SCATTERING OF MASSIVE STATES IN PURE SPINOR SUPERSTRINGS

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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DEDICATION

Dedicated to my parents Sankar and Sipra Chakrabarti, who taught me without integrity no achievement is meaningful, to my older brother Sankhaneel, who taught me what people call impossible is often quite simple when viewed from other perspectives

and

To Madhurima, who always makes me strive to be a better human being.

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Contents

Summary

Over the past few decades superstring theory ([1], [2]) has firmly established itself as by far the most promising theory describing all known interactions in nature, including gravity, in a consistent manner. One of the most crucial feature of superstring theory is that scattering amplitudes computed are always ultraviolet finite as opposed to amplitudes computed in ordinary quantum field theories. Therefore explicit evaluation of string scattering amplitudes and studying their various properties are of paramount importance. The two oldest way of computing string scattering amplitudes are Ramond-Neveu-Schwarz (RNS) ([1], [2], [3]) and Green-Schwarz (GS) ([1]) formalism. However both formulations, despite success in some aspects also suffer from some limitations. While RNS formulation breaks manifest spacetime supersymmetry at the intermediate stages of the computation, computation in GS formulation breaks manifest Poincaré invariance. At the turn of last century, Nathan Berkovits introduced the pure spinor (PS) formulation ([4], [5]) of superstring theory which maintains manifest spacetime supersymmetry as well as Poincaré invariance at all stages of the computation. In addition, evaluation of loop amplitudes in RNS formalism requires insertion of picture changing operators and summing over various spin structures, both of which leads to pragmatic difficulty in explicit evaluation of superstring scattering amplitudes at two loops or higher. The PS formulation provides a prescription for evaluating amplitudes ([6], [7], [8]) which can circumvent both of these difficulties and simplifies evaluation of higher loop amplitudes for superstrings by a considerable margin. However, unlike RNS formalism, the PS formalism so far lacks an explanation starting from a gauge invariant worldsheet action. Furthermore, the present amplitude prescription for PS formalism becomes ill defined as one goes to higher loops. Therefore the equivalence of PS and RNS formalism is a non-trivial statement ([9], [10], [11]) and explicit demonstrations exhibiting such equivalence are extremely crucial.

The massless states and scattering amplitudes involving massless states in PS formalism have been

studied thoroughly ([7], [12], [13], [14], also see [15], [47]) and explicitly shown to be equivalent to the results obtained from RNS formalism ([10]). For first massive states of open superstrings ($m^2 = \frac{1}{\alpha'}$), the unintegrated vertex was constructed in [18]. However, to explicitly compute scattering amplitudes involving massive states one needs further ingredients, such as, covariant theta expansion of all superfields appearing in the vertex and integrated vertex operator for massive states.

The main motivation of this thesis is twofold. Firstly, one needs to develop a systematic procedure for performing fully covariant theta expansion of all superfields describing the first massive states of open superstrings. Secondly, one needs to explicitly compute all tree level three point amplitudes involving two massless states and one massive state. This allows us to directly compare the results of PS formalism with that obtained using RNS formalism.

In this thesis we develop a systematic procedure based on representation theory of SO(9) group (which is the little group for massive states in open superstring) for performing the theta expansion for all massive superfields in a covariant manner [19]. Further the procedure, by construction, is guaranteed to have only physical fields appearing in each order of theta expansion. This procedure also sheds extremely crucial insight into developing a strategy for finding massive integrated vertex [20]. Furthermore our analysis can be argued to be independent of mass level in question and therefore indicates that this systematic procedure of performing theta expansion and constructing vertex operators can be readily generalized to higher massive states.

With the vertex and their theta expansion completely known, one now possesses all the ingredients needed to compute any string amplitude involving massless and first massive states in PS formalism. In this thesis, all massless-massless-massive tree level three point amplitudes have been explicitly computed in PS formalism for the first time [21]. The same amplitudes can be independently computed in RNS formalism and allows one to directly compare RNS and PS formalism for first massive states. We find that PS formalism agrees with RNS formalism for first massive states as well, extending the explicit check of their equivalence from only massless states to include first massive states as well.

Chapter 1

Introduction to Pure Spinor Superstrings

The first half of 20th century saw physicists identify the four fundamental interactions that govern the observed physical world. These four fundamental interactions, viz., gravitational, electromagnetic, weak and strong interactions, have all been described successfully by theories that agree remarkably well with experiments conducted so far. However, while all three non-gravitational interactions could be described together quantum mechanically in the form of the "Standard Model", gravitation so far has been well understood only at the classical level (General theory of relativity). It soon became apparent that to have a complete unified description of all fundamental forces it is crucial to have a quantum theory of gravity first and foremost. At the phenomenological level, current energy scales accessible to modern day and near future colliders, can safely ignore quantum effects of gravity. On the other hand studies of early universe and black holes, some of which are indeed being probed by present observations, crucially hinges on the nature of quantum effects of gravity. Theoretically, however, it has always been clear that at high enough energies (near the Planck scale) one will start seeing the effects of quantum gravity amongst other interactions as well. This requires also to find an "ultraviolet (UV) completion" of the quantum field theory which describes the standard model.

For the last four decades or so, (super)string theory ([1], [2]) has emerged as the sole candidate which is able to consistently and successfully quantize gravity as well as unify it with all other interactions in a single coherent framework. Moreover, as expected of any such candidate for an unified theory, string theory has been demonstrated to be free of all ultraviolet divergences and thereby providing a potential

UV completion of standard model physics seen at lower energies [1]. This remarkable achievement starts with the simplest of ideas - all physics are thought to arise due to oscillations of a quantum, relativistic one dimensional object, or simply put, a string. In this introductory chapter we briefly summarize the evolution of this strikingly simple idea and its far reaching consequences. We start with bosonic strings, move on to superstrings and describe the two formalisms of superstrings that are perhaps more commonly known in literature (both were developed prior to pure spinor formalism). This walk down the memory lane naturally leads to pure spinor formalism which is introduced and reviewed succinctly in this chapter as well.

1.1 Bosonic Strings

A relativistic classical string will trace out a two dimensional surface, commonly referred to as a worldsheet, embedded in the ambient spacetime. The classical dynamics of such a string, in analogue with its point particle cousin, was proposed to be described by a geometrical action named after their proposers Y.Nambu and T.Goto -

$$S_{Nambu-Goto} \propto Area\ of\ worldsheet \tag{1.1}$$

$$S_{Nambu-Goto} = -\frac{1}{2\pi\alpha'}\int d^2\xi\sqrt{-det[h_{ab}(\xi))]} \tag{1.2}$$

where α' is the regge slope parameter, ξ^1, ξ^2 are worldsheet co-ordinates and $h_{\mu\nu}(\xi)$ is the worldsheet metric (with the worldsheet indices μ, ν taking values 1 and 2).

This action, although intuitively obvious and elegant, due to its nonlinear nature posits difficulty in covariant quantization. This was remedied by A.Polyakov by proposing another action which is classically equivalent to Nambu-Goto action but allows covariant quantization readily.

$$S_{Polyakov} = \frac{1}{4\pi\alpha'}\int d^2\xi\, \eta_{mn}\, \partial_\mu X^m(\xi)\, \partial_\nu X^n(\xi)\, h^{\mu\nu}(\xi) \tag{1.3}$$

where, $X^m(\xi)$ are the spacetime co-ordinates describing the embedding of the worldsheet in the d-dimensional spacetime (m, n runs from $0, 1, \cdots (d-1)$). Throughout the thesis we talk of strings propagating in flat spacetime described by the Minkowski metric $\eta_{mn} = diag(-1, +1, +1\cdots)$ in Cartesian co-ordinates. The generalization of the action for propagation in curved target spacetime is straightfor-

[1] How to exactly obtain the standard model starting from superstring theory is as of now an actively pursued research topic in the community with many promising developments over the years.

ward but will not be needed in this thesis.

The Polyakov action also brought into light a very crucial aspect of string theory that continues to dominate how the community approaches the subject. The Polyakov action revealed that while as a spacetime theory string theory is different from a QFT, as a worldsheet theory, string theory is a conformal field theory (CFT) in 2 dimensions. This in conjunction with rapid development of 2D CFT amounted to rapid evolution in the knowledge of string theory. Several crucial features were soon unearthed, viz. the quantum string theory is well defined (in the sense the Weyl anomaly vanishes) only at a critical spacetime dimension (which is 26 for bosonic string theory). The analysis of the spectrum soon revealed that open strings contain gauge bosons as massless states and closed strings contain graviton as its massless states. This was the first time gauge theory and gravity is seen to emerge from a single theory.

Two major shortcomings of bosonic string theory were

- There were no fermionic states in the spectrum of the theory (hence the name bosonic string theory).
- The spectrum contained a tachyon which signaled that the vacuum about which we are doing bosonic string perturbation theory is an unstable one.

Both of these shortcomings are naturally overcome in what has been come to be known as superstring theory. The two earliest ways of describing superstrings were Ramond-Neveu-Schwarz (RNS) formalism which introduces a worldsheet supersymmetry and Green-Schwarz (GS) formalism which introduces spacetime supersymmetry. We discuss the success and the shortcomings of these two formalisms next.

1.2 RNS Superstrings

The gauge fixed worldsheet action for bosonic string reads (we always give the action for open strings in flat background here onwards since that is the primary subject matter under investigation in this thesis),

$$S = \frac{1}{2\pi}\int d^2z\left(\frac{1}{\alpha'}\partial X^m\bar{\partial}X_m + b\bar{\partial}c\right) \tag{1.4}$$

where $b(z)$, $c(z)$ are anti-commuting holomorphic ghost fields that arises out of gauge fixing the worldsheet diffeomorphism. z and $\bar{z}$ are complex co-ordinates obtained from worldsheet co-ordinates ξ_1 and ξ_2.

The central idea of RNS formalism, is to introduce local worldsheet supersymmetry for the Polyakov action. Consequentially, the worldsheet theory now resembles a superconformal field theory (SCFT). The gauge fixed action reads

$$S_{RNS} = \frac{1}{2\pi}\int d^2z \left(\frac{1}{\alpha'}\partial X^m(z,\bar{z})\bar{\partial}X_m(z,\bar{z}) + \psi^m(z)\bar{\partial}\psi_m(z) + b(z)\bar{\partial}c(z) + \beta(z)\bar{\partial}\gamma(z)\right) \tag{1.5}$$

Here, $\psi_m(z)$ is an anti-commuting worldsheet field (it is a conformal primary) which transforms as a spacetime vector under spacetime Lorentz transformation. The $\beta(z)$ and $\gamma(z)$ are two commuting holomorphic ghost fields which arise due to gauge fixing the local worldsheet supersymmetry. The powerful techniques of SCFT immediately allows one to extract a huge amount of physics from the quantum RNS superstring. In particular, the spectrum now contains at each mass level a supermultiplet which has both bosons and fermions present. The Weyl anomaly vanishing condition for superstrings translates to the statement that the dimension of spacetime must be equal to 10.

The RNS formalism has proven to be extremely powerful over the years. It is by no means an overstatement to say it is, for most purposes in the existing literature, synonymous with superstring theory. The major advantages of this formalism are

- Manifest Poincaré covariance at all stages.
- Well defined methodology based on SCFT to construct all possible vertex operators representing external states.
- Well defined amplitude prescription to all loop order for arbitrary number of external states based on theory of (super) Riemann surfaces and their moduli space.
- Provides all basic ingredients to formulate an off-shell string field theory based on on-shell RNS formalism.
- Helped in listing all possible consistent superstring theories, viz., Type-I , Heterotic $SO(32)$, Heterotic $E_8 \times E_8$, Type IIA and Type IIB. It further enabled a detailed study of the spectrum of each of these theories, their low energy supergravity limits and was pivotal in realizing non-trivial web of duality relations via which all of these superstring theories (along with the unique 11 dimensional supergravity theory) are related to each other non-perturbatively. In turn it also lead to discovery of various D-branes which also belong to the spectrum of superstrings.

While the RNS formalism lends great conceptual depth and clarity to the theory of superstrings, it does have some shortcomings, mostly pragmatic in nature. Firstly, the spacetime supersymmetry is obfuscated in RNS and only appears after one performs the GSO projection [23]. Furthermore, SCFT involving both periodic and anti-periodic boundary conditions for conformal fields (which is the case in RNS formalism) leads to many equivalent description of a single SCFT vacua (known in literature as different "pictures", see [3]). This picture changing operation is part of the amplitude prescription and leads to spurious singularities in evaluating superstring amplitudes. Even without these unphysical poles, the direct evaluation of superstring scattering amplitude for 2-loops and beyond prove to be a mathematically challenging and daunting task. Mostly the challenge comes in summing over spin structures (which arises due to existence of worldsheet spinor). So while there is a well defined prescription, it does not lead to a feasible calculation for amplitudes at higher loops.

RNS formalism also suffers from the drawback that it does not offer any clue regarding how to generalize it in presence of a background involving nonzero Ramond-Ramond fluxes. However, in this thesis we only focus on superstrings in flat backgrounds and therefore we will refrain from going into details of this issue as well as how pure spinor formalism can potentially resolve them (see, for example [24]).

1.3 Green-Schwarz Superstrings

There exists another approach to superstring theory which starts with manifest spacetime supersymmetry. This approach,named Green-Schwarz formalism, has the following worldsheet action in conformal gauge

$$S_{GS} = \frac{1}{\pi\alpha'}\int d^2z \left(\frac{1}{2}\partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial}\theta^\alpha\right) \tag{1.6}$$

here θ^α is a right handed Majorana-Weyl spinor in 10 dimension which serves as the superpartner of the co-ordinate field X^m. The conjugate momenta to the θ^α is p_α which is related to the other fields via what is known as the Green-Schwarz constraint

$$d_\alpha = 0 \quad , \quad d_\alpha \equiv p_\alpha - \frac{1}{2}\gamma^m{}_{\alpha\beta}\theta^\beta \partial X_m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\sigma\delta}\theta^\beta\theta^\sigma\partial\theta^\delta\ . \tag{1.7}$$

The constraint consists of both first class as well as second class constraint due to the OPE

$$d_\alpha(z) d_\beta(w) = -\frac{\alpha' \gamma^m_{\alpha\beta}}{2(z-w)}\Pi_m(w) + \cdots\ , \tag{1.8}$$

$$\Pi^m \equiv \partial X^m + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\alpha\partial\theta^\beta \tag{1.9}$$

This formalism has the distinct advantage of manifest spacetime supersymmetry at all stages (The quantities Π^m and d_α are manifestly spacetime supersymmetric combinations). It has also proved to be extremely useful in establishing the seminal result of anomaly cancellation which triggered what is referred popularly as "first superstring revolution". However this formalism suffers from a serious drawback in that the existence of this second class constraint prevents us from performing a straightforward quantization in a covariant manner. The difficulty however can be gotten rid of by going to special gauges (e.g. light-cone gauge) where the theory can be quantized readily. However, this automatically implies manifest Poincaré covariance is sacrificed and for each calculation performed in this formalism, one must establish the covariance painstakingly at the end.

1.4 Pure Spinor Superstrings

At this stage, the question that begs to be answered is - can we develop a formalism of superstring theory which has the following three characteristics?

1. It maintains manifest spacetime supersymmetry at all stages.
2. It maintains manifest Poincaré covariance at all stages.
3. It possesses an amplitude prescription which actually enables one to compute higher loop string amplitudes in practice and not just in principle.

Roughly two decades ago Nathan Berkovits gave the pure spinor formalism in [4] (also see [5] for a review) which answered this question in affirmative as far as the first two characteristics are considered. As for the third characteristics, the answer can also be considered to be an affirmative with certain caveats that we will discuss later in this chapter.

1.4.1 Worldsheet CFT

In this section, we introduce the pure spinor formalism. The original formulation is now dubbed as the minimal pure spinor formalism ([4]). The world sheet action of the minimal formalism is given by [2].

$$S = \frac{1}{\pi\alpha'} \int d^2z \left(\frac{1}{2} \partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial} \theta^\alpha - w_\alpha \bar{\partial} \lambda^\alpha \right) \tag{1.10}$$

[2] Some numerical factors for pure spinor formalism in this thesis will differ from literature. This is entirely due to different choices of various conventions which we summarize in appendix A

where, $m = 0, 1, , \cdots, 9$ and $\alpha = 1, \cdots, 16$.

The p_α and w_α are the conformal weight one conjugate momenta of the conformal weight zero fields θ^α and λ^α respectively. The fields with upper (lower) spinor indices transform as right (left) handed Weyl spinor. The λ^α satisfy the pure spinor constraint (first written down by the great french mathematician Elle Cartan almost 90 years ago)

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 \tag{1.11}$$

Due to pure spinor constraint, the conjugate momentum to the pure spinor, viz. w_α enjoys the following gauge transformation under which the action is invariant

$$\delta w_\alpha = \Lambda_m \gamma^m_{\alpha\beta} \lambda^\beta \tag{1.12}$$

where Λ_m is the gauge transformation parameter. Therefore, the worldsheet fields can only appear in the following gauge invariant combinations [3]

$$N_{mn} = \frac{1}{2} w_\alpha (\gamma_{mn})^\alpha{}_\beta \lambda^\beta \quad , \quad J = w_\alpha \lambda^\alpha \quad , \quad T = w_\alpha \partial \lambda^\alpha$$

Any other gauge invariant combination can be expressed in terms of these objects. The ghost current is given by $J = \lambda^\alpha w_\alpha$ which implies that the λ^α carries the ghost number +1 and the rest of the fields carry zero ghost number.

1.4.2 OPE

Before proceeding ahead, we note down some important OPEs of the theory which arise frequently in calculations

$$d_\alpha(z) d_\beta(w) = -\frac{\alpha' \gamma^m_{\alpha\beta}}{2(z-w)} \Pi_m(w) + \cdots \quad , \qquad d_\alpha(z) \Pi^m(w) = \frac{\alpha' \gamma^m_{\alpha\beta}}{2(z-w)} \partial\theta^\beta(w) + \cdots$$

[3] The coincident operators are normal ordered via

$$: A(z)B(z) :\equiv \frac{1}{2\pi i} \oint_z \frac{dw}{w-z} A(w)B(z) \tag{1.13}$$

where, A and B are any two operators and the contour surrounds the point z. However, we shall often suppress the normal ordering symbol : : throughout this thesis with the understanding all coincident operators are always normal ordered.

$$d_\alpha(z)V(w) = \frac{\alpha'}{2(z-w)}D_\alpha V(w) + \cdots \quad , \qquad \Pi^m(z)V(w) = -\frac{\alpha'}{(z-w)}\partial^m V(w) + \cdots$$

$$\Pi^m(z)\Pi^n(w) = -\frac{\alpha'\eta^{mn}}{2(z-w)^2} + \cdots \quad , \qquad N^{mn}(z)\lambda^\alpha(w) = \frac{\alpha'(\gamma^{mn})^\alpha{}_\beta}{4(z-w)}\,\lambda^\beta(w) + \cdots$$

$$J(z)J(w) = -\frac{(\alpha')^2}{(z-w)^2} + \cdots \quad , \qquad J(z)\lambda^\alpha(w) = \frac{\alpha'}{2(z-w)}\lambda^\alpha(w) + \cdots$$

$$N^{mn}(z)N^{pq}(w) = -\frac{3(\alpha')^2}{2(z-w)^2}\eta^{m[q}\eta^{p]n} - \frac{\alpha'}{(z-w)}\Big(\eta^{p[n}N^{m]q} - \eta^{q[n}N^{m]p}\Big) + \cdots \tag{1.14}$$

In the above OPEs, ∂_m is the derivative with respect to the spacetime coordinate X^m, ∂ is the derivative with respect to the worldsheet coordinate, V denotes an arbitrary superfield. The d_α and Π^m denote the spacetime supersymmetric combinations

$$\begin{aligned} d_\alpha &= p_\alpha - \frac{1}{2}\gamma^m{}_{\alpha\beta}\theta^\beta\partial X_m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\sigma\delta}\theta^\beta\theta^\sigma\partial\theta^\delta \\ \Pi^m &= \partial X^m + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\alpha\partial\theta^\beta \end{aligned} \tag{1.15}$$

The D_α is the supercovariant derivative given by

$$D_\alpha \equiv \partial_\alpha + \gamma^m_{\alpha\beta}\theta^\beta\partial_m \, , \quad \partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} \qquad \Longrightarrow \qquad \{D_\alpha, D_\beta\} = 2(\gamma^m)_{\alpha\beta}\partial_m \tag{1.16}$$

Finally, the remarkably simple BRST operator is defined to be[4]

$$Q = \oint_C dz\ \big(\lambda^\alpha d_\alpha\big)(z)\ . \tag{1.17}$$

To see the nilpotency of the BRST operator, first make use of the OPE between $d_\alpha(z)d_\beta(w)$ to express

$$Q^2 = \oint_{C_1} dz \oint_{C_2} dw \lambda^\alpha(z)\lambda^\beta(w)d_\alpha(z)d_\beta(w) \propto (\lambda^\alpha\lambda^\beta\gamma^m_{\alpha\beta})\ , \tag{1.18}$$

[4]In all subsequent occurrences of $\oint dz$ in this thesis we will use the notation to imply the usual factor of $\frac{1}{2\pi i}$ is absorbed within the definition of the measure dz.

which trivially vanishes due to pure spinor constraint.

1.4.3 Non-minimal Pure Spinor formalism

With the minimal formalism one can finally write down an action for superstrings which in some sense forms a union of the best parts of both RNS and Green-Schwarz formalism without their respective drawbacks. However, the amplitude prescription for minimal formalism at loop levels still requires one to make use of picture changing operation. For this and some other reasons, soon the formalism was extended by introducing an additional set of variables which we will describe now (for details see [6]).

The action describing the worldsheet CFT for non-minimal pure spinor formalism is given as

$$S^{non-min} = \frac{1}{\pi\alpha'}\int d^2z\left(\frac{1}{2}\partial X^m\bar{\partial}X_m + p_\alpha\bar{\partial}\theta^\alpha - w_\alpha\bar{\partial}\lambda^\alpha - \bar{w}^\alpha\bar{\partial}\bar{\lambda}_\alpha + s^\alpha\bar{\partial}r_\alpha\right) \tag{1.19}$$

where the new variables are a bosonic left handed pure spinor $\bar{\lambda}_\alpha$ and a fermionic spinor r_α satisfying the following constraints

$$\left(\bar{\lambda}\gamma^m\bar{\lambda}\right) = 0 \quad , \quad \left(\bar{\lambda}\gamma^m r\right) = 0 \; . \tag{1.20}$$

These constraints imply the following gauge invariance for the conjugate momentum variables

$$\delta\bar{w}^\alpha = \bar{\Lambda}^m(\gamma_m\bar{\lambda})^\alpha - \phi^m(\gamma_m r)^\alpha \quad , \quad \delta s^\alpha = \phi^m(\gamma_m\bar{\lambda})^\alpha \; . \tag{1.21}$$

For arbitrary $\bar{\Lambda}^m$ and ϕ^m.

This in turn implies, that only the following gauge invariant combinations can introduce the conjugate momenta of the new non-minimal variables

$$\bar{N}_{mn} = \frac{1}{2}[(\bar{w}\gamma_{mn}\bar{\lambda}) - (s\gamma_{mn}r)] \quad , \quad \bar{J} = \bar{w}^\alpha\bar{\lambda}_\alpha - s^\alpha r_\alpha \; ,$$

$$S_{mn} = \frac{1}{2}(s\gamma_{mn}\bar{\lambda}) \quad , \quad S = s^\alpha\bar{\lambda}_\alpha \; . \tag{1.22}$$

It is crucial that the introduction of this new non-minimal variables do not affect the cohomology.

This is ensured by defining the non-minimal BRST charge as follows

$$Q_{non-min} = Q + \oint_C dz \bar{w}^\alpha r_\alpha \; . \tag{1.23}$$

The additional term is invariant under the gauge transformation (1.21). This implies via the quartet argument (see [25]) that the cohomology of $Q_{non-min}$ is same as that of Q. We will not require non-minimal formalism in the new results being presented in this thesis. So without going into unnecessary details let us point out certain crucial improvements that this offers over the minimal formalism. In turn this will also justify why our results presented in this thesis, while obtained using minimal variables, are equally valid for non-minimal formalism.

1.4.4 Vertex Operators

First and foremost, as we have just discussed that since the cohomology is unaffected by introduction of non-minimal variables,i.e. for unintegrated vertex operator V and integrated vertex operator U,

$$QV = 0 \quad \& \quad QU = \partial_{\mathbb{R}} V \quad \Longrightarrow \quad Q_{non-min} V = 0 \quad \& \quad Q_{non-min} U = \partial_{\mathbb{R}} V \; , \tag{1.24}$$

there is a possibility that all physical states can be potentially constructed using minimal variables alone. For massless states, this indeed is the case as shown in [4] (also see appendix D for derivation of massless vertex in our conventions). Let us briefly describe why using this result we can rigorously establish that all higher massive vertex operators can also be constructed solely in terms of minimal variables.

We recall from the RNS formalism that the massive states also appear in the OPEs of the massless vertex operators. This allows us, in principle, to construct the massive vertex operators from the knowledge of the massless vertex operators. More specifically for open strings, this construction, pointed out to the author and his collaborators by Nathan Berkovits, goes as follows. If V_1, V_2 are unintegrated and U_1, U_2 are integrated massless vertex operators respectively, then we have

$$QU_1 = \partial_{\mathbb{R}} V_1 \qquad \text{and} \qquad QU_2 = \partial_{\mathbb{R}} V_2 \tag{1.25}$$

We now take the contour integral of U_1 around the integrand of U_2 and define

$$U_3(z) \equiv \oint \frac{dw}{2\pi i} U_1(w) U_2(z) \tag{1.26}$$

Acting on this with the BRST operator Q and using (1.25), we obtain

$$QU_3 = \oint \frac{dw}{2\pi i} U_1(w) QU_2(z) = \oint \frac{dw}{2\pi i} U_1(w)\partial_z V_2(z) \equiv \partial_z V_3 \tag{1.27}$$

where,

$$V_3(z) \equiv \oint \frac{dw}{2\pi i} U_1(w) V_2(z) \tag{1.28}$$

and in the first equality in (1.27), we have used the fact that $\oint dw\ \partial_{\mathbb{R}} V_1(w)$ is zero.

