RESOURCE THEORIES OF QUANTUM COHERENCE AND ENTANGLEMENT

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Uttam Singh

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Uttam Singh

List of Publications arising from the thesis

Journals

1. Maximally coherent mixed states: Complementarity between maximal coherence and mixedness, U. Singh, M. N. Bera, H. S. Dhar, A. K. Pati, *Phys. Rev. A*, **2015**, *91*, 052115-1-052115-8.

2. Measuring quantum coherence with entanglement, A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, G. Adesso, *Phys. Rev. Lett.*, **2015**, *115*, 020403-1-020403-6.

3. Average coherence and its typicality for random pure states, U. Singh, L. Zhang, A. K. Pati, *Phys. Rev. A*, **2016**, *93*, 032125-1-032125-8.

4. Catalytic coherence transformations, K. Bu, U. Singh, J. Wu, *Phys. Rev. A*, **2016**, *93*, 042326-1-042326-9.

5. Average subentropy, coherence and entanglement of random mixed quantum states, L. Zhang, U. Singh, A. K. Pati, *Annals of Phys.*, **2017**, *377*, 125-146.

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1. Erasing quantum coherence: An operational approach, U. Singh, M. N. Bera, A. Misra, and A. K. Pati, arXiv:1506.08186 [quant-ph].

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1. August 2013: Oral presentation titled "Weak measurement can resurrect lost quantumness", The 13th Asian Quantum Information Science Conference (AQIS-13), Institute of Mathematical Sciences, Chennai, India.

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3. December 2013: Meeting On Quantum Simulations (QS-2013), Indian Institute of Science, Bangalore, India.

4. February 2014: Information Processing and Quantum Information (IPQI-14), Institute of Physics, Bhubaneshwar, India.

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Journals

1. Quantum discord with weak measurements, U. Singh and A. K. Pati, *Annals of Phys.*, **2014**, *343*, 141-152.

2. Enhancing robustness of multiparty quantum correlations using weak measurement, U. Singh, U. Mishra, H. S. Dhar, *Annals of Phys.*, **2014**, *350*, 50-.

3. Measuring non-Hermitian operators via weak values, A. K. Pati, U. Singh, U. Sinha, *Phys. Rev. A*, **2015**, *92*, 052120-1-052120-8.

4. From Rényi relative entropic generalization to quantum thermodynamical universality, A. Misra, U. Singh, M. N. Bera, A. K. Rajagopal, *Phys. Rev. E*, **2015**, *92*, 042161-1-042161-8.

5. Energy cost of creating quantum coherence, A. Misra, U. Singh, S. Bhattacharya, A. K. Pati, *Phys. Rev. A*, **2016**, *93*, 052335-1-52335-8.

6. Coherence breaking channels and coherence sudden death, K. Bu, Swati, U. Singh, J. Wu, *Phys. Rev. A*, **2016**, *94*, 052335-1-052335-12.

7. Uncertainty relations for quantum coherence, U. Singh, A. K. Pati, M. N. Bera, *Mathematics*, **2016**, *4*, 47-1-47-12.

Preprints

1. Max- relative entropy of coherence: an operational coherence measure, K. Bu, U. Singh, S.-M. Fei, A. K. Pati, J. Wu, arXiv:1707.08795 [quant-ph].

2. Asymmetry and coherence weight of quantum states, K. Bu, N. Anand, U. Singh, arXiv:1703.01266 [quant-ph].

3. Average distance of random pure states from maximally entangled and coherent states, K. Bu, U. Singh, L. Zhang, J. Wu, aarXiv:1603.06715 [quant-ph].

4. Weak values with remote postselection and shared entanglement, A. K. Pati, U. Singh, arXiv:1310.6002 [quant-ph].

5. Weak measurement induced super discord can resurrect lost quantumness, U. Singh , A. K. Pati, arXiv:1305.4393 [quant-ph].

Conferences

1. November 2015: Visit and oral presentation titled “Catalytic transformations”, Department of Mathematics, Zhejiang University, Hangzhou, China.

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Uttam Singh

DEDICATIONS

This thesis is dedicated to my mommy and papa.

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CONTENTS

SYNOPSIS

In the field of quantum information science [1–3], quantum entanglement [4] is a well known resource allowing for implementation of various quantum information protocols that are otherwise impossible in the classical realm. The examples include quantum teleportation [5], super dense coding [6], quantum key distribution [7], and remote state preparation [8, 9] among others. The quantitative understanding of entanglement as a resource is of utmost importance to harness entanglement effectively and therefore a formal resource theory of entanglement is developed in recent years [4]. However, quantum resource theory is not exclusive to entanglement and can be developed for other non-classical resources at our disposal. In general, a quantum resource theory comprises three basic ingredients (not all of them are independent): (i) a set of allowed or free operations, (ii) a set of allowed or free states, and (iii) a set of resource or non-free states [10].

With our ever increasing ability to control and manipulate systems at smaller and smaller scales and need of miniaturization, resources other than entanglement have been identified. Recent developments in the field of quantum thermodynamics [11–16] and quantum biology [17–20] suggest quantum coherence as one of the key resources in various applications within these areas of research. This necessitates a deeper and quantitative understanding of quantum coherence which is a basic manifestation of the linearity of quantum mechanics. Though there exists a formal theory of coherence in terms of correlation functions in the context of quantum optics [21], however, for discrete quantum systems and in the new light of quantum coherence as a resource, formal and general resource theories of coherence are developed very recently. This has been a very vibrant and fruitful area of research since then and various applications of coherence based on these theories have been reported till date [22]. The proposed thesis is a substantial contribution to these resource theories of coherence consolidating their foundations as well providing various applications of the same.

The crucial results obtained in the proposed thesis are listed below.

- We provide a set of coherence quantifiers based on well known entanglement quantifiers. This establishes a quantitative equivalence of entanglement and coherence resource theories. These results are published in Ref. [23].

- We provide an operational meaning to a key quantifier of coherence, namely, the relative entropy of coherence which is vital for its justification as a physically meaningful resource. This result has been reported in Ref. [24].

- We provide the trade off of coherence as a resource in noisy environments, thereby, providing a set of maximally coherent mixed states in Ref. [25].

- We provide detailed results on coherence manipulations in physical processes including the possibility of catalysis in Ref. [26].

- Using typicality analysis, we provide generic aspects of coherence as a resource and establish that the most of the randomly sampled pure states are not maximally coherent. This analysis also helps in reducing the computational complexity of certain entanglement measures on a specific class of bipartite mixed quantum states. These results are published in Refs. [27, 28].

The thesis is divided into eight Chapters and a few appendices.

Chapter 1 (Introduction) forms the basic platform that is necessary to the understand the crux of the thesis. We provide a short pedagogical account of various developments that led to our research. This chapter includes various definitions and a brief sketch of results in the context of resource theories of coherence with a comprehensive list of references.

Quantum coherence and entanglement are two fundamental manifestations of quantum theory and key resources for quantum technologies. Now we have resource theoretic description for both the notions. It is the aim of Chapter 2 (Measuring quantum coherence with entanglement) to provide a clear quantitative and operational connection between coherence and entanglement. We start with showing that all quantum states displaying coherence in some reference basis are useful resources for the creation of entanglement via incoherent operations. We then define a general class of measures of coherence for a quantum system in terms of the maximum bipartite entanglement that can be created via incoherent operations applied to the system and an incoherent ancilla. The resulting measures are proven to be valid coherence monotones satisfying all the requirements dictated by the resource theory of quantum coherence. Thus, given an entanglement monotone (measure of entanglement) we can always construct a coherence monotone (measure of coherence).

The resource theory of coherence provides a solid platform for quantitative understanding of the notion of coherence. However, coherence quantifiers in this theory lack operational significance. Providing operational meaning to coherence quantifiers is insightful for our understanding of the notion of coherence. To achieve this aim, in Chapter 3 (Erasing quantum coherence: An operational approach), we introduce the concept of erasing of coherence via injecting noise (decohering process) into the system of interest. In particular, we find that in the asymptotic limit, the minimum amount of noise that is required to fully decohere a quantum system, is equal to the relative entropy of coherence. This holds even if we allow for the nonzero small errors in the decohering process.

In realistic implementation of quantum technologies the resourcefulness of quantum coherence is severely restricted by environmental noise, which is indicated by the loss of information in a quantum system, measured in terms of its purity. In Chapter 4 (Maximally coherent mixed states: Complementarity between maximal coherence and mixedness), we obtain an analytical trade-off between the coherence and mixedness. Using this we find the upperbound on the coherence for fixed mixedness in a system. This gives rise to a class of quantum states, "maximally coherent mixed states", whose coherence cannot be increased further under any purity-preserving operation. For the above class of states, quantum coherence and mixedness satisfy a complementarity relation, which is crucial to understand the interplay between a resource and noise in open quantum systems.

In Chapter 5 (Catalytic coherence transformations), we study the physical processes that are allowed in the resource theory of coherence with/without aid of catalysts. Catalytic coherence transformations allow the otherwise impossible state transformations using only incoherent operations with the aid of an auxiliary system with finite coherence which is not being consumed in anyway. We show that the simultaneous decrease of a family of Rényi entropies of the diagonal parts of the states under consideration are necessary and sufficient conditions for the deterministic catalytic coherence transformations. Similarly, for stochastic catalytic coherence transformations we find the necessary and sufficient conditions for achieving higher optimal probability of conversion. We, thus, completely characterize the coherence transformations amongst pure quantum states under incoherent operations. We give numerous examples to elaborate our results. We also explore the possibility of the same system acting as a catalyst for itself and find that indeed self catalysis is possible.

Generic aspects of the properties such as entanglement of closed quantum systems have been established using the concentration of measure phenomenon on the set of random pure states [29]. It serves as a beautiful example in the context of the reduction of computational complexity of various entanglement measures for bipartite mixed states. In Chapter 6 (Average coherence and its typicality for random pure states), we establish the typicality of the relative entropy of coherence using the concentration of measure phenomenon. In particular, we prove that the coherence content of an overwhelming majority of Haar distributed random pure states is equal to the average relative entropy of coherence (within an arbitrarily small error) in higher dimensional Hilbert spaces. We find the dimension of a random subspace of the total Hilbert space such that all pure states that reside on it have the relative entropy of coherence arbitrarily close to the typical value. Moreover, we establish that randomly chosen pure states are not typically maximally coherent (within an arbitrarily small error).

In Chapter 6 we have explored coherence content of random pure states. However, mixed states are encountered more naturally in experimental scenarios due to the interaction between the system of interest and the environment. Therefore, consideration of typicality of coherence content of random mixed states is of great importance in practical scenarios and this is the topic of Chapter 7 (Typicality of coherence for random mixed states). In this chapter we establish the typicality of the relative entropy of coherence for random mixed states sampled from the induced measure via partial tracing of random bipartite pure states, invoking again the concentration of measure phenomenon. As an important application towards the reduction of computational complexity of entanglement measures, we establish the typicality of the relative entropy of entanglement and distillable entanglement for a specific class of random bipartite mixed states. In particular, most of the random states in this specific class have relative entropy of entanglement and distillable entanglement equal to some fixed number (to within an arbitrary small error).

We provide the crux of our thesis and possible future research directions inspired from our investigations in chapter 8.

Finally, in appendices A, B, C and D, we provide some old mathematical results that are used in the thesis together with proofs of some of the results of the thesis.

The results obtained in this thesis enrich our understanding of the notion of quantum coherence at a fundamental and quantitative level. The proposed thesis is a significant contribution to the nascent field of the resource theory of coherence and resource theories, in general. We believe that the results presented here can motivate and lead to further research in the area of quantum resource theories and quantum information science.

List of Figures

List of Tables

CHAPTER 1

Introduction

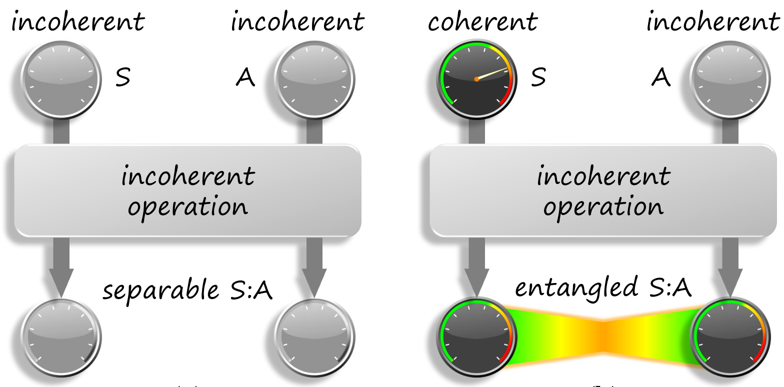
The storage, transmission and processing of information form the building blocks of information theory and these building blocks rely on physical carriers. When these physical carriers are single quantum systems such as single atoms or single photons, it provides the basis for quantum information theory. The consideration of single quantum systems as the physical carriers of information stems from the rapid miniaturization of devices at atomic scales. For example, the storage and processing units in a classical computer are becoming smaller and smaller reaching nano or atomic scales in which quantum effects become crucial. Moreover, with the current technological advancements, it has been possible to manipulate single quantum systems [2]. At these small scales, the quantum theory describes the physical systems and the governing laws for the manipulation of these systems are completely different than their counterparts in classical physics. However, the probabilistic nature of quantum theory has baffled the greatest minds in science and philosophy to the extent that it was deemed an incomplete theory to explain the physical reality [30]. Over the years, quantum physics has emerged as a very successful theory that explains microscopic phenomena and its predictions match with the experiments to a great accuracy. Quantum theory opens up new possibilities for information processing and communication tasks. The enhanced performance of quantum systems in information processing and communication tasks is believed to be rooted mainly in quantum superposition and entanglement present in quantum systems. Some noteworthy applications include quantum teleportation [5], super dense coding [6], quantum key distribution [7], factorization of large numbers [31] and searching a large database on a quantum computer [32]. However, the role of quantum entanglement is not very explicit in many applications. Since entanglement stems from the basic yet extremely nontrivial superposition principle of quantum physics, the superposed quantum states are good candidates that can provide the reason behind the enhancement of performance in information processing and communication tasks. Also, recent developments in thermodynamics of nano scale systems suggest that the quantum superposition or quantum coherence plays an essential role in determining the quantum state transformations and more importantly, in providing a family of second laws of thermodynamics [11–14, 16, 33–39]. Further, the phenomenon of quantum coherence has been arguably attributed to the efficient functioning of some complex biological systems [17–20, 40, 41].

Miniaturization [42] and technological advancements to handle and control systems at smaller and smaller scales necessitate the deeper understanding of concepts such as quantum coherence, entanglement and correlations [11–14, 16–20, 33, 34, 41]. On a purely theoretical level it is important to understand what kinds of tasks may be achieved with quantum systems. While the theory of quantum coherence is historically well developed in quantum optics [21, 43, 44] in terms of quasiprobability distributions and higher-order correlation functions, a rigorous framework to quantify coherence for general states adopting the language of quantum information theory has only been attempted in recent years [41, 45–48]. These frameworks fall in the subject of quantum resource theories. Based on these theories, various quantifiers of coherence have been proposed and explored in detail. Moreover, a plethora of applications of coherence under these frameworks are obtained (see for example Ref. [22] for a recent review of coherence).

Quantum resource theories (QRTs) are formalisms that identify properties of physical systems such as entanglement, coherence, athermality and asymmetry as resources and provide rigorous framework for manipulation of physical systems (quantum states) under relevant restrictions. QRTs consist of three main ingredients: (a) the restricted set of allowed or free states, (b) the restricted set of allowed or free operations, and (c) the restricted set of non free or resource states. For example, for the QRT of entanglement these three ingredients are: (a) the set of separable states as the restricted set of allowed states, (b) the set of local operations and classical communication as the set of allowed operations, and (c) the set of entangled states as the set of resource states. However, these three ingredients are not independent of each other and affecting one of the constituents affects the others. The examples of QRTs include the resource theory of athermality in quantum thermodynamics [11–13, 16, 33, 49, 50] (which led to the introduction of many second laws of quantum thermodynamics [11]), the resource theories of coherence [41, 45–48], the resource theory of asymmetry [45, 46, 51–61] (which led to generalization of Noether's theorem [60]), the resource theory of steering [62] and the resource theory of noncontextuality [63]. Recently, it is obtained that a QRT is asymptotically reversible if its allowed operations are the set of all operations that do not generate asymptotically a resource [10].

This thesis is a contribution towards the resource theoretic frameworks of quantum coherence, especially, towards the QRT of coherence based on incoherent operations. It deals mainly with the resource character of quantum superposition in particular with its quantification, mathematical characterization, operational justification and manipulation in the absence and presence of environmental noise.

Coherence is a fundamental aspect of quantum physics that encapsulates the defining features of the theory [64] ranging from the superposition principle to quantum correlations. It is a key ingredient in various quantum information and estimation protocols, and is primarily accountable for the advantage offered by quantum tasks versus classical ones [2, 65]. In general, quantum coherence is an important physical resource in low-temperature thermodynamics [12, 13, 15, 16, 66], for exciton and electron transport in



(a) (b)

Figure 1.1: (a) Incoherent operations cannot create entanglement from incoherent input states. (b) Entanglement can instead be created by incoherent operations if at least one of the inputs is coherent. We show that all coherent input states of a system S are useful for entanglement creation via incoherent operations on S and an incoherent ancilla A. Input coherence and output entanglement are quantitatively equivalent: For every entanglement monotone E, the maximum entanglement that can be created between S and A by incoherent operations defines a faithful measure of quantum coherence C_E in the initial state of the system S.

biomolecular networks [17–20, 40, 41], and for applications in nanoscale physics [38, 39].

Intuitively, both coherence and entanglement capture the quantumness of a physical system, and it is well known that entanglement stems from the superposition principle, which is also the essence of coherence. It is then legitimate to ask how can one resource emerge *quantitatively* from the other. In chapter 2, we investigate whether the parallel between coherence and entanglement, apparent at the formal level of resource theories, can be upgraded to a more solid conceptual premise. We provide a mathematically rigorous approach to resolve the above question. Our approach is based on using a common frame to quantify quantumness in terms of coherence and entanglement. In particular, in our central result we show that any nonzero amount of coherence in a system S can be 'activated' into (distillable) entanglement between S and an initially incoherent ancilla A, by means of incoherent operations (see Fig. 1.1). This establishes coherence as a universal resource for entanglement creation. In quantitative terms, given a distance-based pair of quantifiers for coherence and entanglement, we show that the initial degree of coherence of S bounds from above the entanglement that can be created between S and A by any incoherent operation. Conversely, our scheme also reveals a novel, general quantification of coherence in terms of entanglement creation. Namely we prove that, given an arbitrary set of entanglement monotones $\{E\}$, one can define a corresponding class of coherence monotones $\{C_E\}$ that satisfy all the requirements of Ref. [47]. The input coherence C_E of S is specifically defined as the maximum output entanglement E over all incoherent operations on S and A. Fundamentally, these results demonstrate a quantitative *equivalence* between coherence and entanglement, and provide an intuitive operational scheme to interchange these two nonclassical resources for suitable applications in quantum technologies.

Despite the resource theoretic framework of coherence is well grounded, it lacks an operational significance as such. Chapter 3 is aimed at filling this gap. In the quantum information theory, to equip a particular "resource" of interest with an operational meaning, consideration of thermodynamic cost of destroying (erasing) the "resource", turns out to be very fruitful and far reaching [67–73]. For example, the Landauer erasure principle [67] has been a central one in laying the foundation of physics of information theory. Similarly, an operational definition of total correlation, classical correlation and quantum correlation is obtained independently in Refs. [74] and [75], considering the thermodynamic cost to erase the same. Additionally, it has been shown that the thermodynamic cost of erasing quantum correlation has to be associated with entropy production in the environment [76]. This approach has also been successfully applied to private quantum decoupling [77] and recently to markovianization [78]. Importantly, this approach can suitably be used for the quantification of any other quantum resource [74]. In these tasks, quantum state randomization [79–81] plays a pivotal role. The resource theory of quantum coherence is still in its infancy as our understanding about it is limited from both qualitative and quantitative perspectives. Following the aforementioned operational approach, we quantify quantum coherence in terms of the amount of noise that has to be injected into the system such that the system decoheres completely. This, in turn, will provide operational meaning of the coherence. We consider two different measures to quantify the

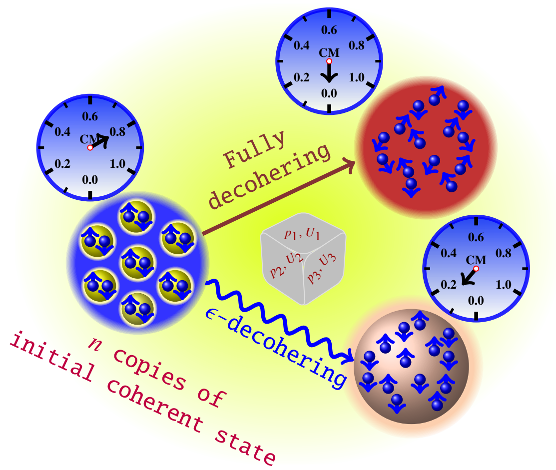


Figure 1.2: Fully decohering and ϵ-decohering maps: If we start with n copies of any state (coherent or incoherent) and pass them through some decohering map, then the n copies decohere completely if the map is fully decohering and if the map is ϵ-decohering then the n copies come very close to the fully decohered state keeping some amount of coherence which is close to zero. We show that in both the cases the minimum amount of noise that is required is same and is equal to relative entropy of coherence in the asymptotic limit.

amount of noise in the process of decohering a quantum system: the entropy exchange between system and environment during the decohering operation [82, 83] and the memory required to store the information about the decohering operation [74]. We show that in the asymptotic limit, both these measures yields the same minimal cost of erasing coherence (the minimal noise required to fully decohere the system) and it turns out to be equal to the relative entropy of coherence [47] (see Fig. 1.2). This result is valid provided we take the decohering operations as the random unitary operations considered by us. Relative entropy of coherence has been already identified as bona fide measure of coherence in the resource theory of coherence [47], by considering the allowed operations to be incoherent operations and free states to be incoherent states. Here, to quantify coherence we followed a approach that is very much different compared to the other measures existing in resource theory but interestingly this approach yields the same quantifier as the relative entropy of coherence. Thus, our results provide an operational meaning of the relative entropy of coherence which in turn strengthens the basis of the coherence resource theory, in general.

A significant aspect in the dynamics of quantum systems is the role of environmental noise and the unavoidable phenomenon of decoherence. It is known that decoherence is detrimental to amount of information contained in a quantum state, as measured by its purity. To effectively characterize the role of decoherence in erasing information [67] one needs to quantify the purity or, its complementary property, the mixedness of the state. A faithful measure of mixedness is the normalized linear entropy [84]. From the perspective of resource theory of purity [71, 85], mixedness can be obtained as a complementary quantity to global information. Since, noise tends to increase the mixedness of a quantum system, it emerges as an intuitive parameter to understand decoherence. A natural question that arises is: How do important physical quantities in quantum information theory, such as entanglement [4], fare against mixedness of quantum systems? An interesting direction is to obtain the maximum amount of entanglement for a given mixedness, which leads to the notion of maximally entangled mixed states [86–90]. The amount of entanglement in such states cannot be increased further under any global unitary operation. Also, the form of the maximally entangled mixed states depends on the measures employed to quantify entanglement and mixedness in the system [89]. Such states have also been investigated in Gaussian quantum systems [91–93]. In chapter 4, we investigate the limits imposed by mixedness of a quantum system on the amount of quantum coherence present in the system. Since, we consider quantum systems where the missing phase-reference frame is apparently lacking, the formalism based on the resource theory of asymmetry [46] becomes over-restrictive [a]. Hence, we use the theoretical approach based on the set of incoherent operations and states [47], to characterize and quantify coherence. We derive an analytical trade-off between the two quantities that allows us to upperbound the maximum coherence in a given mixed quantum state and vice-versa. Using the l_1 norm of coherence [47] as a measure of quantum coherence and normalized linear entropy [84] as a

[a]The discussion on coherence, in Ref. [46], states that the set of phase insensitive operations restricts transformations allowed by incoherent operations. However, restrictive operations are not necessarily physically more relevant or significant under all contexts. For example, in a recent article on the resource theories of thermodynamics based on Gibbs preserving operations and thermal operations [50], the authors have argued how Gibbs preserving operations outperform thermal operations even though the former is restricted by the latter set of operations.

measure of mixedness, we prove that for a general d-dimensional quantum system the sum of the (scaled) squared coherence and the mixedness is always less than or equal to unity. This allows us to derive a class of quantum states, viz. "maximally coherent mixed states" (MCMS), that have maximal coherence, up to incoherent unitaries, for a fixed mixedness. These states are parametrized mixtures of a d-dimensional pure maximally coherent state and maximally mixed state. Interestingly, for different values of mixedness the analytical form of MCMS remains unchanged and, unlike maximally entangled mixed states, is not dependent on the choice of the measure of coherence and mixedness, as observed for the l_1 norm, the relative entropy, and the geometric measures of coherence. The obtained analytical results, show an important trade-off between a relevant quantum resource and noise in open quantum systems and a complementary behavior between coherence and mixedness in the class of MCMS, which may be crucial from the perspective of quantum resource theories and thermodynamics. Significantly, since the mixedness of a quantum system can be experimentally measured using quantum interferometric setups [94, 95], without resorting to complicated state tomography, our results provide a mathematical framework to experimentally determine the maximal coherence in a quantum state.

A major concern of any resource theory is to describe and uncover the intricate structure of the physical processes (state transformations) within the set of allowed operations. The possibility of catalysis is one such phenomenon which allows the otherwise impossible state transformations via the set of allowed operations in a given resource theory. This is very natural as the additional systems (catalysts) are always available and importantly, in such transformations the additional resources are not consumed in anyway. The catalysis in quantum resource theories was first introduced in Refs. [96, 97] in the context of entanglement. The consideration of catalysts in the resource theory of thermodynamics turned out to be very surprising and extremely important that has led to the introduction of many second laws of quantum thermodynamics [11] compared to the single second law in the macroscopic thermodynamics [98]. Recently, in the context of entanglement it is

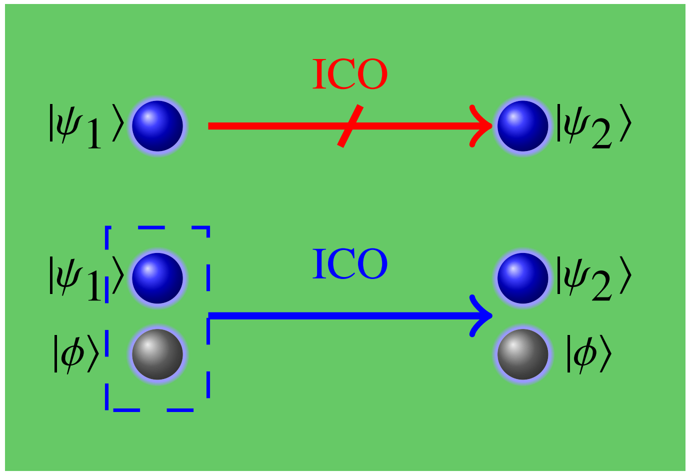


Figure 1.3: The schematic for catalytic coherence transformations. Consider a finite dimensional quantum system in state $|\psi_1\rangle$. Let $|\psi_2\rangle$ be an incomparable state to $|\psi_1\rangle$, i.e., using only incoherent operations one cannot convert $|\psi_1\rangle$ into $|\psi_2\rangle$ with certainty. However, if one has temporary access to another coherent state $|\phi\rangle$, one can always achieve the transformation from $|\psi_1\rangle$ to $|\psi_2\rangle$. The state $|\phi\rangle$ is not consumed in any way and can, therefore, be viewed as a catalyst for this transformation.

found that a quantum system can act as a catalyst for itself, allowing further the possibility of *self catalysis* [99]. Catalysis in the resource theory of coherence was first considered in Ref. [66] and is developed since then (see Refs. [100, 101], also see Fig. 1.3). In chapter 5, we further delineate the phenomenon of catalysis in the resource theory of coherence both in the deterministic and stochastic scenarios and completely characterize the coherence transformations amongst pure quantum states under incoherent operations. This is an important step towards a complete theory of quantum coherence based on incoherent operations as the allowed operations [47]. In particular, we obtain the necessary and sufficient conditions for the deterministic and stochastic coherence transformations in the presence of catalysts. We first find the necessary and sufficient conditions for the enhancement of the optimal probability of conversion while the catalysts are available. Then we go on to find the necessary and sufficient conditions for the deterministic catalytic coherence transformations and show that these are given by the simultaneous decrease of a family of Rényi entropies of the diagonal parts of the states under consideration in a fixed basis. This result is very similar in nature to the many second laws of quantum thermodynamics. We also provide a dedicated discussion on the practicality of these necessary and sufficient conditions. Further, for the cases where no catalytic coherence transformation is possible we consider the possibility of entanglement assisted coherence transformations and find the necessary and sufficient conditions for the same. Furthermore, we find that *self catalysis* in the context of coherence resource theory is possible for the transformations of certain states. We hope that our results will be useful for coherence transformations in the resource theory of coherence, in situations where processing of coherence is limited by additional restrictions from quantum thermodynamics and in context of *single-shot* information theory [102].

Random pure quantum states offer new insights for various phenomena in quantum physics and quantum information theory by exploiting the strong mathematical tools of probability theory and random matrix theory [103]. These states play a fundamental role in providing a satisfactory explanation to the postulate of *equal a priori probability* of statistical physics [104, 105]. Moreover, various properties of complex quantum systems become typical for these states allowing to infer general structures on the set of states on the Hilbert space [29, 81, 103]. In particular, the entanglement properties of pure bipartite quantum states sampled from the uniform Haar measure have been studied extensively [29, 81, 106–115]. It has been shown that the overwhelming majority of random pure quantum states sampled from the uniform Haar measure are extremely close to the maximally entangled state [29] which seems very counterintuitive. Notably, Lévy's lemma and in general, the concentration of measure phenomenon, used in proving the above result paved the way to construct counterexamples to the conjecture of the additivity of minimum output entropy [116–118] among other important implications [119]. Also, the physical relevance of generic entanglement has been established by showing that it can be generated efficiently [120]. In chapter 6, we find the behavior of the quantum coherence for a system in a pure quantum state chosen randomly from the uniform Haar measure. We show that for higher dimensional systems the coherence behaves generically, i.e., most of the systems in these random pure states posses almost the same amount of coherence. We demonstrate that the generic nature of coherence of these states holds for various measures of coherence such as the relative entropy of coherence [47] which is also equal to the distillable coherence, the coherence of formation [121] and the l_1 norm of coherence [47]. In these situations the coherence is solely determined by a few generic parameters that appear in the "concentration of measure phenomenon", such as the dimension of the Hilbert space. We find a large concentrated subspace of the full Hilbert space with the property that the relative entropy of coherence [47] of every pure state in this subspace is almost always lower bounded by a fixed number that is very close to the typical value of coherence. Moreover, for all the states (pure or mixed) in this subspace, the coherence of formation [121] is also lower bounded by the same fixed number. These subspaces are of immense importance in situations where quantum coherence is a useful resource as they guarantee a lower bound on the amount of coherence that may be used. Furthermore, we find that most of the pure states sampled randomly from the Haar measure are not typically maximally coherent. This is in sharp contrast to the fact that most of the bipartite pure state sampled randomly from the Haar measure are typically maximally entangled [29]. Since the quantum coherence quantifies the wave nature of a particle [122, 123], one may ask how 'wavy' is a quantum particle if the state of the particle is chosen at random from the uniform Haar measure? Our result shows that the 'typical wave nature' of a quantum particle such as a qudit is directly related to d-th harmonic number.

As we approach towards the realistic implementations of quantum technology, mixed states are encountered naturally due to the interaction between the system of interest and the external world. Therefore, consideration of average entanglement and coherence content of random mixed states is of great importance in realistic scenarios. In chapter 7, we aim at finding the average relative entropy of coherence of random mixed states sampled from various induced measures including the one obtained via the partial tracing of the Haar distributed random bipartite pure states. We first find the exact expression for the average subentropy of random mixed states sampled from induced probability measures and use it to find the average relative entropy of coherence of random mixed states. We note that the subentropy is a nonlinear function of state and therefore, it is expected that the average subentropy of a random mixed state should not be equal to the subentropy of the average state (the maximally mixed state). Surprisingly, we find that the average subentropy of a random mixed state approaches exponentially fast towards the maximum value of the subentropy, which is achieved for the maximally mixed state [124]. As one of the applications of our results, we note that the average subentropy may also serve as the state independent quality factor for ensembles of states to be used for estimating accessible information. Interestingly, we find that the average coherence of random mixed states, just like the average coherence of random pure states, shows the concentration phenomenon. This means that the relative entropies of coherence of most of the random mixed states are equal to some fixed number (within an arbitrarily small error) for larger Hilbert space dimensions. It is well known that the exact computation of the most of the entanglement measures for bipartite mixed states in higher dimensions is almost impossible [4]. However, using our results, we compute the average relative entropy of entanglement and distillable entanglement for a specific class of random bipartite mixed states and show their typicality for larger Hilbert space dimensions. It means that for almost all random states of this specific class, both the measures of entanglement are equal to a fixed number (that we calculate) within an arbitrarily small error, reducing hugely the computational complexity of both the measures for this specific class of bipartite mixed states. This is a very important practical application of the results obtained in this thesis.

Finally, in chapter 8, we provide a brief summary and implications of our investigations along with the possible future directions and open questions.

CHAPTER 2

Measuring quantum coherence with entanglement

2.1 Introduction

The linearity and probabilistic nature are fundamental features of quantum theory that are the root causes of various surprises offered by the quantum theory. For example, a quantum system may simultaneously be in a superposition of more than one of the possible states and it has a definite nonzero probability of being in any of the superposed states. Quantum coherence and entanglement [4] are distinctive traits encompassing the defining features of the quantum theory [64]. Quantum entanglement is provably a key resource in various quantum information processing tasks including superdense coding [6], remote state preparation [8, 9] and quantum teleportation [5]. Recent developments in thermodynamics of nano scale systems suggest that the quantum coherence plays an essential role in determining the quantum state transformations and more importantly, in providing a family of second laws of thermodynamics [11–14, 16, 33–37]. Similarly, coherence is deemed to be an important resource in exciton and electron transport in biomolecular networks [17–20, 40, 41].

While the theory of quantum coherence is historically well developed in quantum optics [21, 43, 44] in terms of quasiprobability distributions and higher-order correlation functions, a rigorous framework to quantify coherence for general states adopting the language of quantum information theory has only been attempted in recent years [41, 47, 48]. This framework is based on the characterization of the set of incoherent states and a class of 'free' operations, named incoherent quantum channels, that map the set onto itself [41, 47]. The resulting resource theory of quantum coherence is in direct analogy with the resource theory of quantum entanglement [4], in which local operations and classical communication are identified as the 'free' operations that map the set of separable states onto itself [49]. Within such a framework for coherence, one can define suitable measures that vanish for any incoherent state, and satisfy specific monotonicity requirements under incoherent quantum channels. Measures that respect these conditions gain the attribute of coherence monotones, in analogy with entanglement monotones [125].

Motivated by the fact that the cause underlying the notion of quantum coherence and entanglement is same, and noting that entanglement stems from the superposition principle (which is also the essence of coherence), in this chapter, we investigate the parallel between quantum coherence and entanglement. The parallel between these two notions is apparent at the formal level of resource theories, here, we aim to upgrade this formal parallelity to a more solid conceptual premise. Intuitively, both coherence and entanglement capture the quantumness of a physical system, it is then legitimate to ask how can one resource emerge *quantitatively* from the other. We provide a mathematically rigorous approach to answer the above question. We base our research on treating both the notions of coherence and entanglement on the similar footing by using a common frame to quantify quantumness. The results of this chapter clearly show qualitative and quantitative *equivalence* between coherence and entanglement at a fundamental level, and provide an intuitive operational scheme to interchange these two nonclassical resources for suitable applications in quantum technologies.

This chapter is organized as follows. In Sec. 2.2, we introduce and discuss the resource theory of coherence and give a brief overview of various coherence monotones relevant for this thesis. Then we consider the problem of creation of entanglement from coherence and show that any nonzero amount of coherence in a system can be 'activated' into (distillable) entanglement between the system and an initially incoherent ancilla, by using only incoherent operations, in Sec. 2.3. In Sec. 2.4, we prove that, given a family of entanglement monotones, one can define a corresponding family of coherence monotones. Towards the end of the chapter, in Sec. 2.5, we summarize the results obtained in this chapter.

2.2 Characterizing coherence

As mentioned in the chapter 1, a resource theory is essentially constructed from the restrictions imposed on the quantum operations. These restrictions may arise from variety of reasons. For example, it may arise from our practical limitations to carry out some quantum operations such as global quantum operations in the case where two or more parties are separated and this gives rise to the very well studied free operations, namely, the local operations and classical communication (LOCC). Or these restrictions may arise as a consequence of symmetries in the form of superselection rules and conservation laws. A resource theory may be formulated by starting from a set of allowed (or free) states and then defining the allowed (or free) operations as the quantum operations that map the set of allowed states into itself. Or a resource theory may also be given its full structure by defining the set of allowed (or free) states starting from the maximally mixed state and applying only the allowed (or free) operations. We will follow the former procedure to develop the resource theory of coherence in this thesis.

In the following we discuss the resource theory of quantum coherence based incoherent operations following Ref. [47]. For the QRT of coherence the three basic ingredients are: (a) the set of incoherent states as the restricted set of allowed states, (b) the set of incoherent operations as the set of allowed operations, and (c) the set of coherent states as the set of resource states. It is to be noted that these three elements need not be independent from one another. The concept of quantum coherence is intrinsically basis dependent and it is parallel to the fact that quantum entanglement is dependent on global basis changes. Therefore, we start by fixing a reference basis. For an arbitrary fixed reference basis $\{|i\rangle\}$, the incoherent states are defined as

$$\sigma = \sum_i p_i |i\rangle \langle i| , \tag{2.1}$$

where p_i are nonnegative probabilities. Any state which cannot be written in the above form is referred to as a coherent state. For instance, the maximally coherent state for a system of dimension d is given by $|\phi_d\rangle = \sum_{i=0}^{d-1} |i\rangle /\sqrt{d}$ [47]. A completely positive trace preserving map Λ is said to be an incoherent operation if it can be written as

$$\Lambda[\rho] = \sum_l K_l \rho K_l^\dagger, \tag{2.2}$$

where the corresponding Kraus operators K_l map every incoherent state to some other incoherent state. If $\mathcal{I}$ is the set of incoherent states, then each of the Kraus operator K_l satisfies $K_l \mathcal{I} K_l^\dagger \subseteq \mathcal{I}$. Such operators K_l will be called incoherent Kraus operators in the following. Following established notions from entanglement theory [4, 126–128], Baumgratz *et al.* proposed the following postulates for a measure of coherence $C(\rho)$ in Ref. [47]:

(C1) $C(\rho) \geq 0$, and $C(\rho) = 0$ if and only if $\rho \in \mathcal{I}$.

(C2) $C(\rho)$ is nonincreasing under incoherent operations, i.e., $C(\rho) \geq C(\Lambda[\rho])$ with $\Lambda[\mathcal{I}] \subseteq \mathcal{I}$.

