ZERO ANGULAR MOMENTUM CONJECTURE FOR BPS BLACK HOLES IN STRING THEORY

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A thesis submitted to the
Board of Studies in Physical Sciences
In partial fulfillment of requirements
for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

June, 2016

Homi Bhabha National Institute[1]

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. "BPS State Counting in N=8 Supersymmetric String Theory for Pure D-brane Configurations", A. Chowdhury, R. S. Garavuso, S. Mondal, and A. Sen, *J. High Energ. Phys. 1410*, **2014**, *186* (2014), [arXiv:1405.0412 [hep-th]

Others

1. "Do All BPS Black Hole Microstates Carry Zero Angular Momentum?", A. Chowdhury, R. S. Garavuso, S. Mondal, and A. Sen, *JHEP 1604 (2016) 082*, **2016**, [arXiv:1511.06978 [hep-th]]

Swapnamay

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To

Ma, Baba & Kaka

Acknowledgements

First and foremost, I would like to express my profound gratitude towards my advisor and mentor, Prof. Ashoke Sen, for his continuous support during my Ph.D. years. It was a great facility to be able to approach him with any question (however trivial) anytime. Everytime he had listened to those questions patiently and explained the answer (which he always has) with equal patience. His guidance, patience (immense!), motivation and deep knowledge have proved invaluable to me in my struggle to make the very difficult transition from a student to a researcher.

I am thankful to the members of my thesis advisory committee for their support in letting me continue my research without any obstructions. I would like to thank Prof. A. Raychaudhuri and Prof. J. K. Bhattacharjee for giving me an opportunity to work at HRI. I am grateful to all the teachers who have taught us during the coursework. It is this detailed coursework that has prepared the ground for all my future research activies.

I have very much enjoyed the free academic environment of string group at HRI, in which anybody could work with any other person. I would like to thank the faculties of the string group, Prof. Ashoke Sen, Prof. Rajesh Gopakumar, Prof. Dileep Jatkar, Prof. Satchidananda Naik, Prof. Sudhakar Panda, Dr. Anshuman Maharana and Dr. Anirban Basu, to whom we owe this environment. My gratitude to Shweta Srivastava, who has been very helpful as the secretray of string group.

My deepest gratitude to workers of Fighting Four (or its new avatar), for they are the ones who keep HRI running (despite of their very humble salary)! I would also like to thank Ravindra Singh, Yashpal Singh Bhadouriya, Archana Singh, Ajay Srivastava, Manish Sharma, Sanjay Verma, Anju Verma, Umakant Dwivedi, Swapna madam, Dharmendra Malhotra, Amit Roy, Raj Gulati, Archana Tandon, Seema Agrawal, Rachna Pareek, Sushant and all other members of HRI administration for their help in various occasions. I am thankful for the HPC clusters at HRI, on which most of the numerical calculations relevant to this thesis were done.

The long list of friends, to whom I owe all the fun and happiness during my stay at HRI, first comes the ones to enter HRI with me. Thank you Avijit, Masud, Dharma, Raghu, Abhishek, Nyayabanta, Shrobona, Akansha, Mohan, Akhilesh. I am very much thankful to my seniors who had embraced us warmly and made us feel home at HRI. Thank you JD, Sanjoy-da, Ujjal-da, Arjun-da, Ayan-da, Subhaditya-da, Niyogi-da, Arunabha-da, Manoj-da, Anushree-di, Tarat-da, Paromita-di, Satya-da, Rambha-da, Atri-da, Ipsita-di, Dhiraj-da, Bhuru-da, Shailesh Bhaiya and many others. Similar gratitude is deserved by the ones to come to HRI afterwards. Thank you Shankha, Ushosi, Roji, Taushif, Rana, Aritra, Dibya, Sauri, Arijit, Nabarun, Subhroneel, Titas, Mrityunjay, Sitender, Ritabrata, Avinanda-di, Sayantani-di, Madhuparna-di, Tapomay-da, Nilay-da, Subhadeep-da, Arindam-da, Animesh-da, Imtak, Takimi, Masazumi, Sourav-da (Bhattacharya), Manab-da, Avirup-da, Anirban-da (Biswas and Basu), Nandan-da, Tapasi-di and all others.

Without the sacrifices made by my parents, I could never have reached here. I am especially thankful to them for never imposing their own dreams about my career and always letting me pursue my own interests. I will take the opportunity to express my gratitude to my *kaka*, whose love for me can only be compared with that of a parent.

Finally its the turn of my beloved Saheli. I would never have gathered the motivation of doing whatever I have done, unless you were there, half a world away, but still in my beating heart, at each and every moment with your constant quarreling!

CONTENTS

SYNOPSIS

From their very inception to their recent detection, black holes have always captured physicists intrigue as well as popular imagination. One major theoretical puzzle offered by black holes is the black hole entropy problem. Second law of thermodynamics tells that entropy can not go down in any physical process. Now think of a black hole being formed through gravitational collapse of a star, which would carry some entropy. Second law in this case would imply the resultant black hole must have entropy as well. A statistical understanding of this entropy would require the black hole to have microstates. However the usual understanding of black holes, based on Einstein's theory of gravity, gives no clue where these microstates may come from. Works of Hawking and others however strongly suggest that entropy of a black hole should be proportional to its surface area with the proportionality constant carrying signature of quantum gravity. This suggests that this is actually a problem to be handled in the framework of quantum gravity. Put another way, a successfull theory of quantum gravity must explain the origin of black hole microstates. String theory being leading candidate for the theory of quantum gravity, has naturally been put to this test and has been extremely sucessful for supersymmetric black holes. However the way string theory addresses this question is somewhat indirect and one counts the microstates in a regime away from black hole description, which is allowed by supersymmetry. Main goal of this thesis is to understand the nature of black hole microstates as we move away from the black hole description.

Major result of our studies is to gather evidence for the conjecture that at a generic point of moduli space, all microstates of a single centered supersymmetric black hole carry zero angular momentum. In the regime where black hole description is valid, it is known to be the case. This in particular means that the number of ground states for such a black hole is same as Witten index, i.e. the difference between the number of bosonic and fermionic states. By virtue of supersymmetry, Witten index remains unchanged as one changes the theory smoothly. Exploiting this feature one goes to a regime in moduli space where Witten index can be evaluated and compares the result with Bekenstein-Hawking formula. Till now in each case studied, there is an excellent match. However this match by itself does not tell how does various properties of the microstates change as one moves in moduli space. One key feature of the microstates, when black hole description is valid, is that they carry zero angular momentum. It is natural to wonder whether this remains true as one moves away from the black hole regime of moduli space. We gather evidence supporting the conjecture that at a generic point of moduli space, microstates of a single centered supersymmetric black hole continue to carry zero angular momentum.

CHAPTER 1

INTRODUCTION

String theory attempts to be the theory of everything, i.e. a theory that describes all matters and forces within a single theoretical framework. Experience in physics has repeatedly shown that apparently very different physical phenomenon often stems from same basic physics. This has been the case with electricity and magnetism, which naively does look very different, nevertheless can be described together by Maxwell's equations. We had similar experience with weak force, that is responsible for beta decay. Till now three fundamental forces of nature, electromagnetic force, weak force and strong force along with all the matter particles that has still been detected experimentally[a]. are described by standard model of particle physics, with extremely high accuracy. Standard model uses the theoretical framework of quantum field theory and various fundamental matter particles and three forces all appear as quantum fields. So a rule of thumb that works in this case is each fundamental particle or force corresponds to a quantum field. One would naively expect that if we just introduce a quantum field corresponding to gravitational force, we could describe all forces and matter in a single framework. However this fails disastrously due to renormalization problems. Thus the first and probably the biggest challenge to develop a unified theory is to develop a theory of quantum gravity.

String theory has the almost magical feature that it automatically provides a quantum theory of gravity, with no hard work at all! Hard work is needed to reproduce standard model of particle physics from string theory and this has not yet be achieved. But we would not concern ourselves with this aspect. Even the claim of being "the theory of quantum gravity" is a highly non trivial one. Although string theory does have the theoretical qualification for being a theory of quantum gravity, to establish this we need experimental verification. On general grounds one can argue that the natural energy scale associated with any theory of quantum gravity is Planck energy $\sim 10^{16}$ TeV. This means to verify a theory of quantum gravity, we need to perform experiments at energy

[a]We will not include dark energy and dark matter in our discussion, which constitute most of the universe, but still far away from human understanding.

scale 10^{16} TeV. Given that with the integrated effort of almost whole human civilization, we have only been able to reach energy scale 10 TeV in LHC, reaching 10^{16} TeV is certainly not something that is going to happen in near future.

In such a situation either we can sit idle or we can check whether a candidate theory of quantum gravity passes various theoretical consistency tests. Luckily black hole entropy puzzle serves as a highly non trivial theoretical consistency check. The puzzle is the following. Classically black holes are described as solutions of Einstein's equations and are completely described by only a few parameters, namely the charges carried by the black hole. In other words, once all the charges are specified, a black hole does not have scope for any microstates. However for the second law of thermodynamics to hold in presence of black holes, they must have entropy. The "laws of black hole mechanics", based on classical general relativity, happen to have peculiar resemblance with laws of thermodynamics. In particular area of event horizon behaves much like entropy and surface gravity at the horizon behaves much like temperature, both up to some proportionality constant. Further analysis of quantum black holes has proved that this temperature is indeed the physical temperature of the black hole. This fixes the proportionality constants and shows the black hole entropy to be $\frac{1}{4}\times$ area of event horizon, in natural units ($\hbar = 1, c = 1, G = 1$). This is called Bekenstein-Hawking entropy. Since entropy requires microstates and as explained above, classical black holes do not have any scope of having microstates, we are left absolutely clueless. However a strong hint comes from the fact that the proportionality constant has Planck length, which is the characteristic length of any theory of quantum gravity, in it. This suggests that this question can be answered in a theory of quantum gravity. We can reverse the logic and say that if a candidate theory of quantum gravity is the correct one, it must explain the origin of black hole microstates. In other words black hole entropy problem can serve as a litmus test for a candidate theory of quantum gravity. It should be stressed that the question is not only a qualitative one, but also a quantitative one, i.e. it does not suffice to explain the origin of black hole microstates, one also have to count them to produce Bekenstein Hawking entropy, with correct numerical coefficient. Further, a theory of quantum gravity can have many phases (i.e. vacua) other than the one we happen to live in. Many such phases may admit black hole solutions. The correct theory quantum gravity should be able to address the black hole entropy puzzle for all such black holes, a single failure is enough to disqualify the theory as a correct theory of quantum gravity[b].

String theory has been able to address this question with high accuracy for various supersymmetric black holes. Along with producing Bekenstein-Hawking entropy, one has also been able to address quantum corrections to Bekenstein-Hawking formula. This enormous success certainly provides a strong hint that string theory is "the theory of quantum gravity". However to count the

[b] In practice all phases may not be easy to deal with due to various difficulties. Then the ones that can be dealt with must reproduce Bekenstein-Hawking entropy correctly.

black hole microstates, one deforms the system to go to a regime where such counting is conceptually clear as well as computationally possible. By merit of supersymmetry this does count the number of black hole microstates, despite being deformed. So despite the computational success, one does not really have a direct hold over black hole microstates.

To get a better hold over the nature of black hole microstates, a good strategy would be to ask how various properties of black hole microstates change as we move away from black hole description. In particular it would be interesting to find properties that do not change as we move in moduli space, for these properties can be thought as characteristic features black hole microstates. A particularly interesting property of single cenetered supersymmetric black holes is that they carry zero angular momentum. It is only natural to wonder what happens to this property as one moves away from black hole description, or for that matter anywhere in moduli space. Existing literature only explores special points in moduli space and in those points black hole microstates are seen to carry non zero angular momentum as well. However those being only special points, one can still ask what happens at a generic point in moduli space? Our works suggest that at a generic point of moduli space microstates of a single centered supersymmetric black hole still carry zero angular momentum.

1.1 Black Hole Thermodynamics

Black hole is a region of space time from which nothing, not even light, can escape. More technically speaking, black hole is the part of space time that is not included in the chronological past of future null infinity. Black holes arise as solutions of Einstein's equations. Due to such strange properties in the initial days people (including Einstein himself!) were somewhat sceptic whether such objects can exist in real world. However there have been strong indications for the existence of astronomical black holes, e.g. there is strong speculation that there is a supermassive black hole at the center of our galaxy. Nevertheless decisive observational evidence for existence of black holes have been gathered only very recently [1]. Now that we know black holes do exist in real world, any question about black holes is of some relevance to real world.

In [2] it was shown that if two black holes collide and settle down to form a new one, area of the event horizon can not decrease in this process. This was shown based on fairly mild assumptions, such as absence of naked singularities, null energy condition ($T_{\mu\nu}k^\mu k^\nu \geq 0 \,\forall$ null k^μ) and therefore is a very general result for black holes. One can not but note the striking similarity between black hole area and entropy, which never goes down in a physical process. This might seem only a coincidence. However, Bekenstein has argued [3], [4] that in order for second law to hold in presence of black holes, black holes must have some entropy. The argument is fairly simple. Let us throw a bucket of hot water in a black hole, black hole will eat it up and would soon settle down to a little bigger black hole. But if black hole does not have any entropy then the entropy in the hot water would be lost and hence second law would be violated. In the light of Hawking's finding [2] that area of the event horizon of a black hole never goes down, Bekenstein proposed that black hole entropy is proportional to its area, the proportionality constant being an universal constant. Bekenstein proposed a "Generalized Second Law": the sum of the ordinary entropy of matter outside of a black hole plus a suitable multiple of the area of a black hole never decreases.

This speculation was put on a firmer ground by the work of Bardeen, Carter and Hawking [5], in which "laws of black hole mechanics" were derived. In particular clear analogs of zeroth and first law were found, with surface gravity κ behaving as temperature. The analogy between surface gravity and temperature was further proven to be physical in [6]. There quantum effects were taken into account and it was shown that a black hole behaves much like a black body and emits thermal radiation at temperature $\hbar\kappa/(2\pi)$. From the first law one can fix the proportionality constant in the entropy and the black hole entropy is found to be

$$S_{BH} = \frac{c^3 A}{4G\hbar}\,. \tag{1.1.1}$$

This is referred as Bekenstein-Hawking entropy[c].

This establishes that black holes indeed carry entropy given by S_{BH}. But now there is another more serious puzzle. Statistical mechanics is known to be the underlying physics of thermodynamics. In particular entropy is known to arise from the fact that for specified macroscopic properties, a thermodynamic system has many microstates available, in which it can stay. One would like to understand statistical origin of black hole entropy. For this it is essential to understand the microstates of a black hole. Here we are left absolutely clueless, since in classical gravity black holes are completely specified by their mass, angular momentum and charges and thus there is no scope for microstates!

However Bekenstein-Hawking formula (1.1.1) itself gives s strong hint regarding where we should look for an explanation. Interestingly the expression (1.1.1) contains Planck length $l_p := \sqrt{\frac{G\hbar}{c^3}}$. It contains both Newton's constant G (which is signature of gravity) and Planck's constant $\hbar$ (which is signature of quantum physics), and hence is a natural length scale of the theory of quantum gravity (although we do not know the theory itself). This tells that the theory of quantum gravity is the place to look for a statistical explanation of black hole microstates. We can reverse the argument to argue that the correct theory of quantum gravity must successfully give a statistical explanation of black hole entropy. Given that we are nowhere near testing a theory of quantum

[c] All these were for Einstein's theory of gravity. One may ask what happens if one considers more general diffeomorphism invariant theory of gravity. In fact such theories are essential in string theory, since stringy effects give rise to corrections to Einstein-Hilbert action. For such general diffeomorphism invariant theories, the analog of Bekenstein-Hawking entropy is given in [7], [8], [9], [10].

gravity experimentally, such theoretical tests are of immense importance.

Few more words on stringency of this test. Firstly the challenge is not only qualitative but also quantitative. Namely if a candidate theory of quantum gravity gives a qualitative account of black hole microstates, but fails to reproduce the Bekenstein Hawking entropy with correct numerical coefficients, that is still a failure. Secondly, this question can be asked for all black holes and the correct theory must address this issue for all black holes. In particular, a theory of quantum gravity can have many phases[d], although they are not realized in the universe we live in. Nevertheless each such phase is a mathematically consistent possibility, that just happens not to be realized in real world. Many such phases may contain black hole solutions. Although these are not the black holes that we see in our universe, they are nevertheless consistent possibilities. Thus the black hole entropy puzzle has to be addressed for all such black holes.

1.2 How does string theory address black hole entropy problem

Given that string theory is the leading candidate for the theory of quantum gravity, it is only natural to put string theory to this test. The first ray of hope comes from the observation that the number of string states at a given mass level grows rapidly with the mass, which qualitatively resembles rapid growth in the number of black hole microstates with mass. However this resemblance is only qualitative since the number of string states at a given mass level M grows like e^M for large M, whereas the number of black hole microstates at a given mass level M grows like e^{M^2}. Despite of this mismatch, there is still some hope due to the following possibility. String spectrum is usually computed in flat spacetime, which is very different from black hole spacetime. To get to black hole spacetime, one has to increase Newton's constant G significantly so that GM becomes large enough to have considerable gravitational backreaction and therefore create a black hole out of an elementary string. In such case, large renormalizations of black hole masses might fix the e^M versus e^{M^2} mismatch, as has been suggested in [11], [12], [13]. In spite of reigniting our hope, the above arguments certainly do not account for black hole entropy convincingly. We would be able to say something with certainty only if we had black holes which do not receive any mass renormalization.

Luckily such states do exist in a supersymmetric theory with $\mathcal{N} > 1$ supersymmetry. These are well known BPS states [14], [15]. They occur when "charge equals the mass" (in appropriate units). The reason for their stability is that for these special values of charges and masses, one gets an entirely different representation of supersymmetry algebra, usually referred to as short representation, since it has less number of states than non-BPS ones. Now as we change the coupling, a state can not suddenly change which representation it sits in. Black holes for which charge equals the mass

[d]E.g. in string theory one expects the number of vacua to be of the order of 10^{500}.

are called extremal black holes. Certain such elementary string states and corresponding black hole have been considered in [16] and an order of magnitude match in the number of microstates has indeed been found.

A more profound and quantitative match is found when one considers black holes that corresponds to not only elementary strings but also various solitons in string theory. Let us restrict to type IIA string theory for definiteness. The qualitative picture is same for all string theories. The bosonic part of the massless spectrum in this theory contains the graviton $G_{\mu\nu}$ (symmetric traceless), B field $B_{\mu\nu}$ (antisymmetric), and RR fields $C^{(1)}_{\mu}$,$C^{(3)}_{\mu\nu\rho}$ (totally antisymmetric). String theory is naturally defined in 10 space time dimensions. To get to real world, one has to compactify 6 of these directions. Let the compact directions form a six-torus T^6. In the discussion below, the Roman indices a, b, c will denote the compact directions and the Greek indices α, β, γ will denote 4 non-compact directions. After compactification the set of fields $G_{\alpha a}, B_{\alpha a}, C^{(1)}_{\alpha}, C^{(3)}_{\alpha ab}$ behave as gauge fields. This way one gets 28 gauge fields in the low energy effective theory, which is t$\mathcal{N} = 8$ supergravity. The black holes in this theory may be charged under these gauge fields. Now one can track back the gauge fields to various tensor fields in string spectrum. In full string theory one knows which objects are charged under these tensor fields[e]. These may be elementary strings with momentum/winding along the compact directions as well as non perturbative solitons such as D branes. Then one can think of the black hole as some collection of say D branes spread along the compact directions. This is referred to as microscopic description of the system[f]. From the perspective of the non compact directions, the collection of elementary strings and solitonic branes looks like a point, namely the black hole. It should be emphasised that the black hole description and microscopic description are valid in two very different regimes. When the string coupling g is very small and hence Newton's constant G is very small, one can neglect gravitational backreactions of various constituents of microscopic description and consider them in flat space background. This is the regime when microscopic description is a good one. However when the string coupling g and hence Newtons constant G is large enough so that gravitational backreactions are important then such description is not a good one, since they are constructed in flat space background. In this regime they are better described as black holes. Another way to put this is to consider the Schwarzschild radius $\sim GM$. When string coupling g is small enough so that GM is comparable to the string length scale, curved space time is not a good enough description and one must consider this as a quantum system[g]. On the other hand when g is large enough so that the Schwarzschild

[e]Electric charges of $G_{\alpha a}, B_{\alpha a}, C^{(1)}_{\alpha}, C^{(3)}_{\alpha ab}$ are respectively carried by momentum along x^a, elementary string winding along x^a, D0 brane and D2 brane along x^a–x^b. Magnetic charges of $G_{\alpha a}, B_{\alpha a}, C^{(1)}_{\alpha}, C^{(3)}_{\alpha ab}$ are respectively carried by KK monopole along x^a, NS5 brane spanned along 5 compact directions except x^a, D6 brane spanning all the T^6 directions, D4 brane spanning the 4 compact directions except x^a and x^b.

