MONOGAMY OF QUANTUM CORRELATIONS

By

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

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2. "Conditions for monogamy of quantum correlations in multipartite systems", Asutosh Kumar, *Phys. Lett. A*, **2016**, *380*, 3044-3050.

3. "Forbidden regimes in the distribution of bipartite quantum correlations due to multiparty entanglement", Asutosh Kumar, H. S. Dhar, R. Prabhu, A. Sen(De), and U. Sen, *Phys. Lett. A*, **2017**, *381*, 1701-1709.

4. "Information complementarity in multipartite quantum states and security in quantum cryptography", A. Bera, Asutosh Kumar, D. Rakshit, R. Prabhu, A. Sen(De), and U. Sen, *Phys. Rev. A*, **2016**, *93*, 032338. [Only part of the results used in this thesis. Will possibly be used in other theses by other authors.]

5. "Lower bounds on violation of monogamy inequality for quantum correlation measures", Asutosh Kumar and H. S. Dhar, *Phys. Rev. A*, **2016**, *93*, 062337.

6. "Conclusive identification of quantum channels via monogamy of quantum correlations", Asutosh Kumar, S. S. Roy, A. K. Pal, R. Prabhu, A. Sen(De), and U. Sen, *Phys. Lett. A*, **2016**, *380*, 3588-3594.

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Journal

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9. "Cohering power of quantum operations", K. Bu, Asutosh Kumar, L. Zhang, and J. Wu, *Phys. Lett. A*, **2017**, *381*, 1670-1676.

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Asutosh Kumar

Asutosh Kumar

to my
parents: Usha Devi & Shrawan Kumar Tiwari
grandparents: Rambha Devi & Jagarnath Tiwari

purnamadah purnamidam purnat purnamudachyate |
purnasya purnamadaya purnamevavashishyate ||
(that is perfect. this is perfect. out of perfect comes only perfect. even after taking perfect out of perfect, what remains is perfect.–Vedic Scriptures)

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Contents

SYNOPSIS

Quantification of quantum correlations [1–3] is crucial because entanglement [1] and other quantum correlations [2, 3] are indispensable resources in quantum information processing [4–6] and analysis and detection of co-operative quantum phenomena in various interacting quantum many-body systems. [7–11]. Although a number of quantum correlation measures for bipartite systems have been studied extensively in last few decades, there has not been much investigation of multipartite quantum correlations, owing at least partially to the difficulty in defining and dealing of multipartite correlations. The notion of "monogamy" [12–14] has proved to be a useful tool in exploring multipartite non-classical correlations, and it appears to be *the trait* of multipartite quantum systems. Several measures of quantum correlations like squared concurrence, squared negativity, squared quantum discord, global quantum discord, squared entanglement of formation, Bell inequality violation, EPR steering, contextual inequalities, etc. exhibit the monogamy property. By monogamy of quantum correlations, we mean a certain restriction on distribution of quantum correlations of one fixed party with other parties of a multipartite system. In particular, if party A in a tripartite system ABC is maximally quantum correlated with party B, then A cannot be correlated at all to the third party C. In other words, monogamy forbids arbitrary sharing of quantum correlations among the constituents of a multipartite quantum system. This is a non-classical property in the sense that such constraints are not observed for classical correlations. Monogamy is a vibrant area of research because of its potential applications in several areas in quantum information. Monogamy of non-classical correlations has found important applications in quantum information including quantum key distribution, classification of quantum states, co-operative phenomena in many-body systems, black hole physics, identification of quantum channels, distinguishing phases of many-body systems, etc.

We state below the main results obtained in the proposed thesis.

- We investigate which non-classical correlation measures and classes of quantum states, under what conditions, satisfy monogamy in Refs. [15,16].
- We obtain upper as well as lower bounds on "monogamy scores" [17–19].
- We illustrate that monogamy can be used to "conclusively" identify quantum channels in Ref. [20].
- We study complementarity and distribution of quantum coherence measures [21].

The content of the thesis is split into nine Chapters.

In Chapter 1 (Introduction), we briefly review the concept of "information" and its inevitable role in the development of humankind. We understand that the knowledge of correlations is a useful information, and a resource. In Chapter 2 (Quantum Correlations), we discuss measures of entanglement and other quantum correlations useful for our purpose. We succintly recall the notion of monogamy of quantum correlations in Chapter 3 (Monogamy of Quantum Correlations).

In Chapter 4 (Effect of Large Number of Parties on Monogamy), we establish a sufficient condition for monogamy of arbitrary quantum correlation measures of multipartite quantum states, and find that higher number of parties enforce monogamy of quantum correlations for almost all states. The result is generic and holds for a large class of quantum correlation measures. Nonetheless, we identify important zero Haar measure classes of pure states that remain non-monogamous with respect to quantum discord and quantum work-deficit, irrespective of the number of qubits. In Chapter 5 (Conditions for Monogamy of Quantum Correlations), we investigate conditions under which monogamy is preserved for functions of quantum correlation measures. We prove that a monogamous measure remains monogamous on raising its power, and a non-monogamous measure remains non-monogamous on lowering its power. We also prove that monogamy of a convex quantum correlation measure for arbitrary multipartite pure quantum state leads to its monogamy for mixed states in the same Hilbert space. Monogamy of squared negativity for mixed states and that of entanglement of formation follow as corollaries of our results.

Monogamy poses a fundamental restriction on the sharability and distribution of quantum resources. Moreover, the "monogamy score" has been used as an important figure of merit in studies on multipartite quantum states, identification of quantum channels, and distinguishing phases of many-body systems. Since the monogamy score is a difficult quantity to compute and estimate for generic quantum states and generic quantum correlations, it is interesting to derive both upper and lower bounds on the monogamy score. The existence of non-trivial bounds on the monogamy score is an important aspect in the study of various quantum information protocols. In Chapter 6 (Bounds on Monogamy Scores), we show that the monogamy scores for different quantum correlation measures are bounded above by functions of genuine multipartite entanglement for a large majority of multiqubit pure states. We analytically show that the bound is universal for three-qubit states and identify the conditions for its validity in higher number of qubits. The results show that the distribution of bipartite quantum correlations in a multipartite system is restricted by its genuine multipartite entanglement content. We also obtain lower bounds on monogamy scores, for quantum correlation measures that violate monogamy inequality, using the complementarity relation between a measure of bipartite quantum correlation and purity of a part of the system in question.

Conclusive identification of quantum channels via monogamy of quantum correlations is discussed in Chapter 7 (Conclusive Identification of Quantum Channels via Monogamy). We investigate the action of local and global noise on monogamy of quantum correlations, when monogamy scores are considered as the measurable quantities, and three-qubit systems are subjected to global noise and various local noisy channels, namely, amplitude-damping, phase-damping, and depolarizing channels. We show that the dynamics of monogamy scores corresponding to negativity and quantum discord, in the case of generalized W states as inputs to the noisy channels, can exhibit non-monotonic dynamics, which is in contrast to the monotonic decay of monogamy scores when generalized GHZ states are exposed to noise. We quantify the persistence of monogamy against noise via a characteristic value of the noise parameter, and show that depolarizing noise destroys monogamy of quantum correlation faster compared to other noisy channels. We demonstrate that the negativity monogamy score is more robust than the discord monogamy score, when the noise is of phase-damping type. We also investigate the variation of monogamy with increasing noise for arbitrary three-qubit pure states as inputs. Finally, depending on these results, we propose a two-step protocol, which can conclusively identify the type of noise applied to the quantum system, by using generalized GHZ and generalized W states as resource states.

Quantum coherence, arising from the superposition principle of quantum mechanics, is a potentially important resource for quantum information processing tasks. In Chapter 8 (Quantum Coherence: Reciprocity and Distribution), we observe that the reciprocity between coherence and mixedness of a quantum state is a general extensive feature in the sense that it is satisfied by large spectra of measures of coherence and those of mixedness. Numerical investigation reveals that the percentage of quantum states satisfying the additivity relation of coherence increases with increasing number of parties, with increment in rank of the quantum states, and with raising of power of the coherence measures under investigation. We also study distribution of coherence in "X"-states. For Dicke states, while the normalized measures of coherence violate the additivity relation, the unnormalized ones satisfy the same.

In Chapter 9 (Summary and Conclusion), we provide a brief summary of all the results presented in the thesis.

We believe that the results obtained in the proposed thesis will be important at the fundamental level to understand both the qualitative and quantitative aspects of quantum correlations in multipartite systems, and will be useful in quantum information processings.

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Chapter 1

Introduction

Mathematics is a part of **informatics** *that is a part of anthropology that is a part of biology that is a part of physics. Informatics refers to all aspects of information, the role of information in human affairs as well as the various techniques used to build useful information systems [22].*

Because information plays an important role in almost every human activity, humans have been amassing information since ages, in every possible form, to develop and better their lives day by day. We owe our existence and survival to our information-handling skills. Sailing through different ages–stone age, bronze age, iron age–the primitive human has learnt to exercise her/his mind, and identify and exploit the resources to enliven her/his life, and enlighten herself/himself. Peeping into the era of science, engineering and technology, the primitive human paved her/his ways, and pioneered the remotest aspects of the mysterious ambience and the "fathomless" universe. Despite centuries of relentless, untiring efforts of numerous inquisitive minds across the globe, the story is far from complete. Even the fundamental space-time is daunting. The quest is on for the missing elusive information in an attempt to amend, repair, and settle the tale.

The twenty-first century may have marked the onset of the "formal" information age. It is information that is being materialized and commercialized. Information is all potent. We frequently hear the words *Data*, *Information* and *Knowledge*, and use them interchangeably, though they are apparently not the same [1]. Various information systems–differing wildly in form and application–

[1] *Data* refers to raw or unprocessed facts and figures without any added interpretation or analysis. Because of its raw and possibly unorganized form, data may sometimes appear random, overly simple, or abstract.

essentially serve a common purpose which is to convert data into meaningful information which in turn enables us to build knowledge. But what is information? "Information" comes from Latin word *informationem* or *informare* (in + formare), which means "to give form, shape, or character" to something. The Oxford English Dictionary defines information as "the action of informing; formation or molding of the mind or character, training, instruction, teaching; communication of instructive knowledge". It is processed and analyzed data that has meaning, and serves a purpose for the recipient. Information is a valuable resource for development, and is power when used and applied effectively. The lack of information can impact negatively on the development process. In the wake of information and communication technologies, information has become a commodity. The successful use of information as a resource for development depends in large part on knowledge of the nature of information. We, therefore, need to identify the attributes of information. What is the nature of information [22–25]? Is it physical, engineered, material or non-material? The debate is still on.

Information, much like temperature, is physical because it resides or manifests in a physical body. Landauer discusses the physical nature of information in Ref. [26]. He argues that information is inevitably tied to a physical representation and therefore to restrictions and possibilities related to the laws of physics and the parts available in the universe. Shannon's classical information theory [27], long back in 1948, already shocked and revolutionized the world. Its time to move ahead and embrace the "weirdness" of quantum information theory, and be prepared to absorb further shocks.

Quantum mechanics, being the theory of microparticles, is deep-rooted in all branches of physics, and has had an enormous theoretical and technological impact since its advent. To appreciate this point, consider the invention of the diode, the transistor, the laser, the electron microscope, and the magnetic resonance imaging among numerous remarkable applications of quantum mechanics. The invention of the transistor, in particular, has imparted huge impetus to com-

Information is processed and analyzed data that has meaning, and serves a purpose for the recipient.

The concepts of *knowledge* is important. Data becomes information, which in turn is processed as knowledge, then finally manifested in a physical way as decisions and actions. Knowledge is a combination of information, experience and insight that may benefit the individual or the organisation. Each of above three concepts are integral to the other two and without one, the others would cease to exist.

putational power. Who can deny the enormous impact of computers in scientific investigations, and on everyday life! Considerations of portability issue, increase in computational power [2], and technological advancements have profusely boosted the silicon industry, in its efforts to build computers with reduced size of integrated circuits. This miniaturization, in particular, is related to speedup of the computations. It requires cramming more components (transistors or logic gates) on a single integrated-circuit chip of fixed dimension. Miniaturization has been quantified empirically by *Moore's law [28, 29]: the number of transistors that may be placed on a single integrated-circuit chip doubles approximately every 18-24 months*. At the present time, this exponential growth has not yet saturated and Moore's law still enjoys its validity. However, we will soon reach a regime when quantum effects will become unavoidably dominant. Thus, despite affordable cost of manufacturing chips owing to technological breakthroughs, quantum effects could bring Moore's law to an end. In the 1980's, Feynman suggested that a computer based on quantum logic would be ideal for simulating quantum-mechanical systems, and quantum mechanics can help in the solution of basic problems of computer science. The synergy between computer science and quantum physics has marked an unprecedented revolution, and has led us into the *quantum information age*. This synergy offers completely new opportunities and promises exciting advances in both fundamental science and technological application. *A quantum computer is the conception of a computing device based on quantum logic, that is, it would process the information and perform logic operations by exploiting the laws of quantum mechanics.* The unit of quantum information is *qubit*. Physically, a qubit is a two-level quantum system, like the two spin states of a quantum spin-$\frac{1}{2}$ particle, the horizontal and vertical polarization states of a single photon, or the ground and excited states of an atom. A quantum computer is a many-qubit system, whose evolution can be controlled, and a quantum computation is a unitary operation that acts on the many-qubit state of the quantum computer, preceded by an initialization stage of the quantum device and succeeded by a measurement of the same. Classical computers work on *Boolean logic*–a form of algebra in which all values are reduced to either TRUE or FALSE. The Boolean logic gates take two or more inputs and give a single output. Thus one cannot figure out the inputs from the output. Hence information is erased, and the algebra is *irreversible*. It

[2]Computational power is a measure of the number of floating-point operations per second (flops) a computer can perform. A floating-point operation is any mathematical operation (such as addition, subtraction, multiplication, division) or assignment that involves floating-point numbers. The execution or running time of floating-point operations is typically larger than binary integer operations.

causes dissipation of energy. According to *Landauer's principle [30–32]:* each time a single bit of information is erased, at least $k_B T$ ln2 amount of energy is dissipated into the environment, where k_B is Boltzmann's constant and T is the temperature of the environment in which the computer resides. This can be replaced by a reversible logic that involves storing of the same information in an ancillary "junk". The situation is similar in the quantum case, except that there are far many operations allowed.

A quantum computer offers numerous remarkable advantages over a classical computer in terms of potentially efficient quantum algorithms, secure cryptography, etc. The power of quantum information and quantum computers is due to a few typical quantum concepts [4–6], such as "*indeterminism*, *interference*, *uncertainty*, *superposition* and *entanglement*". Quantum theory is *indeterministic* because it makes probabilistic predictions, even for pure inputs. *Quantum interference* requires a wave description, in the sense of quantum superposition, for every fundamental particle of matter, such as an electron. Uncertainty is at the heart of quantum physics, and the *uncertainty principle* states that it is impossible for a quantum state to exist for which we know the precise values of two *incompatible* observables, that is observables without a common eigenstate. In quantum information science, cryptography protocols [12, 33–35] exploit the *no-cloning theorem* [36, 37], an atomic version of the uncertainty principle, to detect an eavesdropper, and to ascertain the security of quantum key distribution. The *superposition principle*–a result of the linearity of quantum theory–states that a quantum particle can be in a linear combination of two or more allowed states. There is an inherent *quantum parallelism* associated with the superposition principle–a quantum computer can process a large number of classical inputs, represented as different terms of a superposition, in a single run.

After two classical systems have interacted, they are in well-defined individual states, if we have complete information about the whole system. On the other hand, after two quantum particles have interacted, in general, they cannot be described independently of each other. There can be purely quantum correlations [1–3] between two such, in principle spatially-separated, particles. Quantum correlations, in particular entanglement, are central to many quantum information and communication protocols [4–6], such as quantum dense coding [38, 39], which allows transmission of two bits of classical information through the (local) manipulation and sending of only one of two entangled qubits, and quantum teleportation [40], which allows the transfer of an unknown state of one quantum system to another through an entangled channel. Quantum teleportation and dense coding have been realized experimentally over relatively long distances [41–45].

From everyday experience, we learn that arbitrary operations cannot do an assigned job. Specific resources–like allowed operations, "free assets" that one can use at will, and some "force" or catalyst in a prescribed amount–are needed to carry out a particular task. Therefore, to establish a quantitative theory of any physical resource, one needs to address the following fundamental issues: (i) the characterization or unambiguous definition of resource, (ii) the quantification of the resource, and (iii) the transformation or manipulation of quantum states under the imposed constraints [46–49]. Several useful quantum resources like purity [50], entanglement [1,51–55], reference frames [56,57], thermodynamics [58, 59], asymmetry [60], etc. have been identified and quantified until now. Recently, Baumgratz *et al.* in Ref. [61], provided a quantitative theory of coherence as a new quantum resource, using the formalism already established for entanglement, thermodynamics and reference frames. Quantum correlations [1–3] play an integral role in quantum information tasks [4–6]. Hence, quantification of quantum correlations, entanglement [62] in particular, is crucial in quantum information processing and quantum computation [4], in describing area laws [11], for understanding quantum phase transitions, and detecting cooperative quantum phenomena [7–10]. Although a number of correlation measures for bipartite systems have been studied extensively in the last few decades, there has not been much investigation of multipartite quantum correlations owing to the inherent difficulty in defining multipartite correlations. *The notion of "monogamy" [14, 63] has proved to be a useful tool in exploring multipartite non-classical correlations, and it appears, like entanglement, to be the trait of multipartite quantum systems.* Several measures of quantum correlations like squared concurrence [14, 63], squared negativity [16, 64–66], squared quantum discord [67], global quantum discord [68, 69], squared entanglement of formation [70, 71], Bell inequality [72–74], EPR steering [75, 76], contextual inequalities [77, 78], etc. exhibit the monogamy property. *By monogamy of quantum correlations, we mean certain restrictions on distribution of quantum correlations of one fixed party with other parties of a multipartite system.* In other words, monogamy forbids arbitrary sharing of quantum correlations among the constituents of a multipartite quantum system. This is a non-classical property in the sense that such constraints are maximally violated for classical correlations. Monogamy of non-classical correlations has found important applications in quantum information, ranging through quantum key distribution or quantum cryptography [79–82], classification of quantum states [83–86], black hole physics [87,88], etc.

In Chapter 2, we discuss measures of entanglement and other quantum correlations useful for our purposes. In Chapter 3, we succintly recall the notion of monogamy of quantum correlations. In Chapter 4, we study the effect of considering a large number of parties on monogamy of quantum correlations [15], and in Chapter 5, we investigate conditions under which monogamy is preserved for functions of quantum correlation measures [16].

Monogamy poses a fundamental restriction on the sharability and distribution of quantum resources, and can be quantified by using the notion of monogamy score [14, 89]. However, the monogamy score is a difficult quantity to compute and estimate for generic quantum states and generic quantum correlations. It is therefore interesting to derive both upper and lower bounds on the monogamy score. The existence of non-trivial bounds on the monogamy score is an important aspect in the study of various quantum information protocols. We obtain both upper and lower bounds on monogamy scores in Chapter 6. These results are published in Refs. [17–19].

Conclusive identification of quantum channels via monogamy of quantum correlations is discussed in Chapter 7. We investigate the action of local and global noise on monogamy of quantum correlations, when monogamy scores are considered as the "observables"–physical quantities– and three-qubit systems are subjected to global noise and various local noisy channels, namely, amplitude-damping, phase-damping, and depolarizing channels. We also investigate the variation of monogamy with increasing noise, when the input states are generalized GHZ state, generalized W state, and arbitrary three-qubit pure states. Finally, depending on these results, we propose a two-step protocol, which can conclusively identify the type of noise applied to the quantum system, by using generalized GHZ and generalized W states as resource states. These results are published in Ref. [20].

Moreover, since quantum coherence (a resource in quantum information theory like entanglement) arises from the superposition principle, and is the premise of quantum correlations in multipartite systems, it is important to study its quantification, its relationship with other properties like mixedness, and the distribution of coherence in multipartite systems. In Chapter 8, we observe that the reciprocity between coherence and mixedness of a quantum state is a general feature in the sense that it is satisfied by a large spectra of measures of coherence and of mixedness. We show analytically that a measure of coherence that satisfies the additivity relation–monogamy-type relation between coherences of different parts of the system–does satisfy the same on raising its power, and a measure of coherence that violates the additivity relation does violate the same on lowering its power. Numerical investigation reveals that the percentage of quantum states satisfying the additivity relation of coherence increases with increasing number of parties, with increment in the rank of quantum states, and with raising of the power of coherence measures under investigation. We also study distribution of coherence in "X"-states. For Dicke states, while the normalized measures of coherence violate the additivity relation, the unnormalized ones satisfy the same. These results are published in Ref. [21].

We provide a brief summary of all the results presented in the thesis in Chapter 9.

Chapter 2

Quantum Correlations

2.1 Introduction

Before we embark upon classical and quantum correlations and the concerned measures in considerable detail, let us first try to understand, in general terms, what we mean by correlation. "Correlation" has its etymological origin in the Latin word *correlatio* derived from 'cor: together' + 'relatio: relation'. The concept of correlation can be defined as *a statistical measure that indicates the degree or extent to which two or more quantities (variables) are interdependent or fluctuate together*. It is a *reciprocal* relation between two or more variables such that systematic changes in the value of one variable are accompanied by systematic changes in the other.

Let us begin by considering two classical systems, observed respectively by Alice (A) and Bob (B). Each system consists in drawing a coin from a box containing two coins C_1 and C_2 (say, a rupee and a yen). Suppose also that Alice and Bob are stationed far apart, before they are allowed to look at their coins. Alice now looks at her coin. She is then able to know about the coin with Bob. The same implication holds if Bob looks at his coin. This is a typical situation where two systems are connected. Note however that this does not entail that the no-signalling principle, which states that information cannot go faster than the speed of light, is violated. This is because Alice's observation does not let Bob instantly know or expect his coin. If Alice and Bob both agree to repeat this experiment many times with "$C_1 = +1$" and "$C_2 = -1$", then they would observe, under fair shuffling of the coins, the following facts hold:
(i) On average, C_1 and C_2 will be drawn equal number of times for both Alice and Bob: $\langle C_A \rangle = 0 = \langle C_B \rangle$.
(ii) The product $C_A C_B$ always equals -1: $\langle C_A C_B \rangle = -1$.

Clearly, $\langle C_A C_B \rangle \neq \langle C_A \rangle \langle C_B \rangle$, indicating that Alice's and Bob's observations are correlated. The quantity, $\langle C_A C_B \rangle - \langle C_A \rangle \langle C_B \rangle$, called the "statistical" correlation, is nonzero for correlated observations. The correlations remained intact even after Alice and Bob got spatially separated because the coins didn't change during the trip. The quantity $\langle C_A C_B \rangle - \langle C_A \rangle \langle C_B \rangle = -1$, in the example with the coins, imply "perfect anticorrelation". Speaking in terms of a probability distribution $p_{AB}(a, b)$ for two variables $A = a$ and $B = b$, the variables are completely uncorrelated iff the probability factorizes, i.e., $p_{AB}(a, b) = p_A(a) p_B(b)$. Otherwise, they are correlated.

In quantum mechanics, situations in physical systems are described by state vectors in a Hilbert space. [We are assuming here that complete information about the physical situation is available so that it can be representated by a state vector. We will soon consider the more formal case.] Assume that Alice's system is described by a Hilbert space $\mathcal{S}_A$, and similarly Bob's system is described by a Hilbert space $\mathcal{S}_B$. The two systems can be combined into a single composite system using *tensor product*: $\mathcal{S}_{AB} = \mathcal{S}_A \otimes \mathcal{S}_B$. To define the tensor product space $\mathcal{S}_{AB}$, it is enough to specify its basis vectors. Alice may possess apparatuses to implement a set of operators corresponding to physical observables that act on her system, and Bob may possess a similar set for his system. They may also possess apparatuses for preparation and measurement of quantum states, and implementation of observables that can only be measured when both parties are together. The above framework can be generalized to multipartite systems. Systems with N components can be described by tensor products of N state spaces, and so on.

Correlations, classical as well as quantum, present in quantum systems play a significant role in quantum physics. A composite quantum system is, for many purposes, characterized by the correlations between its constituting parts. Differentiating quantum correlations [1–3] from classical ones has arrested a lot of attention, since it has been established that quantum ones can be a useful resource for quantum information processing tasks including those in quantum communication and possibly quantum computation. Moreover, it turns out to be an effective tool to detect cooperative quantum phenomena in many-body physics [9, 10]. Current technological developments ensure detection of quantum correlations in several physical systems [90–92]. However, the boundary between classical and quantum correlations is not sharp [93]. Quantum systems possess, in addition to classical correlation, a number of non-classical correlations [1–3] including entanglement [62], steerability [94–96], (Bell) nonlocality [97–99], and information-theoretic correlations beyond entanglement [2,3]. Before we discuss these, let us briefly recall the description of quantum states.

A *product state* is the result of completely independent preparations by the comprising parties of a bipartite or multipartite quantum system, in which case each uses her/his own apparatus to prepare, say, a spin [1]. By preparation, each subsystem of a product state behaves independently of the other. That is, if one constituent party does an experiment on her/his own subsystem, the result, even if sent to the other parties, does not affect the actions at other subsystems. Employing the principles of quantum mechanics, we can superpose basis vectors to get more general state vectors than just product states. The property that a composite state cannot be expressed as a product of individual states is referred to as *entanglement*. The two factors responsible for entanglement in the quantum world are the superposition principle [2] and the tensor product structure for composite systems.

The simplest quantum system is a qubit or a quantum bit, an arbitrary state $|\varphi\rangle$ of which can be representated as a superposition of "classical" bits 0 and 1, i.e., $|\varphi\rangle = c_0|0\rangle + c_1|1\rangle$ with normalization $|c_0|^2 + |c_1|^2 = 1$. The most general state of an n-qubit system in the *computational basis*, then, has the form

$$|\psi^{(n)}\rangle = \sum_{\substack{0\leq x<2^n \\ x_i\in\{0,1\}}} \alpha_x |x_n x_{n-1}\cdots x_1\rangle \equiv \sum_{x=0}^{2^n-1} \alpha_x |x^{(n)}\rangle, \tag{2.1}$$

with normalization $\sum_x |\alpha_x|^2 = 1$. If $|\psi^{(n)}\rangle$ is product, then $|\psi^{(n)}\rangle = \otimes_{i=1}^n (a_i|0\rangle + b_i|1\rangle)$. This imposes severe constraints on the expansion coefficients, α_x, as shown below. For $n = 2$ qubits,

$$|\psi^{(2)}\rangle = \sum_{x=0}^{3} \alpha_x |x^{(2)}\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle. \tag{2.2}$$

[1]Due to the existence of the phenomenon of *entanglement swapping* [42, 100–102], it is necessary to define the concept of independent preparations carefully. In particular, it is disallowed that the comprising parties, in the two-party case, meet two other parties, with the latter two given the option to interact.

[2]The superposition principle, a consequence of the *linearity* of quantum mechanics, states that if $\{|\phi_i\rangle\}_{i=1}^n$ are solutions of a quantum system, then the (complex) linear combination, $|\psi\rangle = \sum_{i=1}^n c_i|\phi_i\rangle$, such that $\sum_{i=1}^n |c_i|^2 = 1$, is also a valid solution of the system.

Assuming that $|\psi^{(2)}\rangle$ is product, we get

$$\begin{aligned} |\psi^{(2)}\rangle &= (a_1|0\rangle + b_1|1\rangle)(a_2|0\rangle + b_2|1\rangle) \\ &= a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle. \end{aligned} \tag{2.3}$$

This is possible when and only when $\alpha_0\alpha_3 = \alpha_1\alpha_2$. A similar argument is also true for an n-qubit system. The state $|\psi^{(n)}\rangle$ in Eq. (2.1) is product if and only if

$$\alpha_0\alpha_{2^n-1} = \alpha_1\alpha_{2^n-2} = \cdots = const. \tag{2.4}$$

The quantity *const* is fixed by the normalization. The condition in Eq. (2.4) is hardly ever met, and hence causes the number of entangled states to grossly outnumber that of product states. That is, the volume of product states is vanishingly small.

The two-qubit states, $|\psi\rangle_{AB} = \alpha|01\rangle_{AB} + \beta|10\rangle_{AB}$ where $\alpha,\ \beta \neq 0$ and $|\alpha|^2+|\beta|^2 = 1$, are entangled states because they cannot be written as a product of two independent qubit states: $|\psi\rangle_{AB} \neq (\alpha_A|0\rangle + \beta_A|1\rangle)(\alpha_B|0\rangle + \beta_B|1\rangle)$ with $|\alpha_i|^2 + |\beta_i|^2 = 1$ ($i = A, B$). An example of a bipartite "maximally" entangled state is the celebrated *singlet* [3] state

$$|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{AB}. \tag{2.5}$$

A maximally entangled state is fascinating, in the sense that we know as much as can possibly be quantum mechanically known about the composite system, and yet know nothing about the individual subsystems [4]. Entanglement has many astonishing uses from unbreakable encryption scheme in quantum world [4–6] to teleportation [40–43] to deep issues in a possible unification of general relativity and quantum mechanics, such as the black hole information paradox [103–113] and the firewall paradox [114–116].

An observer can have either complete or incomplete information about a physical system, and thereby its quantum mechanical state can be either *pure* or *mixed*. A convenient language of representing general quantum states is the

[3] This is one of the four orthonormal maximally entangled Bell states, and is anti-symmetric with respect to exchange of subsystems A and B. The remaining Bell states, $|\psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)_{AB}$ and $|\phi^{\pm}\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{AB}$, are symmetric with respect to exchange of subsystems A and B, and constitute the *triplets*. The Bell states are eigenvectors of the operator $\vec{\sigma_A}.\vec{\sigma_B}$ with eigenvalues ± 1.

[4] This property of maximally entangled states can also be seen as their definition.

density matrix [5] (operator) ρ. An n-party pure state, in terms of density matrix, is given by $\rho^{(n)} = |\psi^{(n)}\rangle\langle\psi^{(n)}|$, and $\mathrm{Tr}((\rho^{(n)})^2) = 1$. A mixed state, on the other hand, is described by an ensemble $\{p_k, |\psi_k^{(n)}\rangle\}$ with $p_k \geq 0$ and $\sum_k p_k = 1$. This may, for instance, be an outcome of state preparation. Its density matrix is given by $\rho^{(n)} = \sum_k p_k |\psi_k^{(n)}\rangle\langle\psi_k^{(n)}|$. For (non-pure) mixed states, $\mathrm{Tr}((\rho^{(n)})^2) < 1$. A (non-pure) mixed state always has an infinite number of pure-state decompositions.

A two-party state, which is possibly mixed, is said to be entangled if it cannot be expressed as a convex sum of product pure states. Convex sums of pure product states are referred to as separable states [118]. The motivation for this definition is that separable states are exactly those states which can be prepared by local (quantum) operations and classical communication [6] (LOCC). The case of more than two parties is far richer. An n-party possibly mixed state will be called entangled if it cannot be written as a convex sum of n-party product pure states. However, it is clear that there can be a large number of sub-classes within this class of entangled states with respect to their preparation procedures and entanglement content.

2.2 Measures of quantum correlations

Quantum information theory provides a plethora of quantum correlation measures [1–3] (see also Chapter 15 of Ref. [117]). The general properties that these measures should exhibit, have been studied extensively [120]. These measures are broadly categorized into two classes. One is the "entanglement-separability" class, encompassing various measures of quantum entanglement in both bipartite and multipartite domain [1]. The other is the quantum information-theoretic regime [2, 3], consisting of quantum correlations such as quantum dis-

[5] The density matrix ρ, for both pure and mixed states, is (i) Hermitian: $\rho^\dagger = \rho$, (ii) non-negative: $\rho \geq 0$, and (iii) has unit trace: $\mathrm{Tr}(\rho) = 1$. The unit trace emphasizes the fact that the total probability of outcomes in a potential measurement is conserved. On the hyperball of states, while pure states lie on the surface, mixed states correspond to points in the bulk. For more details on the *geometry of quantum states*, see Ref. [117].

[6] *Local operations and classical communication* (LOCC) are a special class of operations that were introduced to understand entanglement and other quantum correlations from the resource perspective in quantum information tasks. In a two- or more-party quantum system, a general LOCC protocol consists of both local quantum operations and classical communication. LOCC is a subset of *separable operations* (SEP) [119]. A separable operation takes the form $\Lambda(.) = \sum_k M_k(.)M_k^\dagger$, where $M_k = A_k^{(1)} \otimes A_k^{(2)} \otimes \cdots \otimes A_k^{(n)}$, and n is the number of the parties, with $\sum_k M_k^\dagger M_k = I$, I being the identity on the space on which the argument is defined.

cord [121, 122], and various "discord-like" measures, that are more general in nature than entanglement. In this section, we present brief descriptions of various quantum correlation measures that we will employ in our investigations.

2.2.1 Entanglement measures

Here, we briefly review the quantum correlation measures belonging to *entanglement-separability* paradigm. These measures vanish for separable states. Moreover, they do not increase, on average, under local quantum operations and classical communication. We restrict ourselves to short discussions on the following entanglement measures: entropy of entanglement, entanglement of formation (EoF), concurrence, logarithmic negativity, and generalized geometric measure (GGM).

Entropy of Entanglement

The von Neumann entropy [123, 124] for an arbitrary density matrix ρ on $\mathbb{C}^d$ is defined by

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho) = -\sum_i \lambda_i \log_2 \lambda_i \leq \log_2 d, \tag{2.6}$$

where λ_i's are the eigenvalues of ρ. Hence the (quantum) von Neumann entropy is the (classical) Shannon entropy [27] of the spectrum of ρ. It varies from zero for pure states to $\log_2 d$ for the maximally mixed state $\rho = \mathbb{I}/d$. The von Neumann entropy is also the smallest possible *mixing entropy* [7] [117]. It (i) is *nonnegative*: $S(\rho) \geq 0$, (ii) is *concave*: if $\rho = \sum_i p_i \varrho_i$, with $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$, then $S(\rho) \geq \sum_i p_i S(\varrho_i)$, (iii) is *subadditive*: $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$, with equality iff $\rho_{AB} = \rho_A \otimes \rho_B$, (iv) satisfies the *Araki-Lieb inequality*: $|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB})$, and (v) obeys *strong subadditivity*: $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$, or equivalently $S(\rho_A) + S(\rho_B) \leq S(\rho_{AC}) + S(\rho_{BC})$. The Araki-Lieb inequality, when combined with subadditivity, becomes a triangle inequality.

[7] The Shannon entropy [27], in classical probability theory, is a distinguished function of probability distribution as it represents a unique classical state. However, a quantum state (density matrix ρ) can be associated to many probability distributions because in addition to its affiliation to many possible measurements including generalized *positive-operator-valued measurements* (POVMs), a density matrix can also be expressed as a mixture of pure normalized states in numerous ways. A special mixture of ρ is its eigenstate decomposition, $\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|$, where $\{\lambda_i, |e_i\rangle\}$ is the eigenspectrum of ρ. Any probability vector $\vec{p}$, in $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, is related to the spectral vector $\vec{\lambda}$ by $\vec{p} = B\vec{\lambda}$, where B is a *bistochastic* matrix ($B_{ij} \geq 0$, $\sum_i B_{ij} = 1$ and $\sum_j B_{ij} = 1$), and Shannon entropy is a *Schur concave* function, so that we have $H(\vec{p}) = -\sum_i p_i \log_2 p_i \geq -\sum_i \lambda_i \log_2 \lambda_i = H(\vec{\lambda}) = S(\rho)$.

The entropy of entanglement of a two-party *pure* state, $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$, is given by the von Neumann entropy of the either reduced density matrix $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ or $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. Both yield the same result, as can be seen by using the *Schmidt decomposition* [125–130] [8] for arbitrary two-party pure states. For Bell states, $S(\rho_A^{\mathrm{Bell}}) = 1$. The motivation for this definition comes, for example, from the fact that the asymptotic rate at which singlets can be extracted by LOCC from a pure bipartite state ρ_{AB} is given by $S(\rho_A) = S(\rho_B)$. The operation is asymptotically reversible.

Concurrence

The concurrence [132, 133] of any two-qubit state, ρ_{AB}, is given by,

$$\mathcal{C}(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{2.7}$$

where λ_i's are the square roots of the eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$, arranged in a decreasing order. σ_y is the Pauli spin matrix and $\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$, with the complex conjugation being done in the computational basis. For two-qubit pure states, $|\psi\rangle_{AB}$, the concurrence is given by $2\sqrt{\det \rho_A}$. In case of pure states in $\mathbb{C}^2 \otimes \mathbb{C}^d$, the concurrence is again given by $2\sqrt{\det \rho_A}$. For mixed states in $\mathbb{C}^2 \otimes \mathbb{C}^d$, one can use the convex roof extension to define the same. The importance of concurrence lies in the fact that for arbitrary two-qubit states, it can be used to derive a closed form of the entanglement of formation [132–134] defined below.

Entanglement of Formation

Let ρ_{AB} be a bipartite density matrix. The entanglement of formation (EoF) [132–134] of ρ_{AB} is defined as $\mathcal{E}(\rho_{AB}) = \min \sum_i p_i S(\rho_A^i)$, where $S(\rho_A^i)$ is the entropy of entanglement of $|\psi_i\rangle_{AB}$, with $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i| = \sum_i p_i \rho_{AB}^i$ being a pure-state decomposition of ρ_{AB}. The minimization in the above expression is over all possible pure-state decompositions of ρ_{AB}. In general, this minimization

[8] An arbitrary two-party pure state $|\psi\rangle_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can be re-written as, $|\psi\rangle_{AB} = \sum_{i=1}^{\min\{d_A, d_B\}} \sqrt{p_i}|e_A^i\rangle|f_B^i\rangle$, where $\{|e_A^i\rangle\}$ and $\{|f_B^i\rangle\}$ are orthonormal *Schmidt bases* for subsystems A and B respectively, and $\{p_i\}$–*Schmidt coefficients*–are nonnegative real numbers satisfying $\sum_i p_i = 1$. The number of nonzero p_is is called the *Schmidt number*. If the Schmidt number is 1, then $|\psi\rangle_{AB}$ is product, otherwise it is entangled. For the Schmidt decomposition with Schmidt number k, the amount of entanglement can also be quantified by the *participation ratio*, $\xi = \frac{1}{\sum_{i=1}^k p_i^2}$ [131]. Clearly, ξ is bounded between 1 (for product state) and k (for maximally entangled state): it is close to 1 if a single term dominates the Schmidt decomposition, whereas it equals k if all terms in the decomposition have the same weight (that is, all $p_i = \frac{1}{k}$).

is an uphill task. For an arbitrary two-qubit state, ρ_{AB}, there exists a closed form for the entanglement of formation [132, 133], in terms of the concurrence, $\mathcal{C}$, as

$$\mathcal{E}(\rho_{AB}) = h\left(\frac{1+\sqrt{1-\mathcal{C}^2(\rho_{AB})}}{2}\right), \tag{2.8}$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the Shannon (binary) entropy. Note that while EoF is a concave function of $\mathcal{C}^2$, the squared EoF is a convex function of $\mathcal{C}^2$, lies between 0 and 1, and vanishes for separable states. $\mathcal{E}$ and $\mathcal{C}$ are monotonically increasing functions of each other, and hence for two-qubit states, they can be interchangeably used as measures of entanglement.

Logarithmic Negativity

Logarithmic negativity (LN), an entanglement monotone for deterministic LOCC maps, but that is not convex, is defined in terms of negativity [135, 136]. Negativity is based on the fact that the *partial transpose* of a separable bipartite state preserves positivity [137]. However, there may be positive partial transposed (PPT) *bound entangled* states for which negativity is zero [138, 139]. For a bipartite state, ρ_{AB}, the negativity, $\mathcal{N}(\rho_{AB})$, is defined as the absolute value of the sum of the negative eigenvalues of $\rho_{AB}^{T_A}$, where $\rho_{AB}^{T_A}$ denotes the partial transpose of ρ_{AB} with respect to the subsystem A. It is expressed as

$$\mathcal{N}(\rho_{AB}) = \frac{\|\rho_{AB}^{T_A}\|_1 - 1}{2}, \tag{2.9}$$

where $\|M\|_1 \equiv \mathrm{tr}\sqrt{M^\dagger M}$ is the trace-norm of the matrix M. Negativity is an easily computable convex entanglement measure that is not a monotone. It has been shown in Ref. [140] that the negativity of entangled two-qubit mixed states can never exceed its concurrence, and respects the following inequality

$$\sqrt{(1-\mathcal{C})^2+\mathcal{C}^2} - (1-\mathcal{C}) \le \mathcal{N} \le \mathcal{C}. \tag{2.10}$$

Negativity equals the lower bound iff the state is a rank-2 "quasi-distillable" state. Further, since partial transposition of an arbitrary bipartite quantum state, ρ_{AB} on $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$, cannot have more than $(d_A-1)(d_B-1)$ number of negative eigenvalues, and since all the eigenvalues of partial transposition lie within $\left[-\frac{1}{2}, 1\right]$ [141, 142], we have the following upper bound on negativity: $\mathcal{N}(\rho_{AB}) \le \frac{(d_A-1)(d_B-1)}{2}$. The logarithmic negativity, in terms of negativity, is defined as

$$E_{\mathcal{N}}(\rho_{AB}) = \log_2[2\mathcal{N}(\rho_{AB}) + 1]. \tag{2.11}$$

For two-qubit states, a strictly positive LN implies that the state is entangled and "distillable" [137–139], whereas vanishing LN implies that the state is separable [138].

