MASSIVE STATES IN PURE SPINOR SUPERSTRING AND EVADING AN IMPLICATION OF STRING THEORY – VACCUM DECAY

By

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. "Integrated Massive Vertex Operator in Pure Spinor Formulation", S. Chakrabarti, S. P. Kashyap and M. Verma, *JHEP*, **10 (2018)** 147
2. "Surviving in a Metastable de Sitter Space-Time", S. P. Kashyap, S. Mondal, A. Sen and M. Verma, *JHEP*, **09 (2015)** 139

Conferences

1. Equivalence of amplitudes involving massive string states in pure spinor and RNS formalisms, **Strings 2018**, 25-29 June 2018, *Okinawa Institute of Science and Technology, Japan*
2. First Integrated Massive Vertex in Pure Spinor Formalism of Superstring, **Kavli Asian Winter School (KAWS) On Strings, Particles and Cosmology 2018,** 08-18 January 2018, *ICTS-TIFR, India*

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CONTENTS

SUMMARY

String theory comes in various formulations [1–7]. Of these, the pure spinor formulation [7] achieves the quantization keeping superPoincaré invariance manifest. This feature results in a more efficient tool for computation of scattering amplitudes as well as offers a promising avenue for quantization of superstring in curved backgrounds with Ramond-Ramond fields. Pure spinor formalism has delivered various remarkable results on both of these fronts. In this thesis we construct the integrated massive vertex operator at first excited level of open superstring [8]

$$
\begin{aligned}
U \;=\; & :\Pi^m\Pi^n F_{mn}: \;+\; :\Pi^m d_\alpha F_m{}^\alpha: \;+\; :\Pi^m\partial\theta^\alpha G_{m\alpha}: \;+\; :\Pi^m N^{pq} F_{mpq}: \\
& +\; :d_\alpha d_\beta K^{\alpha\beta}: \;+\; :d_\alpha \partial\theta^\beta F^\alpha{}_\beta: \;+\; :d_\alpha N^{mn} G^\alpha{}_{mn}: \;+\; :\partial\theta^\alpha\partial\theta^\beta H_{\alpha\beta}: \\
& +\; :\partial\theta^\alpha N^{mn} H_{mn\alpha}: \;+\; :N^{mn}N^{pq} G_{mnpq}:
\end{aligned}
$$

where, the superfields e.g. $F_{mn}, F_m{}^\alpha, G_{m\alpha}$ are all expressed in terms of the basic superfields G_{mn}, B_{mnp} and $\Psi_{m\alpha}$ that appear in the unintegrated vertex operator [9]. This work forms the major part of this thesis. The procedure developed in this work can be generalized to construction of the massive states at higher mass level and is applicable for unintegrated vertex operators as well. Further it can be trivially generalized to type II and heterotic superstring theories.

Independent of any of the formulations, it has been a very difficult problem to reproduce the standard model of physics as a low energy limit of string theory. A major part of the reason is that string theory is defined in 10 dimensions whereas our world is four dimensional. In order to arrive at the four dimensional world, we need to compactify the six of the spatial directions which form an internal six dimensional manifold at each spacetime point in the four dimensions. Different choices of the six dimensional compact manifold give rise to different particle content and interactions. As of today we do not know what is *the choice* of six compact dimensional manifold which gives rise to the standard model of physics. Nonetheless, we cannot discard a vacuum solution of string theory

just because it does not gives rise to our universe. Whenever a quantum field theory has multiple vacua, there is a possibility of vacuum decay. From the cosmological data [10,11], we know that the cosmological constant of our universe is small but positive. In string theory we can have positive, zero and even negative cosmological constants [12–14]. So, if string theory is indeed correct, our universe will undergo a vacuum decay and end up in a vacuum with smaller cosmological constant. If that happens, everything will be annihilated. The second part of this thesis looks into the problem maximizing the lifetime of human civilization in a metastable de-Sitter spacetime. The vacuum decay occurs via a bubble nucleation procedure. The bubble of stable vacuum grows till it converts all the available space into stable vacuum. In order to enhance the collective lifetime of set of objects, we separate them spatially and use the expansion of universe to drive them out of causal contact, forbidding destruction by a single vacuum bubble [15]. We find that for a even a small initial separation the collective life expectancy reaches a value close to the maximum possible if the decay rate is less than 1% of the expansion rate [16].

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Chapter 1

Introduction

What is/are the most basic entity/entities out of which everything we know of is made of? The answer to this question has undergone revisions from being *atoms* during the times of Dalton to quarks and leptons in the modern times. Dalton defined the atoms to the *indivisible* particles. However, modern physics has come a long way from Dalton's atom. Today we understand that the answer is very much *dependent* on how much energy we can *pack* into a localized regions of spacetime. In order to discover new particles we build accelerators that can collide particles with ever increasing energy, the Large Hadron Collider (LHC) [17] being the state of art today. Theoretical physics concerns itself with classifying these particles and figuring out the laws that govern the interactions among these particles. There are four different kinds of interactions among all the particles that we know today, namely - strong, weak, electromagnetism and gravity. The current understanding of these interactions can be put together in the form of standard model (SM) [18–21] of physics and the general relativity (GR) [22]. While the former describes to a very high degree the particles that interact among themselves via strong, electromagnetic or weak interactions, the latter tells how these particles interact gravitationally.

Both of these theories came in being in the past century and were applied to describe universe at vastly different scales. While the GR is used to describe universe at the large scales (motion of planets around stars, the motion of galaxies and universe as a whole - cosmology), SM has been tremendously successful at describing what happens at around $(10^{-15}m - 10^{18}m)$. Both of these theories have done exceptionally in making predictions that have been verified to a very high degree. In fact in a remarkable feat, gravitation waves which are a prediction of GR (1916) got verified recently by LIGO [23, 24] nearly a century later. Similarly the Higgs boson which is a requirement of SM got discovered in 2012, nearly four decades later after it was predicted.

It appeared as if we know all the content of the universe because of the success stories of these theories. However, there are certain observations that cannot be accounted for based on SM and GR. Assuming that GR is a correct description at large scales, we are lead to the discovery of *dark energy*[1] and *dark matter* [25–27]. The existence of dark matter (based on astronomical data) suggests that there is matter in the universe that cannot be accounted for based on SM. In fact these new findings have changed our view of the composition of the observable universe, summarized in the figure 1.1[2] [28]. We can classify all the matter and the energy content of the universe into three parts - the visible matter which exists in the form of atoms, the invisible matter which goes by the name *dark matter* and a *repulsive energy density* responsible for the expansion of the universe which goes by the popular name *dark energy*. Further, SM assumes that neutrinos are massless, but, recently it has been discovered that this is not so. So, it appears that SM is not the complete story.

Thus, despite the unparalleled success of SM for the last 4 decades, we know that SM cannot be a complete story in the wake of discoveries such as non-zero mass of neutrinos [29, 30], existence of dark matter, matter-antimatter asymmetry [31] and so on. This calls for beyond standard model physics (BSM). None of the BSM models have so far been been vindicated by experiments. However, a common framework that is employed in formulation of SM or BSM is the quantum field theory

[1]The dark energy can be thought of as a constant that appears in the Einstein equations without requiring any further justification.

[2]Image taken from https://wmap.gsfc.nasa.gov/universe/uni_matter.html

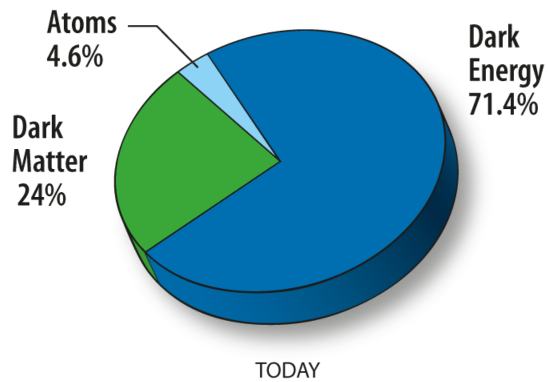


Figure 1.1: The present composition of the universe as of Today (defined as January 2013).

(QFT). There are various kinds of QFT possible. The QFTs that have been found to be useful for describing our universe so far are *gauge theories*. The specific gauge theory that has been used to formulate the standard model of physics has the gauge symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$ [18–21]. QFTs provide a very elegant formalism for the description of particles and interactions among them. In particular in gauge theories, the gauge particles are interpreted as the force carriers. The matter particle interact via exchange of the gauge particles which act as the *force* carriers. The particle and anti-particle pairs get annihilated into gauge bosons and can then reemerge as different particles and anti-particles. Figure 1.2 shows illustrations of these processes.

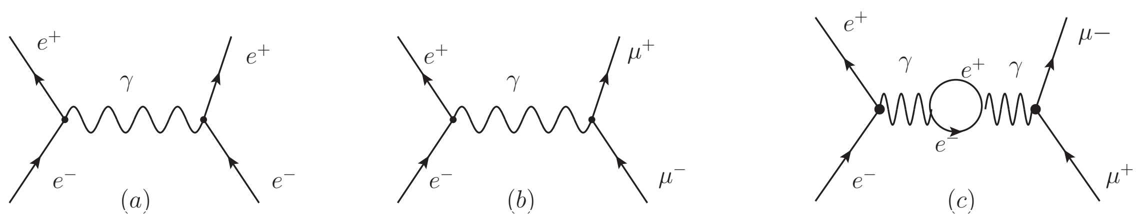


Figure 1.2: Some processes in SM. In (a) two an electron (e^-) and positron (e^+) interact via exchange of a photon (γ). Fig. (b) shows how an electron-positron pair gets converted into a muon and anti-muon pair at the tree level and figure (b) shows the one loop contribution to this process in the quantum theory.

As given by Einstein, general relativity is a classical theory. A classical theory differs from a quantum theory in a very fundamental way. All the observables in a classical theory can be measured with arbitrary precision *in principle*. In actual measurements there is always uncertainty. But this comes because of the resolution/limitation of the experimental setup. The uncertainties can be reduced by making use of better apparatus. In a quantum theory the canonically conjugate observables cannot be measured to arbitrary precision. They must satisfy the Heisenberg uncertainty relations. For example in non-relativistic quantum mechanics in one dimension, the position and the momentum satisfy

$$\sigma_x \, \sigma_p \geq \frac{\hbar}{2} \tag{1.0.1}$$

where, $\sigma_O := \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}$ is the standard deviation for any operator O and $\hbar$ is the reduced Planck constant $\frac{h}{2\pi}$. QFTs give a framework which coherently takes care of the principles of quantum mechanics and finiteness of speed of light in special relativity. Standard model is a QFT in the flat spacetime. In a flat spacetime the gravitation effects are absent. However, there are scenarios where we need to formulate quantum mechanics in a curved background i.e. in presence of gravity. When the gravitational effects are small, we can describe quite well quantum mechanics. However, when the gravity is strong the classical description of gravity itself will breakdown (see [32] and references therein). Such scenarios do occur in nature, for instance in the interior of the black-holes. This calls for the description of GR relativity as a quantum theory.

Given the success of QFTs its quite natural to ask if general relativity can be formulated in the language of a QFT. Any naive attempts to do so lead to a non-renormalizable QFT. Prior to Wilson, non-renormalizable theories were rejected on the premise that they cannot lead to sensible predictions. But, after Wilson [33,34] we know that we can make use of non-renormalizable theories to make predictions in the form of effective field theories. However, if one is aiming at the perfect sensibility of a non-renormalizable theory at all length scales, its not possible. So, its seems we must look for better methods to understand a quantum theory of gravity.

If we could formulate a quantum theory of gravity then the starting question of thesis can have a provisional answer as follows - there are certain particles called matter particles and they interact via exchange of the gauge particles. The gauge particle for the gravity will be called graviton. However, the question still remains - what are these matter and gauge particles made up of? What are the electrons, photons, quarks, gluons etc (currently known fundamental particle) made up of? One answer can be that these are the fundamental particles and they are not made up of anything more fundamental. It might be the case that this interpretation is indeed true, but, given the fact that we know that neither SM nor GR explain everything, we must look for alternatives to explain the questions that remain unanswered.

String theory is a theory based on an idea that all of the particles that we regard as fundamental are nothing but *fundamental string* in its various states of vibrations. The standard view taken in formulation of SM for example is that particles are zero dimensional objects - points. This is contrasted in figure 1.3. If one formulates a quantum theory of strings, that contains fermions and bosons in a mathematically consistent manner, one is lead to superstrings. Not only does string theory unify all the interactions and particles (all matter and gauge particles are nothing but strings), it also offers a quantum description of gravity. This is because a spin 2 massless particle, the graviton, is present in its spectrum. In fact Einsteins equation naturally arises in string theory. It further predicts various corrections to Einstein theory. String theory, if correct can be an answer to the question - what is everything made up of?

However, so far string theory has not made any direct or indirect contact with experiments, nor has it been shown that SM emerges out of string theory. The reason for this has been that even though we live in four dimensional spacetime, superstring theory requires for its consistency dimension of spacetime to be 10. In that case, the 6 of the dimensions must be compact. But, there are many consistent ways in which we can compactify the six spatial dimensions and this number is astonishingly

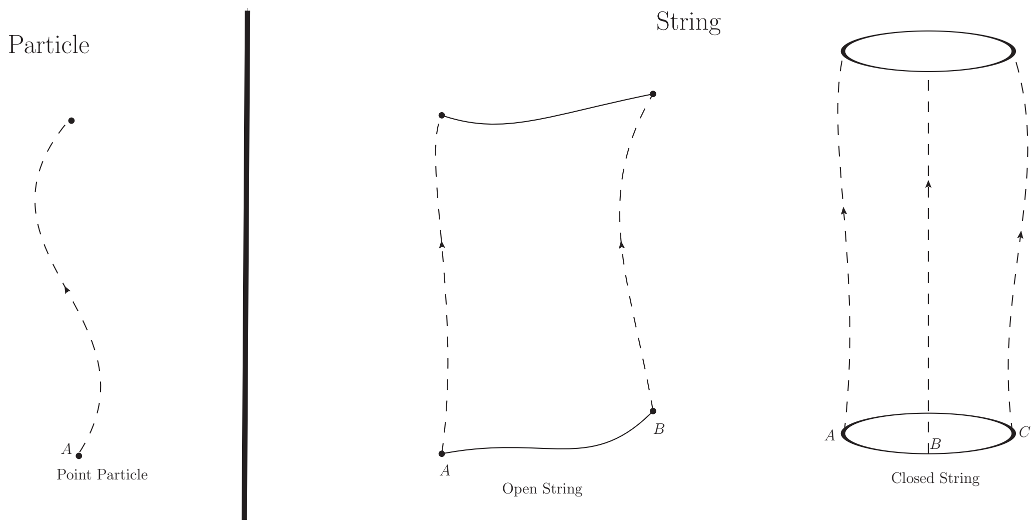


Figure 1.3: Trajectory of a point vs string. The point particle traces a *worldline* whereas a string traces a *worldsheet*. The worldsheet depends on whether the string is open or closed. Open string trace out sheets and closed string trace out cylinders when evolving in spacetime. The arrows denote the evolution in time.

large and keeps increasing (from $O(10^{500})$ [35–37] to $10^{272,000}$ in [38] for type II). The consistent set of vacua following from string theory is called *landscape*. A brute force search that involves checking each vacuum of the landscape for the SM is clearly out of question. Also, right now there does not seem to be available any analytic method to tackle this problem. Recent developments in *deep learning* might be a useful tool to make the machine learn features about the landscape and figure out the possible pockets therein where we can hope to find the physics relevant our universe [39, 40]. In this thesis we shall not be involved in studying the details of the landscape. Rather, we shall take it as a fact that the landscape exists and that it has vacua with positive, zero or negative energy densities.

1.1 String theory and its various formulations

String theory formulations come with their own set of fields. All of these share a common feature - they all can be framed in the language of conformal field theories (CFTs) in two dimensions describing the motion of a string in spacetime. Conformal group in two dimensions becomes very special in the sense that it has infinite number of generators. Since, there is conformal invariance, it makes sense to work with conformally invariant quantities. In two dimensions, the conformal symmetry is so restrictive that the functional form of all of the Greens functions is fixed to a large degree. The various formulations of string theory correspond to different worldsheet conformal theories in the sense that they are formulated in terms of fields that differ in number and conformal properties. There have been three most popular settings in which strings have been quantized (the order in which they were discovered)[3]

1. Ramond-Neveu-Schwarz (RNS) [1–3]
2. Green-Schwarz (GS) [4,5]
3. Pure spinor formalism (PS) [7]

We shall very briefly describe the first two here. The third one will form the heart of this thesis. We shall concentrate on strings in 10 dimensional flat spacetime where the theory is invariant under supersymmetry and Poincaré transformations. RNS formalism keeps the Poincaré covariance but does not have a manifest spacetime supersymmetry. GS on the other had manifest supersymmetry, but, is fails to have Lorentz covariance. Its only the pure spinor formulation that preserves the superPoincaré covariance. Let us look at the first two and at which stage they break

[3]There have been some attempts to build upon pure spinor formalism and provide and extension of it [41–45]. Further, [46] presents a covariant quantization without use of pure spinor formalism. All of these are pure spinor inspired formalisms and hence we shall not discuss them.

Lorentz/supersymmetry covariance. We shall keep on hold the conventions that we shall follow for the pure spinor part of this thesis.

1.1.1 The RNS formulation of superstring

The starting point of the RNS formalism in a 10 dimensional flat spacetime is a reparametrization invariant action given by

$$S_{RNS} = \int d\sigma d\tau \sqrt{g}\left[\frac{1}{2}g^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{1}{2}i\psi^\mu\gamma^a\nabla_a\psi^\mu + \frac{1}{2}i(\chi_a\gamma^b\gamma^a\psi^\mu)(\partial_b X^\mu - \frac{1}{4}i\chi_b\psi^\mu)\right] \quad (1.1.2)$$

where, (σ, τ) denote the worldsheet coordinates. In the above

1. $X^\mu(\sigma,\tau)$ is the string spacetime coordinate and $\psi^\mu(\sigma,\tau)$ is its worldsheet supersymmetry partner.
2. g^{ab} is a metric on the worldsheet and is a nondynamical Lagrange multiplier. g denotes the determinant of the worldsheet metric g_{ab}
3. χ^a gauges the worldsheet supersymmetry.
4. γ^a are the two dimensional gamma matrices.

The local worldsheet supersymmetry of the above action is given by

$$\begin{aligned}
\delta g_{ab} &= 2i\epsilon\gamma_{(a}\chi_{b)} \ , \\
\delta\chi_a &= 2\nabla_a\epsilon \ , \\
\delta X^\mu &= i\epsilon\psi^\mu \ , \\
\delta\psi^\mu &= \gamma^a(\partial_a X^\mu - \frac{1}{2}i\chi_a\psi^\mu)\epsilon
\end{aligned} \quad (1.1.3)$$

After choosing the covariant superconformal gauge

$$g_{ab} = \rho\delta_{ab} \ , \quad \chi_a = \gamma_a \zeta \tag{1.1.4}$$

the action 1.1.2 reduces to

$$S^{g.f}_{RNS} = \int d\sigma d\tau \left(\frac{1}{2}(\partial_a X^\mu)^2 - \frac{1}{2} i\psi^\mu \gamma^a \partial_a \psi_\mu \right) \tag{1.1.5}$$

On working out the spectrum one finds that the spectrum not only contains spacetime bosons but spacetime fermions as well. Thus, unlike the theory of bosonic strings, this theory can serve as a candidate to model realistic physical theories. The full spectrum, however leads to an inconsistent theory. The solution to this problem is to project out some states and work with the partial Hilbert space. This projection goes by the name of GSO projection [47, 48]. After the GSO projections, tachyons get removed and the resultant theories have spacetime supersymmetry[4]. The spacetime supersymmetry however is not manifest. This problem was solved by constructing the Green-Schwarz superstring as we shall very briefly explain in the next subsection.

The construction of fermionic vertex operators is somewhat involved and was solved in [51] by making use of the spin field S^α

$$S_\alpha = \prod_{j=0}^{4} e^{\pm iH_j} \tag{1.1.6}$$

where, H_j *bosonize* ψ^m

$$\psi^{2j} \pm \psi^{2j-1} = e^{\pm iH_j} \tag{1.1.7}$$

[4]There are other consistent GSO projections that do not yield spacetime supersymmetry yet give rise to consistent theories i.e. theories without tachyons and are modular invariant. More details can be found in [49, 50] and reference therein. In thesis we shall however be concerned with superstrings and hence not pursue these theories further. We thank Anirban Basu for bringing such theories to our notice

This allows one to construct the fermionic vertex operators and perform the various computations. However, since the ψ^m are worldsheet spinors, the loop amplitudes require summing over various spin structures for loop amplitudes. Because of this it can so happen that physically finite results arise only after summing over various terms that may diverge individually. Further, the formalism lacks manifest spacetime supersymmetry which leads to difficulties in quantization in curved backgrounds with Ramond-Ramond flux.

There are additional difficulties associated with spurious singularities associated with the PCOs. The PCOs can be inserted at arbitrary positions naively. But, it has been long known that a more careful analysis is much more complicated. Recently there have been been some progress in tackling this problem in the form of *vertical integration* [52, 53].

This description has been quite successful for performing simple computations and obtaining a lot of results. This formalism gave us the first taste of superstrings. However, as we increase the number of external legs and number of loops, computations become difficult especially when the external strings carry fermionic degrees of freedom.

1.1.2 The Green-Schwarz formulation of superstring

The GS formalism improves upon the difficulty in the description of spacetime fermion by introducing spacetime spinors from the very beginning. In this formalism the basic worldsheet fields are a spacetime vector $X^m(\sigma, \tau)$ and a spacetime spinor $\theta^\alpha(\sigma, \tau)$. These both are worldsheet scalars. The price one pays for this simplification is that the action is quite complicated [4]

$$S_{GS} = \frac{1}{\pi} \int d\tau \int_0^\pi d\sigma \left(L_1 + L_2\right) \tag{1.1.8}$$

where,

$$
\begin{aligned}
L_1 &= -\frac{1}{2}\sqrt{-g}g^{\alpha\beta}\Pi^\mu_\alpha\Pi_{\beta\mu}, && (1.1.9)\\
L_2 &= -i\varepsilon^{\alpha\beta}\partial_a X^\mu\left[\bar{\theta}^1\gamma_\mu\partial_\beta\theta^1-\bar{\theta}^2\gamma_\mu\partial_\beta\theta^2\right]+\varepsilon^{\alpha\beta}(\bar{\theta}^1\gamma^\mu\partial_\alpha\theta^1)(\bar{\theta}^2\gamma_\mu\partial_\beta\theta^2) && (1.1.10)
\end{aligned}
$$

where,

$$
\Pi^\mu_\alpha = \partial_\alpha X^\mu - i\bar{\theta}^A\gamma^\mu\partial_\alpha\theta^A \quad (1.1.11)
$$

In the above action

- $A = 1, 2$ is the worldsheet spinor index, $a = 1, 2, \cdots, 32$ is the spacetime spinor index in 10 dimensions.
- $\varepsilon^{\alpha\beta}$ is the two-dimensional Levi-Civita tensor density.
- X^μ with $\mu = 0, 1, 2, \cdots, 9$ is the spacetime coordinate, θ^{1a} and θ^{2a} are 32-component Majorana-Weyl spinors (and can have same or opposite chiralities)
- ρ^α_{AB} and γ^μ_{ab} are Dirac matrices satisfying $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$ and $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$. The former is in the two dimensions while the latter in ten dimensions.
- It is worth noting that equation (1.1.10) is particular to the flat space. It has a nice geometric interpretation as a pullback from the superspace to the world-sheet in any on-shell background[5].

This action is impossible to quantize covariantly, at least all the attempts have failed so far. However, in the light cone gauge (the gauge transformation that allows this can be found in [4]) the action reduces to [54]

[5]We thank one of our referees to bring this to our notice.

$$S_{GS}^{l.c.} = \int d\tau \int_0^\pi d\sigma \left(-\frac{1}{2\pi} \partial_\alpha X^i \partial^\alpha X^i + \frac{i}{4\pi} \bar{S} \gamma^- \rho^\alpha \partial_\alpha S \right) \tag{1.1.12}$$

where,

$$\bar{S}^{Aa} := S^{\dagger Bb} (\gamma^0)^{ba} (\rho^0)^{BA} \tag{1.1.13}$$

In the above action, X^i with $i = 1, 2, \cdots, 8$ represent transverse directions and $\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9)$ (similar definitions exist for all other light-cone quantities). Further the new variable S is related to θ as

$$S^{Aa} := 2\sqrt{\rho^+} \theta^{Aa} \tag{1.1.14}$$

If we look at the (1.1.12) we see that we have lost $SO(9,1)$ Lorentz covariance. Pure spinor formulation essentially builds upon the GS formalism by supplying additional sets of worldsheet fields - the pure spinor. From now on we shall not be concerned with the RNS or the GS formalism.

1.2 Organization of the thesis

The rest of the thesis is organized as follows. There are two parts I and II and their contents are

- Part I will be the chapter 2 and will present the details of the computation of the integrated massive vertex operator at the first excited level of the open string. We shall further find that the construction can be generalized to other mass levels and other types of superstrings.
- Part II which comprises chapter 3 will essentially provide a means of calculating the enhancement of the collective lifetime of a civilization in a metastable de-Sitter spacetime if the civi-

lization spreads into various casually disconnect pockets of the universe. This analysis will be done without any knowledge of the microscopic physics of our universe (i.e. the vacuum decay rate which at present is not known).

We shall close this thesis with the chapter 4 and various appendices necessary and/or useful for the discussion in the main body of the thesis.

Chapter 2

The Integrated Vertex Operator

The pure spinor formalism is a super-Poincaré covariant formalism [7, 55, 56] (for review, see [57–62]) of superstrings. This feature allows for an efficient way of computing the scattering amplitudes [55, 63–67] making computations simpler. The equivalence between the pure spinor and the other superstring formalisms has been verified in many examples [68–70].

As in the Ramond-Neveu-Schwarz (RNS) formalism, the scattering amplitudes in the pure spinor formalism also involve computing worldsheet correlation functions of unintegrated as well as integrated vertex operators. However, unlike the RNS formalism, the gauge fixed worldsheet action in the pure spinor formalism does not arise from the gauge fixing of a reparametrization invariant action. Due to this, there is no elementary b ghost in the pure spinor formalism. This makes the relation between the unintegrated and the integrated vertex operators in the pure spinor formalism less direct. So, even though the computation of amplitudes are easier to carry out in pure spinor formalism, the construction of the vertex operators (integrated as well as unintegrated) are considerably more involved as compared to the RNS formalism (see e.g., [71, 72]). In this chapter, we propose an ansatz for the integrated vertex and explicitly show that it satisfies the relevant BRST condition demonstrating that it is the correct integrated vertex operator. We shall also give the arguments as to how to arrive at the ansatz. This chapter gives an explicit construction of the integrated vertex for the first massive states in open superstrings and is based on our paper [8]. The same method can also be used for the construction of the massive integrated as well as unintegrated vertex operators in the pure spinor formalism.

Restricting to the open strings for simplicity, the vertex operators in pure spinor formalism in ten dimensional flat spacetime are constructed in the super-Poincaré covariant manner using $\mathcal{N} = 1$ superfields. In particular, to construct the unintegrated vertex operator V for the states at mass level n, i.e. for $m^2 = \frac{n}{\alpha'}$, one first needs to construct "basis elements" with ghost number 1 and conformal weight n using the world sheet pure spinor fields. These basis elements[1] are then multiplied with an arbitrary 10 dimensional $\mathcal{N} = 1$ superfield. The unintegrated vertex operator is the most general linear combination of such objects. The superfields appearing in this unintegrated vertex operator are fixed using the on-shell condition $QV = 0$, where Q is the BRST operator of the theory. The integrated vertex operator U can then be determined using the relation $QU = \partial_{\mathbb{R}} V$ where the subscript $\mathbb{R}$ in the right hand side denotes the fact that the derivative is taken along the real axis.