[f]It should be mentioned that the microscopic description is far from being unique. Given one microscopic description, one can use various string dualities to get to many other equivalent microscopic description.

[g]One subtlety comes from the fact that for solitons, the mass M also depends on g. In fact it grows as g approaches

radius is much larger than string length, black hole is a good description.

Now in the line of our discussions for elementary string, one considers BPS black holes. BPS states do not change as we change the string coupling, therefore given a BPS black hole we may well count the number of BPS states in the microscopic description and regard this as counting microstates of the black hole. Actually one has to be a little more careful due to the following reason. We know that the number supersymmetric ground states in a supersymmetric theory may change due to creation of Bose-Fermi pairs, as we smoothly change the Lagrangian (without affecting any symmetry or normalizability of wave-functions). What remains unchanged is the difference between number of bosonic and fermionic states, famously known as Witten index. For a BPS state there are two kinds of supersymmetries. 1^{st} kind of supersymmetries annihilate the BPS state, hence with respect to these supersymmetries a BPS state is like a supersymmetric ground state (in appropriate sector). 2^{nd} kind of supersymmetries do not annihilate the BPS state and hence generate the BPS supermultiplet structure. These supersymmetries can readily be identified with fermionic Goldstones in the quantum mechanics describing the dynamics of the BPS particle, henceforth referred as Goldstino fermions or simply Goldstinos. Since in each supermultiplet there are equal number of bosonic and fermionic states, the Witten index is identically zero and therefore useless, despite of being protected. However if we throw the Goldstinos along with its superpartner Goldstones (which is possible since they are by definition non-interacting) then each BPS supermultiplet is replaced by a bosonic/fermionic supersymmetric ground state. In this reduced theory Witten index may well be non zero. Thus we will throw the Goldstinos and compute Witten index in the theory thus obtained[h]. This would be same as the number of BPS states only if the BPS states sans the supermultiplet structure, are all bosonic. This is luckily the case for all single centered supersymmetric black hole (whether or not it descends from string theory). All microstates of such black holes carry zero angular momentum[i]. Thus the appropriate index does count the degeneracy. Now one can go to the microscopic description where it is relatively straight forward to compute the index and check whether it gives rise to the degeneracy as required by Bekenstein-Hawking formula.

First quantitative success in this direction was achieved in [17]. It is instructive to briefly go through the basic logic of this paper. In this paper one considers five dimensional $\mathcal{N} = 4$ theory,

0. Since $G \sim g^2$, we need GM to be proportional to some positive power of g. This indeed is the case for D branes, whose mass goes like $1/g$.

[h] In the full theory, which includes Goldstinos, this corresponds to computing a certain index, to be described later.

[i] This is because near horizon geometry of any single centered black hole has a AdS^2 factor and and the closure of the AdS^2 symmetry algebra and the algebra generated by the unbroken supersymmetries include also an $SO(3)$ algebra. Hence a supersymmetric single centered black hole must be spherically symmetric and consequently carry zero average angular momentum. Further the black hole defines a microcanonical ensemble. This is because for the near horizon AdS^2 path integral (which is needed to define quantum analog of Bekenstein-Hawking formula) to be well-defined, one has to fix all the charges, which in particular includes the angular momentum. Thus all microstates of such a black hole must carry zero angular momentum.

obtained by $K3 \times S^1$ compactification of type II theory (or equivalently T^5 compactification of heterotic string theory). This theory contains two 2-form field strengths $\tilde{H}$ and F. We will work in type IIB description. Then $\tilde{H}$ comes from the Kaluza Klein gauge potential $G_{\mu\theta}$, θ being the coordinate of the S^1 and μ denoting non-compact directions. The charge of this field, which we call Q_H, is the momentum along the S^1. Similarly F can be thought as collection of RR field strengths, after reduction along compact directions. The charge vector of F is called Q_F. Components of Q_F count the number of D branes wrapped along different cycles of $K3$ times S^1. Black holes[j] carrying both Q_H and Q_F charges are considered. It is useful to consider the limit when $K3$ size is small and as a result the D brane world volume theory is described by a sigma model living on $S^1 \times R$, where R stands for the time direction. Existing results indicate that this sigma model has central charge $(\frac{1}{2}Q_F^2 + 1)$, where Q_F^2 is computed with the intersection matrix of the cycles of $K3$. Now the supersymmetry requires the contributing states must be annihilated by either $L_0 \equiv \frac{1}{2}(H+P)$ or $\bar{L}_0 \equiv \frac{1}{2}(H-P)$, where H stands for the Hamiltonian and P for momentum along S^1. Taking $\bar{L}_0 = 0$ for definiteness one has $L_0 = P = Q_H$. Thus the task in hand is to count the number of states[k] with $L_0 = Q_H, \bar{L}_0 = 0$ in a CFT of central charge $(\frac{1}{2}Q_F^2 + 1)$. For large Q_H this is known [25] to go like

$$d_{microscopic} \sim e^{2\pi\sqrt{Q_H(\frac{1}{2}Q_F^2+1)}} . \tag{1.2.2}$$

On the other hand, for such black holes, Bekenstein-Hawking formula gives[l]

$$S_{BH} = 2\pi\sqrt{\frac{Q_H Q_F^2}{2}} . \tag{1.2.3}$$

Thus one finds precise match between $\ln d_{microscopic}$ and S_{BH}, including numerical coefficients, for large charges.

Following [17], there has been a plethora of explorations in this direction and for a wide class of extremal supersymmetric black holes in string theory, microscopic considerations have successfully reproduced Bekenstein-Hawking entropy to leading order in the charges. This is old story by now and attention of the community has moved towards matching subleading corrections in the microscopic formula to quantum and higher derivative correcions to Bekenstein-Hawking formula in the macroscopic side. For $\mathcal{N} = 8$ theory, there are logarithmic corrections to Bekenstein-Hawking

[j] Here are some early works on various other black holes. These include five dimensional extreme black holes with rotation [18], slightly nonextremal five dimensional black holes [19], [20], [21], four dimensional extreme black holes [22], [23], and slightly nonextremal four dimensional black holes [24].

[k] It was argued that to leading order, which was the interest of the paper, the number of states is same as the appropriate index in this case.

[l] Since this is five dimensional black hole, now area stands for five dimensional area and Planck length is also five dimensional Planck length.

entropy coming from quantum effects and in $\mathcal{N} = 4$ theory, there are non-logarithmic corrections to Bekenstein-Hawking entropy coming from higher derivative terms in the effective Lagrangian. Both these corrections have been explained from microscopic considerations. However for $\mathcal{N} = 2$ theories enough microscopic results, needed to account for logarithmic corrections to Bekenstein-Hawking entropy, are still lacking. There are no known counter-examples though. Thus no doubt we can interpret these results as strong hints that string theory indeed is the theory of quantum gravity.

1.3 Summary of our works

Despite enormous success of string theory in giving a microscopic account of the Bekenstein-Hawking entropy for BPS black holes, the procedure is somewhat indirect. We do get to count the microstates but we do not really learn what the black hole microstates actually look like. Namely what are the various properties of black hole microstates. Clearly this is made difficult by the simple fact the we have a better hold of the microscopic picture in a regime away from the black hole regime. One strategy could be to study which properties are robust as we move in moduli space. This would be of some help in understanding the black hole microstates because given a proposed description of black hole microstates, e.g. the fuzzball proposal, we will have a litmus test for the proposal.

Based on the analysis of the near horizon geometry of a BPS black hole it has been argued that as long as the black hole carries four unbroken supersymmetries, the microstates of single centered black holes will all carry zero angular momentum [26, 27]. It is natural to wonder whether this property continues to hold as we move away from the black hole regime. Indirect tests of this have been performed by examining the BPS index – given by number of states weighted by $(-1)^{2J_3}$ after factoring out the goldstino fermion zero mode contributions – which is expected to be independent of the moduli. On the black hole side if all the microstates carry zero angular momentum, then the BPS index will be positive. This implies that on the microscopic side also we must always have positive index. This has been tested in many examples [28, 29], and so far no counterexample has been found. Nevertheless existing microscopic countings typically give states carrying different angular momenta (even after factoring out the the goldstino zero mode contribution), some of which give positive contribution to the index and others give negative contribution. However existing computations are usually done at rather special points in the moduli space. So we can still ask whether this property holds at a generic point in the moduli space. To be able to answer this question or at least to get some strong hint, we need to be able to turn on various moduli in our microscopic description and also to make sure that there are no multi-centered black holes for the charges considered (this is because one expects only the microstates of a single centered black hole to carry zero angular momentum). Both these can be accomplished if we consider $1/8$ BPS black holes carrying only RR charges in $\mathcal{N} = 8$ theory. The microscopic description contains various D branes wrapping various cycles of the compact T^6. We find for small charges that all microstates of this system carry zero angular momentum. Due to practical difficulties we are unable to address the problem for large charges. Nevertheless the mathematical structure that led to this result for small charges, remains the same even for large charges. Thus we are led to the conjecture that at a generic point of moduli space, all microstates of a single centered supersymmetric black hole carry zero angular momentum.

CHAPTER 2

THE SYSTEM

2.1 The Theory

In the microscopic description, one gets all microstates for a given charge. As one increases the Newton's constant to go to the black hole regime, some of these microstates may become single centered black holes and some may become multi-centered (if such black holes exist and they generically do). In the microscopic description, it is very difficult to identify which microstates correspond to single centered and which ones to multi-centered black holes. But the conjecture we wish to verify is strictly a statement about single centered[a] black holes and thus such an identification is a must for us.

In $\mathcal{N} = 2$ supersymmetric theories in 3+1 dimensions the contribution from the multi-centered black holes is very complicated [30] and it is not easy to disentangle it from the contribution from single centered black holes, although some progress has been made on this front [31]. In this context the results of [32–36] can be taken as providing partial support to this conjecture. The situation is somewhat better in $\mathcal{N} = 4$ supersymmetric theories where the contribution from multi-centered black holes are better understood [37, 38, 41, 42].

The situation is best for $\mathcal{N} = 8$ theories. In this theory, although a $1/8$ BPS black hole can sometimes breaks up into two half-BPS[b] black holes across walls of marginal stability, for this to be allowed a certain inequality[c] needs to be satisfied by the charge vector. Thus if we choose our charge vector such that this inequality is not satisfied, then only contribution to the index is

[a] Multi-centered black holes carry non-zero angular momentum.

[b] A $1/8$ BPS states breaks $32 - 4 = 28$ supersymmetries. Correspondingly there are 28 fermionic Goldstones (usually called Goldstino). On the other hand each half-BPS state breaks 16 supersymmetries and hence have 16 Goldstinos associated to this breaking. Two half BPS states seems to have 32 Goldstinos. However 4 of these are actually interacting, hence one has only 28 Goldstinos, same as a $1/8$ BPS state.

[c] Such black holes satisfy $Q^2P^2 - (Q \cdot P)^2 < 0$, where Q and P are electric and magnetic charges of the black hole, in a certain duality frame.

from single centered black holes [43]. In this case the conjecture means that at a generic point of moduli space all micostates must carry zero angular momentum. Now we will set out to test this conjecture.

2.2 The System

Various microscopic descriptions can be cooked up to describe such black holes. All such descriptions would be related to each other by U-duality. Previously $1/8$ BPS black holes in $\mathcal{N} = 8$ theory has been considered in [47]. Their microscopic description contains a D1-D5-momentum system in KK monopole background. However their analysis is carried out at a special point in moduli space and hence fermionic microstates also contribute to the index. So we need to consider a different system where we can go to a generic point in moduli space. Nevertheless the results of [47] will serve as a cross check.

Regarding the choice of such a system, we take a hint from recent results [32], [33], [34]. There it has been found that when half-BPS black holes in $\mathcal{N} = 2$ theory are described as D branes wrapping various cycles of Calabi-Yau, one gets exactly zero angular momentum for microstates of a single centered black hole. Thus we go to a duality frame where all the charges carried by the black hole are RR charges, as a result the microscopic system comprises of only D branes and no other kind of charges (e.g. momentum or winding). As we will see, in this system turning on various moduli is quite easy, in fact desirable.

$\mathcal{N} = 8$ theory is obtained by compactifying 10 dimensional type II theory on T^6. We take the T^6 to span 4-5-6-7-8-9 directions and 1-2-3 to be the non compact directions. Our system contains a stack of N_1 D2 branes wrapped along 4-5 cycle, a stack of N_2 D2 branes wrapped along 6-7 cycle, a stack of N_3 D2 branes wrapped along 8-9 cycle and a stack of N_4 D6 branes wrapping all the T^6 directions. We will however restrict to the case $N_1 = N_2 = N_3 = 1$.

2.3 The relevant index

The system considered preserves 4 superchargres, which means there are 28 Goldstinos. These are non interacting massless fermions[d]. In the presence of such fermions, Witten index vanishes identically. Thus we have to find something that is non-vanishing as well as protected under continuous deformations. This is given by B_{14}, 14-th helicity supertrace, defined as

$$B_{14} = -\frac{1}{14!}\mathrm{Tr}[(-1)^{2J_3}(2J_3)^{14}]\,. \tag{2.3.1}$$

[d] All massless non-interacting fermions are not necessarily Goldstinos though.

The case of interest is $0+1$ dimensional theory, i.e. quantum mechanics, obtained by restriciting to the 0 mode sector of a 4 dimensional theory. J_3 has the following interpretation in the quantum mechanics. Supersymmetric ground states of a supersymmetric quantum mechanics can be identified with elements of cohomology of the classical vacuum moduli space. Lefschetz $SU(2)$ (to be discussed later) has a natural action on this space and is idenfied [51] with the $SU(2)$ of spatial rotation.

To understand (2.3.1) better, let us first consider the case of 4 Goldstinos. They lead to 4 supersymmetries and they can in turn be arranged into 2 fermionic oscillators, corresponding to $J_3 = \pm 1/2$ respectively. Let us denote corresponding creation operators as $a^\dagger_\uparrow$ and $a^\dagger_\downarrow$. Now suppose if one removes the Goldstinos (which is possible since they are non-interacting), rest of the theory has non-zero Witten index. Let $|\psi\rangle$ be a state contributing to Witten index, i.e. annihilated by supersymmetry generators (the ones not coming from the Goldstinos). When the Goldstinos are included, we can have the following 4 states with same energy

$$|\psi\rangle, a^\dagger_\uparrow|\psi\rangle, a^\dagger_\downarrow|\psi\rangle, a^\dagger_\uparrow a^\dagger_\downarrow|\psi\rangle\,.$$

Two of these states are fermionic and two of these are bosonic. Thus together they have vanishing contribution to Witten index. Same holds true for all states previously contributing to Witten index and thus the Witten index vanishes.

The cure is to compute something that amounts to computing Witten index in the absence of the Goldstinos. Let the Hilbert space spanned by the Goldstinos be $\mathcal{H}_g$ and the rest of the Hilbert space be $\mathcal{H}_r$. The total Hilbert space is their tensor product $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_g$.

Consider the quantity

$$\begin{aligned} I_n &= \mathrm{Tr}[(-1)^{2J_3}(2J_3)^n] = \mathrm{Tr}[(-1)^{2J_3^{(g)}+2J_3^{(r)}}(2J_3^{(g)}+2J_3^{(r)})^n] \\ &= \sum_{k=0}^{n}\binom{n}{k}\mathrm{Tr}_{\mathcal{H}_r}[(-1)^{2J_3^{(r)}}(2J_3^{(r)})^k] \times \mathrm{Tr}_{\mathcal{H}_g}\left[(-1)^{2J_3^{(g)}}(2J_3^{(g)})^{n-k}\right]\,. \end{aligned}$$

Now note,

$$\mathrm{Tr}_{\mathcal{H}_g}\left[(-1)^{2J_3^{(g)}}(2J_3^{(g)})^m\right] = [0^m + 0^m] - [(-1)^m + 1^m]\,,$$

which is 0 for odd m and -2 for even non-zero m. This means

$$
\begin{aligned}
I_2 &= \mathrm{Tr}_{\mathcal{H}_r}[(-1)^{2J_3^{(r)}}] \times \mathrm{Tr}_{\mathcal{H}_g}\left[(-1)^{2J_3^{(g)}}(2J_3^{(g)})^2\right] + 2\mathrm{Tr}_{\mathcal{H}_r}[(-1)^{2J_3^{(r)}}(2J_3^{(r)})] \times \mathrm{Tr}_{\mathcal{H}_g}\left[(-1)^{2J_3^{(g)}}(2J_3^{(g)})\right] \\
&+ \mathrm{Tr}_{\mathcal{H}_r}[(-1)^{2J_3^{(r)}}(2J_3^{(r)})^2] \times \mathrm{Tr}_{\mathcal{H}_g}\left[(-1)^{J_3^{(g)}}\right] \\
&= \mathrm{Tr}_{\mathcal{H}_r}[(-1)^{2J_3^{(r)}}] \times \mathrm{Tr}_{\mathcal{H}_g}\left[(-1)^{2J_3^{(g)}}(2J_3^{(g)})^2\right] \\
&= -2\mathrm{Tr}_{\mathcal{H}_r}\left[(-1)^{2J_3^{(r)}}\right] .
\end{aligned}
$$

This is just Witten index on $\mathcal{H}_r$ (up to the factor -2) and hence protected. The lesson is that we needed to insert $(2J_3)^2$ inside the trace to soak up four Goldstinos. When there are n Goldstinos one needs a factor $(2J_3)^{n/2}$. Taking care of all normalizations one can show the following quantity

$$
B_{2n} := \frac{(-1)^n}{(2n)!}\mathrm{Tr}\left[(-1)^{2J_3}(2J_3)^{2n}\right] \tag{2.3.2}
$$

equals Witten index in the theory obtained after throwing all the Goldstinos.

For our case, 28 supersymmetries are broken. Thus the relevant index is B_{14}.

