can undertake an orthogonal measurement to determine which of the four operators Alice used, and hence determine what the original two bits of Alice's message are.

The previous result is possible because the two parties share initially an entangled state. Unfortunately, in real world applications of quantum computation, one generally has imperfect knowledge and noisy quantum operations, therefore the resulting shared quantum states are usually mixed and non-maximally entangled states and described in terms of density matrices. Therefore, it is interesting to study coding protocols for Alice to send classical information to Bob, directly acting on copies of a shared mixed state ρ_{AB}. We describe the protocol of dense coding with CPTP map as follows. Alice and Bob share a mixed state ρ_{AB}. The protocol to be optimized is the following.
(1). Alice performs a local CPTP map Λ_i with a priori probability p_i on her part of ρ_{AB}. She therefore transforms ρ_{AB} into the ensemble $\{(p_i, {\rho^{AB}}_i)\}$, with ${\rho^{AB}}_i = (\Lambda_i \otimes I)[\rho_{AB}]$.
(2). Alice sends her part of the ensemble state to Bob.
(3). Bob, having at disposal the ensemble $\{(p_i, {\rho^{AB}}_i)\}$, extracts the maximal possible information about the index i.

Each letter in an alphabet is associated to a completely positive trace preserving TPCP map.

As stated in the paper [41], the quantum advantage of dense coding as stated above shares a monogamy relation with the entanglement of purification. As a result, any results that can be obtained from entanglement of purification may be used to obtain results for the quantum advantage of dense coding. We exploit this monogamy relation to reveal some useful and important properties of quantum advantage of dense coding. These results are mentioned in the next section in more details.

Mathematically, the quantum advantage of dense coding of the above scheme for a quantum state ρ_{AB} is defined in terms of the coherent information as

$$\Delta(A\rangle B) = S(B) - inf_{\Lambda_A} S[(\Lambda_A \otimes I_B)\rho_{AB}] = sup_{\Lambda_A} I'(A\rangle B), \tag{2.13}$$

where the infimum or supremum is performed over all the maps Λ_A acting on the state ρ_{AB} and $I'(A\rangle B) = S(B) - S(AB)$ is the coherent information of ρ_{AB}. There, it was proved that the quantum advantage of dense coding is a non-negative quantity. Again, a quantum state is said to be dense codeable if the above quantity $\Delta(A\rangle B)$ is strictly positive.

It was shown in the paper by Horodecki [41] that it suffices to consider only the extremal TPCP maps in evaluating the infimum or supremum for the above quantity, owing to the concavity of the von-Neumann entropy. It was also shown that the quantum advantage of dense coding may be non-additive, though not proved definitely. Apart from the aforementioned properties, the quantum advantage of dense coding was shown to obey a monogamy relation with the entanglement of purification as

$$S(B) \geq \Delta(A\rangle B) + E_p(B : C), \tag{2.14}$$

for any tripartite state ρ_{ABC}, with equality for pure tripartite states [41]. This monogamy inequality in particular is an useful inequality and it will help us to derive various of the quantum advantage of dense coding.

2.1.8 Quantum Correlations

The states in quantum world consisting of more than one parties have correlations between their different parties which have both the classical as well as the quantum origin. Therefore, the quantum systems can be correlated in non-classical ways and the existence of nonclassical correlations in a quantum system can be seen as a proof that subsystems of the composite quantum system are genuinely quantum.

2.1.8.1 Bipartite entanglement

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a finite dimensional Hilbert space. Then, $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ will generally have different dimensions.I_A denotes the identity in $\mathcal{B}(\mathcal{H}_A)$, the set of bounded linear operators acting on $\mathcal{H}_{\mathcal{A}}$, while I_B is the identity linear map acting on $\mathcal{H}_{\mathcal{B}}$. A distinction between elements in $\mathcal{B}(\mathcal{H})$ and matrices is neglected, though it is always pointed out when a particular basis is chosen. The term positive operator is used everywhere to refer to positive semi-definite operators and thus not just strictly positive. First, an entangled state is defined as follows:

Definition:

A quantum state represented by a density operator ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called separable if it can be written as a convex combination $\rho = \sum_i p_i \rho_{Ai} \otimes \rho_{Bi}$, where $\rho_{A/B} i$ acts on $H_{A/B}, \sum_i p_i = 1$ and $pi > 0$. If this is not possible, the state is inseparable or entangled.

The quantum entanglement in bipartite pure states can be described completely by the use of the Schmidt decomposition. An example of the above is the entanglement entropy or the von-Neumann entropy [1]. One can step from pure state entanglement measures to mixed ones by the use of convex roof extension [1]. A well-known example of this is the entanglement of formation [1], which is the extension of the entanglement entropy.

2.1.8.2 Genuine multipartite entanglement

In the case of systems composed of $n > 2$ subsystems the classification and varities of entangled states is much richer than in the bipartite case. Indeed, in multipartite entanglement apart from fully separable states and fully entangled states, there also exists the notion of partially separable states. In particular, one calls a state genuinely multipartite entangled when it is not separable across any bipartition, in contrast to the partially separable states.

Multipartite entanglement is much more complicated than the entanglement of systems consisting of two subsystems due to some various problems such as the non-existence of the Schmidt decomposition in the multipartite case, several types of separability and entanglement in the multipartite case and non-existence of maximally entangled state on some multipartite systems. Another striking property is that there exist n-particle states which contain n-partite entanglement, but are separable if at least one party is traced out. For example, cat states have such a property. In multipartite entanglement, interesting property of entanglement appears - the monogamy of entanglement. While bipartite entanglement is already understood quite well, there are many open questions in the field of multipartite entanglement.

2.1.8.3 Measures of Entanglement

Entanglement can be thought of as a resource for various tasks. It is natural to ask how good a given state is to perform those tasks in terms of its entanglement content. Therefore, quantification of entanglement seems to be very crucial for trying to answer such questions. Quantification of entanglement is done by measures of entanglement. Specifically, a measure of entanglement is a function E from states to real numbers which satisfies some specific postulates, as discussed below.

Postulates for measures of entanglement

In order to quantify entanglement, it should be noted first that the quantum operations that may be physically performed are from the LOCC (Local operations and classical communications) class. Quantum correlations including entanglement are those correlations which cannot be created by LOCC. There are some tasks, for example, quantum teleportation or superdense coding, which cannot be achieved only by LOCC operations, they need entanglement. Therefore, entanglement can be understood as a resource for performing these kind of nonclassical tasks.

- **Monotocity under LOCC**:

  Entanglement cannot increase under local operations and classical communication. This postulate has some consequences. Firstly, it implies that local unitary operators do not change entanglement, as if ρ_1, ρ_2 are local unitarily equivalent, they may be prepared one from another by LOCC, so $E(\rho_1) \geq E(\rho_2)$ and

$E(\rho_2) \geq E(\rho_1)$, so $E(\rho_1) = E(\rho_2)$. Another consequence of this postulate is that entanglement is minimum on separable states. Therefore, it is natural to set entanglement of separable (unentangled) states to zero. This is the convention that ones uses in the field of quantum information and computation.

- **Vanishes on separable states**:
  ρ is separable implies $E(\rho) = 0$.

The first two postulates together imply that $E(\rho) \geq 0$ for all ρ.

In the bipartite case, where maximally entangled states exist, it is natural to normalize the amount of entanglement content of quantum states. Let $|\Psi_d\rangle^+$ be defined as $|\Psi_d\rangle^+ = \sqrt{\frac{1}{d}}(|00...0\rangle... + |11...1\rangle + ...|dd...d\rangle)$. Then, for this state we have the following

- $E(|\Psi_d\rangle^+) = log_2 d$
- For any bipartite pure state, the measure of entanglement on should be equal to the entropy of entanglement.

Sometimes some additional properties may be required, as they may be useful and are quite natural. However, it is not always easy to satisfy all of them, so they are taken as optional, i.e., there are some measures which do not satisfy all of these properties simultaneously. These properties are listed as below:

- **Convexity**:
$$E(\sum_i p_i\rho_i) \leq \sum_i p_i E(\rho_i) \tag{2.15}$$
- **Additivity**:
$$E(\rho^{\otimes n}) = nE(\rho) \tag{2.16}$$
- **Strong additivity**:
$$E(\rho \otimes \sigma) = E(\rho) + E(\sigma) \tag{2.17}$$
- **Asymptotic continuity for pure states**:
$$\frac{E(|\Phi\rangle_n) - E(|\Psi\rangle_n)}{1 + log(dim(\mathcal{H}_n))} \to 0 \tag{2.18}$$
  whenever, one has $Tr(||\Phi\rangle_n\langle\Phi|_n - |\Psi\rangle_n\langle\Psi|_n||)$ between two sequence of states $|\Phi\rangle_n\langle\Phi|_n$ and $|\Psi\rangle_n\langle\Psi|_n$ on sequence of Hilbert space $\mathcal{H} \otimes \mathcal{H}$ tends to zero as $n \to 0$.

2.1.8.4 Bipartite pure states

The only simple case to define a measure of entanglement is that of bipartite pure states. That measure of entanglement is based on von Neumann entropy of reduced states and is also called the entanglement entropy $E(\rho) := tr(\rho_A log_2 \rho_A) = tr(\rho_B log_2 \rho_B)$, where $\rho_{A(B)}$ is obtained from ρ_{AB} by tracing out the subsystem $B(A)$. The equality between von Neumann entropies of ρ_A and ρ_B is a simple consequence of the Schmidt decomposition.

2.1.8.5 Bipartite mixed states

Entropy of entanglement is not a proper measure of entanglement for mixed bipartite states. In this section, we will review some measures of entanglement and principles of constructing them.

Convex roof construction:

Convex roof construction [1] is a method of deriving a measure for mixed states from a measure on pure states. It is defined in the following way:
Let $E1$ be a measure of entanglement on pure states. Then $E2$, the measure of entanglement for mixed states is defined as

$$E2(\rho) = inf\{\sum_i p_i E1(\rho_i) | \rho = \sum_i p_i \rho_i\}, \tag{2.19}$$

where the infimum is taken over all possible pure state decompositions of ρ as $\rho = \sum_i p_i \rho_i$, each ρ_i being a pure state.

2.1.8.6 Generalized geometric measure

The generalized geometric measure (GGM) [52] (cf. [49]) of an N-party pure quantum state, $|\psi_N\rangle$, is a computable entanglement measure that can quantify genuine multiparty entanglement. It is defined as an optimized distance of the given state from the set of all states that are not genuinely multiparty entangled. Mathematically, it is given by

$$\mathcal{E}(|\psi_N\rangle) = 1 - \Lambda^2_{max}(|\psi_N\rangle), \tag{2.20}$$

where $\Lambda_{\max}(|\psi_N\rangle) = \max|\langle\chi|\psi_N\rangle| = \max F(|\psi_N\rangle, |\chi\rangle)$, with the maximization being performed over all $|\chi\rangle$ that are not genuinely multiparty entangled. Here $F(|\psi_N\rangle, |\chi\rangle)$ is the fidelity [85] between two pure states $|\psi_N\rangle$ and $|\chi\rangle$. Additionally, one can show that the GGM for a pure state $|\psi_N\rangle$, is exactly equal to the square of the minimum trace distance of $|\psi_N\rangle$ from pure states that are not genuinely multiparty entangled. It can also be expressed as functions of the minimal Hilbert-Schmidt distances of $|\psi_N\rangle$ from the same set of states [85].

An equivalent form of the above equation is

$$\mathcal{E}(|\psi_n\rangle) = 1 - \max\{\lambda^2_{I:L} | I \cup L = \{A_1, \ldots, A_N\}, I \cap L = \emptyset\}, \tag{2.21}$$

where $\lambda_{I:L}$ is the maximal Schmidt coefficient in the bipartite split $I : L$ of $|\psi_N\rangle$ [52].

Let us enumerate some properties of the GGM which establish it as a bona fide measure of genuine multiparty entanglement [52]. $\mathcal{E}(|\psi_N\rangle) \geq 0$, for all $|\psi_N\rangle$, $\mathcal{E}(|\psi_N\rangle) = 0$ iff $|\psi_N\rangle$ is not genuinely multiparty entangled, and $\mathcal{E}(|\psi_N\rangle)$ is non-increasing under local quantum operations at the N parties and classical communication between them.

We can now define the GGM of a general mixed quantum state, in terms of the convex roof construction. For an arbitrary N-party mixed state, ρ_N, the GGM can be defined as

$$\mathcal{G}(\rho_N) = \min_{\{p_i, |\psi^i_N\rangle\}} \sum_i p_i \mathcal{E}(|\psi^i_N\rangle), \tag{2.22}$$

where the minimization is over all pure state decompositions of ρ_N i.e., $\rho_N = \sum_i p_i |\psi^i_N\rangle\langle\psi^i_N|$. It is difficult to find the optimal decomposition and the computation of GGM is in general impossible even for moderate-sized systems.

2.2 Entanglement measures under symmetry

The evaluation of entanglement measures can be simplified when the quantum states possess some symmetry properties. A quantum state is said to possess some symmetry if it remains invariant under some kind of transformation. In the paper by [60], the authors have explored the technique of using symmetry to simplify the calculation of the measures of entanglement. Therefore, in this section, we explore this idea, namely looking at sets of states which are invariant under a group of local unitaries. The authors in [60] note that this idea itself is not new, and goes back to the first studies of entanglement [86, 87]. Symmetry has also been used in this way to study tripartite entanglement [88, 89]. Similarly, a paper by Rains [90] discusses distillible entanglement under symmetry. Now we move onto the details of exploiting the symmetry for the evaluation of an entanglement measure.

2.2.0.1 Local symmetry groups

Two states ρ, ρ' are regarded as equally entangled if they differ only by a choice of basis in $\mathcal{H}_1$ and $\mathcal{H}_2$ or, equivalently, if there are unitary operators U_i acting on $\mathcal{H}_i$ such that $\rho' = (U1 \otimes U2)\rho(U1 \otimes U2)^\dagger$. If in this equation $\rho' = \rho$, we call $U = (U1 \otimes U2)$ a (local) symmetry of the entangled state ρ. Clearly, the set of symmetries forms a closed group of unitary operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Now one follows a method opposite to the stated above. Specifically, one fixes the symmetry group and studies the set of states left invariant by it. Therefore, from now on, let G be a closed group of unitary operators of the form $U = (U1 \otimes U2)$. As a closed subgroup of the unitary group, G is compact, hence carries a unique measure which is normalized and invariant under right and left group translation. Integrals with respect to this Haar measure are denoted by $\int dU$, and are also considered as averages over the group. In particular, when $\mathcal{G}$ is a finite group, we have $\int dU f(U) = |G| \sum_{U \in G} f(U)$. An important operator used in the application of this theory is the operator $\mathbf{P}(A) = \int dUUAU^\dagger$, for any operator A on $\mathcal{H}_1 \otimes \mathcal{H}_2$, which is sometimes referred to as the twirl operation. It is a completely positive operator, and is doubly stochastic since it takes density operators to density operators and the identity operator to itself. Using the invariance of the Haar measure it is immediately clear that $\mathbf{P}A = A$ is equivalent to $[U, A] = 0$ for all $U \in G$. The set of all A with this property is called the commutant of $\mathcal{G}$. This is denoted by G'. Computing the commutant is always the first step in applying this theory. However, the main focus of the theory proposed by authors in [60] lies in the G-invariant density operators ρ with $P(\rho) = \rho$. These are precisely the states that have the symmetry that we will use to simplify the evaluation of the entanglement measures.

2.2.0.2 Computation of an entanglement measure using symmetry argument

The method for computing the entanglement measure can be explained in the setting of the convex hull construction.