For the massless states, both the unintegrated as well as the integrated vertex operators are explicitly known. This allows us to calculate tree as well as loop amplitudes involving massless states in the pure spinor formalism. In this chapter, we shall focus on the first massive states. The open string spectrum at first massive level comprises of 128 bosonic and 128 fermionic degrees of freedom. These states form a massive spin 2 supermultiplet of 10 dimensional $\mathcal{N} = 1$ supersymmetry. The 128 fermionic degrees of freedom are encoded in a spin $3/2$ field $\psi_{m\alpha}$. On the other hand, the 128 bosonic degrees of freedom are encoded in a 3-form field b_{mnp} carrying 84 degrees of freedom

[1] We shall refer to the products of worldsheet pure spinor variables which appear in the vertex operators, multiplied by some superfield, as basis elements. So, e.g., $\partial\theta^\beta\lambda^\alpha$ in equation (2.1.21) will be referred as basis element which multiplies the superfield $B_{\alpha\beta}$.

and a symmetric traceless field g_{mn} carrying 44 degrees of freedom (see, e.g. [73]).

To describe the first massive states in a super-Poincaré covariant manner, we introduce three basic superfields $\Psi_{m\alpha}, B_{mnp}$ and G_{mn} whose theta independent components are $\psi_{m\alpha}, b_{mnp}$ and g_{mn} respectively. The higher theta components of these superfields contain the same physical fields in a more involved manner [74]. The unintegrated vertex operator describing these states was constructed in [9] and its theta expansion was done in [74]. The superfields appearing in this vertex operator can be expressed in terms of the basic superfields G_{mn}, B_{mnp} or $\Psi_{m\alpha}$.

In this chapter, our goal will be to construct the integrated form of the vertex operator for the first massive states. We shall use the defining relation $QU = \partial_{\mathbb{R}} V$ for this purpose. As we shall see, the superfields appearing in U can also be expressed in terms of the basic superfields $\Psi_{m\alpha}, B_{mnp}$ and G_{mn}.

Rest of the chapter is organized as follows. In section 2.1, we briefly review some of the elements of the pure spinor formalism and the first massive unintegrated vertex operator which are used in our analysis. In section 2.2, we give our general strategy and the main results of this chapter. The equations (2.2.35) and (2.2.36) are our main equations which give the first massive integrated vertex operator in terms of the basic superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$. In section 2.3, we give the details of our construction following the strategy given in section 2.2. Finally, we conclude with discussion in section 2.4. While our ansatz once verified to be a solution does not require any further justification, we summarize the chain of reasoning in the appendix C that led us to our proposed ansatz. Even though the solution does not depend on how we arrive at this ansatz, the arguments presented in the appendix C are nonetheless of value since they imply that one can replicate the same method quite readily for all higher massive states.

2.1 Review of some pure spinor elements

In this section, we briefly recall some of the results of the minimal pure spinor formalism and the first massive states which will be needed in our analysis. We shall also describe some results regarding open strings which will be needed in this chapter.

2.1.1 Some pure spinor results

We start by recalling some results about the open string world-sheet theory in the pure spinor formalism. We shall follow the conventions used in [74]. The open string world-sheet CFT in the pure spinor formalism in flat spacetime is described by the action

$$S = \frac{1}{\pi\alpha'}\int_{UHP} d^2z\left(\frac{1}{2}\partial X^m\bar{\partial}X_m + p^L_\alpha\bar{\partial}\theta^\alpha_L - w^L_\alpha\bar{\partial}\lambda^\alpha_L + p^R_\alpha\partial\theta^\alpha_R - w^R_\alpha\partial\lambda^\alpha_R\right) \tag{2.1.1}$$

where, $m = 0, 1, , \cdots , 9$ and $\alpha = 1, \cdots , 16$. Further, we use the acronym UHP and LHP for upper and lower half of the complex plane. The L and R denote the left and right moving fields respectively on the world-sheet which will be related through the boundary conditions. All the worldsheet fields $X^m, p^L_\alpha, w^L_\alpha, \theta^\alpha_L$ and λ^α_L and the corresponding right moving fields (with script R) are function of both $(z, \bar{z})$ off-shell. However, on making use of the equations of motion, namely

$$\begin{gathered}\bar{\partial}\partial X^m(z,\bar{z}) = 0\\ \bar{\partial}\theta^\alpha_L(z,\bar{z}) = 0 \quad , \quad \partial\theta^\alpha_R(z,\bar{z}) = 0 \quad , \quad \bar{\partial}p^L_\alpha(z,\bar{z}) = 0 \quad , \quad \partial p^R_\alpha(z,\bar{z}) = 0\\ \bar{\partial}\lambda^\alpha_L(z,\bar{z}) = 0 \quad , \quad \partial\lambda^\alpha_R(z,\bar{z}) = 0 \quad , \quad \bar{\partial}w^L_\alpha(z,\bar{z}) = 0 \quad , \quad \partial w^R_\alpha(z,\bar{z}) = 0,\end{gathered} \tag{2.1.2}$$

we find that the fields with subscript L and R become holomorphic and anti-holomorphic respectively. The X^m fields satisfy the harmonic equation and hence it can be written as sum of holomorphic and anti-holomorphic fields. This means that ∂X^m and $\bar{\partial}X^m$ are holomorphic and anti-holomorphic respectively. Besides the above equations of motion, we have to impose appropriate boundary conditions. These boundary conditions for the open strings are

$$\begin{aligned}
\partial X^m(z,\bar{z}) &= \bar{\partial} X^m(z,\bar{z}) \\
\theta_L^\alpha(z,\bar{z}) &= \theta_R^\alpha(z,\bar{z}) \\
p_\alpha^L(z,\bar{z}) &= p_\alpha^R(z,\bar{z})\ , \quad \text{at } z=\bar{z} \\
\lambda_L^\alpha(z,\bar{z}) &= \lambda_R^\alpha(z,\bar{z}) \\
w_\alpha^L(z,\bar{z}) &= w_\alpha^R(z,\bar{z})
\end{aligned} \tag{2.1.3}$$

Taking these boundary conditions into account and using the action (2.1.1), we can derive various OPEs. The OPE between the various matter sector fields can be worked out to be

$$\begin{aligned}
\partial X^m(z,\bar{z})\partial X^n(w,\bar{w}) &= -\frac{\alpha'\eta^{mn}}{2(z-w)^2} + \cdots \\
p_\alpha^L(z,\bar{z})\theta_L^\beta(w,\bar{w}) &= \frac{\alpha'}{2}\frac{\delta_\alpha{}^\beta}{z-w} + \cdots \\
p_\alpha^R(\bar{z},z)\theta_R^\beta(\bar{w},w) &= \frac{\alpha'}{2}\frac{\delta_\alpha{}^\beta}{\bar{z}-\bar{w}} + \cdots
\end{aligned} \tag{2.1.4}$$

It is cumbersome to work with both left and right moving fields and impose the boundary conditions each time. Fortunately, using the "doubling trick", we can combine the left and right moving fields into a single field. The left and right moving fields considered so far are defined only in the upper half plane with their values agreeing on the real axis. Using the doubling trick, we construct a field defined in the whole complex plane. Moreover, this requires only the boundary conditions and not the on-shell conditions following from the equations of motion. For example, the boundary condition

(2.1.3) allows us to combine $\theta^\alpha_L(z,\bar{z})$ and $\theta^\alpha_R(\bar{z},z)$ into a single field as

$$\theta^\alpha(z,\bar{z}) \equiv \begin{cases} \theta^\alpha_L(z,\bar{z}) & \text{for } z \;\in\; UHP \\ \theta^\alpha_R(\bar{z},z) & \text{for } z \;\in\; LHP \end{cases} \tag{2.1.5}$$

We can similarly define p_α, w_α and λ^α in the whole complex plane. Furthermore, all of the holomorphically factorized quantities such as the vertex operators and the stress tensor can be defined in a similar manner. The θ^α as defined in (2.1.5) is holomorphic in the whole complex plane. It is instructive to see this explicitly. For this, we need to show that $\bar{\partial}\theta^\alpha = 0$ for $z \in \mathbb{C}$. For $z \in UHP$, we have

$$\bar{\partial}\theta^\alpha(z,\bar{z})|_{UHP} = \bar{\partial}\theta^\alpha_L(z,\bar{z}) = 0, \tag{2.1.6}$$

by virtue of equation of motion for θ^α_L. On the other hand, for $z \in LHP$, we have

$$\bar{\partial}\theta^\alpha(z,\bar{z})|_{LHP} = \bar{\partial}\theta^\alpha_R(\bar{z},z) = 0 \tag{2.1.7}$$

where, we have used the fact that the equation of motion for $\theta^\alpha_R(\bar{z},z)$ in (2.1.2) implies that it is independent of the first argument. This completes the proof that $\theta^\alpha(z)$ is indeed a holomorphic function in the whole complex plane. Identical proofs can also be given for other fields or their derivatives. Moreover, the OPEs involving $\theta^\alpha_{L,R}$ and $p^{L,R}_\alpha$ which follow from (2.1.5) can be combined into a single OPE as

$$p_\alpha(z)\theta^\beta(w) = \frac{\alpha'}{2}\frac{\delta_\alpha{}^\beta}{z-w} + \cdots \tag{2.1.8}$$

From now on, we shall work with the fields defined using the doubling trick. However, one can always go back to the expressions involving the original fields using equation (2.1.5) and similar relations for other fields. The worldsheet fields $p_\alpha, w_\alpha, \theta^\alpha$ and λ^α carry the conformal weights $1, 1, 0, 0$ respectively. The field λ^α satisfies the pure spinor constraint

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 \tag{2.1.9}$$

The γ^m in above equation are the 16×16 gamma matrices. The antisymmetrized product of these gamma matrices are referred as forms. So, e.g., $\gamma^{\alpha\beta}_{mnp}$ is called 3-form and so on.

The field λ^α and w_β carry the ghost numbers 1 and -1 respectively. All other worldsheet fields carry the 0 ghost number. Due to the pure spinor constraint, we shall work with the following gauge invariant combinations instead of bare w_α[2]

$$N^{mn} = \frac{1}{2} w_\alpha (\gamma^{mn})^\alpha{}_\beta \lambda^\beta \quad , \qquad J = w_\alpha \lambda^\alpha \tag{2.1.10}$$

All the components of these variables are not independent. This fact is captured by the following non-trivial constraint between the currents N_{mn} and J [9]

$$: N^{mn} \lambda^\alpha : (z) (\gamma_m)_{\alpha\beta} - \frac{1}{2} : J \lambda^\alpha : (z) (\gamma^n)_{\alpha\beta} - \alpha' \gamma^n_{\alpha\beta} \partial \lambda^\alpha (z) \ = \ 0 \tag{2.1.11}$$

Two other important supersymmetric invariant combinations of the theory are given by

$$\begin{aligned} d_\alpha &= p_\alpha - \frac{1}{2} \gamma^m{}_{\alpha\beta} \theta^\beta \partial X_m - \frac{1}{8} \gamma^m_{\alpha\beta} \gamma_{m\sigma\delta} \theta^\beta \theta^\sigma \partial \theta^\delta \\ \Pi^m &= \partial X^m + \frac{1}{2} \gamma^m_{\alpha\beta} \theta^\alpha \partial \theta^\beta \end{aligned} \tag{2.1.12}$$

[2]Here by gauge invariance, we mean invariance under $w_\alpha \to w_\alpha + \Lambda^m (\gamma_m \lambda)_\alpha$.

The BRST operator of the theory is given in terms of λ^α and d_α to be[3]

$$Q = \oint dz\ \lambda^\alpha(z) d_\alpha(z) \tag{2.1.13}$$

The BRST operator $\oint dz\lambda^\alpha d_\alpha$ is a postulate or a clever guess at best and does not follow from an action with a local worldsheet symmetry. However, one can intuitively get some justifications by closely following the construction of supersymmetric gauge theories in the superspace. One can see that pure spinor constraint leads to and is implicitly present in the superspace gauge theory. This however gives rise to massless states i.e. to superyang mills/supergravity theories. We are interested in massive states of strings in this thesis, so we shall not dwell deeper into this area as it is a research topic by itself. For further reading please refer to [75,76] and references therein.

However, we can very easily see the important role that the pure spinor constraint plays in the definition of the BRST charge above. In order to see this we recall that one of the fundamental requirements of the BRST is that it must be order two nilpotent operator i.e. $Q^2 = 0$. Let us see if/when this is it true

$$\begin{aligned} Q^2 &= \oint dw\lambda^\alpha(w)d_\beta(w) \oint dz\lambda^\beta(z)d_\beta(z) \\ &= \oint dw\lambda^\alpha(w) \oint dz\lambda^\beta(z)d_\alpha(w)d_\beta(z) \\ &= -\frac{\alpha'}{2}\oint dw\lambda^\alpha(w)(\gamma^m)_{\alpha\beta}\lambda^\beta(w)\Pi_m(w) \end{aligned} \tag{2.1.14}$$

This will be zero if λ^α has the property that $(\lambda\gamma^m\lambda) = 0$. Thus, we see that if λ^α satisfies pure spinor constraint then Q is nilpotent.

[3] Having holomorphic fields defined in the whole complex plane using doubling trick means that we can use the closed contour integrals $\oint$ in the usual manner even for the open string.

The OPE between various worldsheet operators is given by[4]

$$d_\alpha(z)d_\beta(w) = -\frac{\alpha'\gamma^m_{\alpha\beta}}{2(z-w)}\Pi_m(w) + \cdots \quad , \qquad d_\alpha(z)\Pi^m(w) = \frac{\alpha'\gamma^m_{\alpha\beta}}{2(z-w)}\partial\theta^\beta(w) + \cdots$$

$$d_\alpha(z)V(w) = \frac{\alpha'}{2(z-w)}D_\alpha V(w) + \cdots \quad , \qquad \Pi^m(z)V(w) = -\frac{\alpha'}{(z-w)}\partial^m V(w) + \cdots$$

$$\Pi^m(z)\Pi^n(w) = -\frac{\alpha'\eta^{mn}}{2(z-w)^2} + \cdots \quad , \qquad N^{mn}(z)\lambda^\alpha(w) = \frac{\alpha'(\gamma^{mn})^\alpha{}_\beta}{4(z-w)}\,\lambda^\beta(w) + \cdots$$

$$J(z)J(w) = -\frac{(\alpha')^2}{(z-w)^2} + \cdots \quad , \qquad J(z)\lambda^\alpha(w) = \frac{\alpha'}{2(z-w)}\lambda^\alpha(w) + \cdots$$

$$N^{mn}(z)N^{pq}(w) = -\frac{3(\alpha')^2}{2(z-w)^2}\eta^{m[q}\eta^{p]n} - \frac{\alpha'}{(z-w)}\Big(\eta^{p[n}N^{m]q} - \eta^{q[n}N^{m]p}\Big) + \cdots \qquad (2.1.15)$$

In the above OPEs, ∂_m is the derivative with respect to the spacetime coordinate X^m, ∂ is the derivative with respect to the world-sheet coordinate, V denotes an arbitrary superfield and D_α is the supercovariant derivative given by

$$D_\alpha \equiv \partial_\alpha + \gamma^m_{\alpha\beta}\theta^\beta\partial_m \qquad (2.1.16)$$

This supercovariant derivative satisfies the identity

$$\{D_\alpha, D_\beta\} = 2(\gamma^m)_{\alpha\beta}\partial_m \quad \Longrightarrow \quad (\gamma_m)^{\alpha\beta}D_\alpha D_\beta = \frac{1}{16}\partial_m \qquad (2.1.17)$$

[4]Note the minus sign in front of the single pole in $N^{mn}N^{pq}$ OPE. There is a typo regarding this sign in [9]. We thank Nathan Berkovits for confirming this.

The matter and the ghost stress energy tensors of the theory are given by

$$T_m = -\frac{1}{\alpha'} : \Pi^m \Pi_m : \; - \frac{2}{\alpha'} : d_\alpha \partial\theta^\alpha : \qquad , \quad T_g = \frac{2}{\alpha'} w_\alpha \partial\lambda^\alpha \tag{2.1.18}$$

The total stress tensor T is given by the sum of T_m and T_g[5]. The Lorentz current N^{mn} is a primary operator with respect to the stress energy tensor. This follows due to the OPE

$$T_g(z) N^{mn}(w) = \frac{N^{mn}(w)}{(z-w)^2} + \frac{\partial N^{mn}(w)}{z-w} + \cdots \tag{2.1.19}$$

and the fact that the matter and the ghost sector fields do not have any non trivial OPE between them.

After briefly reviewing the basics, we now turn to the first massive unintegrated vertex operator [9, 74]. There are 128 fermionic and 128 bosonic degrees of freedom at the first massive level of the open string spectrum. The fermionic degrees of freedom are contained in a spin-$3/2$ field $\psi_{m\alpha}$ whereas the bosonic degrees of freedom are contained in a traceless symmetric tensor g_{mn} and a 3-form field b_{mnp}. These fields are demanded to satisfy

$$\partial^m \psi_{m\alpha} = 0 \quad ; \quad \gamma^{m\alpha\beta}\psi_{m\beta} = 0 \quad ; \quad \partial^m b_{mnp} = 0 \quad ; \quad \eta^{mn} g_{mn} = 0 \quad ; \quad \partial^m g_{mn} = 0 \tag{2.1.20}$$

These constraints ensure that the number of independent components in the fields $\psi_{m\beta}, b_{mnp}$ and g_{mn} are $128, 84$ and 44 respectively. These fields form a massive spin-2 supermultiplet in 10 dimensions. To describe the system in a supersymmetric invariant manner, we introduce basic superfields $\Psi_{m\alpha}, B_{mnp}$ and G_{mn} whose theta independent components are $\psi_{m\alpha}, b_{mnp}$ and g_{mn} respectively. The higher components of these basic superfields contain the same physical fields in a more involved

[5] It is possible to express the ghost stress tensor T_g in terms of the currents N^{mn} and J (see, e.g., [58]). However, we shall not need this expression. For our purposes, equation (2.1.19) is sufficient.

manner.

At the first mass level, the unintegrated vertex operator of the open string is given by [9]

$$V = :\partial\theta^\beta\lambda^\alpha B_{\alpha\beta}: + :d_\beta\lambda^\alpha C^\beta_{\ \alpha}: + :\Pi^m\lambda^\alpha H_{m\alpha}: + :N^{mn}\lambda^\alpha F_{\alpha mn}: \tag{2.1.21}$$

where, the superfields appearing in the above expression are given in terms of the basic superfields B_{mnp} and $\Psi_{m\alpha}$ to be [9]

$$H_{s\alpha} = \frac{3}{7}(\gamma^{mn})_\alpha^{\ \beta} D_\beta B_{mns} = -72\Psi_{s\alpha}\ , \quad C_{mnpq} = \frac{1}{2}\partial_{[m}B_{npq]}\ ,$$
$$F_{\alpha mn} = \frac{1}{8}\left(7\partial_{[m}H_{n]\alpha} + \partial^q(\gamma_{q[m})_\alpha^{\ \beta}H_{n]\beta}\right) \tag{2.1.22}$$

The normal ordering : : is defined as

$$:AB:(z) \equiv \frac{1}{2\pi i}\oint_z \frac{dw}{w-z}A(w)B(z) \tag{2.1.23}$$

where, A and B are any two operators and the contour surrounds the point z.

The basic superfields at the first massive level, namely, $B_{mnp}, \Psi_{m\alpha}$ and G_{mn} satisfy the superspace equations[6] [74]

$$D_\alpha G_{sm} = 16\ \partial^p(\gamma_{p(s}\Psi_{m)})_\alpha \tag{2.1.24}$$

$$D_\alpha B_{mnp} = 12(\gamma_{[mn}\Psi_{p]})_\alpha - 24\alpha'\partial^t\partial_{[m}(\gamma_{|t|n}\Psi_{p]})_\alpha \tag{2.1.25}$$

[6]To go from position to momentum space and vice versa, we use the convention $\partial_m \to ik_m$ and $k_m \to -i\partial_m$. We shall do calculations mostly in the momentum space but express the final result in the position space using this rule.

$$D_\alpha \Psi_{s\beta} = \frac{1}{16} G_{sm} \gamma^m_{\alpha\beta} + \frac{1}{24} \partial_m B_{nps} (\gamma^{mnp})_{\alpha\beta} - \frac{1}{144} \partial^m B^{npq} (\gamma_{smnpq})_{\alpha\beta} \tag{2.1.26}$$

and the constraints

$$(\gamma^m)^{\alpha\beta} \Psi_{m\beta} = 0 \quad ; \quad \partial^m \Psi_{m\beta} = 0 \quad ; \quad \partial^m B_{mnp} = 0 \quad ; \quad \partial^m G_{mn} = 0 \quad ; \quad \eta^{mn} G_{mn} = 0 \tag{2.1.27}$$

2.1.2 Some results regarding open strings

For the open strings, the vertex operators live on the boundary, i.e., on the real axis in the complex plane. This means that in the BRST equation $QU = \partial_{\mathbb{R}} V$, the derivative in the right hand side is along the real axis (represented by the subscript $\mathbb{R}$. For comparing the left and right hand side of this equation, we shall need to express the partial derivative in the right hand side to derivative with respect to the world-sheet fields X^m and θ^α. For this, we first convert the derivative along the real axis into the holomorphic derivative as follows. If x denotes the coordinate along the real axis, then the derivative of an arbitrary function f along the real axis can be written as[7]

$$\partial_{\mathbb{R}} f = \frac{\partial f}{\partial x} = \left(\frac{\partial z}{\partial x} \frac{\partial f}{\partial z} + \frac{\partial \bar{z}}{\partial x} \frac{\partial f}{\partial \bar{z}} \right) \Big|_{\bar{z}=z} = \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \right) \Big|_{\bar{z}=z} \tag{2.1.28}$$

Now, for the open strings, the left and right moving fields living on the world-sheet are identified along the real axis as in (2.1.3). Thus, any function along the real axis (such as the vertex operator) can be expressed only in terms of either left moving or right moving fields (or the fields defined using the doubling trick as in (2.1.5)). Working with the left moving fields, we can use the chain rule to write

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial X^m} \frac{\partial X^m}{\partial z} + \frac{\partial f}{\partial \theta^\alpha} \frac{\partial \theta^\alpha}{\partial z} \quad , \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial X^m} \frac{\partial X^m}{\partial \bar{z}} + \frac{\partial f}{\partial \theta^\alpha} \frac{\partial \theta^\alpha}{\partial \bar{z}} \tag{2.1.29}$$

[7] We define $z = x + iy$ and $\bar{z} = x - iy$ with $x, y \in \mathbb{R}$.

Using the equation of motion for p_α, namely, $\bar\partial\theta^\alpha = 0$ and the above equations, we obtain

$$
\begin{aligned}
\partial_{\mathbb{R}} f &= \frac{\partial f}{\partial X^m}\frac{\partial X^m}{\partial z} + \frac{\partial f}{\partial X^m}\frac{\partial X^m}{\partial \bar z}\Big|_{\bar z = z} + \frac{\partial f}{\partial\theta^\alpha}\frac{\partial\theta^\alpha}{\partial z} \\
&= 2\frac{\partial f}{\partial X^m}\frac{\partial X^m}{\partial z} + \frac{\partial f}{\partial\theta^\alpha}\frac{\partial\theta^\alpha}{\partial z}
\end{aligned}
\tag{2.1.30}
$$

Now, for the left moving fields, the SUSY momenta and the supercovariant derivatives are given by

$$
\Pi^m_L = \partial X^m + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\alpha\partial\theta^\beta \quad , \qquad D^L_\alpha = \frac{\partial}{\partial\theta^\alpha} + \gamma^m_{\alpha\beta}\theta^\beta_L\partial_m \tag{2.1.31}
$$

Using these, we obtain

$$
\partial_{\mathbb{R}} f = 2\Pi^m_L\partial_m f + \partial\theta^\alpha D^L_\alpha f \tag{2.1.32}
$$

If we had worked with the right moving fields, instead of the above equation, we would have obtained

$$
\partial_{\mathbb{R}} f = 2\Pi^m_R\partial_m f + \bar\partial\tilde\theta^\alpha D^R_\alpha f \tag{2.1.33}
$$

where, Π^m_R and D^R_α are given by definitions similar to (2.1.31) but with θ^α and ∂ replaced by $\tilde\theta^\alpha$ and $\bar\partial$ respectively.

Since we are on the real axis, we can replace the left moving variables of (2.1.32) or the right moving variables of (2.1.33) in terms of fields defined on the whole complex plane using the doubling trick. Doing this, we obtain

$$
\partial_{\mathbb{R}} f = 2\Pi^m\partial_m f + \partial\theta^\alpha D_\alpha f \tag{2.1.34}
$$

where Π^m and D_α are given in (2.1.12) and (2.1.16) respectively.

We shall make use of the identity (2.1.34) while computing the right hand side of the BRST equation $QU = \partial_{\mathbb{R}} V$. Moreover, throughout the draft, the world-sheet derivatives ∂ will denote the holomorphic derivative. In the places where it is derivative along the real axis (e.g., the right hand side of $QU = \partial_{\mathbb{R}} V$), it can be easily converted to the holomorphic derivative using the identity (2.1.28).

2.2 General strategy and the main result

The integrated massive vertex U is constructed following a series of steps which can be summarized quite succinctly. In this section, we give the general strategy as a series of steps while the subsequent section will provide the details of these steps. First let us state our goal clearly. All vertex operators (integrated or unintegrated) are schematically of the form $\hat{O}A$, where $\hat{O}$ is a worldsheet operator of appropriate conformal weight and ghost number constructed out of ($\Pi^m, d_\alpha, \partial\theta^\alpha, N^{mn}, J, \lambda^\alpha$) and their worldsheet derivatives and A is a superfield whose tensor-spinor structure is such that $\hat{O}A$ is Lorentz invariant. As mentioned in footnote 1, the operators $\hat{O}$ will be referred as basis elements. We know the expression of the unintegrated vertex (2.1.21) in terms of the superfields which describe the massive supermultiplet (i.e. any one of $\Psi_{s\alpha}, B_{mnp}$ or G_{mn}). Our goal is to find U in terms of the same superfields describing the massive multiplet such that it satisfies $QU = \partial_{\mathbb{R}} V$.

The steps for the construction of the first massive vertex operator U are as follows :

- **Step 1 :** Write all possible worldsheet operators with conformal weight 2 and ghost number zero using $\Pi^m, d_\alpha, \partial\theta^\alpha, N^{mn}, J, \lambda^\alpha$, noting that worldsheet derivative (denoted by ∂) can increase the weight of any operator on worldsheet by 1. Contract each of these operators by an arbitrary superfield with appropriate index structure to obtain a Lorentz invariant combination. The most general U is the sum of all these possible terms.

- **Step 2 :** Compute QU using the OPEs given in (2.1.15). Also compute the worldsheet derivative $\partial_{\mathbb{R}} V$ of the unintegrated vertex operator.

- **Step 3 :** The pure spinor constraint (2.1.9) and the OPEs (2.1.15) imply several non trivial identities relating a specific subset of the basis operators of a given conformal weight and ghost number. List all such identities and express them in the form $I = 0$.

- **Step 4 :** To take into account the constraint identities, introduce Lagrange multipliers and set up the equation $QU - \partial_{\mathbb{R}} V - IK = 0$ (where K denotes the Lagrange multiplier). The inclusion of I ensures that all operator basis constructed in step 1 now can be treated as linearly independent. Instead of introducing the Lagrange multipliers, one can also directly eliminate some basis operators in favor of others.

- **Step 5 :** Express each of the arbitrary superfields in U as a generic linear combination of $\Psi_{m\alpha}, B_{mnp}$ and G_{mn} and their space time derivatives. The correct number of terms in each ansatz can be determined by using the representation theory of SO(9) which is the little group for the massive states in 10 dimensions. The number of times $\Psi_{m\alpha}, B_{mnp}$ and G_{mn} will appear in a given ansatz is same as the number of **128**, **84** and **44** representations of SO(9) respectively in the superfield. This can be figured out by analyzing the index structure of the superfield in the rest frame.

- **Step 6 :** Substitute the ansatz of step 5 in the equations obtained in step 4. These lead to a set of linear algebraic equations for the unknown co-efficients appearing in the ansatz.

