PERTURBATIVE ASPECTS OF STRING THEORY AND GAUGE/STRING DUALITY

By
ROJI PIUS
PHYS08201004003

Harish-Chandra Research Institute, Allahabad

A thesis submitted to the
Board of Studies in Physical Sciences
In partial fulfillment of requirements
for the Degree of
DOCTOR OF PHILOSOPHY
of
HOMI BHABHA NATIONAL INSTITUTE

May, 2015

Homi Bhabha National Institute[1]

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Mr. Roji Pius entitled "Perturbative Aspects of String Theory and Gauge/String Duality" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Chairman – Prof. Ashoke Sen Ashoke Sen	**Date:** February 25, 2016
Guide / Convener – Prof. Rajesh Gopakumar Rajesh Gopakumar	**Date:** February 25, 2016
Co-guide -	**Date:**
Examiner – Prof. Shiraz Minwalla Minwalla	**Date:** February 25, 2016
Member 1- Prof. B. Mukhopadhyaya B Mukhopadhyaya	**Date:** February 25, 2016
Member 2- Prof. Dileep Jatkar Jatkar	**Date:** February 25, 2016

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I/We hereby certify that I/we have read this thesis prepared under my/our direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: 25/2/16.

Place: Bangalore, India

Rajesh Gopakumar
Signature

Guide

[1] This page is to be included only for final submission after successful completion of viva voce.

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Roji Pius

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Roji Pius

List of Publications arising from the thesis

Journal

1. " Mass Renormalization in String Theory: Special States", Roji Pius, Arnab Rudra, Ashoke Sen, *Journal of High Energy Physics*, **2014**, *Vol. 1407(2014)058*.

2. " Mass Renormalization in String Theory: General States", Roji Pius, Arnab Rudra, Ashoke Sen, *Journal of High Energy Physics*, **2014**, *Vol. 1407(2014)062*.

3. "S-duality Improved Perturbation Theory in Compactified Type I/Heterotic String Theory.", Roji Pius, Ashoke Sen, *Journal of High Energy Physics*, **2014**, *Vol. 1406(2014)068*.

4. " Mass Renormalization in String Theory: General States", Rajesh Gopakumar, Roji Pius, *Journal of High Energy Physics*, **2013**, *Vol. 1303(2013)175*.

Roji Pius

DEDICATIONS

To my wife Molin and daughter Neha.

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my advisor Rajesh Gopakumar for the invaluable guidance which made this thesis possible. I am thankful for all wise advices and academic freedom that he provided. I have no words to thank Ashoke Sen. I am indebted to him for being such an incredible teacher, a great inspirer and a wonderful mentor. I would always be thankful to him for changing my way of understanding and doing theoretical physics. I thank Dileep Jatkar for the helpful advices both on academic and non-academic matters. I wish to thank all members in string theory group at Harish-Chandra Research Institute for many insightful discussions. It is my pleasure to thank my collaborator Arnab Rudra for the invaluable and enjoyable discussions. His contributions are essential for shaping this thesis into this form. I would like to thank all academic, non-academic and visiting members in Harish-Chandra Research Institute for making these years fruitful both academically and non-academically. I am grateful to my parents and brother for their unconditional support in pursuing research. Finally I thank my wife Molin for greatly enriching my life and for the love and support she gave me in these years.

CONTENTS

SYNOPSIS

0.1 Introduction

String theory is an attempt to provide a framework to unify everything we know about nature, including all particles and forces between them, in a consistent quantum theory. But surprising devolepments in string theory from the time of its inception suggests that it should be thought of as a unified frame work in a more broader sense which has the potential to unfold mysterious unity of many seemingly different arenas in theoretical physics and mathematics. A prime example for such a deep relation uncovered by this frame work is the equivalence between quantum field theories and quantum theory for gravity known as gauge/gravity duality.

Main goal of this thesis is an improved understanding of string theory and gauge/gravity duality by looking into the perturbative aspects of it. Below we list the major results of the studies that will be presented in this thesis.

- We found a consistent prescription for computing the renormalised masses and S-matrix elements of string states whose masses are not protected by any symmetry [1], [2].
- We determined the stability of massive non-BPS SO(32) spinor in SO(32) heterotic string theory compactified on circle in the entire moduli space which is parametrised by the radius of compactification and string coupling [3].
- As a modest step towards understanding the inner working of gauge/string duality we studied a simple example of gauge/string duality put forward in [67]. We found the matching of leading order terms in a class of correlators of this conjectured duality between topological A-model string theory on $\mathcal{CP}^1$ and gaussian matrix models. We also found a nice interpretation for the mismatch in the sub-leading terms [4].

In the following sections we give a brief description of these results.

0.2 Mass Renormalisation in String Theory

String perturbation theory gives a well defined procedure for computing S-matrix elements only for external states whose masses are protected by symmetry to any order in perturbation theory which is free from ultraviolet divergences. This is because string perturbation theory based on world sheet conformal invariance require external states to satisfy the tree level on-shell condition. But generic states in string theory receive mass renormalisation. It is possible to address this apparent conflict directly if we generalise Polyakov prescription for computing string amplitudes for off-shell vertex operators. Off-shell amplitudes computed this way are the analogue of off-shell Green's functions in quantum field theory. So once we have off-shell amplitudes we can use them to compute physical masses and S-matrix elements following LSZ prescription. But off-shell string amplitudes defined this way are ambiguous because they depend on the local coordinates on the world sheet. However following the spirit of gauge theories we showed that physical quantities defined using off-shell string amplitudes like S-matrix elements and physical masses do not depend on the choice of local coordinates [1], [2]. We also showed that when a state appears as single particle intermediate state in the S-matrix of massless / BPS states, the mass renormalisation obtained from our prescription agrees with that obtained by factorising S-matrix of massless / BPS states.

String perturbation theory is a degenerate perturbation theory with high amount of degeneracy. Because of this degeneracy renormalization leads to the mixing of physical states with unphysical states. To understand the situation we studied a degenerate quantum fields theory example. We chose to study degenerate abelian Higgs model because it is a simple and at the same time complicated enough setup to demystify the secrets of string perturbation theory. We found a prescription for perturbatively computing the loop corrected physical states in degenerate quantum field theories and successfully lifted it to string theory. We also argued that only the squares of renormalized physical masses appear as the locations of the poles of the S-matrix of other physical states [2].

0.3 S-duality Improved String Perturbation Theory

Our current understanding of string theory is based mostly on perturbation expansion in the string coupling. Furthermore this perturbation expansion is believed to be an asymptotic

expansion. For this reason one might worry that our ability to compute anything in string theory may be limited to very narrow corners of the full string theory landscape - regions in which the theory admits a description as a very weakly coupled string/M/F-theory. Ref. [44] suggested making use of string duality and suitable interpolation formula to translate the weak coupling results in string theory to approximate results for physical quantities over the entire range of string coupling constant. As a specific example, the mass of the stable non-BPS particle in ten dimensional type I / SO(32) heterotic string theory was considered. Using a suitable formula that interpolates between the result for this mass in weakly coupled SO(32) heterotic string theory and weakly coupled type I string theory, an approximate formula for the mass of this state was derived over the entire range of string coupling of SO(32) heterotic / type I string theory compactified on a circle. Now if we compactify SO(32) heterotic string theory on a circle then that will introduce BPS winding states into the theory whose total charge is the same as that of the charge carried by the non-BPS state under study. Thus the latter can decay into the former if the mass of the non-BPS state is larger than the sum of the masses of the BPS states to which it could possibly decay. If we do not switch on any Wilson line so that the SO(32) gauge group is unbroken then the moduli space is two dimensional, parametrized by the string coupling and the radius of compactification. We used perturbative results in the compactified string theory and a suitable generalization of the interpolation formula used in [44] to derive expression for the mass of the non-BPS state in the full two dimensional moduli space. With the help of the approximate formula for the mass we determined the part of the region of the two dimensional moduli space in which the non-BPS state is unstable.

0.4 Correlators in the Simplest Gauge-String Duality

Although different examples of gauge/string duality are well studied by now, the underlying mechanism is still not well understood. A simple example of how the Feynman diagrams for an n-point gauge correlator glue up into an n-point string scattering amplitude in a dual space-time can potentially provide a lot of insight into how gauge-string duality works. A candidate proposal for the "simplest gauge-string duality" was put forward in [67]. It relates the Gaussian one matrix integral in a large N 'tHooft limit to the A-model topological string theory on $\mathcal{CP}^1$ [68], [69]. This conjectured duality was constructed using Gopakumar's prescription for systematically constructing the string theory dual of free field theory [76], [77], [79] with the modification suggested by Razamat for the case of matrix models [80], [81]. If this duality is

correct then gauge invariant correlators of the single trace operators can be expected to be related to physical vertex operator scattering amplitudes in the dual topological string theory. We computed a class of n-point correlators for single trace operators in Gaussian matrix model and found leading order agreement with the corresponding correlators in the dual topological string theory. But the sub-leading terms showed mismatch. However we found that correlators of topological string theory can be expressed as combination of the class of matrix model correlators in which we are interested. Interestingly using this relation the mismatch can be interpreted as due to the contact terms corresponds to the collision of physical vertex operators present in the topological string correlators which is absent in the Gaussian matrix model.

List of Figures

CHAPTER 1

Introduction

Physics is a collective effort to explain and predict wide range of physical phenomena using minimal set of fundamental laws. The tenets of physics relevant to sub-atomic scale are quantum mechanics and special theory of relativity. Quantum mechanics describes the nature of physical objects and special theory of relativity explains the nature of spacetime where these objects exist. In order to explain phenomena relevant to sub-atomic scale we require a frame-work which can combine the principles of quantum mechanics and special theory of relativity. Quantum field theory provides such a frame work to combine the principles of quantum mechanics and special theory of relativity. Within the framework of quantum field theory we could successfully develop and unify the theories of electromagnetic interaction, weak interaction and strong interaction. This unified theory is known as the standard model of particle physics. Phenomena occurring in the cosmological scale require deep understanding of gravity and is based on the principles of general relativity. Unfortunately quantum field theory could not facilitate the consistent inclusion of general relativity in a straight forward manner.

String theory attempts to provide a fundamental description of nature by providing a framework to unify everything we know about nature, including all particles and forces between them, in a consistent quantum theory. Basic objects in string theory are one dimensional extended objects known as strings. Harmonics of these vibrating strings correspond to elemen-

tary particles with different masses and quantum numbers. Though string theory has passed several non-trivial mathematical consistency checks, so far there is no experimental evidence for string theory. This can be attributed to the lack of experimental facilities which can detect quantum effects of gravity. Such a situation is common for any theory of quantum gravity.

Interestingly string theory can play crucial role in explaining phenomena which are experimentally verifiable using current experimental facilities. Consider the phenomena of quantum phase transition. Quantum phase transitions are believed to be important in describing superconducting-insulator transitions in thin metallic films. In some cases near critical point quantum phase transition can be described using strongly interacting field theories. In order to to explain quantum phase transition at critical point we need to solve strongly interacting field theories. But traditional methods in quantum field theory can not solve strongly interacting quantum field theories. Surprisingly string theory framework unfolded the magical equivalence between certain conformal field theories and quantum theory of gravity in anti-de Sitter space-time known as AdS/CFT or gauge/gravity correspondence. Gauge/gravity correspondence is a unique approach to strongly coupled field theories. This correspondence states that every sensible quantities in strongly interacting quantum field theory can be calculated by doing a perturbative computation in string theory. In this sense a detailed understanding of perturbative string theory is crucial in understanding real world phenomena.

Our current understanding of string theory is based on three aspects: perturbative string theory, non-perturbative string dualities and gauge/gravity correspondence. This thesis delve into all these three aspects from the perturbative side with the aim of an improved understanding of string theory. Below we give brief introduction to the issues addressed and summary of the results reported in this thesis.

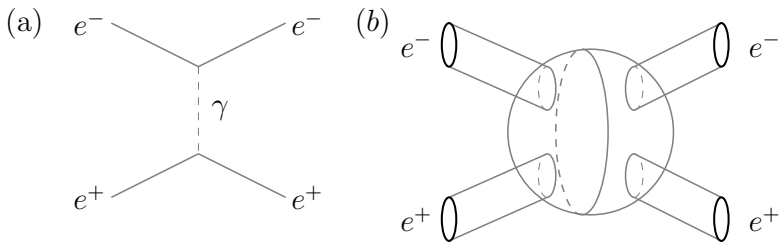

Figure 1.1: Scattering of electron and positron from the point of view of (a) quantum field theory and (b) string theory.

1.1 Mass Renormalization in String Theory

Primary quantity of interest in string theory are string amplitudes which are the proposed scattering amplitudes in string theory. Scattering amplitudes give the probability of producing different final products at the end of scattering of different particles. For example consider scattering of an electron and a positron. A Feynman diagram corresponding to this process is shown in figure 1.1.a. According to string theory electron and positron are specific vibration of strings. So from the point of view of string theory Feynman diagram looks as figure1.1.b. In quantum field theory we compute the scattering amplitude corresponding to the scattering of electron and positron by evaluating the Feynman diagrams using Feynman rules or Feynman prescription. In string theory we compute this scattering amplitude by evaluating string diagrams using string theory analogue of Feynman prescription known as Polyakov prescription.

String theory contains infinite number of elementary particles because string has infinite number of harmonics. But Polyakov prescription computes scattering amplitudes directly only for those particles which do not receive mass renormalization. Examples of such states are massless gauge particles and BPS states. But a generic state in string theory receives correction to its mass from quantum effects. Interestingly for many states this is not a problem since they appear as single particle intermediate states in the S-matrix of massless and/or BPS external states. Renormalized masses and S-matrix elements of those states can be found by examining the locations and residues of the poles of the S-matrix of massless and/or BPS states. For

this reason direct computation of the S-matrix of massive string states has not received much attention. However this does not always work, *e.g.* if the massive state under consideration carries a conserved charge that is not carried by any of the massless or BPS states, then the former cannot appear as a single particle intermediate state in the S-matrix of the latter. For this reason it seems important to find a more direct approach to computing the mass renormalization and S-matrix elements of massive string states.

There are two related but independent problems which arise in the computation of mass renormalization in string theory. First we have to define the analog of the off-shell Green's function in string theory. This requires giving up the conformal invariance of vertex operators and hence is ambiguous. Second is related to the fact that string perturbation theory is a degenerate perturbation theory with high amount of degeneracy. Because of this degeneracy renormalization leads to the mixing of physical states with unphysical states and hence the definition of the physical state needs to be modified carefully. In chapter 2 we will address this issue by restricting ourselves to a special class of states which under renormalization do not mix with unphysical states due to some global symmetries and will show that renormalized mass and S-matrix elements of these special states are unambiguous [1]. In chapter 3 we will generalize this procedure to general states in string theory [2]. We will give systematic prescription for perturbatively finding the modified physical states and show that renormalized mass and S-matrix elements computed using the modified physical states are unambiguous. Concise summary of background that may be needed for a clear understanding of these two chapters are given in appendix A and appendix B with the aim of making the discussion as self contained as possible.

1.2 S-duality Improved String Perturbation Theory

Our current understanding of string theory is based mostly on perturbation expansion in the string coupling. Furthermore this perturbation expansion is believed to be an asymptotic ex-

pansion. For this reason one might worry that our ability to compute anything in string theory may be limited to very narrow corners of the full string theory landscape - regions in which the theory admits a description as a very weakly coupled string/M/F-theory. Ref. [44] suggested making use of S-duality of string theory and suitable interpolation formula to translate the weak coupling results in string theory to approximate results for physical quantities over the entire range of string coupling constant. As a specific example, the mass of the stable non-BPS particle in ten dimensional type I / SO(32) heterotic string theory [54] was considered. Using a suitable formula that interpolates between the results for this mass in weakly coupled SO(32) heterotic string theory and weakly coupled type I string theory, an approximate formula for the mass of this state was derived over the entire range of string coupling of SO(32) heterotic / type I string theory compactified on a circle.

Consider the compactification of SO(32) heterotic string theory on a circle. Compactification will introduce BPS winding states into the theory whose total charge is the same as that of the charge carried by the non-BPS state under study. Thus the latter can decay into the former if the mass of the non-BPS state is larger than the sum of the masses of the BPS states to which it could possibly decay. If we do not switch on any Wilson line so that the SO(32) gauge group is unbroken then the moduli space is two dimensional, parametrized by the string coupling and the radius of compactification. In chapter 4 we will address this problem of determining the stability of non-BPS particle using this interpolation technique which exploit the S-duality between type I and SO(32) heterotic string theory [3]. We will use perturbative results in the compactified string theory and a suitable generalization of the interpolation formula used in [44] to derive expression for the mass of the non-BPS state in the full two dimensional moduli space. With the help of the approximate formula for the mass we will determine the part of the region of the two dimensional moduli space in which the non-BPS state is unstable.

1.3 Simplest Gauge-String Duality

Gauge/string duality is a statement of equivalence between certain quantum field theories and quantum theory of gravity in certain spaces. Attractive feature of this equivalence is that it gives a non-perturbative definition of quantum theory of gravity in certain spaces. Also it provides a unique method for solving strongly interacting quantum field theory. Although different examples of gauge/string duality are well studied by now, the underlying mechanism is still not well understood. A simple example of how the Feynman diagrams for an n-point gauge correlator glue up into an n-point string scattering amplitude in a dual space-time can potentially provide a lot of insight into how gauge-string duality works.

A candidate proposal for the "simplest gauge-string duality" was put forward in [67]. It relates the Gaussian one matrix integral in a large N 'tHooft limit to the A-model topological string theory on $\mathcal{CP}^1$ [68], [69]. This conjectured duality was constructed using Gopakumar's prescription for systematically constructing the string theory dual of free field theory [76], [77], [79] with the modification suggested by Razamat for the case of matrix models [80], [81]. If this duality is correct then gauge invariant correlators of the single trace operators can be expected to be related to physical vertex operator scattering amplitudes in the dual topological string theory. In chapter 5 we will compute a class of n-point correlators for single trace operators in Gaussian matrix model and compare them with the corresponding correlators in the dual topological string theory [4]. Brief discussion on gauge/string duality and Gopakumar prescription is given in appendix C. Detailed calculations are given in appendix D.

CHAPTER 2

Mass Renormalization in String Theory: Special States

2.1 Introduction

String theory unlike conventional quantum field theory is not based on a Lagrangian. Keystone of string theory is a prescription for directly computing string amplitudes, which are the proposed S-matrix elements of string theory, known as Polyakov prescription.

Path integral formulation of string theory requires summing over all possible world-sheets traced by the relativistic strings in space-time with each surface weighted by a unimodular complex number whose phase is the area of the corresponding world-sheet. Polyakov in his seminal papers [11, 12] showed that this summation over random two dimensional surfaces reduces to two dimensional exactly solvable conformal field theory on Riemann surfaces. This connection was established by introducing a metric on each world-sheet which matches with the metric on world-sheet induced from space-time only when string propagates according to the laws of classical mechanics. Net effect of this is to introduce an unphysical degree of freedom, Liouville mode, into the problem which disappears from the world-sheet action due to it's Weyl invariance. Demanding Weyl invariance even at the quantum level makes sure that

Liouville mode will not reappear in the action due to quantum correction. This demand impose stringent condition on the dimension and geometry of the background space-time through which string propagates.

In conformal field theory language an asymptotic state of string can be mapped to an operator known as vertex operator constructed using fields in conformal field theory. Using these vertex operators $V_{a_i},\ i = 1, ..., n$ we can compute the n-point string amplitude corresponding to scattering of n-asymptotic states having quantum numbers $a_1, ..a_n$ as follows,

$$A^{(n)}_{a_1,...,a_n} = \sum_g \int d(moduli) \ \langle \ (Ghost\ insertions) \ \prod_{i=1}^{n} V_{a_i} \ \rangle_{\Sigma_g} \tag{2.1.1}$$

where subscript Σ_g indicates that we need to compute the correlator of conformal field theory on genus g Riemann surface Σ_g, the integration is over all distinct Riemann surfaces having genus g and the summation is over the genus of world-sheet. String amplitudes defined in this way are independent of Liouville mode only if we demand that the vertex operators corresponding to physical asymptotic states are conformal $(0, 0)$ primaries. In BRST formalism this is equivalent to the demand that vertex operator corresponding to physical states should be annihilated by BRST charge. Space time interpretation of this demand is that physical states should satisfy classical mass-shell condition. Thus Polyakov prescription gives a well defined procedure for computing S-matrix elements involving BPS or a class of massless external states -whose masses are protected from renormalization - to any order in perturbation theory. Attractive feature of string theory is that this perturbation expansion is free from ultraviolet divergences [5–10].

However this procedure for computing S-matrix elements breaks down for generic states in string theory. This is due to the fact that for general states loop corrections generate (ultraviolet finite) mass renormalizations, and hence in order to compute the physical S-matrix elements we have to shift the external momenta to their renormalized on-shell values. On the

other hand string perturbation theory, which is based on world-sheet conformal invariance, requires the vertex operators representing external states to carry momenta that satisfy the tree level on-shell condition.

For many states this is not a problem since they appear as single particle intermediate states in the S-matrix of massless and/or BPS external states and hence their renormalized masses and S-matrix elements can be found by examining the locations and residues of the poles of the S-matrix of massless and/or BPS states. For this reason direct computation of the S-matrix of massive string states has not received much attention. However this does not always work, e.g. if the massive state under consideration carries a conserved charged that is not carried by any of the massless or BPS states, then the former cannot appear as a single particle intermediate state in the S-matrix of the latter. For this reason it seems important to find a more direct approach to computing the mass renormalization and S-matrix elements of massive string states.

There are two related but independent problems which arise in the computation of mass renormalization in string theory. First we have to define the analog of the off-shell Green's function in string theory. This requires giving up the conformal invariance of vertex operators and hence is ambiguous. Second is related to the fact that string perturbation theory is a degenerate perturbation theory with high amount of degeneracy. Because of this degeneracy renormalization leads to the mixing of physical states with unphysical states and hence the definition of the physical state needs to be modified carefully. By choosing a special class of states we avoid the second problem. These special class of states do not mix with unphysical states due to some global symmetries. However we still need to deal with the first problem, i.e. the ambiguity in the definition of the off-shell Greens function. we will show that although the off-shell Greens functions are ambiguous, the renormalized mass and S-matrix elements of special states computed from them are free from these ambiguities. For general states we need to address both problems. We will give systematic prescription for perturbatively finding the modified physical

states and show that renormalized mass and S-matrix elements computed using the modified physical states are unambiguous.

The chapter is organised as follows. In §2.2 we make precise the problem associated with mass renormalization in string theory. In §2.3 we introduce the special class of states for which we address the problem in this chapter. In §2.4 we describe how to compute the renormalized mass of these special states and also show that this renormalized mass is free from any ambiguity. In §2.5 we show how to compute the S-matrix elements of these special states, and demonstrate that they are also free from ambiguities. We end in §2.6 with a discussion of our results, extensions to heterotic and superstring theories and possible generalizations. Concise summary of string perturbation theory and closed bosonic string field theory is given in appendix A and appendix B. This chapter is based on [1].

Various other approaches to studying mass renormalization in string theory can be found in [13–26]

2.2 The question

Consider a string theory amplitude with n-external states representing particles carrying momenta $k_1, \cdots k_n$ and other discrete quantum numbers $a_1, \cdots a_n$ with tree level masses $m_{a_1}, \cdots m_{a_n}$. Then the momenta k_i satisfy the *tree level* on-shell condition $k_i^2 = -m_{a_i}^2$, this is needed to ensure the BRST invariance of the vertex operators in the world sheet theory. The world-sheet computation, involving correlation functions of these vertex operators integrated over the moduli spaces of (punctured) Riemann surfaces, yields the result for what in a quantum field theory

can be called 'truncated Green's function on classical mass shell'.[1]

$$R^{(n)}_{a_1\cdots a_n}(k_1,\cdots k_n) \equiv \lim_{k_i^2\to -m^2_{a_i}} F^{(n)}_{a_1\cdots a_n}(k_1,\cdots k_n)\,,$$
$$F^{(n)}_{a_1\cdots a_n}(k_1,\cdots k_n) \equiv G^{(n)}_{a_1\cdots a_n}(k_1,\cdots k_n)\prod_{i=1}^{n}(k_i^2+m^2_{a_i})\,, \tag{2.2.2}$$

where $G^{(n)}_{a_1\cdots a_n}(k_1,\cdots k_n)$ correspond to the momentum space Green's function in the quantum field theory. This is similar to but not the same as the combination that appears in the expression for the S-matrix in a quantum field theory

$$S^{(n)}_{a_1\cdots a_n}(k_1,\cdots k_n) = \lim_{k_i^2\to -m^2_{a_i,p}} G^{(n)}_{a_1\cdots a_n}(k_1,\cdots k_n)\prod_{i=1}^{n}\{Z^{-1/2}(k_i,a_i)(k_i^2+m^2_{a_i,p})\}\,, \tag{2.2.3}$$

where $m_{a_i,p}$ is the physical mass of the i-th particle, defined as the location of the pole as a function of $-k^2$ in the untruncated two point Green's function $G^{(2)}$ and $Z(k_i,a_i)$'s are the residues at these poles.

For simplicity we have ignored the mixing between different states under wave-function renormalization in writing down (2.2.3), but we shall discuss the general case now. If we consider the set of all fields whose tree level masses are all equal to m then the two point Green's function $G^{(2)}_{ab}(k,k')$ for all these fields is described by the matrix

$$G^{(2)}_{ab}(k,k') = (2\pi)^{D+1}\delta^{(D+1)}(k+k')\,Z^{1/2}(k)_{ac}(k^2+M_p^2)^{-1}_{cd}(Z^{1/2}(-k))^T_{db}\,, \tag{2.2.4}$$

where M_p^2 is the mass^2 matrix and $Z^{1/2}(k)$ is the wave-function renormalization matrix, the latter being free from poles near $k^2+m^2\simeq 0$. The sum over c,d are restricted to states which have the same tree level mass m as the states labelled by the indices a,b. $D+1$ is the total number of non-compact space-time dimensions. We can diagonalize M_p^2 and absorb

[1] We have absored all factors of $\mathbf{i}\equiv\sqrt{-1}$ and minus signs into the definition of $G^{(n)}$ and $R^{(n)}$.

the diagonalizing matrices into the wave-function renormalization factor $Z^{1/2}(k)$ to express M_p^2 as a diagonal matrix. These eigenvalues, which we shall denote by $m_{a,p}^2$, are the squares of the physical masses. Taking into account the non-diagonal nature of the wave-function renormalization factor Z, (2.2.3) is modified to

$$S^{(n)}_{a_1\cdots a_n}(k_1,\cdots k_n) = \lim_{k_i^2\to -m^2_{a_i,p}} G^{(n)}_{b_1\cdots b_n}(k_1,\cdots k_n)\prod_{i=1}^n \{Z_i^{-1/2}(k_i)_{a_i,b_i}(k_i^2+m^2_{a_i,p})\}\,, \tag{2.2.5}$$

where $Z_i^{-1/2}$ is the inverse of the matrix $Z^{1/2}$ introduced in (2.2.4) for the i-th external state. In this expression we can interpret the sum over b_i's as sum over all fields in the theory if we define $Z^{1/2}(k)_{ab}$ and $Z^{-1/2}(k)_{ab}$ to be zero when a, b label fields with different classical mass.

At tree level $Z = 1$, $M_p^2 = m^2\, I$ and hence the $R^{(n)}$ defined in (2.2.2) and $S^{(n)}$ defined in (2.2.5) agree. In general however $R^{(n)}$ and $S^{(n)}$ are different. While $S^{(n)}$ defined in (2.2.5) is the physically relevant quantity, string theory directly computes $R^{(n)}$ defined in 2.2.2. This is a serious trouble for states receiving mass renormalization. Say external state with quantum number a_i and tree level mass m_{a_i} receives mass renormalization. Then beyond one loop radiative correction introduces series of $\frac{1}{k_i^2+m^2_{a_i}}$ with $k_i^2+m^2_{a_i}=0$ in $R^{(n)}$ which makes the string amplitude ill-defined. Thus the question arises: how can we use string theory to compute on-shell S-matrix elements beyond tree level? At a more basic level: how can we use string theory to calculate the physical mass $m_{a_i,p}$ of the i-th particle?

When the external strings represent massless gauge particles, the situation improves dramatically. In this case gauge symmetry prevents mass renormalization and hence we have $m^2_{a_i,p} = m^2_{a_i} = 0$. As a result $R^{(n)}$ and $S^{(n)}$ differ only by the wave-function renormalization factor Z. This can be fixed by using analyticity property of the S-matrix, *e.g.* the S-matrix should factorize into the product of lower point S-matrices when the external momenta are such that some internal line could become on-shell. Thus string world-sheet computation can

be used to compute the S-matrix of massless external states.

Now typically in string theories many massive string states appear as one particle intermediate states in the scattering of massless states and as a result the S-matrix of the massless states can have poles when the square of appropriate combination of external momenta approaches the squared mass of a massive state. The location of this pole gives information about the mass of the massive state while the residue at this pole contains information about the S-matrix involving massive external states. However this procedure does not always work. Some string theories contain massive states which do not appear as one particle intermediate states in the scattering of massless particles. We shall now describe some examples of such situations.

- Consider bosonic string theory compactifed on a circle S^1. In this case a state carrying a winding number (and/or momentum) along S^1 cannot be produced as single particle intermediate state in the scattering of massless states which do not carry any momentum and winding charge.[2]

- Another notable example is SO(32) heterotic string theory which contains massive states belonging to the spinor representation of SO(32). They cannot appear as single particle intermediate states in the scattering of massless external states which are all in the adjoint or singlet representation of SO(32). Thus the S-matrix element involving these particles cannot be computed by examining any massless S-matrix element near its poles.

In order to deal with these cases we shall try to develop a different strategy – compute the mass renormalization directly. We shall focus on a special class of states – which we shall call special states – for which the analysis simplifies. In the following section we shall be describing these special states and their relevance to the problems mentioned above.

[2]Such states could still appear in pairs in the intermediate channel, producing a cut in the S-matrix of massless states, and by examining where the cut begins, we can find the mass of the intermediate state. But it is much harder to identify cuts than poles in the S-matrix, and we shall not explore this option.

2.3 Special States

Let us suppose that we are dealing with a string theory with $D+1$ non-compact dimensions, with $SO(D,1)$ Lorentz invariance. Then while discussing the mass renormalization of a massive state we can go to the rest frame of the particle so that the spatial component $\vec{k}$ of the momentum vanishes. In this frame we consider physical states described by vertex operators of the form

$$c\,\bar{c}\,e^{\pm \mathbf{i} k_0 X^0}\,V \tag{2.3.6}$$

where $c, \bar{c}$ are ghost fields and V is a dimension (h,h) primary made of the compact coordinates and the oscillators of the non-compact spatial coordinates. The on-shell condition on k_0 is

$$k^0 = m, \quad m^2 = 4(h-1)\,, \tag{2.3.7}$$

in the $\alpha' = 1$ units. The operators V will form a finite dimensional representation of the $SO(D)$ little group. If the world-sheet theory has additional global symmetry group G associated with the compact directions then the operators V will also belong to finite dimensional representation of this symmetry group.

Now consider all operators of the form $e^{\pm \mathbf{i} k_0 X^0}\mathcal{O}$ where $\mathcal{O}$'s are dimension $(h-1,h-1)$ operators made of the ghost fields, compact coordinates and oscillators of X^0 and the non-compact spatial coordinates. They can be organised into irreducible representations of $SO(D)\times G$. Among them the operators which are not of the form (2.3.6) will be called unphysical vertex operators at mass level m.[3] We shall define *special vertex operators* to be a set of vertex operators of the form given in (2.3.6) belonging to those irreducible representations of the symmetry group $SO(D)\times G$ such that there are no unphysical vertex operator at mass level m transforming in these representations. Put another way, if the unphysical vertex operators at mass level m

[3]Technically the unphysical operators described here can the divided into two kinds, BRST trivial ones and states which are not invariant under BRST transformation. The former are called pure gauge and the latter are called unphysical. We shall not need to make this distinction, and call all such states unphysical.

transform in certain irreducible representations $R_1, R_2, \cdots$ then the special vertex operators are those physical states which transform in representations other than $R_1, R_2, \cdots$. In this case the two point function of any special vertex operator and an unphysical operator on any Riemann surface will vanish.

We shall now give some examples of special vertex operators.

- Consider bosonic string theory in $25+1$ dimensions. We consider vertex operators of the form (2.3.6) with V given by

$$S\left[\partial X^{i_1}\cdots\partial X^{i_n}\bar{\partial}X^{j_1}\cdots\bar{\partial}X^{j_n}\right]\,. \tag{2.3.8}$$

  where X^i for $1 \le i \le 25$ are the spatial coordinates and S denotes the operation of taking the symmetric traceless part of the product. This belongs to a rank $2n$ symmetric traceless representation of $SO(25)$ – also known as the leading Regge trajectory. In order to get an unphyical state at the same mass level we have to replace some of the ∂X^i or $\bar{\partial}X^j$ by ghost or X^0 excitations and/or replace the product of some of the ∂X^{i}'s and/or $\bar{\partial}X^{i}$'s by higher derivatives of X^{i}'s. This clearly reduces the rank of the tensor and hence the unphysical states cannot belong to the rank $2n$ symmetric tensor representation of $SO(25)$. Thus vertex operators of the form (2.3.8) are special.

- Consider bosonic string theory compactified on a circle. Let Y be the coordinate along the compact direction, and Y_L and Y_R be its left and right-moving components on the world-sheet. We now consider the vertex operator of the form (2.3.6) with

$$V = e^{\pm\mathbf{i}(n/R-wR)Y_L/2}e^{\pm\mathbf{i}(n/R+wR)Y_R/2}\,S\left[\bar{\partial}X^{i_1}\cdots\bar{\partial}X^{i_p}\partial X^{j_1}\cdots\partial X^{j_q}\right]\,,\quad p-q=nw\,, \tag{2.3.9}$$

  where X^i for $i = 1, \cdots 24$ denote the non-compact directions and S stands for the pro-

jection into rank $p+q$ symmetric traceless representation of $SO(24)$. Following the same argument as before it follows that there are no unphysical vertex operators at this mass level carrying n units of momentum, w units of winding and belonging to the rank $p+q$ symmetric traceless representation of $SO(24)$. Thus these are also special states according to the definitions given above.

- Finally we note that the stable non-BPS states of SO(32) heterotic string theory- which correspond to the lowest mass states in the spinor representation of SO(32)-are also special states. Besides the ghost fields and the $e^{\pm ik_0 X^0}$ factor, the left-moving part of the vertex operator is given by the SO(32) spin field of dimension 2, and has no further oscillator excitations. Level matching requires that the right-moving part of the Neveu-Schwarz (NS) sector vertex operator corresponds to level 3/2 excitations above the NS-sector ground state. We can take this to be $\psi^i\psi^j\psi^k$ where ψ^i for $1 \le i \le 9$ are the world-sheet superpartners of the 9 non-compact bosonic coordinates. This belongs to the totally anti-symmetric rank 3 tensor representation **84** of SO(9). It is easy to see that any other unphysical state at this mass level, obtained by replacing some of the ψ^i's by ghost or ψ^0 oscillators or derivatives of ψ^i or bosonic coordinates cannot belong to the **84** representation of $SO(9)$. Thus these states are special states.

The reader might have noticed that there is a close relationship between special states which are prevented from mixing with the unphysical states due to global symmetry on the world-sheet and the states which cannot appear as poles in the scattering of massless states due to conserved charges. Indeed the lowest mass states in each of the examples of the latter kind given earlier also correspond to special states. On general grounds one expects that in every charge sector we can construct a set of special states by saturating the required oscillator levels by (anti-)symmetric products of bosonic (fermionic) fields associated with the non-compact coordinates. For this reason we shall focus on computation of physical mass and S-matrix elements involving these special states and massless states, since the renormalized mass and S-matrix elements of all other states can be obtained from the locations of the poles of the

S-matrix involving the special states and massless states.

2.4 Mass renormalization

If we work in the rest frame, then the off-shell continuation of a special vertex operator would correspond to deforming k^0 away from m. This keeps the vertex operator primary but it no longer has dimension 0. Thus in order to define the correlation functions of such vertex operators on a Riemann surface we need to make a choice of local coordinate around every point on the Riemann surface. If z denotes some reference cordinate system on the Riemann surface then the local coordinate w around some point $z = z_0$ is described by some function $f(w; z_0)$ that maps the w plane to the z plane around $z = z_0$. We take $f(0; z_0) = z_0$ and $f(w; z_0)$ to be analytic around $w = 0$. Thus f depends on both, the choice of the reference coordinate system z and the choice of the local coordinate system w. The vertex operator at z_0 is inserted using the local coordinate w, which corresponds to inserting its conformal transformation by the function $f(w; z_0)$ in the z coordinate system [27]. Thus if the off-shell vertex operator is a primary operator of dimension (δ, δ) then we multiply it by $(f'(0; z_0))^\delta (\overline{f'(0; z_0)})^\delta$ while inserting it into the correlation function in the z coordinate system, f' being the derivative of the function $f(w; z_0)$ with respcet to w. For more general vertex operator representing general off-shell string state the same procedure would work although the conformal transform of the vertex operator will be more complicated.[4]. This definition makes the correlation function invariant under a change of reference coordinate system z, but dependent on the choice of local coordinates w, i.e. the function $f(w; z_0)$. As a result, if we define off-shell strng amplitudes by integrating such correlation functions over the moduli space, the result will depend on the choice of local coordinates.

In a sense the situation in string theory is not very different from that in a gauge theory. In gauge theory for computing the mass renormalizaton of a massive charged particle we have

[4]For related approaches to defining off-shell amplitudes in string theory, see [28–31]

to first compute the off-shell propagator carrying momentum $k = (k^0, \vec{0})$ and then look for its poles in the k^0 plane. The off-shell propagator is not gauge invariant; however the location of its pole in the k^0 plane is gauge invariant and leads to a gauge invariant definition of the renormalized mass. Thus a possible strategy in string theory will be to consider off-shell propagator that depends on the choice of local coordinates, look for its poles in the k^0 plane and prove that the location of the pole is independent of the choice of local coordinates even though the propagator itself is not gauge invariant. If we had an underlying string field theory then this analysis will be parallel to that in an ordinary gauge theory. This can be done in principle for bosonic string theory where a complete closed string field theory is known [32].[5]

At present there is no known string field theory for closed heterotic and superstring theories except a closed heterotic string field theory at tree level [33]. Nevertheless we can try to extract the relevant features of the off-shell string theory amplitudes from a bosonic string field theory and then develop a general proof of indpendence of the renormalized mass of the choice of local coordinates that does not require the existence of an underlying string field theory. The essential features seem to be the following:

1. Bosonic string field theory gives a triangulation of the punctured Riemann surface equipped with local coordinate system at each puncture. Using this local coordinate system we can define off-shell amplitudes.

2. Near boundaries of the moduli space where a Riemann surface of genus n degenerates into two Riemann surfaces of genus n_1 and $n_2 = n - n_1$ connected by a long handle, the choice of the local coordinates of the original Riemann surface matches with the choice of the local coordinates of the lower genus surfaces. The precise meaning on 'near boundaries of the moduli space' will be made clear later (see item 6 in the discussion in §2.4.2 (above eq.(3.4.2)).

[5] Due to the presence of tachyon, the mass renormalization in this case is infrared dvergent.

For an off-shell amplitude induced from string field theory the above requirements are automatically satisfied, but even in the absence of string field theory we could try to choose local coordinates at the punctures consistent with the above criteria. Indeed even before the construction of fully covariant closed string field theory, such choices of local coordinates were explored (see *e.g.* [34]). Given such a choice of local coordinates, we can define off-shell two point functions in string theory and define the mass to be the location of the pole in the k^0 plane. The important point is to show that this definition is independent of the choice of local coordinates.

From now on we shall restrict our analysis to bosonic string theory. We discuss possible generalizations to superstring and heterotic string theories in §2.6.

2.4.1 Analysis of poles of off-shell two point function

Let us denote the set of all the special vertex operators by $c\bar{c}V_i\, e^{\mathbf{i}k_0X^0}$ and the corresponding states as

$$c_1\bar{c}_1|V_i\rangle \otimes |k^0, \vec{k}=0\rangle\,. \tag{2.4.1}$$

In the zero mode sector of non-compact bosons labelled by $(k^0, \vec{k})$, the states satisfy the usual δ-function normalization. The operaors V_i will be chosen so that in the rest of the matter-ghost CFT, they satisfy the orthonormality relation

$$\langle V_i|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1|V_j\rangle = \delta_{ij}\,. \tag{2.4.2}$$

Let $F(k)$ be the off-shell two point function of special states obtained by summing over all genera. If there are n_p special states at mass level m then $F(k)$ is an $n_p \times n_p$ matrix satifying

$$F(k) = F(-k)^T\,, \tag{2.4.3}$$

where F^T denotes transpose of F. Then the off-shell propagator of special states is given by[6]

$$\frac{1}{k^2+m^2} + \left(\frac{1}{k^2+m^2}\right)^2 F(k)\,, \tag{2.4.4}$$

where m is the tree level mass. The first term represents the tree level propagator whereas the first factor of the second term is the effect of the two external propagators. $F(k)$ admits a genus expansion of the form $\sum_n F_n g^{2n}$ in string coupling g, with higher genus contributions having higher order poles at $k^2+m^2=0$ from regions of the moduli space where the Riemann surface degenerates into two or more Riemann surfaces of lower genera connected by long handles, with the two external vertices lying on the two lower genus Riemann surfaces at the two ends. We expect that after resummation, (2.4.4) may be written as $Z^{1/2}(k)\,(k^2+M_p^2)^{-1}(Z^{1/2}(-k))^T$ for some physical mass^2 matrix M_p^2 and wave-function renormalization matrix $Z^{1/2}(k)$ which has no pole near $k^2=-m^2$. This will be seen explicitly in (2.4.11)-(2.4.15) below. We can take M_p^2 to be diagonal by absorbing the diagonalizing matrix into the definition of $Z^{1/2}(k)$. If $m_{a,p}^2$ for $a=1,2\cdots n_p$ are the eigenvalues of the mass^2 matrix M_p^2 then the physical poles of the propagator are at $k^2=-m_{a,p}^2$.

Now consider the effect of changing the local coordinate system by an infinitesimal amount. Let the change in F to first order be δF. Then in order that the location of the poles of the propagator in the k^2 plane does not shift, the net change in (2.4.4) must be of the form of an overall multiplicative factor that renormalizes $Z^{1/2}(k)$. Thus we require

$$\begin{aligned} & \frac{1}{k^2+m^2} + \left(\frac{1}{k^2+m^2}\right)^2 (F(k)+\delta\, F(k)) \\ = \;& (1+\delta Y(k))\left\{\frac{1}{k^2+m^2} + \left(\frac{1}{k^2+m^2}\right)^2 F(k)\right\}(1+\delta Y(-k))^T\,, \end{aligned} \tag{2.4.5}$$

[6]We have removed an overall factor of $-\mathbf{i}$ and also absorbed a factor of $-\mathbf{i}$ into the definition of $F(k)$.

for some matrix $\delta Y(k)$ whose genus expansion is free from any poles at $k^2+m^2=0$. Equivalently we can write

$$\delta F(k) = (k^2+m^2)\,\delta Y(k) + (k^2+m^2)\,(\delta Y(-k))^T + \delta Y(k)\,F(k) + F(k)\,\delta Y(-k)^T\,. \tag{2.4.6}$$

At each genus the two point function δF receives two contributions – from the change of local coordinates at the vertex carrying momentum k and the change in local coordinates at the vertex carrying momentum $-k$. Both these contributions have an explicit factor of k^2+m^2 due to the fact that when $k^2+m^2=0$ the vertex is on-shell and hence there is no dependence on local coordinates. In concrete terms, since the off-shell vertex operator of a special state is a primary of dimension $((k^2+m^2)/4,(k^2+m^2)/4)$, if we insert such an operator at the origin $w=0$ of the local cordinate system, and then change the local coordinate from w to $w+\epsilon(w)$ then we pick up a net multiplicative factors of $(1+\epsilon'(0))^{(k^2+m^2)/2} \simeq (1+(k^2+m^2)\epsilon'(0)/2)$. Thus we introduce the function δH via the relations

$$\delta F(k) = (k^2+m^2)\,\delta H(k) + (k^2+m^2)\,(\delta H(-k))^T\,, \tag{2.4.7}$$

where the first term is the effect of the change of local coordinates at the vertex carrying momentum k and the second term is the effect of change of local coordinates at the vertex carrying momentum $-k$. The rules for computing δH are the same as that of F except that at one of the punctures the vertex $c\bar{c}V_i$ is replaced by $\epsilon'(0)c\bar{c}V_i/2$. We shall call the puncture where the effect of change of local coordiantes is inserted the 'special puncture'. Eq.(2.4.6) can now be satisfied by choosing $\delta Y(k)$ such that

$$\delta H(k) = \delta Y(k) + (k^2+m^2)^{-1}\,\delta Y(k)\,F(k)\,. \tag{2.4.8}$$

Our goal will be to show the existence of $\delta Y(k)$ satisfying (2.4.8) such that the genus expansion of $\delta Y(k)$ does not have any pole at $k^2+m^2=0$.

We now claim that there exist quantities $\widetilde{F}$ and $\delta\widetilde{H}$ with the properties that the genus expansion of neither of them has any poles near $k^2+m^2=0$, both have genus expansion starting at one loop and F and δH can be expressed in terms of $\widetilde{F}$ and $\delta\widetilde{H}$ as

$$F = \widetilde{F}(1-(k^2+m^2)^{-1}\widetilde{F})^{-1}\,, \tag{2.4.9}$$

$$\delta H = \delta\widetilde{H}(1-(k^2+m^2)^{-1}\widetilde{F})^{-1}\,. \tag{2.4.10}$$

Let us first proceed assuming this to be true. From eqs.(2.4.9) and (2.4.4) we see that the full propagator is given by

$$(k^2+m^2)^{-1}+(k^2+m^2)^{-2}\widetilde{F}\,(1-(k^2+m^2)^{-1}\widetilde{F})^{-1} = (k^2+m^2-\widetilde{F}(k))^{-1}\,. \tag{2.4.11}$$

If we choose a real basis of fields in position space then we have $\widetilde{F}(k)^\dagger = \widetilde{F}(k)$ and $\widetilde{F}(k)^T = \widetilde{F}(-k)$. In this case by choosing suitable unitary matrix $U(k)$ satisfying $U(-k)^T = U(k)^\dagger$ we can express $\widetilde{F}(k)$ as $U(k)\widetilde{F}_d(k)U(k)^\dagger$ where $\widetilde{F}_d(k)$ is a diagonal matrix satisfying $\widetilde{F}_d(-k) = \widetilde{F}_d(k)$. Furthermore the genus expansion of $U(k)$ is free from poles at $k^2+m^2=0$ since $\widetilde{F}(k)$ has this property. We can now express (2.4.11) as

$$U(k)(k^2+m^2-\widetilde{F}_d(k))^{-1}U(k)^\dagger = U(k)(k^2+m^2-\widetilde{F}_d(k))^{-1}U(-k)^T\,. \tag{2.4.12}$$

Let M_p^2 denote the diagonal matrix that describes the locations of the zeroes of the eigenvalues of the diagonal matrix $k^2+m^2-\widetilde{F}_d(k)$ in the $-k^2$ plane. We can solve for this iteratively starting with the leading order solution $k^2=-m^2$. Then we can write

$$(k^2+m^2-\widetilde{F}_d(k))^{-1} = X_d(k)(k^2+M_p^2)^{-1}\,, \tag{2.4.13}$$

where $X_d(k)$ is a diagonal matrix whose genus expansion does not have any pole near $k^2 = -m^2$ and satisfies $X_d(-k) = X_d(k)$. Defining

$$Z^{1/2}(k) = U(k)\sqrt{X_d(k)}\,, \tag{2.4.14}$$

satisfying $Z^{1/2}(k)^\dagger = Z^{1/2}(-k)^T$ we can express the propagator (2.4.11) as

$$Z^{1/2}(k)\,(k^2 + M_p^2)^{-1}\,Z^{1/2}(-k)^T\,. \tag{2.4.15}$$

The genus expansion of $Z^{1/2}(k)$ does not have any poles at $k^2 + m^2 = 0$ since neither $U(k)$ nor $X_d^{1/2}(k)$ has such poles.

Now using eq.2.4.9 we can express (2.4.10) as

$$\delta H = \delta\widetilde{H}(1 + (k^2 + m^2)^{-1}F)\,. \tag{2.4.16}$$

Comparing this with (2.4.8) we get

$$\delta Y = \delta\widetilde{H}\,. \tag{2.4.17}$$

Since $\delta\widetilde{H}$ does not have any pole near $k^2 + m^2 = 0$ this establishes that δY also does not have any pole near $k^2 + m^2 = 0$. This in turn establishes the desired result that the locations of the poles of (2.4.4) in the k^2 plane do not change under change in local coordinates.

2.4.2 Explicit construction of $\widetilde{F}$ and $\delta\widetilde{H}$

It now remains to prove the existence of pole free $\widetilde{F}$ and $\delta\widetilde{H}$ satisfying 2.4.9 and 2.4.10. We shall do this in steps.

1. First we extend the definitions of δH and F where we allow the external states inserted at the punctures (except at the special puncture) to be general string states of ghost

number two,[7] inserted using the same local coordinate system as before. This makes F into an infinite dimensional square matrix which we shall call $\mathcal{F}$ and δH into an $n_p\times$ infinite dimensional matrix (since one of its two punctures is special) which we shall call $\delta\mathcal{H}$.

2. We now use another insight from string field theory [32]: it provides us with a triangulation of the moduli space in which the full moduli space of a genus n Riemann surface with two punctures can be decomposed into a 'one particle irreducible' region R_n and the rest. The region R_n has the property that it does not contain any boundary of the moduli space in which a genus n Riemann surface degenerates into a pair of two punctured lower genus Riemann surfaces connected by a long handle, with each side containing one of the original punctures. The rest of the moduli space is obtained by gluing in all possible ways lower genus punctured Riemann surfaces corresponding to regions $R_{n'}$ by the plumbing fixture procedure [34, 35] (see appendix A.5.1). If we denote by $\widehat{\mathcal{F}}$ and $\delta\widehat{\mathcal{H}}$ the contributions to $\mathcal{F}$ and $\delta\mathcal{H}$ from integration over the one particle irreducible regions R_n of the moduli spaces, then $\widehat{\mathcal{F}}$ and $\delta\widehat{\mathcal{H}}$ have no poles at $k^2+m^2=0$ since the region of integration does not include the degenerating Riemann surfaces. We shall shortly discuss how to define $\widehat{\mathcal{F}}$ and $\delta\widehat{\mathcal{H}}$ in the absence of a string field theory underlying the choice of local coordinates we have made. There is also an additional subtle point in the definition of $\delta\widehat{\mathcal{H}}$ which will be discussed in point 11 of this discussion.

3. We can regard $\widehat{\mathcal{F}}$ and $\mathcal{F}$ as maps from $\mathcal{H}\times\mathcal{H}$ to $\mathbb{C}$ where $\mathcal{H}$ denotes the space of string states of ghost number 2. However since string states of ghost number 4 form the dual vector space of string states of ghost number 2 via the inner product in the CFT, we can also regard $\mathcal{F}$ and $\widehat{\mathcal{F}}$ as maps from states of ghost number 2 to string states of ghost number 4. We shall in fact include left multiplication by the operator $\bar{b}_0 b_0$ – the zero modes of the b and $\bar{b}$ ghost fields – to regard $\mathcal{F}$ and $\widehat{\mathcal{F}}$ as maps from states of ghost

[7] As will become clear later, we need to extend this definition only to those states which are annihilated by $L_0-\bar{L}_0$, b_0 and $\bar{b}_0$.

number 2 to states of ghost number 2. This is the way we shall interpret $\mathcal{F}$ and $\widehat{\mathcal{F}}$ from now on. By including similar factor in the definition of $\delta\mathcal{H}$ and $\delta\widehat{\mathcal{H}}$, they can be regarded as maps from string states of ghost number 2 to the space of special states.