For this purpose we first define the entanglement measure for mixed states using the convex hull construction method. Therefore, let K be a compact convex set, let $M \subset K$ be an arbitrary subset, and let $f : M \to R \cup +\infty$. We then define a function $cof :\to R \cup +\infty$ by

$$cof(x) = inf_{\{\lambda_i, s_i\}} \{\sum_i \lambda_i f(s_i) | \sum_i \lambda_i s_i = x\} \tag{2.23}$$

where the infimum is over all convex combinations with $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, and by convention the infimum over the empty set is $+\infty$. Thus along the line of this definition of convex hull, the entanglement measure defined for a mixed state by the convex hull construction is just given by the following.

$$E(\rho) = coE(\rho). \tag{2.24}$$

With the above definition in hand, we now proceed to the method of applying symmetry to simplify the evaluation of the above convex hull construction to evaluate the entanglement measure E for a mixed state ρ.

So in an addition to a subset $M \in K$ of a compact convex set and a function $f : M \to R \cup \infty$, one considers a compact group $\mathcal{G}$ of symmetries acting on K by transformations $\alpha_U : K \to K$, which preserve convex combinations. It is also assumed that $\alpha_U M \subset M$, and $f(\alpha_U s) = f(s)$ for $s \in M$. All of this assumption is readily verified for $\alpha_U(A) = UAU^\dagger$ and f the entanglement defined on the subset $M \subset K$ of pure bipartite states. Now, the task at hand is to compute $cof(x)$ for all G-invariant $x \in K$, i.e., those with $\alpha_U(x) = x$ for all $U \in G$.

Since the integral with respect to the Haar measure is itself a convex combination, one can define, as before, the projection $\mathbf{P} : K \to K$ by $\mathbf{P}x = \int dU \alpha_U(x)$. The set of projected points $\mathbf{P}x$ will be denoted by $\mathbf{P}K$. Usually, this is of a much lower dimensional object than K, so one will try to reduce the computation of the infimum, which involves a variation over all convex decompositions of x in the high dimensional set K to a computation, which can be done entirely in $\mathbf{P}K$. To this end, one can define the function $\epsilon : \mathbf{P}K \to R \cup +\infty$ by

$$\epsilon(x) = inf(f(s) | s \in M, Ps = x). \tag{2.25}$$

, again with the convention that the infimum over the empty set is $+\infty$. Then the main result as stated in [60] is that, for

$$x \in \mathbf{P}K, \quad cof(x) = co\epsilon(x) \tag{2.26}$$

. But the convex hull on the right is now to be computed in the convex subset $\mathbf{P}K$. The authors in [60] thus arrive at the following recipe for computing the entanglement of formation of G-invariant states:

- Find, for every state $\rho \in \mathbf{P}S$, the set M_ρ of pure states σ such that $P\sigma = \rho$.
- Compute

$$\epsilon(\rho) := inf(E(\sigma), \sigma \in M_\rho). \tag{2.27}$$

- For later use try to get a good understanding of the pure states achieving this minimum.
- Compute the convex hull of the above function given by Eq.(2.27).

The remainder of this subsection is now devoted to the proof of the Eq.(2.26). This is done by first showing that both sides are equal to

$$Z = inf\{\sum_i \lambda_i f(s_i) | s_i \in M, \sum_i \lambda_i \mathbf{P} s_i = x\}. \tag{2.28}$$

Indeed, the only difference between Eq.(2.28) and Eq.(2.23) is that in this equation a weaker condition is demanded on the s_i. Hence more s_i are admissible, and therefore this infimum is smaller, $Z \leq cof(x)$. On the other hand, if s_i satisfying the constraint for Z are given, inserting the definition of $\mathbf{P}$ produces a convex combination giving x, namely the combination of the states $\alpha_U(si)$, labeled by the pair (i, U), and weighted with $\sum_i \lambda_i \int dU$. This convex combination is again admissible for the infimum defining cof, and gives the value $\sum_i \lambda_i \int dU f(\alpha_U(s_i)) = \sum_i \lambda_i \int dU f(s_i) = \sum_i \lambda_i f(s_i)$, where the authors in [60] have used the invariance property of f and the normalization of the Haar measure. Hence all numbers arising in the infimum defined in Eq.(2.28) also appear in the infimum in Eq.(2.23), which proves that $Z \geq cof(x)$, hence $Z = cof(x)$. Now, in order to prove the equality $Z = co\epsilon(x)$ just note that in the infimum in Eq.(2.28) the constraint is only in terms of $\mathbf{P}s_i$, whereas the functional to be minimized involves

$f(s_i)$. Therefore one can compute the infimum given by Eq.(2.28) in stages, by first fixing all $\mathbf{P}s_i$ and minimizing each $f(s_i)$ under this constraint, which amounts to replacing f by ϵ, and then varying over the $\mathbf{P}s_i$, which is the infimum defining $co\epsilon$. Hence $co\epsilon(x) = Z = cof(x)$.

Therefore, from the above theory we now have the technique to exploit the symmetry properties of quantum states to simplify the calculation of the entanglement measures for mixed states.

Chapter 3

Main results I: Entanglement of purification

The entanglement of purification is a measure of total correlation in a quantum state defined in the entanglement separability paradigm. As discussed in the previous section, it lacks a closed form for analytical calculation and involves an optimization over a Hilbert space of large dimension. Therefore, many of its properties have remained unexplored. In this chapter we prove some of its important properties as well as give some improved lower bounds as well as exact values for some classes of states. We also shed light on the open problem of the additivity of entanglement of purification.

3.0.1 Entanglement of purification under discarding quantum systems

The entanglement of purification can be rewritten in terms of the quantum mutual information. For the pure state $|\Psi\rangle_{ABA'B'}$, which is the optimally purified state for the mixed state ρ_{AB} for evaluating the entanglement of purification, the quantum mutual information between parties AA' and BB' is given by $I(AA' : BB') = S(AA') + S(BB') - S(AA'BB')$. Since $|\Psi\rangle_{ABA'B'}$ is a pure state, we have

$$E_p(A : B) = \frac{I(AA' : BB')}{2}.$$

Therefore, the entanglement of purification is actually half of the optimised quantum mutual information of the purified version of the mixed density matrix. The above equations are then used to prove a better lower bound for the entanglement of purification. Before that, we prove an important property of entanglement of purification, an attribute of a measure of total correlation.

Proposition 1: The entanglement of purification never increases upon discarding of quantum system, i.e.,

$$E_p(A:BC) \geq E_p(A:B). \tag{3.1}$$

Proof: If ρ_{ABC} is pure, then $E_p(A:BC) = S(A)$. Also, we know that $E_p(A:B) \leq S(A)$. This leads to $E_p(A:BC) \geq E_p(A:B)$. In case of mixed states ρ_{ABC}, we note that the set of all the pure states for calculating $E_p(A:BC)$ is a subset of the set of all pure states taken for calculating $E_p(A:B)$. This clearly implies that

$$min[I(AA':BB')] \leq min[I(AA':BC(BC)')]. \tag{3.2}$$

From here we thus conclude that $E_p(A:BC) \geq E_p(A:B)$. Thus, like the quantum mutual information, the entanglement of purification also never increases upon discarding of quantum systems. This is a desired property that the total correlation should not increase upon discarding of quantum system. It is easily seen that the equality condition holds when ρ_{AB} is supported in the symmetric or antisymmetric subspace.

3.0.2 Lower bounds for entanglement of purification

Here, we state some simple inequalities for entanglement of purification which will be later used for deriving the monogamy and polygamy conditions for it. Let $|\Psi\rangle_{ABA'B'}$ be the optimal pure state for evaluating the entanglement of purification of ρ_{AB}. Using the sub-additivity of conditional entropy [85] for a composite quantum system of four parties, i.e., $S(AB|A'B') \leq S(A|A') + S(B|B')$, we get $S(ABA'B') - S(AB) \leq S(AA') - S(A) + S(BB') - S(B)$. But we know $E_p(A:B) = S(AA') = S(BB')$ and $S(ABA'B') = 0$, since according to the definition of entanglement of purification $\rho_{ABA'B'}$ is a pure state. Using this in the above inequality, we get $2E_p(A:B) \geq I(A:B)$. Therefore, we have the following lower bound on $E_p(A:B)$

$$E_p(A:B) \geq \frac{I(A:B)}{2}. \tag{3.3}$$

Extending this to the asymptotic limit, one easily obtains $E_p^\infty(A:B) = E_{LO_q}(A:B) \geq \frac{I(A:B)}{2}$, by using the fact that the quantum mutual information is additive on tensor product of quantum states. The above lower bound was known for the entanglement of purification, but only in the asymptotic limit, and it was obtained from an operational point of view in [34]. Here we obtain this bound for a single copy of ρ_{AB}, and easily extend this to the asymptotic limit as $E_{LO_q}(A:B) \geq \frac{I(A:B)}{2}$ and get back the result given in Ref.[34]. Also, the lower bound given in [34] for a single copy of ρ_{AB} is $E_f(A:B)$. However, we know that for some states one has $E_f(A:B) \leq \frac{I(A:B)}{2}$. Therefore, for these states we get a better lower bound for a single copy of ρ_{AB}. Now, we use the equation for entanglement of purification in terms of quantum mutual information to derive a lower bound for tripartite mixed states which is different from half of its quantum mutual information.

Proposition 2: For any pure or mixed tripartite quantum state, it holds that

$$E_p(A:BC) \geq S(A) - \frac{1}{2}[S(A|B) + S(A|C)]. \tag{3.4}$$

Proof: Let $|\Psi\rangle_{ABCA'D'}$ be the optimal pure state for evaluating the entanglement of purification of ρ_{ABC}. Therefore, we have $E_p(A:BC) = \frac{I(AA':BCD')}{2}$. Note that the quantum mutual information of pure states satisfy the monogamy equality condition. Therefore, $E_p(A:BC) = \frac{I(AA':B)}{2} + \frac{I(AA':CD')}{2}$. Again, the mutual information is non-increasing upon discarding of quantum systems, hence we have

$$E_p(A:BC) \geq \frac{I(A:B)}{2} + \frac{I(A:C)}{2}. \tag{3.5}$$

This implies $E_p(A:BC) \geq S(A) - (\frac{S(A|B)}{2} + \frac{S(A|C)}{2})$. In general, from the previous literature we know that $E_p(A:BC) \geq \frac{I(A:BC)}{2}$. However, for the states with $I(A:BC) \leq I(A:B) + I(A:C)$, i.e, with the negative interaction information, we then have $E_p(A:BC) \geq \frac{I(A:B)}{2} + \frac{I(A:C)}{2} \geq \frac{I(A:BC)}{2}$. Therefore, for these class of states, the entanglement of purification is upper and lower bounded as $S(A) \geq E_p(A:BC) \geq S(A) - (\frac{S(A|B)}{2} + \frac{S(A|C)}{2})$. Extending this to the asymptotic limit we obtain

$$E_{LO_q}(A:BC) \geq \frac{I(A:B)}{2} + \frac{I(A:C)}{2}, \tag{3.6}$$

using the fact that quantum mutual information is additive on tensor product of density matrices. We note that the tripartite quantum states with negative interaction information are always polygamous for the quantum mutual information. Therefore, for these states, the above bound is always greater than the previous bound $\frac{I(A:BC)}{2}$. This may give a better lower bound than $\frac{I(A:B)}{2}$ or the regularised classical mutual information [34] for states consisting of quantum as well as classical correlations, depending on the negativity of interaction information. One may extend this to the case of n parties as well, such that for a n partite density matrices $\rho_{A_1A_2...A_n}$, we get

$$E_p(A_1 : A_2A_3..A_n) \geq \max[\frac{(I(A_1 : A_iA_j..)}{2} + \frac{(I(A_1 : A_kA_l..)}{2}] \tag{3.7}$$

etc. where one takes all possible combinations of bipartitions between $A_1A_2...A_n$ (keeping the node A_1 same for the reduced density matrices) to achieve the maximum value of the lower bound. Therefore, the quantum states with negative interaction information across any bipartition will have either the regularized classical mutual information or this as the better lower bound than half of its quantum mutual information.

Corollary: The entanglement of purification for the class of tripartite mixed states satisfying the sub-additivity equality condition is given by $S(A)$.

Proof: From the previous paragraph we see that when $S(A|B) + S(A|C) = 0$, we get $E_p(A : BC) \geq S(A)$. But again, from the upper bound of entanglement of purification we have $E_p(A : BC) \leq S(A)$. Therefore combining the above two equations, one obtains $E_p(A : B) = S(A)$ for the states which satisfy the strong sub-additivity equality condition. Also, we know that mixtures of the tripartite mixed states each satisfying the strong sub-additivity equality condition and satisfying an additional constraint of biorthogonality if the third party is traced out, satisfy the strong sub-additivity equality, and hence their entanglement of purification is also $S(A)$. Hence the proof. The structure of the states obeying the sub-additivity equality condition has been precisely given in Ref.[91]. There it was shown that every separable state can be extended to a state that obeys the sub-additivity equality condition. Therefore, from these observations we can comment that all separable states can be extended to a tripartite mixed state which has the maximum amount of total correlation as $S(A)$. From the viewpoint of the structure of the states [91], the structure states satisfying the SSA equality has been given as $\rho_{ABC} = \bigoplus_j q_j \rho_{Ab_j^L} \otimes \rho_{b_j^R C}$, with states $\rho_{Ab_j^L}$ on Hilbert space $H_A \otimes H_{b_j^L}$ and $\rho_{b_j^R C}$

on $H_{b_j^R} \otimes H_C$ with probability distribution q_j. Thus, all states of this form and all extensions of this class of states have the maximal amount of total correlation given by the entanglement of purification as $S(A)$. Now we discuss the lower bound and exact values with some specific examples as given below.

3.0.2.1 Lower bounds for entanglement of purification for bipartite states

One can use the polygamy of the quantum mutual information to lower bound the entanglement of purification in higher dimensional bipartite states. If a sub-party is of higher dimension, and if the quantum mutual information is polygamous for the lower dimensional subparts obtained by breaking the higher dimensional subparty, then it gives a better lower bound for the entanglement of purification than just half of the quantum mutual information of the state ρ_{AB}.

Suppose for a 2^n dimensional party B in ρ_{AB}, we break it down into two lower dimensional subparties B_1 and B_2 [92]. Then, from Eq(3.5) we have

$$E_p(A:B) \geq \frac{1}{2}[I(A:B_1) + I(A:B_2)]. \tag{3.8}$$

For negative interaction information between B_1 and B_2, i.e., $S(AB_1) + S(AB_2) + S(B_1B_2) - S(A) - S(B_1) - S(B_2) - S(AB_1B_2) < 0$, the R.H.S is greater than $\frac{I(A:B)}{2}$ [21]. Thus it gives a better lower bound. We may say that this better lower bound arises as a result of a second order polygamy relation of quantum mutual information. One can easily extend to the asymptotic limit as well, thus we obtain the lower bound

$$E_{LO_q}(A:B) \geq \frac{1}{2}[I(A:B_1) + I(A:B_2)] > \frac{I(A:B)}{2}. \tag{3.9}$$

For these states $E_{LO_q}(A:B)$ quantifies more correlation than $\frac{I(A:B)}{2}$ as given in the original paper. For these states, one now has to compare the quantity $\frac{1}{2}[I(A:B_1)+I(A:B_2)]$ with the classical mutual information for obtaining a better lower bound. The above equation can also be written as

$$E_p(A:B) \geq S(A) - \frac{1}{2}[S(A|B_1) + S(A|B_2)].$$

From this equation we can say that for the 2^n dimensional party B in the bipartite state ρ_{AB}, if the internal structure of B is such that across any subpartition inside it, the subadditivity equality condition is satisfied then the entanglement of purification of that state is $S(A)$. Therefore, with the aid of the new lower bound as half of the summation of the quantum mutual information of the subparties, we are able to conclude about the new exact values of entanglement of purification for these classes of the higher dimensional bipartite states.