- **Step 7 :** Solve these linear equations. Plugging the solutions back allows us to express U completely in terms of the superfields that describe the massive supermultiplet.

Following this procedure, the final form of the first massive integrated vertex operator is obtained to be

$$
\begin{aligned}
U \;\; = \;\; & :\Pi^m \Pi^n F_{mn}: \; + \; :\Pi^m d_\alpha F_m{}^\alpha: \; + \; :\Pi^m \partial\theta^\alpha G_{m\alpha}: \; + \; :\Pi^m N^{pq} F_{mpq}: \\
& + \; :d_\alpha d_\beta K^{\alpha\beta}: \; + \; :d_\alpha \partial\theta^\beta F^\alpha{}_\beta: \; + \; :d_\alpha N^{mn} G^\alpha{}_{mn}: \; + \; :\partial\theta^\alpha \partial\theta^\beta H_{\alpha\beta}: \\
& + \; :\partial\theta^\alpha N^{mn} H_{mn\alpha}: \; + \; :N^{mn} N^{pq} G_{mnpq}: \qquad (2.2.35)
\end{aligned}
$$

where, the superfields appearing in (2.2.35) are given in position space by

$$
\begin{aligned}
F_{mn} &= -\frac{18}{\alpha'} G_{mn} \quad , \quad F_m{}^\alpha = \frac{288}{\alpha'} (\gamma^r)^{\alpha\beta} \partial_r \Psi_{m\beta} \quad , \quad G_{m\alpha} = -\frac{432}{\alpha'} \Psi_{m\alpha} \\
F_{mpq} &= \frac{12}{(\alpha')^2} B_{mpq} - \frac{36}{\alpha'} \partial_{[p} G_{q]m} \quad , \quad K^{\alpha\beta} = -\frac{1}{(\alpha')^2} \gamma^{\alpha\beta}_{mnp} B^{mnp} \\
F^\alpha{}_\beta &= -\frac{4}{\alpha'} (\gamma^{mnpq})^\alpha{}_\beta \partial_m B_{npq} \quad , \quad G^\alpha_{mn} = \frac{48}{(\alpha')^2} \gamma^{\alpha\sigma}_{[m} \Psi_{n]\sigma} + \frac{192}{\alpha'} \gamma^{\alpha\sigma}_r \partial^r \partial_{[m} \Psi_{n]\sigma} \\
H_{\alpha\beta} &= \frac{2}{\alpha'} \gamma^{mnp}_{\alpha\beta} B_{mnp} \quad , \quad H_{mn\alpha} = -\frac{576}{\alpha'} \partial_{[m} \Psi_{n]\alpha} - \frac{144}{\alpha'} \partial^q (\gamma_{q[m})_\alpha{}^\sigma \Psi_{n]\sigma} \\
G_{mnpq} &= \frac{4}{(\alpha')^2} \partial_{[m} B_{n]pq} + \frac{4}{(\alpha')^2} \partial_{[p} B_{q]mn} - \frac{12}{\alpha'} \partial_{[p} \partial_{[m} G_{n]q]} \qquad (2.2.36)
\end{aligned}
$$

It can be explicitly verified that the integrated vertex operator constructed here is a primary operator with respect to the stress energy tensor of the theory[8]. The 3rd and the 4th order poles of the OPE between the total stress tensor T and the vertex operator U given in (2.2.35) vanish identically for the solution given in (2.2.36) on using the conditions (2.1.27). The full computation, on using the expression of the matter stress tensor given in (2.1.18) and the OPE between T_g and N^{mn} given in

[8]We thank Nathan Berkovits for raising this issue.

(2.1.19), gives

$$T(z)U(w) = \frac{2U(w)}{(z-w)^2} + \frac{\partial U(w)}{z-w} + \cdots \tag{2.2.37}$$

which confirms that the integrated vertex operator U is a world-sheet primary operator of conformal weight 2 with respect to the stress energy tensor.

2.3 Details of the derivation

In this section, we give the details of the procedure outlined in the previous section. To construct the integrated vertex operator for the massive states, we start by noting that the relation between the integrated and unintegrated vertex operator is given by

$$QU(z) = \partial_{\mathbb{R}} V(z) \qquad \Longrightarrow \qquad \frac{1}{2\pi i}\oint_z dw\ \lambda^\alpha(w) d_\alpha(w) U(z) = \partial_{\mathbb{R}} V(z) \tag{2.3.38}$$

We shall derive the integrated vertex by first writing down the most general form of the integrated vertex in terms of arbitrary superfields and then use the above equation to determine these superfields.

2.3.1 Ingredients of equation of motion

As mentioned earlier, the integrated vertex operator describing the physical states at mass level n, i.e., $m^2 = \frac{n}{\alpha'}$ is constructed out of objects with ghost number 0 and conformal dimension $n+1$. These Lorentz and SUSY invariant objects are constructed using the pure spinor variables $\Pi^m, \partial\theta^\alpha, d_\alpha, \lambda^\alpha, J$ and N^{mn}. Moreover, as argued in appendix C.0.2, we can choose the integrated vertex to be independnet of the $\bar{\lambda}\lambda$ factors. Consequently, the most general integrated vertex operator at first massive level ($n = 1$) of the open string can be written as[9]

$$
\begin{aligned}
U \;=\;& :\partial^2\theta^\alpha C_\alpha: \;+\; :\partial\Pi^m C_m: \;+\; :\partial d_\alpha E^\alpha: \;+\; :(\partial J)C: \;+\; :\partial N^{mn}C_{mn}: \\
&+\; :\Pi^m\Pi^n F_{mn}: \;+\; :\Pi^m d_\alpha F_m{}^\alpha: \;+\; :\Pi^m N^{pq}F_{mpq}: \;+\; :\Pi^m J F_m: \;+\; :\Pi^m\partial\theta^\alpha G_{m\alpha}: \\
&+\; :d_\alpha d_\beta K^{\alpha\beta}: \;+\; :d_\alpha N^{mn}G^\alpha{}_{mn}: \;+\; :d_\alpha J F^\alpha: \;+\; :d_\alpha\partial\theta^\beta F^\alpha{}_\beta: \\
&+\; :N^{mn}N^{pq}G_{mnpq}: \;+\; :N^{mn}JP_{mn}: \;+\; :N^{mn}\partial\theta^\alpha H_{mn\alpha}: \\
&+\; :JJH: \;+\; :J\partial\theta^\alpha H_\alpha: \;+\; :\partial\theta^\alpha\partial\theta^\beta H_{\alpha\beta}:
\end{aligned}
\tag{2.3.39}
$$

The terms in the first line involve derivatives of fields to produce objects of conformal weight 2. The terms in the last 4 lines involve products of fields with conformal weights 1 to produce objects of conformal weight 2. Note that the superfields contain the expansion in θ^α. Hence, there are no explicit θ^α dependent terms in the above expression.

To set up the equation of motion (2.3.38), we now need to compute QU. Before stating the result, we note that the superfields appearing in (2.3.39) must be expressible in terms of the basic superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$. Moreover, we shall argue below that the superfields whose theta independent components can't contain the physical fields b_{mnp}, g_{mn} and $\psi_{m\alpha}$ must be zero. These superfields are $C_\alpha, C_m, E^\alpha, C, C_{mn}, F_m, F^\alpha, P_{mn}, H$ and H_α. Keeping this in mind, the action of the BRST operator Q on the 10 non zero terms of (2.3.39) can be computed to be[10]

[9]Inside a normal ordering, the order of the operators matters if they have non trivial OPE between them (see e.g., chapter 6 of [77]). Hence, for comparing various expressions (e.g., LHS and RHS of $QU = \partial_{\mathbb{R}}V$), we need to have the same ordering of the world-sheet operators inside normal ordering. However, during the intermediate stages of the calculation, the operators may not occur in the same order and we need to bring them in a given fixed order. We shall use the following convention for the ordering of the world-sheet operators from left to right if more than one of them appear inside normal ordering : $\Pi^m, d_\alpha, \partial\theta^\alpha, N^{mn}, J, \lambda^\alpha$. If the operators in some terms are not in this order, we shall bring them in this order using OPEs. An example of this is given in equation (2.3.44).

[10]These computations were also checked using the Mathematica package OPEDefs [78].

1. $\underline{\Pi^m \Pi^n F_{mn}}$

$$Q\,(:\Pi^m \Pi^n F_{mn}:) \;=\; \frac{\alpha'}{2}\Big[:\Pi^m \Pi^n \lambda^\alpha D_\alpha F_{mn}: \;+\; :\Pi^m (\gamma^n_{\alpha\beta})\partial\theta^\beta \lambda^\alpha \Big(F_{mn}+F_{nm}\Big):\Big]$$

2. $\underline{\Pi^m d_\alpha F_m{}^\alpha}$

$$\begin{aligned} Q\left(:\Pi^m d_\beta F_m{}^\beta:\right) \;=\;& -\frac{\alpha'}{2}\Big[:\Pi^m d_\beta \lambda^\alpha D_\alpha F_m{}^\beta: \;+\; :d_\beta(\gamma^m_{\alpha\sigma})\partial\theta^\sigma \lambda^\alpha F_m{}^\beta: \\ & +\; :\Pi^m(\gamma^n_{\alpha\beta})\Pi_n \lambda^\alpha F_m{}^\beta:\Big] - \frac{1}{2}\left(\frac{\alpha'}{2}\right)^2 \partial^2\lambda^\alpha \gamma^m_{\alpha\sigma} F_m{}^\sigma \\ & + \frac{(\alpha')^2}{2} :\Pi^m(\gamma^n_{\alpha\beta})\partial\lambda^\alpha \partial_n F_m{}^\beta: \end{aligned}$$

3. $\underline{\Pi^m N^{pq} F_{mpq}}$

$$\begin{aligned} Q\,(:\Pi^m N^{pq} F_{mpq}:) \;=\;& \frac{\alpha'}{2}\Big[:\Pi^m N^{pq}\lambda^\alpha D_\alpha F_{mpq}: \;+\; :\partial\theta^\sigma N^{pq}(\gamma^m_{\alpha\sigma})\lambda^\alpha F_{mpq}:\Big] \\ & -\frac{\alpha'}{4}\;:\Pi^m d_\alpha (\gamma^{pq})^\alpha{}_\beta \lambda^\beta F_{mpq}: -\frac{1}{2}\left(\frac{\alpha'}{2}\right)^2 :\Pi^m \partial\lambda^\beta (\gamma^{pq})^\alpha{}_\beta D_\alpha F_{mpq}: \\ & -\frac{1}{2}\left(\frac{\alpha'}{2}\right)^2 \Big[\partial^2\theta^\sigma \lambda^\beta \gamma^m_{\alpha\sigma}(\gamma^{pq})^\alpha{}_\beta F_{mpq} + \partial\theta^\sigma \partial\lambda^\beta \gamma^m_{\alpha\sigma}(\gamma^{pq})^\alpha{}_\beta F_{mpq}\Big] \end{aligned}$$

4. $\underline{\Pi^m \partial\theta^\beta G_{m\beta}}$

$$\begin{aligned} & Q\left(:\Pi^m \partial\theta^\beta G_{m\beta}:\right) \\ & \quad = \; -\frac{\alpha'}{2} :\Pi^m \partial\theta^\beta \lambda^\alpha D_\alpha G_{m\beta}: \;+\; \frac{\alpha'}{2} :\partial\theta^\sigma \partial\theta^\beta \lambda^\alpha \gamma^m_{\alpha\sigma} G_{m\beta}: \;+\; \frac{\alpha'}{2}\; :\Pi^m \partial\lambda^\beta G_{m\beta}: \end{aligned}$$

5. $\underline{d_\alpha d_\beta K^{\alpha\beta}}$

$$
\begin{aligned}
Q\left(: d_\alpha d_\beta K^{\alpha\beta} :\right) &= \frac{\alpha'}{2} : d_\sigma d_\beta \lambda^\alpha D_\alpha K^{\sigma\beta} : - \frac{\alpha'}{2} : \Pi_m d_\beta(x) \lambda^\alpha \gamma^m_{\alpha\sigma} \left[K^{\sigma\beta}(z) - K^{\beta\sigma}\right] : \\
&\quad + \frac{\alpha'^2}{2} : d_\beta \partial\lambda^\alpha \gamma^m_{\alpha\sigma} \partial_m \left[K^{\sigma\beta} - K^{\beta\sigma}\right] : + \left(\frac{\alpha'}{2}\right)^2 \partial\theta^\delta \partial\lambda^\alpha \gamma_{m\beta\delta} \gamma^m_{\alpha\sigma} K^{\sigma\beta} \\
&\quad + \left(\frac{\alpha'}{2}\right)^2 : \gamma_{n\sigma\rho} \partial^2 \theta^\rho(x) \lambda^\alpha(z) \gamma^n_{\alpha\beta} K^{\sigma\beta}
\end{aligned}
$$

6. $\underline{d_\beta N^{mn} G^\beta_{mn}}$

$$
\begin{aligned}
& Q\left(: d_\beta N^{mn} G^\beta_{mn} :\right) \\
&\quad = \frac{\alpha'}{2}\Big[- : d_\beta N^{mn} \lambda^\alpha D_\alpha G^\beta_{mn} : - : \Pi^p N^{mn} \lambda^\alpha \gamma_{p\alpha\beta} G^\beta_{mn} : + \alpha' : N^{mn} \partial\lambda^\alpha \gamma_{p\alpha\beta} \partial^p G^\beta_{mn} : \\
&\qquad + \frac{\alpha'}{4} (\gamma_p \gamma^{mn})_{\beta\sigma} \Big(: \partial\Pi^p \lambda^\sigma G^\beta_{mn} + : \Pi^p \partial\lambda^\sigma G^\beta_{mn} : - \frac{\alpha'}{2} : \partial^2 \lambda^\sigma \partial^p G^\beta_{mn} : \Big) \\
&\qquad + \frac{(\gamma^{mn})^\alpha{}_\sigma}{2} \Big(: d_\beta d_\alpha \lambda^\sigma G^\beta_{mn} : (z) + \frac{\alpha'}{2} : d_\beta \partial\lambda^\sigma D_\alpha G^\beta_{mn} : \Big)\Big] \qquad (2.3.40)
\end{aligned}
$$

7. $\underline{d_\beta \partial\theta^\delta F^\beta{}_\delta}$

$$
\begin{aligned}
Q\left(: d_\beta \partial\theta^\delta F^\beta{}_\delta :\right) &= \frac{\alpha'}{2}\left[: d_\beta \partial\theta^\delta \lambda^\alpha D_\alpha F^\beta{}_\delta : \; - \; : d_\beta \partial\lambda^\alpha F^\beta{}_\alpha :\right] - \frac{\alpha'}{2} : \Pi_m \partial\theta^\delta \lambda^\alpha \gamma^m_{\alpha\beta} F^\beta{}_\delta : \\
&\quad + \frac{(\alpha')^2}{2} : \partial\theta^\delta \partial\lambda^\alpha \gamma^m_{\alpha\beta} \partial_m F^\beta{}_\delta :
\end{aligned}
$$

8. $\underline{N^{mn}N^{pq}G_{mnpq}}$

$$
\begin{aligned}
&Q\left(:N^{mn}N^{pq}G_{mnpq}:\right)\\
&\quad= \left(\frac{\alpha'}{4}\right)^2\left[\frac{8}{\alpha'}:N^{mn}N^{pq}\lambda^\alpha D_\alpha G_{mnpq}:-\frac{4}{\alpha'}:d_\alpha N^{pq}\lambda^\beta(\gamma^{mn})^\alpha{}_\beta G_{mnpq}:\right.\\
&\qquad -2:N^{pq}\partial\lambda^\beta(\gamma^{mn})^\alpha{}_\beta D_\alpha G_{mnpq}:+(\gamma^{mn}\gamma^{pq})^\alpha{}_\beta\left(:\partial d_\alpha\lambda^\beta G_{mnpq}:+:d_\alpha\partial\lambda^\beta G_{mnpq}:\right)\\
&\qquad -\frac{4}{\alpha'}:d_\alpha N^{mn}\lambda^\beta(\gamma^{pq})^\alpha{}_\beta G_{mnpq}:-2:N^{mn}\partial\lambda^\beta(\gamma^{pq})^\alpha{}_\beta D_\alpha G_{mnpq}:\\
&\qquad \left.+\frac{\alpha'}{4}\left(\partial^2\lambda^\beta D_\alpha(\gamma^{mn}\gamma^{pq})^\alpha{}_\beta G_{mnpq}\right)\right] \qquad (2.3.41)
\end{aligned}
$$

9. $\underline{N^{mn}\partial\theta^\beta H_{mn\beta}}$

$$
\begin{aligned}
&Q\left(:\partial\theta^\beta N^{mn}H_{mn\beta}:\right)\\
&\quad= \frac{\alpha'}{2}\left[-:\partial\theta^\beta N^{mn}\lambda^\alpha D_\alpha H_{mn\beta}:+:N^{mn}\partial\lambda^\beta H_{mn\beta}:-\frac{\alpha'}{8}:\partial^2\lambda^\alpha(\gamma^{mn})^\beta{}_\alpha H_{mn\beta}:\right.\\
&\qquad \left.-\frac{1}{2}:d_\alpha\partial\theta^\beta\lambda^\sigma(\gamma^{mn})^\alpha{}_\sigma H_{mn\beta}:+\frac{\alpha'}{4}:\partial\theta^\beta\partial\lambda^\sigma(\gamma^{mn})^\alpha{}_\sigma D_\alpha H_{mn\beta}:\right](z) \qquad (2.3.42)
\end{aligned}
$$

10. $\underline{\partial\theta^\beta\partial\theta^\delta H_{\beta\delta}}$

$$
Q\left(:\partial\theta^\beta\partial\theta^\delta H_{\beta\delta}:\right) = \frac{\alpha'}{2}\left[:\partial\theta^\beta\partial\theta^\delta\lambda^\alpha D_\alpha H_{\beta\delta}:-:\partial\theta^\beta\partial\lambda^\delta\left(H_{\beta\delta}-H_{\delta\beta}\right):\right]
$$

The BRST equation of motion also involves the world-sheet derivative of the unintegrated vertex operator, namely, $\partial_{\mathbb{R}}V$. Making use of the equation (2.1.21) and the operator identity (2.1.34), we obtain

$$
\begin{aligned}
\partial_{\mathbb{R}} V \;=\; & :\partial\theta^\beta \partial\lambda^\alpha B_{\alpha\beta}: \;+\; :\Pi^m \partial\lambda^\alpha H_{m\alpha}: \;+\; :\partial^2\theta^\alpha \lambda^\beta \left(B_{\beta\alpha} + \alpha' \gamma^m_{\sigma\alpha} \partial_m C^\sigma{}_\beta\right): \\
& + \;:\partial\theta^\beta \partial\theta^\delta \lambda^\alpha D_\delta B_{\alpha\beta}: \;+\; :\Pi^m \partial\theta^\beta \lambda^\alpha \big(2\partial_m B_{\alpha\beta} + D_\beta H_{m\alpha}\big): \;+\; :\partial d_\beta \lambda^\alpha C^\beta{}_\alpha: \\
& + \;:d_\beta \partial\lambda^\alpha C^\beta{}_\alpha: \;+\; :d_\beta \partial\theta^\sigma \lambda^\alpha D_\sigma C^\beta{}_\alpha: \;+\; :2\Pi^m d_\beta \lambda^\alpha \partial_m C^\beta{}_\alpha: \;+\; :\partial\Pi^m \lambda^\alpha H_{m\alpha}: \\
& + \;:2\Pi^m \Pi^n \lambda^\alpha \partial_n H_{m\alpha}: \;+\; :\partial N^{mn} \lambda^\alpha F_{\alpha mn}: \;+\; :N^{mn} \partial\lambda^\alpha F_{\alpha mn}: \\
& + \;:\partial\theta^\beta N^{mn} \lambda^\alpha D_\beta F_{\alpha mn}: \;+\; :2\Pi^p N^{mn} \lambda^\alpha \partial_p\, F_{\alpha mn}:
\end{aligned}
\tag{2.3.43}
$$

where, we have used

$$
:2 d_\beta \Pi^m \lambda^\alpha \partial_m C^\beta{}_\alpha: \;=\; :2\Pi^m d_\beta \lambda^\alpha \partial_m C^\beta{}_\alpha: + \alpha' :\gamma^m_{\beta\sigma} \partial^2\theta^\sigma \lambda^\alpha \partial_m C^\beta{}_\alpha: \tag{2.3.44}
$$

We now need to equate QU and $\partial_{\mathbb{R}} V$. A convenient way to do this is to compare the same basis elements in both sides. For the conformal weight 2 and ghost number 1 pure spinor objects (which appear in QU and $\partial_{\mathbb{R}} V$), naively, we have following 26 basis elements

$$
\begin{gathered}
\Pi^m \Pi^n \lambda^\alpha \,,\; \Pi^m d_\alpha \lambda^\beta \,,\; \Pi^m \partial\theta^\beta \lambda^\gamma \,,\; \Pi^m J \lambda^\alpha \,,\; \Pi^m N^{np} \lambda^\alpha \,,\; \partial\Pi^m \lambda^\alpha \,,\; \Pi^m \partial\lambda^\alpha \\
d_\alpha d_\beta \lambda^\gamma \,,\; d_\alpha \partial\theta^\beta \lambda^\gamma \,,\; d_\alpha J \lambda^\alpha \,,\; d_\alpha N^{mn} \lambda^\alpha \,,\; \partial d_\alpha \lambda^\beta \,,\; d_\alpha \partial\lambda^\beta \\
\partial\theta^\alpha \partial\theta^\beta \lambda^\gamma \;,\; \partial\theta^\alpha J \lambda^\beta \,,\; \partial\theta^\alpha N^{mn} \lambda^\alpha \,,\; \partial^2\theta^\alpha \lambda^\beta \,,\; \partial\theta^\alpha \partial\lambda^\beta \\
N^{mn} N^{pq} \lambda^\alpha \,,\; N^{mn} J \lambda^\alpha \,,\; \partial N^{mn} \lambda^\alpha \,,\; N^{mn} \partial\lambda^\alpha \\
J J \lambda^\alpha \,,\; \partial J \lambda^\alpha \,,\; J \partial\lambda^\alpha \\
\partial^2 \lambda^\alpha
\end{gathered}
\tag{2.3.45}
$$

As mentioned earlier, all of these basis elements are not independent. There are non trivial relations among some of these bases. We turn to these constraint relations between the basis elements in the next subsection.

2.3.2 Constraint identities

As mentioned in section 2.1, due to pure spinor constraint, the Lorentz current N^{mn} and the ghost current J satisfy the identity [9]

$$: N^{mn}\lambda^\alpha : (z)(\gamma_m)_{\alpha\beta} - \frac{1}{2} : J\lambda^\alpha : (z)(\gamma^n)_{\alpha\beta} - \alpha'\gamma^n_{\alpha\beta}\partial\lambda^\alpha(z) \ = \ 0 \tag{2.3.46}$$

This constraint is relevant if one is interested in the quantities involving conformal weight 1 and ghost number 1. However, in the expressions for QU and $\partial_{\mathbb{R}}V$, we encounter quantities with conformal weight 2 and ghost number 1. For this case, there are several identities which can be obtained from the above identity (2.1.11) by taking the OPE of this with the objects of conformal weight 1 and demanding the normal order terms in the OPE to vanish (the pole terms of the OPE vanish automatically as expected). Since the derivative and the normal ordering commute, the world-sheet derivative of (2.3.46) also gives a constraint. We list these constraint identities below.

$$(I_1)^n_\beta \ \equiv \ : N^{mn}J\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} : JJ\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha' : J\partial\lambda^\alpha : \gamma^n_{\alpha\beta} \ = \ 0 \tag{2.3.47}$$

$$(I_2)^{mnq}_\beta \ \equiv \ : N^{mn}N^{pq}\lambda^\alpha : (\gamma_p)_{\alpha\beta} - \frac{1}{2} : N^{mn}J\lambda^\alpha : (\gamma^q)_{\alpha\beta} - \alpha' : N^{mn}\partial\lambda^\alpha : \gamma^q_{\alpha\beta} \ = \ 0 \tag{2.3.48}$$

$$(I_3)^n_{\sigma\beta} \ \equiv \ : d_\sigma N^{mn}\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} : d_\sigma J\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha' : d_\sigma\partial\lambda^\alpha : \gamma^n_{\alpha\beta} \ = \ 0 \tag{2.3.49}$$

$$(I_4)^{pn}_\beta \ \equiv \ : \Pi^p N^{mn}\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} : \Pi^p J\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha' : \Pi^p\partial\lambda^\alpha : \gamma^n_{\alpha\beta} \ = \ 0 \tag{2.3.50}$$

$$(I_5)^{\sigma n}_\beta \ \equiv \ : \partial\theta^\sigma N^{mn}\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} : \partial\theta^\sigma J\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha' : \partial\theta^\sigma\partial\lambda^\alpha : \gamma^n_{\alpha\beta} \ = \ 0 \tag{2.3.51}$$

The above 5 identities follow from taking the OPE of (2.3.46) with the object of conformal weight one, namely $J, N^{mn}, d_\sigma, \Pi^p$ and $\partial\theta^\sigma$ respectively. The identity which can be obtained by taking the derivative of (2.3.46) is given by

$$(I_6)^n_\beta \equiv \; :\partial N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta} + :N^{mn}\partial\lambda^\alpha:(\gamma_m)_{\alpha\beta} - \frac{1}{2}:\partial J\lambda^\alpha:(\gamma^n)_{\alpha\beta} - \frac{1}{2}:J\partial\lambda^\alpha:(\gamma^n)_{\alpha\beta} - \alpha'\gamma^n_{\alpha\beta}\partial^2\lambda^\alpha = 0 \tag{2.3.52}$$

Apart from these, there are two more constraint identities which follow from the OPEs given in section 2.1. The OPE of d_α with d_β implies

$$:d_\alpha d_\beta: + :d_\beta d_\alpha: + \frac{\alpha'}{2}\partial\Pi^t(\gamma_t)_{\alpha\beta} = 0 \tag{2.3.53}$$

Similarly, the OPE of N^{mn} with N^{pq} implies

$$:N^{mn}N^{pq}: - :N^{pq}N^{mn}: = -\frac{\alpha'}{2}\Big[\eta^{np}\partial N^{mq} - \eta^{nq}\partial N^{mp} - \eta^{mp}\partial N^{nq} + \eta^{mq}\partial N^{np}\Big] \tag{2.3.54}$$

One way to think about these two identities is to note that we are working with a given ordering of the pure spinor variables inside the normal ordering. However, for $:d_\alpha d_\beta:$ and $:N^{mn}N^{pq}:$, there is no preferred ordering. The above two identities (2.3.53) and (2.3.54) are a reflection of this fact[11].

For later purpose, we multiply (2.3.53) with 5-form $\gamma^{\alpha\beta}_{mnpqr}$ to obtain

$$\gamma^{\alpha\beta}_{mnpqr}\Big(:d_\alpha d_\beta: + :d_\beta d_\alpha: + \frac{\alpha'}{2}\partial\Pi^t(\gamma_t)_{\alpha\beta}\Big) = 0 \qquad \Longrightarrow \qquad \gamma^{\alpha\beta}_{mnpqr}:d_\alpha d_\beta: = 0 \tag{2.3.55}$$

where, we have used the fact that the trace of product of 5-form and 1-form is zero and the 5-form is

[11] Note that there are OPE between Π^m and Π^n as well as between J and J. However, no pure spinor fields appear in these OPE and hence they do not lead to any non trivial constraint between basis elements.

symmetric in its spinor indices.

For solving the equations of motion, we shall need to take into account all of these constraint relations between the pure spinor variables.

2.3.3 Setting up the equations

We shall now equate QU and $\partial_{\mathbb{R}}V$ and solve the resulting equations of motion. As mentioned earlier, a convenient way to do this is to equate the terms with the same basis elements taking into account the constraint identities given above.