Generalized Geometric Measure

Thus far, we have discussed bipartite entanglement measures. Below we describe a genuine multiparty entanglement measure for multiparty pure states called *generalized geometric measure* (GGM) [143, 144] (cf. [145–147]). A multiparty pure state is said to be genuinely multipartite entangled if it cannot be expressed as a product across any bipartition of the system. The Greenberger-Horne-Zeilinger (GHZ) [148] and the W [83, 149] states are quintessential examples of genuinely multipartite entangled states. The GGM ($\mathcal{G}$) of the state $|\psi\rangle$ can be defined as

$$\mathcal{G}(|\psi\rangle) = 1 - \max_{\{|\phi\rangle\}} |\langle\phi|\psi\rangle|^2, \tag{2.12}$$

where the maximization is performed over the set of states $\{|\phi\rangle\}$ that are not genuinely multiparty entangled. From the definition, it follows that the quantity $\mathcal{G}(|\psi\rangle)$ vanishes for all states that are bi-separable across any partition and is non-zero otherwise. Further, it is a valid entanglement monotone in that it is non-increasing, on average, under local operations and classical communication. The optimization in defining GGM can be simplified in terms of the maximization of the Schmidt coefficients across all possible bi-partitions, allowing the quantity to be calculated for arbitrary pure states in arbitrary dimensions, and for any number of parties. In terms of the Schmidt coefficients, the GGM of $|\psi\rangle$ can be defined as [143, 144]

$$\mathcal{G}(|\psi\rangle) = 1 - \max\left\{\lambda^2_{A:B} | A \cup B = \{1, 2, ..., n\}, A \cap B = \emptyset\right\}, \tag{2.13}$$

where, $\lambda_{A:B}$ is the maximal Schmidt coefficient across the bipartition $A : B$ of the state $|\psi\rangle$. This allows one to compute $\mathcal{G}(|\psi\rangle)$ in terms of the eigenvalues of its different reduced density matrices.

2.2.2 Information-theoretic measures

Although many of the quantum information protocols are assisted by entanglement, there are several related phenomena for which presence of entanglement is not required or believed to be unnecessary [119, 150–164]. Information-theoretic quantum correlations–quantum correlations beyond entanglement–may potentially explain such phenomena. Thus, information-theoretic correlations are more general in nature than entanglement. In this section, we succinctly review the quantum correlation measures belonging to the *information-theoretic* paradigm relevant to our study. We confine ourselves to the two popular information-theoretic measures, namely, quantum discord and quantum work-deficit.

Quantum Discord

Quantum discord for a bipartite state is defined as the difference between the "total correlation" and the "classical correlation" of the state. The total correlation is defined as the quantum mutual information of ρ_{AB}, which is given by [165] (see also [166–168])

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{2.14}$$

where $S(\varrho) = -\mathrm{tr}(\varrho \log_2 \varrho)$ is the von Neumann entropy of the quantum state ϱ. The classical correlation is based on the conditional entropy, and is defined as

$$\mathcal{J}^{\leftarrow}(\rho_{AB}) = S(\rho_A) - S(\rho_{A|B}). \tag{2.15}$$

Here,

$$S(\rho_{A|B}) = \min_{\{B_i\}} \sum_i p_i S(\rho_{A|i}) \tag{2.16}$$

is the conditional entropy of ρ_{AB}, conditioned on a measurement performed by B with a rank-one projection-valued measurement $\{B_i\}$, producing the states $\rho_{A|i} = \frac{1}{p_i}\mathrm{tr}_B[(\mathbb{I}_A \otimes B_i)\rho(\mathbb{I}_A \otimes B_i)]$, with probability $p_i = \mathrm{tr}_{AB}[(\mathbb{I}_A \otimes B_i)\rho(\mathbb{I}_A \otimes B_i)]$. $\mathbb{I}$ is the identity operator on the Hilbert space of A. Quantum discord is given by [121, 122]

$$\mathcal{D}^{\leftarrow}(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \mathcal{J}^{\leftarrow}(\rho_{AB}). \tag{2.17}$$

Here, the superscript "$\leftarrow$" on $\mathcal{J}^{\leftarrow}(\rho_{AB})$ and $\mathcal{D}^{\leftarrow}(\rho_{AB})$ indicates that the measurement is performed on the subsystem B of the state ρ_{AB}. Similarly, if measurement is performed on the subsystem A of the state ρ_{AB}, one can define quantum discord as

$$\mathcal{D}^{\rightarrow}(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \mathcal{J}^{\rightarrow}(\rho_{AB}). \tag{2.18}$$

In the absence of any ambiguity, we will simply use

$$\mathcal{D}(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \mathcal{J}(\rho_{AB}). \tag{2.19}$$

Quantum Work-deficit

Quantum work-deficit [169–172] is another quantum correlation measure belonging to the information-theoretic paradigm. It is defined as the difference between the amount of pure states that can be extracted under global operations and pure product states that can be extracted under local operations, in closed systems for which addition of the corresponding pure states as ancillas are not allowed. The number of pure qubits that can be extracted from ρ_{AB} by *closed global operations* (CGO) is given by

$$I_G(\rho_{AB}) = N - S(\rho_{AB}), \tag{2.20}$$

where $N = \log_2(\dim \mathcal{H})$. Here, CGO are any sequence of unitary operations and dephasing of the given state ρ_{AB} by using a set of projectors $\{P_i\}$, i.e., $\rho \to \sum_i P_i \rho_{AB} P_i$, where $P_i P_j = \delta_{ij} P_i$, $\sum_i P_i = \mathbb{I}$, with $\mathbb{I}$ being the identity operator on the Hilbert space $\mathcal{H}$ on which ρ_{AB} is defined. The number of qubits that can be extracted from a bipartite quantum state ρ_{AB} under *closed local operations and classical communication* (CLOCC), is given by

$$I_L(\rho_{AB}) = N - \inf_{\Lambda \in CLOCC} [S(\rho'_A) + S(\rho'_B)], \tag{2.21}$$

where $S(\rho'_A) = S(\mathrm{tr}_B(\Lambda(\rho_{AB})))$ and $S(\rho'_B) = S(\mathrm{tr}_A(\Lambda(\rho_{AB})))$. Here CLOCC consists of local unitaries, local dephasings, and sending dephased states from one party to another. The quantum work-deficit is the difference between the "work", $I_G(\rho_{AB})$, extractable by CGO, and that by CLOCC, $I_L(\rho_{AB})$:

$$\mathcal{W}(\rho_{AB}) = I_G(\rho_{AB}) - I_L(\rho_{AB}). \tag{2.22}$$

Since it is difficult to compute this quantity for arbitrary states, we restrict our analysis only to CLOCC, where measurement is done at any one of the subsystems.

Remark. The difficulty in the computation of quantum discord or quantum work-deficit arises due to the optimization involved in the definition of classical correlation of the state ρ_{AB} [173]. In the case of a pure bipartite state ρ_{AB}, quantum discord is shown to be equal to $S(\rho_A)$, the von Neumann entropy of the local density matrix ρ_A [174]. The same is true for quantum work-deficit. On the other hand, there are only a few examples of mixed bipartite states, for which they can be obtained analytically [175–179]. For an arbitrary mixed bipartite state ρ_{AB}, computation of quantum discord and quantum work-deficit involves adaptation of numerical optimization techniques [180–183]. In the case of a $\mathbb{C}^2 \otimes \mathbb{C}^d$ system, where measurement is performed on the qubit, the rank-1 projectors, $\{P_i = |\phi_i\rangle\langle\phi_i|\}$, $i = 1, 2$, can be parametrized as

$$\begin{aligned}|\phi_1\rangle &= \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle,\\ |\phi_2\rangle &= -e^{-i\phi}\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle.\end{aligned} \tag{2.23}$$

The optimization, in this case, is to be performed over the space of the real parameters (θ, ϕ), where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

2.3 Summary

In this Chapter, we have discussed measures of entanglement and other quantum correlations that we will use in the following chapters. Entanglement measures vanish for product and separable states, and do not increase, on average, under local quantum operations and classical communication. We have briefly discussed entanglement measures such as entropy of entanglement, entanglement of formation, concurrence, logarithmic negativity, and generalized geometric measure. We have also presented short discussions on information-theoretic quantum correlations, which are quantum correlations beyond entanglement.

Chapter 3

Monogamy of Quantum Correlations

3.1 Introduction

An important characterization of composite quantum systems is by the correlations, both classical and quantum, between its constituting parts. Correlations present in quantum systems play a significant role in quantum physics. Identifying classical and non-classical correlations in composite quantum systems has arrested a lot of attention in the quantum information community, since it has been established that quantum ones can be a useful resource for quantum information processing tasks including those in quantum communication and possibly quantum computation [4–6]. A fundamental property that distinguishes quantum correlations [1–3] from classical correlations is the set of restrictions on their distribution among several parties in a quantum system. Classical correlations can be distributed among any number of parties, with each pair attaining the maximum possible correlation. For instance, for the classical mixture

$$\rho_{ABC} = \frac{1}{2}|000\rangle\langle 000| + \frac{1}{2}|111\rangle\langle 111|, \tag{3.1}$$

the reduced density matrices $\rho_{XY} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$ $(X, Y \in \{A, B, C\})$ share the maximal classical correlation. However, for quantum correlations, there exist strong constraints on its sharability among the different parties of a multipartite system. For example, if quantum systems with Alice and Bob have a perfect quantum correlation, it can quite generally be proven that the two-party state is local unitarily related to the singlet state, $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{AB}$. In that case, neither of the parties can be correlated at all to a third system Charlie, even classically. This is because $|\psi^-\rangle_{AB}$ being a pure state has null entropy, and therefore zero correlation with any other system. That is, the quantum system AB, in this case, can only be in a product state with any third system C. This indicates that there is a drastic limitation on the distribution of entanglement for maximally entangled pure states. A perfect quantum correlation precludes the possibility of classical correlations to other systems, and conversely, a perfect classical correlation between two systems will forbid either of the system from being entangled to other systems.

However, when Alice and Bob are not maximally quantum correlated, the restriction is not so severe. In particular, one system may be entangled with two others simultaneously. This feature of constrained distribution of correlations form the basis for the concept of monogamy of quantum correlations.

Monogamy, derived from 'mono: one, only, single' + 'gamous: marriage, engagement', in normal terms means allegiance or loyalty to one. For example, in an election we cannot cast vote to two or more parties, the usual social practice of having one spouse at a time, etc. Monogamy forms a connecting theme in the exquisite variety in the space of quantum correlation measures. As discussed earlier, monogamy is a "non-classical" concept and classical correlation can violate monogamy to the maximal extent. In general, monogamy implies that if two systems are strongly correlated with respect to a nonclassical quantity, they can only be weakly correlated, with respect to the same quantity, to a third system. These qualitative statements have been quantified [12–14, 63], and while quantum correlations, in general, are expected to be qualitatively monogamous, they may violate the proposed quantitative monogamy relations for some states. The monogamy property is an important feature in quantum information theory [4] and in essence captures the "trade-offs" between various quantifiers of quantum and classical properties [174]. Quantification of the monogamy property of quantum correlations in multiparty systems is not very straightforward. This is in part due to the fact that quantum correlation shared among arbitrary parties in a multiparty system is not always computable, making the study of its distribution among several parties extremely difficult. However, there are ongoing efforts to overcome this constraint [184–188]. Further, characterization of both bipartite and multipartite quantum correlations in higher-dimensional mixed states is not well-developed [1–3]. Nevertheless, various attempts have been made to systematically quantify the monogamy of quantum correlations. Recent developments on the monogamy relation of quantum correlations [13–15,63,64,68,70–74,77–79,84,85,174,189–200] have provided an effective tool to characterize the multipartite nature of quantumness present in a composite quantum system.

A seminal result on monogamy was obtained in [14] for squared concurrence [132, 133] in three-qubit states, and this led to an important quantitative monogamy relation. This relation is schematically captured in Fig. 3.1. It was

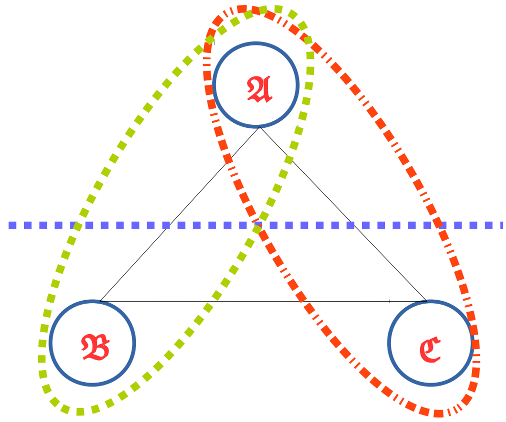


Figure 3.1: Schematic illustration of the concept of monogamy: $\mathcal{Q}(\rho_{AB}) + \mathcal{Q}(\rho_{AC}) \leq \mathcal{Q}(\rho_{A:BC})$. This inequality is satisfied by, e.g., squared concurrence.

demonstrated that, for a three-qubit state, ρ_{ABC}, the sum of the squared concurrence between qubits A and B, and that between qubits A and C, is bounded above by the squared concurrence between qubit A and the joint subsystem BC. That is,

$$\mathcal{C}^2(\rho_{AB}) + \mathcal{C}^2(\rho_{AC}) \leq \mathcal{C}^2(\rho_{A:BC}). \tag{3.2}$$

Hence, the monogamy property was captured in the form of an inequality, known as the "monogamy inequality". An advantage of this inequality is that it is a multiparty property expressed in terms of bipartite quantum correlation measures, with the latter being relatively well-understood, at least for two-qubit systems. The inequality was later shown to hold for squared concurrence in multiqubit quantum states [63] [1]. Recent results on monogamy of quantum correlation have shown that the monogamy inequality is satisfied for any state for a sufficiently large power of a large class of quantum correlation measure [200]. It is also satisfied for almost all states for a large number of quantum correlation measures where a moderately large number of parties are considered [15].

The monogamy property of quantum correlations has been shown to be important in several aspects in quantum information theory, like foundations

[1]Ref. [63] also demonstrated the inequality for arbitrary mixed three-qubit states. Ref. [14] contained the demonstration for pure three-qubit states.

of quantum mechanics [73, 74], security of quantum key distribution (QKD) [12, 80–82, 201–203], teleportation [204, 205], quantum dense coding [206], quantum steering [75, 76], many-body physics [207–209], and black-hole information theory [87, 88]. Further, it has been used to characterize three-qubit genuinely multiparty entangled states [85, 89] and distinguish Bell-like orthonormal bases [86]. See also Refs. [210]. Experimental investigation of this property has also been initiated [211].

3.2 Definitions

In this section, we briefly discuss about the monogamy inequality and the corresponding "monogamy score" for any quantum correlation measure.

Suppose that $\mathcal{Q}$ is a bipartite quantum correlation measure. If for a multipartite quantum system described by state $\rho_{AB_1B_2\ldots B_n}$, the inequality [14]

$$\sum_{j=1}^{n} \mathcal{Q}(\rho_{AB_j}) \leq \mathcal{Q}(\rho_{A(B_1\cdots B_n)}), \tag{3.3}$$

holds, then the state $\rho_{AB_1B_2\ldots B_n}$ is said to be monogamous under the quantum correlation measure $\mathcal{Q}$. Otherwise, it is non-monogamous. Here we call the party A as a "nodal" observer. Moreover, the deficit between the two sides is referred to as *monogamy score* [89], and is given by

$$\delta_{\mathcal{Q}} = \mathcal{Q}(\rho_{A(B_1\cdots B_n)}) - \sum_{j=1}^{n} \mathcal{Q}(\rho_{AB_j}). \tag{3.4}$$

Quantifying the monogamy relation via the monogamy score [212] by using bipartite measures of quantum correlations provides a pathway for understanding and quantifying multiparty quantum correlations with bipartite ones. The monogamy score can be interpreted as "residual" quantum correlation of the bipartition $A : B_1 \cdots B_n$ of the $n+1$-party state that cannot be accounted for by the conjunction of the quantum correlations of their two-qubit reduced density matrices, ρ_{AB_j}, separately [14].

The monogamy inequality in (3.3) is satisfied by a host of nonclassical quantities including those related to Bell [72–74] and contextual [77, 78] inequalities, quantum steering witnesses [75, 76], and dense-coding capacities [206]. It is satisfied by entanglement monotones such as squared-concurrence [14,63], squared entanglement of formation [70, 71], and squared-negativity [16, 64], for multi-qubit systems and by squashed entanglement [174, 189] in arbitrary dimensions (also see [190–192, 204, 205, 213]). However, the monogamy inequality is not obeyed by all quantum correlation measures even for three-qubit quantum states [16,84,85,195,197,214]. Entanglement measures, such as entanglement of formation [134] and logarithmic negativity [135,136] do not satisfy monogamy [16,214]. Further, information-theoretic measures of quantum correlation, such as quantum discord [121, 122] and quantum work-deficit [169–172], are also known to violate the monogamy inequality, even for three-qubits [84, 85, 195, 197].

It should be noted here that the monogamy inequality in (3.3) is just one constraint on the distribution of quantum correlations. While the status of monogamy and its violation have traditionally been seen with respect to the relation (3.3), it is clear that a constraint of the form

$$\sum_{j=1}^{n} \mathcal{Q}(\rho_{AB_j}) \leq b, \tag{3.5}$$

also implies monogamy, in some form, of the state $\rho_{AB_1 \cdots B_n}$ for the quantum correlation $\mathcal{Q}$, whenever b, which can be state-dependent, is at least sometimes strictly less than n. Here we are assuming that the value of $\mathcal{Q}$ lies between zero and unity. Numerical evidence of such a limitation was observed for entanglement of formation and concurrence in Ref. [71] for three-qubit systems. There can be more general and tighter monogamy relations than in (3.3). Attempts have been made to address this question from different perspectives [16, 186, 197, 215] recently.

3.3 Summary

Monogamy is a distinctive trait of quantum states. In this Chapter, we have defined the notion of monogamy of quantum correlations, and discussed ways to quantify it.

Chapter 4

Effect of Large Number of Parties on Monogamy

4.1 Introduction

In Chapter 3, we reviewed the notion of monogamy of quantum correlations. In majority of the previous works on monogamy, only the case of tripartite pure states have been investigated. In this Chapter, we address the question of monogamy for quantum states of an arbitrary number of parties for arbitrary quantum correlation measures. We first provide a sufficient condition for a given state of an arbitrary number of parties to satisfy the monogamy relation. Using this analytical result and further numerical justification, we provide evidence that entanglement measures like distillable entanglement [134] and relative entropy of entanglement [51, 52], which are as yet intractable analytically as well as numerically while being physically vital, are monogamous for almost all quantum states for a moderate number of parties. The results show that even though there is violation of monogamy in the case of three-qubit systems, most of the quantum correlation measures are generically monogamous for almost all states in any multipartite quantum system of a moderately high number of parties. Furthermore, we show that if a measure is monogamous for all tripartite pure states, it is monogamous for all quantum states of an arbitrary number of parties, provided the measure is convex. We next go on to identify classes of multiparty pure quantum states that are non-monogamous for an arbitrary number of parties for certain quantum correlations. These classes, which have zero Haar volumes and hence are not covered in the random Haar searches, include the multiparty W [83, 149], the Dicke states [216], and the symmetric states, and the corresponding quantum correlations are quantum discord and quantum work-deficit. We provide sufficient conditions for a multiparty quantum state to be non-monogamous. Precisely, we show that the multiqubit states with vanishing *tangle* [1] [14] violate the monogamy relation for quantum discord with a certain nodal observer, provided the sum of the unmeasured conditional entropies of the nodal observer, conditioned on the non-nodal observers, is negative. Monogamy plays an important role for security of quantum cryptography [79, 217] and can be a useful tool to estimate multipartite quantum correlation present in a many-body system [211, 218, 219]. The results obtained in this Chapter shed light on secure communication in moderately large to large quantum networks.

This Chapter is arranged as follows. In Sec. 4.2, we numerically establish that a large class of quantum correlation measures eventually become monogamous for almost all pure states with the increase in the number of parties. The zero Haar volume regions containing non-monogamous states are identified in Sec. 4.3 where we also find sufficient conditions for violation of monogamy for given multisite quantum states. We present a brief summary of the Chapter in Sec. 4.4.

4.2 Appearance of monogamy as generic in large systems

In most of the previous works on monogamy, the status of the monogamy relation for different quantum correlation measures has been considered only for three-qubit pure states. Exceptions include squared concurrence [63] and squared negativity [64] which are monogamous for any number of qubits. Here we address the question of monogamy for an arbitrary number of parties for any bipartite quantum correlation measure. Let us first prove here a sufficient condition that has to be satisfied by any entanglement measure for it to be monogamous. In Ref. [120], bipartite entanglement measures satisfying certain reasonable axioms, were referred to as "good" measures. Here, we slightly broaden the definition, and call an entanglement measure as "good" if it is lower than or equal to the entanglement of formation and it is equal to the local von Neumann entropy for pure bipartite states.

Theorem 4.2.1 Monogamy for given states: *If entanglement of formation is monogamous for a pure quantum state of an arbitrary number of parties, any bipartite "good" entanglement measure is also monogamous for that state.*

[1]Here, tangle is not squared concurrence but the monogamy score of the same.

Proof. Let $\delta_{\mathcal{E}}$, the monogamy score of entanglement of formation, be non-negative for any n-party pure quantum state $|\psi_{12\cdots n}\rangle$. Consider now any "good" bipartite entanglement measure $\mathcal{Q}$. Therefore, when entanglement of formation is monogamous, we have

$$\mathcal{Q}(|\psi_{1(2\cdots n)}\rangle) = S(\rho_1^{\psi}) = \mathcal{E}(|\psi_{1(2\cdots n)}\rangle) \geq \sum_{j=2}^{n} \mathcal{E}_{1j} \geq \sum_{j=2}^{n} \mathcal{Q}_{1j}, \tag{4.1}$$

where $\rho_1^{\psi} = \mathrm{tr}_{2\cdots n}|\psi\rangle\langle\psi|$. Hence the proof. ■

The status of the monogamy relation, as obtained via the numerical simulations, for all computable entanglement measures are summarized below in Table 4.1, which clearly indicates that several entanglement measures which are non-monogamous for three-qubit pure states, become monogamous, when one increases the number of parties by a relatively moderate amount. Some of the results from Table 4.1 are also depicted in Fig. 4.1. Moreover, Table 4.1 shows

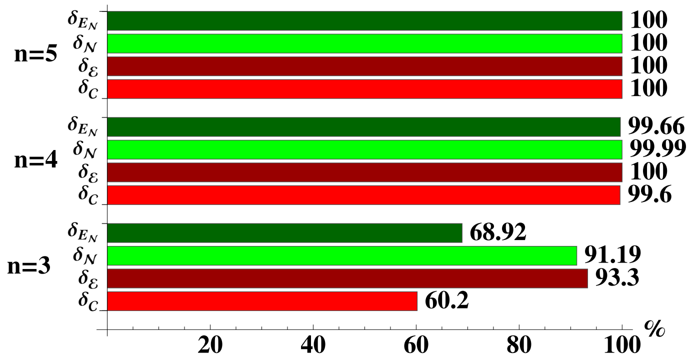


Figure 4.1: Percentage bar-diagram for monogamy scores of entanglement measures for pure n-party states. 10^5 random pure quantum states are generated Haar-uniformly for each n. All notations are the same as for Table 4.1.

that entanglement of formation is monogamous for almost all pure states of four qubits. Utilizing this result along with that from Theorem 4.2.1, we obtain that relative entropy of entanglement [51, 52], regularized relative entropy of entanglement [220], entanglement cost [134, 221], distillable entanglement [134, 222], all of which are not generally computable, are monogamous for almost all pure states of four or more qubits. Just like for entanglement measures, percentages of randomly chosen pure states satisfying monogamy, increase also for all information-theoretic quantum correlation measures with the increase in the number of parties (see Table 4.2 and Fig. 4.2). Note here that uniform Haar searches may tend to become inefficient when we consider a large number of

n	$\delta_{\mathcal{C}}$	$\delta_{\mathcal{E}}$	$\delta_{\mathcal{E}^2}$	$\delta_{\mathcal{N}}$	$\delta_{\mathcal{N}^2}$	$\delta_{E_{\mathcal{N}}}$	$\delta_{E^2_{\mathcal{N}}}$
3	60.2	93.3	100	91.186	100	68.916	100
4	99.6	100	100	99.995	100	99.665	100
5	100	100	100	100	100	100	100

Table 4.1: Monogamy percentage table for entanglement measures. We randomly choose 10^5 pure quantum states uniformly according to the Haar measure over the complex 2^n-dimensional Hilbert space for each n, for $n = 3, 4,$ and 5. Here n, therefore, denotes the number of qubits forming the system from which the pure quantum states are chosen. $\delta_{\mathcal{C}}, \delta_{\mathcal{E}}, \delta_{\mathcal{N}}, \delta_{E_{\mathcal{N}}}$ respectively denote the monogamy scores of concurrence, entanglement of formation, negativity and logarithmic negativity, while $\delta_{\mathcal{E}^2}, \delta_{\mathcal{N}^2}, \delta_{E^2_{\mathcal{N}}}$ are the monogamy scores of the squares of these measures. The numbers shown are percentages of the randomly chosen states that are monogamous for that case.

parties (cf. [223]). However, they are efficient for the few qubit systems that we consider here, especially for $n = 3$ and $n = 4$.

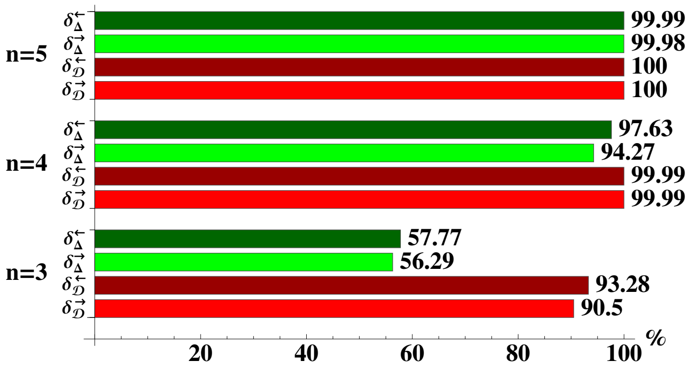


Figure 4.2: Percentage bar-diagram for monogamy scores of information-theoretic quantum correlation measures [quantum discord ($\mathcal{D}$) and quantum work-deficit ($\triangle$)] for pure n-party states. 10^5 random pure quantum states are generated for each n. See Table 4.2.

Let us now specify certain sets of properties that are sufficient for any quantum correlation measure to satisfy the monogamy relation for an arbitrary number of parties in arbitrary dimensions, when it is monogamous for smaller number of parties [63]. Let us consider an n-party state, $|\psi_{12\ldots n}\rangle$ in d dimensions, in which we make the partition, in such a way that the final state is always tripartite. In this case, we have the following theorem.

Theorem 4.2.2 Monogamy for given measures: *If $\mathcal{Q}$ is monogamous for all tripartite quantum states in $d \otimes d \otimes d_C$ where $d_C = d^m$, $m \leq n-2$, then $\mathcal{Q}$ is monogamous for pure quantum states of the n parties.*

n	$\delta_{\mathcal{D}}^{\rightarrow}$	$\delta_{\mathcal{D}^2}^{\rightarrow}$	$\delta_{\mathcal{D}}^{\leftarrow}$	$\delta_{\mathcal{D}^2}^{\leftarrow}$	$\delta_{\triangle}^{\rightarrow}$	$\delta_{\triangle^2}^{\rightarrow}$	$\delta_{\triangle}^{\leftarrow}$	$\delta_{\triangle^2}^{\leftarrow}$
3	90.5	100	93.28	100	56.29	88.10	57.77	89.56
4	99.997	100	99.99	100	94.27	99.99	97.63	100
5	100	100	100	100	99.98	100	99.99	100

Table 4.2: Percentage table for quantum states satisfying the monogamy relation for information-theoretic paradigm measures. We randomly chose 10^5 pure quantum states uniformly according to the Haar measure over the complex 2^n-dimensional Hilbert space. The numbers shown are percentages of the randomly chosen states that are monogamous for that case.

Proof. Suppose that the dimension of the third party is d^{n-2}, consisting of $(n-2)$ parties, say $3, \dots, n$, each being of dimension d. The monogamy of such a 3-party state, $|\psi_{1:2:3\dots n}\rangle$, implies that

$$\begin{aligned}
\mathcal{Q}(|\psi_{1(23\dots n)}\rangle) &\geq \mathcal{Q}(\rho^{\psi}_{12}) + \mathcal{Q}(\rho^{\psi}_{1(3\dots n)}) \\
&\geq \mathcal{Q}(\rho^{\psi}_{12}) + \mathcal{Q}(\rho^{\psi}_{13}) + \mathcal{Q}(\rho^{\psi}_{1(4\dots n)}) \\
&\geq \mathcal{Q}(\rho^{\psi}_{12}) + \mathcal{Q}(\rho^{\psi}_{13}) + \sum_{k=4}^{n} \mathcal{Q}(\rho^{\psi}_{1k}),
\end{aligned} \tag{4.2}$$

where the second inequality is obtained by applying the monogamy relation for the tripartite state $\rho^{\psi}_{1:3:4\dots n}$, with the third party having $(n-3)$ parties, and the third inequality is by applying such monogamy relations recursively. Here, the local density matrices are denoted by ρ^{ψ} with the appropriate suffixes determined by the parties that are not traced out. ■

Theorem 4.2.3 *If $\mathcal{Q}$ is monogamous for all tripartite pure quantum states in $d \otimes d \otimes d_C$ where $d_C = d^m,\ m \leq n-2$, then $\mathcal{Q}$ is monogamous for all quantum states, pure or mixed, of the n parties, provided $\mathcal{Q}$ is convex, and $\mathcal{Q}$ for mixed states is defined through the convex roof approach.*

Proof. Consider a mixed state $\rho_{123\dots n}$ in the tri-partition $1 : 2 : 3 \dots n$ and let $\{p_i, |\psi^i_{1(2\dots n)}\rangle\}$ be the optimal decomposition that attains the convex roof of $\mathcal{Q}(\rho_{1(2\dots n)})$. Therefore,

$$\begin{aligned}
\mathcal{Q}(\rho_{1(23\dots n)}) &= \mathcal{Q}\left(\sum_i p_i |\psi^i_{1(2\dots n)}\rangle\langle\psi^i_{1(2\dots n)}|\right) \\
&= \sum_i p_i \mathcal{Q}(|\psi^i_{1(2\dots n)}\rangle).
\end{aligned} \tag{4.3}$$

Due to the assumed monogamy over pure states, we have

$$\begin{aligned}\mathcal{Q}(\rho_{1(2\ldots n)}) &\geq \sum_i p_i \left(\mathcal{Q}(\rho_{12}^{\psi^i}) + \mathcal{Q}(\rho_{1(3\ldots n)}^{\psi^i})\right) \\ &= \sum_i p_i \mathcal{Q}(\rho_{12}^{\psi^i}) + \sum_i p_i \mathcal{Q}(\rho_{1(3\ldots n)}^{\psi^i}),\end{aligned} \tag{4.4}$$

which, due to convexity of $\mathcal{Q}$, reduces to

$$\mathcal{Q}(\rho_{1(2\ldots n)}) \geq \mathcal{Q}(\rho_{12}) + \mathcal{Q}(\rho_{1(3\ldots n)}). \tag{4.5}$$

The result follows now by applying Theorem 4.2.2 and convexity of $\mathcal{Q}$. ■

4.3 A zero measure class of non-monogamous states

In Sec. 4.2, we presented evidence that almost all multiparty states for even a moderate number of parties are monogamous with respect to all quantum correlation measures. The qualification "almost" is important and necessary, firstly because the uniform Haar searches do not take into account of violations of the corresponding property (monogamy, here) on hypersurfaces (more generally, on sets of zero Haar measure). Secondly, and more constructively, we identify a class of multiparty states that are non-monogamous with respect to information-theoretic quantum correlation measures for an arbitrary number of parties.

For an n-party pure state $|\psi_{12\cdots n}\rangle$, we have the relation

$$\mathcal{S}(\rho_{1|j}^{\psi}) + D^{\leftarrow}(\rho_{1j}^{\psi}) \geq \mathcal{E}(\rho_{1i}^{\psi}), \tag{4.6}$$

with $i \neq j$, for the corresponding three-party portion of $|\psi_{12\ldots n}\rangle$ [174]. If we choose $i = j+1$ for all n except for $j = n$ and choose $i = 2$ for $j = n$, we have

$$\sum_{j=2}^{n} \mathcal{E}(\rho_{1j}^{\psi}) \leq \sum_{j=2}^{n} \left(\mathcal{S}(\rho_{1|j}^{\psi}) + \mathcal{D}^{\leftarrow}(\rho_{1j}^{\psi})\right). \tag{4.7}$$

The above inequalities between EoF, quantum discord and conditional entropy hold for arbitrary states with arbitrary number of parties.

Theorem 4.3.1 *Multiparty pure states with vanishing tangle violate the monogamy relation for quantum discord with a certain nodal observer, provided the sum of the unmeasured conditional entropies, conditioned on all non-nodal observers, is* *negative.*

Proof. Let us consider an n-party state $|\psi_{12\cdots n}\rangle$ for which the tangle, $\tau(\rho_{12\ldots n}) = \mathcal{C}^2(\rho_{1(2\cdots n)}) - \sum_{j=2}^n \mathcal{C}^2(\rho_{1j})$, vanishes, i.e., the state saturates the monogamy relation for $\mathcal{C}^2$. Hence $\sum_{j=2}^n \mathcal{C}^2(\rho_{1j}^\psi) = \mathcal{C}^2(|\psi_{1(2\cdots n)}\rangle)$, whereby we have

$$\sum_{j=2}^{n} \mathcal{E}(\rho_{1j}^\psi) \geq \mathcal{E}(|\psi_{1(2\cdots n)}\rangle). \tag{4.8}$$

Since $\mathcal{E}(|\psi_{1(2\cdots n)}\rangle) = S(\rho_1^\psi) = \mathcal{D}^{\leftarrow}(|\psi_{1(2\cdots n)}\rangle)$, by using the inequality between entanglement of formation and quantum discord in (4.7), we have

$$\begin{aligned} \sum_{j=2}^{n} \left(\mathcal{S}(\rho_{1|j}^\psi) + \mathcal{D}^{\leftarrow}(\rho_{1j}^\psi) \right) \ &\geq \ \sum_{j=2}^{n} \mathcal{E}(\rho_{1j}^\psi) \\ &\geq \ \mathcal{E}(|\psi_{1(2\cdots n)}\rangle) = S(\rho_1^\psi) = \mathcal{D}^{\leftarrow}(|\psi_{1(2\cdots n)}\rangle). \end{aligned} \tag{4.9}$$

Therefore, the discord monogamy score has the following bound:

$$\delta_{\mathcal{D}}^{\leftarrow} = \mathcal{D}^{\leftarrow}(|\psi_{1(2\cdots n)}\rangle) - \sum_{j=2}^{n} \mathcal{D}^{\leftarrow}(\rho_{1j}^\psi) \leq \sum_{j=2}^{n} \mathcal{S}(\rho_{1|j}^\psi). \tag{4.10}$$

Hence, if states with vanishing tangle, additionally satisfies $\sum_{j=2}^n \mathcal{S}(\rho_{1|j}^\psi) < 0$, quantum discord is non-monogamous for those states. ■

The non-monogamy of quantum discord for three-qubit W states [84, 85] is a special case of Theorem 4.3.1, given in Eq. (4.10).

It can be easily checked that the n-qubit W state, $|W_n\rangle$ [83, 149], given by

$$|W_n\rangle = \frac{1}{\sqrt{n}}(|0\ldots 1\rangle + \ldots + |1\ldots 0\rangle), \tag{4.11}$$

remain non-monogamous with respect to quantum discord for an arbitrary number of parties. Let us also consider the Dicke state [216]

$$|W_n^r\rangle = \frac{1}{\sqrt{\binom{n}{r}}} \sum_{permuts} |\underbrace{00\ldots0}_{n-r}\underbrace{11\ldots1}_{r}\rangle, \tag{4.12}$$

where $\sum_{permuts}$ represents the unnormalized equal superposition over all $\binom{n}{r}$ combinations of $(n-r)$ $|0\rangle$'s and r $|1\rangle$'s. Using the property that the Dicke state is permutationally invariant with respect to the subsystems, an analytic

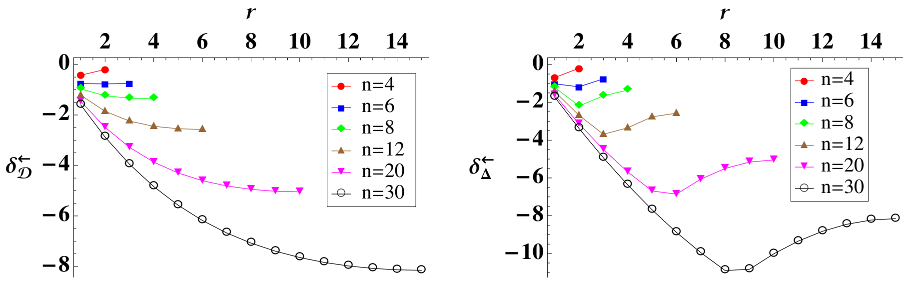


Figure 4.3: Left panel: Non-monogamy of Dicke states with respect to quantum discord ($\mathcal{D}$). For fixed n (> 6), $\delta_{\mathcal{D}}^{\leftarrow}$ decreases with increasing r and it decays exponentially for large n. Right panel: Non-monogamy of Dicke states with respect to quantum work-deficit (Δ). For fixed n (> 6), the trajectory of $\delta_{\Delta}^{\leftarrow}$ resembles an anharmonic potential well. It decays with increasing r (upto $r \leq \left[\frac{n}{4}\right]$) and rises in the interval $\left[\frac{n}{4}\right] < r \leq \left[\frac{n}{2}\right]$. In both the panels, the horizontal axes are in terms of the number of excitations in the corresponding Dicke state, while the vertical axis in the top panel is in bits and that in the bottom panel is in qubits.

expression [175, 179, 224] of discord score for the Dicke states is given by

$$\delta_{\mathcal{D}}^{\leftarrow}(|W_n^r\rangle) = S_1 - (n-1)\Big(S_2 - S_{12} + H(\{\lambda_\pm\})\Big), \tag{4.13}$$

where

$$\begin{aligned} S_1 &= -\frac{r}{n}\log_2\frac{r}{n} - (1-\frac{r}{n})\log_2(1-\frac{r}{n}), \\ S_2 &= -(a+b)\log_2(a+b) - (b+c)\log_2(b+c), \\ S_{12} &= -a\log_2 a - 2b\log_2 2b - c\log_2 c, \\ \lambda_\pm &= (1 \pm \sqrt{1-4(ab+bc+ca)})/2, \end{aligned} \tag{4.14}$$

with $a = \frac{(n-r)(n-r-1)}{n(n-1)}$, $b = \frac{r(n-r)}{n(n-1)}$ and $c = \frac{r(r-1)}{n(n-1)}$. Note here that the tangle vanishes for the Dicke states for $r = 1$. However it is non-vanishing for $r \neq 1$ and hence the previous theorem cannot be applied for the Dicke states with $r \neq 1$. The quantum discord and work-deficit scores of the Dicke states for various choices of n with respect to excitations, r, are plotted in Fig. 4.3. The tangle of the Dicke state for various choices of n with respect to excitations, r, are plotted in Fig. 4.4.

Although the Dicke states, for arbitrary r and n, are non-monogamous with respect to discord and work-deficit, (see Fig. 4.3 (bottom panel)), the generalized

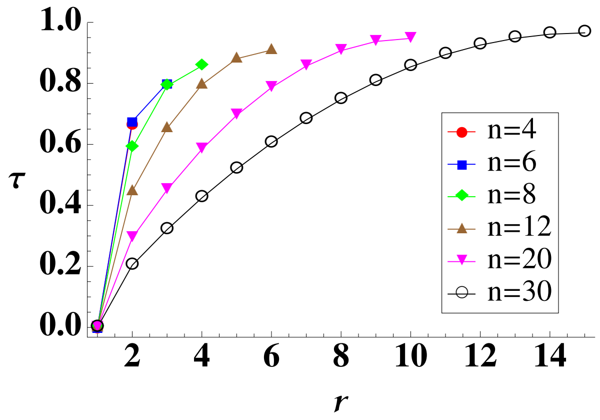


Figure 4.4: Tangle (τ) for Dicke states is plotted against the number of excitations, r, present in the n-party system. The tangle vanishes for all Dicke states with $r = 1$ (i.e., for the W states) and remains positive for Dicke states with $r > 1$. Tangle is measured in ebits.

Dicke states given by

$$|W_n^r(\alpha_i^r)\rangle = \sum_i \alpha_i^r |\underbrace{00...0}_{n-r}\underbrace{11...1}_{r}\rangle, \tag{4.15}$$

with the normalization $\sum_i |\alpha_i^r|^2 = 1$, for $r > 1$, are largely monogamous. In Table 4.3, we list the percentage of randomly chosen states with positive quantum discord score and quantum work-deficit score of the above state for $n = 3, 4, 5, 6$ qubits for different excitations, r.

n	3	4	5	6	6
r	1	2	2	2	3
$\delta_{\mathcal{D}}^{\rightarrow}$	0	94.86	99.29	99.80	100
$\delta_{\triangle}^{\rightarrow}$	27.19	66.77	71.46	72.01	88.25

Table 4.3: Percentage table for the generalized Dicke states that satisfies monogamy for quantum discord ($\mathcal{D}$) and quantum work-deficit ($\triangle$) for 10^5 randomly chosen pure states in each of the cases, according to the Haar measure over the corresponding space.