3.0.2.2 Lower bounds for entanglement of purification for tripartite states

Among other examples, for the tripartite states of the form

$$\rho_{ABC} = p|W\rangle\langle W| + (1-p)[a|000\rangle\langle 000| + (1-a)|111\rangle\langle 111|] \tag{3.10}$$

, where $|W\rangle = \frac{1}{\sqrt{3}}[|100\rangle + |010\rangle + |001\rangle]$ is the $|W\rangle$ state, a better lower bound is provided by $\frac{1}{2}[I(A:B) + I(A:C)] \geq \frac{1}{2}I(A:BC)$, since the quantum mutual information is polygamous for these classes of states. This holds even for the regularised version of the entanglement of purification, i.e., $E_{LO_q}(A:BC) \geq \frac{1}{2}[I(A:B) + I(A:C)]$, owing to the additivity of the quantum mutual information on tensor product of density matrices. The difference Δ_{LB} between the two lower bounds equal to $\frac{1}{2}[I(A:B)+I(A:C)-I(A:BC)]$ is plotted in Fig 3.1, which shows that it is always positive. Again we may consider the state

$$\rho_{ABC} = p|W\rangle\langle W| + \frac{(1-p)}{8}I_3 \tag{3.11}$$

and the difference between the lower bounds are plotted in Fig 3.2.

3.0.3 Exact values of entanglement of purification

First we state the value of entanglement of purification for the following class of bipartite mixed states. The entanglement of purification of the states satisfying the Araki-Lieb equality condition is $S(A)$. We know $S(A) \geq E_p(A:B) \geq \frac{1}{2}I(A:B)$. But $\frac{1}{2}I(A:B) = S(A) + \frac{1}{2}[S(B) - S(A) - S(AB)]$. The states satisfying the Araki-Lieb equality condition have $S(B) - S(A) = S(AB)$. Then, we have $S(A) \geq E_p(A:B) \geq S(A)$. Therefore, $E_p(A:B) = S(A)$ for these states. The structure of states satisfying the Araki-Lieb equality condition is given in Ref.[93]. There, it was shown that the states satisfy the Araki-Lieb equality condition if and only if the following conditions are satisfied. First, $\mathcal{H}_\mathcal{A}$ can be factorized as $\mathcal{H}_\mathcal{L} \otimes \mathcal{H}_\mathcal{R}$ and secondly $\rho_{AB} = \rho_L \otimes |\Psi_{RB}\rangle\langle\Psi_{RB}|$, where $|\Psi_{RB}\rangle \in \mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{B}$. The structure of such states that satisfy the Araki-Lieb equality condition is therefore of the form $\rho_{AB} = \rho_L \otimes |\Psi_{RB}\rangle\langle\Psi_{RB}|$. Therefore, the value of entanglement of purification for these states is $S(A)$.

For the case of tripartite states, the entanglement of purification of states of the form

$$\rho_{ABC} = p|GHZ\rangle\langle GHZ|^{\pm} + (1-p)[b|000\rangle\langle 000| + (1-b)|111\rangle\langle 111|] \tag{3.12}$$

is $S(A)$ for all values of $\{p, a, b\} \in [0, 1]$, where $|GHZ\rangle^{\pm} = \sqrt{a}|000\rangle \pm \sqrt{(1-a)}|111\rangle$ is the generalized GHZ state [64]. This holds for n party as well, i.e., for the following state

$$\rho_{ABC} = p|GHZ_n\rangle\langle GHZ_n|^{\pm} + (1-p)[b|0\rangle\langle 0|^{\otimes n} + (1-b)|1\rangle\langle 1|^{\otimes n}] \tag{3.13}$$

where $|GHZ_n\rangle^{\pm} = \sqrt{a}|0\rangle^{\otimes n} \pm \sqrt{(1-a)}|1\rangle^{\otimes n}$.

Proof: The proof is as follows. We know that for tripartite states $E_p(A : BC) \geq \frac{1}{2}[I(A : B) + I(A : C)]$. For the state given above, $I(A : B) + I(A : C) = 2I(A : B) = 2[S(A) + S(B) - S(AB)] = 2S(A)$. The first equality follows owing to the symmetry of the state between parties B and C. The third equality follows from the fact that the nonzero eigenvalues of the density matrices ρ_{AB} and ρ_B are exactly equal. Therefore, for the given state $S(A) \geq E_p(A : BC) \geq S(A)$. Thus, $E_p(A : BC) = S(A)$. Let us consider another example. The tripartite mixed state as a mixture of the $|GHZ\rangle^{+}$ and $|GHZ\rangle^{-}$, i.e., if $\rho_{ABC} = p|GHZ\rangle\langle GHZ|^{+} + (1-p)|GHZ\rangle\langle GHZ|^{-}$ then it also has $E_p(A : B) = S(A)$ according to our previous argument. Here, the states are generalized $|GHZ\rangle$ states. And similar to the above, this is also true for the arbitrary mixture of n partite generalized $|GHZ\rangle$ states.

3.0.4 Monogamy and polygamy relations for entanglement of purification

As we have discussed in the last chapters, we explore the monogamy or polygamy relations of entanglement of purification hereafter for the tripartite as well as the multipartite case.

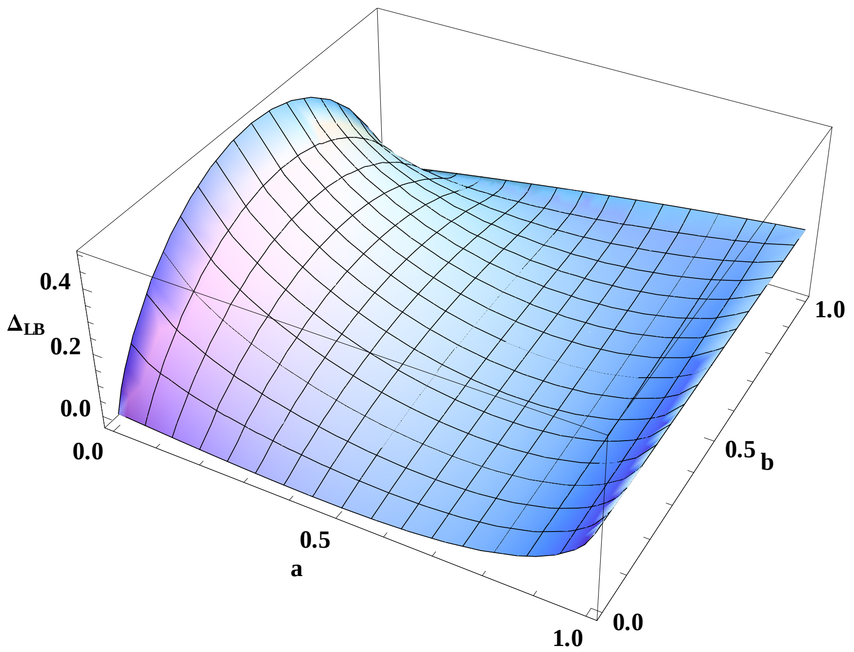

FIGURE 3.1: Difference between lower bounds for state $p|W\rangle\langle W| + (1-p)[a|000\rangle\langle 000| + (1-a)|111\rangle\langle 111|]$. The difference between the new lower bound and the previous one is always positive in this case.

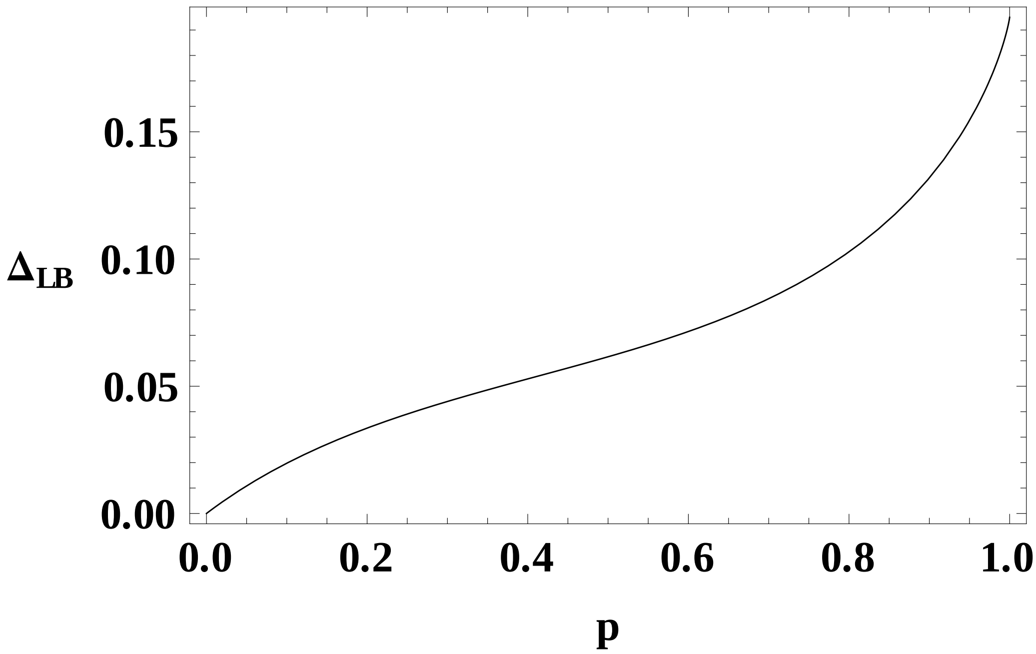

FIGURE 3.2: Difference between lower bounds for state $\rho = p|W\rangle\langle W| + \frac{(1-p)}{8}I_3$. The difference between the new and the old lower bound is always positive here. The difference in lower bounds is given by the amount of polygamy of quantum mutual information.

3.0.4.1 Monogamy and polygamy relations for entanglement of purification for tripartite pure states

Proposition : The entanglement of purification is polygamous for a tripartite pure state ρ_{ABC}:

$$E_p(A:B) + E_p(A:C) \geq E_p(A:BC). \tag{3.14}$$

Proof: From Eq(3.3) we know that $E_p(A:B) \geq \frac{I(A:B)}{2}$. Therefore, we have

$$E_p(A:B) + E_p(A:C) \geq \frac{I(A:B)}{2} + \frac{I(A:C)}{2}. \tag{3.15}$$

In case of the tripartite pure state ρ_{ABC} the right hand side of the inequality just gives $S(A)$. This implies that

$$E_p(A:B) + E_p(A:C) \geq S(A).$$

Since for pure tripartite state ρ_{ABC}, $E_p(A:BC) = S(A)$, we obtain:

$$E_p(A:B) + E_p(A:C) \geq E_p(A:BC).$$

This shows the polygamous nature of the entanglement of purification for pure tripartite state ρ_{ABC}. One can directly see that the same relation holds for the regularised entanglement of purification: $E_{LO_q}(A:B) + E_{LO_q}(A:C) \geq E_{LO_q}(A:BC)$, i.e., the regularised entanglement of purification is also a polygamous quantity. This proves that entanglement of purification for any tripartite pure state is in general a polygamous quantity. An implication of this is that the sum of the asymptotic entanglement cost of preparing ρ_{AB} and ρ_{AC} will not be restricted by the asymptotic cost of preparing $\rho_{A:BC}$.

The polygamy inequality above shows that there can be states satisfying the equality condition in the inequality. To analyse the states that may satisfy the equality condition we find a following relation to the monogamy of entanglement of formation for those states. Given a pure state ρ_{ABC}, if entanglement of formation violates monogamy, then entanglement of purification will violate monogamy equality for the same. However the converse is not true. The proof is as follows. If entanglement of formation $E_f(A:BC)$ violates monogamy for some pure state ρ_{ABC}, then we have $E_f(A:BC) < E_f(A:B) + E_f(A:C)$. But for a pure state ρ_{ABC}, we know that $E_f(A:BC) = E_p(A:BC)$. Therefore, replacing this in the above equation we get $E_p(A:BC) < E_f(A:B) + E_f(A:C)$. Also, it is known that for any state ρ_{AB}, we have $E_f(A:B) \leq E_p(A:B)$. This implies $E_p(A:BC) < E_p(A:B) + E_p(A:C)$ which shows that the entanglement of purification also violates monogamy. Hence the proof. However the vice versa may not be true. We know that for pure states the monogamy of entanglement of formation is equivalent to the monogamy of quantum discord [21]. Therefore, we conclude that the polygamy of quantum discord will also imply the polygamy of entanglement of purification likewise. In other words, monogamy of entanglement of formation or quantum discord is a necessary condition for the tripartite state ρ_{ABC} to satisfy the monogamy equality condition for entanglement of purification. Now let us try to compare the monogamy inequality of the entanglement of formation with the entanglement of purification for mixed tripartite state ρ_{ABC}. Before that, we define a quantity called correlation of classical and quantum origin $E_{cq}(A:B)$ of the state ρ_{AB} as

$$E_{cq}(A:B) = E_p(A:B) - E_f(A:B).$$

This quantity is positive for mixed states and vanishes for pure bipartite states. Intuitively, this may contain some classical correlation and some amount of quantum correlation beyond entanglement that is captured by the entanglement of formation. From the definition it is clear that for a given mixed state ρ_{ABC}, if $E_{cq}(A:B)$ and $E_f(A:B)$ are monogamous or polygamous, then the entanglement of purification will be monogamous or polygamous correspondingly. One can also show that for three-qubit states if the the correlation of classical and quantum origin obeys monogamy and entanglement of formation satisfies [94]

$$E_f(A:B) + E_f(A:C) \leq 1.18$$

then the entanglement of purification will obey a weak monogamy relation as given by

$$E_p(A:B) + E_p(A:C) \leq E_p(A:BC) + 1.18. \tag{3.16}$$

3.0.4.2 Monogamy and polygamy relations for entanglement of purification for tripartite mixed states

The entanglement of purification $E_p(A:BC)$ of a mixed tripartite state ρ_{ABC} is $\frac{I(AA':BC(BC)')}{2}$, where the optimal pure state of ρ_{ABC} is $|\Psi_{ABCA'(BC)'}\rangle$. Similarly, the entanglement of purification $E_p(A:B)$ of ρ_{AB} is $\frac{I(AA'':BB'')}{2}$, where the optimal pure state for ρ_{AB} is $|\Phi_{ABA''B''}\rangle$, and the entanglement of purification $E_p(A:C)$ of ρ_{AC}

is $\frac{I(AA''':CC''')}{2}$, where the optimal pure state for ρ_{AC} is $|\xi_{ACA'''C'''}\rangle$. Therefore, the monogamy inequality for a mixed tripartite state ρ_{ABC} is

$$I(AA' : BC(BC)') \geq I(AA'' : BB'') + I(AA''' : CC'''). \tag{3.17}$$

But owing to the largely difficult optimization needed, we may not be able to check this equation directly. Instead, we analyze some specific cases of mixed states that are polygamous for entanglement of purification as follows.

At first we note that the tripartite mixed states satisfying the strong sub-additivity equality condition are polygamous for entanglement of purification. To see this, let $|\Psi_{ABA'B'}\rangle$ and $|\Psi_{ACA''C''}\rangle$ be the optimal pure states for ρ_{AB} and ρ_{AC} respectively. Then $E_p(A : B) + E_p(A : C) \geq \frac{1}{2}[I(A : B) + I(A : C) + I(AA' : B') + I(AA'' : C'')]$. But $I(A : B) + I(A : C) = 2S(A) - (S(A|B) + S(A|C))$, and if the strong sub-additivity equality condition is satisfied then we have $S(A|B) + S(A|C) = 0$. Putting these in the equation, we get $E_p(A : B) + E_p(A : C) \geq S(A) + \frac{1}{2}[I(AA' : B') + I(AA'' : C'')]$. But the last two terms on the R.H.S are positive in general, as the quantum mutual information is always positive and vanishes only for the maximally mixed state. Also, we know $E_p(A : BC) = S(A)$. Thus, combining these inequalities together we obtain $E_p(A : B) + E_p(A : C) \geq E_p(A : BC)$. Thus, the entanglement of purification is polygamous for the class of states that satisfy the strong sub-additivity equality. Among other classes of states, if anyone of the reduced density matrices ρ_{AB}, ρ_{AC} of a mixed state ρ_{ABC} are entirely supported on the symmetric or antisymmetric subspaces, then the state will violate monogamy of entanglement of purification. This follows from the result by Winter *et al.*[18]. The entanglement of purification of such bipartite density matrices (with the same dimension for both parties) is $S(A)$. But the entanglement of purification of the tripartite mixed state is also $S(A)$ and in general $E_p(A : C) \geq 0$. Therefore, the polygamy inequality follows directly by combining the above observations. Also, any tripartite extension of bipartite mixed states that satisfy the Araki-Lieb equality condition for their von-Neumann entropy is polygamous for entanglement of purification. We know that the states that satisfy the Araki Lieb equality condition have $E_p(A : B) = S(A)[S(B)]$. However, the other reduced density matrix has some non zero correlation and therefore non-zero entanglement of purification. Thus, in this case we have $E_p(A : B) + E_p(A : C) \geq S(A) = E_p(A : BC)$, making entanglement of purification a polygamous measure of total correlation. Though for pure tripartite states we could prove the general polygamy inequality, for mixed states it is not clear whether such general inequality exists or not.