Now, if we choose the momentum k_1 and k_2 of U_1 and U_2 to satisfy

$$(k_1 + k_2)^2 = 2k_1 \cdot k_2 \equiv (k_3)^2 = -m^2 = -\frac{n}{\alpha'} \tag{1.29}$$

then, by construction, the V_3 and U_3 will be unintegrated and integrated massive vertex operators respectively of open string states at mass level n.

One might ask how do we know that the U_3 and V_3 as defined in (1.26) and (1.28) do not vanish. To answer this question, we recall that the OPE of two massless vertex operators necessarily contain the massive vertex operators (this is necessary for the consistency of the theory and is well known from the RNS formalism). Now, the integrated vertices U_1 and U_2 have conformal weight one. Hence, by dimensional analysis, it is easy to see that the integrand involving the integrated massive vertex operator can only appear at the first order pole in (1.26) and hence its contour integral can't vanish. By a similar argument, we see that V_3 as defined in (1.28) can't vanish.

Since the massless vertices can be chosen to be independent of all non-minimal variables as shown in [4], this construction shows that the massive vertices can also be constructed without using the non-minimal variables.

While this result establishes that all higher massive vertex operators can always be constructed by taking OPE of massless states, this algorithm turns out to be not very pragmatic as of yet. In fact, prior to work presented in this thesis, no covariant methodology of constructing vertex operators existed for pure spinor formalism. This is usually thought to be a criticism of pure spinor formalism that this thesis, in part, wishes to nullify.

1.4.5 Amplitude prescriptions for pure spinor formalism

Minimal Formalism

The tree level N-point amplitude prescription is given by

$$A_N = \langle V_1 V_2 V_3 \int U_4 \cdots \int U_N \rangle \, . \tag{1.30}$$

where as in RNS and bosonic strings, we use $SL(2,\mathbb{R})$ symmetry to fix three external states at three fixed points on the disk (these are therefore represented by unitegrated vertex V_i, $i = 1, 2, 3$). The rest of the external states are represented by integrated vertex U_i, $i = 4, \cdots, N$. The measure is defined in such a way that it is non-vanishing if and only if

$$\big\langle (\lambda\gamma^s\theta)(\lambda\gamma^t\theta)(\lambda\gamma^u\theta)(\theta\gamma_{stu}\theta) \big\rangle = 1 \, . \tag{1.31}$$

Later in chapter 5 we will give step by step details of how to evaluate such amplitudes.

We will not need to evaluate loop amplitudes in this thesis, but nonetheless a few comments are necessary for completeness. As discussed in [6] (see also [7], [12], [13], [14], [15]), the multiloop amplitude prescription in minimal PS formalism involves insertion of a picture lowering operator Y_C and two picture raising operators Z_B and Z_J, to absorb the 11 zero modes of λ^α and $11g$ zero modes of w_α at genus g. These operators are given by

$$Y_C = C_\alpha \theta^\alpha \delta(C_\beta \lambda^\beta) \; , \;\; Z_B = \frac{1}{2} B_{mn} (\lambda\gamma^{mn} d) \delta(B^{pq} N_{pq}) \; , \;\; Z_J = (\lambda^\alpha d_\alpha) \delta(J) \, . \tag{1.32}$$

The crucial point here is that C_α is a constant spinor and B_{mn} is a constant tensor. Even though one can show that dependence of the amplitude on these two constant spinor/tensor is BRST trivial, there presence nonetheless breaks manifest Lorentz invariance. While this instance of breaking is by no means as severe as that of Green-Schwarz formalism, it still leaves one craving for a better amplitude prescription which maintains manifest Lorentz covariance *at all stages*.

Non-minimal formalism

The non-minimal formalism can be identified with $\hat{c} = 3$, $N = 2$ string theory. Therefore one can directly use the amplitude prescription for the topological strings to write down an amplitude prescription for non-minimal pure spinor formalism (see [6], also [15]).

The tree level prescription is given as

$$A_N = \langle \mathcal{N} V_1 V_2 V_3 \int U_4 \cdots \int U_N \rangle \; . \tag{1.33}$$

Following standard methods in any CFT, one first needs to integrate out all conformal weight 1 variables by using their OPEs to reduce the amplitude to an expression consisting of only zero modes of λ^α and θ^α. The prescription looks almost identical to that of minimal, except for the presence of the factor $\mathcal{N}$, which serves as a regularization. The regularization is necessary since the integral over λ and $\bar{\lambda}$ may diverge due to their bosonic non-compact nature. At first glance it may seem demoralizing, but a neat method for regularization, originally formulated in [26] (also see [6]) for any BRST invariant system, proposes

$$\mathcal{N} = exp(\{Q, \chi\}) \; . \tag{1.34}$$

It was shown in [26], since $\mathcal{N} = 1 + Q\Omega$, for some Ω, the functional integral which gives the amplitude is independent of the choice of the regularization $\mathcal{N}$.

The precise form of the measures were given in [6] and there it was also shown explicitly that for tree level and using the fact that all vertex operators are expressed solely in terms of minimal variables, *both minimal and non-minimal prescription coincides exactly.*

Finally, the g-loop, N-point amplitude in non-minimal formalism is given by

$$A_{N,g} = \int d^{3g-3}\tau \langle \mathcal{N}(y) \prod_{i=1}^{3g-3} (\int dw_i \mu_i(w_j) b(w_j)) \prod_{j=1}^{N} \int dz_j U(z_j) \rangle \; . \tag{1.35}$$

Here, τ are the complex Teichmüller parameters and μ_j are the associated Beltrami differentials, b is a composite operator which serves as the "b-ghost" in non-minimal pure spinor formalism. This expression does not involve any PCOs and as a result is *completely Lorentz covariant at all stages.*

However, as mentioned earlier, things are not as perfect as they may seem at first glance. This amplitude prescription comes with its own problems which we briefly discuss now. The expression of the b ghost is quite complicated in non-minimal formalism, but the important thing to note is that it diverges $\sim \frac{\bar{\lambda}}{(\bar{\lambda}\lambda)^4}$ as $\bar{\lambda}\lambda \to 0$. The entire measure as goes as $\sim \lambda^{8+3g}\bar{\lambda}^{11}$ as $\bar{\lambda}\lambda \to 0$. This suggests that the amplitude prescription can give a meaningful premise of calculation provided the integrand diverges slower than $\sim \lambda^{-(8+3g)}\bar{\lambda}^{-11}$ as $\bar{\lambda}\lambda \to 0$. Since all our vertex are polynomials in λ, the only possible divergence can come from the b-ghosts. This suggests that we can have at most three number of b-ghosts, or the prescription, with the current regularization, works only up to 2 loops.

While this situation leaves one unsated, there are still plenty of causes for optimism. Firstly, the fact that this prescription, for the massless amplitudes ([7], [13], [12], [14], [15], [16], [17]) have been shown to reproduce the RNS results (for tree as well as loop) gives confidence that one is along the right path. Furthermore, for the cases where the computation is well defined, this prescription is much more practical than the corresponding RNS computation. Also, in [8], general arguments were given to demonstrate, that in principle one can always find a regulated b-ghost which has a better divergence and will therefore allow one to compute loop amplitudes beyond genus 2 (in principle even up to all orders). The only snag is that so far no one has managed to find a concrete realization of the proposed regularization methodology.

1.5 Outline of the rest of the thesis

For now we set our sight into the problem at hand. We will like to develop a systematic strategy to construct massive vertex operators which also provides a recipe for performing their full covariant θ expansion in terms of the physical fields. The first construction of massive unintegrated vertex was done in [18], which is reviewed in chapter 2 . In chapter 3 we give our methodology of performing the full covariant θ expansions of massive vertex. We also give the explicit θ expansion results for the various superfields appearing in the vertex. In chapter 4 we use the insights developed in chapter 3 to propose a generalized methodology based on group theory to construct all massive vertex operators. As an evidence for our methodology, we re-derive the unintegrated massive vertex using this novel technique and also state the result for the integrated vertex which was constructed for the first time in literature using this methodology. Once vertex operators are constructed for first massive states, we would like to establish the equivalence of RNS and PS formalism for first massive states as well. This has been done in chapter 5 by computing all massless-massless-massive amplitudes in PS formalism and directly comparing them with their RNS counterparts. We conclude with brief summary and some interesting future directions that this thesis leads to in chapter 6 .

Chapter 2

Review of construction of first massive unintegrated vertex

2.1 Vertex Operators in pure spinor formalism

Pure spinor formalism expresses every vertex for a given mass level in a following schematic form

$$V \sim \sum_n B_n F_n \tag{2.1}$$

here B_n are composite operators of appropriate conformal weight and ghost number (same as that of the vertex V) constructed out of the objects $\Pi^m, \partial\theta^\alpha, d_\alpha, N^{mn}, J$ and λ^α. We often use the name "basis"[1] to denote such an object B_n appearing in a vertex. The F_n are suitable spacetime superfields whose index structure is such that after contraction with B_n it yields a scalar. All of the superfields F_n must describe the same supermultiplet corresponding to the vertex operator V.

Construction of an unintegrated vertex operator V for a given mass level therefore requires one to find at least one superfield F describing the massive supermultiplet in question such that following conditions hold -

- All F_n can be expressed in terms of F or its supercovariant derivative $D_\alpha F$.
- The BRST condition must hold : $QV = 0$.

[1] The name is slightly misleading, since these objects B_n for a given conformal weight and ghost number are really not linearly independent due to pure spinor constraint. But nonetheless one can take care of the pure spinor constraint in a systematic way by either solving the constraints or using Lagrange multiplier.

The notion of a b-ghost is not a natural one in pure spinor since one does not start from a worldsheet diffeomorphism invariant theory to arrive at the pure spinor wordsheet CFT. Nonetheless in non-minimal pure spinor, one can construct a composite operator which serves the purpose of a b-ghost[2] . However this b-ghost has multiple poles at $\lambda^\alpha = 0$ and any integrated vertex constructed using b-ghost are mostly not very useful when it comes to computing scattering amplitudes. Instead one defines the integrated vertex U via the following descent equation

$$QU = \partial_{\mathbb{R}} V \ , \tag{2.2}$$

where $\partial_{\mathbb{R}}$ denotes derivative along the real line $\mathbb{R}$ (since we are talking of open strings, for closed strings it will be just the holomorphic derivative across the entire $\mathbb{C}$ plane).

Constructing an integrated vertex therefore first requires construction of the unintegrated vertex V. Then it must be expressible in the schematic form

$$U \sim \sum_m \tilde{B}_m G_m \tag{2.3}$$

with the following conditions holding true

- All superfield G_m can be expressed in terms of F or its supercovariant derivative $D_\alpha F$ (This F is must be the same superfield in terms of which the unintegrated massive vertex was expressed).
- The BRST condition must hold : $QU - \partial_{\mathbb{R}} V = 0$.

For massless states, the unintegrated as well as integrated vertex operator were explicitly constructed in [4]. We give a brief review of this construction in our convention in appendix D. For the first massive states of open superstrings, the unintegrated vertex was constructed in [18]. In this chapter we review that construction, in our convention. Then we discuss what are the ingredients lacking in this construction that prevent us from using it to compute scattering of first massive states. The conclusion of this chapter will complete the review part of this thesis and lay the groundwork for the new results which form the subsequent chapters of this thesis.

[2]There is a b-ghost in minimal PS formalism as well, but it breaks manifest Lorentz covariance by introducing a constant right handed spinor.

2.1.1 Unintegrated massive vertex operator at $(mass)^2 = \frac{1}{\alpha'}$

In this subsection, we focus on the open string states at first mass level, i.e. $m^2 = \frac{1}{\alpha'}$. The unintegrated vertex operator for these states was constructed in [18]. We review this construction below. At the first mass level, the open string spectrum comprises of 128 bosonic and 128 fermionic degrees of freedom contained in a traceless symmetric tensor g_{mn}, a three-form field b_{mnp} and a spin-3/2 field $\psi_{m\alpha}$. These fields satisfy the following constraints

$$\eta^{mn} g_{mn} = 0 \quad ; \quad \partial^m g_{mn} = 0 \quad ; \quad \partial^m b_{mnp} = 0 \quad ; \quad \partial^m \psi_{m\alpha} = 0 \quad ; \quad \gamma^{m\alpha\beta} \psi_{m\beta} = 0 \tag{2.4}$$

Due to these constraints, the number of independent components in g_{mn}, b_{mnp} and $\psi_{m\alpha}$ is $44, 84$ and 128 respectively. Further, these form a massive spin-2 supermultiplet in 10 dimensions.

The unintegrated vertex operator describing the physical states at mass level n, i.e., $m^2 = \frac{n}{\alpha'}$ is constructed out of objects[3] with ghost number 1 and conformal dimension n. Recall, in the list of objects using which we will construct our basis, only the pure spinor ghost λ^α has non-zero ghost number (but zero conformal weight). And all other operators have zero ghost number and conformal weight one. So for n-th massive level each of the basis will be of the form $\sim \sum_{m=0}^{m=n} \partial^m \lambda^\alpha \hat{B}_{n-m}$, where $\hat{B}_{n-m}$ is a conformal weight $(n-m)$ operator constructed by multiplying suitable number of all weight 1 operators and their worldsheet derivatives (for example, a subset of conformal weight 3 candidates for basis can be $\lambda^\beta \Pi^m d_\sigma \partial\theta^\alpha$, $\lambda^\beta \Pi^m \partial d_\sigma$, $\lambda^\beta \partial^2 \Pi^m$,$\partial^3 \lambda^\alpha$, $\partial^2 \lambda^\alpha \Pi^m$ and so on). Consequently, the most general unintegrated vertex operator at first massive level $(n = 1)$ of the open string can be written as

$$\begin{aligned} V \;\; = \;\; & \partial\lambda^\alpha A_\alpha(X,\theta) + :\partial\theta^\beta \lambda^\alpha B_{\alpha\beta}(X,\theta): + :d_\beta \lambda^\alpha C^\beta{}_\alpha(X,\theta): + :\Pi^m \lambda^\alpha H_{m\alpha}(X,\theta): \\ & + :J\lambda^\alpha E_\alpha(X,\theta): + :N^{mn} \lambda^\alpha F_{\alpha mn}(X,\theta): \end{aligned} \tag{2.5}$$

where $A_\alpha, B_{\alpha\beta}, C^\beta{}_\alpha, H_{m\alpha}, E_\alpha$ and $F_{\alpha mn}$ are general superfields, unconstrained as of now. In accordance with [18], the normal ordering : : is defined as follows

$$:AB:(z) \equiv \frac{1}{2\pi i} \oint_z \frac{dw}{w-z} A(w) B(z) \tag{2.6}$$

[3]These objects are constructed using $\Pi^m, \partial\theta^\alpha, d_\alpha, \lambda^\alpha, J$ and N^{mn}.

where, A and B are any two operators.

The equation of motion for the superfields in (2.5) is determined by the BRST condition $QV = 0$. Let us explicitly see how one evaluates QV using the OPEs given in 1.4.2 for one of the terms.

$$Q(\partial\theta^{\alpha}\lambda^{\beta}B_{\alpha\beta}(w)) = \oint_C dz\lambda^{\rho}d_{\rho}(z)\partial\theta^{\alpha}\lambda^{\beta}B_{\alpha\beta}(w) \tag{2.7}$$

Where the contour C encloses the point w. The only relevant OPE needed to evaluate this expression are that between d_ρ and $\partial\theta^\alpha$ and d_ρ and $B_{\alpha\beta}$,

$$d_{\rho}(z)\partial\theta^{\alpha}(w) \sim \frac{\alpha'}{2}\frac{\delta_{\rho}{}^{\alpha}}{(z-w)^2} \quad , \quad d_{\rho}(z)B_{\alpha\beta}(w) \sim \frac{\alpha'}{2}\frac{D_{\rho}B_{\alpha\beta}(w)}{(z-w)}. \tag{2.8}$$

This on performing the standard complex integrals using Cauchy formula gives,

$$Q(\partial\theta^{\alpha}\lambda^{\beta}B_{\alpha\beta}(w)) = \frac{\alpha'}{2}\partial\theta^{\alpha}\lambda^{\rho}\lambda^{\beta}D_{\rho}B_{\alpha\beta}(w) \tag{2.9}$$

Similarly one can evaluate the action of Q on the other terms of the vertex. One obtains [4]

$$\begin{aligned}
\frac{2}{\alpha'}QV \quad &= -\partial\theta^{\sigma}\lambda^{\alpha}\lambda^{\beta}\left[D_{\alpha}B_{\beta\sigma} - \gamma^{s}_{\alpha\sigma}H_{s\beta}\right] + \Pi^{m}\lambda^{\alpha}\lambda^{\beta}\left[D_{\alpha}H_{s\beta} - \gamma_{s\alpha\sigma}C^{\sigma}{}_{\beta}\right] \\
&-d_{\sigma}\lambda^{\alpha}\lambda^{\beta}\left[D_{\alpha}C^{\sigma}{}_{\beta} + \delta^{\sigma}{}_{\alpha}E_{\beta} + \frac{1}{2}(\gamma^{st})^{\sigma}{}_{\alpha}F_{\beta st}\right] \\
&+\lambda^{\alpha}\partial\lambda^{\beta}\left[D_{\alpha}A_{\beta} + B_{\alpha\beta} + \alpha'\gamma^{s}_{\beta\sigma}\partial_{s}C^{\sigma}{}_{\alpha} - \frac{\alpha'}{2}D_{\beta}E_{\alpha} + \frac{\alpha'}{4}(\gamma^{st}D)_{\beta}F_{\alpha st}\right] \\
&+J\lambda^{\alpha}\lambda^{\beta}\left[D_{\alpha}E_{\beta}\right] + N^{st}\lambda^{\alpha}\lambda^{\beta}\left[D_{\alpha}F^{st}_{\beta}\right]
\end{aligned} \tag{2.10}$$

At this point we would like to set each term in square bracket to zero, but there are two primary obstacles in doing so, both stemming from pure spinor constraint. Firstly, any operator quadratic in λ^β is constrained by pure spinor constraint. This is easily remedied by using the bi-spinor decomposition A.23, the pure spinor constraint $(\lambda\gamma^m\lambda) = 0$ and the fact that λ^β is a Grassmann even object. We can express

$$\lambda^{\alpha}\lambda^{\beta} = \frac{1}{5!\,32}(\lambda\gamma^{mnpqr}\lambda)(\gamma_{mnpqr})^{\alpha\beta}. \tag{2.11}$$

[4]Note that some of the numerical factors in various expressions in this thesis will not agree with that in [18] since we are using different convention for (anti)symmetrization. See appendix A for details.

This is done for each of the term in QV, for example,

$$\frac{2}{\alpha'}\, Q(\partial\theta^\alpha\lambda^\beta B_{\alpha\beta}(w)) = \frac{1}{5!\,32}\,\partial\theta^\alpha(\lambda\gamma^{mnpqr}\lambda)\left[(\gamma_{mnpqr})^{\rho\beta}D_\rho B_{\alpha\beta}(w)\right]. \tag{2.12}$$

Similarly, using the fact $(\lambda\gamma^m\partial\lambda)=0$ (which is readily obtained by taking ∂ on pure spinor constraint), we can express

$$\lambda^\alpha\partial\lambda^\beta = \frac{1}{3!\,16}(\lambda\gamma^{mnp}\partial\lambda)(\gamma_{mnp})^{\alpha\beta} + \frac{1}{5!\,32}(\lambda\gamma^{mnpqr}\partial\lambda)(\gamma_{mnpqr})^{\alpha\beta}. \tag{2.13}$$

The second obstacle comes from the fact that not all of the "basis" are linearly independent (see footnote 1). In particular, one obtains as a consequence of the pure spinor constraint the following identity

$$:N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta} - \frac{1}{2}:J\lambda^\alpha:(\gamma^n)_{\alpha\beta} = \alpha'\partial\lambda^\alpha(\gamma^n)_{\alpha\beta} \tag{2.14}$$

It is assumed henceforth that all failure of linear independence of the basis operators due to pure spinor constraint stem from this identity and anything directly derivable from this identity [5]. For the mass level in question, the relevant identity that can be derived from (2.14) is

$$:N_{st}\lambda^\alpha\lambda^\beta:\gamma^s_{\beta\gamma} - \frac{1}{2}:J\lambda^\alpha\lambda^\beta:(\gamma_t)_{\beta\gamma} - \frac{5\alpha'}{4}\lambda^\alpha\partial\lambda^\beta(\gamma_t)_{\beta\gamma} + \frac{\alpha'}{4}\lambda^\delta\partial\lambda^\beta(\gamma_{st})^\alpha{}_\delta(\gamma^s)_{\beta\gamma} = 0. \tag{2.15}$$

This clearly demonstrates that 3 of our basis elements are related by a single relation. To take into account of this constraint, we contract (2.15) by an arbitrary superfield with appropriate index structure and treat the arbitrary superfield as a Lagrange multiplier.

$$\begin{aligned} -2:N_{st}\lambda^\alpha\lambda^\beta:(\gamma^{vwxy}\gamma^{[s})_{\alpha\beta}K^{t]}{}_{vwxy} + :J\lambda^\alpha\lambda^\beta:(\gamma^{vwxy}\gamma_s)_{\alpha\beta}K^s{}_{vwxy} \\ +\alpha'\lambda^\alpha\partial\lambda^\beta\Big[2(\gamma^{vwxys})_{\alpha\beta}\eta_{st}K^t{}_{vwxy} + 16(\gamma^{wxy})_{\alpha\beta}K^s{}_{wxys}\Big] = 0 \end{aligned} \tag{2.16}$$

Therefore, we must solve (2.10) subjected to the constraint (2.16). Which yields the following set of

[5] At the moment of writing this thesis, we are unaware of any explicit proof of this assumption. But given the successful construction of unintegrated and integrated vertex for first massive states, which crucially depends on taking into account all constraints correctly, provides strong evidence that this assumption is correct. It is perhaps not too optimistic to forecast probably this assumption can be proven exactly in the future.

equations [6]

$$(\gamma_{mnpqr})^{\alpha\beta}\left[D_\alpha B_{\beta\sigma} - \gamma^s_{\alpha\sigma}H_{s\beta}\right] = 0 \tag{2.17}$$

$$(\gamma_{mnpqr})^{\alpha\beta}\left[D_\alpha H_{s\beta} - \gamma_{s\alpha\sigma}C^\sigma{}_\beta\right] = 0 \tag{2.18}$$

$$(\gamma_{mnpqr})^{\alpha\beta}\left[D_\alpha C^\sigma{}_\beta + \delta^\sigma{}_\alpha E_\beta + \frac{1}{2}(\gamma^{st})^\sigma{}_\alpha F_{\beta st}\right] = 0 \tag{2.19}$$

$$\begin{aligned}(\gamma_{mnpqr})^{\alpha\beta}&\left[D_\alpha A_\beta + B_{\alpha\beta} + \alpha'\gamma^s_{\beta\sigma}\partial_s C^\sigma{}_\alpha - \frac{\alpha'}{2}D_\beta E_\alpha + \frac{\alpha'}{4}(\gamma^{st}D)_\beta F_{\alpha st}\right]\\ &= 2\alpha'\gamma^{\alpha\beta}_{mnpqr}\gamma^{vwxys}_{\alpha\beta}\eta_{st}K^t{}_{vwxy}\end{aligned} \tag{2.20}$$

$$\begin{aligned}(\gamma_{mnp})^{\alpha\beta}&\left[D_\alpha A_\beta + B_{\alpha\beta} + \alpha'\gamma^s_{\beta\sigma}\partial_s C^\sigma{}_\alpha - \frac{\alpha'}{2}D_\beta E_\alpha + \frac{\alpha'}{4}(\gamma^{st}D)_\beta F_{\alpha st}\right]\\ &= 16\alpha'\gamma^{\alpha\beta}_{mnp}\gamma^{wxy}_{\alpha\beta}\eta_{st}K^s{}_{wxys}\end{aligned} \tag{2.21}$$

$$(\gamma_{mnpqr})^{\alpha\beta}D_\alpha E_\beta = (\gamma_{mnpqr}\gamma^{vwxy}\gamma_s)^\alpha{}_\alpha K^s{}_{vwxy} \tag{2.22}$$

$$(\gamma_{mnpqr})^{\alpha\beta}D_\alpha F^{st}_\beta = -2(\gamma_{mnpqr}\gamma^{vwxy}\gamma^{[s})^\alpha{}_\alpha K^{t]}{}_{vwxy} \tag{2.23}$$

Above identity (2.14) implies that the vertex operator V remains invariant under the following field redefinition for an arbitrary tensor spinor Λ^β_m

$$\delta F_{\alpha mn} = \gamma_{m\alpha\beta}\Lambda^\beta_n - \gamma_{n\alpha\beta}\Lambda^\beta_m \quad , \quad \delta E_\alpha = -\gamma^m_{\alpha\beta}\Lambda^\beta_m \quad , \quad \delta A_\alpha = -2\alpha'\gamma^m_{\alpha\beta}\Lambda^\beta_m \tag{2.24}$$

Due to the nilpotency of the BRST operator Q, any vertex operator V also enjoys a gauge freedom given

[6] Since the operators which are used to build each of the basis have non-trivial OPEs amongst themselves, it is essential one adopts a fixed ordering as a convention before comparing two terms with same basis. For example, $J\lambda^\alpha$ is not same as $\lambda^\alpha J$ because both operators have non-trivial OPE amongst themselves. Our convention is from left to right whenever they appear, the ordering must be $\{\Pi^m, d_\alpha, \partial\theta^\alpha, N^{mn}, J, \lambda^\beta\}$.

by the transformation

$$V(z) \to V(z) + Q\Omega(z) \tag{2.25}$$

This gauge freedom along with the field redefinition freedom (2.24) can be used to impose the following algebraic conditions on the superfields

$$B_{\alpha\beta} = (\gamma^{mnp})_{\alpha\beta} B_{mnp} \quad , \quad C^{\alpha}{}_{\beta} = (\gamma^{mnpq})^{\alpha}{}_{\beta} C_{mnpq}$$

$$\gamma^{m\alpha\beta} F_{\beta mn} = 0 \quad , \quad (\gamma^m H_m)_\alpha = 0 \tag{2.26}$$

Using these and the equations of motion (2.17) - (2.23), all the superfields can be solely expressed in terms of the single superfield B_{mnp} as

$$H_{s\alpha} = \frac{3}{7}(\gamma^{mn})_\alpha{}^\beta D_\beta B_{mns} \ , \quad C_{mnpq} = \frac{1}{2}\partial_{[m} B_{npq]} \ , \quad E_\alpha = 0 = A_\alpha$$

$$F_{\alpha mn} = \frac{1}{8}\left(7\partial_{[m} H_{n]\alpha} + \partial^q (\gamma_{q[m})_\alpha{}^\beta H_{n]\beta}\right) \tag{2.27}$$

Further, one also finds

$$K^m{}_{npqr} = \frac{1}{1920}\left[(\gamma_{npqrs})^{\alpha\beta} D_\alpha F^{ms}_\beta - \frac{1}{3}(\gamma_{su[npq})^{\alpha\beta}\delta^m_{r]} D_\alpha F^{su}_\beta\right] \tag{2.28}$$

The above solution, when substituted in (2.21), simplifies to

$$(\partial^2 - \frac{1}{\alpha'})B_{mnp} = 0 \tag{2.29}$$

which demonstrates that the states described by B_{mnp} are indeed massive with $(mass)^2 = \frac{1}{\alpha'}$. Next, we must demonstrate that the superfield B_{mnp} indeed describes the states with the desired mass. We now need only show that this superfield contains the correct physical fields as well.