Finally, we consider a general n-qubit symmetric state, given by $|\psi_{GS}\rangle = \sum_{r=0}^n a_r |W_n^r\rangle$. We generate 10^5 states randomly in the space of n-qubit symmetric states for $n = 3, 4, 5, 6$ uniformly chosen according to the Haar measure (see Table 4.4).

n	3	4	5	6
$\delta_{\mathcal{D}}^{\rightarrow}$	97.47	98.37	86.12	49.71
$\delta_{\mathcal{D}}^{\leftarrow}$	97.15	97.69	86.77	64.35
$\delta_{\triangle}^{\rightarrow}$	81.40	81.49	56.41	26.40
$\delta_{\triangle}^{\leftarrow}$	78.97	77.77	61.19	41.48

Table 4.4: Percentage table for the n-qubit symmetric states that satisfies monogamy for quantum discord ($\mathcal{D}$) and quantum work-deficit ($\triangle$) for 10^5 randomly chosen pure states, for $n = 3, 4, 5, 6$.

Comparing now with the Tables 4.1 and 4.2, where general quantum states (not necessarily symmetric) in the same multiqubit spaces were considered, we find that in drastic contrast to those cases, the frequency of states which satisfies monogamy actually decreases with increasing number of qubits for the symmetric case. We have performed a finite-size scaling analysis, by assuming that all symmetric states will be monogamous for sufficiently large n. We find log-log scaling, with the scaling law being $p_n \approx p_c + n^{-\alpha}$, where p_n is the percentage of symmetric n-qubit states that are monogamous with respect to a given measure, $\mathcal{Q}$, $p_c \equiv p_{n\to\infty}$ (being assumed to be vanishing), and α is the critical exponent of the scaling law. Based on the percentages obtained in the Haar searches, we calculated the critical exponents in Table 4.5.

Note here that all the classes of pure states considered above fall in a set of zero Haar measure in the space of all pure quantum states, for a given n-qubit space, including all symmetric pure states. It is plausible that symmetric mixed states form a non-zero, perhaps fast-decaying, volume of monogamous multiparty quantum states within the space of all quantum states, for large systems.

$\mathcal{Q}$	$\delta_{\mathcal{D}}^{\rightarrow}$	$\delta_{\mathcal{D}}^{\leftarrow}$	$\delta_{\triangle}^{\rightarrow}$	$\delta_{\triangle}^{\leftarrow}$
α	0.8715	0.5523	1.5351	0.8960

Table 4.5: The critical exponents of the scaling law of monogamous states among the symmetric states, for quantum discord ($\mathcal{D}$) and quantum work-deficit ($\triangle$).

4.4 Summary

In quantum communication protocols, in particular in quantum key sharing, it is desirable to detect and control external noise like eavesdropping. In this case, the concept of monogamy comes as a savior, as it does not allow an arbitrary sharing of quantum correlation among subsystems of a larger system. Thus identifying and quantifying which quantum correlation measures are monogamous for the given states, and under what conditions, become extremely important. It is well-known that bipartite quantum correlation measures are, in general, not monogamous for arbitrary tripartite pure states. In this Chapter, we have shown that a quantum correlation measure which is non-monogamous for a substantial section of tripartite quantum states, typically becomes monogamous for almost all quantum states of n-party systems, with n being only slightly higher than 3. We have also identified sets of zero Haar measure in the space of all multiparty quantum states that remain non-monogamous for an arbitrary number of parties.

Apart from providing an understanding on the structure of the space of quantum correlation measures, and their relation to the underlying space of multiparty quantum states, our results may shed more light on the methods for choosing quantum systems for secure quantum information protocols, especially in large quantum networks.

The results of this Chapter are based on the following paper:

1. *Effect of a large number of parties on the monogamy of quantum correlations*, **Asutosh Kumar**, R. Prabhu, A. Sen(De), and U. Sen, Phys. Rev. A **91**, 012341 (2015).

Chapter 5

Conditions for Monogamy of Quantum Correlations

5.1 Introduction

In recent developments on monogamy, we have seen that exponent of a quantum correlation measure and multipartite quantum states play a remarkable role in characterization of monogamy [15, 200]. A non-monogamous quantum correlation measure can become monogamous, for three or more parties, when its power is increased. For instance, concurrence, entanglement of formation, negativity, quantum discord are non-monogamous for three-qubit states, but their squared versions are monogamous. Also, it has been shown that *any quantum correlation measure that is non-monogamous for a multiparty state, can be made monogamous for that state by considering a monotonically increasing function of the measure* [200]. We note that the increasing function of the correlation measure under consideration satisfies all the necessary properties for being a quantum correlation measure including *posititvity* and *monotonicity under local operations*. Furthermore, the function can be so chosen that it is reversible [1,225], such that the information about quantum correlation in the state under consideration, after applying the function on the quantum correlation remains intact. The power of a correlation measure is an example of such a function. It is interesting to note that the power function $f(x) = x^{\alpha}$ is *concave* for $0 < \alpha \leq 1$ and *convex* for $1 \leq \alpha \leq \infty$ on the interval $(0, \infty)$. The power function has an intrinsic geometric interpretation. The power defines the slope of the graph. The higher power, the graph is nearer to the vertical axis. It has been found that the squares of several measures of quantum correlations like concurrence, negativity, quantum discord, entanglement of formation, etc. exhibit monogamy property [14,63–78]. Thus, we observe that the convexity plays a key role in establishing monogamy of quantum correlations. In another case, non-monogamous quantum correlation measures become monogamous, for moderately large number of parties [15].

The motivation behind the considerations in this Chapter is three-fold. In this Chapter, we have asked (i) under what conditions monogamy property of quantum correlations is preserved, (ii) does monogamy for arbitrary pure multipartite state lead to monogamy of mixed states, and (iii) are there more general and stronger monogamy relations different from the standard one,

$$\mathcal{Q}(\rho_{A(B_1\cdots B_n)}) \geq \sum_{j=1}^{n} \mathcal{Q}(\rho_{AB_j}). \tag{5.1}$$

We prove that while a monogamous measure remains monogamous on raising its power, a non-monogamous measure remains non-monogamous on lowering its power. We also prove that monogamy of a convex quantum correlation measure for an arbitrary multipartite pure quantum state leads to its monogamy for the mixed state in the same Hilbert space. Monogamy of squared negativity for mixed states and that of entanglement of formation follow as direct corollaries. Authors of Ref. [188] have proposed following two conjectures regarding monogamy of squared entanglement of formation in multiparty systems: *the squared entanglement of formation may be monogamous for multipartite (i)* $2\otimes d_2\otimes\cdots\otimes d_n$, *and (ii) arbitrary d-dimensional, quantum systems.* Our previous result partially answers these conjectures in the sense that it now only remains to prove the monogamy of the squared entanglement of formation for pure states in arbitrary dimensions. We have further given hierarchical monogamy relations, and a strong monogamy inequality

$$\mathcal{Q}^{\alpha}(\rho_{AB}) \geq \frac{1}{2^{n-1}-1} \sum_{X} \mathcal{Q}^{\alpha}\left(\rho_{AX}\right) \geq \sum_{j} \mathcal{Q}^{\alpha}\left(\rho_{AB_j}\right), \tag{5.2}$$

where $X = \{B_{i_1}, \cdots, B_{i_k}\}$ is a nonempty proper subset of $B \equiv \{B_1, B_2, \cdots, B_n\}$, and $\alpha \geq 1$ is some positive real number.

This Chapter is arranged as follows. In Sec. 5.2, we prove that a monogamous measure remains monogamous on raising its power, a non-monogamous measure remains non-monogamous on lowering its power, and monogamy of a convex quantum correlation measure for arbitrary multipartite pure quantum states leads to its monogamy for the mixed states. We also examine tighter monogamy inequalities compared to the standard one. We present a summary in Sec. 5.3.

5.2 Results

In this section, we prove that a monogamous measure remains monogamous on raising its power, a non-monogamous measure remains non-monogamous on lowering its power, and monogamy of a convex quantum correlation measure for arbitrary multipartite pure quantum states leads to its monogamy for the mixed states. We also examine tighter monogamy inequalities compared to the standard one in Eq. (5.1), and hierarchical monogamy relations. Throughout our discussion we denote the multipartite quantum state $\rho_{AB_1B_2\cdots B_n}$ by ρ_{AB}, unless stated otherwise.

Theorem 5.2.1 (Monogamy preserved for raising of power) *For an arbitrary multipartite quantum state ρ_{AB}, if $\mathcal{Q}^r(\rho_{AB}) \geq \sum_j \mathcal{Q}^r(\rho_{AB_j})$ then $\mathcal{Q}^\alpha(\rho_{AB}) \geq \sum_j \mathcal{Q}^\alpha(\rho_{AB_j})$ for $\alpha \geq r \geq 1$.*

Proof. We have the inequalities $(1+x)^t \geq 1+x^t$ and $(\sum_i x_i^t)^{\frac{s}{t}} \geq \sum_i x_i^s$ where $0 \leq x, x_i \leq 1$ and $s \geq t \geq 1$. Now there exists $1 \leq k \leq n$ such that $\sum_{j\neq k} \mathcal{Q}^r(\rho_{AB_j}) \geq \mathcal{Q}^r(\rho_{AB_k})$. Now $\mathcal{Q}^r(\rho_{AB}) \geq \sum_j \mathcal{Q}^r(\rho_{AB_j})$ implies

$$\begin{aligned}
\mathcal{Q}^\alpha(\rho_{AB}) &\geq \left(\sum_j \mathcal{Q}^r(\rho_{AB_j})\right)^{\frac{\alpha}{r}} \\
&= \left(\sum_{j\neq k} \mathcal{Q}^r(\rho_{AB_j})\right)^{\frac{\alpha}{r}} \left(1 + \frac{\mathcal{Q}^r(\rho_{AB_k})}{\sum_{j\neq k} \mathcal{Q}^r(\rho_{AB_j})}\right)^{\frac{\alpha}{r}} \\
&\geq \left(\sum_{j\neq k} \mathcal{Q}^r(\rho_{AB_j})\right)^{\frac{\alpha}{r}} \left(1 + \left(\frac{\mathcal{Q}^r(\rho_{AB_k})}{\sum_{j\neq k} \mathcal{Q}^r(\rho_{AB_j})}\right)^{\frac{\alpha}{r}}\right) \\
&= \left(\sum_{j\neq k} \mathcal{Q}^r(\rho_{AB_j})\right)^{\frac{\alpha}{r}} + \mathcal{Q}^\alpha(\rho_{AB_k}) \\
&\geq \left(\sum_{j\neq k} \mathcal{Q}^\alpha(\rho_{AB_j})\right) + \mathcal{Q}^\alpha(\rho_{AB_k}) \\
&= \sum_j \mathcal{Q}^\alpha(\rho_{AB_j}),
\end{aligned} \tag{5.3}$$

thus proving the theorem. ■

The first and second inequalities respectively follow from the fact that $(1+x)^t \geq 1+x^t$ and $(\sum_i x_i^t)^{\frac{s}{t}} \geq \sum_i x_i^s$ where $0 \leq x, x_i \leq 1$ and $s \geq t \geq 1$. This theorem can be viewed as an extension of the key result in Ref. [200] that a non-monogamous quantum correlation measure will become monogamous for some value when its power is raised.

Theorem 5.2.2 (Non-monogamy preserved for lowering of power) *For an arbitrary multipartite quantum state* ρ_{AB}*, if* $\mathcal{Q}^r(\rho_{AB}) \leq \sum_j \mathcal{Q}^r(\rho_{AB_j})$ *then* $\mathcal{Q}^\alpha(\rho_{AB}) \leq \sum_j \mathcal{Q}^\alpha(\rho_{AB_j})$ *for* $\alpha \leq r$*.*

Proof. The inequality $\mathcal{Q}^r(\rho_{AB}) \leq \sum_j \mathcal{Q}^r(\rho_{AB_j})$ implies $\mathcal{Q}^\alpha(\rho_{AB}) \leq \left(\sum_j \mathcal{Q}^r(\rho_{AB_j})\right)^{\frac{\alpha}{r}}$. Now above theorem can be proved by using the inequality $(1+x)^t \leq 1+x^t$, for $x > 0$ and $t \leq 1$, repeatedly. ∎

Remark 5.2.3 Theorems 5.2.1 and 5.2.2 ensure that varying the exponent preserves monogamy (non-monogamy) relations of monogamous (non-monogamous) correlation measures. Recently, it was shown in Ref. [226] that, for multiqubit systems, the r^{th}-power of concurrence is monogamous for $r \geq 2$ while non-monogamous for $r \leq 0$, and the r^{th}-power of entanglement of formation (EoF) is monogamous for $r \geq \sqrt{2}$. These observations are consistent with above theorems. Similarly, negativity, quantum discord for three-qubit pure states, contextual inequalities, etc., will remain monogamous for $r \geq 2$. Also, quantum work-deficit, for all three-qubit pure states, will remain monogamous for the fifth power and higher [200].

Remark 5.2.4 Note that, at first sight, it seems that Theorems 5.2.1 and 5.2.2 are rather about properties of abstract functions that can describe not only non-classical correlations but any other property. We wish to note however that the theorems are not true for an arbitrarily chosen physical property. For instance, for the mixture $\rho_{ABC} = \frac{1}{2}\big(|000\rangle\langle 000| + |111\rangle\langle 111|\big)$, the classical correlation, irrespective of the definition used, in all the bi-partitions A:(BC), A:B, and A:C is unity, after a suitable normalization. In this case, raising the power to any value, however large, of the classical correlation, won't make it monogamous. This example illustrates the fact why raising or lowering of powers to nonclassical correlations is important and necessary. Thus, the above two theorems should be seen mainly in the context of nonclassical correlations.

Remark 5.2.5 A particularly interesting scenario is the following. Suppose that a quantum correlation measure $\mathcal{Q}$ is monogamous for its r^{th}-power. It is important to know the least power, r^*, for which the monogamy relation of $\mathcal{Q}$ is preserved. That is, for what power does a monogamous measure become non-monogamous, and vice-versa? This situation is extremely demanding for generic quantum correlation measures and generic quantum states. Moreover, if

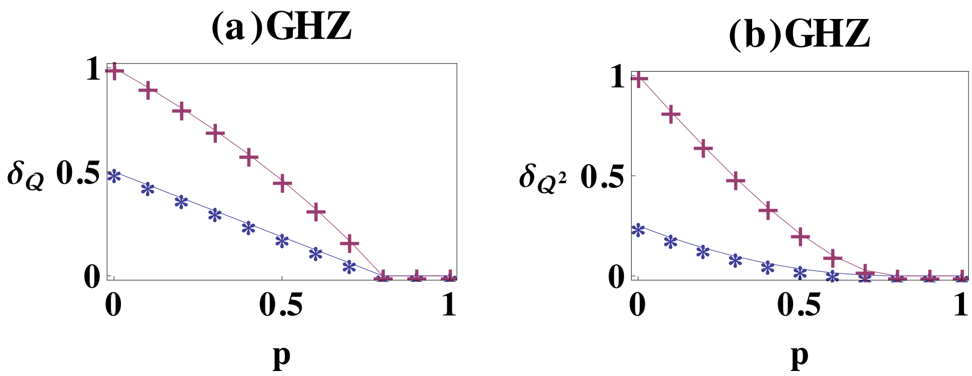


Figure 5.1: Plots of monogamy scores, $\delta_{\mathcal{Q}^r}(\rho_{ABC}) = \mathcal{Q}^r(\rho_{A(BC)}) - \mathcal{Q}^r(\rho_{AB}) - \mathcal{Q}^r(\rho_{AC})$, of negativity (stars) and logarithmic-negativity (pluses) against noise parameter p of GHZ state mixed with white noise, $\rho_{ABC} = (1-p)|GHZ\rangle\langle GHZ| + p\frac{\mathbb{I}}{8}$. For illustration, power of the quantum correlation measures we have considered is (a) $r = 1$, and (b) $r = 2$. Here we see that, for GHZ state, both negativity and logarithmic-negativity are monogamous for $r \geq 1$. While x-axis is dimensionless, y-axis is in ebits.

quantum measure $\mathcal{Q}$ is monogamous for $r \geq 1$ power, then it will become non-monogamous for $\alpha \leq 0$. That is, if $\mathcal{Q}^r(\rho_{AB}) \geq \sum_{j=1}^{n} \mathcal{Q}^r(\rho_{AB_j})$, then $\mathcal{Q}^\alpha(\rho_{AB}) \leq \sum_{j=1}^{n} \mathcal{Q}^\alpha(\rho_{AB_j})$ for $\alpha \leq 0$. As specific examples, we give plots of monogamy scores, $\delta_{\mathcal{Q}^r}(\rho_{ABC}) = \mathcal{Q}^r(\rho_{A(BC)}) - \mathcal{Q}^r(\rho_{AB}) - \mathcal{Q}^r(\rho_{AC})$, of negativity [136] and logarithmic-negativity [135] against noise parameter p, of GHZ state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and W state, $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, mixed with white noise in Figs. 5.1 and 5.2. Power of negativity and logarithmic-negativity that we have considered for illustartion is (a) $r = 1$, and (b) $r = 2$. We see that for GHZ state, both negativity and logarithmic negativity are monogamous for $r \geq 1$ (see Fig. 5.1). On the other hand for W state, while negativity is monogamous for $r \geq 1$, logarithmic negativity is monogamous for $r \geq 2$ (see Fig. 5.2). From Fig. 5.2(a), we see that logarithmic-negativity is non-monogamous for W state ($p = 0$) when $r = 1$. However, from Fig. 5.3, we find that it remains non-monogamous upto $r^* \approx 1.06$ (upto second decimal point), and becomes monogamous when $r \gtrsim 1.06$.

Sometimes an entanglement measure $\mathcal{Q}$ can be a function of another entanglement measure q, say, $\mathcal{Q} = f(q^r)$. Depending on the nature of function f and monogamy of q, the monogamy properties of $\mathcal{Q}$ can be derived. For instance, in the seminal paper of CKW [14], it was already pointed out that any monotonic convex function of squared concurrence would also be a monogamous measure of entanglement. We extend this observation for a general quantum correlation measure in the following theorem.

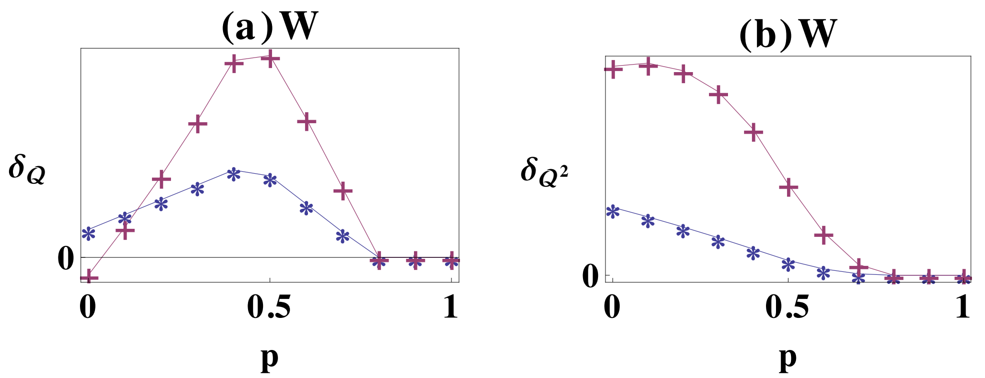


Figure 5.2: Plots of monogamy scores, $\delta_{\mathcal{Q}^r}(\rho_{ABC}) = \mathcal{Q}^r(\rho_{A(BC)}) - \mathcal{Q}^r(\rho_{AB}) - \mathcal{Q}^r(\rho_{AC})$, of negativity (stars) and logarithmic-negativity (pluses) against noise parameter p of W state mixed with white noise, $\rho_{ABC} = (1-p)|W\rangle\langle W| + p\frac{\mathbb{I}}{8}$. For demonstration, power of the quantum correlation measures that we have considered is (a) $r = 1$, and (b) $r = 2$. For W state, unlike GHZ state, while negativity is monogamous for $r \geq 1$, logarithmic-negativity is monogamous for $r \geq 2$. While x-axis is dimensionless, y-axis is in ebits.

Theorem 5.2.6 *For an arbitrary multipartite quantum state ρ_{AB}, given that $q^r(\rho_{AB}) \geq \sum_j q^r(\rho_{AB_j})$ and $\mathcal{Q} = f(q^r)$, where f is a monotonically increasing convex function for which $f^m\left(\sum_j q_j^r\right) \geq \sum_j f^m(q_j^r)$, we have $\mathcal{Q}^m(\rho_{AB}) \geq \sum_j \mathcal{Q}^m(\rho_{AB_j})$, where r and m are some positive numbers.*

Proof. Let $\rho_{AB} = \sum_i p_i |\psi^i\rangle_{AB}\langle\psi^i|$ be the optimal decomposition of ρ_{AB} for $\mathcal{Q}$. Then

$$
\begin{aligned}
\mathcal{Q}(\rho_{AB}) &= \sum_i p_i \mathcal{Q}(|\psi\rangle^i_{AB}) \\
&= \sum_i p_i f(q^r_{ABi}) \\
&\geq f\left(\sum_i p_i q^r_{ABi}\right) \\
&\geq f\left(\left(\sum_i p_i q_{ABi}\right)^r\right) \\
&\geq f\left(q^r(\rho_{AB})\right) \\
&\geq f\left(\sum_j q^r_{AB_j}\right) \\
&\geq \left(\sum_j f^m(q^r_{AB_j})\right)^{\frac{1}{m}}
\end{aligned}
$$

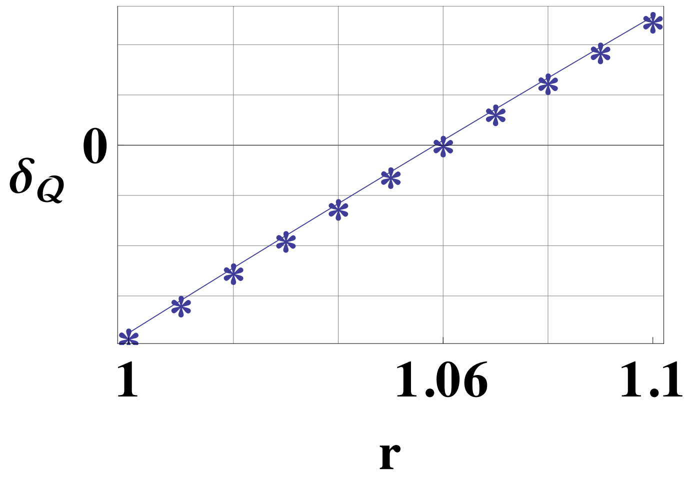


Figure 5.3: Plot showing the transition from non-monogamy to monogamy of logarithmic-negativity for W state. Logarithmic-negativity is non-monogamous for W state at $r = 1$. It remains non-monogamous upto $r \approx 1.06$ (upto second decimal point), beyond which it becomes monogamous. That is, the minimum power, r^*, for which logarithmic-negativity becomes monogamous for W state is 1.06. While x-axis is dimensionless, y-axis is in ebits.

$$= \left(\sum_j \mathcal{Q}_{AB_j}^m \right)^{\frac{1}{m}}, \tag{5.4}$$

where the first inequality is due to convexity of f, the second is due to monotonically increasing nature of f and $\sum_i p_i x_i^r \geq (\sum_i p_i x_i)^r$, the third is due to monotonicity of f and because $\rho_{AB} = \sum_i p_i |\psi^i\rangle_{AB}\langle\psi^i|$ may not be the optimal decomposition of ρ_{AB} for q (that is, $q(\rho_{AB}) \leq \sum_i p_i q(|\psi\rangle_{AB}^i)$), the fourth is due to monogamy of q^r, and the fifth inequality follows from the constraint $f^m(\sum_j q_j^r) \geq \sum_j f^m(q_j^r)$. Hence the theorem is proved. ■

The monogamy of squared EoF can be stated as a corollary of Theorem 5.2.6.

Corollary 5.2.7 *The square of entanglement of formation is monogamous.*

Proof. EoF is a concave function of squared concurrence given by

$$\mathcal{E}(\rho_{AB}) = h\left(\frac{1 + \sqrt{1 - \mathcal{C}^2(\rho_{AB})}}{2} \right), \tag{5.5}$$

where $h(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the Shannon (binary) entropy. However, squared EoF is a convex monotonic function of squared concurrence and satisfies $\mathcal{E}^2(\sum_j \rho_{AB_j}) \geq \sum_j \mathcal{E}^2(\rho_{AB_j})$. The corollary follows because squared concurrence is monogamous [14, 63]. ■

Independent proofs of monogamy of squared EoF have been provided previously in Refs. [70, 71].

Remark 5.2.8 Using the same line of proof as in Theorem 5.2.6, we can show that for an arbitrary multipartite quantum state ρ_{AB}, given that $q^r(\rho_{AB}) \geq \sum_j q^r(\rho_{AB_j})$ and $\mathcal{Q} = f(q^r)$, where f be a monotonically decreasing concave function for which $f^m\left(\sum_j q_j^r\right) \geq \sum_j f^m(q_j^r)$, we have $\mathcal{Q}^m(\rho_{AB}) \leq \sum_j \mathcal{Q}^m(\rho_{AB_j})$, where r and m are some positive numbers.

Next we ask whether there is any correspondence between monogamy of a quantum correlation measure for arbitrary pure and that of mixed states. The answer of this question led us to the result in Theorem 5.2.9, and the remarks and corollary following it.

Theorem 5.2.9 *If a convex bipartite quantum correlation measure $\mathcal{Q}$ when raised to power $r = 1, 2$ is monogamous for pure multipartite states, then $\mathcal{Q}^r$ is also monogamous for the mixed states in the given Hilbert space.*

Proof. Convexity of $\mathcal{Q}$ implies that if $\rho = \sum_i p_i \rho^i$ then $\mathcal{Q}(\rho) \leq \sum_i p_i \mathcal{Q}(\rho^i)$. Assume that $\mathcal{Q}^r$, $(r = 1, 2)$, is monogamous for arbitrary multipartite pure state $|\psi\rangle_{AB} = |\psi\rangle_{AB_1B_2\cdots B_n}$ in some Hilbert space of dimension $d_A \otimes d_{B_1} \otimes d_{B_2} \cdots \otimes d_{B_n}$. That is,

$$\mathcal{Q}^r(|\psi\rangle_{AB}) \geq \sum_{j=1}^{n} \mathcal{Q}^r(\rho^{\psi}_{AB_j}). \tag{5.6}$$

Let $\rho_{AB} = \sum_i p_i |\psi^i\rangle_{AB}\langle\psi^i| = \sum_i p_i \rho^i_{AB}$ be the optimal decomposition of ρ_{AB} for $\mathcal{Q}$, and $\rho^i_{AB_j} = \mathrm{tr}_{\overline{AB_j}} \rho^i_{AB}$, $\rho_{AB_j} = \mathrm{tr}_{\overline{AB_j}} \rho_{AB}$ be the reduced density matrices obtained after partial-tracing the sub-systems except A *and* B_j $(j = 1, 2, \cdots, n)$.

When $r = 1$, we have $\mathcal{Q}(\rho_{AB}) = \sum_i p_i \mathcal{Q}(|\psi^i\rangle_{AB}) \geq \sum_i p_i \sum_j \mathcal{Q}(\rho^i_{AB_j}) = \sum_j \left(\sum_i p_i \mathcal{Q}(\rho^i_{AB_j})\right) \geq \sum_j \mathcal{Q}\left(\sum_i p_i \rho^i_{AB_j}\right) = \sum_j \mathcal{Q}(\rho_{AB_j})$, where the first inequality is due to monogamy of $\mathcal{Q}$ for pure states and the second inequality is due to the convexity of $\mathcal{Q}$.

When $r = 2$, let us write

$$\mathcal{Q}(\rho_{AB}) = \sum_i p_i \mathcal{Q}(|\psi^i\rangle_{AB}) = \sum_i \mathcal{Q}_{ABi} \tag{5.7}$$

$$\begin{aligned}\mathcal{Q}'(\rho_{AB_j}) = \sum_i p_i \mathcal{Q}(\rho^i_{AB_j}) = \sum_i \mathcal{Q}_{AB_j i} \\ \geq \mathcal{Q}(\rho_{AB_j})\end{aligned} \tag{5.8}$$

The above inequality follows from convexity of $\mathcal{Q}$. We, then, have the following inequality

$$\begin{aligned} & \mathcal{Q}^2(\rho_{AB}) - \sum_j \mathcal{Q}'^2(\rho_{AB_j}) \\ = & \left(\sum_i \mathcal{Q}_{ABi}\right)^2 - \sum_j \left(\sum_i \mathcal{Q}_{AB_j i}\right)^2 \\ = & \sum_i \left(\mathcal{Q}^2_{ABi} - \sum_j \mathcal{Q}^2_{AB_j i}\right) \\ & +2\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\left(\mathcal{Q}_{ABi}\mathcal{Q}_{ABk} - \sum_j \mathcal{Q}_{AB_j i}\mathcal{Q}_{AB_j k}\right) \\ \geq 0, & \end{aligned} \tag{5.9}$$

because, in the second equation, the first term is non-negative due to monogamy of $\mathcal{Q}^2$ for pure states and the second term is non-negative as shown below. We have, for arbitrary pure states $|\psi^i\rangle_{AB}$ and $|\psi^k\rangle_{AB}$,

$$\begin{aligned}\mathcal{Q}^2_{ABi}\mathcal{Q}^2_{ABk} &\geq \left(\sum_j \mathcal{Q}^2_{AB_j i}\right)\left(\sum_j \mathcal{Q}^2_{AB_j k}\right) \\ &\geq \left(\sum_j \mathcal{Q}_{AB_j i}\mathcal{Q}_{AB_j k}\right)^2,\end{aligned} \tag{5.10}$$

where the first inequality is due to monogamy of $\mathcal{Q}^2$ for pure states while the second inequality follows from the Cauchy-Schwarz inequality, $\sum_i a_i b_i \leq \sqrt{\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)}$. Hence,

$$\mathcal{Q}_{ABi}\mathcal{Q}_{ABk} - \sum_j \mathcal{Q}_{AB_j i}\mathcal{Q}_{AB_j k} \geq 0. \tag{5.11}$$

Since $\mathcal{Q}'(\rho_{AB_j}) \geq \mathcal{Q}(\rho_{AB_j})$ (due to convexity of $\mathcal{Q}$ as shown in Eq. (5.8)), we obtain the desired monogamy relation for mixed state,

$$\mathcal{Q}^2(\rho_{AB}) \geq \sum_j \mathcal{Q}^2(\rho_{AB_j}). \tag{5.12}$$

■

Remark 5.2.10 Theorem 5.2.9 cannot be stated conclusively for $r > 2$ by using the same line of proof as shown below. Thus, for $r > 2$, the monogamy of $\mathcal{Q}^r$ is as yet inclusive for mixed states, even though monogamy holds for pure states.

Multinomial expansion is given by

$$\left(\sum_i x_i\right)^r = \sum_{\{r_k\}} \frac{r!}{\prod_k r_k!} \prod_i x_i^{r_k} \tag{5.13}$$

where $\{r_k | 0 \le r_k \le r \;\; \& \sum_k r_k = r\}$ is the integer partition of r, and the summation is over all permutations of such integer partitions of r. Then, as in Theorem 5.2.9, we have

$$\begin{aligned}
& \mathcal{Q}^r(\rho_{AB}) - \sum_j \mathcal{Q}'^r(\rho_{AB_j}) \\
= & \left(\sum_i \mathcal{Q}_{ABi}\right)^r - \sum_j \left(\sum_i \mathcal{Q}_{AB_j i}\right)^r \\
= & \sum_{\{r_k\}} \frac{r!}{\prod_k r_k!} \left(\prod_i \mathcal{Q}^{r_k}_{ABi} - \sum_j \prod_i \mathcal{Q}^{r_k}_{AB_j i}\right) \\
= & \sum_i \left(\mathcal{Q}^r_{ABi} - \sum_j \mathcal{Q}^r_{AB_j i}\right) \\
& + \sum_{\{r_k \neq r\}} \frac{r!}{\prod_k r_k!} \left(\prod_i \mathcal{Q}^{r_k}_{ABi} - \sum_j \prod_i \mathcal{Q}^{r_k}_{AB_j i}\right).
\end{aligned} \tag{5.14}$$

Although, in the third equality, the first term is non-negative due to the monogamy of $\mathcal{Q}^r$ for pure states, we cannot say anything with certainty about the second term as we do not have *Holder-type inequality for multi-variables*. However, the other way is always true, i.e., if $\mathcal{Q}^r$ is monogamous for mixed states then it is certainly monogamous for pure states. ■

Remark 5.2.11 In Ref. [15], it was shown numerically that entanglement measures become monogamous for pure states with increasing number of qubits. It was also figured out that "good" entanglement measures [120] like relative entropy of entanglement, regularized relative entropy of entanglement [220], entanglement cost [134,221], distillable entanglement, all of which are not generally computable, are monogamous for almost all pure states of four or more qubits. Theorem 5.2.9 then implies that such "good" convex entanglement measures will become monogamous for multiqubit mixed states also.

Corollary 5.2.12 *The squared negativity is monogamous for n-qubit mixed states.*

Proof. Negativity is a convex function [136], and it has been proven that the square of negativity is monogamous for n-qubit pure states [64]. Hence the proof. ■

Further, we explored if we could obtain general and tighter monogamy relations other than the standard one in Eq. (5.1). This led us to the results in Theorem 5.2.13 and Theorem 5.2.15, and the remarks following the same.

Theorem 5.2.13 (Hierarchical monogamy relations) $\mathcal{Q}^r(\rho_{AXY}) \geq \mathcal{Q}^r(\rho_{AX}) + \mathcal{Q}^r(\rho_{AY})$ *for an arbitrary state* ρ_{AXY} *implies* $\mathcal{Q}^\alpha(\rho_{ABC}) \geq \mathcal{Q}^\alpha(\rho_{AB}) + \sum_{j=1}^{k-1} \mathcal{Q}^\alpha(\rho_{AC_j}) + \mathcal{Q}^\alpha(\rho_{AC_k\cdots C_m})$ *for* $\rho_{ABC} \equiv \rho_{ABC_1\cdots C_m}$ *when* $\alpha \geq r$.

Proof. First, we will prove the hierarchical monogamy relations using the given condition, $\mathcal{Q}^r(\rho_{AXY}) \geq \mathcal{Q}^r(\rho_{AX}) + \mathcal{Q}^r(\rho_{AY})$ for arbitrary state ρ_{AXY}, and thereafter we will show that these relations are also valid for $\alpha \geq r$. For multiparty state $\rho_{ABC} \equiv \rho_{ABC_1\cdots C_m}$, applying the given condition repeatedly, we obtain a family of hierarchical monogamy relations as given below

$$\begin{aligned}
\mathcal{Q}^r(\rho_{ABC}) &\geq \mathcal{Q}^r(\rho_{AB}) + \mathcal{Q}^r(\rho_{AC}) \\
&\geq \mathcal{Q}^r(\rho_{AB}) + \mathcal{Q}^r(\rho_{AC_1}) + \mathcal{Q}^r(\rho_{AC_2\cdots C_m}) \\
&\quad\vdots \\
&\geq \mathcal{Q}^r(\rho_{AB}) + \sum_{j=1}^{k-1} \mathcal{Q}^r(\rho_{AC_j}) + \mathcal{Q}^r(\rho_{AC_k\cdots C_m}) \\
&\quad\vdots \\
&\geq \mathcal{Q}^r(\rho_{AB}) + \sum_{j=1}^{m} \mathcal{Q}^r(\rho_{AC_j}).
\end{aligned} \tag{5.15}$$

Now, we will show that these hierarchical monogamy relations are also valid for $\alpha \geq r$. From Theorem 5.2.1, $\mathcal{Q}^r(\rho_{ABC}) \geq \mathcal{Q}^r(\rho_{AB}) + \mathcal{Q}^r(\rho_{AC})$ implies that

$$\mathcal{Q}^\alpha(\rho_{ABC}) \geq \mathcal{Q}^\alpha(\rho_{AB}) + \mathcal{Q}^\alpha(\rho_{AC}). \tag{5.16}$$

Now,

$$\begin{aligned}
\mathcal{Q}^\alpha(\rho_{AC}) &= \left\{\mathcal{Q}^r(\rho_{AC_1(C_2\cdots C_m)})\right\}^{\frac{\alpha}{r}} \\
&\geq \left\{\mathcal{Q}^r(\rho_{AC_1}) + \mathcal{Q}^r(\rho_{A(C_2\cdots C_m)})\right\}^{\frac{\alpha}{r}}
\end{aligned}$$

$$\geq \sum_{j=1}^{k-1} \mathcal{Q}^{\alpha}(\rho_{AC_j}) + \mathcal{Q}^{\alpha}(\rho_{AC_k \cdots C_m}) \tag{5.17}$$

$$\geq \sum_{j=1}^{m} \mathcal{Q}^{\alpha}(\rho_{AC_j}). \tag{5.18}$$

Thus, we obtain inequalities

$$\mathcal{Q}^{\alpha}(\rho_{ABC}) \geq \mathcal{Q}^{\alpha}(\rho_{AB}) + \sum_{j=1}^{k-1} \mathcal{Q}^{\alpha}(\rho_{AC_j}) + \mathcal{Q}^{\alpha}(\rho_{AC_k \cdots C_m}), \tag{5.19}$$

and

$$\mathcal{Q}^{\alpha}(\rho_{ABC}) \geq \mathcal{Q}^{\alpha}(\rho_{AB}) + \sum_{k=1}^{m} \mathcal{Q}^{\alpha}(\rho_{AC_k}). \tag{5.20}$$

■

Remark 5.2.14 Using Theorem 5.2.2 and the same line of proof as in Theorem 5.2.13, one can prove *hierarchical non-monogamy relations.* $\mathcal{Q}^r(\rho_{XYZ}) \leq \mathcal{Q}^r(\rho_{XY}) + \mathcal{Q}^r(\rho_{XZ})$ for an arbitrary state ρ_{XYZ} implies $\mathcal{Q}^{\alpha}(\rho_{ABC}) \leq \mathcal{Q}^{\alpha}(\rho_{AB}) + \sum_{j=1}^{k-1} \mathcal{Q}^{\alpha}(\rho_{AC_j}) + \mathcal{Q}^{\alpha}(\rho_{AC_k \cdots C_m})$ for $\rho_{ABC} \equiv \rho_{ABC_1 \cdots C_m}$ when $\alpha \leq r$.

Theorem 5.2.15 (Strong monogamy inequality) *If* $\mathcal{Q}^r(\rho_{AB}) \geq \sum_j \mathcal{Q}^r(\rho_{AB_j})$ *for an arbitrary multipartite quantum state* $\rho_{AB_1 \cdots B_n} \equiv \rho_{AB}$*, then* $\mathcal{Q}^{\alpha}(\rho_{AB}) \geq \frac{1}{2^{n-1}-1} \sum_X \mathcal{Q}^{\alpha}(\rho_{AX}) \geq \sum_j \mathcal{Q}^{\alpha}(\rho_{AB_j})$ *for* $\alpha \geq r \geq 1$*, where* X *is the composite system corresponding to some nonempty proper subset of* $B = \{B_1, B_2, \cdots, B_n\}$*.*

Proof. Here again we can split the proof in two parts, as in the proof of Theorem 5.2.13. For instance, first we can obtain the monogamy relation, $\mathcal{Q}(\rho_{AB}) \geq \frac{1}{2^{n-1}-1} \sum_X \mathcal{Q}\left(\rho_{AX}\right) \geq \sum_j \mathcal{Q}\left(\rho_{AB_j}\right)$, and then show that such a monogamy relation is also true for the α-th power. However, for the sake of brevity, we will start with the α-th power. Let $B = \{B_1, B_2, \cdots, B_n\}$ be the set of subsystems B_i's, and $X = \{B_{i_1}, \cdots, B_{i_k}\}$ and $X^c = B - X$ be nonempty proper subsets of B. Thus $\rho_{AB} = \rho_{AXX^c}$. Applying monogamy inequality and Theorem 5.2.1, we get

$$\mathcal{Q}^{\alpha}\left(\rho_{AB}\right) \geq \mathcal{Q}^{\alpha}\left(\rho_{AX}\right) + \mathcal{Q}^{\alpha}\left(\rho_{AX^c}\right). \tag{5.21}$$

Since the set of all nonempty proper subsets of B is same as the set of their complements, i.e., $\{X | X \subset B\} = \{X^c | X \subset B\}$, summing over all possible nonempty proper subsets X's of B leads to the following inequality,

$$\begin{aligned}\mathcal{Q}^{\alpha}(\rho_{AB}) &\geq \frac{1}{2^n-2}\sum_X\left(\mathcal{Q}^{\alpha}(\rho_{AX})+\mathcal{Q}^{\alpha}(\rho_{AX^c})\right)\\ &= \frac{1}{2^{n-1}-1}\sum_X \mathcal{Q}^{\alpha}(\rho_{AX}).\end{aligned} \tag{5.22}$$

We also have

$$\mathcal{Q}^{\alpha}(\rho_{AX})+\mathcal{Q}^{\alpha}(\rho_{AX^c}) \geq \sum_j \mathcal{Q}^{\alpha}\left(\rho_{AB_j}\right). \tag{5.23}$$

Again summing over all possible nonempty proper subsets X's of B, we obtain

$$\frac{1}{2^{n-1}-1}\sum_X \mathcal{Q}^{\alpha}(\rho_{AX}) \geq \sum_j \mathcal{Q}^{\alpha}\left(\rho_{AB_j}\right). \tag{5.24}$$

Combining inequalities (5.22) and (5.24), we obtain the desired *strong monogamy inequality* for arbitrary multi-party quantum state $\rho_{AB_1B_2\cdots B_n}$. ■

Remark 5.2.16 It was shown in Ref. [227] that entanglement of assistance [228] follows strong non-monogamy relation. Using Theorem 5.2.2 and the same line of proof as in Theorem 5.2.15, we can prove that if $\mathcal{Q}^r(\rho_{AB}) \leq \sum_j \mathcal{Q}^r(\rho_{AB_j})$ for any multipartite quantum state $\rho_{AB_1\cdots B_n} \equiv \rho_{AB}$, then $\mathcal{Q}^{\alpha}(\rho_{AB}) \leq \frac{1}{2^{n-1}-1}\sum_X \mathcal{Q}^{\alpha}(\rho_{AX}) \leq \sum_j \mathcal{Q}^{\alpha}(\rho_{AB_j})$ for $\alpha \leq r$.