4. With this convention the full contribution to $\mathcal{F}$ and $\delta\mathcal{H}$ is obtained by gluing $\widehat{\mathcal{F}}$ and $\delta\widehat{\mathcal{H}}$ using the string propagator

$$\Delta = \frac{1}{4\pi}\int_0^{2\pi} d\theta \int_0^\infty ds\, e^{-s(L_0+\bar{L}_0)+i\theta(L_0-\bar{L}_0)} = \frac{1}{2}\,\delta_{L_0,\bar{L}_0}\int_0^\infty ds\, e^{-s(L_0+\bar{L}_0)}\,. \tag{2.4.18}$$

The normalization of Δ has been chosen such that acting on special states at mass level m it gives $(k^2+m^2)^{-1}$. We can now express $\mathcal{F}$ and $\delta\mathcal{H}$ as

$$\begin{aligned}\mathcal{F} &= \widehat{\mathcal{F}} + \widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}} + \widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}} + \cdots = \widehat{\mathcal{F}}(1-\Delta\widehat{\mathcal{F}})^{-1} = (1-\widehat{\mathcal{F}}\Delta)^{-1}\widehat{\mathcal{F}}\,,\\ \delta\mathcal{H} &= \delta\widehat{\mathcal{H}} + \delta\widehat{\mathcal{H}}\Delta\widehat{\mathcal{F}} + \delta\widehat{\mathcal{H}}\Delta\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}} + \cdots = \delta\widehat{\mathcal{H}}(1-\Delta\widehat{\mathcal{F}})^{-1}\,.\end{aligned} \tag{2.4.19}$$

Note that each factor of Δ is accompanied by a hidden factor of $\bar{b}_0 b_0$ coming from $\widehat{\mathcal{F}}$; these are required to provide the correct integration measure on the moduli space. Eqs.(2.4.19) provide us with explicit implementation of plumbing fixture, building a higher genus Riemann surface from gluing of lower genus punctured Riemann surfaces.

In the world-sheet description, $\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}$ contains integration over those Riemann surfaces, which can be obtained by gluing two Riemann surfaces corresponding to regions of the moduli space included in the definition of $\widehat{\mathcal{F}}$, at one each of their punctures by the relation

$$w_1 w_2 = e^{-s+i\theta}, \quad 0\le s<\infty, \quad 0\le\theta<2\pi\,, \tag{2.4.20}$$

where w_1 and w_2 are the local coordinates at the punctures. Similar interpretation holds for terms like $\delta\widehat{\mathcal{H}}\Delta\widehat{\mathcal{F}}$.

5. In the absence of an underlying string field theory we can use (2.4.19) to define $\widehat{\mathcal{F}}$ and

$\delta\widehat{\mathcal{H}}$. Consider for example $\widehat{\mathcal{F}}$. Let $\mathcal{F}_n$ and $\widehat{\mathcal{F}}_n$ denote the genus n contribution to $\mathcal{F}$ and $\widehat{\mathcal{F}}$ respectively. Since both $\mathcal{F}$ and $\widehat{\mathcal{F}}$ have genus expansion beginning at genus one, the genus expansion of (2.4.19) tells us that $\widehat{\mathcal{F}}_1$ is identical to $\mathcal{F}_1$. Now at genus two the right hand of the first equation in (2.4.19) gets a contribution from the $\widehat{\mathcal{F}}_1\Delta\widehat{\mathcal{F}}_1$ term. This represents integration over certain region of the genus two moduli space with the same integrand as that in the expression for $\mathcal{F}_2$. Then $\widehat{\mathcal{F}}_2$ is given by the integral of the same integrand over the complementary region of the genus two moduli space. The same process can now be repeated for higher genus, $\widehat{\mathcal{F}}_n$ being given by an integration over certain region of the genus n moduli space with the same integrand as that of $\mathcal{F}_n$. The region of integration is the region that is not covered by gluing the lower genus $\widehat{\mathcal{F}}_m$'s by Δ. By construction $\widehat{\mathcal{F}}_n$ defined this way does not include integration over any region of the moduli space that corresponds to degeneration of the Riemann surface of the kind discussed before, since these regions are already included from the gluing of lower genus contributions. Since the structure of the second equation in (2.4.19) is similar to that of the first equation, the genus n contribution to $\delta\widehat{\mathcal{H}}$ will be given by integration over the same region of the genus n moduli space as that for $\widehat{\mathcal{F}}_n$, with the integrand being the same as that of $\delta\mathcal{H}$.

6. Note however that for this procedure to be consistent it is essential that for those Riemann surfaces which are built by gluing lower genus Riemann surfaces, represented in the genus expansion of the right hand side of (2.4.19) by product of lower genus contributions connected by Δ, the choice of local coordinates at the punctures must coincide with those on the lower genus Riemann surfaces. We shall assume that the local coordinates have been chosen this way even if they are not inherited from an underlying string field theory. We also need to assume that the Riemann surfaces produced by the gluing procedure are all distinct, i.e. the same Riemann surface should not be produced by two different gluing procedure. This can be achieved with an appropriate choice of local coordinates, *e.g.* by scaling the local coordinates by a sufficiently small number λ we can

ensure that the gluing produces only Riemann surfaces close to degeneration and hence different gluing produces different Riemann surfaces.

7. We define P_T to be the projection operator into all states of momentum k – physical and unphysical – with $L_0 = \bar{L}_0 = (k^2+m^2)/4$, and define

$$\bar{\Delta} \equiv \Delta - \frac{1}{k^2+m^2} P_T \,, \tag{2.4.21}$$

$$\begin{aligned} \bar{\mathcal{F}} &\equiv \widehat{\mathcal{F}} + \widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}} + \widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}} + \cdots = \widehat{\mathcal{F}}(1-\bar{\Delta}\widehat{\mathcal{F}})^{-1} = (1-\widehat{\mathcal{F}}\bar{\Delta})^{-1}\widehat{\mathcal{F}} \,, \\ \delta\bar{\mathcal{H}} &\equiv \delta\widehat{\mathcal{H}} + \delta\widehat{\mathcal{H}}\bar{\Delta}\widehat{\mathcal{F}} + \delta\widetilde{\mathcal{H}}\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}} + \cdots = \delta\widehat{\mathcal{H}}(1-\bar{\Delta}\widehat{\mathcal{F}})^{-1} \,. \end{aligned} \tag{2.4.22}$$

Physically $\bar{\mathcal{F}}$ and $\delta\bar{\mathcal{H}}$ denote 'one particle irreducible' contribution to appropriate two point functions of fields at mass level m *after integrating out the fields at other mass levels*. Using (2.4.22) we can rewrite 2.4.19 as

$$\begin{aligned} \mathcal{F} &= \bar{\mathcal{F}}(1-(k^2+m^2)^{-1}P_T\bar{\mathcal{F}})^{-1} \\ &= \bar{\mathcal{F}} + \bar{\mathcal{F}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}} + \bar{\mathcal{F}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}} + \cdots \,, \\ \delta\mathcal{H} &= \delta\bar{\mathcal{H}}(1-(k^2+m^2)^{-1}P_T\bar{\mathcal{F}})^{-1} \\ &= \delta\bar{\mathcal{H}} + \delta\bar{\mathcal{H}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}} + \delta\bar{\mathcal{H}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}} + \cdots \,. \end{aligned} \tag{2.4.23}$$

8. We now define

$$P = c_1\bar{c}_1|V_i\rangle\langle V_i|c_{-1}\bar{c}_{-1}c_0\bar{c}_0 \otimes I_{zero} \,, \tag{2.4.24}$$

as the projection operator into the special states with tree level mass m. Here I_{zero} corresponds to identity operator acting on the zero mode sector of non-compact bosons, labelled by $(k^0, \vec{k})$. In the following we shall omit explicit mention of the operator I_{zero} as the various operators we shall work with will always act as identity operator in this

sector. Applying the projection operator P on both sides of the first equation in (2.4.23) and from the right in the second equation in (2.4.23), and noting that

$$P\,\mathcal{F}\,P = F, \quad \delta\mathcal{H}\,P = \delta H, \tag{2.4.25}$$

we get

$$\begin{aligned} F &= P\bar{\mathcal{F}}\,P + P\,\bar{\mathcal{F}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}\,P + P\,\bar{\mathcal{F}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}\,P + \cdots\,, \\ \delta H &= \delta\bar{\mathcal{H}}\,P + \delta\bar{\mathcal{H}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}\,P + \delta\bar{\mathcal{H}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}\,P + \cdots\,. \end{aligned} \tag{2.4.26}$$

9. Now P denotes projection operator into special states which transform in certain representations of the symmetry group $SO(D)\times G$. $P_T - P$ denotes projection operator into states at the same mass level which are not special, and hence by definition transform in representations of $SO(D)\times G$ other than those in which special states transform. Thus the two point function of special and non-special states on any Riemann surface vanishes, leading to $(P_T-P)\widehat{\mathcal{F}}P = 0$, $(P_T-P)\Delta P = 0$. This in turn gives

$$(P_T - P)\,\bar{\mathcal{F}}\,P = 0, \qquad P\,\bar{\mathcal{F}}\,(P_T - P) = 0\,. \tag{2.4.27}$$

Using this we can replace the $P_T\bar{\mathcal{F}}\,P$ factors in (2.4.26) by $P\,\bar{\mathcal{F}}\,P$. Defining

$$\widetilde{F} = P\,\bar{\mathcal{F}}\,P, \quad \delta\widetilde{H} = \delta\bar{\mathcal{H}}\,P, \tag{2.4.28}$$

we get

$$\begin{aligned} F &= \widetilde{F} + \widetilde{F}(k^2+m^2)^{-1}\widetilde{F} + \widetilde{F}(k^2+m^2)^{-1}\widetilde{F}(k^2+m^2)^{-1}\widetilde{F} + \cdots \\ \delta H &= \delta\widetilde{H} + \delta\widetilde{H}(k^2+m^2)^{-1}\widetilde{F} + \delta\widetilde{H}(k^2+m^2)^{-1}\widetilde{F}(k^2+m^2)^{-1}\widetilde{F} + \cdots\,. \end{aligned} \tag{2.4.29}$$

10. This reproduces 2.4.9, 2.4.10. Furthermore since $\bar{\mathcal{F}}$ and $\delta\bar{\mathcal{H}}$ have no poles at $k^2+m^2=0$ it follows that $\widetilde{F}$ and $\delta\widetilde{H}$ defined in (2.4.28) also have no poles at $k^2+m^2=0$. This is the desired result.

11. We now come to a subtle point in the definition of $\delta\widehat{\mathcal{H}}$ alluded to earlier. First consider the contributions $\delta\widehat{\mathcal{H}}$ to the right hand side of the second equation in (2.4.19). Naively, $(k^2+m^2)\delta\widehat{\mathcal{H}}$ represents the difference between two contributions, both given by integrating over the same subspace of the moduli space that is used to define $\widehat{\mathcal{F}}$. In one of them we use the original local coordinate encoded in the function f at the puncture carrying momentum k, while in the other one we use the local coordinates encoded in the function $f+\delta f$ at the puncture carrying momentum k. This difference is clearly what we need to compute the contribution to $(k^2+m^2)\delta\mathcal{H}$ from these Riemann surfaces. For reasons that will become clear soon, let us denote this contribution to $\delta\widehat{\mathcal{H}}$ by $\delta_0\widehat{\mathcal{H}}$.

    Now consider the contribution $(k^2+m^2)\delta_0\widehat{\mathcal{H}}\Delta\widehat{\mathcal{F}}$. Again this gives the difference between two contributions: $B-A$. The first contribution A is obtained by gluing the Riemann surfaces corresponding to $\widehat{\mathcal{F}}$ to those corresponding to $\widehat{\mathcal{F}}$ at one each of their punctures using the original coordinate system f, with the coordinate at the external punctures also given by the original local coordinate system f. This induces a specific local coordinate system at the external punctures on the Riemann surfaces represented by $\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}$ (see *e.g.* [37]). By the compatibiity condition discussed in point 6, this is the correct choice of coordinate system on the Riemann surfaces in the original system. The second contribution B is obtained by gluing Riemann surfaces represented by $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}$ at one each of their punctures using the original coordinate system f, with the coordinate at the external puncture carrying momentum k given by the deformed local coordinate system $f+\delta f$. This induces a specific local coordinate system at the external punctures carrying momentum k on the Riemann surfaces represented by $\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}$, but this is not the correct choice of coordinate system as prescribed in the deformed system since we are

still using the original local coodinate system f for the gluing. Let $f+\delta_1 f$ denote the coordinate at the external puncture carrying momentum k that we get using the gluing procedure described above and $f+\delta f$ be the local coordinate at the external puncture for the deformed system which we would get by using the coordinate system $f+\delta f$ both for external puncture and for the punctures we are using for gluing.[8] Let us denote by $(k^2+m^2)\delta_1\widehat{\mathcal{H}}$ the difference between the two contributions, the (k^2+m^2) factor being there due to the fact that the external vertex represents a dimension (0,0) primary in the $k^2+m^2\to 0$ limit, and hence $f+\delta_1 f$ and $f+\delta f$ acting on the external vertex gives the same result in the $k^2+m^2\to 0$ limit. Then $(k^2+m^2)\delta_0\widehat{\mathcal{H}}\Delta\widehat{\mathcal{F}}+(k^2+m^2)\delta_1\widehat{\mathcal{H}}$ gives the desired difference between the off-shell amplitudes computed using the deformed system and the original system. We can then add the error term $\delta_1\widehat{\mathcal{H}}$ to $\delta_0\widehat{\mathcal{H}}$ to define a corrected $\delta\widehat{\mathcal{H}}$ so that the net contribution to $\delta\mathcal{H}$ can still be written as the right hand side of the second equation in (2.4.19). The only possible caveat with this is that since the definition of $\delta_1\widehat{\mathcal{H}}$ involves integration over moduli spaces of Riemann surfaces corresponding to $\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}$, this involve a degeneration limit where the parameter s appearing in the definition of Δ in (3.3.10) goes to ∞. Integration over s from this region could produce a pole at $k^2=-m^2$. We shall now argue that this does not happen. For this note that in the $s\to\infty$ limit the Riemann surface degenerates into two Riemann surfaces, and the local coordinates induced at external punctures are inherited from the local coordinates at the external punctures of the Riemann surfaces which are being glued, and independent of the local coordinates at the punctures which we use to glue the two Riemann surfaces. Thus the functions $f+\delta f$ and $f+\delta_1 f$ should become identical as $s\to\infty$. From this we conclude that for large s they should differ by a term proportional to $q=e^{-s+i\theta}$. As a result the expression for $\delta_1\widehat{\mathcal{H}}$, which involves difference in the contributions with

[8]Two points may need clarification here. First we are using the same symbol f for the coordinates on the component Riemann surface and the Riemann surface we get by gluing these components since f stands for the original choice of local coordinates on *all* Riemann surfaces. Similar remark applies to $f+\delta f$. The second point is that while comparing the coordinate systems $f+\delta f$ and $f+\delta_1 f$ we work at the same point in the moduli space of the glued Riemann surface.

local coordinates $f + \delta_1 f$ and $f + \delta f$ at the external puncture carrying momentum k, will have an extra factor of q and/or $\bar{q}$ in the integrand. Since the leading contribution to the integrand in (3.3.10) in the $s \to \infty$ limit comes from states of mass level m and is proportional to $e^{-(k^2+m^2)s/2} \sim |q|^{(k^2+m^2)/2}$, we see that an extra factor of q and/or $\bar{q}$ in the integrand will kill the pole at $k^2 = -m^2$. Thus $\delta_1\widehat{\mathcal{H}}$ is free from poles at $k^2+m^2 = 0$.

To summarize, $(k^2 + m^2)(\delta_1\widehat{\mathcal{H}} + \delta_0\widehat{\mathcal{H}}\Delta\widehat{\mathcal{F}})$, added to $P\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}$, produces correctly the contribution to off-shell Green's function with the deformed coordinate system from those Riemann surfaces which correspond to $\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}$. Furthermore $\delta_1\widehat{\mathcal{H}}$ does not contain any pole at $k^2 = -m^2$. Defining $\delta\widehat{\mathcal{H}} = \delta_0\widehat{\mathcal{H}}+\delta_1\widehat{\mathcal{H}}$, we ensure the equality of two sides of the second equation of (2.4.19) to this order. We can then move on to the term $(\delta_1\widehat{\mathcal{H}} + \delta_0\widehat{\mathcal{H}}\Delta\widehat{\mathcal{F}})\Delta\widehat{\mathcal{F}}$ and carry out similar analysis, generating further correction $\delta_2\widehat{\mathcal{H}}$ to $\delta\widehat{\mathcal{H}}$. After carrying out this procedure to the desired order in perturbation theory we can ensure that (2.4.19) and hence all subsequent equations still hold with this new definition of $\delta\widehat{\mathcal{H}}$.

2.5 S-matrix elements of Special States

The on-shell S-matrix element for massive external string states can be analyzed byfollowing a procedure similar to the one used for mass renormalization. Again we shall restrict to S-matrix elements of special states (and possibly massless states for which there is no mass renormalization); the S-matrix elements of other states can be found in principle from the above by computing its residues at appropriate poles. Using the given local coordinate system for n-punctured Riemann surfaces we compute the off-shell n-point function $F^{(n)}_{a_1\cdots a_n}(k_1, \cdots k_n)$ of n external legs. With the help of (2.2.5), (2.2.2) we can then define the on-shell S-mtatrix elements via

$$S^{(n)}_{a_1\cdots a_n}(k_1, \cdots k_n) = \lim_{k_i^2 \to -m^2_{a_i,p}} F^{(n)}_{b_1\cdots b_n}(k_1, \cdots k_n) \prod_{i=1}^{n} \left\{ Z_i^{-1/2}(k_i)_{a_i b_i} (k_i^2 + m^2_{a_i,p}) \, (k_i^2 + m^2_{a_i})^{-1} \right\} . \tag{2.5.1}$$

We shall now prove that $S^{(n)}$ defined this way is invariant under a change of local coordinates even though $F^{(n)}$'s themselves transform under such changes. The change in $S^{(n)}$ comes from two sources: the change in $F^{(n)}$ and the change in $Z_i^{-1/2}(k_i)$. We begin by computing the change in $Z_i^{-1/2}(k_i)$. First of all comparing (2.2.4) with the transformation law (2.4.5) of the propagator under a change of local coordinates, we get

$$\delta Z_i^{1/2}(k_i) = \delta Y_i(k_i) Z_i^{1/2}(k_i)\,, \tag{2.5.2}$$

where δY_i is the same as δY introduced in (2.4.5) and computed in (2.4.17) for the i-th external state. The multiplication on the right hand side of (2.5.2) should be regarded as a matrix multiplication. This gives

$$\delta Z_i^{-1/2}(k_i) = -Z_i^{-1/2}(k_i)\delta Y_i(k_i) = -Z_i^{-1/2}(k_i)\delta\widetilde{H}_i(k_i)\,, \tag{2.5.3}$$

where in the last step we have used the equality of δY and $\delta\widetilde{H}(k)$given in (2.4.17).

Next we shall analyze the contribution to $\delta F^{(n)}_{b_1\cdots b_n}$. This can be expressed as

$$\delta F^{(n)}_{b_1\cdots b_n} = \sum_j \delta_j F^{(n)}_{b_1\cdots b_n}\,, \tag{2.5.4}$$

where δ_j denotes the effect of the change of local coordinates at the j-th puncture. We shall later show that there exist quantities $\widetilde{F}^{(n)}_{j;b_1\cdots b_n}$ and $\delta_j\widetilde{H}^{(n)}_{b_1\cdots b_n}$ whose perturbation expansions have no poles at $k_j^2 + m_{a_j}^2 = 0$ and in terms of which we have the relations

$$F^{(n)}_{b_1\cdots b_n} = \Big(1 - (k_j^2 + m_{a_j}^2)^{-1}\widetilde{F}_j(k_j)\Big)^{-1}_{b_j c_j}\ \widetilde{F}^{(n)}_{j,b_1\cdots b_{j-1}c_j b_{j+1}\cdots b_n}\,, \tag{2.5.5}$$

and

$$\delta_j F^{(n)}_{b_1\cdots b_j} \;=\; (k_j^2 + m_{a_j}^2)\left[\delta_j\widetilde{H}^{(n)}_{b_1\cdots b_n} + \delta\widetilde{H}_j(k_j)_{b_j c_j}\Big(1 - (k_j^2 + m_{a_j}^2)^{-1}\widetilde{F}_j(k_j)\Big)^{-1}_{c_j d_j}\right.$$

$$(k_j^2+m_{a_j}^2)^{-1}\widetilde{F}^{(n)}_{j,b_1\cdots b_{j-1}d_jb_{j+1}\cdots b_n}\Big]\,, \tag{2.5.6}$$

where the quantities $\widetilde{F}_j(k_j)$ and $\delta\widetilde{H}_j(k_j)$ are the same matrices which were called $\widetilde{F}(k_j)$ and $\delta\widetilde{H}(k_j)$ in eqs.(2.4.9), (2.4.10), with the subscript j indicating that we have to use appropriate matrices $(\widetilde{F}_j(k_j))_{b_jc_j}$ and $(\delta\widetilde{H}_j(k_j))_{b_jc_j}$ relevant for the j-th external leg. The various products and inverses appearing in (2.5.5), (2.5.6) are then interpreted as matrix products and matrix inverses acting on the j-th leg.

We shall prove the existence of $\widetilde{F}^{(n)}_{j;b_1\cdots b_n}$ and $\delta_j\widetilde{H}^{(n)}_{b_1\cdots b_n}$ with the desired properties later; for now we shall proceed assuming this to be true. Using (2.5.5) we can express (2.5.6) as

$$\delta_j F^{(n)}_{b_1\cdots b_j} = (k_j^2+m_{a_j}^2)\,\delta_j\widetilde{H}^{(n)}_{b_1\cdots b_n} + \delta\widetilde{H}_j(k_j)_{b_jc_j}\,F^{(n)}_{b_1\cdots b_{j-1}c_jb_{j+1}\cdots b_n}\,. \tag{2.5.7}$$

We are now in a position to calculate $\delta S^{(n)}$. Using eqs.(2.5.1), (2.5.3) and (2.5.7) we get

$$\begin{aligned}
\delta S^{(n)}_{a_1\cdots a_n} &= \lim_{k_i^2\to -m^2_{a_i,p}\,\forall\, i}\sum_{j=1}^n\prod_{\substack{\ell=1\\ \ell\neq j}}^n\Big\{Z_\ell^{-1/2}(k_\ell)_{a_\ell b_\ell}(k_\ell^2+m^2_{a_\ell,p})\,(k_\ell^2+m^2_{a_\ell})^{-1}\Big\}\,(k_j^2+m^2_{a_j,p})\,(k_j^2+m^2_{a_j})^{-1}\\
&\qquad\Big[\delta Z_j^{-1/2}(k_j)_{a_jb_j}F^{(n)}_{b_1\cdots b_j}+Z_j^{-1/2}(k_j)_{a_jb_j}\delta_jF^{(n)}_{b_1\cdots b_n}\Big]\\
&= \lim_{k_i^2\to -m^2_{a_i,p}\,\forall\, i}\sum_{j=1}^n\prod_{\ell=1}^n\Big\{Z_\ell^{-1/2}(k_\ell)_{a_\ell b_\ell}(k_\ell^2+m^2_{a_\ell,p})\,(k_\ell^2+m^2_{a_\ell})^{-1}\Big\}\\
&\quad\times\sum_{j=1}^n\Big[-\delta\widetilde{H}_j(k_j)_{b_jc_j}F^{(n)}_{b_1\cdots b_{j-1}c_jb_{j+1}\cdots b_n}+(k_j^2+m^2_{a_j})\delta_j\widetilde{H}^{(n)}_{b_1\cdots b_n}\\
&\qquad+\delta\widetilde{H}_j(k_j)_{b_jc_j}F^{(n)}_{b_1\cdots b_{j-1}c_jb_{j+1}\cdots b_n}\Big]\\
&= \lim_{k_i^2\to -m^2_{a_i,p}\,\forall\, i}\prod_{\ell=1}^n\Big\{Z_\ell^{-1/2}(k_\ell)_{a_\ell b_\ell}(k_\ell^2+m^2_{a_\ell,p})\Big\}\sum_{j=1}^n\prod_{\substack{\ell=1\\ \ell\neq j}}^n(k_\ell^2+m^2_{a_\ell})^{-1}\,\delta_j\widetilde{H}^{(n)}_{b_1\cdots b_\ell}\,.
\end{aligned} \tag{2.5.8}$$

Now note that the genus expansion of the j-th term in the sum has no poles at $k_j^2+m^2_{a_j}=0$ since there is no explicit factor of $(k_j^2+m^2_{a_j})^{-1}$ and the genus expansion of $\delta_j\widetilde{H}^{(n)}$ does not

contain any poles at $(k_j^2 + m_{a_j}^2) = 0$. As a result after resummation this term will have no pole at $k_j^2 + m_{a_j,p}^2 = 0$, and after being multiplied by the $(k_j^2 + m_{a_j,p}^2)$ term, will give vanishing contribution in the $k_j^2 \to -m_{a_j,p}^2$ limit. Since this analysis can be repeated for every j, we see that $\delta S^{(n)}$ vanishes. Thus the S-matrix is invariant under a change in the local coordinates.

It remains to prove the existence of $\widetilde{F}^{(n)}_{j;b_1\cdots b_n}$ and $\delta_j \widetilde{H}^{(n)}_{b_1\cdots b_n}$ satisfying (2.5.5), (2.5.6) and having no poles at $k_j^2 = -m_{a_j}^2$ in their genus expansion. For this we first define $\mathcal{F}_j^{(n)}$ by allowing the j-th external state of $F^{(n)}$ to be an arbitrary string state. We also use the fact that the change in local coordinates generates a vertex proportional to $(k^2 + m^2)$ to introduce the quantity $\delta_j H^{(n)}$ via

$$\delta_j H^{(n)}_{b_1\cdots b_n} = (k_j^2 + m_{a_j}^2)^{-1}\, \delta_j F^{(n)}_{b_1\cdots b_n}\,. \tag{2.5.9}$$

Then in the same spirit as the $\widehat{\mathcal{F}}$ and $\delta\widehat{\mathcal{H}}$ defined in (2.4.19) we introduce $\widehat{\mathcal{F}}_j^{(n)}$ and $\delta_j \widehat{H}^{(n)}$ via the expansion:

$$\begin{aligned}
\mathcal{F}_j^{(n)} &= \widehat{\mathcal{F}}_j^{(n)} + \widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}_j^{(n)} + \widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}_j^{(n)} + \cdots = (1 - \widehat{\mathcal{F}}\Delta)^{-1}\,\widehat{\mathcal{F}}_j^{(n)}\,, \\
\delta_j H^{(n)} &= \delta_j \widehat{H}^{(n)} + \delta\widehat{\mathcal{H}}_j\Delta\widehat{\mathcal{F}}_j^{(n)} + \delta\widehat{\mathcal{H}}_j\Delta\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}_j^{(n)} + \delta\widehat{\mathcal{H}}_j\Delta\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}_j^{(n)} + \cdots \\
&= \delta_j \widehat{H}^{(n)} + \delta\widehat{\mathcal{H}}_j\Delta(1 - \widehat{\mathcal{F}}\Delta)^{-1}\,\widehat{\mathcal{F}}_j^{(n)}\,,
\end{aligned} \tag{2.5.10}$$

where $\widehat{\mathcal{F}}$ has been defined via (2.4.19) and $\delta\widehat{\mathcal{H}}_j$ is the same as $\delta\widehat{\mathcal{H}}$ defined in (2.4.19), but for the j-th external state. All multiplications in (2.5.10) are matrix multiplications on the j-th external leg with fixed indices b_i for $i \neq j$ on all other legs. $\widehat{\mathcal{F}}_j^{(n)}$ and $\delta_j \widehat{H}^{(n)}$ represent contributions to $\mathcal{F}_j^{(n)}$ and $\delta_j H^{(n)}$ which are one particle irreducible on the j-th external leg. Thus they are given by integration over subregions of the moduli space of Riemann surface with the same integrands as $\mathcal{F}^{(n)}$ and $\delta_j H^{(n)}$, and these subregions have the property that they do not include any degeneration of the j-th external leg.[9] Thus the genus expansions of $\widehat{\mathcal{F}}_j^{(n)}$

[9]The definition of $\delta_j \widehat{H}^{(n)}$ suffers from subtleties of the same kind that affects the definition of $\delta\widehat{\mathcal{H}}$, and these are dealt with in the same way as in the case of $\delta\widehat{\mathcal{H}}$, following the procedure discussed in point 11 at the end of §2.4.

and $\delta_j \widehat{H}^{(n)}$ do not have any pole at $k_j^2 + m_{a_j}^2 = 0$. Now we define

$$\begin{aligned}
\bar{\mathcal{F}}_j^{(n)} &= \widehat{\mathcal{F}}_j^{(n)} + \widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}_j^{(n)} + \widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}_j^{(n)} + \cdots = (1-\widehat{\mathcal{F}}\bar{\Delta})^{-1}\,\widehat{\mathcal{F}}_j^{(n)} \\
\delta_j \widetilde{H}^{(n)} &= \delta_j \widehat{H}^{(n)} + \delta\widehat{\mathcal{H}}_j\bar{\Delta}\widehat{\mathcal{F}}_j^{(n)} + \delta\widehat{\mathcal{H}}_j\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}_j^{(n)} + \delta\widehat{\mathcal{H}}_j\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}_j^{(n)} + \cdots \\
&= \delta_j \widehat{H}^{(n)} + \delta\widehat{\mathcal{H}}_j\bar{\Delta}(1-\widehat{\mathcal{F}}\bar{\Delta})^{-1}\widehat{\mathcal{F}}_j^{(n)} = \delta_j \widehat{H}^{(n)} + \delta\widehat{\mathcal{H}}_j(1-\bar{\Delta}\widehat{\mathcal{F}})^{-1}\bar{\Delta}\widehat{\mathcal{F}}_j^{(n)}\,, \qquad (2.5.11)
\end{aligned}$$

where $\bar{\Delta}$ has been defined in (3.4.2). Since $\bar{\Delta}$ has no poles at $k_j^2 + m_{a_j}^2 = 0$, the genus expansions of $\bar{\mathcal{F}}_j^{(n)}$ and $\delta_j \widetilde{H}^{(n)}$ also do not have any poles at $k_j^2 + m_{a_j}^2 = 0$. Using (2.4.22), (2.5.10) and (2.5.11) we get

$$\begin{aligned}
\mathcal{F}_j^{(n)} &= (1 - \bar{\mathcal{F}}(k_j^2 + m_{a_j}^2)^{-1}P_T)^{-1}\bar{\mathcal{F}}_j^{(n)} \\
\delta_j H^{(n)} &= \delta_j \widetilde{H}^{(n)} + \delta\bar{\mathcal{H}}_j(k_j^2 + m_{a_j}^2)^{-1}P_T(1 - \bar{\mathcal{F}}(k_j^2 + m_{a_j}^2)^{-1}P_T)^{-1}\bar{\mathcal{F}}_j^{(n)}\,. \qquad (2.5.12)
\end{aligned}$$

We now define

$$\widetilde{F}_j^{(n)} = P\bar{\mathcal{F}}_j^{(n)}\,. \qquad (2.5.13)$$

Since the genus expansion of $\bar{\mathcal{F}}_j^{(n)}$ has no poles at $k_j^2 = -m_{a_j}^2$, the genus expansion of $\widetilde{F}_j^{(n)}$ also has no poles at $k_j^2 = -m_{a_j}^2$. It follows from the definition of special states that $\delta\bar{\mathcal{H}}_j P_T = \delta\bar{\mathcal{H}}_j P$. Using this and (2.5.13), multiplying the first equation of (2.5.12) by P from the left, using $P\mathcal{F}_j^{(n)} = F^{(n)}$ and eqs.(2.4.27), (2.4.28) we can write the two equations in (2.5.12) as

$$\begin{aligned}
F^{(n)} &= (1 - (k_j^2 + m_{a_j}^2)^{-1}\widetilde{F}_j)^{-1}\,\widetilde{F}_j^{(n)}\,, \\
\delta_j H^{(n)} &= \delta_j \widetilde{H}^{(n)} + \delta\widetilde{H}_j(k_j^2 + m_{a_j}^2)^{-1}(1 - (k_j^2 + m_{a_j}^2)^{-1}\widetilde{F}_j)^{-1}\widetilde{F}_j^{(n)}\,. \qquad (2.5.14)
\end{aligned}$$

This reproduces (2.5.5) and (2.5.6) after using (2.5.9).

2.6 Discussion and generalizations

In this chapter we have given an algorithm for computing renormalized mass and S-matrix elements for a special class of massive states in bosonic string theory, and have shown that these are independent of the specific off-shell continuation that we use for computing them. While the results are in the same spirit as the proof of gauge invariance of physical mass and S-matrix elements in a gauge theory, in many sense the analysis here is simpler than in gauge theories. In the latter the gauge invariance results from cancellation between the contributions from different Feynman diagrams, while here we do not require any such cancellations. In fact if we had been trying to prove gauge invariance of renormalized mass and S-matrix elements in string field theory, we would still need cancellation between different Feynman diagrams.[10]

The simplicity in string theory of course is a consequence of the fact that in string theory there is only one contribution from every genus. Technically the difference between our analysis and the corresponding analysis in string field theory can be traced to the fact that in string field theory a change in local coordinates will change the local coordinates not only at the external punctures, but also at the internal punctures that we use to glue two Riemann surfaces using the plumbing fixture procedure. As a result each Feynman diagram gets additional contribution from the change in local coordinates at the internal punctures which cancel between different Feynman diagrams.

For general external states we expect new complications even in the bosonic string theory. This is due to the fact that under quantum correction the physical states would begin mixing with the unphysical states and we need to take into account this mixing for defining an appropriate off-shell continuation. For example from genus two onwards $\widehat{\mathcal{F}}$ will have non-zero matrix element between a physical state and a BRST trivial state from the boundary of the

[10] A change of local coordinates correspond to a field redefinition of the string field [36] followed by a gauge transformation that is needed to bring the transformed fields to the Siegel gauge.

region of integration of the moduli space that defines $\widehat{\mathcal{F}}$, forcing us to change the definition of the physical state. Furthermore the required mixing will depend on the particular off-shell continuation we choose i.e. on the choice of local coordinates at the punctures. In the next chapter we will study general states and will prove that the renormalized mass and S-matrix elements are independent of the off-shell continuation for all physical states, suitably defined.

In fact it seems to us that the off-shell formalism could be a useful way of studying string perturbation theory both for massive and massless external states, and can be used to give alternate proofs of well known results in string theory. For example in the standard on-shell approach the proof of decoupling of pure gauge states, corresponding to trivial elements of the BRST cohomology, involves first showing that the result is given by a total derivative in the moduli space and then showing that the boundary terms arising from the integration of the total derivative terms vanish. In the off-shell formalism the boundary terms can be ignored altogether since they can be made to vanish by appropriate off-shell continuation of the external momenta. The price we pay is that due to BRST non-invariance of the external off-shell states there will be additional terms proportional to one or more powers of $(k_i^2 + m_{a_i}^2)$ associated with the external states. In individual terms these may be cancelled by inverse powers of $(k_i^2 + m_{a_i}^2)$ coming from integration over moduli near the boundaries. Thus the task will be to show that the final result vanishes nevertheless in the on-shell limit.

CHAPTER 3

Mass Renormalization in String Theory: General States

3.1 Introduction

In chapter 2 we described a systematic procedure for computing the renormalized masses and S-matrix elements of a special class of states in bosonic string theory which do not mix with unphysical states under renormalization. Our goal in this chapter will be to generalize this procedure to states in bosonic string theory which do mix with unphysical states under renormalization. We shall also briefly discuss extensions to the Neveu-Schwarz (NS) sector states in superstring and heterotic string theories.

We shall now summarize the contents of the rest of the sections. The reason that we had to restrict our analysis to a special class of states in chapter 2 was to avoid the mixing between physical and unphysical states which are degenerate at tree level. In §3.2 we construct an example of a gauge theory where the tree level spectrum in a particular gauge has accidental degeneracy between physical and unphysical states. We then develop an algorithm for extracting the quantum corrected physical mass in this theory, with the aim of generalizing this to string theory later.

In §3.3 we review some basic results for on-shell states in closed bosonic string theory, dividing them into physical, unphysical and pure gauge states and discuss their off-shell generalization. We also review the prescription for defining off-shell amplitudes in string theory which depend on the choice of local coordinates at the punctures where the vertex operators are inserted. Finally we discuss the constraints imposed on the choice of local coordinate system from the requirement that they be compatible with the plumbing fixture procedure for gluing two Riemann surfaces to form a third one. This allows us to express an off-shell amplitude as sums of products of one particle irreducible contributions and propagators.

§3.4-§3.6 contains our main results. In §3.4 we generalize the method of §3.2 for systematically computing the renormalized physical masses in string theory. We also show that at one loop order the renormalized physical masses are independent of the choice of local coordinate system but the renormalized masses in the unphysical / pure gauge sector do depend on the choice of local coordinates. In §3.5 we examine the locations of the poles in the scattering amplitudes of external massless / BPS / special states in the complex $-k^2$ plane where k is given by the sum of some specific subset of external momenta. We find that the possible locations of the poles are precisely at the squares of physical and unphysical masses found using the general algorithm of §3.4. We also show that at the leading order the residues at the physical poles are non-vanishing in general but the residues at the poles associated with the unphysical / pure gauge sector states vanish. In §3.6 we combine the results of §3.4, §3.5 with the result of chapter 2 that the S-matrices of massless / BPS / special states are independent of the choice of local coordinate system, to argue that to all orders in string perturbation theory the renormalized physical masses are independent of the choice of local coordinate system and that the residues at the poles associated with the unphysical / pure gauge sector states vanish. In other words the poles in the S–matrix elements of massless / BPS / special states in the $-k^2$ plane occur only at the renormalized physical mass^2 defined in §3.4. Finally in §3.7 we

briefly discuss generalization of our analysis to Neveu-Schwarz sector states in heterotic and superstring theories. This chapter is based o [2].

3.2 A field theory example

String perturbation theory is a degenerate perturbation theory with high amount of degeneracy. To understand the situation it is a good idea to look into a degenerate quantum fields theory example which share most of the complications that occur in string theory. Degenerate abelian Higgs model is a tractable and complicated enough setup to demystify the mysteries of string perturbation theory. In this section we shall illustrate the problem of mixing between physical and unphysical states in this gauge theory. We shall also provide an algorithm for extracting the renormalized physical mass in this theory. This algorithm will be generalized to string theory in §3.4.

3.2.1 The model

Consider a quantum field theory in $D+1$ dimensions containing an abelian gauge field A_μ and a pair of complex scalars ϕ, χ, each carrying charge q under the gauge field. We consider a gauge invariant Lagrangian density of the form

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu\phi^* + iqA_\mu\phi^*)(\partial^\mu\phi - iqA^\mu\phi) - c\,(\phi^*\phi - v^2)^2 \\ &\quad -(\partial_\mu\chi^* + iqA_\mu\chi^*)(\partial^\mu\chi - iqA^\mu\chi) - V(\phi,\chi)\,, \\ F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,, \end{aligned} \tag{3.2.1}$$

where $V(\phi,\chi)$ is a potential whose detailed properties will be discussed shortly, but for now we just mention that it plays no role in the breaking of the U(1) gauge symmetry. Minimizing the potential in the first line we see that $|\phi| = v$ is the minimum of the potential. We choose

$\phi = v$ as the vacuum expectation value of ϕ. We now define $\phi_{R,I}, \chi_{R,I}$ via

$$\phi = v + \frac{1}{\sqrt{2}}(\phi_R + i\phi_I), \quad \chi = \frac{1}{\sqrt{2}}(\chi_R + i\chi_I), \tag{3.2.2}$$

and

$$m \equiv \sqrt{2}\, q\, v\,. \tag{3.2.3}$$

We now describe the choice of the potential $V(\phi, \chi)$. We require it to have the property that when expanded around the point $(\phi = v, \chi = 0)$, it has an expansion of the form

$$-\frac{1}{2}m_0^2\chi_R^2 - \frac{1}{2}m^2\chi_I^2 + \text{cubic and higher order terms in } \phi_R, \phi_I, \chi_R, \chi_I\,, \tag{3.2.4}$$

where m_0 is an arbitrary mass parameter but m has been chosen to be the same quantity defined in (3.2.3). Using this we get, after throwing away total derivative terms,

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}(\partial_\mu A_\mu - m\,\phi_I)^2 - \frac{1}{2}\partial_\mu\phi_I\partial^\mu\phi^I - \frac{1}{2}m^2\phi_I^2 - \frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R \\ & -2\,c\,v^2\,\phi_R^2 - \frac{1}{2}\partial_\mu\chi_I\partial^\mu\chi^I - \frac{1}{2}m^2\chi_I^2 - \frac{1}{2}\partial_\mu\chi_R\partial^\mu\chi^R - \frac{1}{2}\,m_0^2\,\chi_R^2 + \text{interaction terms}\,.\end{aligned} \tag{3.2.5}$$

To this we add a gauge fixing term

$$\mathcal{L}_{gf} = -\frac{1}{2}(\partial^\mu A_\mu - m\,\phi_I)^2\,, \tag{3.2.6}$$

so that the third term in $\mathcal{L}$ is cancelled by $\mathcal{L}_{gf}$ in the total Lagrangian density $\mathcal{L} + \mathcal{L}_{gf}$. The resulting Lagrangian has the fields A_μ, ϕ_I and χ_I all carrying mass m, whereas ϕ_R and χ_R carry different masses.

Now if we work in the momentum space and are at the rest frame $k = (k^0, \vec{0})$ then the fields A_i transform in the vector representation of the little group SO(D) whereas the fields

A_0, ϕ_I and χ_I transform in the scalar representation of the same group. At tree level the fields A_i and χ_I are physical whereas the fields A_0 and ϕ_I are unphysical.[1] In particular by choosing unitary gauge we can remove A_0 and ϕ_I from the spectrum. Alternatively by choosing another gauge fixing term *e.g.* $-(\partial^\mu A_\mu - m\,\xi\,\phi_I)^2/(2\xi)$ with $\xi \neq 1$ we could make the unphysical fields A_0 and ϕ_I have mass different from m and hence non-degenerate with the physical fields. We shall however work with $\xi = 1$ and address the problems associated with the degeneracy directly since this is what we shall need to do in string theory. Our main goal will be to disentangle the physical and unphysical states after inclusion of loop corrections.

Now it is clear that under loop corrections the SO(D) vector fields A_i cannot mix with the unphysical fields and hence they remain physical states. Interestingly as we will explain in §2.3 such special states exists in string theory also. However the state χ_I can now mix with A_0 and ϕ_I. To see what kind of mixing is possible, we note that according the general principle of gauge theory the corrections must take the form of a gauge invariant term written in terms of the original variables ϕ, χ, A_μ together with a possible renormalizaton of the gauge fixing term. Let us suppose that quantum corrections generate a gauge invariant mass term for χ of the form $-\alpha\chi^*\chi$ and changes the gauge fixing term (3.2.6) to $-(\partial^\mu A_\mu - m\,\phi_I + \beta\phi_I + \gamma\chi_I)^2/2$.[2]. Here, α, β and γ are in principle computable constants which arise from loop corrections. Adding these to (3.2.1) we can express the quadratic terms involving A_μ, ϕ_I and χ_I as

$$\begin{aligned}&-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{2}\partial_\mu\phi_I\partial^\mu\phi^I - \frac{1}{2}m^2\phi_I^2 - \frac{1}{2}\partial_\mu\chi_I\partial^\mu\chi^I - \frac{1}{2}m^2\chi_I^2\\ &-\frac{1}{2}\,\alpha\,\chi_I^2 - \beta\phi_I\partial_\mu A^\mu + \frac{1}{2}(2m\beta - \beta^2)\phi_I^2 - \gamma\chi_I\partial_\mu A^\mu - \frac{1}{2}\gamma^2\chi_I^2 + (m-\beta)\,\gamma\,\phi_I\,\chi_I. \qquad (3.2.7)\end{aligned}$$

[1] In the language that we shall develop shortly, one linear combination of these fields will be called unphysical and the other will be called pure gauge.

[2] We could have also changed the coefficient of the $\partial_\mu A^\mu$ inside the gauge fixing term and added other gauge invariant terms, but the corrections we have taken are sufficiently general to illustrate the basic points.

In momentum space, up to overall multiplication and momentum conserving delta functions, the quadratic Lagrangian density in the $\vec{k} = 0$ sector can be written as

$$\frac{1}{2}A_i(-k)\{(k^0)^2 - m^2\}A_i(k) + \frac{1}{2}\begin{pmatrix} A_0(-k) & \phi_I(-k) & \chi_I(-k) \end{pmatrix} M \begin{pmatrix} A_0(k) \\ \phi_I(k) \\ \chi_I(k) \end{pmatrix}, \tag{3.2.8}$$

where

$$M = \begin{pmatrix} -(E^2 - m^2) & \mathbf{i}\,E\,\beta & \mathbf{i}\,E\,\gamma \\ -\mathbf{i}\,E\,\beta & E^2 - (m-\beta)^2 & (m-\beta)\gamma \\ -\mathbf{i}\,E\,\gamma & (m-\beta)\gamma & E^2 - m^2 - \gamma^2 - \alpha \end{pmatrix}, \quad E \equiv k^0\,. \tag{3.2.9}$$

As expected $A_i(k)$'s, being special states, do not mix with other fields. In this example its mass is not affected by the quantum corrections, but this is just a consequence of the limited number of terms we have added, *e.g.* this could change if we had added a gauge invariant term proportional to $F_{\mu\nu}F^{\mu\nu}$ in the quantum corrections to the Lagrangian density.

Let us define the matrices

$$\mathcal{I} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{F}_T = \begin{pmatrix} 0 & \mathbf{i}\,E\,\beta & \mathbf{i}\,E\,\gamma \\ -\mathbf{i}\,E\,\beta & 2m\beta - \beta^2 & (m-\beta)\gamma \\ -\mathbf{i}\,E\,\gamma & (m-\beta)\gamma & -\gamma^2 - \alpha \end{pmatrix}, \tag{3.2.10}$$

so that we can write

$$M = -\{(m^2 - E^2)\mathcal{I} - \widetilde{F}_T\}\,. \tag{3.2.11}$$

The full propagator (up to overall sign and factors of i) is then given by

$$\mathcal{P}_T = -M^{-1} = \{(m^2 - E^2)\mathcal{I} - \widetilde{F}_T\}^{-1}\,, \tag{3.2.12}$$

and the renormalized squared masses are the locations of the poles of this matrix in the E^2 plane. Only one of these poles is physical. We need to find a systematic algorithm for determining which one is physical and calculate its location. This will be done in §3.2.2, but to facilitate the analysis we shall now introduce a few notations.

Let us introduce a set of basis states as follows:

$$|p\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |g\rangle = \frac{1}{|E|\sqrt{2}} \begin{pmatrix} -\mathbf{i}E \\ |E| \\ 0 \end{pmatrix}, \quad |u\rangle = \frac{1}{|E|\sqrt{2}} \begin{pmatrix} \mathbf{i}E \\ |E| \\ 0 \end{pmatrix}. \tag{3.2.13}$$

The conjugate basis $\langle p|$, $\langle g|$ and $\langle u|$ are defined by taking transpose together with a change of sign of the momentum vector. The latter operation changes the sign of E and hence effectively the conjugate basis corresponds to hermitian conjugates of the vectors (3.2.13). Then we have the following identities

$$\begin{pmatrix} \langle g|\mathcal{I}|g\rangle & \langle g|\mathcal{I}|u\rangle & \langle g|\mathcal{I}|p\rangle \\ \langle u|\mathcal{I}|g\rangle & \langle u|\mathcal{I}|u\rangle & \langle u|\mathcal{I}|p\rangle \\ \langle p|\mathcal{I}|g\rangle & \langle p|\mathcal{I}|u\rangle & \langle p|\mathcal{I}|p\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2.14}$$

We shall call $|p\rangle$, $|g\rangle$ and $|u\rangle$ as tree level physical, pure gauge and unphysical states respectively. The name pure gauge for $|g\rangle$ stems from the fact that on-shell (at $|E| = m$) this describes a pure gauge deformation of the vacuum at the linearized level and the name physical originates from the fact that the χ_I field represented by the vector $|p\rangle$ is the physical field at the tree level.

3.2.2 The algorithm for computing the physical mass

Our goal will be to develop an algorithm for finding the corrected physical state and the physical mass after taking into account the quantum correction to M represented by $\widetilde{F}_T$. Furthermore

instead of aiming at the exact result we want to do this perturbatively in the parameters α, β, γ since this is what we need in string theory. The problem is made complicated by the fact that the full matrix M is expected to have zero eigenvalue at more than one value of E near m, and we expect only one of these to represent physical mass. Let m_p be the quantum corrected physical mass, and $|p\rangle'$ be the eigenvector with zero eigenvalue at $E = m_p$. Then naively we might expect that as we switch off the perturbation parameters α, β, γ, the vector $|p\rangle'$ should approach the unperturbed physical state $|p\rangle$ and we can use this as a criterion for identifying the quantum corrected physical state. The problem however is that since the unperturbed matrix has three different eigenvectors with zero eigenvalue at $E = m$, what we have here is an analog of degenerate perturbation theory and there is no guarantee that the eigenvectors of the quantum corrected matrix will approach a particular unperturbed eigenvector in the limit of switching off the perturbation. Indeed, we shall see that in general it is not possible to construct an eigenvector with zero eigenvalue in the perturbed theory that approaches the particular vector $|p\rangle$ in the limit $\alpha, \beta, \gamma \to 0$. The best we can do is to find such an eigenvector that approaches a linear combination of the unperturbed physical state $|p\rangle$ and the unperturbed pure gauge state $|g\rangle$ as we switch off the perturbation. We shall take this as the criterion for identifying the quantum corrected physical state and look for an algorithm for constructing such a state.