The superfield B_{mnp} also satisfies [18]

$$D_\alpha B^{mnp} = 6\gamma^{[m}_{\alpha\beta} Z^{np]\beta} - \frac{1}{8}(\gamma^{[mn})_\alpha{}^\beta H^{p]}_\beta \tag{2.30}$$

The $Z^{mn\beta}$ can be chosen to be anti symmetric in the indices m and n and furthermore it satisfies $Z^{mn\beta}\gamma_{m\alpha\beta} = 0$.

In order to show that the superfields contain the physical fields g_{mn}, b_{mnp} and $\psi_{m\alpha}$, it is convenient to go to the rest frame $\vec{k} = 0$, where $\vec{k}$ denotes the spatial momenta. In the following, we shall label the spatial indices using the beginning Latin alphabets, namely, a, b, c etc.

As argued in [18] using the supersymmetry transformation properties, $Z^{mn\alpha}$, B_{mnp} and $H_{m\alpha}$ satisfy in the rest frame,

$$Z^{ab\alpha} = \frac{1}{2}\left(\gamma^{[a}\Psi^{b]}\right)^{\alpha} \qquad ; \qquad Z^{0b\alpha} = -\frac{7}{4}\left(\gamma^{0}\Psi^{b}\right)^{\alpha} \tag{2.31}$$

$$D_\alpha B^{abc} = 12(\gamma^{[ab}\Psi^{c]})_\alpha \qquad ; \qquad B^{0ab} = 0 \tag{2.32}$$

and

$$H^c_\alpha = -72\,\Psi^c_\alpha \tag{2.33}$$

where, Ψ^c_α is an arbitrary tensor-spinor superfield satisfying

$$(\gamma_a)^{\beta\alpha}\Psi^a_\alpha = 0 \tag{2.34}$$

The spin-3/2 field ψ^a_α is defined to be the θ independent component of Ψ^a_α i.e.

$$\psi^a_\alpha = \Psi^a_\alpha\Big|_{\theta=0} \tag{2.35}$$

Furthermore, the physical fields g_{ab} and b_{abc} are defined to be the θ independent components of G_{ab} and B_{abc} respectively i.e.

$$g^{ab} \equiv G^{ab}\Big|_{\theta=0} \quad ; \quad b_{abc} \equiv B_{abc}\Big|_{\theta=0} \tag{2.36}$$

where, the superfield G^{ab} is defined to be

$$G_{ab} \equiv 2\, D_\alpha\gamma^{\alpha\beta}_{(a}\Psi_{b)\beta} \qquad , \qquad \eta_{ab}G^{ab} = 0 \tag{2.37}$$

Due to the fact that b_{abc} is anti-symmetric in all the indices and equations (2.34) and (2.37), the fields b_{abc}, g_{ab} and ψ^a_α contain precisely the desired number of degrees of freedoms, namely, $84, 44$ and 128 respectively. This shows that in the rest frame we have the correct counting for the degrees of freedom in the massive superfields.

2.2 Why this vertex is not enough for computing scattering of massive states?

For these superfields to represent the massive spin-2 supermultiplet, the higher θ components of the superfields $B_{abc}, \Psi_{a\alpha}$ and G_{ab} must be determined completely in terms of b_{mnp}, g_{mn} and $\psi_{m\alpha}$. The construction laid out so far can ensure that only in the rest frame. However, this construction is not enough to compute all scattering amplitudes involving first massive states for the following two reasons.

1. The relations derived in [18] and reviewed in this chapter provides enough information only to obtain a θ expansion of the superfield B_{mnp}, but only in a rest frame. Since knowing the full θ expansion is imperative to compute any scattering amplitude, only scattering processes involving a single massive states can be evaluated using this vertex (by going to the rest frame of the massive state). Not only is this incomplete once we have more than one massive states, it is also displeasing to use pure spinor formalism which boasts of manifest super-Poincaré covariance at all stages to perform a calculation in a specific choice of frame (viz. rest frame). Therefore we must develop a way of performing the full θ expansion of the superfield in a completely covariant manner.

2. As is well known, typically all superfields contain some auxiliary fields in higher components of θ expansion. It is expected that any naive covariantization of the rest frame results will lead to superfields which possesses auxiliary fields in its θ expansion. While this is not a very serious problem, it is preferable to have a vertex which is by construction such that only the physical fields appear at all stages of θ expansion. This ensures that such a vertex can be readily used to compute scattering of any number of massive states.

In the next chapter we will give a prescription which solves both these problems simultaneously.

Chapter 3

Group theory and covariant θ expansion of Massive superfields

The goal of this chapter is to basically covariantize the rest frame results described in last chapter. We seek to covariantize not only the superfield B_{mnp}, but also the superfields $\Psi_{m\alpha}$ and G_{mn}. Furthermore we will postulate three relations which expresses the supercovariant derivative of $\Psi_{m\alpha}$ in terms of B_{mnp} and G_{mn}, and supercovariant derivative of B_{mnp} and G_{mn} in terms of $\Psi_{m\alpha}$. These differential relations along with certain other on-shell conditions will provide us with necessary ingredients to achieve the following two goals

- Perform the θ expansion of all 3 superfields in a fully covariant manner.
- Establish firmly that the θ expansion thus performed will only contain physical fields b_{mnp}, g_{mn} and ψ_α as co-efficients and no auxiliary fields will appear.

Prima facie introducing three superfields, all of which describe the same spin 2 massive supermultiplet may seem redundant. However, the redundancy is completely taken care of by our proposed relations which allow us to express any superfields in terms of the others by making use of the supercovariant derivative. Moreover we will see in the next chapter that this method of obtaining the covariant θ expansion immediately provides crucial insight into developing a general methodology of constructing massive vertex for any mass level.

There are two consistency checks for our proposed relations. Firstly, they must reduce in the rest frame to the results already described in the previous chapter (which are the results of [18]). Secondly, all proposed relations must be consistent among themselves as well as with the BRST condition $QV = 0$. We will see at the end of this chapter that both these checks are satisfied by our relations. We conclude this chapter by giving the explicit θ expansion results for the relevant superfields obtained by our proposed relations ([19], [21]). There has been a previous attempt to covariantize the rest frame results (in [27]). As will be pointed out later in this chapter, our result differ from that paper. In fact, as we will demonstrate, the results of that paper are in conflict with both of the consistency checks described above.

3.1 Ingredients for θ expansion

As mentioned in the last chapter, one drawback of working in the superspace formalism is that a given superfield contains usually auxiliary degrees of freedom in surplus to the actual physical degrees of freedom of the theory. In our case also, by looking at the coefficients in the θ expansion of the superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$, we can easily convince ourselves that they will generically contain much more degrees of freedom than 128+128 provided by the physical fields g_{mn}, b_{mnp} and $\psi_{m\alpha}$. Thus, it is imperative that we express the higher θ components of these superfields in terms of the physical fields thereby removing the redundant degrees of freedom.

To ensure that g_{mn}, b_{mnp} and $\psi_{m\alpha}$ are the only physical degrees of freedom, there must be relations expressing $D_\alpha \Psi_{m\beta}$ in terms of G_{mn}, B_{mnp} and $D_\alpha G_{mn}$, $D_\alpha B_{mnp}$ in terms of $\Psi_{m\alpha}$. These will provide the recursive relations[1] relating the higher θ components of the superfields to the lowest components g_{mn}, b_{mnp} and $\psi_{m\alpha}$. Along with these, the algebraic constraints such as $k^m B_{mnp} = 0$ are also needed to remove the extra degrees of freedom at the zeroth order in θ expansion. We need to ensure that all these relations are consistent with the on-shell condition $QV = 0$ (or equivalently equations(2.17) to (2.23)).

In this section, we give the above mentioned relationships among the superfields. In the process, we also give the covariant generalizations of the rest frame results (2.32) - (2.37) given in section 2.1.1. We shall be very brief and just state the result. One can check the validity of these by writing them in the rest frame and verifying that they agree with those in the subsection 2.1.1 and satisfy all the equations. In appendix 3.2, we indicate how to check this systematically. For simplicity, we work in the momentum space in what follows[2].

We start by recalling the rest frame results (2.31), (2.33), (2.34) and (2.37). We claim that the

[1] Note that the gauge invariance (2.25) has already been completely exploited in writing down the solution (2.27).

[2] i.e. we replace all the ∂_m by $i\, k_m$.

covariant generalization of these results is given by

$$H_{m\beta} = -72\Psi_{m\beta} \quad , \qquad (\gamma_m)^{\alpha\beta}\Psi^m_\beta \;=\; 0 \tag{3.1}$$

and,

$$G_{mn} = 2D_\alpha \gamma^{\alpha\beta}_{(m}\Psi_{n)\beta} \quad , \qquad \eta^{mn}G_{mn} = 0 \tag{3.2}$$

These results have the correct limit in the rest frame. In fact the covariant expression of (2.32) as proposed in [27], after substituting in $H_{s\beta}$ as given in (2.27) reduces in the rest frame to $H^b_\beta = -96\Psi^b_\beta$. This is in clear contradiction with (2.33) .

Further, we also claim that the relations between the various superfields and all the necessary constraints, which are needed to ensure that superfields contain only the physical degrees of freedom, are given by

$$D_\alpha G_{sm} = 16ik^p(\gamma_{p(s}\Psi_{m)})_\alpha \tag{3.3}$$

$$D_\alpha B_{mnp} = 12(\gamma_{[mn}\Psi_{p]})_\alpha + 24\alpha' k^t k_{[m}(\gamma_{|t|n}\Psi_{p]})_\alpha \tag{3.4}$$

$$D_\alpha \Psi_{s\beta} = \frac{1}{16}G_{sm}\gamma^m_{\alpha\beta} + \frac{i}{24}k_m B_{nps}(\gamma^{mnp})_{\alpha\beta} - \frac{i}{144}k^m B^{npq}(\gamma_{smnpq})_{\alpha\beta} \tag{3.5}$$

$$(\gamma^m)^{\alpha\beta}\Psi_{m\beta} = 0 \quad ; \quad k^m\Psi_{m\beta} = 0 \quad ; \quad k^m B_{mnp} = 0 \quad ; \quad k^m G_{mn} = 0 \;\;\&\;\; \eta^{mn}G_{mn} = 0 \tag{3.6}$$

We shall now argue as to how to arrive at above equations. To see that equations (3.1) and (3.2) are the correct covariant generalizations of the rest frame results given in [18], we note that the 128 fermionic degrees of freedom in the rest frame are contained in the spatial components of $\Psi_{m\beta}$, namely in $\Psi_{a\beta}$. Similarly, the 44 Bosonic degrees of freedom are contained in G_{ab}. This means that $\Psi_{0\beta}, G_{0b}$ and G_{00} must either be zero or should be determined in terms of $\Psi_{a\beta}, B_{abc}$ and G_{ab}, since otherwise, we shall need to impose further constraints on $\Psi_{a\beta}$ and G_{ab}. However, it is easy to see that due to the rest frame constraints

$$(\gamma_a)^{\alpha\beta}\Psi^a_\beta = 0 \quad , \qquad \eta^{ab}G_{ab} = 0 \qquad , \qquad \text{when} \qquad k^a = 0 \tag{3.7}$$

we can't construct $\Psi_{0\beta}$ and G_{00} in terms of $\Psi_{a\beta}$ and G_{ab} consistent with rotational invariance in the rest frame. Another way to argue this is to note that G_{00}, G_{0a} and $\Psi_{0\beta}$ belong to the singlet, **9** and **16** representations of SO(9) group (which is the little group in the rest frame). This means that these fields can not be expressed as linear combinations of the existing fields $\Psi_{a\beta}, B_{abc}$ and G_{ab} (which form **128**, **84** and **44** representations respectively of SO(9)). Hence, $\Psi_{0\beta}$, G_{00} and G_{0a} must be zero in the rest frame[3], i.e.

$$\Psi_{0\beta} = 0 \quad , \qquad G_{00} = 0 \quad , \qquad G_{0a} = 0 \quad , \qquad\qquad \text{when} \quad k^a = 0 \tag{3.8}$$

Now, the proposed generalizations (3.1) and (3.2) can be written as

$$H_{m\beta} + 72\Psi_{m\beta} = 0 \ \ , \ \ (\gamma_m)^{\alpha\beta}\Psi^m_\beta = 0 \ \ , \ \ G_{mn} - 2D_\alpha\gamma^{\alpha\beta}_{(m}\Psi_{n)\beta} = 0 \ \ , \ \ \eta^{mn}G_{mn} = 0 \tag{3.9}$$

which can alternatively be written in terms of their spatial and temporal components as

$$\begin{aligned}
&H_{a\beta} + 72\Psi_{a\beta} = 0 \quad , \qquad H_{0\beta} + 72\Psi_{0\beta} = 0 \quad , \qquad (\gamma_a)^{\alpha\beta}\Psi^a_\beta + (\gamma_0)^{\alpha\beta}\Psi^0_\beta = 0 \\
&G_{ab} - 2D_\alpha\gamma^{\alpha\beta}_{(a}\Psi_{b)\beta} = 0 \quad , \quad G_{00} - 2D_\alpha\gamma^{\alpha\beta}_{(0}\Psi_{0)\beta} = 0 \quad , \quad G_{0a} - 2D_\alpha\gamma^{\alpha\beta}_{(0}\Psi_{a)\beta} = 0 \\
&\eta^{ab}G_{ab} + \eta^{00}G_{00} = 0
\end{aligned} \tag{3.10}$$

All of these equations are identically satisfied in the rest frame by the rest frame equations[4] given in [18] and the above equation (3.8). This means that if we think of the left hand side of the four equations in (3.9) as tensors, then all the components of these tensors vanish in the rest frame. This means that these

[3] For G_{0a}, also see the footnote (4).

[4] It should be noted that the covariant derivative of $\Psi_{m\beta}$ was not given in [18]. Hence, the equation involving G_{0a} in (3.10) can't be fully derived using only the results given there. However, it easily follows from the above equations (3.5) and (3.6). The constraint $k^m G_{mn} = 0$ of (3.6) implies that G_{0a} is identically zero in the rest frame. On the other hand, the equation (3.5) implies that the second term of the equation involving G_{0a} in equation (3.10) is also identically zero. In fact, using (2.31), we see that this second term is proportional to $D_\beta Z^{0a\beta}$ in the rest frame whose result in terms of the basic superfields was not given in [18].

tensors must vanish identically in all the frames. This proves equations (3.1) and (3.2).

The equation (3.4) can be derived from equation (2.30). However, this requires knowing the covariant form for $Z^{mn\alpha}$. It is easy to see that this is given by

$$Z^{mn\alpha} = \frac{1}{2}\left(\gamma^{[m}\Psi^{n]}\right)^{\alpha} + 4\alpha' k^t(\gamma_t^{\alpha\beta})k^{[m}\Psi_\beta^{n]} \tag{3.11}$$

It reduces to the correct rest frame results given in equation (2.31) and satisfies $Z^{mn\beta}\gamma_{m\alpha\beta} = 0$ as required. By putting this expression for $Z^{mn\alpha}$ in equation (2.30) and using (3.1), we precisely obtain equation (3.4).

A simple procedure to covariantize a given rest frame expression is to make use of two projectors which serve the purpose projecting out the temporal and spatial parts of a given vector in the rest frame. Recall in the rest frame, on-shell momentum for 1st massive states can be expressed as $k = (k^0, \vec{k})$ with $(k^0)^2 = \frac{1}{\alpha'}$ and $\vec{k} = 0$.

Suppose one has a rest frame equation for some vector as $V^a = 0$. It can be written covariantly using the tensor

$$P^{mn} = \eta^{mn} + \alpha' k^m k^n \quad , \quad P^{mn}P_{nr} = P^m{}_r. \tag{3.12}$$

It is quite easy to see in the rest frame,

$$P^{00} = 0\ ,\ P^{0a} = 0\ ,\ P^{ab} = \eta^{ab}. \tag{3.13}$$

So this quantity is indeed a projector which projects out only the spatial part of a vector in the rest frame. So the equation $V^a = 0$ in rest frame can be expressed covariantly in rest frame as $P^{mn}V_n = 0$. Since we have managed expressing a covariant tensor to be zero at rest frame, it must be zero as a tensor at all frames and therefore we have in general at any frame the following equation to be true $P^{mn}V_n = 0$.

Similarly to covariantize a rest frame equation which involves the temporal part, i.e. equation of form (in rest frame) $V^0 = 0$, we can use as a projector

$$\tilde{P}^{mn} = \alpha' k^m k^n \tag{3.14}$$

which in rest frame reduces to

$$\tilde{P}^{00} = 1 \quad , \quad \tilde{P}^{0a} = \tilde{P}^{ab} = 0. \tag{3.15}$$

Alternatively, in order to arrive at the equations[5] (3.3)-(3.5), we can write an ansatz with arbitrary coefficients using the superfields, gamma matrices, Minkowski metric and momentum vector k^m. The coefficients can be fixed by demanding that the ansatz reduces to the rest frame result and it is consistent with other equations in which it appears. For instance, to obtain the relation (3.5), we expand $D_\alpha \Psi_{s\beta}$ in the basis of linearly independent gamma matrices using the identity (A.23). The superfields appearing in the expansion can be fixed by using the equations of motion $QV = 0$ and demanding its consistency with the solution (2.27). Similarly, the coefficients appearing in the ansatz for (3.3) and (3.4) can be fixed by using equations (2.27), (3.1), (3.2) and (3.5). We will discuss this particular procedure in its full glory later in chapter 4. There we will also see how it allows a potential generalization to all higher massive levels.

The constraints given in (3.6) are necessary to ensure that the lowest components of the superfields contain only the physical degrees of freedom. To see this, we note that these conditions, on the θ independent components of superfields, imply

$$(\gamma^m)^{\alpha\beta}\psi_{m\beta} = 0 \quad ; \quad k^m\psi_{m\beta} = 0 \quad ; \quad k^m b_{mnp} = 0 \quad ; \quad k^m g_{mn} = 0 \ \ \& \ \ \eta^{mn} g_{mn} = 0 \tag{3.16}$$

These conditions are the momentum space version of the constraints given in (2.4) and guarantee that $\psi_{s\beta}, g_{mn}$ and b_{mnp} have the correct number of degrees of freedom, namely 128, 44 and 84 respectively.

Once the equations (3.3) - (3.6) and (2.26) - (2.29) hold, all the equations resulting from $QV = 0$, namely (2.17) - (2.23) are satisfied identically as we indicate in section 3.2. The equations (3.3) - (3.6) are the equations which are needed to do the θ expansion of the superfields completely which is done in section 3.3.

[5]The equation 67 of [27] is to be contrasted with the equation (3.4) given above. Note that the term proportional to α' is not present in [27].

3.2 Consistency of the proposed covariant relations with $QV = 0$

In this section we give an outline of the proof that the relations given in section 3.1 are consistent with the equations of motion (2.17) to (2.23) and among themselves. We indicate the steps for the consistency with equations (2.17) and (2.18). The rest of the equations can be verified in a similar fashion.

We first show that our expression for $D_\alpha B_{mnp}$ is consistent with the two expressions of $H_{s\beta}$ given in equations (2.27) and (3.1). For this, we note that by putting the expression of $D_\alpha B_{mnp}$ from (3.4) into the expression of $H_{s\alpha}$ given in (2.27), we obtain

$$\begin{aligned} H_{s\alpha} &= \frac{3}{7}(\gamma^{mn})_\alpha{}^\beta D_\beta B_{mns} \\ &= \frac{3}{7}(\gamma^{mn})_\alpha{}^\beta \Big[12(\gamma_{[mn}\Psi_{p]})_\beta + 24\alpha' k^t k_{[m}(\gamma_{|t|n}\Psi_{p]})_\beta\Big] \\ &= -96\Psi_{s\alpha} + 24\Psi_{s\alpha} \\ &= -72\Psi_{s\alpha} \end{aligned}$$

In the rest frame, this matches with the corresponding result given in (2.33). If we only consider the momentum independent term in the covariant expression of $D_\alpha B_{mnp}$ as done in [27], then the two expressions of $H_{s\alpha}$ do not match with each other (see also 5). In fact the covariant expression of (2.32) as proposed in [27], after substituting in $H_{s\beta}$ as given in (2.27) reduces in the rest frame to $H^b_\beta = -96\Psi^b_\beta$. This is in clear contradiction with (2.33)

Next, on substituting (3.1) and (3.4) in (2.17) and using (2.29) along with $k^m\Psi_{m\alpha} = 0$ of (3.6), we obtain

$$\begin{aligned} LHS &= (\gamma_{mnpqr})^{\alpha\beta}\left[D_\alpha B_{\beta\sigma} - \gamma^s_{\alpha\sigma}H_{s\beta}\right] \\ &= -12(\gamma^{stu}\gamma_{mnpqr}\gamma_{st})_\sigma{}^\alpha\Psi_{u\alpha} - 24\alpha' k^v k_s(\gamma^{stu}\gamma_{mnpqr}\gamma_{vt})_\sigma{}^\alpha\Psi_{u\alpha} + 72(\gamma^s\gamma_{mnpqr})_\sigma{}^\beta\Psi_{s\beta} \\ &= 0 \end{aligned}$$

This shows that the equation (2.17) is identically satisfied.

Next, we consider equation (2.18). Using the expression of C_{mnpq} from (2.27), expression of $H_{s\beta}$ from equation (3.1), equation (3.5) and noting that the trace of product of 5 form with 1 and 3 form is zero, we obtain for the left hand side of (2.18)

$$\begin{aligned} LHS &= (\gamma_{mnpqr})^{\alpha\beta}\left[D_\alpha H_{s\beta} - (\gamma_s)_{\alpha\sigma} C^{\sigma}{}_{\beta}\right] \\ &= \frac{i}{2}(\gamma_{mnpqr}\gamma_{stuvw})^{\beta}{}_{\beta} k^t B^{uvw} - \frac{i}{2}(\gamma_{mnpqr}\gamma_s\gamma^{tuvw})^{\beta}{}_{\beta} k_t B_{uvw} \\ &= 0 \end{aligned}$$

Hence, (2.18) is also identically satisfied.

To verify the rest of the equations resulting from $QV = 0$, one can follow similar steps as in the above two cases. All one needs to use are the various gamma matrix identities and equations (2.27) - (2.29), (3.1) and (3.3) - (3.6). Using these, we can show that all the remaining equations, viz. (2.19) to (2.23) are satisfied identically. This establishes the consistency of our proposed relations in section 3.1 with $QV = 0$.

Finally, we also need to verify that all the relations given in section 3.1 are consistent with each other. One way to verify this is to take the supercovariant derivative of both sides of equations (3.3)-(3.5). Using the identity (A.6), the left hand side of these equations will become proportional to a single superfield, whereas the right hand sides will now involve the supercovariant derivatives of the superfields. The RHS can be shown to be identical to LHS using equations (3.3) - (3.6) and various gamma matrix identities. The consistency of (3.6) with the equations (3.3) - (3.5) is also easy to verify. The consistency of our proposed relations with $QV = 0$ and among themselves is therefore established.

3.3 θ Expansion

As mentioned in the previous sections, the lowest components of the superfields $\Psi_{s\alpha}, B_{mnp}$ and G_{mn} are the physical fields $\psi_{s\alpha}, b_{mnp}$ and g_{mn} respectively. The higher θ components of the superfields contain the same physical fields in a more involved manner. In this section, we shall outline the procedure to determine the θ expansion of these superfields in terms of the physical fields exclusively. We recall the key equations from previous section which we shall need below

$$D_\alpha \Psi_{s\beta} = \frac{1}{16} G_{sm} \gamma^m_{\alpha\beta} + \frac{i}{24} k_m B_{nps} (\gamma^{mnp})_{\alpha\beta} - \frac{i}{144} k^m B^{npq} (\gamma_{smnpq})_{\alpha\beta} \tag{3.17}$$

$$D_\alpha B_{mnp} = 12(\gamma_{[mn}\Psi_{p]})_\alpha + 24\alpha' k^t k_{[m}(\gamma_{|t|n}\Psi_{p]})_\alpha \tag{3.18}$$

$$D_\alpha G_{sm} = 16ik^p(\gamma_{p(s}\Psi_{m)})_\alpha \tag{3.19}$$

These equations are sufficient to obtain the θ expansion of all the superfields to all orders in θ once we specify the θ independent components of B_{mnp}, G_{sm} and $\Psi_{s\alpha}$. Intuitively, we can see how these equations will determine the higher θ components once the lowest components are specified - if we equate the θ^ℓ components on both sides of these equations, we shall have θ^ℓ components of the superfields on the right hand side. But on the left hand side, because we have a covariant derivative D_α, we shall always have $\theta^{\ell+1}$ and $\theta^{\ell-1}$ component of the superfield on which D_α acts. Thus, the higher components can be determined in terms of the lower components.