5.3 Summary

We have explored the conditions under which monogamy of functions of quantum correlation measures is preserved. We have shown that a monogamous measure remains monogamous on raising its power, and a non-monogamous measure remains non-monogamous on lowering its power. We have also proven that monogamy of a convex quantum correlation measure for arbitrary multipartite pure quantum states leads to its monogamy for the mixed states. This significantly simplifies the task of establishing the monogamy relations for mixed states. Our study partially answers the two conjectures in Ref. [188] in the sense that it now only remains to prove the monogamy of the squared entanglement of formation for pure states in arbitrary dimensions. Monogamy of squared negativity for mixed states and that of squared entanglement of formation turn out to be special cases of our results. Furthermore, we have examined hierarchical monogamy relations and tighter monogamy inequalities compared to the standard one. Given that a quantum correlation measure $\mathcal{Q}$ is monogamous for its r^{th}-power, it is important to know the least power r^* for which the monogamy relation of $\mathcal{Q}$ is preserved. In this Chapter, we have tried to give partial answers to this question from different perspectives.

The results of this Chapter are based on the following paper:

1. *Conditions for monogamy of quantum correlations in multipartite systems*, **Asutosh Kumar**, Phys. Lett. A **380**, 3044 (2016).

Chapter 6

Bounds on Monogamy Scores

6.1 Introduction

Since monogamy score is a difficult quantity to compute and estimate for generic quantum states and generic quantum correlations, it is interesting to derive both upper and lower bounds on the monogamy score. The existence of non-trivial bounds on the monogamy score is an important aspect in the study of various quantum information protocols. In this Chapter, we aim to obtain non-trivial upper and lower bounds on the monogamy score.

We establish a relation between monogamy of quantum correlation, quantified by using the concept of "monogamy score" [85, 89, 212], and genuine multiparty entanglement measure for n-qubit pure states. The connection holds irrespective of the number of parties and is independent of the choice of the bipartite quantum correlation measure used in the conceptualization of monogamy. We show that for a large majority of pure multiqubit states, the monogamy score for a broad range of quantum correlation measures is upper-bounded by a function of genuine multiparty entanglement in the system, as quantified by the generalized geometric measure (GGM) [143, 144] (see also [145–147]). Considering the squared concurrence and squared negativity as measures from the entanglement-separability paradigm, and quantum discord and quantum work-deficit as information-theoretic measures of quantum correlation, we analytically show that the bound is universally satisfied for all pure three-qubit states [212] and find conditions for its validity for an arbitrary number of qubits. We also identify a set of necessary conditions to be satisfied in order to violate the bound for more than three qubits. We numerically observe that these conditions are only satisfied for an extremely small set of n-qubit quantum states. In fact, by numerically generating random 4- and 5-qubit pure quantum states, using a uniform Haar distribution, we find that the bound is virtually never violated. The results show that the sharability of arbitrary bipartite quantum correlations in multisite quantum states is nontrivially limited by the multiparty entanglement content of the states irrespective of the number of parties. We note that the monogamy inequality, as a physical characteristic, is dependent on the quantum correlation measure under consideration. A direct consequence of our work, as formulated more precisely in later sections, is that the monogamy score is now bounded above by a measure-independent quadratic or entropic function of GGM. Significantly, this allows us to readily obtain the upper bound of monogamy score, even for quantum correlation measures, such as distillable entanglement and entanglement cost, where the monogamy score is either intractable or not computable. We also obtain a lower bound on monogamy scores in terms of purity for quantum correlations that violate monogamy inequality.

This Chapter is organized as follows. In Sec. 6.2, we consider the multiparty entanglement bounds on monogamy score in terms of the genuine multiparty entanglement. In Sec. 6.3, we analytically show how the bound is satisfied for a host of n-qubit ($n > 3$) symmetric and many-body ground states. In Sec. 6.4, we numerically find that the bound is satisfied for randomly generated four- and five-qubit quantum states. Lower bounds on monogamy scores, in terms of purity, for quantum correlation measures that violate the monogamy inequality are obtained in Sec. 6.5. We summarize in Sec. 6.6.

6.2 Monogamy score and genuine multiparty entanglement

In this section, we connect the monogamy score with genuine multiparty entanglement measure. In particular, we show that the monogamy score of a quantum correlation measure for any multiqubit pure state is upper-bounded by the genuine multiparty entanglement of the state, quantified using the generalized geometric measure [143, 144]. Let us consider an n-qubit pure state $|\psi\rangle$. The corresponding k-qubit reduced states are given by $\rho^{(k)} = \text{Tr}_{n-k}(|\psi\rangle\langle\psi|)$, where $n - k$ parties have been traced out. From the definition of GGM, we know that $\mathcal{G} = 1 - \max_{k\in[1,n/2]}\left[\left\{\xi_m(\rho^{(k)})\right\}\right]$, where $\left\{\xi_m(\rho^{(k)})\right\}$ is the set of maximum eigenvalues corresponding to all possible k-qubit reduced states of the $k : n - k$ bipartitions, for k ranging from 1 to $\frac{n}{2}$. Let us consider the squares of concurrence ($\mathcal{C}^2$) and negativity ($\mathcal{N}^2$), which are from the entanglement-separability paradigm, quantum discord ($\mathcal{D}$) and quantum work-deficit ($\mathcal{W}$), from the information-theoretic paradigm, as quantum correlation measures. Let us now establish the connection between GGM and monogamy of bipartite measures $\mathcal{Q}$ in the following theorem.

Theorem 6.2.1 *For all multiqubit pure states, $|\psi\rangle$, the monogamy score, $\delta_{\mathcal{Q}}(|\psi\rangle)$, of a quantum correlation measure, $\mathcal{Q}$, based on entanglement-separability (information theoretic) criteria, is bounded above by a function of the generalized geometric measure, $\mathcal{G}(|\psi\rangle)$, provided the maximum eigenvalue in obtaining $\mathcal{G}$ emerges from any single-qubit reduced density matrix.*

Proof. Let $a = \max\{\xi_m(\rho^{(1)})\}$ be the maximum eigenvalue corresponding to all possible single-qubit reduced density matrices, $\rho^{(1)}$ of $|\psi\rangle$, obtained from, say qubit j. The monogamy score for node j is given by $\delta_{\mathcal{Q},j} = \mathcal{Q}(\rho_{j:\text{rest}}) - \sum_{k \neq j} \mathcal{Q}(\rho^{(2)}_{jk})$. Therefore, one obtains $\delta_{\mathcal{Q},j} \leq \mathcal{Q}(\rho_{j:\text{rest}})$. Since $\mathcal{Q}$ is local unitary invariant, the quantity $\mathcal{Q}(\rho_{j:\text{rest}})$ is a function of the maximum eigenvalue a, $\mathcal{F}_{\mathcal{Q}}(a)$. Since, the maximum eigenvalue in obtaining the generalized geometric measure $\mathcal{G}$ emerges from a single-qubit reduced density matrix, we have $\mathcal{G} = 1 - a$. Thus we have $\mathcal{F}_{\mathcal{Q}}(a) = \mathcal{F}_{\mathcal{Q}}(\mathcal{G})$, which gives us the bound

$$\delta_{\mathcal{Q},j} \leq \mathcal{F}_{\mathcal{Q}}(a) = \mathcal{F}_{\mathcal{Q}}(\mathcal{G}). \tag{6.1}$$

Now the monogamy score, $\delta_{\mathcal{Q}}$, is defined as the minimum score over all possible nodes. Hence, $\delta_{\mathcal{Q}} \leq \delta_{\mathcal{Q},j}$, and thus we obtain an upper-bound on the monogamy score in terms of a function of generalized geometric measure, as given by, $\delta_{\mathcal{Q}}(|\psi\rangle) \leq \mathcal{F}_{\mathcal{Q}}(\mathcal{G}(|\psi\rangle))$. ■

From the proof of Theorem 6.2.1 it may appear that the $n-1$ bipartite correlation terms, $\mathcal{Q}(\rho^{(2)}_{jk})$, are not given due priviledge. However, this is not the case. This is observed, for example, in Figs. 6.1-6.3, where the region just above the curved boundaries contain representatives of a large number of multiparty states. On the boundary the bipartite terms are vanishing. As we move away from the boundary, the bipartite contributions come into the picture. The bound in Theorem 6.2.1, states that only the region above the boundary is populated. An important implication of Theorem 6.2.1 is that the above bound is valid for even those quantum correlation measures that can not be explicitly computed for arbitrary states using any analytical or numerical methods, such as distillable entanglement, entanglement cost, and relative entropy of entanglement. The theorem implies that any possible value of these measures will always result in monogamy scores that lie on or above the boundary. Moreover, GGM is a well-defined measure of genuine multipartite entanglement, which is monotonically non-increasing under local operations and classical communication, and thus provides a strong physical connection between the monogamy score bound and multiparty entanglement.

Let us consider the squares of concurrence ($\mathcal{C}^2$) and negativity ($\mathcal{N}^2$), from the entanglement-separability paradigm, quantum discord ($\mathcal{D}$) and quantum work-deficit ($\mathcal{W}$), from the information-theoretic paradigm, as quantum correlation measures. In the corollary below, we prove that for the above measures, the quantity $\mathcal{F}_{\mathcal{Q}}(\mathcal{G}(|\psi\rangle))$ can be analytically expressed in terms of a quadratic or an entropic function.

Corollary 6.2.2 *For all $|\psi\rangle$ that satisfy Theorem 6.2.1, the monogamy score for the quantum correlation measures $\mathcal{C}^2$ and $\mathcal{N}^2$ ($\mathcal{D}$ and $\mathcal{W}$), based on the entanglement-separability (information-theoretic) criteria, is bounded above by a quadratic (entropic) function, $\mathcal{F}_{\mathcal{Q}}(\mathcal{G})$, of the generalized geometric measure, $\mathcal{G}(|\psi\rangle)$.*

Proof. For $\mathcal{C}^2$ and $\mathcal{N}^2$, the quantity $\mathcal{Q}(\rho_{j:\text{rest}})$ is equal to $4\det(\rho_j^{(1)})$ and $\det(\rho_j^{(1)})$, respectively, when $\rho_{j:\text{rest}}$ is pure. For $\mathcal{D}$ and $\mathcal{W}$, the quantity $\mathcal{Q}(\rho_{j:\text{rest}})$ reduces to the von Neumann entropy, $S(\rho_j^{(1)})$, for pure states. Therefore, we obtain

$$\mathcal{Q}(\rho_{j:\text{rest}}) = z\ a(1-a) = z\ \mathcal{G}(1-\mathcal{G}) \equiv z\ f(\mathcal{G}), \tag{6.2}$$

where $z = 4$ and 1 for $\mathcal{C}^2$ and $\mathcal{N}^2$, respectively. Similarly,

$$\mathcal{Q}(\rho_{j:\text{rest}}) = S(\rho_j^{(1)}) = h(a) = h(\mathcal{G}), \tag{6.3}$$

for $\mathcal{D}$ and $\mathcal{W}$. $f(x) = x(1-x)$ and $h(x)$ is the Shannon entropy of the variable x. Hence, $\mathcal{F}_{\mathcal{Q}}(\mathcal{G})$ is a quadratic function equal to $z\ f(\mathcal{G})$ or an entropic function $h(\mathcal{G})$ of the generalized geometric measure, depending on whether the quantum correlation is entanglement-based or information-theoretic, and thus satisfies the upper-bound on the monogamy-score given in Theorem 6.2.1. ■

The applicability of the bound obtained in Theorem 6.2.1 is limited, in the sense that it is only valid for genuinely multipartite entangled pure states for which the maximum Schmidt coefficient contributing to the GGM of the state comes from the j : *rest* bipartition, where j is the single qubit. However, as we shall observe in the following analysis, the bound in Theorem 6.2.1 holds for a large number of randomly generated multiqubit states. There is only a small fraction of randomly generated states for which the maximal Schmidt coefficient comes from other bipartitions than the bipartition containing single qubit. Nevertheless, the results of Theorem 6.2.1 can be extended to other states with specific restrictions.

Proposition 6.2.3 *For n-qubit pure states $|\psi\rangle$, where $n > 3$ and the maximum eigenvalue in calculating the generalized geometric measure $\mathcal{G}(|\psi\rangle)$ emerges from a reduced density matrix, $\rho^{(k)}$, with $k \neq 1$, the upper-bound of monogamy score, $\delta_{\mathcal{Q}}(|\psi\rangle)$, of a quantum correlation, $\mathcal{Q}$, is a function of $\mathcal{G}$, provided the function $\mathcal{H}_{\mathcal{Q}}(|\psi\rangle)$, in Eq.(6.5), is nonnegative.*

Proof. From Theorem 6.2.1, we know that the monogamy score satisfies the relation, $\delta_{\mathcal{Q}} \leq \mathcal{F}_{\mathcal{Q}}(a)$, where $a = \max\{\xi_m(\rho^{(1)})\}$ is the maximum eigenvalue corresponding to all possible single-qubit reduced density matrices. However, in this case, the generalized geometric meausre, $\mathcal{G} = 1 - b$, where $b = \{\xi_m(\rho^{(k)})\}$ is the maximum eigenvalue corresponding to all possible non-single qubit bi-partitions, i.e., $k \neq 1$. Hence, the premise implies that $b > a$, and we have $\mathcal{F}_{\mathcal{Q}}(a) \neq \mathcal{F}_{\mathcal{Q}}(\mathcal{G})$. Therefore, though we have $\delta_{\mathcal{Q}} \leq \mathcal{F}_{\mathcal{Q}}(a)$, it does not necessarily imply $\delta_{\mathcal{Q}} \leq \mathcal{F}_{\mathcal{Q}}(\mathcal{G})$. Let us define the quantity, $\beta = b - a > 0$. Now, $\mathcal{F}_{\mathcal{Q}}(a) = \mathcal{F}_{\mathcal{Q}}(b - \beta)$. Expanding the above expression, one can show that $\mathcal{F}_{\mathcal{Q}}(b - \beta) = \mathcal{F}_{\mathcal{Q}}(b) - \mathcal{R}_{\mathcal{Q}}(b, \beta)$, where $\mathcal{R}_{\mathcal{Q}}(b, \beta)$ being function of b, β and $\mathcal{Q}$ can be different for various quantum correlation measures, as shown in Table 6.1. Since, $\mathcal{G} = 1 - b$, the bound on monogamy score can be written as

$$\delta_{\mathcal{Q}} \leq \mathcal{F}_{\mathcal{Q}}(b) - \mathcal{R}_{\mathcal{Q}}(b, \beta) = \mathcal{F}_{\mathcal{Q}}(\mathcal{G}) - \mathcal{R}_{\mathcal{Q}}(b, \beta). \tag{6.4}$$

Therefore, we obtain the bound $\delta_{\mathcal{Q}} \leq \mathcal{F}_{\mathcal{Q}}(\mathcal{G})$, provided $\mathcal{R}_{\mathcal{Q}}(b, \beta) \geq 0$. However, for $b > a \geq 1/2$, it can be easily shown that the function $\mathcal{R}_{\mathcal{Q}}(b, \beta)$ is always negative. Hence, to satisfy the bound, we look at the function,

$$\mathcal{H}_{\mathcal{Q}}(|\psi\rangle) = \sum_{k \neq j} \mathcal{Q}(\rho^{(2)}_{j:k}) + \mathcal{R}_{\mathcal{Q}}(b, \beta), \tag{6.5}$$

where j corresponds to the node for which $\delta_{\mathcal{Q},j}$ is minimal. We note that the quantities b, β, and $\mathcal{R}_{\mathcal{Q}}$ are independent of the node j. The monogamy score can be written as

$$\begin{aligned} \delta_{\mathcal{Q}} &= \mathcal{F}_{\mathcal{Q}}(b) - \left(\mathcal{R}_{\mathcal{Q}}(b, \beta) + \sum_{k \neq j} \mathcal{Q}(\rho^{(2)}_{j:k}) \right) \\ &= \mathcal{F}_{\mathcal{Q}}(\mathcal{G}) - \mathcal{H}_{\mathcal{Q}}(|\psi\rangle). \end{aligned} \tag{6.6}$$

Therefore, provided $\mathcal{H}_{\mathcal{Q}}(|\psi\rangle) \geq 0$, we obtain the bound, $\delta_{\mathcal{Q}} \leq \mathcal{F}_{\mathcal{Q}}(\mathcal{G})$[1]. ■

At this point, we note that for a system composed of a large number of qubits or arbitrary quantum correlation measures, the function $\mathcal{H}^{\mathcal{Q}}(|\psi\rangle)$, is in general, not easily accessible as it requires explicit calculation of the terms $\mathcal{Q}(\rho_{lk}^{(2)})$. However, for specific states and measures, including symmetric states, or for small number of qubits these quantities can be efficiently computed. In Section 6.3, we analytically estimate the quantities in Proposition 6.2.3, for several important classes of states, viz., the n-qubit Dicke states, the superposition of generalized GHZ and W states, and ground states of interesting physical systems, such as quantum spin-1/2 lattices. Moreover, in Section 6.4, we numerically estimate these quantities for randomly generated Haar-uniform four and five qubit pure states. All these states provide important instances where the upper bound on monogamy score in terms of the GGM are exemplified. Proposition 6.2.3 thus provides the generic mathematical apparatus to estimate the bound on monogamy score. Analytical and numerical analyses of symmetric and random pure states show that the fraction of states that satisfy the above conditions, and thus may violate the bound on monogamy score, is extremely small. Table 6.2 indicates the percentages of states that satisfy each of these conditions for different classes of states [2]. It is evident that a large majority of four and five qubit states satisfy the bound on monogamy score.

For symmetric n-qubit pure states, $|\psi\rangle$, as discussed in following sections, the expression for $\mathcal{H}_{\mathcal{Q}}(|\psi\rangle)$ simplifies significantly, as all the $n-1$ bipartite quantum correlation contributions, $\mathcal{Q}(\rho_{j:k}^{(2)})$, are equivalent. Hence, the expression for

[1]The condition $\mathcal{H}_{\mathcal{Q}}(|\psi\rangle) \geq 0$, implies that the distribution of bipartite entanglement between the nodal party, l, and each of the parties, k, given by $\sum_{k\neq l} \mathcal{Q}(\rho_{lk}^{(2)})$, must be larger than the absolute value of $\mathcal{R}^{\mathcal{Q}}(b,\beta)$ for the bound to be satisfied.

[2]For three-qubit pure states, there exist two inequivalent classes of states under stochastic local operation and classical communication (SLOCC), namely the GHZ and the W class of states [83]. However, for four-qubits, there exist infinitely many inequivalent SLOCC classes of states [229]. A useful classification into nine classes for four-qubit was obtained in [230, 231]. It was observed that up to permutation of the qubits, any four-qubit pure state can be transformed into one of the nine classes of states $\{|G^x\rangle\}$, as shown in Table 6.3.
Also, a general symmetric state can be written as a linear combination of the Dicke states as

$$|\psi_{sym}^n\rangle = \sum_{r=0}^{n} a_r \, |D_r^n\rangle, \tag{6.7}$$

where $|D_r^n\rangle$ is an n-qubit Dicke state [216] with r excitations, given in Eq. (6.9). The normalization condition is satisfied by demanding $\sum_{r=0}^{n} |a_r|^2 = 1$. Any general symmetric state can be generated by randomly choosing a set of coefficients a_r that satisfy the normalization.

$\mathcal{H}_{\mathcal{Q}}(|\psi\rangle)$ reduces to

$$\mathcal{H}_{\mathcal{Q}}(|\psi\rangle) = \mathcal{R}_{\mathcal{Q}}(b,\beta) + (n-1)\mathcal{Q}(\rho^{(2)}_{j:k}). \tag{6.8}$$

$\mathcal{Q}$	$\mathcal{F}_{\mathcal{Q}}(b)$	$\mathcal{R}_{\mathcal{Q}}(b,\beta)$
$\mathcal{C}^2$	$4b(1-b)$	$4\beta(1-2b+\beta)$
$\mathcal{N}^2$	$b(1-b)$	$\beta(1-2b+\beta)$
$\mathcal{D}$, $\mathcal{W}$	$h(b)$	$-((b-\beta)\log_2 b + (1-b-\beta)\log_2(1-b))$ $-h(b-\beta) - \beta\log_2(\frac{b}{1-b})$

Table 6.1: Expressions of $\mathcal{F}_{\mathcal{Q}}(b)$ and $\mathcal{R}_{\mathcal{Q}}(b,\beta)$ for squared concurrence ($\mathcal{C}^2$), squared negativity ($\mathcal{N}^2$), quantum discord ($\mathcal{D}$) and quantum work-deficit ($\mathcal{W}$), where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$, is the binary Shannon entropy.

Hence, to violate the bound on momogamy score, an n-qubit pure state $|\psi\rangle$, where $n > 3$, must simultaneously satisfy the following necessary conditons: $\beta > 0$ and $\mathcal{H}_{\mathcal{Q}}(|\psi\rangle) < 0$.
An important question to consider is – *When does a multiparty state violate the bound on monogamy score?* From previous discussions, it is clear that $\mathcal{H}^{\mathcal{Q}}(|\psi\rangle) < 0$ ensures that the state will not satisfy Proposition 6.2.3. Moreover, it has also been established that $\mathcal{H}^{\mathcal{Q}}(|\psi\rangle)$ may not always be analytically accessible, or even numerically tractable, for all states and quantum correlation measures. Hence, it is impossible to analytically characterize a multiparty state that may violate the bound in Proposition 6.2.3. However, one can construct examples of a genuinely multiparty separable state, to highlight certain properties of states that may violate the bound on monogamy score. For instance, consider the six-qubit state, $|\psi\rangle = |\psi_g\rangle \otimes |\psi_g\rangle$, where $|\psi_g\rangle = \frac{1}{2}(|000\rangle + |111\rangle)$ is the 3-qubit Greenberger-Horne-Zeilinger (GHZ) state. First, note that the state is not genuinely multiparty entangled and hence, $\mathcal{G}(\psi) = 0$ ($b = 1$). Interestingly, for a GHZ state, $\mathcal{Q}(\rho^{(2)}_{lk}) = 0$, $\forall\ l, k$. It implies that choosing any qubit as the nodal site, we obtain, $\mathcal{H}^{\mathcal{Q}}(|\psi\rangle) < 0$, since $\mathcal{R}^{\mathcal{Q}} = -1$. Therefore, the state $|\psi\rangle$ can violate Proposition 6.2.3. This is true since the monogamy score, for the measure $\mathcal{C}^2$, is equal to 1, for $|\psi\rangle$. Thus, $\delta^{\mathcal{Q}} > \mathcal{G}$. We note that the crux of the question lies in the quantity, $\mathcal{Q}(\rho^{(2)}_{lk})$ and its relation to the function, $-\mathcal{R}^{\mathcal{Q}} = f^{\mathcal{Q}}(b) - f^{\mathcal{Q}}(a)$. For states with low $\mathcal{G}$ (high b) and low aggregate of $\mathcal{Q}(\rho^{(2)}_{lk})$, the term $-\mathcal{R}^{\mathcal{Q}}$ can dominate in $\mathcal{H}^{\mathcal{Q}}(|\psi\rangle)$, leading to the violation of Proposition 6.2.3.

Corollary 6.2.4 *For all three-qubit pure states $|\psi\rangle$ the monogamy score, $\delta_{\mathcal{Q}}(|\psi\rangle)$, is upper-bounded by a quadratic (an entropic) function of the generalized geo-*

State	β	$\mathcal{H}_{\mathcal{C}^2}$	$\mathcal{H}_{\mathcal{N}^2}$	$\mathcal{H}_{\mathcal{D}}$	$\mathcal{H}_{\mathcal{W}}$
$\lvert G^1\rangle$	99.87	0.007	0	0	0
$\lvert G^2\rangle$	91.65	0	0	0	0
$\lvert G^3\rangle$	57.04	0	0	0	0
$\lvert G^4\rangle$	97.58	0	0	0	0
$\lvert G^5\rangle$	60.77	0	0	0	0
$\lvert G^6\rangle$	4.97	0	0	0	0
$\lvert\psi^4_{sym}\rangle$	6.37	0	0	0	0
$\lvert\psi^5_{sym}\rangle$	0.303	0	0	0	0
$\lvert\psi^4_{gen}\rangle$	4.44	0	0	0	0
$\lvert\psi^5_{gen}\rangle$	0.26	0.12	0.125	0	0

Table 6.2: Percentages of states that satisfy the necessary conditions, $\beta > 0$ and $\mathcal{H}_{\mathcal{Q}}(|\psi\rangle) < 0$. We randomly generate 2.5×10^5 states belonging to different classes using uniform Haar distributions. States that simultaneously satisfy all three inequalities may violate the monogamy bound. $|G^i\rangle$ ($i = 1$ to 6) are parameterized inequivalent classes under stochastic local operation and classical communication (SLOCC) for four-qubit states defined in [230, 231]. $|\psi^k_{sym}\rangle$ and $|\psi^k_{gen}\rangle$ are randomly generated symmetric and general k-qubit states, respectively.

metric measure, $\mathcal{G}(|\psi\rangle)$, for a quantum correlation, $\mathcal{Q}$, based on entanglement-separability (information-theoretic) criteria.

Proof. For any three-qubit pure state $|\psi\rangle$, for all bipartitions, the relevant reduced density matrices are the single-qubit reduced density matrices $\{\rho^{(1)}\}$. Hence, the maximum eigenvalue contributing to the generalized geometric measure, $\mathcal{G}(|\psi\rangle)$, always comes from $\{\rho^{(1)}\}$, thus satisfying the premise of Theorem 6.2.1. ■

We note that in Corollary 6.2.4, which is a special case of the generalized n-qubit statement presented in Theorem 6.2.1 of our study, was formally shown in [212]. However, the results presented therein do not contain any information about those pure states where the maximum eigenvalue contributing to the calculation of GGM does not arise from the j : rest bipartition, as conjectured in Proposition 6.2.3.

In subsequent sections we analyze the statements made in Theorem 6.2.1 and Proposition 6.2.3, through the quantum correlation measures $\mathcal{C}^2$ and $\mathcal{N}^2$, based on entanglement-separability, and $\mathcal{D}$ and $\mathcal{W}$, based on information-theoretic criteria. Table 6.1 provides the specific form of the functions $\mathcal{F}_{\mathcal{Q}}(b)$ and $\mathcal{R}_{\mathcal{Q}}(b, \beta)$ for the measures, $\mathcal{C}^2$, $\mathcal{N}^2$, $\mathcal{D}$ and $\mathcal{W}$. We refer to the bounds obtained on the monogamy scores as the *multiparty entanglement bounds.*

$\vert G^x\rangle$ (unnormalized)
$\vert G^1_{abcd}\rangle = \frac{a+d}{2}(\vert 0000\rangle + \vert 1111\rangle) + \frac{a-d}{2}(\vert 0011\rangle + \vert 1100\rangle) + \frac{b+c}{2}(\vert 0101\rangle + \vert 1010\rangle) + \frac{b-c}{2}(\vert 0110\rangle + \vert 1001\rangle)$
$\vert G^2_{abc}\rangle = \frac{a+b}{2}(\vert 0000\rangle + \vert 1111\rangle) + \frac{a-b}{2}(\vert 0011\rangle + \vert 1100\rangle) + c(\vert 0101\rangle + \vert 1010\rangle) + \vert 0110\rangle$
$\vert G^3_{ab}\rangle = a(\vert 0000\rangle + \vert 1111\rangle) + b(\vert 0101\rangle + \vert 1010\rangle) + \vert 0110\rangle + \vert 0011\rangle$
$\vert G^4_{ab}\rangle = a(\vert 0000\rangle + \vert 1111\rangle) + \frac{a+b}{2}(\vert 0101\rangle + \vert 1010\rangle) + \frac{a-b}{2}(\vert 0110\rangle + \vert 1001\rangle) + \frac{i}{\sqrt{2}}(-\vert 0001\rangle - \vert 0010\rangle + \vert 0111\rangle + \vert 1011\rangle)$
$\vert G^5_{a}\rangle = a(\vert 0000\rangle + \vert 0101\rangle + \vert 1010\rangle + \vert 1111\rangle) + i(\vert 0001\rangle - \vert 1011\rangle) + \vert 0110\rangle$
$\vert G^6_{a}\rangle = a(\vert 0000\rangle + \vert 1111\rangle) + \vert 0011\rangle + \vert 0101\rangle + \vert 0110\rangle$
$\vert G^7\rangle = \vert 0000\rangle + \vert 0101\rangle) + \vert 1000\rangle + \vert 1110\rangle$
$\vert G^8\rangle = \vert 0000\rangle + \vert 1011\rangle) + \vert 1101\rangle + \vert 1110\rangle$
$\vert G^9\rangle = \vert 0000\rangle + \vert 1111\rangle$

Table 6.3: Normal-form representatives of the nine four-qubit SLOCC inequivalent classes defined in [230, 231]. Here a, b, c, d are complex parameters with non-negative real parts. The first six classes are parameterized.

6.3 Analyzing the upper bounds for special multiqubit states

In this section, we study some important classes of multipartite states for which the multiparty entanglement bound on monogamy score holds. If in the evaluation of GGM, the maximum eigenvalue is obtained from the $1 : rest$ bipartition, then monogamy score is always bounded above by the GGM via Theorem 6.2.1. However, Proposition 6.2.3 holds when a state obeys certain conditions. We consider several paradigmatic states for which we check whether the criteria required for Proposition 6.2.3 to hold are satisfied.

6.3.1 Dicke states

Let us consider an n-qubit Dicke state [216] with r excitations, given by the equation

$$|D^n_r\rangle = \binom{n}{r}^{-\frac{1}{2}} \sum \mathcal{P}\left(|0\rangle^{\otimes(n-r)} \otimes |1\rangle^{\otimes r}\right), \tag{6.9}$$

where the summation is over all possible permutations ($\mathcal{P}$) of the product state having r qubits in the excited state, $|1\rangle$, and $n-r$ qubits in the ground state, $|0\rangle$. The state $|D^n_1\rangle$ is the well-known W state [83, 149, 232]. Since, the Dicke state is symmetric, all $k : rest$ bipartitions are equivalent, and the reduced density matrix can be written as

$$\rho^{(k)}_D = \binom{n}{r}^{-1} \sum_{i=0}^{k} \binom{k}{i}\binom{n-k}{r} |D^k_i\rangle\langle D^k_i|. \tag{6.10}$$

Using Eq. (6.10), the maximum eigenvalue contributing to GGM can be obtained. For $r \neq \frac{n}{2}$, the maximum eigenvalue comes from the $1 : rest$ $(k = 1)$ bipartition and is equal to a, where

$$\begin{aligned} a = \binom{n-1}{r-1} \Big/ \binom{n}{r} &= \frac{r}{n}, \qquad \text{for} \;\; r > \frac{n}{2}, \\ \text{and} \;\; a = \binom{n-1}{r} \Big/ \binom{n}{r} &= 1 - \frac{r}{n}, \;\; \text{for} \;\; r < \frac{n}{2}. \end{aligned} \tag{6.11}$$

Hence, for the n-qubit Dicke state $|D_r^n\rangle$ with $r \neq \frac{n}{2}$, the bound on monogamy score is satisfied via Theorem 6.2.1. However, for the Dicke state with $r = \frac{n}{2}$, the situation is involved. The maximum eigenvalue comes from the $2 : rest$ $(k = 2)$ bipartition and is equal to $b = 2\binom{n-2}{r-1}/\binom{n}{r} = \frac{n}{2(n-1)}$. Hence, for $r = \frac{n}{2}$, the quantities $\beta = b - a = \frac{1}{2(n-1)} > 0$, $\mathcal{R}_{\mathcal{C}^2(\mathcal{N}^2)} = z\beta(1 - 2b + \beta) = -\frac{z}{4(n-1)^2} < 0$, where $z = 4$ (1) for $\mathcal{C}^2$ $(\mathcal{N}^2)$, and $\mathcal{R}_{\mathcal{D}(\mathcal{W})} = -\frac{1}{2}\left(\log_2 \frac{n(n-2)}{(n-1)^2} + \frac{1}{n-1}\log_2 \frac{n}{n-2}\right) < 0$. Hence, for all even $n \geq 4$ and $r = \frac{n}{2}$, for the monogamy score bound to be satisfied, we must have $\mathcal{H}_{\mathcal{Q}}(|D_r^n\rangle) \geq 0$.

For the symmetric Dicke states, analytical forms for $\mathcal{C}^2$, $\mathcal{N}^2$ and $\mathcal{D}$, for any two-qubit density matrices, are known and can be written as [15]

$$\mathcal{C}^2(\rho_{ij}^{(2)}) = 4(v - \sqrt{uw})^2, \tag{6.12}$$

$$\mathcal{N}^2(\rho_{ij}^{(2)}) = \frac{1}{4}|(u + w) - \sqrt{(u - w)^2 + 4v^2}|^2, \tag{6.13}$$

$$\mathcal{D}(\rho_{ij}^{(2)}) = S' - S'' + h(l), \tag{6.14}$$

where $l = \frac{1}{2}\left(1 + \sqrt{1 - 4(uv + vw + wu)}\right)$, $S' = -(u + v)\log_2(u + v) - (v + w)\log_2(v + w)$, $S'' = -u\log_2 u - 2v\log_2 2v - w\log_2 w$, and $u = (n - r)(n - r - 1)/(n^2 - n), v = r(n - r)/(n^2 - n)$, and $w = r(r - 1)/(n^2 - n)$.

For the Dicke state with $r = \frac{n}{2}$, all these quantities become functions of a single parameter, the size of the state, n. Using Eq. (6.8) for symmetric pure states, it can be easily shown that the quantity, $\mathcal{H}_{\mathcal{Q}}(|D_{n/2}^n\rangle) = (n - 1)\mathcal{Q}(\rho_{ij}^{(2)}) + \mathcal{R}_{\mathcal{Q}} \geq 0$, for the quantum correlation measures $\mathcal{C}^2$, $\mathcal{N}^2$, and $\mathcal{D}$. Thus, from Proposition 6.2.3, the multiparty entanglement bound on monogamy score is satisfied for these states.

6.3.2 Superposition of generalized GHZ and W states

Let us consider the permutationally invariant states defined by a superposition of generalized Greenberger-Horne-Zeilinger (GHZ) [148] state and W state [83,149], given by

$$|\Psi^n_{\alpha,\gamma}\rangle = \alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n} + \gamma|W^n\rangle, \tag{6.15}$$

where $(\alpha, \beta, \gamma) \in \mathbb{C}$, $|\beta| = \sqrt{1 - |\alpha|^2 - |\gamma|^2}$ and $|W^n\rangle$ is the normalized n-qubit W state. For $\gamma = 0$, $|\Psi^n_{\alpha,\gamma}\rangle$ becomes the generalized GHZ state. To obtain the reduced density matrices, one can notice that the state can be rewritten in the form

$$\begin{aligned} |\Psi^n_{\alpha,\gamma}\rangle &= \alpha|0\rangle^{\otimes k}|0\rangle^{\otimes n-k} + \beta\sqrt{\frac{n-k}{n}}|0\rangle^{\otimes k}|W^{n-k}\rangle \\ &+ \beta\sqrt{\frac{k}{n}}|W^k\rangle|0\rangle^{\otimes n-k} + \gamma|1\rangle^{\otimes k}|1\rangle^{\otimes n-k}. \end{aligned} \tag{6.16}$$

Therefore, the reduced k-qubit ($k \geq 2$) density matrix can be written as

$$\rho^{(k)}_{\alpha,\gamma} = \begin{pmatrix} |\alpha|^2 + |\beta|^2\frac{n-k}{n} & \alpha^*\beta\sqrt{\frac{k}{n}} & 0 \\ \alpha\beta^*\sqrt{\frac{k}{n}} & |\beta|^2\frac{k}{n} & 0 \\ 0 & 0 & |\gamma|^2 \end{pmatrix}, \tag{6.17}$$

in the orthogonal basis formed by $|0\rangle^{\otimes k}$, $|W^k\rangle$, and $|1\rangle^{\otimes k}$. By evaluating the eigenvalues of the above matrix, we find that the maximum eigenvalue corresponds to the $1 : rest$ ($k = 1$) bipartition and is given by

$$a = \frac{1}{2}\left(1 + \sqrt{1 - 4|\alpha|^2|\gamma|^2 + \frac{4(n-1)}{n}|\beta|^2\left(|\gamma|^2 + \frac{|\beta|^2}{n}\right)}\right). \tag{6.18}$$

Hence for $|\Psi^n_{\alpha,\gamma}\rangle$, the multiparty entanglement bound on monogamy score is satisfied via Theorem 6.2.1.