To take into account the constraint identities, we have two options - eliminate some basis in terms of others or introduce Lagrange multipliers. We shall make use of both of these options. We shall use the elimination method for taking care of (2.3.53) and (2.3.54) constraints. More specifically, we shall eliminate the basis involving $\partial\Pi^m$ in favour of the basis involving $d_\alpha d_\beta$ and similarly we shall eliminate the anti-symmetric part of the basis involving $N^{mn}N^{pq}$ (in simultaneous $m \leftrightarrow p$ and $n \leftrightarrow q$ exchange) in the favor of basis involving ∂N^{mn}. On the other hand, we shall introduce Lagrange multipliers for the six constraints (2.3.47)-(2.3.52) which follow from the pure spinor constraint and involve the pure spinor ghost. This means that we add a very specific zero to $QU = \partial_{\mathbb{R}}V$ equation so that we have

$$QU = \partial_{\mathbb{R}}V + \sum_{a=1}^{6} I_a K_a \tag{2.3.56}$$

The $I_a K_a$ involve contraction of the six identities (2.3.47)-(2.3.52) with appropriate Lagrange multiplier superfields. We denote these arbitrary superfields by $K_i \ \ (i = 1, \cdots 6)$. Thus,

$$\sum_{a=1}^{6} I_a K_a \equiv (I_1)^n_\beta (K_1)^\beta_n + (I_2)^{mnq}_\beta (K_2)^\beta_{mnq} + (I_3)^n_{\sigma\beta} (K_3)^{\sigma\beta}_n + (I_4)^{pn}_\beta (K_4)^\beta_{pn} + (I_5)^{\sigma n}_\beta (K_5)^\beta_{\sigma n} + (I_6)^n_\beta (K_6)^\beta_n \tag{2.3.57}$$

The Lagrange multiplier superfields K_i will also be determined in terms of the basic superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$ as we shall see.

We can now write down the equations of motion. Using equations (2.3.57), (2.3.47)-(2.3.52) and the expressions of QU and $\partial_{\mathbb{R}} V$, we obtain the following equations after comparing the same basis elements in both sides of (2.3.56)

1. $\underline{\Pi^m \Pi^n \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[D_\alpha F_{mn} - \gamma_{n\alpha\beta} F_m{}^\beta\right] = 2\partial_n H_{m\alpha}$$

2. $\underline{\Pi^m \partial\theta^\beta \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[\gamma^n_{\alpha\beta}(F_{mn} + F_{nm}) - D_\alpha G_{m\beta} - \gamma^m_{\alpha\delta} F^\delta{}_\beta\right] = 2\partial_m B_{\alpha\beta} + D_\beta H_{m\alpha}$$

3. $\underline{d_\alpha \partial\theta^\beta \lambda^\sigma}$

$$\frac{\alpha'}{2}\left[-\gamma^m_{\sigma\beta} F_m{}^\alpha + D_\sigma F^\alpha_{\ \beta} - \frac{1}{2}(\gamma^{mn})^\alpha{}_\sigma H_{mn\beta}\right] = D_\beta C^\alpha{}_\sigma$$

4. $\underline{\Pi^m d_\beta \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[-D_\alpha F_m{}^\beta - \frac{1}{2}(\gamma^{pq})^\beta{}_\alpha F_{mpq} - \gamma^m_{\alpha\sigma}\left(K^{\sigma\beta} - K^{\beta\sigma}\right)\right] = 2\partial_m C^\beta{}_\alpha$$

5. $\underline{\partial\theta^\alpha\partial\theta^\beta\lambda^\sigma}$

$$\frac{\alpha'}{2}\left[\gamma^m_{\sigma[\alpha}G_{m\beta]} + D_\sigma H_{\alpha\beta}\right] \quad = \quad D_{[\beta}B_{|\sigma|\alpha]}$$

6. $\underline{\partial\Pi_m\lambda^\alpha}$

$$\frac{(\alpha')^2}{8}(\gamma_m\gamma^{pq})_{\beta\alpha}G^\beta_{pq} = H_{m\alpha}$$

7. $\underline{d_\alpha d_\beta\lambda^\sigma}$

$$\frac{\alpha'}{2}\left[D_\sigma K^{\alpha\beta} + \frac{1}{2}(\gamma^{mn})^\beta{}_\sigma G^\alpha_{mn}\right] = 0$$

8. $\underline{\partial^2\theta^\beta\lambda^\alpha}$

$$\frac{\alpha'}{2}\left[-\frac{\alpha'}{4}\gamma^m_{\beta\sigma}(\gamma^{pq})^\sigma{}_\alpha F_{mpq} + \frac{\alpha'}{2}\gamma^m_{\delta\beta}\gamma_{m\alpha\sigma}K^{\delta\sigma}\right] = B_{\alpha\beta} + \alpha'\gamma^m_{\sigma\beta}\partial_m C^\sigma{}_\alpha$$

9. $\underline{\Pi^m N^{pq}\lambda^\alpha}$

$$\frac{\alpha'}{2}\left[D_\alpha F_{mpq} - \gamma_{m\alpha\beta}G^\beta{}_{pq}\right] = 2\partial_m F_{\alpha pq} + (\gamma_{[p})_{\alpha\beta}(K_4)^\beta{}_{|m|q]}$$

10. $\underline{\Pi^m J\lambda^\alpha}$

$$0 = -\frac{1}{2}\gamma^q{}_{\alpha\beta}(K_4)^\beta{}_{mq}$$

11. $\underline{\Pi^m \partial\lambda^\alpha}$

$$\frac{\alpha'}{2}\left[\alpha'\gamma^n_{\alpha\beta}\partial_n F_m{}^\beta - \frac{\alpha'}{4}(\gamma^{pq})^\beta{}_\alpha D_\beta F_{mpq} + G_{m\alpha} + \frac{\alpha'}{4}(\gamma_m\gamma^{pq})_{\beta\alpha}G^\beta_{pq}\right]$$

$$= H_{m\alpha} - \alpha'\gamma^q_{\alpha\beta}(K_4)^\beta{}_{mq}$$

12. $\underline{\partial\theta^\alpha N^{mn}\lambda^\beta}$

$$\frac{\alpha'}{2}\left[\gamma^p_{\alpha\beta}F_{pmn} - D_\beta H_{mn\alpha}\right] \;=\; D_\alpha F_{\beta mn} + (\gamma_{[m})_{\beta\sigma}(K_5)^\sigma{}_{\alpha n]}$$

13. $\underline{\partial\theta^\alpha J\lambda^\beta}$

$$0 \;=\; -\frac{1}{2}\gamma^n_{\beta\sigma}(K_5)^\sigma{}_{\alpha n}$$

14. $\underline{\partial\theta^\alpha\partial\lambda^\beta}$

$$\frac{\alpha'}{2}\left[-\frac{\alpha'}{4}\gamma^m_{\alpha\sigma}(\gamma^{pq})^\sigma{}_\beta F_{mpq} + \frac{\alpha'}{2}\gamma^m_{\delta\alpha}\gamma_{m\beta\sigma}K^{\sigma\delta} + \alpha'\gamma^m_{\beta\sigma}\partial_m F^\sigma{}_\alpha + \frac{\alpha'}{4}(\gamma^{mn})^\sigma{}_\beta D_\sigma H_{mn\alpha} - 2H_{\alpha\beta}\right]$$

$$= B_{\beta\alpha} - \alpha'\gamma^n_{\beta\sigma}(K_5)^\sigma{}_{\alpha n}$$

15. $\underline{\partial^2\lambda^\alpha}$

$$\frac{\alpha'}{2}\left[-\frac{\alpha'}{4}\gamma^m_{\alpha\beta}F_m{}^\beta - \frac{(\alpha')^2}{8}(\gamma_m\gamma^{pq})_{\beta\alpha}\partial^m G^\beta_{pq} + \frac{\alpha'^2}{32}(\gamma^{mn}\gamma^{pq})^\beta{}_\alpha D_\beta G_{mnpq} - \frac{\alpha'}{8}(\gamma^{mn})^\beta{}_\alpha H_{mn\beta}\right]$$

$$= -\alpha'\gamma^n_{\alpha\beta}(K_6)^\beta_n$$

16. $\underline{\partial J\lambda^\alpha}$

$$0=-\frac{1}{2}\gamma^n_{\alpha\beta}(K_6)^\beta_n$$

17. $\underline{J\partial\lambda^\alpha}$

$$0=-\alpha'\gamma^n_{\alpha\beta}(K_1)^\beta_n-\frac{1}{2}\gamma^n_{\alpha\beta}(K_6)^\beta_n$$

18. $\underline{JJ\lambda^\alpha}$

$$0 \;=\; -\frac{1}{2}\gamma^n{}_{\alpha\beta}(K_1)^\beta_n$$

19. $\underline{\partial d_\alpha\lambda^\beta}$

$$\frac{(\alpha')^2}{16}(\gamma^{mn}\gamma^{pq})^\alpha{}_\beta G_{mnpq}=C^\alpha{}_\beta$$

20. $\underline{d_\alpha N^{mn}\lambda^\beta}$

$$\frac{\alpha'}{2}\left[-D_\beta G^\alpha_{mn}-\frac{1}{2}(\gamma^{pq})^\alpha{}_\beta\Big(G_{mnpq}+G_{pqmn}\Big)\right]=(\gamma_{[m})_{\beta\sigma}(K_3)^{\sigma\alpha}{}_{n]}$$

21. $\underline{d_\alpha J\lambda^\beta}$

$$0 \;=\; -\frac{1}{2}\gamma^n{}_{\beta\sigma}(K_3)^{\sigma\alpha}{}_n$$

22. $\underline{d_\alpha \partial\lambda^\beta}$

$$\frac{\alpha'}{2}\left[\alpha'\gamma^n_{\beta\sigma}(\partial_n K^{\sigma\alpha}-\partial_n K^{\alpha\sigma})+\frac{\alpha'}{4}(\gamma^{mn})^\sigma{}_\beta D_\sigma G^\alpha_{mn}-F^\alpha_\beta+\frac{\alpha'}{8}(\gamma^{mn}\gamma^{pq})^\alpha{}_\beta G_{mnpq}\right]$$
$$=C^\alpha{}_\beta-\alpha'\gamma^n{}_{\beta\sigma}(K_3)^{\sigma\alpha}_n$$

23. $\underline{N^{mn}\partial\lambda^\alpha}$

$$\frac{\alpha'}{2}\left[\alpha'\gamma_{p\alpha\beta}\partial^p G^\beta{}_{mn}-\frac{\alpha'}{4}(\gamma^{pq})^\beta{}_\alpha D_\beta\Big(G_{mnpq}+G_{pqmn}\Big)+H_{mn\alpha}\right]$$
$$= \quad F_{\alpha mn}-\alpha'\gamma^q_{\alpha\beta}(K_2)^\beta{}_{mnq}+(\gamma_{[m})_{\alpha\beta}(K_6)^\beta_{n]}$$

24. $\underline{JN^{mn}\lambda^\alpha}$

$$0 \;=\; (\gamma_{[m})_{\alpha\beta}(K_1)^\beta_{n]}-\frac{1}{2}\gamma^q_{\alpha\beta}(K_2)^\beta{}_{mnq}$$

25. $\underline{\partial N^{mn}\lambda^\alpha}$

$$0=F_{\alpha mn}+(\gamma_{[m})_{\alpha\beta}(K_6)^\beta_{n]}$$

26. $\underline{N^{mn}N^{pq}\lambda^\alpha}$

$$\frac{\alpha'}{2}\left[D_\alpha G_{mnpq}\right]=(\gamma_{[p})_{\alpha\beta}(K_2)^\beta{}_{|mn|q]}$$

We have not yet taken into account the constraints imposed by (2.3.53) and (2.3.54) on the basis elements. We do this now and first consider (2.3.53) which will relate 6th and the 7th equations of the above 26 equations. Eliminating $\partial\Pi^m$ in 6th equation in favor of $d_\alpha d_\beta$ using (2.3.53) and combining it with the 7th equation gives following equation for the coefficient of $d_\alpha d_\beta \lambda^\sigma$

$$\frac{\alpha'}{2}\left[D_\sigma K^{\alpha\beta} - \frac{1}{2}(\gamma^{mn})^\alpha{}_\sigma G^\beta_{mn} - \frac{36}{(\alpha')^2}\gamma_m^{\alpha\beta}\Psi^m_\sigma\right] = 0 \tag{2.3.58}$$

Next, we consider (2.3.54) which relates the basis involving ∂N^{mn} with the anti symmetric part of the basis involving $N^{mn}N^{pq}$. This will relate 25th and the 26th equations. We first seperate the symmetric and the anti symmetric parts of $N^{mn}N^{pq}$ of 26th equation and then combine the anti symmetric part with 25th equation using (2.3.54).

The anti symmetric part of $QU - \partial_{\mathbb{R}}V - \sum_i I_i$ side of the 26th equation is given by

$$\begin{aligned} & \frac{1}{2} : \Big(N^{mn}N^{pq} - N^{pq}N^{mn}\Big)\lambda^\alpha\left[\frac{\alpha'}{2}D_\alpha G_{mnpq} - (\gamma_p)_{\alpha\beta}(K_2)^\beta{}_{mnq}\right]: \\ = \quad & -\frac{\alpha'}{2}\partial N^{mn}\lambda^\alpha\left[\alpha'\eta^{pq}D_\alpha G_{mpqn} + (\gamma^p)_{\alpha\beta}(K_2)^\beta{}_{pmn} - \eta^{pq}(\gamma_m)_{\alpha\beta}(K_2)^\beta{}_{npq}\right] \end{aligned}$$

where, we have used equation (2.3.54) in going from the first to second line.

Combining this with the 25th equation and demanding the coefficient of $\partial N^{mn}\lambda^\alpha$ to vanish gives the following equation

$$\begin{aligned} \frac{\alpha'}{2}\left[-\alpha'\eta^{pq}D_\alpha G_{[m|pq|n]} - (\gamma^p)_{\alpha\beta}(K_2)^\beta{}_{p[mn]} + \eta^{pq}(\gamma_{[m})_{\alpha\beta}(K_2)^\beta{}_{n]pq}\right] - F_{\alpha mn} - (\gamma_{[m})_{\alpha\beta}(K_6)^\beta_{n]} \\ = 0 \end{aligned} \tag{2.3.59}$$

On the other hand, the symmetric part of $QU - \partial_{\mathbb{R}}V - \sum_i I_i$ side of the 26th equation is given by

$$\begin{aligned} & \frac{1}{2} : \Big(N^{mn}N^{pq} + N^{pq}N^{mn}\Big)\lambda^\alpha\left[\frac{\alpha'}{2}D_\alpha G_{mnpq} - (\gamma_p)_{\alpha\beta}(K_2)^\beta{}_{mnq}\right]: \\ = \quad & \frac{1}{2} : N^{mn}N^{pq}\lambda^\alpha\left[\frac{\alpha'}{2}\Big(D_\alpha G_{mnpq} + D_\alpha G_{pqmn}\Big) - (\gamma_p)_{\alpha\beta}(K_2)^\beta{}_{mnq} - (\gamma_m)_{\alpha\beta}(K_2)^\beta{}_{pqn}\right]: \end{aligned}$$

Demanding the coefficient of $N^{mn}N^{pq}\lambda^\alpha$ to vanish gives the following equation

$$\frac{\alpha'}{2}\Big(D_\alpha G_{mnpq} + D_\alpha G_{pqmn}\Big) - (\gamma_{[p})_{\alpha\beta}(K_2)^\beta{}_{|mn|q]} - (\gamma_{[m})_{\alpha\beta}(K_2)^\beta{}_{|pq|n]} = 0 \tag{2.3.60}$$

Our goal now is to find the superfields (and Lagrange multipliers) which satisfy the 26 equations listed earlier (except 5, 6, 25 and 26) and (2.3.58), (2.3.59) and (2.3.60). If our superfields satisfy these equations, then they will automatically satisfy the BRST equation of motion $QU = \partial_{\mathbb{R}}V$.

2.3.4 The ansatz for various superfields

The equations of motion arising from $QU = \partial_{\mathbb{R}}V$, in general, are very complicated due to the presence of gamma matrices and the super covariant derivatives. A direct approach based on comparing the different theta components of the superfields soon becomes messy and intractable. Due to this reason, we shall follow an alternative approach in which we directly propose an ansatz for the superfields and verify that they indeed satisfy the equations given in the previous section. These ansatz follow from the requirement of Lorentz invariance, equations of motion given in (2.1.24)-(2.1.27) and demanding that the superfields appearing in the integrated vertex should be expressible in terms of the 3 basic superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$. This allows us to work with the full covariant superfields instead of working with their theta components as required by the presence of super covariant derivatives. More details about how to arrive at these ansatz in given in appendix C.

Our proposed ansatz for expressing various superfields appearing in the integrated vertex in terms of the 3 basic superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$ and a set of unknown constant co-efficients are as follows

$$
\begin{aligned}
C_\alpha &= C_m = E^\alpha = C = C_{mn} = F_m = F^\alpha = P_{mn} = H = H_\alpha = 0 \\
F_{mn} &= f_1 G_{mn} \qquad , \qquad G_{m\alpha} \;=\; g_1 \Psi_{m\alpha} \\
K^{\alpha\beta} &= a\, \gamma^{\alpha\beta}_{mnp} B^{mnp} \qquad , \qquad H_{\alpha\beta} \;=\; h_1 \gamma^{mnp}_{\alpha\beta} B_{mnp} \\
F^\alpha{}_\beta &= f_5 (\gamma^{mnpq})^\alpha{}_\beta k_m B_{npq} \qquad , \qquad F^\alpha_m \;=\; f_2 k^r (\gamma_r)^{\alpha\beta} \Psi_{m\beta} \\
F_{mpq} &= f_3 G_{m[p} k_{q]} + f_4 B_{mpq} \qquad , \qquad G^\beta_{pq} \;=\; g_2 \gamma^{\beta\sigma}_{[p} \Psi_{q]\sigma} + g_3 k^r \gamma^{\beta\sigma}_r k_{[p} \Psi_{q]\sigma} \\
H_{mn\alpha} &= h_2\, k_{[m} \Psi_{n]\alpha} + h_3 k^q (\gamma_{q[m})_\alpha{}^\sigma \Psi_{n]\sigma} \\
G_{mnpq} &= g_4 k_{[m} B_{n]pq} + g_5 k_{[p} B_{q]mn} + g_6 k_{[m} G_{n][p} k_{q]} + g_7\, \eta_{[m[p} G_{q]n]}
\end{aligned} \tag{2.3.61}
$$

We also need similar ansatz for the Lagrange multipliers in terms of the basic superfields. We propose

$$
\begin{aligned}
(K_1)^\alpha_m &= c_1 k^r (\gamma_r)^{\alpha\beta} \Psi_{m\beta} \\
(K_2)^\alpha_{mnq} &= c_2 k_{[m} \gamma^{\alpha\beta}_{n]} \Psi_{q\beta} + c_3 k_q \gamma^{\alpha\beta}_{[m} \Psi_{n]\beta} + c_4 \gamma^{\alpha\beta}_q k_{[m} \Psi_{n]\beta} + c_5 k^r \gamma^{\alpha\beta}_{rmn} \Psi_{q\beta} + c_6 k^r \gamma^{\alpha\beta}_{rq[m} \Psi_{n]\beta} \\
&\quad + c_7 k^r k_q \gamma^{\alpha\beta}_r k_{[m} \Psi_{n]\beta} + c_8 k^r \gamma^{\alpha\beta}_r \eta_{q[m} \Psi_{n]\beta} \\
(K_3)^{\alpha\beta}_m &= c_9 G_{mn} (\gamma^n)^{\alpha\beta} + c_{10} k_m B_{stu} (\gamma^{stu})^{\alpha\beta} + c_{11} k_s B_{tum} (\gamma^{stu})^{\alpha\beta} + c_{12} k_s B_{tuv} (\gamma_m{}^{stuv})^{\alpha\beta} \\
(K_4)^\alpha_{mn} &= c_{13} (\gamma_n)^{\alpha\beta} \Psi_{m\beta} + c_{14} (\gamma_m)^{\alpha\beta} \Psi_{n\beta} + c_{15} k^r k_m (\gamma_r)^{\alpha\beta} \Psi_{n\beta} + c_{16} k^r k_n (\gamma_r)^{\alpha\beta} \Psi_{m\beta} \\
(K_5)^\alpha{}_{\beta m} &= c_{17} k_p G_{qm} (\gamma^{pq})^\alpha{}_\beta + c_{18} B_{mpq} (\gamma^{pq})^\alpha{}_\beta + c_{19} B_{pqr} (\gamma_m{}^{pqr})^\alpha{}_\beta + c_{20} k_m k_p B_{qrs} (\gamma^{pqrs})^\alpha{}_\beta \\
(K_6)^\alpha_m &= c_{21} k^r (\gamma_r)^{\alpha\beta} \Psi_{m\beta}
\end{aligned}
\tag{2.3.62}
$$

Our job has now reduced to finding the unknown coefficients appearing in above ansatz. If we put these ansatz for the superfields in the equation of motion given above, we shall obtain a system of linear algebraic equations for the unknown coefficients which are much easier to solve. However, before doing this, we shall now see that there are some restriction on some of the coefficients which follow from the constraint identities given earlier and also directly from pure spinor condition.

We start by noting that the superfield G_{mnpq} appears in the expression of the integrated vertex operator as $N^{mn}N^{pq}G_{mnpq}$. We want to find the consequence of the identity (2.3.54) on G_{mnpq}. For this, we consider the quantity $(N^{mn}N^{pq} - N^{pq}N^{mn})G_{mnpq}$. Using the identity (2.3.54) and the ansatz for G_{mnpq} given in (2.3.61), we find that the right hand side of the identity (2.3.54) vanishes identically after contraction with G_{mnpq} and hence

$$
:(N^{mn}N^{pq} - N^{pq}N^{mn})G_{mnpq}: = 0 \quad \Longrightarrow \quad :N^{mn}N^{pq}(G_{mnpq} - G_{pqmn}): = 0 \tag{2.3.63}
$$

This shows that G_{mnpq} is symmetric under the exchange of simultaneous $m \leftrightarrow p$ and $n \leftrightarrow q$ indices. Now, the last two terms in the expression of G_{mnpq} are already consistent with this property. However, this is not the case with the first two terms for which the tensor structures multiplying the coefficients g_4 and g_5 get exchanged. Thus, for G_{mnpq} to be symmetric under the exchange of $m \leftrightarrow p$ and $n \leftrightarrow q$ indices, we must have

$$g_4 = g_5 \tag{2.3.64}$$

Next, we show that the term involving g_7 in the G_{mnpq} superfield vanishes identically. For this, we first note that the term involving g_7 appears in the integrated vertex operator as

$$g_7 N^{mn} N^{pq} \eta_{mp} G_{qn} = -g_7 N^{mn} N^{nq} G_{mq} \tag{2.3.65}$$

Using the definition of N^{mn}, we obtain classically

$$N^{mn} N^{nq} G_{mq} \;\; = \;\; \frac{1}{4} w_\alpha w_\sigma (\gamma^{mn})^\alpha{}_\beta (\gamma^{nq})^\sigma{}_\rho \lambda^\beta \lambda^\rho G_{mq}$$

The right hand side vanishes after using the fierz relation (which follows from the pure spinor condition)

$$\lambda^\beta \lambda^\rho = \frac{1}{32 \times 5!} (\lambda \gamma_{stuvw} \lambda) \gamma^{\beta\rho}_{stuvw} \quad , \tag{2.3.66}$$

the identities involving the product of gamma matrices and the symmetry and tracelessness properties of G_{mq}. We shall now show that this holds true even at the quantum level. The normal ordering piece which arises at quantum level is given by the right hand side of the identity (2.3.54) contracted with

η^{np}. So that the quantum version of the classical equation $N^{mn}N^{pq}\eta_{np}G_{mq} = 0$ is given by

$$: N^{mn}N^{pq}\eta_{np}G_{mq} := \; c : \Big[\eta^{np}\partial N^{mq} - \eta^{nq}\partial N^{mp} - \eta^{mp}\partial N^{nq} + \eta^{mq}\partial N^{np}\Big]\eta_{np}G_{mq} : \quad (2.3.67)$$

where c is an arbitrary coefficient which needs to be determined. But, a little algebra shows that the right hand side is proportional to $: \partial N^{mq}G_{mq} :$ which is zero identically (since N^{mq} is anti symmetric whereas G_{mq} is symmetric in their indices). This means that the term involving g_7 vanishes identically even at the quantum level. Hence, g_7 does not enter in our equations of motion and thus we can drop this term from the expression of G_{mnpq} given in (2.3.61).

Next, we consider the Lagrange multipliers. The first constraint identity I_1 is given by

$$: N^{mn}J\lambda^\alpha(\gamma_m)_{\alpha\beta}(K_1)^\beta_n : -\frac{1}{2} : JJ\lambda^\alpha(\gamma^n)_{\alpha\beta}(K_1)^\beta_n : -\alpha' : J\partial\lambda^\alpha\gamma^n_{\alpha\beta}(K_1)^\beta_n := \; 0 \qquad (2.3.68)$$

Using the expression of $(K_1)^\beta_n$ given in (2.3.62), we find that the last two terms in the left hand side of the above expression vanish identically and the equation reduces to

$$c_1 k^r : N^{mn}J\lambda^\alpha(\gamma_m)_{\alpha\beta}(\gamma_r)^{\beta\sigma}\Psi_{n\sigma} := \; 0 \qquad (2.3.69)$$

Again following the similar steps as described after equation (2.3.65) and noting that J and N^{mn} have trivial OPE, we find that this equation is identically satisfied and hence c_1 does not enter into our equations of motion. Thus, we can drop $(K_1)^\beta_n$ from the equations given in the previous subsection.

Finally, we consider the term involving c_9 in the Lagrange multiplier $(K_3)^{\alpha\beta}_n$. After contracting $(I_3)^n_{\alpha\beta}$ with the term involving c_9 of $(K_3)^{\alpha\beta}_n$, we find that the last two terms of the constraint identity I_3 vanish identically whereas the first term vanishes by using the similar argument as given below equation (2.3.65). Thus, we can also drop the term involving c_9 from our equation of motions.

We are now ready to solve the equations of motion and determine the unknown coefficients appearing in the superfields.

2.3.5 Solving for unknown coefficients

To determine the unknown coefficients in superfields, we put (2.3.61) and (2.3.62) in the equations of motion given in subsection 2.3.3 and analyze them one by one. Some of the equations will determine the unknown coefficients while others will be satisfied identically. The Mathematica package GAMMA is very helpful for doing these calculations [79].

The first five equations[12] of the previous subsection give[13]

$$
\begin{aligned}
f_1 &= -\frac{18}{\alpha'} \quad , \quad f_2 = \frac{288i}{\alpha'} \quad , \quad f_3 = \frac{36i}{\alpha'} \quad , \quad f_4 = \frac{12}{(\alpha')^2} \quad , \quad f_5 = -\frac{4i}{\alpha'} \\
h_1 &= \frac{2}{\alpha'} \quad , \quad h_2 = -\frac{576i}{\alpha'} \quad , \quad h_3 = -\frac{144i}{\alpha'} \quad , \quad a = -\frac{1}{(\alpha')^2} \\
g_1 &= -\frac{432}{\alpha'}
\end{aligned}
\tag{2.3.70}
$$

Next, we contract the combined 6th and 7th equation (2.3.58) with $\gamma^p_{\alpha\beta}$ and $\gamma^{pqr}_{\alpha\beta}$ and use (2.3.61) to find

$$
g_2 = \frac{48}{(\alpha')^2} \quad , \quad g_3 = -\frac{192}{\alpha'} \tag{2.3.71}
$$

[12] To extract the information from the 3rd equation, it is convenient to contract it with 1-form, 3-form and 5-forms. This gives rise to 3 different equations which determine f_2, f_5, h_2 and h_3. Similarly, g_1 and h_1 can be determined from the 5th equation by contracting it with the 3-form.

[13] In general, some of the coefficients appearing in the superfields are determined by more than one equations. But, their values always agree. This also shows the consistency of the equations with our ansatz.

Multiplying with a 5-form $\gamma_{\alpha\beta}^{pqrst}$ does not give any new information due to (2.3.55).