(C3) $C(\rho)$ is nonincreasing on average under selective incoherent operations, i.e., $C(\rho) \geq \sum_l p_l C(\sigma_l)$, with probabilities $p_i = \mathrm{Tr}[K_l \rho K_l^\dagger]$, quantum states $\sigma_l = K_l \rho K_l^\dagger / p_l$, and incoherent Kraus operators K_l satisfying $K_l \mathcal{I} K_l^\dagger \subseteq \mathcal{I}$.

(C4) $C(\rho)$ is a convex function of density matrices, i.e., $C(\sum_i p_i \rho_i) \leq \sum_i p_i C(\rho_i)$.

At this point we note that conditions (C3) and (C4) automatically imply condition (C2). The reason why we listed all conditions above is that – similar to entanglement measures – there might exist meaningful quantifiers of coherence which satisfy conditions (C1) and (C2), but for which conditions (C3) and (C4) are either violated or cannot be proven. Following the analogous notion from entanglement theory, we call a quantity which satisfies conditions (C1), (C2), and (C3) a *coherence monotone*. It is important to note that the set of incoherent operations is not uniquely defined and there exists various sets of incoherent operations leading to the different resource theories of coherence. For a good exposition of these resource theories we refer the readers to a very recent review article [22] on the resource theories of coherence.

Various (convex) coherence monotones have been obtained in recent years. We will list a few of them that are relevant to this thesis. These include the l_1 norm of coherence, the relative entropy of coherence and the coherence of formation. For a density matrix ρ of dimension d and a fixed reference basis $\{|i\rangle\}$, the l_1 norm of coherence $\mathscr{C}_{l_1}(\rho)$ [47] is defined as

$$\mathscr{C}_{l_1}(\rho) = \sum_{\substack{i,j=1 \\ i\neq j}}^{d} |\langle i|\, \rho\, |j\rangle\, |. \tag{2.3}$$

The relative entropy of coherence $\mathscr{C}_r(\rho)$ [47] is defined as

$$\mathscr{C}_r(\rho) = S(\rho^{(\mathrm{d})}) - S(\rho), \tag{2.4}$$

where $\rho^{(\mathrm{d})}$ is the diagonal part of the density matrix ρ in the fixed reference basis and S is the von Neumann entropy defined as $\mathrm{S}(\rho) = -\operatorname{Tr}(\rho \ln \rho)$. Here and in the rest of the thesis, all the logarithms are taken with respect to the base e. The coherence of formation

$\mathscr{C}_f(\rho)$ [121] is defined as

$$\mathscr{C}_f(\rho) = \min_{\{p_a, |\psi_a\rangle\langle\psi_a|\}} \sum_a p_a S(\rho^{(\mathrm{d})}(\psi_a)), \tag{2.5}$$

where $\rho^{(\mathrm{d})}(\psi_a)$ is the diagonal part of the pure state $|\psi_a\rangle$, $\rho = \sum_a p_a |\psi_a\rangle \langle\psi_a|$ and minimum is taken over all such decompositions of ρ. We emphasize that in this thesis, we consider the notion of coherence applicable to finite dimensional quantum systems. See Ref. [22] for a recent review on coherence.

Bipartite coherence.—We first extend the framework of the resource theory of coherence to the bipartite scenario (see also [129]). In particular, for a bipartite system with two subsystems X and Y, and with respect to a fixed reference product basis $\{|i\rangle^X \otimes |j\rangle^Y\}$, we define bipartite incoherent states as follows:

$$\rho^{XY} = \sum_k p_k \sigma_k^X \otimes \tau_k^Y. \tag{2.6}$$

Here, p_k are nonnegative probabilities and the states σ_k^X and τ_k^Y are incoherent states on the subsystem X and Y respectively, i.e. $\sigma_k^X = \sum_i p'_{ik} |i\rangle \langle i|^X$ and $\tau_k^Y = \sum_j p''_{jk} |j\rangle \langle j|^Y$ for probabilities p'_{ik} and p''_{jk}. Note that bipartite incoherent states as given in Eq. (2.6) are always separable.

We next define bipartite incoherent operations as follows:

$$\Lambda^{XY}[\rho^{XY}] = \sum_l K_l \rho^{XY} K_l^\dagger \tag{2.7}$$

with incoherent Kraus operators K_l such that $K_l \mathcal{I} K_l^\dagger \subseteq \mathcal{I}$, where $\mathcal{I}$ is now the set of bipartite incoherent states defined in Eq. (2.6). It is straightforward to extend the definitions in Eqs. (2.6) and (2.7) to arbitrary multipartite states.

An important example of a bipartite incoherent operation is the two-qubit CNOT gate U_{CNOT}. It is not possible to create coherence from an incoherent two-qubit state by using the CNOT gate, since it takes any pure incoherent state $|i\rangle \otimes |j\rangle$ to another pure incoherent state,

$$U_{\mathrm{CNOT}} \left(|i\rangle \otimes |j\rangle \right) = |i\rangle \otimes |\mathrm{mod}\,(i + j, 2)\rangle \, . \tag{2.8}$$

It is important to mention that—despite being incoherent—the CNOT gate can instead be used to create entanglement. In particular, note that the state $U_{\mathrm{CNOT}}(|\psi\rangle \otimes |0\rangle)$ is entangled for any coherent state $|\psi\rangle$. This observation will be crucial for the results presented in this chapter.

2.3 Coherence and entanglement creation

Referring to Fig. 1.1 for an illustration, we say that a (finite-dimensional) system S in the initial state ρ^S can be used for the task of "entanglement creation via incoherent operations" if, by attaching an ancilla A initialized in a reference incoherent state $|0\rangle\langle 0|^A$, the final system-ancilla state $\Lambda^{SA}[\rho^S \otimes |0\rangle\langle 0|^A]$ is entangled for some incoherent operation Λ^{SA}. Note that incoherent system states ρ^S cannot be used for entanglement creation in this way, since for any incoherent state ρ^S the state $\Lambda^{SA}[\rho^S \otimes |0\rangle\langle 0|^A]$ will be of the form given in Eq. (2.6), and thus separable.

However, the situation is different if coherent states are considered. In particular, entanglement can in general be created by incoherent operations, if the underlying system state ρ^S is coherent. This phenomenon was exemplified above by using the two-qubit CNOT gate. In the light of these observations, it is natural to ask the following question: *Are all coherent states useful for entanglement creation via incoherent operations?*

In order to answer this question, we will first consider distance-based quantifiers of entanglement E_D and coherence C_D as presented in [47, 126, 128, 129]:

$$E_D(\rho) = \min_{\sigma \in \mathcal{S}} D(\rho, \sigma), \qquad C_D(\rho) = \min_{\sigma \in \mathcal{I}} D(\rho, \sigma). \tag{2.9}$$

Here, $\mathcal{S}$ is the set of separable states and $\mathcal{I}$ is the set of incoherent states. Moreover, we demand that the distance D be contractive under quantum operations,

$$D(\Lambda[\rho], \Lambda[\sigma]) \leq D(\rho, \sigma) \tag{2.10}$$

for any completely positive trace preserving map Λ. This implies that E_D does not increase under local operations and classical communication [126, 127], and C_D does not increase under incoherent operations [47]. Equipped with these tools we are now in position to present the first important result of this chapter.

Theorem 1. *For any contractive distance D, the amount of (distance-based) entanglement E_D created from a state ρ^S via an incoherent operation Λ^{SA} is bounded above by its (distance-based) coherence C_D:*

$$E_D^{S:A}\left(\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right]\right) \leq C_D\left(\rho^S\right). \tag{2.11}$$

Proof. Let σ^S be the closest incoherent state to ρ^S, i.e., $C_D(\rho^S) = D(\rho^S, \sigma^S)$. The contractivity of the distance D further implies the equality

$$D\left(\rho^S, \sigma^S\right) = D\left(\rho^S \otimes |0\rangle\langle 0|^A, \sigma^S \otimes |0\rangle\langle 0|^A\right). \tag{2.12}$$

In the final step, note that the application of an incoherent operation Λ^{SA} to the incoherent state $\sigma^S \otimes |0\rangle\langle 0|^A$ brings it to another incoherent – and thus separable – state. Applying Eq. (2.10) and combining the aforementioned results we arrive at the desired inequality:

$$\begin{aligned} C_D\left(\rho^S\right) &\geq D\left(\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right], \Lambda^{SA}\left[\sigma^S \otimes |0\rangle\langle 0|^A\right]\right) \\ &\geq E_D^{S:A}\left(\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right]\right). \end{aligned} \tag{2.13}$$

This completes the proof of the Theorem. □

The result just presented provides a strong link between the frameworks of entanglement on the one hand and coherence on the other hand. An even stronger statement can be made for the specific case of D being the quantum relative entropy. The corresponding quantifiers in this case are the relative entropy of entanglement E_{r} [126], and the relative entropy of coherence $\mathscr{C}_{\mathrm{r}}$ [47] already introduced in Eq. (2.4). As we will now show, the inequality (2.11) can be saturated for these measures if the dimension of the ancilla is not smaller than the dimension of the system, $d_A \geq d_S$. In this case there always exists an incoherent operation Λ^{SA} such that

$$E_{\mathrm{r}}^{S:A}\left(\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right]\right) = \mathscr{C}_{\mathrm{r}}\left(\rho^S\right). \tag{2.14}$$

To prove this statement, we consider the unitary operation

$$U = \sum_{i=0}^{d_S-1}\sum_{j=0}^{d_S-1} |i\rangle\langle i|^S \otimes |\mathrm{mod}(i+j, d_S)\rangle\langle j|^A + \sum_{i=0}^{d_S-1}\sum_{j=d_S}^{d_A-1} |i\rangle\langle i|^S \otimes |j\rangle\langle j|^A. \tag{2.15}$$

Note that for two qubits this unitary is equivalent to the CNOT gate with S as the control qubit and A as the target qubit. It can further be seen by inspection that this unitary is incoherent, i.e., the state $\Lambda^{SA}[\rho^{SA}] = U\rho^{SA}U^\dagger$ is incoherent for any incoherent state ρ^{SA}. Moreover, this operation takes the state $\rho^S \otimes |0\rangle\langle 0|^A$ to the state

$$\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right] = \sum\nolimits_{i,j} \rho_{ij} |i\rangle\langle j|^S \otimes |i\rangle\langle j|^A, \tag{2.16}$$

where ρ_{ij} are the matrix elements of $\rho^S = \sum_{i,j} \rho_{ij} |i\rangle\langle j|^S$. In the next step we use the fact that for any quantum state τ^{SA} the relative entropy of entanglement is bounded below as follows [130]: $E_{\mathrm{r}}^{S:A}(\tau^{SA}) \geq H(\tau^S) - H(\tau^{SA})$. Applied to the state $\Lambda^{SA}[\rho^S \otimes |0\rangle\langle 0|^A]$, this inequality reduces to

$$E_{\mathrm{r}}^{S:A}\left(\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right]\right) \geq H\left(\sum\nolimits_i \rho_{ii} |i\rangle\langle i|^S\right) - H(\rho^S). \tag{2.17}$$

Noting that the right-hand side of this inequality is equal to the relative entropy of coherence $\mathscr{C}_{\rm r}(\rho^S)$ [47], we obtain $E_{\rm r}^{S:A}(\Lambda^{SA}[\rho^S \otimes |0\rangle\langle 0|^A]) \geq \mathscr{C}_{\rm r}(\rho^S)$. The proof of Eq. (2.14) is complete by combining this result with Theorem 1.

This shows that the degree of (relative entropy of) coherence in the initial state of S can be exactly *activated* into an equal degree of (relative entropy of) entanglement created between S and the incoherent ancilla A by a suitable incoherent bipartite operation, that is a generalized CNOT gate.

With these results we are now in position to tackle the central question whether all coherent states are useful for entanglement creation via incoherent operations. The (affirmative) answer is provided by the following Theorem.

Theorem 2. *A state ρ^S is useful for entanglement creation via incoherent operations if and only if ρ^S is coherent.*

Proof. If ρ^S is incoherent, it cannot be used for entanglement creation via incoherent operations due to Theorem 1. On the other hand, if ρ^S is coherent, it also has nonzero relative entropy of coherence $\mathscr{C}_{\rm r}(\rho^S) > 0$. Due to Eq. (2.14) there exists an incoherent operation Λ^{SA} leading to nonzero relative entropy of entanglement $E_{\rm r}^{S:A}(\Lambda^{SA}[\rho^S \otimes |0\rangle\langle 0|^A]) > 0$. This completes the proof of the Theorem. □

Let us mention that the results presented above also hold for the distillable entanglement $E_{\rm d}$. In particular, the relative entropy of coherence $\mathscr{C}_{\rm r}$ also serves as an upper bound for the creation of distillable entanglement via incoherent operations:

$$E_{\rm d}^{S:A}\left(\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right]\right) \leq \mathscr{C}_{\rm r}\left(\rho^S\right). \tag{2.18}$$

This inequality follows from Theorem 1, together with the fact that the relative entropy of entanglement is an upper bound on the distillable entanglement [131]: $E_{\rm d} \leq E_{\rm r}$. Moreover, it can also be shown that the inequality (2.18) is saturated for the unitary incoherent operation presented in Eq. (2.15). This can be seen using the same reasoning as below

Eq. (2.15), together with the fact that the distillable entanglement is also bounded below as follows [132]: $E_{\mathrm{d}}^{S:A}(\tau^{SA}) \geq H(\tau^{S}) - H(\tau^{SA})$.

2.4 Quantifying coherence with entanglement

Somehow reversing the perspective, the result presented in Theorem 1 can also be regarded as providing a lower bound on distance-based measures of coherence via the task of entanglement creation. In particular, the amount of coherence C_D of a state ρ^S is always bounded below by the maximal amount of entanglement E_D generated from this state by incoherent operations. Going now beyond the specific setting of distance-based quantifiers, we will show that such a maximization of the created entanglement, for any given (completely general) entanglement monotone, leads to a quantity which can be used as a valid quantifier of coherence in its own right.

Namely, we introduce the family of *entanglement-based coherence measures* $\{C_E\}$ as follows:

$$C_E(\rho^S) = \lim_{d_A \to \infty} \left\{ \sup_{\Lambda^{SA}} E^{S:A} \left(\Lambda^{SA} \left[\rho^S \otimes |0\rangle \langle 0|^A \right] \right) \right\}. \tag{2.19}$$

Here, E is an arbitrary entanglement measure, the supremum is taken over all incoherent operations Λ^{SA}, and d_A is the dimension of the ancilla [a].

It is crucial to benchmark the validity of $\{C_E\}$ as proper measures of coherence. Remarkably, we will now show that C_E satisfies all the aforementioned conditions (C1)–(C3) given any entanglement monotone E, with the addition of (C4) if E is convex as well. We namely get the following result:

Theorem 3. *C_E is a (convex) coherence monotone for any (convex) entanglement monotone E.*

Proof. Using the arguments presented above it is easy to see that C_E is nonnegative, and

[a] Note that the limit $d_A \to \infty$ in Eq. (2.19) is well defined, since the supremum $\sup_{\Lambda^{SA}} E^{S:A} \left(\Lambda^{SA} \left[\rho^S \otimes |0\rangle \langle 0|^A \right] \right)$ cannot decrease with increasing dimension d_A.

zero if and only if the state ρ^S is incoherent. Moreover, C_E does not increase under incoherent operations Λ^S performed on the system S. This can be seen directly from the definition of C_E in Eq. (2.19), noting that an incoherent operation Λ^S on the system S is also incoherent with respect to SA. The proof that C_E further satisfies condition (C3) is presented in the Appendix A. There we also show that C_E is convex for any convex entanglement monotone E, i.e. (C4) is fulfilled as well in this case. □

This powerful result completes the parallel between coherence and entanglement, *de facto* establishing their full quantitative equivalence within the respective resource theories.

2.5 Chapter summary

In this chapter we have shown that the presence of coherence in the state of a quantum system yields a necessary and sufficient condition for its ability to generate entanglement between the system and an incoherent ancilla using incoherent operations (see Fig. 1.1). Building on the above connection, we proposed a family of coherence quantifiers in terms of the maximal amount of entanglement that can be created from the system by incoherent operations. The proposed coherence quantifiers satisfy all the necessary criteria for them to be *bona fide* coherence monotones [47].

The framework presented in this chapter should also be compared to the scheme for activating distillable entanglement via premeasurement interactions [133–135] from quantum discord, a measure of nonclassical correlations going beyond entanglement [136, 137]. The latter approach has attracted a large amount of attention recently [136, 138–140], and it is reasonable to expect that several (theoretical and experimental) results obtained in that context also carry over to the concept presented here, even taking into account the close relationship between bipartite coherence and discord [129]. Exploring these connections further will be the subject of future research.

The theory of entanglement has been the cornerstone of major developments in quantum information theory and, in recent years, it has immensely contributed to the advancement of quantum technologies. A complete characterization of coherence may improve our perception of quantumness at its most essential level, and further guide our understanding of nascent fields such as quantum biology and nanoscale thermodynamics. Hence, it is of primary importance to construct a physically meaningful and mathematically rigorous quantitative theory of quantum coherence. By effectively realizing a unification between the notions of coherence and entanglement from a quantum informational viewpoint, we believe the present chapter delivers a substantial step in this direction.

This chapter is based on the following paper:

1. *Measuring quantum coherence with entanglement*,
   A. Streltsov, **U. Singh**, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).

CHAPTER 3

Erasing quantum coherence: An operational approach

3.1 Introduction

With our ever increasing abilities to control systems at smaller and smaller scales, the quantum properties like quantum coherence and quantum entanglement make their presence felt more and more prominently. As mentioned in chapter 1, recent developments in thermodynamics of nano scale systems [11–14, 16, 33–37] and in the field of complex biological systems suggest the pivotal role played by quantum coherence in these contexts. Given the importance of quantum coherence, a formal structure of coherence resource theories is developed in recent years [23, 45–48, 60, 100, 129, 141–144]. The resource theories of coherence are still in a very active phase of development and our understanding of these resource theories is severely restricted both at quantitative and qualitative levels. Although mathematically sound, the resource theories of coherence lack operational significance as such. There are some notable works, e.g. Refs. [23, 121], which have contributed towards filling this gap. In this chapter, we follow a thermodynamical approach to provide an operational significance to the resource theory of coherence based on the incoherent operations.

The thermodynamic cost of destroying (erasing) a particular resource of interest has been central in quantum information theory to equip the resource with an operational meaning. This approach has been very fruitful, far reaching [67–73], and has been applied successfully in the contexts of private quantum decoupling [77], markovianization [78] and operationally defining total correlation, classical correlation and quantum correlation [74, 75]. Additionally, the thermodynamic cost of erasing quantum correlation has shown to be associated with entropy production in the environment [76].

In this chapter, we quantify coherence in terms of the cost of erasing the same. In particular, we define the coherence of a quantum system in terms of the minimal amount of noise that has to be injected into the system such that the system decoheres completely utilizing random unitary operations as the decohering operations. This, in turn, will provide operational meaning of the coherence. The obtained measure of coherence depends on method of decohering process (here we consider an ensemble of random unitary operations) and on the quantifiers of the injected noise during the decohering process. We consider the entropy exchange between system and environment during the decohering operation [82, 83] and the memory required to store the information about the decohering operation [74] as quantifiers of injected noise. Interestingly, we find that in the asymptotic limit, both these noise quantifiers yield the same minimal cost of decohering the systems completely (utilizing our method) and it turns out to be equal to the relative entropy of coherence [47]. We emphasize here that in order to quantify coherence we followed an approach that is very much different compared to the other measures existing in resource theory of coherence but we get the same quantifier as the relative entropy of coherence. Our results show that the relative entropy of coherence is endowed with an operational meaning.

The chapter is organized as follows: In Sec. 3.2, we give a brief outline of the concepts required to understand the process of erasure of quantum coherence with illustrious examples. We present our main result of obtaining minimal cost of erasing coherence of a quantum system or of decohering a quantum system completely, in Sec. 3.3. We conclude in Sec. 3.4 with overview and implications of the results presented in this chapter. In the

Appendix B, we provide some definitions like that of typical subspaces and present some well known theorems for the sake of completeness.

3.2 Preliminaries and various definitions

Before we proceed further, we would like to give an illustration of a process of fully decohering a qubit quantum system in state $|\psi_2\rangle = \frac{1}{\sqrt{2}}\sum_{a=0}^{1}|a\rangle$, which is a maximally coherent state. Suppose we want to erase the coherence of this state. This can be achieved by applying two incoherent unitary transformations $\mathbb{I}_2$ and σ_z with equal probability, i.e.,

$$|\psi_2\rangle\langle\psi_2| \rightarrow \rho = \frac{1}{2}|\psi_2\rangle\langle\psi_2| + \frac{1}{2}\sigma_z|\psi_2\rangle\langle\psi_2|\sigma_z = \frac{1}{2}\mathbb{I}_2. \tag{3.1}$$

Note that the final state is an incoherent state. This means that the application of two incoherent unitary operations with equal probability suffices to erase the coherence of the maximally coherent state. The same holds for the class of maximally coherent mixed states in two dimensions [25]. Also, it can be seen that for a d dimensional quantum system, an ensemble of unitary transformations $\{\frac{1}{d^2}, \hat{X}^k\hat{Z}^j\}_{jk}$ exists, where $\hat{X}|j\rangle = |j\oplus 1\rangle, \hat{Z}|j\rangle = e^{\frac{2\pi i j}{d}}|j\rangle$ and $\oplus$ denotes addition modulo d, that can randomize any state ρ of the system completely [81], i.e.,

$$\rho \rightarrow \frac{1}{d^2}\sum_{j=1}^{d}\sum_{k=1}^{d}\hat{X}^k\hat{Z}^j\rho\hat{Z}^{j\dagger}\hat{X}^{k\dagger} = \frac{1}{d}\mathbb{I}_d. \tag{3.2}$$

But what is the cost to be paid in order to implement this probabilistic incoherent operation or how much noise does this operation inject into the system? One possibility, is to consider the amount of information (memory) needed to implement this (erasing) operation, which is related to the probabilities associated with the unitaries, and is equal to the Shannon entropy, $S(p = 1/2) = 1$ bit for above qubit example. Therefore, one can say that applying an operation consisting of two elements, with equal probability, costs one bit of information or injects one bit of noise in the system. Similarly, for a qudit system, we can achieve the exact randomization via a map of the form Eq. (3.2). The entropy that this map injects into the system as quantified by the amount of information needed to implement it, is given by $S(p = 1/d^2) = 2\log_2 d$ bits. Clearly, the state independent randomization over-estimates the amount of noise that is necessary to decohere the state (cf. qubit and qudit cases). Also, this cost is independent of the nature of the operation, i.e., whether the operation is incoherent, unitary etc. The other choice to quantify the amount of noise injected into the system can be obtained based on exchange entropy as in Refs. [74, 82, 83]. As we show below, the exchange entropy is smaller than $S(p)$.

Exchange entropy:— The exchange entropy [82, 83] is defined as the amount of entropy that any channel R injects into the system S which passes through R. To define exchange entropy, we purify the system state ρ^S by a reference system Z such that $\rho^S = \mathrm{Tr}_Z \left|\psi\right\rangle\left\langle\psi\right|^{SZ}$. Now the entropy that R injects into the system is defined as $H_e(R, \rho^S) := H\left((R \otimes \mathbb{I}^Z)[\psi^{SZ}]\right)$, where $\mathbb{I}^Z$ is the identity operator on the reference system Z and H is the von Neumann entropy. The exchange entropy has been successfully employed in gaining insights in security of cryptographic protocols [82, 83], in determining cost of erasing total, classical and quantum correlations [74]. Let R be comprised of random unitary ensemble $\{p_i, U_i\}_{i=1}^N$. Then exchange entropy satisfies, $H_e(R, \rho^S) \leq H(p) \leq \log N$. For the example of maximally coherent qubit state, the entropy exchange is equal to one bit which is equal to the memory required to implement the erasing operation, as obtained in the preceding paragraph. Next we define general decohering map which can decohere any system and then ϵ-decohering map that decoheres any state with small error $\epsilon > 0$.

Decohering and ϵ-decohering maps:— Let the decohering be achieved by an ensemble of incoherent unitaries $\{p_i, U_i^I\}_{i=1}^N$. We associate the map $\mathcal{R} : \rho \mapsto \sum_{i=1}^N p_i U_i^I \rho U_i^{I\dagger}$, to the ensemble of these incoherent unitaries. We call this class of incoherent completely positive trace preserving (ICPTP) maps on system S as the decohering maps. A decohering map $\mathcal{R}$ acting on a state ρ, is defined to be ϵ-decohering map if there exists an incoherent state

τ such that $||\mathcal{R}(\rho) - \tau||_1 \leq \epsilon$, where $||\cdot||_1$ is the trace norm [2, 3] and for a matrix A, the trace norm is defined as $||A||_1 = \mathrm{Tr}\sqrt{A^\dagger A}$. Note that the map $\mathcal{R}$ need not be comprised of incoherent unitaries. In fact, we show that a map $\mathcal{R}_c$, comprised of general unitaries, is equally suitable for decohering process. With these definitions in hand, we now proceed to present our results.

3.3 Cost of erasing quantum coherence

We will mainly be concerned with the asymptotic case of the decohering procedure. But before going to the asymptotic case, let us consider the single copy scenario. Suppose a CPTP map Υ decoheres the system in any state ρ and maps it to some incoherent state $\rho_I = \sum_a p_a |a\rangle\langle a|$, where $\{|a\rangle\}$ is the fixed reference basis, $p_a \geq 0$ and $\sum_a p_a = 1$, i.e., $\rho \to \Upsilon[\rho] = \rho_I$. The entropy exchange of this map is given by $H_e(\Upsilon, \rho) = H\left((\Upsilon \otimes \mathbb{I}^Z)[|\psi\rangle\langle\psi|^{SZ}]\right)$, where Z is a reference system used to purify ρ. Now from monotonicity of the mutual information, i.e., $I(\Upsilon[\rho^{SZ}]) \leq I(\rho^{SZ})$, we have $H_e(\Upsilon, \rho) \geq H(\rho_I) - H(\rho)$. Based on this expression, the minimum exchange entropy in this case is defined as $H_e^{\min} = \min_{\{p_a\}} H(\rho_I) - H(\rho)$ (Note that the minimum exchange entropy is defined and not derived). Next, we will compute the minimum exchange entropy in the asymptotic limit when the CPTP map decoheres the state ρ with some nonzero small error.

Lemma 1: Consider an ϵ-decohering map $\mathcal{R}$ on the n copies of the system S in the state ρ as $\mathcal{R} : \rho^{\otimes n} \mapsto \sum_{i=1}^N p_i U_i^I \rho^{\otimes n} U_i^{I\dagger}$, where U_i^I is an incoherent unitary operator. Then, the amount of entropy that is injected into the system is lower bounded as $H_e(\mathcal{R}, \rho^{\otimes n}) \geq n[\mathscr{C}_r(\rho) - \epsilon \log d - H_2(\epsilon)]$, where $\mathscr{C}_r(\rho)$ is the relative entropy of coherence for the state ρ and $H_2(\epsilon) = -\epsilon \ln \epsilon - (1-\epsilon)\ln(1-\epsilon)$ is the binary Shannon entropy. In the asymptotic limit, the minimum entropy exchange, i.e., the minimum cost for erasing coherence, is given by

$$\sup_{\epsilon>0} \liminf_{n\to\infty} \frac{1}{n} \min \left\{ H_e(\mathcal{R}, \rho^{\otimes n}) : \mathcal{R} \;\; \epsilon\text{-decohering} \right\} = \mathscr{C}_r(\rho). \tag{3.3}$$

The relative entropy of coherence of any state ρ is given by $\mathscr{C}_r(\rho) = H(\rho^{(\mathrm{d})}) - H(\rho)$, where $H(\rho) = -\mathrm{Tr}(\rho \ln \rho)$, is the von Neumann entropy and $\rho^{(\mathrm{d})} = \sum_a \langle a|\rho|a\rangle |a\rangle\langle a|$ is the diagonal part of ρ in the reference basis $\{|a\rangle\}$.

Proof.—First of all, define

$$R_D := \mathcal{P}(\mathcal{R}[\rho^{\otimes n}]) = \sum_{\boldsymbol{k}} \Pi_{\boldsymbol{k}} \mathcal{R}[\rho^{\otimes n}] \Pi_{\boldsymbol{k}}, \tag{3.4}$$

where $\{\Pi_{\boldsymbol{k}}\}$ are the projectors on the product subspaces written in the reference basis for the n copies of the system. Any incoherent state under the projective measurement in the reference basis remains intact. Now utilizing the monotonicity of the trace norm under CPTP maps [2, 3], we have

$$\begin{aligned} ||R_D - \tau||_1 &= ||\mathcal{P}(\mathcal{R}[\rho^{\otimes n}]) - \mathcal{P}(\tau)||_1 \\ &\leq ||\mathcal{R}[\rho^{\otimes n}] - \tau||_1 \leq \epsilon, \end{aligned} \tag{3.5}$$

where in the last line we have used the fact that the map $\mathcal{R}$ is an ϵ-decohering map. Now consider the following quantity

$$||\mathcal{R}[\rho^{\otimes n}] - R_D||_1 \leq ||\mathcal{R}[\rho^{\otimes n}] - \tau||_1 + ||\tau - R_D||_1 \leq 2\epsilon, \tag{3.6}$$

where we have used the triangle inequality for the trace distance and made use of Eq. (3.5) together with the fact that the map $\mathcal{R}$ is an ϵ-decohering map. Now, since $||\mathcal{R}[\rho^{\otimes n}] - R_D||_1 \leq 2\epsilon$, in the worst case one has $||\mathcal{R}[\rho^{\otimes n}] - R_D||_1 = 2\epsilon$. From the Fannes-Audenaert inequality [145] (see also Appendix B, Sec. B.1), we have

$$\begin{aligned}|H(\mathcal{R}[\rho^{\otimes n}]) - H(R_D)| &\leq \epsilon \ln(d^n - 1) + H_2(\epsilon) \\ &\leq \epsilon n \log d + H_2(\epsilon),\end{aligned} \tag{3.7}$$

where in the last line we have used $\ln(d^n - 1) \leq n \log d$ and $H_2(\epsilon) = -\epsilon \ln \epsilon - (1-\epsilon)\ln(1-\epsilon)$. Noting the fact that R_D is the diagonal part of $\mathcal{R}[\rho^{\otimes n}]$ and $H(R_D) \geq H(\mathcal{R}[\rho^{\otimes n}])$, we have

$$H(\mathcal{R}[\rho^{\otimes n}]) \geq H(R_D) - n\epsilon \log d - H_2(\epsilon). \tag{3.8}$$

Here, we pause to look at entropy of R_D more closely. The incoherent unitary operations cannot change the diagonal parts of any density matrix except permuting the diagonal elements (of course they can change phases of off diagonal terms). This can be seen from the fact that any incoherent unitary U^I can be written as a product of a unitary diagonal matrix V and a permutation matrix Π, i.e., $U^I = V\Pi$. Therefore, we have $U^I \rho U^{I\dagger} = V \sum_{ij} \rho_{ij} |\Pi(i)\rangle \langle \Pi(j)| V^\dagger$. In the following, a superscript (d) on a state ρ will mean the diagonal part of the density matrix in the fixed product reference basis. Now the diagonal part of the density matrix $U^I \rho U^{I\dagger}$ is given by

$$\begin{aligned}(U^I \rho U^{I\dagger})^{(\mathrm{d})} &= \sum_l \langle l| V \sum_{ij} \rho_{ij} |\Pi(i)\rangle \langle \Pi(j)| V^\dagger |l\rangle |l\rangle \langle l| \\ &= \sum_i \rho_{\Pi(i)\Pi(i)} |\Pi(i)\rangle \langle \Pi(i)| .\end{aligned} \tag{3.9}$$

Therefore, we have $H((U^I \rho U^{I\dagger})^{(\mathrm{d})}) = H(\rho^{(\mathrm{d})})$. Making use of this fact for R_D, we have

$$\begin{aligned} H(R_D) &\geq \sum_i p_i H\left(\left(U_i^I \rho^{\otimes n} U_i^{I\dagger}\right)^{(\mathrm{d})}\right) \\ &= \sum_i p_i H\left(\rho^{(\mathrm{d})\otimes n}\right) = nH\left(\rho^{(\mathrm{d})}\right). \end{aligned} \tag{3.10}$$

From the Eq. (3.8), we have

$$\begin{aligned} H(\mathcal{R}[\rho^{\otimes n}]) &\geq nH(\rho^{(\mathrm{d})}) - n\epsilon \log d - H_2(\epsilon) \\ &\geq n[H(\rho^{(\mathrm{d})}) - \epsilon \log d - H_2(\epsilon)], \end{aligned} \tag{3.11}$$

where in the last line, we have used $-H_2(\epsilon) \geq -nH_2(\epsilon)$. Now, we come to the question of finding the cost of decohering operation, i.e., the entropy that we have injected in the system. For this (as in the definition), we will consider the purification of ρ which is given by ψ such that $\rho^{\otimes n} = \mathrm{Tr}_Z(|\psi\rangle\langle\psi|^{\otimes n})$. Let us define

$$\Omega_{S^n Z^n} := (\mathbb{I}_Z^{\otimes n} \otimes \mathcal{R})[|\psi\rangle\langle\psi|^{\otimes n}]. \tag{3.12}$$

Since, $\mathcal{R}$ does not act on the reference system Z, $H(\Omega_{Z^n}) = H\left(\mathrm{Tr}_S(|\psi\rangle\langle\psi|^{\otimes n})\right) = H(\rho^{\otimes n}) = nH(\rho)$. Now,

$$\begin{aligned} H_e(\mathcal{R}, \rho^{\otimes n}) = H(\Omega_{S^n Z^n}) &\geq H(\Omega_{S^n}) - H(\Omega_{Z^n}) \\ &\geq H(\mathcal{R}[\rho^{\otimes n}]) - nH(\rho), \end{aligned} \tag{3.13}$$

where in the first line, we have made use of the Araki-Lieb inequality [2, 3, 146]. Using Eq. (3.11) in the above equation, we get

$$\begin{aligned} H_e(\mathcal{R}, \rho^{\otimes n}) &\geq n[H(\rho^{(\mathrm{d})}) - H(\rho) - \epsilon \log d - H_2(\epsilon)] \\ &= n[\mathscr{C}_r(\rho) - \epsilon \log d - H_2(\epsilon)]. \end{aligned} \tag{3.14}$$

Therefore, in the asymptotic limit, the minimal entropy exchange is equal to the relative entropy of coherence as in Eq. (3.3) (see Fig. 1.2). This completes the proof of the Lemma 1.

Next we consider the question of cost of erasing coherence while the amount of noise injected into the system is quantified by $\log N$, where N is the number of unitaries in the ensemble comprising the ϵ-decohering map.

Lemma 2: For any state ρ and $\epsilon > 0$ there exists, for all sufficiently large n, a map $\mathcal{R}_c : \rho \mapsto \frac{1}{N}\sum_{i=1}^{N} U_i \rho U_i^\dagger$ on system with U_i being a unitary operator on the system, which ϵ-decoheres it, and with $\log N \leq n\left(\mathscr{C}_r(\rho) + \epsilon\right)$, where $\mathscr{C}_r(\rho)$ is the relative entropy of coherence of the state ρ. In the asymptotic limit, the minimal amount of noise as quantified by $\log N$, that is injected into the system is given by

$$\sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{n} \min\left\{\log N : \mathcal{R}_c \;\; \epsilon\text{-decohering}\right\} = \mathscr{C}_r(\rho). \tag{3.15}$$

Proof.—Let us consider n copies of the system in the state ρ. Also, consider the typical projector Π that projects the system onto its typical subspace. Let $\tilde{\rho} = \Pi\rho^{\otimes n}\Pi$. By definition of the typical projector, we have $\mathrm{Tr}(\Pi\rho^{\otimes n}) \geq (1-\epsilon)$. Therefore, using the "gentle operator lemma" [3] (see also Appendix B, Sec. B.1), we have

$$||\rho^{\otimes n} - \tilde{\rho}||_1 \leq 2\sqrt{\epsilon}. \tag{3.16}$$

Now consider an ensemble of unitaries with some probability density function $p(dU)$, i.e. $\{U, p(dU)\}$ such that, for any state γ on the typical subspace of $\rho^{\otimes n}$, $\int_U p(dU) U\gamma U^\dagger = \frac{1}{D}\mathcal{I}_\Pi$, where $D = 2^{n(H(\rho)-\epsilon)}$ and $\mathcal{I}_\Pi$ is the identity supported on the typical subspace of the system. Therefore, we have

$$\int_U p(dU) U\tilde{\rho}U^\dagger = \frac{1}{D}\mathcal{I}_\Pi := \tau \geq \frac{1}{D_d}\mathcal{I}_\Pi, \tag{3.17}$$

where $D_d = 2^{n(H(\rho^{(\mathrm{d})})+\epsilon)}$. Then, using the "operator Chernoff bound" [147, 148] (see also Appendix B, Sec. B.2), we show that we can select a subensemble of these unitaries which suffices the approximation. To this end, we consider $X := DU\tilde{\rho}U^\dagger$ as random operators with the distribution $p(dU)$. Here $X \geq 0$. Using $\tilde{\rho} \leq \Pi/D$, we have $X = DU\tilde{\rho}U^\dagger \leq U\Pi U^\dagger \leq \mathcal{I}$. Now, the average value $\mathbb{E}X$ of the random operator X is given by

$$\mathbb{E}X = D\int_U p(dU)U\tilde{\rho}U^\dagger \geq \frac{D}{D_d}\mathcal{I}_\Pi = 2^{-n(\mathscr{C}_r(\rho)+2\epsilon)}\Pi, \tag{3.18}$$

where $\mathscr{C}_r(\rho)$ is the relative entropy of coherence of the state ρ. If $X_1, .., X_N$, where $X_i = DU_i\tilde{\rho}U_i^\dagger$ $(i = 1, .., n)$, are N independent realizations of X, then using the operator Chernoff bound, we have

$$\begin{aligned}\Pr\left((1-\epsilon)\mathbb{E}X \leq \frac{1}{N}\sum_{i=1}^{N} X_i \leq (1+\epsilon)\mathbb{E}X\right) \\ \geq 1 - 2\dim(\Pi)\exp[-\frac{N\epsilon^2}{4\ln 2}2^{-n(\mathscr{C}_r(\rho)+2\epsilon)}].\end{aligned} \tag{3.19}$$

For $N = 2^{n(\mathscr{C}_r(\rho)+3\epsilon)}$ or higher, we have the corresponding probability on LHS of Eq. (3.19) nonzero for sufficiently large n. For this case, we have $(1-\epsilon)\mathbb{E}X \leq \frac{1}{N}\sum_{i=1}^{N} X_i \leq (1+\epsilon)\mathbb{E}X$. This can be recast as $\left|\left|\frac{1}{N}\sum_{i=1}^{N} U_i\tilde{\rho}U_i^\dagger - \tau\right|\right|_1 \leq \epsilon$. Now, we have

$$\left|\left|\frac{1}{N}\sum_{i=1}^{N} U_i\rho^{\otimes n}U_i^\dagger - \tau\right|\right|_1 \leq \epsilon + ||\rho^{\otimes n} - \tilde{\rho}||_1 \leq \epsilon + 2\sqrt{\epsilon}. \tag{3.20}$$

Therefore, there indeed exists decohering map $\mathcal{R}_c$ that $(\epsilon + 2\sqrt{\epsilon})$-decoheres any state with, $N = 2^{n(\mathscr{C}_r(\rho)+3\epsilon)} \leq 2^{n\left(\mathscr{C}_r(\rho)+\epsilon+2\sqrt{\epsilon}\right)}$, i.e., $\log N \leq n\left(\mathscr{C}_r(\rho) + \epsilon + 2\sqrt{\epsilon}\right)$. Thus, in the asymptotic limit, the minimal cost of erasing coherence is given by Eq. (3.15) (see Fig. 1.2). This concludes the proof of the Lemma 2.