6.3.3 The Majumdar-Ghosh model

Let us now consider a physical system that is useful in studying quantum phenomena in strongly-correlated quantum spin systems. The Majumdar-Ghosh (MG) model [233] is a one-dimensional, antiferromagnetic frustrated system, with a Hamiltonian given by

$$H_{\text{MG}} = J_1 \sum_{\langle i,j\rangle} \vec{\sigma_i} \cdot \vec{\sigma_j} + \frac{J_1}{2} \sum_{\langle\langle i,j\rangle\rangle} \vec{\sigma_i} \cdot \vec{\sigma_j}, \tag{6.19}$$

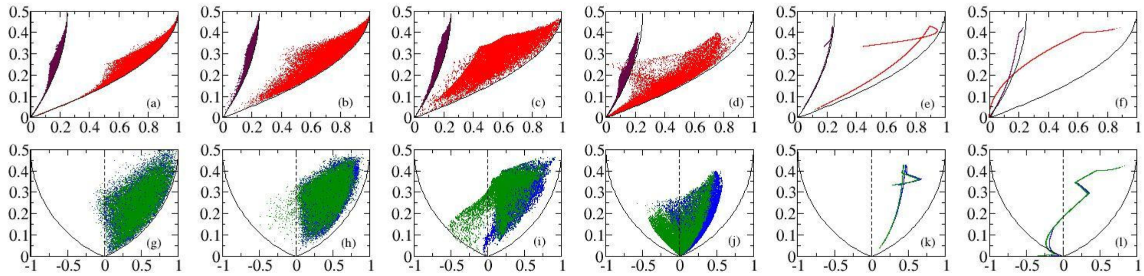


Figure 6.1: Genuine multiparty entanglement versus quantum monogamy scores for the SLOCC inequivalent classes. The figure exhibits plots of quantum monogamy scores ($\delta_{\mathcal{Q}}$), as the abscissae, against the generalized geometric measure ($\mathcal{G}$), as the ordinates. Monogamy scores for squared-concurrence (red dots) and squared-negativity (maroon dots) are shown in the first row (Figs. 6.1(a-f)), and quantum discord (blue dots) and quantum work-deficit (green dots) are shown in the second row (Figs. 6.1(g-l)). Each of the six columns represents plots for 2.5×10^5 random states, generated through uniform Haar distribution, for the normal-form representatives of the six four-qubit SLOCC inequivalent classes ($|G^1\rangle$ (Figs. 6.1(a,g)) through $|G^6\rangle$ (Figs. 6.1(f,l)) given in Table 6.3). The multiparty entanglement bounds on the monogamy scores are given by the equation, $\delta_{\mathcal{Q}}(|\psi\rangle) = \mathcal{F}_{\mathcal{Q}}(\mathcal{G}(|\psi\rangle))$, as proposed in Sec. 6.2. Monogamy scores for quantum discord and quantum work-deficit, in the second row, can be negative but are not bounded by the negative of the entropic function, i.e., the mirror image of the equation $\delta_{\mathcal{Q}} = \mathcal{F}_{\mathcal{Q}}(\mathcal{G})$, for $\delta_{\mathcal{Q}} > 0$, about the $\delta_{\mathcal{Q}} = 0$ axis. The abscissae are measured in ebits. The ordinates are measured in ebits for Figs. 6.1(a-f) and in bits for Figs. 6.1(g-l).

where $\langle i, j\rangle$ and $\langle\langle i, j\rangle\rangle$ refer to the nearest and the next-nearest neighbors interactions respectively. $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli spin operators, and $J_1 > 0$. Here, we assume that the number of spins, n, is even, and the chain is periodic. The MG model is a special case of the more general $J_1 - J_2$ model for which the ground state is exactly known for $J_2 = \frac{J_1}{2}$ [233]. The n-qubit ground state is doubly degenerate and frustrated. The ground state space is spanned by

$$|\psi^n_{(\pm)}\rangle = \frac{1}{2^{n/4}} \prod_{i=1}^{n/2} (|0_{2i}1_{2i\pm1}\rangle - |1_{2i}0_{2i\pm1}\rangle). \tag{6.20}$$

Let us consider the ground state

$$|\Psi^n_{\rm MG}\rangle = |\psi^n_{(+)}\rangle + |\psi^n_{(-)}\rangle. \tag{6.21}$$

It is known to be genuinely multipartite entangled and is rotationally invariant [144,234]. For $n \geq 4$, the maximum eigenvalue is known to come from the 2 : *rest* nearest-neighbor bipartition, where the reduced two-qubit density matrix is the rotationally invariant Werner state. The maximum eigenvalue from the 1 : *rest* bipartition is $a = \frac{1}{2}$. The maximum eigenvalue from the nearest-neighbor $2 : rest$ bipartition is $b = \frac{1+3p}{4}$, where p is the Werner parameter, given by

$$p = \frac{1 + 2^{\frac{n}{2}-2}}{1 + 2^{\frac{n}{2}-1}}. \tag{6.22}$$

Hence for $n > 4$, we have $p > \frac{1}{3}$, which implies $\beta = b - a = \frac{3p-1}{4} > 0$. For the reduced two-site density matrix, the exact analytical forms for $\mathcal{C}^2$, $\mathcal{N}^2$, and $\mathcal{D}$ are known in terms of the Werner parameter p, and can be written as

$$\begin{aligned}
\mathcal{C}^2(\rho_{ij}^{(2)}) &= \max\left[0, \frac{3p-1}{2}\right]^2, \\
\mathcal{N}^2(\rho_{ij}^{(2)}) &= \left|\frac{1-3p}{4}\right|^2, \\
\mathcal{D}(\rho_{ij}^{(2)}) &= \frac{p_-}{4}\log_2(p_-) - \frac{p_+}{2}\log_2(p_+) + \frac{p'}{4}\log_2(p'),
\end{aligned} \tag{6.23}$$

where $p_\pm = 1 \pm p$, $p' = 1 + 3p$, and i and j are the nearest-neighbors. To prove that the multiparty entanglement bounds on monogamy scores hold for these quantum correlations for the ground states of the MG model, we need to show that either of the quantities, $\mathcal{R}_\mathcal{Q}$ or $\mathcal{H}_\mathcal{Q}(|\Psi^n_{\text{MG}}\rangle)$, is positive. Using Eq. (6.22), we can derive that $\mathcal{R}_{\mathcal{C}^2(\mathcal{N}^2)} = -z\left(\frac{1-3p}{4}\right)^2 < 0$, for $p > \frac{1}{3}$. Similarly, one can show that $\mathcal{R}_{\mathcal{D}(\mathcal{W})} < 0$. Hence, we need to show that the quantity $\mathcal{H}_\mathcal{Q}(|\Psi^n_{\text{MG}}\rangle) > 0$. For the ground state, $|\Psi^n_{\text{MG}}\rangle$, only the nearest-neighbor spins are entangled and $\mathcal{C}^2(\rho_{ij}^{(2)}) = \mathcal{N}^2(\rho_{ij}^{(2)}) = 0$, for $j \neq i \pm 1$. $\mathcal{D}(\rho_{ij}^{(2)})$ for non-nearest-neighbor qubits is finite but two orders of magnitude lower than the nearest-neighbor values. Hence, using Eq. (6.8), one obtains $\mathcal{H}_\mathcal{Q}(|\Psi^n_{\text{MG}}\rangle) = \mathcal{R}_\mathcal{Q} + \mathcal{Q}(\rho_{i(i+1)}^{(2)}) + \mathcal{Q}(\rho_{i(i-1)}^{(2)})$, where we approximate $\mathcal{D}(\rho_{ij}^{(2)}) = \mathcal{W}(\rho_{ij}^{(2)}) \approx 0$, for $j \neq i \pm 1$. For $\mathcal{C}^2$ and $\mathcal{N}^2$, $\mathcal{H}_{\mathcal{C}^2(\mathcal{N}^2)} = \frac{z}{16}(1-3p)^2 > 0$. Similarly, for $\mathcal{D}$ and $\mathcal{W}$, one can show that $\mathcal{H}_{\mathcal{D}(\mathcal{W})} > 0$, for all n. Hence, the monogamy score bound is satisfied via Proposition 6.2.3.

6.3.4 The Ising model

In this section, we consider two paradigmatic Hamiltonians belonging to the Ising group of models [235] that give us multipartite entangled ground states. We first consider the highly frustrated Ising model with long-range antiferromagnetic interactions, also called the Ising gas model. The Hamiltonian for an n-spin Ising gas is given by

$$H_{\text{gas}}(x) = \frac{J}{n}(S - nx)^2, \quad J > 0, \tag{6.24}$$

where $S = \sum_i \sigma_i^z$, $J > 0$, and $0 \leq x \leq 1$. The quenched unnormalized ground state of the Ising gas Hamiltonian is given by [233]

$$|\Psi_{\text{gas}}^n\rangle = \sum_{\{[0,1]\}} |0\rangle^{\otimes n(1+x)/2} \otimes |1\rangle^{\otimes n(1-x)/2}, \tag{6.25}$$

where $\{[0,1]\}$ indicates that the summation is over all possible combinations of $|0\rangle$ and $|1\rangle$ that satisfy the density $\frac{1+x}{1-x}$. For maximally frustrated ground states, the density is unity ($x = 0$), and the ground state reduces to the Dicke state, given by Eq. (6.9), for $r = \frac{n}{2}$. For these states, as discussed in Sec. 6.3.1, the multiparty entanglement always gives the upper bound on the monogamy of quantum correlation.

We next consider the weakly frustrated, periodic Ising spin chain with nearest-neighbor interactions, also called the Ising ring. All the nearest-neighbor interactions are ferromagnetic, except one that is antiferromagnetic. The Hamiltonian is given by

$$H_{\text{ring}} = -J \sum_{i=1}^{n-1} \sigma_i^z \sigma_{i+1}^z + J\sigma_n^z \sigma_1^z, \quad J > 0. \tag{6.26}$$

The quenched ground state of the Hamiltonian is given by [233]

$$\begin{aligned} |\Psi_{\text{ring}}^n\rangle &= \sum_{k=0}^{n-1} \Big(|0^{\otimes n-k} 1^{\otimes k}\rangle + |1^{\otimes n-k} 0^{\otimes k}\rangle \\ &+ |1^{\otimes k+1} 0^{\otimes n-k-1)}\rangle + |0^{\otimes k+1} 1^{\otimes n-k-1)}\rangle \Big) . \end{aligned} \tag{6.27}$$

For the ground state given in Eq. (6.27), the reduced density matrix from the $2:rest$ bipartition can be written as

$$\rho_{\text{ring}}^{(2)} = \frac{1}{2n} \begin{pmatrix} n-1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & n-1 \end{pmatrix}. \tag{6.28}$$

The maximum eigenvalue for the $2 : rest$ bipartition is given by $b = \frac{1}{4n}(n + 2 + \sqrt{n^2+16})$, and for the $1 : rest$ bipartition is given by $a = \frac{1}{2}(1 + \frac{1}{n})$. These eigenvalues are highest among all bipartitions. For any finite number of spins n, $a \geq b$. Therefore the bound on monogamy is satisfied via Theorem 6.2.1. Interestingly, for $n \to \infty$, the maximum eigenvalues, $a = b = \frac{1}{2}$, and maximum GGM is achieved.

6.4 Numerical results

The n-qubit states, considered in the analytical study of the bound on quantum monogamy in the previous section, constitute some special classes of multiparty state of arbitrary number of qubits. To visualize the multiparty entanglement bound obtained in Theorem 6.2.1 and Proposition 6.2.3, we now randomly generate four- and five-qubit states. The random states are chosen using Haar uniform distribution.

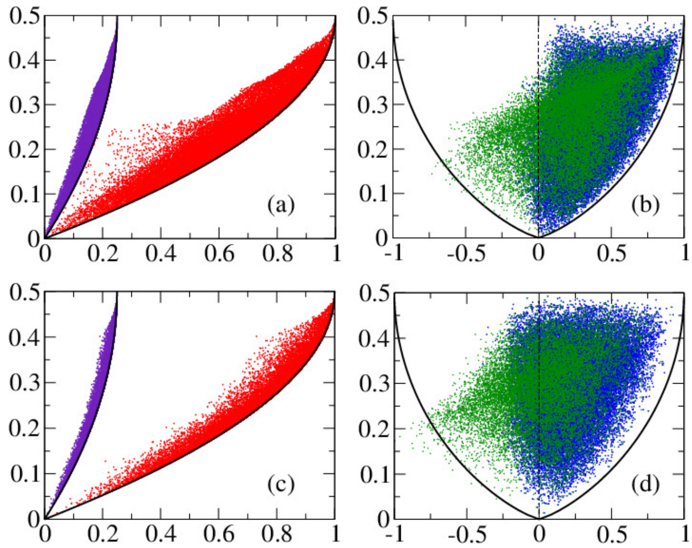


Figure 6.2: Genuine multiparty entanglement versus quantum monogamy scores for **symmetric** states. The figure shows the plot of quantum monogamy scores, along the abscissae, against the generalized geometric measure, along the ordinates, for symmetric four-qubit (Figs. 6.2(a,b)) and five-qubit (Figs. 6.2(c,d)) states generated using random superposition of Dicke states. The description of quantum correlation measures, random state generation, bounds, and the axes are the same as in Fig. 6.1.

Fig. 6.1 depicts the behavior of genuine multiparty entanglement with respect to the quantum monogamy scores for $\mathcal{C}^2$ and $\mathcal{N}^2$ (Figs. 6.1(a-f)) and for $\mathcal{D}$ and $\mathcal{W}$ (Figs. 6.1(g-l)) for the randomly generated four-qubit states corresponding to the parametrized six SLOCC inequivalent classes ($|G^i\rangle$, for $i = 1$ to 6). The nine SLOCC inequivalent classes of four-qubit states are discussed in Ref. [230] and their exact forms are given in Table 6.3. Fig. 6.1 shows that the quantum monogamy scores are bounded by the quadratic and entropic functions of generalized geometric measure for the set of states belonging to the SLOCC inequivalent classes for four-qubits. It is known that quantum discord and quantum work-deficit can have negative monogamy scores for certain states, i.e., the measures are not monogamous [84,85]. This is observed by the negative regions Figs. 6.1(g-l).

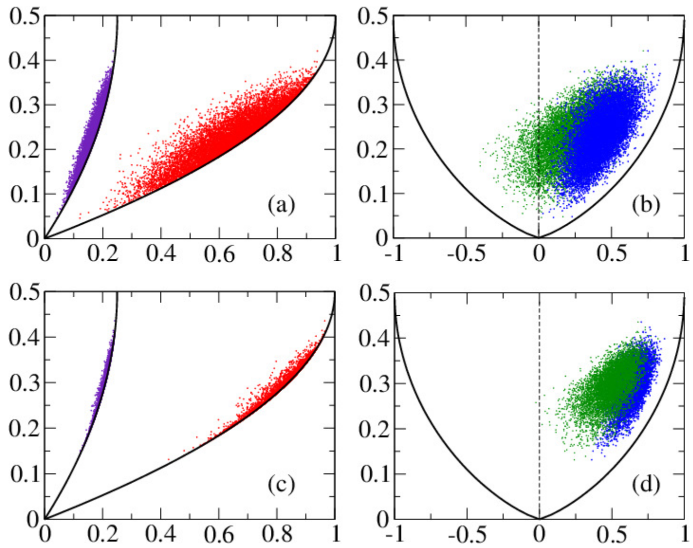


Figure 6.3: Genuine multiparty entanglement versus quantum monogamy scores for **generic** four-qubit (Figs. 6.3(a,b)) and five-qubit (Figs. 6.3(c,d)) states. The description is same as in Fig. 6.2.

Fig. 6.2 shows the bound on monogamy scores for randomly generated symmetric four- and five-qubit states. The symmetric states are generated using a random superposition of Dicke states, with different excitations, by using a uniform Haar distribution. The figure shows that the generated symmetric four- and five-qubit states satisfy the multiparty entanglement bound on quantum monogamy in terms of the functions of the generalized geometric measure. Fig. 6.3 exhibits the bound for randomly generated four- and five-qubit states using a uniform Haar distribution.

From the analytical and numerical results obtained in the preceding and this sections, it is observed that the bound on quantum monogamy scores for the quantum correlation measures $\mathcal{C}^2$, $\mathcal{N}^2$, $\mathcal{D}$, and $\mathcal{W}$, in terms of derived functions of the generalized geometric measure, is satisfied for a large majority of multiqubit quantum states.

6.5 Lower bounds on monogamy scores for non-monogamous measures

An important aspect of the monogamy score, $\delta_{\mathcal{Q}} = \mathcal{Q}(\rho_{AB}) - \sum_{k=1}^{n} \mathcal{Q}(\rho_{AB_k})$, is that the quantity has a distinct upper bound given by the term $Q(\rho_{AB})$, where we use $\rho_{AB} = \rho_{A(B_1 \cdots B_n)}$. This implies that for monogamous $\mathcal{Q}$, the amount of distributed bipartite entanglement in a multipartite state is bounded by the amount of block entanglement shared by a single party with the rest of the system. We have seen above that the upper bound of $\delta_{\mathcal{Q}}$, for a large number of pure n-qubit states, is determined by a set of entropic or quadratic functions of the genuine multipartite entanglement of the state ρ_{AB} (see Ref. [17]). Moreover, for monogamous $\mathcal{Q}$, there exists a definite lower bound on $\delta_{\mathcal{Q}}$, given by $\delta_{\mathcal{Q}} \geq 0$. However, for $\mathcal{Q}$ that does not satisfy monogamy inequality, in general, the situation is not straightforward. Though the upper bound remains the same, a non-trivial lower bound is not obvious. For cases where ρ_{AB} is not monogamous with respect to $\mathcal{Q}$, $\delta_{\mathcal{Q}}$ can be negative. One of the primary motivation of our work is to find out the non-trivial degree or limit to which this violation of monogamy inequality occurs. In other words, we question, what is the worst negative value of the quantity $\delta_{\mathcal{Q}}$? In this section, we aim at obtaining lower bounds on monogamy scores, for non-mongamous quantum correlations, in terms of "purity" [3]. To achieve this, we will use the *complementarity relation* (CR) between purity of a subsystem and a bipartite quantum correlation measure $\mathcal{Q}$ of the whole quantum system. The aforementioned complementarity relation was introduced in Ref. [18], and has been used to obtain a security proof of quantum cryptography for individual attacks via a variation [236] of the Ekert key distribution [12] protocol, in the same paper. For the sake of completeness, we reproduce the complementarity relation in Theorem 6.5.1.

6.5.1 Complementarity relation

Theorem 6.5.1 *If a bipartite quantum correlation measure $\mathcal{Q}$, for quantum state $\rho_{XY} \in \mathbb{C}^{d_X} \otimes \mathbb{C}^{d_Y}$, satisfies*

$$\mathcal{Q}(\rho_{XY}) \leq S(\rho_X), \tag{6.29}$$

and

$$0 \leq \mathcal{Q}(\rho_{XY}) \leq \log_2 d_Y, \tag{6.30}$$

[3] In literature, purity of quantum state $\rho \in \mathbb{C}^d$ is given by $\mathrm{Tr}(\rho^2)$. Here, however, we define purity as $\mathbb{P}(\rho) = \frac{\log_2 d - S(\rho)}{\log_2 d}$. For pure quantum states, both expressions yield unity.

the following nontrivial complementarity relation between purity and quantum correlation holds:

$$\mathbb{P}(\rho_X) + \mathbb{Q}(\rho_{XY}) \leq b \begin{cases} = 1 & \textit{if } d_X \leq d_Y \\ = 2 - \frac{\log_2 d_Y}{\log_2 d_X} & \textit{if } d_X > d_Y, \end{cases} \tag{6.31}$$

where

$$\mathbb{P}(\rho_X) = \frac{\log_2 d_X - S(\rho_X)}{\log_2 d_X}, \tag{6.32}$$

quantifies the normalized purity of the system in the X-part, and

$$\mathbb{Q}(\rho_{XY}) = \frac{\mathcal{Q}(\rho_{XY})}{\min\{\log_2 d_X, \log_2 d_Y\}}, \tag{6.33}$$

represents the normalized non-classical correlation of the system in the $X : Y$ bipartition.

Proof. The condition, $\mathcal{Q}(\rho_{XY}) \leq S(\rho_X)$, can be rewritten, using appropriate normalized quantities, as

$$\begin{aligned} &\frac{\log_2 d_X - S(\rho_X)}{\log_2 d_X} + \frac{\mathcal{Q}(\rho_{XY})}{\min\{\log_2 d_X, \log_2 d_Y\}} \\ \leq\ & 1 + \mathcal{Q}(\rho_{XY}) \left(\frac{1}{\min\{\log_2 d_X, \log_2 d_Y\}} - \frac{1}{\log_2 d_X} \right). \end{aligned} \tag{6.34}$$

While the above relation only needs $\mathcal{Q}(\rho_{XY}) - S(\rho_X) \leq 0$, we may note that the choice of the denominators in the terms on the left hand side have been guided by the fact that $0 \leq S(\rho_X) \leq \log_2 d_X$, so that $0 \leq \log_2 d_X - S(\rho_X) \leq \log_2 d_X$, and the often-true relation $0 \leq \mathcal{Q}(\rho_{XY}) \leq \min\{\log_2 d_X, \log_2 d_Y\}$. Now we consider the two cases, $d_X \leq d_Y$ & $d_X > d_Y$, separately.

Case 1: $d_X \leq d_Y$
In this case, the right-hand side of Ineq. (6.34) is trivially 1.

Case 2: $d_X > d_Y$
In this case, the right-hand side of Eq. (6.34), using the condition 6.30, becomes

$$\begin{aligned} & 1 + \mathcal{Q}(\rho_{XY}) \left(\frac{1}{\log_2 d_Y} - \frac{1}{\log_2 d_X} \right) \\ \leq\ & 1 + \log_2 d_Y \left(\frac{1}{\log_2 d_Y} - \frac{1}{\log_2 d_X} \right) \\ =\ & 2 - \frac{\log_2 d_Y}{\log_2 d_X}. \end{aligned} \tag{6.35}$$

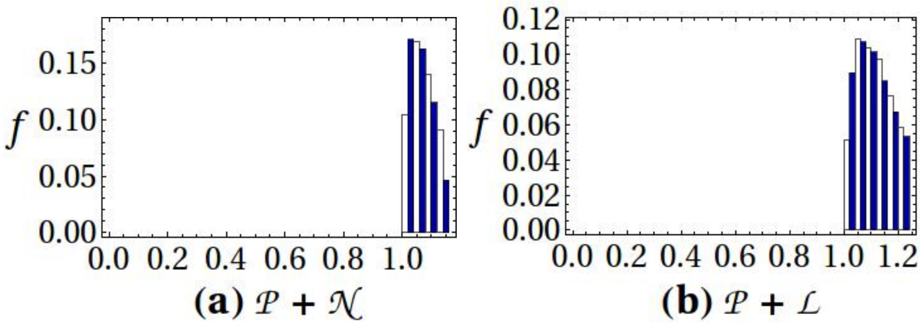


Figure 6.4: Histograms depicting the relative frequency (f) of states for different values of the quantity, $\mathbb{P}_A + \mathbb{Q}_{A:BC}$, for different quantum correlation measures $\mathcal{Q}$. The graph corresponds to 2×10^4 random Haar uniform generated **rank-1** three-qubit states, for the quantum correlation measures (a) negativity and (b) logarithmic negativity. We note that for rank-1 states, $\mathbb{P}_A + \mathbb{Q}_{A:BC} = 1$, for quantum discord and quantum work-deficit. We observe that $\mathbb{P}_A + \mathbb{Q}_{A:BC}$ is well below the trivial value, 2, for all states.

Combining together the two cases above, we obtain the desired complementarity relation. ■

Inequality (6.29) holds for several bipartite measures, viz., entanglement of formation, entanglement cost, distillable entanglement, relative entropy of entanglement, one-way distilled key rate. We note that there is a host of quantum correlation measures $\mathcal{Q}$ that may not satisfy one or both the conditions, $\mathcal{Q}(\rho_{XY}) \leq S(\rho_X)$ and $0 \leq \mathcal{Q}(\rho_{XY}) \leq \log_2 d_Y$, but still satisfy the complementarity relation, in Eq. (6.31), non-trivially, i.e.,

$$\mathbb{P}(\rho_X) + \mathbb{Q}(\rho_{XY}) = x \leq b(< 2), \tag{6.36}$$

as shown in Figs. 6.4 & 6.5 (cf. [18]).

For quantum states $\rho_{ABC} \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, the complementarity relation

$$\mathbb{P}(\rho_{AB}) + \mathbb{Q}(\rho_{AB:C}) \leq \frac{3}{2}, \tag{6.37}$$

is independent of dimension, and is saturated by the Greenberger-Horne-Zeilinger (GHZ) state.

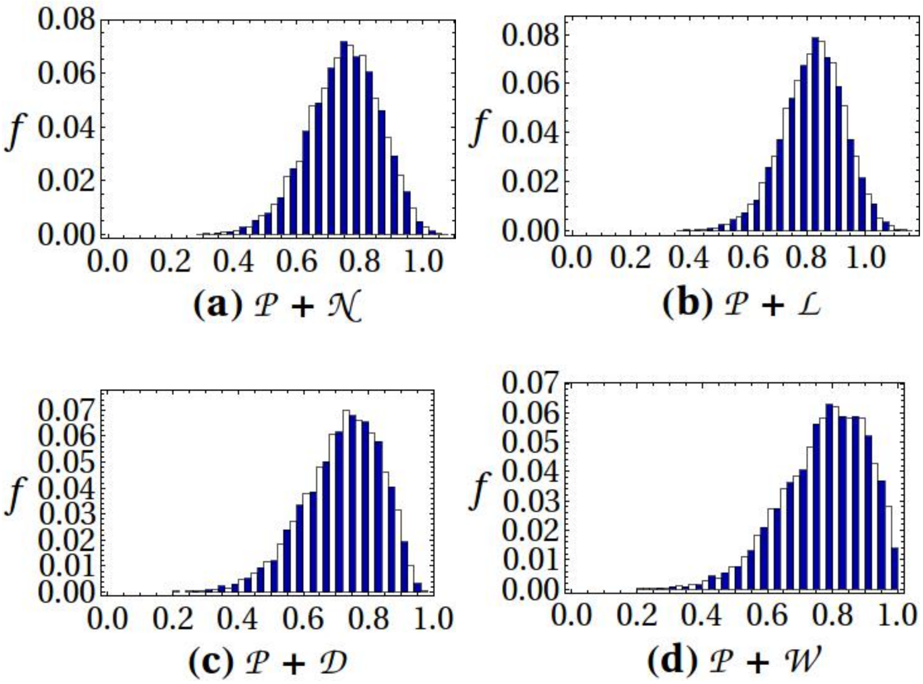


Figure 6.5: Histograms depicting the relative frequency (f) of states for different values of the quantity, $\mathbb{P}_A + \mathbb{Q}_{A:BC}$, for different quantum correlation measures $\mathcal{Q}$. The graph corresponds to 2×10^4 random Haar uniform generated **rank-2** three-qubit states, for the quantum correlation measures (a) negativity and (b) logarithmic negativity, (c) quantum discord, and (d) quantum work-deficit. Note that $\mathbb{P}_A + \mathbb{Q}_{A:BC} < 2$ for all states.

6.5.2 Lower bounds

In this section, we obtain a lower bound on monogamy scores, for quantum correlation measures $\mathcal{Q}$ that violate monogamy inequality, by making use of the complementarity relation, given in Eq. (6.31). In deriving the lower bounds on monogamy scores below, we will use the notation $\mathcal{Q}_{XY} \equiv \mathcal{Q}(\varrho_{XY})$, $\mathbb{Q}_{XY} \equiv \mathbb{Q}(\varrho_{XY})$, $\mathbb{P}_X \equiv \mathbb{P}(\varrho_X)$, etc., where $\varrho_X = \text{Tr}_Y \varrho_{XY}$. Consider an arbitrary multipartite quantum state $\rho_{AB} \equiv \rho_{A(B_1 \cdots B_n)}$. For a normalized bipartite quantum correlation measure $\mathbb{Q}$, the trivial lower bound on monogamy score, assuming that quantum correlation is non-increasing under discarding of parties, is

$$\delta_{\mathbb{Q}} = \mathbb{Q}_{AB} - \sum_{k=1}^{n} \mathbb{Q}_{AB_k} \geq -(n-1), \tag{6.38}$$

considering that value of $\mathbb{Q}$ is unity for each bipartite quantum state. However, we can obtain a tighter lower bound on monogamy scores using the complementarity relation, Eq. (6.31), in terms of purity as shown below.

For ρ_{AB}, using Eq. (6.31), we can write

$$\begin{aligned} \mathbb{P}_A + \mathbb{Q}_{AB} &= x_0 \leq b_0 \\ or, \;\; \mathbb{Q}_{AB} &= x_0 - \mathbb{P}_A. \end{aligned} \tag{6.39}$$

Similarly, for the reduced density matrices, $\rho_{AB_k} = \mathrm{Tr}_{\overline{AB_k}} \rho_{A(B_1B_2 \ldots B_n)}$, we can obtain

$$n\mathbb{P}_A + \sum_{k=1}^{n} \mathbb{Q}_{AB_k} = \sum_{k=1}^{n} x_k \leq \sum_{k=1}^{n} b_k. \tag{6.40}$$

When $d_A \leq d_{B_k}$ (say, $d_A = d_{B_k} = d$), Eq. (6.40) becomes

$$\begin{aligned} n\mathbb{P}_A + \sum_{k=1}^{n} \mathbb{Q}_{AB_k} &= \sum_{k=1}^{n} x_k \leq n \\ or, \;\; \sum_{k=1}^{n} \mathbb{Q}_{AB_k} &\leq n(1 - \mathbb{P}_A). \end{aligned} \tag{6.41}$$

Subtracting Eq. (6.39) from Eq. (6.41), and arranging, we obtain [4]

$$\delta_{\mathbb{Q}} = \mathbb{Q}_{AB} - \sum_{k=1}^{n} \mathbb{Q}_{AB_k} \geq -(n-1)(1 - \mathbb{P}_A) - (1 - x_0). \tag{6.43}$$

The lower bound on monogamy score obtained above can be expressed as

$$\delta_{\mathbb{Q}} = \mathbb{Q}_{AB} - \sum_{k=1}^{n} \mathbb{Q}_{AB_k} \geq -(n-1)\left(1 - \mathbb{P}_A - \frac{1 - x_0}{n-1}\right). \tag{6.44}$$

It is evident from Eqs. (6.43 & 6.44) that either when $x_0 \geq 1$ or when n is large, the lower bound on monogamy score reduces to

$$\delta_{\mathbb{Q}} \geq -(n-1)(1 - \mathbb{P}_A). \tag{6.45}$$

where $\mathbb{P}_A$ is the normalized purity defined in Eq. (6.32), and ranges from $0 \leq \mathbb{P}_A \leq 1$. For the important three-qubit states, which have received a lot of attention in the study of monogamy, we observe that the lower bound on the monogamy score is given by, $\delta_{\mathbb{Q}} \geq -S(\rho_A)$, where $S(\rho_A)$ is the von Neumann entropy of the subsystem A. This shows that the lower bound satisfies the limits

[4] Also, for the reduced density matrices, $\rho_{A\overline{B_k}} = \mathrm{Tr}_{B_k} \rho_{A(B_1B_2 \ldots B_n)}$, one can obtain

$$\delta_{\mathbb{Q}} = \mathbb{Q}_{AB} - \sum_{k=1}^{n} \mathbb{Q}_{A\overline{B_k}} \geq -(n-1)(1 - \mathbb{P}_A) - (1 - x_0). \tag{6.42}$$

for monogamy ($\delta_{\mathbb{Q}} \gtrsim 0$) for states that are weakly entangled along the $A : rest$ bipartition, such that $S(\rho_A) \approx 0$.

6.5.3 Analyzing the lower bound

We now examine the lower bound of the monogamy score, $\delta_{\mathbb{Q}}$, for certain quantum states. For example, we consider $(n+1)$-party quantum states $\varrho_{AB_1\cdots B_n} \in \mathbb{C}^2 \otimes \mathbb{C}^{d_{B_1}} \otimes \cdots \otimes \mathbb{C}^{d_{B_n}}$. For these states, the lower bound in relation (6.45) becomes $\delta_{\mathbb{Q}} \geq -(n-1)S(\varrho_A)$. For $d_{B_k} = 2$, $\forall\ k$, the above expression for the lower bound of $\delta_{\mathbb{Q}}$ holds for all multiqubit quantum states.

Let us consider again the superposition of generalized GHZ [148] state and W state [83, 149], which are permutationally invariant multiqubit states given by

$$|\Psi_{\alpha,\gamma}\rangle = \alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n} + \gamma|W^n\rangle, \tag{6.46}$$

where $|W^n\rangle$ is the normalized W state for n qubits, (α, β, γ) are complex numbers that satisfy $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. The n-qubit state, $|\Psi_{\alpha,\gamma}\rangle$, satisfies the lower bound for the monogamy score, given by $\delta_{\mathbb{Q}} \geq -(n-2)S(\rho_{\alpha,\gamma})$, where $\rho_{\alpha,\gamma} = \text{Tr}_{n-1}(|\Psi_{\alpha,\gamma}\rangle\langle\Psi_{\alpha,\gamma}|)$ is obtained by tracing out $n-1$ qubits. $\rho_{\alpha,\gamma}$ can be written as

$$\varrho_{\alpha,\gamma} = \begin{pmatrix} |\alpha|^2 + \frac{n-1}{n}|\gamma|^2 & \alpha^*\gamma\sqrt{\frac{1}{n}} \\ \alpha\gamma^*\sqrt{\frac{1}{n}} & \frac{1}{n}|\gamma|^2 + |\beta|^2 \end{pmatrix}. \tag{6.47}$$

The largest eigenvalue of the single qubit state $\varrho_{\alpha,\gamma}$ is given by

$$e = \frac{1}{2}\left(1 + \sqrt{1 - 4|\alpha|^2|\beta|^2 - \frac{4(n-1)}{n}|\gamma|^2(|\beta|^2 + \frac{|\gamma|^2}{n})}\right).$$

Hence, $S(\rho_{\alpha,\gamma})$ is equal to $h(e)$, where $h(x)$ is the Shannon (binary) entropy [27] of the variable x. The lower bound of the monogamy score is given by, $\delta_{\mathbb{Q}} \geq -(n-2)h(e)$. Hence, a tight lower bound on the monogamy score is obtained for low values of $h(e)$.

If we set, $\gamma = 0$, then we obtain the generalized GHZ state, $|\Psi_{GHZ}\rangle = \alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$, and the largest eigenvalue of the single qubit reduced state is $\frac{1}{2}\left(1 + \sqrt{1 - 4|\alpha|^2|\beta|^2}\right)$, which is independent of n. For states with $|\alpha|^2 = |\beta|^2 = 1/2$, $e = 1/2$, and $h(e) = 1$, which gives us a weak bound. However, for states with $|\alpha|^2|\beta|^2 \approx 0$, $e \approx 1$, which implies $h(e) \approx 0$. For these states, the lower bound of the monogamy score is given by, $\delta_{\mathbb{Q}} \geq -\epsilon$, where $\epsilon \to 0$. Alternately, if one sets, $\alpha = \beta = 0$, such that $\gamma = 1$, one obtains the n-qubit W state, given by

$|\Psi_W\rangle = \frac{1}{\sqrt{n}} \sum \mathcal{P}\left(|0\rangle^{\otimes(n-1)} \otimes |1\rangle\right)$, where the sum in the expression is over all $\mathcal{P}$ permutations of the product states with a single $|1\rangle$. The maximum eigenvalue of the reduced state is then given by, $e = \frac{n-1}{n}$, and $h(e) = h(1/n)$. Hence, as n increases, the quantity $h(e) \to 0$, and hence we obtain, $\delta_{\mathbb{Q}} \gtrsim 0$.

To expand the study on W states, we consider the n-qubit Dicke state [216] in Eq. (6.9). For $r = 1$, the Dicke state is the same as an n-qubit W state, $|W^n\rangle$, mentioned earlier. The single qubit reduced state eigenvalue, e, is equal to $\frac{r}{n}$ (or $\frac{n-r}{n}$), and hence $h(e) = h(r/n)$. Hence, as the $\frac{r}{n}$ ratio decreases, $h(e)$ decreases, and we obtain, $\delta_{\mathbb{Q}} \gtrsim 0$. However, for $r = n/2$, $h(e) = 1$, and the lower bound on the monogamy score is weak.

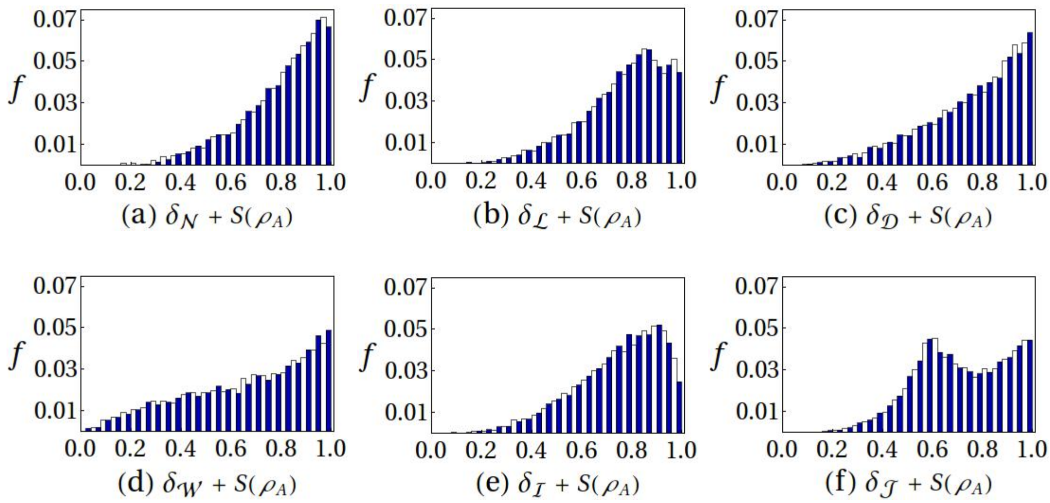


Figure 6.6: Histograms depicting the relative frequency (f) of states against the quantity, $\delta_{\mathbb{Q}} + S(\rho_A)$, for different correlation measures $\mathcal{Q}$. The graph corresponds to a set ($\sim 10^6$) of random Haar uniform generated rank-1 and rank-2 three-qubit states. The monogamy score is calculated for the quantum correlation measures, given by (a) negativity ($\mathcal{N}$), (b) logarithmic negativity ($\mathcal{L}$), (c) quantum discord ($\mathcal{D}$), (d) quantum work-deficit ($\mathcal{W}$), (e) quantum mutual information ($\mathcal{I}$), and (f) conditional mutual information ($\mathcal{J}$). The histogram shows that $\delta_{\mathbb{Q}} + S(\rho_A) \geq 0$, and thus satisfies $\delta_{\mathbb{Q}} \geq -S(\rho_A)$.

As noted earlier, the complementarity relation in Eq. (6.31) is satisfied by a host of quantum correlation measures, irrespective of whether one or both the conditions $\mathcal{Q}(\rho_{XY}) \leq S(\rho_X)$ and $0 \leq \mathcal{Q}(\rho_{XY}) \leq \log_2 d_Y$ are satisfied. It is, therefore, natural to investigate whether the lower bounds on monogamy scores of these quantum correlation measures comply with that obtained in Eq. (6.45). We generate a large number of random Haar uniform three-qubit states, and compute $\delta_{\mathbb{Q}}$ for a set of monogamy inequality violating measures such as negativity ($\mathcal{N}$), logarithmic negativity ($\mathcal{L}$), quantum discord ($\mathcal{D}$), quantum work-deficit ($\mathcal{W}$), quantum mutual information ($\mathcal{I}$), and conditional mutual in-

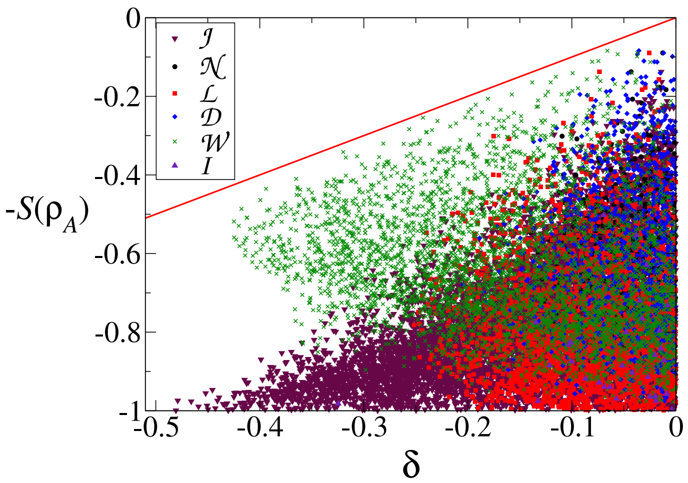


Figure 6.7: Tightness of the lower bound. The score, $\delta \equiv \delta_{\mathbb{Q}}$, of monogamy inequality violating quantum correlation measures for a set of randomly generated Haar uniform three-qubit states, plotted along the x-axis, with respect to the negative von Neumann entropy ($-S(\rho_A)$) of party A, plotted along the y-axis. The measures are negativity (black-circle), logarithmic negativity (red-square), quantum discord (blue-diamond), quantum work-deficit (green-cross), and conditional mutual information (violet-triangle). The figure shows that $\delta_{\mathbb{Q}} \geq -S(\rho_A)$. It is evident that the lower bound is tight for states with low reduced entropy. For the ease of viewing and without affecting the results, the plot is provided for a set of 2×10^4 rank-1 and rank-2 three-qubit states, drawn from a larger sample set ($\sim 10^6$) of randomly generated Haar uniform states. Along with Fig. 6.6, this figure numerically reasserts the lower bound on monogamy score, given by $\delta_{\mathbb{Q}} \geq -S(\rho_A)$.

formation ($\mathcal{J}$). For the generated pure and mixed three-qubit states, and for the above measures, it is observed that the monogamy score satisfies the lower bound, $\delta_{\mathbb{Q}} \geq -S(\rho_A)$, as shown in Fig. 6.6, which shows an histogram of the relative frequency of states corresponding to different values of the quantity $\delta_{\mathbb{Q}} + S(\rho_A)$. The figure shows that this quantity is always positive. To check the tightness of the bound, we plot the monogamy score, $\delta_{\mathbb{Q}}$, against $-S(\rho_A)$ in Fig. 6.7. It is evident from the figure that the lower bound is tight for states with low reduced von-Neumann entropy.

6.6 Summary

Monogamy score captures the amount by which the monogamy inequality is satisfied or violated. However, the monogamy score is a difficult quantity to compute and estimate for generic quantum states and for arbitrary measures of quantum correlations. It is therefore interesting to derive bounds on the monogamy score. This would allow a better understanding of the constraints on quantum correlations in many-body systems even when the exact monogamy inequalities are not accessible. Our work provides an easily estimable upper bound on the monogamy score.

All monogamy inequality satisfying measures have a positive monogamy score, which is bounded above by a well defined value that can be derived as functions of the genuine multipartite entanglement of the quantum state. We have shown that the bound is a quadratic function for the entanglement-based measures and an entropic function for the information-theoretic measures, and is universally satisfied for all three-qubit states. The upper bound holds also for an arbitrary number of qubits provided the states satisfy certain conditions. We have provided a set of necessary conditions to characterize the set of states that may violate the bound, and numerically observe that the set is extremely small. Moreover, we analytically investigated several important classes of multiparty quantum states with arbitrary number of parties for which we found that the conditions required to have the upper bound on monogamy scores of computable bipartite measures are satisfied. The obtained monogamy score bound due to the genuine multiparty entanglement in the system shows a forbidden regime in the distribution of bipartite quantum correlation measures among different parties in a multiparty system and limits the amount by which the monogamy inequality can be satisfied.

However, not all quantum correlation measures satisfy the monogamy inequality, and in general, the monogamy score can be negative. We obtained a non-trivial lower bound of the monogamy score for measures and states where the monogamy inequality is violated. This is achieved using a complementarity relation between the normalized purity of a subsystem and the bipartite quantum correlation in the system. Subsequently, we analyze the strengths and weaknesses of the lower bound for different quantum states and measures of quantum correlation. We observe, in particular, that monogamy score of quantum correlation measures for three-qubit quantum states is bounded from below by negative (von Neumann) entropy of the single-qubit reduced density matrix.

The results provide a unifying framework to study monogamy relations in both entanglement and information-theoretic quantum correlations.

The results of this Chapter are based on the papers in items 1 and 3. The paper in item 2 is used partially.

1. *Forbidden regimes in the distribution of bipartite quantum correlations due to multiparty entanglement*, **Asutosh Kumar**, Himadri Shekhar Dhar, R.