We denote the θ expansion of the superfields as

$$\Psi_{s\beta} = \sum_{n=0}^{16} \psi_{s\beta\alpha_1\alpha_2\cdots\alpha_n} \theta^{\alpha_1}\theta^{\alpha_2}\cdots\theta^{\alpha_n} \tag{3.20}$$

$$B_{mnp} = \sum_{n=0}^{16} b_{mnp\alpha_1\alpha_2\cdots\alpha_n} \theta^{\alpha_1}\theta^{\alpha_2}\cdots\theta^{\alpha_n} \tag{3.21}$$

$$G_{mn} = \sum_{n=0}^{16} g_{mn\alpha_1\alpha_2\cdots\alpha_n} \theta^{\alpha_1}\theta^{\alpha_2}\cdots\theta^{\alpha_n} \tag{3.22}$$

For each of the superfields, the fermionic and bosonic degrees of freedom occur either at even or odd θ components only. For example in $\Psi_{s\beta}$, the fermionic field $\psi_{s\beta}$ appears at even θ components and the bosonic fields g_{mn} and b_{mnp} appear at odd θ components respectively. In the case of B_{mnp} and G_{mn}, the bosonic fields appear at the even θ components and the fermionic field appear at odd θ components. While moving the fermionic objects such as θ^α across these various components, it is helpful to keep in mind the aforementioned points.

On substituting (3.20)-(3.22) into (3.17)-(3.19) and comparing the $(\ell-1)^{th}$ component on both sides, we find

$$
\begin{aligned}
&(-1)^{\ell-1}\Big[\ell\,\psi_{s\beta\alpha_1\alpha_2\cdots\alpha_\ell}+ik_r(\gamma^r)_{\alpha_1\alpha_2}\psi_{s\beta\alpha_3\cdots\alpha_\ell}\Big]\\
=\;&\Big[\frac{1}{16}(\gamma^m)_{\beta\alpha_1}\;g_{sm\alpha_2\cdots\alpha_\ell}-\frac{i}{24}k_m b_{nps\alpha_2\cdots\alpha_\ell}(\gamma^{mnp})_{\beta\alpha_1}-\frac{i}{144}k_m b_{npq\alpha_2\cdots\alpha_\ell}(\gamma_s{}^{mnpq})_{\beta\alpha_1}\Big]
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\ell\,b_{mnp\alpha_1\alpha_2\cdots\alpha_\ell}+ik_r(\gamma^r)_{\alpha_1\alpha_2}b_{mnp\alpha_3\cdots\alpha_\ell}\\
&=(-1)^\ell\Big[12(\gamma_{[mn})_{\alpha_1}^{\;\;\sigma}\psi_{p]\sigma\alpha_2\cdots\alpha_\ell}+24\alpha' k^t k_{[m}(\gamma_{|t|n})_{\alpha_1}^{\;\;\sigma}\psi_{p]\sigma\alpha_2\cdots\alpha_\ell}\Big]
\end{aligned}
\tag{3.24}
$$

$$
\ell\,g_{mn\alpha_1\alpha_2\cdots\alpha_\ell}+ik_r(\gamma^r)_{\alpha_1\alpha_2}g_{mn\alpha_3\cdots\alpha_\ell}=(-1)^\ell 16\,i\,k^p(\gamma_{p(s})_{\alpha_1}^{\;\;\sigma}\psi_{m)\sigma\alpha_2\cdots\alpha_\ell}
\tag{3.25}
$$

where, $(\ell-1)=1,2,\cdots,16$.

The higher θ components of the superfields can now be fixed using the above equations recursively. We start with the $\ell=1$ component of (3.23) which fixes the $O(\theta)$ component of $\Psi_{s\beta}$ to be

$$
\psi_{s\beta\alpha_1}\theta^{\alpha_1}=\frac{1}{16}(\gamma^m\theta)_\beta\;g_{sm}-\frac{i}{24}k_m b_{nps}(\gamma^{mnp}\theta)_\beta-\frac{i}{144}k_m b_{npq}(\gamma_s{}^{mnpq}\theta)_\beta
\tag{3.26}
$$

Next, using the fact that 1-form $\gamma^m_{\alpha\beta}$ is symmetric in its spinor indices, the equation (3.23) gives for $\ell=2$

$$
\psi_{s\beta\alpha_1\alpha_2}\theta^{\alpha_1}\theta^{\alpha_2}=\frac{1}{16}(\gamma^m\theta)_\beta\;g_{sm\alpha_2}\theta^{\alpha_2}+\frac{i}{24}k_m(\gamma^{mnp}\theta)_\beta b_{nps\alpha_2}\theta^{\alpha_2}+\frac{i}{144}k_m(\gamma_s{}^{mnpq}\theta)_\beta b_{npq\alpha_2}\theta^{\alpha_2}
\tag{3.27}
$$

where, we have used the fact that $b_{mnp\alpha}$ and $g_{mn\alpha}$ are Grassmann odd. They can be determined using the $\ell=1$ components of equations (3.24) and (3.25) respectively as

$$
\begin{aligned}
b_{mnp\alpha_2}&=-12(\gamma_{[mn})_{\alpha_2}^{\;\;\sigma}\psi_{p]\sigma}-24\alpha' k^t k_{[m}(\gamma_{|t|n})_{\alpha_2}^{\;\;\sigma}\psi_{p]\sigma}\\
g_{mn\alpha_2}&=-16\,i\,k^p(\gamma_{p(m})_{\alpha_2}^{\;\;\sigma}\psi_{n)\sigma}
\end{aligned}
\tag{3.28}
$$

These can be substituted in equation (3.27) to fix the $O(\theta^2)$ component of $\Psi_{s\beta}$ completely.

At the next oder in θ, the $\psi_{s\beta\alpha_1\alpha_2\alpha_3}$ will require $g_{mn\alpha_2\alpha_3}$ and $b_{mnp\alpha_2\alpha_3}$. These can again be determined using equations (3.24) and (3.25) along with order θ result (3.26).

Executing this process recursively determines all the θ components of the superfields in terms of the physical fields. Equation (3.23) relates the ℓ^{th} component of $\Psi_{s\beta}$ to the $(\ell-2)^{th}$ component of $\Psi_{s\beta}$ and $(\ell-1)^{th}$ components of B_{mnp} and G_{sm}. Equation (3.24) relates the ℓ^{th} component of B_{mnp} to the $(\ell-2)^{th}$ component of B_{mnp} and $(\ell-1)^{th}$ component of $\Psi_{s\beta}$. Finally, the equation (3.25) determines ℓ^{th} component of G_{mn} in terms of $(\ell-2)^{th}$ component of G_{mn} and $(\ell-1)^{th}$ component of $\Psi_{s\alpha}$. These statements can be schematically represented as

$$\begin{aligned} D^{(\ell+1)}\Psi_{s\beta} &\sim D^{\ell}G_{sm} + D^{\ell}B_{mnp} \\ D^{\ell}B_{mnp} &\sim D^{(\ell-1)}\Psi_{s\beta} \\ D^{\ell}G_{mn} &\sim D^{(\ell-1)}\Psi_{s\beta} \end{aligned}$$

where for any superfield V, $DV \equiv (\theta^{\alpha}D_{\alpha}V)\big|_{\theta=0}$, so that D^{ℓ} picks up the ℓ^{th} θ component of the superfield it acts on.

Furthermore, the only seed that we need to provide to these recursive relations are the zeroth component of each of the three superfields, which by definition are the three physical fields in the massive spin 2 supermultiplet. This guarantees that by construction the θ expansion performed using these recursion relations will only have physical fields appearing at all orders.

For calculating any amplitude involving the massive states in pure spinor formalism, we shall need theθ expansion result of the above vertex operator. Clearly, the θ expansion of the basic superfields B_{mnp} and $\Psi_{m\alpha}$ automatically implies the θ expansion of the full vertex operator. For our purposes, we shall need the θ expansion results of $\Psi_{m\alpha}$ upto order θ^3 and that of $B_{\alpha\beta}$ upto order θ^4. We used the mathematica package GAMMA to do this computation [28]. Using equations (3.3), (3.4) and (3.5), we obtain (for details, please see [19])

$$
\begin{aligned}
B_{\alpha\beta} \;=\; \gamma^{mnp}_{\alpha\beta}\Big[& b_{mnp} + 12(\psi_p\gamma_{mn}\theta) + 24\alpha' k^r k_m(\psi_p\gamma_{rn}\theta) + \frac{3}{8}(\theta\gamma_{mn}{}^q\theta)\; g_{pq} - \frac{3i}{4}(\theta\gamma^{tu}{}_m\theta)k_t b_{unp} \\
& + \frac{3}{4}\alpha' k^r k_m(\theta\gamma_{rn}{}^{q}\theta)\; g_{pq} - \frac{i}{24}(\theta\gamma_{tuvwmnp}\theta)k^t b^{uvw} - \frac{1}{6}ik_s\,(\psi_v\gamma_{tu}\theta)\,(\theta\gamma_{stuvmnp}\theta) \\
& - 4i\alpha k_s k_t k_m\,(\theta\gamma_{tun}\theta)\,(\psi_p\gamma_{su}\theta) + ik_s\,(\theta\gamma_{tmn}\theta)\,(\psi_p\gamma_{st}\theta) + ik_s\,(\theta\gamma_{tmn}\theta)\,(\psi_t\gamma_{sp}\theta) \\
& + 2ik_s\,(\theta\gamma_{stm}\theta)\,(\psi_n\gamma_{tp}\theta) - ik_s\,(\theta\gamma_{stm}\theta)\,(\psi_t\gamma_{np}\theta) + \frac{1}{64\alpha'}(\theta\gamma_{smn}\theta)(\theta\gamma_{tup}\theta)b_{stu} \\
& - \frac{1}{288\alpha'}(\theta\gamma_{stu}\theta)(\theta\gamma_{mnp}\theta)b_{stu} + \frac{1}{64\alpha'}(\theta\gamma_{stu}\theta)(\theta\gamma_{unp}\theta)b_{stm} \\
& + \frac{1}{32}(\theta\gamma_{sux}\theta)(\theta\gamma_{txp}\theta)b_{smn}k_t k_u - \frac{1}{16}(\theta\gamma_{sun}\theta)(\theta\gamma_{txp}\theta)b_{stm}k_u k_x \\
& + \frac{1}{64}(\theta\gamma_{stx}\theta)(\theta\gamma_{unp}\theta)b_{stm}k_u k_x + \frac{1}{192}(\theta\gamma_{xzm}\theta)(\theta\gamma_{stuyznp}\theta)b_{stu}k_x k_y \\
& + \frac{1}{192}(\theta\gamma_{uyz}\theta)(\theta\gamma_{stxzmnp}\theta)b_{stu}k_x k_y + \frac{1}{3456}(\theta\gamma_{stuwxyz}\theta)(\theta\gamma_{vxyzmnp}\theta)b_{stu}k_v k_w \\
& + \frac{1}{32}(\theta\gamma_{svn}\theta)(\theta\gamma_{tup}\theta)b_{stu}k_v k_m + \frac{1}{64}(\theta\gamma_{tuv}\theta)(\theta\gamma_{snp}\theta)b_{stu}k_v k_m \\
& - \frac{1}{96}(\theta\gamma_{stu}\theta)(\theta\gamma_{vnp}\theta)b_{stu}k_v k_m - \frac{1}{32}(\theta\gamma_{stv}\theta)(\theta\gamma_{uvp}\theta)b_{stm}k_u k_n \\
& + \frac{1}{384}i\,(\theta\gamma_{tvw}\theta)\,(\theta\gamma_{suvwmnp}\theta)\,k_u g_{st} + \frac{1}{32}i\,(\theta\gamma_{sun}\theta)\,(\theta\gamma_{tup}\theta)\,k_t g_{sm} \\
& + \frac{1}{64}i\,(\theta\gamma_{stu}\theta)\,(\theta\gamma_{unp}\theta)\,k_t g_{sm} + \frac{1}{64}i\,(\theta\gamma_{smn}\theta)\,(\theta\gamma_{tup}\theta)\,k_u g_{st} \\
& + \frac{1}{64}i\,(\theta\gamma_{sum}\theta)\,(\theta\gamma_{tnp}\theta)\,k_u g_{st} - \frac{i\alpha'}{16}\,(\theta\gamma_{suv}\theta)\,(\theta\gamma_{tvp}\theta)\,k_t k_u k_n g_{sm} \;+\; O(\theta^5)\Big]
\end{aligned}
\tag{3.29}
$$

Similarly, the θ expansion of the superfield $\Psi_{s\alpha}$ is given by

$$
\begin{aligned}
&\Psi_{s\beta} \\
= \;& \psi_{s\beta} + \frac{1}{16}(\gamma^m\theta)_\beta\; g_{sm} - \frac{i}{24}(\gamma^{mnp}\theta)_\beta k_m b_{nps} - \frac{i}{144}(\gamma_s{}^{npqr}\theta)_\beta k_n b_{pqr} \\
& -\frac{i}{2}k^p(\gamma^m\theta)_\beta(\psi_{(m}\gamma_{s)p}\theta) - \frac{i}{4}k_m(\gamma^{mnp}\theta)_\beta(\psi_{[s}\gamma_{np]}\theta) - \frac{i}{24}(\gamma_s{}^{mnpq}\theta)_\beta k_m(\psi_q\gamma_{np}\theta) \\
& -\frac{i}{6}\alpha' k_m k^r k_s(\gamma^{mnp}\theta)_\beta(\psi_p\gamma_{rn}\theta) + \frac{i}{288}\alpha'(\gamma^{mnp}\theta)_\beta k_m k^r k_s(\theta\gamma^q{}_{nr}\theta)\; g_{pq} \\
& -\frac{i}{192}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^q{}_{[np}\theta)g_{s]q} - \frac{i}{1152}(\gamma_{smnpq}\theta)_\beta k^m(\theta\gamma_{npt}\theta)\; g^{qt} \\
& -\frac{i}{96}k^p(\gamma^m\theta)_\beta(\theta\gamma_{pq(s}\theta)\; g_{m)q} - \frac{1}{1728}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^{tuvw}{}_{nps}\theta)k_t b_{uvw} \\
& -\frac{1}{864\alpha'}(\gamma_s\theta)_\beta(\theta\gamma^{npq}\theta)b_{npq} - \frac{1}{10368}(\gamma_s{}^{mnpq}\theta)_\beta k_m(\theta\gamma_{tuvwnpq}\theta)k^t b^{uvw} \\
& -\frac{1}{864}(\gamma^m\theta)_\beta(\theta\gamma^{npq}\theta)b_{npq}k_m k_s - \frac{1}{576}(\gamma_{smnpq}\theta)_\beta k^m(\theta\gamma^{tun}\theta)b_u{}^{pq}k_t \\
& -\frac{1}{96\alpha'}(\gamma^m\theta)_\beta(\theta\gamma^{qr}{}_{(s}\theta)b_{m)rq} + \frac{1}{96}(\gamma^m\theta)_\beta(\theta\gamma^{nqr}\theta)k_n k_{(s}b_{m)qr} \\
& +\frac{1}{96}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^r{}_{q[n}\theta)b_{ps]r}k^q + O(\theta^4)
\end{aligned}
\tag{3.30}
$$

where,

$$
b_{mnp} = e_{mnp}e^{ik\cdot X} \quad , \qquad g_{mn} = e_{mn}e^{ik\cdot X} \quad , \qquad \psi_{m\alpha} = e_{m\alpha}e^{ik\cdot X} \tag{3.31}
$$

Using these θ expansion results in (2.5) we get the θ expansion of the unintegrated vertex operator. In this form, the vertex operator can be readily used to compute scattering of massive states and also to construct the integrated vertex for first massive states.

Chapter 4

A general methodology for constructing massive vertex in pure spinor formalism

In the last chapter we saw how to covariantly perform the θ expansion of the massive unintegrated vertex. This was achieved in particular by expressing the massive spin 2 supermultiplet in three ways using three superfields. The defining features of these superfields lie in the fact that the zeroth component of each of this superfields are the physical fields that constitute the massive spin 2 supermultiplet. The seeming redundancy of introducing multiple superfields to describe the same supermultiplet was accounted for by our proposed differential relations which are self-consistent and do not affect the BRST condition for the vertex. These differential relations lead to a set of recursive relations which allow us to perform the fully covariant θ expansion. Of course, once we have the full expansion and it satisfies the consistency check, it does not need any more justification regarding the origin of these differential relations. However, in this chapter we will see that our results strongly hints at a much more powerful methodology based on representation theory of SO(9) group (which is the little group for all massive states) which not only allows us to systematically derive our proposed differential relations, but also provide us with a completely systematic way of constructing the vertex. This in particular is of huge significance, since as discussed in chapter 1 , a common critique of PS formalism is its lack of a clear principle for constructing all vertex operators.

In this chapter, we will first illustrate our methodology by re-deriving the massive unintegrated vertex of chapter 2 while simultaneously deriving our proposed differential relations. Then we will hypothesize a generalized construction methodology for any massive vertex (unintegrated as well as integrated). As further evidence for our hypothesized methodology, we will briefly describe its success in constructing the integrated vertex for the first massive states (Which was done for the first time in literature). We will conclude by briefly discussing the case for massless states and also how to obtain the answers for closed strings once the vertex is known for open superstrings.

4.1 A re-derivation of massive unintegrated vertex

Imagine a situation, where the construction of [18] and the rest frame analysis performed therein is not known of. In such a case one can start constructing the vertex operator based on certain number of very reasonable assumptions in a systematic fashion. Let us describe this in a step by step manner.

Recall, to construct a vertex, is to basically express all the superfields in terms of a single superfield and its supercovariant derivatives such that $QV = 0$ is satisfied. We modify this as follows -

- We seek to express the vertex not only in terms of a single superfield and its supercovariant derivative, but in terms of multiple superfields, *without* any supercovariant derivatives.

- The precise number and nature (by that we mean its index structure and grassmanality of its zeroth component) of these superfields are fixed by requiring that zeroth component of each of these superfields contain precisely the physical field.

$$B_{mnp}\Big|_{\theta=0} \equiv b_{mnp} \quad , \quad G_{mn}\Big|_{\theta=0} = g_{mn} \quad , \quad \Psi_{m\alpha}\Big|_{\theta=0} = \psi_{m\alpha}. \tag{4.1}$$

  Promote all the on-shell conditions for the physical fields to hold true for these superfields as well.

$$\begin{aligned} k^m\psi_{m\alpha} = 0 \rightarrow k^m\Psi_{m\alpha} \equiv 0 \quad , \quad (\gamma^m\psi_{(m)})_\alpha = 0 \rightarrow (\gamma^m\Psi_{(m)})_\alpha \equiv 0 \\ k^m b_{mnp} = 0 \rightarrow k^m B_{mnp} \equiv 0 \quad , \quad k^m g_{mn} = 0 \rightarrow k^m G_{mn} \equiv 0 \end{aligned} \tag{4.2}$$

  (4.1) and (4.2) together constitutes the definitions of these superfields. We will refer these superfields often as physical superfields.

- Next we re-express all unconstrained superfields of the vertex (2.5) and the Lagrange multiplier superfield of equation (2.16) as a linear combination of these 3 superfields with unknown constant co-efficients. The guiding principle behind constructing this ansatz is representation theory of the little group $SO(9)$. For example, let us see how to write down the ansatz for the superfield $F_{\alpha mn}$.

First, we express the superfield separately for different possible spatial and temporal index. Recall that $F_{\alpha mn}$ is antisymmetric under $m \leftrightarrow n$, it gives us -

$$F_{\alpha mn} \Rightarrow \begin{cases} F_{\alpha\, 0a} \\ F_{\alpha\, ab} \end{cases}$$

Now consider the component $F_{\alpha\, 0a}$, under $SO(9)$ it can potentially transform as the following irreps $\mathbf{16} \otimes \mathbf{9} = \mathbf{16} \oplus \mathbf{128}$. We recognize the irrep $\mathbf{128}$ as the physical irrep. Similarly for $F_{\alpha\, ab}$, it can potentially transform as $\mathbf{16} \otimes \mathbf{36} = \mathbf{16} \oplus \mathbf{128} \oplus \mathbf{432}$. Again we recognize only $\mathbf{128}$ to be the physical representation. Since we are after a solution of superfield for which at all stages the co-efficient should be physical fields, we want to write an ansatz for $F_{\alpha mn}$ solely in terms of the physical superfields. The representation theory analysis tells us that there are two independent tensor structures linear in the superfield $\Psi_{m\alpha}$ (which contains the irrep $\mathbf{128}$ in its zero-th component). At this point by simple inspection we can write down the following ansatz

$$F_{\alpha mn} = a\, k_{[m} \Psi_{n]\alpha} + b\, k^s (\gamma_{s[m]} \Psi_{n]})_\alpha \tag{4.3}$$

where a,$b \in \mathbb{C}$ are constants to be determined. Solving for $F_{\alpha mn}$ now reduce to solving for a and b.

- The redundancy of describing a single supermultiplet by three physical superfields suggest that there must be relations amongst these three superfields. In particular we are after a set of relations which relates the supercovariant derivative of bosonic superfields in terms of linear combination of fermionic superfields and vice versa. Once again we employ representation theory to write down the ansatz for such differential relations. Let us see how it works for $D_\alpha \Psi_{m\beta}$.

  First we use A.23 to express

$$D_\alpha \Psi_{m\beta} = (\gamma^s)_{\alpha\beta} S_{sm} + (\gamma^{stu})_{\alpha\beta} A_{mstu} + (\gamma^{stuvw})_{\alpha\beta}\, S_{mstuvw} \tag{4.4}$$

Now we do similar group theoretical analysis for each of the superfields S_{sm}, A_{mstu} and S_{mstuvw}. Note that since in 10 dimensions, the gamma 5-form is self dual, only the self dual part of S_{mstuvw}

should be kept. This leads to the ansatz

$$D_\alpha \Psi_{m\beta} = (\gamma^s)_{\alpha\beta}\,\hat{a}\,G_{sm} + (\gamma^{stu})_{\alpha\beta}\,[\hat{a}_1 k_{[s}B_{tu]m} + \hat{b}_1 k_m B_{stu}] + (\gamma^{mstuv})_{\alpha\beta}\,\hat{c}_1 k_s B_{tuv} \tag{4.5}$$

Once again, the complex co-efficients $\hat{a}$, $\hat{a}_1$ etc. are to be determined. In the last chapter we fixed this co-efficients by demanding they reduce to rest frame results derived in [18]. Then we went on to show that they lead to self-consistent relations without affecting the BRST condition $QV = 0$. Here, we *do not assume* any prior knowledge of rest frame analysis. We fix these co-efficients by demanding that they are

1. Consistent with $QV = 0$.
2. Consistent with the definitions of the physical superfields (equations (4.1) and (4.2)).
3. Consistent among themselves (self-consistency).

- All such superfields expressed in ansatz forms are substituted in $QV = 0$ equations (i.e. equations (2.17)-(2.23)) and we set the co-efficients of each independent tensor structure to zero which gives us a system of linear algebraic equations for the unknown co-efficients. Solving for the co-efficients gives us the solution for the massive unintegrated vertex.

4.1.1 Ansatz for the vertex and the differential relations

We note that the superfields B_{mnp} and $H_{m\alpha}$ appearing in 2.5 along with the gauge fixing conditions 2.25 satisfy the definitions of two of the three physical superfields. So instead of introducing $\Psi_{m\alpha}$, we will work with $H_{m\alpha}$ in this chapter [1].

Differential relations :

$$D_\alpha H_{s\beta} = a(\gamma^m)_{\alpha\beta}G_{ms} + s_1(\gamma^{mnp})_{\alpha\beta}k_{[m}B_{np]s} + b(\gamma^{mnp})_{\alpha\beta}k_s B_{mnp} + s_2(\gamma^{smnpq})_{\alpha\beta}k_m B_{npq} \tag{4.6}$$

$$(\gamma^{mnp})^{\rho\sigma} D_\alpha B_{mnp} = (\gamma^{mnp})^{\rho\sigma}[b_1(\gamma_{mn}H_p)_\alpha + b_2 k^s K_m(\gamma_{sn}H_p)_\alpha] \tag{4.7}$$

[1] In fact, as we have seen both superfields are related to each other just by an overall numerical factor.

$$D_\alpha G_{mn} = \hat{a} k^p (\gamma_{p(m} H_{n)})_\alpha \tag{4.8}$$

Consistency of these ansatz with the definition (4.2) of the superfields gives us the following equations,

$$b_2 = 2\alpha' b_1 \quad , \quad s_2 = -\frac{s_1}{6} \quad , \quad b = 0. \tag{4.9}$$

Next, using the equation $k^m (\gamma_m)^{\alpha\beta} D_\alpha D_\beta = -\frac{16\,i}{\alpha'}$ (see (A.6)), we obtain further,

$$s_1 = 18i\, b_1 \quad , \quad a\hat{a} = i\,. \tag{4.10}$$

Superfields :

We take B_{mnp} and $H_{m\alpha}$ to be the two physical superfields corresponding to irreps **84** and **128** respectively. The rest we express in the following ansatz

$$C_{mnpq} = c_1 k_{[m} B_{npq]}, \tag{4.11}$$

$$E_\alpha = A_\alpha = 0, \tag{4.12}$$

$$F_{\alpha mn} = c_2\, k_{[m} H_{n]\alpha} + c_3\, k^s (\gamma_{s[m]} H_{n]})_\alpha, \tag{4.13}$$

$$K^m{}_{npqr} = e_1 k^m k_{[n} B_{pqr]} + e_2 \delta^m{}_{[n} B_{pqr]}. \tag{4.14}$$

1st equation of $QV = 0$

On substituting the ansatz in (2.17), we obtain

$$80 b_1 - \frac{10}{\alpha'} b_2 = -10\,. \tag{4.15}$$

2nd equation of $QV = 0$

On substituting the ansatz in (2.18), we obtain

$$c_1 = s_2 \quad , \quad c_1 = -\frac{1}{6}s_1 \ . \tag{4.16}$$

3rd equation of $QV = 0$

On substituting the ansatz in (2.19), we obtain

$$12b_1c_1 + c_2 + c_3 = 0 \ . \tag{4.17}$$

4th equation of $QV = 0$

On substituting the ansatz in (2.20), we get identically zero just based on the tensor structure of the ansatz and it does not give any relation between the unknown co-efficients.

5th equation of $QV = 0$

On substituting the ansatz in (2.21), we obtain

$$4 + 4ic_2 + 8e_1 - c_2s_1 + 4c_3s_1 + 6c_2s_2 - 56e_2\alpha' = 0 \ . \tag{4.18}$$

6th equation of $QV = 0$

On substituting the ansatz in (2.22), we once again get identically zero just based on the tensor structure of the ansatz and it does not give any relation between the unknown co-efficients.