Before we go into computing B_{14} for our system, let us briefly review some relevant results of existing literature. By a series of duality transformations described in appendix D, our system (with an additional stack of D4 branes along 6-7-8-9 directions) is related to a system of N_1 KK monopoles associated with the 4-direction, $-N_2$ units of momentum along the 5-direction, N_3 D1-branes along the 5-direction, N_4 D5-branes along 5-6-7-8-9 directions and $-N_5$ units[e] of momentum along the 4-direction. The microscopic index of this system was computed explicitly in [47] for $N_1 = 1$. By a further series of U-duality transformations reviewed *e.g.* in [50], this system can be mapped to a system in type IIA string theory on T^6 with only NS-NS sector charges, containing $-N_2$ units of momentum along the 5-direction, N_1 fundamental strings wound along the 5-direction, N_4 KK monopoles associated with the 4-direction, $-N_3$ NS 5-branes wrapped along 5-6-7-8-9 directions and N_5 NS 5-branes along 4-6-7-8-9 directions. In the notation of [50], the electric charge vector Q and magnetic charge vector P for this state in the latter description are represented as

$$
Q = \begin{pmatrix} 0 \\ -N_2 \\ 0 \\ -N_1 \end{pmatrix}, \quad P = \begin{pmatrix} N_3 \\ N_5 \\ N_4 \\ 0 \end{pmatrix} . \tag{2.3.3}
$$

The T-duality invariant inner product matrix between charges was given by $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$. With this

[e]We will however set $N_5 = 0$.

we get

$$Q^2 = 2\, N_1 N_2, \quad P^2 = 2\, N_3 N_4, \quad Q \cdot P = -N_1 N_5\,. \tag{2.3.4}$$

We also define

$$\begin{aligned} \ell_1 &= \gcd\{Q_i P_j - Q_j P_i\} = \gcd\{N_1N_3, N_1N_4, N_2N_3, N_2N_4, N_5N_1\}, \\ \ell_2 &= \gcd\{Q^2/2, P^2/2, Q\cdot P\} = \gcd\{N_1N_2, N_3N_4, N_1N_5\}\,. \end{aligned} \tag{2.3.5}$$

We shall consider configurations for which

$$\gcd\{\ell_1, \ell_2\} = 1, \quad i.e. \quad \gcd\{N_1N_3, N_1N_4, N_2N_3, N_2N_4, N_1N_2, N_3N_4, N_1N_5\} = 1\,. \tag{2.3.6}$$

In this case, following [38,39] one can show that there is a further series of duality transformations that map this system to one with $N_1 = 1$ [40] for which the microscopic index is known from the analysis of [47]. Expressed in terms of the more general set of variables $(N_1, \cdots, N_5)$, the result for the BPS index for this system, which in this case corresponds to the 14-th helicity supertrace B_{14}, takes the form [48]

$$B_{14} = (-1)^{Q\cdot P+1} \sum_{s|\ell_1\ell_2} s\, \widehat{c}(\Delta/s^2)\,, \quad \Delta \equiv Q^2P^2 - (Q\cdot P)^2 = 4\, N_1N_2N_3N_4 - (N_1N_5)^2\,, \tag{2.3.7}$$

where $\widehat{c}(u)$ is defined through the relation [46,47]

$$-\,\vartheta_1(z|\tau)^2\, \eta(\tau)^{-6} \equiv \sum_{k,l} \widehat{c}(4k - l^2)\, e^{2\pi i(k\tau + lz)}\,. \tag{2.3.8}$$

$\vartheta_1(z|\tau)$ and $\eta(\tau)$ are respectively the odd Jacobi theta function and the Dedekind eta function.

We consider configurations with $N_1 = N_2 = N_3 = 1, N_4 = N, N_5 = 0$. Then the expected result for the index B_{14} is [47][f]

$$-\,\hat{c}(4N)\,. \tag{2.3.9}$$

Explicit computation (shown in C) gives

$$-\,\hat{c}(4) = 12, \quad -\hat{c}(8) = 56, \quad -\hat{c}(12) = 208, \quad \cdots \tag{2.3.10}$$

As we will see in upcoming chapters, our computations for these three cases [g] gives respectively 12, 56 and 208 isolated solutions for the $V = 0$ equation. Using (2.3.10) we see that this is in

[f] Various macroscopic tests of this formula beyond the one provided by the Bekenstein-Hawking formula have been carried out in [54–58].

[g] With our present methods it seems difficult to handle higher charges.

perfect agreement with the results from the dual description.

2.4 The Lagrangian

The brane system considered by us, intersects at a point. We have to compute B_{14} for the theory living in the worldline of this intersection. For this purpose only on the low energy modes are relevant. So we may concentrate only on the massless spectrum of the strings stuck to various branes. Further we may ignore spatial dependence of these massless fields and concentrate only on the dynamics of the zero modes of these massless fields. This gives the field content of quantum mechanics living in the worldline of the brane intersection.

Once we have the field content, supersymmetry fixes most[h] of the interaction. One may worry about all the terms coming due to stringy corrections. Although they are certainly there, they would not be very consequencial for our purpose. This is because in this regime the physics is controlled by low energy dynamics and stringy effects do not matter much.

We mention that although our theory is really a quantum mechanics, we arrange our fields according to four dimensional $\mathcal{N} = 1$ supersymmetry. It is understood that the fields depend only on time though.

2.4.1 String spectrum and physical interpretation of the Bosons

First let us determine the massless string spectrum. There are two kinds of strings:

1. strings with both ends on same D brane. We will be calling them “on brane strings”.

2. strings with two end points on different branes. We will be calling them “mixed strings”.

On brane strings: First let us consider the massless spectrum of strings stuck to a D3 brane. It is essentially the same as a 10 dimensional open string with NN boundary condition on all directions, only difference being that they are allowed to move only on the D3 brane. We know that the massless spectrum of a 10 dimensional open string with NN boundary condition along all directions contains a ten dimensional gauge field and a ten dimensional Majorana-Weyl fermion. When dimensionally reduced to the the D3 world volume, this gives 4 Weyl fermions, 1 gauge field and 3 complex scalars (equivalently 6 real scalars). This is just the field content of $\mathcal{N} = 4$ super Yang-Mills theory. For a stack of N D3 branes, the $\mathcal{N} = 4$ SYM has gauge group $U(N)$.

Although our branes are not quite D3 branes, for each brane we can perform a T duality to get a D3 brane[i] and field content does not change under this operation. Thus the quantum mechanics

[h]As we will see, certain important terms will not be fixed by supersymmetry and have to be guessed.

[i]In fact by performing T dualities along say 4-6-8 directions we can change all the branes into D3 branes.

obtained by dimensionally reducing would be the same. So we have 4 copies of dimensionally reduced $\mathcal{N}=4$ SYM for four stacks, with gauge groups $U(1), U(1), U(1)$ and $U(N)$ respectively.

We will use 4 dimensional $\mathcal{N}=1$ multiplets to arrange the fields. Dimensional reduction to $0+1$ dimension would be implicit. $\mathcal{N}=4$ SYM contains one $\mathcal{N}=1$ vector multiplet $\mathcal{V}$ and three $\mathcal{N}=1$ chiral multiplets Φ_1, Φ_2, Φ_3. The standard interpretation of the bosons is as follows: The 6 real scalars (coming from three $\mathcal{N}=1$ chiral multiplets) describe D brane collective coordinate for 6 transverse directions and the gauge field in the vector multiplet describe the gauge field living in D3 brane world volume. We denote by $X_i^{(k)}$ for $1 \leq k \leq 4$, $1 \leq i \leq 3$ the i-th spatial component of these four sets of gauge fields. $k=1,2,3$ will stand for the U(1) factors and $k=4$ will stand for the $U(N)$ factor.[j] Therefore $X_i^{(4)}$ denotes an $N \times N$ Hermitian matrix for each i whereas $X_i^{(k)}$ for $1 \leq k \leq 3, 1 \leq i \leq 3$ are real numbers. Since we have a quantum mechanics, these fields appear in same footing as six real scalars and this gives us some freedom in their interpretation. We interpret $X_i^{(k)}$ for $1 \leq k \leq 3$ as the position of the k-th D2-brane along the three non-compact directions and the diagonal elements of $X_i^{(4)}$ as the positions of the N D6-branes along the non-compact directions. The off-diagonal components of $X_i^{(4)}$ arise from open strings stretched between the D6-branes. The rest of the bosonic fields can be organized into chiral multiplet scalars. First of all, for each of the four stacks of D-branes we have three chiral multiplets in the adjoint representation. We shall denote them by $\Phi_i^{(k)}$ for $1 \leq k \leq 4$, $1 \leq i \leq 3$. $\Phi_i^{(k)}$ for $1 \leq k \leq 3$ describe 1×1 complex matrices whereas $\Phi_i^{(4)}$ will describe an $N \times N$ complex matrix, transforming in the adjoint representation of the respective gauge groups. Physically the $\Phi_i^{(k)}$'s for $1 \leq k \leq 3$ label the position of the k-th D2-brane along the directions of T^6 transverse to the D2-brane, and also the Wilson lines on the k-th D2-brane along directions tangential to the D2-brane. On the other hand the diagonal elements of $\Phi_i^{(4)}$ label the Wilson lines along the D6-branes. The off-diagonal components of $\Phi_i^{(4)}$ arise from open strings stretched between the D6-branes.

Mixed strings: Besides these fields, for every pair of D-brane stacks labelled by (k, ℓ) we have two chiral superfields $Z^{(k\ell)}$ and $Z^{(\ell k)}$ arising from open strings stretched between the two D-brane stacks, transforming respectively in the bi-fundamental and anti-bi-fundamental representation of the corresponding gauge groups. Therefore $Z^{(k\ell)}$ for $1 \leq k, \ell \leq 3$ are 1×1 matrices, $Z^{(k4)}$ for $1 \leq k \leq 3$ are $1 \times N$ matrices and $Z^{(4k)}$ for $1 \leq k \leq 3$ are $N \times 1$ matrices.

2.4.2 Supersymmetry

As mentioned earlier, supersymmetry works as an important guideline to write down possible terms in the Lagrangian. Hence we must first understand the supersymmetry preserved by the brane

[j] There are also non-dynamical fields $X_0^{(k)}$ which implement the gauge invariance constraints. Their effect will be discussed later.

system.

For type II theory, one has the supercharges

$$Q = \bar{\epsilon}_L Q_L + \bar{\epsilon}_R Q_R \,, \tag{2.4.11}$$

where $Q_L(Q_R)$ is left (right) moving supercharges. ϵ_L, ϵ_R are two ten dimensional Majorana-Weyl[k] fermions. The spinor indices are contracted and suppressed. Since a ten dimensional Majorana-Weyl fermion has 16 real off shell degrees of freedom, ϵ_L and ϵ_R together gives rise to 32 independent supercharges.

In the presence of a D brane, ϵ_L and ϵ_R are no more independent and they satisfy the following relation

$$\epsilon_L = \pm \Gamma^0 \Gamma^1 \ldots \Gamma^p \epsilon_R \,, \tag{2.4.12}$$

if the brane spans directions $x^1, x^2, \ldots, x^p$. The $\pm$ sign depends on whether we have a brane or an anti-brane. However this convention only has a relative meaning, i.e. for any given stack we can choose which sign corresponds to branes, then the opposite sign corresponds to anti-branes. We choose $+$ sign for D2 branes and $-$ sign for D6 brane.

Out of ϵ_L and ϵ_R, (2.4.12) leaves only one of them independent. It follows from (2.4.11) number of supercharges reduces to 16. This corresponds to the statement that a D brane is a half-BPS state.

But we have 4 stacks of D branes, so ϵ_L, ϵ_R need to satisfy 4 conditions, namely

$$\epsilon_L = \Gamma^0 \Gamma^4 \Gamma^5 \epsilon_R \,, \tag{2.4.13}$$

$$\epsilon_L = \Gamma^0 \Gamma^6 \Gamma^7 \epsilon_R \,, \tag{2.4.14}$$

$$\epsilon_L = \Gamma^0 \Gamma^8 \Gamma^9 \epsilon_R \,, \tag{2.4.15}$$

$$\epsilon_L = -\Gamma^0 \Gamma^4 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 \epsilon_R \,. \tag{2.4.16}$$

(2.4.13) and (2.4.14) together imply

$$\epsilon_R = \Gamma^5 \Gamma^4 \Gamma^6 \Gamma^7 \epsilon_R \,. \tag{2.4.17}$$

Now half of the eigenvalues of $\Gamma^5\Gamma^4\Gamma^6\Gamma^7$ are $+1$ and half are -1. So this equation tells that ϵ_R lies on the $+1$ eigenvalue subspace, which is $16/2 = 8$ dimensional. Thus the first two intersecting D2 branes preserve only 8 supercharges and hence a quarter-BPS state. This corresponds to $\mathcal{N} = 2$ supersymmetry in four dimensional language.

[k] In type IIA theory they have opposite chirality, whereas in type IIB theory they have same chirality.

Similarly, (2.4.13) and (2.4.15) imply

$$\epsilon_R = \Gamma^5\Gamma^4\Gamma^8\Gamma^9\epsilon_R\,. \tag{2.4.18}$$

This condition, reduces the number of components of ϵ_R by a further half and preserves $\mathcal{N} = 1$ supersymmetry in four dimensions.

(2.4.14) and (2.4.15) implies

$$\epsilon_R = \Gamma^7\Gamma^6\Gamma^8\Gamma^9\epsilon_R\,. \tag{2.4.19}$$

This alongwith (2.4.13) implies

$$\epsilon_L = -\Gamma^0\Gamma^4\Gamma^5\Gamma^6\Gamma^7\Gamma^8\Gamma^9\epsilon_R\,. \tag{2.4.20}$$

This is just same as (2.4.16). Hence (2.4.16) is not an independent condition.

So we see each pair of D branes preserves $\mathcal{N} = 2$ supersymmetry. But different $\mathcal{N} = 2$ subalgebras are preserved for different pairs and altogether they preserve only $\mathcal{N} = 1$ supersymmetry.

Note if we alter the orientation of odd number of branes, no supersymmetry will be preserved. But if we alter the orientation of even number of branes, we still have the same amount of supersymmetry preserved.

2.4.3 Goldstones and Goldstinos

Goldstones can be identified just by physical intuition and without the detailed knowledge about the Lagrangian. Furthermore, since the whole system preserves $\mathcal{N} = 1$ supersymetry, each Goldstone is accompanied by a Goldstino (i.e. fermionic Goldstone) and fit together in a $\mathcal{N} = 1$ supermultiplet. There are two advantages of identifying these:

1. By definition they do not appear in interaction in the low energy limit. So we also get to know which fields appear in interaction terms of the Lagrangian.

2. Since Goldstinos are associated with broken supersymmetries, counting Goldstinos we can tell the amount of supersymmetry preserved.

As a warm up, let us first take the case of a single D3 brane. Moving the D3 brane along 6 transverse directions does not cost any energy. These are Goldstones. Similarly exciting Wilson lines along the world volume of the D3 brane (assuming it wraps a compact direction) does not cost any energy. These are Goldstones as well. The 6 real scalars or equivalently the 3 complex scalars are superpartners of 3 Weyl fermions and the gauge field is superpartner of 1 Weyl fermion.

Thus we have 4 Weyl fermions as Goldstinos. These have 16 off shell degrees of freedom, which indicates 16 broken supersymmetries and hence $32-16=16$ conserved supercharges. This means the world volume theory should be a $\mathcal{N}=4$ theory. We know this indeed is the case.

Now we list the Goldstone multiplets in our system:

1. $\mathcal{V}^{(1)}+\mathcal{V}^{(2)}+\mathcal{V}^{(3)}+\frac{1}{N}\mathrm{Tr}\ \mathcal{V}^{(4)}$ **:** This corresponds to the center of mass movement of the whole system along the flat non-compact directions.

2. $\Phi_3^{(1)}+\Phi_3^{(2)}$ **:** This corresponds to moving first and second stacks together along their common transverse direction 8-9.

3. $\Phi_2^{(1)}+\Phi_2^{(3)}$ **:** This corresponds to moving first and third stacks together along their common transverse direction 6-7.

4. $\Phi_1^{(2)}+\Phi_1^{(3)}$ **:** This corresponds to moving second and third stacks together along their common transverse direction 4-5.

5. $\Phi_1^{(1)}+\frac{1}{N}\mathrm{Tr}\ \Phi_1^{(4)}$ **:** This corresponds to turning on "center of mass Wilson lines" along the common world volume directions 4-5 of first and fourth stacks. In the T dual (along 4-5 directions) torus, this corresponds to an actual center of mass movement.

6. $\Phi_2^{(2)}+\frac{1}{N}\mathrm{Tr}\ \Phi_2^{(4)}$ **:** This corresponds to turning on "center of mass Wilson lines" along the common world volume directions 6-7 of second and fourth stacks. In the T dual (along 6-7 directions) torus, this corresponds to an actual center of mass movement.

7. $\Phi_3^{(3)}+\frac{1}{N}\mathrm{Tr}\ \Phi_3^{(4)}$ **:** This corresponds to turning on "center of mass Wilson lines" along the common world volume directions 8-9 of first and fourth stacks. In a T dual (along 8-9 directions) torus, this corresponds to an actual center of mass movement.

Correspondingly there are 7 Goldstino Weyl fermions, therefore 28 broken supersymmetries and therefore 4 preserved supersymmetries. Thus our Lagrangian should be a $\mathcal{N}=1$ Lagrangian, in four dimensional language.

Another lesson is that the remaining linear combinations of $\mathcal{V}$-s and Φ-s should appear in the interaction with the mixed strings. E.g. consider $\Phi_2^{(1)}-\Phi_2^{(3)}$. This is the separation of the first and second brane along 8-9 directions. The strings stretched between these two branes should certainly be sensitive to this separation.

2.4.4 The Lagrangian

The Lagrangian can be written as a sum of different parts.

$$L = \sum_{i=1}^{4} L_i + \sum_{i<j;\ i,j=1}^{4} L_{ij} + L_W \,. \tag{2.4.21}$$

L_i denotes the Lagrangian of $\mathcal{N} = 4$ super Yang Mills theory living on the i-th stack. L_{ij} denotes the standard $\mathcal{N} = 2$ interaction between i-th and j-th stack. This involves $Z^{(ij)}, Z^{(ji)}$ and certain on brane fields of i-th and j-th brane. Different L_{ij}-s preserve different $\mathcal{N} = 2$ subalgebra of the full $\mathcal{N} = 4$ algebra. As a result, together they preserve only $\mathcal{N} = 1$ supersymmetry. This allows one to add new $\mathcal{N} = 1$ superpotentials. $\{L_i, L_{ij}\}$ are uniquely fixed by supersymmetry but not L_W. We will guess some crucial pieces of L_W that would allow us to go to a generic point in moduli space and as we will see, this would suffice.

First let us consider the pieces $\{L_{ij}\}$. We have previously seen that two intersecting branes preserve $\mathcal{N} = 2$ supersymmetry. So each L_{ij} is a $\mathcal{N} = 2$ Lagrangian. Let us determine the field content first. It should definitely involve $Z^{(ij)}, Z^{(ji)}$ and they fit in a $\mathcal{N} = 2$ hypermultiplet. To see which on brane fields should appear let us consider the specific case of L_{12}. The strings stretched between these stacks should only feel their separation, which can be along common transverse directions. Common flat transverse directions are 1-2-3 and common compact transverse directions are 8-9. Looking at which supermultiplets they are in, we see the combinations $\mathcal{V}^{(1)} - \mathcal{V}^{(2)}$ and $\Phi_3^{(1)} - \Phi_3^{(2)}$ should appear in interaction with $Z^{(12)}, Z^{(21)}$. $\mathcal{V}^{(1)} - \mathcal{V}^{(2)}$ and $\Phi_3^{(1)} - \Phi_3^{(2)}$ fit in a $\mathcal{N} = 2$ vector multiplet. So we have to write down a $\mathcal{N} = 2$ Lagrangian involving one $\mathcal{N} = 2$ vector and one $\mathcal{N} = 2$ hypermultiplet. Given a $\mathcal{N} = 2$ vector multiplet (V, Φ) and one $\mathcal{N} = 2$ hypermultiplet $(Q, \tilde{Q})$, one can write down a standard $\mathcal{N} = 2$ Lagrangian [49] in terms of $\mathcal{N} = 1$ superfields

$$\int d^4\theta \left(Q^\dagger e^{-2V} Q + \tilde{Q} e^{2V} \tilde{Q}^\dagger \right) + \sqrt{2} \int d^2\theta \tilde{Q} \Phi Q \,, \tag{2.4.22}$$

the first term being the kinetic term and the second term being the superpotential. To get L_{12} we simply make the following substitutions in 2.4.22:

$$V \to \left(\mathcal{V}^{(1)} - \mathcal{V}^{(2)}\right),\ \Phi \to \left(\Phi_3^{(1)} - \Phi_3^{(2)}\right),\ Q \to Z^{(12)},\ \tilde{Q} \to Z^{(21)} \,.$$

Similar procedure can be applied for all other pairs of branes. Together they give the following superpotential

$$\begin{aligned}\mathcal{W}_1 &= \sqrt{2}\Bigg[\sum_{k,\ell,m=1}^{3}\varepsilon^{k\ell m}\mathrm{Tr}\left(\Phi_m^{(k)}\,Z^{(k\ell)}Z^{(\ell k)}\right)+\sum_{k=1}^{3}\mathrm{Tr}\left(Z^{(4k)}\Phi_k^{(k)}Z^{(k4)}\right)\\ &\qquad -\sum_{k=1}^{3}\mathrm{Tr}\left(\Phi_k^{(4)}Z^{(4k)}Z^{(k4)}\right)\Bigg]\\ &= \sqrt{2}\Big[(\Phi_3^{(1)}-\Phi_3^{(2)})Z^{(21)}Z^{(12)}+(\Phi_1^{(2)}-\Phi_1^{(3)})Z^{(32)}Z^{(23)}+(\Phi_2^{(3)}-\Phi_2^{(1)})Z^{(31)}Z^{(13)}\\ &\quad +\mathrm{Tr}\Big((\Phi_1^{(1)}I_N-\Phi_1^{(4)})Z^{(41)}Z^{(14)}\Big)+\mathrm{Tr}\Big((\Phi_2^{(2)}I_N-\Phi_2^{(4)})Z^{(42)}Z^{(24)}\Big)\\ &\quad +\mathrm{Tr}\Big((\Phi_3^{(3)}I_N-\Phi_3^{(4)})Z^{(43)}Z^{(34)}\Big)\Big]\,, \end{aligned} \qquad (2.4.23)$$

Each L_i has the standard superpotential of $\mathcal{N}=4$ SYM. But this vanishes unless the stack contains more than one branes. Hence in our case, only the fourth stack contributes to this piece.

$$\mathcal{W}_4=-\sqrt{2}\,\mathrm{Tr}\Big(\Phi_1^{(4)}\left[\Phi_2^{(4)},\Phi_3^{(4)}\right]\Big)=-\sqrt{2}\,\mathrm{Tr}\Big(\Phi_1^{(4)}\Phi_2^{(4)}\Phi_3^{(4)}-\Phi_1^{(4)}\Phi_3^{(4)}\Phi_2^{(4)}\Big)\,, \qquad (2.4.24)$$

It is being called $\mathcal{W}_4$ to match conventions used in our paper.