Prabhu, Aditi Sen De, and Ujjwal Sen, Phys. Lett. A **381**, 1701 (2017).

2. *Information complementarity in multipartite quantum states and security in cryptography*, Anindita Bera, **Asutosh Kumar**, Debraj Rakshit, R. Prabhu, Aditi Sen De, and Ujjwal Sen, Phys. Rev. A **93**, 032338 (2016).

3. *Lower bounds on violation of monogamy inequality for quantum correlation measures*,
   Asutosh Kumar and Himadri Shekhar Dhar, Phys. Rev. A **93**, 062337 (2016).

Chapter 7

Conclusive Identification of Quantum Channels via Monogamy

7.1 Introduction

An important aspect in the characterization of a composite quantum system is to understand the correlations between its constituting parts. Quantum information theory provides a collection of measures of quantum correlations [1–3], which can be broadly categorized into two classes: the "entanglement-separability" class [1], and the "information-theoretic" ones [2, 3]. Both entanglement and quantum information-theoretic correlations have been proposed to be resources for several quantum protocols (see e.g. Refs. [12, 38–43, 119, 150–157, 163, 170, 217, 237–260]), a number of which have been successfully observed in the laboratory (see e.g. Refs. [91, 261–266]). However, quantum correlations can be fragile under decoherence [93, 267–269]. Naturally, due to their immense importance in quantum information processing tasks, investigating the behavior of quantum correlations under various kinds of environmental noise has been a topic of utmost importance in quantum information theory.

Most of the available literature that deals with decoherence of quantum correlations consider bipartite quantum correlation measures due to their relative computational simplicity [181, 270–302]. It has been shown that the bipartite entanglement measures tend to decay rapidly with increasing noise, and vanish when a threshold noise level is crossed. This phenomenon is known as "entanglement sudden death", and has been studied extensively in the case of bipartite systems under different types of environment [271–278]. In stark contrast to this behavior, quantum information theoretic measures, namely, quantum discord [121, 122], quantum work-deficit [169–172], and several geometric measures [270, 303–313], have been found to undergo an asymptotic decay with increasing noise strength [279–292, 314], indicating a higher robustness against noise than that of entanglement. It has also been shown that special two- as well as multiqubit mixed quantum states can be engineered for which "discord-like" quantum correlations may remain invariant over a finite range of noise strength [181, 270, 293–301], while the entanglement measures calculated for those states exhibit no such property (cf. [302]). Although behavior of bipartite quantum correlations under decoherence is a well-investigated topic, similar studies in the multipartite scenario [315–320] have been only a few due to the lack of computable measures of quantum correlations for mixed mltipartite states. Recent developments on the monogamy relation of quantum correlations [13–15,63,64,68,70–74,77–79,84,85,174,189–200] have provided an effective tool to characterize the multipartite nature of quantumness present in a composite quantum system, while still computing only bipartite measures of quantum correlations. The monogamy property is important in several aspects in quantum information theory. Therefore, it has become crucial to investigate how the monogamy property of quantum correlation behaves when subjected to noisy environments. Recall that the "monogamy score" [14,89] with respect to a bipartite quantum correlation measure $\mathcal{Q}$, for the multipartite state $\rho_{AB_1B_2\ldots B_n} \equiv \rho_{AB}$, is defined as

$$\delta_{\mathcal{Q}} = \mathcal{Q}(\rho_{AB}) - \sum_{j=1}^{n} \mathcal{Q}(\rho_{AB_j}). \tag{7.1}$$

Non-negativity of $\delta_{\mathcal{Q}}$ for a given quantum state is considered to imply monogamy of quantum correlation measure $\mathcal{Q}$ for that state.

This Chapter has two different aspects that are complementary to each other. In one, we use the monogamy of negativity [137,138,321], a measure of bipartite entanglement, and quantum discord, a quantum correlation measure from the information-theoretic domain, to study the dynamics of monogamy of quantum correlations. As different models of environmental noise, we choose the global noise, and three local noisy channels, namely, the amplitude-damping (AD), the phase-damping (PD), and the depolarizing (DP) channels. We demonstrate how the dynamics of monogamy, in the case of three-qubit systems, exhibit qualitatively different behavior if the input quantum state is chosen from the set of generalized Greenberger-Horne-Zeillinger (gGHZ) state, and the generalized W (gW) state [83,148,232,322], which are not equivalent under stochastic local operations and classical communications (SLOCC). More specifically, we show that while monogamy scores of negativity as well as quantum discord exhibit a monotonic decay with respect to the corresponding noise parameter when gGHZ state is subjected to these noise models, there exist non-monotonic dynamics when the input state is the gW state. We also investigate the trends of monogamy scores against noise, when arbitrary three-qubit pure states belonging to the two inequivalent SLOCC classes of three-qubit pure states, namely, the GHZ and the W classes [83], are chosen as inputs. Moreover, we introduce a concept called the "dynamics terminal", which quantify the durability of quantum correlation measures under decoherence, and show that it can distinguish between different quantum correlation measures as well as different types of noise. The study also reveals that for the PD channel, the negativity monogamy score can exhibit a more robust behavior against noise strength than that observed for the monogamy score of quantum discord, which we call the "discord monogamy score".

Besides characterizing the dynamical features of quantum correlations under decoherence, it is also interesting to address the reverse question of whether the modes of environmental noise can be characterized by the properties of quantum correlation. Although a few studies have been motivated by similar goal [323, 324], the literature regarding this issue is extremely limited. While most of the studies have tried to distinguish different types of noise by the different dynamical behavior of different quantum correlations, concrete protocol to conclusively identify the type of noise to which the quantum state is exposed is yet to be introduced. As the second directive of this Chapter, we propose a protocol to use monogamy relation of tripartite states, which have been exposed to an unknown noisy environment, to distinguish between the types of the noise. More specifically, we use a highly entangled gGHZ state, and an arbitrary gW state as resource, and design a two-step protocol to conclusively identify the type of noise applied to the quantum state, where the noise models include global noise, and local channels, namely, AD, PD, and DP channels.

The Chapter is organized as follows. In Sec. 7.2, we give short descriptions of different types of noise relevant to our study. The dynamical behavior of the negativity and quantum discord [1] monogamy scores, when gGHZ and gW states are subjected to different types of noise are characterized in Sec. 7.3. The behavior of monogamy against noise, when arbitrary three-qubit pure states are considered as input, is also studied. In Sec. 7.4, the two-step channel

[1] The choice of measurement, in the definition of quantum discord, puts an inherent asymmetry in the measure. In this Chapter, unless otherwise stated, the measurement is applied on subsystem A of the bipartite quantum system AB.

discrimination protocol with monogamy scores as observables is presented. Sec. 7.5 presents the summary of the Chapter.

7.2 Decoherence under global and local noise

A closed quantum system is an ideal concept. Every quantum system inevitably interacts and builds undesirable correlations with its environment and eventually decoheres loosing its quantum correlations. Decoherence [269] drives an initial quantum state to become more mixed and causes degradation of the quantum correlations. Noisy quantum channels are the examples of such environment. The degree of decoherence may vary from one system to another. Consequently a given quantum system may be more robust to the decohering effects of the environment compared to other quantum systems or vice-versa. As presence of quantum correlations is vital for exotic quantum information tasks in both communication and computation [4–6], investigating the behaviour of quantum states in noisy environment becomes extremely important. Its better understanding may provide some insight and ways to manipulate and control the degradation of quantum correlations against the environment and also help us identify the degree of robustness of quantum states that retain quantum correlations in them. Many efforts have been put towards the study of dynamics of quantum correlations under decoherence. Wang *et al.* [325] showed that non-maximally entangled states are more robust than maximally entangled states for quantum correlation distribution and storage under the local amplitude-damping [4, 5]. In Refs. [326] authors have shown, considering that noise acts on only part of the bipartite system, that the depolarizing channel [4, 5] is far effective over other noisy quantum channels, in causing entanglement sudden death [271–278] in pure maximally entangled states. In this section, we study the effect of local and global noise on three-qubit pure states. In particular, we study the dynamics of the monogamy of negativity and quantum discord of generalized GHZ and generalized W states and those of quantum states belonging to GHZ class and W class.

The dynamics of a quantum system, ρ, can be described by a quantum operation, $\mathcal{E}$, that transforms ρ as [5, 269, 314, 327, 328]

$$\rho \rightarrow \rho' = \mathcal{E}(\rho). \tag{7.2}$$

Unitary transformations and measurements are examples of quantum operations.

The quantum operation describes the dynamical change to a state which occurs as the result of some physical process. The noise can act either globally, or locally on each subsystem of the system of interest. Below we succinctly describe a number of noise models in both scenarios.

7.2.1 Global noise

If the environment acts globally on a system of dimension d^n in a state ρ, the resulting state is given by

$$\rho' = \frac{p}{d^n} I + (1-p)\rho, \tag{7.3}$$

where p is the mixing parameter ($0 \leq p \leq 1$), and I is the identity operator in the Hilbert space of the system. Note that $p = 0$ stands for the noiseless case, while $p = 1$ corresponds to the fully decohered state.

7.2.2 Local noise

For a composite quantum system having n spatially apart subsystems, it is reasonable to assume that the environment to each of the subsystems are independent of each other, and acts locally on each subsystem. Here we describe how independent local environments can be modeled via various local noisy channels.

The dynamics of a closed quantum system are described by a unitary transformation. To describe the dynamics of an open quantum system, which is interacting with its environment, one can assume that the system and the environment together form a closed quantum system, whose state, ρ, is given by $\rho = \rho_s \otimes \rho_e$. Here, ρ_s and ρ_e are respectively the states of the system and the environment. The next step would be to apply unitary transformation to the given composite system and finally trace out the environment part to obtain the reduced state of the changed system of interest. In this case, quantum operations can be considered in the *operator-sum representation* [327], written explicitly in terms of operators on the Hilbert space of the system of interest as follows:

$$\rho_s' = \mathrm{tr}_e[U(\rho_s \otimes \rho_e)U^\dagger] = \sum_k E_k \rho_s E_k^\dagger, \tag{7.4}$$

where the operators $\{E_k\}$ are known as Kraus operators [5, 314, 327, 328] and satisfy $\sum_k E_k^\dagger E_k \leq I$, the equality holding for the trace-preserving maps. For a system "s" of dimension d, there can be at most d^2 Kraus operators, such that $k = 0, 1, \cdots, d^2 - 1$. For the n-partite system $\rho_{A_1 A_2 \cdots A_n}$ in arbitrary dimensions, the action of all the local environments together can be obtained through the Kraus operators for each of the subsystems, such that the evolved state, $\rho'_{A_1 A_2 \cdots A_n}$, can be written as

$$\rho'_{A_1 A_2 \cdots A_n} = \sum_{k_1, k_2, \cdots, k_n} E_{k_1 k_2 \cdots k_n} \rho_{A_1 A_2 \cdots A_n} E^{\dagger}_{k_1 k_2 \cdots k_n}, \tag{7.5}$$

with $E_{k_1 k_2 \cdots k_n} = E^{(1)}_{k_1} \otimes E^{(2)}_{k_2} \otimes \cdots \otimes E^{(n)}_{k_n}$ corresponding to the composite system. Here, $E^{(j)}_{k_j}$, $j = 1, 2, \cdots, n$, is the Kraus operator for the subsystem A_j with dimension d_j so that $0 \leq k_j \leq d_j^2 - 1$. Now, we describe the Kraus operators of a number of single-qubit quantum channels, namely, AD, PD, and DP channels [5, 314].

Amplitude-damping channel

The AD channel represents a scenario where energy dissipation from a quantum system is allowed. It applies to several processes like, vacuum single-photon qubit with photon loss, decay of atomic qubit due to spontaneous emission of a photon, and superconducting qubit with zero-temperature energy relaxation. The Kraus operators for a single-qubit AD channel are given by

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \tag{7.6}$$

with $0 \leq p \leq 1$, where p corresponds to the strength of the noise acting on the input (qubit) state, and the single-qubit density matrix ρ evolves as

$$\rho' = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix}. \tag{7.7}$$

Phase-damping channel

As an example of a non-dissipative channel, we consider the PD channel. A state, after passing through the PD channel, or the dephasing channel, decays its off-diagonal elements, resulting in information loss about its coherence without loss of energy. The single qubit Kraus operators for PD channel are given by

$$E_0 = \sqrt{1-p} I, \;\; E_1 = \frac{\sqrt{p}}{2}(I + \sigma_3), \;\; E_2 = \frac{\sqrt{p}}{2}, (I - \sigma_3), \tag{7.8}$$

where I is the identity matrix in the qubit Hilbert space, and p is the noise-strength. The initial state which is acted upon by the above evolution operators, transforms to

$$\rho' = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}. \tag{7.9}$$

Depolarizing channel

In DP channel, the input qubit is depolarized, that is, replaced by the completely mixed state $\frac{I}{2}$, with probability p and left untouched with probability $(1-p)$. Such an operation on the single-qubit state ρ is represented by

$$\rho' = \frac{p}{2}I + (1-p)\rho. \tag{7.10}$$

Note that the form in Eq. (7.10) is not in the operator-sum representation. The operation given in Eq. (7.10) is often parametrized as

$$\rho' = (1-p)\rho + \frac{p}{3}(\sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2 + \sigma_3\rho\sigma_3), \tag{7.11}$$

leading to single qubit Kraus operators of the following form.

$$E_0 = \sqrt{1-p}I,\ E_1 = \sqrt{\frac{p}{3}}\sigma_1,\ E_2 = \sqrt{\frac{p}{3}}\sigma_2,\ E_3 = \sqrt{\frac{p}{3}}\sigma_3. \tag{7.12}$$

Remark 1. All the above noisy quantum channels are trace-preserving. Similar to the case of global noise, in the case of local noisy channels also, the noiseless case is denoted by $p = 0$, while at $p = 1$, full decoherence takes place. Thus the relevant range of the noise parameter is $0 \leq p \leq 1$.

Remark 2. By computing the quantum correlation $\mathcal{Q}(p)$ as a function of p, corresponding to the decohered quantum state ρ', one can determine the behavior of a specific quantum correlation, $\mathcal{Q}$, when the state is subjected to a particular type of noise.

7.3 Monogamy of quantum correlations under decoherence

In this section, we investigate the behavior of monogamy scores of negativity and quantum discord for three-qubit quantum states under the influence of global as well as local noise. Before considering arbitrary three-qubit pure states, we examine the generalized GHZ (gGHZ), and the generalized W (gW) state as the input states to various types of noise.

7.3.1 Generalized GHZ states

The generalized GHZ state, shared between three qubits, 1, 2, and 3, reads as

$$|\Psi\rangle = a_0|000\rangle + a_1|111\rangle, \tag{7.13}$$

where a_0 and a_1 are the complex parameters satisfying $|a_0|^2 + |a_1|^2 = 1$. In this Chapter, we consider qubit 1 as the nodal observer while computing monogamy scores for negativity ($\delta_{\mathcal{N}}$), and quantum discord ($\delta_{\mathcal{D}}$). Note that the monogamy scores for the gGHZ state, in the noiseless scenario, is always positive for all quantum correlation measures including negativity and quantum discord. This is due to the fact that the two-qubit reduced density matrix $\rho_{12} = \rho_{13} = |a_0|^2|00\rangle\langle 00| + |a_1|^2|11\rangle\langle 11|$, obtained from the gGHZ state, is a classically correlated two-qubit state having vanishing quantum correlations, while the state $|\Psi\rangle$ in the $1:23$ bipartition always has a non-zero value of quantum correlation for non-zero values of a_0 and a_1.

Let us first consider the case of global noise acting on the gGHZ state. The final state, ρ^{gGHZ}, as a function of the mixing parameter, p, can be obtained from Eq. (7.3), which leads to two-party reduced states ρ_{12}^{gGHZ} and ρ_{13}^{gGHZ} of the form

$$\begin{aligned} \rho_{12}^{\text{gGHZ}} &= \rho_{13}^{\text{gGHZ}} \\ &= (1-p)(|a_0|^2|00\rangle\langle 00| + |a_1|^2|11\rangle\langle 11|) + \frac{p}{4}I, \end{aligned} \tag{7.14}$$

with I being a 4×4 identity matrix. They still remain classically correlated with vanishing entanglement and quantum discord. In case of AD, PD, and DP channels, the resulting states ρ^{gGHZ} are obtained as

$$\begin{aligned} \rho^{\text{gGHZ}} &= \sum_{i=0}^{1} |a_i|^2 \left(u_i^p|0\rangle\langle 0| + v_i^p|1\rangle\langle 1|\right)^{\otimes 3} \\ &\quad + w^p(a_0 a_1^*|000\rangle\langle 111| + h.c.). \end{aligned} \tag{7.15}$$

Here the functions u_i^p, v_i^p, and w^p, for the three channels, are given by

$$\text{AD channel} \ : \ u_i^p = \delta_{0i} + p\delta_{1i}, v_i^p = (1-p)\delta_{1i}, w^p = (1-p)^{\frac{3}{2}},$$

$$\begin{aligned}
\text{PD channel} \quad &: \quad u_i^p = \delta_{0i}, v_i^p = \delta_{1i}, w^p = (1-p)^3, \\
\text{DP channel} \quad &: \quad u_i^p = q\delta_{0i} + (1-q)\delta_{1i}, v_i^p = (1-q)\delta_{0i} + q\delta_{1i}, \\
& \quad\;\; w^p = (2q-1)^3, \qquad (7.16)
\end{aligned}$$

with $q = 1 - \frac{2p}{3}$. From the above expressions, it can be shown that the two-qubit reduced density matrices, in case of the PD channel, do not depend on the noise parameter p, and remain classically correlated. On the other hand, ρ_{12}^{gGHZ} and ρ_{13}^{gGHZ} remains diagonal in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, resulting in vanishing entanglement as well as quantum discord for the other two channels also. In effect, even for $p \neq 0$, $\delta_{\mathcal{N}}$ and δ_D are given by $\delta_{\mathcal{N}} = \mathcal{N}\big(\rho_{1:23}^{\text{gGHZ}}\big)$ and $\delta_{\mathcal{D}} = \mathcal{D}\big(\rho_{1:23}^{\text{gGHZ}}\big)$ respectively, when the gGHZ state is subjected to these four types of noise. Hence, both negativity and quantum discord are always monogamous in the present scenario, which can be applied to discriminate channels as we shall see in Sec. 7.4. Note that all the above discussions hold for the gGHZ state of arbitrary number of parties subjected to different types of local and global noise considered in this Chapter.

Using Eqs. (7.15), analytical expressions of $\delta_{\mathcal{N}}$, as functions of the noise parameter, p, and the state parameter, $|a_0|$, can be obtained for different types of noise. In the case of the global noise, it is given by

$$\delta_{\mathcal{N}}^{g} = \left|\min\left[0, \frac{1}{2}\left\{\frac{p}{4} - 2|a_0||a_1|(1-p)\right\}\right]\right|, \qquad (7.17)$$

while in the case of PD channel,

$$\delta_{\mathcal{N}}^{pd} = |a_0||a_1|(1-p)^3. \qquad (7.18)$$

The expressions of negativity monogamy score in the case of AD channel ($\delta_{\mathcal{N}}^{ad}$) and DP channel ($\delta_{\mathcal{N}}^{dp}$) are given by

$$\begin{aligned}
\delta_{\mathcal{N}}^{ad} &= \left|\min\left[0, \frac{1}{2}\left\{|a_1|^2 p(1-p) - \sqrt{f_1^{ad} + f_2^{ad}}\right\}\right]\right|, \qquad (7.19) \\
\delta_{\mathcal{N}}^{dp} &= \left|\min\left[0, \frac{1}{2}\left\{q(1-q) - \sqrt{f_1^{dp} + f_2^{dp}}\right\}\right]\right|, \qquad (7.20)
\end{aligned}$$

with the functions f_1^{ad}, f_2^{ad}, f_1^{dp}, and f_2^{dp} defined as

$$\begin{aligned}
f_1^{ad} &= |a_1|^4(4p^6 - 12p^5 + 13p^4 - 6p^3 + p^2), \\
f_2^{ad} &= 4|a_0|^2|a_1|^2(1-p)^3, \\
f_1^{dp} &= q^2(1-q)^2(1-2q)^2,
\end{aligned}$$

$$f_2^{dp} = 4|a_0|^1|a_1|^2(1-2q)^2(1-3q+3q^2)(1-5q+5q^2). \tag{7.21}$$

Note here that in all the above expressions, one can replace $|a_1|$ by $\sqrt{1-|a_0|^2}$.

On the other hand, analytically determining the discord monogamy score, $\delta_{\mathcal{D}}$, for all the types of noise, is in general hard due to the optimization required to compute quantum discord for ρ^{gGHZ} in the $1:23$ split [173]. So far, analytical determination of quantum discord has been possible only for very restricted class of mixed states [175–179]. Hence, we employ numerical optimization over the real parameters (θ, ϕ) of measurement involved in the definition of quantum discord. The behavior of the monogamy scores corresponding to negativity and quantum discord for different types of noise are depicted in Fig. 7.1, where the top panels are for $\delta_{\mathcal{N}}$, and the bottom panels correspond to $\delta_{\mathcal{D}}$. For all the noise models considered in this Chapter, $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ monotonically decreases with increasing values of p for a fixed value of $|a_0|$, and vanishes when noise is considerably high, as can be clearly seen from the figures. In the case of discord, gGHZ state gives non-zero discord monogamy score when the noise in the PD and the DP channels are less than 80% and 60%, respectively. In the rest of the cases, the corresponding monogamy scores persist to be non-zero for higher values of the noise-parameter ($p \geq 0.9$).

It is clear from Fig. 7.1 that for global noise as well as for the AD channel, the decay of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ with increasing p is much slower than that in the case of the PD and the DP channels. Note also that in the case of the PD channel, $\delta_{\mathcal{N}}$ persists to be non-zero up to a higher value of noise parameter than that in the case of $\delta_{\mathcal{D}}$. We shall elaborate on this issue in Sec. 7.3.3. The regions (in Fig. 7.1) marked by "**R**", and enclosed by the boxes, in the case of $\delta_{\mathcal{N}}$ under global noise and DP channel, and $\delta_{\mathcal{D}}$ under AD and PD channels, are defined by the parameter ranges $0.65 \leq |a_0| \leq 0.7071$, and $0.4 \leq p \leq 0.6$, respectively. Note that in the marked areas, $\delta_{\mathcal{N}} > 0$ for global noise, while $\delta_{\mathcal{N}} = 0$ under DP channel. On the other hand, in the region **R**, $\delta_{\mathcal{D}} > 0$ for both AD and PD channels. The implications of these values are discussed in Sec. 7.4.

7.3.2 Generalized W states

Let us now move to the monogamy scores of negativity and quantum discord for the gW state, given by

$$|\Phi\rangle = a_0|001\rangle + a_1|010\rangle + a_2|100\rangle, \tag{7.22}$$

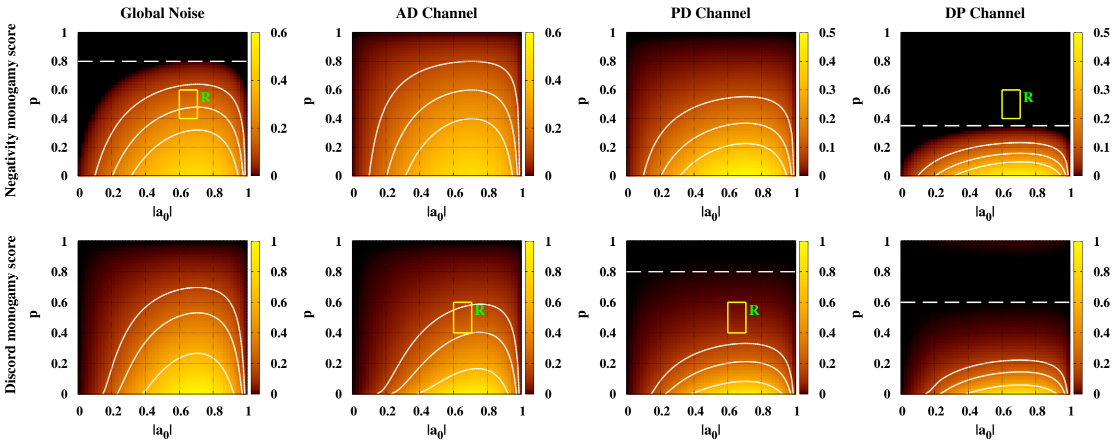


Figure 7.1: Variations of $\delta_{\mathcal{N}}$ (top panels), and $\delta_{\mathcal{D}}$ (bottom panels) as functions of the noise parameter, p, and the state parameter, $|a_0|$, when gGHZ states are subjected to (from left to right) global noise, AD channel, PD channel, and DP channel. The absolute value of the other state parameter, $|a_1|$, is determined by normalization. The solid lines in the plots are the contours obtained by joining the points corresponding to a fixed value of either $\delta_{\mathcal{N}}$, or $\delta_{\mathcal{D}}$. In the case of $\delta_{\mathcal{N}}$, the lines, from low to high values of p, correspond to $\delta_{\mathcal{N}} = 0.3, 0.2$, and 0.1, while for $\delta_{\mathcal{D}}$, they represent the contours of $\delta_{\mathcal{D}} = 0.6, 0.3$, and 0.15. The dashed lines, depicted in the case of $\delta_{\mathcal{N}}$ under global noise ($p = 0.8$ line), $\delta_{\mathcal{D}}$ under PD channel ($p = 0.8$ line), $\delta_{\mathcal{N}}$ under DP channel ($p = 0.35$ line), and $\delta_{\mathcal{D}}$ under DP channel ($p = 0.6$ line), represent the boundaries above which the corresponding monogamy score vanishes for most of the states. The regions marked by "**R**", and enclosed by the boxes are defined by the ranges $0.65 \leq |a_0| \leq 0.7071$, and $0.4 \leq p \leq 0.6$. The implications of these ranges of values are discussed in Sec. 7.4. All quantities plotted are dimensionless, except for $\delta_{\mathcal{N}}$, which is in ebits, and for $\delta_{\mathcal{D}}$, which is in bits.

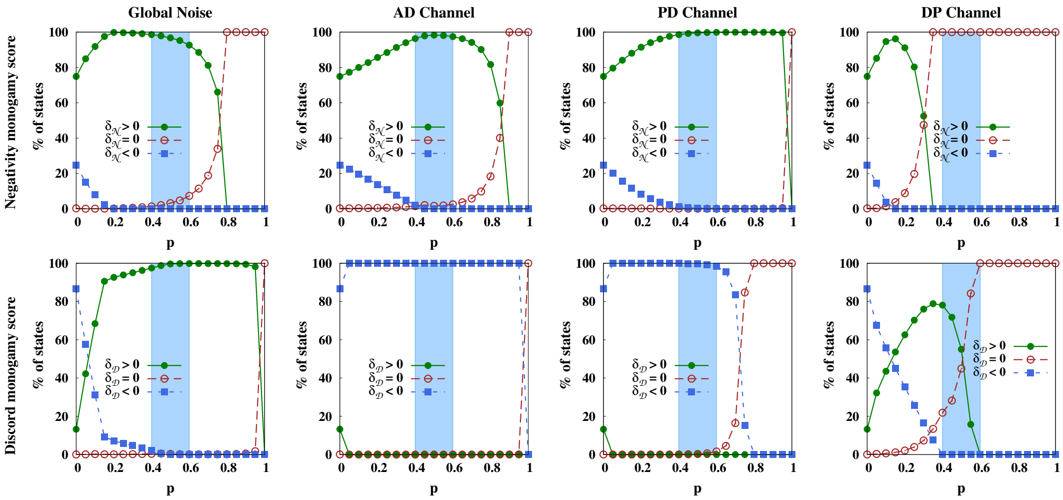


Figure 7.2: Variations of the percentages of states for which $\delta_{\mathcal{N}}$ (top panels), and $\delta_{\mathcal{D}}$ (bottom panels) are greater than (solid lines with filled circles), equal to (dashed lines with empty circles), and less than (dot-dashed lines with filled squares) zero, for different types of noise considered in this Chapter. The range of moderate noise, given by $0.4 \leq p \leq 0.6$, is shown by the shaded region in each figure. All quantities plotted are dimensionless.

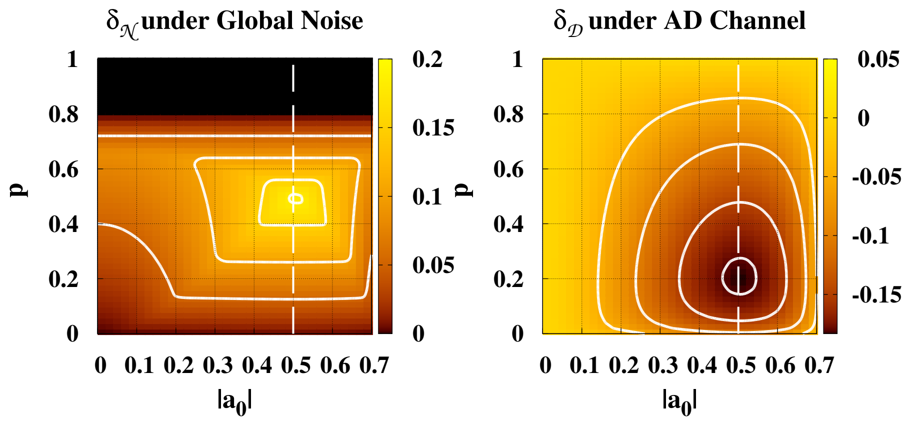


Figure 7.3: Variation of monogamy score corresponding to (a) $\delta_{\mathcal{N}}$ in the case of global noise, and (b) $\delta_{\mathcal{D}}$ in the case of amplitude damping channel, as functions of the state parameter $|a_0|$, and the noise parameter p, when gW state is subjected to noise. The value of $|a_2|$ is fixed at 0.7, while the value of $|a_1|$ is determined via normalization. The dynamics of monogamy score along the dashed line at $|a_0| = 0.5$ is non-monotonic in both the cases. The solid lines represent the contours obtained by joining the points at which $\delta_{\mathcal{N}}$, or $\delta_{\mathcal{D}}$ has a fixed value. From outside to inside, the closed contours correspond to (a) $\delta_{\mathcal{N}} = 0.05, 0.10, 0.15, 0.18$, and (b) $\delta_{\mathcal{D}} = -0.05, -0.10, -0.15, -0.18$. All the quantities plotted are dimensionless, except $\delta_{\mathcal{D}}$, which is in bits.

where a_0, a_1 and a_2 are complex numbers, satisfying $|a_0|^2 + |a_1|^2 + |a_2|^2 = 1$. When the gW state is subjected to global noise, the evolved three-qubit state, ρ^{gW} leads to the two-qubit reduced density matrix, ρ_{12}^{gW}, of the form

$$\rho_{12}^{\mathrm{gW}} = \frac{p}{4} I_4 + (1-p)\left(|a_0|^2 P[|00\rangle] + P[|\psi\rangle]\right) \tag{7.23}$$

in the computational basis, where $P[|x\rangle] = |x\rangle\langle x|$, and $|\psi\rangle = a_1|01\rangle + a_2|10\rangle$. The reduced state of qubits 1 and 3 can be determined from ρ_{12}^{gW} by interchanging a_0 and a_1. One should note here that unlike the gGHZ state, the reduced states with, as well as without, noise in the current case, are no more "classical-classical" states, and possess non-vanishing entanglement as well as quantum discord. For the AD channel, the three-qubit resulting state, starting from $|\Phi\rangle$, is given by $\rho^{\mathrm{gW}} = pP[|000\rangle] + (1-p)P[|\Phi\rangle]$, leading to $\rho_{12}^{\mathrm{gW}} = [p + (1-p)|a_0|^2]P[|00\rangle] + (1-p)P[|\psi\rangle]$, while ρ_{13}^{gW} is obtained by interchanging a_0 and a_1 in ρ_{12}^{gW}. In case of the PD channel, we define the states $|\tilde{\psi}\rangle = (1-p)(h_0 a_0|001\rangle + h_1 a_1|010\rangle + h_2 a_2|100\rangle)$, and $|\tilde{\phi}\rangle = h_1 a_1|01\rangle + h_2 a_2|10\rangle$, so that

$$h_i h_j = \begin{cases} (1-p)^{-2} & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases} \tag{7.24}$$

In terms of $|\tilde{\psi}\rangle$ and $|\tilde{\phi}\rangle$, $\rho^{\mathrm{gW}} = P[|\tilde{\psi}\rangle]$, and $\rho_{12}^{\mathrm{gW}} = |a_0|^2 P[|00\rangle] + (1-p)P[|\tilde{\phi}\rangle]$, respectively. Again, ρ_{13}^{gW} can be obtained from ρ_{12}^{gW} by interchanging a_0 and a_1. The form of ρ^{gW} in the case of the DP channel is given by

$$\begin{aligned} \rho^{\mathrm{gW}} &= \sum_{i=1}^{3} |a_{i-1}|^2 \varrho^{\otimes(i-1)} \otimes \varrho' \otimes \varrho^{\otimes(3-i)} \\ &\quad + (2q-1)^2 \Big[\Big(a_0 a_1^* \varsigma_1 + a_0 a_2^* \varsigma_2 + a_1 a_2^* \varsigma_3\Big) + h.c.\Big], \end{aligned} \tag{7.25}$$

where $\varrho = qP[|0\rangle] + (1-q)P[|1\rangle]$, $\varrho' = (1-q)P[|0\rangle] + qP[|1\rangle]$, and

$$\begin{aligned} \varsigma_1 &= \varrho \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0|, \\ \varsigma_2 &= |0\rangle\langle 1| \otimes \varrho \otimes |1\rangle\langle 0|, \\ \varsigma_3 &= |0\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes \varrho. \end{aligned} \tag{7.26}$$

The two-qubit reduced states, ρ_{12}^{gW} and ρ_{13}^{gW}, can be obtained from Eq. (7.25) by tracing out qubit 3 and 2 respectively. As in the case of global noise, local density matrices, up to certain value of the noise parameter, remains quantum

	(I) % of states $\in \mathcal{S}_x$		(II) % of states $\in \mathcal{S}_z$		(III) % of states $\notin \mathcal{S}$		(IV) Values of ε_{max}	
Noise-types	$\rho^{\text{gW}}_{1:23}$	ρ^{gW}_{12}	$\rho^{\text{gW}}_{1:23}$	ρ^{gW}_{12}	$\rho^{\text{gW}}_{1:23}$	ρ^{gW}_{12}	$\rho^{\text{gW}}_{1:23}$	ρ^{gW}_{12}
Global	13.4×10^{-2}	76.632	99.866	23.367	0	1×10^{-3}	0	1.23×10^{-3}
AD	99.937	99.9027	6.3×10^{-2}	9.73×10^{-2}	0	0	0	0
PD	9.73×10^{-2}	9.617	99.9027	90.373	0	1×10^{-2}	0	2.75×10^{-3}
DP	96.479	95.735	3.521	4.265	0	0	0	0

Table 7.1: Percentages of the states of the form ρ^{gW} and ρ^{gW}_{12}, belonging to the sets $\mathcal{S}_x$ and $\mathcal{S}_z$, are given in the columns **(I)** and **(II)** for gW states subjected to different noise models. The fraction of states of the form ρ^{gW} and ρ^{gW}_{12}, which do not belong to either of $\mathcal{S}_x$ or $\mathcal{S}_z$, are given in the column **(III)**. The upper bound of the absolute error, ε_{max}, is given in column **(IV)** for different types of noise considered in this Chapter. In each column, the first sub-column corresponds to the states of the form ρ^{gW}, while the second is for ρ^{gW}_{12}.

correlated.

Negativity monogamy score

The evaluation of negativity monogamy score, in the present case, becomes involved, in comparison to the case of the gGHZ state, due to the non-zero contribution of all the terms in the negativity monogamy score. However, analytical expressions for $\delta_{\mathcal{N}}$ can be determined for the global as well as different types of local noise considered in this Chapter. In case of the global noise, negativity score, $\delta^g_{\mathcal{N}}$, is given by

$$\delta^g_{\mathcal{N}} = \Big|\min\Big[0, s^g\Big]\Big| - \Big|\min\Big[0, s^g_{12}\Big]\Big| - \Big|\min\Big[0, s^g_{13}\Big]\Big|, \quad (7.27)$$

with $s^g = \frac{p}{8} - (1-p)|a_2|\sqrt{1-|a_2|^2}$, and $s^g_{12} = \frac{1}{4}\Big[p + 2(1-p)(|a_0|^2 - \sqrt{|a_0|^4 + 4|a_1|^2|a_2|^2})\Big]$, while for the PD channel, $\delta^{pd}_{\mathcal{N}}$ is obtained as

$$\delta^{pd}_{\mathcal{N}} = s^{pd} - \frac{1}{2}\Big(s^{pd}_{12} + s^{pd}_{13} + |a_2|^2 - 1\Big), \quad (7.28)$$

where $s^{pd} = (1-p)^2|a_2|^2([1-|a_2|^2)^{\frac{1}{2}}$, $s^{pd}_{12} = [|a_0|^2 + 4|a_1|^2|a_2|^2(1-p)^4]^{\frac{1}{2}}$. In both the cases, s^g_{13} and s^{pd}_{13} are obtained from s^g_{12} and s^{pd}_{12}, respectively, by interchanging $|a_0|$ and $|a_1|$. The expressions for negativity score, $\delta^{ad}_{\mathcal{N}}$, in the case of AD channel, is given by

$$\delta^{ad}_{\mathcal{N}} = \frac{1}{2}\Big[(s^{ad} - p) - (s^{ad}_{12} - \tilde{p}_0) - (s^{ad}_{13} - \tilde{p}_1)\Big], \quad (7.29)$$

where $s^{ad} = \sqrt{p^2 + 4(1-|a_2|^2)|a_2|^2(1-p)^2}$, and $s^{ad}_{12} = \sqrt{\tilde{p}_0^2 + 4|a_1|^2|a_2|^2(1-p)^2}$, with $\tilde{p}_j = p + (1-p)(\delta_{j0}|a_0|^2 + \delta_{j1}|a_1|^2)$. Here also, the function s^{ad}_{13} is obtained from s^{ad}_{12} by interchanging a_0 and a_1. The expression for $\delta^{dp}_{\mathcal{N}}$, in the case of the

DP channel, can also be obtained following the same procedure as in the cases of other three types of noise. However, the expression is rather involved, and we choose not to include the same.

Discord monogamy score

The computation of $\delta_{\mathcal{D}}$, in case of the gW states under noise, requires more numerical resources than that in the case of the gGHZ states, since both $\mathcal{D}\big(\rho_{12}^{\text{gW}}\big)$ and $\mathcal{D}\big(\rho_{13}^{\text{gW}}\big)$ do not vanish for almost all p. In the present case, $\delta_{\mathcal{D}}$ can be written as

$$\delta_{\mathcal{D}} = S - S(\rho_1^{\text{gW}}) - S_c, \tag{7.30}$$

where $S = S(\rho_{12}^{\text{gW}}) + S(\rho_{13}^{\text{gW}}) - S(\rho^{\text{gW}})$, and $S_c = S(\rho_{2|1}^{\text{gW}}) + S(\rho_{3|1}^{\text{gW}}) - S(\rho_{23|1}^{\text{gW}})$. The determination of $\delta_{\mathcal{D}}$ for a single three-qubit state requires, in principle, three separate optimizations for the terms in S_c. However, information acquired via numerical analysis may result in considerable reduction of the computational complexity [180–183]. Let us first concentrate on the computation of $\mathcal{D}(\rho_{1:23}^{\text{gW}})$ under four types of noise considered in this Chapter. We perform extensive numerical search by Haar uniformly generating a set of 3×10^6 random three-qubit states of the form ρ' for each of the types of noise considered in this Chapter. We find that for all such states, considering two sets of values of the real parameters, (θ, ϕ), in projection measurements involved in $\mathcal{D}(\rho_{1:23}^{\text{gW}})$, is enough. These sets are given by **(i)** $\theta = \pi/2, 0 \leq \phi < 2\pi$, and **(ii)** $\theta = 0, \pi, 0 \leq \phi < 2\pi$, which correspond to projection measurements on the (x, y) plane, and along the z axis of the Bloch sphere, respectively. Without any loss of generality, one can consider a projection measurement corresponding to the observable σ_x in the former case, while a projection measurement corresponding to σ_z in the latter. We refer to the set of states of the form ρ^{gW}, for which measurement corresponding to σ_x, or σ_z provides the optimal measurement, as the "special" set, denoted by $\mathcal{S}$. In the present case, the set $\mathcal{S}$ represents the set of all states of the form ρ^{gW}, for each of the types of noise, according to our numerical analysis. The set of states for which the optimization occurs for σ_x, is denoted by $\mathcal{S}_x$, while $\mathcal{S}_z$ represents the set of ρ^{gW} for which optimal measurement corresponds to σ_z. Note that $\mathcal{S} = \mathcal{S}_x \cup \mathcal{S}_z$, while $\mathcal{S}_x \cap \mathcal{S}_z = \Phi$, the null set.