With this goal in mind, we now seek a change of basis of the form

$$|p\rangle' = A|p\rangle + B|g\rangle + C|u\rangle, \quad |g\rangle' = |g\rangle + D|p\rangle, \quad |u\rangle' = |u\rangle + K|p\rangle\,, \tag{3.2.15}$$

such that the following conditions hold

$$\begin{pmatrix} {}'\langle g|\mathcal{I}|g\rangle' & {}'\langle g|\mathcal{I}|u\rangle' & {}'\langle g|\mathcal{I}|p\rangle' \\ {}'\langle u|\mathcal{I}|g\rangle' & {}'\langle u|\mathcal{I}|u\rangle' & {}'\langle u|\mathcal{I}|p\rangle' \\ {}'\langle p|\mathcal{I}|g\rangle' & {}'\langle p|\mathcal{I}|u\rangle' & {}'\langle p|\mathcal{I}|p\rangle' \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.2.16}$$

and

$$\begin{pmatrix} {}'\langle g|\widetilde{F}_T|g\rangle' & {}'\langle g|\widetilde{F}_T|u\rangle' & {}'\langle g|\widetilde{F}_T|p\rangle' \\ {}'\langle u|\widetilde{F}_T|g\rangle' & {}'\langle u|\widetilde{F}_T|u\rangle' & {}'\langle u|\widetilde{F}_T|p\rangle' \\ {}'\langle p|\widetilde{F}_T|g\rangle' & {}'\langle p|\widetilde{F}_T|u\rangle' & {}'\langle p|\widetilde{F}_T|p\rangle' \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} . \tag{3.2.17}$$

where $*$ denotes unconstrained numbers. Notice that (3.2.15) is not the most general change of basis. In fact the most general change of basis is related to the one given in (3.2.15) by arbitrary mixing between the states $|u\rangle'$ and $|g\rangle'$ without involving $|p\rangle'$. However all the conditions demanded in (3.2.16), (3.2.17) are invariant under such a change of basis and hence by taking convenient linear combinations of $|u\rangle'$ and $|g\rangle'$ satisfying (3.2.16), (3.2.17) we can always ensure that the change of basis is of the form given in (3.2.15). We now substitute (3.2.15) into (3.2.16), (3.2.17) and use (3.2.14) to get

$$\begin{aligned} &A^*A + B^*C + C^*B = 1, \quad D^*A + C = 0, \quad K^*A + B = 0\,, \\ &A\langle u|\widetilde{F}_T|p\rangle + B\langle u|\widetilde{F}_T|g\rangle + C\langle u|\widetilde{F}_T|u\rangle + K^*A\langle p|\widetilde{F}_T|p\rangle + K^*B\langle p|\widetilde{F}_T|g\rangle + K^*C\langle p|\widetilde{F}_T|u\rangle = 0 \\ &A\langle g|\widetilde{F}_T|p\rangle + B\langle g|\widetilde{F}_T|g\rangle + C\langle g|\widetilde{F}_T|u\rangle + D^*A\langle p|\widetilde{F}_T|p\rangle + D^*B\langle p|\widetilde{F}_T|g\rangle + D^*C\langle p|\widetilde{F}_T|u\rangle = 0\,. \end{aligned} \tag{3.2.18}$$

We shall soon discuss how to construct A, B, C, D, K perturbatively satisfying (3.2.18) and the criteria mentioned at the beginning of this subsection. However let us first examine the consequences of (3.2.16) and (3.2.17). Using these equations we see that in the primed basis the matrices $\mathcal{I}$ and $\widetilde{F}_T$ are exactly block diagonal, with the $|p\rangle'$ block having no mixing with the $|u\rangle'$ and $|g\rangle'$ blocks. Of course the basis we have chosen is not orthonormal in the $(|u\rangle', |g\rangle')$ sector, but this can be rectified by appropriate linear transformation in the $(|u\rangle',\ |g\rangle')$ space without affecting the $|p\rangle'$-$|p\rangle'$ element. Thus we get

$${}'\langle p|\mathcal{P}_T|p\rangle' = \{(m^2 - E^2) - \widetilde{F}(E)\}^{-1}\,, \tag{3.2.19}$$

where

$$\begin{aligned}\widetilde{F}(E) &\equiv {}'\langle p|\widetilde{F}_T|p\rangle' \\ &= A^*A\langle p|\widetilde{F}_T|p\rangle + A^*B\langle p|\widetilde{F}_T|g\rangle + A^*C\langle p|\widetilde{F}_T|u\rangle + B^*A\langle g|\widetilde{F}_T|p\rangle + B^*B\langle g|\widetilde{F}_T|g\rangle \\ &\quad + B^*C\langle g|\widetilde{F}_T|u\rangle + C^*A\langle u|\widetilde{F}_T|p\rangle + C^*B\langle u|\widetilde{F}_T|g\rangle + C^*C\langle u|\widetilde{F}_T|u\rangle\,. \end{aligned} \tag{3.2.20}$$

The pole of (3.2.19) can be constructed iteratively by expressing this equation as

$$E^2 = m^2 - \widetilde{F}(E), \tag{3.2.21}$$

and solving the equation iteratively by starting with $E^2 = m^2$. We can identify this as the physical pole provided the following two conditions hold:

1. Let us introduce a perturbation parameter λ and take

$$\alpha \sim \lambda, \quad \beta \sim \lambda, \quad \gamma \sim \lambda\,. \tag{3.2.22}$$

   In particular if α, β, γ arise at one loop order then the power of λ counts the number of loops. We need to ensure that the coefficient of λ^n in the expressions for $A, \cdots K$ and $\widetilde{F}(E)$ are free from any pole at $E \simeq m$ for every n. Otherwise the iterative procedure for finding the solution that starts with $E = m$ will break down.

2. We also need to ensure that the coefficient C approaches 0 in the limit $\lambda \to 0$ and $E \to m$ so that the state $|p\rangle'$ approaches a linear combination of the tree level physical state and tree level pure gauge state in this limit. $|p\rangle'$ will then satisfy the criteria mentioned at the beginning of this subsection.

We shall now discuss how to solve (3.2.18) satisfying these conditions. Since each matrix element of $\widetilde{F}_T$ is of order λ, we can factor out the overall factor of λ from the last two equations in (3.2.18), take the $\lambda \to 0$ limit, and regard (3.2.18) as a set of λ independent equations which

can be solved to determine the leading order result for the coefficients $A, \cdots K$. It is easy to check that leaving aside an overall phase there are as many unknowns as the number of equations, and hence we expect these equations to have solutions. Solving the leading order equations can in fact be facilitated by using another expansion parameter, namely $(E^2 - m^2)$. For this we note that (3.2.10), (3.2.13) gives

$$\begin{aligned}
&\lambda^{-1}\langle p|\widetilde{F}_T|g\rangle \sim \mathcal{O}(E^2-m^2) + \mathcal{O}(\lambda), \quad \lambda^{-1}\langle g|\widetilde{F}_T|p\rangle \sim \mathcal{O}(E^2-m^2) + \mathcal{O}(\lambda), \\
&\lambda^{-1}\langle g|\widetilde{F}_T|g\rangle \sim \mathcal{O}(E^2-m^2) + \mathcal{O}(\lambda)\,,
\end{aligned} \tag{3.2.23}$$

while the other matrix elements of $\lambda^{-1}\widetilde{F}_T$ are of order unity as $E \to m$ and $\lambda \to 0$. Making use of (3.2.23), let us look for a leading order in λ solution in which

$$A, B, K \sim 1, \quad C, D \sim (E^2 - m^2)\,. \tag{3.2.24}$$

Using (3.2.23), (3.2.24) we see that to the leading order in λ, (3.2.18) gives

$$\begin{aligned}
&A^*A = 1 + \mathcal{O}(E^2-m^2), \qquad D^*A + C = 0, \quad K^*A + B = 0, \\
&\lambda^{-1}\left\{A\langle u|\widetilde{F}_T|p\rangle + B\langle u|\widetilde{F}_T|g\rangle + K^*A\langle p|\widetilde{F}_T|p\rangle\right\} = \mathcal{O}(E^2-m^2)\,, \\
&\lambda^{-1}\left\{A\langle g|\widetilde{F}_T|p\rangle + B\langle g|\widetilde{F}_T|g\rangle + C\langle g|\widetilde{F}_T|u\rangle + D^*A\langle p|\widetilde{F}_T|p\rangle\right\} = \mathcal{O}((E^2-m^2)^2)\,.
\end{aligned} \tag{3.2.25}$$

Each term in the left hand side of the first, third and fourth equations is of order unity and each term in the left hand side of the third and fifth equations is of order $(E^2 - m^2)$. The solution is

$$\begin{aligned}
&A = 1 + \mathcal{O}(E^2-m^2), \quad K^* = \{\langle u|\widetilde{F}_T|g\rangle - \langle p|\widetilde{F}_T|p\rangle\}^{-1}\langle u|\widetilde{F}_T|p\rangle + \mathcal{O}(E^2-m^2), \\
&D^* = \{\langle g|\widetilde{F}_T|u\rangle - \langle p|\widetilde{F}_T|p\rangle\}^{-1}\{\langle g|\widetilde{F}_T|p\rangle - K^*\langle g|\widetilde{F}_T|g\rangle\} + \mathcal{O}((E^2-m^2)^2)\,, \\
&B = -K^* + \mathcal{O}(E^2-m^2), \quad C = -D^* + \mathcal{O}((E^2-m^2)^2)\,.
\end{aligned} \tag{3.2.26}$$

Using (3.2.23) and the comments below it, we see that as long as the order λ contribution to $\{\langle u|\widetilde{F}_T|g\rangle - \langle p|\widetilde{F}_T|p\rangle\}$ does not vanish (and in particular does not have zero at $E^2 = m^2$), A, B and K given in (3.2.26) are of order unity, while C and D are of order $(E^2 - m^2)$, in agreement with our assumption (3.2.24). The reader may be surprised by the appearance of the one loop term $\{\langle u|\widetilde{F}_T|g\rangle - \langle p|\widetilde{F}_T|p\rangle\}$ in the denominator in a perturbation theory, but this is simply a consequence of the degenerate perturbation theory that we need to carry out in this case. Requiring $\{\langle u|\widetilde{F}_T|g\rangle - \langle p|\widetilde{F}_T|p\rangle\}$ to be non-zero is equivalent to demanding that the degeneracy between the physical and the unphysical / pure gauge states is lifted at the first order. Starting with (3.2.26) we can now iteratively solve the system of equations in a power series in λ and $(E^2 - m^2)$. For this we choose A to be real,[3] express eqs.(3.2.18) as

$$
\begin{aligned}
&A = \sqrt{1 - B^*C - C^*B}\,, \\
&K^* = \{\langle u|\widetilde{F}_T|g\rangle - \langle p|\widetilde{F}_T|p\rangle\}^{-1}\Big[A\langle u|\widetilde{F}_T|p\rangle + (B + K^*)\langle u|\widetilde{F}_T|g\rangle + C\langle u|\widetilde{F}_T|u\rangle \\
&+K^*(A-1)\langle p|\widetilde{F}_T|p\rangle + K^*B\langle p|\widetilde{F}_T|g\rangle + K^*C\langle p|\widetilde{F}_T|u\rangle\Big]\,, \\
&D^* = \{\langle g|\widetilde{F}_T|u\rangle - \langle p|\widetilde{F}_T|p\rangle\}^{-1}\Big[A\langle g|\widetilde{F}_T|p\rangle + B\langle g|\widetilde{F}_T|g\rangle + (C + D^*)\langle g|\widetilde{F}_T|u\rangle \\
&+D^*(A-1)\langle p|\widetilde{F}_T|p\rangle + D^*B\langle p|\widetilde{F}_T|g\rangle + D^*C\langle p|\widetilde{F}_T|u\rangle\Big]\,, \\
&C = -D^*A, \quad B = -K^*A\,,
\end{aligned}
\tag{3.2.27}
$$

and evaluate the right hand sides of these equations iteratively, beginning with the leading order solution. To get a perturbation expansion we also need to expand $\{\langle u|\widetilde{F}_T|g\rangle - \langle p|\widetilde{F}_T|p\rangle\}^{-1}$ in a power series in λ starting with the leading order solution. Each power of λ will be free from any pole near $E^2 = m^2$ as long as the leading order result for $\lambda^{-1}\{\langle u|\widetilde{F}_T|g\rangle - \langle p|\widetilde{F}_T|p\rangle\}$ does not have any zero near $E^2 = m^2$. Once we determine the coefficients $A, \cdots K$ we can also determine $\widetilde{F}(E)$ using (3.2.20).

[3] Eqs.(3.2.18) have a symmetry under which the constants A, B, C are multiplied by an overall phase. We have chosen this phase appropriately to make A real.

Note that in this scheme even in a fixed order in λ we need to iterate the procedure infinite number of times to generate all powers of $E^2 - m^2$. However eventually we are interested in computing these coefficients at the physical mass^2 which differs from m^2 by order λ. Similarly when we solve (3.2.21) to find the location of the pole, we need to know the expansion of $\widetilde{F}(E)$ to order $(E^2 - m^2)^n$ for computing the correction to mass^2 to order λ^{n+1}. Thus for computing physical quantities to any given order in λ we need to run the iteration only a finite number of times.

We now observe that since eq.(3.2.26) gives $B \simeq -K^* \sim 1$, it follows from (3.2.15) that $|p\rangle'$ differs from $|p\rangle$ by an order one term proportional to the pure gauge states. This is a consequence of having degenerate eigenvalues at the tree level and will continue to be true in string theory as well. On the other hand since $C \sim E^2 - m^2$ which is of order λ when E is set equal to the corrected physical mass, the coefficient of $|u\rangle$ in $|p\rangle'$ vanishes as $\lambda \to 0$. Thus the quantum corrected physical state approaches a linear combination of the unperturbed physical state and the unperturbed pure gauge state in the limit in which we switch off the perturbation. This is consistent with the criteria for identifying the quantum corrected physical state that we set out at the beginning of this subsection.

3.2.3 Explicit evaluation of the physical mass

Let us now explicitly evaluate the coefficients $A, \cdots K$ and $F(E)$ for the problem at hand and from this find the location of the physical pole. From (3.2.10), (3.2.13) it follows that here

$$\begin{aligned}
&\langle p|\widetilde{F}_T|p\rangle = -\gamma^2 - \alpha, \quad \langle g|\widetilde{F}_T|g\rangle = \frac{\beta}{2}(2m - \beta - 2|E|), \quad \langle u|\widetilde{F}_T|u\rangle = \frac{\beta}{2}(2m - \beta + 2|E|), \\
&\langle p|\widetilde{F}_T|g\rangle = \langle g|\widetilde{F}_T|p\rangle = \frac{1}{\sqrt{2}}(-|E| + m - \beta)\gamma, \quad \langle p|\widetilde{F}_T|u\rangle = \langle u|\widetilde{F}_T|p\rangle = \frac{1}{\sqrt{2}}(|E| + m - \beta)\gamma, \\
&\langle g|\widetilde{F}_T|u\rangle = \langle u|\widetilde{F}_T|g\rangle = \frac{\beta}{2}(2m - \beta),
\end{aligned} \tag{3.2.28}$$

This gives the leading order solutions (3.2.26) to be

$$A = 1, \quad B = -K = -\frac{1}{\sqrt{2}}\,(\beta m + \alpha)^{-1}\gamma(m + |E|),$$
$$C = -D = -(\beta m + \alpha)^{-1}\left\{\frac{1}{\sqrt{2}}(-|E| + m)\gamma - \frac{\beta\gamma}{\sqrt{2}}(m^2 - E^2)(\beta m + \alpha)^{-1}\right\}. \tag{3.2.29}$$

There are corrections to these solutions of order λ and also of order $(E^2 - m^2)$ ($(E^2 - m^2)^2$ in C and D), but these will not be needed for computing the leading correction to the physical mass. Since α, β, γ are each of order λ we see that $B \simeq -K^* \sim 1$ and $C \simeq -D^* \sim (|E| - m)$ in the $\lambda \to 0$ limit, in agreement with the general results quoted earlier. Substituting these into (3.2.20) and using (3.2.28) we get

$$\widetilde{F}(E) = -\alpha + \mathcal{O}(\lambda^2) + \mathcal{O}(\lambda)(|E| - m)\,. \tag{3.2.30}$$

The iterative procedure (3.2.21) now gives the leading order correction to the physical mass

$$E^2 = m^2 + \alpha + \mathcal{O}(\lambda^2)\,. \tag{3.2.31}$$

The physical state at leading order in λ, obtained from (3.2.13), (3.2.15), (4.2.3) and (3.2.29) is given by, for $E = \sqrt{m^2 + \alpha}$

$$\begin{pmatrix} i\gamma E/(\alpha + \beta m) \\ -\gamma m/(\alpha + \beta m) \\ 1 \end{pmatrix} + \mathcal{O}(\lambda)\,. \tag{3.2.32}$$

Let us compare this with the exact result. We have from (3.2.9)

$$\det M = -(E^2 - m^2 + m\beta)^2(E^2 - m^2 - \alpha)\,. \tag{3.2.33}$$

This has zeroes at $E^2 = m^2 + \alpha$ and $E^2 = m^2 - m\beta$. Since we know that β enters through the renormalized gauge fixing term, the physical mass should not depend on β. This determines the physical pole to be at

$$E^2 = m^2 + \alpha \,, \tag{3.2.34}$$

which agrees with the perturbative result (3.2.31). Furthermore at $E = \sqrt{m^2 + \alpha}$ we can easily compute the zero eigenvector of M and it is given by

$$v = \begin{pmatrix} i\gamma E/(\alpha + \beta m) \\ -\gamma m/(\alpha + \beta m) \\ 1 \end{pmatrix} . \tag{3.2.35}$$

This agrees with the perturbative result (3.2.32) up to corrections of order λ.

3.2.4 Masses of the unphysical / pure gauge states

For completeness we shall also describe the computation of the masses in the unphysical / pure gauge sector using perturbation theory. For this we define the matrices

$$\mathcal{I}' = \begin{pmatrix} {}'\langle g|\mathcal{I}|g\rangle' & {}'\langle g|\mathcal{I}|u\rangle' \\ {}'\langle u|\mathcal{I}|g\rangle' & {}'\langle u|\mathcal{I}|u\rangle' \end{pmatrix} , \quad \widetilde{F}' = \begin{pmatrix} {}'\langle g|\widetilde{F}_T|g\rangle' & {}'\langle g|\widetilde{F}_T|u\rangle' \\ {}'\langle u|\widetilde{F}_T|g\rangle' & {}'\langle u|\widetilde{F}_T|u\rangle' \end{pmatrix} . \tag{3.2.36}$$

Then the unphysical / pure gauge sector masses will be at the zeroes of the eigenvalues of the matrix

$$(m^2 - E^2)\mathcal{I}' - \widetilde{F}'(E) \,, \tag{3.2.37}$$

as a function of E.[4] For computing the first subleading correction to the unphysical mass we can use the ansatz that the zero eigenvalue of (3.2.37) will occur at $(E - m) \sim \lambda$ and evaluate each matrix element to order λ using this ansatz. Since $(m^2 - E^2) \sim \lambda$ we have to evaluate $\mathcal{I}'$ to order unity. It follows from (3.2.15) and the fact that at the leading order $D \sim (E^2 - m^2) \sim \lambda$

[4]It follows from (3.2.15), (3.2.24) that $\mathcal{I}'$ is a non-singular matrix near $E \sim m$ and hence the zero eigenvalue of (3.2.37) occurs at the same value of E as that of $m^2 - E^2 - (\mathcal{I}')^{-1}\widetilde{F}'(E)$.

that $\mathcal{I}'$ to order λ^0 has the structure

$$\begin{pmatrix} 0 & 1 \\ 1 & K^*K \end{pmatrix}, \tag{3.2.38}$$

with K given in (3.2.26). On the other hand (3.2.15), (3.2.23) and (3.2.24) shows that $\widetilde{F}'$ to order λ has the structure

$$\begin{pmatrix} 0 & \langle g|\widetilde{F}_T|u\rangle \\ \langle u|\widetilde{F}_T|g\rangle & {}'\langle u|\widetilde{F}_T|u\rangle' \end{pmatrix}. \tag{3.2.39}$$

Thus for computing order λ correction to the unphysical / pure gauge sector masses we need to look for zero eigenvalue of the matrix

$$\begin{pmatrix} 0 & (m^2-E^2) - \langle g|\widetilde{F}_T|u\rangle \\ (m^2-E^2) - \langle u|\widetilde{F}_T|g\rangle & (m^2-E^2)K^*K - {}'\langle u|\widetilde{F}_T|u\rangle' \end{pmatrix}. \tag{3.2.40}$$

Now in order that a matrix has zero eigenvalue, its determinant must vanish. From the structure of the matrix given above it is clear that this requires one of the off-diagonal elements to vanish. Since the off-diagonal elements are conjugates of each other and hence vanish at the same value of E, the condition for zero eigenvalue of the (3.2.40) can be stated as

$$(m^2-E^2) - \langle u|\widetilde{F}_T|g\rangle = 0\,. \tag{3.2.41}$$

Using the value of $\langle u|\widetilde{F}_T|g\rangle$ quoted in (3.2.28) we see that to order λ the renormalized masses in the unphysical / pure gauge sector occur at the zero of

$$E^2 - m^2 + m\beta = 0\,. \tag{3.2.42}$$

This is in agreement with the exact result quoted below (3.2.33).

3.3 Organization of off-shell amplitudes in string theory

In this section we shall discuss some general aspects of off-shell states and off-shell amplitudes in closed bosonic string theory.

3.3.1 Off-shell string states and a basis

We begin by describing the space of off-shell string states with which we shall work and reviewing some well known results about the choice of basis for off-shell states. Off-shell string states are required to satisfy the following conditions:

1. They have ghost number 2 where we count the c, $\bar{c}$ ghosts to have ghost number 1, b, $\bar{b}$ ghosts to have ghost number -1 and SL(2,C) invariant vacuum to have ghost number 0.

2. They are annihilated by the b, $\bar{b}$ ghost zero modes b_0 and $\bar{b}_0$ and $L_0 - \bar{L}_0$ where $\bar{L}_n$ and L_n are the total left and right moving Virasoro generators.

This is also the space of off-shell states in covariant closed string field theory in the Siegel gauge [32]. The requirement of annihilation by $(L_0 - \bar{L}_0)$ and $(b_0 - \bar{b}_0)$ is needed for consistently defining off-shell amplitude [27] whereas the condition $(b_0 + \bar{b}_0)\,|\text{state}\rangle = 0$ is needed to make the kinetic operator invertible.[5] In this space we can introduce a non-degenerate inner product between states $|s\rangle$ and $|s'\rangle$ via

$$\langle s|s'\rangle \equiv \langle s|c_0\bar{c}_0|s'\rangle_{BPZ} \tag{3.3.1}$$

[5] In contrast the off-shell states in gauge invariant closed string field theory of [32] are only annihilated by $(b_0 - \bar{b}_0)$ and $(L_0 - \bar{L}_0)$. Like in all gauge theories, the kinetic operator in this theory is not invertible till we fix a gauge and the Siegel gauge condition of annihilation by $(b_0 + \bar{b}_0)$ precisely does that. In quantum closed string field theory we also need to relax the constraint on the ghost number and allow states of all ghost numbers to propagate in the loop. In our analysis we shall dump all the loop contributions into one particle irreducible (1PI) amplitudes and express the full amplitude as sum of tree diagrams constructed out of 1PI amplitudes as vertices and tree level propagators. Thus the only place where we have to explicitly introduce off-shell states is as the external lines of the 1PI amplitudes and as the states propagating along the propagator in the tree amplitudes. These states always carry ghost number two when we compute physical amplitudes relevant for mass renormalization or S-matrix elements, and hence we have put that restriction on the definition of off-shell states.

where $\langle r|r'\rangle_{BPZ}$ is the BPZ inner product. In defining the bra $\langle r|$ corresponding to a given ket $|r\rangle$ we reverse the sign of the momentum. We also remove the momentum conserving delta function from the definition of the inner product. The fact that the inner product is non-degenerate follows from the Fock space representation of the basis states.

On-shell condition for the string state $|s\rangle$ takes the form

$$L_0|s\rangle = 0\,, \tag{3.3.2}$$

which also implies $\bar{L}_0|s\rangle = 0$. On-shell we can divide the states into physical, pure gauge and unphysical states as follows. First of all pure gauge states are of the form

$$Q_B|r\rangle \tag{3.3.3}$$

where Q_B is the total BRST charge (left moving plus right moving) and $|r\rangle$ is a state of ghost number 1 annihilated by b_0, $\bar{b}_0$, L_0 and $\bar{L}_0$. Since Q_B has ghost number 1, commutes with L_n, $\bar{L}_n$ and $\{Q_B, b_0\} = L_0$ and $\{Q_B, \bar{b}_0\} = \bar{L}_0$, it follows that $Q_B|r\rangle$ has ghost number 2 and is annihilated by b_0, $\bar{b}_0$, L_0 and $\bar{L}_0$.

Physical states are defined to be states of ghost number two which are annihilated by Q_B, b_0, $\bar{b}_0$, L_0 and $\bar{L}_0$ but cannot be written in the form $Q_B|r\rangle$ with $|r\rangle$ annihilated by b_0, $\bar{b}_0$, L_0 and $\bar{L}_0$. It follows from this that the physical states are orthogonal to pure gauge states. The main point to note is that $\{Q_B, c_0\}$ and $\{Q_B, \bar{c}_0\}$ do not have any c_0 or $\bar{c}_0$ factor, and hence the matrix elements of $\{Q_B, c_0\}$ and $\{Q_B, \bar{c}_0\}$ between states, satisfying condition 2 above, vanish. The same argument, together with the relation $Q_B^2 = 0$, shows that the pure gauge states also have vanishing inner product with pure gauge states. A linearly independent basis of physical states is the maximal set of physical states satisfying the condition that no linear combination of these basis states is a pure gauge state.

Now since the inner product is non-degenerate there must exist states which have non-vanishing inner product with the pure gauge states. These states are annihilated by b_0, $\bar{b}_0$, L_0 and $\bar{L}_0$, but not by Q_B. We shall call them unphysical states. We can choose a linearly independent basis of unphysical states such that no linear combination is annihilated by Q_B. The number of such basis states must be at least equal to the number of pure gauge states so that we have a non-degenerate inner product matrix. We shall now argue that the number is actually equal to the number of pure gauge states. For this let us temporarily relax the constraint on the ghost number and consider states of all ghost number annihilated by b_0, $\bar{b}_0$, L_0 and $\bar{L}_0$. Then since for every unphysical state $|s\rangle$ of ghost number g, $Q_B|s\rangle$ is a pure gauge state of ghost number $g+1$, we conclude that the number of pure gauge states at ghost number $g+1$ is the same as the number of unphysical states at ghost number g. On the other hand, since the inner product (3.3.1) pairs states of ghost number g and $4-g$, we know from our previous argument that the number of unphysical states at ghost number $3-g$ must be at least equal to the number of pure gauge states at ghost number $g+1$ and hence the number of unphysical states at ghost number g. Taking $g \to 3-g$ we can arrive at the reverse conclusion. This shows that the number of unphysical states at ghost number $3-g$ should be equal to the number of unphysical states at ghost number g and hence the number of pure gauge states at ghost number $g+1$. Taking $g=1$ we see that the number of unphysical states at ghost number 2 must be equal to the number of pure gauge states at ghost number 2. This is the promised result.

Let us now return to states of ghost number 2 only. We have already seen that the inner product pairs unphysical states with pure gauge states by a non-degenerate matrix and that the pure gauge states are orthogonal to themselves as well as physical states. By adding appropriate linear combinations of pure gauge states and physical states to the unphysical states we can ensure that the latter are orthonormal to the physical states and unphysical states.

Taking further linear combinations within physical states and within unphysical states we can ensure that the physical states form an orthonormal basis and that the pure gauge states and the unphysical states are paired in a one to one fashion. Thus at any mass level the inner product matrix will have a block diagonal structure of the form

$$\mathcal{I} = \begin{pmatrix} & I & \\ I & & \\ & & I \end{pmatrix} \tag{3.3.4}$$

where I denotes identity matrix of appropriate dimensions. The first set of rows/columns stand for pure gauge states, the second set of rows/columns stand for unphysical states and the last set of rows/columns stand for physical states. At non-zero momentum, it is in fact possible to choose a basis satisfying this requirement with physical states of the form

$$|\alpha\rangle = c_1\bar{c}_1|\Phi_\alpha\rangle \tag{3.3.5}$$

where Φ_α are dimension (1,1) primary in the matter sector satisfying

$$\langle\alpha|\beta\rangle \equiv \langle\Phi_\alpha|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1|\Phi_\beta\rangle_{BPZ} = \delta_{\alpha\beta}\,. \tag{3.3.6}$$

Physical states of the form (3.3.5) are dimension zero primaries and hence transform as scalars under conformal transformation.

So far we have reviewed well known results, but now we shall make a small jump and discuss the off-shell continuation of these results. At a given mass level m we can go off-shell (satisfying the two conditions mentioned at the beginning of this section) by deforming the momentum k such that $k^2 + m^2$ is deformed away from zero. We shall require the deformed basis to still satisfy the inner product structure described in (3.3.4), but will need to relax the various other requirements by terms of order $(k^2 + m^2)$. For example if we take a state $|s\rangle$ of

ghost number 1 that is annihilated by b_0, $\bar{b}_0$ and $(L_0 - \bar{L}_0)$, and apply the BRST charge Q_B on it, the resulting state will not be annihilated by b_0 and $\bar{b}_0$. The part that is not annihilated by b_0 and $\bar{b}_0$ is given by $(c_0 + \bar{c}_0)L_0|s\rangle = \frac{1}{4}(k^2 + m^2)(c_0 + \bar{c}_0)|s\rangle$. Hence the off-shell 'pure gauge' states will have to be defined as $Q_B|s\rangle - \frac{1}{4}(k^2 + m^2)(c_0 + \bar{c}_0)|s\rangle$. These are not annihilated by Q_B but under the action of Q_B give states proportional to $(k^2 + m^2)$. Similarly physical states will now be defined by first continuing the momentum off-shell and then by adding appropriate linear combination of unphysical states proportional to $(k^2 + m^2)$ so that they remain orthonormal to the pure gauge states. These will only be BRST invariant up to terms of order $(k^2 + m^2)$ and transform under a conformal transformation as scalars up to terms of order $(k^2 + m^2)$. Similar procedure can be used to define the unphysical states off-shell so that they remain orthogonal to physical states and themselves.

We shall denote by $|\alpha\rangle_p$, $|s\rangle_g$ and $|s\rangle_u$ an appropriate basis of off-shell physical, pure gauge and unphysical states at mass level m, satisfying the identities

$$ {}_p\langle\alpha|\beta\rangle_p = \delta_{\alpha\beta}, \quad {}_g\langle r|s\rangle_u = {}_u\langle r|s\rangle_g = \delta_{rs}, \quad {}_p\langle\alpha|s\rangle_u = {}_p\langle\alpha|s\rangle_g = 0, \quad {}_g\langle r|s\rangle_g = {}_u\langle r|s\rangle_u = 0\,. \tag{3.3.7} $$

Note that this preserves the inner product matrix $\mathcal{I}$ given in (3.3.4). We shall see that at higher loop order we need to redefine the physical, unphysical and pure gauge states by making a further rotation of the basis.

3.3.2 Off-shell amplitudes

In this subsection we shall describe the construction of off-shell amplitudes in string theory, which in turn was inspired by bosonic string field theory [32] and other earlier work (*e.g.* [27, 34]). In order to define off-shell amplitudes in string theory we need to introduce local coordinate system around the punctures on the Riemann surface where the vertex operators are inserted [27] (see also [28–31]). Let us denote by z a reference coordinate system on a

Riemann surface, possibly consisting of several coordinate charts. Let z_i denote the location of the i-th puncture in the z-coordinate system and w_i denote the local coordinate system around the i-th puncture, related to z by some functional relation $z = f_i(w_i)$ such that the $w_i = 0$ point gets mapped to $z = z_i$: $f_i(0) = z_i$. Then the contribution to the n-point off-shell amplitude from the genus g Riemann surfaces can be expressed as

$$\int_{\mathcal{M}_{g;n}} \left\langle \prod_{i=1}^{n} f_i \circ V_i(0) \times \text{ghost insertions} \right\rangle , \tag{3.3.8}$$

where $f \circ V(0)$ denotes the conformal transformation of the vertex operator V by the function $f(w)$, the correlator $\langle\ \rangle$ is evaluated in the reference z-coordinate system and $\int_{\mathcal{M}_{g;n}}$ denotes integration over the moduli space of Riemann surfaces of genus g with n punctures with appropriate measure. A detailed description of how to construct the integration measure (or equivalently the rules for inserting b-ghosts into the correlation function) for a given choice of local coordinate system can be found in [27, 32]. The off-shell amplitudes defined this way depend on the choice of local coordinate system w_i but are independent of the choice of the reference coordinate system z.

We shall work with a class of local coordinate systems satisfying the following properties:[6]

1. The local coordinate system is taken to be symmetric in all the puncture, i.e. the function $f_i(w)$ should depend on i only via the location z_i of the puncture.

2. On 3-punctured sphere and 1-punctured tori the choice of the local coordinate system is arbitrary subject to condition 1. We declare all 3-punctured spheres and 1-punctured

[6] We note that the choice of local coordinates which appear in the Siegel gauge amplitudes in closed bosonic string field theory of [32] automatically satisfies these requirements. Thus all our subsequent discussions hold for this theory. In particular our analysis shows that the renormalized physical masses are the same in different versions of closed string field theory using different vertices satisfying Batalin-Vilkovisky equations. Since these different versions are related to each other by field redefinitions together with a change in the gauge fixing condition [36] this indirectly tests gauge invariance of the renormalized physical masses in closed string field theory.

tori to be one particle irreducible (1PI) contributions to genus zero 3-point amplitudes and genus one 1-point amplitudes respectively.

3. We can construct a set of 4-punctured spheres by gluing a 3-punctured sphere with another 3-punctured sphere at one each of their punctures by the plumbing fixture procedure

$$w_1 w_2 = e^{-s+i\theta} \quad 0 \le \theta < 2\pi, \quad 0 \le s < \infty \,. \tag{3.3.9}$$

Here w_1 and w_2 are the local coordinates at the punctures used for gluing. We choose the local coordinates on these 4-punctured spheres to be the ones induced from the local coordinates on the original 3-punctured spheres [37], and declare the contribution from these 4-punctured spheres to off-shell four point amplitudes to be the one particle reducible (1PR) contributions to the genus zero four point amplitudes. On the rest of the genus zero four punctured Riemann surfaces we choose the local coordinate system arbitrarily subject to condition 1 and continuity and declare them to be 1PI contributions to genus zero four point amplitude. We shall use a shorthand notation calling the corresponding Riemann surfaces 1PI Riemann surfaces. Similarly by gluing a 3-punctured sphere to a 1-punctured torus we can generate a set of 2-punctured tori. We choose the local coordinates on these 2-punctured tori to be the ones induced from the local coordinates of the 3-punctured sphere and the 1-punctured torus, and declare their contribution to be the 1PR contribution to the genus one 2-point function. On the rest of the 2-punctured tori we choose the local coordinates arbitrarily subject to condition 1 and the requirement of continuity, and declare them to be 1PI contribution to the genus one 2-point amplitude.

4. We now repeat this process to Riemann surfaces of higher genus and/or higher number of punctures. At any stage, Riemann surfaces which can be obtained by gluing two or more 1PI Riemann surfaces to each other using the plumbing fixture procedure are declared to be contributions to 1PR amplitudes and on these Riemann surfaces the choice of local coordinates is induced from the local coordinates of the 1PI Riemann surfaces which have

been glued. The rest of the Riemann surfaces are declared as 1PI contributions and the local coordinates at the punctures on these Riemann surfaces can be chosen arbitrarily subject to condition 1 and continuity.

We shall call the choice of local coordinates satisfying the criteria described above *'gluing compatible local coordinate system'*. In the language of string field theory this has been called off-shell factorization, – a brief discussion and relevant references can be found in [39].

For our analysis it will also be useful to introduce the notion of amplitudes which are *1PI in a given momentum* k, where k is the sum of a subset of the momenta carried by the external states of that amplitude. Riemann surfaces 1PI in the leg carrying momentum k are defined to be those Riemann surfaces which cannot be obtained by gluing two or more 1PI or 1PR Riemann surfaces at punctures carrying momenta k and $-k$. Thus this set of Riemann surfaces include the usual 1PI Riemann surfaces but also many 1PR Riemann surfaces which are obtained by gluing two or more 1PI Riemann surfaces at punctures carrying momenta other than k or $-k$. The total contribution to an amplitude 1PI in momentum k is then obtained by integrating over the moduli spaces of all Riemann surfaces which are 1PI in momentum k.

As an example consider genus one 2-point function with external vertex operators carrying momentum k and $-k$. This receives contribution from 1PI Riemann surfaces and also 1PR Riemann surfaces obtained by gluing 1-punctured torus to 3-punctured sphere. However all of these are counted as 1PI in the momentum k since the 1PR Riemann surfaces are obtained by gluing punctures carrying zero momentum, and not momentum $\pm k$.

3.3.3 Off-shell amplitudes from 1PI amplitudes

As we shall now discuss, the off-shell amplitudes constructed with the help of such choice of local coordinates can be organized in the same way that the full amplitudes in a quantum field theory can be organized as sums over tree level Feynman diagrams with 1PI amplitudes as

vertices. As in [38] we begin our discussion with the propagator. We shall work with general off-shell string states of ghost number 2, as defined in §3.3.1. If $\pm k$ denote the momenta carried by the external legs, then let $\widehat{\mathcal{F}}$ be the contribution to the off-shell two point amplitude from Riemann surfaces which are 1PI in momentum k. This includes sum over different genera starting from genus 1. As discussed in [38], this can be regarded as a map from $\mathcal{H}\times\mathcal{H}$ to $\mathbb{C}$ where $\mathcal{H}$ denotes the Hilbert space of off-shell states of ghost number two as defined in §3.3.1, but using the duality between ghost number two and ghost number four states by the BPZ inner product we can also regard this as a map from states of ghost number two to states of ghost number four which are annihilated by c_0 and $\bar{c}_0$. We can include a further action by $\bar{b}_0 b_0$ to regard $\widehat{\mathcal{F}}$ as a map from $\mathcal{H}$ to $\mathcal{H}$. This is the viewpoint we shall adopt from now. The factor of $\bar{b}_0 b_0$ in fact arises naturally in the tree level propagator of the string, which after being stripped of this factor, has the form

$$\Delta = \frac{1}{4\pi}\int_0^\infty ds \int_0^{2\pi} d\theta\, e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} = \frac{1}{2(L_0+\bar{L}_0)}\delta_{L_0,\bar{L}_0}\,. \tag{3.3.10}$$

With this convention the full propagator is given by

$$\Pi = \Delta + \Delta\widehat{\mathcal{F}}\Delta + \Delta\widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}}\Delta + \cdots = \Delta(1-\widehat{\mathcal{F}}\Delta)^{-1} = (1-\Delta\widehat{\mathcal{F}})^{-1}\Delta\,. \tag{3.3.11}$$

Pictorially this contribution can be represented as in Fig. 4.3 with the horizontal line denoting Δ and the blob marked 1PI denoting the contribution $\widehat{\mathcal{F}}$ from the Riemann surfaces that are 1PI in momentum k. If $\mathcal{F}$ is the full off-shell two point function, then $\mathcal{F}$ and Π are related by

$$\Pi = \Delta + \Delta\mathcal{F}\Delta\,. \tag{3.3.12}$$

Also $\mathcal{F}$ and $\widehat{\mathcal{F}}$ are related by

$$\mathcal{F} = \widehat{\mathcal{F}} + \widehat{\mathcal{F}}\Delta\widehat{\mathcal{F}} + \cdots = \widehat{\mathcal{F}}(1-\Delta\widehat{\mathcal{F}})^{-1} = (1-\widehat{\mathcal{F}}\Delta)^{-1}\widehat{\mathcal{F}} = \widehat{\mathcal{F}} + \widehat{\mathcal{F}}(\Delta^{-1}-\widehat{\mathcal{F}})^{-1}\widehat{\mathcal{F}}\,. \tag{3.3.13}$$

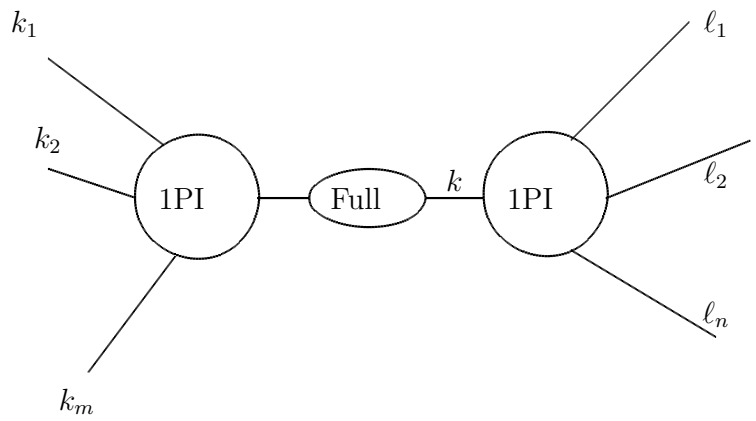

Figure 3.1: Pictorial representation of the second terms on the right hand sides of eq.(3.3.14). Here 1PI means sum of contributions which are 1PI in the leg carrying momentum k, whereas Full means sum of all contributions to the 2-point function shown in Fig.4.3.

Like $\widehat{\mathcal{F}}$, $\mathcal{F}$, Π and Δ can be regarded as maps from $\mathcal{H}$ to $\mathcal{H}$.

As described in [38], we can use (3.3.13) to define $\widehat{\mathcal{F}}$ in terms of $\mathcal{F}$. At genus one $\widehat{\mathcal{F}} = \mathcal{F}$. Starting with this, we define $\widehat{\mathcal{F}}$ at genus two so as to satisfy (3.3.13) up to genus two. Physically the contribution to $\widehat{\mathcal{F}}$ at genus two is given by integrating over those Riemann surfaces which cannot be obtained by plumbing fixture of a pair of genus one Riemann surfaces. This definition of course depends on the choice of local coordinates at the punctures that we use to glue the two genus one Riemann surfaces. This procedure can be continued to define $\widehat{\mathcal{F}}$ at higher orders.

As another example let us consider an $m+n$ point amplitude Γ with external momenta $k_1, \cdots k_m$, $\ell_1, \cdots \ell_n$ satisfying $\sum_{i=1}^m k_i = -\sum_{j=1}^n \ell_j = k$, and other quantum numbers $a_1, \cdots a_m$, $b_1, \cdots b_n$. Our goal is to express the amplitude in a way that makes manifest the poles in the momentum k. For this we introduce two auxiliary quantities: Γ_1^a describing the contribution to $(m+1)$-point functions with external states carrying quantum numbers $a_1, \cdots a_m$, a and momenta $k_1, \cdots k_m$, $-k$ and Γ_2^b describing the contribution to $(n+1)$-point functions with external states carrying quantum numbers $b_1, \cdots b_n$, b and momenta $\ell_1, \cdots \ell_n$, k. Here the quantum numbers a and b run over all off-shell string states of ghost number 2. Note that

we have not explicitly exhibited the dependence of Γ_1 on the indices $a_1, \cdots a_m$ and momenta $k_1, \cdots k_m$ for brevity; a similar comment holds for Γ_2. We shall also introduce the quantities $\widehat{\Gamma}_1^a$ and $\widehat{\Gamma}_2^b$ which describe contributions to Γ_1^a and Γ_2^b from those Riemann surfaces which are 1PI in the leg carrying momentum k (in the sense described at the end of §3.3.2). Then the full contribution to Γ can be expressed as

$$\begin{aligned}\Gamma &= \widehat{\Gamma} + \widehat{\Gamma}_1^a \mathcal{I}_{ac} \Pi_{cb} \widehat{\Gamma}_2^b \\ &= \widehat{\Gamma} + \widehat{\Gamma}_1^T \mathcal{I} \Delta (1 - \widehat{\mathcal{F}} \Delta)^{-1} \widehat{\Gamma}_2 \\ &= \widehat{\Gamma} + \widehat{\Gamma}_1^T \mathcal{I} (1 - \Delta \widehat{\mathcal{F}})^{-1} \Delta \widehat{\Gamma}_2 \,, \end{aligned} \tag{3.3.14}$$

where $\widehat{\Gamma}$ represents contributions to Γ which are 1PI in the leg carrying momentum k and $\mathcal{I}$ is the inner product matrix (3.3.4) over the full space of off-shell string states. The equality between different expressions on the right hand sides of (3.3.14) follows from (3.3.11). A pictorial representation of the second term on the right hand side of the first line of (3.3.14) has been shown in Fig. 3.1

3.4 Physical state propagator in string theory

In this section we shall generalize the gauge theory analysis of §3.2 to give an iterative procedure for constructing physical state propagator in string theory. From this we can compute the masses of physical states.

3.4.1 Renormalized propagator at a given mass level

Since string theory contains infinite number of states, the quantities Π, Δ, $\mathcal{F}$ and $\widehat{\mathcal{F}}$ introduced in §3.3.3 are all infinite dimensional matrices. Our first step will be to 'integrate out' all states except the ones at mass level m so that we can work with finite dimensional matrices with rows

and columns labelled by states at mass level m.[7] For this we denote by P_T the total projection operator at mass level m,

$$P_T = \{|\alpha\rangle_p \, {}_p\langle\alpha| + |s\rangle_g \, {}_u\langle s| + |s\rangle_u \, {}_g\langle s|\}\,, \tag{3.4.1}$$

and define

$$\bar\Delta = \Delta - (k^2+m^2)^{-1} P_T\,, \tag{3.4.2}$$

$$\bar{\mathcal{F}} = \widehat{\mathcal{F}} + \widehat{\mathcal{F}}\bar\Delta\widehat{\mathcal{F}} + \cdots = \widehat{\mathcal{F}}(1-\bar\Delta\widehat{\mathcal{F}})^{-1} = (1-\widehat{\mathcal{F}}\bar\Delta)^{-1}\widehat{\mathcal{F}} = \widehat{\mathcal{F}} + \widehat{\mathcal{F}}(\bar\Delta^{-1}-\widehat{\mathcal{F}})^{-1}\widehat{\mathcal{F}}\,, \tag{3.4.3}$$

where $\widehat{\mathcal{F}}$ has been defined in §3.3.3. It is clear from the definition of $\bar\Delta$ and $\widehat{\mathcal{F}}$ that their genus expansions do not have any poles at $k^2=-m^2$. Hence $\bar{\mathcal{F}}$ defined in (3.4.3) also does not have such poles. From (3.3.13), (3.4.3) we get

$$\mathcal{F} = \bar{\mathcal{F}}\{1-(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}\}^{-1} = \{1-\bar{\mathcal{F}}\,(k^2+m^2)^{-1}P_T\}^{-1}\bar{\mathcal{F}}\,. \tag{3.4.4}$$

We now define

$$\mathcal{P}_T = P_T\,\mathcal{I}\,\Pi\,P_T, \quad \widetilde{F}_T = P_T\,\mathcal{I}\,\bar{\mathcal{F}}P_T, \quad F_T = P_T\mathcal{I}\mathcal{F}P_T\,, \tag{3.4.5}$$

where Π has been defined in (3.3.11). Physically F_T denotes the two point amplitude restricted to external states of mass level m, $\widetilde{F}_T$ is the contribution to F_T that is 1PI in momentum k *after integrating out all states other than those at mass level* m, and $\mathcal{P}_T$ denotes the off-shell two point Green's function restricted to external states of mass level m. It follows from (3.4.4), (3.4.5) that

$$\begin{aligned} F_T &= \widetilde{F}_T(1-(k^2+m^2)^{-1}\mathcal{I}\widetilde{F}_T)^{-1}, \\ \mathcal{P}_T &= (k^2+m^2)^{-1}\mathcal{I}P_T + (k^2+m^2)^{-2}\,F_T = P_T\{(k^2+m^2)\mathcal{I}-\widetilde{F}_T\}^{-1}\,, \end{aligned} \tag{3.4.6}$$

[7] Throughout this paper we shall denote by states at mass level m all states which have tree level mass m, i.e. states which are annihilated by L_0 and $\bar L_0$ when $k^2=-m^2$.

where it is understood that the inverse on the right hand sides is being taken in the finite dimensional subspace of mass level m states only. We shall label the matrices $\widetilde{F}_T$ and $\mathcal{P}_T$ as

$$\begin{pmatrix} {}_g\langle r|\widetilde{F}_T|r'\rangle_g & {}_g\langle r|\widetilde{F}_T|s'\rangle_u & {}_g\langle r|\widetilde{F}_T|\alpha'\rangle_p \\ {}_u\langle s|\widetilde{F}_T|r'\rangle_g & {}_u\langle s|\widetilde{F}_T|s'\rangle_u & {}_u\langle s|\widetilde{F}_T|\alpha'\rangle_p \\ {}_p\langle \alpha|\widetilde{F}_T|r'\rangle_g & {}_p\langle \alpha|\widetilde{F}_T|s'\rangle_u & {}_p\langle \alpha|\widetilde{F}_T|\alpha'\rangle_p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} {}_g\langle r|\mathcal{P}_T|r'\rangle_g & {}_g\langle r|\mathcal{P}_T|s'\rangle_u & {}_g\langle r|\mathcal{P}_T|\alpha'\rangle_p \\ {}_u\langle s|\mathcal{P}_T|r'\rangle_g & {}_u\langle s|\mathcal{P}_T|s'\rangle_u & {}_u\langle s|\mathcal{P}_T|\alpha'\rangle_p \\ {}_p\langle \alpha|\mathcal{P}_T|r'\rangle_g & {}_p\langle \alpha|\mathcal{P}_T|s'\rangle_u & {}_p\langle \alpha|\mathcal{P}_T|\alpha'\rangle_p \end{pmatrix} \tag{3.4.7}$$

respectively.

$\mathcal{P}_T$ and $\widetilde{F}_T$ and the inner product matrix $\mathcal{I}$ are the exact analogs of the corresponding quantities defined in §3.2. In particular the genus expansion of $\widetilde{F}_T$ is free from any poles at $k^2 = -m^2$ at every order. In §3.4.3 we shall generalize the procedure of §3.2 to construct the propagator of physical states.

One point worth emphasizing is that for our analysis we do not really need the gluing compatibility condition discussed in §3.3.2 to be valid for the whole range $0 \le s < \infty$ with s defined in (3.3.9); it is sufficient if the compatibility condition holds in a small neighborhood of degeneration points, *e.g.* for $s \ge s_0$ for some constant s_0. One way to see this is that we can rescale all the local coordinates w_i to bring the range $s \ge s_0$ in (3.3.9) to $s \ge 0$. This will have the effect of rescaling all the off-shell amplitudes by some power of $e^{-s_0(k^2+m^2)}$. But we can also proceed with the original choice of local coordinates and repeat the whole analysis by changing the definition of 1PI and 1PR amplitudes so that two or more 1PI amplitudes glued together using (3.3.9) for $s \ge s_0$ are now declared as 1PR. We also have to modify the definition of Δ given in (3.3.10), with the integral over s now running from s_0 to ∞. This will produce a multiplicative factor of $e^{-s_0(L_0+\bar{L}_0)}$ in the definition of Δ. But the rest of the analysis is not affected by this. In particular we can continue to define $\bar{\Delta}$ and $\bar{\mathcal{F}}$ via eqs.(3.4.2) and (3.4.3). The contribution to $\bar{\Delta}$ from states of mass level m now gives $P_T(k^2+m^2)^{-1}(e^{-s_0(k^2+m^2)/2}-1)$. Since this does not have a pole at $(k^2+m^2)=0$, $\bar{\Delta}$ and $\bar{\mathcal{F}}$ will continue to be free from poles

at $k^2 + m^2 = 0$.

3.4.2 An alternate definition of $\widetilde{F}_T$

The definition of $\widetilde{F}_T$ given in §3.4.1 looks complicated, since we first need to define the 1PI amplitudes $\widehat{\mathcal{F}}$, then construct $\bar{\mathcal{F}}$ via (3.4.3) and finally project onto the mass level m sector as in (3.4.5). In particular the definition of $\widehat{\mathcal{F}}$ requires dividing up the moduli space of Riemann surfaces into 1PI and 1PR parts. Since $\widetilde{F}_T$ will play a crucial role in the definition of the physical renormalized mass, we shall now give an alternate definition of $\widetilde{F}_T$ which does not require us to explicitly identify the 1PI subspace in the moduli space of Riemann surfaces. For this we note from (3.4.6) that

$$\widetilde{F}_T = F_T(1+(k^2+m^2)^{-1}\mathcal{I}F_T)^{-1} = F_T - F_T\mathcal{I}(k^2+m^2)^{-1}F_T + F_T\mathcal{I}(k^2+m^2)^{-1}F_T\mathcal{I}(k^2+m^2)^{-1}F_T + \cdots . \tag{3.4.8}$$

Now F_T has a simple interpretation since it denotes the full off-shell 2-point function restricted to mass level m. Thus we can regard (3.4.8) as the definition of $\widetilde{F}_T$. In this way of defining $\widetilde{F}_T$ we never have to divide the contribution to an amplitude into 1PI and 1PR parts. The only price we pay is that from (3.4.8) it is not obvious that $\widetilde{F}_T$ is free from poles at $k^2 + m^2 = 0$, since each term on the right hand side of (3.4.8) does contain such poles. Nevertheless our previous arguments guarantee that all such poles cancel.

It may seem that $\widetilde{F}_T$ defined this way requires less information than in the earlier definition, but this is not the case. The definition of $\widetilde{F}_T$ requires information on the choice of local coordinate system, which in turn completely fixes the division of the amplitudes into 1PI and 1PR parts. Thus even though we do not explicitly use this division in defining $\widetilde{F}_T$, the data used in the construction of $\widetilde{F}_T$ is sufficient to determine the division of an amplitude into 1PI and 1PR parts.