Note that we have not assumed any measure of coherence to start with, rather we have used two different quantifiers of the amount of noise, that are very important and well accepted in information theory (see also [74]). We find that the relative entropy of coherence emerges as the minimal amount of the noise that has to be added to the system to erase the coherence by considering our method of erasing via random unitary transformations. Thus, our result provides an operational interpretation of the relative entropy of coherence developed in the resource theory of coherence [47]. Moreover, our result is robust, i.e. even if we allow for nonzero errors in the erasing process, we still get the same answer in the asymptotic limit. Operational interpretations of various quantities in both classical and quantum information theory have been very striking with far-reaching impact on our understanding about the subject that has lead to explore different avenues. Operational interpretations of total and quantum correlations [74] are worth mentioning in this regard. However, computing the operational quantifiers is a formidable task in general. Thanks to the relative entropy of coherence that computing operational quantifier of coherence proposed by us is not a difficult task. Moreover, to the best of our knowledge, no other measure of coherence except the relative entropy of coherence, the coherence of formation [121] and the l_1 norm of coherence [149] is endowed with such an operational interpretation.

3.4 Chapter summary

To summarize, we have provided an operational quantifier of quantum coherence in terms of the amount of noise that is to be injected into a quantum system in order to fully decohere it. In the asymptotic limit, it is equal to the relative entropy of coherence provided one uses our method of erasing coherence via random unitary transformations. This provides the cost of erasing quantum coherence. It is worth mentioning that we have not assumed any of the measures of coherence to start with in order to prove our results. The relative entropy of coherence emerges naturally as the minimal erasing cost of coherence. Moreover, our result is robust, i.e., if we allow for nonzero error in the erasing process, it still gives the same answer in the asymptotic limit. In an independent work, Winter and Yang [121] have shown that the relative entropy of coherence, emerges as the asymptotic rate at which one can distill maximally coherent states. This is very surprising as the same quantity, namely, the relative entropy of coherence comes up from two (apparently) completely different tasks such as the erasure and distillation of coherence. The resource theory of coherence starts with the premise that the allowed operations are the incoherent ones and the free states are the incoherent states, and thereby proposes the relative entropy of coherence as a valid measure of coherence along with the other measures like l_1 norm of coherence. The formalism used in this chapter and in Ref. [121] is well established and has far reaching implications in providing operational meaning to a resource, similar to other resources like quantum entanglement and correlations, in general. In this regard, our results along with results of Ref. [121] further escalate the significance of the relative entropy of coherence as a bona fide measure of coherence.

In future one may ask the converse, i.e., in a complete protocol, what is the cost to keep a state coherent? Some partial results in a specific situation is provided in Ref. [150]. It is worth proving that whether this cost is also equal to the relative entropy of coherence of ρ, in the asymptotic limit. However, we leave it for future explorations. It is interesting to find a quantitative connection of our results to Landauer's erasure principle [67] along with its improved and generalized versions [151, 152]. Moreover, it will also be very interesting to further explore the quantitative relation between the no-hiding theorem [153, 154] and coherence erasure, both being very fundamental in their nature, as the no-hiding theorem applies to any process of hiding a quantum state, whether by randomization, thermalization or any other procedure. This will be the subject of future work. We hope that our results provide deep insights to the nature of coherence and interplay of information within the realm of quantum information and thermodynamics.

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CHAPTER 4

Maximally coherent mixed states: Complementarity between maximal coherence and mixedness

4.1 Introduction

Recent developments in modern science have shown that quantum coherence plays an important role in low temperature physics starting from the formulation of the basic laws of thermodynamics to work extraction [11–16, 33, 34, 49, 66, 155, 156]. Furthermore, it plays an important role in investigating nanoscale systems [38, 39] and understanding efficient energy transfer in complex biological systems [18–20, 40, 41]. In recent years, researchers have attempted to develop a framework to formalize the theory of quantum coherence within the realms of quantum information and quantum resource theories [10, 45–49, 58, 60, 129, 141–143]. The resource theories of coherence have been discussed briefly in chapter 2 of this thesis.

While dealing with quantum dynamical systems, a careful consideration of environmental noise and the unavoidable phenomenon of decoherence is needed as it is a well known fact that the noise is worst enemy in realizing the quantum technologies for practical uses. The presence of noise deteriorates the amount of information contained in a quantum state as measured by the purity, during the information processing tasks. Therefore, an effective characterization of decoherence requires the quantification of the purity or, its complementary property, the mixedness of a quantum state. The mixedness of a quantum state acts as an intuitive parameter to understand decoherence as noise tends to increase it. At this point, it becomes a natural question that: How does an important resource in quantum information theory trade against the mixedness of quantum systems? This question has attracted a lot of attention in the context of quantum entanglement and has led to the notion of the maximally entangled mixed states [86–90]. For these states, one cannot increase their entanglement further under any global unitary operation and the form of these states depend on the quantifiers used to measure entanglement and mixedness. These states have also been explored in the context of Gaussian quantum information theory [91–93].

In this chapter, we explore the limitations on processing of coherence in the noisy scenarios. In particular, we find analytical trade off relations between the coherence and the mixedness of a quantum systems at our disposal. For quantifiers of coherence, we use the quantifiers obtained in the resource theory of coherence based on incoherent operations [47]. These relations are then utilized to find the upper bound on the amount of coherence of a given mixed quantum state (the mixedness of the state is fixed) and vice-versa. Based on the above results, we derive a class of quantum states and call them "maximally coherent mixed states" (MCMS), that have maximal coherence, up to incoherent unitaries, for a fixed mixedness. Interestingly, unlike for the case of entanglement, the analytical form of MCMS is independent of measures employed to quantify the coherence and mixedness as confirmed for the l_1 norm, the relative entropy, and the geometric coherence (as measures of coherence) [23]. It is known that quantum interferometric setups [94, 95] can be utilized to measure the mixedness of a quantum system experimentally, our results provide a mathematically rigorous framework to experimentally measure the maximal coherence in a quantum state, without invoking the state tomography.

This chapter is organized as follows. In Sec. 4.2, we briefly discuss the quantification of coherence and mixedness in the realm of quantum information theory. In Sec. 4.3, we theorize the trade-off between coherence and mixedness in d-dimensional systems. In Sec. 4.4, we define a class of maximally coherent mixed states that satisfy a complementarity relation between coherence and mixedness. In Sec. 4.5, we investigate the allowed set of transformations within classes of fixed coherence or mixedness. We conclude with a discussion of the main results of this chapter in Sec. 4.6.

4.2 Quantifying coherence and mixedness

In this section we present a brief overview of the concepts of quantum coherence and mixedness of quantum systems. To characterize the coherence in a quantum system, we follow the theoretical approach developed in Ref. [47]. All mathematical formulations and results that are subsequently presented and discussed are valid within the framework of the above theory of quantum coherence.

4.2.1 Quantum coherence

Quantum coherence, an essential feature of quantum mechanics arising from the superposition principle, is inherently a basis dependent quantity. Therefore, any quantitative measure of it must depend on a reference basis. We have discussed the resource theories of coherence in chapter 1 and introduced the l_1 norm, the relative entropy of coherence [47] and a few other measures of coherence. Relevant to this chapter is one more quantifier of coherence—the geometric coherence [23]—and is given by $\mathscr{C}_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho, \sigma)$, where $\mathcal{I}$ is the set of all incoherent states and $F(\rho, \sigma) = \left(\mathrm{Tr}[\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}]\right)^2$ is the fidelity of the states ρ and σ. It is important to note that quantum coherence, by definition, is not invariant under general unitary operation but does remain unchanged under incoherent unitaries. Furthermore, the maximally coherent pure state is defined by $|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$, for which $\mathscr{C}_{l_1}(|\psi_d\rangle \langle\psi_d|) = d - 1$ and $\mathscr{C}_r(|\psi_d\rangle \langle\psi_d|) = \ln d$.

4.2.2 Mixedness

For every quantum state, the ubiquitous interaction with environment or decoherence affects its purity. Noise introduces mixedness in the quantum system leading to loss of information, and hence, its characterization is an important task in quantum information protocols. The mixedness, which represents nothing but the disorder in the system, can be quantified in terms of entropic functionals, such as the linear and the von Neumann entropy of the quantum state. For an arbitrary d-dimensional state, the mixedness, based on the normalized linear entropy [84], is given by

$$M_l(\rho) = \frac{d}{d-1}\left(1 - \mathrm{Tr}\rho^2\right). \tag{4.1}$$

Therefore, for each quantum system, mixedness varies between 0 and 1, i.e. $0 \leq M_l(\rho) \leq 1$. Furthermore, since $\mathrm{Tr}\rho^2$ describes the purity of quantum system, mixedness expectedly emerges as a complementary quantity to the purity of the given quantum state. The other operational measure of mixedness of a quantum state ρ is the von Neumann entropy, $H(\rho) = -\mathrm{Tr}(\rho \ln \rho)$. Moreover, in a manner similar to quantum coherence, a geometric measure of mixedness can also be defined, which is given by $M_g(\rho) := F(\rho, \mathbb{I}/d) = \frac{1}{d}\left(\mathrm{Tr}\sqrt{\rho}\right)^2$ and lies between 0 and 1.

4.3 Trade-off between quantum coherence and mixedness

In this section, we investigate the restrictions imposed by the mixedness of a system on the maximal amount of quantum coherence. We prove analytically, that there exists a trade-off between the two quantities and for a fixed amount of mixedness the maximal amount of coherence is limited. The results allow us to derive a class of states that are the most resourceful, in terms of quantum coherence, under a fixed amount of noise, characterized by its mixedness.

The important trade-off between quantum coherence, as quantified by the l_1 norm, and mixedness, in terms of the normalized linear entropy, is captured by the following theorem.

Theorem 4. *For any arbitrary quantum system, ρ, in d-dimensions, the amount of quantum coherence, $\mathscr{C}_{l_1}(\rho)$, in the state is restricted by the amount of mixedness, $M_l(\rho)$, through the the inequality*

$$\frac{\mathscr{C}_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) \leq 1. \tag{4.2}$$

Proof. Using the parametric form of an arbitrary density matrix, the state of a d-dimensional quantum system can be written in terms of the generators, $\hat{\Lambda}_i$, of $SU(d)$ [2, 157–160], as

$$\rho = \frac{\mathbb{I}}{d} + \frac{1}{2}\sum_{i=1}^{d^2-1} x_i \hat{\Lambda}_i, \tag{4.3}$$

where $x_i = \mathrm{Tr}[\rho\hat{\Lambda}_i]$. The condition of positivity can be stated in terms of the coefficients of the characteristic equation for the density matrix ρ. Specifically, the Eq. (4.3) is positive iff all the coefficients of the polynomial $\det(\lambda\mathbb{I} - \rho) = \sum_{i=0}^{d}(-1)^i A_i \lambda^{d-i}$=0, $A_i \geq 0$ for $1 \leq i \leq d$ ($A_0 = 1$). This criterion can be verified simply by calculating traces of various powers of ρ [159, 160]. The generators $\hat{\Lambda}_i$ ($i = 1, 2, .., d^2 - 1$) satisfy (1) $\hat{\Lambda}_i = \hat{\Lambda}_i^\dagger$, (2) $\mathrm{Tr}(\hat{\Lambda}_i) = 0$, and (3) $\mathrm{Tr}(\hat{\Lambda}_i\hat{\Lambda}_j) = 2\delta ij$. These generators are defined by the structure constants f_{ijk} (a completely antisymmetric tensor) and g_{ijk} (a completely symmetric tensor), of Lie algebra $su(d)$ [158, 159]. The generators can be conveniently written as $\{\hat{\Lambda}_i\}_{i=1}^{d^2-1} = \{\hat{u}_{jk}, \hat{v}_{jk}, \hat{w}_l\}$. Here $\hat{u}_{jk} = (|j\rangle\langle k| + |k\rangle\langle j|)$, $\hat{v}_{jk} = -i(|j\rangle\langle k| - |k\rangle\langle j|)$, and $\hat{w}_l = \sqrt{\frac{2}{l(l+1)}}\sum_{j=1}^{l}(|j\rangle\langle j| - l|l+1\rangle\langle l+1|)$, where $j < k$ with $j, k = 1, 2, .., d$ and

$l = 1, 2, ..(d-1)$ [158, 159]. The generators can be labelled as

$$\{\hat{\Lambda}_1, .., \hat{\Lambda}_{\frac{(d^2-d)}{2}}, \hat{\Lambda}_{\frac{(d^2-d)}{2}+1}, .., \hat{\Lambda}_{(d^2-d)}, \hat{\Lambda}_{(d^2-d)+1}, .., \hat{\Lambda}_{(d^2-1)}\}$$
$$= \{\hat{u}_{12}, .., \hat{u}_{(d-1)d}, \hat{v}_{12}, .., \hat{v}_{(d-1)d}, \hat{w}_1, .., \hat{w}_{(d-1)}\}.$$

The l_1 norm of coherence of a d-dimensional system, given by Eq. (4.3), can be written as

$$\mathscr{C}_{l_1}(\rho) = \sum_{i=1}^{(d^2-d)/2} \sqrt{x_i^2 + x_{i+(d^2-d)/2}^2}. \tag{4.4}$$

Furthermore, the mixedness is given by

$$M_l(\rho) = \frac{d}{d-1}(1 - \mathrm{Tr}\rho^2) = 1 - \frac{d}{2(d-1)} \sum_{i=1}^{d^2-1} x_i^2. \tag{4.5}$$

Using the expressions for $\mathscr{C}_{l_1}(\rho)$ and $M_l(\rho)$, we obtain

$$\begin{aligned}
&\frac{\mathscr{C}_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) \\
&= \frac{1}{(d-1)^2}\left(\sum_{i=1}^{(d^2-d)/2}\sqrt{x_i^2 + x_{i+(d^2-d)/2}^2}\right)^2 + 1 - \frac{d}{2(d-1)}\sum_{i=1}^{d^2-1} x_i^2 \\
&= 1 - \frac{1}{(d-1)^2}\sum_{i=1}^{d^2-1} x_i^2 \\
&+ \frac{1}{(d-1)^2}\left(\left(\sum_{i=1}^{(d^2-d)/2}\sqrt{x_i^2 + x_{i+(d^2-d)/2}^2}\right)^2 - (\frac{d^2-d}{2} - 1)\sum_{i=1}^{d^2-1} x_i^2\right) \\
&= 1 - \frac{1}{(d-1)^2}\sum_{i=1}^{d^2-1} x_i^2 - \frac{((d^2-d)/2-1)}{(d-1)^2}\sum_{i=d^2-d}^{d^2-1} x_i^2 \\
&+ \frac{1}{(d-1)^2}\left(\left(\sum_{i=1}^{(d^2-d)/2}\sqrt{x_i^2 + x_{i+(d^2-d)/2}^2}\right)^2 - (\frac{d^2-d}{2} - 1)\sum_{i=1}^{d^2-d} x_i^2\right) \\
&\leq 1 - \frac{d}{2(d-1)}\sum_{i=d^2-d}^{d^2-1} x_i^2,
\end{aligned} \tag{4.6}$$

where, in the last step, we have used the inequality $2\sqrt{xy} \leq (x+y)$. Since $\frac{d}{2(d-1)}\sum_{i=d^2-d}^{d^2-1} x_i^2 \geq 0$, we have $\frac{\mathscr{C}_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) \leq 1$, which concludes our proof. □

Theorem 4 proves that the scaled coherence, $\frac{\mathscr{C}_{l_1}(\rho)}{(d-1)}$, of a quantum system with mixedness $M_l(\rho)$, is bounded to a region below the parabola $\frac{\mathscr{C}_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) = 1$ (see Fig. 4.1). The quantum states with (scaled) quantum coherence that lie on the parabola are the maximally coherent states corresponding to a fixed mixedness and vice-versa. The trade-off obtained between coherence and mixedness can be neatly presented for a qubit system. Let us consider an arbitrary single-qubit density matrix of the form

$$\rho = \begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix}. \tag{4.7}$$

The eigenvalues of the above density matrix are given by

$$\lambda_{\pm} = \left(1 \pm \sqrt{1 - 4[a(1-a) - 4|c|^2]}\right)/2. \tag{4.8}$$

The positivity and Hermiticity of the density matrix implies that $0 \leq a(1-a) - 4|c|^2 \leq 1/4$. Now, the mixedness of the state ρ is given by $M_l(\rho) = 4a(1-a) - 4|c|^2$. The l_1 norm of coherence is $\mathscr{C}_{l_1}(\rho) = 2|c|$. Using the expressions of coherence and mixedness, we obtain $\mathscr{C}_{l_1}^2(\rho) + M_l(\rho) = 4a(1-a)$. Since, $4a(1-a) \leq 1$, we have $\mathscr{C}_{l_1}^2(\rho) + M_l(\rho) \leq 1$, with the equality holding if and only if $a = 1/2$.

From Theorem 4, we know that the maximum coherence permissible in an arbitrary quantum state with a fixed mixedness, are the values that lie on the parabola $\frac{\mathscr{C}_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) = 1$. The same holds for the maximum mixedness allowed in a quantum state with fixed coherence (see Fig. 4.1). A natural question arises: What are the quantum states that correspond to the maximal coherence and satisfy the equality in Eq. (4.2)? The above question is addressed in the following section.

4.4 Maximally coherent mixed states and complementarity

Let us find the quantum states with maximal l_1 norm of coherence for a fixed amount of mixedness, say, M_f. For this, we need to maximize the coherence under the constraint that the mixedness M_f as quantified by normalized linear entropy, is invariant. Here we provide the form of maximally coherent mixed state for a general d-dimensional system.

Theorem 5. *An arbitrary d-dimensional quantum system with maximal coherence for a fixed mixedness, M_f, up to incoherent unitaries, is of the following form*

$$\rho_m = \frac{1-p}{d}\mathbb{I}_{d\times d} + p\,|\psi_d\rangle\langle\psi_d|, \tag{4.9}$$

where $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle$*, is the maximally coherent state in the computational basis,* $\mathbb{I}_{d\times d}$ *is the* d*-dimensional identity operator and the mixedness, in terms of normalized linear entropy, is equal to,* $M_f = 1-p^2$*.*

Proof. Using the parametric form of the density matrix given in Eq. (4.3), the expressions for coherence and mixedness of any d-dimensional system was obtained in Eqs. (4.4) and (4.5). To prove the above theorem, we seek the maximal coherence for a fixed mixedness, say M_f, i.e. we maximize the function $\mathscr{C}_{l_1}$, under the constraint

$$M_f = 1 - \frac{d}{2(d-1)}\sum_{i=1}^{d^2-1} x_i^2. \tag{4.10}$$

Hence, we need to maximize the Lagrange function

$$\mathcal{L} = \sum_{i=1}^{k/2}\sqrt{x_i^2 + x_{i+k/2}^2} + \lambda\left(1 - \frac{d}{2(d-1)}\sum_{i=1}^{k+d-1} x_i^2 - M_f\right), \tag{4.11}$$

where $D = d^2 - d$ and λ is the Lagrange multiplier. The stationary points, $\{x'_j\}$, of $\mathscr{C}_{l_1}(\rho)$ imply vanishing of

$$\frac{\partial \mathcal{L}}{\partial x_j}\Big|_{\{x'_j\}} = \begin{cases} \frac{x'_j}{\sqrt{{x'_j}^2 + {x'_{j+D/2}}^2}} - \frac{\lambda d}{d-1}x'_j\,, & \text{for } j \le D/2 \\ -\frac{\lambda d}{d-1}x'_j\,, & \text{for } j > D \end{cases}. \tag{4.12}$$

Therefore, we have $x'_j = 0$ for all $j > D$ and $\sqrt{{x'_j}^2 + {x'_{j+D/2}}^2} = \frac{d-1}{\lambda d}$ for $j \le D/2$. This implies that

$${x'_1}^2 + {x'_{1+D/2}}^2 = {x'_2}^2 + {x'_{2+D/2}}^2 = ... = {x'_{D/2}}^2 + {x'_D}^2 = \left(\frac{d-1}{\lambda d}\right)^2. \tag{4.13}$$

Putting these values of x'_j's in the constraint equation Eq. (4.10) we get, $\lambda = (d-1)/(2\sqrt{(1-M_f)})$. The positive value of λ is chosen because negative value leads to negative coherence, which is not desired. The value of coherence for the stationary states, is given by

$$\mathscr{C}_{l_1}(\rho) = \sum_{j=1}^{D/2} \sqrt{{x'_j}^2 + {x'_{j+D/2}}^2} = (d-1)\sqrt{(1-M_f)}. \tag{4.14}$$

This is the maximal value of coherence that a state can have for a fixed value of mixedness M_f. Therefore, the states with $x_j^2 + x_{j+D/2}^2 = 4(1-M_f)/d^2$ for $j \leq D/2$ and $x_j = 0$ for $j > D$, are the states that have maximum coherence for a given mixedness M_f. These states can be written as

$$\rho_m = \frac{\mathbb{I}}{d} + \frac{R}{2}\sum_{i=1}^{D/2}(\cos\theta_i \hat{\Lambda}_i + \sin\theta_i \hat{\Lambda}_{i+D/2}), \tag{4.15}$$

where $R = \frac{2\sqrt{(1-M_f)}}{d}$ and $\theta_i = \tan^{-1}(x_{i+D/2}/x_i)$. We observe that the diagonal part of these states is maximally mixed and the points, $\left\{x_i, x_{i+D/2}\right\}_{i=1}^{D/2}$, that define the offdiagonal elements, lie on the circle of radius R in the real $(x_i, x_{i+D/2})$–plane. An equivalent form of above states can be written, by identifying $\{\theta_1, .., \theta_{d-1}, \theta_d .., \theta_{\frac{(d^2-d)}{2}}\} = \{\phi_{12}, .., \phi_{1d}, \phi_{23}, .., \phi_{(d-1)d}\}$, as

$$\rho_m = \frac{\mathbb{I}}{d} + \frac{R}{2}\sum_{\substack{i,j=1\\ i<j}}^{d}(e^{i\phi_{ij}}|i\rangle\langle j| + e^{-i\phi_{ij}}|j\rangle\langle i|). \tag{4.16}$$

Now, the phases appearing in the off diagonal components can be removed by applying an incoherent unitary of the form $U = \sum_{n=1}^{d} e^{-i\gamma_n}|n\rangle\langle n|$, which keeps the coherence invariant. To this end by choosing $\phi_{ij} = \gamma_i - \gamma_j$ we get,

$$\rho_m = \frac{\mathbb{I}}{d} + \frac{R}{2}\sum_{\substack{i,j=1\\ i<j}}^{d}(|i\rangle\langle j| + |j\rangle\langle i|). \tag{4.17}$$

Now, setting $R = 2p/d$, we obtain the state given in Eq. (4.9). Therefore, up to incoherent unitary transformations, the states with maximal coherence for a fixed mixedness are those that take the form given by Eq. (4.9). This completes the proof. $\square$

For a single-qubit quantum system, the proof can be mathematically elaborated. For the density matrix, given in Eq. (4.7), we need to maximize the coherence under the constraint that, $M_f = 4a(1-a) - 4|c|^2$, is invariant. Hence, we need to maximize, $\mathscr{C}_{l_1}(\rho) = 2|c| + \lambda(4a(1-a) - 4|c|^2 - M_f)$, where λ is the Lagrange multiplier. Upon optimization, the stationary points are given by $a = 1/2$ and $|c| = 1/(4\lambda)$. Using constraint equation, we get $\lambda = \pm 1/(2\sqrt{1-M_f})$. Choosing the positive value of λ, we obtain $|c| = \sqrt{1-M_f}/2$. Thus, the maximum value of coherence is equal to, $\mathscr{C}_{l_1}(\rho) = \sqrt{1-M_f}$ and the corresponding states, are given by

$$\rho_m(\phi) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{1-M_f}\exp[i\phi] \\ \sqrt{1-M_f}\exp[-i\phi] & 1 \end{pmatrix}, \tag{4.18}$$

where ϕ is an arbitrary phase. The phase can be removed through incoherent unitaries which keeps the coherence invariant. The density matrix in Eq. (4.18), up to incoherent unitaries, has the form $\rho_m = \frac{1-p}{2}\mathbb{I}_{2\times 2} + p|\psi_2\rangle\langle\psi_2|$, where $|\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ is the maximally coherent state and $\mathbb{I}_{2\times 2}$ is the identity operator, in two dimensions. $p = \sqrt{1-M_f}$.

From Theorem 5, the l_1 norm of coherence of the maximally coherent mixed state, given in Eq. (4.9), is $\mathscr{C}_{l_1}(\rho_m) = (d-1)p$, and the mixedness is equal to $M_l(\rho_m) = \frac{d}{d-1}(1 - \text{Tr}[\rho_m^2]) = 1 - p^2$. Therefore, we obtain a *complementarity* relation between coherence and mixedness,

$$\frac{\mathscr{C}_{l_1}^2(\rho_m)}{(d-1)^2} + M_l(\rho_m) = 1, \tag{4.19}$$

which satisfy the equality in Eq. (4.2), and thus lie on the parabola, $\frac{\mathscr{C}_{l_1}^2(\rho_m)}{(d-1)^2} + M_l(\rho_m) = 1$, in the coherence-mixedness plane (see Fig. 4.1). We call the parametrized class of states, defined by Eq. (4.9), that satisfy the complementarity between coherence and mixedness,

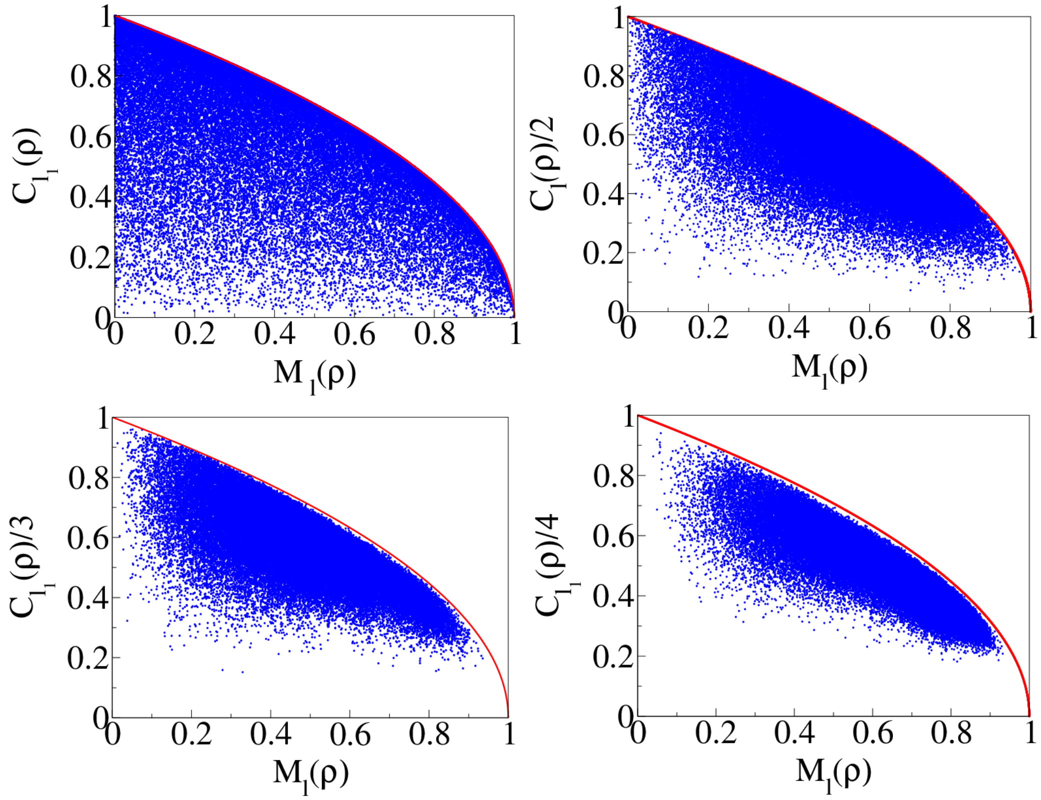

Figure 4.1: Plot showing the trade-off between the (scaled) coherence, $\mathscr{C}_{l_1}(\rho)/(d-1)$, and mixedness, $M_l(\rho)$, as obtained from Eq. (4.2). The redline represents the extremal parabola in Eq. (4.19), which corresponds to the MCMS class that satisfy a complementarity relation between coherence and mixedness. The figure plots the (scaled) coherence, along the Y-axis, and mixedness, along the X-axis, for 1×10^5 randomly generated states in d = 2, 3, 4, and 5 dimensions, using a specific mathematica package [162].

i.e., any change in coherence leads to a complementary change in mixedness, as the "maximally coherent mixed states". The MCMS class consists of pseudo-pure states, which are an admixture of the maximally coherent pure state and an incoherent state. Incidentally, states of the form given by Eq. (4.9), have also been discussed as states of fixed purity that maximize the sum of quantum uncertainties [161].

Similarly, one can derive a class of states with maximal mixedness for fixed coherence. Using an approach similar to Theorem 5, one can show that the set of maximally mixed coherent states also satisfy the complementarity relation and thus lie on the parabola given by Eq. (4.19), and hence, are of the same form as MCMS class.

Interestingly, we note that the form of MCMS remains the same if we employ a different set of measures for characterizing coherence and mixedness. For example, let us consider, the relative entropy of coherence, $\mathscr{C}_r(\rho)$, and von Neumann entropy $H(\rho)$, as our respective measures of coherence and mixedness. It can be shown, using the formalism employed in Theorems 4 and 5, that the trade-off relation, $\mathscr{C}_r(\rho) + H(\rho) \leq 1$ (for qubits), and the subsequent form of MCMS remains the same. Similarly, if one considers geometric coherence and geometric mixedness for qubit systems, as the measures of coherence and mixedness, one can obtain an identical trade off relation between the two quantities. To elaborate, the analytical form of geometric coherence for any arbitrary qubit state (Eq.(4.7)) is given by [23],

$$\mathscr{C}_g(\rho) = \frac{1}{2}[1 - \sqrt{1 - 4|c|^2}], \tag{4.20}$$

where c is the offdiagonal element of the qubit density matrix ρ in the computational basis. Further, for an arbitrary qubit state, the geometric mixedness is given by

$$M_g(\rho) = \frac{1}{2}[1 + \sqrt{4(a(1-a) - |c|^2)}]. \tag{4.21}$$

From Eq. (4.20) and Eq. (4.21), we have

$$\begin{aligned} \mathscr{C}_g(\rho) + M_g(\rho) &= 1 + \frac{1}{2}[\sqrt{4(a(1-a) - |c|^2)} - \sqrt{1 - 4|c|^2}] \\ &\leq 1, \end{aligned} \tag{4.22}$$

where in the last line we have used the fact that $4a(1-a) \leq 1$. Hence, we observe that the trade off relation is the same in Theorem 4. For arbitrary qubit systems, the form of MCMS, given in Eq. (4.9), remains the same for the geometric coherence and geometric mixedness considered as the measures of coherence and mixedness, respectively, and the complementarity relation, $\mathscr{C}_g(\rho)+M_g(\rho) = 1$, is satisfied. Hence there is a strong sense of universality about the form of MCMS, within the framework of the considered theory of coherence, in contrast to the measure dependent class of maximally entangled mixed states derived in the context of entanglement theory [86–90]. However, the form of MCMS for geometric coherence in general qudit systems needs to be further investigated. We note that the question of universality of the class MCMS for all equivalent sets of measures for coherence and mixedness, in any dimension, is still open.

4.5 Transformations within classes of state

The trade-off between coherence and mixedness, as established in Theorem 4, along with the complementarity relation given by Eq. (4.19) for MCMS class, lead to the question of convertibility within the classes of fixed mixedness or coherence. In other words, given a class of states with fixed mixedness what are the transformations that allow one to vary the coherence, while keeping the mixedness invariant, or vice-versa. The importance of transformation and interconversion between classes of states lies in the predominant role it plays in resource theories [10, 58, 60] and its central status in the formulation of the second law(s) of thermodynamics in quantum regime [11, 15, 16, 33, 34, 49, 66]. In this section, we investigate the set of operations that allow for such transformations for qubit states. Here, we exclusively consider the l_1 norm of coherence and normalized linear entropy as the measures of coherence and mixedness, respectively.

4.5.1 States with fixed coherence

For a fixed value of coherence, say α, in a fixed reference basis, say the computational basis, the states with varying mixedness, up to incoherent unitaries, are given by

$$\rho(a) = \begin{pmatrix} a & \alpha \\ \alpha & 1-a \end{pmatrix}. \tag{4.23}$$

Now, let us consider two states, $\rho(a_1)$ and $\rho(a_2)$, that have the same coherence but different mixedness. For the conditions, $(1-a_1) \geq a_2 \geq a_1$ or $(1-a_1) \leq a_2 \leq a_1$, the inequality, $a_1(1-a_1) \leq a_2(1-a_2)$ is satisfied. For this case, it is easy to see that $\rho(a_2)$ is majorized [3, 163–166] by $\rho(a_1)$, i.e., $\rho(a_2) \prec \rho(a_1)$. Therefore, using Uhlmann's theorem [3, 164–166], we can write

$$\rho(a_2) = \sum_i p_i U_i \rho(a_1) U_i^\dagger, \tag{4.24}$$

where U_i's are unitaries and $p_i \geq 0$, $\sum_i p_i = 1$. For qubit case, to keep the coherence invariant, we only allow incoherent unitaries. In the following, we shall see that the map,

$$\Phi[\rho] = p\rho + (1-p)\sigma_x \rho \sigma_x, \tag{4.25}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is sufficient to convert the state from $\rho(a_1)$ to $\rho(a_2)$, keeping the coherence unchanged. Specifically, we can achieve $\rho(a_2)$ from $\rho(a_1)$ using Eq. (4.25), by setting $p = (1-a_1-a_2)/(1-2a_1)$, which is a valid probability for the case we are considering. Similarly, in the opposite case with the conditions $(1-a_2) \geq a_1 \geq a_2$ or $(1-a_2) \leq a_1 \leq a_2$, one can find a similar map, as in Eq. (4.25), from $\rho(a_2)$ to $\rho(a_1)$.

Therefore, given two qubit density matrices ρ and σ with the same coherence, if $\rho \prec \sigma$ ($\sigma \prec \rho$), then there will always exist a probability distribution and incoherent unitaries, leading to a transformation $\sigma \to \rho$ ($\rho \to \sigma$). An interesting observation of the above analysis arises from considering maps related to open quantum systems. For noisy operations, for example the maps in Eq. (4.25), the transformation between states with the same coherence is reminiscent of the phenomenon of freezing of quantum coherence [129].

4.5.2 States with fixed mixedness

In the same vein, we explore the transformations which convert one state to other with the same mixedness, but varying amount of coherence. The states of the form

$$\rho(a) = \begin{pmatrix} a & \sqrt{\frac{4a(1-a)-M}{4}} \\ \sqrt{\frac{4a(1-a)-M}{4}} & 1-a \end{pmatrix}, \tag{4.26}$$

have the same mixedness M but can have different coherences. Now, let us consider two different states $\rho(a_1)$ and $\rho(a_2)$. Since, these states have same mixedness, and hence same eigenvalues, they must be related to each other by a unitary similarity transformation. This similarity transformation can be easily found, once we get the eigenvectors of both the states. Let $\rho(a_1)\,|e_i^{(1)}\rangle = \lambda_i\,|e_i^{(1)}\rangle$ and $\rho(a_2)\,|e_i^{(2)}\rangle = \lambda_i\,|e_i^{(2)}\rangle$ $(i = 1, 2)$. Now, the unitary similarity transformation S, such that $\rho(a_2) = S\rho(a_1)S^\dagger$ can be obtained from the definition $S\,|e_i^{(1)}\rangle = |e_i^{(2)}\rangle$. Thus, for two states of given fixed mixedness, one can always find a reversible similarity transformation between them. For an example, consider two states,

$$\rho_1 = \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & 0.7 \end{pmatrix}; \quad \rho_2 = \begin{pmatrix} 0.9 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}, \tag{4.27}$$

of the mixedness $M = 0.2$. The similarity transformation from ρ_2 to ρ_1, i.e., $\rho_2 = S\rho_1 S^\dagger$, using eigenvectors of both the states, is given by $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, which is a coherent unitary. In general, the state with identical mixedness but with varying coherence are connected through coherent unitaries.

4.6 Chapter summary

In this chapter, we show that there exists an intrinsic trade-off between the resourcefulness and the degree of noise in an arbitrary d-dimensional quantum system, as quantified by its coherence and mixedness, respectively. The obtained results are important from the perspective of resource theories as it allows us to quantify the maximal amount of coherence that can be harnessed from quantum states with a predetermined value of mixedness. Thus, we are able to analytically derive a class of maximally coherent mixed states, up to incoherent unitaries, that satisfy a complementarity relation between coherence and mixedness, in any quantum system. Due to the experimental ease with which the measurement of purity is feasible [95], our results can be utilized to experimentally determine the maximal l_1 norm of coherence for any general d-dimensional quantum state. For qubit systems, the above conclusions can also be extended to the relative entropy and geometric measures of coherence. Importantly, the theoretical formulation and results provided in this chapter are valid within the framework of the resource theory of coherence, as defined in [47], and cannot be mathematically extended directly to the quantification of coherence based on the theory of asymmetry [46]. Developing a framework that can operationally connect the two resource-theoretical perspectives is an important direction for future research.

The results presented in this chapter provide interesting insights on other aspects of the theory of coherence and may help in understanding the connection between the resource theories of coherence and entanglement. It was shown in a recent paper [23], that the maximum amount of entanglement that can be created between a system and an incoherent ancilla, via incoherent operations, is equal to the coherence present in the system. Using the formalism presented in [23] and the complementarity relations derived in this chapter, one can prove that the maximum entanglement that can be created between a quantum system and an incoherent ancilla, via incoherent operations, is bounded from above by the mixedness present in the system. Another significant aspect of the results is to address the question of order and interconvertibility between classes of quantum states, which is the fundamental premise for developing quantum resource theory and thermodynamics. Our analysis shows that for qubit systems with a fixed coherence, majorization provides a total order on the states based on their degree of mixedness, while for fixed mixedness, all the qubit states with varying degree of coherences are interconvertible. As a future direction, it will be very interesting to investigate if there exists such a total order in d-dimensional states with fixed coherence based on their degree of mixedness. We note that the total order on the states is only possible for a specific class of states and provided one works within the framework of the resource theory of coherence considered in our study. It is known that total order between states of fixed coherence is not possible within the resource theory of asymmetry [12, 45, 58].

To summarize, the present chapter deals with an important aspect of quantum physics, in particular, it addresses the question of how much a resource can be extracted from any arbitrary quantum system subjected to decoherence. We prove that there is a fundamental limit on the amount of coherence that can be extracted from mixed quantum systems and also derive the class of states that are most resourceful under decoherence. The results presented in this chapter provide impetus and new directions to the study of important physical quantities in open quantum systems and the effect of noise on quantum resources.