The situation is a little different in the case of the two-qubit states ρ_{12}^{gW} and ρ_{13}^{gW}, obtained from ρ^{gW}. We generate 3×10^6 states Haar uniformly, which are of the form ρ_{12}^{gW}, and we find that, like in the case of $(\rho_{1:23}^{\text{gW}})$, there exists, for each type of noise, a "special" set, $\mathcal{S}$, of states ρ_{12}^{gW}, for which optimization occurs

	a		b		c		d	
Noise-types	$\delta_{\mathcal{N}}$	$\delta_{\mathcal{D}}$	$\delta_{\mathcal{N}}$	$\delta_{\mathcal{D}}$	$\delta_{\mathcal{N}}$	$\delta_{\mathcal{D}}$	$\delta_{\mathcal{N}}$	$\delta_{\mathcal{D}}$
Global	75.009	13.330	0.186	0.003	24.805	83.536	0.000	0.131
AD	17.532	13.321	57.663	0.012	23.964	82.902	0.841	3.765
PD	56.145	13.323	19.050	0.010	24.784	53.272	0.021	33.395
DP	53.531	8.708	21.664	4.625	24.802	68.710	0.003	17.957

Table 7.2: The percentage of gW states exhibiting **a**, **b**, **c**, and **d**-type dynamics for $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ under the application of different types of noise.

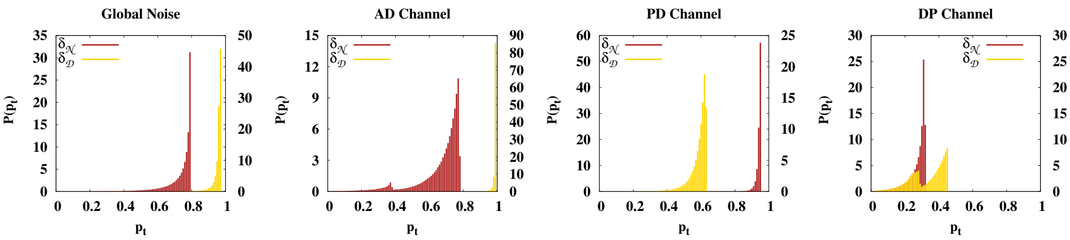


Figure 7.4: Variations of the normalized probability density function, $P(p_t)$, against the dynamics terminal, p_t, for $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$, when gW states are subjected to different types of noise. All quantities plotted are dimensionless.

corresponding to either σ_x, or σ_z. However, in the case of global noise and PD channel, a small fraction of ρ_{12}^{gW} does not belong to $\mathcal{S}$, and the optimization of $\mathcal{D}(\rho_{12}^{\text{gW}})$, for these states, occur for other values of (θ, ϕ). Let the maximum absolute error, resulting from the assumption that all the three-qubit states of the form ρ_{12}^{gW} belong to $\mathcal{S}$, in the case of the global noise and PD channel, is ε. Our numerical analysis provides an upper bound of ε, denoted by ε_{max}, which is of the order of 10^{-3} in the case of both types of noise. Table 7.1 displays our findings regarding the percentages of states of the form ρ^{gW} and ρ_{12}^{gW}, that belong to the sets $\mathcal{S}_x$, $\mathcal{S}_z$, and do not belong to $\mathcal{S}$ for all the four types of noise. The last column (column **(IV)**) tabulates the values of $\varepsilon_{\max}$ in the relevant cases. From now on, unless otherwise mentioned, we determine the values of $\delta_{\mathcal{D}}$ by computing quantum discord with the assumption that the states either belong to $\mathcal{S}_x$ or $\mathcal{S}_z$.

Behavior of monogamy under moderate noise

Let us now quantitatively study the behavior of monogamy scores, $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$, of the gW states for a fixed noise parameter. We determine the fractions of the set of ρ^{gW}, for which the monogamy score corresponding to the chosen quantum correlation measure is strictly greater than, equal to, and strictly less than zero. We study the variation of these fractions with the change in values of the noise parameter for the specified type of noise. The variations of the three different

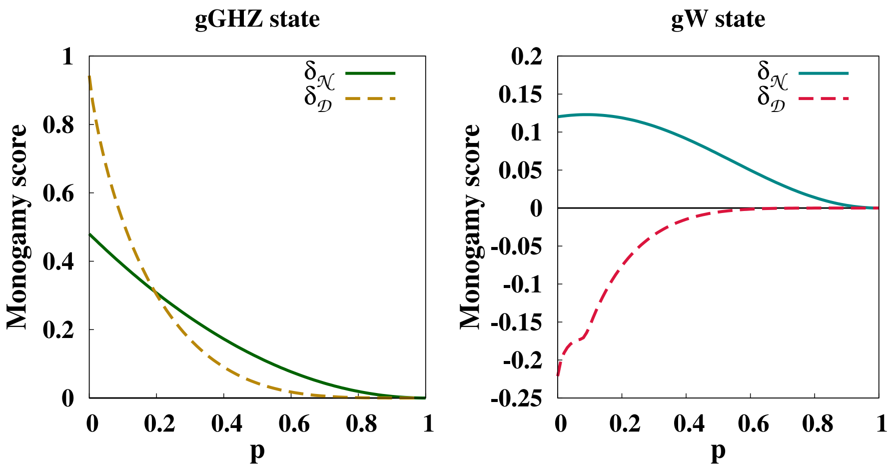


Figure 7.5: Dynamics profiles of $\delta_{\mathcal{N}}$ (solid line) and $\delta_{\mathcal{D}}$ (dashed line) in the case of the input gGHZ state given by $|a_0| = 0.7$ (left panel), and the input gW state given by $a_0 = -0.287 - 0.552i$, and $a_1 = 0.637 + 0.23i$ (right panel). In both the cases, $\delta_{\mathcal{N}}$ remains positive for higher values of p ($p \geq 0.6$), while $\delta_{\mathcal{D}}$ vanishes. Note that for the gW states considered here, $\delta_{\mathcal{N}}$ exhibits a type-**a** dynamics, while that of $\delta_{\mathcal{D}}$ is of type-**c**. All the quantities plotted are dimensionless, except for $\delta_{\mathcal{D}}$, which is in bits.

fractions, as described above, with respect to p, are depicted in Fig. 7.2. Let us now investigate the effect of moderate noise on the monogamy scores. In the present study, we choose a range of p given by $0.4 \leq p \leq 0.6$ (marked by the shaded regions in the panels in Fig. 7.2), which is moderate in comparison to the lower and upper bounds of p. From Fig. 7.2, it is clear that for moderate values of p (viz. $0.4 \leq p \leq 0.6$), in the case of global noise as well as the AD channel, most of the states have $\delta_{\mathcal{N}} > 0$, while for the DP channel, $\delta_{\mathcal{N}} = 0$ for 100% of the states. Remarkably, for the PD channel, all the states have $\delta_{\mathcal{N}} > 0$ when the noise parameter is in the moderate range.

The situation is different for discord monogamy score. It is found that $\delta_{\mathcal{D}} \geq 0$ for almost the entire range of moderate values of p, when gW states are subjected to global noise. In the case of the AD channel, $\delta_{\mathcal{D}} < 0$ for the entire range $0.4 \leq p \leq 0.6$. In this scenario, $\delta_{\mathcal{D}} < 0$ for the entire range of p, except only at $p = 1$, the fully decohered states. Also, for the PD channel, $\delta_{\mathcal{D}} < 0$ for moderate p except when $p \approx 0.6$. However, in the case of the DP channel, $\delta_{\mathcal{D}} \geq 0$ for $0.4 \leq p \leq 0.6$. Hence it is clear that the monogamy of negativity behaves differently than the monogamy of quantum discord, in the case of global noise, and local channels considered in this Chapter. These results are of prime importance to our goal of channel discrimination, which will be discussed in Sec. 7.4.

Types of dynamics: Monotonic vs. non-monotonic

In case of the gW state subjected to global or local noise, the set of different types of dynamics that $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ undergo is far richer compared to that for gGHZ states. While only monotonic decay of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ with increasing p is found in the latter case, non-monotonic dynamics of monogamy scores emerges in the former. As an example, consider the set of states of the form $|\Phi\rangle$ (of Eq. (7.22)) having a fixed value of $|a_2|$. The states in the set can be represented by the different allowed values of the absolute value of the free parameter, a_0. Fig. 7.3 depicts the landscapes of $\delta_{\mathcal{N}}$, in the case of global noise, and of $\delta_{\mathcal{D}}$, in the case of the AD channel, as functions of $|a_0|$ and p, for $|a_2| = 0.7$. The solid lines in the figures represent contours obtained by joining the points having a constant value of either $\delta_{\mathcal{N}}$, or $\delta_{\mathcal{D}}$. Note that the contours form closed curves, and from outside to inside, the lines represent increasing values of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$. The dashed lines in the plots represent the dynamics of $\delta_{\mathcal{N}}$, in the case of global noise, and $\delta_{\mathcal{D}}$, in the case of the AD channel, when the input gW state is taken with $|a_0| = 0.5$. The behavior of the monogamy scores with increasing values of p are non-monotonic, as clearly indicated from the values of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$, represented by different shades in Fig. 7.3. An increase in the monogamy scores can be argued to be a signature of increase in quantumness. Although noise destroys quantum correlations, here we see the opposite by obtaining non-monotonicity of monogamy score with the increase of p.

Types of dynamics. Now we catalog four "typical" dynamics profiles observed for both $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ for global noise as well as for AD, PD, and DP local channels. **a.** In the first profile, $\delta_{\mathcal{Q}}(p=0) \geq 0$, and $\delta_{\mathcal{Q}}(p)$ goes to zero non-monotonically as $p \to 1$. **b.** For the second one, $\delta_{\mathcal{Q}}(p)$ monotonically goes to zero when p increases, with $\delta_{\mathcal{Q}}(p=0) \geq 0$. **c.** In contrast to the first two profiles, $\delta_{\mathcal{Q}}(p=0) < 0$ for the third profile. With an increase of p, $\delta_{\mathcal{Q}}$ vanishes non-monotonically. **d.** Similar to the third profile, the fourth and the final profile starts with a non-monogamous scenario ($\delta_{\mathcal{Q}}(p=0) < 0$). However, with increasing p, $\delta_{\mathcal{Q}}(p)$ goes to zero monotonically as $p \to 1$.

Evidently, the frequencies of occurrence of the dynamics types **a**, **b**, **c**, and **d** must vary for different types of noise, and for different observables, viz., $\delta_{\mathcal{N}}$, and $\delta_{\mathcal{D}}$. To estimate these, we prepare a sample of 10^6 Haar-uniformly generated gW states as input, which can be subjected to each of the types of noise, and study the dynamics profiles of the states. We find that at $p = 0$, 75.195% of the gW states are monogamous when negativity is considered, while only 13.333% of them are monogamous with respect to quantum discord. When the value of p is increased, the four types of dynamics are found to occur with different frequencies in the case of the global noise and the local channels (see Table. 7.2). Note that for $\delta_{\mathcal{N}}$, type-**a** is more frequent in the case of global noise as well as for the PD and DP channels, while type-**b** occurs mostly in the case of the AD channel. The frequency of occurrence of **d** is much less compared to that of **a**, **b**, and **c** for the negativity monogamy score. Among all the noisy channels, the non-monotonic decay of $\delta_{\mathcal{N}}$ occurs close to 100% of times when global noise acts on the gW state, irrespective of the sign of $\delta_{\mathcal{N}}$ at $p = 0$. On the other hand, in the case of $\delta_{\mathcal{D}}$, frequency of occurrence of **c** and **d** is high in the cases of global noise and the AD channel, while the same is moderate in the case of the PD and the DP channels.

Dynamics terminal

So far, we have qualitatively discussed and characterized the dynamics of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ under the application of global and local noise to the gW state. It is observed as well as intuitively clear that the persistence of the monogamy scores, when subjected to noise, must be different for different types of noise considered in this Chapter. To analyze this quantitatively, for a given state, ρ, we define the "dynamics terminal", p_t, which is given by the value of the noise parameter, p, at which the monogamy score vanishes, and remains so for $p_t \leq p \leq 1$. The value of p_t is characteristic to the input state, $|\Phi\rangle$, and the type of noise applied to it. A high value of p_t implies a high persistence of the monogamy score for the state $|\Phi\rangle$ against the particular type of noise applied to it. It is clear that for gW states as the input states, p_t may assume a range of values since the dynamics terminal will clearly assume different values for different input gW states, when the type of noise is fixed. However, for a specific type of noise, the average value of p_t, denoted by $\langle p_t \rangle$, and defined by

$$\langle p_t \rangle \quad = \quad \int_0^1 p_t P(p_t) dp_t, \tag{7.31}$$

provides a scale for the "high" values of the noise parameter. Here, $P(p_t)$ is the normalized probability density function (PDF) such that $P(p_t)dp_t$ provides the probability that for an arbitrary three-qubit gW state under the fixed type of noise, the value of p_t lies between p_t and $p_t + dp_t$. Note that the full range of the allowed values of p_t is given by $0 \leq p_t \leq 1$, which follows from the definition of the noise parameter.

It is important to check whether $\langle p_t \rangle$ can distinguish between different types of noise. In order to determine $P(p_t)$, we Haar uniformly generate 10^6 gW states for each of the four kinds of noise, and study their dynamics profiles to determine $\langle p_t\rangle$. The variations of $P(p_t)$ against p_t are given in Fig. 7.4. It is clear from the figure that the maximum possible value of p_t is considerably different in the case of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$, when the type of noise is fixed. The values of $\langle p_t\rangle$ corresponding to $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$, calculated from Eq. (7.31), for global noise, AD channel, PD channel, and DP channel are given in Table 7.3. Note that the dynamics terminal corresponding to $\delta_{\mathcal{D}}$ is higher than that corresponding to $\delta_{\mathcal{N}}$ in the case of the global noise, AD channel, and the DP channel, while the trend is reversed in the case of the PD channel. Note also that the behavior of $\langle p_t\rangle$ corresponding to $\delta_{\mathcal{N}}$, in the results obtained previously in Sec. 7.3.2 in the case of the PD channel, is in good agreement with the value of $\langle p_t\rangle$ corresponding to $\delta_{\mathcal{N}}$, which is maximum for the PD channel, and minimum for the DP channel. The implication of this result is elaborated in Sec. 7.3.3. It is interesting to note here that there can not be any bipartite state for which $\mathcal{N}>0$ while $\mathcal{D}=0$. However, such a situation is possible in the case of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$. We shall discuss this issue in Sec. 7.3.3 in detail.

Note. Due to the extensive numerical effort required for determining the values of $\mathcal{D}(\rho^{\text{gW}}_{1:23})$, $\mathcal{D}(\rho^{\text{gW}}_{12})$, and $\mathcal{D}(\rho^{\text{gW}}_{13})$ in computing $\delta_{\mathcal{D}}$ when $p>0$, we employ the constrained optimization corresponding to σ_x and σ_z only, as discussed in Sec. 7.3.2, to obtain several important statistics reported in Sec. 7.3.2, 7.3.2, and 7.3.2. However, the error in the various statistics obtained for different channels, due to this approximation, is insignificant, and does not change the qualitative aspects of the results. Note that in all the occasions in this Chapter, where actual value of $\delta_{\mathcal{D}}$ has been plotted, or reported, exact optimization has been carried out using numerical techniques.

7.3.3 Robustness of negativity monogamy score

As already mentioned in the Introduction, in the bipartite domain, it has been observed that quantum discord vanishes asymptotically with increasing noise

Noise	$\langle p_t\rangle$ for $\delta_{\mathcal{N}}$	$\langle p_t\rangle$ for $\delta_{\mathcal{D}}$
Global	0.733	0.947
AD	0.667	0.986
PD	0.940	0.584
DP	0.274	0.331

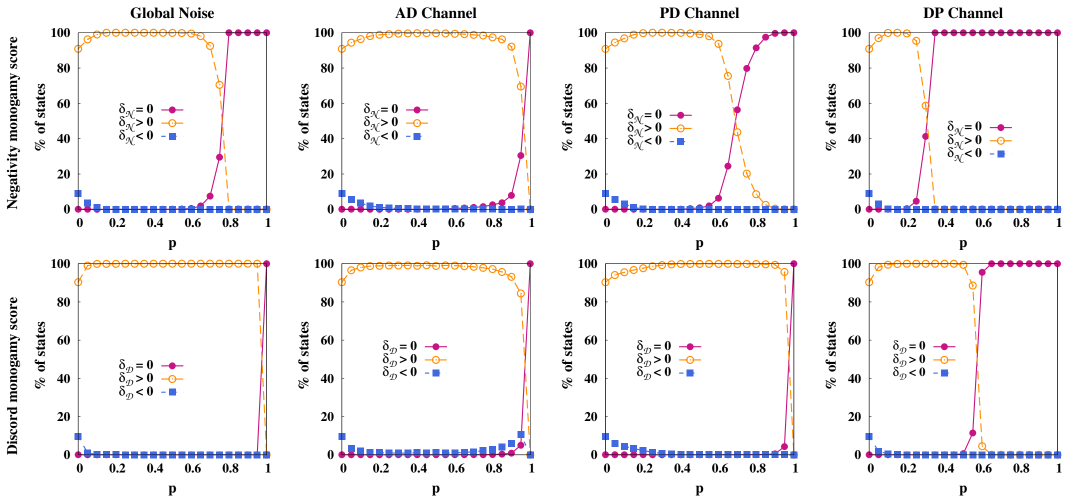
Table 7.3: The average values of dynamics terminal, $\langle p_t\rangle$, for $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$, when gW states are subjected to various kinds of noise. The profiles of the probability density function, $P(p_t)$, corresponding to different types of noise, for both $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$, are given in Fig. 7.4.

strength, p, when quantum states are exposed to local noise. On the other hand, entanglement measures undergo a "sudden death" at a finite value of p under similar noise, indicating a more fragile behavior than quantum discord. Interestingly, an opposite trend is observed when monogamy of quantum correlations are subjected to local noisy channels. The variation of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ with $|a_0|$ and p in the case of PD channels with gGHZ states as input states (Fig. 7.1) indicates that there exists gGHZ states for which $\delta_{\mathcal{N}}$ persists longer than $\delta_{\mathcal{D}}$ for higher values of the noise parameter, p ($p \geq 0.8$). In Sec. 7.3.2, it has been pointed out that the value of $\langle p_t \rangle$ for negativity monogamy score, in the case of the DP channel, is much larger compared to that of the discord monogamy score. Also, Fig. 7.2 indicates that for higher values of p ($0.6 \leq p \leq 0.9$), 100% of gW states have $\delta_{\mathcal{N}} > 0$ when the noise is of PD type. Note that for all such states, $\delta_{\mathcal{D}} \leq 0$. This implies that there is a finite probability of finding gW states which, when subjected to PD channel, will evolve into a state ρ^{gW} with $\delta_{\mathcal{N}} > 0$, but $\delta_{\mathcal{D}} = 0$.

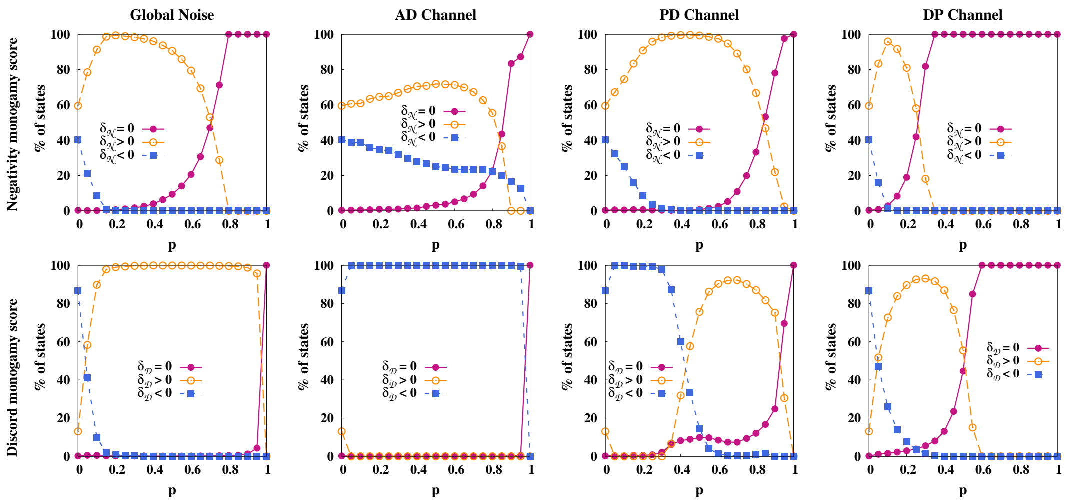
We present two specific examples to establish such observations. Our first example is the gGHZ states represented by $|a_0| = 0.6$, while the second example is the gW states given in Eq. (7.22) with $a_0 = -0.287 - 0.552i$, and $a_1 = 0.637 + 0.23i$. The behavior of $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ against p are plotted in Fig. 7.5, where the quantum discord components of $\delta_{\mathcal{D}}$ are computed via exact numerical optimization. It is clear from the figure that in both the cases, $\delta_{\mathcal{N}}$ persists longer than $\delta_{\mathcal{D}}$ at higher end of noise parameter. One must note here that the quantum discord components of $\delta_{\mathcal{D}}$ cancel each other at higher noise, while being individually non-zero. Hence the observation of $\delta_{\mathcal{N}} > 0$ in situations where $\delta_{\mathcal{D}} = 0$ is consistent with the fact that entanglement measures vanish for zero discordant states in bipartite systems. Therefore, it is evident that negativity monogamy score, in the presence of PD noise, exhibits a more robust behavior compared to that of the discord monogamy score. This is in contrast to the usual observation for bipartite quantum discord and entanglement measures.

7.3.4 Arbitrary tripartite pure states

Hitherto, we have investigated gGHZ and gW states, for which the effects of various noisy channels on monogamy scores can be addressed analytically up to certain extent. To complete the investigation for three-qubit states as input, we now consider the two mutually exclusive and exhaustive classes of three-qubit states, viz., the GHZ class and the W class [83]. These two classes, inequivalent under stochastic local operations and classical communication (SLOCC), together span the entire set of three-qubit pure states [83]. An arbitrary three-qubit pure



(a)



(b)

Figure 7.6: Variations of the percentages of arbitrary three-qubit pure states chosen from (a) GHZ class and (b) W class, for which monogamy scores corresponding to negativity and quantum discord are strictly greater than, equal to, and strictly less than zero, with the noise parameter, p. All quantities plotted are dimensionless.

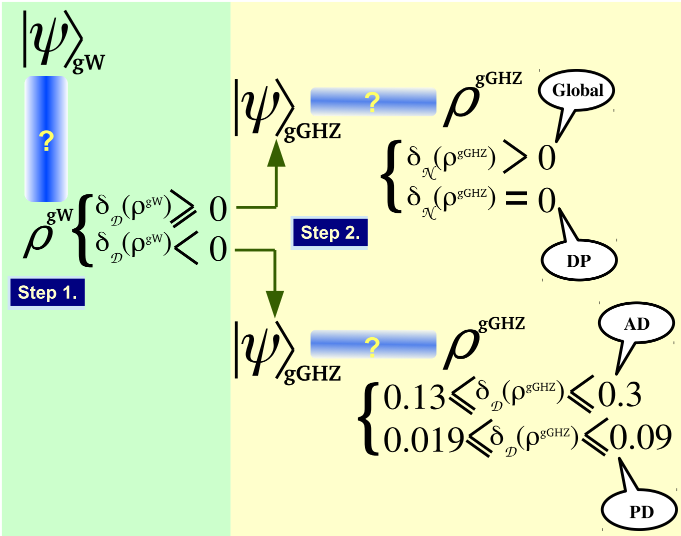


Figure 7.7: Schematic representation of the two-step channel discrimination protocol.

state from the GHZ class, up to local unitary operations, can be parametrized as $|\psi_{GHZ}\rangle = \sqrt{K}(c_\delta|000\rangle + s_\delta e^{i\varphi}|\varphi_\alpha\rangle|\varphi_\beta\rangle|\varphi_\gamma\rangle)$, where $|\varphi_k\rangle = c_k|0\rangle + s_k|1\rangle$ with $c_k = \cos k$, $s_k = \sin k$, $k = \alpha, \beta, \gamma$, and $K = (1 + 2c_\delta s_\delta c_\alpha c_\beta c_\gamma c_\varphi)^{-1} \in (\frac{1}{2}, \infty)$ is the normalization factor. Here, the ranges for the five real parameters are $\delta \in (0, \pi/4], \alpha, \beta, \gamma \in (0, \pi/2]$ and $\varphi \in [0, 2\pi)$. On the other hand, a three-qubit pure state from the W class, up to local unitaries, can be written in terms of three real parameters as $|\psi_W\rangle = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{1-(a+b+c)}|000\rangle$, where $a, b, c \geq 0$. Due to higher number of state parameters in arbitrary three-qubit pure states chosen from these classes, determining compact forms for $\delta_{\mathcal{N}}$ as well as $\delta_{\mathcal{D}}$ is difficult. Also, the constrained optimization, as discussed in Sec. 7.3.2, is not applicable due to the high absolute error in the value of quantum discord. Therefore, we employ exact numerical optimization technique to compute quantum discord in discord monogamy scores of these states. We Haar-uniformly generate 10^4 states from each of the two classes – the GHZ class and the W class – for a chosen value of the noise parameter, p, when a specific type of noise is applied to it. We then determine the percentage of states for which negativity and discord monogamy scores are greater than, equal to, and less than zero, and study the variation of these percentages with varying noise parameter.

The variation of the percentages of three-qubit pure states from the GHZ class, for which $\delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}}$ are $>$, $=$, and < 0, against p is given in Fig. 7.6(a), while the same for the W class states are presented in Fig. 7.6(b). The percentages vary non-monotonically with varying noise parameter, and the percentage of states for which the monogamy scores corresponding to negativity and quantum discord are equal to zero, for both classes of states, tend to become 100% with increasing p, as expected. For both the classes, this trend is considerably slower in the case of global noise, AD channel, and PD channel, in comparison to that for the DP channel. The patterns in the W class states are similar to those in the case of gW states, except for discord monogamy score under PD channel. While no gW states have a strictly positive $\delta_{\mathcal{D}}$ for higher values of p, in the case of W class states, the corresponding fraction increases with increasing p, reaches a maximum value at moderately high p, and then, as expected, decreases to zero as $p \to 1$.

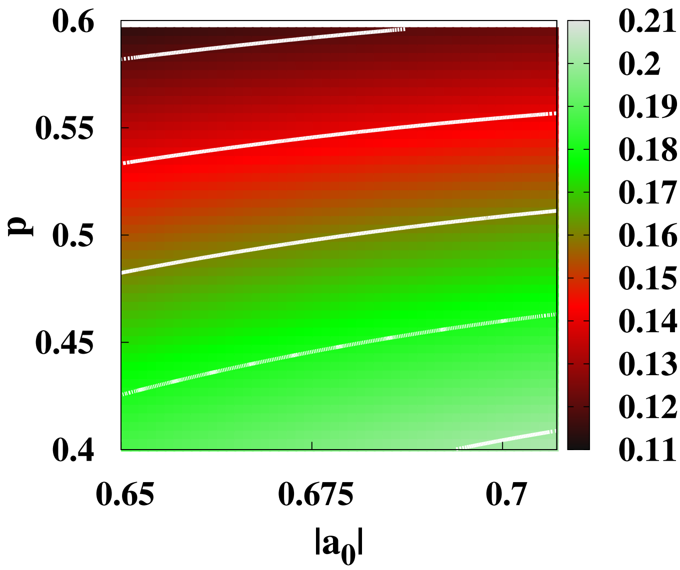


Figure 7.8: Variation of Δ, the difference between the values of $\delta_{\mathcal{D}}$ for a fixed gGHZ state under AD and PD channels, with the state parameter, $|a_0|$, and the noise parameter, p, in the region "**R**" marked in Fig. 7.1. The solid lines are obtained by joining constant values of Δ, where from low to high value of p, the lines stand for $\Delta = 0.12, 0.14, 0.16, 0.18$, and 0.2. All quantities plotted are dimensionless, except for Δ, which is in bits.

7.4 Channel discrimination via monogamy

In this section, we investigate the second objective of this Chapter, and address the question whether monogamy of quantum correlations can be applied to conclusively detect the type of noise to which the quantum state is exposed. In particular, we propose a two-step protocol to discriminate global noise as well as local channels, namely, AD, PD, and DP channels, via negativity and discord monogamy score, by using a gW state and a gGHZ state as resources. The choice of observable in the second step is determined according to the outcome of the first step. The assumptions required for the success of the protocol are **(i)** that the strength of the noise is moderate, viz., $0.4 \leq p \leq 0.6$, and **(ii)** that the given noisy channel can be used twice. Below, by an unknown channel, we shall mean one of th four channels, among global noise, AD, PD, and DP channels.

Two-step discrimination protocol. The two steps constituting the protocol are as follows. **1.** Given an unknown channel, the first step is to send an arbitrary gW state through that channel and to measure the value of $\delta_{\mathcal{D}}$ for the output state. **2.** The next step is to send a gGHZ state with high entanglement (e.g., $0.65 \leq |a_0| \leq 0.7071$) through the channel, and to measure the monogamy score corresponding to either negativity, or quantum discord, subject to the outcome of the first step. If $\delta_{\mathcal{D}} \geq 0$ in the first step, $\delta_{\mathcal{N}}$ is chosen as the observable, while for $\delta_{\mathcal{D}} < 0$ in step **1**, discord monogamy score can conclusively identify the type of noise in the channel (as shown schematically in Fig. 7.7).

Now we explain the implications of the output of the protocol. If $\delta_{\mathcal{D}} < 0$ in step **1** for moderate values of p, then the original gW state was subjected to either the AD, or the PD channel, while a nonnegative $\delta_{\mathcal{D}}$ implies that the noise was either global, or DP. This is clear from the variation of the percentages of states for which $\delta_{\mathcal{D}} \geq 0$ and < 0 in the range $0.4 \leq p \leq 0.6$, as depicted in Fig. 7.2. Hence, the first step divides the four types of noise in a block – the duo of AD and PD channels, and global noise and DP channel.

First, let us assume that $\delta_{\mathcal{D}} \geq 0$ in the first step, which leads one to choose $\delta_{\mathcal{N}}$ as observable in the second step of the strategy. For $\delta_{\mathcal{N}} > 0$ in the second step, the type of noise that acts on the gGHZ state is the global noise, while $\delta_{\mathcal{N}} = 0$ implies that the channel is DP. This can be understood from the boxed regions marked "**R**" in Fig. 7.1, where $\delta_{\mathcal{N}} = 0$ for the DP channel, while $\delta_{\mathcal{N}} > 0$ for global noise. On the other hand, if the outcome of the first step is $\delta_{\mathcal{D}} < 0$, the channel is either AD, or PD. In this situation, $\delta_{\mathcal{D}}$ is always positive when the

Step **1**: input gW	Step **2**: input gGHZ	Conclusion
$\delta_{\mathcal{D}} \geq 0$	$\delta_{\mathcal{N}} > 0$	Global noise
$\delta_{\mathcal{D}} < 0$	$0.13 \leq \delta_{\mathcal{D}} \leq 0.3$	AD Channel
$\delta_{\mathcal{D}} < 0$	$0.019 \leq \delta_{\mathcal{D}} \leq 0.09$	PD Channel
$\delta_{\mathcal{D}} \geq 0$	$\delta_{\mathcal{N}} = 0$	DP channel

Table 7.4: Encoding of the outcomes of the two-step channel discrimination protocol using monogamy scores of negativity and quantum discord.

input state is the gGHZ state with a specific value of $|a_0|$ in the range mentioned before, and so the discrimination protocol is more involved. In particular, we observe that in the marked region "**R**" in Fig. 7.1, $0.13 \leq \delta_{\mathcal{D}}^{ad} \leq 0.3$ for the AD channel, while for the PD noise, $0.019 \leq \delta_{\mathcal{D}}^{pd} \leq 0.09$. The variation of $\Delta = \delta_{\mathcal{D}}^{ad} - \delta_{\mathcal{D}}^{pd}$, the difference between the values of discord monogamy score in the case of AD channel ($\delta_{\mathcal{D}}^{ad}$) and PD channel ($\delta_{\mathcal{D}}^{pd}$), with $|a_0|$ and p in the region "**R**" in Fig. 7.1 is plotted in Fig. 7.8. We notice that there is no overlap between the allowed ranges of $\delta_{\mathcal{D}}$ for the two channels (as also indicated by the absence of the value $\Delta = 0$ in Fig. 7.8), implying that $\delta_{\mathcal{D}}$ can conclusively distinguish between the AD and the PD channels. The possible encoding of the outcomes of the two-step protocol, and their implications are tabulated in Table 7.4.

Remark 1. The first step of our channel discrimination protocol requires not the value, but only the sign of $\delta_{\mathcal{D}}$, while the second step requires an estimation of the discord monogamy score, $\delta_{\mathcal{D}}$, for AD and PD channels.

Remark 2. Although the range of values of $\delta_{\mathcal{D}}$ are non-overlapping for AD and PD channels when the state parameter is in the range $0.6 < |a_0| < 0.7071$, the difference between the values corresponding to the lower bound of $\delta_{\mathcal{D}}^{ad}$, and upper bound of $\delta_{\mathcal{D}}^{pd}$ decreases with relaxing the lower bound of $|a_0|$. Hence, the lower bound of the allowed range of $|a_0|$ can be relaxed depending on the accuracy with which $\delta_{\mathcal{D}}^{ad}$ and $\delta_{\mathcal{D}}^{pd}$ can be estimated with the current technology in hand. The best result is obtained for the three-qubit GHZ state, for which $|a_0| = 1/\sqrt{2}$.

Remark 3. In the presence of high noise ($p > 0.6$), our protocol may fail to distinguish the type of noise applied to the quantum state. This is because both $\delta_{\mathcal{D}}$ and $\delta_{\mathcal{N}}$ may vanish in the case of both gGHZ state and gW state when the noise strength is high. It is also clear that the above distinguishing protocol fails when $p \approx 0$.

7.5 Summary

In this Chapter, we have investigated the dynamical features of the monogamy property of quantum correlations using monogamy score as the observable, when three-qubit systems are subjected to global noise as well as local noisy channels, including amplitude-damping, phase-damping, and depolarizing channels, As the quantum correlation measures, we choose negativity and quantum discord to show that the dynamics of monogamy score, when generalized GHZ states are subjected to different types of noise, is qualitatively different than that of the generalized W state as input. While monogamy score corresponding to both the quantum correlation measures exhibit a monotonic decay with increasing noise in the former case, non-monotonic dynamics takes place in the latter, giving rise to a rich set of dynamics profiles. We catalog the different dynamics profiles that take place when a generalized W state is exposed to different types of noise, and determine the frequency of occurrence of them for all the noise models considered. We define a characteristic noise scale, called the "dynamics terminal", that quantifies the persistence of the monogamy score corresponding to a particular measure of quantum correlation, when the state is subjected to a specific type of noise. We show that the dynamics terminal can distinguish between the different noise models, and indicates that the depolarizing channel destroys monogamy scores faster compared to the other types of noise. To investigate how the monogamy property behaves against increasing noise, we investigate the variation of the fraction of states, for which the monogamy score is strictly greater than, equal to, and strictly less than zero, with increasing value of the noise parameter, when the input states are chosen from generalized GHZ state, generalized W state, and arbitrary three-qubit pure states. We also show that the negativity monogamy score may exhibit a more robust behavior against phase damping noise, compared to the discord monogamy score, which is in contrast to the usual observation regarding bipartite entanglement and quantum discord. Using the dynamical properties of the monogamy scores corresponding to generalized GHZ and generalized W states as input states, we have proposed a two-step channel discrimination protocol that can conclusively identify the different types of noise, by using monogamy scores as the observables.

The results of this Chapter are based on the following paper:

1. *Conclusive identification of quantum channels via monogamy of quantum correlations*, **Asutosh Kumar**, Sudipto Singha Roy, Amit Kumar Pal, R. Prabhu, Aditi Sen De, and Ujjwal Sen, Phys. Lett. A **380**, 3588 (2016).

Chapter 8

Quantum Coherence: Reciprocity and Distribution

8.1 Introduction

From everyday life experiences, we learn that arbitrary operations cannot do an assigned job. That is, specific resources–like allowed operations, "free assets" that one can use at will, and some "force" or catalyst in a prescribed amount–are needed to carry out a particular task. Therefore, to establish a quantitative theory of any physical resource, one needs to address the following fundamental issues: (i) the characterization or unambiguous definition of resource, (ii) the quantization or valid measures, and (iii) the transformation or manipulation of quantum states under the imposed constraints [46–49]. Several useful quantum resources like purity [50], entanglement [51–55], reference frames [56, 57], thermodynamics [58, 59], asymmetry [60], etc. have been identified and quantified until now. Recently, Baumgratz *et al.* in Ref. [61], provided a quantitative theory of coherence as a new quantum resource, borrowing the formalism already established for entanglement [51–55], thermodynamics [58, 59] and reference frames [56, 57].

Coherence arises from the superposition principle, and is defined for single as well as multipartite systems. Quantum coherence is identified by the presence of off-diagonal terms in the density matrix, and hence is a basis-dependent quantity. It being a basis-dependent quantity, local and nonlocal unitary operations can alter the amount of coherence in a quantum system. A density matrix has zero coherence with respect to a specific basis if it is diagonal in that basis. Diagonal density matrices, in the above sense, therefore represent essentially the classical mixtures. A coherent quantum state is considered as a resource in thermodynamics as it allows non-trivial transformations [329]. Quantum superposition is the most fundamental feature of quantum mechanics. Quantum coherence is a direct consequence of the superposition principle. Moreover, combined with the tensor product structure of quantum state space, it gives rise to the novel concepts such as entanglement and quantum correlations. It, being the premise of quantum correlations in multipartite systems, has attracted the attention of quantum information community significantly, and in addition to its quantification [330–332], other developments like the freezing phenomena [333], the coherence transformations under incoherent operations [334], establishment of geometric lower bound for a coherence measure [335], the complementarity between coherence and mixedness [336], its relation with other measures of quantum correlations and creation of coherence using unitary operations [337,338], erasure of quantum coherence [339], and catalytic transformations of coherence [340] have been reported recently.

In this Chapter, we revisit the complementarity between coherence and mixedness of a quantum state, and the distribution of coherence in multipartite systems in considerable detail. We provide analytical and numerical results in this regard. This Chapter is organized as follows. In Sec. 8.2, we briefly define the measures that quantify quantum coherence and mixedness. In Sec. 8.3, we show that the reciprocity between coherence and mixedness in quantum systems is an extensive feature in the sense that it holds for large spectra of measures of coherence and of mixedness. In Sec. 8.4, we discuss the distribution of coherence in multipartite quantum systems. Numerical investigation unravels the fact that the percentage of quantum states satisfying the additivity relation of coherence increases with increasing number of parties, with increment in the rank of quantum states, and with raising of the power of coherence measures under investigation. We provide conditions for the violation of the additivity relation of the relative entropy of coherence. In Sec. 8.5, we investigate the distribution of coherence in a special type of quantum states called “X”-states, and provide examples. Finally, we conclude our findings in Sec. 8.6.

8.2 Quantifying coherence and mixedness of a quantum state

In this section, we briefly review the axiomatic approach to characterize and quantify coherence, as proposed in Ref. [61], and mixedness of a quantum system.

8.2.1 Quantum coherence

In the framework of Ref. [61], all the diagonal states, in a given reference basis, constitute a set of incoherent states, denoted by $\mathcal{I}$. And a completely positive and trace preserving (CPTP) map is an incoherent operation if it possesses a Kraus operator decomposition $\{K_t\}$ such that $K_t \rho K_t^\dagger$ is incoherent for every incoherent state $\rho \in \mathcal{I}$. A function, $C(\rho)$, is a valid measure of quantum coherence of the state ρ if it satisfies the following conditions [61]: (1) $C(\rho) = 0$ iff $\rho \in \mathcal{I}$. (2a) Monotonicity under the incoherent operations, i.e., $C\left(\Phi_I(\rho)\right) \leq C(\rho)$. (2b) Monotonicity under the selective incoherent operations on an average, i.e., $\sum_k p_k C(\rho_k) \leq C(\rho)$, where $\rho_k = M_k \rho M_k^\dagger / p_k$, $p_k = \mathrm{Tr} M_k \rho M_k^\dagger$, and M_k are the incoherent Kraus operators as described above. That is, $C(\rho)$ is non-increasing on an average under the selective incoherent operations. (3) Convexity or nonincreasing under mixing of quantum states, i.e., $C(\sum_k p_k \rho_k) \leq \sum_k p_k C(\rho_k)$. That is, coherence cannot increase under mixing.