The definition of $\widetilde{F}_T$ given in this subsection will be useful when we generalize the analysis to super and heterotic string theories.

3.4.3 Renormalized physical state propagator and masses

Following the analysis of §3.2.2 we now seek a change of basis

$$|\alpha\rangle'_p = A_{\beta\alpha}|\beta\rangle_p + B_{s\alpha}|s\rangle_g + C_{s\alpha}|s\rangle_u, \quad |r\rangle'_g = |r\rangle_g + D_{\beta r}|\beta\rangle_p, \quad |r\rangle'_u = |r\rangle_u + K_{\beta r}|\beta\rangle_p\,, \tag{3.4.9}$$

such that the following conditions hold

$${}'_p\langle\alpha|\beta\rangle'_p = \delta_{\alpha\beta}, \quad {}'_p\langle\alpha|s\rangle'_u = {}'_p\langle\alpha|s\rangle'_g = {}'_u\langle r|\beta\rangle'_p = {}'_g\langle r|\beta\rangle'_p = 0\,, \tag{3.4.10}$$

and

$${}'_p\langle\alpha|\widetilde{F}_T|s\rangle'_u = {}'_p\langle\alpha|\widetilde{F}_T|s\rangle'_g = {}'_u\langle r|\widetilde{F}_T|\beta\rangle'_p = {}'_g\langle r|\widetilde{F}_T|\beta\rangle'_p = 0\,. \tag{3.4.11}$$

We now substitute (3.4.9) into (3.4.10), (3.4.11) and use (3.3.7) to get[8]

$$\begin{aligned}
&(A^\dagger A + B^\dagger C + C^\dagger B)_{\alpha\beta} = \delta_{\alpha\beta}, \quad (D^\dagger A + C)_{r\alpha} = 0, \quad (K^\dagger A + B)_{r\alpha} = 0\,,\\
&{}_u\langle r|\widetilde{F}_T|\alpha\rangle_p A_{\alpha\beta} + {}_u\langle r|\widetilde{F}_T|s\rangle_g B_{s\beta} + {}_u\langle r|\widetilde{F}_T|s\rangle_u C_{s\beta} + (K^\dagger)_{r\gamma\,p}\langle\gamma|\widetilde{F}_T|\alpha\rangle_p A_{\alpha\beta}\\
&\qquad +(K^\dagger)_{r\gamma\,p}\langle\gamma|\widetilde{F}_T|s\rangle_g B_{s\beta} + (K^\dagger)_{r\gamma\,p}\langle\gamma|\widetilde{F}_T|s\rangle_u C_{s\beta} = 0\\
&{}_g\langle r|\widetilde{F}_T|\alpha\rangle_p A_{\alpha\beta} + {}_g\langle r|\widetilde{F}_T|s\rangle_g B_{s\beta} + {}_g\langle r|\widetilde{F}_T|s\rangle_u C_{s\beta} + (D^\dagger)_{r\gamma\,p}\langle\gamma|\widetilde{F}_T|\alpha\rangle_p A_{\alpha\beta}\\
&\qquad +(D^\dagger)_{r\gamma\,p}\langle\gamma|\widetilde{F}_T|s\rangle_g B_{s\beta} + (D^\dagger)_{r\gamma\,p}\langle\gamma|\widetilde{F}_T|s\rangle_u C_{s\beta} = 0\,.
\end{aligned} \tag{3.4.12}$$

[8]We seek a change of basis that is real in the position space. In momentum space this implies that changing the momentum from k to $-k$ has the effect of complex conjugating the coefficients $A_{\alpha\beta}, \cdots K_{\beta r}$. This has been used in (3.4.12).

Let us first count the number of independent variables and the number of independent equations. The number of real components in the variables $A_{\alpha\beta}$, $B_{s\alpha}$, $C_{s\alpha}$, $D_{\beta r}$ and $K_{\beta r}$ are

$$2n_p^2 + 4 \times 2n_p n_g \,, \tag{3.4.13}$$

where n_p is the number of physical states and $n_g = n_u$ is the number of pure gauge / unphysical states at mass level m. On the other hand the number of independent equations can be counted as follows. Since both sides of the first equation in (3.4.12) are hermitian matrices, this gives n_p^2 real equations, whereas each of the rest gives $2n_p n_g$ real equations. Thus the total number of equations is

$$n_p^2 + 4 \times 2n_p n_g \,. \tag{3.4.14}$$

Thus we see that we have n_p^2 extra variables compared to the number of equations. This can be traced to the freedom of multiplying A, B and C by a unitary matrix from the right which is a symmetry of the equations (3.4.12) (and represent the freedom of a unitary rotation in the subspace of physical states $|\alpha\rangle'_p$). Up to this freedom we can determine the matrices $A, \cdots K$ by solving (3.4.12).

We shall now describe an iterative procedure for solving these equations. For this we note that the leading (genus one) contribution to $\widetilde{F}_T$ satisfies the property

$${}_p\langle\alpha|\widetilde{F}_T|s\rangle_g \sim \lambda\,(k^2+m^2), \quad {}_g\langle r|\widetilde{F}_T|s\rangle_g \sim \lambda\,(k^2+m^2), \quad {}_g\langle r|\widetilde{F}_T|\beta\rangle_p \sim \lambda\,(k^2+m^2)\,, \tag{3.4.15}$$

where λ now stands for the genus expansion parameter given by the square of the string coupling. These properties follow from the fact that at genus one $\widetilde{F}_T$ includes the full contribution to the torus two point function. Representing a pure gauge state as $Q_B|n\rangle$ plus a term of order (k^2+m^2), deforming the contour of integration of the BRST current so that it acts on the other vertex operator, and then using that fact that acting on an off-shell physical or pure gauge state Q_B gives a term proportional to (k^2+m^2), we arrive at (3.4.15). This in turn

allows us to look for solutions where at order λ^0,

$$A, B, K \sim 1, \quad C, D \sim (k^2 + m^2)\,. \tag{3.4.16}$$

The solution to order λ^0 and leading order in $k^2 + m^2$ are given by

$$\begin{aligned}
&A_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(k^2 + m^2), \quad C = -D^\dagger + \mathcal{O}(k^2 + m^2), \quad B = -K^\dagger + \mathcal{O}(k^2 + m^2), \\
&\lambda^{-1}\left\{\delta_{\beta\gamma}\,{}_u\langle r|\widetilde{F}_T|s\rangle_g - \delta_{rs}\,{}_p\langle\gamma|\widetilde{F}_T|\beta\rangle_p\right\}K^\dagger_{s\gamma} = \lambda^{-1}\,{}_u\langle r|\widetilde{F}_T|\beta\rangle_p + \mathcal{O}(k^2 + m^2) \\
&\lambda^{-1}\left\{\delta_{\beta\gamma}\,{}_g\langle r|\widetilde{F}_T|s\rangle_u - \delta_{rs}\,{}_p\langle\gamma|\widetilde{F}_T|\beta\rangle_p\right\}D^\dagger_{s\gamma} \\
&\qquad = \lambda^{-1}\,{}_g\langle r|\widetilde{F}_T|\beta\rangle_p + \lambda^{-1}\,{}_g\langle r|\widetilde{F}_T|s\rangle_g\,B_{s\beta} + \mathcal{O}\left((k^2 + m^2)^2\right)\,.
\end{aligned} \tag{3.4.17}$$

This gives a sensible solution satisfying (3.4.16) provided the $n_p n_g \times n_p n_g$ matrix

$$S_{r\beta,s\gamma} \equiv \lambda^{-1}\{\,{}_u\langle r|\widetilde{F}_T|s\rangle_g\;\delta_{\beta\gamma} - \;{}_p\langle\gamma|\widetilde{F}_T|\beta\rangle_p\;\delta_{rs}\}, \tag{3.4.18}$$

is invertible. Starting with this solution we can solve for the matrices A, B, C, D, K iteratively in powers of the genus expansion parameter λ and $(k^2 + m^2)$ exactly as in §3.2. As long as the matrix defined in (3.4.18) is invertible, the coefficient of λ^n for any n is free from poles near $k^2 = -m^2$. Physically, invertibility of $S_{r\beta,s\gamma}$ is the condition that the degeneracy between the masses of physical states and the unphysical / pure gauge states is lifted at one loop order. If this condition fails then we need to go to higher order in perturbation theory to lift the degeneracy. We expect that in principle there should be no difficulty in carrying out this procedure, although in practice the analysis is likely to become more complicated.

The coefficients $A, \cdots K$ satisfying (3.4.12) ensures, via eqs.(3.4.9)-(3.4.11) that the matrices $\mathcal{I}$ and $\widetilde{F}_T$ expressed in the primed basis have block diagonal form, with no cross terms

between the states $|\alpha\rangle'_p$ and $(|r\rangle'_u, |r\rangle'_g)$. As in §3.2 we define

$$\widetilde{F}_{\alpha\beta}(k) = {}'_p\langle\alpha|\widetilde{F}_T|\beta\rangle'_p\,. \tag{3.4.19}$$

Then the propagator restricted to the modified physical sector is given by

$$\mathcal{P}_{\alpha\beta} \equiv {}'_p\langle\alpha|\mathcal{P}_T|\beta\rangle'_p = \Big((k^2+m^2-\widetilde{F}(k))^{-1}\Big)_{\alpha\beta}\,. \tag{3.4.20}$$

From here onwards we proceed as in [38]. We can diagonalize $\widetilde{F}(k)$ as

$$\widetilde{F}(k) = U(k)\widetilde{F}_d(k)U(k)^\dagger, \qquad U(k)^\dagger = U(k)^{-1} = U(-k)^T\,, \tag{3.4.21}$$

so that we have

$$\mathcal{P} = U(k)(k^2+m^2-\widetilde{F}_d(k))^{-1}U(k)^\dagger\,. \tag{3.4.22}$$

We can now determine the solutions to the equation $k^2+m^2-\widetilde{F}_d(k)=0$ iteratively for each of the diagonal entries of $\widetilde{F}_d(k)$, starting with $k^2=-m^2$ as the leading order solution. This gives the physical masses. Let M_p^2 denotes the diagonal matrix with the diagonal elements being equal to the squares of the physical masses. Then we can express $(k^2+m^2-\widetilde{F}_d(k))^{-1}$ as

$$X_d(k)(k^2+M_p^2)^{-1}\,, \tag{3.4.23}$$

where $X_d(k)$ is a diagonal matrix which has no poles near $k^2=-m^2$. Eq.(3.4.22) now allows us to express the physical propagator $\mathcal{P}_{\alpha\beta}$ as

$$\mathcal{P} = Z^{1/2}(k)(k^2+M_p^2)^{-1}Z^{1/2}(-k)^T, \qquad Z^{1/2}(k) \equiv U(k)X_d(k)^{1/2}\,. \tag{3.4.24}$$

In §3.6 we shall argue that the squares of the physical masses given by the diagonal elements of M_p^2 do not depend on the choice of local coordinates at the punctures, although the wave-function renormalization matrix $Z^{1/2}(k)$ does depend on the choice of local coordinates.

Finally we would like to note that since $B_{s\alpha}$ is of order unity, the corrected physical state $|\alpha\rangle'_p$ differs from the tree level physical state $|\alpha\rangle_p$ by a pure gauge state with coefficient of order unity. Thus even in the $\lambda \to 0$ limit, $|\alpha\rangle'_p$ does not approach $|\alpha\rangle_p$.

3.4.4 Renormalized masses in the unphysical / pure gauge sector

We shall now briefly describe the computation of the renormalized masses in the unphysical / pure gauge sector by generalizing the procedure described in §3.2.4. For this we define

$$\mathcal{I}' = \begin{pmatrix} {}'_g\langle r|s\rangle'_g & {}'_g\langle r|s\rangle'_u \\ {}'_u\langle r|s\rangle'_g & {}'_u\langle r|s\rangle'_u \end{pmatrix}, \quad \widetilde{F}' = \begin{pmatrix} {}'_g\langle r|\widetilde{F}_T|s\rangle'_g & {}'_g\langle r|\widetilde{F}_T|s\rangle'_u \\ {}'_u\langle r|\widetilde{F}_T|s\rangle'_g & {}'_u\langle r|\widetilde{F}_T|s\rangle'_u \end{pmatrix}. \tag{3.4.25}$$

Then the renormalized mass^2's in the unphysical / pure gauge sector will be given by the zeroes of the eigenvalues of the matrix

$$(k^2 + m^2)\mathcal{I}' - \widetilde{F}'(k)\,, \tag{3.4.26}$$

in the complex $-k^2$ plane. To evaluate the order λ correction to these masses, we shall assume as in §3.2.4 that $k^2 + m^2$ is of order λ when $-k^2$ is equal to the renormalized mass^2 and keep terms in (3.4.26) up to order λ. Using (3.4.9), (3.4.15) and (3.4.17) one finds that to order unity

$${}'_g\langle r|s\rangle'_g = 0, \quad {}'_g\langle r|s\rangle'_u = {}'_u\langle r|s\rangle'_g = \delta_{rs}, \tag{3.4.27}$$

and to order λ,

$${}'_g\langle r|\widetilde{F}_T|s\rangle'_g = 0, \quad {}'_g\langle r|\widetilde{F}_T|s\rangle'_u = {}_g\langle r|\widetilde{F}_T|s\rangle_u, \quad {}'_u\langle r|\widetilde{F}_T|s\rangle'_g = {}_u\langle r|\widetilde{F}_T|s\rangle_g\,. \tag{3.4.28}$$

Hence to order λ (counting k^2+m^2 as order λ)

$$(k^2+m^2)\mathcal{I}' - \widetilde{F}'(k) = \begin{pmatrix} 0 & (k^2+m^2)\,\delta_{rs} - {}_g\langle r|\widetilde{F}_T|s\rangle_u \\ (k^2+m^2)\,\delta_{rs} - {}_u\langle r|\widetilde{F}_T|s\rangle_g & (k^2+m^2)\,{}'_u\langle r|s\rangle'_u - {}'_u\langle r|\widetilde{F}_T|s\rangle'_u \end{pmatrix}. \tag{3.4.29}$$

Using the fact that the vanishing of an eigenvalue of a matrix is equivalent to requiring the vanishing of its determinant, we see that the required condition is the vanishing of the determinant of the upper right (or lower left) block. This in turn is equivalent to requiring the vanishing of an eigenvalue of

$$(k^2+m^2)\,\delta_{rs} - {}_g\langle r|\widetilde{F}_T|s\rangle_u \tag{3.4.30}$$

as a function of $-k^2$. Starting with this first order solution one can iteratively compute higher order corrections to the renormalized mass^2 in the unphysical / pure gauge sector by looking for zero eigenvalues of (3.4.26).

3.4.5 Dependence on choice of local coordinates

An important question is: how do the physical masses depend on the choice of local coordinates? We shall postpone a full discussion on this till §3.6, but at this stage we can derive the result at order λ. The locations of the physical mass squares are determined by the zeroes of $k^2+m^2-\widetilde{F}_d(k)$ in the $-k^2$ plane. Let us focus on the one loop, i.e. order λ correction to the mass^2. For this we need to determine the function $\widetilde{F}_d(k)$ and hence $\widetilde{F}(k)$ to order λ at $k^2=-m^2$. It follows from (3.4.9), (3.4.15), (3.4.17), (3.4.19) and the fact that the leading contribution to $\widetilde{F}_T$ is of order λ that to order λ and at $k^2+m^2=0$

$$\widetilde{F}_{\alpha\beta} = \langle\alpha|\widetilde{F}_T|\beta\rangle\,. \tag{3.4.31}$$

At order λ this represents the full two point function of the tree level physical states $|\alpha\rangle$ and $|\beta\rangle$ on the torus. Since $|\alpha\rangle$ and $|\beta\rangle$ are both dimension zero primaries at $k^2=-m^2$, we see that to this order $\widetilde{F}_{\alpha\beta}$ at $k^2=-m^2$ is independent of the choice of local coordinates. Hence

the renormalized physical masses are also independent of the choice of local coordinates to this order.

We can also consider the fate of the masses in the unphysical / pure gauge sector under a change in the local coordinate system. To order λ the mass^2's in this sector are given by the zeroes of the eigenvalues of the matrix (3.4.30) in the $-k^2$ plane. Since the matrix elements ${}_g\langle r|\widetilde{F}_T|s\rangle_u$ involve unphysical and pure gauge states, which are generically not dimension zero primaries, we see that in the generic case the order λ contribution to the masses of the unphysical and pure gauge states will depend on the choice of local coordinates.[9] Higher order contributions can correct these results but cannot cancel the order λ corrections. This we conclude that the unphysical / pure gauge sector masses do depend on the choice of local coordinate system.

3.5 Poles of S-matrix elements of massless / BPS / special states

In this section we shall show that if we consider an S-matrix of external massless, BPS and/or special states then the poles in this S-matrix in any channel are the same ones as those which appear in the analysis of §3.4.3.[10] Let us denote by k the total momentum carried in some particular internal channel, being equal to the sum of momenta of two or more external states, and look for poles in the $-k^2$ plane. Our starting point will be the expression (3.3.14) for the $(m+n)$-point amplitude. The S-matrix elements are obtained from this by multiplying this by appropriate renormalization factors on the external legs and then setting the external momenta on-shell. Since multiplicative factors on the external legs do not affect the locations of the poles in the k^2 plane, we can directly use Γ to examine these poles. Our interest will

[9] If the vertex operator involves ghost excitations then the integration measure provided by b-ghost insertions also depend on the choice of local coordinates [27, 32].

[10] This generalizes the result of [35] in the absence of mass renormalization.

be to look for those poles which arise from states at mass level m. For this it will be useful to 'integrate out' the states at other mass levels. With this goal in mind, we define

$$\begin{aligned}\bar{\Gamma}_1^T\mathcal{I} &= \widehat{\Gamma}_1^T\mathcal{I}(1+\bar{\Delta}\widehat{\mathcal{F}}+\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}+\cdots) = \widehat{\Gamma}_1^T\mathcal{I}(1-\bar{\Delta}\widehat{\mathcal{F}})^{-1}\,,\\ \bar{\Gamma}_2 &= (1+\widehat{\mathcal{F}}\bar{\Delta}+\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}+\cdots)\,\widehat{\Gamma}_2 = (1-\widehat{\mathcal{F}}\bar{\Delta})^{-1}\,\widehat{\Gamma}_2\,,\end{aligned} \tag{3.5.1}$$

where $\bar{\Delta}$ has been defined in (3.4.2). We also define

$$\bar{\Gamma} = \widehat{\Gamma}+\widehat{\Gamma}_1^T\mathcal{I}(\bar{\Delta}+\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}+\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}\widehat{\mathcal{F}}\bar{\Delta}+\cdots)\widehat{\Gamma}_2 = \widehat{\Gamma}+\widehat{\Gamma}_1^T\mathcal{I}\bar{\Delta}(1-\widehat{\mathcal{F}}\bar{\Delta})^{-1}\widehat{\Gamma}_2 = \widehat{\Gamma}+\widehat{\Gamma}_1^T\mathcal{I}(1-\bar{\Delta}\widehat{\mathcal{F}})^{-1}\bar{\Delta}\widehat{\Gamma}_2\,. \tag{3.5.2}$$

Using (3.3.14), (3.4.2), (3.4.3), (3.5.1) and (3.5.2) we now get

$$\begin{aligned}\Gamma &= \bar{\Gamma}+\bar{\Gamma}_1^T\mathcal{I}\left\{1-(k^2+m^2)^{-1}P_T\bar{\mathcal{F}}\right\}^{-1}P_T(k^2+m^2)^{-1}\bar{\Gamma}_2\\ &= \bar{\Gamma}+\bar{\Gamma}_1^T\mathcal{I}P_T(k^2+m^2)^{-1}\left\{1-(k^2+m^2)^{-1}\bar{\mathcal{F}}P_T\right\}^{-1}\bar{\Gamma}_2\\ &= \bar{\Gamma}+\bar{\Gamma}_1^T\mathcal{I}P_T(k^2+m^2-P_T\bar{\mathcal{F}}P_T)^{-1}P_T\bar{\Gamma}_2\\ &= \bar{\Gamma}+\bar{\Gamma}_1^T\mathcal{P}_T\bar{\Gamma}_2\,,\end{aligned} \tag{3.5.3}$$

where $\mathcal{P}_T$ has been defined in (3.4.5). Now the genus expansions of $\bar{\Gamma}$, $\bar{\Gamma}_1^T$ and $\bar{\Gamma}_2$ are free from poles at $-k^2=m^2$. Thus the only poles near $-k^2=m^2$ can come from the poles of matrix $\mathcal{P}_T$. These are precisely the renormalized physical and unphysical squared masses as discussed in §3.4.

— For later use, it will be useful to isolate the contribution from the physical states from that of the unphysical and pure gauge states. For this we insert the projection operator P_T on both sides of $\mathcal{P}_T$ on the right hand side of (3.5.3) using the identity $P_T\mathcal{P}_TP_T=\mathcal{P}_T$. Now using (3.4.1) and (3.4.9), P_T may be expressed as

$$P_T = \sum_\alpha |\alpha\rangle'_{p\ \ p}\langle\alpha| + \sum_{r,s}\left[\widetilde{A}_{rs}|r\rangle'_{g\ \ g}\langle s| + \widetilde{B}_{rs}|r\rangle'_{g\ \ u}\langle s| + \widetilde{C}_{rs}|r\rangle'_{u\ \ g}\langle s| + \widetilde{D}_{rs}|r\rangle'_{u\ \ u}\langle s|\right]\,, \tag{3.5.4}$$

where $\widetilde{A}_{rs}$, $\widetilde{B}_{rs}$, $\widetilde{C}_{rs}$ and $\widetilde{D}_{rs}$ are constants which can be computed from (3.4.9), (3.4.12). The first term on the right hand side of (3.5.4) describes the contribution from renormalized physical states whereas the other terms represent contribution from renormalized unphysical and pure gauge states.

Let us now examine the residues at the poles in (3.5.3) at leading order in string perturbation theory. First consider the residue at a physical pole. This is given by the products of the components of $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ along the corresponding physical state $|\alpha\rangle'_p$. At the tree level the relevant component of $\bar{\Gamma}_1$ is given by the contribution to the full $(m+1)$ point tree amplitude with external states $|\alpha\rangle'_p$ and m other massless / BPS / special states, and similarly the relevant component of $\bar{\Gamma}_2$ is given by the contribution to the full $(n+1)$ point tree amplitude with external states $|\alpha\rangle'_p$ and n other massless / BPS / special states. Since in the leading order $|\alpha\rangle'_p$ is given by a linear combination of tree level physical state $|\alpha\rangle_p$ and a pure gauge state, and since the pure gauge states decouple in the on-shell tree level amplitude, we can replace $|\alpha\rangle'_p$ by $|\alpha\rangle_p$ in computing the leading order contribution to the relevant components of $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. Thus in the leading order the residue at the physical pole is given by the product of two tree level S-matrix elements – one with $(m+1)$ external states and the other one with $(n+1)$ external states. As long as these are non-zero, the residue at the corresponding physical pole will be non-zero. Higher order contributions can correct the residue but cannot make this vanish in perturbation theory. Thus even after including higher order corrections, the corresponding physical mass^2's will appear as the locations of the poles in the $-k^2$ plane of the original S-matrix element involving $(m+n)$ external massless / BPS / special states.

Let us now turn to the contribution from the unphysical / pure gauge states. It follows from (3.4.9), (3.4.15), (3.4.17) and (3.5.4) that for $k^2 = -m^2$ and leading order in λ, the coefficients $\widetilde{D}_{rs}$ vanish. On the other hand the same equations show that in this approximation $|s\rangle'_g = |s\rangle_g$. Thus the residue is given by a sum of products of appropriate components of $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$, and

in each of these terms either the component of $\bar{\Gamma}_1$ or the component of $\bar{\Gamma}_2$ (or both) is aligned along a tree level pure gauge state $|s\rangle_g$. Thus this factor is given by a tree level amplitude, one of whose external states is $|s\rangle_g$ and the other states are on-shell massless / pure gauge / special states. Since this vanishes due to BRST invariance, we conclude that at least at leading order in λ the unphysical states do not contribute to the poles in the S-matrix elements of massless / BPS / special states.

— Before concluding this section we would like to note that the various quantities which appear in (3.5.3) – *e.g.* $\bar{\Gamma}$, $\bar{\Gamma}_1^T \mathcal{I} P_T$, $P_T \bar{\Gamma}_2$ etc. – can be defined without having to explicitly identify the 1PI Riemann surfaces by following the same strategy as in §3.4.2. For example we have the relations

$$\begin{aligned}
\bar{\Gamma}_1^T \mathcal{I} P_T &= \Gamma_1^T \mathcal{I} \left(1 - (k^2+m^2)^{-1} \mathcal{I} \widetilde{F}_T\right) P_T \,, \\
P_T \bar{\Gamma}_2 &= \left(1 - (k^2+m^2)^{-1} \mathcal{I} \widetilde{F}_T\right) P_T \, \Gamma_2 \,, \\
\bar{\Gamma} &= \Gamma - \Gamma_1^T \mathcal{I} \, (k^2+m^2)^{-1} \left(1 - (k^2+m^2)^{-1} \mathcal{I} \widetilde{F}_T\right) P_T \, \Gamma_2 \,.
\end{aligned} \tag{3.5.5}$$

Since Γ, Γ_1 and Γ_2 are full amplitudes, their definitions do not require us to divide the moduli space of Riemann surfaces into 1PI and 1PR parts. The definition of $\widetilde{F}_T$ given in §3.4.2 also does not require this division. Thus $\bar{\Gamma}_1^T \mathcal{I} P_T$, $P_T \bar{\Gamma}_2$ and $\bar{\Gamma}$ defined via (3.5.5) also do not require this divison. This observation will be useful when we generalize the analysis to super and heterotic string theories.

3.6 All order results

We shall now combine the results of §3.4.5 and §3.5 to prove some all order results in a generic situation. For this we need to first explain what we mean by a generic situation. The conditions under which our arguments will hold are listed below.

1. We assume that the degeneracies between physical and unphysical masses are lifted

at first order in perturbation theory. Otherwise our prescription of §3.4 of computing renormalized physical masses will have to be modified.

2. We have seen that to leading order the residue at a particular physical mass^2 of an S-matrix element of external massless / BPS / special states is proportional to the product of S-matrix elements of two lower point tree level S-matrix elements each of which contains, as one of the external states, the physical state whose mass we are interested in. We shall assume that it is possible to choose the external massless / BPS / special states of the original amplitude in such a way that both these lower point S-matrix elements are non-vanishing at tree level. Had we restricted the external states to be only massless or BPS states then this fails in some cases, as was illustrated in [38]. (A particular example of this is the SO(32) spinor states of ten dimensional SO(32) heterotic string theory; these cannot appear as one particle intermediate states in the scattering of massless external states which are all in the adjoint or singlet representation of SO(32).) However at present we do not know of an example where it fails even after we allow as external states the special states introduced in [38]. Once the residue at the pole can be made non-vanishing at leading order, higher order corrections can modify the residue but cannot make it vanish in perturbation theory.

3. We have seen in §3.4.5 that the renormalized masses of unphysical / pure gauge states do in general depend on the choice of local coordinates. We shall assume that this is true in all cases, i.e. there is no renormalized mass corresponding to unphysical / pure gauge states which is accidentally independent of the choice of local coordinates.

Next we shall combine the genericity assumption with some of the relevant results in §3.4.5, §3.5 and ref. [38] to draw the following conclusions:

1. In a generic situation the renormalized masses of the unphysical / pure gauge states depend on the choice of local coordinates.

2. It was shown in §3.4.5 the renormalized masses of physical states do not depend on the choice of local coordinates at least to order λ.

3. In a generic situation the mass^2 of physical states appear as poles in the $-k^2$ plane of some S-matrix of massless / BPS / special states.

4. It was also shown near the end of §3.5 that the unphysical /pure gauge states do not contribute poles in the S-matrix of massless / BPS / special states at least to leading order in λ.

5. The S-matrix involving external massless / BPS / special states do not depend on the choice of local coordinates to all orders in λ [38].

Let us now combine these results. Points 1 and 5 show, to all orders in λ, that the unphysical / pure gauge states cannot appear as intermediate states in the S-matrix of massless / BPS / special states. This is consistent with the leading order result mentioned in point 4. On the other hand points 3 and 5 show, to all orders in λ, that the mass^2 of physical states cannot depend on the choice of local coordinates. This is consistent with the leading order result described in point 2.

— We can also extend this argument to prove the invariance of the S-matrix elements of general external physical states under a change of local coordinates. For this we note that as long as each of the external states have non-zero tree level amplitude with some set of massless / BPS / special states, we can replace each of the massive, non-BPS and non-special external physical states by the corresponding combination of massless / BPS / special states and examine the corresponding S-matrix for values of momenta where the intermediate physical states of interest go on-shell. The desired S-matrix can then be found by examining the residue at the pole.[11] Since the S-matrix of massless / BPS / special states is invariant under a change in the local coordinate system, its residues at various poles must also be invariant under a change of local

[11] If the physical state under consideration is unstable then this is the only way to define its 'S-matrix' since the state does not exist as asymptotic state.

coordinates. This establishes the invariance of the S-matrix elements involving general external physical states under a change of the local coordinate system.

3.7 Generalizations to heterotic and super string theories

We shall now briefly discuss generalizations to heterotic and superstring theories. We shall restrict our discussion to the Neveu-Schwarz (NS) sector, and work with picture number -1 states. In this case the discussion of §3.3 can be adapted with few changes:

1. The discussion in §3.3.1 remains valid without any change.

2. In the analysis of §3.3.2 we need to choose local superconformal coordinate system (w, ξ) around every puncture for defining off-shell amplitudes, and $f_i \circ V_i$ will label the transform of the vertex operator V_i by the superconformal transformation f_i that relates the local coordinates near the i-th puncture to the reference superconformal coordinates on the super Riemann surface. The detailed analysis of the integration measure (ghost insertions) for off-shell amplitudes can be carried out by combining the description of the measure for on-shell amplitudes in super and heterotic string theories given in [5] with the description of the measure for off-shell amplitudes in bosonic string theory given in [27, 32].

3. The gluing of two Riemann surfaces is implemented via the identification [5]

$$w_1 w_2 = q_{NS}, \quad w_2 \xi_1 = \varepsilon \xi_2, \quad w_1 \xi_2 = -\varepsilon \xi_1, \quad \xi_1 \xi_2 = 0, \quad \varepsilon = \pm \sqrt{-q_{NS}}\,, \tag{3.7.1}$$

   and we need to sum over both choices of the sign of ε, leading to GSO projection.

4. The choice of local superconformal coordinates should be compatible with gluing in the same way as in the case of bosonic string theory. We shall also require that the contours

in the supermoduli space over which we integrate [5] should be compatible with gluing. In particular this means that in situations where we can integrate out the odd moduli at the expense of inserting picture changing operators [40, 41], the locations of the picture changing operators on the glued Riemann surface should be those induced from the locations of the picture changing operators on the lower genus Riemann surfaces which are being glued. A consistent super or heterotic string field theory should automatically satisfy this property in a Siegel like gauge. Construction of classical superstring field theory satisfying these requirements can be found in [33, 42].

5. We can now define $\widetilde{F}_T$ following the procedure outlined in §3.4.2. This avoids having to divide the super Riemann surfaces into 1PI and 1PR surfaces, and directly gives us the expression for $\widetilde{F}_T$ in terms of the full off-shell two point function F_T of mass level m states. Similarly generalization of the analysis of §3.5 can also be carried out by defining $\bar{\Gamma}$, $\bar{\Gamma}_1^T \mathcal{I} P_T$, $P_T \bar{\Gamma}_2$ etc. as in (3.5.5) instead of in terms of 1PI super Riemann surfaces.

The rest of the analysis can be carried out in a straightforward matter and we arrive at the same conclusions as in the case of bosonic string theory.

The difficulty in the Ramond sector stems from the fact that there is no natural inner product between states in the $-1/2$ picture since the inner product pairs states in the $-1/2$ picture to states in the $-3/2$ picture. Thus generalization of the analysis of §3.3.1 will require us to work with picture number $-1/2$ and $-3/2$ states together. On the other hand superstring perturbation theory naturally uses $-1/2$ picture vertex operators [5]. This issue has been resolved in [43].

CHAPTER 4

S-duality Improved Perturbation Theory in Compactified Type I / Heterotic String Theory

4.1 Introduction

Our current understanding of string theory is based mostly on perturbation expansion in the string coupling [5]. Furthermore this perturbation expansion is believed to be an asymptotic expansion. For this reason one might worry that our ability to compute anything in string theory may be limited to very narrow corners of the full string theory landscape – regions in which the theory admits a description as a very weakly coupled string/M/F-theory.

Ref. [44] suggested making use of duality and suitable interpolation formula to translate the weak coupling results in string theory to approximate results for physical quantities over the entire range of string coupling constant. As a specific example, the mass of the stable non-BPS particle in ten dimensional type I / SO(32) heterotic string theory was considered. Using a suitable formula that interpolates between the result for this mass in weakly coupled SO(32) heterotic string theory and weakly coupled type I string theory, an approximate formula for the mass of this state was derived over the entire range of string coupling. Furthermore by

comparing the results in different orders it was estimated that this approximate formula lies within 10% of the exact result over the entire range of coupling. Generalization of this analysis has since been discussed in [45–47].

Since a generic string theory moduli space is multi-dimensional it is natural to ask if this interpolation technique can be used to find approximate expressions for various physical quantities over the full multi-dimensional moduli space.[1] In this chapter we explore this in the context of SO(32) heterotic / type I string theory compactified on a circle. If we do not switch on any Wilson line so that the SO(32) gauge group is unbroken then the moduli space is two dimensional, parametrized by the string coupling and the radius of compactification. We use perturbative results in the compactified string theory and a suitable generalization of the interpolation formula used in [44] to derive expression for the mass of the non-BPS state in the full two dimensional moduli space. Comparison between different orders of approximation again indicates that the approximate formula derived here lies within 10% of the exact formula over the entire two dimensional moduli space.

In type I / SO(32) heterotic string theory compactified on S^1, one can identify a set of BPS states whose total charge is the same as that of the charge carried by the non-BPS state under study. Thus the latter can decay into the former if the mass of the non-BPS state is larger than the sum of the masses of the BPS states to which it could possibly decay. With the help of the approximate formula for the mass we determine the part of the region of the two dimensional moduli space in which the non-BPS state is unstable. Again we find that the region determined this way is only mildly sensitve to the order of the approximation that we use.

[1] This issue arose aleady in the analysis of [45] (*albeit* in the context of a supersymmetric gauge theory instead of string theory) where the interpolation technique together with perturbative results were used to determine approximate formulæ for anomalous dimension of non-BPS operators in the complex coupling constant plane.

The rest of the chapter is organized as follows. In §4.2 we review the interpolation technique and the normalization conventions of [44], and the additional ingredients we need for dealing with the compactified theory. In §4.3 we carry out the computation of the one loop correction to the mass of stable non-BPS state in type I string theory compactified on S^1. In §4.4 we carry out a similar calculation in SO(32) heterotic string theory compactified on S^1. In §4.5 we construct the interpolation formula for the mass of the non-BPS state in various approximation, and compare the results in different orders of approximation. In §4.6 we use the interpolation formula to analyze the region of stability of the non-BPS state. In §4.7 we discuss extension of our analysis to the case of compactification on higher dimensional tori. We conclude in §4.8 with a brief summary of our results and their possible relation to related developments. This chapter is based on [3].

4.2 Normalization conventions and tree level results

Let g_H and g_I be the string coupling in ten dimensional heterotic and type I string theories respectively. We introduce a new coupling parameter g in terms of which g_H and g_I are given by

$$g_H = 2^{7/2}\pi^{7/2} g, \quad g_I = 2^{3/2}\pi^{7/2} g^{-1} \,. \tag{4.2.1}$$

We normalize the heterotic and type I metric such that the heterotic string tension in heterotic metric and the type I string tension in type I metric are both given by $1/2\pi$. The mass of the non-BPS state, measured in the ten dimensional Einstein metric, is parametrized by

$$M(g) = 2^{15/8}\pi^{7/8} F(g) \,. \tag{4.2.2}$$

The tree level weak and strong coupling values of $F(g)$, computed respectively from tree level heterotic and type I string theories, are

$$F_0^W(g) = g^{1/4}, \qquad F_0^S(g) = g^{3/4} \,. \tag{4.2.3}$$

Upon compactification on a circle the tree level masses will continue to be given by (4.2.3) if we measure it in the canonical metric in ten dimensions. We shall follow this convention. Let r_I, r_H and r_E denote the radii of the compact circle measured in the type I, heterotic and canonical metric. Then

$$\begin{aligned} r_E &= (g_I)^{-1/4} r_I = 2^{-3/8}\pi^{-7/8} g^{1/4} r_I \,, \\ r_H &= (g_H)^{1/4} r_E = 2^{7/8}\pi^{7/8} g^{1/4} r_E = 2^{1/2} g^{1/2} r_I \,. \end{aligned} \tag{4.2.4}$$

We expect the quantum corrections in the heterotic and type I string theories to modify the weak coupling results to

$$F^W(g) = g^{1/4}\left[1 + \sum_{k=1}^{\infty} A_{2k}(r_H) g^{2k}\right], \qquad F^S(g) = g^{3/4}\left[1 + \sum_{n=1}^{\infty} B_n(r_I) g^{-n}\right], \tag{4.2.5}$$

where the functions $A_{2k}(r_H)$ and $B_n(r_I)$ have finite $r_H \to \infty$ and $r_I \to \infty$ limits respectively, corresponding to the results in the non-compact theory. We now introduce the interpolating function[2]

$$\begin{aligned} F_{m,n}(g) &= g^{1/4}\Big[1 + a_1(r_H) g + \cdots a_m(r_H) g^m + b_n(r_I) g^{m+1} + b_{n-1}(r_I) g^{m+2} \\ &\qquad + \cdots + b_1(r_I) g^{m+n} + g^{m+n+1}\Big]^{1/\{2(m+n+1)\}} , \end{aligned} \tag{4.2.6}$$

where the functions $a_k(r_H)$ and $b_k(r_I)$ are determined as follows. We determine $a_k(r_H)$ by demanding that after setting the b_k's to zero, the expansion of (4.2.6) in powers of g at fixed r_H agrees with that of $F^W(g)$ up to order $g^{\frac{1}{4}+m}$. Similarly the functions $b_k(r_I)$ are determined

[2]There are various other possible interpolation schemes (see *e.g,* [48,49]), but the one given in (4.2.6), called the fractional power of polynomial (FPP) scheme in [45], seems to be most suitable for our purpose as this gives a clear separation between the coefficients which are determined using weak coupling expansion and the coefficients which are determined using strong coupling expansion. This is needed to ensure that the weak coupling expansions *at fixed* r_H and strong coupling expansion *at fixed* r_I match the perturbation expansions. The difficulty in achieving this with other approximation schemes, *e.g.* 2-point Padé approximant, is similar to the difficulties faced in [45] in getting a duality invariant approximation scheme beyond four loops using 2-point Padé approximant.

by demanding that after setting the $a_k(r_H)$'s to zero, the expansion of (4.2.6) in powers of g^{-1} at fixed r_I agrees with that of $F^S(g)$ up to order $g^{\frac{3}{4}-n}$.

We shall now argue that the weak coupling expansion at fixed r_H of the full function $F_{m,n}$ keeping both a_k's and b_k's non-zero coincides with that of F^W up to order $g^{\frac{1}{4}+m}$, and similary the strong coupling expansion at fixed r_I of the full function $F_{m,n}$ coincides with that of F^S up to order $g^{\frac{3}{4}-n}$. From eq.(4.2.4) it follows that $r_I = 2^{-1/2} g^{-1/2} r_H$, and hence as $g \to 0$ keeping r_H fixed, $r_I \to \infty$. In this limit the coefficients B_k appearing in the strong coupling expsnsion should approach finite values given by the results in the non-compact theory. Thus the coefficients b_k, determined in terms of the coefficients B_ℓ for $k \leq n$, should also approach finite values in this limit. This shows that the expansion of b_k in powers of g at fixed r_H contains non-negative powers of g. Substituting this into (4.2.6) we now see that the coefficents b_k do not affect the weak coupling expansion of $F_{m,n}$ to order $g^{\frac{1}{4}+m}$, and hence the weak coupling expansion of $F_{m,n}$ to this order agrees with that of F^W. A similar analysis shows that the expansion of a_k in powers of g^{-1} at fixed r_I contains non-positive powers of g. Hence the expansion of $F_{m,n}$ in powers of g^{-1} at fixed r_I to order $g^{\frac{3}{4}-n}$ is insensitive to the coefficients a_k and coincides with that of $F^S(g)$.

From (4.2.3), (4.2.6) we can find the following interpolating functions for the mass of the non-BPS particle

$$\begin{aligned} F_{0,0}(g) &= g^{1/4}\,(1+g)^{1/2}\,, \\ F_{1,0}(g) &= g^{1/4}\,(1+g^2)^{1/4}\,. \end{aligned} \tag{4.2.7}$$

4.3 Strong coupling expansion

Denote by ΔM the first order correction to the mass formula from the strong coupling end, i.e. in weakly coupled type I string theory compactified on a circle S^1 of radius r_I. This can

be obtained by calculating the one loop correction to the energy of the non-BPS D0-brane of type I string theory compactified on S^1. This calculation differs from the corresponding calculation in [44] by having to include extra contribution from open string winding modes along the circle, begining on a D0-brane and ending on one of its images. The result takes the form [50–53]

$$-\Delta M = \frac{1}{2} g_I^{\frac{1}{4}} (8\pi^2)^{-\frac{1}{2}} \int_0^\infty s^{-\frac{3}{2}} ds [Z_{NS;D0D0} - Z_{R;D0D0} + Z_{NS;D0D9} - Z_{R;D0D9}], \tag{4.3.1}$$

where $Z_{NS;D0D0}, Z_{R;D0D0}, Z_{NS;D0D9}, Z_{R;D0D9}$ denote respectievely the contributions from the NS and R sector open strings with both ends on the D0-brane and NS and R open strings with one end on the D0-brane and the other end on the D9-brane wrapped on S^1 of radius r_I. Explicit computation gives

$$\begin{aligned}
Z_{NS;D0D0} &= \frac{1}{2}\left(\sum_n \tilde{q}^{2n^2 r_I^2}\right) \frac{f_3(\tilde{q})^8}{f_1(\tilde{q})^8} + 2^{\frac{5}{2}}(1-i)\frac{f_3(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_4(i\tilde{q})} - 2^{\frac{5}{2}}(1+i)\frac{f_4(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_3(i\tilde{q})}, \\
Z_{R;D0D0} &= \frac{1}{2}\left(\sum_n \tilde{q}^{2n^2 r_I^2}\right) \frac{f_2(\tilde{q})^8}{f_1(\tilde{q})^8}, \\
Z_{NS;D0D9} &= 16\sqrt{2}\frac{f_2(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_3(\tilde{q})}, \\
Z_{R;D0D9} &= 16\sqrt{2}\frac{f_3(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_2(\tilde{q})},
\end{aligned} \tag{4.3.2}$$

where n is the quantum number corresponding to the winding number of the fundamental open string along the compact direction and

$$\tilde{q} \sim e^{-\pi s}, \tag{4.3.3}$$

$$\begin{aligned}
f_1(q) &= q^{1/12}\prod_{n=1}^\infty (1-q^{2n}) = \eta(2\tau), \quad q \sim e^{2\pi i \tau}, \\
f_2(q) &= \sqrt{2}\, q^{1/12}\prod_{n=1}^\infty (1+q^{2n}) = \sqrt{2}\,\frac{\eta(4\tau)}{\eta(2\tau)},
\end{aligned}$$

$$
\begin{aligned}
f_3(q) &= q^{-1/24}\prod_{n=1}^{\infty}(1+q^{2n-1}) = \frac{\eta(2\tau)^2}{\eta(\tau)\eta(4\tau)},\\
f_4(q) &= q^{-1/24}\prod_{n=1}^{\infty}(1-q^{2n-1}) = \frac{\eta(\tau)}{\eta(2\tau)}.
\end{aligned}
\tag{4.3.4}
$$

Individual terms in 4.3.1 are both IR and UV divergent. Using the prescription for the IR and UV regularization described in [44] we can express 4.3.1 as follows,

$$
\Delta M = \tilde{K}_S (g_I)^{1/4}\,,
\tag{4.3.5}
$$

$$
\begin{aligned}
\tilde{K}_S \;\sim\; & -\frac{1}{2}(8\pi^2)^{-\frac{1}{2}}\lim_{\Lambda\to\infty}\lim_{\epsilon\to 0}\Bigg[\int_{\epsilon}^{\Lambda} s^{-\frac{3}{2}}ds\Bigg\{\frac{1}{2}\Bigg(\sum_n \tilde{q}^{2n^2r_I^2}\Bigg)\Bigg(\frac{f_3(\tilde{q})^8}{f_1(\tilde{q})^8}-\frac{f_2(\tilde{q})^8}{f_1(\tilde{q})^8}\Bigg)\\
& +16\sqrt{2}\frac{f_2(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_3(\tilde{q})} - 16\sqrt{2}\frac{f_3(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_2(\tilde{q})}\Bigg\}\\
& +\int_{\epsilon/4}^{\Lambda} s^{-\frac{3}{2}}ds\Bigg\{2^{\frac{5}{2}}(1-i)\frac{f_3(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_4(i\tilde{q})} - 2^{\frac{5}{2}}(1+i)\frac{f_4(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_3(i\tilde{q})}\Bigg\}\Bigg]\,.
\end{aligned}
\tag{4.3.6}
$$

Note that for $r_I < 1/\sqrt{2}$, the $n=1$ term in the sum behaves as $\tilde{q}^{2r_I^2-1} = e^{\pi s(1-2r_I^2)}$ and hence the integral over s has a divergence from the large s region. This reflects the appearance of the open string tachyon in the spectrum for $r_I < 1/\sqrt{2}$ [54]. For this reason the open string loop corrections to the mass of stable non-BPS state makes sense only for $r_I \geq 1/\sqrt{2}$, and in the rest of this section we shall focus on this region. Using (4.2.4) we see that in terms of the radius r_H in the heterotic metric, this condition takes the form

$$
r_H > g^{1/2}\,.
\tag{4.3.7}
$$

It is possible to convert expression 4.3.6 in the 'closed string channel' using the modular tranformation laws of f_i's :

$$
\tilde{K}_S = -\lim_{\Lambda\to\infty}\lim_{\epsilon\to 0}\frac{1}{4\pi}(8\pi^2)^{-\frac{1}{2}}\Bigg[\int_{\pi/\Lambda}^{\pi/\epsilon} dt(C_{00}+C_{09}+C_{09}^*)+\int_{\pi/4\Lambda}^{\pi/\epsilon} dt(\mathcal{M}+\mathcal{M}^*)\Bigg]
\tag{4.3.8}
$$

where

$$\begin{aligned}
C_{00} &= \left(\frac{\pi}{t}\right)^4 \left(\sum_n \frac{q^{\frac{n^2}{2r_I^2}}}{\sqrt{2}\, r_I}\right)\left(\frac{f_3(q)^8}{f_1(q)^8} - \frac{f_4(q)^8}{f_1(q)^8}\right), \\
C_{09} &= 2^{\frac{9}{2}}\left(\frac{f_4(q)^9 f_1(q)}{f_2(q)^9 f_3(q)} - \frac{f_3(q)^9 f_1(q)}{f_2(q)^9 f_4(q)}\right), \\
\mathcal{M} &= 2^{\frac{9}{2}}\left(\frac{f_3(iq)^9 f_1(iq)}{f_2(iq)^9 f_4(iq)} - \frac{f_4(iq)^9 f_1(iq)}{f_2(iq)^9 f_3(iq)}\right), \\
q &\sim e^{-t}.
\end{aligned} \tag{4.3.9}$$

C_{00} denotes the cylinder amplitude with boundaries lying on the D0-brane, given by the inner product between the boundary states of D0-brane. C_{09} denotes the cylinder amplitude with one boundary lying on the D0-brane and the other boundary on the D9-brane wrapped on S^1, given by the inner product between the boundary states of D0-brane and the D9-brane wrapped on S^1. $\mathcal{M}$ denotes the möbius strip amplitude with boundary lying on the D0-brane, given by the inner product between the boundary states of D0-brane and the crosscap.

Using this and eqs.(4.2.1), (4.2.2) we can write the corrected strong coupling expression for $F^S(g, r_I)$ to order $g^{\frac{3}{4}-1}$ as,

$$F_1^S(g, r_I) = g^{\frac{3}{4}}\Big(1 + K_S(r_I) g^{-1}\Big), \quad K_S(r_I) \sim 2^{-\frac{3}{2}}\tilde{K}_S\,. \tag{4.3.10}$$

$K_S(r_I)$ can be obtained by integrating expression 4.3.6 numerically for different values of r_I. We find that the result of this numerical evaluation fits well with the function

$$K_S(r_I) \simeq 0.351 - 0.048\, \exp\left[-10\,(r_I - 2^{-1/2})^{2/3}\right]\,, \tag{4.3.11}$$

within 1% accuracy over the entire range $1/\sqrt{2} \le r_I < \infty$.

4.4 Weak coupling expansion

First order correction to M in the weakly coupled heterotic string theory given by

$$\delta M = M K_W(r_H) g^2, \tag{4.4.1}$$

where $K_W(r_H)$ can be calculated by doing a one loop heterotic string calculation similar to that in [44], but including the effect of closed heterotic string winding and momentum modes along the circle. The result is

$$\begin{aligned} K_W(r_H) = \quad & -\frac{1}{64\pi}\int d^2\tau \int d^2 z \Bigg[\Bigg\{ \sum_{\nu'} \{\overline{\vartheta_{\nu'}(\frac{z}{2})^{16}}\} (\overline{\eta(\tau)})^{-18} (\eta(\tau))^{-6} \left(\frac{\vartheta_{11}(z)}{\overline{\vartheta_{11}(z)}}\right)^2 \Bigg\} \\ & \exp\left(-\frac{4\pi z_2^2}{\tau_2}\right) (\tau_2)^{-9/2} \frac{1}{r_H} \Bigg\{ \sum_{n,w} \exp\Big(-\frac{\pi i \overline{\tau}}{2} (\frac{n}{r_H} + w r_H)^2 + \frac{\pi i \tau}{2} (\frac{n}{r_H} - w r_H)^2 \Big) \Bigg\} \Bigg], \end{aligned} \tag{4.4.2}$$

with τ denoting the modular parameter of the torus, ν denoting the spin structure on the torus taking values 00, 01, 10 and 11, ϑ are the Jacobi theta functions, r_H radius of S^1 on which heterotic string theory is compactified and n, w representing the momentum and winding number along the compactified direction. Since this expression is invariant under T-duality transformation $r_H \to 1/r_H$ (except for the overall factor of $1/r_H$ that is taken care of by the transformation law of g^2 multiplying it), we can focus on the region $r_H \geq 1$. In this region the evaluation of the integral can be facilitated using a Poisson resummation in the variable n. This yields

$$\begin{aligned} K_W(r_H) = \quad & -\frac{1}{64\pi}\int d^2\tau \int d^2 z \Bigg[\Bigg\{ \sum_{\nu'} \{\overline{\vartheta_{\nu'}(\frac{z}{2})^{16}}\} (\overline{\eta(\tau)})^{-18} (\eta(\tau))^{-6} \left(\frac{\vartheta_{11}(z)}{\overline{\vartheta_{11}(z)}}\right)^2 \Bigg\} \\ & \exp\left(-\frac{4\pi z_2^2}{\tau_2}\right) (\tau_2)^{-5} \Bigg\{ \sum_{k,w} \exp\Big(-\frac{\pi}{\tau_2} r_H^2 |k - w\tau|^2 \Big) \Bigg\} \Bigg]. \end{aligned} \tag{4.4.3}$$

In the $r_H \to \infty$ limit only $k = w = 0$ term in the sum survives, giving back the ten dimensional result. For finite r_H numerically integrating expression 4.4.3 for different values of r_H we find that the result can be fitted approximately with the function,

$$K_W(r_H) \simeq 0.23 \left(1 + \frac{1}{r_H^7}\right)^{2/7}. \tag{4.4.4}$$

Then the corrected weak coupling expression for $F^W(g, r_H)$ to order g^2 is given as,

$$F_2^W(g, r_H) = g^{\frac{1}{4}} \left(1 + K_W(r_H) g^2\right). \tag{4.4.5}$$

Notice that $K_W(r_H)$ diverges in the $r_H \to 0$ limit. This is easily understood using the known T-duality invariance $r_H \to 1/r_H$ in the heterotic string theory. Under this the ten dimensional string coupling transforms to g/r_H. Defining

$$\widetilde{r}_H = \frac{1}{r_H}, \quad \widetilde{g} = \frac{g}{r_H}, \tag{4.4.6}$$

we can express (4.4.5) as

$$F_2^W(g, r_H) = (\widetilde{r}_H)^{-1/4} \widetilde{g}^{\frac{1}{4}} \left(1 + .23 \left(1 + (\widetilde{r}_H)^{-7}\right)^{2/7} \widetilde{g}^2\right). \tag{4.4.7}$$

Except for the overall factor of $(\widetilde{r}_H)^{-1/4}$ which reflects the overall scale factor relating the ten dimensional Einstein metric in the dual pair of heterotic string theories, we see that this has a perfectly good $\widetilde{r}_H \to \infty$ ($r_H \to 0$) limit at fixed $\widetilde{g}$. For this reason, for $r_H < 1$ it is more natural to use the coupling constant $\widetilde{g}$ of the T-dual theory as an expansion parameter.