7th equation of $QV = 0$

On substituting the ansatz in (2.23), we obtain

$$e_2 = 0 \quad , \quad ic_2 - 4e_1 = 0 \ . \tag{4.19}$$

All of these equations for unknown co-efficients can be uniquely solved to give the solution

$$
\begin{aligned}
&b_1 = \frac{1}{6} \quad , \quad b_2 = -\frac{\alpha'}{3} \quad , \quad b = 0 \, , \\
&s_1 = -3i \quad , \quad s_2 = \frac{i}{2} \, , \\
&c_1 = \frac{i}{2} \quad , \quad c_2 = \frac{7i}{8} \quad , \quad c_3 = \frac{i}{8} \, , \\
&e_1 = \frac{7}{32} \quad , \quad e_2 = 0 \, .
\end{aligned} \tag{4.20}
$$

It is evident from the ansatz, that none of the superfields for first massive unintegrated vertex contains **44** at their zeroth component, therefore the co-efficient a and $\hat{a}$ do not get individually determined from $QV = 0$ equations. This is not troubling, since the only use of G_{mn} superfield is now to perform the covariant θ expansions of all the physical superfields. From the structure of the recursion relations discussed in section 3.3 it is obvious that neither a nor the $\hat{a}$ enters the θ expansion separately. Instead they will always appear in the product $a\hat{a}$ which is already determined in equation (4.10) as required by self consistency. In chapter 3 we fixed these by comparing with rest frame definition. However note that the rest frame definition of G_{ab} is not unique and can always be redefined by an overall numerical factor (see also footnote 4 of chapter 3).

The solution can now be easily verified to be exactly same as that obtained in chapter 2 and the differential relations are same as that proposed in chapter 3 with the following definitions $H_{m\alpha} = -72\Psi_{m\alpha}$ and $a = -\frac{72}{16}, \hat{a} = -\frac{16i}{72}$.

This method of constructing the vertex has two distinct advantages over the method described in chapter 2. Firstly, the consistency of the solution with BRST condition $QV = 0$ is manifest and does not need separate attention. Secondly, this method can be easily generalized for construction of unintegrated and integrated vertex for all higher massive levels. Next we propose such a general methodology for constructing vertex operator for any massive states.

4.2 A general method to construct massive vertex in pure spinor formalism

From the RNS formalism, the field content of each mass level for open superstrings are known. Suppose in n-th mass level, there are total n_B number of bosonic fields b_i and n_F number of fermionic fields f_i (we suppress the index structure throughout this section). To construct the unintegrated and integrated vertex for this mass level the following steps must be followed.

Unintegrated Vertex

- **Step 1 :** Introduce a superfield for each of the physical superfields such that

$$B_i\Big|_{\theta=0} \equiv b_i \quad , \quad \forall i = 1(1)n_B \tag{4.21}$$

$$F_i\Big|_{\theta=0} \equiv f_i \quad , \quad \forall i = 1(1)n_F \; . \tag{4.22}$$

  Also elevate all kinematic conditions on the physical fields b_i and f_i to their respective superfields (eg. for first mass level $k^m g_{mn} = 0 \to k^m G_{mn} = 0$ and so on.)

- **Step 2 :** Write down the unintegrated vertex in the form $V \sim \sum_n \hat{O}_n S_n$, where $\hat{O}_n$ are all possible conformal weight n and ghost number 1 object constructed out of $\Pi^m, d_\alpha, \partial\theta^\alpha, N^{mn}, J, \lambda^\alpha$ and their worldsheet derivatives. S_n are appropriate superfields which are as of now unconstrained.

- **Step 3 :** Derive all the constraints relevant at conformal weight n and ghost number 1 level by taking suitable OPEs and derivative of the identity (2.14). Introduce a Lagrange multiplier superfield K_n for each of these constraints and add them to equations obtained in $QV = 0$. For conformal weight 2 or greater operators, there can be additional constraints due to non-trivial OPE between the constituent conformal operators (e.g. $\Pi^m\Pi^n$, $\Pi^m d_\alpha$,$N^{mn}\lambda^\alpha$ etc.). One can handle them directly by eliminating some of the basis elements in favor of the others. We give some such instances later in this chapter. For details see [20](also see [22]).

- **Step 4 :** Use representation theory to write down ansatz for all superfields S_n and Lagrange multiplier superfields K_n in terms of the linear combination of physical superfields.

$$\begin{aligned} &S_n^{bosonic} = \sum_i c_i^{(n)} B_i \quad , \quad S_n^{fermionic} = \sum_i d_i^{(n)} F_i \; , \\ &K_n^{bosonic} = \sum_i \hat{c}_i^{(n)} B_i \quad , \quad K_n^{fermionic} = \sum_i \hat{d}_i^{(n)} F_i \; . \\ &c_i^{(n)} \; , d_i^{(n)} \; , \hat{c}_i^{(n)} \; , \hat{d}_i^{(n)} \in \mathbb{C} \; , \forall \; i \; \& \; n \end{aligned} \tag{4.23}$$

- **Step 5 :** Also use representation theory to write down ansatz for differential relations amongst the physical superfields of the following form

$$D_\alpha F_i = \sum_j s_j^{(i)} B_j \ \ \forall i = 1(1)n_F \quad , \tag{4.24}$$

$$D_\alpha B_i = \sum_j \hat{s}_j^{(i)} F_j \ \ \forall i = 1(1)n_B \tag{4.25}$$

$$s_j^{(i)} \ , \hat{s}_j^{(i)} \in \mathbb{C} \ \forall \ i \ \& \ n$$

- **Step 6 :** Substitute these ansatz in $QV = 0$ equations subjected to the constraints and use the consistency with the definitions of physical superfields and self-consistency to solve for all of the unknown co-efficients. This completes the construction of the unintegrated vertex.

Integrated vertex

- **Step 1 :** Once the unintegrated vertex is known, write down ansatz for integrated vertex of the form $U \sim \sum_n \tilde{O}_n T_n$, where now $\tilde{O}_n$ are all possible basis operators of conformal weight $(n+1)$ and ghost number zero constructed using the basic constituent operators. T_n are appropriate superfields which are to be determined in terms of the physical superfields.

- **Step 2 :** Take into account all of the constraints at this conformal weight and ghost number using similar procedure as discussed for unintegrated vertex.

- **Step 3 :** Use representation theory to write down ansatz for all superfields T_n and Lagrangian superfields in terms of the physical superfields.

- **Step 4 :** Substitute the ansatz in the BRST condition $QU - \partial_{\mathbb{R}} V = 0$ subjected to the constraints and solve for the unknown co-efficients to obtain the solution.

This is the first time in literature to the best of our knowledge, a general super-Poincaré covariant prescription for constructing vertex operator for massive states in pure spinor has been proposed. While as of now a direct and explicit proof of this methodology is lacking, there are two non-trivial pieces of evidence that suggest this methodology is correct. First is the re-derivation of the unintegrated vertex presented in this chapter. Second is the successful application of this methodology to construct the integrated vertex operator for the first time in [20]. The details of the construction form a part of a separate thesis (see [22]). Here we will only state the final result for sake of completeness.

4.3 Integrated vertex for first massive state : results

We give the following ansatz for the integrated vertex which has conformal weight = 2 and ghost number = 0,

$$\begin{aligned} U &= :\partial^2\theta^\alpha C_\alpha: + :\partial\Pi^m C_m: + :\partial d_\alpha E^\alpha: + :(\partial J)C: + :\partial N^{mn}C_{mn}: \\ &+ :\Pi^m\Pi^n F_{mn}: + :\Pi^m d_\alpha F_m{}^\alpha: + :\Pi^m N^{pq}F_{mpq}: + :\Pi^m J F_m: + :\Pi^m\partial\theta^\alpha G_{m\alpha}: \\ &+ :d_\alpha d_\beta K^{\alpha\beta}: + :d_\alpha N^{mn}G^\alpha{}_{mn}: + :d_\alpha J F^\alpha: + :d_\alpha\partial\theta^\beta F^\alpha{}_\beta: \\ &+ :N^{mn}N^{pq}G_{mnpq}: + :N^{mn}JP_{mn}: + :N^{mn}\partial\theta^\alpha H_{mn\alpha}: \\ &+ :JJH: + :J\partial\theta^\alpha H_\alpha: + :\partial\theta^\alpha\partial\theta^\beta H_{\alpha\beta}: \end{aligned} \tag{4.26}$$

The ansatz for each of these superfields using representation theory of $SO(9)$ are,

$$\begin{aligned} C_\alpha &= C_m = E^\alpha = C = C_{mn} = F_m = F^\alpha = P_{mn} = H = H_\alpha = 0 \\ F_{mn} &= f_1 G_{mn} \quad , \quad G_{m\alpha} = g_1\Psi_{m\alpha} \\ K^{\alpha\beta} &= a\,\gamma^{\alpha\beta}_{mnp}B^{mnp} \quad , \quad H_{\alpha\beta} = h_1\gamma^{mnp}_{\alpha\beta}B_{mnp} \\ F^\alpha{}_\beta &= f_5(\gamma^{mnpq})^\alpha{}_\beta k_m B_{npq} \quad , \quad F^\alpha_m = f_2 k^r(\gamma_r)^{\alpha\beta}\Psi_{m\beta} \\ F_{mpq} &= f_3 G_{m[p}k_{q]} + f_4 B_{mpq} \quad , \quad G^\beta_{pq} = g_2\gamma^{\beta\sigma}_{[p}\Psi_{q]\sigma} + g_3 k^r\gamma_r^{\beta\sigma}k_{[p}\Psi_{q]\sigma} \\ H_{mn\alpha} &= h_2\, k_{[m}\Psi_{n]\alpha} + h_3 k^q(\gamma_{q[m})_\alpha{}^\sigma\Psi_{n]\sigma} \\ G_{mnpq} &= g_4 k_{[m}B_{n]pq} + g_5 k_{[p}B_{q]mn} + g_6 k_{[m}G_{n][p}k_{q]} + g_7\,\eta_{[m[p}G_{q]n]} \end{aligned} \tag{4.27}$$

As mentioned, all the basis elements obtained in QU will not be independent, they will be subjected to the following constraints for each of which we introduce a Lagrange multiplier superfield,

$$(I_1)^n_\beta \equiv \,:N^{mn}J\lambda^\alpha:(\gamma_m)_{\alpha\beta} - \frac{1}{2}:JJ\lambda^\alpha:(\gamma^n)_{\alpha\beta} - \alpha':J\partial\lambda^\alpha:\gamma^n_{\alpha\beta} = 0 \tag{4.28}$$

$$(I_2)^{mnq}_\beta \equiv \,:N^{mn}N^{pq}\lambda^\alpha:(\gamma_p)_{\alpha\beta} - \frac{1}{2}:N^{mn}J\lambda^\alpha:(\gamma^q)_{\alpha\beta} - \alpha':N^{mn}\partial\lambda^\alpha:\gamma^q_{\alpha\beta} = 0 \tag{4.29}$$

$$(I_3)^n_{\sigma\beta} \equiv \,:d_\sigma N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta} - \frac{1}{2}:d_\sigma J\lambda^\alpha:(\gamma^n)_{\alpha\beta} - \alpha':d_\sigma\partial\lambda^\alpha:\gamma^n_{\alpha\beta} = 0 \tag{4.30}$$

$$(I_4)^{pn}_\beta \equiv \,:\Pi^p N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta} - \frac{1}{2}:\Pi^p J\lambda^\alpha:(\gamma^n)_{\alpha\beta} - \alpha':\Pi^p\partial\lambda^\alpha:\gamma^n_{\alpha\beta} = 0 \tag{4.31}$$

$$(I_5)^{\sigma n}_\beta \equiv \,:\partial\theta^\sigma N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta} - \frac{1}{2}:\partial\theta^\sigma J\lambda^\alpha:(\gamma^n)_{\alpha\beta} - \alpha':\partial\theta^\sigma\partial\lambda^\alpha:\gamma^n_{\alpha\beta} = 0 \tag{4.32}$$

The above 5 identities follow from taking the OPE of (2.14) with the object of conformal weight one, namely $J, N^{mn}, d_\sigma, \Pi^p$ and $\partial\theta^\sigma$ respectively. The identity which can be obtained by taking the derivative of (2.14) is given by

$$\begin{aligned}(I_6)^n_\beta \equiv\; & :\partial N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta} + :N^{mn}\partial\lambda^\alpha:(\gamma_m)_{\alpha\beta} - \frac{1}{2}:\partial J\lambda^\alpha:(\gamma^n)_{\alpha\beta} - \frac{1}{2}:J\partial\lambda^\alpha:(\gamma^n)_{\alpha\beta} \\ & - \alpha'\gamma^n_{\alpha\beta}\partial^2\lambda^\alpha = 0 \end{aligned} \tag{4.33}$$

Apart from these, there are two more constraint identities which follow from the OPEs given in section 1.4.2. The OPE of d_α with d_β implies

$$:d_\alpha d_\beta: + :d_\beta d_\alpha: + \frac{\alpha'}{2}\partial\Pi^t(\gamma_t)_{\alpha\beta} = 0 \tag{4.34}$$

Similarly, the OPE of N^{mn} with N^{pq} implies

$$:N^{mn}N^{pq}: - :N^{pq}N^{mn}: = -\frac{\alpha'}{2}\Big[\eta^{np}\partial N^{mq} - \eta^{nq}\partial N^{mp} - \eta^{mp}\partial N^{nq} + \eta^{mq}\partial N^{np}\Big] \tag{4.35}$$

One way to think about these two identities is to note that we are working with a given ordering of the pure spinor variables inside the normal ordering. However, for $:d_\alpha d_\beta:$ and $:N^{mn}N^{pq}:$, there is no preferred ordering. The above two identities (4.34) and (4.35) are a reflection of this fact[2].

[2] Note that there are OPE between Π^m and Π^n as well as between J and J. However, no pure spinor fields appear in these OPE and hence they do not lead to any non trivial constraint between basis elements.

We also need similar ansatz for the Lagrange multipliers in terms of the basic superfields. We propose

$$
\begin{aligned}
(K_1)^\alpha_m &= c_1 k^r (\gamma_r)^{\alpha\beta}\Psi_{m\beta} \\
(K_2)^\alpha_{mnq} &= c_2 k_{[m}\gamma^{\alpha\beta}_{n]}\Psi_{q\beta} + c_3 k_q \gamma^{\alpha\beta}_{[m}\Psi_{n]\beta} + c_4\gamma^{\alpha\beta}_q k_{[m}\Psi_{n]\beta} + c_5 k^r\gamma^{\alpha\beta}_{rmn}\Psi_{q\beta} + c_6 k^r\gamma^{\alpha\beta}_{rq[m}\Psi_{n]\beta} \\
&\quad + c_7 k^r k_q \gamma^{\alpha\beta}_r k_{[m}\Psi_{n]\beta} + c_8 k^r\gamma^{\alpha\beta}_r \eta_{q[m}\Psi_{n]\beta} \\
(K_3)^{\alpha\beta}_m &= c_9 G_{mn}(\gamma^n)^{\alpha\beta} + c_{10}k_m B_{stu}(\gamma^{stu})^{\alpha\beta} + c_{11}k_s B_{tum}(\gamma^{stu})^{\alpha\beta} + c_{12}k_s B_{tuv}(\gamma_m{}^{stuv})^{\alpha\beta} \\
(K_4)^\alpha_{mn} &= c_{13}(\gamma_n)^{\alpha\beta}\Psi_{m\beta} + c_{14}(\gamma_m)^{\alpha\beta}\Psi_{n\beta} + c_{15}k^r k_m(\gamma_r)^{\alpha\beta}\Psi_{n\beta} + c_{16}k^r k_n(\gamma_r)^{\alpha\beta}\Psi_{m\beta} \\
(K_5)^\alpha{}_{\beta m} &= c_{17}k_p G_{qm}(\gamma^{pq})^\alpha{}_\beta + c_{18}B_{mpq}(\gamma^{pq})^\alpha{}_\beta + c_{19}B_{pqr}(\gamma_m{}^{pqr})^\alpha{}_\beta + c_{20}k_m k_p B_{qrs}(\gamma^{pqrs})^\alpha{}_\beta \\
(K_6)^\alpha_m &= c_{21}k^r(\gamma_r)^{\alpha\beta}\Psi_{m\beta}
\end{aligned}
\tag{4.36}
$$

Following the steps outlined in previous section, the final form of the first massive integrated vertex operator is obtained to be

$$
\begin{aligned}
U &= \;:\Pi^m\Pi^n F_{mn}: \;+\; :\Pi^m d_\alpha F_m{}^\alpha: \;+\; :\Pi^m\partial\theta^\alpha G_{m\alpha}: \;+\; :\Pi^m N^{pq}F_{mpq}: \\
&\quad + \;:d_\alpha d_\beta K^{\alpha\beta}: \;+\; :d_\alpha\partial\theta^\beta F^\alpha{}_\beta: \;+\; :d_\alpha N^{mn}G^\alpha{}_{mn}: \;+\; :\partial\theta^\alpha\partial\theta^\beta H_{\alpha\beta}: \\
&\quad + \;:\partial\theta^\alpha N^{mn}H_{mn\alpha}: \;+\; :N^{mn}N^{pq}G_{mnpq}:
\end{aligned}
\tag{4.37}
$$

where, the superfields appearing in (4.37) are given in position space by

$$
\begin{aligned}
F_{mn} &= -\frac{18}{\alpha'}G_{mn} \quad , \quad F_m^{\ \alpha} = \frac{288}{\alpha'}(\gamma^r)^{\alpha\beta}\partial_r\Psi_{m\beta} \quad , \quad G_{m\alpha} = -\frac{432}{\alpha'}\Psi_{m\alpha} \\
F_{mpq} &= \frac{12}{(\alpha')^2}B_{mpq} - \frac{36}{\alpha'}\partial_{[p}G_{q]m} \quad , \quad K^{\alpha\beta} = -\frac{1}{(\alpha')^2}\,\gamma^{\alpha\beta}_{mnp}B^{mnp} \\
F^\alpha{}_\beta &= -\frac{4}{\alpha'}(\gamma^{mnpq})^\alpha{}_\beta\partial_m B_{npq} \quad , \quad G^\alpha_{mn} = \frac{48}{(\alpha')^2}\gamma^{\alpha\sigma}_{[m}\Psi_{n]\sigma} + \frac{192}{\alpha'}\gamma^{\alpha\sigma}_r\partial^r\partial_{[m}\Psi_{n]\sigma} \\
H_{\alpha\beta} &= \frac{2}{\alpha'}\gamma^{mnp}_{\alpha\beta}B_{mnp} \quad , \quad H_{mn\alpha} = -\frac{576}{\alpha'}\,\partial_{[m}\Psi_{n]\alpha} - \frac{144}{\alpha'}\partial^q(\gamma_{q[m})_\alpha{}^\sigma\Psi_{n]\sigma} \\
G_{mnpq} &= \frac{4}{(\alpha')^2}\partial_{[m}B_{n]pq} + \frac{4}{(\alpha')^2}\partial_{[p}B_{q]mn} - \frac{12}{\alpha'}\partial_{[p}\partial_{[m}G_{n]q]}
\end{aligned}
\tag{4.38}
$$

This completes the construction of the integrated vertex.

There are few comments in order before we move on. It is quite obvious that as one goes on to higher and higher massive states, the tensor structures of the superfields appearing in vertex will involve more and more indices. This can lead to a computational nightmare without a guiding principle like the one given in this chapter based on symmetry arguments. Furthermore, in outlining the general methodology and its successful implementation for first massive level (in both unintegrated and integrated vertex) nowhere did we crucially make use of the mass level in question. In fact for n-th massive states only changes are in the mass-shell condition $k^2 = -\frac{n}{\alpha'}$ and in the irreps which constitutes the higher spin supermultiplets. The general methodology and its successful use so far has never depended on these two details. This gives us enough confidence to propose that this general methodology is correct and we hope to rigorously establish that fact in the future.

4.4 Massless states

At this stage, it is but only natural to wonder whether a similar strategy can work for the massless case also. Sadly, that is not the case however. A crucial feature of our proposed differential relations hinge on the fact that $\not{k}^{\alpha\beta}\{D_\alpha, D_\beta\} \propto k^2$. For $k^2 \neq 0$, i.e. for massive states, this lead to the fact that our differential relations could be inverted. For massless case however, $k^2 = 0$, and this methodology breaks down. This is not worrying for two reasons. Firstly, there is only one massless supermultiplet (spin 1) and the vertex for those states have been explicitly constructed and its covariant θ expansion completely determined in [4] (also see [29] and see appendix D for the derivation in our convention). Since there is only 1 massless supermultiplet but an infinite tower of massive supermultiplets, it is of paramount importance that one develops a methodology for the latter. Secondly, even though the methodology outlined in this chapter will not work for massless case, one can nonetheless construct the vertex by an ansatz. For the massless states every superfield appearing in integrated vertex (the unintegrated vertex is rather trivial for massless case being uniquely fixed by conformal weight and ghost number requirements) can be expressed in terms of a single superfield A_α which satisfies $N = 1, d = 10$ super Yang-Mills equations as a consequence of $QV = 0$. The structure of the set of equations obtained from $QU - \partial_{\mathbb{R}} V = 0$ then is such that each equation determines only one unknown superfield in terms of a superfield determined at the previous equation (with the first equation determining an unknown superfield in terms of A_α). This leads to a systematic procedure for constructing the massless vertex if one so wishes. But as already discussed above, this is not at all necessary for constructing the massless vertex.

4.5 Closed Superstrings

Before concluding this chapter, let us briefly comment on how one can construct the vertex for closed superstrings once the open superstring answer is known. The idea is to use the fact that for closed superstrings, the left moving and the right moving sector is completely decoupled. Therefore each vertex operator can be cast into a tensor product of a left moving and a right moving sector. For heterotic strings, the right moving sector is just the bosonic strings, for which we can use the bosonic vertex operator already known in literature. For type II superstrings, one can use the open superstring vertex for each sector with different labels distinguishing the right handed fields from the left handed fields and take a tensor product. There are some minor changes expected due to change in mass level conditions (for closed superstring n-th mass level has $m^2 = \frac{4n}{\alpha'}$) and also from the fact that vertex operators no longer live only on the boundary but rather on any point of the Riemann surface. However, both these changes can be easily accounted for by suitable redefinitions of various conformal primaries defined on the worldsheet.

Chapter 5

Equivalence of Pure Spinor and RNS superstrings for first massive states

Now that we have the integrated and unintegrated vertex for massive states of open superstrings, as well as their full θ expansion, we have all the ingredients necessary to compute scattering of massive states. A crucial question in pure spinor formalism is whether it is completely equivalent to RNS formalism? Some progress was made in [9], [10] to answer this question (in affirmative) based on cohomology methods (also see [11] for relating pure spinor with Green-Schwarz formalism). For massless states, the explicit equivalence between pure spinor and RNS formalism was firmly established in [13] (also see [7], [12], [14]). Such explicit check for massive states however has been lacking so far. In this chapter we explicitly compute all massless-massless-massive 3-point correlation function for open superstrings in pure spinor formalism. We then directly compare the result thus obtained with the same computation done in RNS formalism and explicitly extend the equivalence between the two formalism from massless states to the first massive states as well. The result of this chapter first appeared in the paper [21].

5.1 Tree amplitude prescription in pure spinor formalism

Both minimal as well as the non minimal pure spinor formalisms give the same amplitude prescription at tree level as we already mentioned in chapter 1. In this section, we shall review this prescription [4,6]. In particular, we shall focus on 3-point functions on the disk with a specific ordering of vertex operators on the boundary. To evaluate any 3-point correlator, the knowledge of the unintegrated form of the vertex operator for external states is sufficient. All the amplitudes of interest in this chapter are of the form

$$\mathcal{A}_3 \;=\; \langle V_1 V_2 V_3 \rangle \tag{5.1}$$

where $\langle V_1 V_2 V_3 \rangle$ denotes 3-point function on the disk with a fixed ordering.

In this chapter, V_1 and V_2 will be taken to be unintegrated vertex operators creating massless states (gluon or gluino) and V_3 will be the unintegrated vertex operator creating a massive state (b_{mnp}, g_{mn} or $\psi_{m\alpha}$). However, we must emphasize, none of the strategy that we shall outline below is particular to this specific kind of 3-point functions. The tree-level amplitude prescription will be equally valid for any 3-point functions of open strings states (massive or massless).

Our choice of normalization of pure spinor measure is the standard one in the literature

$$\left\langle (\lambda\gamma^s\theta)(\lambda\gamma^t\theta)(\lambda\gamma^u\theta)(\theta\gamma_{stu}\theta) \right\rangle_{PSS} = 1 \tag{5.2}$$

The process of 3-point amplitude computation can be succinctly summarized in a series of steps that is given below.

- **S**tep 1 : Assume a particular order of the vertex operator inserted on the disk and make use of the OPEs between various conformal weight 1 objects to reduce the three point function to a correlation function involving only the three superfields and three pure spinor ghost λ^α coming from the three vertices. The correlation function which contains $\partial\theta^\alpha$ terms do not contribute since they always fail to have the correct number of θ zero modes to give non-zero answer as required by 5.2. Now perform the θ expansion for each of the vertex, viz. V_1, V_2 and V_3, explicitly. Although one can do the full θ expansion, but that is rendered redundant due to 5.2 which states that only terms involving exactly five θs can give a non-zero contribution. The relevant order of θ expansion for each of the vertex for a given amplitude can therefore be deduced from this consideration.

- **S**tep 2 : From the product $V_1 V_2 V_3$, retain only the terms which have precisely five θs as all other terms will give zero contribution trivially due to 5.2. [1]

- **S**tep 3 : Expand the physical fields appearing in various vertices in a basis of plane waves with

[1]Notice that 5.2 also suggests we should retain only terms with exactly three number of λs. But this is automatically ensured by the fact that by construction all unintegrated vertex in PS formalism have ghost number 1 and therefore they each carry exactly a single factor of λ. Consequentially the product $V_1 V_2 V_3$ always goes as λ^3.

polarizations as the coefficients

$$H^{\beta_1 \ldots \beta_k}_{p_1 \ldots p_n}(X) = h^{\beta_1 \ldots \beta_k}_{p_1 \ldots p_n}\, e^{ik \cdot X}$$

Here $h^{\beta_1 \ldots \beta_k}_{p_1 \ldots p_n}$ are the constant tensor-spinor of appropriate index structure denoting the polarizations for the physical field $H^{\beta_1 \ldots \beta_k}_{p_1 \ldots p_n}(X)$.