The equation 8 gives

$$f_4 = \frac{4}{(\alpha')^2} - 8a$$

which is identically satisfied by the values of f_4 and a given in equation (2.3.70). Next, using (2.3.70), the 9th equation determines

$$c_{13} = \frac{24}{\alpha'} \quad , \qquad c_{14} = -\frac{24}{\alpha'} \quad , \qquad c_{15} = -30 \quad , \qquad c_{16} = 192 \tag{2.3.72}$$

The equation 10 gives

$$10c_{13} + 2c_{14} - \frac{1}{\alpha'}c_{16} = 0$$

Similarly, equation 11 gives

$$\begin{aligned} -\frac{if_2\alpha'}{2} + -11i\alpha' f_3 - 21(\alpha')^2 f_4 + \frac{g_1\alpha'}{2} - 2(\alpha')^2 g_2 + \frac{\alpha' g_3}{4} \\ = \quad -72 - \alpha' \left(10c_{13} + 2c_{14} - \frac{1}{\alpha'}c_{16} \right) \end{aligned}$$

Both of these equations are identically satisfied by (2.3.70) and (2.3.72).

Next, the 12th equation gives

$$c_{17} = \frac{63i}{16} \quad , \qquad c_{18} = \frac{3}{8\alpha'} \quad , \qquad c_{19} = -\frac{9}{16\alpha'} \quad , \qquad c_{20} = -\frac{57}{16} \tag{2.3.73}$$

Using (2.3.73) and (2.3.70), the equations resulting from 13th and 14th equations, namely,

$$c_{18} + 7c_{19} - \frac{c_{20}}{\alpha'} = 0$$

and

$$a(\alpha')^2 - \frac{f_4(\alpha')^2}{8} + \frac{if_5\alpha}{2} + \frac{i\alpha'}{32}\left(\frac{h_2}{3} - h_3\right) - h_1\alpha' = -1 + \alpha'\left(c_{18} + 7c_{19} - \frac{c_{20}}{\alpha'}\right)$$

are identically satisfied.

Further, the equations 15, 16, 17 and 18 are identically satisfied by the ansatz in (2.3.61) and (2.3.62) without putting any restriction on the coefficients.

Next, the 19th equation on using (2.3.64) gives

$$g_4 \;=\; g_5 \;=\; \frac{4i}{(\alpha')^2} \tag{2.3.74}$$

Similarly, on dropping the terms involving g_7 and c_9 as discussed in the previous subsection and using equation (2.3.74), the 20th equation gives

$$g_6 \;=\; -\frac{12}{\alpha'} \quad , \quad c_{10} \;=\; \frac{i}{\alpha'} \quad , \quad c_{11} \;=\; 0 \quad , \quad c_{12} \;=\; -\frac{i}{6\alpha'} \tag{2.3.75}$$

Next, using equations (2.3.70), (2.3.74) and (2.3.75), the equations resulting from 21st and 22nd equations, namely

$$c_{10} - c_{11} + 6c_{12} = 0 \tag{2.3.76}$$

and,

$$ia(\alpha')^2 + \frac{i\alpha'}{48}\Big(g_2\alpha' + \frac{g_3}{2}\Big) - \frac{f_5\alpha'}{2} + \frac{(\alpha')^2}{16}(g_4 + g_5) = \frac{i}{2} - \alpha'\Big(c_{10} - c_{11} + 6c_{12}\Big)$$

are identically satisfied.

Finally, the 23rd, 24th equations along with (2.3.59) and (2.3.60) determine the Lagrange multiplier superfields $(K_2)^{\beta}_{mnp}$ and $(K_6)^{\beta}_{m}$. On dropping the terms involving g_7 and the Lagrange multiplier $(K_1)^{\beta}_{n}$ from these equations as discussed in the previous subsection and using the other coefficients determined so far, these 4 equations give

$$\begin{aligned} c_2 &= -\frac{96i}{5\alpha'} \quad , \quad c_3 = -\frac{72i}{5\alpha'} \quad , \quad c_4 = \frac{72i}{5\alpha'} \quad , \quad c_5 = \frac{8i}{5\alpha'} \\ c_6 &= -\frac{8i}{5\alpha'} \quad , \quad c_7 = 96i \quad , \quad c_8 = \frac{24i}{5\alpha'} \quad , \quad c_{21} = -9i \end{aligned} \tag{2.3.77}$$

We have now determined all the coefficients appearing in the ansatz for superfields and the Lagrange multipliers. We have also exhausted all the equations of motion. With these coefficients, the BRST equation of motion $QU = \partial_{\mathbb{R}}V$ is now identically satisfied. This establishes that our ansatz for various superfields with the values of coefficients determined in this section indeed gives the correct integrated vertex for the first massive states. The final expression of the integrated vertex operator U including the numerical coefficients in the ansatz is given in equations (2.2.35) and (2.2.36).

2.4 Discussion

We have constructed the integrated form of the first massive vertex operator of open strings in the pure spinor formalism. Since the vertex operator is solely expressed in terms of the superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$, using the theta expansion results given in [74], one can readily obtain the theta expansion of the integrated vertex in terms of only the physical fields b_{mnp}, g_{mn} and $\psi_{m\alpha}$. This, therefore, demonstrates that the integrated vertex operator thus constructed is in terms of the physical degrees of freedom only.

This construction can also be used to obtain the first massive integrated vertex operator in the Heterotic string. For this, one simply need to take the tensor product of the vertex operator constructed here with the anti-holomorphic integrated vertex of the bosonic string. However some normalisation factors need to be accounted for while going to closed superstrings from open superstrings.

Previously, with only the unintegrated form of the massive vertex being known, the possible scattering amplitudes involving massless and first massive states that could be explicitly computed, were severely restricted. Knowing the integrated vertex now enables one to compute any amplitude upto two loop order involving arbitrary number of the massless and first massive states in the pure spinor formalism[14]. The results of some amplitude calculations involving massive states were reported in [82].

The pure spinor constraints as well as the OPEs imply that the basic worldsheet operators satisfy non-linear constraints. This fact leads to several subtleties. In particular, it implies non-trivial identities which a subset of all worldsheet operators at a given conformal weight and ghost number will satisfy. We showed how to take into account all such constraints systematically in section 2.3.2. This line of reasoning was based on its successful role in determining the unintegrated vertex [9] and is now further strengthened by the successful construction of the integrated form of the vertex. These evidences therefore suggest that we have indeed adopted the correct way of incorporating the effect of all such constraints at higher mass levels.

The general strategy outlined in section **2.2** and the method given in appendix C.0.3 for writing the ansatz do not explicitly or implicitly depend on the conformal weight and ghost number for which

[14]It is the understanding of the authors that at present there are no unanimous consensus on computing full multiloop amplitudes in pure spinor formalism. But, also see [63, 80, 81].

we eventually employed it. It is also to be noted that an identical strategy can be applied to construct even unintegrated vertex for any massive state, the only difference being the equation that one now needs to solve is $QV = 0$. This leads us to conjecture that our strategy is very general and can be successfully implemented to determine integrated as well as unintegrated form of vertex operators for all higher massive states in pure spinor formalism.

Chapter 3

Vacuum Decay and Survival

The possibility that we may be living in a universe that is in a metastable vacuum has been explored for more that fifty years [83–88]. Discovery of the accelerated expansion of the universe [89, 90] and subsequent developments in string theory leading to the construction of de Sitter vacua [14, 35, 91] suggest that the vacuum we are living in at present is indeed metastable. Unfortunately our understanding of string theory has not reached a stage where we can make a definite prediction about the decay rate of our vacuum. The only information we have about this is from the indirect observation that our universe is about 1.38×10^{10} years old. Therefore, assuming that we have not been extremely lucky we can conclude that our inverse decay rate[1] is at least of the same order.[2]

Typically the decay of a metastable vacuum proceeds via bubble nucleation [83–88] (see [94] for a recent survey). In a small region of space-time the universe makes transition to a more stable vacuum, and this bubble of stable vacuum[3] then expands at a speed that asymptotically approaches

[1]For exponential decay the inverse decay rate differs from half-life by a factor of $\ln 2$. In order to simplify terminology, we shall from now on use only inverse decay rate and life expectancy – to be defined later – as measures of longevity.

[2]We shall in fact see in §3.4.3 that the actual lower bound for the current inverse decay rate is weaker by a factor of 3.7, making it comparable to the time over which the earth will be destroyed due to the increase in the size of the sun. Allowing for the possibility that we could have been extremely lucky reduces the lower bound on the inverse decay rate by about a factor of 10 [92, 93].

[3]We shall refer to the more stable vacuum as the stable vacuum, even if this vacuum in turn could decay to other vacua of lower energy density. In any case since this vacuum will have negative cosmological constant, the space-time

the speed of light, converting the rest of the region it encounters also to this stable phase. Due to this rapid expansion rate it is impossible to observe the expanding bubble before encountering it – it reaches us when we see it. However, due to the existence of the future horizon in the de Sitter space, even a bubble expanding at the speed of light cannot fill the whole space at future infinity. Indeed, it has been known for quite some time that in de Sitter space if the expansion rate of the universe exceeds the decay rate due to phase transition then even collectively the bubbles of stable vacuum cannot fill the whole space [95] and there will always be regions which will continue to exist in the metastable vacuum. Nevertheless, any single observer in the metastable vacuum will sooner or later encounter an expanding bubble of stable vacuum, and the probability of this decay per unit time determines the inverse decay rate of the observer in the metastable vacuum.

This suggests that while any single observer will always have a limited average life span determined by the microscopic physics, a civilization could collectively increase its longevity by spreading out and establishing different civilizations in different parts of the universe [15]. If the bubble of stable vacuum hits the civilization – henceforth refered to as object – in the initial stages of spreading out then it does not help since the same bubble will most likely destroy all the objects. However, with time the different objects will go outside each other's horizon and a single bubble of stable vacuum will not be able to destroy all of them. This will clearly increase the life expectancy of the objects collectively – defined as the average value of the time at which the last surviving object undergoes vacuum decay – although there will be no way of telling *a priori* which one will survive the longest. A simple calculation shows that if we could begin with 2 objects already far outside each other's horizon so that their decay probabilities can be taken to be independent, then the life expectancy of the combined system increases by a factor of 3/2 compared to the life expectancy of a single isolated object. In the case of n copies the life expectancy increases by a factor given by the n-th harmonic

inside the bubble will undergo a gravitational crunch [88]. We shall ignore the possibility of decay to Minkowski vacua or other de Sitter vacua of lower cosmological constant since the associated decay rates are very small due to smallness of the cosmological constant of our vacuum.

number. However, in actual practice we cannot begin with copies of the object already outside each other's horizon. As a result the increase in the life expectancy is expected to be lower.

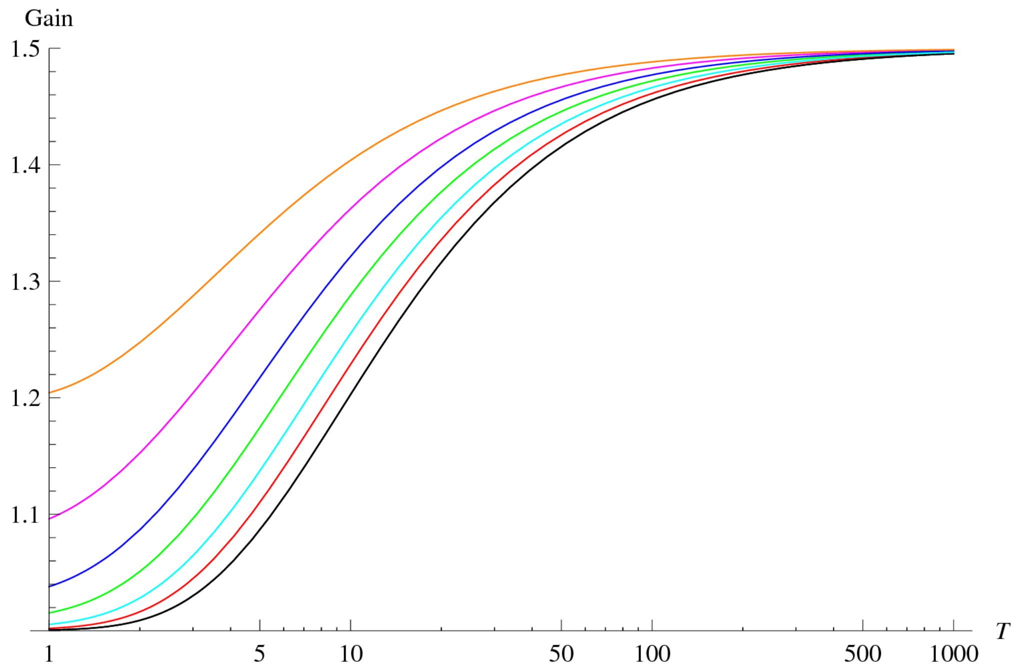


Figure 3.1: The figure showing the 'gain' in the life expectancy for two objects compared to that of one object as a function of T for $r = .0003$, .001, .003, .01, .03, .1 and .3.

The goal of this chapter will be to develop a systematic procedure for computing the increase in the life expectancy of the object as a result of making multiple copies of itself. For two objects we obtain explicit expression for the life expectancy in terms of three parameters: the Hubble constant H of the de Sitter space-time determined by the cosmological constant, the vacuum decay rate or equivalently the life expectancy T of a single isolated object and the initial separation r between the two objects. In fact due to dimensional reasons the result depends only on the combination HT and Hr, so we work by setting $H = 1$. In Fig. 3.1 we have shown the result for the ratio of the life expectancy of two objects to that of a single object – called the 'gain' – as a function of T for different choices of r. From this we see that even for a modest value of $r = 3 \times 10^{-4}$ the gain in the life expectancy reaches close to the maximum possible value of 1.5 if T is larger that 100 times the horizon size of the de Sitter space, i.e. the decay rate is less than 1% of the expansion rate. $T = 100$ corresponds to about 1.7×10^{12} years. $r = 3 \times 10^{-4}$ corresponds to a physical distance of the order of 5×10^{6} light years and is of the order of the minimal distance needed to escape the local gravitationally bound system of galaxies. If $T = 10$ – i.e. of order 1.7×10^{11} years – the gain is about 20% for $r = 3 \times 10^{-4}$. These time scales are shorter than the time scale by which all the stars in the galaxy will die. Therefore, if T lies between 10^{11} years and the life span of the last star in the local group of galaxies which will be gravitationally bound and will remain inside each other's horizon, then we gain a factor of 1.2 - 1.5 in life expectancy even by making one additional copy of the object at a distance larger than about 10^{7} light years from us. On the other hand if T is larger than the life span of the last star in the galaxy then our priority should be to plan how to survive the death of the galaxy rather than vacuum decay. Some discussion on this can be found in [96].

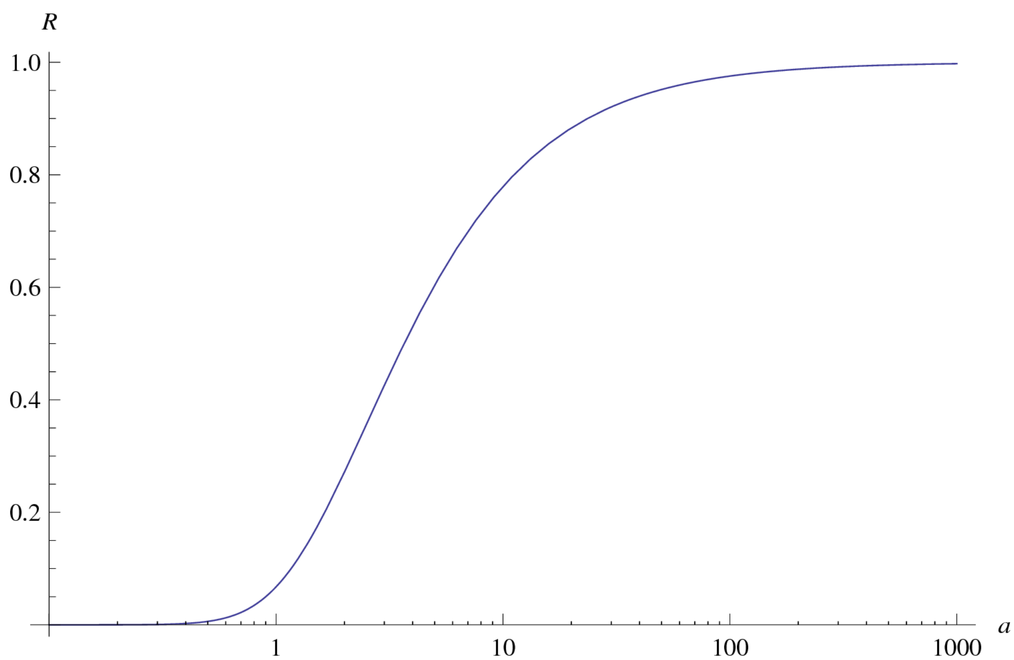


Figure 3.2: Growth of the relative decay rate R – defined as the ratio of the decay rate to its asymptotic value – of a single object in FRW space-time as a function of the scale factor a. Value of R at $a = 1$ represents the decay rate today relative to what it would be in the cosmological constant dominated epoch.

Even though most of our analysis focusses on the case of a pair of objects in de Sitter space at fixed comoving coordinates, our method is quite general and can be applied to arbitrary number of objects in a general FRW metric moving along general trajectories. We discuss these generalizations in §3.4. In particular considering the case of a single object in an FRW metric dominated by matter and cosmological constant, as is the case with the current state of our universe, we find that the vacuum decay rate increases as a function of time due to accelerated expansion of the volume of the past light cone. This has been shown in Fig. 3.2. This rate approaches a constant value as the universe enters the cosmological constant dominated era, but we find for example that this asymptotic decay rate is about 15 times larger than the decay rate today, which in turn is about 3.7 times larger than the average decay rate in the past. Now given that the universe has survived for about 1.38×10^{10} years, we can put a lower bound of this order on the inverse of the average decay rate in the past.[4] This translates to a lower bound of order 3.7×10^9 years on the inverse decay rate today and 2.5×10^8 years on the asymptotic inverse decay rate.

The rest of the chapter is organised as follows. In §3.1 we describe the case of the decay of n objects assuming that their decay probabilities are independent of each other, and show that the life expectancy of the combined system goes up by a factor equal to the n-th harmonic number. In §3.2 we carry out the complete analysis for two observers in 1+1 dimensional de Sitter space. The final result for the life expectancy of the combined system can be found in (3.2.30). This is generalized to the case of two observers in 3+1 dimensional de Sitter space-time in §3.3. Eq.(3.3.17) together with (3.3.16) and (3.3.11) gives the probability that at least one of the two objects survives till time t, which can then be used to compute the life expectancy of the combined system using (3.3.18). In §3.4 we discuss various generalizations including the case of multiple observers, general trajectories and general FRW type metric. We conclude in §3.5 with a discussion of how in future we could improve our knowledge of possible values of the parameters r and T which enter our calculation. In

[4] One must keep in mind that this is not a strict bound since we could have survived till today by just being lucky.

appendix D we compute the time dependence of the decay rate for a general equation of state of the form $p = w\rho$.

3.1 Independent decay

Let us suppose that we have two independent objects, each with a decay rate of c per unit time. We shall label them as C_1 and C_2. If we begin with the assumption that both objects exist at time $t = 0$ then the probability that the first object exists after time t is

$$P_1(t) = e^{-c\,t}\,. \tag{3.1.1}$$

Therefore, the probability that it decays between time t and $t + \delta t$ is $-\dot{P}_1(t)\delta t$ where $\dot{P}_1$ denotes the derivative of P_1 with respect to t, and its life expectancy, is

$$\bar{t}_1 = -\int_0^\infty t\dot{P}_1(t)dt = \int_0^\infty P_1(t)dt = c^{-1}\,. \tag{3.1.2}$$

Independently of this the probability that the second object exists after time t is also given by $\exp[-c\,t]$ and it has the same life expectancy.

Now let us compute the life expectancy of both objects combined, defined as the average of the larger of the actual life time of C_1 and C_2. To compute this note that since the two objects are independent, the probability that both will decay by time t is given by $(1 - P_1(t))(1 - P_2(t)) = (1 - P_1(t))^2$. Therefore, the probability that the last one to survive decays between t and $t + \delta t$ is $\frac{d}{dt}(1 - P_1(t))^2\delta t$. This gives the life expectancy of the combined system to be

$$\bar{t}_{12} = \int_0^\infty t\,\frac{d}{dt}(1 - P_1(t))^2 dt = \frac{3}{2}c^{-1}\,. \tag{3.1.3}$$

Therefore, we see that by taking two independent objects we can increase the life expectancy by a factor of $3/2$. A similar argument shows that for n independent objects the life expectancy will be

$$\bar{t}_{12\cdots n} = \int_0^\infty t\,\frac{d}{dt}(1 - P_1(t))^n dt = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) c^{-1} \,. \tag{3.1.4}$$

3.2 Vacuum decay in 1+1 dimensional de Sitter space

Consider 1+1 dimensional de Sitter space

$$ds^2 = -dt^2 + e^{2t} dx^2 \,. \tag{3.2.1}$$

Note that we have set the Hubble constant of the de Sitter space and the speed of light to unity so that all other time / lengths appearing in the analysis are to be interpreted as their values in units of the inverse Hubble constant. We introduce the conformal time τ via

$$\tau = -e^{-t} \tag{3.2.2}$$

in terms of which the metric takes the form

$$ds^2 = \tau^{-2}(-d\tau^2 + dx^2) \,. \tag{3.2.3}$$

At $t = 0$ we have $\tau = -1$ and comoving distances coincide with the physical distances.

We shall use this space-time as a toy model for studying the kinematics of vacuum decay. We shall assume that in this space-time there is a certain probability per unit time per unit volume of producing a bubble of stable vacuum, which then expands at the speed of light causing decay of the metastable vacuum. We shall not explore how such a bubble is produced; instead our goal will be to

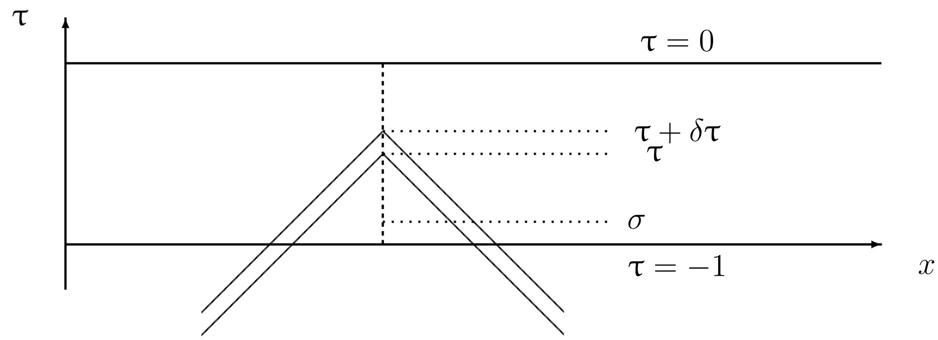


Figure 3.3: A comoving object in de Sitter space and its past light cones at conformal times τ and $\tau + \delta\tau$.

study its effect on the life expectancy of the objects living in this space. In §3.3 we shall generalize this analysis to 3+1 dimensional de Sitter space.

3.2.1 Isolated comoving object

Consider a single object in de Sitter space at rest in the comoving coordinate x (say at $x = 0$), shown by the vertical dashed line in Fig. 3.3. We start at $t = 0$ ($\tau = -1$) and are interested in calculating the probability that it survives at least till conformal time τ. If we denote this by $P_0(\tau)$ then the probability that it will decay between τ and $\tau + \delta\tau$ is $-P_0'(\tau)\delta\tau$ where $'$ denotes derivative with respect to τ. On the other hand this probability is also given by the product of $P_0(\tau)$ and the probability that a vacuum bubble is produced somewhere in the past light cone of the object between τ and $\tau + \delta\tau$, as shown in Fig.3.3. The volume of the past light cone of this interval can be easily calculated to be

$$2\,\delta\tau \int_{-\infty}^{\tau} \frac{d\sigma}{\sigma^2} = -\frac{2}{\tau}\,\delta\tau\,. \tag{3.2.4}$$

Therefore, if K is the probability of producing the bubble per unit space-time volume then the probability of producing a bubble in the past light cone of the object between τ and $\tau + \delta\tau$ is given by

$-2K\delta\tau/\tau$. The previous argument then leads to the equation

$$P_0'(\tau) = 2K\tau^{-1}P_0(\tau)\,. \tag{3.2.5}$$

This equation, together with the boundary condition $P_0(\tau = -1) = 1$, can be integrated to give

$$\ln P_0(\tau) = 2K\ln(-\tau)\,. \tag{3.2.6}$$

In terms of physical time t we have[5]

$$P_0(t) = e^{-2Kt}\,. \tag{3.2.7}$$

From this we can calculate the life expectancy, defined as the integral of t weighted by the probability that the object undergoes vacuum decay between t and $t + dt$. Since the latter is given by $-\dot{P}_0(t)dt$, we have the life expectancy

$$T = -\int_0^\infty t\,\dot{P}_0(t)\,dt = \int_0^\infty P_0(t)\,dt = \frac{1}{2K}\,, \tag{3.2.8}$$

where in the second step we have used integration by parts. We shall express our final results in terms of T instead of K.

3.2.2 A pair of comoving objects

Next we shall consider two comoving objects C_1 and C_2 in de Sitter space separated by physical distance r at $t = 0$ or equivalently $\tau = -1$. We shall take $r < 1$, i.e. assume that the two objects

[5]Note that by an abuse of notation we have used the same symbol P_0 to denote the probability as a function of t although the functional form changes. We shall continue to follow this convention, distinguishing the function by its argument (t or τ). Derivatives with respect to τ and t will be distinguished by using P_0' to denote τ-derivative of P_0 and $\dot{P}_0$ to denote t-derivative of P_0.

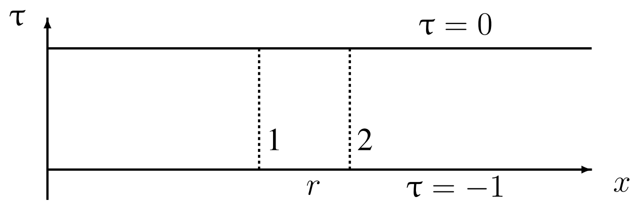


Figure 3.4: Two comoving objects in de Sitter space separated by physical distance r at $\tau = -1$.