4.5 Interpolating functions

We now turn to the construction of the interpolating functions. For definiteness we shall treat g and r_H as independent variables. We shall divide up the g-r_H plane into several regions

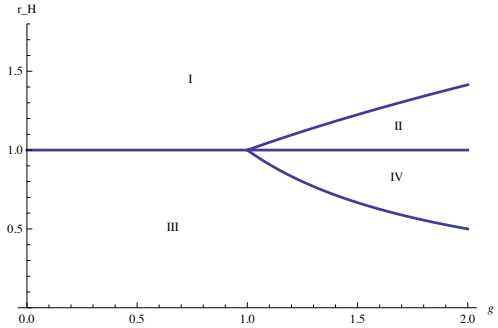

Figure 4.1: The four regions in the g-r_H plane. The curves bounding these regions are $r_H = 1$, $r_H = g^{1/2}$ and $r_H = g^{-1}$.

shown in Fig. 4.1 and use different interpolating functions in these different regions.

Region I: First consider the region I defined by

$$\mathbf{I} \;:\; r_H \geq 1, \quad r_H \geq g^{1/2}\,. \tag{4.5.1}$$

In this region $r_I \geq 1/\sqrt{2}$ (see (4.3.7)) and both the heterotic perturation theory in powers of g and type I perturbation theory in powers of g^{-1} are well defined at small and large g respectively. Thus we can use standard interpolation formula described in §4.2:

$$F_{0,0}(g, r_H) = g^{1/4}\,(1+g)^{1/2}\,, \tag{4.5.2}$$

$$F_{1,0}(g, r_H) = g^{1/4}\,(1+g^2)^{1/4}\,, \tag{4.5.3}$$

$$F_{0,1}(g, r_H) = g^{1/4}\,(1+4\,K_S(r_I)g+g^2)^{1/4}\,, \tag{4.5.4}$$

$$F_{1,1}(g, r_H) = g^{1/4}\,(1+6\,K_S(r_I)g^2+g^3)^{1/6}\,, \tag{4.5.5}$$

$$F_{2,0}(g, r_H) = g^{1/4}\,(1+6\,K_W(r_H)g^2+g^3)^{1/6}\,, \tag{4.5.6}$$

$$F_{2,1}(g, r_H) = g^{1/4}\,(1+8\,K_W(r_H)g^2+8K_S(r_I)g^3+g^4)^{1/8}\,, \tag{4.5.7}$$

$$F_{3,0}(g, r_H) = g^{1/4} \, (1 + 8 \, K_W(r_H) g^2 + g^4)^{1/8} \, , \tag{4.5.8}$$

and

$$F_{3,1}(g, r_H) = g^{1/4} \, \left(1 + 10 \, K_W(r_H) g^2 + 10 \, K_S(r_I) g^4 + g^5\right)^{1/10} \, . \tag{4.5.9}$$

Region II: Region II is defined by

$$\mathbf{II} \quad : \quad 1 \le r_H \le g^{1/2} \, . \tag{4.5.10}$$

In this region heterotic perturbation theory in power of g is still valid for small g but type I perturbation theory in powers of g^{-1} breaks down at large g due to the presence of the tachyon. Thus the only interpolating functions we can use are $F_{m,0}$ for $0 \le m \le 3$.

Region III: Region III is defined by

$$\mathbf{III} \quad : \quad r_H \le 1, \quad r_H \le g^{-1} \, . \tag{4.5.11}$$

The significance of this region can be understood by reexpressing (4.5.11) in terms of T-dual variables $\widetilde{r}_H$, $\widetilde{g}$ introduced in (4.4.6). This corresponds to

$$\widetilde{r}_H \ge 1, \quad \widetilde{r}_H \ge \widetilde{g}^{1/2} \, . \tag{4.5.12}$$

In this region we shall use the interpolating functons of region I with (g, r_H) replaced by $(\widetilde{g}, \widetilde{r}_H)$ and with an overall multiplicative factor of $(\widetilde{r}_H)^{-1/4} = (r_H)^{1/4}$ to account for the rescaling of the canonical metric discussed below (4.4.7). Thus we use the functions

$$\widetilde{F}_{m,n}(g, r_H) = (r_H)^{1/4} \, F_{m,n}(g/r_H, 1/r_H) \, . \tag{4.5.13}$$

Physically this corresponds to using an interpolating formula between the T-dual heterotic string theory and its strong coupling dual type I string theory (obtained in the $\widetilde{g} \to \infty$ limit at

fixed $\widetilde{r}_H/\widetilde{g}^{1/2}$). This is not the original type I string theory, but related to it via a strong-weak coupling duality transformation. This is apparent from the fact that while in the original type I string theory the non-BPS D-brane develops a tachyon for $r_H < g^{1/2}$, in the new theory the non-BPS D-brane is tachyon free for $r_H < g^{-1}$.

Region IV: Region IV is defined by

$$\mathbf{IV} \quad : \quad g^{-1} \le r_H \le 1\,. \tag{4.5.14}$$

In the $(\widetilde{g}, \widetilde{r}_H)$ variables this corresponds to $1 \le \widetilde{r}_H \le \widetilde{g}^{-1/2}$, i.e. this is the heterotic T-dual image of region II. Thus we use the interpolation formulæ of region II with (g, r_H) replaced by $(\widetilde{g}, \widetilde{r}_H)$:

$$\widetilde{F}_{m,0}(g, r_H) = (r_H)^{1/4}\, F_{m,0}(g/r_H, 1/r_H)\,. \tag{4.5.15}$$

Note that the results in regions III and IV can be obtained from those in regions I and II by heterotic T-duality transformation (4.4.6). For this reason we shall focus on regions I and II from now on.

In Fig.4.2 we have plotted the ratios of $F_{m,n}$ to $F_{3,1}$ as a function of g for four different values of r_H in region I. As we can see, except for $F_{0,0}$, all other $F_{m,n}$'s remain within about 10% of $F_{3,1}$ over the entire allowed range of g in region I. This suggests that $F_{3,1}$ gives the actual mass of the particle within about 10% error over the entire range of parameter space of region I (and hence also of region III). We shall return to a discussion of regions II and IV in the next section.

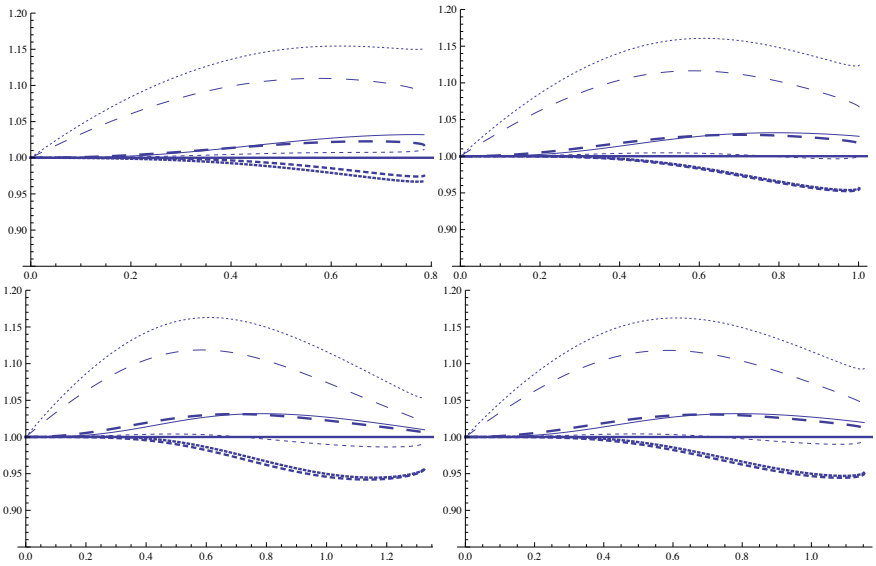

Figure 4.2: Graph of $F_{m,n}(g)/F_{3,1}(g)$ vs. $\tan^{-1} g$ for various (m,n) in region I. The labels are as follows: thin dots for $F_{0,0}$, thick dots for $F_{1,0}$, small thin dashes for $F_{2,0}$, small thick dashes for $F_{3,0}$, large thin dashes for $F_{0,1}$, large thick dashes for $F_{1,1}$, continuous thin line for $F_{2,1}$ and continuous thick line for $F_{3,1}$. The four graphs, clockwise from top left, correspond to $r_H = 1, 1.25, 1.5$ and 2 respectively.

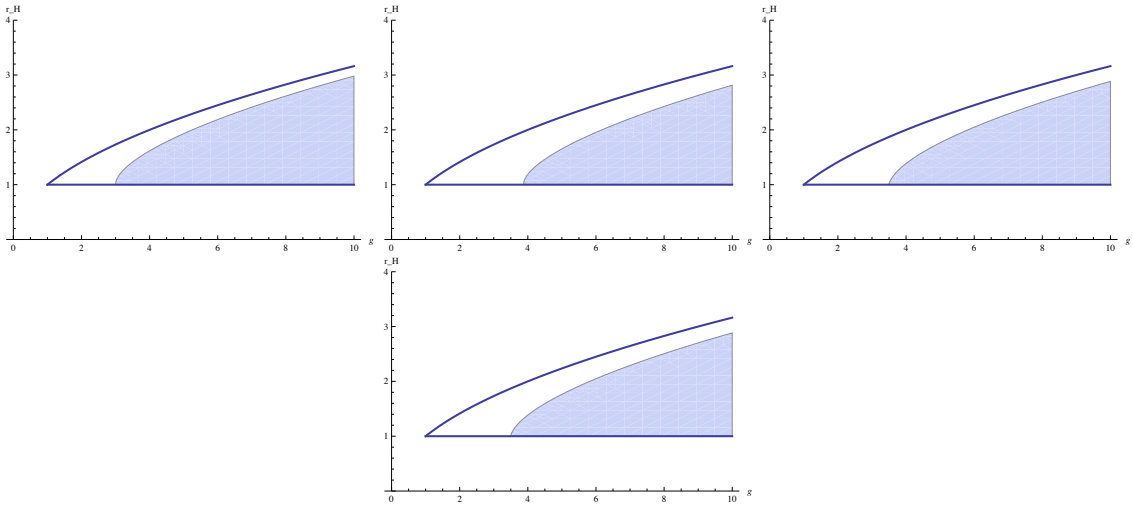

Figure 4.3: Region of instability of $F_{m,0}$ for $0 \leq m \leq 3$ in the region II of the $g - r_H$ plane. Clockwise from top left the diagrams are based on the interpolating function $F_{0,0}$, $F_{1,0}$, $F_{2,0}$ and $F_{3,0}$ respectively.

4.6 Stability analysis

Let (n, w) denote the momentum and winding numbers of a heterotic string state along the compact circle.The non-BPS state carries the same quantum numbers as that of an heterotic string state with quantum numbers $(n, w) = (1, 1)$ in the spinor representation of SO(32), a state with quantum number $(n, w) = (0, -1)$ in the adjoint/singlet representation of SO(32) and a state with quantum number $(n, w) = (-1, 0)$ in the singlet/adjoint representation.[3] The total mass of this state in the heterotic string metric for $r_H > 1$ is

$$M^H_{BPS} = \left(r_H + \frac{1}{r_H}\right) + r_H + \frac{1}{r_H} = 2\left(r_H + \frac{1}{r_H}\right) . \tag{4.6.1}$$

In the Einstein metric this is given by

$$M^E_{BPS} = (g_H)^{1/4} M^H_{BPS} = 2^{15/8}\pi^{7/8} g^{1/4} \left(r_H + \frac{1}{r_H}\right) . \tag{4.6.2}$$

After taking into account the normalization (4.2.2) we get

$$F_{BPS}(g) \sim 2^{-15/8}\pi^{-7/8} M^E_{BPS} = g^{1/4} \left(r_H + \frac{1}{r_H}\right) = 2^{1/2} g^{3/4} \left(r_I + \frac{1}{2\, g\, r_I}\right) . \tag{4.6.3}$$

This expression is not renormalized. Furthermore it is manifestly invariant under heterotic T-duality transformation (4.4.6).

In the (m, n) approximation the non-BPS particle is stable when its mass is less than the total mass of the BPS constituents to which it can decay. In regions I and II this requires

$$F_{m,n}(g) < g^{1/4} \left(r_H + \frac{1}{r_H}\right) . \tag{4.6.4}$$

[3]We could also consider decay into $(n, w) = (1, 1)$ in the spinor representation and $(n, w) = (-1, -1)$ in the adjoint or singlet representation, or $(n, w) = (1, 1)$ in the spinor representation, $(n, w) = (1, -1)$ in the singlet representation and $(n, w) = (-2, 0)$ in the singlet or adjoint representation. In each of these cases the total mass of the decay products is the same as that given on the right hand side of (4.6.1).

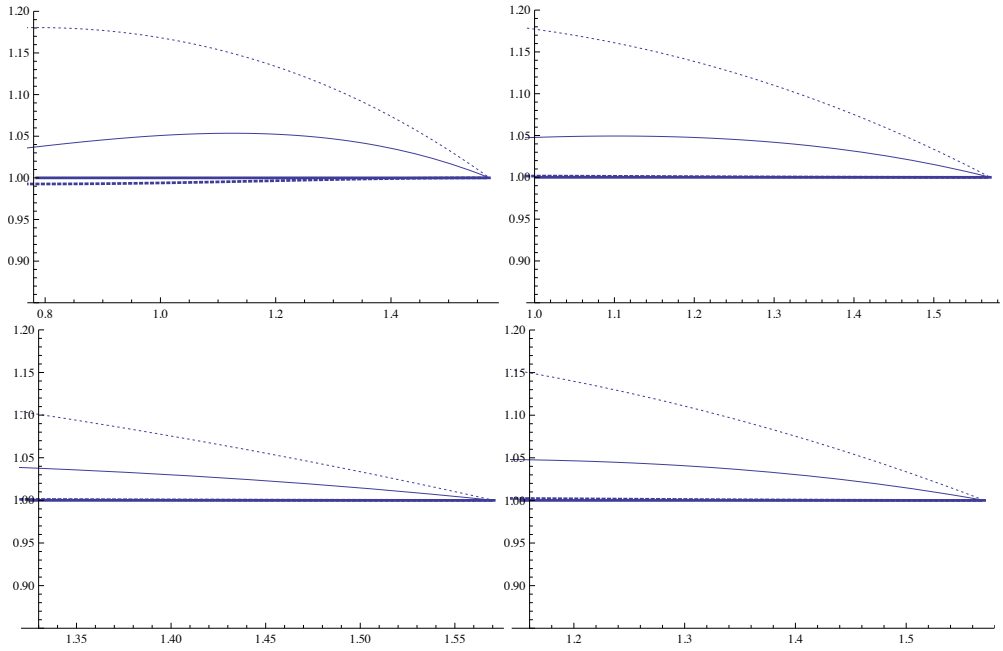

Figure 4.4: Graph of $F_{m,0}(g)/F_{3,0}(g)$ vs. $\tan^{-1} g$ for various m in region II. The labels are as follows: thin dots for $F_{0,0}$, thick dots for $F_{1,0}$, continuous thin line for $F_{2,0}$ and continuous thick line for $F_{3,0}$. The four graphs, clockwise from top left, correspond to $r_H = 1, 1.25, 1.5$ and 2 respectively.

In regions III and IV the left hand side is replaced by $\widetilde{F}_{m,n}(g)$, but the results in these regions are related to those in regions I and II respectively by heterotic T-duality. Now one can check explicitly that in region I all the $F_{m,n}(g)$'s satisfy (4.6.4), showing that whatever approximation we use, the non-BPS state is stable in this region. By heterotic T-duality the same result holds in region III. In region II only $F_{m,0}$ approximations make sense. In Fig. 4.3 we have shown by the shaded region the region of instability of the non-BPS state in different approximations. As we can see, these regions are not too different from each other, indicating that this is a fairly good approximation to the true region of instabiity of the non-BPS state in region II. The region of instability in region III can be found from this using heterotic T-duality transformation.

One point worth noticing is that in each of these plots, there is a narrow strip of region II where the non-BPS state is stable. This may seem a bit surprising at first since in the whole of region II perturbative open string theory describing the non-BPS D0-brane develops a tachyon. Note however that this is true only in tree level open string theory which corresponds to $g \to \infty$

limit of this diagram. Indeed in this limit the strip width reduces to zero showing that the D0-brane becomes unstable as soon as we cross the upper boundary of region II. At finite g however the tachyon mass^2 itself may get corrected and hence the tachyon may not develop as soon as we cross the upper boundary. We cannot do this analysis directly in type I string theory since at present it is not understood how to carry out open string perturbation theory in the presence of a tree level tachyon. Instead we have checked the stability by comparing the mass of the unstable brane with the total mass of the decay products, and arrived at Fig. 4.3.

It is natural to ask whether one can reliably determine the mass of the non-BPS particle in the white part of region II where it is expected to be stable. This can be done via the interpolating function $F_{m,0}$. Fig.4.4 shows the ratios of $F_{m,0}$ to $F_{3,0}$ in region II. As we can see from this graph, the ratios remain within about 10% of unity except for $F_{0,0}$, indicating that the interpolating formulæ based on $F_{3,0}$ determines the actual mass of the stable non-BPS particle even inside region II to within about 10% accuracy.

4.7 Compactification on higher dimensional tori

In this section we shall briefly discuss the generalization of the above analysis to type I / SO(32) heterotic string theory compactified on T^d – a d dimensional torus. We shall refrain from switching on any gauge field background so as to keep SO(32) gauge group unbroken, but allow generic values of the other moduli. This corresponds to choosing arbitrary constant metric G_{Hmn} and NS-NS 2-form field B_{Hmn} along T^d in heterotic description, and arbitrary constant metric G_{Imn} and RR 2-form field C_{Imn} in the type I description. Generalization of (4.2.4) relating the two sets of moduli are

$$G_{Hmn} = 2\, g\, G_{Imn}, \quad B_{Hmn} = C_{Imn}\,. \tag{4.7.1}$$

The weak and strong coupling expansions take the same form as in (4.2.5) and the interpolation formula takes the form of (4.2.6) with the dependence on r_H now generalized to dependence on G_{Hmn} and B_{Hmn} and the dependence on r_I generalized to dependence on G_{Imn}. The procedure for constructing the coefficients a_i and b_i in (4.2.6) is the same as that for S^1 compactification.

The analog of the strong coupling expansion (4.3.6), (4.3.10) now takes the form:

$$\begin{aligned}
F_1^S(g) &= g^{\frac{3}{4}}\Big(1+K_S\, g^{-1}\Big), \\
K_S &\sim -2^{-5/2}(8\pi^2)^{-\frac{1}{2}} \lim_{\Lambda\to\infty}\lim_{\epsilon\to 0}\Bigg[\int_\epsilon^\Lambda s^{-\frac{3}{2}}ds\Bigg\{\frac{1}{2}\Bigg(\sum_{\vec{n}}\tilde{q}^{2G_{Ik\ell}n^k n^\ell}\Bigg)\Bigg(\frac{f_3(\tilde{q})^8}{f_1(\tilde{q})^8}-\frac{f_2(\tilde{q})^8}{f_1(\tilde{q})^8}\Bigg) \\
&\quad +16\sqrt{2}\frac{f_2(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_3(\tilde{q})}-16\sqrt{2}\frac{f_3(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_2(\tilde{q})}\Bigg\} \\
&\quad +\int_{\epsilon/4}^\Lambda s^{-\frac{3}{2}}ds\Bigg\{2^{\frac{5}{2}}(1-i)\frac{f_3(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_4(i\tilde{q})}-2^{\frac{5}{2}}(1+i)\frac{f_4(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_3(i\tilde{q})}\Bigg\}\Bigg], \qquad (4.7.2)
\end{aligned}$$

where the sum over $\vec{n}$ refers to sum over d integers $(n^1,\cdots n^d)$ labelling the winding numbers of open strings along the d circles. Similarly the weak coupling result (4.4.2) now takes the form

$$\begin{aligned}
K_W = \quad & -\frac{1}{64\pi}\int d^2\tau\int d^2 z\Bigg\{\sum_{\nu'}\{\overline{\vartheta_{\nu'}(\frac{z}{2})^{16}}\}(\overline{\eta(\tau)})^{-18}(\eta(\tau))^{-6}\Bigg(\frac{\vartheta_{11}(z)}{\overline{\vartheta_{11}(z)}}\Bigg)^2\Bigg\} \\
& \exp\Bigg(-\frac{4\pi z_2^2}{\tau_2}\Bigg)(\tau_2)^{(d-10)/2}(\det G_H)^{-1/2} \\
& \sum_{\vec{n},\vec{w}}\exp\Big[-\pi\tau_2\Big\{(G_H^{-1})_{k\ell}n^k n^\ell+(G_H-B_H G_H^{-1}B_H)_{k\ell}w^k w^\ell+2(G_H^{-1}B_H)_{k\ell}n^k w^\ell\Big\} \\
& -2\pi i\tau_1 n^k w^k\Big], \qquad (4.7.3)
\end{aligned}$$

where the sum over $\vec{n}$ and $\vec{w}$ respresent the sum over d momentum quantum numbers $(n^1,\cdots n^d)$ and d winding numbers $(w^1,\cdots w^d)$.

As in the case of S^1 compactification, we shall find that on the strong coupling side the

computation of K_S suffers from tachyonic divergence when $2G_{Ik\ell}n^k n^\ell$ becomes less than 1 for any non-zero $\vec{n}$. Inside this region we need to use only the zeroeth order result on the strong coupling side. On the weak coupling side, when the size of the torus T^d is small, K_W computed from (4.7.3) becomes large signalling an apparent breakdown of perturbation theory. The remedy is to use a T-duality transformation and use the T-dual variables. In fact here we have a large group $O(6,6;\ \mathbb{Z})$ of T-duality transformation acting on the moduli space. We need to identify the analog of the regions I and II in Fig. 4.1 in which we carry out the actual computation and interpolation and then extend the result to the rest of the moduli space using heterotic T-duality invarinace. Since in the compactified heterotic string theory the effective coupling constant is given by $g^2/\sqrt{\det G_H}$, the natural analog of regions I and II will be to pick that domain in the moduli space for which $\sqrt{\det G_H}$ takes the maximum possible value – i.e. given any point inside such a domain, any of its T-dual image should have lower value of $\sqrt{\det G_H}$. Once we have identified such a domain we then divide this into the two regions I and II depending on whether $2G_{Ik\ell}n^k n^\ell$ lies above 1 for all $\vec{n}$ or not. The rest of the analysis would then proceed as in the case of S^1 compactification.

4.8 Discussion

In this chapter we have analyzed the mass formula for stable non-BPS state in type I / SO(32) heterotic string theory compactified on a circle using the interpolation formula between the strong and weak coupling results. Our analysis indicates that the interpolation formula determines the mass of the state within 10% accuracy over the entire moduli space. We also determine the region of stability of the particle based on the mass formula and discuss generalization of the analysis for generic toroidal compactification.

In recent times there has been significant developments in resumming perturbation theory [55–66]. It will be interesting to see if the interpolation between strong and weak coupling results can be combined efficiently with these resummation techniques to get a better under-

standing of physical quantities at intermediate values of coupling.

CHAPTER 5

Simplest Gauge-String Duality

5.1 Introduction

Arguably for last two decades gauge/string duality is the most fascinating and studied topic in theoretical physics. Gauge/string duality is a statement of equivalence between certain quantum field theories and quantum theory of gravity in certain spaces. Although different examples of gauge/string duality are well studied by now, the underlying mechanism is still not well understood. A simple example of how the Feynman diagrams for an n-point gauge correlator glue up into an n-point string scattering amplitude in a dual space-time can potentially provide a lot of insight into how gauge-string duality works.

A candidate proposal for the "simplest gauge-string duality" was put forward in [67]. It relates the Gaussian one matrix integral in a large N 'tHooft limit to the A-model topological string theory on $\mathbb{P}^1$ [68,69]. Gauge invariant correlators of the single trace operators $\mathrm{Tr}M^p$ can then be expected to be related to physical vertex operator scattering amplitudes in the dual topological string theory, where M is an $N \times N$ hermitian matrix.

Two pieces of evidence in favor of this (for planar connected correlators of $\mathrm{Tr}M^{2p}$) were discussed in [67]. The first was a nontrivial agreement of the degree of the covering map (from

the worldsheet $\mathbb{P}^1$ to the target $\mathbb{P}^1$) which contributes to a given correlator. The second was a matching of the two point function $\langle \mathrm{Tr} M^{2k_1} \mathrm{Tr} M^{2k_2} \rangle_{conn}$ (for arbitrary k_1, k_2). While this was encouraging, one needs stronger checks. Fortunately, it is possible to carry these out explicitly for a large class of correlators and this is the aim of this note. We find, from the explicit computations on both sides of the putative duality, that there is a simple relation between the two sides which is a natural realization of gauge-string duality in this context.

Before outlining this relation, we briefly recap the thread of logic followed in [67]. The starting point was the observation by de Mello Koch and Ramgoolam [70] (see also the earlier work [71–73]) that the combinatorics for computing Gaussian correlators $\langle \prod_{i=1}^{n} \mathrm{Tr} M^{2k_i} \rangle_g$ is suggestive of a sum over branched covers from a genus g worldsheet to a target $\mathbb{P}^1$, with *three* branchpoints[1]. Such holomorphic maps are known as Belyi maps. This interpretation of the combinatorics was given a concrete realization in [67] (with a crucial modification, though, which restricts one for the time being to planar worldsheets) in terms of a specific prescription to glue Feynman diagrams. This also enables one to give an explicit form for the Belyi maps in question using results of [75]. In essence, this prescription to glue Feynman diagrams is a special application of the general approach to open-closed string duality put forward in [76–79][2] as adapted to the matrix model case by Razamat [80, 81].

Furthermore, the target space $\mathbb{P}^1$ was identified [67] with the riemann surface canonically associated to the Gaussian matrix model. This is the master field geometry which captures the complexified eigenvalue distribution ("Wigner semicircle"). One then sees a skeletal version of AdS/CFT in the scattering on the target $\mathbb{P}^1$. Finally, since the conventional worldsheet theory describing holomorphic maps to a target is the A-model topological string theory, an attempt was made to directly link this to the Gaussian matrix model. As described above,

[1] The authors of [74] have also proposed an intriguing interpretation of this in terms of a three dimensional target space: a sphere with three holes times S^1.

[2] For further elaborations on this proposal and related aspects see [82–89].

some limited evidence was found in favor of this last proposal.

The additional evidence we provide here is much stronger and can be summarized as follows. On the matrix model side one can explicitly compute the planar correlators ($n \geq 2$)

$$\langle \frac{1}{k_1} \mathrm{Tr} M^{2k_1} \frac{1}{k_2} \mathrm{Tr} M^{2k_2} \dots \frac{1}{k_n} \mathrm{Tr} M^{2k_n} \rangle_{conn} = \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i! k_i!} \tag{5.1.1}$$

where $d = \sum_{i=1}^{n} k_i$. We will then compare with the genus zero topological string correlators, of gravitational descendants of the kahler class operator, which turn out to be ($n \geq 2$)

$$\langle (2\sigma_{2k_1-1})(2\sigma_{2k_2-1}) \prod_{i=3}^{n} \sigma_{2k_i} \rangle_{g=0} = d^{n-3} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i! k_i!}. \tag{5.1.2}$$

Firstly note that the RHS is symmetrical in the k_i despite the LHS not being manifestly symmetric (k_1, k_2 singled out). We can view these two vertex operators as being at some fixed positions e.g. $z = 1$ and $z = \infty$ on the worldsheet. Secondly, we note that 5.1.1 and 5.1.2 exactly match for $n = 2, 3$ for any k_i. As we will also see, this matching of two and three point functions holds for a general single trace operator $\mathrm{Tr} M^p$ i.e. we do not have to restrict to even powers.

In a nontrivial AdS/CFT duality, typically, one compares two and three point functions on both sides since higher point functions are determined in terms of these through factorisation. In that sense we have made a successful comparison of both sides. But here we also have the luxury to see how things might work for higher point functions (at the planar level).

We note that the higher n-point functions of the matrix model are close to the string answer though not exactly the same. In fact, for $n > 4$ the prefactor

$$\frac{(d-1)!}{(d-n+2)!} = d^{n-3} - \frac{(n-2)(n-3)}{2} d^{n-2} - \dots \tag{5.1.3}$$

Thus the leading piece (for large degree d) is indeed that of the string correlator but there also are some corrections which are subleading. Can we interpret these subleading correction terms as well?

It turns out that there is a natural interpretation. We have the inverse relation

$$d^{n-3} = \sum_{m=3}^{n} \tilde{S}_{n-2}^{(m-2)} \frac{(d-1)!}{(d-m+2)!} \tag{5.1.4}$$

where the (positive) coefficients $\tilde{S}_{n-2}^{(m-2)}$ turn out to count the number of ways in which to partition $(n-2)$ elements into $(m-2)$ non-empty subsets (see Sec. 9.74 of [90], for example). If we consider the $(n-2)$ vertex operators $\sigma_{2k_3}, \ldots \sigma_{2k_n}$ then their insertions on the worldsheet can come close to each other. Let's say we have $(m-2)$ such groups of these operators where $(m-2)$ can vary between $(n-2)$ (all of them separate) and 1 (all operators together at $z=0$, say). $\tilde{S}_{n-2}^{(m-2)}$ simply counts the number of such groupings.

The interpretation of 5.1.4 is then that there are contributions from "contact terms" in the topological string theory when these operators collide (see for e.g. [91]) which must correspond to lower m-point function matrix correlators. Thus if σ_{2k_3} and σ_{2k_4} "come together"[3] then by the interpretation of [67] two ramification points on the worldsheet coincide. On the matrix model side this is possible only if we replace $\mathrm{Tr}M^{2k_3}\mathrm{Tr}M^{2k_4} \to \mathrm{Tr}M^{2(k_3+k_4)}$ giving rise to an $(n-1)$-point correlator. Note that this is necessitated by the fact that there is no OPE on the matrix model side corresponding to bringing the matrix operators together. As a result, for four or higher point correlators we have to *separately* consider, in the matrix model, the contribution of these contact terms in the topological string correlators. The combinatorial coefficients in 5.1.4 account for these additional contributions and thus gives a natural way to connect the string answers to that of the matrix model. Note that the operators $\sigma_{2k_1-1}, \sigma_{2k_2-1}$

[3]Strictly speaking, points on the worldsheet never come together, one is simply going to a boundary in moduli space, where a sphere pinches off. Nevertheless we will use this loose terminology.

appear in 5.1.2 on a different footing from the σ_{2k}.

We will therefore also discuss a closely related set of correlators, where we will find a similar relation, but which now treats all the k_i on the same footing. This is in terms of the $(n+2)$-point string correlator $\langle P^2 \prod_{i=1}^{n} \sigma_{2k_i} \rangle_{g=0}$. The dual matrix correlator must now involve an operator corresponding to the puncture operator P. We identify this operator to be $\mathrm{Lim}_{p\to 0} \frac{1}{p} \mathrm{Tr} M^{2p} \sim 2\mathrm{Tr} \ln M$. We note that a similar identification was proposed by Eguchi and Yang [98] in their (closely related) matrix model for the A-model on $\mathbb{P}^1$. We will see that we can compute correlators of this operator both by analytic continuation ($p \to 0$) as well as directly using standard matrix model technology. Once again using 5.1.4 we obtain a relation between matrix and string correlators which is, in addition, symmetric in all the k_i.

In sections 2 and 3 we describe the matrix and string results respectively. Sec. 4 elaborates on the above comparison while Sec. 5 closes with general remarks. Appendix D give more details of the calculation of correlators. This chapter is based on [4].

5.2 Matrix Correlators

We will (mostly) consider the subset of matrix correlators in the Gaussian matrix model with even powers $\mathrm{Tr} M^{2p}$ i.e. $\langle \prod_{i=1}^{n} \mathrm{Tr} M^{2k_i} \rangle_{conn}$ and in the planar limit [92]. These can be obtained from the generating function

$$Z[t] = \int [dM]_{N\times N} e^{-\frac{1}{2} N \mathrm{Tr} M^2 + \sum_k t_p N \mathrm{Tr} M^{2p}} \tag{5.2.5}$$

by differentiating appropriately with respect to the t_{k_i}, taking the logarithm and finally the large N limit. This can be done in a variety of ways. One straightforward approach is to use the technique of orthogonal polynomials to write down a general form for the answer. After taking the logarithm to obtain the connected piece we can then take its large N limit.

The generating function can be expressed in terms of the integral over eigenvalues λ_i of M (see for e.g. [93])

$$\begin{aligned} Z[t] &= \int \prod_i d\lambda_i \Delta^2(\lambda) e^{-N\sum_i \mathrm{V}(\lambda)} \\ \mathrm{V}(\lambda) &= \frac{1}{2}\lambda^2 - \sum_p t_p \lambda^{2p} \\ \Delta(\lambda) &= \det \lambda_i^{j-1}. \end{aligned} \tag{5.2.6}$$

We express the Vandermonde determinant $\Delta(\lambda)$ in terms of orthogonal polynomials $P_m(\lambda)$, which satisfy the orthogonality relation

$$\int d\lambda P_m(\lambda) P_l(\lambda) e^{-N\mathrm{V}(\lambda)} = h_m \delta_{ml}. \tag{5.2.7}$$

Then the generating function reads

$$Z = N! h_0^N \prod_{j=0}^{N-1} R_j^{N-j} \tag{5.2.8}$$

where $R_m = \frac{h_m}{h_{m-1}}$. Using the recursion relations of $P_m(\lambda)$ we can derive the equation which determines R_m

$$R_m(t)[1 - \sum_{k=1}^{\infty} \frac{(2k)!}{k!(k-1)!} t_k R_m^{k-1}(t)] = \frac{m}{N}. \tag{5.2.9}$$

In the planar limit the rescaled index m/N becomes a continuous variable y that take values in $(0,1)$ and $R_m(t)$ becomes a continuous function $R(t,y)$. Then the generating function for connected correlators reduces to a simple one-dimensional integral

$$\begin{aligned} G(t) &= \lim_{N\to\infty} \frac{1}{N^2} \ln(\frac{Z[t_k]}{Z[0]}) \\ &= \int_0^1 dy (1-y) \ln(\frac{R(t,y)}{y}). \end{aligned} \tag{5.2.10}$$

We solve for $R(t,y)$ from the continuum version of 5.2.9 and using 5.2.10 we obtain $G(t)$. We can then extract the connected correlators from $G(t)$. Some of the steps are shown are shown in Appendix A. The final answer is (for $n \geq 2$)

$$\langle \frac{1}{k_1}\mathrm{Tr}M^{2k_1}\frac{1}{k_2}\mathrm{Tr}M^{2k_2}\dots\frac{1}{k_n}\mathrm{Tr}M^{2k_n}\rangle_{conn} = \frac{(d-1)!}{(d-n+2)!}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}. \tag{5.2.11}$$

Here $d = \sum_i k_i$. This agrees with the enumeration of graphs in [94] (see also [95]). The nontrivial d dependence shows that the answers are not, in general, factorised. Note that d also has an interpretation as the degree of the Belyi map that contributes to this correlator [67]. It turns out that one can also evaluate the correlators with two odd powers (correlators with one odd power vanish) using the results of [94]

$$\langle \frac{1}{2k_1+1}\mathrm{Tr}M^{2k_1+1}\frac{1}{2k_2+1}\mathrm{Tr}M^{2k_2+1}\prod_{i=3}^{n}\frac{1}{k_n}\mathrm{Tr}M^{2k_n}\rangle_{conn} = \frac{d!}{(d-n+3)!}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}. \tag{5.2.12}$$

with $d = \sum_i k_i$.

From these results we see that, in particular the two point function is given by

$$\langle \frac{1}{k_1}\mathrm{Tr}M^{2k_1}\frac{1}{k_2}\mathrm{Tr}M^{2k_2}\rangle_{conn} = \frac{1}{k_1+k_2}\frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}. \tag{5.2.13}$$

which agrees with the calculation in [67] since $d = k_1 + k_2$ in this case. We also have

$$\langle \frac{1}{2k_1+1}\mathrm{Tr}M^{2k_1+1}\frac{1}{2k_2+1}\mathrm{Tr}M^{2k_2+1}\rangle_{conn} = \frac{1}{k_1+k_2+1}\frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}. \tag{5.2.14}$$

Interestingly the three point function is the only one which is factorised. We have the non-vanishing ones to be

$$\langle \frac{1}{k_1}\mathrm{Tr}M^{2k_1}\frac{1}{k_2}\mathrm{Tr}M^{2k_2}\frac{1}{k_3}\mathrm{Tr}M^{2k_3}\rangle_{conn} = \frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\frac{(2k_3)!}{(k_3!)^2}. \tag{5.2.15}$$

and

$$\langle \frac{1}{2k_1+1}\mathrm{Tr}M^{2k_1+1}\frac{1}{2k_2+1}\mathrm{Tr}M^{2k_2+1}\frac{1}{k_3}\mathrm{Tr}M^{2k_3}\rangle_{conn} = \frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\frac{(2k_3)!}{(k_3!)^2}. \tag{5.2.16}$$

As mentioned in the introduction we will also compare correlators with (two) insertions of the operator $\mathrm{Tr}\ln M$. These can also be explicitly evaluated as we show in Appendix A.

$$\begin{aligned}\langle (\mathrm{Tr}\ln M)^2 \prod_{i=1}^{n}\frac{1}{k_i}\mathrm{Tr}M^{2k_i}\rangle_{conn} &= \frac{(d-n+2)(d-n+1)}{4}\langle\prod_{i=1}^{n}\mathrm{Tr}M^{2k_i}\rangle_{conn} \\ &= \frac{(d-1)!}{4(d-n)!}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}.\end{aligned} \tag{5.2.17}$$

A heuristic way to obtain this answer is to consider the $(n+2)$ point function

$$\mathrm{Lim}_{\epsilon_{1,2}\to 0}\langle\frac{1}{\epsilon_1}\mathrm{Tr}M^{2\epsilon_1}\frac{1}{\epsilon_2}\mathrm{Tr}M^{2\epsilon_2}\prod_{i=1}^{n}\frac{1}{k_i}\mathrm{Tr}M^{2k_i}\rangle_{conn} = \frac{(d-1)!}{(d-n)!}\prod_{i=1}^{n}\frac{(2k_i)!}{k_i!k_i!}. \tag{5.2.18}$$

Thus analytically continuing in ϵ_i and using $\mathrm{Lim}_{p\to 0}\frac{1}{p}\mathrm{Tr}M^{2p} \sim 2\mathrm{Tr}\ln M$ (the constant piece does not contribute to the connected correlator) we obtain the answer in 5.2.17.

In the description in terms of Belyi maps given in [67] both sets of correlators in 5.2.11 and 5.2.17 get contributions only from maps of degree $d = \sum_i k_i$. For the latter, we can understand this using the description of the logarithmic operator as in 5.2.18.

5.3 String correlators

Correlators in the A-model topological string on $\mathbb{P}^1$ are determined by recursion relations. The main relations are summarized in, for instance, [100]. The observables in the theory are the puncture operator $\mathcal{V}_1 = P$, the operator corresponding to the Kahler class $\mathcal{V}_2 = Q$ and their

gravitational descendants $\sigma_n(P), \sigma_n(Q)$ (for $n > 0$). The recursion relation we will mostly employ is

$$\langle \sigma_n(\mathcal{V}_\gamma) XY \rangle_{g=0} = n \langle \sigma_{n-1}(\mathcal{V}_\gamma) \mathcal{V}_\alpha \rangle \eta^{\alpha\beta} \langle \mathcal{V}_\beta XY \rangle_{g=0} \tag{5.3.19}$$

where X, Y are arbitrary observables and this holds in the so-called large phase space i.e. with arbitrary backgrounds for the descendants turned on as well. Therefore this enables one to express a general n-point function in terms of less complicated ones. In this paper we will restrict ourselves to correlators involving the puncture operator P as well as $\sigma_n(Q)$ which we will henceforth denote by σ_n (for $n > 0$) without hopefully causing confusion.

Using these and other recursion relations we find for $n \geq 2$ (see Appendix B for details)

$$\langle (2\sigma_{2k_1-1})(2\sigma_{2k_2-1}) \prod_{i=3}^{n} \sigma_{2k_i} \rangle_{g=0} = d^{n-3} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i! k_i!}. \tag{5.3.20}$$

Here, as before $d = \sum_i k_i$ and is also the degree of the holomorphic map which contributes to the above correlator, as can be seen from the selection rule given in D.2.22. We note that though the left hand side of 5.3.20 is not manifestly symmetric in all the k_i, the answer on the RHS is nevertheless so. We also record the answer for the correlator with all even powers

$$\langle \prod_{i=1}^{n} \sigma_{2k_i} \rangle_{g=0} = (d+1)^{n-3} \prod_{i=1}^{n} \frac{(2k_i)!}{(k_i!)^2}. \tag{5.3.21}$$

Here we retain the notation $d = \sum_i k_i$ but caution that the degree of the map which contributes to this correlator is actually $(d+1)$.

We will also use the related result that (for $n \geq 1$)

$$\langle P^2 \prod_{i=1}^{n} \sigma_{2k_i} \rangle_{g=0} = d^{n-1} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i! k_i!}. \tag{5.3.22}$$

The selection rule D.2.22 for the correlator in 5.3.22 shows that the contributions come only from holomorphic maps of degree $d = \sum_i k_i$.

5.4 Comparison

We can now compare the results on both sides.

5.4.1 Two and Three Point functions

We firstly note that the two and three point functions agree for arbitrary k_i. We see from 5.2.13 and 5.3.20 for $n = 2$ that

$$\begin{aligned} \langle \frac{1}{k_1} \mathrm{Tr} M^{2k_1} \frac{1}{k_2} \mathrm{Tr} M^{2k_2} \rangle_{conn} &= \frac{1}{k_1 + k_2} \frac{(2k_1)!}{(k_1!)^2} \frac{(2k_2)!}{(k_2!)^2} \\ &= \langle (2\sigma_{2k_1 - 1})(2\sigma_{2k_2 - 1}) \rangle_{g=0}. \end{aligned} \tag{5.4.23}$$

as well as 5.2.14 and 5.3.22 for $n = 2$ that

$$\begin{aligned} \langle \frac{1}{2k_1 + 1} \mathrm{Tr} M^{2k_1 + 1} \frac{1}{2k_2 + 1} \mathrm{Tr} M^{2k_2 + 1} \rangle_{conn} &= \frac{1}{k_1 + k_2 + 1} \frac{(2k_1)!}{(k_1!)^2} \frac{(2k_2)!}{(k_2!)^2} \\ &= \langle \sigma_{2k_1} \sigma_{2k_2} \rangle_{g=0}. \end{aligned} \tag{5.4.24}$$

Similarly, from 5.2.15 and 5.2.16 together with 5.3.20 and 5.3.21 for $n = 3$ we have

$$\begin{aligned} \langle \frac{1}{k_1} \mathrm{Tr} M^{2k_1} \frac{1}{k_2} \mathrm{Tr} M^{2k_2} \frac{1}{k_3} \mathrm{Tr} M^{2k_3} \rangle_{conn} &= \frac{(2k_1)!}{(k_1!)^2} \frac{(2k_2)!}{(k_2!)^2} \frac{(2k_3)!}{(k_3!)^2} \\ &= \langle (2\sigma_{2k_1 - 1})(2\sigma_{2k_2 - 1}) \sigma_{2k_3} \rangle_{g=0}. \end{aligned} \tag{5.4.25}$$

and

$$\langle \frac{1}{2k_1+1}\mathrm{Tr}M^{2k_1+1}\frac{1}{2k_2+1}\mathrm{Tr}M^{2k_2+1}\frac{1}{k_3}\mathrm{Tr}M^{2k_3}\rangle_{conn} = \frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\frac{(2k_3)!}{(k_3!)^2}$$

$$= \langle \sigma_{2k_1}\sigma_{2k_2}\sigma_{2k_3}\rangle_{g=0}. \quad (5.4.26)$$

As mentioned in the introduction, this in itself is a fairly good check that things are on the right track.

5.4.2 General correlators

We can however go on to compare the general n point function. The answers are given in 5.1.1 and 5.1.2 (or 5.2.11 and 5.3.20). We see that for $n \geq 4$ they are not quite identical. But this near agreement is quite remarkable in itself for a couple of reasons. Firstly, apart from the individual factors $\frac{(2k_i)!}{k_i!k_i!}$ which is a dependence that can be absorbed into a redefinition of the individual operators, both sets of correlators could have depended on arbitrary symmetric functions of the k_i's in a complicated way. The fact that both sides should, a priori, have depended only on the particular symmetric combination $d = \sum_i k_i$ is not obvious. Secondly, as seen in 5.1.3, the combination $\frac{(d-1)!}{(d-n+2)!}$ is a polynomial in d of degree $(n-3)$ with the leading term the same as the string answer. Thus there is exact agreement in the large d (or large k_i) regime which is some kind of BMN like limit. The large k_i regime is where the Feynman diagrams are dominated by graphs with a large number of edges and faces. From the point of view of the moduli space one is getting contributions from many more points on the moduli space. Effectively one will have a continuum description of moduli space [80][4].

We will now elaborate on the relation between the general matrix n-point correlator and that of the string theory which was sketched in the introduction. Firstly we will normalize the

[4]See [96,97] for a discussion of the subtleties in how equilateral triangulations capture the continuum measure on moduli space.

operators on both sides so as to get rid of the factors $\frac{(2k_i)!}{k_i!k_i!}$. Namely, we define

$$\tilde{\sigma}_{2k} = \frac{(k!)^2}{(2k)!}\sigma_k; \qquad \mathcal{O}_{2k} = \frac{1}{k}\frac{(k!)^2}{(2k)!}\mathrm{Tr}M^{2k}. \tag{5.4.27}$$

Then we have to compare

$$\langle(2\tilde{\sigma}_{2k_1-1})(2\tilde{\sigma}_{2k_2-1})\prod_{i=3}^{n}\tilde{\sigma}_{2k_i}\rangle_{g=0} = d^{n-3} \tag{5.4.28}$$

with

$$\langle\prod_{i=1}^{n}\mathcal{O}_{2k_i}\rangle_{conn} = \frac{(d-1)!}{(d-n+2)!}. \tag{5.4.29}$$

Using the relation 5.1.4 we can write

$$\langle(2\tilde{\sigma}_{2k_1-1})(2\tilde{\sigma}_{2k_2-1})\prod_{i=3}^{n}\tilde{\sigma}_{2k_i}\rangle_{g=0} = \sum_{m=3}^{n}\tilde{S}_{n-2}^{(m-2)}\langle\mathcal{O}_{2k_1}\mathcal{O}_{2k_2}\prod_{j=1}^{m-2}\mathcal{O}_{2\mu_j}\rangle_{conn}. \tag{5.4.30}$$

Here $\mu_j = \sum_{r\in R_j} k_r$ with $(j = 1\ldots(m-2))$ where $\{R_j\}$ are $(m-2)$ different non-empty groupings of the $(n-2)$ integers $(3,4,\ldots n)$. Thus the $\mathcal{O}_{2\mu_j}$ are essentially operators of the form $\mathrm{Tr}M^{(\sum_{R_j} 2k_{r_j})}$ over different groupings of the set $(k_3,\ldots k_n)$.

The stirling number of the second kind, $\tilde{S}_{n-2}^{(m-2)}$, which appears in 5.1.4 precisely counts the number of ways in which we can partition $(3,\ldots n)$ into the $(m-2)$ sets R_j such that each R_j contains at least one integer. Note that since the matrix correlator only depends on $\sum_i k_i$, we have

$$\langle\mathcal{O}_{2k_1}\mathcal{O}_{2k_2}\prod_{j=1}^{m-2}\mathcal{O}_{2\mu_j}\rangle_{conn} = \frac{(d-1)!}{(d-m+2)!} \tag{5.4.31}$$

independent of the partitioning (i.e. the μ_j). We only need that $k_1 + k_2 + \sum_j \mu_j = \sum_i k_i = d$.

This, in turn, is what enables us to write 5.4.30 or equivalently

$$\langle(2\tilde{\sigma}_{2k_1-1})(2\tilde{\sigma}_{2k_2-1})\prod_{i=3}^{n}\tilde{\sigma}_{2k_i}\rangle_{g=0}=\sum_{m=3}^{n}\sum_{partitions\{R_j\}}\langle\mathcal{O}_{2k_1}\mathcal{O}_{2k_2}\prod_{j=1}^{m-2}\mathcal{O}_{2\mu_j}\rangle_{conn}. \tag{5.4.32}$$

Recall that the interpretation of $\mathrm{Tr}M^{2p}$ in [67] was that it created a ramification point of order p on the worldsheet. The map to the target $\mathbb{P}^1$ therefore locally looks like $X(z)=(z-z_1)^p$ near this vertex operator insertion. In the string theory we can have two ramification points with behavior $(z-z_1)^{p_1}$ and $(z-z_2)^{p_2}$ coming together when $z_1\to z_2$ to create a ramification point of order (p_1+p_2). However, in the matrix model, unlike in a QFT, we do not have an OPE of the two corresponding operators $\mathrm{Tr}M^{2p_1}$ and $\mathrm{Tr}M^{2p_2}$ giving something like $\mathrm{Tr}M^{(2p_1+2p_2)}$. We have to put in the contribution to the string correlator, from the collision of ramification points, *separately* on the matrix model side. They are not contained in the original n-point correlator $\langle\prod_i^n \mathrm{Tr}M^{2k_i}\rangle_{conn}$. But we now see that we can interpret the different terms on the right hand side of 5.4.32 as these additional contributions. Thus, in addition to the n-point matrix correlator (corresponding to $m=n$ or the partition where each of the R_j contain exactly one integer) we also have the lower point functions all the way unto a three point function (where we have all of $(k_3,\dots k_n)$ come together). In general, through these lower point functions we include all the contributions where various of the σ_{2k_i} (for $i=3,\dots n$) come together in different groupings.

Note that we do not have any contribution corresponding to bringing any of the σ_{2k_i} near either of σ_{2k_1-1} or σ_{2k_2-1}. It remains to be understood from the string point of view why this is the case. Admittedly, this is somewhat unsatisfactory in that it treats the k_i in an asymmetric way despite the RHS being symmetric[5]. We will therefore remedy this by looking at an alternate set of correlators where the symmetry is manifest. However, this will be at the

[5]One can make a similar comparison for the string correlator 5.3.21 (which is symmetric in the k_i) with the

expense of introducing additional operators of the form $\mathrm{Tr}\ln M$.