Now we turn to L_W. This has two pieces, which we call $\mathcal{W}_2$ and $\mathcal{W}_3$.

Physically a string stretching from i-th brane to j-th brane can join a string stretching from j-th brane to k-th brane and become a string stretching from i-th brane to k-th brane. Clearly this is a tree level process. But till now we do not have any term in our Lagrangian which allows this to happen at tree level. To allow for this, we add a new piece to the superpotential

$$\begin{aligned}\mathcal{W}_2 &= \sqrt{2}\left[\sum_{\substack{k,\ell,m=1\\ k<\ell,m;\,\ell\neq m}}^{4}(-1)^{\delta_{k1}\delta_{\ell 3}\delta_{m4}}\mathrm{Tr}\Big(Z^{(k\ell)}Z^{(\ell m)}Z^{(mk)}\Big)\right]\\ &= \sqrt{2}\Big[Z^{(31)}Z^{(12)}Z^{(23)}+Z^{(13)}Z^{(32)}Z^{(21)}+\mathrm{Tr}\Big(Z^{(12)}Z^{(24)}Z^{(41)}\Big)+\mathrm{Tr}\Big(Z^{(42)}Z^{(21)}Z^{(14)}\Big)\\ &\quad -\mathrm{Tr}\Big(Z^{(13)}Z^{(34)}Z^{(41)}\Big)+\mathrm{Tr}\Big(Z^{(31)}Z^{(14)}Z^{(43)}\Big)+\mathrm{Tr}\Big(Z^{(34)}Z^{(42)}Z^{(23)}\Big)\\ &\quad +\mathrm{Tr}\Big(Z^{(43)}Z^{(32)}Z^{(24)}\Big)\Big]\,, \end{aligned} \qquad (2.4.25)$$

We have fixed the over all normalization of this term from explicit string theory computation described in appendix B. Second, we have a strange sign $(-1)^{\delta_{k1}\delta_{\ell 3}\delta_{m4}}$ in the expression for $\mathcal{W}_2$ which is responsible for the minus sign in front of the $\mathrm{Tr}\Big(Z^{(13)}Z^{(34)}Z^{(41)}\Big)$ term. This sign must be there for symmetry reasons, as has been explained in appendix B[l]. In appendix B we also discuss generalization of (2.4.23)-(2.4.24) to the case where we have an arbitrary number of D-branes in each of the four stacks.

Lastly, to go to a generic point in moduli space we turn on constant background metric and NS-NS two-form field. This does not affect supersymmetry[m] and gives rise to the following superpotential, as explained in A.

$$
\begin{aligned}
\mathcal{W}_3 &= \sqrt{2}\left[\sum_{k,\ell,m=1}^{3} c^{(k\ell)}\,\varepsilon^{k\ell m}\,\mathrm{Tr}\left(\Phi_m^{(k)}\right) + \sum_{k=1}^{3} c^{(k4)}\left[N_4\,\mathrm{Tr}\left(\Phi_k^{(k)}\right) - \mathrm{Tr}\left(\Phi_k^{(4)}\right)\right]\right] \\
&= \sqrt{2}\Big[c^{(12)}(\Phi_3^{(1)} - \Phi_3^{(2)}) + c^{(23)}(\Phi_1^{(2)} - \Phi_1^{(3)}) + c^{(13)}(\Phi_2^{(3)} - \Phi_2^{(1)}) + c^{(14)}\mathrm{Tr}\Big(\Phi_1^{(1)}\,I_N - \Phi_1^{(4)}\Big) \\
&\quad + c^{(24)}\mathrm{Tr}\Big(\Phi_2^{(2)}\,I_N - \Phi_2^{(4)}\Big) + c^{(34)}\mathrm{Tr}\Big(\Phi_3^{(3)}\,I_N - \Phi_3^{(4)}\Big)\Big]\,,
\end{aligned}
\tag{2.4.26}
$$

where $c^{(k\ell)} = c^{(\ell k)}$ for $1 \le k < \ell \le 4$ are parameters whose values are determined in terms of the background metric and 2-form fields in A, and I_p denotes the $p \times p$ identity matrix.

Besides $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4$ the superpotential may contain other terms, e.g. terms quartic in $Z^{(k\ell)}$'s. However if we consider the limit when all the $c^{(k\ell)}$'s and $c^{(k)}$'s are small, say of order λ for some small parameter λ, then the solutions to the F- and D-term equations (to be discussed soon) occur at $Z^{(k\ell)}, \Phi_i^{(k)} \sim \sqrt{\lambda}$. In this limit the effect of the other terms in the superpotential can be ignored. This limit also allows us to ignore the fact that the $\Phi_i^{(k)}$ fields have periodic identification.

The F-term potential is given by

$$
V_F = \sum_\alpha \left|\frac{\partial \mathcal{W}}{\partial \varphi_\alpha}\right|^2\,, \tag{2.4.27}
$$

where

$$
\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4\,, \tag{2.4.28}
$$

and $\{\varphi_\alpha\}$ denotes the set of all the chiral superfields.

[l]The result for the number of states for $N = 1$ was found to be insensitive to this sign. The overall normalization of this term is also inconsequential for $N = 1$. This is because for $N = 1$ the over all normalization of $\mathcal{W}_2$ could be made into an normalization of the whole superpotential (which do not affect the potential minimization) by scaling Φ-s appropriately. However for $N > 1$, similar rescaling of Φ-s would change the relative normalization of the $\Phi - \Phi - \Phi$ terms.

[m]This is because the supersymmetry transformation rules in supergravity involve the field strength $H = dB$, which remains zero when we turn on constant B field.

Besides the F-term potential, there is a D-term potential given by

$$V_D = \frac{1}{2}\sum_{k=1}^{4}\mathrm{Tr}\Big[\Big(\sum_{\substack{\ell=1\\ \ell\neq k}}^{4} Z^{(k\ell)}Z^{(k\ell)\dagger} - \sum_{\substack{\ell=1\\ \ell\neq k}}^{4} Z^{(\ell k)\dagger}Z^{(\ell k)} + \sum_{i=1}^{3}[\Phi_i^{(k)},\Phi_i^{(k)\dagger}] - c^{(k)} I_{N_k}\Big)^2\Big]\,, \qquad (2.4.29)$$

where $N_1 = N_2 = N_3 = 1$ and $N_4 = N$. The FI parameters $c^{(k)}$ are also determined from the background values of the 2-form field and satisfy

$$\sum_{k=1}^{4} c^{(k)} N_k = 0\,. \qquad (2.4.30)$$

Finally the dimensional reduction of the coupling of the gauge fields (whose spatial components are denoted by $X_i^{(k)}$) to chiral multiplets leads to the potential

$$\begin{aligned} V_{gauge} &= \sum_{k=1}^{4}\sum_{\substack{\ell=1\\ \ell\neq k}}^{4}\sum_{i=1}^{3}\mathrm{Tr}\Big[\Big(X_i^{(k)}Z^{(k\ell)} - Z^{(k\ell)}X_i^{(\ell)}\Big)^\dagger\Big(X_i^{(k)}Z^{(k\ell)} - Z^{(k\ell)}X_i^{(\ell)}\Big)\Big] \\ &\quad + \sum_{k=1}^{4}\sum_{i,j=1}^{3}\mathrm{Tr}\Big([X_i^{(k)},\Phi_j^{(k)}]^\dagger[X_i^{(k)},\Phi_j^{(k)}]\Big) + \frac{1}{4}\sum_{k=1}^{4}\sum_{i,j=1}^{3}\mathrm{Tr}\Big([X_i^{(k)},X_j^{(k)}]^\dagger[X_i^{(k)},X_j^{(k)}]\Big)\,. \end{aligned} \qquad (2.4.31)$$

Therefore the total potential is given by

$$V = V_F + V_D + V_{gauge}\,. \qquad (2.4.32)$$

The potential given above has a shift symmetry

$$\begin{aligned} &\Phi_m^{(k)} \to \Phi_m^{(k)} + \xi_m, \quad \text{for} \quad 1\le k\le 3, \quad k\neq m, \quad 1\le m\le 3, \\ &\Phi_k^{(k)} \to \Phi_k^{(k)} + \zeta_k, \quad \Phi_k^{(4)} \to \Phi_k^{(4)} + \zeta_k I_N, \quad \text{for} \quad 1\le k\le 3\,, \\ &X_i^{(k)} \to X_i^{(k)} + a_i, \quad X_i^{(4)} \to X_i^{(4)} + a_i\, I_N, \quad \text{for} \quad 1\le i\le 3, \quad 1\le k\le 3\,, \end{aligned} \qquad (2.4.33)$$

where $\{\xi_m\}$ and $\{\zeta_k\}$ are arbitrary complex parameters and $\{a_i\}$ are arbitrary real parameters. These shift symmetries generate six complex translations along the compact directions and their duals and three real translations along the non-compact directions.

2.5 The Strategy

Our strategy for determining the spectrum of BPS states will be to regard the low energy dynamics of the system as the motion of a superparticle moving on the classical vacuum manifold defined by the space of $V = 0$ configurations, and quantum BPS states are given by the harmonic forms on this moduli space [44]. Now the shift symmetries (2.4.33) generate flat directions of the potential V. Quantization of the bosonic zero modes associated with these flat directions leads to a unique ground state if we restrict to the sector carrying zero momentum and winding along the internal directions and zero momentum along the non-compact directions. There are also associated fermionic zero modes describing the goldstino modes corresponding to $32-4 = 28$ broken supersymmetries. Quantization of these fermion zero modes produces the supermultiplet describing 1/8 BPS states of $\mathcal{N} = 8$ supersymmetric string theory but has no other effect on the rest of the system. The BPS spectrum is given by a tensor product of this basic supermultiplet with some (possibly reducible) representation of the rotation group $SU(2)$. Our goal will be to determine which representation of $SU(2)$ is tensored with the basic supermultiplet.

The information about the SU(2) representation with which the supermultiplet is tensored is contained in the character $P(y) \equiv Tr(y^{2J_3})$ of the representation. $P(y)$ is computed as follows. We shall show at the end of appendix B that the vanishing of V_{gauge} given in (2.4.31) requires all the $X_i^{(k)}$'s to vanish (up to the shift symmetry described in the last line of (2.4.33)). If the gauge inequivalent solutions to the $V_F = V_D = 0$ condition generate a manifold $\mathcal{M}$ of complex dimension d after factoring out the flat directions of the potential associated with the symmetries given in (2.4.33), and setting the $X_i^{(k)}$'s to zero, then the BPS states of the system are in one to one correspondence with the harmonic forms on $\mathcal{M}$, and the rotational $SU(2)$ is identified with the Lefschetz SU(2) acting on these forms (see e.g. [32, 51]). Therefore if b_p denotes the p-th Betti number of $\mathcal{M}$ and d denotes the complex dimension of $\mathcal{M}$, then we have[n]

$$P(y) = \sum_p b_p y^{p-d} \, . \tag{2.5.34}$$

If $\mathcal{M}$ contains several components then we have to add up the contribution from various components to get the total $P(y)$. The BPS index (which is the 14-th helicity trace $-B_{14}$ [52, 53] from the

[n]Intuitively this identification can be understood as follows. Since the vacuum manifold has $X_i^{(k)} = 0$ for all k, i, the moduli space is spanned by the scalars. The fermionic partners of the scalars take values in the tangent space of $\mathcal{M}$ – in fact for each tangent vector there are two massless fermions which we can denote by ψ^a and $\psi^{a\dagger}$ where a labels independent tangent vectors. We can choose ψ^a and $\psi^{a\dagger}$ such that $\psi^{a\dagger}$ has $J_3 = 1/2$ and ψ^a has $J_3 = -1/2$. Now we can begin with the states annihilated by all the ψ^a's, identify them as the zero forms on $\mathcal{M}$, and build the total space of states by applying $\psi^{a\dagger}$'s on this state. This space is isomorphic to the space of forms on $\mathcal{M}$, and the BPS condition translates to these forms being harmonic. In this notation we see that the p-forms carry J_3 eigenvalue $(p-d)/2$, where the shift $-d/2$ is the J_3 eigenvalue of the zero forms, and is necessary to ensure that the states form a representation of $SU(2)$. This leads to (2.5.34).

space-time viewpoint) is given by $Tr(-1)^{2J_3}$, with the trace running over the states with which the basic BPS supermultiplet is tensored to get the full spectrum of BPS states. Therefore it is given by $P(-1)$, which, according to (2.5.34), is $(-1)^d$ times the Euler character of $\mathcal{M}$.

The conjecture that all BPS states carry zero angular momentum now translates to the requirement that $P(y)$ is y-independent, i.e. the subspace $\mathcal{M}$ consists of isolated points[o]. In this case the BPS index $P(-1)$ is equal to the degeneracy $P(1)$ and just counts the number of gauge inequivalent solutions to the $V = 0$ condition.

2.3.10 implies that the expected number of such solutions for $N = 1, 2, 3$ respectively are $12, 56$ and 208.

[o] A similar result for the D0-D4 system was found in [45].

CHAPTER 3

COMPUTING THE INDEX: ABELIAN CASE

For $N_1 = N_2 = N_3 = N_4 = 1$, all the fields are 1×1 matrices, i.e. just numbers. In particular the gauge group is $U(1) \times U(1) \times U(1) \times U(1)$. Thus we refer to this case as Abelian case. Now Φ-s and Z-s are complex numbers and X-s are real numbers. As a result various commutator terms in superpotential and in D term potential vanish, making the problem considerably simpler.

Various parts of the superpotential now read as follows:

$$\begin{aligned} \mathcal{W}_1 &= \sqrt{2}\Big[(\Phi_3^{(1)} - \Phi_3^{(2)})Z^{(21)}Z^{(12)} + (\Phi_1^{(2)} - \Phi_1^{(3)})Z^{(32)}Z^{(23)} + (\Phi_2^{(3)} - \Phi_2^{(1)})Z^{(31)}Z^{(13)} + \\ &\quad (\Phi_1^{(1)} - \Phi_1^{(4)})Z^{(41)}Z^{(14)} + (\Phi_2^{(2)} - \Phi_2^{(4)})Z^{(42)}Z^{(24)} + (\Phi_3^{(3)} - \Phi_3^{(4)})Z^{(43)}Z^{(34)}\Big], \quad (3.0.1) \end{aligned}$$

$$\begin{aligned} \mathcal{W}_2 &= \sqrt{2}\Big[Z^{(31)}Z^{(12)}Z^{(23)} + Z^{(13)}Z^{(32)}Z^{(21)} + Z^{(12)}Z^{(24)}Z^{(41)} + Z^{(42)}Z^{(21)}Z^{(14)} \\ &\quad -Z^{(13)}Z^{(34)}Z^{(41)} + Z^{(31)}Z^{(14)}Z^{(43)} + Z^{(34)}Z^{(42)}Z^{(23)} + Z^{(43)}Z^{(32)}Z^{(24)}\Big], \quad (3.0.2) \end{aligned}$$

$$\begin{aligned} \mathcal{W}_3 &= \sqrt{2}\Big[c^{(12)}(\Phi_3^{(1)} - \Phi_3^{(2)}) + c^{(23)}(\Phi_1^{(2)} - \Phi_1^{(3)}) + c^{(13)}(\Phi_2^{(3)} - \Phi_2^{(1)}) + c^{(14)}(\Phi_1^{(1)} - \Phi_1^{(4)}) \\ &\quad + c^{(24)}(\Phi_2^{(2)} - \Phi_2^{(4)}) + c^{(34)}(\Phi_3^{(3)} - \Phi_3^{(4)})\Big], \quad (3.0.3) \end{aligned}$$

$$\mathcal{W}_4 = 0\,. \quad (3.0.4)$$

Before analyzing F term potential, let us consider the potential coming from the coupling of Z-s with gauge fields, which we denoted as V_{gauge} in (2.4.31). For Ablian case simplifies to

$$V_{gauge} = \sum_{\substack{k,\ell=1 \\ \ell\neq k}}^{4} \left|Z^{(k\ell)}\right|^2 \left|\vec{X}^{(k)} - \vec{X}^{(l)}\right|^2 . \quad (3.0.5)$$

Thus if $Z^{(kl)}$ is non vanishing then one must have $\vec{X}^{(k)} = \vec{X}^{(l)}$, i.e. k^{th} and l^{th} stacks must coincide along non-compact directions. We will soon see that certain F term equations imply that $Z^{(kl)} \neq 0 \,\forall (kl)$. Using this result beforehand we see that minimising V_{gauge} requires all four stacks to coincide along the non compact directions.

Now we turn to F term equations. First we note that the brane separations along compact directions are completely fixed in terms of Z-s. For example:

$$Z^{(12)} \text{ equation} \Rightarrow \quad \Phi_3^{(2)} - \Phi_3^{(1)} = \frac{Z^{(23)}Z^{(31)} + Z^{(24)}Z^{(41)}}{Z^{(21)}} . \quad (3.0.6)$$

Similarly,

$$Z^{(13)} \text{ equation} \Rightarrow \quad \Phi_2^{(1)} - \Phi_2^{(3)} = \frac{Z^{(32)}Z^{(21)} - Z^{(34)}Z^{(41)}}{Z^{(31)}} , \quad (3.0.7)$$

$$Z^{(14)} \text{ equation} \Rightarrow \quad \Phi_1^{(4)} - \Phi_1^{(1)} = \frac{Z^{(42)}Z^{(21)} + Z^{(43)}Z^{(31)}}{Z^{(21)}} , \quad (3.0.8)$$

$$Z^{(23)} \text{ equation} \Rightarrow \quad \Phi_1^{(3)} - \Phi_1^{(2)} = \frac{Z^{(31)}Z^{(12)} + Z^{(34)}Z^{(42)}}{Z^{(32)}} , \quad (3.0.9)$$

$$Z^{(24)} \text{ equation} \Rightarrow \quad \Phi_2^{(4)} - \Phi_2^{(2)} = \frac{Z^{(41)}Z^{(12)} + Z^{(43)}Z^{(32)}}{Z^{(42)}} , \quad (3.0.10)$$

$$Z^{(34)} \text{ equation} \Rightarrow \quad \Phi_3^{(4)} - \Phi_3^{(3)} = \frac{-Z^{(41)}Z^{(13)} + Z^{(42)}Z^{(23)}}{Z^{(43)}} . \quad (3.0.11)$$

However for such a fixing fails if the Z in the denominator vanishes. Luckily, this is never the case, as guaranteed by the corresponding Φ equation itself. E.g.

$$\Phi_3^{(2)} - \Phi_3^{(1)} \text{ equation} \Rightarrow \quad Z^{(12)}Z^{(21)} = -c^{(12)} . \quad (3.0.12)$$

Thus for $c^{(12)} \neq 0$, neither $Z^{(12)}$ nor $Z^{(21)}$ can vanish. As a result $\Phi_3^{(2)} - \Phi_3^{(1)}$ is uniquely fixed in terms of Z-s everywhere on the vacuum manifold[a]. Similarly all brane separations along compact directions gets fixed in terms of Z-s uniquely.