Next, we discuss the relation to polygamy of quantum mutual information. Suppose $E_p(A:B) + E_p(A:C) = S(B|A) + S(C|A) + I(A:B) + I(A:C)$. This is greater than $S(BC|A) + I(A:B) + I(A:C)$ which is again greater than $E_p(A:BC) + I(A:B) + I(A:C) - I(A:BC)$. From the above equations one can see that if the mutual information is polygamous, then here the entanglement of purification becomes polygamous. Again, a sufficient condition for monogamy of E_p is

$$\frac{I(A:BC)}{2} \geq E_p(A:B) + E_p(A:C). \tag{3.18}$$

This implies $\frac{I(A:BC)}{2} \geq \frac{I(A:B)}{2} + \frac{I(A:C)}{2}$, which is nothing but $I(A:BC) \geq I(A:B) + I(A:C)$, i.e., the monogamy inequality for the quantum mutual information. This says that the states satisfying this particular sufficient condition for monogamy of E_p will also satisfy the monogamy inequality of quantum mutual information.

3.0.4.3 Monogamy and polygamy relations for entanglement of purification for multipartite states

Now we investigate the polygamy of entanglement of purification in case of n partite matrices. The conditions for the polygamy for mixed states also get translated here as sufficient conditions for polygamy. To put it in other words, density the n-partite density matrices, pure or mixed, are polygamous if any one of the reduced density matrices of the subsystem satisfy the Araki-Lieb equality condition, strong sub-additivity equality condition or is supported on the symmetric or antisymmetric subspace. Now we state a simple sufficient condition for the polygamy of entanglement of purification and construct some examples.

Proposition 3: All the n-partite states, pure or mixed with $\sum_{i=1}^{n} I(A:A_i) \geq 2S(A)$ are polygamous for entanglement of purification.

Proof: We have $\sum_{i=1}^{n} E_p(A:A_i) \geq \frac{1}{2}[\sum_{i=1}^{n} I(A:A_i)]$. From this we get

$$\sum_{i=1}^{n} E_p(A:A_i) \geq S(A) + \frac{1}{2}[\sum_{i=1}^{n} I(A:A_i)] - 2S(A). \tag{3.19}$$

Thus, we get the condition in the proposition as the sufficient condition for polygamy of entanglement of purification. A large number of states will satisfy this condition, and thus will be polygamous. However, some states will violate this condition, and it will be inconclusive about the polygamous nature in case of those states.

Using the above relation, we easily see that the n-party generalized $|GHZ\rangle$ and the n-party $|W\rangle$ states are polygamous with respect to the entanglement of purification. We can explicitly see the proofs as follows. We have the generalized GHZ state as

$$|GHZ\rangle = \sqrt{p}|0\rangle^{\otimes n} + \sqrt{1-p}|1\rangle^{\otimes n} \tag{3.20}$$

, where $0 \leq p \leq 1$ [64]. But we have obtained before that for tripartite pure states, $E_p(A : A_1) + E_p(A : A_2) \geq S(A)$. Thus, it holds true for the tripartite generalized $|GHZ\rangle$ states as well. Now for $n \geq 3$, we see that all the reduced density matrices are exactly the same and L.H.S becomes $\sum_{i=1}^{n} E_p(A : A_i)$. This is nothing but $E_p(A : A_1) + E_p(A : A_2) + \sum_{i=3}^{n} E_p(A : A_i)$. Since each of the two party reduced density matrices are exactly the same as the two party reduced density matrices in the case of tripartite pure state, therefore using the above two equations we obtain $\sum_{i=1}^{n} E_p(A : A_i) \geq S(A) + \sum_{i=3}^{n} E_p(A : A_i)$. The last term on R.H.S is always positive. Therefore we obtain $\sum_{i=1}^{n} E_p(A : A_i) \geq S(A)$, rendering the entanglement of purification polygamous for all n in the case of generalized $|GHZ\rangle$ state. This is expected since every reduced density matrices share only classical correlation with the other reduced density matrices.

We now consider

$$|W\rangle = \frac{1}{\sqrt{n}}[|10..0\rangle + |01..0\rangle + ..], \tag{3.21}$$

where there are n terms within the parenthesis. We show that this state is also polygamous for all values of n. To see this, first we note that all the two party reduced density matrices ρ_{AA_i} of this state are exactly same due to the symmetry of the state. Specifically we have

$$\rho_{AA_i} = \frac{1}{n}[(n-2)|00\rangle\langle 00|] + 2|\Phi^+\rangle\langle\Phi^+|], \tag{3.22}$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}[|10\rangle + |01\rangle]$ is the Bell state. Now we calculate $\frac{1}{2}[\sum_{i=1}^{n} I(A : A_i)] = \frac{n}{2} I(A : A_1)$, since all the two party reduced density matrices are same. Evaluating the eigenvalues in terms of n, we find that $S(A) = S(A_1) = 2\log_2 n - \log_2(n-1)$ and $S(AA_1) = 2\log_2 n - 1 - \log_2(n-2)$. Putting these values in the equation above, we get $\frac{n}{2} I(A : A_1) - S(A) = \frac{n}{2} + \frac{n}{2}\log_2(n-2) + (n-1)\log_2 \frac{n}{n-1}$. This value is always positive for all values of $n > 2$. Thus combining the earlier result of tripartite pure state with the above finding, we conclude that the entanglement of purification is polygamous for n party $|W\rangle$ state.

Likewise the case for mixed states, where we state some conditions relating monogamy of entanglement of purification with that of quantum mutual information, we now state a proposition connecting the polygamy of quantum mutual information to the polygamy of entanglement of purification for a pure state of n parties.

Proposition 4: All the n party pure states for which the quantum mutual information is $(n-1)$ partite polygamous for at least any one of the $(n-1)$ party reduced density matrices of the pure state, is n partite polygamous for both the entanglement of purification as well as the quantum mutual information.

Proof: Note that for n partite pure state, we have

$$\sum_{i=2}^{n} E_p(A_1 : A_i) \geq \frac{1}{2}\sum_{i=2}^{n} I(A_1 : A_i). \tag{3.23}$$

Now, let us take a reduced density matrix $\rho_{A_1A_2...A_{n-1}}$ to be polygamous for quantum mutual information, i.e., $\sum_{i=2}^{n-1} I(A_1 : A_i) \geq I(A_1 : A_2...A_{n-1})$. Then, we have $I(A_1 : A_n)+\sum_{i=2}^{n-1} I(A_1 : A_i) \geq I(A_1 : A_n)+I(A_1 : A_2...A_{n-1})$. Since the n partite quantum state we are considering is a pure state, therefore by virtue of monogamy of quantum mutual information, the R.H.S. of this equation is nothing but $I(A_1 : A_2A_3...A_n)$. But, we know for a pure state $I(A_1 : A_2A_3...A_n) = 2S(A_1)$. From here it then follows that $\sum_{i=2}^{n} E_p(A_1 : A_i) \geq S(A_1)$ and also $\sum_{i=2}^{n} I(A_1 : A_i) \geq 2S(A_1)$. These two equations are just the equations of polygamy for the entanglement of purification and the quantum mutual respectively for a n partite pure state. It is easy to see that one could take any one of the possible $(n-1)$ different reduced density matrices possible of the n partite pure state (keeping the node A_1 intact for each reduced density matrix) as the one polygamous for the quantum mutual information and eventually get back the polygamy equation for both the entanglement of purification and quantum mutual information. As a specific example of this proposition, we easily see that all the four party pure states with negative interaction information across any two pair of its bipartite reduced density matrices, are polygamous for entanglement of purification.

3.0.5 Sub-additivity of entanglement of purification

Additivity is a desirable property to hold for a given measure of total correlation. Quantum mutual information is an additive measure of correlation, however entanglement of purification may not be an additive measure. Using strong numerical support this has been shown in Ref.[76]. Here we prove that if it is non-additive then it has to be a sub-additive quantity. We have the following theorem.

Theorem 2: The entanglement of purification is sub-additive in the tensor product of density matrices, i.e., for a tensor product density matrix $\rho_{AB} \otimes \sigma_{CD}$, the following equation holds

$$E_p(AC : BD) \leq E_p(A : B) + E_p(C : D).$$

with equality if and only if the optimal pure state for the tensor product of density matrices is the tensor product of optimal pure states of the corresponding density matrices upto a local unitary equivalence.

Proof: Let us suppose $|\Psi_{ABA'B'}\rangle$ and $|\Phi_{CDC'D'}\rangle$ are the optimal purification for ρ_{AB} and σ_{CD} corresponding to the value of entanglement of purification. Then $|\Psi_{ABA'B'}\rangle \otimes |\Phi_{CDC'D'}\rangle$ is a valid purification for $\rho_{AB} \otimes \sigma_{CD}$, however not generally the optimal one. Now, we know that

$$E_p(A : B) = \frac{I(AA' : BB')}{2}, \qquad E_p(C : D) = \frac{I(CC' : DD')}{2}. \tag{3.24}$$

Adding these two quantities we get $E_p(A : B) + E_p(C : D) = \frac{I(AA':BB')}{2} + \frac{I(CC':DD')}{2}$. But the quantum mutual information is additive on tensor product of quantum states. Therefore,

$$\frac{I(AA' : BB')}{2} + \frac{I(CC' : DD')}{2} = \frac{I(AA'CC' : BB'DD')}{2}$$

where $I(AA'CC' : BB'DD')$ is the quantum mutual information of the state $|\Psi_{ABA'B'}\rangle \otimes |\Phi_{CDC'D'}\rangle$. Thus, we have

$$E_p(A : B) + E_p(C : D) = \frac{I(AA'CC' : BB'DD')}{2}.$$

Since $|\Psi_{ABA'B'}\rangle \otimes |\Phi_{CDC'D'}\rangle$ is only one such purification of $\rho_{AB} \otimes \sigma_{CD}$ and the optimization for $E_p(AC : BD)$ is over all possible purifications of $\rho_{AB} \otimes \sigma_{CD}$ denoted by the set of pure states $\{|\xi_{ABCDA''B''}\rangle\}$, therefore we have

$$\min_{A''B''} \frac{I(ACA'' : BDB'')}{2} \leq \frac{I(ACA'C' : BDB'D')}{2},$$

where $I(ACA'' : BDB'')$ is the quantum mutual information of any such purification $|\xi_{ABCDA''B''}\rangle$ and the minimum is over all such purification of $\rho_{AB} \otimes \sigma_{CD}$ by the addition of ancilla part $A''B''$ to it. Hence we easily see that the above equation is nothing but the following inequality,

$$E_p(AC : BD) \leq \frac{I(ACA'C' : BDB'D')}{2},$$

which directly implies that, $E_p(AC : BD) \leq E_p(A : B) + E_p(C : D)$ for the four partite tensor product density matrix $\rho_{AB} \otimes \sigma_{CD}$. Now, in the following paragraph we check the equality condition.

While checking the equality condition, we now omit the subscripts and write $|\Psi_{ABA'B'}\rangle$ as $|\Psi\rangle$, $|\Phi_{CDC'D'}\rangle$ as $|\Phi\rangle$ and $|\xi_{ABCDA''B''}\rangle$ as $|\xi\rangle$ for simplicity. First, we check that if $|\xi\rangle = |\Psi\rangle \otimes |\Phi\rangle$, then whether the dimensionality of the optimal purifying state agrees with the dimension of the Hilbert space of the ancilla part, as given in Ref.[34]. We note that if $|\xi\rangle = |\Psi\rangle \otimes |\Phi\rangle$, then $d_{A''}(|\xi\rangle) = d_{A'}(|\Psi\rangle)d_{C'}(|\Phi\rangle), d_{B''}(|\xi\rangle) = d_{B'}(|\Psi\rangle)d_{D'}(|\Phi\rangle)$. According to the theorem given in Ref.[34],

$$d_{A'}(|\Psi\rangle) = d_{AB}(\rho_{AB}), \qquad d_{C'}(|\Phi\rangle) = d_{CD}(\sigma_{CD}) \tag{3.25}$$

and

$$d_{A''}(|\xi\rangle) = d_{ABCD}(\rho_{AB} \otimes \sigma_{CD}). \tag{3.26}$$

Similarly by the same theorem, we have

$$d_{B'}(|\Psi\rangle) = d^2_{AB}(\rho_{AB}), \qquad d_{D'}(|\Phi\rangle) = d^2_{CD}(\sigma_{CD}) \tag{3.27}$$

and

$$d_{B''}(|\xi\rangle) = d^2_{ABCD}(\rho_{AB} \otimes \sigma_{CD}). \tag{3.28}$$

Now, we verify if the above two equations are consistent with dimensions proposed in Ref.[34] for $|\xi\rangle$. Putting the values of $d_{A'}$ and $d_{B'}$ in terms of d_{AB}, we get

$$d_{A''}(|\xi\rangle) = d_{AB}(\rho_{AB})d_{CD}(\sigma_{CD}) \tag{3.29}$$

and

$$d_{B''}(|\xi\rangle) = d^2_{AB}(\rho_{AB})d^2_{CD}(\sigma_{CD}). \tag{3.30}$$

These values can be reframed as the dimensions of the tensor product of the corresponding density matrices, i.e.,

$$d_{AB}(\rho_{AB})d_{CD}(\sigma_{CD}) = d_{ABCD}(\rho_{AB} \otimes \sigma_{CD}). \tag{3.31}$$

Similarly

$$d^2_{AB}(\rho_{AB})d^2_{CD}(\sigma_{CD}) = d^2_{ABCD}(\rho_{AB} \otimes \sigma_{CD}). \tag{3.32}$$

This holds true even when $|\xi\rangle = U_{A'C'} \otimes U_{B'D'}|\Psi\rangle \otimes |\Phi\rangle$, since the unitary matrices do not map density matrices from Hilbert space of a given dimension to that of a different dimension. This shows that the dimensions are in agreement with those given by the theorem in Ref.[34].

We now move on to the equality condition for the mutual information. For this purpose, let us note that if $|\xi\rangle = U_{A'C'} \otimes U_{B'D'}|\Psi\rangle \otimes |\Phi\rangle$, then owing to the additivity of quantum mutual information and its invariance under the action of local unitaries, one has $I(ACA'' : BDB'') = I(AA' : BB') + I(CC'' : DD')$, where the mutual information terms are that of $|\xi\rangle$, $|\Psi\rangle$, and $|\Phi\rangle$ respectively. This implies that $E_p(AC : BD) = E_p(A : B) + E_p(C : D)$ for $\rho_{AB} \otimes \sigma_{CD}$. This proves the if part of theorem above.