On the string theory side the correlators we consider are the ones in 5.3.22 rewritten using the redefined operators in 5.4.27 as

$$\langle P^2 \prod_{i=1}^{n} \tilde{\sigma}_{2k_i}(Q)\rangle_{g=0} = d^{n-1}. \tag{5.4.34}$$

With the identification $P \leftrightarrow 2\mathrm{Tr}\ln M \sim \mathcal{P}$ we compare with the matrix correlator 5.2.17

$$\langle \mathcal{P}^2 \prod_{i}^{n} \mathcal{O}_{2k_i}\rangle_{conn} = \frac{(d-1)!}{(d-n)!}. \tag{5.4.35}$$

We see that there is a mismatch for $n \geq 2$. But once again we have a natural relation between the two sides. We use

$$d^{n-1} = \sum_{m=1}^{n} \tilde{S}_n^{(m)} \frac{(d-1)!}{(d-m)!} \tag{5.4.36}$$

to write

$$\begin{aligned} \langle P^2 \prod_{i=1}^{n} \tilde{\sigma}_{2k_i}\rangle_{g=0} &= \sum_{m=1}^{n} \tilde{S}_n^{(m)} \langle \mathcal{P}^2 \prod_{j=1}^{m} \mathcal{O}_{2\mu_j}\rangle_{conn} \\ &= \sum_{m=1}^{n} \sum_{partitions\{R_j\}} \langle \mathcal{P}^2 \prod_{j=1}^{m} \mathcal{O}_{2\mu_j}\rangle_{conn}. \end{aligned} \tag{5.4.37}$$

The partitions $\{R_j\}$ (with $j = 1 \ldots m$) are the m different non-empty groupings of the n integers $(1, 2, \ldots n)$. Thus the $\mathcal{O}_{2\mu_j}$ are essentially operators of the form $\mathrm{Tr}M^{(\sum_{R_j} 2k_{r_j})}$ over different groupings of the set $(k_1, k_2, \ldots k_n)$.

matrix correlator 5.2.12 (which singles out k_1, k_2).

$$\langle \tilde{\sigma}_{2k_1}\tilde{\sigma}_{2k_2} \prod_{i=3}^{n} \tilde{\sigma}_{2k_i}\rangle_{g=0} = \sum_{m=3}^{n} \sum_{partitions\{R_j\}} \langle \mathcal{O}_{2k_1+1}\mathcal{O}_{2k_2+1} \prod_{j=1}^{m-2} \mathcal{O}_{2\mu_j}\rangle_{conn}. \tag{5.4.33}$$

The interpretation of the RHS in 5.4.37 is therefore similar to before. Now we have all the n operators σ_{2k_i} on the same footing on the LHS and so they can all come close to each other leading to the merging of ramification points. The RHS counts the contributions of these separate groupings from the matrix model side where we have lower point functions involving the merged operators $\mathrm{Tr}M^{(\sum_{R_j} 2k_{r_j})}$. Note that the puncture operator $P \leftrightarrow \lim_{\epsilon\to 0}\mathrm{Tr}M^{2\epsilon}$ does not create any branching. In many ways this is a much more neat picture for the correspondence between the correlators in the matrix model and the dual string theory.

5.5 Discussion

What we have learnt from this comparison of correlators in the topological string theory on $\mathbb{P}^1$ with those of the Gaussian matrix model is that they are not identically the same except for the two and three point functions. In fact, in hindsight, we see that for four or higher point functions there was no reason to have expected them to be the same since the matrix correlator does not allow for the possibility of bringing operators together on the worldsheet. Instead, the relation between the two sets of correlators is one in which we add in, on the matrix model side, the separate contributions from the fusing of two matrix operators.

On the topological string theory side it would be good to understand more explicitly these contact terms. Note that in the usual approach, such as [91], the worldsheet is in a gauge where all the curvature is concentrated at the location of the vertex operators. However, as remarked in [67], the matrix model naturally gives rise to a "strebel gauge" on the world sheet where the curvature is localized not just at the location of the insertions but also at the interaction vertices. This is likely to affect the contact term contributions and it would be good to see if it exactly matches what we find here.

We have made a comparison of the general set of even power matrix correlators which is the sector in which the answer is easy to obtain in closed form. It would be good to extend this to correlators involving an arbitrary number of odd powers as well. Significantly larger class of correlators has been computed and compared with the matrix model correlators in [104]. Interestingly they also found the mismatch that we discussed and interpreted the mismatch using contact terms. Another direction to extend the checks is to consider not just the so-called stationary sector of the topological string theory involving the gravitational descendants $\sigma_n(Q)$ but also the $\sigma_n(P)$. Here we have looked at correlators with insertion of the puncture operator insertion and seen that they correspond to insertions of $\mathrm{Trln}M$. It is tempting to guess that $\sigma_n(P) \leftrightarrow \mathrm{Tr}(M^n \mathrm{ln}M)$ following [98]. However, as discussed in [67], there are important differences between the current proposal and the Eguchi-Yang model, which is presumably related to the different relation, proposed here, between the general n-point correlators on both sides.

It is important to extend the relation between both sides beyond the planar/genus zero case. This would require taking into account effects of mixing of single and double trace operators and hence the correspondence between string vertex operators and matrix gauge invariant operators will acquire $\frac{1}{N}$ corrections. Another interesting generalization would involve the gaussian *normal* matrix model for which there is a proposed dual [103]. It would be nice to make some contact between topological strings, such as the one described here, with the imaginary Liouville backgrounds proposed in [103][6]. We leave these explorations for the future.

Finally, the results described here, in addition to their value as a toy model of AdS/CFT, may also be significant in the canonical gauge-string duality between $\mathcal{N} = 4$ super Yang-Mills theory and the string theory on $AdS_5 \times S^5$. The localization arguments of [105] have shown how the half BPS Wilson loops in the gauge theory reduce to a Gaussian matrix integral. Given the duality elucidated here, one might hazard the guess that there is a corresponding

[6]We thank J. McGreevy for drawing our attention to this work and for comments on related issues.

localization of the $AdS_5 \times S^5$ string theory in the half BPS sector which reduces the string sigma model to the A-model topological string theory on $\mathbb{P}^{1}$[7].

[7]See [106] for a proposed relation between the half BPS Wilson loops and a topological sector of the sigma model on $AdS_5 \times S^5$.

APPENDIX A

Brief Review of String Perturbation Theory

Primary objects of interest in a relativistic quantum theory are the scattering amplitudes or S-matrix elements because they provide the necessary bridge between theory and scattering experiments. In a scattering experiment we prepare initial state composed of a collection of well separated simple systems, then we allow them to interact and at a later time we measure the end products of the interaction. It is a convenient experiment to setup theoretically in local theories where the interactions turn off when the basic constituents are well separated. In such theories the asymptotic states, i.e. states at $t = -\infty$ and $t = \infty$, can be prepared by taking direct products of single particle states of the full interacting theory that are determined by the quantum numbers of the one particle states like energy, momentum, spin, charge. S-matrix elements describe the transition probability from asymptotic states at $t = -\infty$ to states at $t = \infty$.

For a local relativistic quantum theory of point particles having a Hamiltonian it is possi-

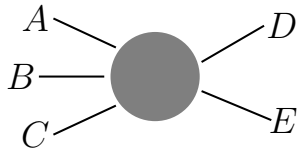

Figure A.1: Scattering experiment where initial state which is a direct product of single particle states A,B,C transform to the final state consists of single particle states D and E.

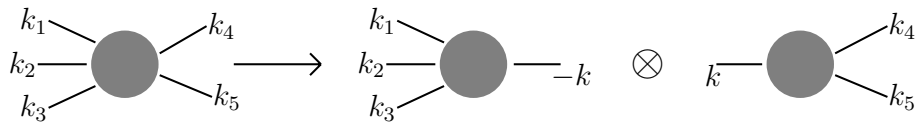

Figure A.2: Factorisation of scattering amplitudes involving 5 external states into product of scattering amplitudes involving 4 and 3 external states when $(k_1 + k_2 + k_3)^2 = (k)^2 = -m_p^2$ where m_p is the mass of physical state in the theory.

ble to split the Hamiltonian into two parts, free part whose eigenstates can be exactly solved and interacting part which can be thought of as determining the interaction. Such splitting of Hamiltonian is very useful due to Gell-Mann and Low theorem [112]. Gell-Mann and Low theorem guarantees that in the interaction picture eigenstates of free Hamiltonian at $t = -\infty$ and $t = \infty$ in full interacting theory evolves and becomes identical to eigenstates of the full Hamiltonian at $t = 0$. Then S-matrix elements can be expressed in terms of the eigenstates of the free part of the full Hamiltonian and interacting part of the Hamiltonian. This can be converted into an expression in terms of Green's function of full theory. Expression of S-matrix elements in a quantum field theory which involves asymptotic states made of n single particle states labelled by quantum numbers and momentum $(a_1, k_1), ..., (a_n, k_n)$ is given as

$$S^n_{a_1,...,a_n}(k_1, ..., k_n) = lim_{k_i^2 \to -m^2_{a_i,p}} G^n_{a_1,...,a_n}(k_1, ..., k_n) \Pi^n_{i=1} \Big\{ Z^{-\frac{1}{2}}(k_i, a_i)(k_i^2 + m^2_{a_i,p}) \Big\} \quad \text{(A.0.1)}$$

where a_i is the discrete quantum number and $m_{a_i,p}$ is the mass of the i^{th} single particle state of full interacting theory. Mass^2, $m^2_{a_i,p}$, of the single particle state of full interacting theory is defined as the pole of 2 point Green's function as a function of k_i^2 and $Z(k_i, a_i)$ is the residue at this pole.

Locality and unitarity are the most important ingredients of a relativistic quantum theory. Locality is inevitable to make the theory casual and unitarity of S-matrix is necessary for a sensible probabilistic interpretation of the theory. Requirement of locality leads to analyticity property of Green's function and S-matrix elements. Demanding unitarity imposes very

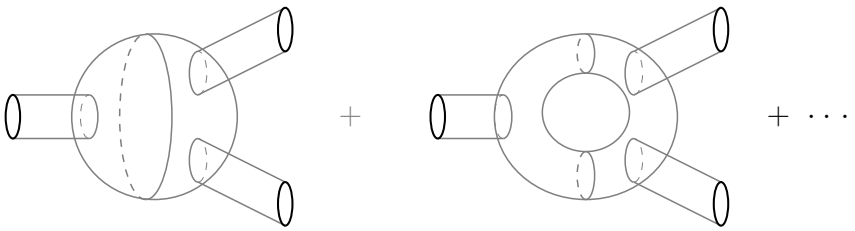

Figure A.3: Pictorial representation of perturbative series corresponds to the scattering of 3 closed strings whose asymptotic states are represented by black thick ellipses.

stringent constraints on the scattering amplitudes. One such constraint is the factorisation property of scattering amplitudes: when square of sum of momenta of a set of $m \leq n-1$ external states becomes equal to the square of mass of a single particle state in the theory then the scattering amplitude involving n external states split into two scattering amplitudes involving $m+1$ and $n-m+1$ external states provided none of the conservation laws are violated.

When we are interested in weakly interacting quantum field theories whose interacting part of Hamiltonian comes with small parameter (coupling constant) then we can taylor expand the Green's function and physical mass in terms of this small parameter. This perturbative expansion of Green's function and physical masses can be expressed diagrammatically using Feynman diagrams. Thus scattering amplitudes can be computed perturbatively using Feynman diagrams.

A.1 Polyakov Prescription

In first quantisation of string theory story is slightly different. This is because formulation of perturbative string theory is not based on a Hamiltonian which we can hope to separate into free and interacting part. Instead what we have is a prescription for directly computing string amplitudes which are the proposed scattering amplitudes of string theory without referring to an underlying Hamiltonian or Lagrangian known as Polyakov prescription.

Underlying logic of Polyakov prescription is the following. First quantisation of free relativistic point particle using path integral formulation involves summing over all possible paths weighted by a complex number of unit modulus whose phase is the invariant length of the world line traced by the point particle in spacetime. Following this logic first quantisation of free relativistic string is straight forward provided we do the following replacements. Instead of world line we need to sum over world sheets traced by the string in spacetime and instead of invariant length we need to consider the area of the world sheets.

Bold step in the Polyakov prescription is the way in which interactions between different strings are introduced and the way in which a perturbative series for the string amplitude is proposed by identifying an expansion parameter. Unlike in usual point particle case interactions are not introduced by generalising the free action by adding an interacting piece. It is introduced by simply stating that when different strings interact they merge to form an intermediate string which at the end again split into a new set of strings. Based on the evolution of intermediate string we can split Polyakov path integral into different terms of a perturbative series. First term in the series corresponds to the free propagation of intermediate string without much happenings and the second term corresponds to an intermediate string which during its propagation splits into two new strings which after a while merge back to form another intermediate string. By repeating this process of splitting the intermediate string into two new strings and merging back we can generate all terms in this series. To identify the expansion parameter we need to analyse the situation a little more carefully which we will do soon.

Summing over all possible world-sheets can be done by introducing a metric on the world-sheet and integrating over all such 2-d metrics and distinct embedding of world-sheets in spacetime. Since the spacetime through which the string propagates has non-Riemannian metric the induced metric on the world-sheet is also non-Riemannian. But by performing a wick rotation on the 2-d world-sheet coordinates we can make it a Riemannian metric and as a result the

closed string world-sheets becomes compact Riemann surfaces with boundaries where boundaries corresponds to initial and final closed strings. Thus the perturbative expansion for string amplitude can be thought of as a series of sum over Riemann surfaces with each terms in the series corresponds to a sum over compact Riemann surfaces having n boundaries corresponding to n closed external states strings with specific number of genus. Polyakov in his seminal paper [11, 12] showed that such a sum over compact Riemann surfaces reduces to 2-d exactly solvable conformal field theory on Riemann surfaces. We will briefly discuss how does this magic happen. Detailed study of relativistic strings, their quantisation and interaction can be found in [107–111]. More recent discussion on perturbative string theory is available in [5].

A.2 Riemann Surfaces

Before getting into a detailed discussion on perturbative string theory let us briefly discuss about Riemann surfaces. Consider a 2-dimensional real manifold Σ and a collection of patches $\mathcal{U}_\alpha$ on it such that union of all these patches completely cover Σ. If Σ is a Riemann surface then it is possible to map every patch $\mathcal{U}_\alpha$ to a patch in complex plane which can be parametrised by a complex coordinate z_α. Also when ever two patches $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ overlaps in the overlapping region the coordinates on each patch can be expressed as holomorphic functions of coordinates on other patches. So the definition of Riemann surface depends on the ways we assign complex coordinates in each patch and how we glue them using transition functions. Each such choice of coordinates and gluing rule is called a specific complex structure. Sphere and torus are the simplest examples of Riemann surfaces.

The realm of Riemann surfaces can be divided into three regimes: hyperbolic, parabolic and elliptic Riemann surfaces. Geometrically, these correspond to negative curvature, zero curvature/flat, and positive curvature Riemann surface. Riemann surface can be further divided based the number of elementary non-contractable closed contours on them denoted by $2h$ and h is called the genus of Riemann surface. In fact the integral of curvature of Riemann surface

just depends on the genus and is proportional to $2h-2$. So only genus 0 Riemann surface, sphere, is hyperbolic. Genus 1 Riemann surfaces, torus, are hyperbolic and all Riemann surfaces with $h \geq 2$ are elliptic.

With in each genus we have a family of inequivalent Riemann surfaces depending on the complex structure that is assigned to it. All inequivalent complex structures forms a non-compact space known as moduli space $\mathcal{M}_h$ and is parametrised by moduli parameters. For rigorous treatment of Riemann surfaces see [113].

A.3 Bosonic String Theory

Bosoinc string theory is the natural generalisation of study of quantum mechanical propagation and interaction of relativistic point particles through a d-dimensional spacetime to the case of relativistic strings propagating through a d-dimensional spacetime.

A.3.0.1 Polyakov Action

Action for a relativistic free string propagating through a flat d-dimensional spacetime is given by the surface area of the world-sheet swept by the string.

$$S_{NG} = T \int dA = T \int d^2\xi \sqrt{-det\ h_{ij}} \tag{A.3.1}$$

here T is the string tension which can be expressed using another constant α' as $T = \frac{1}{2\pi\alpha'}$, $\xi_i,\ i = 1, 2$ are the coordinates on the world-sheet, $h_{ij} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \xi^i}\frac{\partial X^\nu}{\partial \xi^j}$ is the metric induced on the world-sheet with embedding coordinates $X^\mu,\ \mu = 0, 1, \cdots, d-1$ from the flat spacetime. Subscript in the left hand side indicates that this action is known as Nambu-Goto action. Through out the following discussions we will set $\alpha' = 1$ because dimensional analysis can always be used to figure out the correct α' dependence when ever require.

Square roots in action makes quantisation difficult. Instead of Nambu-Goto action we will consider another classically equivalent action without square root known as Polyakov action. Price that we pay for this convenience is that Polyakov action contains an auxiliary field $g^{\alpha\beta}$ which can be interpreted as the intrinsic metric on the world-sheet.

$$S_p = \frac{1}{4\pi}\int d^2\xi\sqrt{-g}\; g^{\alpha\beta}\partial_\alpha X^\nu \partial_\beta X^\mu \eta_{\mu\nu},\ \alpha,\beta = 1,2. \tag{A.3.2}$$

where $g = det\ g^{\alpha\beta}$. Classically $g^{\alpha\beta}$ is proportional to the induced metric on the world-sheet but not in general. We should remember that intrinsic metric on the world-sheet is an auxiliary field so it is important to make sure that physical quantities do not depend on it. Since the choice of local coordinates and metric on the Riemann surface are intimately related the physical quantities should not depend on the choice of coordinates on the Riemann surface.

Polyakov action is invariant under Poincare transformations, reparameterizations of world-sheet coordinates and Weyl transformations of world-sheet metric $g^{\alpha\beta}$. Due to the reparameterization invariance of world-sheet action we need not consider Riemann surfaces with arbitrary metric $g^{\alpha\beta}$. This is because it is always possible to choose coordinates on an arbitrary Riemann surface such that the metric takes the form $\hat{g}_{\alpha\beta} = e^{\omega(\tau,\sigma)}\delta_{\alpha\beta}$ where (τ,σ) is the specific choice of coordinates which achieve this. Such a choice is known as conformal gauge. In conformal gauge action simplifies considerably due to its Weyl symmetry and becomes

$$S_p = \frac{1}{4\pi}\int d\sigma d\tau\ \partial_\sigma X^\nu \partial_\tau X^\mu \eta_{\mu\nu}. \tag{A.3.3}$$

A.3.1 Quantization

Now it is straight forward to write down general classical solution both for open and closed strings by imposing consistent boundary conditions. For closed string we can impose that $X^\mu(\tau,\sigma)$ is periodic along σ direction on world-sheet with a periodicity of 2π. At the end points of open strings we can impose two possible boundary conditions. One is Neumann

boundary condition which demands that canonical momentum with respect to X^μ field along σ direction vanishes at the end points $\sigma = 0, \pi$. Other possibility is Dirichlet boundary condition which demands that the end points of open string is not allowed to move along the spacetime direction X^μ. Most general solution for a closed string is given as,

$$X^\mu(\tau, \sigma) = X^\mu_L(\tau + \sigma) + X^\mu_R(\tau - \sigma) \tag{A.3.4}$$

where $X^\mu_L(\tau + \sigma)$ contains left moving oscillations and $X^\mu_R(\tau - \sigma)$ contains right moving oscillations. Their mode expansion is as follows,

$$\begin{aligned} X^\mu_L(\tau + \sigma) &= \frac{x^\mu}{2} + \frac{p^\mu}{2}(\tau + \sigma) + \frac{i}{\sqrt{2}} \sum_{k \neq 0} \frac{\bar{\alpha}^\mu_k}{k} e^{-ik(\tau+\sigma)} \\ X^\mu_R(\tau - \sigma) &= \frac{x^\mu}{2} + \frac{p^\mu}{2}(\tau - \sigma) + \frac{i}{\sqrt{2}} \sum_{k \neq 0} \frac{\bar{\alpha}^\mu_k}{k} e^{-ik(\tau-\sigma)} \end{aligned} \tag{A.3.5}$$

here x^μ and p^μ are centre of mass position and momentum of the closed string.

To quantise this classical theory we will lift the c-numbers x^μ, p^μ and $\bar{\alpha}^\mu_k, \alpha^\mu_k$ to operator status and impose the following commutation relations.

$$\begin{aligned} \left[x^\mu, p^\nu\right] &= i\eta^{\mu,\nu} \\ \left[\alpha^\mu_m, \alpha^\nu_n\right] &= m\delta_{m+n,0}\eta^{\mu,\nu} \\ \left[\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n\right] &= m\delta_{m+n,0}\eta^{\mu,\nu} \end{aligned} \tag{A.3.6}$$

where m and n are non-zero integers. Now for constructing the Hilbert space of string we choose $\alpha^\mu_m, \bar{\alpha}^\mu_m$ as lowering operators and $\alpha^\mu_{-m}, \bar{\alpha}^\mu_{-m}$ as raising operators with positive m. Ground state $|p\rangle$ which is characterised by the centre of mass momentum is annihilated by all lowering operators. All other states in Hilbert state can be created by acting the raising operators on

the ground state. All of them are spacetime bosons so string theory based on action A.3.3 is known as bosonic string theory.

Stress tensor of the world-sheet theory is given by,

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-g}} \frac{\delta S_p}{\delta g^{\alpha\beta}}. \tag{A.3.7}$$

We should remember that metric on the world-sheet $g^{\alpha\beta}$ is an auxiliary field so we need to make sure that within all physical quantities this auxiliary field satisfies equation of motion. This implies within all physical quantities word-sheet stress tensor should vanish. But reparametrisation invariance automatically guarantees the vanishing of T_{-+} and T_{+-}, where subscript + indicates the coordinate $\tau + \sigma$ and - indicates $\tau - \sigma$. So the only non-trivial components of stress tensor are T_{--} and T_{++}. They receive contribution from each modes of left and right moving oscillations. For closed strings they are given by,

$$\begin{aligned} L_m &= \frac{1}{\pi} \int_0^{2\pi} d\sigma \; T_{--} e^{im(\tau-\sigma)} = \frac{1}{2} \sum_n : \alpha^\mu_{m-n} \alpha^\nu_n : \eta_{\mu\nu} \\ \bar{L}_m &= \frac{1}{\pi} \int_0^{2\pi} d\sigma \; T_{++} e^{im(\tau+\sigma)} = \frac{1}{2} \sum_n : \bar{\alpha}^\mu_{m-n} \bar{\alpha}^\nu_n : \eta_{\mu\nu}. \end{aligned} \tag{A.3.8}$$

Here : : denotes the process of taking all lowering operates to the right. It is not difficult to see that the Hilbert space that we constructed using α^μ_m and $\bar{\alpha}^\mu_m$ satisfying the commutation relations A.3.6 contains negative norm states since $\eta_{00} = -1$. Interestingly the requirement of vanishing of world-sheet stress tensor within all physical quantities save the day. This implies that T_{++} and T_{--} should vanish if we sandwich each of them within ket and bra corresponding to physical states. So the physical subspace of Hilbert space contains only those states which are annihilated by all L_m and $\bar{L}_m$ for every $m \geq 0$.

$$L_m | \; phy \; \rangle = \bar{L}_m | \; phy \; \rangle = 0, \;\; m \geq 0. \tag{A.3.9}$$

For $m = 0$ this condition demands that all physical states in string spectrum carries centre of mass momentum whose square is equal to the tree level mass^2. On the top of it physical states are annihilated by $L_0 - \bar{L}_o$, operator taking a point on closed string to other, which ensures that every points on a closed string are treated equally. This condition demands that physical state can only be created by using a collection of left and right moving oscillators which satisfy level matching condition. Level matching condition says that level numbers, which is the sum of mode numbers (mode number of α^μ_{-m} is m), of left moving oscillators should be equal to that of right moving oscillators. Say a state has level number N, then its mass is given by

$$m^2 = 4\Big(N - \frac{d-2}{24}\Big) \tag{A.3.10}$$

where d is the dimension of spacetime through which the string is propagating.

There are a set of states which has vanishing norm with any states with any physical state. They are known as null states. It is easy to see that they have the following form

$$|\ spur\ \rangle = L_{-n}|\ arbitrary\ state\ \rangle,\ m > 0. \tag{A.3.11}$$

Spurious states with in the physical subspace of Hilbert space are called null states. This implies two physical states which differ by a null state are equivalent. All states which are neither belongs to the equivalence class of a physical state nor a null state are called unphysical states. Let us end our discussion on Hilbert space of string by writing down the first excited physical states of closed string.

- States $\frac{1}{2}\Big(\alpha^i_{-1}\bar{\alpha}^j_{-1} + \alpha^j_{-1}\bar{\alpha}^i_{-1}\Big)|p\rangle,\ i,j = 1,...,d-2$ forms graviton in d-dimensional spacetime, quanta of spacetime metric $G_{\mu\nu},\ \mu,\nu = 0,...,d.$
- States $\frac{1}{2}\Big(\alpha^i_{-1}\bar{\alpha}^j_{-1} - \alpha^j_{-1}\bar{\alpha}^i_{-1} - \frac{2}{d-2}\delta^{ij}\sum_{k=1}^{d-2}\alpha^k_{-1}\bar{\alpha}^k_{-1}\Big)|p\rangle,\ i,j,k = 1,...,d-2$ forms antisymmetric tensor, quanta of spacetime anti-symmetric field $B_{\mu\nu},\ \mu,\nu = 0,...,d.$

- $\frac{1}{d-2}\Big(\sum_{k=1}^{d-2}\alpha_{-1}^{k}\bar{\alpha}_{-1}^{k}\Big)|p\rangle$ forms dilaton a scalar, quanta of spacetime scalar field Φ.

Theses states form irreducible representations of SO(d-2) which is the little group of Lorentz group for the d-dimensional spacetime. But for a theory with Lorentz invariance only massless states can form the irreducible representation of little group of Lorentz group. So requiring Lorentz invariance for quantised free string demands that the first excited states (level number N=1) should be massless. But mass of string states depends on the dimension of the spacetime through which they propagate as shown in equation A.3.10. So consistent quantum mechanical propagation of free relativistic string in flat Minkowskian spacetime is possible only if the dimension of spacetime is 26. This also tells that ground state of string is tachyonic, i.e. mass^2 is negative.

A.3.2 Identifying the Coupling Constant

So far we considered only the free propagation of relativistic string through flat spacetime. Let us consider the propagation of relativistic string through a spacetime having non-zero vacuum expectation value (VEV) for the massless fields $G_{\mu\nu}, B_{\mu\nu}$ and Φ. Just like for an electrically charged particle propagating through a spacetime in the presence of electromagnetic field, Polyakov action gets modified and becomes,

$$S_p = \frac{1}{4\pi}\int d^2\xi\Big[\sqrt{-g}\ g^{ab}G_{\mu\nu}(X) + \epsilon^{ab}B_{\mu\nu}(X)\Big]\partial_a X^\mu \partial_b X^\nu + \frac{1}{8\pi}\int d^2\xi\sqrt{-g}\ R^{(2)}\Phi(x) \tag{A.3.12}$$

where $R^{(2)}$ is the scalar curvature of world-sheet metric g_{ab}. Here consistent propagation of string demands that the VEV of $G_{\mu\nu}, B_{\mu\nu}$ and Φ should satisfy Einstein equations. We are interested in a specific back ground with VEV $B_{\mu\nu} = 0,\ G_{\mu\nu} = \eta_{\mu\nu}$ and $\Phi(X) = \Phi_0$ where Φ_0 is a constant. Corresponding Polyakov action is given by

$$S_p = \frac{1}{4\pi}\int d^2\xi\sqrt{-g}\ g^{ab}\eta_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + \frac{\Phi_0}{2}\chi$$

(A.3.13)

where $\chi = \frac{1}{4\pi}\int d^2\xi\sqrt{-g}\ R^{(2)}$ is the Euler characteristic of world-sheet. For a world-sheet with genus h, n-boundaries and n_c corners, the Euler characteristic is

$$\chi = 2 - 2h - n - \frac{n_c}{4}. \tag{A.3.14}$$

Increasing the genus of world-sheet will change the action by $-\Phi_0$ so we can think of e^{Φ_0} as the expansion parameter of Polyakov prescription for computing string amplitude. Similarly increasing the number of boundaries will change the action by $-\frac{\Phi_0}{2}$ so we can identify $e^{\frac{\Phi_0}{2}}$ as closed string coupling since adding a closed string increase the number of boundaries by one. Finally adding an open string increases the corners of world-sheet by 2 so we identify $e^{\frac{\Phi_0}{4}}$ with open string coupling.

A.3.3 Polyakov Path Integral for String Amplitude

Polyakov prescription defines h-loop closed string amplitude as path integral over all metrics on Riemann surfaces having genus h and n- boundaries with boundary condition corresponds to asymptotic state of n-closed strings and the embedding $X^\mu(\xi^1, \xi^2)$ of this surfaces in spacetime. For example consider the h-loop contribution to the scattering of n-closed strings in their ground state. String at its ground state is specified just by the centre of mass position. Then the h-loop contribution to the n-point amplitude is defined by imposing the boundary condition on the world-sheet that the n-boundaries of the world-sheets are at specific points in the spacetime say $x_I,\ I = 1, ..., n$. Corresponding path integral is given below

$$\int \Big[dg_{ab}\ dX^\mu\Big]_h e^{-S_p[g_{ab}, X^\mu]} \prod_{I=1}^{n} \int d^2\xi_I \sqrt{-g(\xi_I)}\delta\Big(x^\mu(\xi_I) - x_I^\mu\Big). \tag{A.3.15}$$

Then in momentum space the h-loop contribution to n-tachyon string amplitude is given by

$$A_h^{(n)}(k_1,...,k_n) = \int \Big[dg_{ab}\ dX^\mu\Big]_h e^{-S_p[g_{ab},X^\mu]} \prod_{I=1}^{n} \int d^2\xi_I \sqrt{-g(\xi_I)}\ e^{ik_I.x_I(\xi_I)}. \qquad \text{(A.3.16)}$$

We can identify $\int d^2\xi_I \sqrt{-g(\xi_I)}\ e^{ik_I.x_I(\xi_I)}$ as the operator corresponds to tachyonic state of I^{th} closed string with momentum k_I^μ. This operator is known as the integrated form of vertex operator for tachyon. For generic states of the string also it is possible to find the appropriate vertex operators. This one to one correspondence between operators and states is called the state-operator correspondence. We will denote the vertex operator corresponds to a state with quantum numbers represented by index a_I carrying momentum k_I^μ as $\int d^2\xi_I \sqrt{-g(\xi_I)}\ \mathcal{V}_{a_I,k_I}(\xi_I)$. Then the momentum space expression for h-loop string amplitude involving n-string states having quantum numbers a_I and momentum k_I with I running from 1 to n is

$$A_h^{(n)}(a_1,k_1;...;a_n,k_n) = \int \Big[dg_{ab}\ dX^\mu\Big]_h e^{-S_p[g_{ab},X^\mu]} \prod_{I=1}^{n} \int d^2\xi_I \sqrt{-g(\xi_I)}\ \mathcal{V}_{a_I,k_I}(\xi_I). \qquad \text{(A.3.17)}$$

Just like in any gauge theory it is important to carry out gauge fixing to make the path integral well defined. Gauge transformations that we need to fix for making the above path integral well defined are reparametrisations and Weyl transformations. This can be achieved by restricting the path integral over metrics to a class of metrics which are not related to each other by above mentioned gauge transformations. This class of metrics is parametrised a set of parameters which forms a non-compact space known as moduli space.

For $h \geq 2$ story ends here. But on genus 0 and 1 Riemann surfaces there exists a class of reparametrisations which can not be fixed by restricting to a class of metrics because under those reparametrisations metric remains the same. This unfixed reparametrisations can be gauge fixed by fixing the positions of m of the vertex operators where m is the dimension of unfixed space of reparametrisation. Vectors generating these left over transformations are called conformal killing vectors (CKV). There exists a simple relation between the number of

CKV's on genus h Riemann surface with n marked points (where we insert vertex operators) and the dimension of it's moduli space given by,

$$Number\ of\ real\ CKV - Real\ dimension\ of\ moduli\ space = 6 - 6h + 2n. \quad \text{(A.3.18)}$$

Number of real CKV's and real dimension of moduli space for Riemann surfaces without any puncture for different genera are as listed below.

- For genus 0 surface : $number\ of\ real\ CKV = 6,\ dimension\ of\ moduli\ space = 0.$
- For genus 1 surface : $number\ of\ real\ CKV = 2,\ dimension\ of\ moduli\ space = 2.$
- For genus ≥ 2 surface : $number\ of\ real\ CKV = 0,\ dimension\ of\ moduli\ space = 6h-6.$

CKV's generate möbius group for sphere and group of translation for torus.

Gauge fixing can be carried out using Faddeev-Popov procedure by introducing ghost fields to get the gauge fixed Polyakov path integral computing the n-point closed string amplitude,

$$A^{(n)}(a_1, k_1; ...; a_n, k_n) = \sum_h e^{(h-1+\frac{n}{2})\Phi_0} \int_{\mathcal{M}_h} d^s t \prod_{I=1}^{n-m} \int_{\Sigma_h} d^2 z_I$$
$$\Big\{ \int \Big[db\ d\bar{b}\ dc\ d\bar{c}\ dX^\mu \Big]_h e^{-S_p - S_g} \prod_{k=1}^{s} \frac{1}{4\pi} \Big(b, \frac{\partial \hat{g}}{\partial t^k} \Big) \Big(\bar{b}, \frac{\partial \hat{g}}{\partial t^k} \Big) \prod_{i=1}^{m} c(z_i) \bar{c}(z_i) \prod_{I=1}^{n} \sqrt{-\hat{g}(z_I)}\ \mathcal{V}_{a_I, k_I}(z_I) \Big\}. \quad \text{(A.3.19)}$$

Here s is the complex dimension of moduli space of genus h Riemann surfaces $\mathcal{M}_h$ and m is the number of complex CKV's on genus h Riemann surface Σ_h. $(z, \bar{z})$ is the complex coordinates on Σ_h and $\hat{g}$ belongs to the class of metrics on genus h Riemann surface parametrised by the coordinates of moduli space which can not be related each other by gauge transformations. $\Big(b, \frac{\partial \hat{g}}{\partial t^k} \Big)$ represents $\int_\Sigma d^2 z\ \sqrt{-\hat{g}(z)}\ b_{zz} \frac{\partial \hat{g}^{zz}}{\partial t^k}$ and S_g denotes the action for the anti-commuting

bosonic ghost fields b and c given by

$$S_g = \frac{1}{2\pi}\int_{\Sigma_h} d^2z\ \sqrt{-\hat{g}(z)}\ b\bar{\partial}c + \frac{1}{2\pi}\int_{\Sigma_h} d^2z\ \sqrt{-\hat{g}(z)}\ \bar{b}\partial\bar{c}\ . \tag{A.3.20}$$

Interestingly $X^\mu,\ \mu = 0, ..., 25$ fields and $b,\ c$ fields together form a 2 dimensional conformal field theory defined on genus h Riemann surface with vanishing central charge. This means action $S_p + S_g$ and path integral measure are invariant under all analytic coordinate transformations. (So the world-sheet interpretation of spacetime Lorentz invariance is the absence of conformal anomaly in the (X, b, c) conformal field theory.) Conformal dimension of $X^\mu,\ b,\ \bar{b},\ c,\ \bar{c}$ are respectively $(0,0),\ (2,0),\ (0,2),\ (-1,0),\ (0,-1)$. Then for computing terms inside the curly brackets in the right hand side of equation A.3.19, which are nothing but correlators in the above mentioned 2-d CFT, we can use the powerful techniques available in 2-d conformal field theories. In this sense perturbative string theory reduces to 2-d exactly solvable conformal field theory on Riemann surfaces.

A.3.4 BRST Symmetry

Gauge fixed Polyakov action in conformal gauge takes the following form

$$S = S_p + S_g = \frac{1}{4\pi}\int_{\Sigma_h} d^2z\ \partial X^\mu \bar{\partial} X_\mu + \frac{1}{2\pi}\int_{\Sigma_h} d^2z\ b\bar{\partial}c + \frac{1}{2\pi}\int_{\Sigma_h} d^2z\ \bar{b}\partial\bar{c}\ . \tag{A.3.21}$$

Anti-commutation relation of b-c system is as follows

$$\begin{aligned} \{b_m, c_n\} &= \delta_{m,-n} \\ b_m &= \frac{1}{2i\pi}\oint_C z^{m+1} b(z),\ c_m = \frac{1}{2i\pi}\oint_C z^{m-2} c(z). \end{aligned} \tag{A.3.22}$$

Zero modes of b-c system generates two ground states which are annihilated by all of the $n \geq 1$ modes. We call them $|\uparrow\rangle$ and $|\downarrow\rangle$ such that $|\uparrow\rangle$ is annihilated by c_0 and $|\downarrow\rangle$ is annihilated by b_0 and $|\uparrow\rangle = c_0|\downarrow\rangle$.

Though we obtained this action after gauge fixing it still has some left over gauge symmetry. These left over gauge transformations known as BRST transformations are generated by the BRST operator

$$Q_B = \frac{1}{2\pi i}\oint_C (dz\ j_B - d\bar{z}\ \bar{j}_B) \tag{A.3.23}$$

here C denotes a closed counter around the operator on which Q_B acts and j_B is the BRTS current

$$j_B = cT^X + \frac{1}{2} : cT^g : + \frac{3}{2}\partial^2 c \tag{A.3.24}$$

where T^X and T^g are the stress tensor for X^μ system and ghost system respectively. Mode expansion of Q_B is given by

$$Q_B = \sum_m \left(c_n L^X_{-n} + \bar{c}_n \bar{L}^X_{-n}\right) + \sum_{m,n} \frac{m-n}{2} : \left(c_m c_n b_{-m-n} + \bar{c}_m \bar{c}_n \bar{b}_{-m-n}\right) : -c_0 - \bar{c}_0. \tag{A.3.25}$$

Also we have the useful commutation relation

$$\{Q_B, b_m\} = L^X_m + L^g_m. \tag{A.3.26}$$

It is important to note that $Q_B^2 = 0$. Beauty of this left over symmetry is that instead of treating this as an end result we can rewrite the story by treating the gauge fixed action as our starting point and quantize the theory using this symmetry. This approach is known as the BRST formalism. In this formalism physical states satisfy conditions $b_0|\psi\rangle = 0,\ \bar{b}_0|\psi\rangle = 0$ and $Q_B|\psi\rangle = 0$. States which can be written as $Q_B|\tilde{\psi}\rangle$ with $b_0|\tilde{\psi}\rangle = 0$ are the null states. $b_0|\psi\rangle = 0,\ \bar{b}_0|\psi\rangle = 0$ condition is the end result of imposing Siegel gauge condition $(b_0 + \bar{b}_0)|\psi\rangle = 0$ and consistency condition $(b_0 - \bar{b}_0)|\psi\rangle = 0$.

Imposing Siegel gauge condition is important to make sure that string amplitude has only singularities which has concrete physical interpretations. Let us explain this point. As we observed zero modes of ghost system leads to four copies of states. Now we can ask out of these four copies which one can provide states that are annihilated by BRST charge and at the same time sensible string amplitudes. For simplicity we will consider only the ground states of right moving sector. $Q_B|k,\downarrow\rangle = (k^2+m^2)|k,\uparrow\rangle$ and $Q_B|k,\uparrow\rangle = 0$. This means that $|k,\uparrow\rangle$ are physical if $k^2+m^2=0$ and null when $k^2+m^2 \neq 0$. This means that if we include states coming from sector with ground states $|k,\uparrow\rangle$ string amplitude will be proportional to $\delta(k^2+m^2)$ since null states decouples from the amplitudes involving only physical states. But such dependencies are not allowed in S-matrix. By imposing Siegel gauge we can project out states build up of $|k,\uparrow\rangle$ since $b_0|k,\uparrow\rangle \neq 0$. $(b_0 - \bar{b}_0)|\psi\rangle = 0$ condition is to make sure that scattering amplitudes well defined even with the inability to define local coordinates on Riemann surfaces globally in moduli space without any phase ambiguity.

A.4 RNS Formulation of Superstring Theory

Superstring theory is the natural generalisation of supersymmetric quantum mechanics of relativistic point particle to relativistic strings. World-sheet theory of superstring can be obtained by supersymmeterising the bosonic string world-sheet theory. Heterotic string theory can be obtained by combining the right moving part of superstring with the left moving part of bosonic string theory.

A natural way to study the supersymmetric version of bosonic string world-sheet theory is to follow the superfield formalism. Following superfield formalism we will convert the space-time embedding of bosonic string $X^\mu(z,\bar{z}),\ \mu = 0,...,d-1$ to superfield $X^\mu(z,\tilde{z},\theta,\tilde{\theta})$ where θ and $\tilde{\theta}$ are complex grassmannian variables. Interestingly we can think of the world-sheet of superstring as super-Riemann surface which is the super generalisation of ordinary Riemann surfaces. In the following section we will describe the geometry of both super and heterotic

string world sheet more precisely.

A.4.1 Super-Riemann Surface

Super-Riemann surface Σ_R can be locally parameterised by bosonic and fermionic complex coordinates z and θ. Also it is possible to define every where on Σ_R a derivative operator $\mathcal{D}$ such that $\mathcal{D}^2$ is nowhere proportional to $\mathcal{D}$. An example for such a derivative operator is $\mathcal{D}_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$ since $\mathcal{D}_\theta^2 = \frac{\partial}{\partial z}$.

Once we have the derivative operator $\mathcal{D}_\theta$ we can introduce the the notion of super-conformal vector fields. Vector field $\mathcal{W}$ is a super-conformal vector field if $[\mathcal{W}, \mathcal{D}_\theta\} \propto \mathcal{D}_\theta$. Say we have two overlapping patches $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ on Σ with local coordinates $(z_\alpha, \theta_\alpha)$ and (z_β, θ_β) respectively. Then $D_{\theta_\alpha} = (D_{\theta_\alpha}\theta_\beta)D_{\theta_\beta}$. Field A is called a super-conformal primary of dimension $-\frac{n}{2}$ if under a super-conformal transformation $(z, \theta) \to (\hat{z}, \hat{\theta})$ it transform to $(D_\theta\hat{\theta})^{-n}\hat{A}$.

World-sheet of superstring can be thought of as a smooth sub manifold of a complex manifold $\Sigma_L \times \Sigma_R$ where both Σ_L and Σ_R are super-Riemann surfaces and are parametrised by coordinates (z, θ) and $(\tilde{z}, \tilde{\theta})$. Why should we think in this way? For the case of bosonic string it has left moving and right moving oscillations on it and each sector is parametrised by local coordinates z and its complex conjugate $\bar{z}$. Naive generalisation will say that the super analog of left and right moving modes on superstring should be parametrised by (z, θ) and $(\bar{z}, \bar{\theta})$. But this will lead to an inconsistency due to the fact that constraint like $z = \tilde{z}$ are not invariant under super-conformal transformations. Due to this reason we should write $X^\mu(z, \tilde{z}, \theta, \tilde{\theta})$ instead of $X^\mu(z, \bar{z}, \theta, \bar{\theta})$. In the same spirit we should think of Heterotic string world-sheet as a smooth submanifold of a complex manifold $\Sigma_L \times \Sigma_R$ where Σ_L is a super-Riemann surface and Σ_R is a Riemann surface parametrised by coordinates (z, θ) and $\tilde{z}$.

Most general super-conformal transformation takes the following form,

$$\begin{aligned} \hat{z} &= f(z) + \theta \partial f(z) \epsilon(z) \\ \hat{\theta} &= (\partial f)^{\frac{1}{2}} (\theta + \epsilon(z) + \frac{1}{2} \theta \epsilon \partial \epsilon(z)) \end{aligned} \tag{A.4.1}$$

where $f(z)$ is an arbitrary commuting analytic function and $\epsilon(z)$ is an anti-commuting analytic function. So only for special classes of super-Riemann surfaces with transition functions having vanishing $\epsilon(z)$ we can think of the super-Riemann surface as a spinor bundle over ordinary Riemann surface with odd coordinate as section of the spinor bundle. And such super-Riemann surfaces are called split super-Riemann surface otherwise non split.

Since there is no notion of distance along the odd direction of super-Riemann surface, they share the topological characters of the underlying Riemann surface. For example the genus of Super-Riemann surface is exactly same as that of the underlying Riemann surface which can be obtained by setting all odd coordinates to zero. Like ordinary Riemann surfaces inequivalent genus h super-Riemann surfaces also form super analog of moduli space known as super moduli space, $s\mathcal{M}_h$. For $h \geq 2$ super moduli space $s\mathcal{M}_h$ is characterised by $2h-2$ odd and $3h-3$ even complex variables. $3h-3$ even complex variable takes care of the inequivalent underlying ordinary surfaces and $2h-2$ odd variable represents the inequivalent super-Riemann surfaces which can not be thought of as spinor bundle over an ordinary Riemann surface. By this we end our crash course on super-Riemann surface. Detailed beautiful discussion on super manifolds and super-Riemann surfaces and its moduli space can be found in [114].

A.4.2 World-sheet Action

Using this superfield we can write the supersymmetric version of Polyakov action for bosonic string theory as

$$S[X^\mu] = \frac{1}{2}\int DX_\mu \tilde{D}X^\mu. \tag{A.4.2}$$

Most general solution for the equation of motion can be locally written as,

$$\begin{aligned} X^\mu(z,\tilde{z},\theta,\tilde{\theta}) &= X^\mu(z,\theta) + X^\mu(\tilde{z},\tilde{\theta}) \\ X^\mu(z,\theta) &= X^\mu(z) + \theta\psi^\mu(z) \end{aligned} \tag{A.4.3}$$

where $X^\mu(z)$ are same that appeared in the case bosonic string and $\psi^\mu(z)$ are world-sheet spinors. Since $\psi^\mu(z)$ are world-sheet spinors they can have two possible boundary conditions along every non-contractible cycles on the underlying Riemann surface of world-sheet. They are periodic and anti-periodic boundary conditions. Say we look only into right moving modes then creation operators coming from ψ^μ with periodic condition will produce states which are world-sheet bosons and creation operators coming from ψ^μ with periodic condition will produce states which are world-sheet spinors. Thus the spectrum splits into two sectors which we call Neveu-Schwarz (NS) sector where we have spacetime bosons and Ramond (R) sector where we have spacetime spinors. We need to introduce superfield for ghost system also. They are super-conformal primaries $B(z,\theta), \tilde{B}(\tilde{z},\tilde{\theta})$ with dimensions $(\frac{3}{2},0),(0,\frac{3}{2})$ and $C(z,\theta), \tilde{C}(\tilde{z},\tilde{\theta})$ with dimensions $(-1,0),(0,-1)$. Action for this Fadeev-Popov ghost system is

$$S[B,C] = \frac{1}{2}\int B\bar{D}C. \tag{A.4.4}$$

Locally we can split these superfields into $B(z,\theta) = \beta(z) + \theta b(z)$ and $C(z,\theta) = c(z) + \theta\gamma(z)$. Ghost b, c corresponds to reparametrisation and β, γ ghosts are commuting fermions corresponds to local susy transformations. Requiring the absence of super-conformal anomaly in

the combined system of X, B, C forces us to work with 10 dimensional spacetime.

A.4.3 Superstring Amplitudes

Superstring amplitudes are super analog of string amplitudes in bosonic string theory. Instead of all possible Riemann surfaces with vertex operators for string states at marked points or punctures we need to sum over all possible super-Riemann surfaces with vertex operators at punctures. After gauge fixing sum over all surfaces with specific genus h will reduce to an integration over super moduli space s$\mathcal{M}_{h,n}$.

Vertex operators corresponds to states in the NS-sector are inserted on marked points on super-Riemann surface. Since R-sector states creates branch cuts in the θ coordinates, Ramond punctures where vertex operators corresponding Ramond states are inserted are part of super-conformal structure. More precisely at Ramond puncture the special derivative $\mathcal{D}$ which defines the super-conformal structure does not exist because at Ramond punctures $\mathcal{D}^2$ vanishes. Consider a genus h super-Riemann surface with n_R Ramond punctures and n_{NS} NS-punctures. Derivative operator $\mathcal{D}$ defined on such a surface takes the following form,

$$D^*_\theta = \frac{\partial}{\partial\theta} + \Big(\prod_{i=1}^{n_R}(z - z_i)\Big)\theta\frac{\partial}{\partial z}. \tag{A.4.5}$$

In this sense super-Rieman surfaces with Ramond punctures are super-Riemann surfaces with singularity in the super-conformal structure. Complex dimension of the super moduli space of such surfaces is $(3h-3+n_{NS}+n_R|2h-2+n_{NS}+\frac{1}{2}n_R)$. Dimensionality of super moduli space will never be fractional because number of Ramond punctures n_R on a super-Riemann surface is always even. Physical meaning of this fact is that a spacetime fermion can never decay into even number of fermions.

In subsection A.3.4 we argued that only a subspace of the full Hilbert space of bosonic string

theory can give sensible string amplitude. Similarly we should check in superstring theory whether we need any projection other than imposing that states should be annihilated by b_0 and $\tilde{b}_0$. We can find the needed further projection by studying the propagation of on-shell intermediate states in string amplitudes. This corresponds to analysing the contribution of amplitude coming from the integration over part of the moduli space which is very near to it's boundary. As we will see later (when we discuss factorisation of superstring amplitudes) near the boundary of moduli space we can perform a part of super moduli integration (integration of few coordinates of super moduli space is not a well defined operation in general due to the non-split nature of super moduli) which corresponds to summing over spin structure. Summing over spin structure will attach a projection operation to the propagator which measures the fermion parity. This projection, known as GSO projection, assures that all world-sheet OPE's are local. So we should implement GSO projection in the Hilbert space and restrict to the reduced Hilbert space. One should note that this GSO operation can be implemented separately in left moving and right moving sector. Interestingly appropriate combination of GSO-projection in left and right moving sector can give rise to spacetime supersymmetric spectrum. Even more interesting fact is that such theories do not have tachyons in its spectrum. Also supersymmetry guarantees the vanishing of massless tadpoles. All these issues are thoroughly discussed in [5].

A.4.4 Picture Changing Formalism

Superstring amplitudes after gauge-fixing reduces to an integral over super moduli space. Super moduli space of genus h super-Riemann surfaces are parametrised by $3h-3$ even and $2h-2$ odd complex variables. Picture changing formalism gives a prescription for writing down superstring amplitude as an integral over moduli space of ordinary Riemann surface. It tells us to integrate out odd super moduli at the cost of introducing the so called picture changing operator (PCO) on ordinary Riemann surface. Before trying to understand PCO we should discuss the concept of ' picture ' because PCO is defined as the operator which changes

the picture of vertex operator.