This chapter is based on the following paper:

1. *Maximally coherent mixed states: Complementarity between maximal coherence and mixedness*,

   U. Singh, M. N. Bera, H. S. Dhar, and A. K. Pati, Phys. Rev. A **91**, 052115 (2015).

CHAPTER 5

Catalytic coherence transformations

5.1 Introduction

Quantum resource theories [10, 85] have been a corner stone to the development and quantitative understanding of various physical phenomena in quantum physics and quantum information theory. For an introduction to resource theories see chapter 1 and references therein. The main advantages of having a resource theory for some physical quantity of interest are—they provide a succinct understanding of various physical processes and operational quantification of the relevant physical quantities (deemed as resources). Therefore, one of the main goals of a resource theory is to give a detailed account of physical processes allowed within the set of free operations and unfold the structure underlying these processes. The notion of catalysis is very fruitful to this goal—catalysts allow the otherwise impossible physical processes within the set of allowed operations and are not consumed in anyway during the process. The term catalyst is used in similar spirit as in the field of chemical sciences (see Fig. 1.2). Therefore, various catalysts are readily available at one's disposal. In the context of resource theories, catalysts have been employed in the resource theory of entanglement [96, 97], thermodynamics [11], and very recently to decoupling of quantum information [167] and all these have yielded surprising results. Notably, in the resource theory of thermodynamics, the possibility of catalysis [11] resulted in many second laws of thermodynamics at very low temperatures compared to the single second law in the macroscopic thermodynamics [98]. Moreover, the possibility *self catalysis* is explored in the context of entanglement [99].

Ref. [66] is the first one to introduce the possibility of catalysis in the resource theories of coherence and catalysis is developed since then (see Refs. [100, 101], also see Fig. 1.2). The aim of this chapter is to uncover the intricate structure underlying the coherence transformations (physical processes involving coherence) further using only incoherent operations as these are the only allowed operations in the resource theory of coherence based on incoherent operations. We consider the possibility of catalysis in the deterministic and stochastic scenarios both. We find the necessary and sufficient conditions for transformations among pure quantum states under incoherent operations both in the deterministic and stochastic scenarios considering the presence and absence of catalysts. We prove that the simultaneous decrease of a family of Rényi entropies of the diagonal parts of the states under consideration in a fixed basis are the necessary and sufficient conditions for deterministic pure state transformations in the presence of catalysts. This result is very similar in nature to the many second laws of quantum thermodynamics [11]. In the case of pure state stochastic coherence transformations, we find the necessary and sufficient conditions for the enhancement of the optimal probability of conversion with the aid of catalysts. Moreover, when a given pure state transformation is not possible even catalytically, we consider the possibility of entanglement assisted coherence transformations and find the necessary and sufficient conditions for the same. Thus, this chapter is a significant contribution to a complete theory of quantum coherence based on incoherent operations [47] and will be relevant in the scenarios where processing of coherence is limited by additional thermodynamical restrictions and in the context of *single-shot* information theory [102].

This chapter is organized as follows. We start with a discussion on interconversion of quantum states under incoherent operations, measures of coherence and coherence transformations for pure states along with some other preliminaries in Sec. 5.2. In Sec. 5.3, we discuss and obtain various results on catalytic coherence transformations under deterministic and stochastic scenarios. We present the necessary and sufficient conditions for catalytic coherence transformations in Sec. 5.4. We then go beyond catalytic coherence transformations to entanglement assisted incoherent transformations in Sec. 5.5. Finally, we conclude in Sec. 5.6 with overview and implications of the results presented in this chapter. The Appendix C lists some useful results that are obtained earlier by other researchers.

5.2 Preliminaries

***Interconversion of quantum states, the set of allowed operations.*–**The notion of interconvertibility of quantum states is desirable in many situations. For example, if we have a reference quantum state that serves as a basic unit of certain resource such as coherence or entanglement (just like Kilogram serves as a basic unit for mass), then one would like to convert any other given state to the reference state and in this way one can estimate the amount of resource in a given state. The distillable entanglement [168–170] of a bipartite quantum state is one of the measures of entanglement that is obtained via converting the state into the maximally entangled state. The other examples include the distillable coherence and coherence of formation [121] (for single quantum systems), and entanglement of formation [171, 172] (for bipartite quantum systems). We would like to emphasize here that the conversion from one state to the other is achieved by employing the relevant set of allowed operations. As mentioned in chapter 1, there is no common agreement for the definition of quantum coherence and we have more than one resource theories of coherence. We consider the resource theory of coherence based on the set of incoherent operations. In this resource theory of coherence, the set of incoherent operations is the allowed set of operations and any interconversion among quantum states is effected via operations from this set only. In quantum theory, a physically admissible operation Φ is a linear completely positive and trace-preserving map. Such a map Φ can be expressed by a set of Kraus operators $\{K_n\}_{n=1}^N$ such that $\Phi(\rho) = \sum_{n=1}^N K_n \rho K_n^\dagger$, with $\sum_{n=1}^N K_n^\dagger K_n = \mathbb{I}$ [2]. However, as mentioned, in the resource theory of coherence, the allowed operations are only the *incoherent operations*. An operation Φ_I is called an incoherent operation if the Kraus operators $\{K_n\}$ of Φ_I are such that $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$. Here $\mathcal{I}$ is the set of all incoherent states. Given a fixed reference basis, say $\{|i\rangle\}$, any state which is diagonal in the reference basis is called an incoherent state. Let us recall that the bona fide quantifiers of coherence include the l_1 norm of coherence, the relative entropy of coherence [47] and the Rényi entropies for certain range of the Rényi index [101]. For a pure state $|\psi\rangle$, the relative entropy of coherence $\mathscr{C}_r(|\psi\rangle)$ becomes the von Neumann entropy of its diagonal part in the fixed reference basis, i.e., $\mathscr{C}_r(|\psi\rangle) = H(\psi^{(\mathrm{d})})$, where H is the von Neumann entropy and $\psi^{(\mathrm{d})}$ is the diagonal part of the state $|\psi\rangle$ in a fixed reference basis.

***Catalysis*.–**Just like the concept of catalysis in chemical reactions (conversion of a mixture of compounds into mixture of other compounds with the aid of a catalyst), there exists a similar concept in the context of interconversion of quantum states. Let us consider that we need a conversion of an initial state $|\psi_1\rangle$ into a final state $|\psi_2\rangle$ of a quantum system $\mathcal{H}$ by using only the restricted class of operations and assume further that this conversion is not possible. Now, if there exists a pure state $|\phi\rangle$ of the same system $\mathcal{H}$ or any other ancillary system $\mathcal{K}$ such that $|\psi_1\rangle \otimes |\phi\rangle$ can be transformed into $|\psi_2\rangle \otimes |\phi\rangle$ by using only the restricted class of operations, then such a transformation is called a catalytic transformation and $|\phi\rangle$ is called as a catalyst for the transformation $|\psi_1\rangle \rightarrow |\psi_2\rangle$. The state $|\phi\rangle$, just like a catalyst in a chemical process, does not change after the transformation (also see Fig. 4.1). It is also possible that n copies of same initial state $|\psi_1\rangle$ can act as a catalyst for the transformation $|\psi_1\rangle \rightarrow |\psi_2\rangle$, i.e., despite the impossibility of the transformation $|\psi_1\rangle \rightarrow |\psi_2\rangle$, the transformation $|\psi_1\rangle \otimes |\psi_1\rangle^{\otimes n} \rightarrow |\psi_2\rangle \otimes |\psi_1\rangle^{\otimes n}$ may be possible. Here, n is a positive integer and depends on the transformation under consideration. This kind of catalysis is dubbed as *self catalysis* and we elaborate on it further as we go along. For the resource theory of coherence, we take incoherent operations for the restricted class of operations in the above definition, and the transformations then are referred to as catalytic coherence transformations.

Deterministic coherence transformations.–The possibility of transformation of a quantum system from one state with finite coherence to another state is determined by the majorization of the diagonal elements of the corresponding pure states in a fixed basis. This result was first proved in Ref. [100]. We state this result again for brevity:

Theorem 6 ([100]). *Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two pure states with $\psi_1^{(\mathrm{d})}$ and $\psi_2^{(\mathrm{d})}$ being the diagonal parts of $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively in a fixed reference basis. Then a transformation from the state $|\psi_1\rangle$ to $|\psi_2\rangle$ is possible via incoherent operations if and only if $\psi_1^{(\mathrm{d})}$ is majorized by $\psi_2^{(\mathrm{d})}$, i.e., $\psi_1^{(\mathrm{d})} \prec \psi_2^{(\mathrm{d})}$.*

For two probability vectors $p = \{p_i\}$ and $q = \{q_i\}$ $(i = 1, \ldots, d)$ arranged in decreasing order, p is said to be majorized by q, i.e., $p \prec q$ if $\sum_{i=1}^{l} p_i \leq \sum_{i=1}^{l} q_i$ for $l = 1, \ldots, d-1$ and $\sum_{i=1}^{d} p_i = 1 = \sum_{i=1}^{d} q_i$. Theorem 6 is the key ingredient for discussing the incoherent transformations between two pure states and the problem at the hand. However, we find that the proof of the converse part of the Theorem 6, given in Ref. [100], is true *only* for a specific class of incoherent operations. In the original proof, it was claimed that if a pure state $|\psi\rangle$ can be transformed to another pure state $|\phi\rangle$ through an incoherent channel then the elements K_n of the channel can always be written as [100]

$$K_n = P_{\pi_n} \begin{bmatrix} a_1^{(n)} & \delta_{1,i(2)} a_2^{(n)} & \delta_{1,i(3)} a_3^{(n)} \\ 0 & \delta_{2,i(2)} a_2^{(n)} & \delta_{1,i(3)} a_3^{(n)} \\ 0 & 0 & \delta_{3,i(3)} a_3^{(n)} \end{bmatrix}, \tag{5.1}$$

where δ_{ij} is the Kronecker delta function, $a_j^{(n)}$ $(j = 1, 2, 3)$ is the nonzero entry of K_n in the j^{th} column, $i(j)$ is the location of $a_j^{(n)}$ in the i^{th} row and was treated independent of n, and P_{π_n} is the permutation matrix. This means that the channel elements K_n considered in

Ref. [100] were of the same kind (upper triangular matrices) up to permutations. However, this is not the case always. For example let the initial state be $|\psi\rangle = \sum_{i=0}^{2} \sqrt{1/3}\,|i\rangle$, the final state be $|\phi\rangle = |0\rangle$, and define an incoherent operation $\Phi = \{K_n\}_{n=1}^{8}$ with the Kraus elements K_n being given by

$$K_1 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = K_2,$$

$$K_3 = \begin{bmatrix} -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = K_4,$$

$$K_5 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} = K_6 = K_7 = K_8.$$

It is easy to see $K_n\,|\psi\rangle = \alpha_n\,|\phi\rangle$ (for some α_n such that $\sum_n |\alpha_n|^2 = 1$), $\sum_{n=1}^{8} K_n^\dagger K_n = I$ and $\Phi(|\psi\rangle\langle\psi|) = \sum_{n=1}^{8} K_n|\psi\rangle\langle\psi|K_n^\dagger = |\phi\rangle\langle\phi|$. But, obviously, K_1 and K_5 are not related to each other via permutations and therefore, these are different kinds of upper triangular matrices. This means that $i(j)$ that was considered independent of n must depend on n, in general. This discrepancy has already been noticed in Ref. [101] and amended by considering $i(j)$ that explicitly depend on n.

It is important to note that for coherence transformations of pure states via incoherent operations, without loss of generality, we can always assume that the coefficients of the pure states in a fixed reference basis are all real, positive and arranged in the decreasing order [100]. Throughout this chapter we take this for granted and mention this at the places where we think it is necessary.

Stochastic coherence transformations.–We discussed above the necessary and sufficient conditions for the successful transformation of an initial state $|\psi_1\rangle$ into a final state $|\psi_2\rangle$

by incoherent operations under the name of deterministic coherence transformation (as the probability to achieve the transformation is 1). For dimensions strictly greater than two, because the majorization is only a partial order, Theorem 6 leaves us with the possibility that there can be a pair of states $|\psi_1\rangle$ and $|\psi_2\rangle$ such that neither $\psi_1^{(\mathrm{d})} \prec \psi_2^{(\mathrm{d})}$ nor $\psi_2^{(\mathrm{d})} \prec \psi_1^{(\mathrm{d})}$. These states will be called incomparable states in terms of coherence properties. To deal with these incomparable states, the stochastic transformations have been investigated in Refs. [101, 173–175]. Here, a stochastic coherence transformation means that for an incoherent operation Φ_I with Kraus operators $\{ K_n \}_{n=1}^{N}$, although Φ_I cannot transform $|\psi_1\rangle$ into $|\psi_2\rangle$, i.e., $|\psi_2\rangle \langle\psi_2| \neq \Phi_I[|\psi_1\rangle \langle\psi_1|]$, there may exist a subset $\{ K_i \}_{i=1}^{N'}$ of set $\{ K_n \}_{n=1}^{N}$ ($N' < N$) such that $K_i |\psi_1\rangle \propto |\psi_2\rangle$. The maximum value of probability of transforming the initial state into the finial state, i.e., $\langle\psi_1| \left(\sum_{i=1}^{N'} K_i^\dagger K_i\right) |\psi_1\rangle$, has been calculated in [101]. Here $\sum_{i=1}^{N'} K_i^\dagger K_i \neq \mathbb{I}$. Thus, deterministic coherence transformations are the stochastic coherence transformations with optimal probability of transformation being equal to 1.

5.3 Deterministic and stochastic catalytic coherence transformations

Catalysis under deterministic incoherent operations.– We know that there exists pair of incomparable quantum states such that any one of them cannot be transformed to another only using incoherent operations. Such examples can be constructed very easily. Let us consider a qutrit system with the states $|\psi_1\rangle = \sum_{i=0}^{2} \sqrt{\psi_1^i}\, |i\rangle$ and $|\psi_2\rangle = \sum_{i=0}^{2} \sqrt{\psi_2^i}\, |i\rangle$. Choose ψ_1^i and ψ_2^i such that $|\psi_1^0| \leq |\psi_2^0|$ and $|\psi_1^0| + |\psi_1^1| > |\psi_2^0| + |\psi_2^1|$. The diagonal parts of such states will never be majorized by one another. The specific examples are given in Table 5.1. Let us consider d-dimensional incomparable states $|\psi_1\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_1^i}\, |i\rangle$ and $|\psi_2\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_2^i}\, |i\rangle$. Despite the impossibility of transformation from $|\psi_1\rangle$ to $|\psi_2\rangle$ via incoherent operations, it is known that another auxiliary system with coherence (catalyst) can be used to make this transformation possible [100, 101] (via catalytic coherence transformations (see Fig. 4.1)). There are following general properties of catalytic coherence transformations [100]: (a) No incoherent transformation can be catalyzed by a maximally coherent state $|\psi_M\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$. (b) Two states are interconvertible, i.e., $|\psi_1\rangle \rightleftharpoons |\psi_2\rangle$, under catalytic coherence transformations *if and only if* they are equivalent up to a permutation of diagonal unitary transformations. (c) $|\psi_1\rangle \to |\psi_2\rangle$ under catalytic coherence transformations only if both $|\psi_1^0| \leq |\psi_2^0|$ and $|\psi_1^{d-1}| \geq |\psi_2^{d-1}|$ hold. However, the necessary and sufficient conditions are hitherto missing for catalytic coherence transformations. We provide these conditions for stochastic coherence transformations later in this section and for the deterministic coherence transformations for pure quantum states in the next section.

But before going any further, let us consider a specific example of a qutrit system in the state $|\psi_1\rangle = \sqrt{0.4}\,|0\rangle + \sqrt{0.4}\,|1\rangle + \sqrt{0.1}\,|2\rangle + \sqrt{0.1}\,|3\rangle$. We want to make the otherwise impossible transformation from $|\psi_1\rangle$ to $|\psi_2\rangle = \sqrt{0.5}\,|0\rangle + \sqrt{0.25}\,|1\rangle + \sqrt{0.25}\,|2\rangle$ via incoherent operations using a catalyst in state $|\phi\rangle$. It can be seen that we can choose $|\phi\rangle = \sqrt{0.6}\,|0\rangle + \sqrt{0.4}\,|1\rangle$. In this case, we have $|\psi_1\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |\phi\rangle$. Here ICO denotes the incoherent operation. It is important to note that the state $|\phi\rangle$ is not unique. For example in the above case $|\phi\rangle = \sqrt{0.62}\,|0\rangle + \sqrt{0.38}\,|1\rangle$ can also act as a catalyst. So it is a legitimate question to ask that what is the structure of the set of catalysts for a given catalytic transformation, i.e., for fixed $|\psi_1\rangle$ and $|\psi_2\rangle$, what is the set $\{|\phi\rangle\}$ such that $|\psi_1\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |\phi\rangle$? The following proposition answers this question for four dimensional systems.

Proposition 7. *Consider a four dimensional system with states* $|\psi_1\rangle = \sum_{i=1}^{4} \sqrt{\psi_1^i}\,|i\rangle$ *and* $|\psi_2\rangle = \sum_{i=1}^{4} \sqrt{\psi_2^i}\,|i\rangle$ *such that* $|\psi_1\rangle \nrightarrow |\psi_2\rangle$ *under incoherent operations. Without loss of generality we can assume that the coefficients* $\{\psi_1^i\}$, $\{\psi_2^i\}$ *are real and arranged in decreasing order. The necessary and sufficient conditions for the existence of a catalyst* $|\phi\rangle = \sqrt{a}\,|1\rangle + \sqrt{1-a}\,|2\rangle$ $(a \in (0.5, 1))$ *for these states are:* $\psi_1^1 \leq \psi_2^1$; $\psi_1^1 + \psi_1^2 >$

Table 5.1: Some examples of incomparable (via incoherent operations) coherent states in the computational basis.

$\|\psi_1\rangle$	$\|\psi_2\rangle$
$\sqrt{0.5}\|0\rangle+\sqrt{0.4}\|1\rangle+\sqrt{0.1}\|2\rangle$	$\sqrt{0.6}\|0\rangle+\sqrt{0.2}\|1\rangle+\sqrt{0.2}\|2\rangle$
$\sqrt{0.5}\|0\rangle+\sqrt{0.4}\|1\rangle+\sqrt{0.1}\|2\rangle$	$\sqrt{0.6}\|0\rangle+\sqrt{0.25}\|1\rangle+\sqrt{0.15}\|2\rangle$
$\sqrt{0.5}\|0\rangle+\sqrt{0.4}\|1\rangle+\sqrt{0.1}\|2\rangle$	$\sqrt{0.7}\|0\rangle+\sqrt{0.15}\|1\rangle+\sqrt{0.15}\|2\rangle$
$\sqrt{0.4}\|0\rangle+\sqrt{0.4}\|1\rangle+\sqrt{0.2}\|2\rangle$	$\sqrt{0.45}\|0\rangle+\sqrt{0.3}\|1\rangle+\sqrt{0.25}\|2\rangle$
$\sqrt{0.4}\|0\rangle+\sqrt{0.4}\|1\rangle+\sqrt{0.2}\|2\rangle$	$\sqrt{0.5}\|0\rangle+\sqrt{0.25}\|1\rangle+\sqrt{0.25}\|2\rangle$

$\psi_2^1+\psi_2^2$; $\psi_1^1+\psi_1^2+\psi_1^3 \leq \psi_2^1+\psi_2^2+\psi_2^3$, *and*

$$\max\left\{\frac{\psi_1^1+\psi_1^2-\psi_2^1}{\psi_2^2+\psi_2^3}, 1-\frac{\psi_1^4-\psi_2^4}{\psi_2^3-\psi_1^3}\right\} \leq a \leq \min\left\{\frac{\psi_2^1}{\psi_1^1+\psi_1^2}, \frac{\psi_2^1-\psi_1^1}{\psi_1^2-\psi_2^2}, 1-\frac{\psi_2^4}{\psi_1^3+\psi_1^4}\right\}. \tag{5.2}$$

Proof. For $|\psi_1\rangle = \sum_{i=1}^4 \sqrt{\psi_1^i}|i\rangle$, $|\psi_2\rangle = \sum_{i=1}^4 \sqrt{\psi_2^i}|i\rangle$ and $|\phi\rangle = \sqrt{a}|1\rangle + \sqrt{1-a}|2\rangle$, we can define $|\gamma_1\rangle_{AB} = \sum_{i=1}^4 \sqrt{\psi_1^i}|i\rangle|i\rangle$, $|\gamma_2\rangle_{AB} = \sum_{i=1}^4 \sqrt{\psi_2^i}|i\rangle|i\rangle$ and $|\eta\rangle_{AB} = \sqrt{a}|11\rangle + \sqrt{1-a}|22\rangle$. Then $\psi_1^{(d)} \otimes \phi^{(d)} \prec \psi_2^{(d)} \otimes \phi^{(d)}$ is equivalent to $\mathrm{Tr}_A(|\gamma_1\rangle\langle\gamma_1| \otimes |\eta\rangle\langle\eta|) \prec \mathrm{Tr}_A(|\gamma_2\rangle\langle\gamma_2| \otimes |\eta\rangle\langle\eta|)$. Now the proof of our proposition follows from the Theorem 25 of Appendix C which was proved in Ref. [176]. ■

It may be noted that based on the connections between the resource theories of coherence and entanglement, the results of the catalytic transformations in entanglement theory can always be carried over to the coherence theory. Also, it is noted that if for the states $|\psi_1\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_1^i}|i\rangle$ and $|\psi_2\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_2^i}|i\rangle$, $|\phi\rangle$ is a catalyst then $|\phi\rangle$ acts as a catalyst for the states $|\psi_1\rangle = \sum_{i=0}^{d} \sqrt{\tilde{\psi}_1^i}|i\rangle$ and $|\psi_2\rangle = \sum_{i=0}^{d} \sqrt{\tilde{\psi}_2^i}|i\rangle$, where $\tilde{\psi}_k^i = \psi_k^i$ for $k = 1,2$ and $i = 1,\cdots,d-2$. $\tilde{\psi}_k^{d-1} = \psi_k^{d-1} - \epsilon_k$ and $\tilde{\psi}_k^d = \epsilon_k$ for $k = 1,2$. For example, for the states $|\psi_1\rangle = \sqrt{0.4}|0\rangle + \sqrt{0.4}|1\rangle + \sqrt{0.1}|2\rangle + \sqrt{0.1}|3\rangle$ and $|\psi_2\rangle = \sqrt{0.5}|0\rangle + \sqrt{0.25}|1\rangle + \sqrt{0.25}|2\rangle$ the catalyst is $|\phi\rangle = \sqrt{0.6}|0\rangle + \sqrt{0.4}|1\rangle$. Now for the states $|\psi_1\rangle = \sqrt{0.4}|0\rangle + \sqrt{0.4}|1\rangle + \sqrt{0.1}|2\rangle + \sqrt{0.05}|3\rangle + \sqrt{0.05}|4\rangle$ and $|\psi_2\rangle = \sqrt{0.5}|0\rangle + \sqrt{0.25}|1\rangle + \sqrt{0.25}|2\rangle$ the catalyst can again be chosen as $|\phi\rangle = \sqrt{0.6}|0\rangle + \sqrt{0.4}|1\rangle$. Moreover, if $|\psi_1\rangle \nrightarrow |\psi_2\rangle$, it is possible that $|\psi_1\rangle \otimes |\psi_1\rangle^{\otimes N} \rightarrow$

$|\psi_2\rangle \otimes |\psi_1\rangle^{\otimes N}$ for some $N \geq 1$. This means that a state can act as catalyst for itself. For example, take states $|\psi_1\rangle = \sqrt{0.9}\,|0\rangle + \sqrt{0.081}\,|1\rangle + \sqrt{0.01}\,|2\rangle + \sqrt{0.009}\,|3\rangle$ and $|\psi_2\rangle = \sqrt{0.95}\,|0\rangle + \sqrt{0.03}\,|1\rangle + \sqrt{0.02}\,|2\rangle$. $|\psi_1\rangle$ acts as a catalyst here, i.e., $|\psi_1\rangle \nrightarrow |\psi_2\rangle$ but $|\psi_1\rangle \otimes |\psi_1\rangle \rightarrow |\psi_2\rangle \otimes |\psi_1\rangle$. Similarly, if we take $|\psi_1\rangle = \sqrt{0.9}\,|0\rangle + \sqrt{0.088}\,|1\rangle + \sqrt{0.006}\,|2\rangle + \sqrt{0.006}\,|3\rangle$ and $|\psi_2\rangle = \sqrt{0.95}\,|0\rangle + \sqrt{0.03}\,|1\rangle + \sqrt{0.02}\,|2\rangle$ the two copies of $|\psi_1\rangle$ act as a catalyst, i.e., $|\psi_1\rangle \nrightarrow |\psi_2\rangle$ but $|\psi_1\rangle \otimes |\psi_1\rangle^{\otimes 2} \rightarrow |\psi_2\rangle \otimes |\psi_1\rangle^{\otimes 2}$. These examples are taken from Ref. [99] which deals with *self catalysis* in entanglement theory.

Catalysis under stochastic incoherent operations.– Here we explore the possibility of transforming a pure state to another incomparable pure state using stochastic incoherent operations as we already know that for such a pair of states there doest not exist any deterministic incoherent operation that can facilitate this transformation. We consider the transformations both in presence and in absence of catalysts. It is obtained in Ref. [101] that for a pure state $|\psi\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_i}\,|i\rangle$, the functions $\mathscr{C}_l(\psi) = \sum_{i=l}^{d-1} |\psi_i|$, $l = 0, \cdots, d-1$ are valid coherence measures in the sense of the resource theory of coherence [47]. Moreover, in the absence of catalysts, the optimal probability $P\left(|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle\right)$ of converting a pure state $|\psi_1\rangle$ into $|\psi_2\rangle$ is given by [101]

$$P\left(|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle\right) = \min_{l\in[0,d-1]} \frac{\mathscr{C}_l(\psi_1)}{\mathscr{C}_l(\psi_2)}. \tag{5.3}$$

We first prove that the optimal probability of deterministic incoherent state transformations is always one. Consider a pair of pure states $|\psi_1\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_1^i}\,|i\rangle$ and $|\psi_2\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_2^i}\,|i\rangle$ such that $|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle$. From Theorem 6, if $|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle$, then $(\psi_1^0, \ldots, \psi_1^{d-1}) \prec (\psi_2^0, \ldots, \psi_2^{d-1})$. Thus $\sum_{i=0}^{l} \psi_1^i \leq \sum_{i=0}^{l} \psi_2^i$. Due to normalization, we have

$$\sum_{i=l}^{d-1} \psi_1^i \geq \sum_{i=l}^{d-1} \psi_2^i.$$

That is $\mathscr{C}_l(\psi_1) \geq \mathscr{C}_l(\psi_2)$ for all l values. Hence, $P(|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle) = 1$. We note that mere the presence of another quantum system (a catalyst) can enhance the optimal probability of transition given by Eq. (5.3). For example consider the states $|\psi_1\rangle = \sqrt{0.4}\,|0\rangle + \sqrt{0.4}\,|1\rangle + \sqrt{0.2}\,|2\rangle$ and $|\psi_2\rangle = \sqrt{0.5}\,|0\rangle + \sqrt{0.25}\,|1\rangle + \sqrt{0.25}\,|2\rangle$. Here $P\left(|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle\right) = 0.8$ and $P\left(|\psi_2\rangle \xrightarrow{\text{ICO}} |\psi_1\rangle\right) = 0.83$. Now consider another state $|\phi\rangle = \sqrt{0.6}\,|0\rangle + \sqrt{0.4}\,|1\rangle$. We have $P\left(|\psi_1\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |\phi\rangle\right) = 0.8$ and $P\left(|\psi_2\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_1\rangle \otimes |\phi\rangle\right) = 0.92$. Notice that $P\left(|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle\right) = 0.8$ is not increased by the use of $|\phi\rangle$. This is a consequence of our following proposition.

Proposition 8. *If, under the best strategy of ICO, $P\left(|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle\right)$ is equal to $|\psi_1^{d-1}|/|\psi_2^{d-1}|$, then this probability cannot be increased by the presence of any (catalyst) state. Here $|\psi_1\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_1^i}\,|i\rangle$ and $|\psi_2\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_2^i}\,|i\rangle$.*

Proof. If, under the best strategy of ICO, $P(|\psi_1\rangle \to |\psi_2\rangle)$ is equal to $|\psi_1^{d-1}|/|\psi_2^{d-1}|$, then for any catalyst state $|\phi\rangle = \sum_{i=1}^{m} \sqrt{\phi_i}\,|i\rangle$, the minimal coefficients of $|\psi_1\rangle \otimes |\phi\rangle$ and $|\psi_2\rangle \otimes |\phi\rangle$ are $\sqrt{\psi_1^{d-1}\phi_m}$ and $\sqrt{\psi_2^{d-1}\phi_m}$, respectively. Thus, $P(|\psi_1\rangle \otimes |\phi\rangle \to |\psi_2\rangle \otimes |\phi\rangle) = \min_{l\in[0,(d-1)m]} \frac{\mathscr{C}_l(\psi_1\otimes\phi)}{\mathscr{C}_l(\psi_2\otimes\phi)} \leq \frac{\mathscr{C}_{d-1,m}(\psi_1\otimes\phi)}{\mathscr{C}_{d-1,m}(\psi_2\otimes\phi)} = \frac{|\psi_1^{d-1}\phi_m|}{|\psi_2^{d-1}\phi_m|} = |\psi_1^{d-1}|/|\psi_2^{d-1}|$. ■

We note that the above proposition can be strengthened and we provide the necessary and sufficient conditions for the enhancement of the optimal probability for transformations under incoherent operations in the presence of catalysts as our next proposition.

Proposition 9. *For two pure states $|\psi_1\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_1^i}\,|i\rangle$ and $|\psi_2\rangle = \sum_{i=0}^{d-1} \sqrt{\psi_2^i}\,|i\rangle$ there exists a catalyst $|\phi\rangle$ such that $P(|\psi_1\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |\phi\rangle) > P(|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle)$, if and only if*

$$P\left(|\psi_1\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle\right) < \min\left\{\frac{|\psi_1^{d-1}|}{|\psi_2^{d-1}|}, 1\right\}.$$

Proof. The proof follows directly from Theorem 6 of main text and Theorem 26 of the Appendix C. ■

5.4 Necessary and sufficient conditions for deterministic catalytic coherence transformations

We know that under incoherent operations, in the absence of catalysts, the necessary and sufficient conditions for transforming a pure state $|\psi_1\rangle$ to another pure state $|\psi_2\rangle$ are given by Theorem 6, i.e., $\mathscr{C}_r(|\psi_1\rangle) \geq \mathscr{C}_r(|\psi_2\rangle)$. Here $\mathscr{C}_r(|\psi\rangle)$ denotes the relative entropy of coherence of $|\psi\rangle$ [47]. Now if we allow for catalysts, does the decrease of relative entropy of coherence, $\mathscr{C}_r(|\psi_1\rangle) \geq \mathscr{C}_r(|\psi_2\rangle)$, ensure existence of an incoherent operation that maps $|\psi_1\rangle$ to $|\psi_2\rangle$? In the following we prove that this is not the case, i.e., mere decrease of the relative entropy of coherence is not sufficient. We next characterize the necessary and sufficient conditions for catalytic coherence transformations between the initial state $|\psi_1\rangle$ and the target state $|\psi_2\rangle$.

Proposition 10. *For two pure states $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}(d)$, if the coefficients of $|\psi_1\rangle$, in a fixed basis, are all nonzero, then the necessary and sufficient conditions for catalytic coherence transformations are the simultaneous decrease of a family of Rényi entropies which are defined as $S_\alpha(\psi^{(\mathrm{d})}) = \mathrm{sgn}(\alpha) \ln\left(\mathrm{Tr}\left[\left(\psi^{(\mathrm{d})}\right)^\alpha\right]\right)/(1-\alpha)$. Here $\psi^{(\mathrm{d})}$ is the diagonal part of the pure state $|\psi\rangle$ and $\mathrm{sgn}(\alpha) = 1$ for $\alpha \geq 0$, and $\mathrm{sgn}(\alpha) = -1$ when $\alpha < 0$. More precisely, there exists a catalyst state $|\phi\rangle$ such that $|\psi_1\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |\phi\rangle$ if and only if the conditions*

$$\frac{\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right)}{|\alpha|} < \frac{\tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right)}{|\alpha|} \tag{5.4}$$

are satisfied simultaneously for all $\alpha \in (-\infty, +\infty)$, where $\tilde{S}_\alpha\left(\psi^{(\mathrm{d})}\right) = S_\alpha\left(\psi^{(\mathrm{d})}\right) - \ln d$. For $\alpha = 0$, $\tilde{S}_\alpha\left(\psi^{(\mathrm{d})}\right)/|\alpha| = \lim_{\alpha\to 0^+} \tilde{S}_\alpha\left(\psi^{(\mathrm{d})}\right)/|\alpha| = \sum_{i=1}^{d} \ln \psi_i^{(\mathrm{d})}/d$ where $\psi_i^{(\mathrm{d})}$ are components of $\psi^{(\mathrm{d})}$ in a fixed reference basis.

Proof. We use a result from Ref. [177] (which we restate as Lemma 27 in Appendix C for clarity and completeness) to prove our proposition. For $\alpha \neq \{0, 1\}$ note that

$\tilde{S}_\alpha\left(\psi^{(\mathrm{d})}\right)/|\alpha| = \ln A_\alpha\left(\psi^{(\mathrm{d})}\right)/(1-\alpha) + \ln d/\alpha(1-\alpha) - \ln d/|\alpha|$, where $A_\alpha\left(\psi^{(\mathrm{d})}\right) = \left(\frac{1}{d}\sum_{i=1}^{d}\left(\psi_i^{(\mathrm{d})}\right)^\alpha\right)^{1/\alpha}$ (as in Lemma 27). So for $\alpha \neq \{0,1\}$, the proof of our proposition follows from Lemma 27 and Theorem 6. Similarly, for $\alpha = 1$, the proof follows from Lemma 27 and Theorem 6. For $\alpha = 0$ the proof follows again by noting that

$$\lim_{\alpha\to 0^+}\frac{\tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right)}{|\alpha|} = \frac{1}{d}\sum_{i=1}^{d}\ln\psi_1^i = \ln A_0(\psi_1^{(\mathrm{d})}),$$

where, for any probability vector p, $A_0(p) = \left(\prod_{i=1}^{d} p_i\right)^{\frac{1}{d}}$, as in Lemma 27 and $\psi_1^{(\mathrm{d})} = \left(\psi_1^1,\ldots,\psi_1^d\right)^T$. This completes the proof of proposition. ■

We emphasize that Proposition 10 assumes that the initial state $|\psi_1\rangle$ must contain only nonzero entries. But this problem can be remedied by allowing slight perturbation to the initial state. Moreover, the strict inequality in Proposition 10 can be made nonstrict. In this view, we generalize Proposition 10 to the following proposition.

Proposition 11. *For two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, the following two conditions are equivalent:*

1. For a given pure state $|\psi_1\rangle$ there exists a state $|\psi_1^\varepsilon\rangle$ with $\varepsilon > 0$ and a catalyst state $|\phi\rangle$ such that (i) $||\,|\psi_1\rangle - |\psi_1^\varepsilon\rangle\,|| < \varepsilon$; (ii) $|\psi_1^\varepsilon\rangle\otimes|\phi\rangle \xrightarrow{\mathrm{ICO}} |\psi_2\rangle\otimes|\phi\rangle$.

2. For all $\alpha \in (-\infty,+\infty)$

$$\frac{\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right)}{|\alpha|} \leq \frac{\tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right)}{|\alpha|}. \tag{5.5}$$

Proof. We first prove the implication $1 \Rightarrow 2$. Although $|\psi_1^\varepsilon\rangle$ may have zero component, there always exists a state $|\psi_1^{\varepsilon'}\rangle$ close to $|\psi_1^\varepsilon\rangle$ with nonzero components only, which also satisfy (i) and (ii) in the condition 1. Thus, without loss of any generality, we can assume the components of $|\psi_1^\varepsilon\rangle$ are all nonzero. Since $|\psi_1^\varepsilon\rangle\otimes|\phi\rangle \xrightarrow{\mathrm{ICO}} |\psi_2\rangle\otimes|\phi\rangle$, then by

Proposition 10,

$$\frac{\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right)}{|\alpha|} < \frac{\tilde{S}_\alpha\left((\psi_1^\varepsilon)^{(\mathrm{d})}\right)}{|\alpha|}$$

for every $\varepsilon > 0$. Based on the continuity of functions $\tilde{S}_\alpha(\cdot)/|\alpha|$, we have $\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right)/|\alpha| \leq \tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right)/|\alpha|$.

Now we prove the implication $2 \Rightarrow 1$. For $|\psi_1\rangle = \sum_{i=1}^d \sqrt{\psi_1^i}\,|i\rangle$, let

$$|\psi_1^\varepsilon\rangle = \sum_{i=1}^d \sqrt{(1-\varepsilon)\psi_1^i + \varepsilon/d}\,|i\rangle\,.$$

Then $||\,|\psi_1\rangle - |\psi_1^\varepsilon\rangle\,|| \to 0$, when $\varepsilon \to 0$. Due to Lemma 28 of Appendix C, we know the functions $\tilde{S}_\alpha(\cdot)/|\alpha|$ are strictly Schur concave for all $\alpha \in (-\infty, \infty)$, thus $\tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right)/|\alpha| < \tilde{S}_\alpha\left((\psi_1^\varepsilon)^{(\mathrm{d})}\right)/|\alpha|$ for every $\varepsilon > 0$. As $\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right)/|\alpha| \leq \tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right)/|\alpha|$, we have

$$\frac{\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right)}{|\alpha|} < \frac{\tilde{S}_\alpha\left((\psi_1^\varepsilon)^{(\mathrm{d})}\right)}{|\alpha|}.$$

It is easy to see the coefficients of $|\psi_1^\varepsilon\rangle$ are all nonzero. Hence, by Proposition 10, there exists a catalyst in state $|\phi\rangle$ such that $|\psi_1^\varepsilon\rangle \otimes |\phi\rangle \xrightarrow{\mathrm{ICO}} |\psi_2\rangle \otimes |\phi\rangle$. This completes the proof. ∎

To elaborate more about the necessary and sufficient conditions for catalytic coherence transformations we consider various examples. Fig. 5.1 shows that for states $|\psi_1\rangle = \sqrt{0.4}\,|0\rangle + \sqrt{0.4}\,|1\rangle + \sqrt{0.1}\,|2\rangle + \sqrt{0.1}\,|3\rangle$ and $|\psi_2\rangle = \sqrt{0.5}\,|0\rangle + \sqrt{0.25}\,|1\rangle + \sqrt{0.25}\,|2\rangle$ a catalytic transformation is possible but for states $|\psi_1\rangle = \sqrt{0.5}\,|0\rangle + \sqrt{0.4}\,|1\rangle + \sqrt{0.1}\,|2\rangle$ and $|\psi_2\rangle = \sqrt{0.6}\,|0\rangle + \sqrt{0.25}\,|1\rangle + \sqrt{0.15}\,|2\rangle$ no catalytic transformation is possible. Moreover, using the similar techniques as in Ref. [11], we can remove the $(-\infty, 0)$ part with the help of another ancillary qubit.