The correlation function $\langle V_1 V_2 V_3 \rangle$ at this stage factorizes completely into two separate correlation functions, one involving X^m fields and the other involving pure spinor fields. These can be evaluated independently and their result can be multiplied to obtain the final answer for a given ordering.

- **s**tep 4 : Evaluate the 3-point function on disk by the usual methodology of bosonic open strings. Typically these correlation functions can be put into the schematic form $\langle : e^{ik_1 \cdot X(x_1)} :: e^{ik_2 \cdot X(x_2)} :: \mathcal{F}(\partial X^m(x_3))\, e^{ik_3 \cdot X(x_3)} : \rangle_{Disk}$ where $\mathcal{F}(\partial X^m(x)) = \prod_i \partial X^{m_i}(x)$.

- **S**tep 5 : The correlators living on pure spinor superspace can be evaluated following the list of identities first derived in [12,30]. We list in appendix B.0.1 the subset of those identities that were needed in evaluating the 3-point functions in our case.

- **S**tep 6 : Multiply the results obtained from step 4 and step 5 to obtain the full contribution for a given order of the 3 vertices.

Once the answer has been obtained for a given order, the final answer for all other inequivalent orders can be readily obtained by suitable permutation of momenta and polarization labels. Since each ordering carries different Chan-Paton factors, the full amplitude cannot be obtained by simply adding the contribution coming from different ordering before multiplying them by Chan-Paton factors. Therefore, to compare our result with the result obtained in RNS formalism, we shall directly compare each inequivalent ordering separately.

While the algorithm described above is quite straightforward in principle, the growing number of terms in θ expansion of the vertex operators implies that the cleanest way to perform these amplitude calculations beyond a point is to employ the help of an available computer algebra system. We used Cadabra [31,32] which is an open source computer algebra system developed to aid in field-theoretic computations. For our present purpose, we found Cadabra to be very useful in implementing the algorithm described above (Cadabra was also used for amplitude computation involving massless states in [33]).

5.2 Three point functions using pure spinor formalism

In this section, we calculate the 3-point functions involving two massless and one massive state using the pure spinor formalism. In subsection 5.2.1, we simplify the pure spinor correlators and set up the problem at the superfield level and in subsection 5.2.2, we evaluate the resulting correlators.

5.2.1 Simplifying 3-point correlator

Since we are only interested in the 3-point functions on the disk, the location of all the vertex operators can be fixed and hence we only need to consider the unintegrated vertex operators. Thus, the correlator we need to evaluate is given by

$$\mathcal{A}_3 = \left\langle \ V_A^{(1)}(x_1) V_A^{(2)}(x_2) V_{b,g,\psi}^{(3)}(x_3) \ \right\rangle \tag{5.3}$$

where, $V_{b,g,\psi}$ denotes the massive vertex operator (2.5) and V_A denotes the unintegrated vertex operator of the massless states given by[2]

$$V_A = \lambda^\alpha A_\alpha \tag{5.4}$$

The unintegrated vertex operators in (5.3) are fixed at some arbitrary locations x_i. The $SL(2,\mathbb{R})$ invariance on the disc guarantees that the 3-point function is independent of the choices of x_i. Using the expressions of the unintegrated vertex operators, the desired 3-point function is given by

$$\mathcal{A}_3 \quad = \quad \Big\langle \lambda^\alpha A_\alpha^{(1)}(x_1) \lambda^\beta A_\beta^{(2)}(x_2) \Big(\partial\theta^\rho \lambda^\sigma B_{\sigma\rho} + d_\rho \lambda^\sigma C^\rho_{\ \sigma} + \Pi^m \lambda^\sigma H_{m\sigma} + N^{mn} \lambda^\sigma F_{\sigma mn} \Big)(x_3) \Big\rangle \tag{5.5}$$

We have suppressed the superscript 3 from the massive superfields since there is only one massive state and hence there is no chance of any confusion. We now manipulate each term of (5.5) one by one.

[2] The θ expansion of the massless superfields in our conventions is given in appendix D.

First Term

The first term is given by

$$\begin{aligned} T_1 &= \langle \lambda^\alpha A^{(1)}_\alpha(x_1)\lambda^\beta A^{(2)}_\beta(x_2)\partial\theta^\rho\lambda^\sigma B_{\sigma\rho}(x_3)\rangle \\ &= (\gamma^{mnp})_{\sigma\rho}\Big\langle \lambda^\alpha \tilde{A}^{(1)}_\alpha \lambda^\beta \tilde{A}^{(2)}_\beta \partial\theta^\rho\lambda^\sigma \tilde{B}_{mnp}\Big\rangle\Big\langle e^{ik_1\cdot X(x_1)}e^{ik_2\cdot X(x_2)}e^{ik_3\cdot X(x_3)}\Big\rangle \\ &= (\gamma^{mnp})_{\sigma\rho}\Big\langle \lambda^\alpha \tilde{A}^{(1)}_\alpha \lambda^\beta \tilde{A}^{(2)}_\beta \partial\theta^\rho\lambda^\sigma \tilde{B}_{mnp}\Big\rangle \frac{x_{13}x_{23}}{x_{12}}. \end{aligned} \tag{5.6}$$

where, the superfields with tilde denote the same superfields with $e^{ik\cdot X}$ factor stripped off. Thus, e.g., in theθ expansion of $\tilde{B}_{mnp}$, we just write the polarization tensor $e_{m\alpha}, e_{mnp}$ and e_{mn} instead of $\psi_{m\alpha}, b_{mnp}$ and g_{mn} respectively. We have also used the momentum conservation to write

$$\Big\langle e^{ik_1\cdot X(x_1)}e^{ik_2\cdot X(x_2)}e^{ik_3\cdot X(x_3)}\Big\rangle = |x_{12}|^{2\alpha' k_1\cdot k_2}|x_{23}|^{2\alpha' k_2\cdot k_3}|x_{13}|^{2\alpha' k_1\cdot k_3} = \frac{x_{13}x_{23}}{x_{12}}. \tag{5.7}$$

The T_1 does not contribute to any of the amplitude since it does not provide the 5 zero modes of θ^α which is required for the non vanishing of pure spinor correlators.

Second Term

The 2nd term is given by

$$T_2' = \langle \lambda^\alpha A^{(1)}_\alpha(x_1)\lambda^\beta A^{(2)}_\beta(x_2) d_\rho \lambda^\sigma C^\rho_{\ \sigma}(x_3)\rangle$$

To simplify this term, we use the OPE of d_α with superfields to obtain

$$\begin{aligned} T_2' &= \langle \lambda^\alpha A^{(1)}_\alpha(x_1)\lambda^\beta A^{(2)}_\beta(x_2) d_\rho \lambda^\sigma C^\rho_{\ \sigma}(x_3)\rangle \\ &= \oint_{x_3}\frac{dw}{w-x_3}\ \langle \lambda^\alpha(x_1) A^{(1)}_\alpha(x_1)\lambda^\beta(x_2) A^{(2)}_\beta(x_2) d_\rho(w) \lambda^\sigma(x_3) C^\rho_{\ \sigma}(x_3)\rangle \\ &= -\oint_{x_1}\frac{dw}{w-x_3}\ \Big\langle \lambda^\alpha(x_1)\Big[\frac{\alpha'}{2}\frac{D_\rho A^{(1)}_\alpha(x_1)}{w-x_1}\Big]\lambda^\beta(x_2)A^{(2)}_\beta(x_2)\lambda^\sigma(x_3)C^\rho_{\ \sigma}(x_3)\Big\rangle \\ &\quad +\oint_{x_2}\frac{dw}{w-x_3}\ \Big\langle \lambda^\alpha(x_1)A^{(1)}_\alpha(x_1)\lambda^\beta(x_2)\Big[\frac{\alpha'}{2}\frac{D_\rho A^{(2)}_\beta(x_2)}{w-x_2}\Big]\lambda^\sigma(z_3)C^\rho_{\ \sigma}(x_3)\Big\rangle \end{aligned} \tag{5.8}$$

In going to the last line, we have unwrapped the contour to enclose the points x_1 and x_2. This gives a sign. A further sign comes while moving d_α across A_α. The signs in front of the individual terms in the last line are net effect of these sign factors. We now use equations (3.4),(2.27) and (D.1), the identity

$QV = \frac{\alpha'}{2}\lambda^\alpha D_\alpha V$ for an arbitrary superfield V and unwrap the contour of Q. After using the on-shell condition and the pure spinor fierz identity

$$\lambda^\alpha\lambda^\beta = \frac{1}{5!\times 32}\gamma^{\alpha\beta}_{mnpqr}(\lambda\gamma^{mnpqr}\lambda) \tag{5.9}$$

we obtain (after some simplification using the gamma matrix identities)

$$T_2' = \tilde{T}_2 + T_2 \tag{5.10}$$

where,

$$\begin{aligned} \tilde{T}_2 \equiv\ & -\frac{3i\alpha'}{64}\frac{x_{23}}{x_{12}}\Big[(\gamma_{mstuv})^{\alpha\sigma}(k_3)_m + 4(\gamma_{tuv})^{\alpha\sigma}(k_3)_s\Big]\Big\langle \tilde{A}^{(1)}_\alpha\lambda^\beta\tilde{A}^{(2)}_\beta(\lambda\gamma^{stuvw}\lambda)\tilde{\Psi}_{w\sigma}\Big\rangle \\ & -\frac{3i\alpha'}{64}\frac{x_{13}}{x_{12}}\Big[(\gamma_{mstuv})^{\alpha\sigma}(k_3)_m + 4(\gamma_{tuv})^{\alpha\sigma}(k_3)_s\Big]\Big\langle \lambda^\beta\tilde{A}^{(1)}_\beta\tilde{A}^{(2)}_\alpha(\lambda\gamma^{stuvw}\lambda)\tilde{\Psi}_{w\sigma}\Big\rangle \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} T_2 \equiv\ & -\frac{i\alpha'}{2}\frac{x_{23}}{x_{12}}(\gamma_m\gamma^{stuv})_{\alpha\sigma}(k_3)_s\ \Big\langle\lambda^\alpha\tilde{A}^{(1)}_m\lambda^\beta\tilde{A}^{(2)}_\beta\lambda^\sigma\tilde{B}_{tuv}\Big\rangle \\ & +\frac{i\alpha'}{2}\frac{x_{13}}{x_{12}}(\gamma_m\gamma^{stuv})_{\alpha\sigma}(k_3)_s\ \Big\langle\lambda^\beta\tilde{A}^{(1)}_\beta\lambda^\alpha\tilde{A}^{(2)}_m\lambda^\sigma\tilde{B}_{tuv}\Big\rangle \end{aligned} \tag{5.12}$$

Third Term

The 3rd term is given by

$$\begin{aligned} T_3 &= \langle\lambda^\alpha A^{(1)}_\alpha(x_1)\lambda^\beta A^{(2)}_\beta(x_2)\Pi^m\lambda^\sigma H_{m\sigma}(x_3)\rangle \\ &= \oint_{x_3}\frac{dw}{w-x_3}\langle\lambda^\alpha A^{(1)}_\alpha(x_1)\lambda^\beta A^{(2)}_\beta(x_2)\Pi^m(w)\lambda^\sigma(x_3)H_{m\sigma}(x_3)\rangle \\ &= -\oint_{x_1}\frac{dw}{w-x_3}\Big\langle\lambda^\alpha(x_1)\Big[-i\alpha'(k_1)^m\frac{A^{(1)}_\alpha(x_1)}{w-x_1}\Big]\lambda^\beta(x_2)A^{(2)}_\beta(x_2)\lambda^\sigma(x_3)H_{m\sigma}(x_3)\Big\rangle \\ &\quad -\oint_{x_2}\frac{dw}{w-x_3}\Big\langle\lambda^\alpha(x_1)A^{(1)}_\alpha(x_1)\lambda^\beta(z_2)\Big[-i\alpha'(k_2)^m\frac{A^{(2)}_\beta(x_2)}{w-x_2}\Big]\lambda^\sigma(x_3)H_{m\sigma}(x_3)\Big\rangle \\ &= \frac{72i\alpha'}{5!\times 32}(\gamma_{stuvw})^{\alpha\sigma}(k_1)^m\Big\langle\tilde{A}^{(1)}_\alpha\lambda^\beta\tilde{A}^{(2)}_\beta(\lambda\gamma^{stuvw}\lambda)\tilde{\Psi}_{m\sigma}\Big\rangle \end{aligned} \tag{5.13}$$

Again, in going to the 3rd equality, we have unwrapped the contour and used the OPE between Π^m and superfields. In going to the last line, we have performed the contour integration and used the momentum conservation and the identity (5.9).

Fourth Term

Finally, the 4th term is given by

$$\begin{aligned}
T_4 &= \langle \lambda^\alpha A^{(1)}_\alpha(x_1)\lambda^\beta A^{(2)}_\beta(x_2) N^{mn}\lambda^\sigma F_{\sigma mn}(x_3)\rangle \\
&= \oint_{x_3} \frac{dw}{w-z_3}\langle \lambda^\alpha A^{(1)}_\alpha(x_1)\lambda^\beta A^{(2)}_\beta(x_2) N^{mn}(w)\lambda^\sigma(x_3) F_{\sigma mn}(x_3)\rangle \\
&= -\oint_{x_1} \frac{dw}{w-x_3}\Big\langle \Big[\frac{\alpha'}{4}\frac{(\gamma^{mn})^\alpha{}_\rho\lambda^\rho(x_1)}{w-x_1}\Big] A^{(1)}_\alpha(x_1)\lambda^\beta(x_2) A^{(2)}_\beta(x_2)\lambda^\sigma(x_3) F_{\sigma mn}(x_3)\Big\rangle \\
&\quad -\oint_{x_2} \frac{dw}{w-x_3}\Big\langle \lambda^\alpha(x_1) A^{(1)}_\alpha(x_1)\Big[\frac{\alpha'}{4}\frac{(\gamma^{mn})^\beta{}_\rho\lambda^\rho(x_2)}{w-x_2}\Big] A^{(2)}_\beta(x_2)\lambda^\sigma(x_3) F_{\sigma mn}(x_3)\Big\rangle \\
&= \frac{3i\alpha'}{64}\frac{x_{23}}{x_{12}}\Big[(\gamma_{mstuv})^{\alpha\sigma}(k_3)_m + 4(\gamma_{tuv})^{\alpha\sigma}(k_3)_s\Big]\Big\langle \tilde{A}^{(1)}_\alpha\lambda^\beta\tilde{A}^{(2)}_\beta(\lambda\gamma^{stuvw}\lambda)\tilde{\Psi}_{w\sigma}\Big\rangle \\
&\quad +\frac{3i\alpha'}{64}\frac{x_{13}}{x_{12}}\Big[(\gamma_{mstuv})^{\alpha\sigma}(k_3)_m + 4(\gamma_{tuv})^{\alpha\sigma}(k_3)_s\Big]\Big\langle \lambda^\beta\tilde{A}^{(1)}_\beta\tilde{A}^{(2)}_\alpha(\lambda\gamma^{stuvw}\lambda)\tilde{\Psi}_{w\sigma}\Big\rangle
\end{aligned} \tag{5.14}$$

In going to the 3rd equality, we have unwrapped the contour and used the OPE of N^{mn} with λ^α whereas in going to the 4th equality, we have used the expression of $F_{\alpha mn}$ in terms of $\Psi_{m\alpha}$ as given in (2.27). We note that T_4 is exactly minus of the $\tilde{T}_2$ as given in (5.11).

Combining all the terms, we find that the total 3-point function is given by

$$\mathcal{A}_3 \;=\; \langle V_1V_2V_3\rangle \;=\; T_2 + T_3 \tag{5.15}$$

where T_2 and T_3 are given in equations (5.12) and (5.13) respectively. Below, we shall give the results for the correlators $\langle V_1V_2V_3\rangle$ for different choices of the 2 massless and one massive external states.

5.2.2 Evaluation of correlators

The two terms T_2 and T_3 given in (5.15) are at the superfield level. To compute some specific 3-point function, we need to keep only the fields of interest to be non zero in the superfields. After specializing to some specific amplitude, $\mathcal{A}_3$ can be evaluated using the θ expansion results given in section 3.3 and appendix D and the pure spinor correlators listed in appendix B.0.1. We use the symbolic computer programme Cadabra to do this calculation [31,32]. We now give the results for different 3-point correlators.

2 gluon and 1 b_{mnp} field [3]

For this case, we have

$$T_2 = \frac{29i}{840} e^{mnp} e^1_m e^2_n (k_1)_p \quad , \qquad T_3 = \frac{13i}{840} e^{mnp} e^1_m e^2_n (k^1)_p$$

This gives

$$\langle aab \rangle = T_2 + T_3 = \frac{i}{20} e^{mnp} e^1_m e^2_n (k_1)_p \tag{5.16}$$

2 gluon and 1 g_{mn} field

For the $\langle aag \rangle$ amplitude, we have

$$T_2 = -\frac{1}{80}(e^1 \cdot g \cdot e^2) + \frac{\alpha'}{160}(e^2 \cdot k^1)(e^1 \cdot g \cdot k^1) - \frac{\alpha'}{160}(e^1 \cdot k^2)(e^2 \cdot g \cdot k^1)$$

$$T_3 = -\frac{3\alpha'}{160}(e^1 \cdot k^2)(e^2 \cdot g \cdot k^1) - \frac{\alpha'}{40}(e^1 \cdot e^2)(k^1 \cdot g \cdot k^1) + \frac{3\alpha'}{160}(e^2 \cdot k^1)(e^1 \cdot g \cdot k^1)$$

This gives,

$$\begin{aligned}\langle aag \rangle &= T_2 + T_3 \\ &= -\frac{1}{80}\left[2\alpha'(e^1 \cdot k^2)(e^2 \cdot g \cdot k^1) + 2\alpha'(e^2 \cdot k^1)(e^1 \cdot g \cdot k^2) - 2\alpha'(e^1 \cdot e^2)(k^1 \cdot g \cdot k^2) + (e^1 \cdot g \cdot e^2)\right]\end{aligned} \tag{5.17}$$

2 gluino and 1 b_{mnp} field

For the $\langle \chi\chi b \rangle$ amplitude, we have

$$T_2 = \frac{23}{11340}(\xi^1 \gamma^{mnp} \xi^2) e_{mnp} \quad , \qquad T_3 = \frac{1}{18144}(\xi^1 \gamma^{mnp} \xi^2) e_{mnp}$$

[3] This amplitude was also considered in [27]. As already comented, the θ expansion results of that paper are in conflict with the rest frame analysis done [18]. Furthermore, this amplitude was determined only upto kinematic factors without the overall normalization. This particular amplitude, due to polarization condition can be easily seen to have a unique kinematic factor, therefore it is the overall normalization which is of the crucial importance and which was not calculated in [27].

This gives

$$\langle \chi\chi b\rangle \;=\; T_2 + T_3 \;=\; \frac{1}{480}(\xi^1\gamma^{mnp}\xi^2)e_{mnp} \tag{5.18}$$

2 gluino and 1 g_{mn} field

For the $\langle \chi\chi g\rangle$ amplitude, we have

$$T_2 = 0 \quad , \qquad T_3 = \frac{i\alpha'}{80}(\xi^1\gamma^m\xi^2)g_{mn}k_1^n$$

This gives,

$$\langle \chi\chi g\rangle \;=\; T_2 + T_3 \;=\; \frac{i\alpha'}{80}(\xi^1\gamma^m\xi^2)e_{mn}k_1^n \tag{5.19}$$

1 gluon, 1 gluino and 1 $\psi_{m\alpha}$ field

In this case, since all the external states are different, the two orderings $\langle a\chi\psi\rangle$ and $\langle \chi a\psi\rangle$ are different. We give the result for both cases. For the $\langle a\chi\psi\rangle$ correlator, we have

$$T_2 = \frac{4}{21}e_1^m(\xi^2\psi_m) - \frac{31}{210}\alpha'(\xi^2\psi_n)(e^1\cdot k^2)k_2^n - \frac{9}{140}\alpha'(\xi^2\gamma_{mn}\psi_p)e_1^m k_1^n k_2^p$$

$$T_3 = \frac{1}{105}e_1^m(\xi^2\psi_m) - \frac{53}{210}\alpha'(\xi^2\psi_n)(e^1\cdot k^2)k_2^n - \frac{19}{140}\alpha'(\xi^2\gamma_{mn}\psi_p)e_1^m k_1^n k_2^p$$

This gives

$$\begin{aligned}\langle a\chi\psi\rangle &= T_2 + T_3 \\ &= \frac{1}{5}\Big[e_1^m(\xi^2\psi_m) - 2\alpha'(\xi^2\psi_n)(e^1\cdot k^2)k_2^n - \alpha'(\xi^2\gamma_{mn}\psi_p)e_1^m k_1^n k_2^p\Big]\end{aligned} \tag{5.20}$$

On the other hand, for the $\langle \chi a\psi\rangle$ correlator, we have

$$T_2 = \frac{4}{21}e_2^m(\xi^1\psi_m) - \frac{31}{210}\alpha'(\xi^1\psi_n)(e^2\cdot k^1)k_1^n - \frac{9}{140}\alpha'(\xi^1\gamma_{mn}\psi_p)e_2^m k_2^n k_1^p$$

$$T_3 = \frac{1}{105}e_2^m(\xi^1\psi_m) - \frac{53}{210}\alpha'(\xi^1\psi_n)(e^2\cdot k^1)k_1^n - \frac{19}{140}\alpha'(\xi^1\gamma_{mn}\psi_p)e_2^m k_2^n k_1^p$$

This gives

$$\begin{aligned} \langle \chi a \psi \rangle &= T_2 + T_3 \\ &= \frac{1}{5}\Big[e_2^m (\xi^1 \psi_m) - 2\alpha' (\xi^1 \psi_n)(e^2 \cdot k^1) k_1^n - \alpha' (\xi^1 \gamma_{mn} \psi_p) e_2^m k_2^n k_1^p \Big] \end{aligned} \quad (5.21)$$

3 massless fields

For comparing the pure spinor results with the corresponding RNS results, we also need the massless amplitudes. For $\langle aaa \rangle$ correlator, the pure spinor calculation gives the following result in our conventions

$$\langle aaa \rangle = \frac{i}{180}\left[(e^1 \cdot e^2)(e^3 \cdot k^1) + (e^1 \cdot e^3)(e^2 \cdot k^3) + (e^2 \cdot e^3)(e^1 \cdot k^2) \right] \quad (5.22)$$

On the other hand, for the $\langle a \chi \chi \rangle$ correlator, we get

$$\langle a \chi \chi \rangle = \frac{1}{360} (\chi^2 \gamma^m \chi^3) e_m^1 \, . \quad (5.23)$$

From the explicit expression of all massless-massless-massive amplitudes obtained in this section, we observe that all such 3-point functions are symmetric under a change of cyclic order. Therefore the inclusion of Chan-Paton factors (denoted by t^a, t^b t^c) will make the full amplitude proportional to $Tr(t^a, \{t^b, t^c\})$. Compare this with the 3-point functions involving all massless states, which are antisymmetric under a change of cyclic order and therefore after taking into account the Chan-Paton factors, the final answer becomes proportional to $Tr(t^a, [t^b, t^c])$. Another point to note is that all the amplitudes considered in this section are invariant under the gauge transformations $e^i \to e^i + k^i$. We now turn to comparing the RNS and PS results.

5.3 Comparing pure spinor and the RNS results

By comparing the pure spinor results given above with the corresponding RNS results given in appendix (C.0.2), we see that the tensor structures of the 3-point functions match perfectly. Moreover the relative coefficients of the various terms in the correlators $\langle aag \rangle$ and $\langle a \chi \psi \rangle$ which have more than one terms, also match exactly. This is a non trivial test. We shall now show that the overall numerical factors (i.e.

Correlator	**RNS**	**PS**
$\langle aaa\rangle$	$-g_a^3\sqrt{2\alpha'}$	$\frac{i}{180}$
$\langle a\chi\chi\rangle$	$\frac{1}{\sqrt{2}}g_a g_\chi^2$	$\frac{1}{360}$
$\langle aab\rangle$	$6g_a^2 g_b\sqrt{2\alpha'}$	$\frac{i}{20}$
$\langle \chi\chi b\rangle$	$-\frac{1}{2\sqrt{2}}g_\chi^2 g_b$	$\frac{1}{480}$
$\langle aag\rangle$	$-g_a^2 g_g$	$-\frac{1}{80}$
$\langle \chi\chi g\rangle$	$\sqrt{\alpha'}g_\chi^2 g_g$	$\frac{i\alpha'}{80}$
$\langle a\chi\psi\rangle$	$-\frac{16\alpha'}{\sqrt{2}}g_a g_\chi g_\psi$	$\frac{1}{5}$
$\langle \chi a\psi\rangle$	$-\frac{16\alpha'}{\sqrt{2}}g_a g_\chi g_\psi$	$\frac{1}{5}$

normalizations) of the different 3-point functions in pure spinor and RNS are also in perfect agreement with each other.

We have denoted the overall normalization of the vertex operators in the RNS calculations relative to those in PS calculations by g_a, g_χ etc. (see appendix C.0.2). For example, if $\mathcal{N}_{RNS}$ and $\mathcal{N}_{PS}$ denote the normalizations of the gluon vertex operator in the RNS and PS respectively, then for the $\langle V_a V_a V_a\rangle$ correlator, we have (denoting $V \equiv \mathcal{N}\tilde{V}$)

$$(\mathcal{N}_{PS})^3\langle \tilde{V}_a\tilde{V}_a\tilde{V}_a\rangle_{PS} = (\mathcal{N}_{RNS})^3\langle \tilde{V}_a\tilde{V}_a\tilde{V}_a\rangle_{RNS} \tag{5.24}$$

From this, it is clear that only the relative normalization between RNS and PS vertex operators have any physical significance. To exploit this fact, we define

$$\mathcal{N}_{RNS} = g_a\,\mathcal{N}_{PS} \tag{5.25}$$

In our calculations, we have set the overall normalization of the PS vertex operators to be 1 and kept the relative normalization factor g_a, g_χ etc. to be in the RNS vertex operators.