Concept of picture stems from the curious properties of β-γ system. So let us analyse the β-γ system and its representations. This is a commuting fermionic system. Start with the mode expansion of these fields.

$$\beta(z) = \sum_n \beta_n z^{-n-\frac{3}{2}} \quad \gamma(z) = \sum_n \gamma_n z^{-n+\frac{1}{2}} \tag{A.4.6}$$

where n is a half integer for NS sector and integer for Ramond sector. they satisfy the commutation relations

$$[\gamma_m, \beta_n] = \delta_{n,-m}. \tag{A.4.7}$$

Move on to its Hilbert space. One of the main source of complications in superstring theory arises from the fact that β-γ system has infinite number of inequivalent representation of β-γ algebra which can be constructed using the raising operators by acting on infinite number of vacuum states $|q\rangle$ having ghost charge ($Q_{gh} = \sum_n : \beta_n \gamma_{-n} :$) q, where q is integer or half integer. q is called the 'Bose-sea level' of the representation. These set of vacuum sets are inequivalent because unlike the degenerate ground states of b-c system here we can not start from vacuum with one value of q to another by acting with finite number of oscillators. Say there exists operators which can increase or decrease q and call them $\delta(\gamma_m)$ and $\delta(\beta_n)$. Their action on the q-vacua defines them,

$$\begin{aligned} \delta(\beta_{-q-\frac{3}{2}})|q\rangle &= |q+1\rangle \\ \delta(\gamma_{q+\frac{1}{2}})|q\rangle &= |q-1\rangle. \end{aligned} \tag{A.4.8}$$

Similarly we can define operators, spin fields, Σ_+ and Σ_- mapping Ramond states to NS states or vice versa. $q \in Z + \frac{1}{2}$ for Ramond sector. They are also defined by their action on q-vacuua,

$$\Sigma_+(0)|0\rangle = |\frac{1}{2}\rangle, \quad \Sigma_-(0)|0\rangle = |-\frac{1}{2}\rangle. \tag{A.4.9}$$

So each state in superstring theory has infinite number of inequivalent representation based on the q-vacua that we use for building the tower of states. This leads to infinite number of inequivalent vertex operators for any specific state and we discriminate each of them by associating a picture number which indicates the q-vacua used for constructing the states.

Things are more transparent if we represent β-γ system using a free scalar ϕ and a pair of free chiral fermionic fields ξ and η of conformal weight (0,0) and (1,0). This changeover of representation is known as bosonization. Free scalar ϕ is compactified on the circle $R/2\pi Z$ and is coupled to a background charge $Q = 2$. Action for the combined system is

$$S[\phi, \xi, \eta] = \frac{1}{2\pi} \int \left(\partial\phi\bar{\partial}\phi - \frac{1}{2} R\phi \right) + \frac{1}{\pi} \int \eta\bar{\partial}\xi. \tag{A.4.10}$$

Precise mapping between two systems is as follows,

$$\beta = e^{\phi}\partial_z\xi, \ \gamma = e^{-\phi}\eta, \ \delta(\beta) = e^{-\phi}, \ \delta(\gamma) = e^{\phi}, \ \xi = \Theta(\beta), \ \eta = \partial_z\gamma\delta(\gamma). \tag{A.4.11}$$

q-vacua in this representation is defined as

$$\begin{aligned} &\phi_n|q\rangle = \eta_n|q\rangle = \xi_m|q\rangle = 0 \quad for \ n \geq -1, \ m \geq 0 \\ &\phi_0|q\rangle = q|q\rangle. \end{aligned} \tag{A.4.12}$$

Given these it is not difficult to see that

$$\delta(\beta(z)) = e^{\phi(z)}, \quad \delta(\gamma(z)) = e^{-\phi(z)}, \quad \Sigma_+(z) = e^{\frac{1}{2}\phi(z)}, \quad \Sigma_-(z) = e^{-\frac{1}{2}\phi(z)}. \tag{A.4.13}$$

It is important note that only the derivatives of ξ-field is present in the identification A.4.11 between two systems. So the constant zero mode ξ_0 can not be produced from any of the operators acting within β-γ system. This means that (ξ, η, ϕ) - Hilbert space $\mathcal{H}_{\xi\eta\phi}$ is twice as large as the (β, γ)-Hilbert space $\mathcal{H}_{\beta\gamma}$. Precise equivalence is the following

$$\mathcal{H}_{\beta\gamma} = \{|\psi\rangle \in \mathcal{H}_{\xi\eta\phi} \mid \eta_0|\psi\rangle = 0\}. \tag{A.4.14}$$

We saw that it is possible to construct operators which can take us from one q-vacua to another. This means it should be possible to change the Bose-sea charge of a vertex operator and this procedure is called picture changing. A general state in the Hilbert space of Bose-sea level q can be converted to a state in the Hilbert space of Bose-sea level $q+1$ by acting with $G_{-q-\frac{3}{2}}\delta(\beta_{-q-\frac{3}{2}})$ with G_n defined by $T_F(z) = \sum_n z^{-n-\frac{3}{2}}\frac{G_n}{2}$. On physical vertex operators (satisfy $\delta_{BRST}V^{(q)} = \partial(cV^{(q)})$) picture changing operation is represented as :

$$V^{(q)}(w) \to V^{(q+1)}(w) = \oint \frac{dz}{2i\pi} j_{BRST}\Big(\xi V^{(q)}\Big)(w) - \partial\Big(c\xi V^{(q)}\Big) \tag{A.4.15}$$

where $V^{(q)}$ is the vertex operator with picture number q and j_{BRST} is the superstring BRST current. Though the second term in the right hand side looks like a BRST variation actually it is not a BRST exact deformation due to the presence of zero mode of ξ in it (remember ξ_0 is not in $\mathcal{H}_{\beta\gamma}$). We end our discussion on Picture changing formalism by mentioning that for obtaining sensible string amplitude we should introduce enough number of PCO's on the world-sheet to make sure that total picture number is zero. More on picture changing formalism can be found in [40].

A.4.5 Non-Split Super Moduli Space and Picture Changing Formalism

In section A.4.1 we saw that when a super-Riemann surface is non-split the distinction between even and odd coordinates is not well defined because super-conformal transformations can mix

them. In-fact as shown in [9] super modui spaces relevant in superstring perturbation theory are indeed non-split. But Picture changing formalism advises us to integrate out odd coordinates of super moduli at the cost of introducing picture changing operators. Then the natural question would be that how the ambiguity in distinguishing odd and even coordinate is reflected in the amplitude defined using Picture changing formalism. This can be understood by considering general correlation function of (ξ, η, ϕ) system which appears in superstring amplitudes

$$\begin{aligned}\langle\prod_{i}^{n+1}\xi(x_i)\prod_{j}^{n}\eta(y_j)\prod_{k}e^{q_k\phi(z_k)}\rangle_\nu &= \frac{\prod_j^n\vartheta_\nu(-y_j+\sum x-\sum y+\sum q_kz_k-2\Delta)}{\prod_i^{n+1}\vartheta_\nu(-x_i+\sum x-\sum y+\sum q_kz_k-2\Delta)} \\ &\times\frac{\prod_{i<i'}E(x_i,x_{i'})\prod_{j<j'}E(y_j,y_{j'})}{\prod_{i,j}E(x_i,y_j)\prod_{k<l}E(z_k,z_l)^{q_kq_l}\prod_k\sigma(z_k)^{2q_k}}\end{aligned} \tag{A.4.16}$$

with $\sum_k q_k = 2g-2$ and

$$\frac{\sigma(z)}{\sigma(w)} = \frac{\vartheta(z-\sum_i p_i+\Delta)}{\vartheta(w-\sum_i p_i+\Delta)}\prod_i\frac{E(w,p_i)}{E(z,p_i)} \tag{A.4.17}$$

where $p_i, i = 1, ..., g$, are arbitrary points on genus-g Riemann surface and left hand side is independent of p_i. Δ is the Riemann class, naively it is the submanifold (points) of Riemann surface on which Riemann theta function vanishes. $E(z, w)$ is the prime form which behaves as $z - w$ in the $z \to w$ limit and ϑ_ν is the Riemann theta function which depends on the spin structure ν. Whenever we allow operators with in this correlator with singular OPE to collide they produce singularities and that happen precisely when prime forms in the denominator of right hand side vanishes. It is well known that theta function defined any genus Riemann surface always has zeroes. This implies that theta functions in the denominator also produces poles but this time we can not associate sensible physical interpretation with this so they are spurious singularities.

Interestingly these spurious singularities can be interpreted in terms of the non-splitness of

the super moduli space. Let us take a close look into the process of integrating out of odd super moduli. To integrate out odd coordinates, $\hat{m}_a$ first we need to isolate the $\hat{m}_a$ dependence of the string amplitude. As we discussed earlier in section A.4.1 odd super moduli characterises the difference between a split super-Riemann surface, $\Sigma_{\hat{m}}$, and a non-split one, Σ_0 obtained by setting $\hat{m}^a = 0$. So in order to isolate the $\hat{m}_a$ dependence of the string amplitude we need to identify the difference in the integrated on $\Sigma_{\hat{m}}$, and Σ_0. This can be done in the following way. Start with fields on split super-Riemann surface and then do a field redefinition such that the new fields satisfy the conditions of non-split super-Riemann surface. To implement this philosophy we can use the fact that by doing a quasi-conformal coordinate transformation $(z, \theta) \to (\tilde{z}, \tilde{\theta})$ we can convert a split super-Riemann surface to a no-split super-Riemann surface. A general quasi super-conformal transformation is given by

$$\begin{aligned} \tilde{z} &= z + \theta\hat{v}(z, \bar{z}) \\ \tilde{\theta} &= \theta + \hat{v}(z, \bar{z}) + \frac{1}{2}\theta\hat{v}\partial\hat{v}(z, \bar{z}) \end{aligned} \tag{A.4.18}$$

where $\hat{v}(z, \bar{z})$ is a multi-valued, anti-commuting $(-\frac{1}{2}, 0)$ differential on Σ_0. Use this relation between coordinates on split and non-split super-Riemann surface we can fields on non-split world-sheet in terms of fields on split world-sheet. Interestingly world-sheet action of non-spit world-sheet in terms of fields on split world-sheet becomes action for a 2d massless scalar coupled to $\mathcal{N} = 1$ SUGRA in Wess-Zumino gauge if identify $2\bar{\partial}v(z, \bar{z})$ which is a $(-\frac{1}{2}, 1)$ odd differential on Σ_0, as the gravitino field $\hat{\chi}(z, \bar{z})$ on Σ_0. Precise relation between the world-sheet action on $\Sigma_{\hat{m}}$ and Σ_0 is the following

$$S_{\Sigma_{\hat{m}}} = S_{\Sigma_0} + \int_{\Sigma_0} \hat{\chi}\, T_F \tag{A.4.19}$$

where T_F is the super current given by

$$T_F = \frac{1}{2}\psi\partial X + \frac{1}{2}b\gamma + \frac{3}{2}\beta\partial c + \frac{1}{2}\partial\beta c. \tag{A.4.20}$$

This means odd super moduli is contained within the 2-d gravitino field. Choose a basis (gauge) for graviton field on genus h world-sheet, $\chi_a(z,\bar{z}),\ a = 1, 2, ..., 2h-2$. Then we can write

$$\hat{\chi}(z,\bar{z}) = \sum_{a=1}^{2h-2} \hat{m}^a \chi_a(z,\bar{z}). \tag{A.4.21}$$

Now as proved in [10] if the super moduli space is non-split then it is impossible to choose a global choice of gauge (basis) for gravitino. i.e. at some region in the moduli space combination of graviton basis may become pure gauge. At those regions the β-ghost insertions with in the PCO's will not be able to absorb all zero modes of commuting β-ghost fields. Presence of these left over zero modes will make string amplitude divergent and we can identify theses singularities as the spurious poles in the string amplitudes.

A.5 Factorisation of String Amplitudes

As we mentioned earlier unitarity of S-matrix is necessary for a sensible probabilistic interpretation of the theory. Factorisation is one of the most important property of a unitary S-matrix. So it is important to make sure that string amplitudes defined through Polyakov's prescription has factorisation property. Demonstrating the factorisation property of string amplitudes involves several steps. They are

- Factorisation of moduli/super moduli space.
- Factorisation of CFT on Riemann/super-Riemann surface.
- Factorisation of measure.
- Demonstration of propagation of only physical states in the intermediate channels.

We are interested only in the factorisation of moduli space of bosonic string theory and factorisation of super moduli space in superstring theory. For complete discussion refer [5, 35].

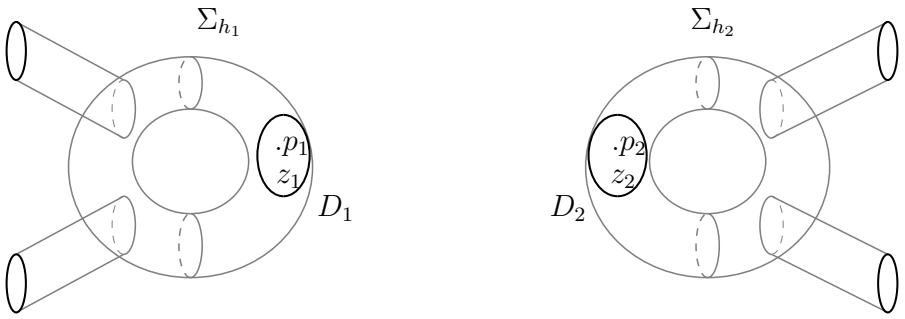

Figure A.4: Riemann surfaces Σ_{h_1} and Σ_{h_2} with unit discs around points p_1 on Σ_{h_1} and p_2 on Σ_{h_2} with respect to the local coordinates z_1 around p_1 and z_2 around p_2.

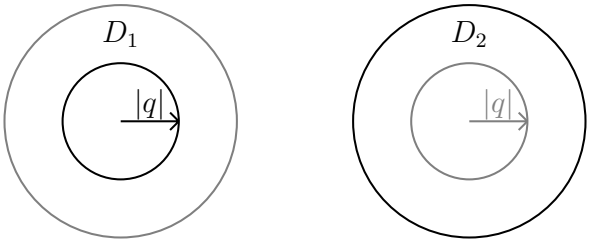

Figure A.5: Two annuli having inner radius $|q|$ and outer radius 1 obtained by removing a disc of radius $|q|$ from D_1 and D_2 where q is a complex parameter.

A.5.1 Plumbing fixture

Since factorisation involves splitting of higher point amplitude into lower point amplitudes it is important to identify a region in the moduli space of genus h Riemann surface with n punctures where it splits into genus h_1 and genus h_2 Riemann surfaces with n_1 and n_2 punctures respectively with $h_1 + h_2 = h$ and $n_1 + n_2 = n + 2$. Such special region in the moduli space of Riemann surfaces can be obtained by gluing a genus h_1 and genus h_2 Riemann surfaces with n_1 and n_2 punctures respectively. Plumbing procedure is an example for a mathematical construction which implements this gluing.

Consider two Riemann surfaces Σ_{h_1} having genus h_1 and Σ_{h_2} having genus h_2. Pick a point p_1 on Σ_{h_1} and p_2 on Σ_{h_2}. Define local coordinates z_1 around p_1 and z_2 around p_2. Then cut out a disc D_1 around p_1 with unit radius in z_1 coordinate and a disc D_2 around p_2 with unit

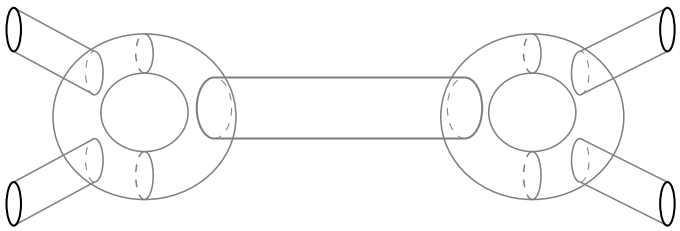

Figure A.6: Plumbing fixture produces Riemann surface Σ_h with genus $h = h_1 + h_2$ with 2 less number of punctures than that total number of punctures which were there on Σ_{h_1} and Σ_{h_2}.

radius in z_2 coordinate as in figure A.4. Remove a disc of radius $|q|$ from D_1 and D_2 where q is a complex parameter. Then we are left with two annuli having inner radius $|q|$ and outer radius 1 as shown in figure A.5. Identify these two annuli as follows,

$$z_1 z_2 = q, \quad q = e^{-s+i\theta} \tag{A.5.1}$$

where s and θ are real variables. This identify the grey circle in D_1 (see figure A.5) with black circle in D_2 and the black circle in D_1 (see figure A.5) with gray circle in D_2 with a phase shift θ. Resulting Riemann surface is a Riemann surface Σ_h with genus $h = h_1 + h_2$ with two less number of punctures than that total number of punctures which were there on Σ_{h_1} and Σ_{h_2}. In this special region of moduli space of genus h Riemann surface with n punctures convenient set of moduli parameters is (m_1^*, m_2^*, q), where m_i^* is the moduli for Σ_{h_i} having $n_i + 1$ marked points. Riemann surfaces with $|q| = 0$ forms the boundary of moduli space and those surfaces are known as degenerate Riemann surface because these surface are not smooth Riemann surfaces. At $|q| = 0$ surface splits into two branches that meet at a point. We call the limit $|q| \to 0$ the degeneration limit. Since $|q| = 0$ points are not nice Riemann surfaces modulus space of Riemann surfaces is not compact. One can do Deligne-Mumford compactification of moduli space by adding these points into the moduli space.

Plumbing fixture procedure is more intuitive if we use local coordinates s_i and ϕ_i related

to z_i as follows

$$z_i = e^{-s_i + i\phi_i},\ i = 1, 2. \tag{A.5.2}$$

This coordinate transformation converts disc D_i around the point p_i on Σ_i into a infinite tube with p_i sits at the ∞ of coordinate s_i. Removal of disc of radius $|q|$ from the interior of disc D_i is equal to cutting the disc into a tube with finite length $s = ln|q|$. In this coordinate plumbing fixture disc identification becomes

$$s_1 + s_2 = s, \quad \phi_1 + \phi_2 = \phi. \tag{A.5.3}$$

which is nothing but the process of inserting one tube into the other after twisting one tube by angle ϕ. Parameter s can be thought of as the proper time elapsed by the intermediate closed string.

Let us take out this plumbing fixture tube. This is well defined if we allow all possible boundary conditions for the intermediate closed string at the ends of the tube. Consider specific state of closed string which is annihilated by both b_0 and $\bar{b}_0$ with specific L_0 and $\bar{L}_0$ at one end of the tube. Evolution of state along the tube is implemented by the operator $e^{-s(L_0+\bar{L}_0)+i\phi(L_0-\bar{L}_0)}$. Since near the boundary of moduli space we can split the Riemann surface into three pieces, two punctured Riemann surfaces and plumbing fixture tube, with only the long plumbing fixture tube depends on the moduli parameter s and ϕ and also states in the restricted Hilbert space which are annihilated by both b_0 and $\bar{b}_0$ can not detect the ϕ we can integrate over s, ϕ with integrand $e^{-s(L_0+\bar{L}_0)+i\phi(L_0-\bar{L}_0)}$. This integration produces the propagator for closed bosonic string. So boundary of the moduli space provides the needed singularity corresponding to the propagation of on-shell intermediate particles.

A.5.2 Super Analog of Plumbing Fixture

Now we will discuss the super analog of plumbing fixture procedure. Again plumbing fixture is implemented by picking two points on two super-Riemann surfaces and identifying the local coordinates around the punctures. As we already discussed there are two kinds of punctures, NS and Ramond punctures. So we can identify local coordinates around either two NS-punctures or two Ramond punctures which leads two possible degenerations in the case of super moduli space and they are

- NS-degeneration: On-shell NS states propagate in the intermediate channel.
- Ramond degeneration: On-shell Ramond states propagates in the intermediate channel.

Consider two super-Riemann surfaces $s\Sigma_{h_i},\ i = 1, 2$ with NS punctures at p_i. Say the local coordinate around puncture p_i is (z_i, θ_i) defined within the disk sD_i. We can map disk sD_i to NS-super tube using following coordinate change,

$$z_i = e^{-s_i + i\phi_i}, \quad \theta_i = e^{-\frac{s}{2} + i\frac{\theta}{2}}\zeta_i. \tag{A.5.4}$$

Like for ordinary Riemann surface insert one NS super tube into other using the following identifications

$$\zeta_2 = \zeta_1 = \zeta,\ s_1 + s_2 = s,\ \phi_1 + \phi_2 = \phi + \pi. \tag{A.5.5}$$

This gives the super analog of gluing formula for NS punctures is

$$\begin{aligned} z_1 z_2 &= q,\ q = e^{-s+i\phi} \\ z_2\theta_1 &= -q^{\frac{1}{2}}\theta_2,\ z_1\theta_2 = \pm q^{\frac{1}{2}}\theta_1 \\ \theta_1\theta_2 &= 0. \end{aligned} \tag{A.5.6}$$

It is not difficult to verify that this gluing can be achieved via following super-conformal coordinate transformation

$$z_2 = \frac{q}{z_1},\ \theta_2 = q^{\frac{1}{2}}\frac{\theta_1}{z_1}. \tag{A.5.7}$$

We can separate out the NS-super tube and perform integration over q to get the string propagator in the NS-sector.

Now assume that we are gluing regions around two Ramond punctures. Salient feature of Ramond puncture is that around it super derivative takes the form $D^*_\theta = \frac{\partial}{\partial\theta} + \theta z\frac{\partial}{\partial z}$. With respect to D^*_θ coordinate transformation A.5.7 is not super-conformal. But the following coordinate transformation is super-conformal with respect to D^*_θ

$$z_2 = \frac{q}{z_1},\ \theta_2 = \alpha \pm i\theta_1 \tag{A.5.8}$$

where α is an odd variable. So gluing formula for Ramond punctures is given by

$$\begin{aligned} z_1 z_2 &= q,\ q = e^{-s+i\phi} \\ \theta_1 &= -\alpha \pm i\theta_2. \end{aligned} \tag{A.5.9}$$

Again we can separate out the Ramond-super tube and perform integration over q and α to get the string propagator in the Ramond sector. Now we need to understand the role of $\pm$ present in the gluing relations for both NS and Ramond punctures. Remember world-sheet parity operator $(-)^F$ has the same effect of changing the sign of odd coordinate. Since averaging over these $\pm$ signs is equivalent to the action of GSO-projection operator $\frac{1+(-)^F}{2}$ integrating over NS or Ramond gluing parameters and averaging over possible $\pm$ signs gives the NS or Ramond propagator with GSO-projection operator. So only states that are invariant under GSO-projection contributes to singularities in superstring amplitude.

APPENDIX B

Brief Review of String Field Theory

String theory based on Polyakov prescription directly defines the perturbative expansion of S-matrix of string theory without referring to any underlying Lagrangian or Hamiltonian. With this formulation we are allowed to ask only question which have a perturbative answer. But there are several interesting non-perturbative questions which requires a complete non-perturbative formulation of string theory. String field theory is an attempt to give a complete non-perturbative definition of string theory. We will restrict ourselves to bosonic string field theory in 26 dimensional space-time since a complete formulation of superstring field theory has not been developed yet.

String field theory hopes to provide an underlying Lagrangian for string theory with a perturbative limit which reproduces Polyakov prescription. String field theory follows the general principles of gauge theories. Transition from string theory to string field theory is analogous to the transition from point particle quantum mechanics to quantum field theory. We go from quantum mechanics to quantum field theory by promoting particle states specified by certain quantum numbers to fields whose quantum fluctuations produces particles with same quantum numbers from vacuum defined all over the space-time. In gauge theories having a Lagrangian description dynamics or time evolution of fields is dictated by the underlying Lagrangian, its symmetries and its local redundancies. Then natural first step in the transition from string

theory to string field theory would be promoting the states in string theory to fields, the string fields. Next step is to look for redundant transformations of these fields which can be identified as gauge transformations. Interestingly covariant description of string theory already carries such redundant transformations. These are the transformations induced on states in string theory by the world-sheet parametrisation. If we restrict to string states which are annihilated by $b_n,\ n \geq 0$ these transformations can be treated as transformations generated by BRST charge Q_{BRST} defined in A.3.25. Unlike in Polyakov prescription we will not demand that string fields satisfy on mass-shell condition. Remaining task is to identify and quantise the Lagrangian with gauge transformation whose linear approximation is generated by Q_{BRST}. We also need to ensure that this reproduces Polyakov prescription in the perturbative limit.

As we will discuss later string field theory has all features of most general gauge theory having a Lagrangian description can have. Quantisation of such a gauge theory requires the sophisticated tools of Batalian-Vilkovsky formalism (BV-formalism) [115–121]. So we will first familiarise ourselves with this powerful formalism before jumping into string field theory.

B.1 Batalian-Vilkovsky Formalism

Let us start by explaining what do we mean by most general gauge theory. Most studied and successful examples of gauge theories are non-abelian Yang-Mills theories which is based on gauge transformation which forms a nice Lie group with the commutators of this Lie-algebra generators produces another Lie-algebra generator with field independent structure constants. This Lie algebra is an associative algebra satisfying Jacobi Identity. Also these are irrespective of whether field configuration satisfy classical equations of motion or not. By most general gauge theory we mean a gauge theory with more flexible gauge group structure. We can consider following generalisations.

- Allow the structure constants to depend on fields involved in the theory with appropri-

ately modified Jacobi Identity.

- Allow gauge invariance for the gauge transformations to make it a reducible system.
- Allow two gauge transformations to produce another gauge transformation plus a term that vanishes only on-shell.

B.1.1 Fields and Antifields

Consider a gauge theory with m_0 gauge invariances whose gauge transformations are not invariant under any other gauge transformation. Then at classical level for each such m_0 gauge invariance we need to introduce ghost fields. Now suppose that gauge theory has m_1 gauge transformations which are invariant under other set of gauge transformations which are not invariant under any further transformations. Then we call it a first-stage reducible. In such theories we need to add m_1 ghost for ghost fields. For a general L^{th} stage reducible theory, the set of fields $\Phi^i,\ i = 1, ..., N$ is

$$\Phi^i = \{\phi^i, C_s^{\alpha_s}\ \alpha_s = 1, ..., m_s;\ s = 0, ..., L\}. \tag{B.1.1}$$

To each of these fields assign a conserved charge known as ghost number. Assignment is as follows. Classical field ϕ^i has ghost number zero and C^{α_s} has ghost number $gh\ [C_s^{\alpha_s}] = s{+}1$. Similarly assign statistics of ghost fields. Statistics (ϵ) of $C_s^{\alpha_s}$ is given by $\epsilon(C_s^{\alpha_s}) = \epsilon_{\alpha_s}{+}s{+}1(mod\ 2)$.

Now for each field Φ^i introduce an antifield Φ_i^*. Assign ghost numbers and statistics as follows,

$$gh\ [\Phi_i^*] = -gh\ [\Phi^i] - 1,\ \epsilon(\Phi_i^*) = \epsilon(\Phi^i) + 1\ (mod\ 2). \tag{B.1.2}$$

Notice that field and antifield have opposite statistics.

B.1.2 Antibracket

Say $X,\ Y$ be two functions of fields Φ^i and Φ^*_i with statistics ϵ_X and ϵ_Y. Then define antibracket as

$$(X,Y) \equiv \frac{\partial_r X}{\partial \Phi^i}\frac{\partial_l Y}{\partial \Phi^*_i} - \frac{\partial_r X}{\partial \Phi^*_i}\frac{\partial_l Y}{\partial \Phi^i} \tag{B.1.3}$$

where subscript r denotes the right derivative and l denotes the left derivative which are defined and related each other as

$$\frac{\partial_l X}{\partial \Phi^i} \equiv \frac{\overrightarrow{\partial} X}{\partial \Phi^i}, \quad \frac{\partial_r X}{\partial \Phi^i} \equiv X\frac{\overleftarrow{\partial}}{\partial \Phi^i}, \quad \frac{\partial_l X}{\partial \Phi^i} = (-)^{\epsilon(\Phi^i)(\epsilon_X+1)}\frac{\partial_r X}{\partial \Phi^i}. \tag{B.1.4}$$

B.1.3 Classical BV-Master Equation

Construct an action fields and antifields $S[\Phi,\Phi^*]$ as a ghost number zero even arbitrary functional of fields and antifields. Demand that $S[\Phi,\Phi^*]$ need to satisfy the classical master equation

$$(S,S) = 2\frac{\partial_r S}{\partial \Phi^i}\frac{\partial_l S}{\partial \Phi^*_i} = 0. \tag{B.1.5}$$

Also impose a regularity condition on $S[\Phi,\Phi^*]$ that upon setting all anti-fields to zero we should get back the classical action for the gauge theory. This is to ensure that we will get back the correct classical limit. We are only interested in solution to the classical master equation which allows the consistent elimination of all antifields Φ^* because antifields are unphysical. Such solutions are called proper solutions. It is guaranteed that classical BV-master equation of a general reducible gauge theory has unique proper solutions satisfying reasonable regularity conditions [122].

To appreciate the meaning of master equation we should expand the proper solution in terms of antifields. Master equation in the zeroth order in antifields is the statement of invariance of

original action under gauge transformation. First order master equation is the algebra satisfied by gauge transformation and second order is the generalised Jacobi identity and so on. In this sense BV-formalism has the nice feature of incorporating the complete structure of gauge symmetry with in the simple looking master equation.

B.1.4 Generalised BRST Symmetry

In the usual BRST-formalism gauge fixed action has residual gauge symmetry (BRST symmetry) whose action is a graded derivation and nilpotent. Interestingly proper solution of classical BV-master equation has a generalised BRST symmetry even before gauge-fixing. Generalised BRST transformation of a functional X of fields and antifields generated by the proper solution S is

$$\delta_B X \equiv (X, S). \tag{B.1.6}$$

S is invariant under this transformation due to the classical BV-master equation. It is straight forward to check that $\delta_B^2 = 0$. A classical observable belongs to the cohomology of δ_B.

B.1.5 Gauge-Fixing

Consider an S that satisfies classical BV-master equation. Then for any function of F of fields and antifields it is true to first order in infinitesimal fermionic constant ϵ that $S' = S + \epsilon((F, S), S)$ also satisfies classical BV-master equation. Assume that $\epsilon(F, S)$ is a fermionic functional $\epsilon\Psi$ which is a function only of fields. Then one can see that

$$S'[\Phi, \Phi^*] = S[\Phi, \Phi^* + \epsilon \frac{\partial \Psi[\Phi]}{\partial \Phi}]. \tag{B.1.7}$$

This is a very useful observation because to calculate S-matrix elements we must first assign a definite value to the antifields. Now we can use the above discussed freedom and gauge fix

the antifields to $\Phi^* = \frac{\partial \Psi[\Phi]}{\partial \Phi}$.

B.1.6 Quantum BV-Master Equation

Consider the partition function of the gauge theory

$$Z_\Psi = \int [\mathcal{D}\Phi] e^{-\frac{1}{\hbar} S[\Phi, \frac{\partial \Psi[\Phi]}{\partial \Phi}]}. \tag{B.1.8}$$

It is important to make sure that physical quantities of the theory do not depend on the choice of gauge fixing function Ψ. Ψ independence of partition function demands that S, quantum master action, should satisfy the quantum BV-master equation

$$\begin{aligned} (S,S) &= -2\hbar \Delta S, \ at \ \Phi^* = \frac{\partial \Psi[\Phi]}{\partial \Phi} \\ \Delta &= \frac{\partial_r}{\partial \Phi^*} \frac{\partial_l}{\partial \Phi}. \end{aligned} \tag{B.1.9}$$

B.2 Closed Bosonic String Field Theory

Closed string field theory is the first example of a gauge theory which require BV-formalism in it's full form. Here we will give a brief summary of [32].

B.2.1 String Fields

In the previous section we discussed how bosonic string theory reduces to conformal field theory of (X^μ, b, c)-fields on Riemann surfaces. Consider the Hilbert space of this conformal field theory. A string field is a vector in this Hilbert space. Choose a set of basis states $\{|\Phi_s\rangle\}$, then an arbitrary string field can be expressed as

$$|\Psi\rangle = \sum_s |\Phi_s\rangle \psi_s. \tag{B.2.10}$$

ψ_s are called the target space fields. Closed string fields need to satisfy the following conditions

$$(b_0 - \bar{b}_0)|\Psi\rangle = (L_0 - \bar{L}_0)|\Psi\rangle = 0. \tag{B.2.11}$$

Restricting to this subspace of string fields is important for the consistent formulation of closed string field theory. While we discussed string perturbation theory in A.3.4 we explained the importance of restring to a subspace of Hilbert satisfying theses conditions in formulating perturbative string theory.

Assign ghost number G for each component of string field which is defined as the ghost number of the first quantised state,

$$G(|\Phi_s\rangle\psi_s) = (G|\Phi_s\rangle)\psi_s = G_s(|\Phi_s\rangle\psi_s). \tag{B.2.12}$$

Similarly assign ghost number for target space fields also. Target space ghost number g^t for a target field associated with state $|\Phi_s\rangle$ is defined as

$$g^t(\psi_s) = 2 - G_s. \tag{B.2.13}$$

Both string field and vacuum $|1, p\rangle$ have even grassmannality. String field satisfy following reality condition

$$(|\Psi\rangle)^\dagger \equiv \langle\Psi_{hc}| = -\langle\Psi| \tag{B.2.14}$$

where $\langle\Psi_{hc}|$ is the hermitian conjugate defined using following rules. Say $|1, p\rangle$ is SL(2,C) vacuum in the asymptotic past and $\langle 1, p|$ is the SL(2,C) vacuum in the asymptotic future. Assume $|\Psi\rangle$ is constructed from the vacuum $|1, p\rangle$ by acting with a set of oscillators $A_i,\ i = 1, ..., n$, i.e. $|\Psi\rangle = A_1...A_n|1, p\rangle$. Then hermitian conjugate $\langle\Psi_{hc}|$ is given by $\langle\Psi_{hc}| = \langle 1, p|A_n^\dagger...A_1^\dagger$. $\langle\Psi|$ is the BPZ conjugate of$|\Psi\rangle$. Say $|\Psi\rangle = \Psi(0)|1\rangle$ with $\Psi(0)$ a normal ordered operator, then BPZ

conjugate is defined as

$$\langle\Psi| \equiv \langle 1|I \circ \Psi(0) \tag{B.2.15}$$

where I denotes the conformal mapping $I(z) = \frac{1}{z}$.

Let us look into a specific string field, the tachyon field

$$|T\rangle = \int \frac{dp}{(2\pi)^d}\phi(p)c_1\bar{c}_1|1,p\rangle. \tag{B.2.16}$$

It's hermitian and BPZ conjugates are

$$\begin{aligned} \langle T_{hc}| &= \int \frac{dp}{(2\pi)^d}\phi^*(p)\langle 1,p|\bar{c}_{-1}c_{-1}, \\ \langle T| &= \int \frac{dp}{(2\pi)^d}\phi(p)\langle 1,-p|c_{-1}\bar{c}_{-1} \end{aligned} \tag{B.2.17}$$

with target space tachyon field satisfying reality condition $\phi^*(p) = \phi(-p)$.

B.2.2 Kinetic Term

Closed string field kinetic term is given by

$$S_{0,2} = \frac{1}{2}\langle\Psi|c_0^- Q|\Psi\rangle. \tag{B.2.18}$$

Since this inner product with normalisation

$$\langle 1,p|c_{-1}\bar{c}_{-1}c_0^+c_0^-c_1\bar{c}_1|1,p'\rangle = (2\pi)^d\delta^d(p-p') \tag{B.2.19}$$

vanishes if total ghost number is not 6, string field should have ghost number 2. Due to the presence of c_0 in the kinetic term gauge invariance of kinetic term requires the dynamical

string field must satisfy conditions $b_0^-|\Psi\rangle = L_0^-|\Psi\rangle = 0$. Gauge transformation of string field is is $\delta|\Psi\rangle = Q|\Lambda\rangle$ where $|\Lambda\rangle$ is a ghost number 1 string field satisfying reality condition and $b_0^-|\Lambda\rangle = L_0^-|\Lambda\rangle = 0$.

For example consider the kinetic term for the tachyon field is

$$\begin{aligned} S_{kin}^{tach} &= \frac{1}{2}\int \frac{dp}{(2\pi)^d}\int \frac{dp'}{(2\pi)^d}\phi(p')\langle -p', 1|c_{-1}\bar{c}_{-1}c_0^- c_0^+(L_0+\bar{L}_0)c_1\bar{c}_1|p,1\rangle\phi(p) \\ &= -\frac{1}{2}\int \frac{dp}{(2\pi)^d}\phi(-p)(p^2-2)\phi(p). \end{aligned} \tag{B.2.20}$$

B.2.3 Physical Spectrum

kinetic term leads to the following equation of motion

$$Q|\Psi\rangle = 0. \tag{B.2.21}$$

Field equation and gauge invariance suggests that physical states are those which have ghost number 2 and annihilated by both Q and b_0^- at the same time not expressible as Q-exact state.

B.2.4 Kinetic Term and Master Equation

Decompose the string field $|\Psi\rangle$ into two pieces $|\Psi_-\rangle, |\Psi_+\rangle$ with $|\Psi_-\rangle$ containing target space fields and $|\Psi_+\rangle$ containing target space antifields,

$$|\Psi\rangle = |\Psi_-\rangle + |\Psi_+\rangle \tag{B.2.22}$$

here both $|\Psi_-\rangle$ and $|\Psi_+\rangle$ are annihilated by b_0^- and L_0^-. Unlike the string field entered in the kinetic term $|\Psi_-\rangle$ and $|\Psi_+\rangle$ contains all possible positive ghost number states annihilated by

b_0^- and L_0^-.

$$|\Psi_-\rangle = \sum_{G(\Phi_s)\leq 2} |\Phi_s\rangle\psi^s,\ |\Psi_+\rangle = \sum_{G(\Phi_s)\leq 2} |\tilde{\Phi}_s\rangle\psi^*_s \tag{B.2.23}$$

here ψ^s denotes target space fields and ψ^*_s denotes target space antifields and

$$|\tilde{\Phi}_s\rangle \equiv b_0^-|\Phi_s^c\rangle. \tag{B.2.24}$$

$|\Phi_s^c\rangle$ is the conjugate state satisfying

$$\langle\Phi_r|\Phi_s^c\rangle = (-)^{\Phi_r}\delta_{rs} \tag{B.2.25}$$

where $(-)^{\Phi_r}$ is for the statistics of the state $|\Phi_r\rangle$. Ghost number for $|\tilde{\Phi}_s\rangle$ is $G(\tilde{\Phi}_s) = 5 - G(\Phi_s)$. From this we can see that sum of target space ghost numbers of field and antifield is -1 and their statistics are also opposite as required in BV-formalism.

Closed string kinetic term which satisfy the classcal BV-master equation is given by the same expression B.2.18 provided we don't impose any restriction on the ghost number of the closed string field. For proof see [32].

B.2.5 String Multilinear Functions

Let us start by defining the notion of string vertices. A string vertex $\mathcal{V}_{g,n}$ is a set of Riemann surfaces of genus g and n punctures which does not include surfaces arbitrarily close to degeneration. Also these set of surfaces are equipped with a specific choice of analytic local coordinates (defined only unto a constant phase) around each of it's punctures. Upto constant phase ambiguity coordinates around each puncture are defined continuously over the set $\mathcal{V}_{g,n}$.

Collection of Riemann surfaces in $\mathcal{V}_{g,n}$ also need to satisfy an important condition. As we

discussed in section A.5.1 a subset of surfaces within $\mathcal{V}_{g,n}$ can be constructed by gluing lower genus surfaces or surfaces with less number of punctures or surfaces with both lower genus and carrying less number punctures. Plumbing fixture also can be implemented on a single surface to obtain higher genius surface. So near the boundary of moduli space we can fill the moduli space either by using surfaces belongs to $\mathcal{V}_{g,n}$ or that can be obtained by plumbing surfaces belongs to $\mathcal{V}_{g_1,n_1}$ and $\mathcal{V}_{g_2,n_2}$ where $g_1 + g_2 = g,\ n_1 + n_2 = n + 2$ or just gluing regions around two punctures of surfaces blonds to $\mathcal{V}_{g-1,n+2}$. *The condition that we wish to impose on $\mathcal{V}_{g,n}$ is that at the boundary of $\mathcal{V}_{g,n}$ both moduli parameters and local coordinates defined around each punctures on the surfaces up-to phase should agree with the surfaces that are obtained by plumbing fixture procedure.* This put a restriction on the way we should construct surfaces to cover $\mathcal{V}_{g,n}$. It is an extremely important condition which with other conditions on string fields help us to construct a quantum action for closed string field theory which satisfy quantum BV-master equation. Zweibach in [32] showed that genus g, n punctured Riemann surfaces with minimal area metric defined on them form such $\mathcal{V}_{g,n}$.

Now we will introduce the notion of surface states. A surface state $\langle\Sigma|$ associated with a Riemann surface Σ with n punctures is a state in the dual space of the n-fold tensor product of the Hilbert space $\mathcal{H}$ of the underlying conformal field theory. It describes the state that is created on the boundaries of $D_i,\ i = 1, .., n$, small discs around punctures, by performing the functional integral over the fields of the conformal field theory on $\Sigma - \bigcup_a D_a$. Assume that $\langle\Sigma|$ is the conjugate state of $\langle\Sigma|$. They have ghost number $6g - 6 + 6n$ and is annihilated by BRST operator belongd to each copy of Hilbert space in $\mathcal{H}^{\otimes}$.

Consider arbitrary string fields $|B_1\rangle, ..., |B_n\rangle$ which are annihilated by both b_0^- and L_0^-. They can have arbitrary ghost number and statistics. Also they don't satisfy reality condition. We will insert $|B_k\rangle$ on the k^{th} puncture of elements belongs to $\mathcal{V}_{g,n}$. Using theses string fields we can make the tensor product state $|\vec{B}\rangle = |B_1\rangle \otimes |B_2\rangle ... \otimes |B_n\rangle \in \mathcal{H}^{\otimes n}$.

Denote the compactified moduli space of genus g Riemann surfaces with n punctures as $\bar{\mathcal{M}}_{g,n}$. Assign local coordinates around each punctures on theses set of surfaces belongs to $\bar{\mathcal{M}}_{g,n}$ up to a constant phase and call this enlarged space $\hat{\mathcal{P}}_{g,n}$. Tangent vectors in $\hat{\mathcal{P}}_{g,n}$ can be described using the concept of Schiffer variations. Say z_j is the local coordinate around j^{th} puncture on an arbitrary Riemann surface $\Sigma \in \hat{\mathcal{P}}_{g,n}$. Assume that the disc around i^{th} puncture D_i within which we define local coordinates z_i and disc around j^{th} puncture D_j within which we define local coordinates z_j overlap in region D_{ij}. Within the region D_{ij} we can relate local coordinates z_i and z_j using transition function f_{ij}, i.e. $z_i = f_{ij}(z_j)$. Now infinitesimally change the local coordinate in the disc D_j from z_j to z_j^ϵ

$$z_i = f_{ij}^\epsilon(z_j^\epsilon),\ (z_{ij}^\epsilon)^{-1}(f_{ij}(z_j)) \equiv z_j + \epsilon v(z_j) \tag{B.2.26}$$

where $v(z_j)$ is a vector field on the Riemann surface that is analytic in D_{ij}. It is not difficult to guess that variation of local coordinates $v(z_j),\ j = 1, ..., n$ can help us to define tangent vector of $\hat{\mathcal{P}}_{g,n}$ along directions corresponding to change in local coordinates around each punctures on Σ. The non-trivial fact is that special choices of $v(z_j),\ j = 1, ..., n$ will even help as to define tangent vectors of $\hat{\mathcal{P}}_{g,n}$ along directions corresponding to changing the moduli parameters of Σ. Then on the tangent space $T(\hat{\mathcal{P}}_{g,n})$ we can define differential form of real degree $6g-6+2n$

$$\Omega^{g,n}_{B_1,...,B_n} = \langle \Sigma | b(v_1)...b(v_{6g-6+2n}) | \vec{B} \rangle. \tag{B.2.27}$$

Here $v_i,\ i = 1, ..., 6g - 6 + 2n$ are the vector fields generating Schiffer variation (for a clear picture on Schiffer variation read [32]) corresponding to $6g-6+2n$ tangent vectors of $T_\Sigma(\hat{\mathcal{P}}_{g,n})$ and

$$b(v) = \oint dz_j v(z_j) b(z_j) + \oint d\bar{z}_j \bar{v}(\bar{z}_j) \bar{b}(\bar{z}) \tag{B.2.28}$$

with the integration contour over $z(\bar{z})$ running along the circle forming the common boundary of D_i and D_j keeping the D_j component to its left (right).

Degree $6g - g + 2n$ form $\Omega^{g,n}_{B_1,...,B_n}$ can be integrated on the section of $\hat{\mathcal{P}}_{g,n}$. Then we can define string field vertex or multilinear function as integral of $\Omega^{g,n}_{B_1,...,B_n}$ over the subset of surfaces defining the string vertex, using the minimal area section,

$$\{B_1, B_2, ..., B_n\}_g = \int_{\mathcal{V}_{g,n}} \Omega^{g,n}_{B_1,...,B_n}. \tag{B.2.29}$$

Also we set

$$\{\cdot\}_g = \int_{\mathcal{V}_{g,0}} \Omega^{g,0}. \tag{B.2.30}$$

For genus zero set $\{\cdot\}_0 = 0$, $\{B\}_0 \equiv 0$ and $\{B_1, B_2\}_0 \equiv \langle B_1|c_0^- Q|B_2\rangle$. String multilinear function has the following symmetry property

$$\{B_1, ..., B_i, B_{i+1}, ..., B_n\}_g = (-)^{B_i B_{i+1}} \{B_1, ..., B_{i+1}, B_i, ..., B_n\}_g. \tag{B.2.31}$$

B.2.6 String Field Products

Let us introduce bra $\langle \Omega^{g,n}|$ via following relation

$$\Omega^{g,n}_{B1,...,B_n} = \langle \Omega^{g,n}|B_1\rangle ...|B_n\rangle. \tag{B.2.32}$$

It is $6g - g + 2n$ form on $T(\hat{\mathcal{P}}_{g,n})$ which is valued in $\mathcal{H}^{\otimes n}$. Using this we can define string field products which take a set of string fields, and give another string field. String field product is defined as follows

$$[B_1, ..., B_{n-1}]_g = \sum_s{}' (-)^{\Phi_s} \int_{\mathcal{V}_{g,n}} \langle \Omega^{g,n}|\Phi_s\rangle|\tilde{\Phi}_s\rangle|B_1\rangle ...|B_n\rangle \tag{B.2.33}$$

where prime over sum indicates that sum extends over basis of states $|\Phi_s\rangle$ complete in the subspace of states annihilated by $(L_0 - \bar{L}_0)$. String multilinear functions can be obtained from string field products by taking the following linear inner product

$$\{B_n, B_2, ..., B_{n-1}\}_g = \langle B_n | c_0^- | [B_1, ..., B_{n-1}]_g \rangle. \tag{B.2.34}$$

String field product satisfy the following crucial identity due to the important condition satisfied by string vertices $\mathcal{V}_{g,n}$ that we discussed in the previous section

$$\begin{aligned} 0 = \quad & Q[B_1, ..., B_n]_g + \sum_{i=1}^{n} (-)^{(B_1 + ... + B_{i-1})} [B_1, ..., QB_i, ..., B_n]_g \\ & + \sum_{g_1, g_1, \{i_l, j_k\}, l, k} \sigma(i_l, j_k) [B_{i_1}, ..., B_{i_l}, [B_{j_1}, ..., B_{j_k}]_{g_2}]_{g_1} \\ & + \frac{1}{2} {\sum_s}' (-)^{\Phi_s} [\Phi_s, \tilde{\Phi}_s, B_1, ..., B_n]_{g-1}. \end{aligned} \tag{B.2.35}$$

It holds for all $n \geq 0$. The second sum in the right hand side runs over all different splittings of the set $\{1, ..., n\}$ into two groups $\{i_1, ..., i_l\}$ and $\{j_1, ..., j_k\}$. This splitting is insensitive to the ordering of elements within each group. g_1, g_2, l and k satisfy the following conditions:

$$\begin{aligned} g1 \quad &\geq \quad 0,\ g_2 \geq 0,\ l \geq 0,\ k \geq 0,\ with\ g_1 + g_2 = g,\ l + k = n \geq 0 \\ l \quad &\geq \quad 1\ when\ g_1 = 0,\ k \geq 2\ when\ g_2 = 0. \end{aligned} \tag{B.2.36}$$

The factor $\sigma(i_l, j_k)$ appearing on the right-hand side of eq. producttofunction is defined to be the sign picked up when one rearranges the sequence $\{Q, B_1, ..., B_n\}$ into the order $\{B_{i_1}, ..., B_{i_l}, Q, B_{j_1}, ..., B_{j_k}\}$. The sum in the last term is over states in a complete basis of the Hilbert space $\mathcal{H}$. This sum is restricted to states satisfying the $L_0 - \bar{L}_0$ constraint. Moreover, since $|\tilde{\Phi}_s\rangle = b_0^- |\Phi_s^c\rangle$, the sum can be restricted to states $|\Phi_s\rangle$ annihilated by b_0^-. This happens because the conjugate of a state $|\Phi_s\rangle$ that is not annihilated by b_0^- must necessarily be annihilated by b_0^-. Then the extra b_0^- in the definition of $|\tilde{\Phi}_s\rangle$ kills the state and the corresponding

term disappears from the sum.

B.2.7 Classical String Field Action

Classical string field theory action is given by

$$\begin{aligned} S(\Psi) &= \frac{1}{\kappa^2}\sum_{n=0}^{\infty}\frac{\kappa^n}{n!}\{\Psi^n\}_0 \\ &= \frac{1}{2}\langle\Psi|c_0^- Q|\Psi\rangle + \sum_{n=3}^{\infty}\frac{\kappa^{n-2}}{n!}\{\Psi^n\}_0. \end{aligned} \tag{B.2.37}$$

Extremise this action to obtain the following field equations

$$0 = \mathcal{F}(\Psi) \equiv Q|\Psi\rangle + \sum_{n=2}^{\infty}\frac{\kappa^{n-1}}{(n-1)!}[\Psi^n]_0. \tag{B.2.38}$$

B.2.8 Gauge Structure

It can be easily verified using the identity B.2.35 that action is invariant user the following gauge transformations

$$\delta_\Lambda|\Psi\rangle = Q|\Lambda + \sum_{n=1}^{\infty}\frac{\kappa^n}{n!}[\Psi^n,\Lambda]_0. \tag{B.2.39}$$

Commutator of two gauge transformation is given by

$$[\delta_{\Lambda_2},\delta_{\Lambda_1}]|\Psi\rangle = \delta_{\Lambda(\Psi)}|\Psi\rangle + \sum_{l=0}^{\infty}\frac{\kappa^{l+2}}{l!}[\Psi^l,\Lambda_2,\Lambda_1,\mathcal{F}(\Psi)]_0 \tag{B.2.40}$$

where $\Lambda(\Psi) = \sum_{n=0}^{\infty}\frac{\kappa^{n+1}}{n!}[\Lambda_1,\Lambda_2,\Psi^n]_0$. This suggest that closed string field theory has the most general gauge structure with field dependent structure constants and on-shell closure. This most general gauge structure is a consequence of the theory not being cubic unlike Witten's open string field theory [123, 124].

B.2.9 Quantum String Field Action

Full quantum action for closed string field theory is given by

$$S(\Psi) = \frac{1}{\kappa^2}\sum_{g\geq 0}(\hbar\kappa^2)^g \sum_{n\geq 0}\frac{\kappa^n}{n!})\{\Psi^n\}_g. \tag{B.2.41}$$

Identity B.2.35 plays crucial role in making sure that this quantum action satisfies quantum BV-master equation [32].

B.2.10 Off-Shell String Amplitudes in Siegel Gauge

Like in any gauge theory we need to choose impose a gauge condition before doing any computation. Standard gauge condition is the Siegel gauge in which we restrict to a subspace Hilbert space which is annihilated by $(b_0 + \bar{b}_0)$. Within this subspace of Hilbert space kinetic term becomes

$$S_{kin} = \frac{1}{2}\langle\Psi|c_0^- c_0^+(L_0 + \bar{L}_0)|\Psi\rangle. \tag{B.2.42}$$

From this we can read out the propagator as

$$\frac{b_0^- b_0^+}{L_0 + \bar{L}_0}\mathcal{P}|R_{12}\rangle \tag{B.2.43}$$

where $\mathcal{P} = \int \frac{d\theta}{2\pi} e^{i\theta(L_0 - \bar{L}_0)}$, $|R_{12}\rangle \equiv \sum_s |\Phi_s\rangle_1 |\Phi_s^c\rangle_2$ and the subscripts on the kets are the labels for the Hilbert spaces. Form of propagator if we identify it with the sum over conformal field theories on plumbing fixture tubes parametrised using $s \in [0, \infty]$ and twist angle $\phi \in [0, 2\pi]$ is given as

$$\frac{1}{2i\pi} b_\phi b_s \int_0^\infty ds \int_0^{2\pi} d\phi \; e^{(-s+i\phi)L_0} e^{(-s-i\phi)\bar{L}_0}|R_{12}\rangle. \tag{B.2.44}$$

We end our review on string field theory by mentioning some of the nice properties of off-sell amplitudes constructed from string field theory in Siegel gauge [39].