Proposition 12. *For pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, the following two conditions are equiva-*

lent:

1. For a given pure state $|\psi_1\rangle$ there exist states $|\psi_1^\varepsilon\rangle \otimes |0^\varepsilon\rangle$ with $\varepsilon > 0$ and $|\phi\rangle$ such that (i) $||\,|\psi_1\rangle \otimes |0\rangle - |\psi_1^\varepsilon\rangle \otimes |0^\varepsilon\rangle\,|| < \varepsilon$; (ii) $|\psi_1^\varepsilon\rangle \otimes |0^\varepsilon\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |0\rangle \otimes |\phi\rangle$.

2. For all $\alpha \in [0, +\infty)$

$$\frac{\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right)}{|\alpha|} \leq \frac{\tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right)}{|\alpha|}. \tag{5.6}$$

Proof. $(1 \Rightarrow 2)$ Similar to the proof of Proposition 11, without loss of generality, we can assume the components of $|\psi_1^\varepsilon\rangle$ and $|0^\varepsilon\rangle$ are all nonzero. Since $|\psi_1^\varepsilon\rangle \otimes |0^\varepsilon\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |0\rangle \otimes |\phi\rangle$, then by Proposition 10,

$$\frac{\tilde{S}_\alpha\left((|\psi_2\rangle\langle\psi_2| \otimes |0\rangle\langle 0|)^{(\mathrm{d})}\right)}{|\alpha|} < \frac{\tilde{S}_\alpha\left((|\psi_1^\varepsilon\rangle\langle\psi_1^\varepsilon| \otimes |0^\varepsilon\rangle\langle 0^\varepsilon|)^{(\mathrm{d})}\right)}{|\alpha|}$$

for every $\varepsilon > 0$. Based on the continuity and additivity of $\frac{\tilde{S}_\alpha(\cdot)}{|\alpha|}$ we have $\frac{\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right)}{|\alpha|} \leq \frac{\tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right)}{|\alpha|}$.

$(2 \Rightarrow 1)$ For $|\psi_1\rangle = \sum_{i=1}^d \sqrt{\psi_1^i}\,|i\rangle$ let $|\psi_1^\varepsilon\rangle = \sum_{i=1}^d \sqrt{(1-\varepsilon)\psi_1^i + \varepsilon/d}\,|i\rangle$, and $|0^\varepsilon\rangle = \sqrt{1-\varepsilon/2}\,|0\rangle + \sqrt{\varepsilon/2}\,|1\rangle$. Then $||\,|\psi_1\rangle \otimes |0\rangle - |\psi_1^\varepsilon\rangle \otimes |0^\varepsilon\rangle\,|| \to 0$, when $\varepsilon \to 0$. Similar to proof of Proposition 11, it is easy to obtain

$$\frac{\tilde{S}_\alpha\left((|\psi_2\rangle\langle\psi_2| \otimes |0\rangle\langle 0|)^{(\mathrm{d})}\right)}{|\alpha|} < \frac{\tilde{S}_\alpha\left((|\psi_1^\varepsilon\rangle\langle\psi_1^\varepsilon| \otimes |0^\varepsilon\rangle\langle 0^\varepsilon|)^{(\mathrm{d})}\right)}{|\alpha|} \tag{5.7}$$

for all $\alpha \in [0, +\infty)$. In the case of $\alpha < 0$, due to the definition of $\tilde{S}_\alpha$, the left side of inequality 5.7 will be $-\infty$, and the right side will be finite. By Proposition 10, there exist catalyst state $|\phi\rangle$ such that $|\psi_1^\varepsilon\rangle \otimes |0^\varepsilon\rangle \otimes |\phi\rangle \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |0\rangle \otimes |\phi\rangle$. This completes the proof. ■

In fact, we need not worry about the rank of the initial state. If $|\psi_1\rangle$ can be transformed to $|\psi_2\rangle$ using the catalyst $|\phi\rangle$, then $\mathscr{C}_s(|\psi_2\rangle \otimes |\phi\rangle) \leq \mathscr{C}_s(|\psi_1\rangle \otimes |\phi\rangle)$ where $\mathscr{C}_s(|\psi\rangle) :=$

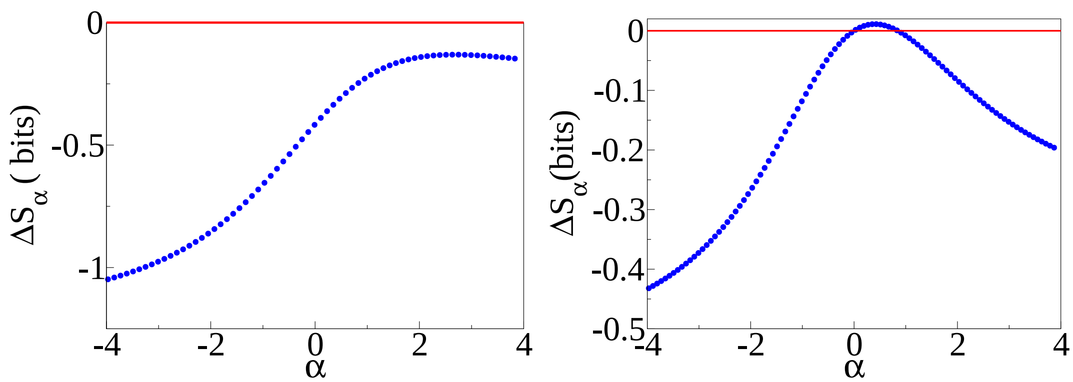

Figure 5.1: The plot shows the variation of $\Delta\tilde{S}_\alpha = \tilde{S}_\alpha(\psi_2^{(\mathrm{d})}) - \tilde{S}_\alpha(\psi_1^{(\mathrm{d})})$ as a function of α. In the left figure, we take $|\psi_1\rangle = \sqrt{0.4}\,|0\rangle + \sqrt{0.4}\,|1\rangle + \sqrt{0.1}\,|2\rangle + \sqrt{0.1}\,|3\rangle$ and $|\psi_2\rangle = \sqrt{0.5}\,|0\rangle + \sqrt{0.25}\,|1\rangle + \sqrt{0.25}\,|2\rangle$. Based on our Proposition 10, the transformation from $|\psi_1\rangle$ to $|\psi_2\rangle$ will be possible with the aid of a catalyst. In the right figure, we take $|\psi_1\rangle = \sqrt{0.5}\,|0\rangle + \sqrt{0.4}\,|1\rangle + \sqrt{0.1}\,|2\rangle$ and $|\psi_2\rangle = \sqrt{0.6}\,|0\rangle + \sqrt{0.25}\,|1\rangle + \sqrt{0.15}\,|2\rangle$. Because for certain values of α, $\Delta\tilde{S}_\alpha$ increases, from Proposition 10, there does not exist a catalyst that can allow the transformation from $|\psi_1\rangle$ to $|\psi_2\rangle$ in this case.

$\mathrm{Rank}(\psi^{(\mathrm{d})})$ and is a proper measure of coherence [178]. $\psi^{(\mathrm{d})}$ is the diagonal part of $|\psi\rangle$ in the fixed reference basis. This implies $\mathscr{C}_s(|\psi_2\rangle) \leq \mathscr{C}_s(|\psi_1\rangle)$, i.e., $\mathrm{Rank}(\psi_2^{(\mathrm{d})}) \leq \mathrm{Rank}(\psi_1^{(\mathrm{d})})$. Therefore, $|\psi_1\rangle$ and $|\psi_2\rangle$ can also be viewed as pure states in $\mathcal{H}(d')$, where $d' = \max\{\mathscr{C}_s(\psi_1), \mathscr{C}_s(\psi_2)\} = \mathscr{C}_s(\psi_1)$; $|\psi_1\rangle$ will be full rank and the above propositions can be used.

All the Propositions 10, 11, and 12 tell us that in order to check whether the transformation under consideration is possible, we need to check infinitely many conditions, thus, making the proposition only of theoretical merit. However, as we show below in Proposition 13, if only a few conditions on Rényi entropy hold, then they suffice to show that all other conditions hold automatically.

Proposition 13. *Consider two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$. Given $\varepsilon > 0$, we can construct two pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ with $\phi_1^{(\mathrm{d})} \in B_\varepsilon(\psi_1^{(\mathrm{d})})$, $\phi_2^{(\mathrm{d})} \in B_\varepsilon(\psi_2^{(\mathrm{d})})$. Here, for two probability vectors x and y, $B_\varepsilon(x)$ is the ε ball around x and is defined as $B_\varepsilon(x) :=$*

$\{ y : \frac{1}{2} \sum_i | y_i - x_i| < \varepsilon \}$. *Consider following conditions:*

$$S_0(\psi_2^{(\mathrm{d})}) \leq S_0(\phi_1^{(\mathrm{d})}) + \frac{\log \varepsilon}{1-\alpha} \quad (\textit{for } 0 < \alpha < 1); \tag{5.8a}$$

$$S_\infty(\phi_2^{(\mathrm{d})}) - \frac{\log \varepsilon}{\alpha - 1} \leq S_\infty(\psi_1^{(\mathrm{d})}) \quad (\textit{for } \alpha > 1); \tag{5.8b}$$

$$\frac{\tilde{S}_\alpha(\psi_2^{(\mathrm{d})})}{|\alpha|} \leq \frac{\tilde{S}_\alpha(\psi_1^{(\mathrm{d})})}{|\alpha|} \quad (\textit{for } \alpha = 0). \tag{5.8c}$$

If conditions (5.8a), (5.8b), and (5.8c) hold, then for any $\alpha \in [0, \infty)$,

$$\frac{\tilde{S}_\alpha(\psi_2^{(\mathrm{d})})}{|\alpha|} \leq \frac{\tilde{S}_\alpha(\psi_1^{(\mathrm{d})})}{|\alpha|}. \tag{5.9}$$

Proof. Based on Lemma 30 of Appendix C, we can construct two pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ with $\phi_1^{(\mathrm{d})} \in B_\varepsilon(\psi_1^{(\mathrm{d})})$, $\phi_2^{(\mathrm{d})} \in B_\varepsilon(\psi_2^{(\mathrm{d})})$, such that

$$S_\alpha(\psi_1^{(\mathrm{d})}) \geq S_0(\phi_1^{(\mathrm{d})}) + \frac{\log \varepsilon}{1-\alpha} \quad (\text{for } 0 < \alpha < 1); \tag{5.10a}$$

$$S_\infty(\phi_2^{(\mathrm{d})}) - \frac{\log \varepsilon}{\alpha - 1} \geq S_\alpha(\psi_2^{(\mathrm{d})}) \quad (\text{for } \alpha > 1). \tag{5.10b}$$

Note that for any probability vector x, $S_\alpha(x) \leq S_\beta(x)$ if $\alpha \geq \beta$. Now for $0 < \alpha < 1$, from conditions (5.10a) and (5.8a), we have

$$\begin{aligned}
\tilde{S}_\alpha(\psi_1^{(\mathrm{d})}) &= S_\alpha(\psi_1^{(\mathrm{d})}) - \ln d \\
&\geq S_0(\phi_1^{(\mathrm{d})}) + \frac{\log \varepsilon}{1-\alpha} - \ln d \\
&\geq S_0(\psi_2^{(\mathrm{d})}) - \ln d \\
&\geq S_\alpha(\psi_2^{(\mathrm{d})}) - \ln d = \tilde{S}_\alpha(\psi_2^{(\mathrm{d})}).
\end{aligned} \tag{5.11}$$

Similarly, for $\alpha > 1$, from conditions (5.8b) and (5.10b), we have

$$\begin{aligned}\tilde{S}_\alpha(\psi_1^{(\mathrm{d})}) &= S_\alpha(\psi_1^{(\mathrm{d})}) - \ln d \\ &\geq S_\infty(\psi_1^{(\mathrm{d})}) - \ln d \\ &\geq S_\infty(\phi_2^{(\mathrm{d})}) - \frac{\log \varepsilon}{\alpha - 1} - \ln d \\ &\geq \tilde{S}_\alpha(\psi_2^{(\mathrm{d})}).\end{aligned} \tag{5.12}$$

Combining the above two equations with the condition (5.8c), we get

$$\frac{\tilde{S}_\alpha(\psi_2^{(\mathrm{d})})}{|\alpha|} \leq \frac{\tilde{S}_\alpha(\psi_1^{(\mathrm{d})})}{|\alpha|},$$

for all values of $\alpha \in [0, \infty)$, where $\alpha = 1$ case comes from the continuity of $\tilde{S}_\alpha(\cdot)/|\alpha|$. □

Hence, we only need to check conditions (5.8a), (5.8b), and (5.8c) to determine whether the transformation between two pure states is possible with the aid of catalysts. This establishes the practicality of Propositions 10, 11, and 12.

5.5 Entanglement assisted coherence transformations

Consider a pair of pure states such that there exists no catalytic incoherent transformation (see Fig. 5.1) between them. Can we find an incoherent operation between such pair of states with some assistance of another physical resource? The following proposition answers this question. We follow the proof techniques of Ref. [179] to prove the proposition.

Proposition 14. *For any pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ there exist a k-partite pure state $|\tilde{\phi}\rangle_{1,\ldots,k}$ and $|\phi_1\rangle \otimes \ldots \otimes |\phi_k\rangle$ such that*

$$|\psi_1\rangle \otimes |\tilde{\phi}\rangle_{1,\ldots,k} \xrightarrow{\mathrm{ICO}} |\psi_2\rangle \otimes |\phi_1\rangle \otimes \ldots \otimes |\phi_k\rangle$$

with $\phi_i^{(\mathrm{d})} = \tilde{\phi}_i^{(\mathrm{d})} := \mathrm{Tr}_{\{1,\ldots,k\}/i}\tilde{\phi}^{(\mathrm{d})}$ *and* $k \leq 3$ *if and only if the following two conditions are satisfied: (1)* $\mathscr{C}_s(|\psi_2\rangle) \leq \mathscr{C}_s(|\psi_1\rangle)$ *and (2)* $\mathscr{C}_r(|\psi_2\rangle) < \mathscr{C}_r(|\psi_1\rangle)$*. Here* $\mathscr{C}_s$ *is a proper coherence measure defined in [178], which for a pure state is equal to the number of nonzero coefficients in the state spanned in the reference basis.*

Proof. Note that from Theorem 6, $|\psi_1\rangle \otimes |\tilde{\phi}\rangle_{1,\ldots,k} \xrightarrow{\text{ICO}} |\psi_2\rangle \otimes |\phi_1\rangle \otimes \ldots \otimes |\phi_k\rangle$ is equivalent to $\psi_1^{(d)} \otimes \tilde{\phi}_{1,\ldots,k}^{(\mathrm{d})} \prec \psi_2^{(d)} \otimes \phi_1^{(\mathrm{d})} \otimes \ldots \otimes \phi_k^{(\mathrm{d})}$. Then the proof of our proposition follows from Lemma 29 of the aAppendix C. ■

Now let us apply Proposition 14 to a numerical example. Consider two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ with $\psi_1^{(\mathrm{d})} = (0.5, 0.4, 0.1)$ and $\psi_2^{(\mathrm{d})} = (0.6, 0.25, 0.15)$. Then, we know that $|\psi_1\rangle$ cannot be transformed to ψ_2 using any catalyst as there exists an α such that $\tilde{S}_\alpha\left(\psi_2^{(\mathrm{d})}\right) - \tilde{S}_\alpha\left(\psi_1^{(\mathrm{d})}\right) < 0$ (see Fig. 5.1). Thus, $|\psi_1\rangle \otimes |\phi\rangle^{\otimes n} \to |\psi_2\rangle \otimes |\phi\rangle^{\otimes n}$ is not possible. However, $|\psi_1\rangle$ can be transformed to $|\psi_2\rangle$ using an entanglement assisted incoherent transformation as $\mathscr{C}_r(|\psi_2\rangle) < \mathscr{C}_r(|\psi_1\rangle)$ and $\mathscr{C}_s(|\psi_2\rangle) = \mathscr{C}_s(|\psi_1\rangle)$ (see Proposition 14). We emphasize here that in the above process of entanglement assisted incoherent transformation, coherence in the ancillary system is not consumed. This can be proved by using the following fact

$$\begin{aligned}
&\mathscr{C}_r(|\phi_1\rangle \otimes \cdots \otimes |\phi_k\rangle) - \mathscr{C}_r(|\tilde{\phi}\rangle_{1,\ldots,k}) \\
&= \sum_{i=1}^{k} H\left(\phi_i^{(\mathrm{d})}\right) - H\left(\tilde{\phi}_{1,\ldots,k}^{(\mathrm{d})}\right) \geq 0.
\end{aligned}$$

Particularly, when $k = 2$, for pure states $|\tilde{\phi}\rangle_{12}$, $|\phi\rangle_1$ and $|\phi\rangle_2$, where $\phi_1^{(\mathrm{d})} = \tilde{\phi}_1^{(\mathrm{d})}, \phi_2^{(\mathrm{d})} = \tilde{\phi}_2^{(\mathrm{d})}$, we have $\mathscr{C}_r\left(|\phi\rangle_1 \otimes |\phi\rangle_2\right) - \mathscr{C}_r\left(|\tilde{\phi}\rangle_{12}\right) = I\left(\left(|\tilde{\phi}\rangle_{12}\right)^{(\mathrm{d})}\right) \leq I\left(|\tilde{\phi}\rangle_{12}\right) = 2E_r\left(|\tilde{\phi}\rangle_{12}\right)$, giving an upper bound on the increased coherence. Here, $I(\rho_{12}) := H(\rho_1) + H(\rho_2) - H(\rho_{12})$ is the mutual information of ρ_{12} and $E_r\left(|\tilde{\phi}\rangle_{12}\right) := H\left(\mathrm{Tr}_2\left[|\tilde{\phi}\rangle\langle\tilde{\phi}|_{12}\right]\right)$ is the entropy of entanglement of the state $|\tilde{\phi}\rangle_{12}$. Further, we generalize Proposition 14 to the following proposition. The proof techniques for the following proposition are adapted from Refs. [11, 180].

Proposition 15. *For two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, the following two conditions are equivalent:*

1. For a given state $|\psi_1\rangle$ there exist states $|\psi_1^\varepsilon\rangle$, $|\tilde{\phi}\rangle_{1,\ldots,k}$ and $|\phi_1\rangle \otimes \ldots \otimes |\phi_k\rangle$ with $\tilde{\phi}_i^{(\mathrm{d})} = \phi_i^{(\mathrm{d})}$ and $\varepsilon > 0$ such that (i) $||\,|\psi_1\rangle - |\psi_1^\varepsilon\rangle\,|| < \varepsilon$; (ii) $|\psi_1^\varepsilon\rangle \otimes |\tilde{\phi}\rangle_{1,\ldots,k} \xrightarrow{\mathrm{ICO}} |\psi_2\rangle \otimes |\phi_1\rangle \otimes \ldots \otimes |\phi_k\rangle$. Here, $k \leq 3$.

2. $\mathscr{C}_r(|\psi_2\rangle) \leq \mathscr{C}_r(|\psi_1\rangle)$.

Proof. Let us first prove the implication $1 \Rightarrow 2$. Since $|\psi_1^\varepsilon\rangle \otimes |\tilde{\phi}\rangle_{1,\ldots,k} \xrightarrow{\mathrm{ICO}} |\psi_2\rangle \otimes |\phi_1\rangle \otimes \ldots \otimes |\phi_k\rangle$ and coherence cannot increase under incoherent operations, we have $\mathscr{C}_r(|\psi_2\rangle \otimes |\phi_1\rangle \otimes \ldots \otimes |\phi_k\rangle) \leq \mathscr{C}_r(|\psi_1^\varepsilon\rangle \otimes |\tilde{\phi}\rangle_{1,\ldots,k})$. As $\mathscr{C}_r(|\phi_1\rangle \otimes \ldots \otimes |\phi_k\rangle) - \mathscr{C}_r(|\tilde{\phi}\rangle_{1,\ldots,k}) = \sum_{i=1}^k H\left(\phi_i^{(\mathrm{d})}\right) - H\left(\tilde{\phi}_{1,\ldots,k}^{(\mathrm{d})}\right) \geq 0$, we have $\mathscr{C}_r(|\psi_2\rangle) \leq \mathscr{C}_r(|\psi_1^\varepsilon\rangle)$. Let $\varepsilon \to 0$, then $\mathscr{C}_r(|\psi_2\rangle) \leq \mathscr{C}_r(|\psi_1\rangle)$.

The implication $2 \Rightarrow 1$ can be proved as follows. For $|\psi_1\rangle = \sum_{i=1}^d \sqrt{\psi_1^i}\,|i\rangle$ let $|\psi_1^\varepsilon\rangle = \sum_{i=1}^d \sqrt{(1-\varepsilon)\psi_1^i + \varepsilon/d}\,|i\rangle$. Then $||\,|\psi_1\rangle - |\psi_1^\varepsilon\rangle\,|| \to 0$, when $\varepsilon \to 0$. Moreover, $\mathscr{C}_r(|\psi_2\rangle) \leq \mathscr{C}_r(|\psi_1\rangle)$ is equivalent to $H\left(\psi_2^{(\mathrm{d})}\right) \leq H\left(\psi_1^{(\mathrm{d})}\right)$. As $H(\cdot)$ is strictly concave, then $H\left(\psi_2^{(\mathrm{d})}\right) \leq H\left(\psi_1^{(\mathrm{d})}\right) < H\left(\tilde{\psi}_1^{(\mathrm{d})}\right)$. It is easy to see $\mathscr{C}_s(|\psi_1^\varepsilon\rangle) = d \geq \mathscr{C}_s(|\psi_2\rangle)$. Now using Proposition 14 we complete the proof. ■

5.6 Chapter summary

In this chapter we find the necessary and sufficient conditions for the deterministic and stochastic coherence transformations between pure quantum states mediated by catalysts using only incoherent operations. We first find the necessary and sufficient conditions for the possibility of the increase of the optimal probability of achieving an otherwise impossible transformation with the aid of a catalyst. Then, we show that for a given pair of pure quantum states, the necessary and sufficient conditions for a deterministic catalytic transformation from one state to another are the simultaneous decrease of a family of Rényi entropies of the corresponding diagonal parts of the given pure states in a fixed reference basis. We also discuss about the practicality of these conditions. Further, we delineate the structure of the catalysts and find that it is possible for a pure quantum state to act as a catalyst for itself for a given otherwise impossible state transformation using incoherent operations. This phenomena may be termed as *self catalysis*. Moreover, for the pair of states which violate the necessary and sufficient conditions for deterministic coherence transformations, we consider the possibility of using an entangled state and show that even though there exists no catalyst for such a pair of states but an entangled state can indeed be used to facilitate the transformation. We dub such transformations as entanglement assisted coherence transformations. Here we emphasize that in entanglement assisted coherence transformations the coherence of the entangled state is *not* consumed at all. We also provide necessary and sufficient conditions for the entanglement assisted coherence transformations. In this way we completely characterize the allowed manipulations of the coherence of pure quantum states and thus, this chapter contributes towards a complete resource theory of coherence based on incoherent operations.

The consideration of catalytic transformations is very natural and has resulted in strikingly nontrivial consequences. One such instance is the introduction of many second laws of quantum thermodynamics superseding the common wisdom of single second law of macroscopic thermodynamics. Now given the importance of quantum coherence in quantum thermodynamics and various other avenues, we hope that our results which provide the limitations on coherence transformations will be extremely helpful in the processing of quantum coherence in such situations and in particular, in the context of *single-shot* quantum information theory. Further, it will be important to analyze the possibility of *self catalysis* in greater detail in future as the catalysts in this case are readily available.

This chapter is based on the following paper:

1. *Catalytic coherence transformations*,

   K. Bu, **U. Singh**, and J. Wu, Phys. Rev. A **93**, 042326 (2016).

CHAPTER 6

Average coherence and its typicality for random pure states

6.1 Introduction

This chapter is devoted mainly to random pure states, matrix integrals and the concentration of measure phenomenon in the context of resource theories of coherence. In classical and quantum information theory, probabilistic methods have been proven to be very useful and far reaching [103]. For example, both the classical and quantum compression theorems employ a well known probabilistic tool—the central limit theorem—for their proofs. One of the main advantages of use of such probabilistic tools is their simplicity together with their generic or typical consequences. Moreover, probabilistic techniques are convenient mathematical tools to prove existence theorems–where non-probabilistic proofs are scarce—however, for some problems like the minimum output entropy additivity problem they are only natural. Therefore, it becomes very important to find the "typical" behaviour of various physical quantities of interest. The technical meaning of the word "typical" will be given in this chapter.

In the entanglement resource theory, the computation complexity of various multipartite entanglement measures grows exponentially with the number of parties. A feasible approach is to consider only typical behaviour of multipartite entanglement and this has been a vibrant research area with several remarkable results [29, 81, 103–115]. These results are naturally linked with probability measures on matrix spaces, random quantum states and a very important tool from probabilistic techniques—the concentration of measure phenomenon [119]. Already, random states and the notion of "typical entanglement" have been used succesfully in providing a satisfactory explanation to the postulate of *equal a priori probability* of statistical physics [104, 105]. In particular, it has been shown that an overwhelming majority of random pure quantum states sampled from the uniform Haar measure are extremely close to the maximally entangled state [29]. Additionally, the usefulness of these results from an experimental viewpoint has also been established [120].

Given the fact that quantum coherence is a useful resource in a wide range of applications arising in quantum thermodynamics [11–14, 16, 33–37] and in quantum biology [17–20, 40, 41], in this chapter, we consider typical behaviour of coherence as quantified in the resource theory of coherence based on incoherent operations. Using the concentration of measure phenomenon, we first show that quantum coherence of random pure states is typical for higher dimensional Hilbert spaces. In particular, we show that for an overwhelming majority of random pure states the coherence is equal to the typical coherence—which is equal to the value of coherence averaged over the probability measure on the state space—to within arbitrarily small error. We show that the typical nature of coherence holds true for various measures of coherence such as the relative entropy of coherence [47] which is also equal to the distillable coherence, the coherence of formation [121] and the l_1 norm of coherence [47]. Further, in contrast to the fact that most of the bipartite pure state sampled randomly from the Haar measure are typically maximally entangled [29], we show that most of the pure states sampled randomly from the Haar measure are not typically maximally coherent.

This chapter is organized as follows. We start with a discussion of random pure quantum states, measures of coherence, concentration of measure phenomenon and some other preliminaries in Sec 6.2. In Sec 6.3, we calculate the average relative entropy of coherence for random pure states sampled from the uniform Haar distribution, establish the typicality of this average amount of coherence and find the dimension of the subspace of the total Hilbert space with the property that all pure states on this subspace have at least a fixed amount of relative entropy of coherence as well as coherence of formation. We then present our results on expected classical purity, its typicality and the upper bound on the root mean square average l_1 norm of coherence in Sec 6.4. Subsequently, in Sec 6.5, we establish that most of the randomly sampled pure states are not typically maximally coherent. Finally, we conclude in Sec 6.6 with overview and implications of the results presented in this chapter.

6.2 Random pure quantum states, measures of coherence and concentration of measure phenomenon

Random pure states.—Before we discuss about random quantum states, let us fix a measure μ on the set of quantum states. Having fixed a measure μ on the set of quantum states one can calculate the desired averages over all states with respect to this measure. Here we are interested in the set of pure quantum states. For a d-dimensional Hilbert space $\mathcal{H}$, the set of pure states is identified as complex projective space $\mathbb{C}P^{d-1}$. On this space there exists a unique natural measure $\mathrm{d}(\psi)$, induced by the uniform Haar measure $\mathrm{d}\mu(U)$ on the unitary group $\mathrm{U}(d)$ [181–185]. This amounts to saying that any random pure state $|\psi\rangle$ is generated equivalently by applying a random unitary matrix $U \in \mathrm{U}(d)$ on a fixed pure state $|\psi_0\rangle$, i.e., $|\psi\rangle = U|\psi_0\rangle$. Now one can define the average value of some function g of pure state as follows:

$$\mathbb{E}_\psi g(\psi) := \int \mathrm{d}(\psi)\, g(\psi) = \int_{\mathrm{U}(d)} \mathrm{d}\mu(U)\, g(U\psi_0).$$

In what follows, by random pure states we mean the states generated by applying random Haar distributed unitaries on some fixed pure state and all the averages are with respect to the Haar measure.

Measures of coherence.—The measures of coherence that we consider in this chapter are the l_1 norm of coherence, the relative entropy of coherence and the coherence of formation. For a density matrix ρ of dimension d and a fixed reference basis $\{|i\rangle\}$, let us recall that the l_1 norm of coherence $\mathscr{C}_{l_1}(\rho)$ [47] is defined as

$$\mathscr{C}_{l_1}(\rho) = \sum_{\substack{i,j=1 \\ i\neq j}}^{d} |\langle i|\rho|j\rangle|. \tag{6.1}$$

The relative entropy of coherence $\mathscr{C}_r(\rho)$ [47] is defined as

$$\mathscr{C}_r(\rho) = H(\rho^{(\mathrm{d})}) - H(\rho), \tag{6.2}$$

where $\rho^{(\mathrm{d})}$ is the diagonal part of the density matrix ρ in the fixed reference basis and H is the von Neumann entropy defined as $H(\rho) = -\operatorname{Tr}(\rho \ln \rho)$. All the logarithms are taken with respect to the base e. The coherence of formation $\mathscr{C}_f(\rho)$ [121] is defined as

$$\mathscr{C}_f(\rho) = \min_{\{p_a, |\psi_a\rangle\langle\psi_a|\}} \sum_a p_a H(\rho^{(\mathrm{d})}(\psi_a)), \tag{6.3}$$

where $\rho^{(\mathrm{d})}(\psi_a)$ is the diagonal part of the pure state $|\psi_a\rangle$, $\rho = \sum_a p_a |\psi_a\rangle\langle\psi_a|$ and minimum is taken over all such decompositions of ρ.

Concentration of measure phenomenon.—For many functions defined over a vector space, the overwhelming majority of vectors take a value of the function very close to the average value as the dimension of the vector space goes to infinity or becomes very large. This observation, collectively, is referred to as the concentration of measure phenomenon. Here we show that several measures of coherence have this property. Let us consider a simple example to demonstrate the concentration of measure phenomenon. Consider the k-sphere $\mathbb{S}^k$ in $\mathbb{R}^{k+1}$ with k being very large. A direct calculation yields that the uniform measure μ on $\mathbb{S}^k$ is almost concentrated around every equator when k is large. Similarly, an explicit calculation [119] of the measure of spherical caps implies that given any measurable set $\mathcal{S}$ with $\mu(\mathcal{S}) \geq 1/2$, for every $r > 0$, $\mu(\mathcal{S}_r) \geq 1 - \exp\{(k-1)r^2/2\}$ where $\mathcal{S}_r = \{x \in \mathbb{S}^k : d(x, \mathcal{S}) < r\}$ and $d(x, y)$ is the Euclidean distance on $\mathbb{R}^{k+1}$. This is one of the first quantitative instances of the concentration of measure phenomenon. For Lipschitz continuous functions on the sphere, Lévy's lemma is the rigorous statement about the concentration of measure phenomenon [119]. Let us first define the Lipschitz continuous functions.

Lipschitz continuous function and Lipschitz constant.—Suppose (M, d_1) and (N, d_2) are metric spaces and $F : M \to N$. If there exists $\eta \in \mathbb{R}^+$ such that $d_2(F(x), F(y)) \leq \eta d_1(x, y)$ for all $x, y \in M$, then F is called a Lipschitz continuous function on M with the Lipschitz constant η. Every real number larger than η is also a Lipschitz constant for F [186]. Next, we introduce a form of Lévy's lemma that will be the key ingredient in this chapter.

Lévy's Lemma (see [119] and [29]).—Consider a sequence $F = \{F_k : \mathbb{S}^k \to \mathbb{R}\}_k$ of Lipschitz continuous functions from the k-sphere to the real line with each function F_k having the same Lipschitz constant η that is independent of k (with respect to the Euclidean norm). Let a point $X \in \mathbb{S}^k$ be chosen uniformly at random. Then, for all $\epsilon > 0$ and k,

$$\Pr\{|F_k(X) - \mathbb{E}(F_k)| > \epsilon\} \leq 2\exp\left(-\frac{(k+1)\epsilon^2}{9\pi^3\eta^2\ln 2}\right). \tag{6.4}$$

Here $\mathbb{E}(F_k)$ is the mean value of F_k. It is insightful to consider $\epsilon = r^{-1/4}$ in Eq. (6.4). With this choice, the bound on the right hand side decreases exponentially as $\exp(-\sqrt{r})$ while the bound on the left hand side decreases like $r^{-1/4}$, making it clear that the probability of being non-typical decreases much faster and hence "essentially zero" for large r. Note that the average over the Haar distributed d-dimensional pure states is equivalent to the average over the k-sphere with $k = 2d - 1$. Importantly, Lévy's lemma has been used in constructing counterexamples to the conjecture of the additivity of minimum output entropy [116–118].

At various places, we use the trace norm and the Euclidean norm for matrices: (1) the trace norm of a matrix A, denoted by $||A||_1$, is defined as $||A||_1 = \mathrm{Tr}\sqrt{A^\dagger A}$, where $\dagger$ is the Hermitian conjugate. (2) the Euclidean norm of a matrix A, denoted by $||A||_2$, is defined as $\sqrt{\mathrm{Tr}(A^\dagger A)}$. The trace distance between two density matrices ρ and σ is defined as $||\rho - \sigma||_1$ [3]. Notice that we follow a definition of trace distance without a factor of half in front of the trace norm. Finally, for proving the existence of concentrated subspaces with fixed amount of coherence we need the notion of *small nets* [81].

Existence of small nets.— It is known [81] that given a Hilbert space $\mathcal{H}$ of dimension d and $0 < \epsilon_0 < 1$, there exists a set $\mathcal{N}$ of pure states in $\mathcal{H}$ with $|\mathcal{N}| \leq (5/\epsilon_0)^{2d}$, such that for every pure state $|\psi\rangle \in \mathcal{H}$ there exists $|\tilde{\psi}\rangle \in \mathcal{N}$ with $||\ |\psi\rangle - |\tilde{\psi}\rangle\ ||_2 \leq \epsilon_0/2$. Such a set is called as an ϵ_0-net.

We emphasize here that all the main results presented below are based on Lévy's lemma and hence are of probabilistic nature. The method to demonstrate the typical properties is always to prove that the opposite is an unlikely event.

6.3 Average relative entropy of coherence and its typicality for random pure states

To show the typicality of coherence of random pure quantum states we first find the average relative entropy of coherence for a random pure state, where average is taken over the uniform Haar measure, and then apply Lévy's lemma to show the concentration effect for quantum coherence. Now consider a pure state $|\psi\rangle$ and denote by $\rho^{(\mathrm{d})}(\psi)$ the diagonal part of $|\psi\rangle$ in the fixed reference basis $\{|i\rangle\}$, i.e., $\rho^{(\mathrm{d})}(\psi) = \sum_{i=1}^{d} |\langle i|\,\psi\rangle|^2 \Pi_i$, where $\Pi_i = |i\rangle\langle i|$. The relative entropy of coherence of the state $|\psi\rangle$ in the fixed reference basis $\{|i\rangle\}$ is $\mathscr{C}_r(\psi) = H(\rho^{(\mathrm{d})}(\psi)) = -\sum_{i=1}^{d} |\langle i|\psi\rangle|^2 \ln |\langle i|\psi\rangle|^2$. If we draw pure states $|\psi\rangle$ from the uniform Haar measure then the expected value of the relative entropy of coherence is given by

$$\mathbb{E}_\psi \mathscr{C}_r(\psi) := -\sum_{i=1}^{d} \int \mathrm{d}(\psi)\, |\langle i|\psi\rangle|^2 \ln |\langle i|\psi\rangle|^2. \tag{6.5}$$

As discussed earlier, we can take $|\psi\rangle = U|1\rangle$ where U is sampled from the Haar distribution and $|1\rangle$ is a fixed state. This allows us to rewrite the above equation as

$$\mathbb{E}_\psi \mathscr{C}_r(\psi) = -\sum_{i=1}^{d} \int \mathrm{d}\mu(U)\, |\langle i|U|1\rangle|^2 \ln |\langle i|U|1\rangle|^2. \tag{6.6}$$

Since the Haar measure is invariant under the left translation, we have

$$\mathbb{E}_\psi \mathscr{C}_r(\psi) = -d \int \mathrm{d}\mu(U)\, |U_{11}|^2 \ln |U_{11}|^2, \tag{6.7}$$

where $U_{11} = \langle 1|U|1\rangle$. Note that all entries U_{ij} of a Haar unitary U has the same distribution [187]: $\frac{d-1}{\pi}(1-r^2)^{d-2} r \mathrm{d}r \mathrm{d}\theta$, where $r = |U_{ij}| \in [0,1]$ and $\theta \in [0, 2\pi]$. We remark here that the distribution of each entry $U_{ij} = re^{\mathrm{i}\theta}$ is just the joint distribution of r and θ. The distribution of $|U_{11}|^2$ is given by $(d-1)(1-r)^{d-2}\mathrm{d}r$, where $0 \le r \le 1$. Now, we have

$$\begin{aligned} \mathbb{E}_\psi \mathscr{C}_r(\psi) &= -d(d-1) \int_0^1 r(1-r)^{d-2} \ln r \,\mathrm{d}r \\ &= -d(d-1) \frac{\partial B(\alpha,\beta)}{\partial \alpha}\Big|_{(\alpha,\beta)=(2,d-1)}, \end{aligned} \tag{6.8}$$

where $B(\alpha,\beta)$ is the Beta function, defined as

$$B(\alpha,\beta) := \int_0^1 r^{\alpha-1}(1-r)^{\beta-1}\mathrm{d}r = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{6.9}$$

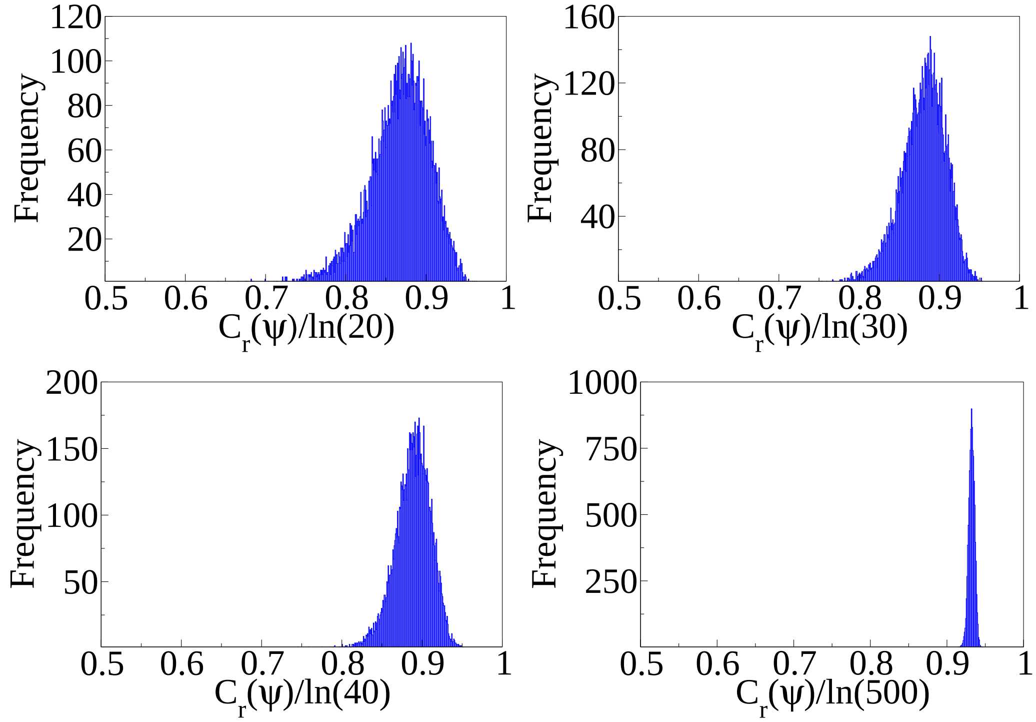

Figure 6.1: The frequency plot showing the (scaled) relative entropy of coherence $\mathscr{C}_r(\psi)/\ln d$ for the Haar distributed random pure states for dimensions $d = 20, 30, 40$ and 500. Here, both the axes are dimensionless. We have $\mathbb{E}_\psi \mathscr{C}_r(\psi)/\ln 20 \approx 0.87$, $\mathbb{E}_\psi \mathscr{C}_r(\psi)/\ln 30 \approx 0.88$, $\mathbb{E}_\psi \mathscr{C}_r(\psi)/\ln 40 \approx 0.89$ and $\mathbb{E}_\psi \mathscr{C}_r(\psi)/\ln 500 \approx 0.93$. The plot shows that the (scaled) relative entropy of coherence is indeed very close to the average value $\sum_{k=2}^{d} 1/k$. As we increase the dimension, the figure shows that more and more states have coherence close to the average value and the variances approach to zero.