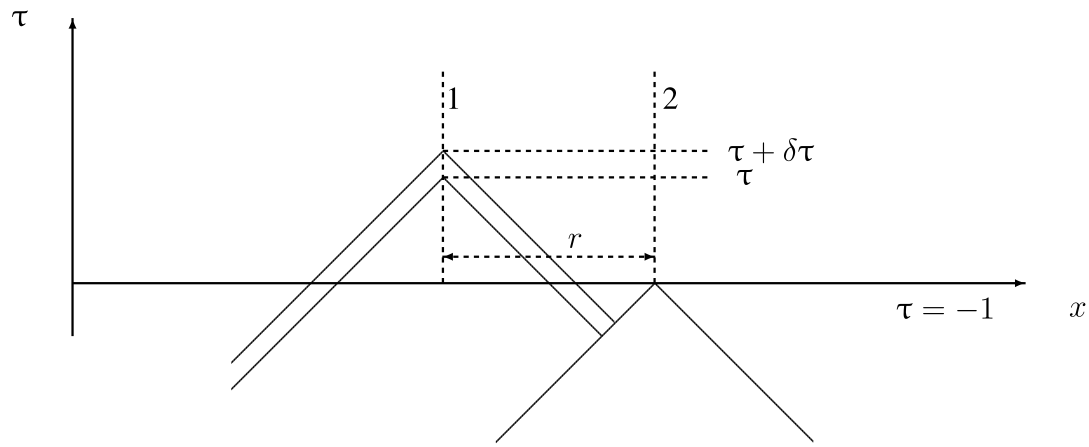


Figure 3.5: The past light-come of C_2 at $\tau = -1$ and the past light cone of C_1 between τ and $\tau + \delta\tau$ for $\tau < r - 1$.

are within each other's horizon at the time they are created. We denote by $P_i(\tau)$ the probability that C_i survives at least till conformal time τ for $i = 1, 2$ and by $P_{12}(\tau_1, \tau_2)$ the joint probability that C_1 survives at least till conformal time τ_1 and C_2 survives at least till conformal time τ_2. The boundary condition will be set by assuming that both objects exist at $\tau = -1$, so that we have

$$P_1(-1) = 1, \quad P_2(-1) = 1, \quad P_{12}(-1, \tau_2) = P_2(\tau_2), \quad P_{12}(\tau_1, -1) = P_1(\tau_1)\,. \qquad (3.2.9)$$

First we shall calculate $P_1(\tau)$ and $P_2(\tau)$. They must be identical by symmetry, so let us focus on $P_1(\tau)$. The calculation is similar to that for $P_0(\tau)$ above for a single isolated object, except that the existence of C_2 at $\tau = -1$ guarantees that no vacuum decay bubble is produced in the past light-come of C_2 at $\tau = -1$, and hence while computing the volume of the past light cone of the C_1 between τ

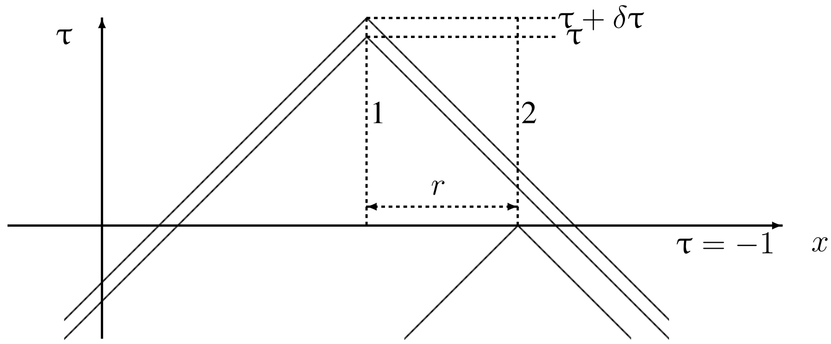


Figure 3.6: The past light-come of C_2 at $\tau = -1$ and the past light cone of C_1 between τ and $\tau + \delta\tau$ for $\tau > r - 1$.

and $\tau + \delta\tau$, we have to exclude the region inside the past light cone of C_2 at $\tau = -1$. This has been shown in Fig. 3.5. This volume is given by

$$\delta\tau \left[2 \int_{-\infty}^{\tau} \frac{d\sigma}{\sigma^2} - \int_{-\infty}^{-1-\frac{r-1-\tau}{2}} \frac{d\sigma}{\sigma^2} \right] = \delta\tau \left[-\frac{2}{\tau} - \frac{2}{r+1-\tau} \right] \quad \text{for } \tau < r-1\,. \tag{3.2.10}$$

However, for $\tau > r - 1$ the past light cone of C_1 between τ and $\tau + \delta\tau$ does not intersect the past light cone of C_2 at $\tau = -1$ (see Fig. 3.6), and we get the volume to be

$$2\,\delta\tau \int_{-\infty}^{\tau} \frac{d\sigma}{\sigma^2} = -2\,\delta\tau\,\frac{1}{\tau} \quad \text{for } \tau > r-1\,. \tag{3.2.11}$$

This leads to the following differential equation for $P_1(\tau)$:

$$\frac{1}{P_1(\tau)} \frac{dP_1}{d\tau} = \begin{cases} -K\left[-2/\tau - 2/(r+1-\tau)\right] & \text{for } \tau < r-1\,, \\ 2\,K\,/\tau \quad \text{for } \tau > r-1\,. \end{cases} \tag{3.2.12}$$

Using the boundary condition $P_1(-1) = 1$ and the continuity of $P_1(\tau)$ across $\tau = r - 1$ we get

$$\ln P_1(\tau) = \begin{cases} 2\,K\,\{\ln(-\tau) - \ln(r+1-\tau) + \ln(r+2)\} & \text{for } \tau < r-1\,, \\ 2\,K\,\{\ln(-\tau) - \ln 2 + \ln(r+2)\} & \text{for } \tau > r-1\,. \end{cases} \tag{3.2.13}$$

Using the symmetry between 1 and 2 we also get the same expression for $P_2(\tau)$. In terms of the physical time t we have

$$P_1(t) = P_2(t) = \begin{cases} e^{-2Kt}(r+1+e^{-t})^{-2K}(r+2)^{2K} & \text{for } t < -\ln(1-r)\,, \\ \Big((r+2)/2\Big)^{2K} e^{-2Kt} & \text{for } t > -\ln(1-r)\,. \end{cases} \tag{3.2.14}$$

Therefore, the life expectancy of C_1 is

$$\begin{aligned} \bar{t}_1 &= -\int_0^\infty t\,\dot{P}_1(t)\,dt = \int_0^\infty P_1(t)dt \\ &= (r+2)^{2K}\left[B\left(\frac{1}{2+r};2K,0\right) - B\left(\frac{1-r}{2};2K,0\right) + \frac{(1-r)^{2K}}{2^{2K+1}K}\right] \end{aligned} \tag{3.2.15}$$

where $B(x;p,q)$ is the incomplete beta function, defined as

$$B(x;p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt = \int_0^{x/(1-x)} \frac{y^{p-1}}{(1+y)^{p+q}}dy\,, \tag{3.2.16}$$

the two expressions being related by the transformation $t = y/(y+1)$. In terms of the life expectancy $T = 1/2K$ of a single isolated object, we have

$$\bar{t}_1 = (r+2)^{1/T}\left[B\left(\frac{1}{2+r};\frac{1}{T},0\right) - B\left(\frac{1-r}{2};\frac{1}{T},0\right) + T\,\frac{(1-r)^{1/T}}{2^{1/T}}\right]\,. \tag{3.2.17}$$

C_2 also has the same life expectancy. (3.2.17) is somewhat larger than T, but that is simply a result of our initial assumption that both objects exist at $t = 0$. If both objects had started at the same space-time point and then got separated following some specific trajectories, then there would have been a certain probability that one or both of them will decay during the process of separation; this possibility has been ignored here leading to the apparent increase in the life expectancy. However, for realistic values of r and T, which corresponds to $r << 1$ and $T \gtrsim 1$, the ratio $\bar{t}_1/T$ remains close

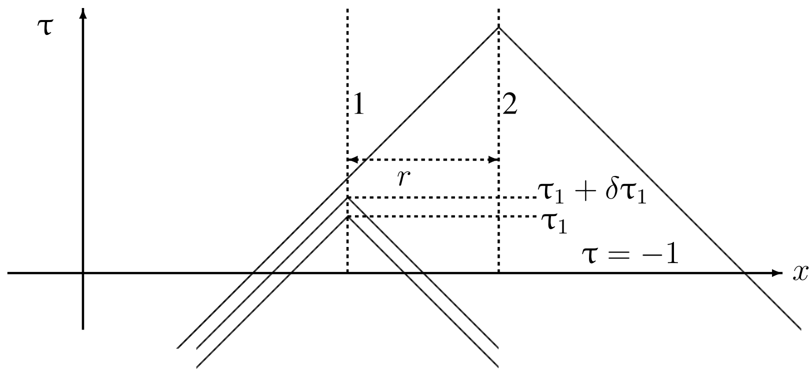


Figure 3.7: The past light-come of C_2 at τ_2 and the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$ for $\tau_1 < \tau_2 - r$.

to unity.

Let us now turn to the computation of the joint survival probability $P_{12}(\tau_1, \tau_2)$. In this case the probability that the first object undergoes vacuum decay between τ_1 and $\tau_1 + \delta\tau_1$ and the second object survives at least till τ_2 is given by $-\delta\tau_1\,(\partial P_{12}(\tau_1, \tau_2)/\partial\tau_1)$. On the other hand the same probability is given by $K \times P_{12}(\tau_1, \tau_2)$ times the volume of the past light-come of C_1 between τ_1 and $\tau_1 + \delta\tau_1$, excluding the region inside the past light cone of C_2 at τ_2. The relevant geometry has been shown in Figs. 3.7, 3.8 and 3.9 for different ranges of τ_1 and τ_2. The results are as follows:

1. For $\tau_1 < \tau_2 - r$ the geometry is shown in Fig. 3.7. In this case C_1 at τ_1 (and hence the whole of the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$) is inside the past light cone of C_2 at τ_2. Therefore, the decay probability is zero and we have the equation:

$$\frac{\partial \ln P_{12}(\tau_1, \tau_2)}{\partial \tau_1} = 0 \quad \text{for } \tau_1 < \tau_2 - r\,. \tag{3.2.18}$$

2. For $\tau_2 - r < \tau_1 < \tau_2 + r$ the geometry is as shown in Fig. 3.8. In this case C_1 at τ_1 and C_2 at τ_2 are space-like separated. The volume of the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$

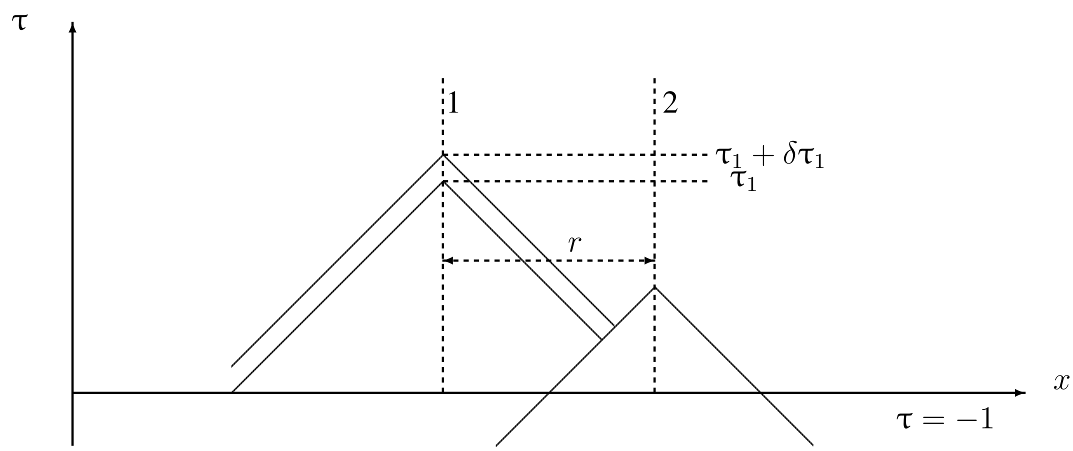


Figure 3.8: The past light-come of C_2 at τ_2 and the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$ for $\tau_2 - r < \tau_1 < \tau_2 + r$.

outside the past light cone of C_2 at τ_2 is given by

$$\int_{-\infty}^{\tau_1} \frac{d\sigma}{\sigma^2} + \int_{\frac{1}{2}(\tau_1+\tau_2-r)}^{\tau_1} \frac{d\sigma}{\sigma^2} = -\frac{2}{\tau_1} - \frac{2}{r-\tau_1-\tau_2}\,. \tag{3.2.19}$$

This gives

$$\frac{\partial \ln P_{12}(\tau_1,\tau_2)}{\partial \tau_1} = 2K\left\{\frac{1}{\tau_1} + \frac{1}{r-\tau_1-\tau_2}\right\} \quad \text{for } \tau_2 - r < \tau_1 < \tau_2 + r\,. \tag{3.2.20}$$

3. For $0 < \tau_2 + r < \tau_1$, the geometry is shown in Fig. 3.9. In this case C_2 at τ_2 is inside the past light cone of C_1 at τ_1 and there is no intersection between the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$ and the past light cone of C_2 at τ_2. Therefore, the volume of the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$ is given by

$$2\int_{-\infty}^{\tau_1} \frac{d\sigma}{\sigma^2} = -\frac{2}{\tau_1}\,, \tag{3.2.21}$$

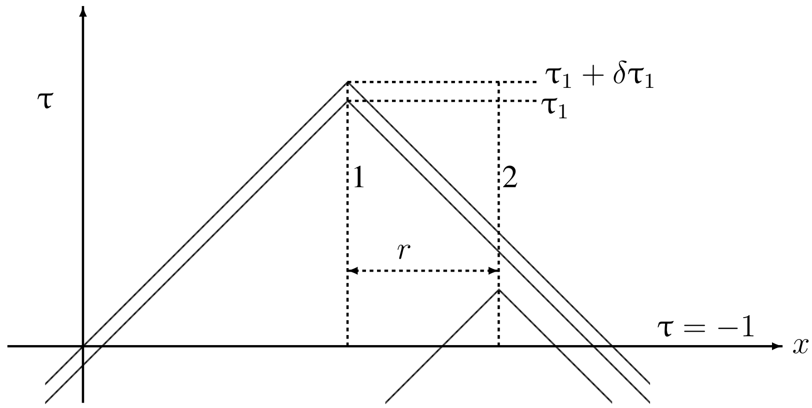


Figure 3.9: The past light cone of C_2 at τ_2 and the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$ for $\tau_2 + r < \tau_1 < 0$.

and we have

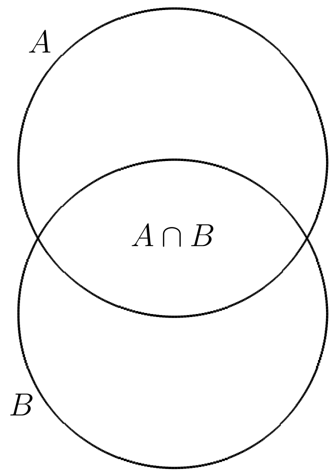
$$\frac{\partial \ln P_{12}(\tau_1, \tau_2)}{\partial \tau_1} = 2K\frac{1}{\tau_1} \quad \text{for } \tau_2 + r < \tau_1 < 0\,. \tag{3.2.22}$$

We can now determine $P_{12}(\tau_1, \tau_2)$ by integrating (3.2.18), (3.2.20), (3.2.22) subject to the boundary condition given in (3.2.9)

$$P_{12}(-1, \tau_2) = P_2(\tau_2) = P_1(\tau_2)\,, \tag{3.2.23}$$

and using the fact that $P_{12}(\tau_1, \tau_2)$ must be continuous across the subspaces defined by $\tau_1 = \tau_2 \pm r$. The result of the integration is

$$\ln P_{12}(\tau_1, \tau_2) = \begin{cases} 2K\left\{\ln(-\tau_2) + \ln(r+2) - \ln 2\right\} & \text{for } \tau_1 < \tau_2 - r\,, \\ 2K\{\ln(-\tau_2) + \ln(-\tau_1) - \ln(r - \tau_1 - \tau_2) + \ln(r+2)\}\,, & \text{for } \tau_2 - r < \tau_1 < \tau_2 + r \\ 2K\{\ln(-\tau_1) + \ln(r+2) - \ln 2\} & \text{for } \tau_2 + r < \tau_1 < 0\,. \end{cases} \tag{3.2.24}$$



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Figure 3.10: Probability rule for N=2 using Venn Diagram.

Note that the result is symmetric under the exchange of τ_1 and τ_2 even though at the intermediate stages of the analysis this symmetry was not manifest.

Expressed in terms of physical time the above solution takes the form:

$$P_{12}(t_1, t_2) = \begin{cases} \{(r+2)/2\}^{2K} e^{-2Kt_2} \quad \text{for } t_1 < -\ln(r + e^{-t_2}) \,, \\ (r+2)^{2K} e^{-2K(t_1+t_2)} (r + e^{-t_1} + e^{-t_2})^{-2K} \quad \text{for } -\ln(r + e^{-t_2}) < t_1 < -\ln(e^{-t_2} - r) \,, \\ \{(r+2)/2\}^{2K} e^{-2Kt_1} \quad \text{for } t_1 > -\ln(e^{-t_2} - r) \,. \end{cases} \tag{3.2.25}$$

If $e^{-t_2} - r$ is negative then the third case is not relevant and in the second case there will be no upper bound on t_1. Physically this can be understood by noting that in this case $\tau_2 > -r$ and C_2 will never come inside the past light cone of C_1 even when τ_1 reaches its maximum value 0.

Our interest lies in computing the probability that at least one of the two objects survives till time t. Let us denote this by $\widetilde{P}_{12}(t)$. This is given by the sum of the probability that C_1 survives till time t and the probability that C_2 survives till time t, but we have to subtract from it the probability that both C_1 and C_2 survive till time t since this will be counted twice otherwise. This can be seen from the Venn diagram of two objects shown in Fig. 3.10. Therefore, we have

$$\widetilde{P}_{12}(t) = P_1(t) + P_2(t) - P_{12}(t,t)\,. \tag{3.2.26}$$

From this we can compute the probability that the last one to survive decays between t and $t+\delta t$ as

$$-\delta t\,\frac{d}{dt}\widetilde{P}_{12}(t)\,. \tag{3.2.27}$$

Therefore, the life expectancy of the combined system is given by

$$\bar{t}_{12} = -\int_0^\infty dt\,t\,\frac{d}{dt}\widetilde{P}_{12}(t,t) = \int_0^\infty dt\,\{P_1(t) + P_2(t) - P_{12}(t,t)\}\,, \tag{3.2.28}$$

where in the second step we have integrated by parts and used (3.2.26). Each of the first two integrals gives the result $\bar{t}_1$ computed in (3.2.17). For the last integral since we have to evaluate $P_{12}(t_1,t_2)$ at $t_1 = t_2 = t$ only the middle expression in (3.2.25) is relevant, and we get

$$\begin{aligned}\int_0^\infty P_{12}(t,t)dt &= \int_0^\infty (r+2)^{2K} e^{-4Kt}(r+2e^{-t})^{-2K}dt \\ &= 2^{-4K} r^{2K}(r+2)^{2K} B\left(\frac{2}{2+r}; 4K, -2K\right)\,. \end{aligned} \tag{3.2.29}$$

Combining this with the result for $\bar{t}_1$ given in (3.2.17) and replacing K by $1/2T$ we get

$$\begin{aligned}\bar{t}_{12} &= 2(r+2)^{1/T}\left[B\left(\frac{1}{2+r}; \frac{1}{T}, 0\right) - B\left(\frac{1-r}{2}; \frac{1}{T}, 0\right) + T\,\frac{(1-r)^{1/T}}{2^{1/T}}\right] \\ &\quad -2^{-2/T} r^{1/T}(r+2)^{1/T} B\left(\frac{2}{2+r}; \frac{2}{T}, -\frac{1}{T}\right)\,. \end{aligned} \tag{3.2.30}$$

We can now check various limits. First of all we can study the $r \to 0$ limit using the result

$$B(x; 2\alpha, -\alpha) \simeq \frac{1}{\alpha}\,(1-x)^{-\alpha}\,, \tag{3.2.31}$$

for x close to 1. This gives $\lim_{r\to 0} \bar{t}_{12} = T$. This is in agreement with the fact that if the two objects remain at the same point then their combined life expectancy is the same as that of individual objects.

If on the other hand we take the limit of large T then, using the result

$$B(x; \alpha, \beta) \simeq \frac{1}{\alpha} \tag{3.2.32}$$

for small α, we get $\bar{t}_{12} \simeq 3T/2$. Therefore, the life expectancy of the two objects together is 3/2 times that of an isolated object. This is consistent with the fact that if the inverse decay rate of individual objects is large then typically there will be enough time for the two objects to go out of each other's horizon before they decay. Therefore, we can treat them as independent objects and recover the result (3.1.3). Mathematically this can be seen from the fact that when T is large and $t \sim T$ then $P_{12}(t,t)$ given in the middle expression of (3.2.25) approaches e^{-4Kt}, which in turn is approximately equal to the square of $P_1(t)$ given in (3.2.14).

3.3 Vacuum decay in 3+1 dimensional de Sitter space

In this section we shall repeat the analysis of §3.2 for 3+1 dimensional de Sitter space-time. Since the logical steps remain identical, we shall point out the essential differences arising in the two cases and then describe the results.

The metric of the 3+1 dimensional de Sitter space is given by

$$ds^2 = -dt^2 + e^{2t}(dx^2 + dy^2 + dz^2) = \tau^{-2}(-d\tau^2 + dx^2 + dy^2 + dz^2), \quad \tau \equiv -e^{-t}\,. \tag{3.3.1}$$

There are of course various other coordinate systems in which we can describe the de Sitter metric, but the coordinate system used in (3.3.1) is specially suited for describing out universe, with (x, y, z) labelling comoving coordinates and t denoting the cosmic time in which the constant t slices have uniform microwave background temperature. This form of the metric uses the observed flatness of the universe. The actual metric at present is deformed due to the presence of matter density, and also there is a lower cut-off on t since our universe has a finite age of the order of the inverse Hubble constant. But both these effects will become irrelevant within a few Hubble time and we ignore them. In §3.4.3 we shall study these effects, but at present our goal is to get an analytic result under these simplifying assumptions.

3.3.1 Isolated comoving object

First consider the case of an isolated object. The calculation proceeds as in §3.2.1. However, in computing the volume of the past light cone in Fig. 3.3 we have to take into account the fact that for each σ, the light cone is a sphere of radius $(\tau - \sigma)$. Since the coordinate radius of the sphere is $(\tau - \sigma)$ and the space-time volume element scales as $1/\sigma^4$ we get the volume of the past light cone of the object between τ and $\tau + \delta\tau$ to be

$$\delta\tau \int_{-\infty}^{\tau} \frac{d\sigma}{\sigma^4} 4\pi(\tau - \sigma)^2 = -\frac{4}{3}\pi\tau^{-1}\delta\tau\,. \tag{3.3.2}$$

This replaces the right hand side of (3.2.4). Therefore, (3.2.5) takes the form

$$P_0'(\tau) = \frac{4}{3}\pi\tau^{-1}K\,P_0(\tau)\,, \tag{3.3.3}$$

with the solution

$$\ln P_0(\tau) = \frac{4}{3}\pi K \ln(-\tau)\,, \tag{3.3.4}$$

$$P_0(t) = \exp\left(-\frac{4}{3}\pi K t\right)\,. \tag{3.3.5}$$

From this we can calculate the life expectancy of the isolated object to be

$$T = \int_0^\infty P_0(t)dt = \frac{3}{4\pi K}\,. \tag{3.3.6}$$

3.3.2 A pair of comoving objects

The additional complication in the case of two objects comes from having to evaluate the contribution of the past light come of the first object between τ_1 and $\tau_1 + \delta\tau_1$ in situations depicted in Figs. 3.5 and 3.8. Let us consider Fig. 3.8 since Fig. 3.5 can be considered as a special case of Fig. 3.8 with $\tau_2 = -1$. Now in Fig. 3.8 which occurs for $\tau_2 - r < \tau_1 < \tau_2 + r$, the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$ lies partly inside the past light cone of C_2. We need to subtract this contribution from the total volume of the past light cone of C_1 between τ_1 and $\tau_1 + \delta\tau_1$, since the assumption that C_2 survives till τ_2 rules out the formation of a bubble inside the past light cone of C_2. Our goal will be to calculate this volume.

Examining Fig. 3.8 we see that the intersection of the past light cones of C_1 at τ_1 and C_2 at τ_2 occur at $\tau = \sigma$ for $\sigma < (\tau_1 + \tau_2 - r)/2$. At a value of σ satisfying this constraint, the past light cone of C_1 at τ_1 is a sphere of coordinate radius $r_1 = (\tau_1 - \sigma)$ and the past light cone of C_2 at τ_2 is a sphere of coordinate radius $r_2 = (\tau_2 - \sigma)$. The centers of these spheres, lying at the comoving coordinates of the two objects have a coordinate separation of r. A simple geometric analysis shows that the coordinate area of the part of the first sphere that is inside the second sphere is given by

$$\pi\frac{r_1}{r}\left\{{r_2}^2-(r_1-r)^2\right\}=\pi\frac{(\tau_1-\sigma)}{r}(\tau_2-\tau_1+r)(\tau_1+\tau_2-r-2\sigma)\,. \tag{3.3.7}$$

Taking into account the fact that physical volumes are given by $1/\sigma^4$ times the coordinate volume we get the following expression for the volume of the past light cone of C_1 between τ_1 and $\tau_1+\delta\tau_1$ that is inside the past light cone of C_2:

$$\begin{aligned} & \frac{\pi}{r}(\tau_2-\tau_1+r)\,\delta\tau_1\int_{-\infty}^{(\tau_1+\tau_2-r)/2}\frac{d\sigma}{\sigma^4}(\tau_1-\sigma)(\tau_1+\tau_2-r-2\sigma) \\ = & \frac{2\,\pi}{3\,r}(\tau_2-\tau_1+r)\,\delta\tau_1\,\frac{(3r-\tau_1-3\tau_2)}{(r-\tau_1-\tau_2)^2}\,. \end{aligned} \tag{3.3.8}$$

As already mentioned the excluded volume in case of Fig. 3.5 can be found by setting $\tau_1=\tau$ and $\tau_2=-1$ in (3.3.8).

We are now ready to generalize all the results of §3.2. Let us begin with (3.2.12). Its generalization to the 3+1 dimensional case takes the form

$$\frac{d}{d\tau}\ln P_1(\tau)=\begin{cases} \dfrac{2\pi K}{3}\left[\dfrac{2}{\tau}+\dfrac{(-1+r-\tau)(3+3r-\tau)}{r(\tau-r-1)^2}\right] & \text{if } \tau<r-1 \\ \dfrac{4\pi K}{3\tau} & \text{if } \tau>r-1 \end{cases} \tag{3.3.9}$$

Its solution is given by

$$\ln P_1(\tau)=\begin{cases} \frac{4\pi K}{3}\left(\ln(-\tau)+\frac{\tau}{2r}-\ln(-\tau+r+1)+\frac{2(r+1)}{r(\tau-r-1)}+\ln(r+2)+\frac{5r+6}{2r(r+2)}\right) & \text{if } \tau<r-1\,, \\ \frac{4\pi K}{3}\left(\ln(-\tau)+\ln(r+2)-\ln 2-\frac{r}{2(r+2)}\right) & \text{if } \tau>r-1\,. \end{cases} \tag{3.3.10}$$

Expressing this in terms of t using $\tau = -e^{-t}$ and $T \equiv 3/(4\pi K)$ we get

$$P_1(t) = \begin{cases} (e^{-t}+r+1)^{-\frac{1}{T}}(r+2)^{\frac{1}{T}} \exp\left[-\frac{t}{T} + \frac{1}{T}\left\{-\frac{e^{-t}}{2r} - \frac{2(r+1)}{r(e^{-t}+r+1)} + \frac{5r+6}{2r(r+2)}\right\}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t < -\ln(1-r)\,, \\ \\ \left(\frac{r+2}{2}\right)^{\frac{1}{T}} \exp\left[-\frac{t}{T} - \frac{r}{2T(r+2)}\right] \qquad\qquad \text{for } t > -\ln(1-r)\,. \end{cases} \tag{3.3.11}$$

The same expression holds for the survival probability $P_2(t)$ of C_2. From this we can find the life expectancy of C_1

$$\bar{t}_1 = \int_0^\infty P_1(t)dt\,. \tag{3.3.12}$$

As in the 1+1 dimensional case, $\bar{t}_1$ is slightly larger than T but this is simply due to the choice of initial condition that both observers are assumed to exist at $t = 0$. In Fig. 3.11 we have plotted the ratio $\bar{t}_1/T$ as a function of T for various values of r, and as we can see the result remains close to 1. More discussion on $\bar{t}_1$ can be found below (3.3.22).