For the only if condition we see that if $|\xi\rangle \neq U_{A'C'} \otimes U_{B'D'}|\Psi\rangle \otimes |\Phi\rangle$, then $I(|\xi\rangle) \neq I(|\Psi\rangle \otimes |\Phi\rangle)$. This is because, the action of a non-local unitary will change the probability distributions of the reduced density matrices and thus will change the value of quantum mutual information across $ACA'C' : BDB'D'$ partition. As a result, the equality holds only if the optimal pure state for the tensor product of the density matrices is the tensor product of corresponding optimal pure states, upto the local unitary equivalence $U_{A'C'} \otimes U_{B'D'}$. Therefore, we see that if the entanglement of purification is non-additive, it is actually sub-additive. Thus, the above theorem rules out the super-additivity of entanglement of purification. The sub-additivity has been shown numerically in Ref.[76] for the Werner states. It is important to note that, according to the result by authors in Ref.[34], one is guaranteed to find the optimal pure state in the Hilbert space of the aforementioned dimensionality. In that case our equality condition holds for the tensor product of the optimal pure states. However it does not rule out the existence of optimal pure states in Hilbert space of other dimensions. Thus, in addition to the optimal pure state in Hilbert space of the dimensions given by the theorem, one may find other optimal pure states in Hilbert space of higher or lower dimension. In particular, one might be able to find optimal pure state in Hilbert space of lower dimension. As an example we have the Werner state and its optimal pure state for entanglement of purification can be found in Hilbert space of dimensions 4×4 as proved numerically in Ref.[34].

Using the results we have obtained on entanglement of purification, we identify the classes of states that are additive on tensor products for the entanglement of purification as follows. We see that the bipartite states satisfying the equality condition in Araki-Lieb inequality, the higher dimensional bipartite states satisfying the equality condition in strong sub-additivity when any party of it can be broken down into two lower dimensional subparties, the tripartite states satisfying the strong sub-additivity equality condition are additive on tensor products for entanglement of purification. Thus, for the above class of states, the regularised entanglement of purification and their optimal visible compression rate is given by the entanglement of purification. Apart from this, we are able to also draw the conclusion that the entanglement of purification is additive on tensor products if and only if it is also super-additive on tensor products for all quantum states. However, whether there can be states $\rho_{AB} \otimes \sigma_{CD}$ for which $E_p(AC : BD) < E_p(A : B)_{\rho_{AB}} + E_p(C : D)_{\sigma_{CD}}$ is still an open question. We note that the question of non-additivity is now reduced to only the sub-additivity condition, ruling out the possibility of $E_p(AC : BD) > E_p(A : B)_{\rho_{AB}} + E_p(C : D)_{\sigma_{CD}}$ for $\rho_{AB} \otimes \sigma_{CD}$.

3.0.6 Implications on quantum advantage of dense coding

3.0.6.1 Upper bounds and exact values

From the lower bound and some of the actual value of entanglement of purification, using the property of monogamy with it and non-negativity of the quantum advantage of dense coding, we can identify some of the quantum states that have no quantum advantage of dense and also put an upper bound on it for some specific cases.

Let ρ_{ABCD} be a quantum state, such that the sub-additivity equality condition is satisfied for the reduced density matrix ρ_{ABC}, i.e., $S(B|A) + S(B|C) = 0$. Then, from the monogamy inequality with entanglement of purification, we get $S(B) \geq \Delta(D\rangle B) + E_p(B : AC)$. But, in this case $E_p(B : AC) = S(B)$. Thus, putting this value, we have $\Delta(D\rangle B) \leq 0$. But, since $\Delta(D\rangle B) \geq 0$, thus we have $\Delta(D\rangle B) = 0$ for the states ρ_{BD}, i.e., the quantum advantage of dense coding vanishes precisely for these states. Similarly, for any tripartite state, pure or mixed ρ_{ABC}, if the state ρ_{BC} satisfies the Araki-Lieb equality condition, then the quantum advantage of dense coding $\Delta(A\rangle B)$ of ρ_{AB} also becomes zero. Apart from the above exact values, the lower bound on entanglement of purification puts an upper bound on the quantum advantage of dense coding via its monogamy relation with the quantum advantage of dense coding.

3.0.6.2 Strict monogamy for tipartite pure states

Therefore, from the monogamy inequality and the polygamy of entanglement of purification for pure tripartite states as well as some of the mixed tripartite states mentioned here previously, it follows that

$$\triangle(B\rangle A) + \triangle(C\rangle A) \leq \triangle(BC\rangle A),$$

implying that the quantum advantage of dense coding is strictly monogamous for the tripartite pure states as well as the other tripartite mixed states mentioned previously. This property is straight forwardly carried over to the asymptotic limit as well. Thus, we have $\triangle^{\infty}(B\rangle A) + \triangle^{\infty}(C\rangle A) \leq \triangle^{\infty}(BC\rangle A)$ for those same set of states. Also, it is easy to see that for the mixed states satisfying SSA equality condition, the symmetric (antisymmetric) subspace condition and the states satisfying the Araki-Lieb equality condition and the cases for the n partite pure states, monogamy is followed.

3.0.6.3 Superadditivity

In the same way as that of the entanglement of purification, we conclude that the quantum advantage of dense coding is super-additive on tensor product of density matrices, i.e., for a four partite tensor product state $\rho_{AB} \otimes \sigma_{CD}$, we have the following equation

$$\Delta(AC\rangle BD) \geq \Delta(A\rangle B) + \Delta(C\rangle D).$$

The proof is as follows. By definition, we have $\Delta(A > B) = sup_{\Lambda_A} I'(A > B)$. Thus, for the density matrix $\rho_{AB} \otimes \sigma_{CD}$ we have $\Delta(A\rangle B) + \Delta(C\rangle D) = sup_{\Lambda_A} I'(A\rangle B) + sup_{\Lambda_C} I'(C\rangle D) = sup_{\Lambda_A \otimes \Lambda_C} I'(AC\rangle BD)$. The second equation follows from the fact that the von-Neumann entropies are additive on tensor products of density matrices. Again for $\rho_{AB} \otimes \sigma_{CD}$, by definition we have $\Delta(AC\rangle BD) = sup_{\Lambda_{AC}} I'(AC\rangle BD)$. However, the optimization for $\rho_{AB} \otimes \rho_{CD}$ is over all Λ_{AC}, and $\{\Lambda_A \otimes \Lambda_C\}$ is only a subset of $\{\Lambda_{AC}\}$. Thus, $sup_{\Lambda_{AC}} I'(AC\rangle BD \geq sup_{\Lambda_A \otimes \Lambda_C} I'(AC\rangle BD)$ for the same four partite product state $\rho_{AB} \otimes \sigma_{CD}$. With the last equation we arrive at the super-additivity equation for the quantum advantage of dense coding for tensor product states of the form $\rho_{AB} \otimes \sigma_{CD}$, i.e., $\Delta(AC\rangle BD) \geq \Delta(A\rangle B) + \Delta(C\rangle D)$ for $\rho_{AB} \otimes \sigma_{CD}$.

With the above results, we finally finish giving or results on the entanglement of purification and therefore also of quantum advantage of dense coding. We hope that these results will be useful in future studies of these quantities.

Chapter 4

Main results II: Generalized Geometric Measure

Multipartite entanglement is a very rich and active area of research in quantum theory. There is a lack of results in this area since the evaluation of multipartite measures of entanglement becomes increasingly difficult with increasing number of parties, dimensions and ranks of the quantum states. However, it has been proposed that symmetry properties of quantum states can be used to simplify such calculations. In this chapter, we have followed this path for the evaluation of the generalized geometric measure, i.e., we have used the symmetry of quantum states to evaluate the genuine multipartite entanglement captured by generalized geometric measure for classes of states possessing symmetry properties in varying number of parties, dimensions and ranks.

4.1 Generalized geometric measure

A pure state is said to be genuinely multiparty entangled if it is not product in any bipartition. The generalized geometric measure (GGM) [52] (cf. [49]) of an N-party pure quantum state, $|\psi_N\rangle$, is a computable entanglement measure that can quantify genuine multiparty entanglement. It is defined as an optimized distance of the given state from the set of all states that are not genuinely multiparty entangled. Mathematically, it is given by

$$\mathcal{E}(|\psi_N\rangle) = 1 - \Lambda^2_{max}(|\psi_N\rangle), \tag{4.1}$$

where $\Lambda_{\max}(|\psi_N\rangle) = \max|\langle\chi|\psi_N\rangle| = \max F(|\psi_N\rangle, |\chi\rangle)$, with the maximization being performed over all $|\chi\rangle$ that are not genuinely multiparty entangled. Here $F(|\psi_N\rangle, |\chi\rangle)$ is the fidelity [85] between two pure states $|\psi_N\rangle$ and $|\chi\rangle$. Additionally, one can show that the GGM for a pure state $|\psi_N\rangle$, is exactly equal to the square of the minimum trace distance of $|\psi_N\rangle$ from pure states that are not genuinely multiparty entangled. It can also be expressed as functions of the minimal Hilbert-Schmidt distances of $|\psi_N\rangle$ from the same set of states [85]. An equivalent form of the above equation is

$$\mathcal{E}(|\psi_n\rangle) = 1 - \max\{\lambda^2_{I:L} | I \cup L = \{A_1, \ldots, A_N\}, I \cap L = \emptyset\}, \tag{4.2}$$

where $\lambda_{I:L}$ is the maximal Schmidt coefficient in the bipartite split $I : L$ of $|\psi_N\rangle$ [52].

Let us enumerate some properties of the GGM which establish it as a bona fide measure of genuine multiparty entanglement [52]. It can be shown that $\mathcal{E}(|\psi_N\rangle) \geq 0$, for all $|\psi_N\rangle$, $\mathcal{E}(|\psi_N\rangle) = 0$ iff $|\psi_N\rangle$ is not genuinely multiparty entangled, and $\mathcal{E}(|\psi_N\rangle)$ is non-increasing under local quantum operations at the N parties and classical communication between them.

We can now define the GGM of a general mixed quantum state, in terms of the convex roof construction. For an arbitrary N-party mixed state, ρ_N, the GGM can be defined as

$$\mathcal{G}(\rho_N) = \min_{\{p_i, |\psi^i_N\rangle\}} \sum_i p_i \mathcal{E}(|\psi^i_N\rangle), \tag{4.3}$$

where the minimization is over all pure state decompositions of ρ_N i.e., $\rho_N = \sum_i p_i |\psi^i_N\rangle\langle\psi^i_N|$. It is difficult to find the optimal decomposition and the computation of GGM is in general impossible even for moderate-sized systems. However, the situation is different if the mixed quantum state under consideration possesses some symmetry [50, 61]. In Ref. [61], Vollbrecht and Werner have provided a general method to compute an entanglement measure, defined via the convex roof extension, of a class of mixed states which are invariant, on average, under a group of local unitaries. Below we briefly outline the same. Suppose $\rho'_N = (U_1 \otimes U_2 \otimes \ldots \otimes U_N)\rho_N(U_1^\dagger \otimes U_2^\dagger \otimes \ldots \otimes U_N^\dagger)$, where U_i are the local unitary operators, acting on Hilbert spaces H_i. The GGM of ρ_N and ρ'_N are the same. If it happens that $\rho_N = \rho'_N$, then $(U_1 \otimes U_2 \otimes \ldots \otimes U_N)$ is called a local symmetry of ρ_N. Let G be a group of unitary operators $U = (U_1 \otimes U_2 \otimes \ldots \otimes U_N)$ and $\mathbf{P}$ be a twirl operator, such that, $A \xrightarrow{\mathbf{P}} \int dU\, U A U^\dagger \equiv \mathbf{P}(A)$, where the integral is carried out Haar uniformly. In case of a mixed state ρ_N, if there exist a twirl operator $\mathbf{P}$ such that $\mathbf{P}(\rho_N) = \rho_N$, then the entanglement, $\mathcal{G}(\rho_N)$, can be obtained from a pure $|\psi\rangle$

which satisfies

$$\mathbf{P}(|\psi\rangle\langle\psi|) = \rho. \tag{4.4}$$

In principle, one can have a set of pure states, $\{|\psi\rangle\} = M_{\rho_N}$, which satisfies Eq. (4.4), and it is sufficient to perform the optimization over this set. A further step is needed where we convexify the optimized quantity over the parameters in ρ_N, if it is not already convex. Moreover, the choice of the set of pure states which satisfy Eq. (4.4) is made by looking at the fact that after the twirl operations, all the off-diagonal terms in the representation of the density matrix should vanish. In this way, the pure states get projected to the initial mixed state. If a given mixed state satisfies the above two properties, the method can be successfully applied to obtain the compact form of GGM for that state.

We now show that the simplified method for convex roof extension [61] can be utilized to evaluate the GGM for several classes of multiparty states with arbitrary number of parties having certain symmetries. We present these classes according to their ranks.

Note that in this method, the GGM of the mixed states reduces to the GGM of certain pure states. Therefore, as mentioned in Sec. 4.1, the value of GGM obtained in this paper is directly connected to other entanglement measures originating from distance-based measures, when optimized over non-genuinely multipartite entangled states. If an entanglement measure E is defined by using a set of multiparty states, over which the optimization is carried out, that is smaller than the set of non-genuinely multiparty entangled states, which for example is the case when the geometric measure [49] is considered, the GGM will form a lower bound for E.

4.1.0.1 Generalized geometric measure for multipartite states for qubits of rank two

The rank 2 mixed state, which we are now going to consider is a mixture of two orthogonal N-party pure states, given by

$$\rho_N^2(x) = x|\psi_N\rangle\langle\psi_N| + (1-x)|\psi_N^\perp\rangle\langle\psi_N^\perp|, \tag{4.5}$$

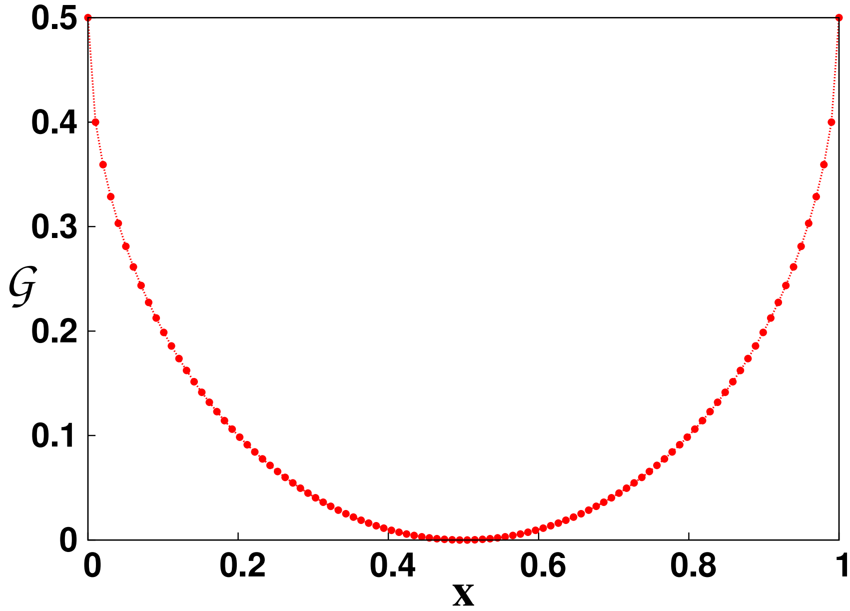

FIGURE 4.1: (Color online.) GGM of $\rho_N^2(x, sym) = x|\psi_N\rangle\langle\psi_N| + (1-x)|\psi_N^{\perp}\rangle\langle\psi_N^{\perp}|$ against x. All the quantities are dimensionless.

where $0 \leq x \leq 1$ and the subscript and superscript of ρ represent the number of qubits and rank respectively. Here, $|\psi_N\rangle$ and $|\psi_N^{\perp}\rangle$ lie in two orthogonal mutually complementary subspaces of the N-party Hilbert space $\mathcal{H}^{\otimes N}$. $|\psi_N\rangle = \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} a_i |D_g^{2i}\rangle$, with

$$|D_g^k\rangle = \sum_{j=1}^{\binom{N}{k}} b_{kj} |\underbrace{00...0}_{N-k}\underbrace{11..1}_{k}\rangle, \tag{4.6}$$

where $|D_g^k\rangle$'s are the generalized Dicke states [65] with k number of excitations i.e. they are the general superpositions of pure states with all permutations of $(N-k)$ $|0\rangle$'s and k $|1\rangle$'s. And

$$|\psi_N^{\perp}\rangle = \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor - 1} a_i' |D_g^{2i+1}\rangle. \tag{4.7}$$

We have chosen the coefficients in all pure and mixed states such that there are properly normalized.