Note that $\partial B(\alpha,\beta)/\partial\alpha = (\Psi(\alpha) - \Psi(\alpha+\beta))B(\alpha,\beta)$, where $\Psi(z) := \Gamma'(z)/\Gamma(z)$ and $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$, with $\mathrm{Re}(z) > 0$, is the Gamma function. In particular, for natural number n, $\Psi(n) = \sum_{k=1}^{n-1} 1/k - \gamma$ with $\gamma \approx 0.57721$ being the Euler constant. Therefore, we get $\partial B(\alpha,\beta)/\partial\alpha|_{(\alpha,\beta)=(2,d-1)} = -(d(d-1))^{-1}\sum_{k=2}^{d} 1/k$. Using this in Eq. (6.8), we have $\mathbb{E}_\psi \mathscr{C}_r(\psi) = \sum_{k=2}^{d} \frac{1}{k}$. Thus, a d-dimensional random pure state has $H_d - 1$ amount of average relative entropy of coherence, where $H_d = \sum_{k=1}^{d} 1/k$ is the d-th harmonic number. Now we are ready to discuss the concentration of measure phenomenon for quantum coherence.

Theorem 16 (**Concentration of the relative entropy of coherence**). *Let $|\psi\rangle$ be a random* *pure state on a d-dimensional Hilbert space $\mathcal{H}$ with $d \geq 3$. Then for all $\epsilon > 0$*

$$\Pr\left\{|\mathscr{C}_r(\psi) - (H_d - 1)| > \epsilon\right\} \leq 2\exp\left(-\frac{d\epsilon^2}{36\pi^3 \ln 2 (\ln d)^2}\right), \tag{6.10}$$

where $H_d = \sum_{k=1}^{d} 1/k$ is the d-th harmonic number.

Proof. We will apply Lévy's lemma, Eq. (6.4) for averages, to prove the theorem. Consider the map $F : |\psi\rangle \to F(\psi) := H(\rho^{(\mathrm{d})}(\psi)) = \mathscr{C}_r(\psi)$. We have already shown that $\mathbb{E}_\psi F = H_d - 1$. We prove the theorem by identifying k with $2d-1$ in Lévy's lemma (Eq. (6.4)). We just need the Lipschitz constant η for the function F such that $|F(\psi) - F(\phi)| \leq \eta || \, |\psi\rangle - |\phi\rangle \, ||_2$. Let $|\psi\rangle = \sum_{i=1}^{d} \psi_i |i\rangle$ and therefore, $\rho^{(\mathrm{d})}(\psi) = \sum_{i=1}^{d} p_i(\psi) |i\rangle \langle i|$ with $p_i(\psi) = |\psi_i|^2$. Now, $F(\psi) = -\sum_{i=1}^{d} p_i(\psi) \ln p_i(\psi)$. The Lipschitz constant for F can be bounded as follows:

$$\begin{aligned}
\eta^2 := \sup_{\langle\psi|\psi\rangle \leq 1} \nabla F \cdot \nabla F &= 4 \sum_{i=1}^{d} p_i(\psi) \left[1 + \ln p_i(\psi)\right]^2 \\
&\leq 4\left(1 + \sum_{i=1}^{d} p_i(\psi)(\ln p_i(\psi))^2\right) \\
&\leq 4\left(1 + (\ln d)^2\right) \leq 8(\ln d)^2,
\end{aligned} \tag{6.11}$$

where the last inequality is true for $d \geq 3$. Therefore, $\eta \leq \sqrt{8}\ln d$ for $d \geq 3$. By definition, any upper bound on the Lipschitz constant can also serve as a valid Lipschitz constant, therefore, we can take $\eta = \sqrt{8}\ln d$ for $d \geq 3$. This concludes the proof of the theorem. ■

The inequality (6.10) means that for large d, the number of pure states with the relative entropy of coherence not very close to $H_d - 1$ are exponentially small, or in other words, most pure states chosen randomly have $H_d - 1$ amount of relative entropy of coherence to within an arbitrarily small error. This is the concentration of relative entropy of coherence around its expected value (the typical value of the relative entropy of coherence). Further, as quantum coherence is a quantifier of the wave nature of a quantum particle [122, 123], Theorem 16 has a nice physical meaning and it quantifies the 'typical wave nature' of a random pure state. Fig. 6.1 plots the relative entropy of coherence for numerically generated Haar distributed random pure quantum states and shows that indeed most of the states have coherences close to the expected value.

Having established the concentration of relative entropy of coherence, it is of great practical importance to delineate the largest subspace of the total Hilbert space such that all the pure states in this subspace have a fixed nonzero amount of coherence. Specifically, we find a large subspace of the total Hilbert space such that the amount of the relative entropy of coherence for every pure state in this subspace can be bounded from below almost always by a number that is arbitrarily close to the typical value of coherence. The following theorem formalizes this.

Theorem 17 (Coherent subspaces). *Let $\mathcal{H}$ be a Hilbert space of dimension $d \geq 3$ of a quantum system. Then, for any positive $\epsilon < \ln d$, there exists a subspace $\mathcal{S} \subset \mathcal{H}$ of dimension*

$$s = \left\lfloor dK \left(\frac{\epsilon}{\ln d}\right)^{2.5} \right\rfloor \qquad (6.12)$$

such that all pure states $|\psi\rangle \in \mathcal{S}$ almost always satisfy $\mathscr{C}_r(\psi) \geq H_d - 1 - \epsilon$. K may be chosen to be $1/16461$. Here $\lfloor\rfloor$ denote the floor function.

Proof. Here we follow the strategy of Ref. [29] which is based on the construction of nets to prove the theorem. Let $\mathcal{S}$ be a random subspace of $\mathcal{H}$ of dimension s. Let $\mathcal{N}_{\mathcal{S}}$ be an ϵ_0-net for states on $\mathcal{S}$, for $\epsilon_0 = \epsilon/(\sqrt{8}\ln d)$. By definition, we have $|\mathcal{N}_{\mathcal{S}}| \leq (5/\epsilon_0)^{2s}$. Note that $\mathcal{S}$ may be thought of as $U\mathcal{S}_0$, with a fixed $\mathcal{S}_0$ and a unitary U distributed according to the Haar measure. We can fix the net $\mathcal{N}_{\mathcal{S}_0}$ on $\mathcal{S}_0$ and let $\mathcal{N}_{\mathcal{S}} = U\mathcal{N}_{\mathcal{S}_0}$. This is a natural way to choose a random subspace. Now, given $|\psi\rangle \in \mathcal{S}$, we can choose $|\tilde{\psi}\rangle \in \mathcal{N}_{\mathcal{S}}$ such that $||\ |\psi\rangle - |\tilde{\psi}\rangle\, ||_2 \leq \epsilon_0/2$. Note that $\mathscr{C}_r(\psi)$ is a Lipschitz continuous function with the

Lipschitz constant $\eta = \sqrt{8}\ln d$. From definition of the Lipschitz function and ϵ_0-net, we have

$$|\mathscr{C}_r(\psi) - \mathscr{C}_r(\tilde{\psi})| \leq \eta || \ |\psi\rangle - |\tilde{\psi}\rangle \ ||_2 \leq \eta\epsilon_0/2 = \epsilon/2.$$

Define $\mathbb{P} = \Pr\left\{\inf_{|\psi\rangle\in\mathcal{S}} \mathscr{C}_r(\psi) < H_d - 1 - \epsilon\right\}$. Now, we have

$$\begin{aligned}\mathbb{P} &\leq \Pr\left\{\min_{|\tilde{\psi}\rangle\in\mathcal{S}} \mathscr{C}_r(\tilde{\psi}) < H_d - 1 - \epsilon/2\right\}\\ &\leq |\mathcal{N}_{\mathcal{S}}| \Pr\left\{\mathscr{C}_r(\psi) < H_d - 1 - \epsilon/2\right\}\\ &\leq 2\left(10\sqrt{2}\ln d/\epsilon\right)^{2s} \exp\left(-\frac{d\epsilon^2}{144\pi^3\ln 2(\ln d)^2}\right),\end{aligned} \tag{6.13}$$

where in the last line we have used our Theorem 16 and the definition of ϵ_0-net. If this probability is smaller than one, a subspace with the stated properties will exist. This can be assured by choosing

$$s < \frac{(d-1)\epsilon^2}{6190(\ln d)^2 \ln\left(\left(10\sqrt{2}\ln d\right)/\epsilon\right)}. \tag{6.14}$$

Now, using the fact that $\ln x \leq \sqrt{x/2}$ for $x \geq 10\sqrt{2}$, we have $\ln\left((10\sqrt{2}\ln d)/\epsilon\right) \leq \sqrt{5\sqrt{2}\ln d/\epsilon}$ with $\epsilon < \ln d$. For a nontrivial dimension s, i.e., $s \geq 2$, we require $d \geq 32921$. Therefore, $s = \left\lfloor \frac{d\epsilon^{2.5}}{16461(\ln d)^{2.5}} \right\rfloor$. This completes the proof of the theorem. ■

The theorem implies that if a subspace of dimension s (which can be appropriately large), given by Eq. (6.12), of total Hilbert space is chosen at random via the Haar distribution then the relative entropy of coherence of any pure state in this subspace is almost always greater than $H_d - 1 - \epsilon$, which is very close to the typical value of coherence. This follows from the fact that the probability that the chosen subspace will not have the above said property is small. Now, for any pure state $|\psi\rangle$ in $\mathcal{S}$, the relative entropy of coherence $\mathscr{C}_r(\psi)$ is typically lower bounded by $H_d - 1 - \epsilon$. Therefore, for all $\rho \in \mathcal{S}$, the coherence of formation which is defined as $\mathscr{C}_f(\rho) = \min \sum_i p_i H(\rho^{(\mathrm{d})}(\psi_i))$ such that $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ [121], is also typically lower bounded by $H_d - 1 - \epsilon$, i.e., $\mathscr{C}_f(\rho) \geq H_d - 1 - \epsilon$.

6.4 Average classical purity of random pure quantum states

In this section, we calculate the average classical purity [188] of random pure quantum states and show its typicality. It is not straightforward to find the expected value of the l_1 norm of coherence for random pure states. Therefore, we resort to an indirect method to obtain an upper bound on it using the expected value of classical purity. The classical purity $P(\Pi(\psi))$ of a state $|\psi\rangle$ is defined as $P(\Pi(\psi)) := \mathrm{Tr}[(\Pi(\psi))^2]$ where $\Pi : \rho \rightarrow \sum_i \langle i| \rho |i\rangle |i\rangle \langle i|$, i.e., it maps any state to its diagonal part in a fixed basis $\{|i\rangle\}$ [188]. For a pure state $|\psi\rangle$, we have $\Pi(\psi) = \rho^{(\mathrm{d})}(\psi)$. The expected classical purity $\mathbb{E}_\psi P(\Pi(\psi))$ can be obtained as follows. For a random pure state $|\psi\rangle$ sampled from the uniform Haar measure the expected classical purity is given by

$$\mathbb{E}_\psi P(\Pi(\psi)) = \int \mathrm{d}(\psi) P(\Pi(\psi)) = \int_{\mathrm{U}(d)} \mathrm{d}\mu(U) P(\Pi(U\psi_0)).$$

Let Φ be a linear super-operator that transforms a random pure state $|\psi\rangle\langle\psi|$ to $\Phi(|\psi\rangle\langle\psi|)$. The purity of the state $\Phi(|\psi\rangle\langle\psi|)$ is defined as $\mathrm{Tr}[\Phi(|\psi\rangle\langle\psi|)^2]$. Therefore, the expected purity for the states $\Phi(|\psi\rangle\langle\psi|)$ is given by

$$\begin{aligned} \mathbb{E}_\psi P(\Phi(\psi)) &= \int \mathrm{d}(\psi) \mathrm{Tr}[\Phi(|\psi\rangle\langle\psi|)^2] \\ &= \int \mathrm{d}(\psi)\, \mathrm{Tr}\left(\Phi^\dagger \circ \Phi(|\psi\rangle\langle\psi|)|\psi\rangle\langle\psi|\right) \\ &= \left\langle \psi_0 \middle| \int \mathrm{d}\mu(U) U^\dagger \Phi^\dagger \circ \Phi(U|\psi_0\rangle\langle\psi_0|U^\dagger)U \middle| \psi_0 \right\rangle, \end{aligned} \tag{6.15}$$

where $\Phi^\dagger$ is the dual of Φ in the sense: $\mathrm{Tr}\,(Y\Phi(X)) = \mathrm{Tr}\left(\Phi^\dagger(Y)X\right)$ for any X, Y and $|\psi_0\rangle$ is a fixed state such that $|\psi\rangle = U|\psi_0\rangle$. We use the following formula from matrix integral [189]

$$\int \mathrm{d}\mu(U) U^\dagger \Upsilon(UXU^\dagger) U = \frac{d\,\mathrm{Tr}\,(\Upsilon(\mathbb{I}_d)) - \mathrm{Tr}\,(\Upsilon)}{d(d^2-1)} \mathrm{Tr}\,(X)\,\mathbb{I}_d + \frac{d\,\mathrm{Tr}\,(\Upsilon) - \mathrm{Tr}\,(\Upsilon(\mathbb{I}_d))}{d(d^2-1)} X, \tag{6.16}$$

where $\mathrm{Tr}\,(\Upsilon)$ is the trace of the super-operator Υ, defined by $\mathrm{Tr}\,(\Upsilon) = \sum_{i,j=1}^{d} \langle i\,| \Upsilon(|i\rangle\,\langle j|)|\,j\rangle$, to simplify Eq. (6.15). Now identifying X with $|\psi_0\rangle\langle\psi_0|$ and Υ with $\Phi^\dagger \circ \Phi$ in Eq. (6.16), we get

$$\begin{aligned}\mathbb{E}_\psi P(\Phi(\psi)) &= \frac{d\,\mathrm{Tr}\left(\Phi^\dagger \circ \Phi(\mathbb{I}_d)\right) - \mathrm{Tr}\left(\Phi^\dagger \circ \Phi\right)}{d(d^2-1)} \\ &\quad + \frac{d\,\mathrm{Tr}\left(\Phi^\dagger \circ \Phi\right) - \mathrm{Tr}\left(\Phi^\dagger \circ \Phi(\mathbb{I}_d)\right)}{d(d^2-1)} \\ &= \frac{1}{d(d+1)}\left[\mathrm{Tr}\left(\Phi^\dagger \circ \Phi(\mathbb{I}_d)\right) + \mathrm{Tr}\left(\Phi^\dagger \circ \Phi\right)\right].\end{aligned}$$

Let $\Phi = \Pi$, then $\Pi^\dagger = \Pi$ and $\Pi \circ \Pi = \Pi$. Moreover, $\mathrm{Tr}\left(\Pi^\dagger \circ \Pi(\mathbb{I}_d)\right) = d$ and $\mathrm{Tr}\left(\Pi^\dagger \circ \Pi\right) = d$. The expected classical purity, therefore, is given by

$$\mathbb{E}_\psi P(\Pi(\psi)) = \frac{2}{d+1}. \tag{6.17}$$

The following theorem establishes that the $\mathbb{E}_\psi P(\Pi(\psi))$ is a typical property of the pure quantum states sampled from the uniform Haar distribution.

Theorem 18 (**Concentration of classical purity**). *Consider a random pure state $|\psi\rangle$ in a d dimensional Hilbert space. The classical purity of any pure state sampled from the Haar distribution, for all $\epsilon > 0$, satisfies*

$$\Pr\left\{\left|P(\Pi(\psi)) - \frac{2}{d+1}\right| > \epsilon\right\} \leq 2\exp\left(-\frac{d\epsilon^2}{18\pi^3 \ln 2}\right). \tag{6.18}$$

Proof. We use Lévy's lemma, Eq. (6.4), to prove the theorem. For this we need the Lipschitz constant for the function $G : |\psi\rangle \to P(\Pi(\psi))$. Noting that $P(\Pi(\psi)) = ||\Pi(\psi)||_2^2$, we have

$$\begin{aligned} |P(\Pi(\psi)) - P(\Pi(\phi))| &= |(||\Pi(\psi)||_2 - ||\Pi(\phi)||_2)(||\Pi(\psi)||_2 + ||\Pi(\phi)||_2)| \\ &\leq ||\Pi(\psi) - \Pi(\phi)||_2 \left(||\Pi(\psi)||_2 + ||\Pi(\phi)||_2\right) \\ &\leq 2||\Pi(\psi) - \Pi(\phi)||_2 \\ &\leq 2|| \, |\psi\rangle - |\phi\rangle \, ||_2. \end{aligned} \tag{6.19}$$

Here in the second line we have used the reverse triangle inequality. In the third line we have used the fact that the purity is upper bounded by 1 and in the last line, we have used the monotonicity of the Euclidean norm under the map Π. Therefore, the Lipschitz constant for the function $G : |\psi\rangle \to P(\Pi(\psi))$ can be chosen to be 2. Now applying Lévy's lemma to the function G and noting $k = 2d - 1$, the proof of the theorem follows. ■

Now we exploit the relation between the l_1 norm of coherence and the classical purity [188] to get an upper bound on the l_1 norm of coherence [a], which is

$$\mathscr{C}_{l_1}(\psi) \leq \sqrt{d(d-1)\left[1 - P(\Pi(\psi))\right]}. \tag{6.20}$$

Since the classical purity of a random pure state is concentrated on its expected value $\mathbb{E}_\psi P(\Pi(\psi)) = 2/(d+1)$ (see Theorem 18), one may replace $P(\Pi(\psi))$ by $2/(d+1)$ in Eq. (6.20) to get an upper bound on the l_1 norm of coherence which depends only on the dimension of the Hilbert space. Thus, $\mathscr{C}_{l_1}(\psi) \leq \sqrt{\frac{d(d-1)^2}{d+1}}$. Although this bound is very close to the trivial bound $(d-1)$, we note that better results on the average l_1 norm of coherence of random pure states and their typical nature can be obtained [b].

[a] Note that some of the results on the average l_1 norm of coherence were mentioned in Ref. [188].
[b] Private communication with Kaifeng Bu.

6.5 Random pure quantum states are not typically maximally coherent

It is well known that random bipartite pure states in higher dimension sampled from the uniform Haar measure are maximally entangled with an overwhelmingly large probability [29]. Our explorations in previous parts suggest that the randomly chosen pure states are not typically maximally coherent (to within an arbitrarily small error) as they have their relative entropy of coherence concentrated around $H_d - 1 \neq \ln d$ (see also Fig. 6.1). Here we make this observation precise by proving that indeed the trace distance between the diagonal part of a random pure state and the maximally mixed state does not typically go to zero in the higher dimension case, instead it is almost always concentrated around a fixed nonzero value. To establish this, we use the following lemma.

Lemma 19. *Let $|\psi\rangle$ be a random pure state in a d dimensional Hilbert space. The average trace distance between the diagonal part of a random pure state and the maximally mixed state is given by $2(1-1/d)^d$, i.e.,*

$$\mathbb{E}_\psi \left|\left|\rho^{(\mathrm{d})}(\psi) - \frac{\mathbb{I}}{d}\right|\right|_1 = 2\left(1-\frac{1}{d}\right)^d.$$

Proof. Consider a pure state $|\psi\rangle = \sum_{j=1}^d \psi_j |j\rangle$ with $\psi_j = \langle j|\psi\rangle = x_j + iy_j$, $i = \sqrt{-1}$ and $x_j, y_j \in \mathbb{R}(j = 1, \ldots, d)$. The unique, normalized, unitary invariant measure $\mathrm{d}(\psi)$ upon the pure state manifold of normalized state vectors $|\psi\rangle$ is realized by the following delta function prescription

$$\frac{\Gamma(d)}{\pi^d}\delta\left(1-\sum_{j=1}^d (x_j^2+y_j^2)\right)\prod_{j=1}^d \mathrm{d}x_j \mathrm{d}y_j,$$

if one is interested in calculating the averages of the functions of the form $f(\langle\psi|\hat{P}|\psi\rangle)$, where $\hat{P}$ is a projector [190]. This is the case for us. Here $\Gamma(d)$, which is equal to $(d-1)!$, is the Gamma function. By performing change of variables, namely, $x_j = \sqrt{r_j}\cos\theta_j$ and $y_j = \sqrt{r_j}\sin\theta_j$ in above for each j with $r_j \geqslant 0$ and $\theta_j \in [0, 2\pi]$, $\mathrm{d}(\psi)$ can also be realized as

$$\frac{\Gamma(d)}{(2\pi)^d}\delta\left(1 - \sum_{j=1}^{d} r_j\right)\prod_{j=1}^{d} \mathrm{d}r_j \mathrm{d}\theta_j.$$

For a fixed reference basis $\{|j\rangle\}(j = 1, \ldots, d)$, we have $\rho^{(\mathrm{d})}(\psi) = \sum_{j=1}^{d} |\psi_j|^2 |j\rangle\langle j|$ with $r_j := x_j^2 + y_j^2 = |\psi_j|^2$. Now

$$\begin{aligned}
&\mathbb{E}_\psi \left\| \rho^{(\mathrm{d})}(\psi) - \frac{\mathbb{I}_d}{d} \right\|_1 \\
&= \int \mathrm{d}(\psi) \left(\sum_{j=1}^{d} \left| |\psi_j|^2 - \frac{1}{d} \right| \right) \\
&= \Gamma(d) \int \left(\sum_{j=1}^{d} \left| r_j - \frac{1}{d} \right| \right) \delta\left(1 - \sum_{j=1}^{d} r_j\right) \prod_{j=1}^{d} \mathrm{d}r_j \\
&= \Gamma(d+1) \int_0^1 \mathrm{d}r_1 \left| r_1 - \frac{1}{d} \right| \int_0^\infty \delta\left((1 - r_1) - \sum_{j=2}^{d} r_j\right) \prod_{j=2}^{d} \mathrm{d}r_j \\
&= \frac{\Gamma(d+1)}{\Gamma(d-1)} \mathcal{K}. \qquad (6.21)
\end{aligned}$$

where $\mathcal{K} = \int_0^1 \mathrm{d}r_1 \left| r_1 - \frac{1}{d} \right| (1 - r_1)^{d-2}$. In what follows, we calculate the integral $\mathcal{K}$.

$$\begin{aligned}
\mathcal{K} &= \int_0^{\frac{1}{d}} \mathrm{d}r_1 \left(\frac{1}{d} - r_1\right)(1 - r_1)^{d-2} + \int_{\frac{1}{d}}^1 \mathrm{d}r_1 \left(r_1 - \frac{1}{d}\right)(1 - r_1)^{d-2} \\
&= \frac{-2}{d(d-1)} \left\{ \left(\frac{d-1}{d}\right)^{d-1} - 1 \right\} - 2\int_0^{\frac{1}{d}} \mathrm{d}r_1 r_1 (1 - r_1)^{d-2}. \qquad (6.22)
\end{aligned}$$

Now

$$\int_0^{\frac{1}{d}} r_1(1 - r_1)^{d-2}\mathrm{d}r_1 = \frac{-1}{d(d-1)} \left[\left(\frac{d-1}{d}\right)^{d-1} + \left(\frac{d-1}{d}\right)^{d} - 1 \right].$$

Putting above in Eq. (6.22), we get $\mathcal{K} = \frac{2}{d(d-1)}\left(1-\frac{1}{d}\right)^d$. Therefore,

$$\mathbb{E}_\psi \left|\left|\rho^{(\mathrm{d})}(\psi) - \frac{\mathbb{I}_d}{d}\right|\right|_1 \;=\; 2\left(1-\frac{1}{d}\right)^d. \tag{6.23}$$

This completes the proof of the lemma. ■

In the following theorem, we establish that most of the Haar distributed pure quantum states are not typically maximally coherent (within an arbitrarily small error). The main idea is to show that the trace distance of the diagonal part of any random pure quantum state from the maximally mixed state is almost always concentrated around a nonzero number, even in $d \to \infty$ limit.

Theorem 20. *Let $|\psi\rangle$ be a random pure state in a d dimensional Hilbert space. The probability that the trace distance between the diagonal part of a random pure state and the maximally mixed state is not close to $2\left(1-\frac{1}{d}\right)^d$ is bounded from above by an exponentially small number in the large d limit, i.e., for all $\epsilon > 0$*

$$\Pr\left\{\left|\ \left|\left|\rho^{(\mathrm{d})}(|\psi\rangle) - \frac{\mathbb{I}}{d}\right|\right|_1 - 2\left(1-\frac{1}{d}\right)^d\right| > \epsilon\right\} \leq 2\exp\left(-\frac{d\epsilon^2}{18\pi^3\ln 2}\right).$$

Proof. The Lipschitz constant for the function $F : |\psi\rangle \to \left|\left|\rho^{(\mathrm{d})}(\psi) - \frac{\mathbb{I}}{d}\right|\right|_1$ is 2 and it can be shown as follows:

$$\begin{aligned}
|F(|\psi\rangle) - F(|\phi\rangle)| &= \left|\ \left|\left|\rho^{(\mathrm{d})}(\psi) - \frac{\mathbb{I}}{d}\right|\right|_1 - \left|\left|\rho^{(\mathrm{d})}(\phi) - \frac{\mathbb{I}}{d}\right|\right|_1\right| \\
&\leq \left|\left|\rho^{(\mathrm{d})}(\psi) - \rho^{(\mathrm{d})}(\phi)\right|\right|_1 \\
&\leq ||\ |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|\ ||_1 \\
&\leq 2\sqrt{2(1 - \mathrm{Re}(\langle\psi|\,\phi\rangle)} = 2\,||\ |\psi\rangle - |\phi\rangle\ ||\,,
\end{aligned}$$

where in the second line we have used the reverse triangle inequality $|||A||_1 - ||B||_1| \leq ||A-B||_1$. Therefore, $F : |\psi\rangle \to \left|\left|\rho^{(\mathrm{d})}(\psi) - \frac{\mathbb{I}}{d}\right|\right|_1$ is a Lipschitz continuous function with the Lipschitz constant $\eta = 2$. Now applying Lévy's lemma for averages to the function $\left|\left|\rho^{(\mathrm{d})}(|\psi\rangle) - \frac{\mathbb{I}}{d}\right|\right|_1$, we obtain

$$\Pr\left\{\left| \left|\left|\rho^{(\mathrm{d})}(|\psi\rangle) - \frac{\mathbb{I}}{d}\right|\right|_1 - \mathbb{E}_\psi \left|\left|\rho^{(\mathrm{d})}(|\psi\rangle) - \frac{\mathbb{I}}{d}\right|\right|_1 \right| > \epsilon\right\} \leq \eta,$$

where $\eta = 2\exp\left(-d\epsilon^2/18\pi^3 \ln 2\right)$. We complete the proof of the theorem by using Lemma 19 in the above expression. ■

Theorem 20 tells us that for majority of pure quantum states the trace distance of the diagonal part from the maximally mixed state is concentrated around $2\left(1 - \frac{1}{d}\right)^d$, which for $d \to \infty$ converges to $2/e = 0.7357$. Therefore, the diagonal part of most of random pure quantum states maintains a fixed finite distance from the maximally mixed state. Thus, Theorem 20 implies that the overwhelming majority of random pure quantum states are not typically maximally coherent (within an arbitrarily small error). Next, we find a lower bound on the relative entropy of coherence of the majority of random pure quantum states, for which $\left|\left|\rho^{(\mathrm{d})}(|\psi\rangle) - \frac{\mathbb{I}}{d}\right|\right|_1 = 2\left(1 - \frac{1}{d}\right)^d$. Utilizing the Fannes-Audenaert inequality [145, 191], we have

$$\begin{aligned}\left|H\left(\frac{\mathbb{I}}{d}\right) - H(\rho^{(\mathrm{d})}(\psi))\right| = \ln d - H(\rho^{(\mathrm{d})}(\psi)) &\leq T\ln(d-1) + H_2(T)\\ &\leq T\ln d + H_2(T),\end{aligned}$$

where $T = \left|\left|\rho^{(\mathrm{d})}(|\psi\rangle) - \frac{\mathbb{I}}{d}\right|\right|_1/2 = \left(1 - \frac{1}{d}\right)^d$ and $H_2(T) = -T\ln T - (1-T)\ln(1-T)$ is the binary Shanon entropy. Therefore,

$$\mathscr{C}_r(\psi) = H(\rho^{(\mathrm{d})}(\psi)) \geq (1-T)\ln d - H_2(T). \tag{6.24}$$

Combining Eq. (6.24) with Theorem 20, we conclude that the relative entropy of coherence of a randomly picked pure state is, with high probability, always greater than

$(1-T)\ln d - H_2(T)$. For $d \to \infty$, we have

$$\lim_{d\to\infty} \mathscr{C}_r(\psi)/\ln d \geq 1 - \frac{1}{e} - \lim_{d\to\infty} H_2(T)/\ln d$$
$$= 1 - \frac{1}{e} \approx 0.6321. \tag{6.25}$$

6.6 Chapter summary

In this chapter, we have established various generic aspects of quantum coherence of random pure states sampled from the uniform Haar measure. We have shown that the amount of relative entropy of coherence of a pure state picked randomly with respect to the Haar measure, with a very high probability, is arbitrarily close to the average relative entropy of coherence, which is given by $\sum_{k=2}^{d} 1/k$ for a d-dimensional system. In other words, an overwhelming majority of the pure states have coherence equal to the expected value, within an arbitrarily small error. This also establishes the typical wave nature of a quantum particle in a random pure state. Further, we find a large subspace (of appropriate dimension) of the total Hilbert space of a quantum system such that for every pure state in this subspace, the relative entropy of coherence (also equal to the distillable coherence [121]) is almost always greater than a fixed number (depending on the dimension of the Hilbert space) that is arbitrarily close to the typical value of coherence. Also, for every state (pure or mixed) in this subspace, the coherence of formation is almost always bounded from below by the same fixed number. Therefore, quantum states in these subspaces can be useful for many coherence consuming protocols. Further, we find the expected value of classical purity of randomly chosen pure states, which is then used to find an upper bound on the l_1 norm of coherence exploiting known relations between coherence and classical purity. Furthermore, we find the average distance of the diagonal part of a randomly chosen pure quantum state from the maximally mixed state. We show that diagonal part of most of random pure states maintains a fixed nonzero distance from the maximally mixed state, thus establishing its typicality. This amounts to stating that most of the randomly chosen pure states are not typically maximally coherent (within an arbitrarily small error).

The results obtained in this chapter show the strong typicality of measures of coherence and establish that the description of coherence properties of the Haar distributed pure states, in larger dimensions, only requires a small number of typical parameters such as the Hilbert space dimension. These parameters appear in the formulation of the concentration of measure phenomenon. This, in turn, reduces a lot the complexity of coherence theory with respect to the Haar distributed pure states. In the future, it will be very interesting, from practical view point, to estimate the dimension of the largest subspace such that it contains no incoherent state, unlike our result, where we find the dimension of the subspace containing at least some fixed nonzero amount of coherence.

This chapter is based on the following paper:

1. *Average coherence and its typicality for random pure states*,
   U. Singh, L. Zhang, and A. K. Pati, Phys. Rev. A **93**, 032125 (2016).

CHAPTER 7

Typicality of coherence for random mixed states

7.1 Introduction

In the previous chapter (chapter 6), we have proved that the coherence of a random pure state sampled from the uniform Haar measure is generic for higher dimensional systems, i.e., most of the random pure states have almost the same amount of coherence [27]. The importance of this result and the similar results for entanglement of random bipartite pure states cannot be overemphasized. From an experimental viewpoint, a system of interest is never free from uncontrollable environmental interactions which lead to the loss of coherence in the quantum system. Therefore, mixed quantum states are encountered naturally in the realistic implementations of quantum technologies. However, to the best of our knowledge, there is no known result on the typicality of coherence for random mixed states.

In the continuation of the chapter 6, here we consider random mixed states sampled from various induced measures on the set of density matrices and establish their typicality in the context of the resource theory of coherence based on the incoherent operations [47]. We pay special attention to the relative entropy of coherence of random mixed states sampled from the induced measure obtained via the partial tracing of the Haar distributed random bipartite pure states. The typicality analysis of these random states is facilitated by finding the exact expression for the average subentropy of random mixed states sampled from induced probability measures. As a side result, we find, surprisingly, that the average subentropy of a random mixed state approaches exponentially fast towards the maximum value of the subentropy—which is achieved for the maximally mixed state [124]—eventhough the subentropy is a nonlinear function of quantum states. Importantly, using the concentration of measure phenomenon, in particular, Lévy's lemma, we show that an overwhelming majority of random mixed states have their relative entropies of coherence equal to the average relative entropy of coherence, within an arbitrarily small error, for larger Hilbert space dimensions. The typicality analysis of coherence of random mixed states helps us in tackling a very important and difficult question in the resource theory of entanglement—the calculation of various entanglement measures for random mixed states in larger Hilbert space dimensions. In particular, we show that for almost all random states of a specific class of bipartite random mixed states, the relative entropy of entanglement and distillable entanglement are equal to a fixed number (that we calculate) within an arbitrarily small error.

This chapter is organized as follows. We start with a discussion of measures of coherence, random mixed states and some other necessary preliminaries in Sec. 7.2. In Sec. 7.3, we calculate the average subentropy of the random mixed states sampled from various induced probability measures on the set of mixed states. We then present our results on the average relative entropy of coherence of random mixed states in Sec. 7.4. Then using the results on the average coherence of random mixed states we elaborate the typicality of the entanglement measures for a specific class of random bipartite mixed states in Sec. 7.5. Finally, we conclude in Sec. 7.6 with overview and implications of the results presented in this chapter. In Appendix D, we present the explicit calculations of various integrals that appear in the main text.

7.2 Quantum coherence and induced measures on the space of mixed states

7.2.1 Quantum coherence

Recently, various coherence monotones that serve as the faithful measures of coherence [23, 24, 47, 121] have been proposed based on the resource theory of coherence [47]. These monotones include the l_1 norm of coherence, relative entropy of coherence [47] and the geometric measure of coherence based on entanglement [23]. In this chapter, unless stated otherwise, by coherence we mean the relative entropy of coherence throughout this chapter. The relative entropy of coherence of a quantum state ρ, acting on an m-dimensional Hilbert space, is defined as [47]: $\mathscr{C}_r(\rho) := H(\Pi(\rho)) - H(\rho)$, where $\Pi(\rho) = \sum_{j=1}^{m} |j\rangle\langle j|\rho|j\rangle\langle j|$ for a fixed basis $\{ \ |j\rangle : j = 1, \ldots, m \}$. $H(\rho) = -\mathrm{Tr}[\rho \ln \rho]$ is the von Neumann entropy of ρ. All the logarithms that appear in this chapter are with respect to natural base.

7.2.2 Induced measures on the space of mixed states

Unlike on the set of pure states, it is known that there exist several inequivalent measures on the set of density matrices, $\mathrm{D}(\mathbb{C}^m)$ (the set of trace one nonnegative $m \times m$ matrices). By the spectral decomposition theorem for Hermitian matrices, any density matrix ρ can be diagonalized by a unitary U. It seems natural to assume that the distributions of eigenvalues and eigenvectors of ρ are independent, implying μ to be product measure $\nu \times \mu_{\mathrm{Haar}}$, where the measure μ_{Haar} is the unique Haar measure on the unitary group and measure ν defines the distribution of eigenvalues but there is no unique choice for it [183, 192].

The induced measures on the $(m^2 - 1)$-dimensional space $\mathrm{D}(\mathbb{C}^m)$ can be obtained by partial tracing the purifications $|\Psi\rangle$ in the larger composite Hilbert space of dimension mn and choosing the purified states according to the unique measure on it. Following Ref.

[183], the joint density $P_{m(n)}(\Lambda)$ of eigenvalues $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ of ρ, obtained via partial tracing, is given by

$$P_{m(n)}(\Lambda) = C_{m(n)} K_1(\Lambda) \prod_{j=1}^{m} \lambda_j^{n-m} \theta(\lambda_j), \tag{7.1}$$

where the theta function $\theta(\lambda_j)$ ensures that ρ is positive definite, $C_{m(n)}$ is the normalization constant and $K_1(\Lambda)$ is given by

$$K_\gamma(\Lambda) = \delta\left(1 - \sum_{j=1}^{m} \lambda_j\right) |\Delta(\lambda)|^{2\gamma}, \tag{7.2}$$

for $\gamma = 1$ with $\Delta(\lambda) = \prod_{1 \leqslant i < j \leqslant m}(\lambda_i - \lambda_j)$. See Refs. [183, 192] for a good exposition of induced measures on the set of density matrices.