Next we consider the generalization of (3.2.18)-(3.2.22). The analysis is straightforward and we get the results

$$\begin{aligned} \frac{\partial \ln P_{12}(\tau_1,\tau_2)}{\partial \tau_1} &= 0 \qquad \text{for} \quad \tau_1 < \tau_2 - r\,, \\ \frac{\partial \ln P_{12}(\tau_1,\tau_2)}{\partial \tau_1} &= \frac{2\pi K}{3}\left[\frac{2}{\tau_1} + \frac{(r-\tau_1+\tau_2)(3r-\tau_1-3\tau_2)}{r\,(r-\tau_1-\tau_2)^2}\right] \qquad \text{for} \quad \tau_2 - r < \tau_1 < \tau_2 + r \\ \frac{\partial \ln P_{12}(\tau_1,\tau_2)}{\partial \tau_1} &= \frac{4\pi K}{3\tau_1} \qquad \text{for} \quad \tau_2 + r < \tau_1 < 0\,. \end{aligned} \tag{3.3.13}$$

The solution to these equations, subject to the boundary condition $P_{12}(\tau_1 = -1, \tau_2) = P_2(\tau_2) =$

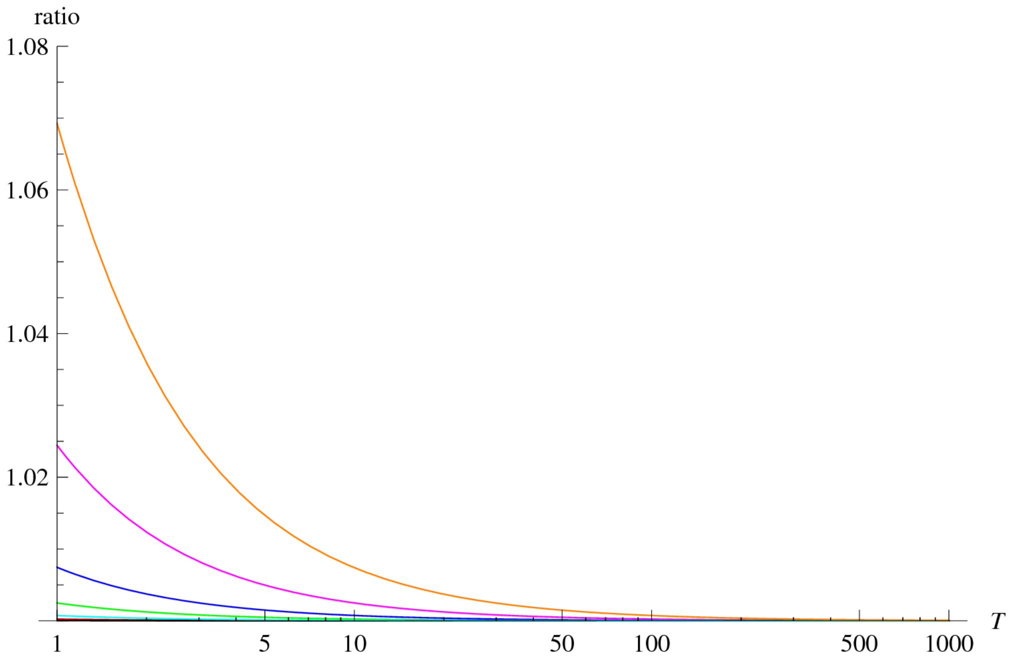


Figure 3.11: The figure showing the ratio $\bar{t}_1/T$ for $r = .0003$, $.001$, $.003$, $.01$, $.03$, $.1$ and $.3$. For $r \leq .003$ the ratio is not distinguishable from 1 in this scale.

$P_1(\tau_2)$ is given by

$$\ln P_{12}(\tau_1,\tau_2) = \begin{cases} \frac{4\pi K}{3}\left[\ln(-\tau_2) + \ln(r+2) - \frac{r}{2(r+2)} - \ln 2\right] & \text{if } \tau_1 < \tau_2 - r \\ \frac{4\pi K}{3}\Big[\ln(-\tau_1) + \ln(-\tau_2) - \ln(-\tau_1 - \tau_2 + r) + \frac{\tau_1+\tau_2}{2r} - \frac{2\tau_1\tau_2}{r(\tau_1+\tau_2-r)} & \\ \qquad + \ln(r+2) + \frac{1}{r+2}\Big] & \text{if } \tau_2 - r < \tau_1 < \tau_2 + r \\ \frac{4\pi K}{3}\left[\ln(-\tau_1) + \ln(r+2) - \frac{r}{2(r+2)} - \ln 2\right] & \text{if } \tau_2 + r < \tau_1 < 0 \end{cases} \tag{3.3.14}$$

In terms of the physical time, and $T = 3/(4\pi K)$, this becomes

$$P_{12}(t_1,t_2) = \begin{cases} \{(r+2)/2\}^{1/T} \exp\left[-\dfrac{r}{2T(r+2)} - \dfrac{t_2}{T}\right] \quad \text{if } \; t_1 < -\ln(r+e^{-t_2}) \\ \\ (r+2)^{1/T}(e^{-t_1}+e^{-t_2}+r)^{-\frac{1}{T}} \\ \quad \times \exp\left[\dfrac{1}{T(r+2)} - \dfrac{1}{T}(t_1+t_2) - \dfrac{1}{2Tr}(e^{-t_1}+e^{-t_2}) + \dfrac{2}{Tr}\dfrac{1}{e^{t_1}+e^{t_2}+re^{t_1+t_2}}\right] \\ \qquad\qquad\qquad\qquad \text{if } \; -\ln(r+e^{-t_2}) < t_1 < -\ln(e^{-t_2}-r) \\ \\ \{(r+2)/2\}^{1/T} \exp\left[-\dfrac{r}{2T(r+2)} - \dfrac{t_1}{T}\right] \quad \text{if } \; t_1 > -\ln(e^{-t_2}-r) \end{cases} \tag{3.3.15}$$

This gives

$$P_{12}(t,t) = (r+2)^{1/T} e^{\frac{1}{T\,(r+2)}} (2e^{-t}+r)^{-\frac{1}{T}} \exp\left[-\frac{2}{T}t - \frac{1}{T\,r}e^{-t} + \frac{2}{T\,r}\frac{1}{2e^t+re^{2t}}\right]. \tag{3.3.16}$$

In terms of this, and the functions $P_1 = P_2$ given in (3.3.11), we can calculate the probability $\widetilde{P}_{12}$ of at least one of the two objects surviving till time t using

$$\widetilde{P}_{12}(t) = P_1(t) + P_2(t) - P_{12}(t,t) \tag{3.3.17}$$

and the combined life expectancy of two objects using the analog of (3.2.28)

$$\bar{t}_{12} = \int_0^\infty \widetilde{P}_{12}(t)dt = \int_0^\infty dt\,\{P_1(t)+P_2(t)-P_{12}(t,t)\} = 2\,\bar{t}_1 - \int_0^\infty dt\,P_{12}(t,t)\,. \tag{3.3.18}$$

For the integral of $P_{12}(t,t)$ one can write down an expression in terms of special functions as follows. Defining y via

$$2 + re^t = \frac{2+r}{y} \tag{3.3.19}$$

for $r \neq 0$, we get

$$\int_0^\infty dt\ P_{12}(t,t) = [r(r+2)^{-1}]^{1/T} e^{\frac{1}{(r+2)T}} \int_0^1 dy\ y^{-1+2/T} \left(1 - \frac{2y}{2+r}\right)^{-1-1/T} \exp\left[-\frac{y}{(2+r)T}\right].$$

Now, using the result

$$\int_0^1 dy\ \frac{y^{a-1}(1-y)^{c-a-1}}{(1-u\,y)^b} e^{v\,y} = B(a, c-a)\Phi_1(a,b,c;u,v) \tag{3.3.20}$$

with $Re\,c > Re\,a > 0, |u| < 1$, B the beta function and Φ_1 the confluent hypergeometric series of two variables (Humbert series), we get

$$\int_0^\infty dt\ P_{12}(t,t) = \frac{T}{2}\left[\frac{r}{(r+2)}\right]^{1/T} e^{\frac{1}{T(r+2)}}\ \Phi_1\left(\frac{2}{T}, 1+\frac{1}{T}, 1+\frac{2}{T}; \frac{2}{2+r}, -\frac{1}{(2+r)T}\right) \tag{3.3.21}$$

Φ_1 has a power series expansion

$$\Phi_1(a,b,c;u,v) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_{m+n} m! n!} u^m v^n\,, \quad |u| < 1 \tag{3.3.22}$$

where $(a)_m \equiv a(a+1)\cdots(a+m-1)$.

Unfortunately we have not been able to find an expression for $\bar{t}_1 = \int_0^\infty dt\, P_1(t)$ in terms of special functions. However, we can write down a series expansion for this that will be suitable for studying its behaviour for small r. The integral of (3.3.11) from $t = -\ln(1-r)$ to ∞ is straightforward and yields

$$T\,(1-r)^{1/T}(r+2)^{1/T}2^{-1/T}\exp\left[-\frac{r}{2T(r+2)}\right]. \tag{3.3.23}$$

The integral of (3.3.11) from $t = 0$ to $-\ln(1-r)$ can be analyzed by making a change of variable from t to y via $e^{-t} = (1 - y\,r)$. In terms of this variable the integral can be expressed as

$$r\int_0^1 dy\left(1-\frac{yr}{r+2}\right)^{-1/T}(1-yr)^{-1+1/T}\exp\left[-\frac{2y^2r(r+1)}{T(2+r)^3}\left(1-\frac{yr}{2+r}\right)^{-1}\right]\exp\left[\frac{yr^2}{2T(2+r)^2}\right].$$

Using series expansion of the second and third terms in the integrand we get

$$\sum_{m,n=0}^{\infty}\frac{1}{m!n!}\left(1-\frac{1}{T}\right)_m(-1)^n\frac{2^nr^{m+n+1}(r+1)^n}{T^n(2+r)^{3n}}$$
$$\int_0^1 dy\,y^{m+2n}\left(1-\frac{yr}{2+r}\right)^{-n-1/T}\exp\left[\frac{yr^2}{2T(2+r)^2}\right]. \tag{3.3.24}$$

The integral over y can be expressed in terms of Φ_1 using (3.3.20). Adding (3.3.23) to this we get

$$\begin{aligned}\bar{t}_1 &= T\,(1-r)^{1/T}(r+2)^{1/T}2^{-1/T}\exp\left[-\frac{r}{2T(r+2)}\right]\\ &+\sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{1}{m+2n+1}\left(1-\frac{1}{T}\right)_m(-1)^n\frac{2^nr^{m+n+1}(r+1)^n}{T^n(2+r)^{3n}}\\ &\qquad\Phi_1\left(m+2n+1,n+\frac{1}{T},m+2n+2;\frac{r}{2+r},\frac{r^2}{2T(2+r)^2}\right)\end{aligned} \tag{3.3.25}$$

It can be checked using (3.3.11), (3.3.12), (3.3.16) and (3.3.18) that for $r \to 0$ we get $\bar{t}_{12}/\bar{t}_1 = 1$ and for $T \to \infty$ we get $\bar{t}_{12}/\bar{t}_1 = 3/2$. The values of 'gain' $\equiv \bar{t}_{12}/\bar{t}_1$ for different values of r have been plotted against T in Fig. 3.1.

3.3.3 The case of small initial separation

Since from practical considerations the small r region is of interest, it is also useful to consider the expansion of $\bar{t}_{12}/\bar{t}_1$ for small r. For this we have to analyze the behaviour of $\bar{t}_1$ as well as that of $\int_0^\infty dt\, P_{12}(t,t)$ for small r. Let us begin with $\bar{t}_1$ given in (3.3.25). It can be easily seen that this is given by $T + \mathcal{O}(r)$ with the contribution T coming from the first term. However, the contribution from $\int_0^\infty dt\, P_{12}(t,t)$ has a more complicated behaviour at small r. This is related to the fact that in the $r \to 0$ limit the fourth argument of Φ_1 in (3.3.21) approaches 1, and in this limit the series expansion (3.3.22) diverges. To study the small r behaviour we shall go back to the original expression for $P_{12}(t,t)$ given in (3.3.16). We change variable to $v = e^{-t}/r$ and write

$$\begin{aligned}
\int_0^\infty dt\, P_{12}(t,t) &= (r+2)^{1/T} e^{1/T(r+2)} r^{1/T} \int_0^{1/r} \frac{dv}{v} (2v+1)^{-1/T} v^{2/T} \exp\left[-\frac{v}{T(2v+1)}\right] \\
&= (r+2)^{1/T} e^{1/T(r+2)} r^{1/T} \int_0^{1/r} \frac{dv}{v} v^{2/T} \left[(2v+1)^{-1/T} \exp\left[-\frac{v}{T(2v+1)}\right]\right. \\
&\quad \left. -(2v)^{-1/T} \exp\left[-\frac{1}{2T}\right]\right] + (r+2)^{1/T} 2^{-1/T} \exp\left[\frac{1}{T(r+2)} - \frac{1}{2T}\right] T\,,
\end{aligned} \tag{3.3.26}$$

where in the last step we have subtracted an integral from the original integral and compensated for it by adding the explicit result for the integral. This subtraction makes the integral convergent even when we replace the upper limit $1/r$ by ∞. Taking the small r limit we get

$$\begin{aligned}
\int_0^\infty dt\, P_{12}(t,t) &= 2^{1/T} r^{1/T} \int_0^\infty \frac{dv}{v} v^{2/T} \left[(2v+1)^{-1/T} \exp\left[-\frac{v}{T(2v+1)} + \frac{1}{2T}\right] - (2v)^{-1/T}\right] \\
&\quad + T + \mathcal{O}(r)\,.
\end{aligned} \tag{3.3.27}$$

Combining this with the earlier result that for $r \to 0$, $\bar{t}_1 \simeq T + \mathcal{O}(r)$ and using (3.3.18) we get

$$\frac{\bar{t}_{12}}{\bar{t}_1} = 2 - \frac{1}{\bar{t}_1}\int_0^\infty dt\, P_{12}(t,t) = 1 + A(T)\, r^{1/T}\,, \tag{3.3.28}$$

where

$$A(T) = T^{-1}2^{1/T}\int_0^\infty dv\, v^{-1+2/T}\left[(2v)^{-1/T} - (2v+1)^{-1/T}\exp\left(-\frac{v}{T(2v+1)} + \frac{1}{2T}\right)\right] \tag{3.3.29}$$

The numerical values of $A(T)$ are moderate – for example $A(5) \simeq 0.439$ and $A(10) \simeq 0.457$. A plot of $A(T)$ as a function of T has been shown in Fig. 3.12. The $1/T$ exponent of r shows that even if we begin with small r, for moderately large T (say $T \sim 5$) we can get moderate enhancement in life expectancy.

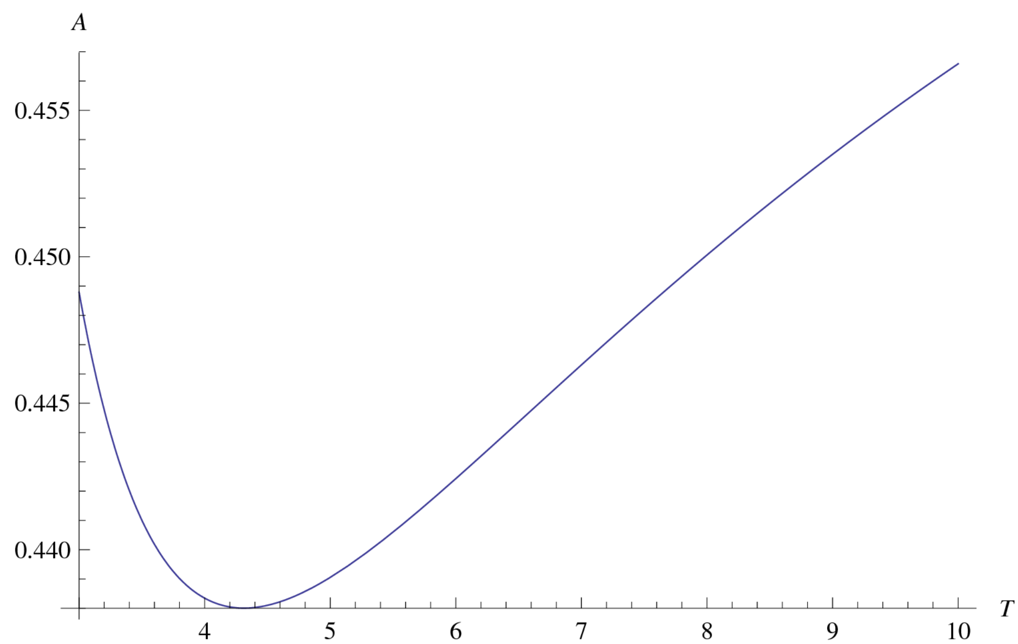


Figure 3.12: The coefficient of the $r^{1/T}$ term in the expression for $\bar{t}_{12}/\bar{t}_1$ as a function of T.

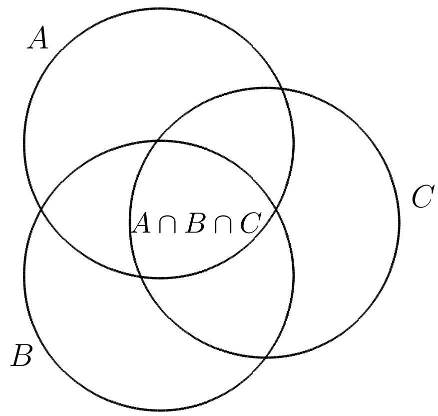


Figure 3.13: The Venn diagram illustrating that the survival probability of one of A, B or C, denoted by $P(A \cup B \cup C)$, is given by $P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$.

3.4 Generalizations

In this section we shall discuss various possible generalizations of our results.

3.4.1 Multiple objects in de Sitter space

We shall begin by discussing the case of three objects C_1, C_2 and C_3 placed at certain points in $3+1$ dimensional de Sitter space-time and analyze the probability that at least one of them will survive till time t. Let $P_{123}(t_1, t_2, t_3)$ denote the probability that C_1 survives till time t_1, C_2 survives till time t_2 and C_3 survives till time t_3. Similarly $P_{ij}(t_i, t_j)$ for $1 \leq i, j \leq 3$ will denote the probability that C_i survives till t_i and C_j survives till t_j and $P_i(t_i)$ will denote the probability that C_i survives till t_i. All probabilities are defined under the prior assumption that all objects are alive at $t = 0$. These probabilities can be calculated by generalizing the procedure described in §3.2 and §3.3 by constructing ordinary differential equations in one of the arguments at fixed values of the other arguments. The geometry of course now becomes more involved due to the fact that the past light cone of one object will typically intersect the past light cones of the other objects which themselves may have overlaps, and one has to carefully subtract the correct volume. But the analysis is straightforward.

The quantity of direct interest is the probability $\widetilde{P}_{123}(t)$ that at least one of the objects survives till time t. With the help of the Venn diagram given in Fig. 3.13 we get

$$\widetilde{P}_{123}(t) = \Big(P_1(t) + P_2(t) + P_3(t) - P_{12}(t,t) - P_{13}(t,t) - P_{23}(t,t) + P_{123}(t,t,t)\Big) . \quad (3.4.1)$$

Using this we can calculate the life expectancy of the combined system as

$$-\int_0^\infty dt\, t\, \frac{d}{dt}\widetilde{P}_{123}(t) = \int_0^\infty dt\, \widetilde{P}_{123}(t) . \quad (3.4.2)$$

The generalization to the case of N objects is now obvious. The relevant formula is

$$\widetilde{P}_{12\cdots N}(t) = \Big(\sum_{i=1}^{N} P_i(t) - \sum_{i<j}^{N} P_{ij}(t,t) + \sum_{i<j<k}^{N} P_{ijk}(t,t,t) + \cdots (-1)^{N+1} P_{12\cdots N}(t,t,\cdots,t)\Big) \quad (3.4.3)$$

where $P_{i_1\cdots i_k}(t_{i_1},\cdots t_{i_k})$ are again computed by solving ordinary differential equations in one of the variables. Once $\widetilde{P}_{12\cdots N}(t)$ is computed we can get the life expectancy of the combined system by using

$$\bar{t}_{12\cdots N} = \int_0^\infty \widetilde{P}_{12\cdots N}(t) dt . \quad (3.4.4)$$

3.4.2 Realistic trajectories

Another generalization involves considering a situation where multiple objects originate at the same space time point and then follow different trajectories, eventually settling down at different comoving coordinates. This has been illustrated in Fig. 3.14. This represents the realistic situation since by definition different civilizations of the same race must originate at some common source. We can now generalize our analysis to take into account the possibility of decay during the journey as well.

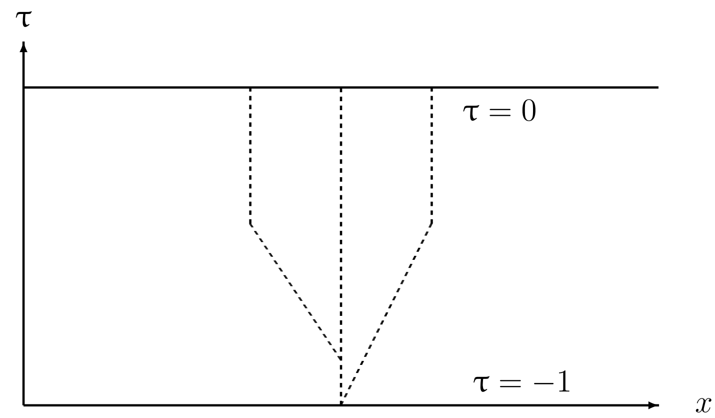


Figure 3.14: Multiple objects originating from the same space-time point. Different dashed lines represent the trajectories followed by different objects.

Eqs.(3.4.3) and (3.4.4) still holds, but the computation of $P_{i_1 \cdots i_k}(t_{i_1}, \cdots t_{i_k})$ will now have to be done by taking into account the details of the trajectories of each object and the overlaps of their past light cones. The principle remains the same, and we can set up ordinary differential equations for each of these quantities. The only difference is that the spatial separation between the i-th object at $\tau = \tau_i$ and the j-th object at $\tau = \tau_j$ will now depend on τ_i and τ_j according to the trajectories followed by them.

This analysis can be easily generalized to the case where each of the descendant objects in turn produces its own descendants which settle away from the parent object and eventually go outside each other's horizon due to the Hubble expansion. If this could be repeated at a rate faster than the vacuum decay rate then we can formally ensure that some of the objects will survive vacuum decay [15]. However, since within a few Hubble periods most of the universe will split up into gravitationally bound systems outside each other's horizon, in practice this is going to be an increasingly difficult task.

3.4.3 Matter effect

A third generalization will involve relaxing the assumption that the universe has been de Sitter throughout its past history. While de Sitter metric will be a good approximation after a few Hubble periods, within the next few Hubble periods we shall still be sensitive to the fact that the universe had been matter dominated in the recent past and had a beginning. This will change the form of the metric (3.3.1) to

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \tag{3.4.5}$$

where $a(t)$ is determined from the Friedman equation

$$\frac{1}{a}\frac{da}{dt} = \sqrt{\frac{8\pi G}{3}\left(\rho_\Lambda + \frac{\rho_m}{a^3}\right)} \tag{3.4.6}$$

in the convention that the value of a is 1 today and ρ_Λ and ρ_m are the energy densities due to cosmological constant and matter today. Since we have chosen the unit of time so that the Hubble parameter in the cosmological constant dominated universe is 1, we have $\sqrt{8\pi G\rho_\Lambda/3} = 1$. Defining[6]

$$c \equiv \rho_m/\rho_\Lambda \simeq 0.45 \tag{3.4.7}$$

we can express (3.4.6) as

$$\frac{1}{a}\frac{da}{dt} = \sqrt{1 + ca^{-3}}\,. \tag{3.4.8}$$

[6]We use cosmological parameters given in [97].

Let τ be the conformal time defined via

$$d\tau = dt/a(t) \tag{3.4.9}$$

with the boundary condition $\tau \to 0$ as $t \to \infty$. Then (3.4.6) takes the form

$$\frac{1}{a^2}\frac{da}{d\tau} = \sqrt{1 + ca^{-3}}\,, \tag{3.4.10}$$

whose solution is

$$\tau = -\int_a^\infty \frac{db}{b^2\sqrt{1+cb^{-3}}} = -\int_0^{1/a} \frac{dv}{\sqrt{1+cv^3}} = -\frac{1}{a}\,{}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{4}{3},-\frac{c}{a^3}\right). \tag{3.4.11}$$

This implicitly determines a as a function of τ. The metric is given by

$$ds^2 = a(\tau)^2(-d\tau^2 + dx^2 + dy^2 + dz^2)\,. \tag{3.4.12}$$

Using the experimental value $c \simeq 0.45$ we get that $\tau \to \tau_0 \simeq -3.7$ as $a \to 0$, showing that the big bang singularity is at $\tau \simeq -3.7$.[7] We also have that at $a = 1$, $\tau \simeq -0.95$. This is not very different from the value $\tau = -1$ for pure de Sitter space-time with which we have worked. However, we shall now show that the decay rate in the matter dominated epoch of the universe differs significantly from that in the cosmological constant dominated epoch. The decay rate of an isolated observer at some

[7] Of course close to the singularity the universe becomes radiation dominated but given the short span of radiation dominated era we ignore that effect for the current analysis.

value of the conformal time τ is given by the following generalization of (3.3.2), (3.3.3):

$$\begin{aligned} \frac{d}{d\tau}\ln P_0(\tau) &= -4\pi K\int_{\tau_0}^{\tau} d\sigma\, a(\sigma)^4\,(\tau-\sigma)^2 \\ &= -4\pi K\int_{b=0}^{a(\tau)}\frac{db}{b^2\sqrt{1+cb^{-3}}}\,b^4\,\left\{\tau+\frac{1}{b}\,{}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{4}{3},-\frac{c}{b^3}\right)\right\}^2 \end{aligned} \tag{3.4.13}$$

where in the second step we have changed the integration variable from σ to $b = a(\sigma)$. From this we can compute the decay rate:

$$\begin{aligned} D(t) &\equiv -\frac{d}{dt}\ln P_0 = -\frac{1}{a(t)}\frac{d}{d\tau}\ln P_0(\tau) \\ &= \frac{4\pi K}{a(t)}\int_{b=0}^{a(t)}\frac{db}{\sqrt{1+cb^{-3}}}\,b^2\,\left\{\tau(t)+\frac{1}{b}\,{}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{4}{3},-\frac{c}{b^3}\right)\right\}^2. \end{aligned} \tag{3.4.14}$$

Using the information that today $a = 1$ and $\tau \simeq -0.95$ we get

$$D(t)|_{\text{today}} \simeq \frac{4\pi K}{3}\times 0.067 \simeq \frac{0.067}{T}\,. \tag{3.4.15}$$

This is lower than the corresponding rate T^{-1} in the de Sitter epoch by about a factor of 15. The growth of the decay rate with scale factor has been shown in Fig. 3.2.

Our analysis of §3.3 can now be repeated for two or more observers and also for general trajectory discussed in §3.4.2 with this general form of the metric to get more accurate computation of the life expectancy. These corrections will be important if $T \lesssim 1$ and the decay takes place within a few Hubble period from now. On the other hand if T is large (say $\gtrsim 10$) then the decay is likely to take place sufficiently far in the future by which time the effect of our matter dominated past will have insignificant effect on the results.

The fact that the decay rate increases with time till it eventually settles down to a constant value in the de Sitter epoch has some important consequences:

1. We have already seen from (3.4.15) that the decay rate today is about 15 times smaller than the decay rate in the de Sitter epoch. Eq.(3.4.14) for $a(t) = 2$ shows that even when the universe will be double its size compared to today, the decay rate will remain at about 27% of the decay rate in the de Sitter epoch. Since most of the journeys to different parts of the universe – if they take place at all – are likely to happen during this epoch, we see that the probability of decay during the journey will be considerably less than that in the final de Sitter phase. This partially justifies our analysis in §3.3 where we neglected the probability of decay during the journey. This also shows that if we eventually carry out a detailed numerical analysis taking into account the effect discussed in §3.4.2, it should be done in conjunction with the analysis of this subsection taking into account the effect of matter.