With the above convention, the overall RNS and the pure spinor numerical factors for each correlator is given in the table above. By comparing the RNS and pure spinor numerical factors for $\langle aaa\rangle$ and $\langle a\chi\chi\rangle$, we find

$$(g_a)^3 = \frac{-i}{180\sqrt{2\alpha'}} \quad , \qquad (g_\chi)^2 = \frac{\sqrt{2}}{360\ g_a}. \tag{5.26}$$

In terms of g_a and g_χ, the numerical factors for $\langle aab\rangle$, $\langle aag\rangle$ and $\langle a\chi\psi\rangle$ give

$$g_b = \frac{i}{120\sqrt{2\alpha'}\ g_a^2} \quad , \qquad g_g = \frac{1}{80\ g_a^2} \quad , \qquad g_\psi = -\frac{\sqrt{2}}{80\alpha' g_a g_\chi} \tag{5.27}$$

The above values of g_b, g_g and g_ψ agree perfectly with the value obtained using $\langle\chi\chi b\rangle$, $\langle\chi\chi g\rangle$ and $\langle\chi a\psi\rangle$ correlators. This is a non trivial consistency check.

This explicitly demonstrates the equivalence of pure spinor and RNS formalism for first massive states of open superstrings.

Chapter 6

Summary of the past and Dreams for the future

6.1 Lessons learnt

Pure spinor formalism, since its inception at the beginning of this century, has experienced plenty of success for massless states. Since the massless states typically enjoy a large amount of protection from supersymmetry, this can somehow mislead a reader into thinking that pure spinor formalism works best only for massless states. The central theme of this thesis has been to persuade its reader of the usefulness of pure spinor formalism even for massive states. This was done by first developing a systematic way of addressing the question of vertex operator construction in pure spinor formalism. This methodology was explicitly applied to re-derive the unintegrated vertex for first massive states and further lead to construction of the corresponding integrated vertex for the first time. This methodology also gave us a way of performing the fully covariant θ expansion of both vertices solely in terms of physical fields for the first time. With both these advances one finally had all the ingredients necessary to compute scattering of massive states in pure spinor formulation.

In particular, explicit computation of all massless-massless-massive amplitudes in pure spinor allowed us to conclusively extend the equivalence of PS and RNS formalism for open superstrings to include the first massive states as well. This result is of great significance since these massive states are not protected by any supersymmetry and therefore should dispel any remaining doubt regarding validity of pure spinor formalism for massive states. The fact that none of our methodology was crucially hinged on the mass level in question also strongly hints at the fact that both RNS and PS formalism are equivalent and a proof may not be in that distant future.

6.2 Looking forward

There are many new questions to consider based on the results of this thesis. Here we summarize a few of them.

Firstly, we would like to subject our general methodology to one more explicit test by constructing the unintegrated vertex operator for 2nd massive level. This should be reasonably straightforward in practice, albeit computationally a bit more challenging. The success of this construction should provide us with a "physicist's proof" of our general methodology and its validity for all higher massive states.

We would also like to explicitly test our integrated vertex operator by using it to evaluate some amplitude which can be compared then with the RNS answer. For this purpose, we can choose to compute 1-loop mass renormalization for first massive states of $SO(32)$ heterotic string theory ([34]). The RNS result was obtained in [35], [36]. The reason behind choosing this particular problem is two-fold. One, this amplitude will also allow us to test the construction for closed superstrings. Two, we will like to use the insight gained for on-shell vertex to devise an off-shell amplitude prescription for pure spinor formalism (A closed string field theory for pure spinor formalism in some sense). This will allow us to compute the 2-loop mass renormalization which seems extremely challenging to do by RNS formalism.

It will also be very interesting to see our methodology adapted to construct vertex operators in $AdS_5 \times S^5$ background. The massless vertex operators are already known in terms of superfields (unintegrated in [37], integrated in [38]). We would like to extend the construction for massive states as well. This can have potentially important implications for detailed study of the conjectured AdS/CFT duality in [39].

The integrated vertex can also be obtained by making use of the b-ghost. From a pragmatic viewpoint this is discouraged since the b-ghost in non-minimal formalism is highly singular (at $\lambda^\alpha = 0$) and the resulting vertex is essentially impotent in computing scattering amplitude. However, there has always been a suggestion that it is possible to find a more regulated b-ghost (see [8], also [40]). While this regulated b-ghost is not essential by any means to construct integrated vertex, it is absolutely indispensable to propose a more regulated and well behaved loop amplitude prescription (especially for higher loops, in principle for all loop order). A direct comparison of massive vertex constructed using b-ghost and the way outlined here must differ only by a BRST exact term $\sim Q\Omega$. A comparison of this two method of obtaining the vertex may potentially suggest a less singular form of the composite b-ghost operator in pure spinor formalism.

It is often quoted, usually attributed to a generic wise person in human history, that every answered question raises a hundred more. However, in light of finitude of human capacity, especially that of the author of this thesis, we can assert that while it is good to look ahead into the unknown horizons once in a while, it seldom serves one well to keep staring too far into the future. The line must be drawn somewhere, so that a finished work can be celebrated and more importantly new unborn works can be brought into the light.

Appendix A

Summary of conventions

In this appendix, we give a summary of the notations and conventions we have used in this thesis.

- Our (anti)symmetrization convention is as follows

$$\text{Anti-symmetrization} \quad : \qquad T^{[m_1 \ldots m_n]} \equiv \frac{1}{n!}(T^{m_1 \ldots m_n} \pm \text{all permutations }) \tag{A.1}$$

$$\text{Symmetrization} \quad : \qquad T^{(m_1 \ldots m_n)} \equiv \frac{1}{n!}(T^{m_1 \ldots m_n} + \text{all permutations}) \tag{A.2}$$

- All antisymmetric products of gamma matrices are defined as

$$\gamma^{m_1 \ldots m_p} \equiv \gamma^{[m_1 \cdots} \gamma^{m_p]} \tag{A.3}$$

  Anti-symmetrized product of p gamma matrices is sometimes referred to as p-form.

- Our convention for super-covariant derivative is

$$D_\alpha = \partial_\alpha + (\gamma^m)_{\alpha\beta}\theta^\beta \partial_m \, ; \text{ where } \quad \partial_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} \tag{A.4}$$

  Therefore, the Clifford identity of gamma matrices implies

$$\{D_\alpha, D_\beta\} = 2(\gamma^m)_{\alpha\beta}\partial_m \qquad \Longrightarrow \qquad (\gamma_m)^{\alpha\beta} D_\alpha D_\beta = 16\, \partial_m \tag{A.5}$$

In momentum space, this implies for the first massive state

$$k^m(\gamma_m)^{\alpha\beta}D_\alpha D_\beta = 16\,i\,k^m k_m = -\frac{16\,i}{\alpha'} \tag{A.6}$$

- All normal ordering of products of operators are considered to be generalized normal ordering defined as follows-

$$:AB:(z) \equiv \frac{1}{2\pi i}\oint_z \frac{dw}{w-z}A(w)B(z)\ , \quad \text{For any two operators } A \text{ and } B. \tag{A.7}$$

A.1 Useful identities involving gamma matrices in d=10

In this appendix, we write down the list of gamma matrix identities that were used in our calculations. A useful reference for the more exhaustive list is [41]. Most of the manipulations involving the gamma matrices were done with the help of the Mathematica package Gamma [28].

We work solely with 16×16 gamma matrices in $d=10$. These are the off-diagonal elements of the 32×32 gamma matrices Γ^m matrices satisfying

$$\{\Gamma^m,\Gamma^n\} = 2\eta^{mn}\mathbb{I}_{32\times32}$$

More specifically,

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)_{\alpha\beta} \\ (\gamma^m)^{\alpha\beta} & 0 \end{pmatrix}$$

- **Spinor index structure of various gamma matrices**

  Following is the spinor index structure for various antisymmetric products of gamma matrices

$$(\gamma^{m_1\dots m_n})^\alpha{}_\beta \qquad \text{or} \qquad (\gamma^{m_1\dots m_n})_\beta{}^\alpha \qquad \text{for } n=0,2,4,6,8,10$$

$$(\gamma^{m_1\dots m_n})_{\alpha\beta} \qquad \text{or} \qquad (\gamma^{m_1\dots m_n})^{\alpha\beta} \qquad \text{for } n=1,3,5,7,9$$

- **Hodge duals**

For 10 dimensional 16×16 gamma matrices, the hodge duality is more than mere duality. It turns out to be an equality. We summarize them below

$$(\gamma^{m_1 \ldots m_{2n}})^{\alpha}{}_{\beta} = \frac{1}{(10-2n)!}(-1)^{(n+1)}\epsilon^{m_1 \ldots m_{2n} p_1 \ldots p_{10-2n}}(\gamma_{p_1 \ldots p_{10-2n}})^{\alpha}{}_{\beta} \tag{A.8}$$

$$(\gamma^{m_1 \ldots m_{2n}})_{\alpha}{}^{\beta} = -\frac{1}{(10-2n)!}(-1)^{(n+1)}\epsilon^{m_1 \ldots m_{2n} p_1 \ldots p_{10-2n}}(\gamma_{p_1 \ldots p_{10-2n}})_{\alpha}{}^{\beta} \tag{A.9}$$

$$(\gamma^{m_1 \ldots m_{2n+1}})^{\alpha\beta} = \frac{1}{(9-2n)!}(-1)^{n}\epsilon^{m_1 \ldots m_{2n+1} p_1 \ldots p_{9-2n}}(\gamma_{p_1 \ldots p_{9-2n}})^{\alpha\beta} \tag{A.10}$$

$$(\gamma^{m_1 \ldots m_{2n+1}})_{\alpha\beta} = -\frac{1}{(9-2n)!}(-1)^{n}\epsilon^{m_1 \ldots m_{2n+1} p_1 \ldots p_{9-2n}}(\gamma_{p_1 \ldots p_{9-2n}})_{\alpha\beta} \tag{A.11}$$

where, $\epsilon^{m_1 \cdots m_9}$ is the 10 dimensional epsilon tensor defined as

$$\epsilon_{0\,1\,\cdots\,9} = 1 \qquad \Longrightarrow \qquad \epsilon^{0\,1\,\cdots\,9} = -1 \tag{A.12}$$

Due to the above dualities, we only take γ^{m_1}, $\gamma^{m_1 m_2}$, $\gamma^{m_1 m_2 m_3}$, $\gamma^{m_1 m_2 m_3 m_4}$ and $\gamma^{m_1 m_2 m_3 m_4 m_5}$ along with the identity matrix $\mathbb{I}_{16\times 16}$ as the linearly independent basis elements for vector spaces of 16×16 complex matrices.

- **Symmetry property of gamma matrices under exchange of Spinor indices**

$$(\gamma^{m})_{\alpha\beta} = (\gamma^{m})_{\beta\alpha} \qquad : \text{Symmetric} \tag{A.13}$$

$$(\gamma^{m_1 m_2})^{\alpha}{}_{\beta} = -(\gamma^{m_1 m_2})_{\beta}{}^{\alpha} \qquad : \text{Anti-Symmetric} \tag{A.14}$$

$$(\gamma^{m_1 m_2 m_3})_{\alpha\beta} = -(\gamma^{m_1 m_2 m_3})_{\beta\alpha} \qquad : \text{Anti-Symmetric} \tag{A.15}$$

$$(\gamma^{m_1m_2m_3m_4})^{\alpha}{}_{\beta} \;=\; (\gamma^{m_1m_2m_3m_4})_{\beta}{}^{\alpha} \qquad : \text{Symmetric} \tag{A.16}$$

$$(\gamma^{m_1m_2m_3m_4m_5})_{\alpha\beta} \;=\; (\gamma^{m_1m_2m_3m_4m_5})_{\beta\alpha} \qquad : \text{Symmetric} \tag{A.17}$$

For 1, 3 and 5 forms, the same (anti) symmetry properties hold when the spinor indices are upstairs.

- **Various Gamma Traces**

$$(\gamma^{m_1\ldots m_n})^{\alpha}{}_{\alpha} \;=\; 0 \quad \text{for} \quad n = 2, 4, 6, 8 \tag{A.18}$$

$$(\gamma^{m_1\ldots m_{10}})^{\alpha}{}_{\alpha} \;=\; -16\,\epsilon^{m_1\ldots m_{10}} \tag{A.19}$$

$$(\gamma^{m})_{\alpha\beta}(\gamma_{n})^{\beta\alpha} \;=\; 16\,\delta^{m}_{n} \tag{A.20}$$

$$(\gamma^{m_1\ldots m_n})_{\alpha\beta}(\gamma_{p_n\ldots p_1})^{\beta\alpha} \;=\; 16 n!\,\delta^{m_1\ldots m_n}_{p_1\ldots p_n} - 16\,\delta^{n}_{5}\,\epsilon^{m_1\ldots m_5}{}_{p_5\ldots p_1}, \text{ for } n \in \text{odd}, \tag{A.21}$$

the second term contributes only when $n = 5$.

$$(\gamma^{m_1\ldots m_n})^{\alpha}{}_{\beta}(\gamma_{p_n\ldots p_1})^{\beta}{}_{\alpha} \;=\; 16 n!\,\delta^{m_1\ldots m_n}_{p_1\ldots p_n}, \text{ for } n \in \text{even} \tag{A.22}$$

- **Bi-Spinor decomposition**

Any Bi-spinor $T_{\alpha\beta}$ can be decomposed as

$$T_{\alpha\beta} = t_m(\gamma^m)_{\alpha\beta} + t_{mnp}(\gamma^{mnp})_{\alpha\beta} + t_{mnpqr}(\gamma^{mnpqr})_{\alpha\beta} \tag{A.23}$$

where, for $r = 1, 3, 5$ [1]

$$t_{m_1\ldots m_r} = \frac{1}{16 r!}(\gamma_{m_1\ldots m_r})^{\alpha\beta}\,T_{\alpha\beta} \tag{A.24}$$

[1] for $r = 5$, the RHS should be multiplied by an extra factor of $\frac{1}{2}$.

Similarly, a tensor-spinor $T^{\alpha}{}_{\beta}$ can be decomposed as

$$T^{\alpha}{}_{\beta} = t\,\delta^{\alpha}{}_{\beta} + t_{mn}(\gamma^{mn})^{\alpha}{}_{\beta} + t_{mnpq}(\gamma^{mnpq})^{\alpha}{}_{\beta} \tag{A.25}$$

where, for $r = 2, 4$

$$t_{m_1 \ldots m_r} = \frac{1}{16r!}(\gamma_{m_1 \ldots m_r})^{\alpha}{}_{\beta}\, T^{\alpha}{}_{\beta} \tag{A.26}$$

- **Tensor index contracted identities involving gamma matrices**

$$(\gamma^{mn})^{\alpha}{}_{\beta}(\gamma_{mn})^{\rho}{}_{\lambda} = 4(\gamma^{m})_{\beta\lambda}(\gamma_{m})^{\alpha\rho} - 2\delta^{\alpha}_{\beta}\delta^{\rho}_{\lambda} - 8\delta^{\alpha}_{\lambda}\delta^{\rho}_{\beta} \tag{A.27}$$

$$(\gamma^{mn})^{\alpha}{}_{\beta}(\gamma_{mnp})^{\rho\lambda} = 2(\gamma^{m})^{\alpha\rho}(\gamma_{pm})^{\lambda}{}_{\beta} + 6(\gamma_{p})^{\alpha\rho}\delta^{\lambda}_{\beta} - (\rho \leftrightarrow \lambda) \tag{A.28}$$

$$(\gamma_{mn})^{\alpha}{}_{\beta}(\gamma^{mnp})_{\rho\lambda} = -2(\gamma_{m})_{\beta\lambda}(\gamma^{pm})^{\alpha}{}_{\rho} + 6(\gamma^{p})_{\beta\lambda}\delta^{\alpha}_{\rho} - (\rho \leftrightarrow \lambda) \tag{A.29}$$

$$(\gamma_{mnp})^{\alpha\beta}(\gamma^{mnp})^{\rho\lambda} = 12[(\gamma_{m})^{\alpha\lambda}(\gamma^{m})^{\beta\rho} - (\gamma_{m})^{\alpha\rho}(\gamma^{m})^{\beta\lambda}] \tag{A.30}$$

$$(\gamma_{mnp})^{\alpha\beta}(\gamma^{mnp})_{\rho\lambda} = 48(\delta^{\alpha}_{\rho}\delta^{\beta}_{\lambda} - \delta^{\alpha}_{\lambda}\delta^{\beta}_{\rho}) \tag{A.31}$$

Appendix B

Some pure spinor results

In this appendix, we note some pure spinor results which are used in this work.

B.0.1 Pure spinor superspace identities

While computing the scattering amplitudes in pure spinor formalism, the last step requires evaluation of integration over the zero modes of λ^α and θ^α. Due to the pure spinor constraints and the symmetry properties of θ^α and λ^α, there are only a finite number of these basic pure spinor correlators which are non zero. Below, we list those pure spinor correlators which are used in this note (see [12] for a complete list, also [30])

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{stu}\theta)\rangle = \frac{1}{120}\delta^{mnp}_{stu} \tag{B.1}$$

$$\langle(\lambda\gamma^{pqr}\theta)(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\theta\gamma_{stu}\theta)\rangle = \frac{1}{70}\delta^{[p}_{[m}\eta_{n][s}\delta^q_t\delta^{r]}_{u]} \tag{B.2}$$

$$\langle(\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_s\theta)(\lambda\gamma_t\theta)(\theta\gamma_{uvw}\theta)\rangle = -\frac{1}{42}\delta^{mnpqr}_{stuvw} - \frac{1}{5040}\epsilon^{mnpqr}{}_{stuvw} \tag{B.3}$$

$$\begin{aligned}\langle(\lambda\gamma_q\theta)(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{rst}\theta)(\theta\gamma_{uvw}\theta)\rangle \quad = \quad & -\frac{1}{280}\Big[\eta_{q[u}\eta^{z[r}\delta^s_v\eta^{t][m}\delta^n_{w]}\delta^{p]}_z - \eta_{q[u}\eta^{z[m}\delta^n_v\eta^{p][r}\delta^s_{w]}\delta^{t]}_z\Big] \\ & +\frac{1}{140}\Big[\delta^{[m}_q\delta^n_{[u}\eta^{p][r}\delta^s_v\delta^{t]}_{w]} - \delta^{[r}_q\delta^s_{[u}\eta^{t][m}\delta^n_v\delta^{p]}_{w]}\Big] \\ & -\frac{1}{8400}\epsilon^{qmnprstuvw}\end{aligned} \tag{B.4}$$

$$
\begin{aligned}
&\langle(\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_{stu}\theta)(\lambda\gamma^{v}\theta)(\theta\gamma_{wxy}\theta)\rangle \\
&\quad= \frac{1}{120}\epsilon^{mnpqr}{}_{ghijk}\left(\frac{1}{35}\eta^{v[g}\delta^{h}_{[s}\delta^{i}_{t}\eta_{u][w}\delta^{j}_{x}\delta^{k]}_{y]} - \frac{2}{35}\delta^{[g}_{[s}\delta^{h}_{t}\delta^{i}_{u]}\delta^{j}_{[w}\delta^{k]}_{x}\delta^{v}_{y]}\right) \\
&\qquad+\frac{1}{35}\eta^{v[m}\delta^{n}_{[s}\delta^{p}_{t}\eta_{u][w}\delta^{q}_{x}\delta^{r]}_{y]} - \frac{2}{35}\delta^{[m}_{[s}\delta^{n}_{t}\delta^{p}_{u]}\delta^{q}_{[w}\delta^{r]}_{x}\delta^{v}_{y]}
\end{aligned}
\tag{B.5}
$$

Appendix C

RNS results

In this section, we summarize the results of the 3-point functions involving two massless and one massive states computed using the RNS formalism. In subsection C.0.1, we state our conventions and in subsection C.0.2, we give the results of 3-point computations. A useful reference for this section is [42] (also see [43])

C.0.1 Conventions for RNS calculations

Due to the picture number anomaly on a genus g Riemann surface, a non vanishing RNS correlator must have the picture number $2g-2$. This is ensured by working with vertex operators in appropriate picture number and the insertions of appropriate number of picture changing operators (PCOs). For the 3-point functions on Riemann sphere, we can avoid the insertions of PCOs by working with vertex operators of the appropriate picture number so that the total picture number adds up to -2 (which is the picture number anomaly on Riemann sphere).

We start by writing down the vertex operators for the massless states in various picture numbers. The gluon vertex operator in the -1 and 0 picture numbers is given by

$$\begin{aligned} V_a^{(-1)}(x) &= g_a\; e_m\psi^m(x)e^{-\phi(x)}e^{ik\cdot X(x)} \\ V_a^{(0)}(x) &= \frac{g_a}{\sqrt{2\alpha'}}e_m\Big(i\partial X^m(x)+2\alpha' k_n\psi^n(x)\psi^m(x)\Big)e^{ik\cdot X(x)} \end{aligned} \tag{C.1}$$

The gluino vertex operator in the $-1/2$ picture number is given by

$$V_\chi^{(-1/2)}(x) = g_\chi\;\xi^\alpha S_\alpha(x)e^{-\phi(x)/2}e^{ik\cdot X(x)} \tag{C.2}$$

The polarization vectors of the gluon and gluino satisfy the transversality conditions

$$e_m k^m \;=\; 0 \quad , \qquad \xi^\alpha \gamma^m_{\alpha\dot\beta} k_m \;=\; 0 \tag{C.3}$$

Now, we turn to the vertex operators for the first massive states [44, 45]. The vertex operators for the anti-symmetric 3-form field b_{mnp} in the picture numbers -1 and 0 are given by

$$\begin{aligned} V_b^{(-1)}(x) &= g_b\; e_{mnp}\psi^m(x)\psi^n(x)\psi^p(x)e^{-\phi(x)}e^{ik\cdot X(x)} \\ V_b^{(0)}(x) &= g_b\sqrt{2\alpha'}\; e_{mnp}\Big(\frac{3i}{2\alpha'}\partial X^m\psi^n\psi^p + k_q\psi^q\psi^m\psi^n\psi^p\Big)e^{ik\cdot X(x)} \end{aligned} \tag{C.4}$$

The vertex operators for the symmetric traceless massive graviton field g_{mn} are given by

$$\begin{aligned} V_g^{(-1)}(x) &= \frac{g_g}{\sqrt{2\alpha'}}\; e_{mn} i\partial X^m(x)\psi^n(x)e^{-\phi(x)}e^{ik\cdot X(x)} \\ V_g^{(0)}(x) &= g_g\; e_{mn}\Big(\frac{1}{2\alpha'}i\partial X^m i\partial X^n + \partial\psi^m\psi^n + i\partial X^m k_p\psi^p\psi^n\Big)e^{ik\cdot X} \end{aligned} \tag{C.5}$$

Finally, the vertex operators for the massive gravitino field $\psi_{m\alpha}$ in the $-1/2$ picture number is given by

$$V_\psi^{(-1/2)}(z) \;=\; \frac{g_\psi}{\sqrt{\alpha'}}\Big(\xi^\alpha_m i\partial X^m - \frac{\alpha'}{4}\xi^\beta_m\psi^m\gamma^n_{\beta\dot\beta}\gamma_p^{\dot\beta\alpha}k_n k^p\Big)S_\alpha e^{-\phi/2}e^{ik\cdot X}(z) \tag{C.6}$$

The polarization vectors of the 1st massive states satisfy the conditions

$$k^m e_{mnp} = k^m e_{mn} = k^m \xi^\alpha_m = 0 \tag{C.7}$$

For comparison with PS result, it is useful to parametrize the massive tensor spinor ξ^α_m as

$$\xi^\alpha_m = -8\alpha'\bar\rho_{m\dot\beta}k^p\bar\gamma_p^{\dot\beta\alpha} \qquad , \qquad k^m\bar\rho_{m\dot\beta} = \bar\rho_{m\dot\beta}\gamma_m^{\dot\beta\alpha} = 0 \tag{C.8}$$

We now turn to the OPEs and correlation functions of the worldsheet matter and ghost sector fields. The important correlators of the open string X^m fields are given by

$$\Big\langle \prod_{j=1}^n e^{ik_j\cdot X(y_j)} \Big\rangle \;=\; \prod_{i<j}^n |y_i - y_j|^{2\alpha' k_i\cdot k_j} \tag{C.9}$$

$$\left\langle \prod_{\ell=1}^{q} i\partial X^{m_\ell}(w_\ell) \prod_{j=1}^{n} e^{ik_j\cdot X(y_j)} \right\rangle = \sum_{i=2}^{p} \frac{2\alpha' \eta^{m_1 m_i}}{(w_1 - w_i)^2} \left\langle \prod_{\substack{\ell=2\\ \ell\neq i}}^{p} i\partial X^{m_\ell}(w_\ell) \prod_{j=1}^{n} e^{ik_j\cdot X(y_j)} \right\rangle + \sum_{\ell=1}^{n} \frac{2\alpha' k_\ell^{m_1}}{(w_1 - y_\ell)} \left\langle \prod_{\ell=2}^{p} i\partial X^{m_\ell}(w_\ell) \prod_{j=1}^{n} e^{ik_j\cdot X(y_j)} \right\rangle \quad \text{(C.10)}$$

This can be evaluated recursively using (C.9). We shall encounter the situation where some w_ℓ may coincide with some y_j. E.g., we shall need the following correlator

$$\begin{aligned} \Gamma^m &\equiv \left\langle e^{ik_1\cdot X(z_1)} e^{ik_2\cdot X(z_2)} \partial X^m(z_3) e^{ik_3\cdot X(z_3)} \right\rangle \\ &= \oint \frac{dw}{w - z_3} \left\langle e^{ik_1\cdot X(z_1)} e^{ik_2\cdot X(z_2)} e^{ik_3\cdot X(z_3)} \partial X^m(w) \right\rangle \\ &= \left(\frac{2i\alpha' k_1^m}{z_{13}} + \frac{2i\alpha' k_2^m}{z_{23}} \right) |z_{12}|^{2\alpha' k_1\cdot k_2} |z_{23}|^{2\alpha' k_2\cdot k_3} |z_{13}|^{2\alpha' k_1\cdot k_3} \end{aligned} \quad \text{(C.11)}$$

In going to the 2nd line, we have used the definition of the normal ordering (1.13) and in going to the 3rd line, we have used (C.10) for $q = 1$ and $n = 3$.