It is emphasized that the incoherency condition, $M \mathcal{I} M^\dagger \in \mathcal{I}$, places a severe constraint on the structure of the incoherent Kraus operator M [337]: there can be at most one nonzero entry in every column of M. Thus, if the incoherent Kraus operator M belongs to the set of $r \times c$ matrices $\mathcal{M}_{r,c}$, then the maximum number of possible structures of M is r^c. Note further that conditions (3) and (2b) together imply condition (2a) [61]:

$$C\left(\Phi_I(\rho)\right) = C\left(\sum_n p_n \rho_n\right) \overset{(3)}{\leq} \sum_n p_n C(\rho_n) \overset{(2b)}{\leq} C(\rho). \tag{8.1}$$

Measures that satisfy the above conditions, include l_1 norm and relative entropy of coherence [61] and the skew information [330]. Coherence can also be quantified through entanglement. It was shown in Ref. [331] that entanglement measures which satisfy above conditions can be used to derive generic monotones of quantum coherence. Recall that quantum coherence is a *basis-dependent* quantity. Yao *et al.* in [337] asked whether a basis-independent measure of quantum coherence can be defined. They observed that the *basis-free* coherence is equivalent to quantum discord [337], supporting the fact that coherence is a form of quantum correlation in multipartite quantum systems. Viewing a d-dimensional quantum state ρ, in the reference basis $\{|i\rangle\}$, as a d^2-dimensional vector, its l_p norm is

$$\|\rho\|_p = \left(\sum_{i,j} |\rho_{ij}|^p\right)^{\frac{1}{p}}, \tag{8.2}$$

where $\rho_{ij} = \langle i|\rho|j\rangle$. The quantity $C_{l_1}(\rho)$, which is based on l_1 norm, and given by

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|, \tag{8.3}$$

is a valid measure of coherence [61]. Another quantity, $C_r(\rho) = \min_{\sigma \in \mathcal{I}} S(\rho||\sigma) = S(\rho_I) - S(\rho)$, is the relative entropy of coherence, where $\mathcal{I}$ is the set of incoherent states in the reference basis, $S(\rho||\sigma) = \mathrm{Tr}\rho(\log \rho - \log \sigma)$ is the relative entropy between ρ and σ, and $\rho_I = \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$. Furthermore, a geometric measure of coherence is also proposed [61, 331, 332] which is a full coherence monotone [331]. The geometric measure is given by $C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho, \sigma)$, where $\mathcal{I}$ is the set of all incoherent states and $F(\rho, \sigma) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right]\right)^2$ is the fidelity [4] of the states ρ and σ. The maximally coherent pure state is defined by $|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$ [61], for which $C_{l_1}(|\psi_d\rangle\langle\psi_d|) = d - 1$ and $C_r(|\psi_d\rangle\langle\psi_d|) = \ln d$.

8.2.2 Mixedness

A quantum system which is not pure is mixed. A pure quantum system described by density matrix ρ is characterized by $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$. $\mathrm{Tr}(\rho^2)$ is called the purity of ρ. Noise in various forms, including inevitable interaction with environment, degrades the purity of a quantum state and renders it mixed. Mixedness characterizes disorder or loss of information, and is a complementary quantity to the purity of a quantum system. There are several ways to quantify the mixedness of a quantum state in the literature. For an arbitrary d-dimensional state, the mixedness, based on normalized linear entropy [341], is given as

$$M_l(\rho) = \frac{d}{d-1}\left(1 - \mathrm{Tr}\rho^2\right). \tag{8.4}$$

Therefore, for every quantum system, mixedness varies between zero and unity. The other operational measures of mixedness of a quantum state ρ include the von Neumann entropy $S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$, and geometric measure of mixedness which is given by $M_g(\rho) := F(\rho, \mathbb{I}/d) = \frac{1}{d}\left(\mathrm{Tr}\sqrt{\rho}\right)^2$. For a d-dimensional pure quantum states $|\phi_d\rangle$, while $M_l(|\phi_d\rangle)$ and $S(|\phi_d\rangle)$ vanish, $M_g(|\phi_d\rangle) = \frac{1}{d}$. Thus, $M_l(|\phi_d\rangle)$ and $S(|\phi_d\rangle)$ lie between 0 and 1, and $M_g(|\phi_d\rangle)$ varies between $\frac{1}{d}$ and 1.

8.3 Reciprocity between quantum coherence and mixedness

As mixedness is complementary to purity and purity is closely related to quantum coherence, it is natural to investigate the restrictions imposed by the mixedness of a system on its quantum coherence. In this section, we show analytically and numerically that there exists a trade-off between the two quantities for different measures of coherence and mixedness.
For any arbitrary quantum system ρ in d dimension, quantum coherence, as quantified by the l_1 norm, and mixedness, in terms of the normalized linear entropy, satisfies the following inequality

$$\frac{C_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) \leq 1. \tag{8.5}$$

Inequality (8.5) dictates that for a fixed amount of mixedness, the maximal amount of coherence is limited, and vice-versa. This important trade-off relation between quantum coherence and mixedness was obtained in Ref. [336] using the parametric form of an arbitrary d-dimensional density matrix, written in terms of the generators, $\mathcal{G}_i$, of $SU(d)$ [4, 342–345], as

$$\rho = \frac{\mathbb{I}}{d} + \frac{1}{2}\vec{x}.\vec{\mathcal{G}} = \frac{\mathbb{I}}{d} + \frac{1}{2}\sum_{i=1}^{d^2-1} x_i \mathcal{G}_i, \tag{8.6}$$

where $x_i = \mathrm{Tr}[\rho \mathcal{G}_i]$. The generators $\mathcal{G}_i$ satisfy (i) $\mathcal{G}_i = \mathcal{G}_i^\dagger$, (ii) $\mathrm{Tr}(\mathcal{G}_i) = 0$, and (iii) $\mathrm{Tr}(\mathcal{G}_i\mathcal{G}_j) = 2\delta_{ij}$. In this representation, three-dimensional state is

$$\rho = \begin{pmatrix} \frac{1}{3} + x_7 + \frac{x_8}{\sqrt{3}} & x_1 - ix_4 & x_2 - ix_5 \\ x_1 + ix_4 & \frac{1}{3} - x_7 + \frac{x_8}{\sqrt{3}} & x_3 - ix_6 \\ x_2 + ix_5 & x_3 + ix_6 & \frac{1}{3} - \frac{2x_8}{\sqrt{3}} \end{pmatrix}. \tag{8.7}$$

The l_1 norm of coherence of a d-dimensional system, given by Eq. (8.6), can be written as [336]

$$C_{l_1}(\rho) = \sum_{i=1}^{(d^2-d)/2} \sqrt{x_i^2 + x_{i+(d^2-d)/2}^2}. \tag{8.8}$$

And, the mixedness is given by

$$M_l(\rho) = 1 - \frac{d}{2(d-1)} \sum_{i=1}^{d^2-1} x_i^2. \tag{8.9}$$

Eq. (8.5) ensures that the normalized coherence, $\frac{C_{l_1}(\rho)}{(d-1)}$, of a quantum system with mixedness $M_l(\rho)$, is bounded to a region below the ellipse $\frac{C_{l_1}^2(\rho)}{(d-1)^2} + \left(\sqrt{M_l(\rho)}\right)^2 = 1$. The quantum states with (normalized) quantum coherence that lie on the conic section are the maximally coherent states corresponding to a fixed mixedness and vice-versa [336].

It is interesting to note that provided $C_{l_2}(\rho) = \left(\sum_{i\neq j} |\rho_{ij}|^2\right)^{\frac{1}{2}}$ were a valid coherence measure, one could easily show that a complementarity relation, analogous to Eq. (8.5), holds:

$$\frac{C_{l_2}^2(\rho)}{\left(\sqrt{1-\frac{1}{d}}\right)^2} + M_l(\rho) \leq 1. \tag{8.10}$$

In Ref. [61], it was shown that the quantity $\tilde{C}_{l_2}(\rho) = C_{l_2}^2(\rho) = \sum_{i\neq j} |\rho_{ij}|^2$ satisfies conditions (1) and (3). However, it fails to satisfy the condition (2b) in general. Thus it is not clear whether $C_{l_2}^2(\rho)$ is a valid coherence measure in the framework of above resource theory.

A natural question that arises is whether the reciprocity between quantum coherence and mixedness is measure specific? Put differently, does complementarity between coherence and mixedness hold for other measures of coherence and mixedness? It is trivial to note that

$$\frac{C_r(\rho)}{\ln d} + \frac{S(\rho)}{\ln d} \leq 1, \tag{8.11}$$

and

$$\begin{aligned} C_g(\rho) + M_g(\rho) &= 1 - \left(\max_{\sigma\in\mathcal{I}} F(\rho,\sigma) - F(\rho, \mathbb{I}/d)\right) \\ &\leq 1. \end{aligned} \tag{8.12}$$

We observe from Eqs. (8.5), (8.11) and (8.12) that for valid coherence measures, there is trade-off between functions of normalized coherence and normalized mixedness. This complementarity between coherence and mixedness appears to be a general feature. It would be an interesting exercise to investigate whether a given measure of coherence respects reciprocity with different measures of mixedness. In particular, we are interested in whether the following relations hold:

$$\frac{C_{l_1}^2(\rho)}{(d-1)^2} + \frac{S(\rho)}{\ln\, d} \leq 1$$
$$\frac{C_r(\rho)}{\ln\, d} + M_l(\rho) \leq 1, \quad \text{etc.} \tag{8.13}$$

For rank-1 (pure) states, the above complementarity relations are trivially satisfied since mixedness for pure states is, by definition, zero (for geometric measure of coherence, one will have to set $M_g(|\psi\rangle) = 0$ by hand). Interestingly, for higher rank quantum states also the above reciprocity relations hold. We provide numerical evidences which suggest that trade-off between coherence and mixedness is indeed an extensive feature of quantum systems (see Figs. 8.1 and 8.2). Though the reciprocity relation, $\frac{C_r(\rho)}{\ln\, d} + M_l(\rho) \leq 1$, is in conflict, it is well below the trivial value 2, for all states. We observe that higher is the rank of quantum states and number of qubits, more is the violation. We found numerically that the reciprocity relation, $\frac{C_r(\rho)}{\ln\, d} + M_l(\rho) \leq 1$, is violated by two-qubit states also. An example of a two-qubit state which violates this relation is given in Eq. (8.14).

$$\rho = \begin{pmatrix} 0.2501 & 0.0490 - 0.0090i & -0.1392 - 0.1148i & -0.2141 - 0.0515i \\ 0.0490 + 0.0090i & 0.2064 & 0.1588 - 0.0438i & 0.0137 + 0.0650i \\ -0.1392 + 0.1148i & 0.1588 + 0.0438i & 0.3001 & 0.1858 + 0.0115i \\ -0.2141 + 0.0515i & 0.0137 - 0.0650i & 0.1858 - 0.0115i & 0.2434 \end{pmatrix} \tag{8.14}$$

Note that $\rho = \rho^\dagger$, $\text{Tr}\rho = 1$, $\text{Tr}\rho^2 = 0.5539$, and eigenvalues of ρ are $\{0.664,\ 0.336,\ 0,\ 0\}$. Hence, ρ is a valid rank-2 density matrix. For this density matrix, $\frac{C_r(\rho)}{2} + M_l(\rho) = 0.5334 + 0.5948 = 1.1282 > 1$.

8.4 Distribution of quantum coherence

Quantum coherence is a resource. Coherence of a multiparty quantum system $\rho_{AB_1B_2\cdots B_n}$ is seen as a quantum correlation amongst the subsystems. We wish to study the distribution of coherence among the constituent subsystems. In particular, we are interested in the following monogamy-type relation [14], which

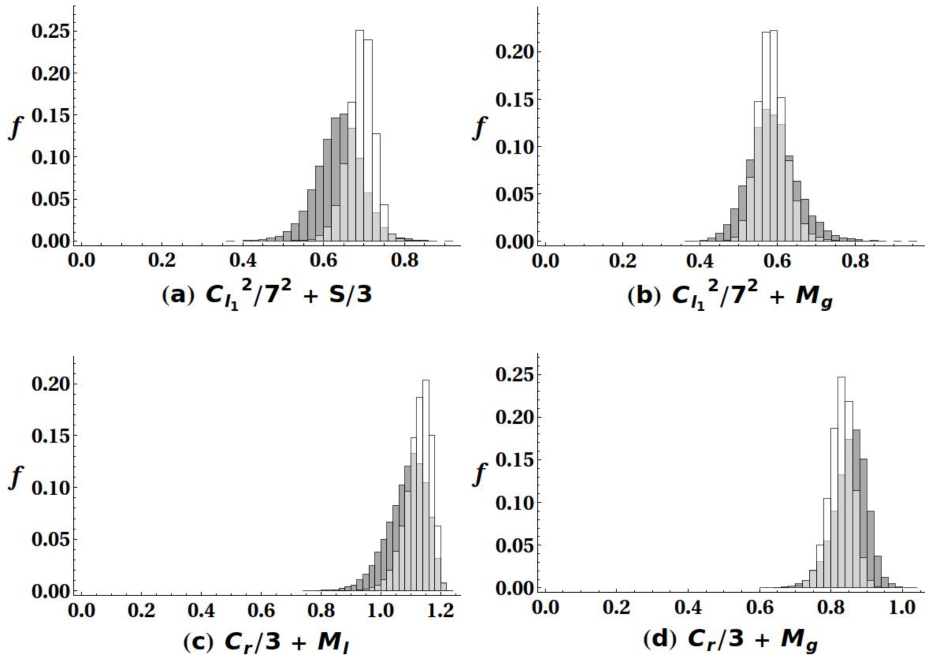


Figure 8.1: Histograms depicting the relative frequency (f) of quantum states against the trade-off between coherence and mixedness for different measures: (a) $\frac{C_{l_1}^2(\rho)}{(d-1)^2}+\frac{S(\rho)}{\ln d}\leq 1$, (b) $\frac{C_{l_1}^2(\rho)}{(d-1)^2}+M_g(\rho)\leq 1$, (c) $\frac{C_r(\rho)}{\ln d}+M_l(\rho)\leq 1$, and (d) $\frac{C_r(\rho)}{\ln d}+M_g(\rho)\leq 1$. For both rank-2 (gray bars) and rank-3 (white bars), 2×10^4 **three-qubit** states are generated Haar uniformly in the computational basis. We see that only the reciprocity relation, $\frac{C_r(\rho)}{\ln d}+M_l(\rho)\leq 1$, is in conflict. However, it is well below the trivial value 2, for all states. Higher is the rank of quantum states, more is the violation.

we refer to as additivity relation:

$$C(\rho_{AB_1B_2\cdots B_n})-\sum_{k=1}^{n}C(\rho_{AB_k})\geq 0, \tag{8.15}$$

where C is some valid coherence measure and ρ_{AB_k} is the two-party reduced density matrix obtained after partial tracing all subsystems but subsystems A and B_k. If the above relation is satisfied for arbitrary quantum system $\rho_{AB_1B_2\cdots B_n}$, we say that the distribution of coherence is "faithful" with respect to subsystem A. In the language of monogamy [14], C is monogamous with respect to pivot A. If the above relation does not hold for any $\rho_{AB_1B_2\cdots B_n}$, C is unfaithful or non-monogamous. Below we provide several interesting results on the distribution of quantum coherence in multipartite quantum systems. Remember that coherence is a basis-dependent quantity. In considering following results and theorems, we assume that quantum system under investigation is described in a fixed reference basis. Let $\{|a_i\rangle\}$ and $\{|b_j^{(k)}\rangle\}$ be the bases of subsystems A

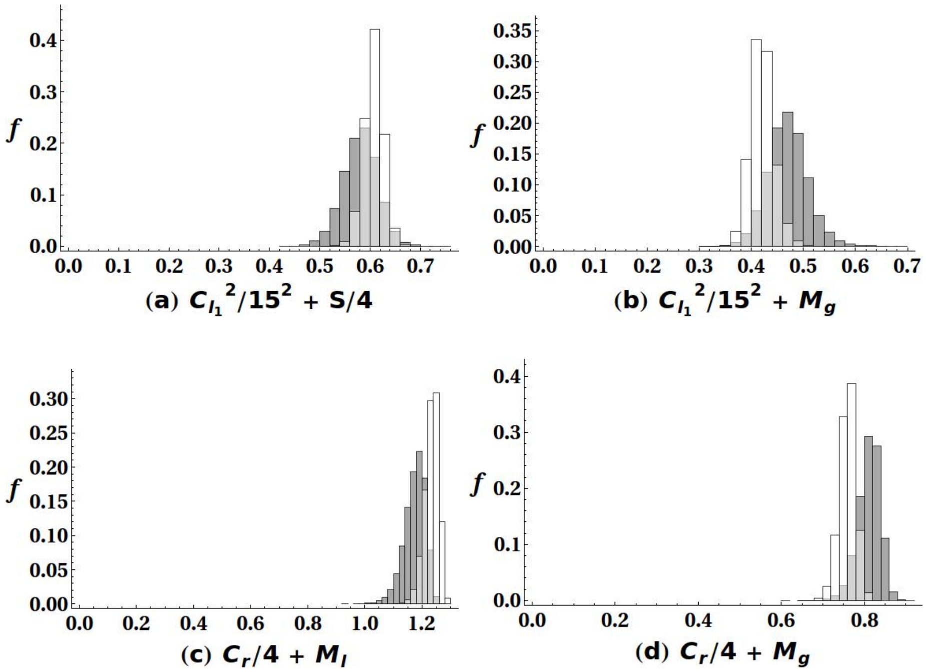


Figure 8.2: Histograms depicting the relative frequency (f) of quantum states against the trade-off between coherence and mixedness for different measures for rank-2 (gray bars) and rank-3 (white bars) **four-qubit** states. Rest of the details are the same as in Fig. 8.1.

and B_k respectively, such that $|\psi\rangle_{AB} \equiv |\psi\rangle_{AB_1\cdots B_n} = \sum c_{a_i b_{k_1}\cdots b_{k_n}} |a_i b^{(1)}_{k_1}\cdots b^{(n)}_{k_n}\rangle$, $\rho_{AB} = \sum_i p_i |\psi^i\rangle_{AB}\langle\psi^i|$, $\rho_{AB_j} = \mathrm{Tr}_{\overline{AB_j}}(\rho_{AB})$, etc.

8.4.1 Numerical results

From numerical findings listed in Table 8.1, we observe a number of important results. The percentage of quantum states satisfying the coherence additivity relation increases with increasing number of parties, the rank of quantum states and raising the power of coherence measures under investigation. Furthermore, for fixed rank and fixed number of qubits, the number of quantum states which satisfy the monogamy condition is larger for entropy-based coherence measure than distance based coherence measure.

Remark 8.4.1 Note that the percentage of quantum states satisfying the additivity relation does increase with increasing number of qubits (for number of qubits 4 onwards) in the third column of Table 8.1. However, the reason behind the dip in percentage from three-qubit states to four-qubit states is not understood.

rank	no. of qubits	$\delta_{\mathcal{C}_{l_1}}$	$\delta_{\mathcal{C}^2_{l_1}}$	$\delta_{\mathcal{C}^3_{l_1}}$	$\delta_{\mathcal{C}_r}$	$\delta_{\mathcal{C}^2_r}$
1	3	0.185	32.045	62.915	5.14	84.56
	4	0.015	64.765	94.445	64.225	99.92
	5	0.035	96.07	99.95	99.02	100.
2	3	0.445	38.245	70.935	75.425	99.685
	4	0.095	75.705	97.74	99.245	100.
	5	0.145	98.715	99.995	99.995	100.
3	3	0.615	41.77	73.885	93.595	99.98
	4	0.14	79.475	98.395	99.975	100.
	5	0.185	99.205	100.	100.	100.
4	3	0.72	42.385	75.155	97.	99.985
	4	0.18	80.845	98.825	100.	100.
	5	0.265	99.385	99.995	100.	100.

Table 8.1: Percentage of quantum states, of varying ranks, satisfying the additivity relation for the "normalized" coherence measures $\mathcal{C}_{l_1}(\rho)$ and $\mathcal{C}_r(\rho)$ for 3, 4 and 5 qubits in the computational basis [346]. The percentage of quantum states satisfying the additivity relation increases with increasing number of parties, with increment in the rank of quantum states, and with raising of the power of coherence measures under investigation. For every rank, 2×10^4 three, four and five qubit states are generated Haar uniformly.

8.4.2 Analytical results

In this section, we provide conditions for the violation of the additivity relation of the relative entropy of coherence C_r.

Theorem 8.4.2 *The relative measure of coherence violates the additivity relation for quantum states $\rho_{AB} \equiv \rho_{AB_1B_2\cdots B_n}$ which satisfy $S(\rho_{AB}) + S(\rho_A) = \sum_{k=1}^n S(\rho_{AB_k})$.*

Proof. Let $\rho_{AB} \equiv \rho_{AB_1B_2\cdots B_n}$ be the density matrix of an $(n+1)$-party quantum system, and $\rho_{AB_k} = \mathrm{Tr}_{\overline{AB_k}}\rho_{AB_1B_2\cdots B_n}$ be the reduced density matrix obtained after partial tracing all subsystems but subsystems A and B_k. Then

$$\begin{aligned}
& C_r(\rho_{AB}) - \sum_{k=1}^n C_r(\rho_{AB_k}) \\
= \;& S(\rho^I_{AB}) - S(\rho_{AB}) - \sum_{k=1}^n \left[S(\rho^I_{AB_k}) - S(\rho_{AB_k})\right] \\
= \;& \left[\sum_{k=1}^n S(\rho_{AB_k}) - S(\rho_{AB}) - S(\rho_A)\right] \\
- \;& \left[\sum_{k=1}^n S(\rho^I_{AB_k}) - S(\rho^I_{AB}) - S(\rho^I_A)\right] - \left[S(\rho^I_A) - S(\rho_A)\right]
\end{aligned}$$

$$= \Delta_1 - \Delta_2 - C_r(\rho_A). \tag{8.16}$$

It can be easily shown that for $\sigma_{AB} \equiv \sigma_{AB_1B_2\cdots B_n}$, $\sum_{k=1}^{n} S(\sigma_{AB_k}) - S(\sigma_{AB}) \geq (n-1)S(\sigma_A) \geq 0$. This bound is a simple consequence of the strong subadditivity relation, $S(\rho_{ABC}) + S(\rho_A) \leq S(\rho_{AB}) + S(\rho_{AC})$, of von Neumann entropy. Hence Δ_1 and Δ_2 are non-negative. When $\sum_{k=1}^{n} S(\rho_{AB_k}) = S(\rho_{AB}) + S(\rho_A)$, i.e., Δ_1 vanishes, $C_r(\rho_{AB}) \leq \sum_{k=1}^{n} C_r(\rho_{AB_k})$. Thus C_r violates the additivity relation.
Very recently, a special case of above result was obtained in Ref. [337]. It was shown that C_r violates the additivity relation for an arbitrary tripartite state ρ_{ABC} which saturates the strong subadditivity relation of von Neumann entropy. Tripartite states satisfying the strong subadditivity relation are reported in Ref. [347].

Analogous result can be obtained for other distributions as well. For instance, the following distribution yields

$$\begin{aligned}
& C_r(\rho_{AB}) - \sum_{k=1}^{n} C_r(\rho_{A\overline{B_k}}) \\
= \;& S(\rho^I_{AB}) - S(\rho_{AB}) - \sum_{k=1}^{n} \left[S(\rho^I_{A\overline{B_k}}) - S(\rho_{A\overline{B_k}})\right] \\
= \;& \left[S(\rho^I_{AB}) - \sum_{k=1}^{n} S(\rho^I_{A\overline{B_k}})\right] + \left[\sum_{k=1}^{n} S(\rho_{A\overline{B_k}}) - S(\rho_{AB})\right] \\
= \;& \left[\sum_{k=1}^{n} S(\rho_{A\overline{B_k}}) - (n-1)S(\rho_{AB})\right] - \left[\sum_{k=1}^{n} S(\rho^I_{A\overline{B_k}}) - (n-1)S(\rho^I_{AB})\right] \\
- \;& (n-2)\left[S(\rho^I_{AB}) - S(\rho_{AB})\right] \\
= \;& \Delta_3 - \Delta_4 - (n-2)C_r(\rho_{AB}),
\end{aligned} \tag{8.17}$$

where $\rho_{A\overline{B_k}} = \mathrm{Tr}_{B_k}\rho_{AB_1B_2\cdots B_n}$ be the reduced density matrix obtained after partial tracing subsystem B_k. Again since $\sum_{k=1}^{n} S(\sigma_{A\overline{B_k}}) - (n-1)S(\sigma_{AB}) \geq S(\sigma_A) \geq 0$ for $\sigma_{AB} \equiv \sigma_{AB_1B_2\cdots B_n}$ [348], Δ_3 and Δ_4 are non-negative. When either $\Delta_3 \leq \Delta_4 + (n-2)C_r(\rho_{AB})$ or $\Delta_3 = 0$, $C_r(\rho_{AB}) \leq \sum_{k=1}^{n} C_r(\rho_{A\overline{B_k}})$. Thus C_r violates the additivity relation. ■

Remark 8.4.3 The proof of the inequality, $\sum_{k=1}^{n} S(\sigma_{A\overline{B_k}}) - (n-1)S(\sigma_{AB}) \geq 0$ for $\sigma_{AB} \equiv \sigma_{AB_1B_2\cdots B_n}$, is given in [348]. However, for the sake of convenience of the readers, we reproduce this bound explicitly for $n = 4$. Let $S(\rho_X) \equiv S_X$, and $S_{X|Y} = S_{XY} - S_Y$ be the conditional entropy. In proving the bound, we will repeatedly use a variant of strong subadditivity relation, $S_{XYZ} + S_Y \leq S_{XY} + S_{YZ}$, which states that *conditioning reduces entropy*, i.e., $S_{X|YZ} \leq S_{X|Y}$. For $n = 4$, we have $\sum_{k=1}^{4} S_{A\overline{B_k}} - 3S_{AB_1B_2B_3B_4} = S_{AB_1B_2B_3} - S_{B_1|AB_2B_3B_4} - S_{B_2|AB_1B_3B_4} - S_{B_3|AB_1B_2B_4} \geq S_{AB_1B_2B_3} - S_{B_1|AB_2B_3} - S_{B_2|AB_1B_3} - S_{B_3|AB_1B_2} \geq S_{AB_1B_2} - S_{B_1|AB_2} - S_{B_2|AB_1} \geq S_{AB_1} - S_{B_1|A} \geq S_A$. Similarly, it can be shown to hold true for arbitrary n.

However, coherence measures are not normalized in general. That is, they do not lie between zero and unity for arbitrary quantum systems. But in investigating monogamy relations for quantum correlation measures, we consider that value of all quantities in the monogamy inequality lies in the same range. Therefore, it is reasonable to consider the normalized coherence. Suppose that $\rho_{AB} \equiv \rho_{AB_1B_2\cdots B_n} \in \left(\mathbb{C}^d\right)^{\otimes(n+1)}$ be a multipartite density operator. Considering the normalized relative entropy of coherence, we have

$$\begin{aligned} & \frac{C_r(\rho_{AB})}{\ln d^{n+1}} - \sum_{k=1}^{n} \frac{C_r(\rho_{AB_k})}{\ln d^2} \\ = & \frac{2(\Delta_1 - \Delta_2) - (n-1)\sum_{k=1}^{n} C_r(\rho_{AB_k})}{2(n+1)\ln d}. \end{aligned} \tag{8.18}$$

Again, when Δ_1 vanishes, the normalized C_r is non-monogamous. Similarly, for the other distribution, we can obtain

$$\begin{aligned} & \frac{C_r(\rho_{AB})}{\ln d^{n+1}} - \sum_{k=1}^{n} \frac{C_r(\rho_{A\overline{B_k}})}{\ln d^n} \\ = & \frac{n(\Delta_3 - \Delta_4) - \sum_{k=1}^{n} C_r(\rho_{A\overline{B_k}}) - n(n-2)C_r(\rho_{AB})}{n(n+1)\ln d}. \end{aligned} \tag{8.19}$$

Thus, when $\Delta_3 = 0$, the normalized C_r violates the additivity relation.

Remark 8.4.4 Several other analytical results can be shown to hold true for quantum coherence: (i) A measure of coherence C that violates the additivity relation for a multipartite system, can be made to satisfy the same by raising its power to some positive real number (> 1), provided C is monotonically decreasing under discarding of subsystems. (ii) A measure of coherence that satisfies the additivity relation does satisfy the same on raising its power, and a measure of coherence that violates the additivity relation does violate the same on lowering its power. However, it is not known whether raising and lowering operations on bonafide measures of coherence will yield another valid coherence measures.

8.5 Coherence in X states

Quantum states having "X"-structure are referred to as X states. Consider an n-qubit X state given by

$$\rho = p|gGHZ\rangle\langle gGHZ| + (1-p)\frac{I_d}{d}, \tag{8.20}$$

where $|gGHZ\rangle = (\alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n})$ with $|\alpha|^2 + |\beta|^2$, I_d is $d \times d$ identity matrix, $d = 2^n$ and $0 \le p \le 1$. It is easy to show that for this state, $C_{l_1}(\rho) = 2p|\alpha\bar{\beta}|$ and $C_r(\rho) = -\left(p|\alpha|^2 + \frac{1-p}{d}\right)\log_2\left(p|\alpha|^2 + \frac{1-p}{d}\right) - \left(p|\beta|^2 + \frac{1-p}{d}\right)\log_2\left(p|\beta|^2 + \frac{1-p}{d}\right) - \frac{1-p}{d}\log_2\frac{1-p}{d} - \left(p + \frac{1-p}{d}\right)\log_2\left(p + \frac{1-p}{d}\right)$.

Theorem 8.5.1 *For an $(n+1)$-party X state $\rho^X_{AB_1B_2\cdots B_n}$ in a given basis, any measure of coherence C satisfies the additivity relation. That is, X states satisfy the relation $C(\rho^X_{AB_1B_2\cdots B_n}) - \sum_{k=1}^{n} C(\rho^X_{AB_k}) \ge 0$, where $\rho^X_{AB_k} = Tr_{\overline{AB_k}}\rho^X_{AB_1B_2\cdots B_n}$ is a two-qubit reduced density matrix.*

Proof. This is because all the two-party reduced density matrices of $(n+1)$-party X states in the given basis are diagonal and any valid measure of quantum coherence vanishes for diagonal states. ■

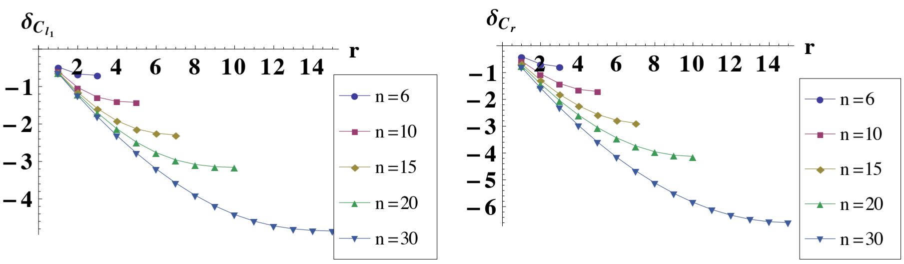


Figure 8.3: Coherence score (y-axis) versus the number of excitations $r \le \frac{n}{2}$ (x-axis) of Dicke states using the "normalized" coherence measures C_{l_1} (left panel) and C_r (right panel). All quantities are dimensionless. The normalized measures of coherence do not satisfy the additivity relation of coherence for the Dicke states.

Now consider the Dicke states [216], which is symmetric with respect to interchange of qubits, given by

$$|D_{n,r}\rangle = \binom{n}{r}^{-\frac{1}{2}} \sum_{\mathcal{P}} \mathcal{P}\left(|0\rangle^{\otimes(n-r)} \otimes |1\rangle^{\otimes r}\right), \tag{8.21}$$

where $\sum_{\mathcal{P}}$ represents sum over all $\binom{n}{r}$ permutations of $(n-r)$ $|0\rangle$s and r $|1\rangle$s. Note that the Dicke state itself, in Eq. (8.21), is not an X state but its all two-qubit reduced density matrices, in the computational basis, are same and has X structure. Again, one can show that for the normalized measures of coherence

$$\begin{aligned}\delta_{C_{l_1}^{(norm)}}\left(|D_{n,r}\rangle\right) &= \frac{C_{l_1}\left(|D_{n,r}\rangle\right)}{2^n-1} - (n-1)\frac{C_{l_1}\left(\rho^{(2)}_{|D_{n,r}\rangle}\right)}{3} \\ &= \frac{\binom{n}{r}-1}{2^n-1} - \frac{2r(n-r)}{3n}, \end{aligned} \tag{8.22}$$

and

$$\begin{aligned}\delta_{C_{r}^{(norm)}}\left(|D_{n,r}\rangle\right) &= \frac{C_{r}\left(|D_{n,r}\rangle\right)}{\log_2 2^n} - (n-1)\frac{C_{r}\left(\rho^{(2)}_{|D_{n,r}\rangle}\right)}{\log_2 4} \\ &= \frac{1}{n}\log_2\binom{n}{r} - \frac{r(n-r)}{n}, \end{aligned} \tag{8.23}$$

where $\rho^{(2)}_{|D_{n,r}\rangle}$ is the two-qubit reduced density matrix of the Dicke state. For $n \geq 3$ and $1 \leq r \leq n$, $\delta_{C_{l_1}}(|D_{n,r}\rangle)$ and $\delta_{C_{l_1}}(|D_{n,r}\rangle)$ are non-positive. Thus, quantum coherence is non-monogamous for the Dicke states in the computational basis (see Fig. 8.3).
Analogous result was obtained in Ref. [15], that the Dicke state is always non-monogamous with respect to quantum discord [121, 122] and quantum work-deficit [169–172], and a Dicke state with more number of parties is more non-monogamous to that with a smaller number of parties.

However, if one considers the unnormalized measures of coherence, then

$$\begin{aligned}\delta_{C_{l_1}}\left(|D_{n,r}\rangle\right) &= C_{l_1}\left(|D_{n,r}\rangle\right) - (n-1)C_{l_1}\left(\rho^{(2)}_{|D_{n,r}\rangle}\right) \\ &= \binom{n}{r} - 1 - \frac{2r(n-r)}{n}, \end{aligned} \tag{8.24}$$

and

$$\begin{aligned}\delta_{C_{r}}\left(|D_{n,r}\rangle\right) &= C_{r}\left(|D_{n,r}\rangle\right) - (n-1)C_{r}\left(\rho^{(2)}_{|D_{n,r}\rangle}\right) \\ &= \log_2\binom{n}{r} - \frac{2r(n-r)}{n}. \end{aligned} \tag{8.25}$$

In this case, when $n \geq 3$ and $1 \leq r \leq n$, $\delta_{C_{l_1}}(|D_{n,r}\rangle)$ and $\delta_{C_{l_1}}(|D_{n,r}\rangle)$ are non-negative. Thus, quantum (unnormalized) coherence measures satisfy the additivity relation for the Dicke states in the computational basis.

8.6 Summary

We have shown that the reciprocity between coherence and mixedness of quantum states is a general feature as this complementarity holds for large spectra of measures of coherence and of mixedness. The numerical investigation of the distribution of coherence in multipartite systems reveals that the percentage of quantum states satisfying the additivity relation increases with increasing number of parties, with increment in the rank of quantum states, and with raising of the power of coherence measures under investigation. We have provided conditions for the violation of the additivity relation of the relative entropy of coherence. We have further shown that the normalized measures of coherence violate the additivity relation for the Dicke states.

The results of this Chapter are based on the following paper:

1. *Quantum coherence: Reciprocity and distribution*, **Asutosh Kumar**, Phys. Lett. A **381**, 991 (2017).

Chapter 9

Summary and Conclusion

Quantum correlations are an important aspect of modern physics and a key enabler in quantum communication and computation technologies. The concept of monogamy is a distinguishing feature of quantum correlations, which sets it apart from classical correlations, and has played a significant role in devising quantum security in secret key generation and multiparty communication protocols. In recent years, a significant amount of research has been devoted to understand the role of monogamy in various quantum phenomena, including violation of Bell inequalities and contextuality, and also in investigating correlation properties beyond quantum mechanics, such as in no-signalling theories. A lot of effort have also been spent in devising stronger monogamy conditions and to extend known features in discrete quantum states to continuous variable systems. An important tool in quantitatively capturing the concept of monogamy of quantum correlations is the monogamy inequality, which bounds from above the distribution of quantum correlations among different parties in a multiparty state. The monogamy inequality can be studied in terms of the *monogamy score*. All monogamy inequality satisfying measures have a positive monogamy score, while those violating the monogamy inequality have a negative monogamy score. This is a non-classical property in the sense that such constraints are maximally violated for classical correlations. Quantifying monogamy relation via the monogamy score leads us to obtain multiparty quantum correlation measures by using bipartite measures. This is because monogamy score can be interpreted as residual quantum correlation of a bi-partition 1 : rest of an $(n+1)$-party state that cannot be accounted for by the conjunction of the quantum correlations of two-party reduced density matrices.

Monogamy of quantum correlations is an active area of research because of its potential applications in several areas in quantum information such as quantum cryptography, identification of quantum states and quantum channels, distinguishing phases of many-body systems, etc. In this thesis, we have investigated which non-classical correlation measures and classes of quantum states, under what conditions, satisfy monogamy. We have obtained upper as well as lower bounds on monogamy scores, and illustrated that monogamy can be used to "conclusively" identify quantum channels. We also studied complementarity and distribution of quantum coherence measures. In this Chapter, we provide a brief summary of the results presented in the thesis.

In Chapter 1, we briefly reviewed the concept of "information". Information has been a useful resource for the development of humankind. In Chapter 2, we recalled measures of entanglement and other quantum correlations that we have employed in the thesis. We visited the notion of monogamy of quantum correlations, and discussed how to quantify it in Chapter 3.

To study monogamy, we need at least a three-party quantum system. However, not all quantum correlation measures satisfy the monogamy inequality for three-party systems. We investigated the effect of increasing the number of parties in studying monogamy in Chapter 4, and found that higher number of parties enforce monogamy of quantum correlations for *almost all* states. We have shown that a quantum correlation measure which is non-monogamous for a substantial section of tripartite quantum states, becomes monogamous for almost all quantum states of n-party systems, with n being only slightly higher than 3. The adjective *almost all* above is important because we have also identified sets of zero Haar measure in the space of all multiparty quantum states that remain non-monogamous for an arbitrary number of parties. We proved a theorem that *if entanglement of formation is monogamous for a pure quantum state of an arbitrary number of parties, then any bipartite "good" entanglement measure is also monogamous for that state.* We call an entanglement measure as "good" if it is lower than or equal to the entanglement of formation and it is equal to the local von Neumann entropy for pure bipartite states. According to this definition, relative entropy of entanglement, regularized relative entropy of entanglement, entanglement cost, and distillable entanglement are good measures. We saw that entanglement of formation is monogamous for almost all pure states of four qubits. Using these observations, we concluded that the above measures, all of which are not generally computable, are monogamous for almost all pure states of four or more qubits. Apart from providing an understanding on the structure of space of quantum correlation measures, and their relation to the underlying space of multiparty quantum states, our results throw light on the methods for choosing quantum systems for secure quantum information protocols, especially in large quantum networks.

In Chapter 5, we explored the conditions under which the monogamy relation is preserved for functions of quantum correlation measures. We proved that a monogamous measure remains monogamous on raising its power, and a non-monogamous measure remains non-monogamous on lowering its power. We also showed that monogamy of a convex quantum correlation measure for arbitrary multipartite pure quantum states leads to its monogamy for mixed states. Monogamy of squared negativity for mixed states and that of squared entanglement of formation follow as corollaries of our results.

Monogamy score is a difficult quantity to compute and estimate for generic quantum states and generic quantum correlations. It is therefore interesting to derive both upper and lower bounds on the monogamy score. The existence of non-trivial bounds on the monogamy score is an important aspect in the study of various quantum information protocols. We obtained upper and lower bounds on monogamy scores in Chapter 6. Monogamous quantum correlation measures have a positive monogamy score, which is bounded above by a well defined value that can be derived as functions of the genuine multipartite entanglement of the quantum state. We showed that such an upper bound holds also for an arbitrary number of qubits provided the states satisfy certain conditions. We derived a set of necessary conditions to characterize the set of states that may violate the bound, and numerically observed that the set is extremely small. Moreover, we analytically investigated several important classes of multiqubit quantum states for which we showed that the conditions required to have the upper bound on monogamy scores of computable bipartite measures are satisfied. However, not all quantum correlation measures satisfy the monogamy inequality, and in general, the monogamy score can be negative. Using a "complementarity relation" between the normalized purity of a subsystem and a bipartite quantum correlation in the system, we obtained a non-trivial lower bound on the monogamy score for measures and states where the monogamy inequality is violated. Subsequently, we analyzed the strengths and weaknesses of the lower bound for different quantum states and measures of quantum correlation, and observed conditions that immediately lead to monogamy.

These bounds on the monogamy score help identify the forbidden regimes in the distribution of bipartite quantum correlation measures among different parties in a multiparty system and limits the amount by which the monogamy inequality can be satisfied. The results provide a unifying framework to study monogamy relations in both entanglement and information-theoretic quantum correlations.

In Chapter 7, we investigated the patterns of the monogamy property of quantum correlations using monogamy score as the "observable", when three-qubit systems are subjected to global noise as well as local noisy channels, viz. amplitude-damping, phase-damping, and depolarizing channels. We defined a characteristic noise scale, called the "dynamics terminal", that quantifies the persistence of the monogamy score corresponding to a particular measure of quantum correlation, when the state is subjected to a specific type of noise. The dynamics terminal can also distinguish between the different noise models, and indicates that the depolarizing channel destroys monogamy scores faster compared to the other types of noise. As an use for such study, we proposed a two-step channel discrimination protocol that can conclusively identify the different types of noise by considering monogamy scores and by using the gGHZ and the gW states as resources.

Quantum coherence arises from the superposition principle, and forms the basis of quantum correlations in multipartite systems. In Chapter 8, we have shown that the reciprocity between coherence and mixedness of quantum states is a general feature as it holds for a plethora of measures of coherence and those of mixedness. We also studied the distribution of coherence in multipartite systems. Numerical investigation revealed that the percentage of quantum states satisfying the additivity relation of coherence increases with increasing number of parties, with increment in the rank of quantum states, and with raising of the power of coherence measures under investigation. We have further shown that the normalized measures of coherence violate the additivity relation for the Dicke states.

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