This allows us to simply forget about Φ-s and concentrate on Z-s only. We have used up half of the Φ equations and half of the Z equations. Remaining Z equations also has the effect of

[a]It is interesting to note that our problem actually simplifies if we go to a generic point in moduli space, i.e. non-zero $c^{(ij)}$-s, which in turn is related to non trivial metric and B field backgrounds (A).

fixing Φ-s only. As a result each brane separation gets fixed by two equations and after fixing the brane separation we are left with a consistency condition on Z-s. E.g. both $Z^{(12)}$ and $Z^{(21)}$ fixes $\Phi_3^{(2)} - \Phi_3^{(1)}$.

$$Z^{(21)} \text{ equation} \Rightarrow \quad \Phi_3^{(2)} - \Phi_3^{(1)} = \frac{Z^{(13)}Z^{(32)} + Z^{(14)}Z^{(42)}}{Z^{(12)}} . \tag{3.0.13}$$

Thus from (3.0.6) and (3.0.13) we get the following consistency condition:

$$Z^{(12)}Z^{(23)}Z^{(31)} + Z^{(12)}Z^{(24)}Z^{(41)} = Z^{(13)}Z^{(32)}Z^{(21)} + Z^{(14)}Z^{(42)}Z^{(21)} . \tag{3.0.14}$$

For each remaining pair of branes, we have one such consistency condition:

$$\begin{aligned}
Z^{(13)}Z^{(32)}Z^{(21)} - Z^{(13)}Z^{(34)}Z^{(41)} &= Z^{(31)}Z^{(12)}Z^{(23)} + Z^{(31)}Z^{(14)}Z^{(43)} , \\
Z^{(14)}Z^{(42)}Z^{(21)} + Z^{(14)}Z^{(43)}Z^{(31)} &= Z^{(41)}Z^{(12)}Z^{(24)} - Z^{(41)}Z^{(13)}Z^{(34)} , \\
Z^{(23)}Z^{(31)}Z^{(12)} + Z^{(23)}Z^{(34)}Z^{(42)} &= Z^{(32)}Z^{(21)}Z^{(13)} + Z^{(32)}Z^{(24)}Z^{(43)} , \\
Z^{(24)}Z^{(41)}Z^{(12)} + Z^{(24)}Z^{(43)}Z^{(32)} &= Z^{(42)}Z^{(21)}Z^{(14)} + Z^{(42)}Z^{(23)}Z^{(34)} , \\
Z^{(34)}Z^{(41)}Z^{(13)} + Z^{(34)}Z^{(42)}Z^{(23)} &= -Z^{(43)}Z^{(31)}Z^{(14)} + Z^{(43)}Z^{(32)}Z^{(24)} .
\end{aligned}$$

It can be checked that out of these equations only 3 are independent, which we take to be

$$Z^{(23)}Z^{(31)}Z^{(12)} + Z^{(23)}Z^{(34)}Z^{(42)} = Z^{(32)}Z^{(21)}Z^{(13)} + Z^{(32)}Z^{(24)}Z^{(43)} , \tag{3.0.15}$$

$$Z^{(24)}Z^{(41)}Z^{(12)} + Z^{(24)}Z^{(43)}Z^{(32)} = Z^{(42)}Z^{(21)}Z^{(14)} + Z^{(42)}Z^{(23)}Z^{(34)} , \tag{3.0.16}$$

$$Z^{(34)}Z^{(41)}Z^{(13)} + Z^{(34)}Z^{(42)}Z^{(23)} = -Z^{(43)}Z^{(31)}Z^{(14)} + Z^{(43)}Z^{(32)}Z^{(24)} . \tag{3.0.17}$$

We also have the six Φ equations, which read

$$Z^{(ij)}Z^{(ji)} = -c^{(ij)} . \tag{3.0.18}$$

So we have 9 independent F term equations, purely in terms of Z-s.

Let us turn to the D term equations, which in the Abelian case are given purely in terms of Z-s. They read

$$|Z^{(12)}|^2 + |Z^{(13)}|^2 + |Z^{(14)}|^2 - |Z^{(21)}|^2 - |Z^{(31)}|^2 - |Z^{(41)}|^2 = c^{(1)} , \tag{3.0.19}$$

$$|Z^{(21)}|^2 + |Z^{(23)}|^2 + |Z^{(24)}|^2 - |Z^{(12)}|^2 - |Z^{(32)}|^2 - |Z^{(42)}|^2 = c^{(2)} , \tag{3.0.20}$$

$$|Z^{(31)}|^2 + |Z^{(32)}|^2 + |Z^{(34)}|^2 - |Z^{(13)}|^2 - |Z^{(23)}|^2 - |Z^{(43)}|^2 = c^{(3)} , \tag{3.0.21}$$

$$|Z^{(41)}|^2 + |Z^{(42)}|^2 + |Z^{(43)}|^2 - |Z^{(14)}|^2 - |Z^{(24)}|^2 - |Z^{(34)}|^2 = c^{(4)} , \tag{3.0.22}$$

with $c^{(1)} + c^{(2)} + c^{(3)} + c^{(4)} = 0$. Further one has to identify Z configurations related by gauge transformation, i.e. impose $Z^{(ij)} \equiv e^{i(\theta_i - \theta_j)} Z^{(ij)}$. With this identification, the D term equations define a toric variety. This is much like projective space. E.g. the equation $|x_1|^2 + \cdots + |x_n|^2 = 1$ along with the identification $(x_1, \ldots, x_n) \equiv (e^{i\theta} x_1, \ldots, e^{i\theta} x_n)$ defines $\mathbb{CP}^{n-1}$. So the Z-s live on a toric variety and various F term equations define hypersurfaces on that toric variety. Thus the vacuum manifold is intersection of hypersurfaces in a toric variety and our task is to compute the Euler characteristic of this manifold. This sounds a not so easy task. But this simplifies a lot.

It is known that D term equations along with gauge identification has the same effect as complexified gauge transformation. Thus the toric variety can also be defined by the following identification:

$$Z^{(ij)} \equiv \lambda_i \lambda_j^{-1} Z^{(ij)}\,, \tag{3.0.23}$$

for $\lambda_i \in \mathbb{C}^*$. This is just like $\mathbb{CP}^{-1}$s, which can also be defined by the identification $(x_1, \ldots, x_n) \equiv (\lambda x_1, \ldots, \lambda x_n)$ for $\lambda \in \mathbb{C}^*$. Now we need to cover the toric variety with various patches and in each patch the coordinates are built out of gauge invariant combinations of Z-s.

The apparent difficulty of the problem is that toric variety has to be covered by various patches and apriori it seems that one would need to consider all the patches. First let us ask for $\mathbb{CP}^{n-1}$ what are the points that are not covered by a given patch? These are essentially the points at which some of the homogeneous coordinates go to zero or infinity. Points where all the homogeneous coordinates are finite, can be accessed by any patch. For us Z-s play the role of homogeneous coordinates. But by (3.0.18), none of the Z-s vanish on the vacuum manifold. This means we can choose just any patch and try to count the solutions of F term equations. Since any patch is just $\mathbb{C}^9$ and we have 9 F term equations, we are guaranteed to have finitely many solutions for generic values of c^{ij}-s.

We choose the following homogeneous coordinates:

$$\begin{aligned}
u_1 &= Z^{(12)}Z^{(21)}\,,\\
u_2 &= Z^{(23)}Z^{(32)}\,,\\
u_3 &= Z^{(31)}Z^{(13)}\,,\\
u_4 &= Z^{(14)}Z^{(41)}\,,\\
u_5 &= Z^{(24)}Z^{(42)}\,,\\
u_6 &= Z^{(34)}Z^{(43)}\,,\\
u_7 &= Z^{(12)}Z^{(24)}Z^{(41)}\,,\\
u_8 &= Z^{(13)}Z^{(34)}Z^{(41)}\,,\\
u_9 &= Z^{(23)}Z^{(34)}Z^{(42)}\,.
\end{aligned} \tag{3.0.24}$$

In terms of these affine coordinates, the equations $Z^{(ij)}Z^{(ji)} = -c^{(ij)}$ simply read

$$u_1 = -c^{(12)}\ ,\ u_2 = -c^{(23)}\ ,\ u_3 = -c^{(13)}\ ,\ u_4 = -c^{(14)}\ ,\ u_5 = -c^{(24)}\ ,\ u_6 = -c^{(34)}\,. \tag{3.0.25}$$

So our problem further simplifies to counting solutions of 3 polynomial equations in 3 variables. Using (3.0.25) we can reexpress (3.0.15),(3.0.16) and (3.0.17) respectively as

$$\begin{aligned}
c^{(13)}u_7u_9^2 + c^{(24)}u_8u_9^2 + c^{(12)}c^{(23)}(c^{(24)})^2u_7 + c^{(23)}(c^{(24)})^2c^{(34)}u_8 &= 0\,,\\
u_7^2u_9 - u_7u_9^2 - c^{(23)}c^{(24)}c^{(34)}u_7 + c^{(12)}c^{(14)}c^{(24)}u_9 &= 0\,,\\
u_8^2u_9 + u_8u_9^2 - c^{(13)}c^{(14)}c^{(34)}u_9 + c^{(23)}c^{(24)}c^{(34)}u_8 &= 0\,.
\end{aligned} \tag{3.0.26}$$

This can simply be put in mathematica [61] and exact solutions[b] can be obtained as functions of $c^{(ij)}$-s. One gets 12 solutions[c].A numerical analysis of the Hilbert series using SINGULAR [59] and Macaulay2 [60], treating the F-term equations as ideals of the ring, also shows that the solution space is a collection of 12 points. Hence our results match the prediction of [47].

This tells that our results are consistent with expectations from string dualities. But we have achieved more. Our vacuum manifold is a zero dimensional space, containing 12 points. Each of these points correspond to a single vacuum state. On a one dimensional vector space, $SU(2)$ can have only one possible action, trivial action[d]. Thus under the space time rotational $SU(2)$, all these

[b]We do not present the exact solutions here, since they are quite clumsy and not very illuminating.

[c]Actually one gets 13 solutions. But one of these solutions is $(u, v, w) = (0, 0, 0)$ and hence should not be counted, since none of the Z-s can vanish implying none of the u-s can vanish either.

[d]One may wonder if $SU(2)$ can have non trivial action on the 12 dimensional vector space spanned by all the vacua? The answer is no. This is because space time rotational $SU(2)$ acts as Lefschetz $SU(2)$ on the space of harmonic forms of a manifold, with raising (lowering) operator acting by increasing (decreasing) the rank of the form by one. Clearly

vacua act as singlets, i.e. carry zero angular momentum. Thus we have tested our conjecture in positive for the Abelian case.

for such a $SU(2)$ to have non trivial action, one needs forms of various ranks. But since our vacuum manifold is a collection of 12 points, we only have 12 zero forms. Hence $SU(2)$ acts trivially on the 12 dimensional space spanned by these forms.

CHAPTER 4

COMPUTING THE INDEX: HIGHER CHARGES

In this chapter we consider charges $(1, 1, 1, N)$ with $N = 2$ and $N = 3$.

4.1 Supersymmetric ground states for $N = 2$

4.1.1 F-term equations

Vanishing of the F-term contribution to the potential requires that $\partial\mathcal{W}/\partial\varphi_\alpha$ vanishes for each α. The $\partial\mathcal{W}/\partial\Phi_m^{(k)} = 0$ equations give

$$\begin{aligned} Z^{(k\ell)} Z^{(\ell k)} &= -c^{(k\ell)} \quad \text{for} \quad 1 \le k, \ell \le 3, \quad k \ne \ell \,, \\ Z^{(k4)} Z^{(4k)} &= -2\, c^{(k4)} \,, \quad 1 \le k \le 3 \,, \\ Z^{(4k)} Z^{(k4)} &= -c^{(k4)}\, I_2 - \sum_{\ell,m=1}^{3} \varepsilon^{k\ell m} \Phi_\ell^{(4)} \Phi_m^{(4)} \,, \quad 1 \le k \le 3 \,. \end{aligned} \tag{4.1.1}$$

The equations in the second line follow from the trace of the equation in the third line, but we have listed them separately as they will be useful in analyzing the solutions. The

$\partial\mathcal{W}/\partial Z^{(k\ell)}=0$ equations give

$$\sum_{m=1}^{3}\varepsilon^{k\ell m}\Big(Z^{(\ell k)}\,\Phi^{(k)}_m-\Phi^{(\ell)}_m\,Z^{(\ell k)}\Big)+\sum_{\substack{m=1\\ m\neq k,\ell}}^{4}Z^{(\ell m)}Z^{(mk)}(-1)^{\delta_{k1}\delta_{\ell 3}\delta_{m4}}=0$$
$$\text{for}\quad 1\le k,\ell\le 3,\quad k\neq\ell\,,$$
$$\Big(\Phi^{(k)}_k Z^{(k4)}-Z^{(k4)}\Phi^{(4)}_k\Big)+\sum_{\substack{\ell=1\\ \ell\neq k}}^{3}Z^{(k\ell)}Z^{(\ell 4)}(-1)^{\delta_{k1}\delta_{\ell 3}}=0\quad\text{for}\quad 1\le k\le 3\,,$$
$$\Big(Z^{(4k)}\,\Phi^{(k)}_k-\Phi^{(4)}_k\,Z^{(4k)}\Big)+\sum_{\substack{m=1\\ m\neq k}}^{3}Z^{(4m)}Z^{(mk)}(-1)^{\delta_{m1}\delta_{k3}}=0\quad\text{for}\quad 1\le k\le 3\,.\tag{4.1.2}$$

As in §**??**, we use the shift symmetries (2.4.33) to choose

$$\Phi^{(1)}_1=0,\quad \Phi^{(1)}_2=0,\quad \Phi^{(1)}_3=0,\quad \Phi^{(2)}_1=0,\quad \Phi^{(2)}_2=0,\quad \Phi^{(3)}_3=0\,.\tag{4.1.3}$$

Next we note that the superpotential is invariant under the complexified gauge transformation

$$Z^{(k\ell)}\to a_k(a_\ell)^{-1}Z^{(k\ell)},\quad Z^{(4k)}\to(a_k)^{-1}M\,Z^{(4k)},\quad Z^{(k4)}\to a_k\,Z^{(k4)}M^{-1}\,,$$
$$\text{for}\quad 1\le k\le 3,\quad k\neq\ell\,,$$
$$\Phi^{(k)}_i\to\Phi^{(k)}_i,\quad \Phi^{(4)}_i\to M\Phi^{(4)}_i M^{-1}\,,\quad\text{for}\quad 1\le i\le 3,\quad 1\le k\le 3\,,\tag{4.1.4}$$

where a_k for $1\le k\le 3$ are complex numbers and M is a 2×2 complex matrix. Therefore solutions to the F-term equations come as orbits of the symmetry group. Using this we shall now make a convenient choice of gauge. Since $Z^{(k\ell)}$'s for $1\le k,\ell\le 3$ are complex numbers the first equation in (4.1.1) shows that neither $Z^{(k\ell)}$ nor $Z^{(\ell k)}$ can vanish as long as $c^{(k\ell)}$'s are chosen to be non-zero. This allows us to fix the gauge corresponding to the transformations generated by a_1 and a_3 by setting

$$Z^{(12)}=1,\quad Z^{(23)}=1\,.\tag{4.1.5}$$

Similarly, since $Z^{(k4)}$ are two component row vectors and $Z^{(4k)}$ are two component column vectors for $1\le k\le 3$, the second equation in (4.1.1) tells us that neither $Z^{(k4)}$ nor $Z^{(4k)}$ can have both components vanishing. This allows us to use the transformation generated by M to set

$$Z^{(14)}=\begin{pmatrix}1 & 0\end{pmatrix}\,.\tag{4.1.6}$$

This does not fix M completely since the choice of M given by

$$\begin{pmatrix} 1 & 0 \\ r & s \end{pmatrix} . \tag{4.1.7}$$

preserves the form of $Z^{(14)}$. Since $Z^{(24)}$ is non-zero, we can use this residual gauge symmetry to set

$$Z^{(24)} = \begin{pmatrix} 0 & 1 \end{pmatrix} . \tag{4.1.8}$$

There is one case where this gauge condition fails, and that is in the case when $Z^{(24)}$ is parallel to $Z^{(14)}$ to begin with. In this case the gauge symmetry described in (4.1.7) cannot be used to bring $Z^{(24)}$ to the form (4.1.8). We shall deal with this case separately. Once we have used a_1, a_3 and M to fix these gauges, we cannot use a_2 any more since its action is determined in terms of the others.

We now substitute the gauge choices (4.1.3), (4.1.5), (4.1.6) and (4.1.8) into the F-term equations and look for solutions to these equations. For this we choose random rational values of the $c^{(k\ell)}$'s for $1 \leq k < \ell \leq 4$. A numerical analysis of the Hilbert series using SINGULAR and Macaulay2, treating the F-term equations as ideals of the ring, shows that the solution space is a collection of 56 points. Explicit numerical solutions in Mathematica also yields precisely 56 solutions for each choice of the constants $\{c^{(k\ell)}\}$.

We also explore the possibility of having solutions with $Z^{(24)}$ proportional to $Z^{(14)}$ for which the gauge choice (4.1.8) will be invalid. In this case the analysis of the Hilbert series shows that there are no solutions. Explicit attempts to find solutions to the F-term equations in Mathematica also gives no results. This shows that there are no solutions for which the gauge choice (4.1.8) breaks down.

4.1.2 D-term equations

The solutions to the F-term equations give orbits of the complexified gauge group generated by the complex numbers a_1, a_2, a_3 and the 2×2 complex matrix M as given in (4.1.4). The D-term equations only respect a subgroup of this symmetry group consisting of physical gauge transformations. Therefore for each of the solutions to the F-term equations we can examine the orbit under (4.1.4) and then try to determine a_1, a_2, a_3 and M by demanding that the D-term equations are satisfied. As before, one combination of a_1, a_2, a_3 and $\det M$ do not act on the variables, and so we can restrict to transformations for which $\det M = 1$. However this is not expected to fix the parameters a_1, a_2, a_3 and M completely since a subgroup of the transformations (4.1.4), corresponding to physical gauge transformations, generates a symmetry of the D-term equations of motion as well. This subgroup is generated by taking the a_i to be phases and M to be an $SU(2)$ matrix. This means that once we have found a set of a_i's and M that give a solution to the D-term equations, we can generate other solutions by performing $U(1)^3 \times SU(2)$ transformations. The effect of these transformations will be to transform the parameters a_1, a_2, a_3 and M by

$$a_k \to e^{i\phi_k} a_k, \quad M \to U\, M\,, \tag{4.1.9}$$

where the ϕ_k are real numbers and U is an SU(2) matrix. Therefore for finding solutions up to gauge transformations, we can use the transformations (4.1.9) to fix the 'gauge' for a_1, a_2, a_3 and M. Using the transformations generated by the ϕ_k's we can make the a_k's real and positive. Furthermore, one can easily check that using the transformations generated by U and the constraint $\det M = 1$, we can bring M to the form

$$M = \begin{pmatrix} 1/a & b \\ 0 & a \end{pmatrix}\,, \tag{4.1.10}$$

where a is a real positive number and b is a complex number.

We now transform each of the 56 solutions to the F-term equations by (4.1.4) with real positive a_k's and M of the form (4.1.10) and try to determine the a_k's, a and b by numerically solving the D-term equations. For each of the 56 cases, we find that there is a unique choice of real positive a_k's and M of the form (4.1.10) that solves the D-term equations. This shows that up to gauge transformations, there are precisely 56 solutions to the F- and D-term equations. More accurately we have 56 different gauge orbits, generated by U(1)$^3\times$SU(2) transformation, as solutions to the F- and D-term equations.