- The amplitudes are symmetric under permutations among scattering states.
- The amplitudes are integrals over sections of fiber bundles with base the moduli spaces of Riemann surfaces and fibers spanning the possible choices of local coordinates at the punctures where the scattering states are inserted. Ignoring coordinates at the punctures, each Riemann surface contributes only once to the amplitude.
- The amplitudes factorizes even without imposing on-shell condition: near poles, all of which must arise from the propagator, the amplitude is a product of the relevant off-shell vertices.

APPENDIX C

Brief Review of Gauge/String Duality and Gopakumar Prescription

String theory is an attempt to provide a framework to unify everything we know about nature, including all particles and forces between them, in a consistent quantum theory. But surprising developments in string theory from the time of its inception suggests that it should be thought of as a unified frame work in a more broader sense which has the potential to unfold mysterious unity of many seemingly different arenas in theoretical physics and mathematics. A prime example for such a deep relation uncovered by this frame work is the equivalence between quantum field theories and quantum theory for gravity known as gauge/gravity duality [125]. Most studied examples of this duality are the equivalence between certain conformal field theories and quantum gravity in anti-de Sitter spaces because of this reason this duality is commonly known as AdS/CFT duality. Since the quantum theory of gravity which are explored in this context are string theories this duality is also known as gauge/string duality.

Although different examples of gauge/string duality are well studied by now, the underlying mechanism is still not well understood. This is mainly because of our limited understanding of string theories which arises in the context of gauge/string duality. But duality between topological string theories and topological quantum field theories are exceptions. There we have

the luxury of solving both sides exactly. Prominent example for such a well studied example is the duality between topological closed string theory on the S^2 blow up of the conifold geometry and $SU(N)$ Chern-Simons theory on S^3 [126] widely known as Gopakumar-Vafa duality. Interestingly it is even possible to derive this duality from the world-sheet perspective [127], because $SU(N)$ Chern-Simons theory on S^3 is can be expressed as a topological string theory of open strings [128]. Underlying mechanism behind this example is identified as the open/closed string duality, holes in the open string world sheets in one side closes to give closed string world sheets of other side.

Given this it would be interesting to have a well defined practical prescription for identifying the string theory dual of a quantum field theory. As a modest step in this direction Gopakumar proposed a prescription [76, 77, 79] for identifying the string dual of free fields. Gopakumar prescription can be understood as a way to implement the open/closed string duality without finding the open string interpretation of the quantum field theory.

In §.1 we will briefly review the canonical example of gauge/string duality which is the equivalence between $\mathcal{N} = 4$ superconformal Yang-Mills theory in 4 dimension and type II B string theory on $AdS_5 \times S^5$. In §.2 we will briefly discuss the proposal of Gopakumar for finding the stringy dual of free field theory. In §.3 we will review the construction of duality between zero dimensional Gaussian Hermitian matrix models and topological A-model string theory on $\mathcal{CP}^1$ [67] as a specific realisation of the general approach to gauge-string duality proposed by Gopakumar.

C.1 $\mathbf{AdS_5/CFT^4}$

Consider N parallel D3 branes of type IIB string theory with string coupling g in a flat 10 dimensional spacetime. Assume that these parallel branes are separated by distance r. We will analyse this configuration in a specific limit where string tension approaches infinity ($\alpha' \to 0$)

keeping the energies involved in the theory and the mass of the strings stretched between parallel D3 branes fixed ($\frac{r}{\alpha'}$ fixed, i.e. $r \to 0$). This limit is known as decoupling limit due to reason which is explained below.

Significant simplifications happens in this limit. In the $\alpha' \to 0$ limit the energy gap between massless and massive excitations of strings become infinite and effectively D3 brane is completely described by massless states in type IIB strings which includes gauge fields and gravitons. It is not difficult to see that the gauge theory involved has U(N) gauge group and $\mathcal{N} = 4$ supersymmetry in 4 dimension with SO(6) R-symmetry. Since there is no special scale in the $r = 0$ limit theory is conformal with conformal group SO(2,4). Full theory describing this limit is given by gauge theory on the world volume of N D3 branes coming from the open strings, gravity coming from the closed strings and their interaction. In this effective theory interaction is proportional to $g\alpha'^2$. Since the limit in which we are interested keep energies involved in the theory fixed, interactions won't be prominent when $\alpha' \to 0$. Due to this vanishing of interaction between gravity and gauge theory the system is described by gravity and $\mathcal{N}$=4 SYM in 4 dimension.

Interestingly this system has nice supergravity description which is trustable in the low energy regime. Supergravity solution with N D3 brane charge in the limit of our interest is given by

$$ds^2 = \alpha'\Big\{\frac{U^2}{\sqrt{4\pi gN}}[-(1-\frac{U_0^4}{U^4})dt^2 + dx_i^2] + \sqrt{4\pi gN}\frac{dU^2}{U^2(1-\frac{U_0^4}{U^4})} + \sqrt{4\pi gN}d\Omega_5^2\Big\} \quad \text{(C.1.1)}$$

where $U = \frac{r}{\alpha'}$ and $U_0^4 = \frac{2^7}{3}\pi^4 g^2\mu$. Here μ is the energy density on the world volume theory which is held fixed in the limit in which we are intersted. (t, x^1, x^2, x^3) are the world volume coordinates and $d\Omega_5^2$ is the metric on the unit five-sphere. $U = 0$ corresponds to horizon and $U > 0$ corresponds to the region outside horizon. Radius of curvature of this geometry is proportional to $\frac{1}{gN^{\frac{1}{4}}}$. So this description is trustable only in the $gN >> 1$. For weak coupling,

i.e. g 0 this implies $N >> 1$.

Now look into the massless low energy excitation in this low energy supergravity description. From the point of view of an observer at infinity there are 2 kinds of massless low energy excitations. They are states which have very low energy from the point of view of proper time observer and states which have finite energy from the point of view of proper time observer but constrained to propagate near the horizon of supergravity geometry.

Comparing the two descriptions of the same system one arrives at the surprising statement that near horizon geometry of supergravity solution describing N D3 brane in the decoupling limit is equivalent to large N limit of $\mathcal{N} = 4$ superconformal Yang-Mills in 4 dimension with gauge group U(N). Near horizon geometry in the decoupling limit is $AdS_5 \times S^5$ with radius $(4\pi gN)^{\frac{1}{4}}$ and flux of 5-form field strength on the 5 sphere. Since $AdS_5 \times S^5$ geometry has boundary data regarding the boundary conditions also becomes part of this duality. Given that we arrived at this conclusion by starting with a consistent string theory configuration one can go ahead and boldly propose the conjecture that type IIB string theory on $AdS_5 \times S^5$ with radius $(4\pi gN)^{\frac{1}{4}}$ and flux of 5-form field strength on the 5 sphere plus some appropriate boundary conditions and possibly some boundary degrees of freedom is dual to $\mathcal{N} = 4$ superconformal Yang-Mills in 4 dimension with gauge group U(N) with Yang-Mills coupling $(2\pi g)^{\frac{1}{2}}$ and the value of RR-scalar as the theta angle. Since N measures the size of geometry in planck units quantum effects in $AdS_5 \times S^5$ have the interpretation of $\frac{1}{N}$ effects in gauge theory

A conjecture is supported by series of checks until it is proved or disproved. So we need to perform a series of checks to validate this conjecture. Till now there is no single example against the claim instead what we have is plethora examples for one to one matching. For instance symmetries in both sides show a perfect agreement. An attractive feature of this canonical example for gauge/string duality is its strong-weak nature. This strong-weak na-

ture is evident because effective coupling of super yang mills which is proportional to $\sqrt{gN}$ is very large when the supergravity description, which can be thought of as the weak coupling description of string theory, is trustable. Because of this one can use this duality as a powerful tool to study strongly interacting quantum field theory. Even one can think of this as a non-perturbative definition a quantum theory of gravity in anti-de Sitter spacetime. To use this duality one need a proper dictionary between objects in both sides of this duality. This can be obtained if we interpret this duality as a duality between quantum theory of gravity in the bulk of spacetime whose asymptotic behaviour matches with that of $AdS_5 \times S^5$ space and a quantum field theory living on the boundary of AdS_5 [129, 130].

C.2 Gopakumar's Prescription

How exactly does a quantum field theory in the large N limit reorganise into a closed string theory? Detailed study of duality between topological string theories and topological quantum field theories taught us that underlying mechanism behind this example is open/closed string duality, holes in the open string world sheets in one side closes to give closed string world sheets of other side.Given this it would be interesting to have a well defined practical prescription for identifying the string theory dual of a quantum field theory. Understanding this may give valuable clue regarding the string dual to a QCD. As a modest step in this direction Gopakumar proposed a prescription [76, 77, 79] for identifying the string dual of free fields.

Gopakumar prescription implements open/closed string duality directly by thinking of it as a field theory limit of open strings theory to find its closed string theory dual. Basic ingredient is the isomorphism between the moduli space of Riemann surfaces and the space of metric graphs. Think of every Feynman graph contributing to a given correlator as a metric graph. Metric graph is a graph with length associated with each of it edges. Length of the edges of Feynman graph is given by the Schwinger parameters. Integration over the Schwinger parameters is then mapped to an integration over the moduli space of Riemann surfaces. Now interpret integrand

as a worldsheet CFT correlator. To obtain skeletal graph with reduced Schwinger parameters from Feynman graphs using electric network analogy. At graphical level, to get the skeletal graph of any Feynman graph merge all homotopically equivalent contractions between any two vertices. Then draw dual graph of skeletal graph and assign length for each edge as the inverse of the Schwinger parameter. Construct Riemann surface from the Feynman graph in the following way. On each dual edge of the correlator attach an infinite strip. Glue all these infinite strips using gluing conditions to get the punctured Riemann surface. Interestingly integral over effective Schwinger parameters together with sum over inequivalent skeleton graphs can be mapped to an integral over the string moduli space $\mathcal{M}_{g,n} \times R_+^n$. Here $\mathcal{M}_{g,n}$ is the moduli space of genus g surfaces with n-punctures. This way we can express the field theory as a closed string theory.

C.3 Simplest Gauge/String Duality

Follow this prescription to obtain the string dual of Gaussian matrix model [67]. But for zero dimensional matrix models there is no natural notion of Schwinger parameter (proper time). To circumvent this Razamat gave an interesting proposal: associate unit length to each contracted edge [80, 81]. Let us restrict to planar graphs contributing to connected correlators of the form $< \prod_{i=1}^{n} TrM^{2k_i} >_c$, where M is an $N \times N$ hermitian matrix. On each dual edge of the correlator attach an infinite strip of width one. Glue all these infinite strips to get the Riemann surface. Resulting surface is a genus zero arithmetic surface. Interestingly arithmetic surfaces are special set of points in the moduli space of genus zero Riemann surface. If a non-singular Riemann surface C is an arithmetic surface then there exists a special map $f : C \to CP^1$ which is branched only at (0,1,∞) of CP^1. Such special maps are known as Beyli maps. Explicit form of Belyi map is given as $X(z) = sin^2(\frac{\pi z}{2})$. This maps each strip to CP^1 and covers CP^1 exactly ones. Since there are d such strips, this map is a degree d map.

Now we can express Gaussian matrix model n-point correlator in terms 3 sets of permuta-

tions as follows.

$$N^n < \prod_{i=1}^{n} TrM^{2k_i} >_g = \sum_{\alpha,\gamma} \delta(\alpha.\beta.\gamma) N^{2-2g(\alpha,\beta,\gamma)}, \sum_{i=1}^{n} k_i = d \tag{C.3.2}$$

where α denotes possible wick contractions among half edges,$[2^d]$, β denotes cyclic permutation of half edges in the diagram, $(2k_1)(2k_2)...(2k_n)$ and γ denotes arrangement of half edges around a face. With the help of Riemann existence counting the number of equivalence class of permutations can be interpreted as counting holomorphic maps. Riemann existence theorem says it is possible to associate equivalence class of ramified coverings to equivalence class of permutation group. Then one can argue that correlator we are considering is a sum of degree d Belyi maps.

Thus using Gopakumar's prescription we expressed Gaussian matrix model as a topological string theory. Now we need to identify a string theory living on CP^1 with correlators getting contributions only from specific degree holomorphic maps. Most suitable candidate is topological A-model string theory on CP^1. Correlators of this theory gets contributions only from specific degree holomorphic maps due to a selection rule.

APPENDIX D

Correlators in Matrix Model and Topological String Theory

D.1 Correlators in the Gaussian Matrix Model

The generating function for arbitrary n point correlators $\langle \prod_i^n \mathrm{Tr} M^{2k_i} \rangle$ of the Gaussian matrix model is

$$Z[t] = \int [dM]_{N\times N} e^{-\frac{1}{2}N\mathrm{Tr}M^2 + \sum_k t_k N \mathrm{Tr} M^{2k}}. \tag{D.1.1}$$

Since M is a hermitian matrix it is possible to parametrize this in terms of a unitary matrix U and eigenvalues λ_k. i.e; we can write $M = U^\dagger \Lambda U$, where $\Lambda = diag(\lambda_1, ..., \lambda_N)$. Then the generating function can be expressed in terms of an integral over eigenvalues λ_k and evaluated using orthogonal polynomials as discussed in the text.

We find the final answer to be given as in 5.2.8

$$Z = N! \prod_{j=0}^{N-1} h_j = N! h_0^N \prod_{j=0}^{N-1} R_j^{N-j} \tag{D.1.2}$$

where $R_m = \frac{h_m}{h_{m-1}}$. Thus the calculation of the generating function reduces to calculating h_j or equivalently R_j. To determine these we need to use the recursion formulae for the P_m. The

orthogonal polynomials satisfy the usual three term recursion formula

$$\lambda P_m(\lambda) = P_{m+1}(\lambda) + S_m P_m(\lambda) + R_m P_{m-1}(\lambda). \tag{D.1.3}$$

For even potentials $V(\lambda)$, where $P_m(-\lambda) = (-1)^m P_m(-\lambda)$, we have $S_m = 0$. On the other hand, we obtain a non-linear recursion formula by looking at $\int d\lambda \lambda P'_m(\lambda) P_m(\lambda) e^{-N\mathrm{V}(\lambda)}$. This gives

$$N R_m \int d\lambda P_m(\lambda) P_{m-1}(\lambda) V'(\lambda) e^{-N\mathrm{V}(\lambda)} = m h_m. \tag{D.1.4}$$

We can now study the large N limit. Assume that n is an integer of order N and k is an integer of order one. Then we can safely assume to leading order

$$R_{m-k} = R_{m-k+1} = ... = R_{m-1} = R_m. \tag{D.1.5}$$

Using repeatedly the orthogonality property and integrating by parts we get

$$\int d\lambda \lambda^{2k-1} P_m(\lambda) P_{m-1}(\lambda) e^{-N\mathrm{V}(\lambda)} = \frac{(2k)!}{k!(k-1)!} h_m R_m^{k-1}(t). \tag{D.1.6}$$

By t we mean the collection of couplings $\{t_k\}$. Then for $V(\lambda)$ given in 5.2.6, relation D.1.4 reduces to

$$R_m(t)(1 - A_m(t)) = \frac{m}{N} \tag{D.1.7}$$

where, $A_m(t) = \sum_k \frac{(2k)!}{k!(k-1)!} t_k R_m^{k-1}(t)$. In the planar limit the rescaled index m/N becomes a continuous variable y that takes values in $(0,1)$, and $R_m(t)$ and $A_m(t)$ become continuous functions $R(t,y)$ and $A(t,y)$. We use this to calculate various correlators explicitly.

D.1.1 Connected correlator $\langle \prod_{i=1}^n \mathrm{Tr} M^{2k_i} \rangle_{conn}$

The generating function for connected correlators $\langle \prod_{i=1}^n \mathrm{Tr} M^{2k_i} \rangle$ in the large N limit is obtained by taking the continuum limit of D.1.2. In this limit, the free energy (up to an irrelevant additive constant) reduces to a simple one-dimensional integral:

$$\begin{aligned} G(t) &= \lim_{N\to\infty} \frac{1}{N^2} \ln\left(\frac{Z[t]}{Z[0]}\right) \\ &= \int_0^1 dy (1-y) \ln\left(\frac{R(t,y)}{y}\right) \\ &= -\int_0^1 dy (1-y) \ln(1 - A(t,y)). \end{aligned} \tag{D.1.8}$$

Here for moving from the second step to the third we have used the following continuum limit of relation D.1.7

$$\begin{aligned} \frac{R(t,y)}{y} &= \frac{1}{1-A(t,y)}, \\ A(t,y) &= \sum_k A(k) t_k R^{k-1}(t,y), \end{aligned} \tag{D.1.9}$$

where $A(k) = \frac{(2k)!}{k!(k-1)!}$. The free energy is the generating function of connected correlators. Thus we obtain the required correlator by suitable differentiation

$$\begin{aligned} \langle \prod_i^n \mathrm{Tr} M^{2k_i} \rangle_{conn} &= \frac{\partial^n}{\partial t_{k_1} \ldots \partial t_{k_n}} G(t_k)|_{t_{k_i}=0} \\ &= \int_0^1 dy (1-y) \Big(\sum_{m=1}^n (m-1)! \sum_{partitions \{R_j\}} \prod_{j=1}^m \prod_{r\in R_j} \partial_{t_{k_r}} A \Big)|_{t_{k_i}=0} \end{aligned} \tag{D.1.10}$$

where $A = A(t, y)$ and we use the fact that $A(t = 0, y) = 0$. The notation is as in Sec. 4: we have partitioned the n integers $(1, 2, \ldots n)$ into m non-empty groupings $\{R_j\}$ where $j = 1 \ldots m$. Examination of D.1.10 then reveals the structure of the correlator to be

$$\langle \prod_{i=1}^{n} \mathrm{Tr} M^{2k_i} \rangle_{conn} = T_{n-1}(k_1, ..., k_n)(\prod_{i=1}^{n} A(k_i)) \int_0^1 dy(1-y) R(t=0, y)^{d-n} \tag{D.1.11}$$

where $d = \sum_{i=1}^{n} k_i$. We also note from D.1.9 that $R(t = 0, y) = y$. And $T_{n-1}(k_1, ..., k_n)$ is a polynomial of order $n-1$ in each k_i which is symmetric in all k_i. It is a combinatorial challenge to work out the form of the polynomial. Surprisingly the answer is simple and given in [94] (see also [95])

$$T_{n-1}(k_1, ..., k_n) = \frac{(d-1)!}{(d-n)!}. \tag{D.1.12}$$

Hence the connected n-point correlator in the large N limit is given by

$$\langle \prod_{i=1}^{n} \mathrm{Tr} M^{2k_i} \rangle_{conn} = \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i!(k_i-1)!}. \tag{D.1.13}$$

D.1.2 Connected correlator $\langle (\mathrm{Tr} \ln M)^2 \prod_{i=1}^{n} \mathrm{Tr} M^{2k_i} \rangle_{conn}$

The generating function for the arbitrary $(n+2)$-point correlator $\langle (\mathrm{Tr} \ln M)^2 \prod_{i=1}^{n} \mathrm{Tr} M^{2k_i} \rangle$ is given by

$$Z[t, \alpha] = \int [dM]_{N \times N} e^{-\frac{1}{2} N \mathrm{Tr} M^2 + \sum_k t_k N \mathrm{Tr} M^{2k} + \alpha N \mathrm{Tr} \ln M^2}. \tag{D.1.14}$$

We can evaluate the required correlator by following the arguments given in previous section. The potential $V(\lambda)$ is slightly modified to have a logarithmic term,

$$\mathrm{V}(\lambda) = \frac{1}{2}\lambda^2 - \sum_k t_k \lambda^{2k} + \alpha \ln \lambda^2. \tag{D.1.15}$$

Here too we can write down a relation like D.1.7, which we derived by taking the large N limit of the relation D.1.4. When m is odd $\lambda^{-1} P_m(\lambda)$ is a polynomial and can be expressed in terms

of $P_{m-1}(\lambda)$ and $P_{m-2}(\lambda)$ using D.1.3. Note that we have an even potential so S_m is again zero. Using this we can derive the following

$$\begin{aligned}\int d\lambda\lambda^{-1}P_m(\lambda)P_{m-1}(\lambda)e^{-N\mathrm{V}(\lambda)} &= h_{m-1}, \quad \text{for m odd} \\ &= 0, \qquad \text{for m even.}\end{aligned} \tag{D.1.16}$$

This leads us to the required relation

$$\begin{aligned}R_m(t,\alpha)[1-\sum_k t_k A(k)R_m^{k-1}(t,\alpha)-2\frac{\alpha}{R_m(t,\alpha)}] &= \frac{m}{N}, \quad \text{for m odd} \\ R_m(t,\alpha)[1-\sum_k t_k A(k)R_m^{k-1}(t,\alpha)] &= \frac{m}{N}, \quad \text{for m even.}\end{aligned} \tag{D.1.17}$$

For large N we can take a continuum limit as in the previous case (we now take an average of the cases with m even and m odd) and get

$$\begin{aligned}\frac{R(t,\alpha,y)}{y} &= \frac{1}{1-A(t,\alpha,y)} \\ A(t,\alpha,y) &= \sum_k A(k)t_k R^{k-1}(t,\alpha,y)+\frac{\alpha}{R(t,\alpha,y)}.\end{aligned} \tag{D.1.18}$$

Thus the generating function for the large N connected correlator $\langle(\mathrm{Tr}\ln M)^2\prod_i^n \mathrm{Tr}M^{2k_i}\rangle_{conn}$ is

$$G(t,\alpha)=-\int_0^1 dy(1-y)\ln[1-A(t,\alpha,y)]. \tag{D.1.19}$$

Now without much difficulty we can extract the correlator from the generating function as we did in the previous case.

$$\begin{aligned}\langle(\mathrm{Tr}\ln M)^2\prod_{i=1}^n \mathrm{Tr}M^{2k_i}\rangle_{conn} &= \frac{\partial^{n+2}}{4\partial\alpha^2\partial t_{k_1}...\partial t_{k_n}}G(t,\alpha)|_{t=0,\alpha=0} \\ &= \frac{\partial^2}{4\partial\alpha^2}\langle\prod_{i=1}^n \mathrm{Tr}M^{2k_i}\rangle_{conn}|_{\alpha=0} \\ &= \frac{(d-1)!}{4(d-n)!}\prod_{i=1}^n A(k_i)\int_0^1 dy(1-y)\frac{\partial^2}{\partial\alpha^2}R^{d-n}(t=0,\alpha,y)|_{\alpha=0}.\end{aligned}$$

(D.1.20)

From D.1.18 it is clear that $R(t=0,\alpha,y) = y+\alpha$. Thus we get

$$\langle (\mathrm{Tr}\ln M)^2 \prod_{i=1}^{n} \mathrm{Tr} M^{2k_i} \rangle_{conn} = \frac{(d-1)!}{4(d-n)!} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i!(k_i-1)!}. \tag{D.1.21}$$

D.2 Correlators of the topological A-model string theory on $\mathbb{P}^1$

Physical observables in topological A-model string theory on $\mathbb{P}^1$ arises from the cohomology of the target manifold $\mathbb{P}^1$. They are the puncture operator P, Kahler class Q and their graviational descendants $\sigma_n(Q)$, $\sigma_n(P)$ (for $n \geq 1$). The partition function of this topological string theory depends on a set of couplings $\{t_k, t'_k\}$ corresponding to these operators. This is the generating function of all the correlators in the theory. The genus g correlation functions $\langle \prod_{i=1}^{n} \sigma_{2k_i}(\mathcal{V}_{\alpha_i}) \rangle_g$ (in the background where all couplings t_k, t'_k vanish) receives contributions only from holomorphic maps (from the world sheet to the target $\mathbb{P}^1$) of degree d satisfying the ghost number conservation law (see for e.g. Eq.(2.24) of [99])

$$2d + 2(g-1) = \sum_{i=1}^{n} (2k_i + q_{\alpha_i} - 1) \tag{D.2.22}$$

where $\mathcal{V}_1 = P, \mathcal{V}_2 = Q$ with $q_1 = 0, q_2 = 1$. Here we will be considering only a specific set of genus zero correlators namely $\langle \sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q) \prod_{i=3}^{n} \sigma_{2k_i}(Q) \rangle_{g=0}$ and $\langle P^2 \prod_{i=1}^{n} \sigma_{2k_i}(Q) \rangle_{g=0}$. This set of correlators defines what is sometimes called the stationary sector of the string theory. For both set of correlators the selection rule D.2.22 reduces to $2d-2 = 2\sum_i^n k_i - 2$.

To compute these correlators we can use the various recursion relations that they satisfy. The important ones are summarized in [100]. These relations help us to express the n-point correlators in terms of low point correlators. Relevant recursion relations are listed below.

Topological recursion relation :

$$\langle \sigma_n(\mathcal{V}_\gamma) XY \rangle_{g=0} = n \langle \sigma_{n-1}(\mathcal{V}_\gamma)\mathcal{V}_\alpha \rangle \eta^{\alpha\beta} \langle \mathcal{V}_\beta XY \rangle_{g=0} \tag{D.2.23}$$

where X, Y are arbitrary observables and this holds in large phase space - where all the couplings $\{t_k, t'_k\}$ are turned on.

Puncture equation :

$$\langle P \prod_{i=1}^{s} \sigma_{n_i}(Q) \rangle_{g=0} = \sum_{i=1}^{s} n_i \langle \sigma_{n_i-1}(Q) \prod_{j\neq i} \sigma_{n_j}(Q) \rangle_{g=0}. \tag{D.2.24}$$

Hori's relation [101] :

$$\langle Q \prod_{i=1}^{s} \sigma_{n_i}(Q) \rangle_{g=0} = d \langle \prod_{i=1}^{s} \sigma_{n_i}(Q) \rangle_{g=0} \tag{D.2.25}$$

with $d = \frac{1}{2}\sum_{i=1}^{s} n_i + 1$.

Eguch-Hori-Yang relation [99]:

$$\begin{aligned} d^2 \langle \sigma_n(\mathcal{V}_\alpha) \prod_{i\in S} \sigma_{n_i}(\mathcal{V}_{\alpha_i}) \rangle_{0,d} &= -2dn \langle \sigma_{n-1}(\mathcal{V}_{\alpha+1}) \prod_{i\in S} \sigma_{n_i}(\mathcal{V}_{\alpha_i}) \rangle_{0,d} \\ + \sum_{X\cup Y = S} \sum_{k=1}^{d} k^2 n \langle \sigma_{n-1}(\mathcal{V}_\alpha) \prod_{i\in X} \sigma_{n_i}(\mathcal{V}_{\alpha_i}) \mathcal{V}_\alpha \rangle_{0,d-k} \langle \mathcal{V}_{\alpha_j} \prod_{j\in Y} \sigma_{n_j}(\mathcal{V}_{\alpha_j}) \rangle_{0,k}. \end{aligned} \tag{D.2.26}$$

Recursion relation among 2-point correlators:

$$\langle \sigma_n(\mathcal{V}_\alpha) \sigma_m(\mathcal{V}_\beta) \rangle = \frac{mnM_{\phi\sigma}\eta^{\delta\phi}\eta^{\gamma\sigma}}{(n+m+q_\alpha+q_\beta)} \langle \sigma_{n-1}(\mathcal{V}_\alpha)\mathcal{V}_\gamma \rangle \langle \sigma_{n-1}(\mathcal{V}_\beta)\mathcal{V}_\beta \rangle \tag{D.2.27}$$

where

$$\begin{aligned} \eta^{PQ} &= \eta^{QP} = 1, \\ \eta^{PP} &= \eta^{QQ} = 0, \\ M_{PP} &= M_{QQ} = 2. \end{aligned} \tag{D.2.28}$$

If we turn off all the couplings then we have $\langle Q \rangle_{0,d} = 0, \langle P \rangle_{0,d} = 0$ except $\langle Q \rangle_{0,1} = 1$. Then setting $n = 2k_i, \alpha = 2$, and $S = 0$ in D.2.26 will give us the following equation

$$(k_i + 1)^2 \langle \sigma_{2k_i}(Q) \rangle_{0,k_i+1} = 2k_i(2k_i - 1) \langle \sigma_{2k_i-2}(Q) \rangle_{0,k_i}. \tag{D.2.29}$$

Therefore

$$\begin{aligned} \langle \sigma_{2k_i}(Q) \rangle_{0,k_i+1} &= \frac{(2k_i)!}{(k_i+1)!(k_i+1)!} \\ \langle \sigma_{2k_i-1}(Q) P \rangle_{0,k_i} &= \frac{(2k_i-1)!}{(k_i!)^2}. \end{aligned} \tag{D.2.30}$$

Also due to the selection rule D.2.22 we have for all d

$$\langle \sigma_{2k_i+1}(Q) \rangle_{0,d} = 0. \tag{D.2.31}$$

Plugging these into D.2.27 will give the following 2-point correlator

$$\langle \sigma_{2k_i-1}(Q) \sigma_{2k_j-1}(Q) \rangle_{0,k_i+k_j} = \frac{1}{4(k_i+k_j)} \frac{(2k_i)!}{(k_i!)^2} \frac{(2k_j)!}{(k_j!)^2}. \tag{D.2.32}$$

To calculate the higher point correlators we can use D.2.23 and reduce them to lower point correlators by remembering the fact that the recursion relations are valid in the large phase

space. For example

$$\begin{aligned}\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\sigma_{2k_3}(Q)\rangle_{0,k_1+k_2+k_3} &= 2k_3\langle\sigma_{2k_3-1}(Q)P\rangle_{0,k_3}\langle Q\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\rangle_{0,k_1+k_2} \\ &= \frac{1}{4}\frac{(2k_1)!}{(k_1!)^2}\frac{(2k_2)!}{(k_2!)^2}\frac{(2k_3)!}{((k_3!)^2}. \end{aligned} \tag{D.2.33}$$

Repeated application of the recursion relations give us the following general correlators

$$\begin{aligned}\langle\sigma_{2k_1-1}(Q)\sigma_{2k_2-1}(Q)\prod_{i=3}^{n}\sigma_{2k_i}(Q)\rangle_{0,d} &= \frac{d^{n-3}}{4}\prod_{i=1}^{n}\frac{(2k_i)!}{(k_i!)^2} \\ \langle\prod_{i=1}^{n}\sigma_{2k_i}(Q)\rangle_{0,d} &= (d+1)^{n-3}\prod_{i=1}^{n}\frac{(2k_i)!}{(k_i!)^2} \\ \langle P^2\prod_{i=1}^{n}\sigma_{2k_i}(Q)\rangle_{0,d} &= d^{n-1}\prod_{i=1}^{n}\frac{(2k_i)!}{(k_i!)^2} \end{aligned} \tag{D.2.34}$$

where $d=\sum_{i=1}^{n}k_i$. These results agree with the ones stated in [102].

Bibliography

[1] R. Pius, A. Rudra and A. Sen, JHEP **1407**, 058 (2014) doi:10.1007/JHEP07(2014)058 [arXiv:1311.1257 [hep-th]].

[2] R. Pius, A. Rudra and A. Sen, JHEP **1407**, 062 (2014) doi:10.1007/JHEP07(2014)062 [arXiv:1401.7014 [hep-th]].

[3] R. Pius and A. Sen, JHEP **1406**, 068 (2014) doi:10.1007/JHEP06(2014)068 [arXiv:1310.4593 [hep-th]].

[4] R. Gopakumar and R. Pius, JHEP **1303**, 175 (2013) doi:10.1007/JHEP03(2013)175 [arXiv:1212.1236 [hep-th]].

[5] E. Witten, "Superstring Perturbation Theory Revisited," arXiv:1209.5461 [hep-th].

[6] A. Belopolsky, "De Rham cohomology of the supermanifolds and superstring BRST cohomology," Phys. Lett. B **403**, 47 (1997) [hep-th/9609220]; "New geometrical approach to superstrings," hep-th/9703183; "Picture changing operators in supergeometry and superstring theory," hep-th/9706033.

[7] E. D'Hoker and D. H. Phong, "Two loop superstrings. I. Main formulas," Phys. Lett. B **529**, 241 (2002) [hep-th/0110247]. "II. The Chiral measure on moduli space," Nucl. Phys. B **636**, 3 (2002) [hep-th/0110283]. "III. Slice independence and absence of ambiguities," Nucl. Phys. B **636**, 61 (2002) [hep-th/0111016]. "IV: The Cosmological constant

and modular forms," Nucl. Phys. B **639**, 129 (2002) [hep-th/0111040]. "V. Gauge slice independence of the N-point function," Nucl. Phys. B **715**, 91 (2005) [hep-th/0501196]. "VI: Non-renormalization theorems and the 4-point function," Nucl. Phys. B **715**, 3 (2005) [hep-th/0501197]. "VII. Cohomology of Chiral Amplitudes," Nucl. Phys. B **804**, 421 (2008) [arXiv:0711.4314 [hep-th]].

[8] E. Witten, "Notes On Supermanifolds and Integration," arXiv:1209.2199 [hep-th]; "Notes On Super Riemann Surfaces And Their Moduli," arXiv:1209.2459 [hep-th]; "More On Superstring Perturbation Theory," arXiv:1304.2832 [hep-th]; "Notes On Holomorphic String And Superstring Theory Measures Of Low Genus," arXiv:1306.3621 [hep-th]; "The Feynman $i\epsilon$ in String Theory," arXiv:1307.5124 [hep-th].

[9] R. Donagi and E. Witten, "Supermoduli Space Is Not Projected," arXiv:1304.7798 [hep-th].

[10] R. Donagi and E. Witten, arXiv:1404.6257 [hep-th].

[11] A. M. Polyakov, Phys. Lett. B **103**, 207 (1981).

[12] A. M. Polyakov, Phys. Lett. B **103**, 211 (1981).

[13] S. Weinberg, "Radiative corrections in string theory," in The Oregon Meeting, Proceedings of the Annual Meeting of the Division of Particles and Fields of the APS, Eugene, Oregon, 1985, edited by R. C. Hwa (World Scientific, Singapore, 1986).

[14] N. Seiberg, "Anomalous Dimensions and Mass Renormalization in String Theory," Phys. Lett. B **187**, 56 (1987).

[15] H. Ooguri and N. Sakai, "String Loop Corrections From Fusion of Handles and Vertex Operators," Phys. Lett. B **197**, 109 (1987).

[16] H. Yamamoto, "One Loop Mass Shifts in O(32) Open Superstring Theory," Prog. Theor. Phys. **79**, 189 (1988).

[17] A. Sen, "Mass Renormalization and BRST Anomaly in String Theories," Nucl. Phys. B **304**, 403 (1988).

[18] S. R. Das, "Renormalizing Handles and Holes in String Theory," Phys. Rev. D **38**, 3105 (1988).

[19] S. -J. Rey, "Unified View of BRST Anomaly and Its Cancellation in String Amplitudes," Nucl. Phys. B **316**, 197 (1989).

[20] B. Sundborg, "Selfenergies of Massive Strings," Nucl. Phys. B **319**, 415 (1989).

[21] N. Marcus, "Unitarity and Regularized Divergences in String Amplitudes," Phys. Lett. B **219** (1989) 265.

[22] K. Amano and A. Tsuchiya, "Mass Splittings and the Finiteness Problem of Mass Shifts in the Type II Superstring at One Loop," Phys. Rev. D **39** (1989) 565.

[23] C. J. -H. Lee, "BRST anomalies and mass renormalization with anomalous U(1) gauge symmetries in string theory," UMI-92-24830.

[24] A. Berera, "The Mass renormalization of string theory," Phys. Rev. D **49**, 6674 (1994).

[25] L. Del Debbio, E. Kerrane and R. Russo, "Mass corrections in string theory and lattice field theory," Phys. Rev. D **80**, 025003 (2009) [arXiv:0812.3129 [hep-th]].

[26] D. Chialva, "String Mass Shifts," Nucl. Phys. B **819**, 225 (2009) [arXiv:0903.3979 [hep-th]].

[27] P. C. Nelson, "Covariant Insertion of General Vertex Operators," Phys. Rev. Lett. **62**, 993 (1989).

[28] A. G. Cohen, G. W. Moore, P. C. Nelson and J. Polchinski, "An Off-Shell Propagator for String Theory," Nucl. Phys. B **267**, 143 (1986).

[29] A. G. Cohen, G. W. Moore, P. C. Nelson and J. Polchinski, "Semi Off-shell String Amplitudes," Nucl. Phys. B **281**, 127 (1987).

[30] L. Alvarez Gaumé, C. Gomez, G. Moore and C. Vafa, "Strings in the operator formalism", Nucl. Phys. B **303**, 455 (1988).

[31] L. Alvarez Gaumé, C. Gomez, P. Nelson, G. Sierra and C. Vafa, "Fermionic strings in the operator formalism", Nucl. Phys. B **311**, 333 (1988).

[32] B. Zwiebach, "Closed string field theory: Quantum action and the B-V master equation," Nucl. Phys. B **390**, 33 (1993) [hep-th/9206084].

[33] N. Berkovits, Y. Okawa and B. Zwiebach, "WZW-like action for heterotic string field theory," JHEP **0411**, 038 (2004) [hep-th/0409018].

[34] P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda and A. Sciuto, "N Point g Loop Vertex for a Free Bosonic Theory With Vacuum Charge Q," Nucl. Phys. B **322**, 317 (1989).

[35] J. Polchinski, "Factorization of Bosonic String Amplitudes," Nucl. Phys. B **307**, 61 (1988).

[36] H. Hata and B. Zwiebach, "Developing the covariant Batalin-Vilkovisky approach to string theory," Annals Phys. **229** (1994) 177 [hep-th/9301097].

[37] A. LeClair, M. E. Peskin and C. R. Preitschopf, "String Field Theory on the Conformal Plane. 1. Kinematical Principles," Nucl. Phys. B **317**, 411 (1989); "String Field Theory on the Conformal Plane. 2. Generalized Gluing," Nucl. Phys. B **317**, 464 (1989).

[38] R. Pius, A. Rudra and A. Sen, "Mass Renormalization in String Theory: Special States," arXiv:1311.1257 [hep-th].

[39] L. Rastelli and B. Zwiebach, "The Off-shell Veneziano amplitude in Schnabl gauge," JHEP **0801**, 018 (2008) [arXiv:0708.2591 [hep-th]].

[40] D. Friedan, E. J. Martinec and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," Nucl. Phys. B **271**, 93 (1986).

[41] E. Verlinde and H. Verlinde, "Multiloop Calculations in Covariant Superstring Theory," Phys. Lett. B **192**, 95 (1987).

[42] R. Saroja and A. Sen, "Picture changing operators in closed fermionic string field theory," Phys. Lett. B **286**, 256 (1992) [hep-th/9202087].

[43] A. Sen, arXiv:1501.00988 [hep-th].

[44] A. Sen, "S-duality Improved Superstring Perturbation Theory," arXiv:1304.0458 [hep-th].

[45] C. Beem, L. Rastelli, A. Sen and B. C. van Rees, "Resummation and S-duality in N=4 SYM," arXiv:1306.3228 [hep-th].

[46] T. Banks and T. J. Torres, "Two Point Pade Approximants and Duality," arXiv:1307.3689 [hep-th].

[47] L. F. Alday and A. Bissi, "The superconformal bootstrap for structure constants," arXiv:1310.3757 [hep-th].

[48] V. Asnin, D. Gorbonos, S. Hadar, B. Kol, M. Levi and U. Miyamoto, "High and Low Dimensions in The Black Hole Negative Mode," Class. Quant. Grav. **24**, 5527 (2007) [arXiv:0706.1555 [hep-th]].

[49] H. Kleinert and V. Schulte-Frohlinde, "Critical properties of ϕ^4-theories," River Edge, USA: World Scientific (2001) 489 p

[50] J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724 [hep-th/9510017].

[51] M. Frau, L. Gallot, A. Lerda and P. Strigazzi, "Stable nonBPS D-branes in type I string theory," Nucl. Phys. B **564**, 60 (2000) [hep-th/9903123].

[52] M. Frau, L. Gallot, A. Lerda and P. Strigazzi, "Stable non-BPS D branes of type I," hep-th/0003022.

[53] M. Frau, L. Gallot, A. Lerda and P. Strigazzi, "D-branes in type I string theory," Fortsch. Phys. **49**, 503 (2001) [hep-th/0012167].

[54] A. Sen, "NonBPS states and Branes in string theory," hep-th/9904207.

[55] M. Stingl, "Field theory amplitudes as resurgent functions," hep-ph/0207349.

[56] U. D. Jentschura and J. Zinn-Justin, "Instantons in quantum mechanics and resurgent expansions," Phys. Lett. B **596**, 138 (2004) [hep-ph/0405279].

[57] I. sAniceto, R. Schiappa and M. Vonk, "The Resurgence of Instantons in String Theory," Commun. Num. Theor. Phys. **6**, 339 (2012) [arXiv:1106.5922 [hep-th]].

[58] P. C. Argyres and M. Unsal, "The semi-classical expansion and resurgence in gauge theories: new perturbative, instanton, bion, and renormalon effects," JHEP **1208**, 063 (2012) [arXiv:1206.1890 [hep-th]].

[59] G. V. Dunne and M. Unsal, "Resurgence and Trans-series in Quantum Field Theory: The CP(N-1) Model," JHEP **1211**, 170 (2012) [arXiv:1210.2423 [hep-th]].

[60] G. V. Dunne and M. Unsal, "Continuity and Resurgence: towards a continuum definition of the CP(N-1) model," Phys. Rev. D **87**, 025015 (2013) [arXiv:1210.3646 [hep-th]].

[61] R. Schiappa and R. Vaz, "The Resurgence of Instantons: Multi-Cuts Stokes Phases and the Painleve II Equation," arXiv:1302.5138 [hep-th].

[62] A. Cherman, D. Dorigoni, G. V. Dunne and M. Unsal, "Resurgence in QFT: Unitons, Fractons and Renormalons in the Principal Chiral Model," arXiv:1308.0127 [hep-th].

[63] G. Basar, G. V. Dunne and M. Unsal, "Resurgence theory, ghost-instantons, and analytic continuation of path integrals," arXiv:1308.1108 [hep-th].

[64] I. sAniceto and R. Schiappa, "Nonperturbative Ambiguities and the Reality of Resurgent Transseries," arXiv:1308.1115 [hep-th].

[65] R. C. Santamara, J. D. Edelstein, R. Schiappa and M. Vonk, "Resurgent Transseries and the Holomorphic Anomaly," arXiv:1308.1695 [hep-th].

[66] M. Garay, A. de Goursac and D. van Straten, "Resurgent Deformation Quantisation," arXiv:1309.0437 [math-ph].

[67] R. Gopakumar, "What is the Simplest Gauge-String Duality?," arXiv:1104.2386 [hep-th].

[68] E. Witten, "On The Structure Of The Topological Phase Of Two-dimensional Gravity," Nucl. Phys. B **340**, 281 (1990).

[69] R. Dijkgraaf, E. Witten, "Mean Field Theory, Topological Field Theory, And Multimatrix Models," Nucl. Phys. **B342**, 486-522 (1990).

[70] R. d. M. Koch and S. Ramgoolam, "From Matrix Models and quantum fields to Hurwitz space and the absolute Galois group", arXiv:1002.1634 [hep-th].

[71] C. Itzykson and J. B. Zuber, "MATRIX INTEGRATION AND COMBINATORICS OF MODULAR GROUPS," Commun. Math. Phys. **134**, 197 (1990).

[72] P. Di Francesco and C. Itzykson, "A Generating Function For Fatgraphs," Annales Poincare Phys. Theor. **59** (1993) 117 [arXiv:hep-th/9212108].

[73] M. Bauer and C. Itzykson, "Triangulations" in *Grothendieck's Theory of Dessins D'Enfants* (ed. L. Schneps), London Mathematical Society Lecture Note Series, Cambridge University Press, UK (1994).

[74] R. de Mello Koch, S. Ramgoolam and C. Wen, "On the refined counting of graphs on surfaces," arXiv:1209.0334 [hep-th].

[75] M. Mulase, M. Penkava, "Ribbon Graphs, Quadratic Differentials on Riemann Surfaces, and Algebraic Curves Defined over $\bar{Q}$," [arXiv:math-ph/9811024].

[76] R. Gopakumar, "From free fields to AdS", Phys. Rev. D **70**, 025009 (2004) [arXiv:hep-th/0308184].

[77] R. Gopakumar, "From free fields to AdS. II," Phys. Rev. D **70**, 025010 (2004) [arXiv:hep-th/0402063].

[78] R. Gopakumar, "Free field theory as a string theory?," Comptes Rendus Physique **5**, 1111 (2004) [hep-th/0409233].

[79] R. Gopakumar, "From free fields to AdS. III," Phys. Rev. D **72**, 066008 (2005) [arXiv:hep-th/0504229].

[80] S. S. Razamat, "On a worldsheet dual of the Gaussian matrix model," JHEP **0807**, 026 (2008) [arXiv:0803.2681 [hep-th]].

[81] S. S. Razamat, "From Matrices to Strings and Back," JHEP **1003**, 049 (2010) [arXiv:0911.0658 [hep-th]].

[82] K. Furuuchi, "From free fields to AdS: Thermal case," Phys. Rev. **D72**, 066009 (2005). [hep-th/0505148].

[83] O. Aharony, Z. Komargodski and S. S. Razamat, "On the worldsheet theories of strings dual to free large N gauge theories," JHEP **0605**, 016 (2006) [arXiv:hep-th/0602226].

[84] J. R. David and R. Gopakumar, "From spacetime to worldsheet: Four point correlators," JHEP **0701**, 063 (2007) [arXiv:hep-th/0606078].

[85] I. Yaakov, "Open and closed string worldsheets from free large N gauge theories with adjoint and fundamental matter," JHEP **0611**, 065 (2006) [arXiv:hep-th/0607244].

[86] O. Aharony, J. R. David, R. Gopakumar, Z. Komargodski and S. S. Razamat, "Comments on worldsheet theories dual to free large N gauge theories," Phys. Rev. D **75**, 106006 (2007) [arXiv:hep-th/0703141].

[87] O. Aharony and Z. Komargodski, "The Space-time operator product expansion in string theory duals of field theories," JHEP **0801**, 064 (2008) [arXiv:0711.1174 [hep-th]].

[88] J. R. David, R. Gopakumar and A. Mukhopadhyay, "Worldsheet Properties of Extremal Correlators in AdS/CFT," JHEP **0810**, 029 (2008) [arXiv:0807.5027 [hep-th]].

[89] T. W. Brown, "Cut-and-join operators and N=4 super Yang-Mills," JHEP **1005**, 058 (2010). [arXiv:1002.2099 [hep-th]].

[90] I. S. Gradshteyn, I. M. Ryzhik, "Tables of integrals, series and products", 7th Ed., Academic press (2007).

[91] E. P. Verlinde and H. L. Verlinde, "A Solution Of Two-dimensional Topological Quantum Gravity," Nucl. Phys. B **348**, 457 (1991).

[92] E. Brezin, C. Itzykson, G. Parisi, J. B. Zuber, "Planar Diagrams," Commun. Math. Phys. **59**, 35 (1978).

[93] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, "2D Gravity and Random Matrices", [arXiv:hep-th/9306153v2]

[94] W. T. Tutte, "A Census of slicings", Can. J. Math. **14**, 708-722 (1962).

[95] J. Ambjorn, M. G. Harris, M. Weis, "On the Connection Between 2d Topological Gravity and the Reduced Hermitian Matrix Model", [arXiv:hep-th/9702188v1].

[96] A. Levin and A. Morozov, "On the foundations of the random lattice approach to quantum gravity," Phys. Lett. B **243**, 207 (1990).

[97] D. -J. Smit, "Summations over equilaterally triangulated surfaces and the critical string measure," Commun. Math. Phys. **143**, 253 (1992).

[98] T. Eguchi, S. -K. Yang, "The Topological CP**1 model and the large N matrix integral," Mod. Phys. Lett. **A9**, 2893-2902 (1994). [hep-th/9407134].

[99] T. Eguchi, K. Hori, S. -K. Yang, "Topological sigma models and large N matrix integral," Int. J. Mod. Phys. **A10**, 4203-4224 (1995). [hep-th/9503017].

[100] T. Eguchi, K. Hori, C. -S. Xiong, "Gravitational quantum cohomology," Int. J. Mod. Phys. **A12**, 1743-1782 (1997). [hep-th/9605225].

[101] K. Hori "Constraints For Topological Strings In D = 1," [arXiv:hep-th/9411135].

[102] P. Norbury, N. Scott, "Gromov Witten Invariants and Eynard-Orantin invariants", [arXiv: 1106.1337v2].

[103] N. Itzhaki and J. McGreevy, "The Large N harmonic oscillator as a string theory," Phys. Rev. D **71**, 025003 (2005) [hep-th/0408180].

[104] R. de Mello Koch and L. Nkumane, JHEP **1503**, 004 (2015) [arXiv:1411.5226 [hep-th]].

[105] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," [arXiv:0712.2824 [hep-th]].

[106] G. Bonelli and H. Safaai, "On gauge/string correspondence and mirror symmetry," JHEP **0806**, 050 (2008) [arXiv:0804.2629 [hep-th]].

[107] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction, Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge, Uk: Univ. Pr. (1987)

[108] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string, Vol. 2: Superstring theory and beyond" Cambridge, UK: Univ. Pr. (1998)

[109] E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60**, 917 (1988).

[110] D. Friedan, E. J. Martinec and S. H. Shenker, Nucl. Phys. B **271**, 93 (1986).

[111] E. P. Verlinde and H. L. Verlinde, IN *TRIESTE 1988, PROCEEDINGS, SUPERSTRINGS '88* 189-250 AND INST. ADV. STUD. PRINCETON - IASSNS-HEP-88-52 (88,REC.MAR.89) 69 P. (906367) (SEE CONFERENCE INDEX)

[112] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

[113] Farkas, Hershel M., and Irwin Kra. Riemann surfaces. Springer New York, 1992.

[114] E. Witten, "Notes On Supermanifolds and Integration," arXiv:1209.2199 [hep-th]; "Notes On Super Riemann Surfaces And Their Moduli," arXiv:1209.2459 [hep-th];

[115] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B **102**, 27 (1981).

[116] I. A. Batalin and G. A. Vilkovisky, Phys. Rev. D **28**, 2567 (1983) [Erratum-ibid. D **30**, 508 (1984)].

[117] G. Barnich, F. Brandt and M. Henneaux, Commun. Math. Phys. **174**, 57 (1995) [hep-th/9405109].

[118] G. Barnich, F. Brandt and M. Henneaux, Commun. Math. Phys. **174**, 93 (1995) [hep-th/9405194].

[119] M. Henneaux, Nucl. Phys. Proc. Suppl. **18A**, 47 (1990).

[120] M. Henneaux and C. Teitelboim, Princeton, USA: Univ. Pr. (1992) 520 p

[121] J. Gomis, J. Paris and S. Samuel, Phys. Rept. **259**, 1 (1995) [hep-th/9412228].

[122] J. M. L. Fisch and M. Henneaux, Commun. Math. Phys. **128**, 627 (1990).

[123] E. Witten, Nucl. Phys. B **268**, 253 (1986).

[124] E. Witten, Nucl. Phys. B **276**, 291 (1986).

[125] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [hep-th/9711200].

[126] R. Gopakumar and C. Vafa, Adv. Theor. Math. Phys. **3**, 1415 (1999) [hep-th/9811131].

[127] H. Ooguri and C. Vafa, Nucl. Phys. B **641**, 3 (2002) [hep-th/0205297].

[128] E. Witten, Prog. Math. **133**, 637 (1995) [hep-th/9207094].

[129] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[130] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [hep-th/9802109].