Now we introduce the family of integrals $\mathcal{I}_m(\alpha, \gamma)$ that will be a key to this chapter, where

$$\begin{aligned}\mathcal{I}_m(\alpha, \gamma) &:= \int_0^\infty \cdots \int_0^\infty K_\gamma(\Lambda) \prod_{j=1}^{m} \lambda_j^{\alpha-1} \mathrm{d}\lambda_j \\ &= b_m(\alpha, \gamma) \prod_{j=1}^{m} \frac{\Gamma(\alpha + \gamma(j-1))\, \Gamma(1 + \gamma j)}{\Gamma(1+\gamma)},\end{aligned}$$

with $\alpha, \gamma > 0$, $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$ is the Gamma function, defined for $\mathrm{Re}(z) > 0$ and $b_m(\alpha, \gamma) = \{\Gamma(\alpha m + \gamma m(m-1))\}^{-1}$. The value of above family of integrals can be obtained using Selberg's integrals [183, 192, 193] (see Appendix D, Sec. D.2 for a quick review of Selberg's integrals). Let us define $C_m^{(\alpha,\gamma)} = 1/\mathcal{I}_m(\alpha, \gamma)$, which are called as normalization constants. A family of probability measures over $\mathbb{R}_+^m$ can be defined as:

$$\mathrm{d}\nu_{\alpha,\gamma}(\Lambda) := C_m^{(\alpha,\gamma)} K_\gamma(\Lambda) \prod_{j=1}^{m} \lambda_j^{\alpha-1} \mathrm{d}\lambda_j. \tag{7.3}$$

Also, $\nu_{\alpha,\gamma}$ is a family of normalized probability measures over the probability simplex

$\Delta_{m-1} := \left\{ \Lambda = \{\lambda_1, \ldots, \lambda_m\} \in \mathbb{R}_+^m : \sum_{j=1}^m \lambda_j = 1 \right\}$, i.e.,

$$\nu_{\alpha,\gamma}\left(\Delta_{m-1}\right) = \int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda) = 1.$$

Now a family of probability measures $\mu_{\alpha,\gamma}$ over the set $\mathrm{D}\left(\mathbb{C}^m\right)$ of all $m \times m$ density matrices on $\mathbb{C}^m$ can be obtained via spectral decomposition of $\rho \in \mathrm{D}\left(\mathbb{C}^m\right)$ with $\rho = U\Lambda U^\dagger$ as follows

$$\mathrm{d}\mu_{\alpha,\gamma}(\rho) = \mathrm{d}\nu_{\alpha,\gamma}(\Lambda) \times \mathrm{d}\mu_{\mathrm{Haar}}(U), \tag{7.4}$$

where $\mathrm{d}\nu_{\alpha,\gamma}(\Lambda) = \mathrm{d}\nu_{\alpha,\gamma}(\lambda_1, \ldots, \lambda_m)$ and μ_{Haar} is the normalized uniform Haar measure. By definition, $\mu_{\alpha,\gamma}$ is a normalized probability measure over $\mathrm{D}\left(\mathbb{C}^m\right)$. In the following, we will use this family of probability measures to calculate the average subentropy and average coherence of randomly chosen quantum states.

7.3 The average subentropy of random mixed states

Let us consider m dimensional random density matrices ρ sampled according to the family of product measures $\mu_{\alpha,\gamma}$, such that $\mathrm{d}\mu_{\alpha,\gamma}(\rho) = \mathrm{d}\nu_{\alpha,\gamma}(\Lambda) \times \mathrm{d}\mu_{\mathrm{Haar}}(U)$. The subentropy of a state ρ with the spectrum $\Lambda = \{\, \lambda_1, \cdots, \lambda_m \,\}$ can be written as [124, 194–196] (see also Appendix D, Sec. D.1)

$$Q(\Lambda) = (-1)^{\frac{m(m-1)}{2}-1} \frac{\sum_{i=1}^m \lambda_i^m \ln \lambda_i \prod_{j\in\widehat{i}} \phi'(\lambda_j)}{\left|\Delta(\lambda)\right|^2}, \tag{7.5}$$

where $\widehat{i} = \{\, 1, \ldots, m \,\} \setminus \{\, i \,\}$, $\phi'(\lambda_j) = \prod_{k\in\widehat{j}}(\lambda_j - \lambda_k)$ and $\left|\Delta(\lambda)\right|^2 = \left|\prod_{1\leqslant i<j\leqslant m}(\lambda_i - \lambda_j)\right|^2$. The average subentropy over the set of mixed states is given by

$$\mathcal{I}_m^Q(\alpha,\gamma) = \int \mathrm{d}\mu_{\alpha,\gamma}(\rho) Q(\rho) = \int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda) Q(\Lambda). \tag{7.6}$$

Proposition 21. *For $\gamma = 1$ and arbitrary α, the average subentropy $\mathcal{I}_m^Q(\alpha, 1)$ is given by*

$$\mathcal{I}_m^Q(\alpha, 1) = \frac{-1}{m(m+\alpha-1)} \sum_{k=0}^{m-1} g_{mk}(\alpha) u_{mk}(\alpha), \tag{7.7}$$

where

$$g_{mk}(\alpha) = \psi(2(m-1)+\alpha+1-k) - \psi(m(m+\alpha-1)+1) \text{ and} \tag{7.8}$$

$$u_{mk}(\alpha) = \frac{(-1)^k \Gamma(2(m-1)+\alpha+1-k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m+\alpha-1-k)}, \tag{7.9}$$

with $\psi(z) = \mathrm{d}\ln\Gamma(z)/\mathrm{d}z$ being the digamma function.

Proof. See Appendix D.2. □

In the remaining, we consider the *induced measure* $\mu_{m(n)}(m \leqslant n)$ over all the $m \times m$ density matrices of the m-dimensional quantum system via partial tracing over the n-dimensional ancilla of uniformly Haar-distributed random bipartite pure states of system and ancilla, which is as follows: for $\rho = U\Lambda U^\dagger$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ and $U \in \mathrm{U}(m)$,

$$\mathrm{d}\mu_{m(n)}(\rho) = \mathrm{d}\nu_{m(n)}(\Lambda) \times \mathrm{d}\mu_{\mathrm{Haar}}(U), \tag{7.10}$$

where $\mathrm{d}\nu_{m(n)}(\Lambda) = C_{m(n)} K_1(\Lambda) \prod_{j=1}^m \lambda_j^{n-m} \mathrm{d}\lambda_j$ [183] is the joint distribution of eigenvalues $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ of the density matrix ρ, and $\mathrm{d}\mu_{\mathrm{Haar}}(U)$ is the uniform Haar measure over unitary group $\mathrm{U}(m)$. Apparently Eq. (7.10) is a special case of Eq. (7.4) when $(\alpha, \gamma) = (n-m+1, 1)$. That is, $\mathrm{d}\mu_{m(n)}(\rho) = \mathrm{d}\mu_{n-m+1,1}(\rho)$ and $\mathrm{d}\nu_{m(n)}(\Lambda) =$

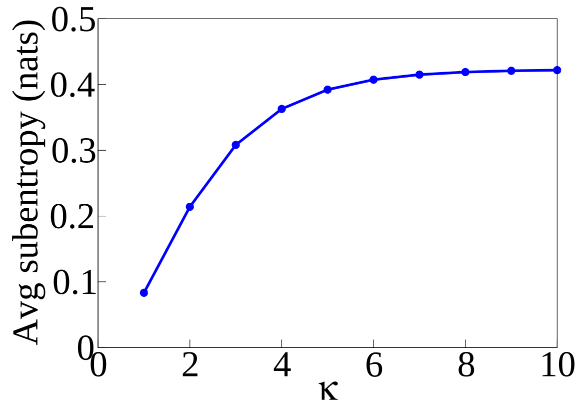

Figure 7.1: The plot shows the average subentropy, obtained in Eq. (7.12), as a function of dimension $m = 2^{\kappa}$. In the first plot x axis is dimensionless. Surprisingly, as we increase κ, the average subentropy approaches to the maximum value that subentropy can take.

$\mathrm{d}\nu_{n-m+1,1}(\Lambda)$. From this, we see that

$$\mathcal{I}_m^Q(n-m+1,1) = -\frac{1}{mn}\sum_{k=0}^{m-1} g_{mk}(n-m+1)u_{mk}(n-m+1). \tag{7.11}$$

The closed compact form of the above equation is obtained by us in the Ref. [28]. If $m = n$, this situation corresponds to the probability measure induced by the Hilbert-Schmidt distance [183], then

$$\mathcal{I}_m^Q(1,1) = -\frac{1}{m^2}\sum_{k=0}^{m-1} g_{mk}(1)u_{mk}(1). \tag{7.12}$$

In Eqs. (7.11) and (7.12), the functions g_{mk} and u_{mk} are given by Eqs. (7.8) and (7.9). If we plot the average subentropy for random mixed states of dimension m, Eq. (7.12), as a function of m (see Fig. 7.1), we find that it approaches exponentially fast towards the maximum value of the subentropy, which is achieved for the maximally mixed state [124]. The maximum value of $Q(\rho)$ is approximately equal to 0.42278 [124]. This is surprising, since $Q(\rho)$ is a nonlinear function of ρ and it is not expected that the average subentropy should match with the subentropy of the average state, which is the maximally mixed state. Table 7.1 lists the values of $\mathcal{I}_m^Q(1,1)$ as a function of m and shows that indeed the subentropy approaches to its maximum value.

Table 7.1: The average subentropy for random mixed states of dimension m. Δ is the difference between successive values in the second column and surprisingly shows an exponential convergence towards the maximum value of subentropy (≈ 0.42278) as we increase m. The difference between the successive values of the average subentropy is almost halved as we increase the number of qubits by one.

m	$\mathcal{I}_m^Q(1,1)$	Δ
2	0.083333	
4	0.214062	-0.130729
8	0.308176	-0.094114
16	0.362886	-0.054710
32	0.392185	-0.029299
64	0.407322	-0.015137
128	0.415012	-0.007690
256	0.418888	-0.003876
512	0.420833	-0.001945
1024	0.421808	-0.000975

7.4 The average coherence of random mixed states and typicality

Now, we are in a position to calculate the average coherence of random mixed states and establish its typicality. Let $\rho = U\Lambda U^\dagger$ be a mixed full-ranked quantum state on $\mathbb{C}^m$ with non-degenerate positive spectra $\lambda_j \in \mathbb{R}^+ (j = 1, \ldots, m)$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$. Then coherence of the state ρ is given by $\mathscr{C}_r(U\Lambda U^\dagger) = H(\Pi(U\Lambda U^\dagger)) - H(\Lambda)$. The average coherence of the isosepectral density matrices can be expressed in terms of the quantum subentropy, von Neumann entropy, and m-Harmonic number as follows [188]:

$$\overline{\mathscr{C}}_r^{\mathrm{iso}}(\Lambda) \quad := \int \mathrm{d}\mu_{\mathrm{Haar}}(U)\mathscr{C}_r(U\Lambda U^\dagger) = H_m - 1 + Q(\Lambda) - H(\Lambda).$$

Here $Q(\Lambda)$ is the subentropy, given by Eq. (7.5), $H(\Lambda)$ is the von Neumann entropy of Λ and $H_m = \sum_{k=1}^m 1/k$ is the m-Harmonic number. From this, we see that the average coherence of isospectral full-ranked density matrices depends completely on the spectrum. Also, it is known that $0 \leqslant Q(\Lambda) \leqslant 1 - \gamma_{\text{Euler}} \approx 0.42278$, where γ_{Euler} is the Euler's constant. Now, using the product probability measures $\mathrm{d}\mu_{\alpha,\gamma} = \mathrm{d}\nu_{\alpha,\gamma} \times \mu_{\text{Haar}}(U)$, the average coherence of random mixed states is given by

$$\begin{aligned} \overline{\mathscr{C}_r}(\alpha,\gamma) &:= \int \mathrm{d}\mu_{\alpha,\gamma}(\rho)\mathscr{C}_r(\rho) = \int \mathrm{d}\mu_{\alpha,\gamma}(U\Lambda U^\dagger)\mathscr{C}_r(U\Lambda U^\dagger) \\ &= H_m - 1 + \mathcal{I}_m^Q(\alpha,\gamma) - \mathcal{I}_m^S(\alpha,\gamma), \end{aligned} \tag{7.13}$$

where $\mathcal{I}_m^Q(\alpha,\gamma) = \int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda)Q(\Lambda)$ and $\mathcal{I}_m^S(\alpha,\gamma) = \int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda)H(\Lambda)$. In the remaining, we again consider the *induced measure* $\mu_{m(n)}(m \leqslant n)$ over all the $m \times m$ density matrices of the m-dimensional quantum system via partial tracing over the n-dimensional ancilla of uniformly Haar-distributed random pure bipartite states of system and ancilla.

Theorem 22. *The average coherence of random mixed states of dimension m sampled from induced measures obtained via partial tracing of Haar distributed bipartite pure states of dimension mn, for $(\alpha,\gamma) = (n-m+1,1)$, is given by*

$$\overline{\mathscr{C}_r}(n-m+1,1) = H_m - 1 - \left(\sum_{k=n+1}^{mn}\frac{1}{k} - \frac{m-1}{2n}\right) - \sum_{k=0}^{m-1}\frac{g_{mk}(n-m+1)u_{mk}(n-m+1)}{mn}.$$

Proof. See Appendix D.2. □

The closed compact form of the above equation is presented in the Ref. [28]. For $m = n$, which corresponds to the probability measure induced by the Hilbert-Schmidt distance, the average coherence of random mixed states is given by

$$\overline{\mathscr{C}_r}(1,1) = H_m - 1 - \mathcal{D}_m, \tag{7.14}$$

with $\mathcal{D}_m = \frac{1}{m^2}\sum_{k=0}^{m-1} g_{mk}(1)u_{mk}(1) + \sum_{k=m+1}^{m^2}\frac{1}{k} - \frac{m-1}{2m}$. To gain insights about above result (Theorem 22), we calculate numerical values of the average coherence, Eq. (7.14), for various values of m and show that the average value approaches to a fixed number very

Table 7.2: The (scaled) average relative entropy of coherence for random mixed states of dimension m. Δ, the difference between successive values in the second column, shows a rather slow convergence as a function of m.

n	$\overline{\mathscr{C}_r}(1,1)/\ln m$	Δ
2	0.360673	
4	0.270505	0.090168
8	0.210393	0.060112
16	0.169065	0.041328
32	0.139761	0.029304
64	0.118346	0.021415
128	0.102244	0.016102
256	0.089816	0.012428
512	0.079993	0.009823
1024	0.072064	0.007929

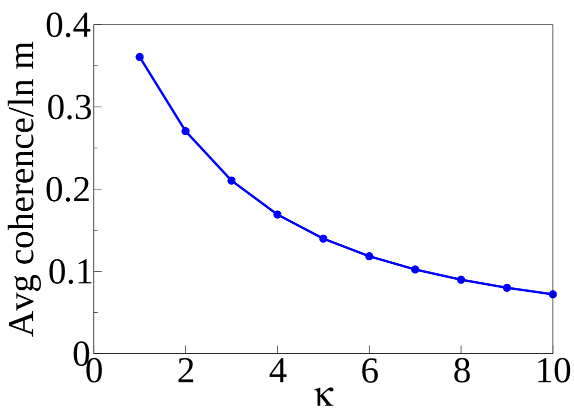

Figure 7.2: The plot shows the (scaled) average relative entropy of coherence, $\overline{\mathscr{C}_r}(1,1)/\ln m$, obtained in Eq. (7.14), as a function of dimension $m = 2^\kappa$. Both the axes in the figure are dimensionless.

slowly (see Fig. 7.2). Table 7.2 lists the values of the (scaled) average relative entropy of coherence.

Now, just like in the case of random pure states where the average coherence is a generic property of all random pure states [27], one may ask if the average coherence of random mixed states is also a generic property of all random mixed states. The following theorem (Theorem 23) establishes that the average coherence is indeed a generic property of all random mixed states, i.e., as we increase the dimension of the density matrix, almost all the density matrices generated randomly have coherence approximately equal to the average relative entropy of coherence, given by Theorem 22. Thus, the average coherence of a random mixed state can be viewed as the typical coherence content of random mixed states.

Theorem 23. *Let ρ^A be a random mixed state on an m dimensional Hilbert space $\mathcal{H}$ with $m \geq 3$ generated via partial tracing of the Haar distributed bipartite pure states on mn dimensional Hilbert space. Then, for all $\epsilon > 0$, the relative entropy of coherence $C_r(\rho^A)$ of ρ^A satisfies the following inequality:*

$$\Pr\left\{|C_r(\rho^A) - \overline{\mathscr{C}_r}(n-m+1,1)| > \epsilon\right\} \leq 2\exp\left(-\frac{mn\epsilon^2}{144\pi^3 \ln 2(\ln m)^2}\right). \tag{7.15}$$

Proof. See Appendix D.2. □

Fig. 7.3 shows that indeed the relative entropy of coherence of most of the randomly generated mixed states concentrate around the average relative entropy of coherence. Next, we present an important consequence of Theorem 23 showing a reduction in computational complexity of certain entanglement measures for a specific class of mixed states.

7.5 Entanglement properties of a specific class of random bipartite mixed states

Consider a specific class $\mathcal{X}$ of random bipartite mixed states χ^{AB} of dimension $m \otimes m$ that are generated as follows. First generate random mixed states for a single quantum system A in an m dimensional Hilbert space via partial tracing the Haar distributed bipartite pure states on an mn dimensional Hilbert space. Now bring in an ancilla B in a fixed state $|0\rangle\langle 0|^B$ on a d_B dimensional Hilbert space and apply the generalized CNOT gate, defined as $\text{CNOT} = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} |i\rangle\langle i|^A \otimes |\text{mod}(i+j,m)\rangle\langle j|^B + \sum_{i=0}^{m-1}\sum_{j=m}^{d_B-1} |i\rangle\langle i|^A \otimes |j\rangle\langle j|^B$, on the composite system AB. The random bipartite mixed states, thus obtained,

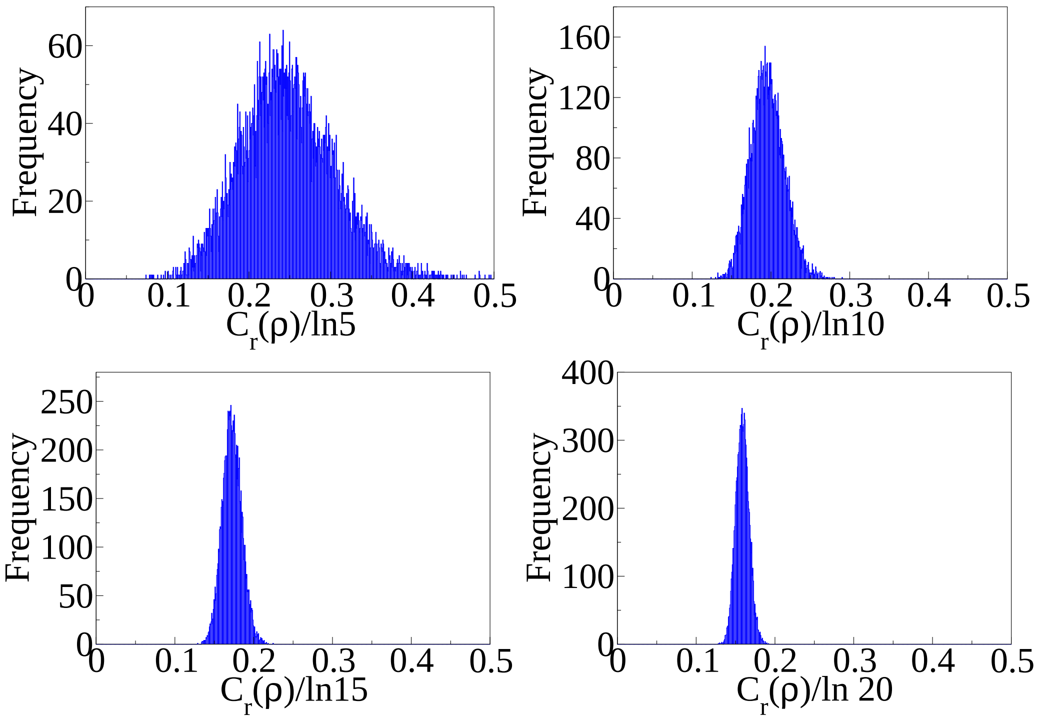

Figure 7.3: The frequency plot showing the (scaled) relative entropy of coherence, $C_r(\rho)/\ln m$ for random mixed states for dimensions $m = 2, 5, 10$ and 20 obtained via partial tracing of bipartite $m \times m$ Haar distributed pure states. Here x and y axes both are dimensionless. We note that as we increase the dimension the figure shows that more and more states have coherence close to a fixed value which is very close to the average value of coherence that we have calculated.

are given by

$$\chi^{AB} := \mathrm{CNOT}\left[\rho^A \otimes |0\rangle\langle 0|^B\right] = \sum_{i,j=0}^{m-1} \rho_{ij} |ii\rangle\langle jj|^{AB},$$

where and $\rho^A := \mathrm{Tr}_{A_0}\left\{|\psi\rangle\langle\psi|^{AA_0}\right\} = \sum_{i,j=0}^{m-1} \rho_{ij} |i\rangle\langle j|^A$ is a random mixed state generated according to an induced measure via partial tracing as mentioned above. Now, using the results on convertibility of coherence into entanglement [23], we can estimate exactly the relative entropy of entanglement E_r [197] and distillable entanglement E_d [168, 170] of random mixed states in the class $\mathcal{X}$. In particular,

$$E_r^{A|B}(\chi^{AB}) = C_r(\rho^A) = E_d^{A|B}(\chi^{AB}). \tag{7.16}$$

We can now use our exact results on the average relative entropy of coherence of random mixed states to find the average entanglement for the specific class of bipartite random mixed states in the class $\mathcal{X}$ as follows:

$$\begin{aligned}\overline{E}_r^{A|B}(\chi^{AB}) &= \int \mathrm{d}\mu_{n-m+1,1}(\rho) E_r^{A|B}\left(\mathrm{CNOT}\left[\rho^A \otimes |0\rangle\langle 0|^B\right]\right) \\ &= \int \mathrm{d}\mu_{n-m+1,1}(\rho) C_r(\rho^A) = \overline{\mathscr{C}_r}(n-m+1,1). \end{aligned} \tag{7.17}$$

Here $\overline{\mathscr{C}_r}(n-m+1,1)$ is given by Theorem 22. Similarly, $\overline{E}_d^{A|B}(\chi^{AB}) = \overline{\mathscr{C}_r}(n-m+1,1)$. The following corollary follows immediately from Theorem 23.

Corollary 24. *Let $\chi^{AB} \in \mathcal{X}$ be a random mixed state on $m \otimes m$ dimensional Hilbert space with $m \geq 3$ generated as mentioned above. Then, for all $\epsilon > 0$*

$$\Pr\left\{|E_r^{A|B}(\chi^{AB}) - \overline{\mathscr{C}_r}(n-m+1,1)| > \epsilon\right\} \leq 2\exp\left(-\frac{mn\epsilon^2}{144\pi^3 \ln 2(\ln m)^2}\right) \tag{7.18}$$

and

$$\Pr\left\{|E_d^{A|B}(\chi^{AB}) - \overline{\mathscr{C}_r}(n-m+1,1)| > \epsilon\right\} \leq 2\exp\left(-\frac{mn\epsilon^2}{144\pi^3 \ln 2(\ln m)^2}\right). \tag{7.19}$$

Corollary 24 establishes that most of the random states in the class $\mathcal{X}$ have almost the same fixed amount of distillable entanglement and relative entropy of entanglement in the large m limit. Thus, our results help in estimating the entanglement content of most of the random states in the class $\mathcal{X}$ (which is an extremely hard task), asymptotically and show the typicality of entanglement for class $\mathcal{X}$ of mixed states.

7.6 Chapter summary

To conclude, we have provided analytical expressions for the average subentropy and the average relative entropy of coherence over the whole set of density matrices distributed according to the family of probability measures obtained via the spectral decomposition. We find that as we increase the dimension of the quantum system, the average subentropy approaches towards the maximum value of subentropy (attained for the maximally mixed state) exponentially fast, which is surprising as the subentropy is a nonlinear function of density matrix. However, the scaled average coherence does not converge quickly to some fixed value. Interestingly, using Lévy's lemma, we prove that the coherence of random mixed states sampled from induced measures via partial tracing show the concentration phenomenon, establishing the generic nature of coherence content of random mixed states. As a very important application of our results, we show a huge reduction in the computational complexity of entanglement measures such as the relative entropy of entanglement and the distillable entanglement. We find the entanglement properties of a specific class random bipartite mixed states, thanks to Theorem 23. Since quantum coherence and entanglement are deemed as useful resources for implementations of various quantum technologies, our results will serve as a benchmark to gauge the resourcefulness of a generic mixed state for a certain task at hand. Furthermore, our results may have some applications in black hole physics as to how much coherence can be there in the Hawking radiation for non-thermal states [198], in thermalization of closed quantum systems and in catalytic coherence transformations.

This chapter is based on the following paper:

1. *Average subentropy, coherence and entanglement of random mixed quantum states*, L. Zhang, **U. Singh**, and A. K. Pati, Annals of Phys. **377**, 125 (2017).

CHAPTER 8

Summary and future directions

The theory of quantum information and computation is arguably one of main theories that put the weirdness of quantum physics into our everyday use in the form of quantum technologies [199, 200] that include quantum key distribution [7], quantum memories (see e.g. Ref. [201] and references therein) and quantum computation [202, 203], among others. Quantum coherence and entanglement are key resources for quantum enhanced technologies and quantum resource theories provide a rigorous and mathematically complete description of various resources such as quantum coherence and entanglement and are the backbones of the theory of quantum information and computation. This thesis is a significant contribution to the quantum resource theories of coherence and entanglement.

In this thesis, we have characterized quantum coherence at quantitative, qualitative and operational levels for single and multipartite quantum systems. We have also investigated noisy scenarios in the context of resource theory of coherence based on incoherent operations and elaborated how coherence of a quantum state trades with mixedness in these scenarios. The salient points of the thesis and possible future directions motivated by our investigations are summarized as follows:

- We find necessary and sufficient conditions for converting coherence of a single quantum system into entanglement between the system and an arbitrary ancilla using incoherent operations. Based on this connection, we introduce a family of *bona fide* coherence monotones in terms of the maximal amount of entanglement that can be created from the system by incoherent operations.

The framework presented here is also comparable to the scheme for activating distillable entanglement via premeasurement interactions [133–135] from quantum discord [136, 137]. Investigating these connections further will be the subject of future research.

- We provide an operational quantifier of quantum coherence in terms of the minimal amount of noise that is required to be injected into a quantum system in order to fully decohere it. This also provides the cost of erasing quantum coherence and in the asymptotic limit, it is equal to the relative entropy of coherence. Our results along with results of Ref. [121] further escalate the significance of the relative entropy of coherence as a bona fide measure of coherence.

  It will be an important future direction to investigate the converse, i.e., what is the cost to keep a state coherent? It is interesting to know if whether this cost is also equal to the relative entropy of coherence of ρ, in the asymptotic limit. Further, a clear quantitative connection of our results to the Landauer's erasure principle [67] along with its improved and generalized versions [151, 152] and the no-hiding theorem [153, 154] will be another research direction that we leave open.

- We consider noisy scenarios in the context of processing coherence as a resource and find an intrinsic trade-off between the resourcefulness and the degree of noise in an arbitrary quantum system. We derive analytically a class of maximally coherent mixed states, up to incoherent unitaries, that satisfy a complementarity relation between coherence and mixedness, in any quantum system. From a resource theoretic point of view, our results quantify the maximal amount of coherence that can be put to our use in quantum technologies in the presence of unavoidable noise. As an immediate application of our investigations here one can prove that the maximum entanglement that can be created between a quantum system and an incoherent ancilla, via incoherent operations, is bounded from above by the mixedness present in the system. We also investigate the order and interconvertibility between classes of quantum states and prove that the for qubit systems with a fixed coherence, majorization provides a total order on the states based on their degree of mixedness, while for fixed mixedness, all the qubit states with varying degree of coherences are interconvertible.

As a future direction, it will be very interesting to investigate if there exists such a total order in d-dimensional states with fixed coherence based on their degree of mixedness.

- We investigate catalytic coherence transformations and find the necessary and sufficient conditions for the deterministic and stochastic coherence transformations between pure quantum states assisted by catalysts using only incoherent operations. Further, we delineate the structure of the catalysts and possibility of *self catalysis*, i.e., possiblity for a pure quantum state to act as a catalyst for itself. Moreover, in the cases where catalysis is not possible, we investigate in detail the possibility of using an entangled state. In this way we completely characterize the allowed manipulations of the coherence of pure quantum states.

  The consideration of catalytic transformations is very natural and has resulted in strikingly nontrivial consequences, e.g., the introduction of many second laws of quantum thermodynamics was made possible by consideration of catalysts in resource theory of quantum thermodynamics. In the similar spirit our results will be useful in the processing of quantum coherence in the context of *single-shot* quantum information theory. An important question for future research will be to analyze the possibility of *self catalysis* in greater detail as the catalysts in this case are readily available.

- We establish generic aspects of quantum coherence of random pure and mixed states sampled from various probability measures on the Hilbert space. We show that an overwhelming majority of the pure states sampled the uniform Haar measure have coherence equal to the expected value, within an arbitrarily small error. We also show that most of the randomly chosen pure states are not typically maximally coherent (within an arbitrarily small error). Similarly, random mixed states sampled from induced measures via partial tracing show the concentration phenomenon. Lévy's lemma has been instrumental in these results. As an important application of our results, we show a huge reduction in the computational complexity of entanglement measures such as relative entropy of entanglement and distillable entanglement for a specific class random bipartite mixed states. Since quantum coherence and entanglement are deemed as useful resources for implementations of various quantum technologies, our results will serve as a benchmark to gauge the resourcefulness of a generic mixed state for a certain task at hand.

  From a practical view point, it will be very important and far reaching to investigate the typical nature of coherence in the constrained Hilbert spaces. These constraints may arise from the conservation laws such as conservation of energy or from other practical limitations. Importantly, this will require a generalization of Lévy's lemma and therefore, will be of independent mathematical interest. Further, it will be important to explore the implications of our typicality analysis in thermalization of closed quantum systems and in catalytic coherence transformations.

APPENDIX A

Proof of monotonicity (C3) and convexity (C4) in Theorem 3

Proof of monotonicity (C3) in Theorem 3

Here we prove that for any entanglement monotone E the coherence quantifier

$$C_E(\rho^S) = \lim_{d_A \to \infty} \left\{ \sup_{\Lambda^{SA}} E^{S:A}\left(\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right]\right)\right\} \tag{A.1}$$

does not increase on average under (selective) incoherent operations:

$$\sum_i p_i C_E(\sigma_i^S) \leq C_E(\rho^S) \tag{A.2}$$

with probabilities $p_i = \mathrm{Tr}[K_i \rho^S K_i^\dagger]$, quantum states $\sigma_i^S = K_i \rho^S K_i^\dagger / p_i$, and incoherent Kraus operators K_i acting on the system S.

Due to the definition of C_E, the amount of entanglement between the system and ancilla cannot exceed C_E for any incoherent operation Λ^{SA}, i.e.,

$$E^{S:A}\left(\Lambda^{SA}\left[\rho^S \otimes |0\rangle\langle 0|^A\right]\right) \leq C_E\left(\rho^S\right). \tag{A.3}$$

Note that this statement is also true if we introduce another particle B in an incoherent state $|0\rangle\langle 0|^B$. Then, for any tripartite incoherent operation Λ^{SAB} it holds:

$$E^{S:AB}\left(\Lambda^{SAB}\left[\rho^S \otimes |0\rangle\langle 0|^A \otimes |0\rangle\langle 0|^B\right]\right) \leq C_E\left(\rho^S\right). \tag{A.4}$$

We will now prove the claim by contradiction, showing that a violation of Eq. (A.2) also implies a violation of Eq. (A.4). If Eq. (A.2) is violated, then by definition of C_E there exists a set of incoherent operations Λ_i^{SA} such that the following inequality is true for d_A large enough:

$$\sum_i p_i E^{S:A}\left(\Lambda_i^{SA}\left[\sigma_i^S \otimes |0\rangle\langle 0|^A\right]\right) > C_E\left(\rho^S\right). \tag{A.5}$$

In the next step we introduce an additional particle B and use the general relation

$$E^{S:AB}\left(\sum_i p_i \rho_i^{SA} \otimes |i\rangle\langle i|^B\right) \geq \sum_i p_i E^{S:A}\left(\rho_i^{SA}\right) \tag{A.6}$$

which is valid for any entanglement monotone E. With this in mind, the inequality (A.5) implies

$$E^{S:AB}\left(\sum_i p_i \Lambda_i^{SA}\left[\sigma_i^S \otimes |0\rangle\langle 0|^A\right] \otimes |i\rangle\langle i|^B\right) > C_E\left(\rho^S\right). \tag{A.7}$$

Recall that the states σ_i^S are obtained from the state ρ^S by the means of an incoherent operation, and thus we can use the relation $p_i\sigma_i^S = K_i\rho^S K_i^\dagger$ with incoherent Kraus operators K_i. This leads us to the following expression:

$$E^{S:AB}\left(\sum_i \Lambda_i^{SA}\left[K_i\rho^S K_i^\dagger \otimes |0\rangle\langle 0|^A\right] \otimes |i\rangle\langle i|^B\right) > C_E\left(\rho^S\right). \tag{A.8}$$

It is now crucial to note that the state on the left-hand side of the above expression can be regarded as arising from a tripartite incoherent operation Λ^{SAB} acting on the total state

$\rho^S \otimes |0\rangle\langle 0|^A \otimes |0\rangle\langle 0|^B$:

$$\Lambda^{SAB}\left[\rho^S \otimes |0\rangle\langle 0|^A \otimes |0\rangle\langle 0|^B\right] = \sum_i \Lambda_i^{SA}\left[K_i \rho^S K_i^\dagger \otimes |0\rangle\langle 0|^A\right] \otimes |i\rangle\langle i|^B. \quad \text{(A.9)}$$

This can be seen explicitly by introducing the Kraus operators M_{ij} corresponding to the operation Λ^{SAB}:

$$M_{ij}^{SAB} = L_{ij}^{SA}\left(K_i^S \otimes \mathbb{I}^A\right) \otimes U_i^B. \quad \text{(A.10)}$$

Here, L_{ij} are incoherent Kraus operators corresponding to the incoherent operation Λ_i^{SA}:

$$\Lambda_i^{SA}\left[\rho^{SA}\right] = \sum_j L_{ij} \rho^{SA} L_{ij}^\dagger. \quad \text{(A.11)}$$

The unitaries U_i^B are incoherent and defined as

$$U_i^B = \sum_{j=0}^{d_B-1} |\mathrm{mod}(i+j, d_B)\rangle\langle j|^B. \quad \text{(A.12)}$$

With these definitions we see that M_{ij} are indeed incoherent Kraus operators. Moreover, it can be verified by inspection that the incoherent operation Λ^{SAB} arising from these Kraus operators also satisfies Eq. (A.9).

Finally, using Eq. (A.9) in Eq. (A.8) we arrive at the following inequality:

$$E^{S:AB}\left(\Lambda^{SAB}\left[\rho^S \otimes |0\rangle\langle 0|^A \otimes |0\rangle\langle 0|^B\right]\right) > C_E\left(\rho^S\right). \quad \text{(A.13)}$$

This is the desired contradiction to Eq. (A.4), and completes the proof of property (C3) for C_E, thus establishing that C_E is a coherence monotone for any entanglement monotone E.

Proof of convexity (C4) in Theorem 3

Here we show that the quantifier of coherence C_E given in Eq. (A.1) is convex for any convex entanglement measure E:

$$C_E\left(\sum_i p_i\rho_i^S\right)\leq\sum_i p_iC_E\left(\rho_i^S\right) \tag{A.14}$$

for any quantum states ρ_i^S and probabilities p_i. For this, note that by convexity of the entanglement quantifier E it follows:

$$E^{S:A}\left(\Lambda^{SA}\left[\sum_i p_i\rho_i^S\otimes|0\rangle\langle 0|^A\right]\right)\leq\sum_i p_iE^{S:A}\left(\Lambda^{SA}\left[\rho_i^S\otimes|0\rangle\langle 0|^A\right]\right). \tag{A.15}$$

Taking the supremum over all incoherent operations Λ^{SA} together with the limit $d_A\rightarrow\infty$ on both sides of this inequality we obtain the following result:

$$C_E\left(\sum_i p_i\rho_i^S\right)\leq\lim_{d_A\rightarrow\infty}\sup_{\Lambda^{SA}}\left\{\sum_i p_iE^{S:A}\left(\Lambda^{SA}\left[\rho_i^S\otimes|0\rangle\langle 0|^A\right]\right)\right\}. \tag{A.16}$$

Finally, note that the right-hand side of this inequality cannot decrease if the supremum over incoherent operations Λ^{SA} and the limit $d_A\rightarrow\infty$ are performed on each term of the sum individually:

$$\begin{aligned}&\lim_{d_A\rightarrow\infty}\sup_{\Lambda^{SA}}\left\{\sum_i p_iE^{S:A}\left(\Lambda^{SA}\left[\rho_i^S\otimes|0\rangle\langle 0|^A\right]\right)\right\}\\&\leq\sum_i p_i\lim_{d_A\rightarrow\infty}\sup_{\Lambda^{SA}}E^{S:A}\left(\Lambda^{SA}\left[\rho_i^S\otimes|0\rangle\langle 0|^A\right]\right)\\&=\sum_i p_iC_E\left(\rho_i^S\right).\end{aligned} \tag{A.17}$$

Together with Eq. (A.16), this completes the proof of convexity in Eq. (A.14).

APPENDIX B

The Fannes-Audenaert Inequality, the gentle operator lemma and the operator Chernoff bound

B.1 The Fannes-Audenaert inequality and the gentle operator lemma

The Fannes-Audenaert inequality :– In the context of continuity of the von Neumann entropy, Audenaert proved a tighter inequality than the Fannes inequality [191], which is now known as the Fannes-Audenaert inequality [145] and can be stated as follows: For any ρ and σ with $T \equiv \frac{1}{2}||\rho - \sigma||_1$, the following inequality holds

$$|H(\rho) - H(\sigma)| \leq T \log(d-1) + H_2(T), \tag{B.1}$$

where d is the dimension of the Hilbert space of the state ρ and $H_2(T) = -T \ln T - (1 - T)\ln(1-T)$ is the binary Shannon entropy.

The gentle operator lemma :– The gentle operator lemma, which was first stated in Ref. [204] and later improved in Ref. [205] is stated as follows: Suppose that a measurement operator Λ ($0 \leq \Lambda \leq I$) has a high probability of detecting a subnormalised state ρ, i.e., $\mathrm{Tr}\{\Lambda\rho\} \geq \mathrm{Tr}(\rho) - \epsilon$, where $1 \geq \epsilon > 0$ and ϵ is close to zero. Then $\sqrt{\Lambda}\rho\sqrt{\Lambda}$ is close to the original state ρ such that,

$$\left\| \rho - \sqrt{\Lambda}\rho\sqrt{\Lambda} \right\|_1 \leq 2\sqrt{\epsilon}, \tag{B.2}$$

where $\|\sigma\|_1 = \mathrm{Tr}\sqrt{\sigma^\dagger \sigma}$.

B.2 The typical subspaces and the operator Chernoff bound

In this section, we give definitions of the typical subspaces and discuss their properties. See Ref. [3] for further reading.

Typical sequence and typical set:– Consider a sequence x^n of n realizations of a random variable X which takes values $\{x\}$ according to probability distribution $\{p_X(x)\}$. A sequence x^n is δ-*typical* if its sample entropy $\overline{H}\left(x^n\right)$, defined as $-\frac{1}{n}\log p_{X^n}(x^n)$, is δ-close to the entropy $H\left(X\right)$ of random variable X, where this random variable is the source of the sequence. The set of all δ-typical sequences x^n is defined as the typical set $T_\delta^{X^n}$, i.e.,

$$T_\delta^{X^n} \equiv \{x^n : |\overline{H}\left(x^n\right) - H(X)| \leq \delta\}. \tag{B.3}$$

Now, consider a quantum state with spectral decomposition as

$$\rho^X = \sum_x p_X(x) \left|x\right\rangle \left\langle x\right|^X. \tag{B.4}$$

Considering n copies of the state ρ^X, we have

$$(\rho^X)^{\otimes n} := \rho^{X^n} = \sum_{x^n} p_{X^n}(x^n) \left|x^n\right\rangle \left\langle x^n\right|^{X^n}, \tag{B.5}$$

where $X^n = (X_1 \ldots X_n)$, $x^n = (x_1 \ldots x_n)$, $p_{X^n}(x^n) = p_X(x_1)\ldots p_X(x_n)$ and $\left|x^n\right\rangle = \left|x_1\right\rangle^{X_1} \otimes \ldots \otimes \left|x_n\right\rangle^{X_n}$.