4.1.3 $X_i^{(k)}$ dependent terms

Next we have to check if the $X_i^{(k)}$ dependent terms in the potential given in (2.4.31) vanish. For each of the 56 solutions there is a simple way to make these terms vanish – we simply choose $X_i^{(k)} = 0$ for $1 \le i \le 3$, $1 \le k \le 4$. The question we shall be interested in is: are there other configurations that make the $X_i^{(k)}$ dependent terms vanish?

Since vanishing of F- and D-terms is a necessary condition for a supersymmetric vacuum, the choice of $Z^{(k\ell)}$'s and $\Phi_i^{(k)}$'s must be restricted to the 56 solutions that we have found. We shall now argue that for each of these solutions, the $X_i^{(k)}$'s for $1 \le k \le 4$ and $1 \le i \le 3$ must vanish identically up to the shift symmetries described in the last line of (2.4.33). For this we turn to the potential (2.4.31) and note that since this is a sum of positive definite terms, in order for V_{gauge} to vanish, each term in the potential must vanish. In particular this will require

$$X_i^{(k)} Z^{(k\ell)} - Z^{(k\ell)} X_i^{(\ell)} = 0\,, \quad \text{for} \quad 1 \le k, \ell \le 4, \quad k \ne \ell, \quad 1 \le i \le 3\,. \tag{4.1.11}$$

For $1 \leq k, \ell \leq 3$, $k \neq \ell$, $X_i^{(k)}$ and $Z^{(k\ell)}$ are numbers and hence, using the result that the $Z^{(k\ell)}$ are non-zero, we get $X_i^{(k)} = X_i^{(\ell)}$. Using the shift symmetry given in the third line of (2.4.33) we can set $X_i^{(1)}$ to zero for $1 \leq i \leq 3$. As a result $X_i^{(2)}$ and $X_i^{(3)}$ must also vanish. Choosing $k = 4$ and $\ell = 1$, 2 or 3 in (4.1.11), and vice versa, we now get

$$X_i^{(4)} Z^{(4\ell)} = 0, \quad Z^{(\ell 4)} X_i^{(4)} = 0\,. \tag{4.1.12}$$

Since we have seen that $Z^{(4\ell)}$ and $Z^{(\ell 4)}$ cannot have their both components vanish, this shows that $Z^{(4\ell)}$ and $Z^{(\ell 4)}$ are right and left eigenvectors of $X_i^{(4)}$ with zero eigenvalues. Let us suppose that $X_i^{(4)}$ is non-zero for at least one i. In that case the $Z^{(\ell 4)}$'s for $1 \leq \ell \leq 3$ must be proportional to each other since each of them is a left eigenvector of the non-vanishing $X_i^{(4)}$ with zero eigenvalue. However as already mentioned, while examining the validity of the gauge choice (4.1.8) we have explicitly checked that there are no solutions to the F-term equations in which $Z^{(14)}$ and $Z^{(24)}$ are parallel to each other. This shows that our initial assumption must have been wrong, and $X_i^{(4)}$ for each i must vanish identically. This shows that the only way to make the potential vanish is to have all the $X_i^{(k)}$'s vanish for $1 \leq k \leq 4$ and $1 \leq i \leq 3$.

4.1.4 Gauss' law constraint

Finally we turn to the Gauss' law constraint. This analysis is identical to that given in §**??** except for one difference: the collective modes associated with the SU(2) gauge transformations have a kinetic term given by that of a rigid rotator with positive definite inertia matrix instead of that of a free particle with positive mass. Positive definiteness of the inertia matrix guarantees that the ground state wave-function is independent of these collective coordinates and hence the ground state is gauge invariant. Thus the Gauss' law constraint is automatically satisfied for each of the 56 gauge orbits.

Therefore we see from (2.5.34) that for $N = 2$ we have

$$P(y) = 56\,. \tag{4.1.13}$$

As in §**??**, the y independence of $P(y)$ is the result of the gauge inequivalent solutions being isolated points, and is consistent with the conjecture that all the microstates carry zero angular momentum. We also see from (2.3.9) and (2.3.10) that the counting in the dual description gives a BPS index 56. Thus there is perfect agreement between our result and that in the dual description.

4.2 Supersymmetric ground states for $N = 3$

The analysis of the equations for the $N = 3$ case proceeds similarly to that for the $N = 2$ case. Up to (4.1.5) there is essentially no change. The gauge conditions (4.1.6) and (4.1.8) are replaced by

$$Z^{(14)} = (1, 0, 0), \quad Z^{(24)} = (0, 1, 0), \quad Z^{(34)} = (0, 0, 1)\,. \tag{4.2.1}$$

Such a gauge choice is always possible if initially the vectors $Z^{(14)}$, $Z^{(24)}$ and $Z^{(34)}$ are linearly independent. The case where they are linearly dependent is analyzed separately and we find no solution in this sector. With the gauge choice (4.2.1) we find 208 distinct solutions to the set of F-term constraints. Furthermore for each of these solutions the $X_i^{(k)}$'s can be shown to vanish identically using arguments identical to those given below (4.1.12). We have not checked explicitly that for each of these solutions we have a unique solution to the D-term constraints up to gauge transformation, but since the D-term constraints usually amount to quotienting by complexified gauge transformations, and since following arguments similar to the one given below (4.1.5) one can argue that none of the vectors $Z^{(4k)}$ and $Z^{(k4)}$ can vanish as a vector, one expects on general grounds that the quotient by complexified gauge transformation will give a unique solution for each solution to the F-term constraints. Therefore we conclude that in this case we have

$$P(y) = 208\,. \tag{4.2.2}$$

Again the y independence of $P(y)$ is the result of the gauge inequivalent solutions being isolated points, and is consistent with the conjecture that all the microstates carry zero angular momentum. (4.2.2) is in perfect agreement with the result in the dual description which, according to (2.3.9) and (2.3.10), gives $P(-1) = 208$.

4.3 Conclusion

In this paper we have provided evidence for the conjecture that all the microstates of single centered BPS black holes carry strictly zero momentum at a generic point in the moduli space. This conjecture is consistent with the near horizon $AdS_2 \times S^2$ geometry of extremal black holes, but there are no direct arguments in the microscopic theory leading to this conjecture. Therefore the test of this conjecture provides evidence that the black hole horizon carries more information than just some average properties of the microstates.

These results put a strong constraint on possible fuzzball solutions describing black hole microstates [62–66]. Typical fuzzball solutions are constructed at special points in the moduli space and carry states of different angular momenta. For demonstrating that they describe genuine black hole microstates, one needs to construct these solutions for generic values of the asymptotic moduli and show that the solutions so obtained carry strictly zero angular momentum.

A different approach to this problem has been suggested in [67] where one constructs solutions with asymptotic AdS_2 boundary conditions. These solutions carry strictly zero angular momentum. However these are not truly in the spirit of the fuzzball program since they exist within the near horizon geometry of the black hole instead of replacing the near horizon geometry by a smooth solution. Furthermore since these solutions have two asymptotic boundaries, they most likely describe an entangled state living on two copies of the black hole Hilbert space instead of the microstates of a single black hole [68].

APPENDIX A

EFFECT OF METRIC AND 2-FORM BACKGROUND

The undeformed system has the six circles of T^6 orthonormal to each other and each having period 2π. The system of D6-D2-D2-D2 branes that we have considered will be described by a set of massless degrees of freedom with action governed by the supersymmetric action with superpotential $\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_4$ and all the FI parameters set to zero. In this appendix we shall show that the effect of small off-diagonal components of the metric and the 2-form field is to generate the superpotential $\mathcal{W}_3$ given in (2.4.26) and the FI terms labelled by the $c^{(k)}$'s.

Our strategy will be to compute the mass of the open strings stretched between different D-branes in the presence of the deformation and compare it with the mass computed from the deformed action given in §2. Consider for example the open string stretched between the D2-brane along the 4-5 directions and the D6-brane along the 4-5-6-7-8-9 directions, labelled by the 0+1 dimensional fields $Z^{(14)}$ and $Z^{(41)}$. For sake of generality we take the stack of D3 branes to have N_1 D2 branes and the stack of D6 branes to have N_4 D6 branes. Now we collect quadratic terms involving these fields. From V_F we get

$$2(N_1+N_4)c_{14}^*\mathrm{Tr}(Z^{(41)}Z^{(14)}) + h.c. = 2(N_1+N_4)\sum_{i=1}^{N_1}\sum_{j=1}^{N_4}\left[c_{14}Z_{ji}^{(41)}Z_{ij}^{(14)} + c_{14}^*(Z_{ij}^{(14)})^*(Z_{ji}^{(41)})^*\right], \tag{A.0.1}$$

and from D term we get

$$(c_1-c_4)\mathrm{Tr}\left[Z^{(14)}(Z^{(14)})^\dagger - (Z^{(41)})^\dagger Z^{(41)}\right] = (c_1-c_4)\sum_{i=1}^{N_1}\sum_{j=1}^{N_4}\left[|Z_{ij}^{(14)}|^2 - |Z_{ji}^{(41)}|^2\right] \tag{A.0.2}$$

For each given i, j we can separately consider the following terms

$$(c_1 - c_4)\left[|Z^{(14)}_{ij}|^2 - |Z^{(41)}_{ji}|^2\right] + 2(N_1+N_4)\left[c_{14}Z^{(41)}_{ji}Z^{(14)}_{ij} + c^*_{14}(Z^{(14)}_{ij})^*(Z^{(41)}_{ji})^*\right] .$$

One gets a 4×4 mass matrix from this, which has two doubly degenerate eigenvalues

$$\pm\sqrt{(c_1-c_4)^2 + 4(N_1+N_4)^2|c_{14}|^2}\,.$$

It is clear that for a string stretched from i-th to j-th brane one gets the following squared masses

$$\pm\sqrt{(c_i-c_j)^2 + 4(N_i+N_j)^2|c_{ij}|^2}\,. \tag{A.0.3}$$

On the other hand, we can calculate the mass of the open string stretched between the D2-brane along the 4-5 directions and the D6-brane along the 4-5-6-7-8-9 directions as follows. First we make a T-duality transformation along the 4-5-directions to convert this into a D0-D4 system. Under a T duality along X^α, to lowest order metric and B fields change as follows:

$$g_{\mu\nu}\to g_{\mu\nu}\ ,\ b_{\mu\nu}\to b_{\mu\nu}\ ,\ g_{\alpha\mu}\to b_{\alpha\mu}\ ,\ g_{\alpha\mu}\to b_{\alpha\mu}\ , \tag{A.0.4}$$

where $g_{\mu\nu}$ is related to the metric as $G_{\mu\nu} = \delta_{\mu\nu} + g_{\mu\nu}$, i.e. $g_{\mu\nu}$ represents deviations from square torus. Thus T duality along 4-5 leaves all the world-volume fields of D0-D4 system unchanged, since they do not have any 4 or 5 index. Now for small values of the background 2-form field, the (mass)^2 of the open string stretched between the D0-brane and the D4-brane along the 6-7-8-9 directions takes the form [70, 71]

$$\pm\sqrt{\frac{1}{2}\sum_{m,n} b_{mn}\left(b^{mn} + \frac{1}{2}\sum_{p,q}\epsilon^{mnpq}b_{pq}\right)} = \pm\sqrt{(b_{67}+b_{89})^2 + (b_{68}-b_{79})^2 + (b_{69}+b_{78})^2}\,, \tag{A.0.5}$$

up to an overall proportionality constant. Here ϵ^{mnpq} denotes the component of the invariant totally anti-symmetric rank 4 tensor along the D4-brane world-volume. Comparing (A.0.3) and (A.0.5) we get

$$4(N_1+N_4)^2\,|c^{(14)}|^2 + (c^{(1)} - c^{(4)})^2 = (b_{67}+b_{89})^2 + (b_{68}-b_{79})^2 + (b_{69}+b_{78})^2\,. \tag{A.0.6}$$

A similar analysis of open strings stretched between other brane pairs, and comparison with the result derived from the deformed action yields the results

$$
\begin{aligned}
4(N_1+N_2)^2\,|c^{(12)}|^2+(c^{(1)}-c^{(2)})^2 &= (g_{47}+g_{56})^2+(b_{45}-b_{67})^2+(g_{46}-g_{57})^2\,,\\
4(N_1+N_3)^2\,|c^{(13)}|^2+(c^{(1)}-c^{(3)})^2 &= (g_{49}+g_{58})^2+(b_{45}-b_{89})^2+(g_{48}-g_{59})^2\,,\\
4(N_2+N_3)^2\,|c^{(23)}|^2+(c^{(2)}-c^{(3)})^2 &= (g_{69}+g_{78})^2+(b_{67}-b_{89})^2+(g_{68}-g_{79})^2\,,\\
4(N_2+N_4)^2\,|c^{(24)}|^2+(c^{(2)}-c^{(4)})^2 &= (b_{45}+b_{89})^2+(b_{48}-b_{59})^2+(b_{49}+b_{58})^2\,,\\
4(N_3+N_4)^2\,|c^{(34)}|^2+(c^{(3)}-c^{(4)})^2 &= (b_{45}+b_{67})^2+(b_{46}-b_{57})^2+(b_{47}+b_{56})^2\,. \qquad \text{(A.0.7)}
\end{aligned}
$$

Note that in the mass formula, only 6 independent combinations of metric components and 9 independent combinations of 2-form field components appear. This gives a total of 15 independent real quantities. On the other hand, in our Lagrangian we have 3 independent FI parameters and six complex parameters $c^{(k\ell)}$. This also gives a total of 15 real parameters. Nevertheless the solutions are not unique since, for example, the left hand sides of eqs. (A.0.6), (A.0.7) are insensitive to the phases of the $c^{(k\ell)}$'s. A similar symmetry exists on the right hand side. However due to various exchange symmetries discussed in B, the following choice seems to be the correct one:

$$
\begin{aligned}
&g_{47}+g_{56}=N_{12}\,c_R^{(12)}, \quad g_{57}-g_{46}=N_{12}\,c_I^{(12)}, \quad g_{49}+g_{58}=N_{13}\,c_R^{(13)}, \quad g_{59}-g_{48}=N_{13}\,c_I^{(13)},\\
&b_{68}-b_{79}=N_{14}\,c_R^{(14)}, \quad b_{69}+b_{78}=N_{14}\,c_I^{(14)}, \quad g_{69}+g_{78}=N_{23}\,c_R^{(23)}, \quad g_{79}-g_{68}=N_{23}\,c_I^{(23)},\\
&b_{48}-b_{59}=N_{24}\,c_R^{(24)}, \quad b_{49}+b_{58}=N_{24}\,c_I^{(24)}, \quad b_{46}-b_{57}=N_{34}\,c_R^{(34)}, \quad b_{47}+b_{56}=N_{34}\,c_I^{(34)}\,,
\end{aligned}
\qquad \text{(A.0.8)}
$$

where $N_{ij}=2(N_i+N_j)$ and the subscripts R and I stand for real and imaginary parts respectively. For completeness we also give the expressions for the FI parameters $c^{(k)}$ for $1\le k\le 4$ in terms of the background fields:

$$
\begin{aligned}
c^{(1)} &= \frac{1}{2}\,(b_{45}-b_{67}-b_{89})+c_0, \quad c^{(2)}=\frac{1}{2}\,(b_{67}-b_{45}-b_{89})+c_0,\\
c^{(3)} &= \frac{1}{2}\,(b_{89}-b_{45}-b_{67})+c_0, \quad c^{(4)}=\frac{1}{2}\,(b_{45}+b_{67}+b_{89})+c_0\,, \qquad \text{(A.0.9)}
\end{aligned}
$$

where c_0 is a constant that is chosen to ensure that $\sum_{k=1}^4 c^{(k)}N_k=0$.

APPENDIX B

NORMALIZATION OF Z-Z-Z COUPLING

In this appendix we shall determine the normalizations and signs of the Z-Z-Z coupling appearing in (2.4.25) by analyzing respectively open string amplitudes and symmetry requirements. The computation will be done around a background in which all the circles of T^6 are orthonormal and have radius $\sqrt{\alpha'}$. Fluctuations away from this background will be parametrized by the constants $c^{(k\ell)}$ and $c^{(k)}$ appearing in (2.4.26) and (2.4.29). Since we work in the region where these constants are small, the corrections to the cubic terms in the superpotential proportional to these constants can be ignored.

The easiest way to determine a cubic term in the superpotential is to examine the Yukawa coupling between two fermions and one boson that arises from this term. For this we need to construct the vertex operators of the corresponding states and compute their three point function on the disk. We shall denote by b and c the usual diffeomorphism ghost fields, by β and γ the superconformal ghosts and by ϕ the scalar that arises from bosonization of the $\beta-\gamma$ system [69], normalized such that

$$\langle c(z_1)e^{-\phi}(z_1)\, c(z_2)e^{-\phi/2}(z_2)\, c(z_3)e^{-\phi/2}(z_3)\rangle = (z_1-z_2)^{1/2}(z_1-z_3)^{1/2}(z_2-z_3)^{3/4}\,, \quad \text{(B.0.1)}$$

up to a sign. In the matter sector, we shall combine the compact spatial coordinates into complex coordinates as

$$w^1 = x^4 + ix^5, \quad w^2 = x^6 + ix^7, \quad w^3 = x^8 + ix^9\,. \quad \text{(B.0.2)}$$

$w^1, \ldots, w^3$ are complex coordinates. For each coordinate w^i we have a complex world-sheet scalar field which we shall denote by W^i. Their superpartners are complex world sheet fermions which we denote by ψ^i. We also introduce the complex spin field s_i that twists ψ^i by a Z_2 transformation $\psi^i \to -\psi^i$ and the real twist field σ_i that twists W^i by a Z_2 transformation $W^i \to -W^i$. Since a world-sheet scalar satisfying a Neumann boundary condition at one end and a Dirichlet boundary condition at the other end has a half-integer mode expansion, the twist fields σ_i will be necessary for constructing the vertex operators for open string states satisfying Neumann-Dirichlet boundary conditions in some directions. The spin fields s_i will be needed for constructing the vertex operators for open string states satisfying Neumann-Dirichlet boundary conditions in some directions and also for constructing vertex operators in the Ramond sector. Finally we shall denote by $s_\alpha^{(nc)}$ the spin fields associated with the non-compact directions carrying spinor index α of $SO(3,1)$. We shall use standard normalizations for the fields ψ^i, s_i and σ_i, e.g.

$$\psi^i(z_1)\bar{\psi}^j(z_2) = \delta_{ij}\,(z_1-z_2)^{-1}, \quad s_i(z_1)\bar{s}_j(z_2) = \delta_{ij}\,(z_1-z_2)^{-1/4},$$
$$\bar{\psi}^i(z_1)s_j(z_2) = \delta_{ij}\,(z_1-z_2)^{-1/2}\bar{s}_j(z_2), \quad \psi^i(z_1)\bar{s}_j(z_2) = \delta_{ij}\,(z_1-z_2)^{-1/2}s_j(z_2),$$
$$\sigma_i(z_1)\sigma_j(z_2) = \delta_{ij}\,(z_1-z_2)^{-1/4}, \quad s_\alpha^{(nc)}(z_1)s_\beta^{(nc)}(z_2) = \epsilon_{\alpha\beta}(z_1-z_2)^{-1/2}\,, \qquad \text{(B.0.3)}$$

up to multiplicative signs and less singular additive terms. The operator products given in the last line are determined by the conformal weights of σ_i, $s_\alpha^{(nc)}$ and their normalizations. The operator products in the first two lines can be shown to be mutually compatible by bosonizing the fermions ψ^i to scalar fields ϕ_i and using the identification

$$\psi^i = e^{i\phi_i}, \quad \bar{\psi}^i = e^{-i\phi_i}, \quad s_i = e^{i\phi_i/2}, \quad \bar{s}_i = e^{-i\phi_i/2}\,. \qquad \text{(B.0.4)}$$

The other ingredient we need for the construction of the vertex operator is Chan-Paton factors. Since we have altogether $N+3$ D-branes, the Chan-Paton factors can be taken to be $(N+3)\times(N+3)$ matrices. We shall choose the convention in which the first three rows and columns represent the branes 1, 2 and 3 and the last N rows and columns label the brane stack 4. In this notation, for $1\le k,\ell\le 3$, $1\le r,s\le N$, the Chan-Paton factors for the vertex operators for $\Phi_\ell^{(k)}$, $(\Phi_\ell^{(4)})_{rs}$, $Z^{(k\ell)}$, $Z_r^{(k4)}$ and $Z_r^{(4k)}$ will be matrices whose only non-zero entries are the (k,k)'th, $(3+r,3+s)$'th, (k,ℓ)'th, $(k,3+r)$'th and $(3+r,k)$'th elements, respectively. In the analysis that follows we shall suppress the Chan-Paton factors – they just accompany the vertex operators as multiplicative matrices. The correlation function of a given set of vertex operators will contain a term proportional to the trace of the ordered product of their Chan-Paton factors.