For $\rho_N^2(x)$, we can find a group of local unitary operators consisting of two unitaries, $U_1 = I$, and $U_2 = \sigma_z$, which, on average, keep $\rho_N^2(x)$ invariant. Here, I is the identity operator on the qubit Hilbert space and σ_x, σ_y, and σ_z are the Pauli operators. One can check that $\rho_N^2(x) = \sum_{k=1}^{2} U_k^{\otimes N} |\psi_N^2(x)\rangle\langle\psi_N^2(x)| U_k^{\dagger\otimes N}$, where $|\psi_N^2(x)\rangle = \sqrt{x}|\psi_N\rangle +$

$e^{i\phi}\sqrt{1-x}|\psi_N^\perp\rangle$ is the only class of pure states that is twirled to $\rho_N^2(x)$ by applying the twirl operator corresponding to those unitaries. Hence, by following the recipe in [61], we can calculate the GGM of $\rho_N^2(x)$. Since it involves several parameters, for illustration, we choose a fully symmetric class of states $\rho_N^2(x, sym)$, where $|\psi_N\rangle$ and $|\psi_N^\perp\rangle$ consist of equal superposition of all possible even and odd excitations respectively. For example, for three particles, $|\psi_3\rangle = \frac{1}{2}(|000\rangle + |110\rangle + |101\rangle + |011\rangle)$ and $|\psi_3^\perp\rangle = \frac{1}{2}(|111\rangle + |001\rangle + |010\rangle + |100\rangle)$ respectively. The GGM of $\rho_N^2(x, sym)$ is the convex hull of the GGM of the pure states $|\psi_N^2(x, sym)\rangle = \sqrt{x}|\psi_N\rangle + \sqrt{1-x}e^{i\phi_{min}}|\psi_N^\perp\rangle$. Here the phase, ϕ_{min}, gives the minimum GGM among all the GGM with different ϕ values. We then find that GGM reaches its minimum for $\phi_{min} = 0$. Therefore, the GGM of $\rho_N^2(x, sym)$ is given by

$$\mathcal{G}(\rho_N^2(x, sym)) = \frac{1}{2}(1 - 2\sqrt{x}\sqrt{1-x}), \tag{4.8}$$

since the right hand side is already convex as depicted in Fig. 4.1. An important point to note here that the GGM of $\rho_N^2(x, sym)$, given in Eq. (4.8), is independent of number of parties, N.

4.1.0.2 Generalized geometric measure for rank three multiqubit states

We now calculate the GGM for different classes of mixed states, of rank 3.

Case 1

Let us now consider a three-qubit rank 3 mixed state, $\rho_3^3(x_1, x_2)$ [50], which is a mixture of known $|GHZ_3^+\rangle$, $|D^1\rangle$, and $|D^2\rangle$. Here, $|GHZ_3^+\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ [64], and $|D^1\rangle$ and $|D^2\rangle$ are given by $|D_g^1\rangle$ and $|D_g^2\rangle$ of Eq. (4.6) respectively, with $b_{kj} = \frac{1}{\sqrt{3}}$ for all j. It reads as

$$\rho_3^3(x_1, x_2) = x_1\, |GHZ_3^+\rangle\langle GHZ_3^+| + x_2\, |D^1\rangle\langle D^1| + (1 - x_1 - x_2)|D^2\rangle\langle D^2|. \tag{4.9}$$

Note that $|D^1\rangle$ is the well-known W-state [95]. The mixture is invariant under twirling operator consisting of local unitaries given by $U_1 = I$, $U_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}$, and $U_3 =$

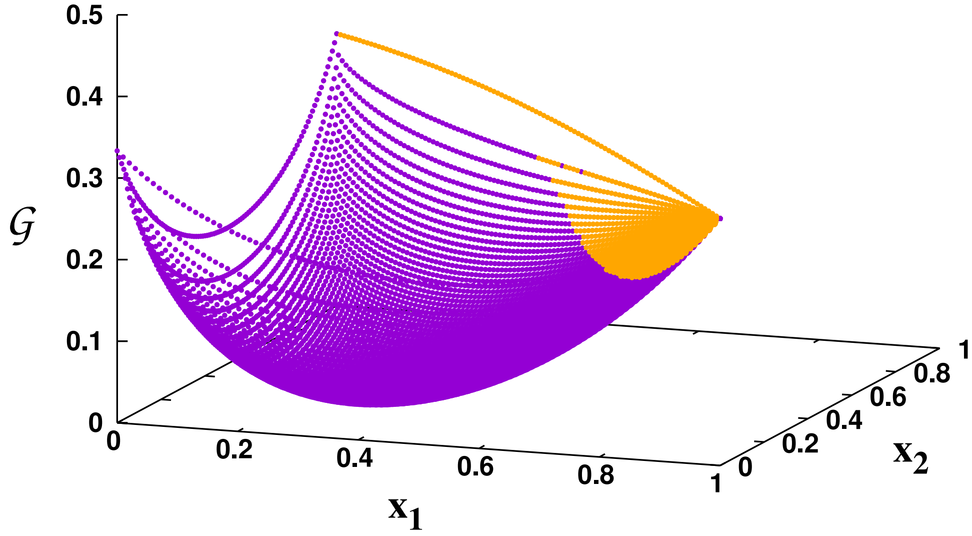

FIGURE 4.2: (Color online.) Plot corresponds to GGM of $|\psi_3^{3,g}\rangle$ vs. the mixing parameters x_1 and x_2. Here, $\alpha = 0.55$ for the $|gGHZ_3\rangle$ state. Both convex and nonconvex regions are seen. The convex part corresponds to the GGM of $\rho_3^{3,g}(x_1, x_2)$. All quantities are dimensionless

$\begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{-2\pi i}{3}} \end{pmatrix}$, that act on each qubit [50]. The corresponding pure states which after twirling operations, leads to $\rho_3^3(x_1, x_2)$, can be written as

$$|\psi_3^3(x_1, x_2)\rangle = \sqrt{x_1}|GHZ^+\rangle + \sqrt{x_2}e^{i\phi_1}|D^1\rangle + \sqrt{1 - x_1 - x_2}e^{i\phi_2}|D^2\rangle. \tag{4.10}$$

The minimum of GGM among $\{\phi_1, \phi_2\}$ is again obtained when $\phi_1 = \phi_2 = 0$. By computing the Hessian matrix, we find both analytically and numerically that the GGM of $|\psi_3^3(x_1, x_2)\rangle$ is convex with respect to x_1 and x_2. Therefore, the GGM of $\rho_3^3(x_1, x_2)$ is given by

$$\begin{aligned} \mathcal{G}\left(\rho_3^3(x_1, x_2)\right) = & \frac{1}{6}\Bigg(3 - \Bigg\{1 - 5x_1^2 - 12x_2(x_2 - 1) + \\ & 8\sqrt{6x_1x_2}\left(1 + \sqrt{x_2(1 - x_1 - x_2)} - x_1 - x_2\right) + \\ & 4x_1\left(1 + 3\sqrt{x_2(1 - x_1 - x_2)} - 3x_2\right)\Bigg\}^{\frac{1}{2}}\Bigg), \end{aligned} \tag{4.11}$$

and is depicted in Fig. 4.2.

Case 2 Let us now move to a more general state while keeping the rank fixed. Precisely, we consider a class of mixed states of the form

$$\rho_3^{3,g}(x_1,x_2) = x_1|gGHZ_3\rangle\langle gGHZ_3| + x_2|D_g^1\rangle\langle D_g^1| + (1-x_1-x_2)|D_g^2\rangle\langle D_g^2|, \tag{4.12}$$

where $|gGHZ_3\rangle = \alpha|000\rangle + \sqrt{1-\alpha^2}|111\rangle$ is the generalized Greenberger-Horne-Zeilinger state with $0 \leq \alpha \leq 1$. The twirling operator that keep $\rho_3^3(x_1,x_2)$ invariant, also keep the state $\rho_3^{3,g}(x_1,x_2)$ invariant, and the class of pure state that are projected to $\rho_3^{3,g}(x_1,x_2)$ is given by

$$|\psi_3^{3,g}(x_1,x_2)\rangle = \sqrt{x_1}|gGHZ_3\rangle + e^{i\phi_1}\sqrt{x_2}|D_g^1\rangle + e^{i\phi_2}\sqrt{1-x_1-x_2}|D_g^2\rangle. \tag{4.13}$$

In this case, we have $\rho_3^{3,g}(x_1,x_2) = \sum_{j=1}^3 U_j^{\otimes 3}|\psi_3^{3,g}(x_1,x_2)\rangle\langle\psi_3^{3,g}(x_1,x_2)|U_j^{\dagger\otimes 3}$, where $\{U_j, j=1,2,3\}$ is the same as in Case 1.

Numerical simulation guarantees that the minimum of $\mathcal{E}(|\psi_3^{3,g}(x_1,x_2)\rangle)$ occurs for $\phi_1 = \phi_2 = 0$. However, unlike the previous cases, we find that $\mathcal{E}(|\psi_3^{3,g}(x_1,x_2)\rangle)$ is not convex for all values of x_1 and x_2. In particular, we plot $\mathcal{E}(|\psi_3^{3,g}(x_1,x_2)\rangle)$ in Fig. 4.2, when $\alpha = 0.55$ and the coefficients in $|D_g^1\rangle$ and $|D_g^2\rangle$ are all equal. For certain regions of the parameter space, the GGM of $|\psi_3^{3,g}(x_1,x_2)\rangle)$ is already convex, and hence GGM of $|\psi_3^{3,g}(x_1,x_2)\rangle)$ in that region is the GGM of $\rho_3^{3,g}(x_1,x_2)$. On the other hand, for the remaining regions, a convexification has to be carried out to obtain the GGM of $\rho_3^g(x_1,x_2)$. Specifically, $\mathcal{E}(|\psi_3^{3,g}(x_1,x_2)\rangle) \neq \mathcal{G}(\rho_3^{3,g}(x_1,x_2))$, when x_1 is high while x_2 is low. To illustrate the convexification, we introduce a new variable, $r = \frac{x_2}{1-x_1}$, and let us consider cases where $r = 0.96$ and 0.98. The convexification of the curves so generated are depicted in Fig. 4.3.

Case 3

Let us move to a class of states which is a multiqubit generalization of $\rho_3^3(x_1,x_2)$. It is given by

$$\rho_N^3(x_1,x_2) = x_1\,|GHZ_N^+\rangle\langle GHZ_N^+| + x_2\,|D^1\rangle\langle D^1| + (1-x_1-x_2)|D^{N-1}\rangle\langle D^{N-1}|, \tag{4.14}$$

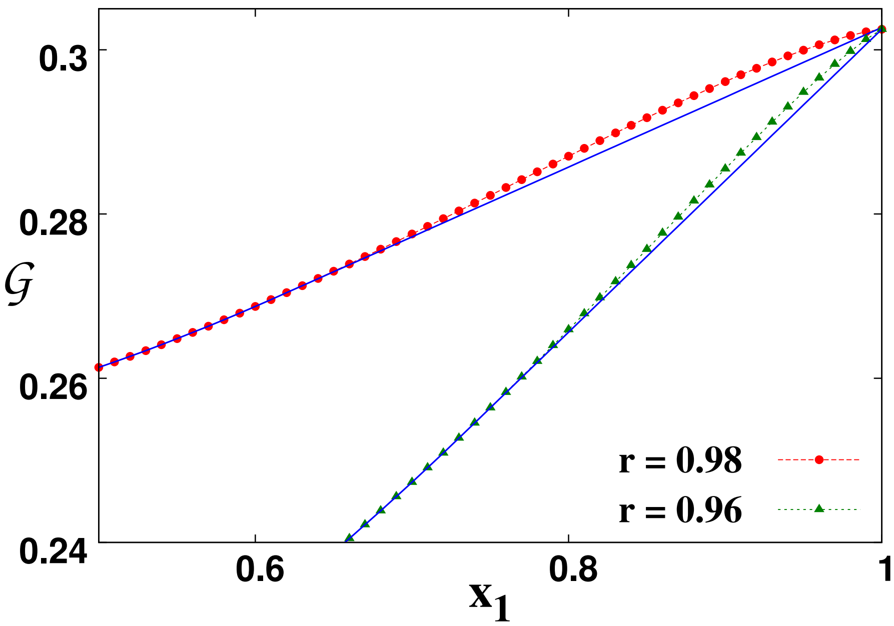

FIGURE 4.3: (Color.) Plot corresponds to GGM of $|\psi_3^{3,g}\rangle$ vs. x_1, for two values of $r = \frac{x_2}{1-x_1}$. Here, $\alpha = 0.55$ for the $|gGHZ_3\rangle$ state. These are given by the dotted lines. The straight lines corresponds to the convexified quantities. All quantities are dimensionless.

where $|GHZ_N^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$, and $|D^{N-1}\rangle$ is given by $|D_g^{N-1}\rangle$ of Eq. (4.6) with $b_{kj} = \frac{1}{\sqrt{\binom{N}{k}}}$. Again, we have

$$\rho_N^3(x_1, x_2) = \sum_{j=1}^{3} U_j^{\otimes N} |\psi_N^3(x_1, x_2)\rangle\langle\psi_N^3(x_1, x_2)| U_j^{\dagger\otimes N}, \tag{4.15}$$

where $|\psi_N^3(x_1, x_2)\rangle$ is given in Eq. (4.10) with $|D^2\rangle$ being replaced by $|D^{N-1}\rangle$, for the same set of unitaries, given in Case 1. Hence, we can compute the GGM of $|\psi_N^3(x_1, x_2)\rangle$ and check its convexity. For $\phi_1 = \phi_2 = 0$ which gives the lowest GGM,

Fig. 4.4 shows the GGM of $|\psi_5^3(x_1, x_2)\rangle$ with respect to the parameters, x_1 and x_2 with $N = 5$. From the figure, it is clear that for example the GGM of $|\psi_5^3(x_1, x_2)\rangle$ is convex for $0.64 \leq x_1 \leq 1.0$ and $0.0 \leq x_2 \leq 0.36$ and hence in that region, we have the GGM of $\rho_5^3(x_1, x_2)$. In the rest of the region, to obtain the GGM of $\rho_5^3(x_1, x_2)$, we have to find the convex hull of $\mathcal{E}(|\psi_5^3(x_1, x_2)\rangle)$.