Typical subspace:– The δ-typical subspace $T_{\rho,\delta}^{X^n}$ is a subspace of the full Hilbert space X_1,

$\ldots$, X_n and is spanned by states $|x^n\rangle^{X^n}$ whose corresponding classical sequences x^n are δ-typical:

$$T_{\rho,\delta}^{X^n} \equiv \textit{span}\left\{|x^n\rangle^{X^n} : x^n \in T_\delta^{X^n}\right\}. \quad \text{(B.6)}$$

Also, one can define a typical projector, which projects a state onto the typical subspace, as

$$\Pi_{\rho,\delta}^{X^n} \equiv \sum_{x^n \in T_\delta^{X^n}} |x^n\rangle \langle x^n|^{X^n}. \quad \text{(B.7)}$$

Properties of typical subspaces:–

(a) The probability that the quantum state ρ^{X^n} is in the typical subspace $T_{\rho,\delta}^{X^n}$ approaches one as n becomes large:

$$\forall \epsilon > 0, \quad \text{Tr}\left\{\Pi_{\rho,\delta}^{X^n} \rho^{X^n}\right\} \geq 1 - \epsilon, \quad \text{(B.8)}$$

for sufficiently large n, where $\Pi_{\rho,\delta}^{X^n}$ is the typical subspace projector.

(b) The dimension $\dim\left(T_{\rho,\delta}^{X^n}\right)$ of the δ-typical subspace satisfies

$$\forall \epsilon > 0, \quad (1-\epsilon)\, 2^{n(H(X)-\delta)} \leq \text{Tr}\left\{\Pi_\delta^{X^n}\right\} \leq 2^{n(H(X)+\delta)}, \quad \text{(B.9)}$$

for sufficiently large n.

(c) For all n the operator $\Pi_\delta^{X^n} \rho^{X^n} \Pi_\delta^{X^n}$ satisfies

$$2^{-n(H(X)+\delta)} \Pi_\delta^{X^n} \leq \Pi_\delta^{X^n} \rho^{X^n} \Pi_\delta^{X^n} \leq 2^{-n(H(X)-\delta)} \Pi_\delta^{X^n}. \quad \text{(B.10)}$$

The operator Chernoff bound:– Let $X_1, \ldots, X_n$ $(\forall m \in [n] : 0 \leq X_m \leq I)$ be n independent and identically distributed random operators with values in the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on some Hilbert space $\mathcal{H}$. Let $\overline{X}$ denote the sample average of the n random variables: $\overline{X} = \frac{1}{n}\sum_{m=1}^{n} X_m$. Suppose that for each operator X_m

$$\mathbb{E}_X \{X_m\} \geq aI, \tag{B.11}$$

where $a \in (0,1)$ and I is the identity operator on $\mathcal{H}$. Then for every ϵ where $0 < \epsilon < 1/2$ and $(1+\epsilon)\, a \leq 1$, the probability that the sample average $\overline{X}$ lies inside the operator interval $[(1 \pm \epsilon)\, \mathbb{E}_X \{X_m\}]$ is bounded as [147, 148],

$$\Pr_X \left\{(1-\epsilon)\, \mathbb{E}_X \{X_m\} \leq \overline{X} \leq (1+\epsilon)\, \mathbb{E}_X \{X_m\}\right\} \geq 1 - 2 \dim(\mathcal{H}) \exp\left(-\frac{n\epsilon^2 a}{4 \ln 2}\right). \tag{B.12}$$

APPENDIX C

Catalytic majorization

Here, for the sake of completeness, we restate various results obtained earlier by other researchers which are useful for this thesis. The following theorem is due to Ref. [176].

Theorem 25 ([176])**.** *For a bipartite qubit system with states* $|\psi_1\rangle = \sum_{i=1}^{4}\sqrt{\alpha_i}\,|ii\rangle$*,* $|\psi_2\rangle = \sum_{i=1}^{4}\sqrt{\beta_i}\,|ii\rangle$ *such that* $|\psi_1\rangle \nrightarrow |\psi_2\rangle$ *under local operations and classical communication (LOCC). Without loss of generality we can assume that the coefficients* $\{\alpha_i\}$*,* $\{\beta_i\}$ *are real and arranged in decreasing order. Then the necessary and sufficient conditions for the existence of a catalyst* $|\phi\rangle = \sqrt{a}\,|11\rangle + \sqrt{1-a}\,|22\rangle$ $(a \in (0.5, 1))$ *for these two states are the following two conditions:* $\alpha_1 \leq \beta_1$*;* $\alpha_1+\alpha_2 > \beta_1+\beta_2$*;* $\alpha_1+\alpha_2+\alpha_3 \leq \beta_1+\beta_2+\beta_3$*, and*

$$\max\left\{\frac{\alpha_1+\alpha_2-\beta_1}{\beta_2+\beta_3}, 1-\frac{\alpha_4-\beta_4}{\beta_3-\alpha_3}\right\} \leq a \leq \min\left\{\frac{\beta_1}{\alpha_1+\alpha_2}, \frac{\beta_1-\alpha_1}{\alpha_2-\beta_2}, 1-\frac{\beta_4}{\alpha_3+\alpha_4}\right\}. \tag{C.1}$$

The following theorem is from Refs. [174, 175].

Theorem 26 ([175])**.** *For two* d*-dimensional probability vectors* p *and* q *with the components arranged in decreasing order, there exists a probability vector* r *such that* $P(p\otimes r \rightarrow$

$q \otimes r) > P(p \to q)$ *if and only if*

$$P\left(p \to q\right) < \min\left\{\frac{p_d}{q_d}, 1\right\}.$$

In the case of catalytic majorization the necessary and sufficient conditions for catalytic transformations are obtained independently in Refs. [177] and [206]. In Ref. [177], the following result was obtained.

Lemma 27 ([177]). *Let p and q be two distinct d-element probability vectors arranged in decreasing order with p having nonzero elements. Then the existence of a vector r such that $p \otimes r \prec q \otimes r$ is equivalent to the following three strict inequalities:*

$$A_\alpha(p) > A_\alpha(q) \text{ for } \alpha \in (-\infty, 1); \tag{C.2}$$

$$A_\alpha(p) < A_\alpha(q) \text{ for } \alpha \in (1, \infty); \tag{C.3}$$

$$S(p) > S(q). \tag{C.4}$$

where $A_\alpha := (\frac{1}{d}\sum_{i=1}^{d} p_i^\alpha)^{\frac{1}{\alpha}}$, and $S(p) = -\sum_{i=1}^{d} p_i \log p_i$ is the Shannon entropy. For $\alpha = 0$, $A_0(p) = (\prod p_i)^{1/d}$. If any component of vector p is zero, then $A_\alpha = 0$ for all $\alpha \leq 0$.

The following lemma is from Ref. [11].

Lemma 28 ([11]). *The Rényi entropies S_α are strictly Schur concave for $\alpha \in (-\infty, 0) \cup (0, \infty)$. The Rényi entropies for $\alpha = 0, \pm\infty$ are Schur concave. Also, the function $\sum_i \log p_i$ is strictly Schur concave.*

Since $\tilde{S}_\alpha(\psi^{(\mathrm{d})})$ is equal to $S_\alpha(\psi^{(\mathrm{d})}) - \ln d$ and $\lim_{\alpha\to 0^+} \tilde{S}_\alpha(\psi^{(\mathrm{d})})/|\alpha| = \frac{1}{d}\sum_i \ln \psi_i^{(\mathrm{d})}$, the above lemma holds for functions $\tilde{S}_\alpha(\cdot)/|\alpha|$ too. That is, $\tilde{S}_\alpha(\cdot)/|\alpha|$ are strictly Schur concave for all $\alpha \in (-\infty, +\infty)$.

The following lemma is from Ref. [179].

Lemma 29 ([179]). *Let p and q be d-dimensional probability vectors with components being arranged in decreasing order and $p \neq q$. Then there exists a k-partite probability distribution $r_{1,\ldots,k}$ such that*

$$q \otimes r_{1,\ldots,k} \prec p \otimes (\otimes r_1 \otimes \ldots \otimes r_k)$$

if and only if $\mathrm{Rank}(p) \leq \mathrm{Rank}(q)$ *and* $S(p) < S(q)$. *Here, we can always choose $k = 3$. $S(p) = -\sum_{i=1}^{d} p_i \log p_i$ is the Shannon entropy.*

Consider a probability vector x. Define another subnormalized probability vector x' from the ε ball $B_\varepsilon(x)$ around x, defined as

$$B_\varepsilon(x) := \{\, y : \frac{1}{2}\sum_i |\, y_i - x_i| < \varepsilon \,\} \tag{C.5}$$

for any $\varepsilon > 0$. Now we have the following lemma from Ref. [11].

Lemma 30 ([11]). *Given any probability vector x, for $0 < \alpha < 1$ and $\varepsilon > 0$, we can construct a probability vector $x' \in B_\varepsilon(x)$ such that*

$$S_\alpha(x) \geq S_0(x') + \frac{\log \varepsilon}{1-\alpha}. \tag{C.6}$$

For $\alpha > 1$, we can construct another probability vector $x'' \in B_\varepsilon(x)$ such that

$$S_\infty(x'') - \frac{\log \varepsilon}{\alpha - 1} \geq S_\alpha(x). \tag{C.7}$$

The explicit construction of x' and x'' from a given probability vector x can be found in Refs. [11, 207].

APPENDIX D

Quantum subentropy, Selberg's Integrals and its consequences

D.1 Quantum subentropy

The von Neumann entropy of a quantum system is of paramount importance in physics starting from thermodynamics [208, 209] to the quantum information theory, e.g., in studies of the classical capacity of a quantum channel and the compressibility of a quantum source [210], and serves as the least upper bound on the accessible information. The von Neumann entropy of an m dimensional density matrix ρ, is defined as $S(\rho) = -\sum_{j=1}^{m} \lambda_j \ln \lambda_j$, where $\lambda = \{\lambda_1, \cdots, \lambda_m\}$ are eigenvalues of ρ. An analogous lower bound on the accessible information, obtained in Ref. [124] and called as the *subentropy* $Q(\rho)$, is defined as $Q(\rho) = -\sum_{i=1}^{m} \lambda_i^m \left(\prod_{j\neq i}(\lambda_i - \lambda_j)\right)^{-1} \ln \lambda_i$. Also, when two or more of the eigenvalues λ_j are equal, the value of Q is determined by taking a limit starting with unequal eigenvalues, unambiguously. The upper bound $S(\rho)$ and the lower bound $Q(\rho)$ on the accessible information are achieved for the ensemble of eigenstates of ρ and the Scrooge ensemble [124], respectively. Thus, the von Neumann entropy and the subentropy together define the range of the accessible information for a given density matrix. For a comparison between the von Neumann entropy and the subentropy, see Refs. [124, 194–196]. Now, we present Selberg's integrals and the calculation of the average subentropy of random mixed states.

D.2 Selberg's Integrals and its consequences

Proposition 31 (Selberg's Integrals, [193]). *If m is a positive integer and α, β, γ are complex numbers such that*

$$\mathrm{Re}(\alpha) > 0, \quad \mathrm{Re}(\beta) > 0, \quad \mathrm{Re}(\gamma) > -\min\left\{\frac{1}{m}, \frac{\mathrm{Re}(\alpha)}{m-1}, \frac{\mathrm{Re}(\beta)}{m-1}\right\},$$

then

$$\begin{aligned} S_m(\alpha,\beta,\gamma) &= \int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^m x_j^{\alpha-1}(1-x_j)^{\beta-1}\right) |\Delta(x)|^{2\gamma}\,[\mathrm{d}x] \\ &= \prod_{j=1}^m \frac{\Gamma(\alpha+\gamma(j-1))\Gamma(\beta+\gamma(j-1))\Gamma(1+\gamma j)}{\Gamma(\alpha+\beta+\gamma(m+j-2))\Gamma(1+\gamma)}, \end{aligned} \tag{D.1}$$

where $\Delta(x) = \prod_{1\leqslant i<j\leqslant m}(x_i - x_j)$ and $[\mathrm{d}x] = \prod_{j=1}^m \mathrm{d}x_j$. Furthermore, if $1 \leqslant k \leqslant m$, then

$$\begin{aligned} &\textstyle\int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^k x_j\right)\left(\prod_{j=1}^m x_j^{\alpha-1}(1-x_j)^{\beta-1}\right) |\Delta(x)|^{2\gamma}\,[\mathrm{d}x] \\ &\qquad = S_m(\alpha,\beta,\gamma)\textstyle\prod_{j=1}^k \frac{\alpha+\gamma(m-j)}{\alpha+\beta+\gamma(2m-j-1)}. \end{aligned} \tag{D.2}$$

The following two integrals (Propositions 32 and 33) are direct consequences of Proposition 31.

Proposition 32 ([193]). *With the same conditions on the parameters α, γ,*

$$\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j}\mathrm{d}x_j = \prod_{j=1}^m \frac{\Gamma(\alpha+\gamma(j-1))\Gamma(1+\gamma j)}{\Gamma(1+\gamma)}. \tag{D.3}$$

Proposition 33 ([193]). *With the same conditions on the parameters α, γ, and $1 \leqslant k \leqslant m$,*

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j\right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} \mathrm{d}x_j \\
&= \left(\prod_{j=1}^k (\alpha+\gamma(m-j))\right) \left(\prod_{j=1}^m \frac{\Gamma(\alpha+\gamma(j-1))\Gamma(1+\gamma j)}{\Gamma(1+\gamma)}\right). \qquad \text{(D.4)}
\end{aligned}$$

In the following, we prove Propositions 34 and 35 from Propositions 32 and 33, respectively, using the Laplace transform.

Proposition 34 ([183, 192]). *It holds that*

$$\begin{aligned}
\frac{1}{C_m^{(\alpha,\gamma)}} &:= \int_0^\infty \cdots \int_0^\infty \delta\left(1 - \sum_{j=1}^m x_j\right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} \mathrm{d}x_j \\
&= \frac{1}{\Gamma(\alpha m + \gamma m(m-1))} \prod_{j=1}^m \frac{\Gamma(\alpha+\gamma(j-1))\Gamma(1+\gamma j)}{\Gamma(1+\gamma)}. \qquad \text{(D.5)}
\end{aligned}$$

Proof. Let

$$F(t) := \int_0^\infty \cdots \int_0^\infty \delta\left(t - \sum_{j=1}^m x_j\right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} \mathrm{d}x_j.$$

Applying the Laplace transform ($t \to s$) to $F(t)$ gives us

$$\begin{aligned}
\widetilde{F}(s) &= \int_0^\infty F(t) e^{-st} \mathrm{d}t \\
&= \int_0^\infty \cdots \int_0^\infty \exp\left(-s \sum_{j=1}^m x_j\right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} \mathrm{d}x_j \\
&= s^{-\alpha m - 2\gamma\binom{m}{2}} \int_0^\infty \cdots \int_0^\infty |\Delta(y)|^{2\gamma} \prod_{j=1}^m y_j^{\alpha-1} e^{-y_j} \mathrm{d}y_j,
\end{aligned}$$

leading to the following via the inverse Laplace transform ($s \to t$) to $\widetilde{F}(s)$:

$$F(t) = \frac{t^{\alpha m + \gamma m(m-1) - 1}}{\Gamma(\alpha m + \gamma m(m-1))} \int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} \mathrm{d}x_j,$$

Therefore, we have

$$\frac{1}{C_m^{(\alpha,\gamma)}} = F(1) = \frac{1}{\Gamma\left(\alpha m + \gamma m(m-1)\right)} \times \int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} \mathrm{d}x_j.$$

Hence the desired identity via Eq. (D.3). □

Proposition 35. *It holds that, for* $1 \leqslant k \leqslant m$*,*

$$\begin{aligned}&\int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j\right) \delta\left(1 - \sum_{j=1}^m x_j\right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} \mathrm{d}x_j \\ &= \frac{1}{\Gamma(\alpha m + \gamma m(m-1) + k)} \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j\right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} \mathrm{d}x_j. \quad \text{(D.6)}\end{aligned}$$

Proof. Similarly, let

$$f(t) := \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j\right) \delta\left(t - \sum_{j=1}^m x_j\right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} \mathrm{d}x_j.$$

Then, the Laplace transform of $f(t)$ is given by

$$\begin{aligned}\widetilde{f}(s) \quad &= \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j\right) \exp\left(-\sum_{j=1}^m s x_j\right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} \mathrm{d}x_j \\ &= s^{-(\alpha m + \gamma m(m-1) + k)} \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k y_j\right) |\Delta(y)|^{2\gamma} \prod_{j=1}^m y_j^{\alpha-1} e^{-y_j} \mathrm{d}y_j.\end{aligned}$$

Therefore, we have

$$f(t) := \frac{t^{\alpha m + \gamma m(m-1) + k - 1}}{\Gamma(\alpha m + \gamma m(m-1) + k)} \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k y_j\right) |\Delta(y)|^{2\gamma} \prod_{j=1}^m y_j^{\alpha-1} e^{-y_j} \mathrm{d}y_j.$$

By setting $t = 1$ in the above equation, we derived the desired identity via Eq. (D.4). □

Proposition 36. *It holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Gamma(t+a)}{\Gamma(t+b)}\right) = (\psi(t+a)-\psi(t+b))\,\frac{\Gamma(t+a)}{\Gamma(t+b)}, \tag{D.7}$$

where $\psi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\Gamma(t)$.

D.2.1 The proof of Proposition 21 of the main text

A family of probability measures over $\mathbb{R}_+^m$ can be defined as:

$$\mathrm{d}\nu_{\alpha,\gamma}(\Lambda) := C_m^{(\alpha,\gamma)} K_\gamma(\Lambda)\prod_{j=1}^m \lambda_j^{\alpha-1}\mathrm{d}\lambda_j, \tag{D.8}$$

where $K_1(\Lambda)$ is given by

$$K_1(\Lambda) = \delta\left(1-\sum_{j=1}^m \lambda_j\right)\left|\Delta(\lambda)\right|^2, \tag{D.9}$$

with $\Delta(\lambda) = \prod_{1\leqslant i<j\leqslant m}(\lambda_i-\lambda_j)$ and $C_m^{(\alpha,\gamma)} = 1/\mathcal{I}_m(\alpha,\gamma)$ with

$$\mathcal{I}_m(\alpha,\gamma) = \frac{1}{\Gamma\left(\alpha m+\gamma m(m-1)\right)}\prod_{j=1}^m \frac{\Gamma\left(\alpha+\gamma(j-1)\right)\Gamma\left(1+\gamma j\right)}{\Gamma\left(1+\gamma\right)}. \tag{D.10}$$

The subentropy of a state ρ with the spectrum $\Lambda = \{\lambda_1,\cdots,\lambda_m\}$ can be written as [124, 194–196]

$$Q(\Lambda) = (-1)^{\frac{m(m-1)}{2}-1}\frac{\sum_{i=1}^m \lambda_i^m \ln\lambda_i \prod_{j\in\hat{i}}\phi'(\lambda_j)}{\left|\Delta(\lambda)\right|^2}, \tag{D.11}$$

where $\widehat{i} = \{ 1, \ldots, m \} \setminus \{ i \}$, $\phi'(\lambda_j) = \prod_{k\in\widehat{j}}(\lambda_j - \lambda_k)$ and

$$|\Delta(\lambda)|^2 = \left| \prod_{1\leqslant i<j\leqslant m} (\lambda_i - \lambda_j) \right|^2 . \tag{D.12}$$

The average subentropy over the set of mixed state is given by

$$\mathcal{I}^Q_m(\alpha,\gamma) = \int \mathrm{d}\mu_{\alpha,\gamma}(\rho)Q(\rho) = \int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda)Q(\Lambda). \tag{D.13}$$

Denote $\phi(x) := \prod_{j=1}^m(x - x_j)$. Then $\phi'(x) = \sum_{i=1}^m \prod_{j\in\widehat{i}}(x - x_j)$. Thus $\phi'(x_i) = \prod_{j\in\widehat{i}}(x_i - x_j)$. Furthermore, we have

$$\prod_{i=1}^m \phi'(x_i) = \prod_{i=1}^m \prod_{j\in\widehat{i}}(x_i - x_j) = (-1)^{\frac{m(m-1)}{2}} |\Delta(x)|^2 . \tag{D.14}$$

Here $|\Delta(x)|^2 = |\Delta(x_1, \ldots, x_m)|^2$ is called the *discriminant* of ϕ [193]. If we expand the polynomial $\phi(x)$, then we have:

$$\phi(x) = x^m - \left(\sum_{j=1}^m x_j\right) x^{m-1} + \cdots + (-1)^m \prod_{j=1}^m x_j = \sum_{j=0}^m (-1)^j e_j x^{m-j}, \tag{D.15}$$

where $e_j (j = 1, \ldots, m)$ is the j-th elementary symmetric polynomial in $x_1, \ldots, x_m$, with $e_0 \equiv 1$.

In what follows, we calculate the integral $\mathcal{I}^Q_m(\alpha,\gamma)$ for $\gamma = 1$. Propositions 34 and 35 will be used frequently for $\gamma = 1$.

$$\begin{aligned}\mathcal{I}^Q_m(\alpha,1) \quad &= -mC^{(\alpha,1)}_m \sum_{k=0}^{m-1}(-1)^k \int_0^1 \mathrm{d}\lambda_1 \lambda_1^{2(m-1)+\alpha-k} \ln \lambda_1 \\ &\times \int_0^\infty \cdots \int_0^\infty e_k \delta\left((1-\lambda_1) - \sum_{j=2}^m \lambda_j\right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j.\end{aligned}$$

It suffices to calculate a family of integrals in terms of the following form: for $k =$

$0, 1, \ldots, m-1$,

$$\int_0^\infty \cdots \int_0^\infty e_k \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j.$$

If $k = 0$, then

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty e_0 \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j \\
&= (1-\lambda_1)^{(m-1)(m+\alpha-2)-1} \int_0^\infty \delta \left(1 - \sum_{j=1}^{m-1} x_j \right) |\Delta(x_1, \ldots, x_{m-1})|^2 \prod_{j=1}^{m-1} x_j^{\alpha-1} \mathrm{d}x_j \\
&= (1-\lambda_1)^{(m-1)(m+\alpha-2)-1} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma((m-1)(m+\alpha-2))}. \qquad \text{(D.16)}
\end{aligned}$$

Here we used Proposition 34 in the last equality.

If $1 \leqslant k \leqslant m-1$, it suffices to calculate the following:

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k \lambda_{j+1} \right) \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j \\
&= (1-\lambda_1)^{(m-1)(m+\alpha-2)+k-1} \times \\
&\quad \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j \right) \delta \left(1 - \sum_{j=1}^{m-1} x_j \right) |\Delta(x_1, \ldots, x_{m-1})|^2 \prod_{j=1}^{m-1} x_j^{\alpha-1} \mathrm{d}x_j \\
&= (1-\lambda_1)^{(m-1)(m+\alpha-2)+k-1} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma((m-1)(m+\alpha-2)+k)} \frac{\Gamma(m+\alpha-1)}{\Gamma(m+\alpha-1-k)}. \qquad \text{(D.17)}
\end{aligned}$$

Here we used Proposition 35. Next, we calculate the integral

$$\int_0^1 \mathrm{d}\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty e_k \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j.$$

(1). If $k = 0$, then

$$
\begin{aligned}
&\int_0^1 \mathrm{d}\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty \delta\left((1-\lambda_1) - \sum_{j=2}^m \lambda_j\right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j \\
&= \frac{\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma((m-1)(m+\alpha-2))} \times \int_0^1 \lambda_1^t (1-\lambda_1)^{(m-1)(m+\alpha-2)-1} \mathrm{d}\lambda_1 \\
&= \frac{\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma((m-1)(m+\alpha-2))} \times \frac{\Gamma(t+1)\Gamma((m-1)(m+\alpha-2))}{\Gamma(t+1+(m-1)(m+\alpha-2))} \\
&= \frac{\Gamma(t+1)\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma(t+1+(m-1)(m+\alpha-2))}.
\end{aligned}
$$

By taking the derivative with respect to t on both sides, we get

$$
\begin{aligned}
&\int_0^1 \mathrm{d}\lambda_1 \lambda_1^t \ln\lambda_1 \int_0^\infty \cdots \int_0^\infty \delta\left((1-\lambda_1) - \sum_{j=2}^m \lambda_j\right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j \\
&= [\psi(t+1) - \psi(t+1+(m-1)(m+\alpha-2))] \frac{\Gamma(t+1)\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma(t+1+(m-1)(m+\alpha-2))}.
\end{aligned}
$$

For $t = 2(m-1)+\alpha$, we have

$$
\begin{aligned}
&\int_0^1 \mathrm{d}\lambda_1 \lambda_1^{2(m-1)+\alpha} \ln\lambda_1 \int_0^\infty \cdots \int_0^\infty \delta\left((1-\lambda_1) - \sum_{j=2}^m \lambda_j\right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j \\
&= [\psi(2(m-1)+\alpha+1) - \psi(m(m+\alpha-1)+1)] \times \\
&\qquad \frac{\Gamma(2(m-1)+\alpha+1)\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma(m(m+\alpha-1)+1)}.
\end{aligned}
$$

(2). If $1 \leqslant k \leqslant m-1$, then

$$\int_0^1 \mathrm{d}\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty e_k \delta\left((1-\lambda_1) - \sum_{j=2}^m \lambda_j\right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j$$

$$= \binom{m-1}{k} \int_0^1 \mathrm{d}\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k \lambda_{j+1}\right) \delta\left((1-\lambda_1) - \sum_{j=2}^m \lambda_j\right) \times$$

$$|\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j$$

$$= \binom{m-1}{k} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma((m-1)(m+\alpha-2)+k)} \prod_{j=1}^k (m+\alpha-j-1) \times$$

$$\int_0^1 \lambda_1^t (1-\lambda_1)^{(m-1)(m+\alpha-2)+k-1} \mathrm{d}\lambda_1$$

$$= \binom{m-1}{k} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma((m-1)(m+\alpha-2)+k)} \prod_{j=1}^k (m+\alpha-j-1) \times$$

$$\frac{\Gamma(t+1)\Gamma((m-1)(m+\alpha-2)+k)}{\Gamma(t+1+(m-1)(m+\alpha-2)+k)}$$

$$= \binom{m-1}{k} \frac{\Gamma(t+1)\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma(t+1+(m-1)(m+\alpha-2)+k)} \prod_{j=1}^k (m+\alpha-j-1)$$

$$= \binom{m-1}{k} \frac{\Gamma(t+1)\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma(t+1+(m-1)(m+\alpha-2)+k)} \frac{\Gamma(m+\alpha-1)}{\Gamma(m+\alpha-1-k)}.$$

By taking the derivative with respect to t, we get

$$\int_0^1 \mathrm{d}\lambda_1 \lambda_1^t \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty e_k \delta\left((1-\lambda_1) - \sum_{j=2}^m \lambda_j\right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j$$

$$= \binom{m-1}{k} [\psi(t+1) - \psi(t+1+(m-1)(m+\alpha-2)+k)]$$

$$\times \frac{\Gamma(t+1)\prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma(t+1+(m-1)(m+\alpha-2)+k)} \frac{\Gamma(m+\alpha-1)}{\Gamma(m+\alpha-1-k)}.$$

For $t = 2(m-1) + \alpha - k$, we have

$$\begin{aligned}
&\int_0^1 \mathrm{d}\lambda_1 \lambda_1^{2(m-1)+\alpha-k} \ln \lambda_1 \\
&\qquad \times \int_0^\infty \cdots \int_0^\infty e_k \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \ldots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} \mathrm{d}\lambda_j \\
&= \binom{m-1}{k} [\psi(2(m-1)+\alpha-k+1) - \psi(m(m+\alpha-1)+1)] \\
&\quad \times \frac{\Gamma(2(m-1)+\alpha-k+1) \prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma(m(m+\alpha-1)+1)} \frac{\Gamma(m+\alpha-1)}{\Gamma(m+\alpha-1-k)}.
\end{aligned}$$

In summary, we get

$$\begin{aligned}
&\mathcal{I}_m^Q(\alpha, 1) \\
&= -mC_m^{(\alpha,1)} \left[\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} [\psi(2(m-1)+\alpha-k+1) - \psi(m(m+\alpha-1)+1)] \right. \\
&\qquad \left. \times \frac{\Gamma(2(m-1)+\alpha-k+1) \prod_{j=1}^{m-1} \Gamma(\alpha+j-1)\Gamma(1+j)}{\Gamma(m(m+\alpha-1)+1)} \frac{\Gamma(m+\alpha-1)}{\Gamma(m+\alpha-1-k)} \right] \\
&= -\frac{1}{m(m+\alpha-1)} \left[\sum_{k=0}^{m-1} (-1)^k [\psi(2(m-1)+\alpha+1-k) - \psi(m(m+\alpha-1)+1)] \right. \\
&\qquad \left. \times \frac{\Gamma(2(m-1)+\alpha+1-k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m+\alpha-1-k)} \right]. \qquad \text{(D.18)}
\end{aligned}$$

Let us define

$$g_{mk}(\alpha) = \psi(2(m-1)+\alpha+1-k) - \psi(m(m+\alpha-1)+1), \qquad \text{(D.19)}$$

and

$$u_{mk}(\alpha) = \frac{(-1)^k \Gamma(2(m-1)+\alpha+1-k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m+\alpha-1-k)}. \qquad \text{(D.20)}$$

Then, from Eq. (D.18), we have

$$\mathcal{I}_m^Q(\alpha,1)=\frac{-1}{m(m+\alpha-1)}\sum_{k=0}^{m-1}g_{mk}(\alpha)u_{mk}(\alpha). \tag{D.21}$$

This completes the proof of Proposition 21 of main text. For $(\alpha,\gamma)=(n-m+1,1)$, we have

$$\mathcal{I}_m^Q(n-m+1,1)=-\frac{1}{mn}\sum_{k=0}^{m-1}g_{mk}(n-m+1)u_{mk}(n-m+1). \tag{D.22}$$

If $m=n$, this situation corresponds to the measure induced by the Hilbert-Schmidt distance [183], then we have

$$\mathcal{I}_m^Q(1,1)=-\frac{1}{m^2}\sum_{k=0}^{m-1}g_{mk}(1)u_{mk}(1). \tag{D.23}$$

In Eqs. (D.22) and (D.23), the functions g_{mk} and u_{mk} are given by Eqs. (D.19) and (D.20).

D.2.2 The proof of Theorem 22 of the main text

For $(\alpha,\gamma)=(n-m+1,1)$ the value of average subentropy $\mathcal{I}_m^Q(n-m+1,1)$ is given by Eq. (D.22). From the results of Page [106] and others [107–109] it is also known that

$$\mathcal{I}_m^S(n-m+1,1)=\sum_{k=n+1}^{mn}\frac{1}{k}-\frac{m-1}{2n}. \tag{D.24}$$

The average coherence of random mixed states is given by

$$\begin{aligned}\overline{\mathscr{C}_r}(\alpha,\gamma) &:= \int \mathrm{d}\mu_{\alpha,\gamma}(\rho)\mathscr{C}_r(\rho)=\int \mathrm{d}\mu_{\alpha,\gamma}(U\Lambda U^\dagger)\mathscr{C}_r(U\Lambda U^\dagger)\\ &= \int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda)\left[\int \mathrm{d}\mu_{\mathrm{Haar}}(U)S(\Pi(U\Lambda U^\dagger))-S(\Lambda)\right]\\ &= H_m-1+\int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda)(Q(\Lambda)-S(\Lambda))\\ &= H_m-1+\mathcal{I}_m^Q(\alpha,\gamma)-\mathcal{I}_m^S(\alpha,\gamma), \end{aligned} \tag{D.25}$$

where $\mathcal{I}_m^Q(\alpha,\gamma) = \int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda)Q(\Lambda)$ and $\mathcal{I}_m^S(\alpha,\gamma) = \int \mathrm{d}\nu_{\alpha,\gamma}(\Lambda)S(\Lambda)$. Also, we have used the fact that the average coherence of the isosepectral density matrices can be expressed in terms of the quantum subentropy, von Neumann entropy, and m-Harmonic number as follows [188]:

$$\begin{aligned}\overline{\mathscr{C}}_r^{\mathrm{iso}}(\Lambda) &:= \int \mathrm{d}\mu_{\mathrm{Haar}}(U)\mathscr{C}_r(U\Lambda U^\dagger)\\ &= H_m - 1 + Q(\Lambda) - S(\Lambda).\end{aligned}$$

Here $Q(\Lambda)$ is the subentropy, given by Eq. (D.11), $S(\Lambda)$ is the von Neumann entropy of Λ and $H_m = \sum_{k=1}^m 1/k$ is the m-Harmonic number. Now using Eqs. (D.22) and (D.24), in Eq. (D.25) completes the proof of the theorem and $\overline{\mathscr{C}_r}(n-m+1,1)$ is given by

$$\overline{\mathscr{C}_r}(n-m+1,1) = H_m - 1 - \left(\sum_{k=n+1}^{mn}\frac{1}{k} - \frac{m-1}{2n}\right) - \sum_{k=0}^{m-1}\frac{g_{mk}(n-m+1)u_{mk}(n-m+1)}{mn}. \tag{D.26}$$

Similarly,

$$\overline{\mathscr{C}_r}(1,1) = H_m - 1 - \left(\sum_{k=m+1}^{m^2}\frac{1}{k} - \frac{m-1}{2m}\right) - \frac{1}{m^2}\sum_{k=0}^{m-1} g_{mk}(1)u_{mk}(1). \tag{D.27}$$

D.2.3 The proof of Theorem 16 of the main text

To prove Theorem 16 of the main text, we use the concentration of measure phenomenon and in particular, Lévy's lemma [29, 119], which can be stated as follows:

Lévy's Lemma: Let $\mathcal{F} : \mathbb{S}^k \to \mathbb{R}$ be a Lipschitz function from k-sphere to real line with the Lipschitz constant η (with respect to the Euclidean norm) and a point $X \in \mathbb{S}^k$ be chosen uniformly at random. Then, for all $\epsilon > 0$,

$$\Pr\{|\mathcal{F}(X) - \mathbb{E}\mathcal{F}| > \epsilon\} \le 2\exp\left(-\frac{(k+1)\epsilon^2}{9\pi^3\eta^2\ln 2}\right). \tag{D.28}$$

Here $\mathbb{E}(\mathcal{F})$ is the mean value of $\mathcal{F}$. But before we present the proof we need to find the Lipschitz constant for the relevant function on $\mathbb{S}^k$ which is $G : \mathbb{S}^{nm} \mapsto \mathbb{R}$, defined as $G\left(|\psi\rangle^{AB}\right) = S\left(\rho^{A(\mathrm{d})}\right) - S\left(\rho^A\right) = \mathscr{C}_r(\rho^A)$ where $\rho^{A(\mathrm{d})}$ is the diagonal part of $\rho^A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|^{AB})$.

Lemma 37. *The function $\tilde{F} : \mathbb{S}^{mn} \mapsto \mathbb{R}$, defined as $\tilde{F}(|\psi\rangle^{AB}) = S(\rho^A)$ where $\rho^A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|^{AB})$ and S is the von Neumann entropy, is a Lipschitz continuous function with Lipschitz constant $\sqrt{8}\ln m$.*

Proof. The proof is given in Ref. [29]. □

Lemma 38. *The function $F : \mathbb{S}^{nm} \mapsto \mathbb{R}$, defined as $F(|\psi\rangle^{AB}) = S(\rho^{A(\mathrm{d})})$ where $\rho^{A(\mathrm{d})}$ is the diagonal part of $\rho^A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|^{AB})$ and S is the von Neumann entropy, is a Lipschitz continuous function with Lipschitz constant $\sqrt{8}\ln m$.*

Proof. We follow the proof strategy of Ref. [29]. Let $|\psi\rangle^{AB} = \sum_{i=1}^{m}\sum_{j=1}^{n}\psi_{ij}|ij\rangle^{AB}$ and therefore, $\rho^{A(\mathrm{d})} = \sum_{i=1}^{d} p_i |i\rangle\langle i|$ with $p_i = \sum_j |\psi_{ij}|^2$. Now, $F(\psi^{AB}) = -\sum_{i=1}^{m} p_i \ln p_i$. The Lipschitz constant for F can be bounded as follows:

$$\begin{aligned}
\eta^2 := \sup_{\langle\psi|\psi\rangle\le 1} \nabla F\cdot\nabla F &= 4\sum_{i=1}^{m} p_i\left[1+\ln p_i\right]^2 \\
&\le 4\left(1+\sum_{i=1}^{m} p_i(\ln p_i)^2\right) \\
&\le 4\left(1+(\ln m)^2\right) \le 8(\ln m)^2,
\end{aligned}$$

where the last inequality is true for $m \ge 3$. Therefore, $\eta \le \sqrt{8}\ln m$ for $d \ge 3$. □

Lemma 39. *The function $G : \mathbb{S}^{nm} \mapsto \mathbb{R}$, defined as $G\left(|\psi\rangle^{AB}\right) = S\left(\rho^{A(\mathrm{d})}\right) - S\left(\rho^A\right)$ where $\rho^{A(\mathrm{d})}$ is the diagonal part of $\rho^A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|^{AB})$ and S is the von Neumann entropy, is a Lipschitz continuous function with the Lipschitz constant $2\sqrt{8}\ln m$.*

Proof. Take $\sigma^A = \mathrm{Tr}_B\left(|\phi\rangle\langle\phi|^{AB}\right)$.

$$\begin{aligned}\left|G\left(|\psi\rangle^{AB}\right) - G\left(|\phi\rangle^{AB}\right)\right| &:= \left|S\left(\rho^{A(\mathrm{d})}\right) - S\left(\sigma^{A(\mathrm{d})}\right) - \left[S\left(\rho^{A}\right) - S\left(\sigma^{A}\right)\right]\right| \\ &\leq \left|S\left(\rho^{A(\mathrm{d})}\right) - S\left(\sigma^{A(\mathrm{d})}\right)\right| + \left|S\left(\rho^{A}\right) - S\left(\sigma^{A}\right)\right| \\ &\leq \sqrt{8}\ln m \left\|\left|\psi\right\rangle^{AB} - \left|\phi\right\rangle^{AB}\right\|_2 + \sqrt{8}\ln m \left\|\left|\psi\right\rangle^{AB} - \left|\phi\right\rangle^{AB}\right\|_2 \\ &\leq 2\sqrt{8}\ln m \left\|\left|\psi\right\rangle^{AB} - \left|\phi\right\rangle^{AB}\right\|_2 .\end{aligned}$$

Thus, G is a Lipschitz continuous function with the Lipschitz constant $2\sqrt{8}\ln m$. □

Now applying Lévy's lemma, Eq. (D.28), to the function $G\left(|\psi\rangle^{AB}\right) = \mathscr{C}_r(\rho^A)$, we have

$$\Pr\left\{\left|\mathscr{C}_r(\rho^A) - \overline{\mathscr{C}_r}(n-m+1,1)\right| > \epsilon\right\} \leq 2\exp\left(-\frac{mn\epsilon^2}{144\pi^3\ln 2(\ln m)^2}\right), \qquad \text{(D.29)}$$

for all $\epsilon > 0$. This completes the proof of Theorem 16 of the main text.

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