The vertex operators for the states corresponding to $\Phi_i^{(k)}$ and $Z^{(k\ell)}$ can be constructed using these operators. For a superfield A we shall denote by $V^f_{A,\alpha}$ and V^b_A the vertex operators of the fermionic and bosonic components of the superfield. Here α denotes a 3+1 dimensional spinor index. In this notation we have, for example

$$V^b_{\Phi_1^{(k)}} = c\,e^{-\phi}\psi^1, \quad V^f_{\Phi_1^{(k)},\alpha} = c\,e^{-\phi/2}s_1\bar{s}_2\bar{s}_3 s_\alpha^{(nc)},$$
$$V^b_{Z^{(12)}} = c\,e^{-\phi}\sigma_1\sigma_2 s_1 s_2, \quad V^f_{Z^{(12)},\alpha} = c\,e^{-\phi/2}\sigma_1\sigma_2\bar{s}_3 s_\alpha^{(nc)}\,, \qquad \text{(B.0.5)}$$

up to multiplicative signs. Here the ψ^1 factor in the expression for $V^b_{\Phi_1^{(k)}}$ reflects that this mode represents position / Wilson line on the brane along w^1, whereas the $s_1\bar{s}_2\bar{s}_3$ factor in $V^f_{\Phi_1^{(k)},\alpha}$ is obtained by starting with the operator $\bar{s}_1\bar{s}_2\bar{s}_3$ that appears in the expression for the supersymmetry generator, and taking the leading term in its operator product with ψ^1 – the same factor that is present in $V^b_{\Phi_1^{(k)}}$. In the expression for $V^b_{Z^{(12)}}$ the $\sigma_1\sigma_2$ and $s_1 s_2$ factors reflect that the fields W^1 and W^2 and their fermionic partners ψ^1 and ψ^2 satisfy Neumann boundary conditions at one end of the open string and Dirichlet boundary conditions at the other end. On the other hand the $\sigma_1\sigma_2\bar{s}_3$ term in the expression for $V^f_{Z^{(12)},\alpha}$ is the leading term in the expression for the operator product of $\bar{s}_1\bar{s}_2\bar{s}_3$ and $\sigma_1\sigma_2 s_1 s_2$. Following this strategy we can find the expressions for all the vertex operators $V^b_{\Phi_i^{(k)}}$, $V^f_{\Phi_i^{(k)},\alpha}$, $V^b_{Z^{(k\ell)}}$ and $V^f_{Z^{(k\ell)},\alpha}$.

Let us now calculate the three point function

$$\langle V^b_{\Phi_1^{(4)}} V^f_{\Phi_2^{(4)},\alpha} V^f_{\Phi_3^{(4)},\beta} \rangle \,, \tag{B.0.6}$$

to fix the normalization of the terms in $\mathcal{W}_4$ in (2.4.24). Using the generalization of (B.0.5) we see that this is computed using the correlation function

$$\langle c\, e^{-\phi}\psi^1(z_1)\, c\, e^{-\phi/2}\bar{s}_1 s_2 \bar{s}_3 s_\alpha^{(nc)}(z_2)\, c\, e^{-\phi/2}\bar{s}_1\bar{s}_2 s_3 s_\beta^{(nc)}(z_3)\rangle \,. \tag{B.0.7}$$

This correlator can be factorized and evaluated using (B.0.1) and (B.0.3) as

$$\begin{aligned} & \langle c\, e^{-\phi}(z_1)\, c\, e^{-\phi/2}(z_2)\, c\, e^{-\phi/2}(z_3)\rangle_{ghost}\, \langle \psi^1(z_1)\bar{s}_1(z_2)\bar{s}_1(z_3)\rangle_{\psi^1} \\ & \langle s_2(z_2)\bar{s}_2(z_3)\rangle_{\psi^2} \langle \bar{s}_3(z_2) s_3(z_3)\rangle_{\psi^3} \langle s_\alpha^{(nc)}(z_2) s_\beta^{(nc)}(z_3)\rangle_{nc} \\ = \; & \epsilon_{\alpha\beta} \,, \end{aligned} \tag{B.0.8}$$

up to a sign. This agrees with the normalization of the three point coupling of $\Phi^{(4)}$ given in (2.4.24) after ignoring the overall $\sqrt{2}$ factor. (Since the $\sqrt{2}$ factor appears universally in front of all terms in the superpotential $\mathcal{W}$, we can ignore this for fixing the relative normalization between different terms.)

Next we compute the Φ-Z-Z three point function. For example the $\Phi_3^{(1)} Z^{(12)} Z^{(21)}$ coupling will be given by the coefficient of $\epsilon_{\alpha\beta}$ in

$$\begin{aligned} \langle V^b_{\Phi_3^{(1)}}(z_1) V^f_{Z^{(12)},\alpha}(z_2) V^f_{Z^{(21)},\beta}(z_3)\rangle &= \langle c\, e^{-\phi}\psi^3(z_1)\, c\, e^{-\phi/2}\sigma_1\sigma_2\bar{s}_3 s_\alpha^{(nc)}(z_2)\, c\, e^{-\phi/2}\sigma_1\sigma_2\bar{s}_3 s_\beta^{(nc)}(z_3)\rangle \\ &= \epsilon_{\alpha\beta} \end{aligned} \tag{B.0.9}$$

up to a sign. In the last step we have used (B.0.1) and (B.0.3) and factorized the correlator into contributions from the ghost sector and different components of the matter sector. This is in agreement with the normalization of the Φ-Z-Z coupling given in (2.4.23).

Finally let us compute the Z-Z-Z three point coupling. For definiteness we focus on the $Z^{(12)}Z^{(23)}Z^{(31)}$ coupling. For this we compute

$$\begin{aligned} & \langle V^b_{Z^{(12)}}(z_1) V^f_{Z^{(23)},\alpha}(z_2) V^f_{Z^{(31)},\beta}(z_3)\rangle \\ = \;& \langle c\, e^{-\phi}\sigma_1\sigma_2 s_1 s_2(z_1)\, c\, e^{-\phi/2}\sigma_2\sigma_3\bar{s}_1 s^{(nc)}_\alpha(z_2)\, c\, e^{-\phi/2}\sigma_3\sigma_1\bar{s}_2 s^{(nc)}_\beta(z_3)\rangle \\ = \;& \epsilon_{\alpha\beta} \end{aligned} \qquad \text{(B.0.10)}$$

Again in the last step we have used (B.0.1) and (B.0.3) and factorized the correlator into contributions from the ghost sector and different components of the matter sector. The result is valid only up to a sign. This agrees with the normalization of the Z-Z-Z coupling in $\mathcal{W}_2$ given in (2.4.25).

Following the same procedure we can show that the coefficients of all the Φ-Φ-Φ, Φ-Z-Z and Z-Z-Z couplings are unity up to signs. In principle these signs can be determined by careful string theory computation but we shall describe an alternative approach based on symmetry considerations. For this we consider a more general system than what has been discussed so far, containing N_1 D2-branes along 4-5 directions, N_2 D2-branes along 6-7 directions, N_3 D2-branes along 8-9 directions and N_4 D6-branes along 4-5-6-7-8-9 directions. We claim that the correct form of the superpotential, up to field redefinition, is given by

$$\begin{aligned} \mathcal{W}_1 \;=\; & \sqrt{2}\,\mathrm{Tr}\Bigg[\left(\Phi^{(1)}_3 Z^{(12)}Z^{(21)} - \Phi^{(2)}_3 Z^{(21)}Z^{(12)}\right) + \left(\Phi^{(2)}_1 Z^{(23)}Z^{(32)} - \Phi^{(3)}_1 Z^{(32)}Z^{(23)}\right) \\ & + \left(\Phi^{(3)}_2 Z^{(31)}Z^{(13)} - \Phi^{(1)}_2 Z^{(13)}Z^{(31)}\right) + \left(\Phi^{(1)}_1 Z^{(14)}Z^{(41)} - \Phi^{(4)}_1 Z^{(41)}Z^{(14)}\right) \\ & + \left(\Phi^{(2)}_2 Z^{(24)}Z^{(42)} - \Phi^{(4)}_2 Z^{(42)}Z^{(24)}\right) + \left(\Phi^{(3)}_3 Z^{(34)}Z^{(43)} - \Phi^{(4)}_3 Z^{(43)}Z^{(34)}\right)\Bigg], \end{aligned} \qquad \text{(B.0.11)}$$

$$\begin{aligned} \mathcal{W}_2 \;=\; & \sqrt{2}\,\mathrm{Tr}\Big[Z^{(31)}Z^{(12)}Z^{(23)} + Z^{(13)}Z^{(32)}Z^{(21)} + Z^{(12)}Z^{(24)}Z^{(41)} + Z^{(42)}Z^{(21)}Z^{(14)} \\ & - Z^{(13)}Z^{(34)}Z^{(41)} + Z^{(31)}Z^{(14)}Z^{(43)} + Z^{(34)}Z^{(42)}Z^{(23)} + Z^{(43)}Z^{(32)}Z^{(24)}\Big], \end{aligned} \qquad \text{(B.0.12)}$$

$$
\begin{aligned}
\mathcal{W}_3 \;=\; \sqrt{2}\,\mathrm{Tr}\Big[& c^{(12)}\left(\Phi_3^{(1)}\otimes I_{N_2} - I_{N_1}\otimes\Phi_3^{(2)}\right) + c^{(23)}\left(\Phi_1^{(2)}\otimes I_{N_3} - I_{N_2}\otimes\Phi_1^{(3)}\right) \\
& + c^{(13)}\left(\Phi_2^{(3)}\otimes I_{N_1} - I_{N_3}\otimes\Phi_2^{(1)}\right) + c^{(14)}\left(\Phi_1^{(1)}\otimes I_{N_4} - I_{N_1}\otimes\Phi_1^{(4)}\right) \\
& + c^{(24)}\left(\Phi_2^{(2)}\otimes I_{N_4} - I_{N_2}\otimes\Phi_2^{(4)}\right) + c^{(34)}\left(\Phi_3^{(3)}\otimes I_{N_4} - I_{N_3}\otimes\Phi_3^{(4)}\right)\Big],
\end{aligned}
\tag{B.0.13}
$$

and

$$
\begin{aligned}
\mathcal{W}_4 \;=\; -\sqrt{2}\Big[& \mathrm{Tr}\Big(\Phi_1^{(1)}\Phi_2^{(1)}\Phi_3^{(1)} - \Phi_1^{(1)}\Phi_3^{(1)}\Phi_2^{(1)}\Big) - \mathrm{Tr}\Big(\Phi_1^{(2)}\Phi_2^{(2)}\Phi_3^{(2)} - \Phi_1^{(2)}\Phi_3^{(2)}\Phi_2^{(2)}\Big) \\
& + \mathrm{Tr}\Big(\Phi_1^{(3)}\Phi_2^{(3)}\Phi_3^{(3)} - \Phi_1^{(3)}\Phi_3^{(3)}\Phi_2^{(3)}\Big) + \mathrm{Tr}\Big(\Phi_1^{(4)}\Phi_2^{(4)}\Phi_3^{(4)} - \Phi_1^{(4)}\Phi_3^{(4)}\Phi_2^{(4)}\Big)\Big].
\end{aligned}
\tag{B.0.14}
$$

This superpotential reduces to the one given in (2.4.23)-(2.4.24) for $N_1 = N_2 = N_3 = 1$. We shall now describe the arguments leading to (B.0.11)-(B.0.14).

Let us begin with the arguments leading to the form of $\mathcal{W}_2$. Since by field redefinitions involving changes of signs of the $Z^{(k\ell)}$'s we can change the relative signs of various terms in $\mathcal{W}_2$, we only have to show that $\mathcal{W}_2$ is given by (B.0.12) up to these field redefinitions. Now these field redefinitions can only change the signs of an even number of terms in $\mathcal{W}_2$ since each $Z^{(k\ell)}$ appears as a factor in two of the terms. Thus an expression for $\mathcal{W}_2$ with an even number of minus signs cannot be turned into an expression with an odd number of minus signs and vice versa. Furthermore, it can be shown by inspection that by these field redefinitions all possible choices of $\mathcal{W}_2$ with an even number of minus signs can be brought to the form in which each term in $\mathcal{W}_2$ has positive sign, and all possible choices of $\mathcal{W}_2$ with an odd number of minus signs can be brought to the form given in (B.0.12). Thus the possible candidates for $\mathcal{W}_2$ can be restricted to either (B.0.12) or the one with all positive signs. We shall argue shortly that requiring symmetry under the exchange of different stacks of D-branes leads to the form given in (B.0.12).

Next we turn to $\mathcal{W}_1$ and $\mathcal{W}_3$. The shift symmetries (2.4.33) fix the relative sign between the pair of terms inside each parenthesis in (B.0.11) and (B.0.13), but do not fix the signs that appear in front of the parentheses. However starting with any arbitrary choice of these signs, we can arrive at (B.0.11) and (B.0.13) by redefinition involving changes of the signs of the fields $\Phi_i^{(k)}$ and the parameters $c^{(k\ell)}$. These field redefinitions change the signs of various terms in $\mathcal{W}_4$, but leave $\mathcal{W}_2$ unchanged.

Finally turning to $\mathcal{W}_4$ we see that the relative sign between the pair of terms inside each parenthesis is fixed by the requirement that these come from the dimensional reduction of $\mathcal{N} = 4$ supersymmetric theories in 3+1 dimensions. We shall see shortly that the relative signs between the different parentheses are fixed by the symmetry under the exchange of brane stacks. This however does not fix the overall sign of $\mathcal{W}_4$ leaving behind a 2-fold ambiguity. We expect that a careful string theory calculation will be able to resolve this ambiguity, but we have not done this. As we have described in the text, the choice of $\mathcal{W}_4$ given in (B.0.14), after restriction to the case $N_1 = N_2 = N_3 = 1$, $N_4 = N$ gives the results 12, 56 and 208 for the index for $N = 1$, 2 and 3, respectively, in agreement with the results in the dual description. In contrast the opposite choice of sign gives the results 12, 60 and 232 for the index for the cases $N = 1$, 2 and 3, respectively. These do not agree with the results computed using the dual description.

What remains is to show how the exchange symmetry constrains the form of $\mathcal{W}_2$ and $\mathcal{W}_4$. For this we need to examine how the exchange symmetry acts on the coefficients $c^{(k\ell)}$, which is straightforward exercise using A.0.8.

Now type IIA string theory on T^6 has an exchange symmetry $x^4 \leftrightarrow x^6$, $x^5 \leftrightarrow x^7$, $N_1 \leftrightarrow N_2$ under which the D2-brane stacks 1 and 2 get exchanged and the stacks 3 and 4 remain unchanged. We see from (A.0.8) that under this transformation $c^{(34)}$ changes sign, $c^{(12)}$ remains unchanged, and $c^{(1i)}$ and $c^{(2i)}$ get exchanged for $i = 3, 4$. Thus there must be an action on the variables $\Phi_i^{(k)}$ and $Z^{(k\ell)}$ which, together with these transformations on the $c^{(k\ell)}$'s and N_i's, transform the $\mathcal{W}_i$'s at most by an overall multiplicative phase.[a] It is easy to verify that for the superpotential given in (B.0.11)-(B.0.14) the following accompanying transformation takes $\mathcal{W}_i \to -\mathcal{W}_i$ for $1 \le i \le 4$:

$$\begin{aligned}
&(\Phi_3^{(4)}, \Phi_3^{(3)}) \to (\Phi_3^{(4)}, \Phi_3^{(3)}), \quad (\Phi_3^{(2)}, \Phi_3^{(1)}) \to (\Phi_3^{(1)}, \Phi_3^{(2)}), \\
&(\Phi_1^{(2)}, \Phi_1^{(3)}) \leftrightarrow (\Phi_2^{(1)}, \Phi_2^{(3)}), \quad (\Phi_1^{(4)}, \Phi_1^{(1)}) \leftrightarrow -(\Phi_2^{(4)}, \Phi_2^{(2)}), \\
&Z^{(34)} \to Z^{(34)}, \quad Z^{(i1)} \leftrightarrow -Z^{(i2)}, \quad Z^{(1i)} \leftrightarrow -Z^{(2i)}, \quad \text{for } i = 3, 4, \\
&Z^{(12)} \leftrightarrow -Z^{(21)}, \quad Z^{(43)} \to -Z^{(43)}\,.
\end{aligned} \tag{B.0.15}$$

On the other hand, for the choice of $\mathcal{W}_2$ without the sign $(-1)^{\delta_{k1}\delta_{\ell 3}\delta_{m4}}$, i.e. all Z-Z-Z coupling coming with positive coefficients, this property will be lost.[b] This shows that the form of $\mathcal{W}_2$ given in (B.0.12) is the correct one, and also fixes the relative signs between the terms in $\mathcal{W}_4$ involving $\Phi_k^{(1)}$ and $\Phi_k^{(2)}$.

We now turn to the second exchange symmetry, generated by the transformation $x^6 \leftrightarrow x^8$, $x^7 \leftrightarrow x^9$ and $N_2 \leftrightarrow N_3$. This exchanges the second and the third D-brane stacks. (A.0.8) shows that under this transformation $c^{(14)} \to -c^{(14)}$, $c^{(23)} \to c^{(23)}$, $c^{(12)} \leftrightarrow c^{(13)}$ and $c^{(24)} \leftrightarrow c^{(34)}$. We can verify that all the $\mathcal{W}_i$'s change sign if we accompany these transformations of $c^{(k\ell)}$ and N_i with the

[a] Multiplying the superpotential by an overall phase leaves the potential invariant.

[b] Since the $c^{(k\ell)}$'s are only determined up to phases in terms of the background fields, one may wonder whether the analysis that led to the conclusion that the choice of all positive signs in $\mathcal{W}_2$ is not allowed could be modified if we use $c^{(k\ell)}$ with different phases. To this end we note that the information that was needed to arrive at this result was that under the exchange $x^4 \leftrightarrow x^6$ and $x^5 \leftrightarrow x^7$, $c^{(12)}$ remains unchanged and $c^{(34)}$ changes sign. These transformation laws of $c^{(12)}$ and $c^{(34)}$ do not depend on the choice of phases of the $c^{(k\ell)}$'s given in (A.0.8). Thus our argument holds.

following transformation on the fields:

$$
\begin{aligned}
&(\Phi_1^{(4)}, \Phi_1^{(1)}) \to (\Phi_1^{(4)}, \Phi_1^{(1)}), \quad (\Phi_3^{(1)}, \Phi_3^{(2)}) \leftrightarrow (\Phi_2^{(1)}, \Phi_2^{(3)}),\\
&(\Phi_1^{(2)}, \Phi_1^{(3)}) \to (\Phi_1^{(3)}, \Phi_1^{(2)}), \quad (\Phi_2^{(4)}, \Phi_2^{(2)}) \leftrightarrow -(\Phi_3^{(4)}, \Phi_3^{(3)}),\\
&Z^{(41)} \to Z^{(41)}, \quad Z^{(2i)} \leftrightarrow -Z^{(3i)}, \quad Z^{(i2)} \leftrightarrow -Z^{(i3)}, \quad \text{for } i = 1, 4,\\
&Z^{(32)} \leftrightarrow -Z^{(23)}, \quad Z^{(14)} \to -Z^{(14)} . \qquad (B.0.16)
\end{aligned}
$$

It follows from (B.0.15) and (B.0.16) that if for $i = 1$ and 3, $\Phi_i^{(i)}$ and $\Phi_i^{(4)}$ are interpreted as Wilson lines along the w^i direction on the respective branes, then $\Phi_2^{(2)}$ and $\Phi_2^{(4)}$ should be interpreted as Wilson lines along the $-w^2$ direction on the D2-brane along the 6-7 directions and the D6-brane, respectively. This analysis also fixes the relative signs between the terms in $\mathcal{W}_4$ involving $\Phi_k^{(2)}$ and $\Phi_k^{(3)}$.