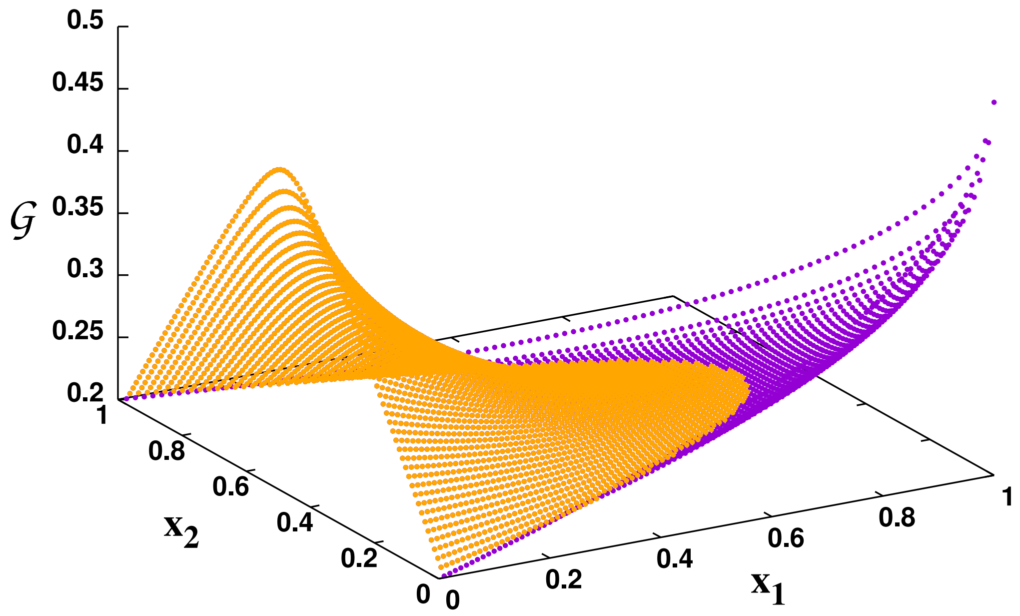

FIGURE 4.4: (Color online.) The plot of GGM for $\rho_5^3(x_1, x_2) = x_1 \, |GHZ_5^+\rangle\langle GHZ_5^+| + x_2 \, |D^1\rangle\langle D^1| + (1 - x_1 - x_2)|D^4\rangle\langle D^4|$ against x_1 and x_2 whenever it is convex. All axes are dimensionless.

4.1.0.3 Generalized geometric measure for multipartite states for qubits of higher ranks

We now consider classes of mixed states with rank more than three. First, we explore a class of multiparty states which can be dealt with symmetric unitaries. In other words, this class of states remain invariant, when the same unitary acts on all the parties, i.e. $\rho_N^N = \sum_j U_j^{\otimes N} \rho_N^N U_j^{\dagger \otimes N}$. We will then find another class of states for which symmetric unitaries do not work.

Symmetric unitary case Let us now consider a class of mixed states with arbitrary number of parties, which can be obtained by generalizing $\rho_3^3(x_1, x_2)$. The state,

$$\rho_N^N(x_1, x_2, \ldots, x_{N-1})$$

, is a mixture of generalized GHZ and all the Dicke states. It reads as

$$\rho_N^N(x_1, x_2, \ldots, x_{N-1}) = (1 - \sum_i x_i)|gGHZ_N\rangle\langle gGHZ_N| + \sum_{i=1}^{N-1} x_i |D_g^i\rangle\langle D_g^i|, \quad (4.16)$$

with $|gGHZ_N\rangle = \alpha|0\rangle^{\otimes N} + \sqrt{1-\alpha^2}|1\rangle^{\otimes N}$. Rank of the above state spans the integers in $[1, N]$. One can check that

$$\rho_N^N(x_1, \ldots, x_{N-1}) = \sum_{j=1}^{N} U_j^{\otimes N} \rho_N^N(x_1, \ldots, x_{N-1}) U_j^{\dagger \otimes N}, \tag{4.17}$$

where the set of local unitaries, $\{U_j\}_{j=1}^N$ consists of I and $\begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i j}{N}} \end{pmatrix}$ with $j = 1, \ldots,$ $(N-1)$. We have to now show that

$$\rho_N^N(x_1, x_2, \ldots, x_{N-1}) = \sum_j U_j^{\otimes N} |\psi_N^N(x_1, \ldots, x_{N-1})\rangle\langle\psi_N^N(x_1, \ldots, x_{N-1}| U_j^{\dagger\otimes N}, \tag{4.18}$$

where $|\psi_N^N(x_1, \ldots, x_{N-1})\rangle = \sqrt{1 - \sum_i x_i}|gGHZ_N\rangle + \sum_{i=1}^{N-1} \sqrt{x_i} e^{\phi_i} |D_g^i\rangle$. To prove this, we note the actions of local unitaries on each off-diagonal terms which e.g. are given by

$$U_j^{\otimes N} |D_g^q\rangle\langle D_g^r| U_j^{\dagger\otimes N} = e^{\frac{2\pi i (q-r)}{N}} |D_g^q\rangle\langle D_g^r|. \tag{4.19}$$

We use the identity $\sum_j e^{\frac{2\pi i (q-r)}{N}} = \delta_{qr}$ in the analysis. Similarly,

$$\sum_j U_j^{\otimes N} |D_g^q\rangle\langle gGHZ_N| U_j^{\dagger\otimes N} = e^{\frac{2\pi i q}{N}} |D_g^q\rangle\langle gGHZ_N| = 0. \tag{4.20}$$

All off-diagonal terms therefore vanish. Therefore, we are now able to calculate the GGM of $|\psi_N^N(x_1, \ldots, x_{N-1})\rangle$ and check whether $\mathcal{E}(|\psi_N^N(x_1, \ldots, x_{N-1})\rangle)$ is convex or not. If it is convex, then $\mathcal{E}(|\psi_N^N(x_1, \ldots, x_{N-1})\rangle) = \mathcal{G}(\rho_N^N(x_1, \ldots, x_{N-1}))$. If it is naturally convex, we perform convexification to obtain the exact value of $\mathcal{G}(\rho_N^N(x_1, \ldots, x_{N-1}))$. To illustrate this example, we consider a five-qubit state which is of the form

$$\rho_5^5 = x_1 |GHZ_5^+\rangle\langle GHZ_5^+| + \frac{x_2}{2}(|D^1\rangle\langle D^1| + |D^2\rangle\langle D^2|) + \frac{1 - x_1 - x_2}{2}(|D^3\rangle\langle D^3| + |D^4\rangle\langle D^4|). \tag{4.21}$$

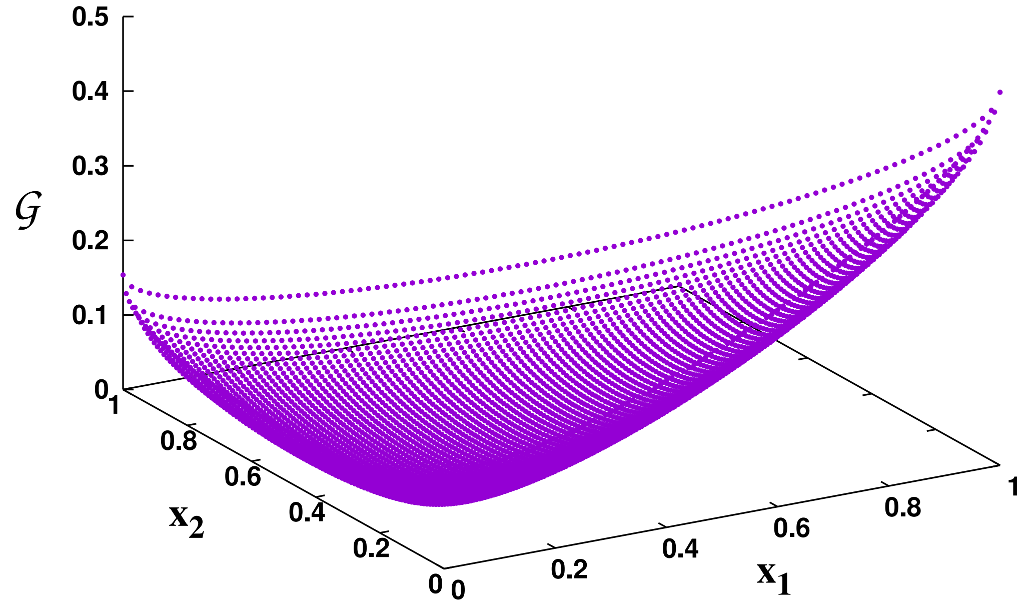

FIGURE 4.5: (Color online.) GGM of $\rho_5^5 = x_1 P[GHZ_5^+] + \frac{x_2}{2}(P[D^1] + P[D^2]) + \frac{1-x_1-x_2}{2}(P[D^3] + P[D^4])$. All axes are dimensionless.

Following the aforementioned prescription, we compute $\mathcal{E}(|\psi_5^5(x_1, x_2)\rangle)$ with

$$|\psi_5^5(x_1, x_2)\rangle = \sqrt{x_1}|GHZ_5^+\rangle + \sqrt{\frac{x_2}{2}}\sum_{k=1}^{2} e^{i\phi_k}|D^k\rangle + \sqrt{\frac{1-x_1-x_2}{2}}\sum_{k=3}^{4} e^{i\phi_k}|D^k\rangle. \tag{4.22}$$

For $\phi_k = 0, k = 1, \ldots, 4$ which gives the infimum of GGM, $\mathcal{E}(|\psi_5^5(x_1, x_2)\rangle)$ is plotted with x_1 and x_2 in Fig. 4.5. By using the Hessian technique, we find that it is convex for the entire range of x_1 and x_2. Therefore, $\mathcal{G}(\rho_5^5)$ is obtained for all x_1 and x_2 and is given by

$$\mathcal{G}(\rho_5^5) = \frac{1}{2}\Bigg(1 - \Bigg(1 - 4\Bigg\{\frac{2x_1 + 4x_2 + 3}{10}\,\frac{7 - 2x_1 - 4x_2}{10} - \Bigg(\sqrt{\frac{x_1 x_2}{20}} + \sqrt{\frac{x_1(1-x_1-x_2)}{20}} + \frac{2x_2}{5\sqrt{2}} + \frac{2(1-x_1-x_2)}{5\sqrt{2}} + \frac{3}{10}\sqrt{x_2(1-x_1-x_2)}\Bigg)^2\Bigg\}\Bigg)^{\frac{1}{2}}\Bigg). \tag{4.23}$$

Comparing Figs. 4.4 and 4.5 with the situations obtained before, it seems that higher rank states, for a fixed total number of qubits of the entire systems, have a greater affinity

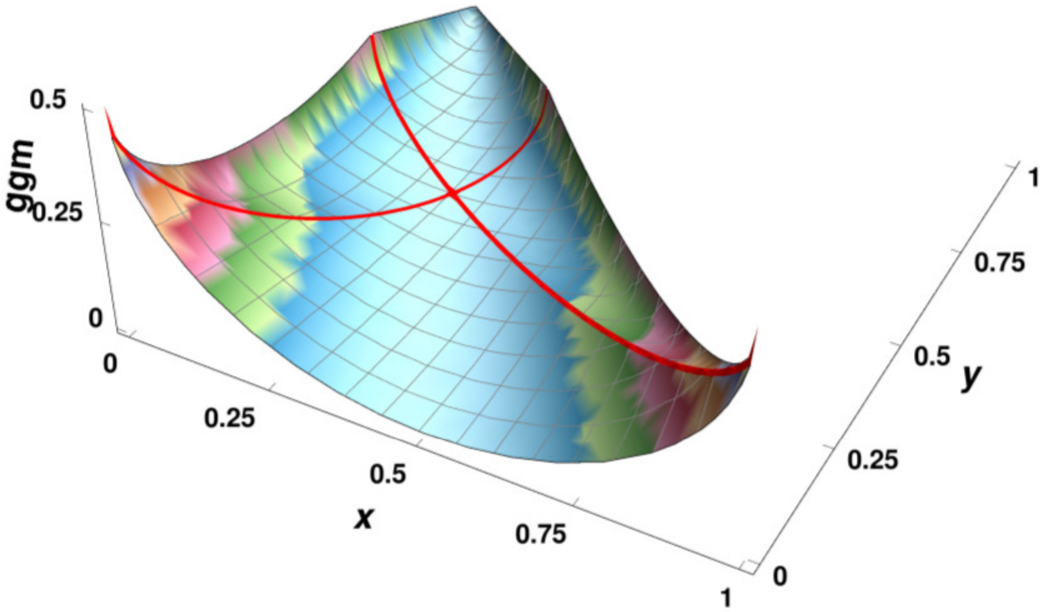

FIGURE 4.6: (Color online.) Plot of GGM of ρ_3^4 with respect to the parameters, x and y. The GGM of the corresponding unique pure state, $|\psi_3^4(x,y)\rangle = \sqrt{x}|\zeta_1\rangle - i\sqrt{y/2}(|\zeta_2\rangle - |\zeta_3\rangle) + \sqrt{1-x-y}|\zeta_4\rangle$ has a kink along the lines shown on the surface, in the plot. The GGM of the pure state is non-convex around these lines, and hence convexifications are required to obtain the GGM of ρ_3^4 along those lines.

for being convex, when their GGMs are considered.

Asymmetric unitary case

Until now, we have considered the states which remain unaltered under twirling operator consisting of local symmetric unitaries of the form $U_i^{\otimes N}$.

Let us now illustrate a class of three-qubit mixed states which remains unchanged under the local unitaries of the form $U_i \otimes U_j \otimes U_k$. The class of mixed state having rank 4, reads

$$\rho_3^4 = \sum_i x_i |\zeta_i\rangle\langle\zeta_i|, \tag{4.24}$$

where

$$
\begin{aligned}
|\zeta_1\rangle &= \frac{1}{2}(|001\rangle + |010\rangle - |100\rangle + |111\rangle), \\
|\zeta_2\rangle &= \frac{1}{2}(-i|000\rangle - i|011\rangle + |100\rangle + |111\rangle), \\
|\zeta_3\rangle &= \frac{1}{2}(i|000\rangle + i|011\rangle + |100\rangle + |111\rangle), \\
\text{and } |\zeta_4\rangle &= \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle - |111\rangle).
\end{aligned}
$$

It is invariant under $\{U_i, i = 1, \ldots 4\}$, which are given by

$$
\begin{aligned}
U_1 &= I \otimes I \otimes I, \\
U_2 &= i\sigma_y \otimes H' \otimes H', \\
U_3 &= I \otimes \sigma_y \otimes \sigma_y, \\
\text{and } U_4 &= -i\sigma_y \otimes H'^T \otimes H'^T,
\end{aligned}
$$

with $H' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Note that these unitaries form a closed group. The only pure states that are twirled to the above mixed states are of the form $|\psi_3^4\rangle = \sum_i \sqrt{x_i} e^{i\phi_i} |\zeta_i\rangle$. We compute the GGM of $|\psi_3^4\rangle$ and minimize it over ϕ_i's. The GGM of ρ_4^3 is given by the minimum of the $\mathcal{E}(|\psi_3^4\rangle)$ for different values of (ϕ_i)s provided the quantity is convex itself.

To visualize its GGM, let us consider, $x_2 = x_3 = \frac{y}{2}$, i.e. the state is of the form

$$
\rho_4^3 = x|\zeta_1\rangle\langle\zeta_1| + \frac{y}{2}(|\zeta_2\rangle\langle\zeta_2| + |\zeta_3\rangle\langle\zeta_3|) + (1 - x - y)|\zeta_4\rangle\langle\zeta_4|. \tag{4.25}
$$

In this case, we find that the minimum GGM of $|\psi_3^4(x, y)\rangle$ for different values of (ϕ_i)s is obtained when $\phi_1 = -\phi_2 = -\frac{\pi}{2}$ and $\phi_3 = 0$. We find the GGM of $\rho_3^4(x, y)$ by convexifying the GGM of $|\psi_3^4(x, y)\rangle)$.