Similarly, we also need

$$\begin{aligned} &\left\langle e^{ik_1\cdot X(z_1)} e^{ik_2\cdot X(z_2)} i\partial X^p(z_3) i\partial X^q(z_3) e^{ik_3\cdot X(z_3)} \right\rangle \\ = \; & 4(\alpha')^2 \left[\frac{k_1^p}{z_{13}} \left(\frac{k_1^q}{z_{13}} + \frac{k_2^q}{z_{23}} \right) + \frac{k_2^p}{z_{23}} \left(\frac{k_1^q}{z_{13}} + \frac{k_2^q}{z_{23}} \right) \right] |z_{12}|^{2\alpha' k_1\cdot k_2} |z_{23}|^{2\alpha' k_2\cdot k_3} |z_{13}|^{2\alpha' k_1\cdot k_3} \end{aligned}$$

Next, we consider the worldsheet correlators involving the ψ^m fields

$$\left\langle \prod_{i=1}^{n} \psi^{m_i}(y_i) \right\rangle = \sum_{j=2}^{n} (-1)^j \frac{\eta^{m_1 m_j}}{(y_1 - y_j)} \left\langle \prod_{\substack{\ell=2\\ \ell\neq j}}^{n} \psi^{m_\ell}(y_\ell) \right\rangle \quad \text{(C.12)}$$

This expression can also be evaluated recursively. In the final step, the two point function can be evaluated by using the OPE

$$\psi^m(z)\psi^n(w) = \frac{\eta^{mn}}{z - w} + \cdots \quad \text{(C.13)}$$

Next, we consider the ghost sector. The basic correlator involving the reparametrization ghost $c(x)$ is given by

$$\langle c(x_1)c(x_2)c(x_3)\rangle \;=\; x_{12}x_{23}x_{13} \tag{C.14}$$

The basic correlator involving the bosonized ghost field $\phi(z)$ is given by

$$\left\langle \prod_{k=1}^{n} e^{q_k\phi(z_k)} \right\rangle \;=\; \prod_{k<\ell}^{n} \frac{1}{(z_k-z_\ell)^{q_kq_\ell}} \qquad ; \qquad \sum_{k=1}^{n} q_k \;=\; -2 \tag{C.15}$$

Finally, we consider the spin fields. The basic OPEs involving the spin fields are given by (a consistent set of these OPEs can be found in [46])

$$\psi^m(z)S_\alpha(w)e^{-\phi(w)/2} \;=\; \frac{(\bar\gamma^m)^{\dot\alpha\beta}S_\beta e^{-\phi(w)/2}}{\sqrt{2}(z-w)^{1/2}} + \cdots \tag{C.16}$$

$$\psi^m(z)S^{\dot\alpha}(w)e^{-\phi(w)/2} \;=\; \frac{(\gamma^m)_{\alpha\dot\beta}S^{\dot\beta}e^{-\phi(w)/2}}{\sqrt{2}(z-w)^{1/2}} + \cdots \tag{C.17}$$

$$S_\alpha(z)e^{-\phi(z)/2}S^{\dot\beta}(w)e^{-3\phi(w)/2} \;=\; \frac{C_\alpha{}^{\dot\beta}e^{-2\phi(w)}}{(z-w)^2} + \cdots \tag{C.18}$$

$$S_\alpha(z)e^{-\phi(z)/2}S_\beta(w)e^{-\phi(w)/2} \;=\; \frac{(\gamma_mC)_{\alpha\beta}\psi^m(w)e^{-\phi(w)}}{\sqrt{2}(z-w)} + \cdots \tag{C.19}$$

$$\psi_m(z)e^{-\phi(z)}S_\alpha(w)e^{-\phi(w)/2} \;=\; \frac{(\bar\gamma_m)_{\alpha\dot\beta}S^{\dot\beta}(w)e^{-3\phi(w)/2}}{\sqrt{2}(z-w)} + \cdots \tag{C.20}$$

$$j^{mn}(z)S_\alpha(w) \;=\; -\frac{(\gamma^{mn})_\alpha{}^\beta S_\beta(w)}{2(z-w)} \quad , \qquad j^{mn} \;\equiv :\psi^m\psi^n: \tag{C.21}$$

Using the above OPEs, we can work out the following useful correlators which are needed in our calculations

$$\langle\psi^m(x_1)e^{-\phi(x_1)}S_\alpha(x_2)e^{-\phi(x_2)/2}S_\beta(x_3)e^{-\phi(x_3)/2}\rangle = \frac{(\gamma^m C)_{\alpha\beta}}{\sqrt{2}z_{12}z_{13}z_{23}} \tag{C.22}$$

$$\xi_q^\beta\Big\langle\psi^m(x_1)e^{-\phi(x_1)}S_\alpha(x_2)e^{-\phi(x_2)/2}\psi^q(x_3)\psi^p(x_3)\gamma_p^{\dot\beta\sigma}S_\sigma(x_3)e^{-\phi(x_3)/2}\Big\rangle = \frac{8\ \xi_m^\beta C_\alpha^{\dot\beta}}{\sqrt{2}z_{13}^2 z_{23}^2} \tag{C.23}$$

C.0.2 3-point functions

We are now ready to give the results of 3-point functions of two massless and one massive field (for the computation of 3-point functions of states in leading Regge trajectory in RNS, see [47]). These are straightforward to evaluate using the vertex operators and the correlators given in the previous subsection. Hence, we just state the final results below. For our calculations, we shall take the bosonic massless and massive fields to be either in the -1 or 0 picture numbers. On the other hand, the fermionic massless or massive fields will always be taken in the $-1/2$ picture. The picture numbers will be shown by a superscript on the vertex operators inside correlators. Thus, the 3-point amplitudes are given by

$$\mathcal{A}_3 = \big\langle\ c(x_1)V_{a,\chi}^{(1)}(x_1)c(x_2)V_{a,\chi}^{(2)}(x_2)c(x_3)V_{b,g,\psi}^{(3)}(x_3)\ \big\rangle \tag{C.24}$$

We start by considering the 3-point functions involving all massless fields. The two possible correlators in this case are $\langle aaa\rangle$ and $\langle a\chi\chi\rangle$. Using the worldsheet correlators given above, these 3-point functions can be evaluated to be

$$\begin{aligned}\langle aaa\rangle &\equiv \Big\langle c(x_1)V_a^{(-1)}(x_1)c(x_2)V_a^{(-1)}(x_2)c(x_3)V_a^{(0)}(x_3)\Big\rangle \\ &= -g_a^3\sqrt{2\alpha'}\Big[(e^1\cdot e^2)(e^3\cdot k^1)+(e^1\cdot e^3)(e^2\cdot k^3)+(e^2\cdot e^3)(e^1\cdot k^2)\Big]\end{aligned} \tag{C.25}$$

As an amplitude this vanishes on summing $(k_1 \leftrightarrow k_2)$ term if the gauge group is abelian.

Similarly, the $a\chi\chi$ correlator can be worked out to be

$$\begin{aligned}\langle a\chi\chi\rangle &\equiv \Big\langle c(x_1)V_a^{(-1)}(x_1)c(x_2)V_\chi^{(-1/2)}(x_2)c(x_3)V_\chi^{(-1/2)}(x_3)\Big\rangle \\ &= \frac{1}{\sqrt{2}}g_a g_\chi^2 e_m^1\Big(\xi^2\gamma^m C\xi^3\Big)\end{aligned} \tag{C.26}$$

Again, for the abelian gauge group, the corresponding amplitude vanishes.

Next, we consider the two massless and one massive field. We start with two gluons and one b_{mnp} field

$$\begin{aligned} \langle aab \rangle &\equiv \Big\langle c(x_1)V_a^{(-1)}(x_1)c(x_2)V_a^{(-1)}(x_2)c(x_3)V_b^{(0)}(x_3)\Big\rangle \\ &= 6g_a^2 g_b \sqrt{2\alpha'} e^1_m e^2_n e_{mnp} k_1^p \end{aligned} \tag{C.27}$$

Next, we consider the 3-point function of two gluons and one massive g_{mn} field which is given by

$$\begin{aligned} \langle aag \rangle &\equiv \Big\langle c(x_1)V_a^{(-1)}(x_1)c(x_2)V_a^{(-1)}(x_2)c(x_3)V_g^{(0)}(x_3)\Big\rangle \\ &= -g_a^2 g_g\Big[2\alpha'\eta^{mn} e^1_m e^2_n e_{pq} k_1^p k_1^q + e^1_p e^2_q e_{pq} + 2\alpha' e^1_m e^2_q e_{pq} k_1^p k_2^m - 2\alpha' e^1_q e^2_m e_{pq} k_1^m k_1^p\Big] \end{aligned} \tag{C.28}$$

Next, we include gluino and consider the $\langle \chi\chi b\rangle$ and $\langle \chi\chi g\rangle$ correlators which can be worked out to be

$$\begin{aligned} \langle \chi\chi b \rangle &\equiv \Big\langle c(x_1)V_\chi^{(-1/2)}(x_1)c(x_2)V_\chi^{(-1/2)}(x_2)c(x_3)V_b^{(-1)}(x_3)\Big\rangle \\ &= -\frac{1}{2\sqrt{2}} g_\chi^2 g_b (\xi^1\gamma^{mnp}C\xi^2)e_{mnp} \end{aligned} \tag{C.29}$$

and,

$$\begin{aligned} \langle \chi\chi g \rangle &\equiv \langle c(x_1)V_\chi^{(-1/2)}(x_1)c(x_2)V_\chi^{(-1/2)}(x_2)c(x_3)V_g^{(-1)}(x_3)\rangle \\ &= \sqrt{\alpha'} g_\chi^2 g_g e_{mn}(\xi^1\gamma^n C\xi^2)k_1^m \end{aligned} \tag{C.30}$$

Finally, we consider the massive fermion. The two different 3-point functions with one massive fermion and two massless fields are $\langle a\chi\psi\rangle$ and $\langle \chi a\psi\rangle$ which are given by

$$\begin{aligned} \langle a\chi\psi \rangle &\equiv \Big\langle c(x_1)V_a^{(-1)}(x_1)c(x_2)V_\chi^{(-1/2)}(x_2)c(x_3)V_\psi^{(-1/2)}(x_3)\Big\rangle \\ &= -\frac{16\alpha'}{\sqrt{2}} g_a g_\chi g_\psi\Big[-\alpha' e^{(1)}_m \xi^\alpha_{(2)} k_2^q k_1^p \bar\rho_{q\dot\beta}(\bar\gamma_{mp})^{\dot\beta}{}_{\dot\sigma} C^{\dot\sigma}{}_\alpha - 2\alpha' k_2^q k_2^m e^{(1)}_m \xi^\alpha_{(2)}\bar\rho_{q\dot\beta}C^{\dot\beta}{}_\alpha + e^{(1)}_m \xi^\alpha_{(2)}\bar\rho_{m\dot\sigma}C^{\dot\sigma}{}_\alpha\Big] \end{aligned} \tag{C.31}$$

and,

$$\begin{aligned}\langle \chi a \psi \rangle &\equiv \left\langle c(x_1) V_\chi^{(-1/2)}(x_1) c(x_2) V_a^{(-1)}(x_2) c(x_3) V_\psi^{(-1/2)}(x_3) \right\rangle \\ &= -\frac{16\alpha'}{\sqrt{2}} g_a g_\chi g_\psi \left[-\alpha' e_m^{(2)} \xi_{(1)}^\alpha k_1^q k_2^p \bar{\rho}_{q\dot{\beta}} (\bar{\gamma}_{mp})^{\dot{\beta}}{}_{\dot{\sigma}} C^{\dot{\sigma}}{}_\alpha - 2\alpha' k_1^q k_1^m e_m^{(2)} \xi_{(1)}^\alpha \bar{\rho}_{q\dot{\beta}} C^{\dot{\beta}}{}_\alpha + e_m^{(2)} \xi_{(1)}^\alpha \bar{\rho}_{m\dot{\sigma}} C^{\dot{\sigma}}_\alpha \right]\end{aligned} \tag{C.32}$$

To compare the results involving the fermionic fields with the PS results, we need to first convert the RNS gamma matrix conventions into the PS gamma matrix conventions. This mainly involves setting the charge conjugation matrix to be the Kronecker delta $\delta^\alpha{}_\beta$ which implies (for details, see e.g., [42, 43])

$$(\gamma^m C)_{\alpha\beta} \to \gamma^m_{\alpha\beta} \quad , \quad (\bar{\gamma}_m C)^{\dot{\alpha}\dot{\beta}} \to \gamma_m^{\alpha\beta} \quad , \quad (\gamma^{mnp} C)_{\alpha\beta} \to (\gamma^{mnp})_{\alpha\beta} \tag{C.33}$$

and so on.

Appendix D

θ expansion of massless vertex operator

θ expansion of the massless vertex operator is known and has been extensively used in the literature (see, e.g., [15]). However, some of our conventions (e.g., equation (1.16)) are different from the literature in which theθ expansion of massless vertex operator is used. Below, we derive the results in our convention briefly indicating the steps. We start by recalling the $\mathcal{N}=1$ SYM equations in 10 dimensions [48] (see also [49], [50]).

$\mathcal{N}=1$ SYM equations in 10 dimensions

In 10 dimensions, the open string massless states are described by the 10 dimensional $\mathcal{N}=1$ super Yang Mills equations. The field strengths describing the theory are given by

$$F_{\alpha\beta}=\{\nabla_\alpha,\nabla_\beta\}-2\gamma^m_{\alpha\beta}\nabla_m \qquad , \qquad F_{\alpha m}=[\nabla_\alpha,\nabla_m]=-F_{m\alpha} \qquad , \qquad F_{mn}=[\nabla_m,\nabla_n]$$

where $\nabla_m\equiv\partial_m+A_m$, $\nabla_\alpha\equiv D_\alpha+A_\alpha$ and D_α is defined in (1.16).

The 10 dimensional Yang-Mills equations of motion follow from

$$F_{\alpha\beta}=0 \qquad \Longrightarrow \qquad D_\alpha A_\beta+D_\beta A_\alpha=2\gamma^m_{\alpha\beta}A_m \tag{D.1}$$

Using this along with the Bianchi identities, we obtain the following equations at linearized level

$$\begin{aligned}
F_{m\alpha} &= (\gamma_m)_{\alpha\beta}W^\beta \quad , \quad D_\alpha W^\beta = -\frac{1}{2}(\gamma^{mn})_\alpha{}^\beta F_{mn} \quad \Longrightarrow \quad D_\alpha W^\alpha = 0 \\
D_\alpha F_{mn} &= \partial_m(\gamma_n W)_\alpha - \partial_n(\gamma_m W)_\alpha \quad , \quad D_\alpha A^m = -\gamma^m_{\alpha\beta}W^\beta + \partial^m A_\alpha \\
\partial_m F^{mn} &= 0 \quad , \quad \gamma^m_{\alpha\beta}\partial_m W^\beta = 0
\end{aligned} \tag{D.2}$$

Further, the superfields A_m, W^α and F_{mn} can be expressed as

$$\begin{aligned}
A_m &= \frac{1}{16}\gamma_m^{\alpha\beta}D_\alpha A_\beta \quad , \quad F_{mn} = \partial_m A_n - \partial_n A_m \\
W^\alpha &= -\frac{1}{10}(\gamma^m)^{\alpha\beta}(D_\beta A_m - \partial_m A_\beta)
\end{aligned} \tag{D.3}$$

$\mathcal{N} = 1$ SYM equations from pure spinor formalism

The pure spinor formalism gives the $\mathcal{N} = 1$ SYM equations through its BRST equations of motion. To obtain the correct normalization of the superfields, we derive these equations using the BRST equation of motion $QU = \partial_{\mathbb{R}} V$ and match with the equations given above. At the massless level, the unintegrated vertex operator is constructed from the ghost number 1 and conformal weight 0 objects. The most general object with this property has the form $V = \lambda^\alpha A_\alpha$. Similarly, the most general integrated vertex operator has the form

$$U = \partial\theta^\alpha \tilde{A}_\alpha + \Pi^m \tilde{A}_m + d_\alpha \tilde{W}^\alpha + N^{mn}\tilde{F}_{mn} \tag{D.4}$$

Using the OPEs given in section 5.1, we obtain

$$\begin{aligned}
Q(\partial\theta^\alpha \tilde{A}_\alpha) &= -\frac{\alpha'}{2}\partial\theta^\alpha\lambda^\beta D_\beta \tilde{A}_\alpha + \frac{\alpha'}{2}\partial\lambda^\alpha \tilde{A}_\alpha \\
Q(\Pi^m \tilde{A}_m) &= \frac{\alpha'}{2}\Pi^m\lambda^\alpha D_\alpha \tilde{A}_m + \frac{\alpha'}{2}\partial\theta^\alpha\lambda^\beta \tilde{A}_m \gamma^m_{\alpha\beta} \\
Q(d_\alpha \tilde{W}^\alpha) &= -\frac{\alpha'}{2}d_\alpha\lambda^\beta D_\beta \tilde{W}^\alpha - \frac{\alpha'}{2}\Pi^m\lambda^\alpha \tilde{W}^\beta\gamma^m_{\alpha\beta} + \frac{(\alpha')^2}{2}\partial\lambda^\alpha\partial_m \tilde{W}^\beta\gamma^m_{\alpha\beta} \\
Q(N^{mn}\tilde{F}_{mn}) &= \frac{\alpha'}{2}N^{mn}\lambda^\beta D_\beta \tilde{F}_{mn} - \frac{\alpha'}{4}d_\alpha\lambda^\beta \tilde{F}_{mn}(\gamma^{mn})^\alpha{}_\beta - \frac{(\alpha')^2}{8}\partial\lambda^\alpha D_\beta \tilde{F}_{mn}(\gamma^{mn})^\beta{}_\alpha
\end{aligned}$$

We also have,

$$\begin{aligned}\partial_{\mathbb{R}}(\lambda^\alpha A_\alpha) &= \partial\lambda^\alpha A_\alpha + \lambda^\alpha \partial A_\alpha \\ &= \partial\lambda^\alpha A_\alpha + \partial\theta^\beta \lambda^\alpha D_\beta A_\alpha + 2\Pi^m \lambda^\alpha \partial_m A_\alpha \qquad \text{(D.5)}\end{aligned}$$

where we have used the pure spinor open string identity

$$\partial_{\mathbb{R}} = \partial\theta^\alpha D_\alpha + 2\Pi^m \partial_m \qquad \text{(D.6)}$$

The BRST equation of motion gives,

$$\begin{aligned}0 &= QU - \partial_{\mathbb{R}} V \\ &= \partial\theta^\alpha \lambda^\beta \Big(-\frac{\alpha'}{2} D_\beta \tilde{A}_\alpha - D_\alpha A_\beta + \frac{\alpha'}{2} \tilde{A}_m \gamma^m_{\alpha\beta} \Big) + \Pi^m \lambda^\alpha \Big(\frac{\alpha'}{2} D_\alpha \tilde{A}_m - \frac{\alpha'}{2} \tilde{W}^\beta \gamma^m_{\alpha\beta} - 2\partial_m A_\alpha \Big) \\ &\quad + d_\alpha \lambda^\beta \Big(-\frac{\alpha'}{2} D_\beta \tilde{W}^\alpha - \frac{\alpha'}{4} \tilde{F}_{mn} (\gamma^{mn})^\alpha{}_\beta \Big) + \frac{\alpha'}{2} N^{mn} \lambda^\beta D_\beta \tilde{F}_{mn} \\ &\quad + \partial\lambda^\alpha \Big(\frac{\alpha'}{2} \tilde{A}_\alpha + \frac{(\alpha')^2}{2} \partial_m \tilde{W}^\beta \gamma^m_{\alpha\beta} - \frac{(\alpha')^2}{8} D_\beta \tilde{F}_{mn} (\gamma^{mn})^\beta{}_\alpha - A_\alpha \Big) \qquad \text{(D.7)}\end{aligned}$$

To match these equations with the 10 dimensional SYM equations, we rescale the fields as

$$\tilde{A}_\alpha = \frac{2}{\alpha'} A_\alpha \quad , \quad \tilde{A}_m = \frac{4}{\alpha'} A_m \quad , \quad \tilde{W}^\alpha = -\frac{4}{\alpha'} W^\alpha \quad , \quad \tilde{F}_{mn} = \frac{4}{\alpha'} F_{mn} \qquad \text{(D.8)}$$

After this rescaling, the BRST equation gives

$$\begin{aligned}& D_\beta A_\alpha + D_\alpha A_\beta = 2 A_m \gamma^m_{\alpha\beta} \quad , \quad D_\alpha A_m = -W^\beta \gamma^m_{\alpha\beta} + \partial_m A_\alpha \quad , \quad D_\beta W^\alpha = -\frac{1}{2} F_{mn} (\gamma^{mn})_\beta{}^\alpha \\ & -\partial_m W^\beta \gamma^m_{\alpha\beta} = \frac{1}{4} D_\beta F_{mn} (\gamma^{mn})^\beta{}_\alpha \quad , \quad N^{mn} \lambda^\beta D_\beta F_{mn} = 0\end{aligned}$$

The first 4 equations are precisely satisfied by the 10 dimensional $\mathcal{N} = 1$ SYM equations given in appendix D whereas the last equation is satisfied by the pure spinor constraint. The correctly normalized vertex operators of the massless states in our conventions thus become

$$\begin{aligned} V &= \lambda^\alpha A_\alpha \\ U &= \frac{2}{\alpha'}\partial\theta^\alpha A_\alpha + \frac{4}{\alpha'}\Pi^m A_m - \frac{4}{\alpha'}d_\alpha W^\alpha + \frac{4}{\alpha'}N^{mn}F_{mn} \\ &= \frac{4}{\alpha'}\left(\frac{1}{2}\partial\theta^\alpha A_\alpha + \Pi^m A_m - d_\alpha W^\alpha + N^{mn}F_{mn}\right) \end{aligned} \tag{D.9}$$

Theta expansion of massless superfields

Finally, we turn to the θ expansion. We shall follow the steps outlined in [29]. We shall need the following equations for doing the θ expansion

$$\begin{aligned} &D_\alpha A_\beta + D_\beta A_\alpha = 2A_m\gamma^m_{\alpha\beta} \quad , \qquad D_\alpha A_m = -W^\beta\gamma^m_{\alpha\beta} + \partial_m A_\alpha \\ &D_\beta W^\alpha = -\frac{1}{2}F_{mn}(\gamma^{mn})_\beta{}^\alpha \quad , \quad F_{mn} = \partial_m A_n - \partial_n A_m \end{aligned} \tag{D.10}$$

Before proceeding to do theθ expansion, we need to fix a gauge. We shall choose the gauge $\theta^\alpha A_\alpha = 0$. In this gauge choice, we have

$$0 = D_\beta(\theta^\alpha A_\alpha) = A_\beta - \theta^\alpha D_\beta A_\alpha \quad \Longrightarrow \quad A_\beta = \theta^\alpha D_\beta A_\alpha \tag{D.11}$$

Now, multiplying by θ^β in the 1st equation and by θ^α in the 2nd and 3rd equations of (D.10) and using the above identity along with the gauge choice $\theta^\alpha A_\alpha = 0$, we obtain

$$(1+D)A_\alpha = 2A_m(\gamma^m\theta)_\alpha \quad , \quad DA_m = -(\theta\gamma_m W) \quad , \quad DW^\alpha = \frac{1}{2}F_{mn}(\gamma^{mn}\theta)^\alpha \tag{D.12}$$

where we have defined $D \equiv \theta^\alpha D_\alpha = \theta^\alpha\partial_\alpha$.

We can use the above 3 equations along with the 4th equation of (D.10) to do theθ expansion. If we denote the ℓ^{th} order component of the superfield M by $M^{(\ell)}$, then we have

$$M^{(\ell)} = m_{\alpha_1\cdots\alpha_\ell}\theta^{\alpha_1}\cdots\theta^{\alpha_\ell} \quad \Longrightarrow \quad DM^{(\ell)} = \ell\ M^{(\ell)} \tag{D.13}$$

Using this, the 3 equations of (D.12) and the 4th equation of (D.10) give the following recursive relations

$$\begin{aligned}
(1+\ell)A_\alpha^{(\ell)} &= 2A_m^{(\ell-1)}(\gamma^m\theta)_\alpha \quad \Longrightarrow \quad A_\alpha^{(\ell)} = \frac{2}{1+\ell}A_m^{(\ell-1)}(\gamma^m\theta)_\alpha \\
\ell A_m^{(\ell)} &= -(\theta\gamma_m W^{(\ell-1)}) \quad \Longrightarrow \quad A_m^{(\ell)} = -\frac{1}{\ell}(\theta\gamma_m W^{(\ell-1)}) \\
\ell W_{(\ell)}^\alpha &= \frac{1}{2}F_{mn}^{(\ell-1)}(\gamma^{mn}\theta)^\alpha \quad \Longrightarrow \quad W_{(\ell)}^\alpha = \frac{1}{2\ell}(\gamma^{mn}\theta)^\alpha \; F_{mn}^{(\ell-1)} \\
F_{mn}^{(\ell)} &= \partial_m A_n^{(\ell)} - \partial_n A_m^{(\ell)}
\end{aligned} \tag{D.14}$$

Denoting theθ independent components of the superfields A_m and W^α to be

$$A_m^{(0)} \equiv a_m \quad , \qquad W_{(0)}^\alpha \equiv \chi^\alpha \tag{D.15}$$

the above recursive relations give

$$\begin{aligned}
A_\alpha &= a_m(\gamma^m\theta)_\alpha - \frac{2}{3}(\gamma^m\theta)_\alpha(\theta\gamma_m\chi) - \frac{1}{8}(\gamma_m\theta)_\alpha(\theta\gamma^{mpq}\theta)f_{pq} - \frac{i}{15}(\gamma_m\theta)_\alpha(\theta\gamma_p\chi)(\theta\gamma^{mpq}\theta)k_q + \cdots \\
A_m &= a_m - (\theta\gamma_m\chi) - \frac{1}{4}(\theta\gamma_{mnp}\theta)f^{np} \; - \; \frac{1}{6}(\theta\gamma_m\gamma^{pq}\theta)(\theta\gamma_{[m}\partial_{n]}\chi) + \frac{1}{48}(\theta\gamma_m\gamma^{rn}\theta)(\theta\gamma_{npq}\theta)\partial_r f^{pq} + \cdots
\end{aligned} \tag{D.16}$$

where $f_{mn} = \partial_m a_n - \partial_n a_m$ and the plane wave expansion of the gluon and gluino are given by

$$a_m \;=\; e_m e^{ik\cdot X} \quad , \qquad \chi^\alpha \;=\; \xi^\alpha e^{ik\cdot X} \tag{D.17}$$

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