The theory under consideration also has a symmetry that exchanges brane stacks 1 and 4, leaving the stacks 2 and 3 unchanged. This is induced by making a T-duality transformation along 6-7-8-9 directions and then performing an exchange $6 \leftrightarrow 8$, $7 \leftrightarrow 9$, and at the same time exchanging N_1 and N_4. We shall use the convention that, under a T-duality transformation along the x^m direction, $g_{mn} \leftrightarrow b_{mn}$ to leading order in g_{mn} and b_{mn} for $n \neq m$. It is easy to see from (A.0.8) that under this exchange $c^{(14)}$ remains unchanged, $c^{(23)}$ changes sign, $c^{(12)} \leftrightarrow i\, c^{(24)}$ and $c^{(13)} \leftrightarrow i\, c^{(34)}$. It can be seen that the following accompanying transformation rules of the fields take $\mathcal{W}_i$ to $-\mathcal{W}_i$ for $1 \leq i \leq 4$:

$$
\begin{aligned}
&(\Phi_1^{(4)}, \Phi_1^{(1)}) \to (\Phi_1^{(1)}, \Phi_1^{(4)}), \quad (\Phi_2^{(1)}, \Phi_2^{(3)}) \leftrightarrow i\,(\Phi_3^{(4)}, \Phi_3^{(3)}),\\
&(\Phi_1^{(2)}, \Phi_1^{(3)}) \to (\Phi_1^{(2)}, \Phi_1^{(3)}), \quad (\Phi_2^{(4)}, \Phi_2^{(2)}) \leftrightarrow i\,(\Phi_3^{(1)}, \Phi_3^{(2)}),\\
&Z^{(12)} \leftrightarrow -Z^{(42)}, \quad Z^{(13)} \leftrightarrow -i\, Z^{(43)}, \quad Z^{(21)} \leftrightarrow -iZ^{(24)}, \quad Z^{(31)} \leftrightarrow -Z^{(34)},\\
&Z^{(14)} \leftrightarrow i\, Z^{(41)}, \quad Z^{(32)} \to Z^{(32)}, \quad Z^{(23)} \to -Z^{(23)} . \qquad (B.0.17)
\end{aligned}
$$

This fixes the relative signs between the terms in $\mathcal{W}_4$ involving $\Phi_k^{(1)}$ and $\Phi_k^{(4)}$. The factors of i in the transformation laws relating $\Phi_i^{(k)}$'s to $\Phi_k^{(k)}$'s and $\Phi_k^{(4)}$'s for $1 \leq i, k \leq 3$, $i \neq k$ show that if we interpret $\Phi_i^{(k)}$ as the position of the k-th D2-brane along w^i, then up to signs, $i\,\Phi_i^{(4)}$ and $i\,\Phi_i^{(i)}$ are to be interpreted as Wilson lines along w^i on the i-th D2-brane and the D6-brane, respectively. Due to the comments below (B.0.16) it follows that there is a further factor of -1 in the definition of $\Phi_2^{(2)}$ and $\Phi_2^{(4)}$ relative to those for $\Phi_i^{(i)}$ and $\Phi_i^{(4)}$ for $i = 1, 3$.

All other exchange symmetries are compositions of the above three transformations and hence invariance of the potential under the former follows as a consequence of their invariance under the latter. Thus we see that the exchange symmetries together with judicious utilization of the field redefinition freedom fixes the signs of all the terms in the superpotential except for the overall sign of $\mathcal{W}_4$ relative to the other terms.

The superpotential determined this way also has other desired symmetries. For example we see from (A.0.8) that under the transformation $x^4 \to x^5$, $x^5 \to -x^4$, $c^{(12)}$, $c^{(13)}$, $c^{(24)}$ and $c^{(34)}$ get multiplied by $-i$ while the other $c^{(k\ell)}$'s remain unchanged. It is easy to see that under this transformation the superpotential gets multiplied by an overall factor of $-i$, and hence leaves the potential invariant, if we transform the various fields as

$$\begin{aligned}
&\Phi_1^{(k)} \to -i\Phi_1^{(k)} \quad \text{for } 1 \le k \le 4, \quad Z^{(12)} \to -iZ^{(12)}, \quad Z^{(13)} \to -iZ^{(13)}, \\
&Z^{(42)} \to -iZ^{(42)}, \quad Z^{(43)} \to -iZ^{(43)},
\end{aligned} \tag{B.0.18}$$

leaving the other fields invariant.

Finally consider the world-sheet parity transformation under which the NS-NS 2-form field changes sign. This by itself is not a symmetry of type IIA string theory, but becomes a symmetry if we accompany this by the parity transformation along the non-compact directions and $(-1)^{F_L}$ – this is simply the symmetry group by which we quotient the theory to generate orientifold 6-planes. Furthermore this transformation leaves the D6- and D2-branes invariant. We see from (A.0.8) that under this transformation $c^{(k\ell)} \to c^{(k\ell)}$ and $c^{(k4)} \to -c^{(k4)}$ for $1 \le k < \ell \le 3$. It is easy to see that the following transformation on the fields combined with the above leaves the superpotential invariant:

$$\begin{aligned}
&\Phi_i^{(4)} \to -(\Phi_i^{(4)})^T, \quad \Phi_i^{(i)} \to -(\Phi_i^{(i)})^T, \quad \Phi_j^{(i)} \to (\Phi_j^{(i)})^T \quad \text{for } 1 \le i, j \le 3, \, j \ne i, \\
&Z^{(k\ell)} \to (Z^{(\ell k)})^T \quad \text{for } 1 \le k, \ell \le 3, \, \ell \ne k, \\
&Z^{(14)} \to (Z^{(41)})^T, \quad Z^{(24)} \to -(Z^{(42)})^T, \quad Z^{(34)} \to (Z^{(43)})^T, \\
&Z^{(41)} \to -(Z^{(14)})^T, \quad Z^{(42)} \to (Z^{(24)})^T, \quad Z^{(43)} \leftrightarrow -(Z^{(34)})^T .
\end{aligned} \tag{B.0.19}$$

The transposition operation involved in the transformation laws is a reflection of the fact that under world-sheet parity transformation open strings change their orientation.

Given the superpotential constructed above, the potential V is given by (2.4.27)-(2.4.32). The shift symmetry (2.4.33) generalizes to

$$\begin{aligned}
&\Phi_m^{(k)} \to \Phi_m^{(k)} + \xi_m I_{N_k}, \quad \text{for} \quad 1 \le k \le 3, \quad k \ne m; \quad 1 \le m \le 3, \\
&\Phi_k^{(k)} \to \Phi_k^{(k)} + \zeta_k I_{N_k}, \quad \Phi_k^{(4)} \to \Phi_k^{(4)} + \zeta_k I_{N_4}, \quad \text{for} \quad 1 \le k \le 3\,, \\
&X_i^{(k)} \to X_i^{(k)} + a_i\, I_{N_k}\,, \quad \text{for} \quad 1 \le i \le 3\,.
\end{aligned} \tag{B.0.20}$$

We shall now argue that at a generic point in the moduli space, the vanishing of V_{gauge} given in (2.4.31) requires all the $X_i^{(k)}$'s to vanish (up to the shift symmetry described in the last line of

(B.0.20)). To see this first note that the vanishing of V_{gauge} requires each of the terms in (2.4.31) to vanish separately since each is a positive definite term. The vanishing of the last term tells us that $X_i^{(k)}$ and $X_j^{(k)}$ commute. Since $X_i^{(k)}$'s are hermitian matrices, this implies that with the help of $U(N_k)$ gauge transformations we can simultaneously diagonalize each $X_i^{(k)}$. Therefore we can take

$$\left(X_i^{(k)}\right)_{mn} = a_{i,m}^{(k)}\,\delta_{mn}\,, \quad \text{for} \quad 1 \le m,n \le N_k, \quad 1 \le k \le 4\,. \tag{B.0.21}$$

Physically $a_{i,m}^{(k)}$ has the interpretation of the position along x^i of the m-th brane in the k-th stack. Now requiring that the first two terms in (2.4.31) vanish we can easily see that

$$\begin{aligned}\left(Z^{(k\ell)}\right)_{mn}\left(a_{i,m}^{(k)} - a_{i,n}^{(\ell)}\right) &= 0 \quad \text{for } 1 \le m \le N_k,\, 1 \le n \le N_\ell,\, 1 \le i \le 3\,,\\ \left(\Phi^{(k)}\right)_{mn}\left(a_{i,m}^{(k)} - a_{i,n}^{(k)}\right) &= 0 \quad \text{for } 1 \le m,n \le N_k,\, 1 \le i \le 3\,.\end{aligned} \tag{B.0.22}$$

This gives

$$\begin{aligned}\left(Z^{(k\ell)}\right)_{mn} &= 0 \quad \text{if } a_{i,m}^{(k)} \ne a_{i,n}^{(\ell)} \text{ for any } i \text{ for } 1 \le m \le N_k,\, 1 \le n \le N_\ell\,,\\ \left(\Phi^{(k)}\right)_{mn} &= 0 \quad \text{if } a_{i,m}^{(k)} \ne a_{i,n}^{(k)} \text{ for any } i \text{ for } 1 \le m,n \le N_k\,.\end{aligned} \tag{B.0.23}$$

Furthermore, once (B.0.23) is satisfied, there is no further constraint on the $a_{i,m}^{(k)}$'s. Physically this means that if the m'th brane in the k-th stack and the n-th brane in the ℓ-th stack are separated along the non-compact directions, then all fields associated with the open strings stretched between the two branes must have zero expectation value. Therefore we can divide the brane stacks into groups, where within each group all branes are coincident along the non-compact directions, and the brane stacks in two different groups are separated from each other along at least one of the non-compact directions. In this case all fields associated with open strings stretching between two different groups must vanish, and once this condition is satisfied we have no further constraint on the locations of the groups along the non-compact directions. In particular we can separate the different groups by arbitrary distance. If such a solution exists, then the mass of the total brane system will be given by the sum of the masses of the individual groups. But at a generic point in the moduli space away from subspaces of marginal stability this violates the BPS bound if the total charge vector is primitive, i.e. $\gcd\{N_1, N_2, N_3, N_4\} = 1$. This shows that it should be impossible to satisfy the F- and D-term equations in this case. Therefore the only way to solve the equations is to set all the $a_{i,m}^{(k)}$'s to be equal for any given i. Using the shift symmetry given in the last line of (B.0.20) we can set these to zero, showing that all the $X_i^{(k)}$'s can be taken to vanish.

We can also see the absence of solutions with more than one group by directly analyzing the F-term equations. Let us consider for example the equations we get by setting the variation of $\mathcal{W}$

with respect to $\Phi_3^{(1)}$ and $\Phi_3^{(2)}$ to zero. Using (B.0.11)-(B.0.14) these equations take the form

$$\begin{aligned} &Z^{(12)}Z^{(21)} + N_2 c^{(12)}\, I_{N_1} - [\Phi_1^{(1)}, \Phi_2^{(1)}] = 0\,, \\ &Z^{(21)}Z^{(12)} + N_1 c^{(12)}\, I_{N_2} + [\Phi_1^{(2)}, \Phi_2^{(2)}] = 0\,. \end{aligned} \tag{B.0.24}$$

Now suppose that we have a solution with multiple groups, with the first group containing M_1 D2-branes along 4-5 directions, M_2 D2-branes along 6-7 directions, M_3 D2-branes along 8-9 directions and M_4 D6-branes along 4-5-6-7-8-9 directions. Let us now take the trace of the first equation over the M_1 D2-branes in the 4-5 directions and the trace of the second equation over the M_2 D2-branes in the 6-7 directions. Since there are no components of $Z^{(k\ell)}$ or $\Phi_m^{(k)}$ with one leg in the first group and another leg in another group, the traces described above restrict the sum to be over the indices in the first group only. Denoting the trace with all indices in the first group by Tr_1, we get

$$\text{Tr}_1(Z^{(12)}Z^{(21)}) + c^{(12)} N_2 M_1 = 0\,, \quad \text{Tr}_1(Z^{(21)}Z^{(12)}) + c^{(12)} N_1 M_2 = 0\,. \tag{B.0.25}$$

Taking the difference between the two equations we see that we must have $N_1 M_2 = N_2 M_1$. Repeating the analysis for other equations we get $N_i M_j = N_j M_i$ for $1 \le i < j \le 4$, or equivalently, M_i/N_i=constant. The same analysis may be repeated for other groups, showing that the charge vector of each group must be proportional to the total charge vector. But this is impossible if the total charge vector is primitive. This shows that for primitive charge vector there are no solutions to the F-term equations with more than one group.

For completeness we also give the expected number of solutions to the F- and D-term constraints from the counting in the dual description [47]. For the N_i's satisfying the restriction

$$\gcd\{N_1N_2, N_1N_3, N_1N_4, N_2N_3, N_2N_4, N_3N_4\} = 1\,, \tag{B.0.26}$$

this is given by

$$-\sum_{s|s_0} s\, \widehat{c}(4N_1N_2N_3N_4/s^2) \tag{B.0.27}$$

where $\widehat{c}(u)$ has been defined in (2.3.8) and

$$s_0 = \prod_{\substack{i,j=1\\ i<j}}^{4} \gcd\{N_i, N_j\}\,. \tag{B.0.28}$$

APPENDIX C

DERIVATION OF THE INDEX

For convenience, we repeat 2.3.7,

$$B_{14} = (-1)^{Q\cdot P+1} \sum_{s|\ell_1\ell_2} s\,\widehat{c}(\Delta/s^2)\,, \quad \Delta \equiv Q^2P^2 - (Q\cdot P)^2 = 4\,N_1N_2N_3N_4 - (N_1N_5)^2\,. \tag{C.0.1}$$

For our choice of charges, i.e. $N_1 = N_2 = N_3 = 1, N_4 = n, N_5 = 0$, using 2.3.4, 2.3.5 and 2.3.7 we have

$$Q\cdot P = 0,\ \Delta = 4n,\ \ell_1 = \gcd\{1,n,1,n,0\} = 1,\ \ell_2 = \gcd\{1,n,0\} = 1\,.$$

This implies $s = 1$ and hence

$$B_{14} = -\widehat{c}(4n)\,. \tag{C.0.2}$$

$\widehat{c}(u)$ is give by 2.3.8

$$-\,\vartheta_1(z|\tau)^2\,\eta(\tau)^{-6} \equiv \sum_{k,l} \widehat{c}(4k - l^2)\,e^{2\pi i(k\tau + lz)}\,, \tag{C.0.3}$$

where

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,;\, q = e^{2\pi i\tau}\,. \tag{C.0.4}$$

$$\vartheta_1(z|\tau) = \theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi(n+1/2)^2\tau + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}\,. \tag{C.0.5}$$

we can get $\widehat{c}(4n)$ by picking the coefficient of the $l = 0, k = n$ term in C.0.3. The choice $l = 0$ means that we only have to look at z independent terms coming from $\vartheta_1(z|\tau)^2$.

$$\vartheta_1(z|\tau)^2|_{z=0} = \sum_{m,n\in\mathbb{Z}} e^{i\pi\tau\left[(n+1/2)^2+(m+1/2)^2\right]+2\pi i(m+n+1)(z+\frac{1}{2})}|_{z=0} = q^{1/4}\sum_{n\in\mathbb{Z}} q^{n(n+1)} \tag{C.0.6}$$

Since we are looking for $\widehat{c}(4n)$ for $n = 1, 2, 3$, we collect terms upto q^3 from $-\vartheta_1(z|\tau)^2\eta(\tau)^{-6}|_{z=0}$.

$$\begin{aligned}
&-\vartheta_1(z|\tau)^2\eta(\tau)^{-6}|_{z=0}\\
&= -\left[q^{1/4}\sum_{n\in\mathbb{Z}} q^{n(n+1)}\right] \times \left[q^{-1/4}\prod_{n=1}^{\infty}(1-q^n)^{-6}\right]\\
&= -2\left[1+q^2+\cdots\right] \times \left[(1-q)^6(1-q^2)^6(1-q^3)^6\cdots\right]^{-1}\\
&= -2\left[1+q^2+\cdots\right] \times \left[(1-6q+15q^2-20q^3+\cdots)(1-6q^2+\cdots)(1-6q^3+\cdots)\cdots\right]^{-1}\\
&= -2\left[1+q^2+\cdots\right] \times \left[1-6q+9q^2+10q^3\cdots\right]^{-1}\\
&= -2\left[1+q^2+\cdots\right] \times \left[1+(6q-9q^2-10q^3)+(6q-9q^2-10q^3)^2+(6q-9q^2-10q^3)^3+\cdots\right]\\
&= -2\left[1+q^2+\cdots\right] \times \left[1+(6q-9q^2-10q^3)+(6q-9q^2-10q^3)^2+(6q-9q^2-10q^3)^3+\cdots\right]\\
&= -2\left[1+q^2+\cdots\right] \times \left[1+6q+27q^2+98q^3+\cdots\right]\\
&= -2-12q-56q^2-208q^3+\cdots
\end{aligned}$$

Comparing with C.0.2, we see

$$B_{14}(1,1,1,1) = 12,\ B_{14}(1,1,1,2) = 56,\ B_{14}(1,1,1,3) = 208, \tag{C.0.7}$$

APPENDIX D

DUALITY TRANSFORMATION

In [47], the counting of BPS states was done for a system consisting of N_1 KK monopoles associated with the 4-direction, $-N_2$ units of momentum along the 5-direction, N_3 D1-branes along the 5-direction, N_4 D5-branes along 5-6-7-8-9 directions and $-N_5$ units of momentum along the 4-direction.[a] Our goal will be to show that via a series of duality transformations this can be related our system. During this analysis we shall ignore all the signs (which can in principle be determined by following some specific sign convention, *e.g.* the one given in appendix A of [50]). This way we shall at most miss the signs of the charges carried by the final D -brane configurations. However, our analysis of the world-line theory of the D-brane system is independent of the signs of these charges as long as the signs are chosen to give a configuration that preserves 4 out of 32 supersymmetries.

Consider the following series of duality transformations:

- T-duality transformations along the 4 and 5 directions: This gives a configuration of N_1 NS-5-branes along 5-6-7-8-9 directions, N_2 fundamental strings along the 5-direction, N_3 D1-branes along the 4-direction, N_4 D5-branes along 4-6-7-8-9 directions and N_5 fundamental strings along the 4-direction.

- T-duality transformation along 8 and 9 directions: This gives a configuration of N_1 NS-5-branes along 5-6-7-8-9 directions, N_2 fundamental strings along the 5-direction, N_3 D3-branes along 4-8-9 directions, N_4 D3-branes along 4-6-7 directions and N_5 fundamental strings along the 4-direction.

- S-duality: This gives a configuration of N_1 D5-branes along 5-6-7-8-9 directions, N_2 D1-branes along the 5-direction, N_3 D3-branes along 4-8-9 directions, N_4 D3-branes along 4-6-7 directions and N_5 D1-branes along the 4-direction.

[a] The actual computation was done for $N_1 = 1$ but we shall consider a more general situation.

- T-duality along 5, 8 and 9 directions: This gives a configuration of N_1 D2-branes along 6-7 directions, N_2 D2-branes along 8-9 directions, N_3 D2-branes along 4-5 directions, N_4 D6-branes along 4-5-6-7-8-9 directions and N_5 D4-branes along the 4-5-8-9 directions.

- Cyclic permutation of 6-7→4-5→8-9→6-7: This gives a configuration of N_1 D2-branes along 4-5 directions, N_2 D2-branes along 6-7 directions, N_3 D2-branes along 8-9 directions, N_4 D6-branes along 4-5-6-7-8-9 directions and N_5 D4-branes along the 6-7-8-9 directions.

For $N_5 = 0$, this reduces to the configuration considered by us.

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