4.1.0.4 Generalized geometric measure for multipartite states for qudits

In the previous sections, we have evaluated the GGM of certain multiqubit systems. We will now show that a similar method can be extended to obtained the analytical expression of GGM of multiqudit mixed states. Specifically, we consider an N-qudit mixed state of rank d, in the Hilbert space $\mathcal{H}_d^{\otimes N}$, of the form

$$\rho_{N,d}^d = \sum_{k=1}^{d} p_k |\Psi\rangle_k \langle\Psi|_k, \tag{4.26}$$

where $|\Psi\rangle_k = \sum_{\{j\}} q_{j_1 j_2 \ldots j_N} |j_1 j_2 \ldots j_N\rangle^{(k)}$ and $(\sum_m j_m)(\text{mod } d) = k$. Our aim is to evaluate the GGM of the state $\rho_{N,d}^d$. Therefore, like previous cases, we construct a twirling operator, consisting of unitary operators Z_d which are d-dimensional, non-hermitian generalization of the σ_z and given by

$$Z_d = \sum_{j=0}^{d-1} e^{\frac{2\pi i j}{d}} |j\rangle\langle j|. \tag{4.27}$$

Here, each of the unitary operators act locally and symmetrically on $\rho_{N,d}^d$ as $Z_d^{\otimes N}$. Note that the set $\left\{ I_d, Z_d^{\otimes N}, \left(Z_d^{\otimes N}\right)^2, .., \left(Z_d^{\otimes N}\right)^{d-1} \right\}$ forms a group and the corresponding twirling operator keeps $\rho_{N,d}^d$ invariant. Now, we have to find the set of all pure states $|\Psi\rangle_{N,d}^d$ that are projected to $\rho_{N,d}^d$ under the action of the aforementioned twirling operator. It can be easily checked that $|\Psi\rangle_{N,d}^d = \sum_{k=1}^{d} e^{i\phi_k} |\Psi\rangle_k$ are the only class of pure states that are mapped to $\rho_{N,d}^d$ under the twirling operator, i.e., we have the following

$$\sum_{q=0}^{d-1} \left(Z_d^{\otimes N}\right)^q |\Psi\rangle_{N,d}^d \langle\Psi|_{N,d}^d \left(Z_d^{\dagger\otimes N}\right)^q = \rho_{N,d}^d. \tag{4.28}$$

In this case also, the minimum of the GGM's of $|\Psi\rangle_{N,d}^d$ over the phases $\{\phi_k\}$ gives the GGM of $\rho_{N,d}^d$ provided the minimum GGM is already a convex function of the state parameters. Otherwise one has to convexify the function to obtain the GGM of $\rho_{N,d}^d$.

Until now, we have considered systems with the same dimensions of the local Hilbert spaces. However, this formalism can be further extended where the local Hilbert spaces' dimensions are not equal, i.e., for quantum systems belonging in $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \ldots \otimes \mathcal{H}_{d_N}$, with $d_1 \neq d_2 \neq \ldots d_N$. In that case, we have two different scenarios. Firstly, $a_1 d_1 = a_2 d_2 = \ldots = d_N$, where $\{a_i\}_{i=1}^{N-1} \in \mathcal{I}^+$. Without loss of generality, d_N is taken to be the largest dimension and the corresponding unitaries are of the form $Z_{d_1} \otimes Z_{d_2} .. \otimes Z_{d_N}$ with its subsequent powers upto $d_N - 1$, such that the composite unitary matrices form a group. Evidently, the case of equal dimensions is a special case of this. Thus, the pure states over which we have to perform the minimization still have the same form, with a slightly different version of the condition given by $\sum_m j_m (\text{mod } d_N) = k$. The second one is the situation when all the dimensions are prime to each other, and in this case, we have to take unitaries upto the power of $(d_1d_2...d_N) - 1$, where the form of pure states remain the same, with the modified condition, $\sum_m j_m \big(\text{mod } d_1d_2...d_N\big) = k$. Therefore, in general, we have to take the maximum power of the unitaries which is the lowest common multiple of $d_1, d_2, ..., d_N$ to apply the similar prescription. In the next paragraph, we illustrate this with an example.

For simplicity, we consider the following three-qutrit state, $\rho^3_{3,3} = \sum_{k=0}^{2} x_k|\Psi\rangle_k\langle\Psi|_k$, where $|\Psi\rangle_k = \sum_j q_{j_1j_2j_3}|j_1j_2j_3\rangle^{(k)}$ and $j_1 + j_2 + j_3(\text{mod } 3) = k$. The exact form of the pure states $\{|\Psi_k\rangle\}_{k=0}^{2}$ reads as

$$\begin{aligned}|\Psi_0\rangle &= \frac{1}{3}(\sum_{i=0}^{2}|iii\rangle + \sum_{perm}|012\rangle),\\ |\Psi_1\rangle &= \frac{1}{3}(\sum_{perm}|001\rangle + \sum_{perm}|022\rangle + \sum_{perm}|112\rangle),\\ \text{and } |\Psi_2\rangle &= \frac{1}{3}(\sum_{perm}|011\rangle \sum_{perm}|002\rangle + \sum_{perm}|122\rangle). \end{aligned} \tag{4.29}$$

For this case, the unitaries which construct the twirling operators are given as $\{I_3, Z_3, Z_3^2\}$. Note that the unitaries of the form $Z_3^i \otimes Z_3^i \otimes Z_3^i$ form a group for i ranging from 0 to 2 and $\rho^3_{3,3}$ is evidently invariant under the corresponding twirling operator. The pure state that is mapped to $\rho^3_{3,3}$ under the action of the aforesaid twirling operator is of the form

$$\begin{aligned}|\Psi^3_{3,3}\rangle &= \sqrt{x_1}|\Psi\rangle_1 + e^{i\phi_2}\sqrt{x_2}|\Psi\rangle_2\\ &+ e^{i\phi_3}\sqrt{1-x_1-x_2}|\Psi\rangle_3. \end{aligned} \tag{4.30}$$

It can be easily found that minimum GGM of $|\Psi^3_{3,3}\rangle$ is obtained for $\phi_2 = \phi_3 = 0$ and it is a convex function of the parameters x_1 and x_2. Hence, the GGM of $\rho^3_{3,3}$ is given by

$$\mathcal{G}(\rho^3_{3,3}) = \frac{2}{3}\{1 - \sqrt{x_1x_2} - \sqrt{x_1\{1-x_1-x_2\}} - \sqrt{x_2\{1-x_1-x_2\}}\}.$$

$\mathcal{G}(\rho^3_{3,3})$ is depicted in Fig. 4.7 and the convexity of the function can be visualized from the same.

Note added: This work is based on a poster presentation [98] at the International Workshop on Quantum In- formation (IWQI-2012), Harish-Chandra Research Institute, Allahabad, India. We thank J. Solomon Ivan for pointing out during a discussion over the poster that the same method as followed here can be used to evaluate the GGM for an arbitrary mixture of $|GHZ_+\rangle and|GHZ_-\rangle$ where $|GHZ_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} \pm |1\rangle^{\otimes N})$.

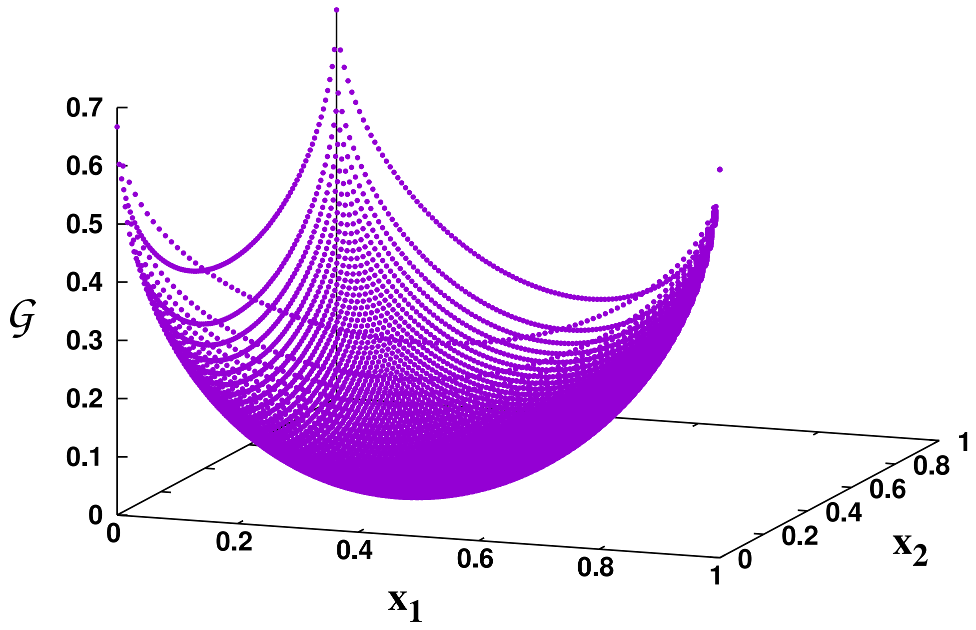

FIGURE 4.7: (Color online.) Plot of GGM of $\rho^3_{3,3}$ against x_1 and x_2. The GGM of the corresponding unique pure state, $|\Psi^3_{3,3}\rangle = \sqrt{x_1}|\Psi\rangle_1 + e^{i\phi_2}\sqrt{x_2}|\Psi\rangle_2 + e^{i\phi_3}\sqrt{1-x_1-x_2}|\Psi\rangle_3$ is plotted with $\phi_1 = \phi_2 = 0$. The GGM of the pure state is convex everywhere, as evident from this plot and hence $\mathcal{E}(|\Psi^3_{3,3}\rangle) = \mathcal{G}(\rho^3_{3,3})$.

We thank Otfried Guehne for informing us about their independent work on evaluating multipartite entanglement [99], by a method that is different from the one followed in this work on generalized geometric measure.

Chapter 5

Conclusions and future directions

5.1 Conclusions

Quantum physics has played an exceptionally important and prominent role in shaping and developing the field of modern theoretical physics and also furthering the cutting edge technology. Though not fully understood, one cannot deny the tremendous impact it has on furthering our understanding of nature. Yet a lot remains to be understood and explored in the field of quantum science. Specifically, an exciting field of quantum information and computation promise us with many new possibilities in theory as well as applications. In this field of quantum information science and technology, the theory of quantum correlations perhaps form one of the most important and impactful area. In this area, research has been done extensively in the past to understand, quantify and use the purely quantum correlations. In particular, the quantum entanglement has been a topic of active research all throughout. Quantum entanglement has also been used to perform various tasks that perform well above that allowed by laws of classical physics. Therefore, our main focus of research here has been on quantum entanglement.

Quantum entanglement has found myriad used in quantum information science and computation. One of the very interesting and unique application of entanglement has also been to quantify total correlations [34]. This total correlation measure also called the entanglement of purification is less explored area compared to other uses and applications of entanglement. Therefore, in the first part of this thesis, we have found out various interesting properties of the entanglement of purification, and through it we have also discovered many interesting properties of the quantum advantage of dense coding as well. Another important and less understood area is the multipartite entanglement and its evaluation for multipartite mixed states. Typically with increasing dimensions, parties and ranks of the quantum states the calculation of the genuine multipartite entanglement measure becomes very difficult and arduous. So, in the second part of our thesis we have focused on evaluation of the genuine multipartite entanglement measure called the generalized geometric measure for arbitrary number of parties, dimensions and ranks of mixed quantum states. We have used the technique of using symmetries inherent in quantum states to simplify such calculations. We have been able to find many closed formulas for this measure for many classes of states, which we have elaborated in the second part of the thesis. Therefore, with the motivation as stated in this paragraph, we now summarize the results that we we obtained as part of this thesis in the next paragraph.

In the first part of the thesis, we find that the monogamous nature of correlations is not unique to quantum correlations, but can also be the case for the total correlations for certain quantum states. Thus, monogamy is not a property of the quantum correlation alone. Contrary to the monogamy nature of the mutual information for tripartite pure states, we have proved that the entanglement of purification can be polygamous for such states. This shows that even though the mutual information and the entanglement of purification are supposed to capture total correlation, the nature of these correlations can be completely opposite at least for tripartite systems. In case of pure and mixed states, the monogamy of entanglement of purification is related to the monogamy of entanglement of formation. Also, we have found a necessary condition for monogamy of entanglement of purification for a special class of mixed states, in terms of the interaction information or the polygamy of the quantum mutual information. A new lower bound of the entanglement of purification has been given for the tripartite mixed states and higher dimensional bipartite systems. Using the formula for the lower bound we have been able to find the exact values of entanglement of purification for some classes of states. Furthermore, in this paper we have also shown that if entanglement of purification is not additive, it has to be a sub-additive quantity. Using these results we have also shown that the quantum advantage of dense coding is strictly monogamous for all tripartite pure states and it is super-additive on tensor products. We have also identified some of the quantum states with no quantum advantage of dense coding. We have brought forward these important aspects of the measure of total correlation as well as that of the quantum advantage of dense coding to the forefront. These will help us understand better the nature of total and quantum correlations of composite quantum states. This calls for more explorations and a deeper understanding of the total correlation present in a composite mixed state. The total correlation quantified by the mutual information can be split into quantum correlation and classical correlation. However, we still do not know whether we can express the entanglement of purification as the sum of quantum and classical correlations. In view of the polygamy nature of entanglement of purification, can it be the case that the entanglement of purification contains more classical like correlation than the quantum correlation. This will be a topic of future investigation.

Computing entanglement of an arbitrary mixed state is a formidable task. The entanglement of mixed states is generally defined by constructing the convex roof over all possible pure states which is practically impossible to compute in most of the cases. Although there exists a few bipartite measures which can be obtained for arbitrary states, the evaluation of entanglement for a mixed state in multiparty domain is still a challenging task. In this thesis, we have computed a genuine multiparty entanglement measure known as generalized geometric measure of some classes of mixed states with arbitrary number of parties and dimensions by using certain symmetries. We evaluate the measure for several classes of multiqubit and multiqudit states having different ranks. The method, we exploited, uses a pure state that contains the same amount of entanglement as the given mixed state, and leads to the mixed state by action of a certain twirling operation. This summarizes our results on the generalized geometric measure aptly.

5.2 Future directions

In this thesis, we have been able to unravel certain important properties of the entanglement of purification as well as we have found the exact values of the generalized geometric measure for different classes of mixed states. Our analysis and results have in turn paved way for new possibilities in this filed. In the next two sections, we therefore discuss some of the future directions to research that arise from our thesis.

The entanglement of purification is a difficult quantity to calculate analytically due to the absence of a closed formula. In our thesis, we have tried to find out its exact values for some of the cases. However, also due to the lack of a closed formula and optimization over a high dimensional Hilbert space, even numerical optimization is a difficult task. Therefore, it is an interesting problem to try to formulate a numerical program for entanglement of purification. This numerical program may use the semi-definite optimization procedure to simplify the method for numerical calculation. Also, even if one is unable to find exact values for entanglement of purification, good lower bounds for general mixed states shall help too. We have found better lower bounds for some states, but better lower bounds can be found as part of future research too. This shall help in answering several other important questions about the entanglement of purification, for example the monogamy and additivity. Thus, apart from the exact values and lower bounds, various other important properties of it remains unknown. In this thesis, we have been able to find a general monogamy inequality for entanglement of purification, however only for the case of pure states. Thus it is still an open question as to whether the entanglement of purification satisfies a general monogamy or polygamy inequality for mixed states as well. This shall help one to compare and contrast its properties with that of the quantum mutual information. Next, the other very important question about the entanglement of purification is the question of additivity. It still remains an open question whether the entanglement of purification is additive or not. Though we have proved the sub-additivity, we have still not been able to find states that satisfy strict sub-additivity condition. Thus it will be interesting to see and find states that satisfies the strict inequality condition. Otherwise, to prove additivity, one just has to prove a general super-additivity condition for entanglement of purification for all states. Therefore, we see that the entanglement of purification is an important area of research and our results in this thesis opens up possibilities of new research in this are.

The generalized geometric measure is a computable measure of genuine multipartite entanglement, that has already been calculated for arbitrary number of parties for pure states. However, like other measures of entanglement, it is a very difficult task to compute it for the mixed states. In this regard, we have used the symmetry properties of the quantum states to simplify such calculation and found out exact values for some classes of mixed states. But, the choice of the unitary operators and the quantum states possessing symmetry are limited. Therefore, it is an interesting direction to find out larger classes of mixed states of varying parties, dimensions and ranks for which this technique will be applicable. Apart from this, we also have chosen only some groups of unitary operators for the process. It will also be a topic of further investigation to find out other groups of unitary operators that can used for applying the same technique to calculated exact values of the generalized geometric measure for more number of classes of mixed states. This will be helpful in other arenas where there are vast applications of the genuine multiparty entanglement measures as well.

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