2. It is also possible to see from (3.4.14), (3.4.15) (or Fig. 3.2) that the decay rate in the past was even smaller than that of today. If $D(t)$ denotes the decay rate at time t defined in (3.4.14), then the average decay rate in our past can be defined as

$$\frac{1}{t_1}\int_0^{t_1} D(t)dt\,, \tag{3.4.16}$$

where t_1 denotes the current age of the universe given by

$$t_1 = \int_0^1 da\left(\frac{da}{dt}\right)^{-1} = \int_0^1 \frac{da}{a\sqrt{1+ca^{-3}}} \simeq 0.79\,. \tag{3.4.17}$$

In physical units t_1 is about 1.38×10^{10} years. The evaluation of (3.4.16) can be facilitated using the observation that $D(t)dt$ is K times the volume enclosed between the past light cones of the object at times t and $t+dt$. Therefore, $\int_0^{t_1} dtD(t)$ must be K times the total volume enclosed by the past light cone of the object at t_1. This can be easily computed, yielding

$$\begin{aligned} \int_0^{t_1} D(t)dt &= \frac{4}{3}\pi K \int_{\tau_0}^{\tau} d\sigma\, a(\sigma)^4 (\tau - \sigma)^3 \\ &= \frac{1}{T}\int_{b=0}^{a(\tau)} \frac{db}{b^2\sqrt{1+cb^{-3}}}\, b^4 \left\{\tau + \frac{1}{b}\, {}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{4}{3},-\frac{c}{b^3}\right)\right\}^3 . \end{aligned} \tag{3.4.18}$$

For $c \simeq 0.45$ and τ given by today's value -0.95 this gives

$$\int_0^{t_1} D(t)dt = \frac{0.014}{T} . \tag{3.4.19}$$

Using (3.4.16), (3.4.17) we get the average decay rate to be $0.018/T$. This is about 3.7 times smaller than the present decay rate given in (3.4.15) and 56 times smaller than the decay rate $1/T$ in the de Sitter epoch.

Integrating the equation $dP_0/dt = -D(t)P_0(t)$ we get

$$\ln P_0(t_1) = -\int_0^{t_1} dt D(t) = -\frac{0.014}{T} . \tag{3.4.20}$$

Requiring this to be not much smaller than -1 (which is equivalent to requiring that the inverse of the average decay rate (3.4.16) be not much smaller than the age of the universe t_1) gives $T \gtrsim 0.014$. This is much lower than what one might have naively predicted by equating the lower bound on T to the age of the universe i.e. $T \gtrsim t_1 \sim 0.79$. Recalling that the unit of time is set by the Hubble period in the de Sitter epoch which is about 1.7×10^{10} years, the bound $T \gtrsim 0.014$ translates to a lower bound of order 2.5×10^8 years. Since the current decay rate is about 15 times smaller than that in the final de Sitter epoch, we see that the lower bound on the current inverse decay rate is of order 3.7×10^9 years. This is comparable to the period over which the earth is expected to be destroyed due to the expanded size of the Sun.

3. Finally we note that the above analysis was based on the assumption that the bubbles continue to nucleate and expand in the FRW metric at the same rate as they would do in the metastable vacuum. This will be expected as long as the matter and radiation density and temperature are small compared to the microscopic scales involved in the bubble nucleation process, *e.g.* the scale set by the negative cosmological constant of the vacuum in the interior of the bubble. Some discussion on the effect of cosmological space-time background on the bubble nucleation / evolution can be found in [98, 99].

3.5 Discussion

We have seen that the result for how much we can increase the life expectancy by spreading out in space depends on the parameters r and K, which in turn are determined by the Hubble parameter of the de Sitter space-time, the initial spread between different objects and the inverse decay rate. Therefore, the knowledge of these quantities is important for planning our future course of action if we are to adapt this strategy for increasing the life expectancy of the human race. In this section we shall discuss possible strategies for determining / manipulating these quantities.

We begin with the Hubble expansion parameter H. This is determined by the cosmological constant which has been quite well measured by now. Assuming that the current expansion rate is of order 68Km/sec/Mpc and accounting for the fact that the cosmological constant accounts for about 69% of the total energy density we get $H^{-1} \simeq 1.7 \times 10^{10}$ years. Future experiments will undoubtedly provide a more accurate determination of this number, but given the uncertainty in the other quantities, this will not significantly affect our future course of action. Of course we may discover that the dark energy responsible for the accelerated expansion of the universe comes from another source, in which case we have to reexamine the whole situation.

Next we turn to the initial separation between different objects which determine the value of r.

Since in order for the Hubble expansion to be effective in separating the objects they have to be unbound gravitationally, a minimum separation between the objects is necessary for overcoming the attractive gravitational force of the home galaxy. For example the size of our local gravitationally bound group of galaxies is of order 5 million light years which correspond to $r \sim 3 \times 10^{-4}$. The question is whether larger values of r can be accessed. An interesting analysis by Heyl [100] concluded that by building a space-ship that can constantly accelerate / decelerate at a value equal to the acceleration due to gravity, we can reach values of r close to unity in less than 100 years viewed from the point of view of the space-traveller. Of course this will be close to about 10^{10} years viewed from earth, and roughly the reduction of time viewed from the space-ship can be attributed to the large time dilation at the peak speed of the space-ship reaching a value close to that of light. However, this large time dilation will also increase the effective temperature of the microwave background radiation in the forward direction and without a proper shield such a journey will be impossible to perform. If one allows a maximum time dilation of the order of 100 then the microwave temperature in the forward direction rises to about the room temperature. Even then we have to worry about the result of possible collisions with intergalactic dust and othe debris in space. Even if these problems are resolved, we shall need a time of order 10^8 years from the point of view of the space-ship to travel a distance of order 10^{10} light years. Even travelling the minimum required distance of order 10^7 light years will take 10^5 years in such a space-ship. Such a long journey in a space-ship does not seem very practical but may not be impossible.

Another interesting suggestion for populating regions of space-time which will eventually be outside each other's horizon has been made by Loeb [101]. Occasionally there are hypervelocity stars which escape our galaxy (and the cluster of galaxies which are gravitationally bound) and so if we could find a habitable planet in such a star we could take a free ride in that planet and escape our local gravitationally bound system. In general of course there is no guarantee that such a star will reach another cluster of galaxies where we could spread out and thrive, but some time we may be lucky. It has been further suggested in [102, 103] that the merger of Andromeda and the Milky Way galaxies in the future [104] could generate a large number of such hypervelocity stars travelling at speeds comparable to that of light and they could travel up to distances of the order of 10^9 light years by the time they burn out. This could allow us to achieve values of r of order 10^{-1} or more.

Let us now turn to the value of T or equivalently the decay rate of the de Sitter vacuum in which we currently live. This is probably the most important ingredient since we have seen that for $T \lesssim 1$ we do not gain much by spreading out, while for large enough T we can achieve the maximum possible gain, given by the harmonic numbers, by spreading out even over modest distances of $r \sim 10^{-3}$. At the same time if T is so large that it exceeds the period over which galaxies will die then vacuum decay may not have a significant role in deciding our end and we should focus on other issues. For this reason estimating the value of T seems to be of paramount importance. Unfortunately, due to the very nature of the vacuum decay process it is not possible to determine it by any sort of direct experiment since such an experiment will also destroy the observer. It may be possible in the future to device clever indirect experiments to probe vacuum instability without actually causing the transition to the stable vacuum, but no such scheme is known at present.

At a crude level the current age of the universe – which is about 0.79 times the asymptotic Hubble period in the cosmological constant dominated epoch – together with the assumption that we have not been extremely lucky to survive this long, suggests that the inverse decay rate of the universe is $\gtrsim 0.8$. However, (3.4.18) shows that this actually gives a lower bound of $T \gtrsim 0.014$ reflecting the fact that the decay rate in the de Sitter epoch will be about 56 times faster than the average decay rate in the past. If we allow for the possibility that we might have been extremely lucky to have survived till today, then we have indirect arguments that lower the bound by a factor of 10 [93]. Clearly these rates are too fast and if T really happens to be less than 1, then there is not much we can do to prolong our collective life.

Is there any hope of computing T theoretically? Unfortunately any bottom up approach based on the analysis of low energy effective field theory is insufficient for this problem. The reason for this is that the vacuum decay rate is a heavily ultraviolet sensitive quantity. Given a theory with a perfectly stable vacuum we can add to it a new heavy scalar field whose effect will be strongly suppressed at low energy, but which can have a potential that makes the vacuum metastable with arbitrarily large decay rate. For this reason the only way we could hope to estimate T is through the use of a top down approach in which we have a fundamental microscopic theory all of whose parameters are fixed by some fundamental principle, and then compute the vacuum decay rate using standard techniques. In the context of string theory this will require finding the vacuum in the landscape that describes our universe. Alternatively, in the multiverse scenario, we need to carry out a statistical analysis that establishes that the overwhelming majority of the vacua that resemble our vacua will have their decay rate lying within a narrow range. This can then be identified as the likely value of the decay rate. There have been attempts in this direction [105–108], but it is probably fair to say that we do not yet have a definite result based on which we can plan our future course of action.

Chapter 4

Conclusion

An advantage of multiple formulations of a theory is that it affirms the proverb that *all roads lead to Rome*. Same is true for string theory. There are various facets of string theory and often it happens that one formulation is more economical than the others in explorations involving certain aspects. This thesis was concerned with pure spinor superstring which offers a superior computational power i.e. superstring perturbation theory. The secret behind this power is the superPoincaré covariance. This results in efficient methods for computation of scattering amplitudes [55, 63–67]. The amplitude computations involving fermions are as simples as those that involve bosons because of manifest supersymmetry. Further pure spinor formalism has been used to prove various non-renomalization theorems [109–111]. It offers an advantage when superstring is to be quantized in a curved background, especially with Ramond-Ramond fluxes [112, 113].

Despite its success, some of the foundational aspects of pure spinor often rely on analogies with RNS/ bosonic string theory. One fundamental reason for this has been the lack of an action with a worldsheet gauge invariance (for instance reparametrization invariance) which when gauge fixed leads to pure spinor formalism as we know it. Consequently, for example, the BRST charge, even though it has a very simple form remains a mysterious object - what is its origin? In [114] Berkovits finds out a possible resolution of origin of the BRST charge. In this paper he starts with a classical worldsheet action whose matter content comprises a $d = 10$ spacetime vector X^m and a pure spinor λ^α which satisfy the constraint $\partial X^m(\gamma_m\lambda)_\alpha = 0$. This equation holds true if $\partial X^m = \lambda\gamma^m\xi$ for some spinor ξ^α, implying a ten-dimensional twistor-like constraint. Further the spacetime spinor, θ^α of pure spinor formalism emerges as a Faddeev-Popov ghost of the gauge fixing of the classical action. The resulting BRST operator that one finds is same can be related to the pure spinor formalism BRST operator $\int dz\lambda^\alpha d_\alpha$ similarity transformation. Further, [115] builds upon [114] and writes down an action which has the pure spinor as the only matter field, and X^m and θ^α arise as Faddeev-Popov ghosts while gauge fixing. From here it appears that pure spinor is much more fundamental than previously thought. The second issue that has been presenting difficulties is the composite nature of the b ghost as it involves terms with inverse powers of $\bar{\lambda}\lambda$ so that there is convergence issue in $\bar{\lambda}\lambda \to 0$. There have been various suggestions for taking care of these issues [63, 80, 81] without however a unanimous result. These issues are worth explorations.

Having indicated the various successful features of pure spinor superstring, we should point out that most of the explicit results were for the massless states[1]. The reason for this is that the massive vertex operators were not very well understood (prior to our the major part of work on which this thesis is based). The only massive vertex that was known was at $(mass)^2 = \frac{1}{\alpha'}$ in the unintegrated form [9]. The non-availability of the integrated vertex severely restricts the number of amplitudes that can be computed. A knowledge of the integrated vertex is a must for computing more general amplitudes. In this thesis we were able to find the integrated massive vertex for open strings at $(mass)^2 = \frac{1}{\alpha'}$ [8]. We found that our construction can be carried out for construction of higher massive states and can also be generalized for the heterotic and the closed superstrings. Given this, we hope that in future certain useful results can be obtained which involve massive states.

[1]There was an attempt do so in [116], however in [74] it was found that certain equations presented in [116] were incorrect. Further, the result in [74] is validated by doing many explicit computations and comparing them with RNS computations [82, 117].

In the second part of this work, we took it as an implication of string theory that we may be living in a metastable vacuum of string theory. It was suggested in [15] that spreading the civilization in distant, causally disconnected pockets of universe will enhance the collective lifetime of the civilization. In [16] we worked out the details of efficiency of this strategy by keeping the decay constant of the universe arbitrary (since we do not know it). Our conclusions are described in 3.5.

Appendix A

Notations and Conventions

We shall be using the following conventions

- The beginning of Roman alphabets $a, b, c, \cdots \in \{1, 2, \cdots 9\}$ will denote space coordinate (this is mostly used when we are doing analysis in the rest frame).

- The middle of Roman alphabets $m, n, p, \cdots \in \{0, 1, 2, \cdots 9\}$ denote spacetime coordinates in general Lorentz frame.

- The small Greek alphabets $\alpha, \beta, \cdots \in \{1, 2, \cdots, 16\}$ will denote the spacetime spinor indices.

- $:=$ will mean definition.

- We shall always represent a superfield by capital letters and its components by corresponding small letters with spinor indices attached. As an example for a super field S, we have

$$S = s + s_{\alpha_1}\theta^{\alpha_1} + s_{\alpha_1\alpha_2}\theta^{\alpha_1}\theta^{\alpha_2} + \cdots s_{\alpha_1\alpha_1\cdots\alpha_{12}}\theta^{\alpha_1}\theta^{\alpha_2}\cdots\theta^{\alpha_{16}}$$

- While giving a reference to a research article, if we refer to precise location it would correspond to the document available on arXiv.

- Our (anti)symmetrization convention is as follows

$$\text{Anti-symmetrization} \quad : \quad T^{[m_1 \dots m_n]} \equiv \frac{1}{n!}(T^{m_1 \dots m_n} \pm \text{all permutations}) \tag{A.0.1}$$

$$\text{Symmetrization} \quad : \quad T^{(m_1 \dots m_n)} \equiv \frac{1}{n!}(T^{m_1 \dots m_n} + \text{all permutations}) \tag{A.0.2}$$

- All antisymmetric products of gamma matrices are defined as

$$\gamma^{m_1 \dots m_p} \equiv \gamma^{[m_1 \dots} \gamma^{m_p]} \tag{A.0.3}$$

Anti-symmetrized product of p gamma matrices is sometimes referred to as p-form.

- Our convention for super-covariant derivative is

$$D_\alpha = \partial_\alpha + (\gamma^m)_{\alpha\beta}\theta^\beta \partial_m \,; \text{ where } \quad \partial_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} \tag{A.0.4}$$

Therefore, the Clifford identity of gamma matrices implies

$$\{D_\alpha, D_\beta\} = 2(\gamma^m)_{\alpha\beta}\partial_m \quad \Longrightarrow \quad (\gamma_m)^{\alpha\beta} D_\alpha D_\beta = 16\partial_m \tag{A.0.5}$$

In momentum space, this implies for the first massive state

$$k^m (\gamma_m)^{\alpha\beta} D_\alpha D_\beta = \frac{i}{16} k^m k_m = -\frac{16\, i}{\alpha'} \tag{A.0.6}$$

- All normal ordering of products of operators are considered to be generalized normal ordering defined as follows-

$$: AB : (z) \equiv \frac{1}{2\pi i} \oint_z \frac{dw}{w-z} A(w) B(z) \ , \quad \text{For any two operators } A \text{ and } B. \qquad \text{(A.0.7)}$$

Appendix B

Gamma Matrices in 10 Dimensions

The Clifford algebra in a d dimensional spacetime is

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}\mathbb{I} \qquad ; \qquad m, n, p, \cdots = 1, 2, \cdots, d \tag{B.0.1}$$

It smallest dimension of the Dirac representation is given by $2^{d/2}$. We shall consider the gamma matrices in 10 dimensions. For a more detailed and exhaustive discussion a good reference is the appendix D of [118]. We work solely with the 16×16 gamma matrices in $d = 10$. These are the off-diagonal elements of the 32×32 gamma matrices Γ^m matrices satisfying

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}\mathbb{I}_{32\times 32}$$

More specifically,

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)_{\alpha\beta} \\ (\gamma^m)^{\alpha\beta} & 0 \end{pmatrix}$$

In terms of the 16×16 gamma matrices the Clifford algebra becomes

$$\{\gamma^m, \gamma^n\}^\alpha{}_\beta = \left(\gamma^m\right)^{\alpha\sigma}\left(\gamma^n\right)_{\sigma\beta} + \left(\gamma^n\right)^{\alpha\sigma}\left(\gamma^m\right)_{\sigma\beta} = 2\eta^{mn}\delta^\alpha_\beta \tag{B.0.2}$$

We define $\gamma^{n_1\cdots n_k}$ as the completely antisymmetric product of gamma matrices $\gamma^{n_1}, \gamma^{n_2}, \cdots \gamma^{n_k}$ i.e.

$$\gamma^{n_1\cdots n_k} \equiv \frac{1}{k!}\Big[\gamma^{n_1}\gamma^{n_2}\cdots\gamma^{n_k} + \text{All antisymmetric permutations}\Big] \tag{B.0.3}$$

It becomes very cumbersome when we need to work with product of various gamma matrices. There are very efficient and easy to use computer algebra systems available which can save tremendous amount of time [79,119,120]. Below we shall provide list of all the useful formulae.

- **Spinor index structure of various gamma matrices**

  Following is the spinor index structure for various antisymmetric products of gamma matrices

$$(\gamma^{m_1\ldots m_n})^\alpha{}_\beta \qquad \text{or} \qquad (\gamma^{m_1\ldots m_n})_\beta{}^\alpha \qquad \text{for } n = 0, 2, 4, 6, 8, 10$$

$$(\gamma^{m_1\ldots m_n})_{\alpha\beta} \qquad \text{or} \qquad (\gamma^{m_1\ldots m_n})^{\alpha\beta} \qquad \text{for } n = 1, 3, 5, 7, 9$$

- **Hodge duals**

  For 10 dimensional 16×16 gamma matrices, the hodge duality is more than mere duality. It turns out to be an equality. We summarize them below

$$(\gamma^{m_1\ldots m_{2n}})^\alpha{}_\beta \quad = \quad \frac{1}{(10-2n)!}(-1)^{(n+1)}\epsilon^{m_1\ldots m_{2n}p_1\ldots p_{10-2n}}(\gamma_{p_1\ldots p_{10-2n}})^\alpha{}_\beta \tag{B.0.4}$$

$$(\gamma^{m_1\ldots m_{2n}})_\alpha{}^\beta = -\frac{1}{(10-2n)!}(-1)^{(n+1)}\epsilon^{m_1\ldots m_{2n}p_1\ldots p_{10-2n}}(\gamma_{p_1\ldots p_{10-2n}})_\alpha{}^\beta \qquad \text{(B.0.5)}$$

$$(\gamma^{m_1\ldots m_{2n+1}})^{\alpha\beta} = \frac{1}{(9-2n)!}(-1)^{n}\epsilon^{m_1\ldots m_{2n+1}p_1\ldots p_{9-2n}}(\gamma_{p_1\ldots p_{9-2n}})^{\alpha\beta} \qquad \text{(B.0.6)}$$

$$(\gamma^{m_1\ldots m_{2n+1}})_{\alpha\beta} = -\frac{1}{(9-2n)!}(-1)^{n}\epsilon^{m_1\ldots m_{2n+1}p_1\ldots p_{9-2n}}(\gamma_{p_1\ldots p_{9-2n}})_{\alpha\beta} \qquad \text{(B.0.7)}$$

where, $\epsilon^{m_1\cdots m_9}$ is the 10 dimensional epsilon tensor defined as

$$\epsilon_{0\,1\,\cdots\,9} = 1 \qquad \Longrightarrow \qquad \epsilon^{0\,1\,\cdots\,9} = -1 \qquad \text{(B.0.8)}$$

Due to the above dualities, we only take γ^{m_1}, $\gamma^{m_1m_2}$, $\gamma^{m_1m_2m_3}$, $\gamma^{m_1m_2m_3m_4}$ and $\gamma^{m_1m_2m_3m_4m_5}$ along with the identity matrix $\mathbb{I}_{16\times16}$ as the linearly independent basis elements for vector spaces of 16×16 complex matrices.

- **Symmetry property of gamma matrices under exchange of Spinor indices**

$$(\gamma^m)_{\alpha\beta} = (\gamma^m)_{\beta\alpha} \qquad : \text{Symmetric} \qquad \text{(B.0.9)}$$

$$(\gamma^{m_1m_2})^\alpha{}_\beta = -(\gamma^{m_1m_2})_\beta{}^\alpha \qquad : \text{Anti-Symmetric} \qquad \text{(B.0.10)}$$

$$(\gamma^{m_1m_2m_3})_{\alpha\beta} = -(\gamma^{m_1m_2m_3})_{\beta\alpha} \qquad : \text{Anti-Symmetric} \qquad \text{(B.0.11)}$$

$$(\gamma^{m_1 m_2 m_3 m_4})^{\alpha}{}_{\beta} \;=\; (\gamma^{m_1 m_2 m_3 m_4})_{\beta}{}^{\alpha} \qquad \text{: Symmetric} \tag{B.0.12}$$

$$(\gamma^{m_1 m_2 m_3 m_4 m_5})_{\alpha\beta} \;=\; (\gamma^{m_1 m_2 m_3 m_4 m_5})_{\beta\alpha} \qquad \text{: Symmetric} \tag{B.0.13}$$

For 1, 3 and 5 forms, the same (anti) symmetry properties hold when the spinor indices are upstairs.

- **Various Gamma Traces**

$$(\gamma^{m_1 \ldots m_n})^{\alpha}{}_{\alpha} \;=\; 0 \quad \text{for} \quad n = 2, 4, 6, 8 \tag{B.0.14}$$

$$(\gamma^{m_1 \ldots m_{10}})^{\alpha}{}_{\alpha} \;=\; -16\, \epsilon^{m_1 \ldots m_{10}} \tag{B.0.15}$$

$$(\gamma^{m})_{\alpha\beta}(\gamma_{n})^{\beta\alpha} \;=\; 16\, \delta^{m}_{n} \tag{B.0.16}$$

$$(\gamma^{m_1 \ldots m_n})_{\alpha\beta}(\gamma_{p_n \ldots p_1})^{\beta\alpha} \;=\; 16 n!\, \delta^{m_1 \ldots m_n}_{p_1 \ldots p_n}, \text{ for } n \in \text{odd} \tag{B.0.17}$$

$$(\gamma^{m_1 \ldots m_n})^{\alpha}{}_{\beta}(\gamma_{p_n \ldots p_1})^{\beta}{}_{\alpha} \;=\; 16 n!\, \delta^{m_1 \ldots m_n}_{p_1 \ldots p_n}, \text{ for } n \in \text{even} \tag{B.0.18}$$

- **Bi-Spinor decomposition**

Any Bi-spinor $T_{\alpha\beta}$ can be decomposed as

$$T_{\alpha\beta} = t_m(\gamma^m)_{\alpha\beta} + t_{mnp}(\gamma^{mnp})_{\alpha\beta} + t_{mnpqr}(\gamma^{mnpqr})_{\alpha\beta} \tag{B.0.19}$$

where, for $r = 1, 3, 5$

$$t_{m_1 \ldots m_r} = \frac{1}{16r!} (\gamma_{m_1 \ldots m_r})^{\alpha\beta} \, T_{\alpha\beta} \tag{B.0.20}$$

Similarly, a tensor-spinor $T^{\alpha}{}_{\beta}$ can be decomposed as

$$T^{\alpha}{}_{\beta} = t \, \delta^{\alpha}{}_{\beta} + t_{mn} (\gamma^{mn})^{\alpha}{}_{\beta} + t_{mnpq} (\gamma^{mnpq})^{\alpha}{}_{\beta} \tag{B.0.21}$$

where, for $r = 2, 4$

$$t_{m_1 \ldots m_r} = \frac{1}{16r!} (\gamma_{m_1 \ldots m_r})^{\alpha}{}_{\beta} \, T^{\alpha}{}_{\beta} \tag{B.0.22}$$

- **Tensor index contracted identities involving gamma matrices**

$$(\gamma^{mn})^{\alpha}{}_{\beta} (\gamma_{mn})^{\rho}{}_{\lambda} = 4 (\gamma^m)_{\beta\lambda} (\gamma_m)^{\alpha\rho} - 2 \delta^{\alpha}_{\beta} \delta^{\rho}_{\lambda} - 8 \delta^{\alpha}_{\lambda} \delta^{\rho}_{\beta} \tag{B.0.23}$$

$$(\gamma^{mn})^{\alpha}{}_{\beta} (\gamma_{mnp})^{\rho\lambda} = 2 (\gamma^m)^{\alpha\rho} (\gamma_{pm})^{\lambda}{}_{\beta} + 6 (\gamma_p)^{\alpha\rho} \delta^{\lambda}_{\beta} - (\rho \leftrightarrow \lambda) \tag{B.0.24}$$

$$(\gamma_{mn})^{\alpha}{}_{\beta} (\gamma^{mnp})_{\rho\lambda} = -2 (\gamma_m)_{\beta\lambda} (\gamma^{pm})^{\alpha}{}_{\rho} + 6 (\gamma^p)_{\beta\lambda} \delta^{\alpha}_{\rho} - (\rho \leftrightarrow \lambda) \tag{B.0.25}$$

$$(\gamma_{mnp})^{\alpha\beta} (\gamma^{mnp})^{\rho\lambda} = 12 [(\gamma_m)^{\alpha\lambda} (\gamma^m)^{\beta\rho} - (\gamma_m)^{\alpha\rho} (\gamma^m)^{\beta\lambda}] \tag{B.0.26}$$

$$(\gamma_{mnp})^{\alpha\beta} (\gamma^{mnp})_{\rho\lambda} = 48 (\delta^{\alpha}_{\rho} \delta^{\beta}_{\lambda} - \delta^{\alpha}_{\lambda} \delta^{\beta}_{\rho}) \tag{B.0.27}$$

B.0.1 Explicit representation

Let us now give an explict representation of the gamma matrices that we have used in the thesis

$$\Gamma^m = \begin{pmatrix} 0_{16\times 16} & (\gamma^m)^{\alpha\beta} \\ \\ (\gamma^m)_{\alpha\beta} & 0_{16\times 16} \end{pmatrix} \tag{B.0.28}$$

Here the γ matrices are given by

$$(\gamma^0)^{\alpha\beta} = \begin{pmatrix} 1_{8\times 8} & 0_{8\times 8} \\ \\ 0_{8\times 8} & 1_{8\times 8} \end{pmatrix} \qquad (\gamma^0)_{\alpha\beta} = -\begin{pmatrix} 1_{8\times 8} & 0_{8\times 8} \\ \\ 0_{8\times 8} & 1_{8\times 8} \end{pmatrix}$$

$$(\gamma^9)^{\alpha\beta} = \begin{pmatrix} 1_{8\times 8} & 0_{8\times 8} \\ \\ 0_{8\times 8} & -1_{8\times 8} \end{pmatrix} \qquad (\gamma^9)_{\alpha\beta} = \begin{pmatrix} 1_{8\times 8} & 0_{8\times 8} \\ \\ 0_{8\times 8} & -1_{8\times 8} \end{pmatrix}$$

$$(\gamma^i)^{\alpha\beta} = \begin{pmatrix} 0_{8\times 8} & \sigma^i_{a\dot{a}} \\ \\ \sigma^i_{\dot{b}b} & 0_{8\times 8} \end{pmatrix} \qquad (\gamma^i)_{\alpha\beta} = \begin{pmatrix} 0_{8\times 8} & \sigma^i_{a\dot{a}} \\ \\ \sigma^i_{\dot{b}b} & 0_{8\times 8} \end{pmatrix}$$

In the above expression the 8×8 σ matrices satisfy the following algebra,

$$\sigma^i_{a\dot{a}}\sigma^j_{\dot{a}b} + \sigma^j_{a\dot{a}}\sigma^i_{\dot{a}b} = 2\delta^{ij}\delta_{ab} \tag{B.0.29a}$$

$$\sigma^i_{\dot{a}a}\sigma^j_{a\dot{b}} + \sigma^j_{\dot{a}a}\sigma^i_{a\dot{b}} = 2\delta^{ij}\delta_{\dot{a}\dot{b}} \tag{B.0.29b}$$

$$\sigma^i_{a\dot{a}}\sigma^j_{\dot{a}b} + \sigma^i_{b\dot{a}}\sigma^j_{\dot{a}a} = 2\delta^{ij}\delta_{ab} \tag{B.0.29c}$$

$$\sigma^i_{\dot{a}c}\sigma^j_{c\dot{b}} + \sigma^i_{\dot{b}c}\sigma^j_{c\dot{a}} = 2\delta^{ij}\delta_{\dot{a}\dot{b}} \tag{B.0.29d}$$

Further, the light cone gamma matrices are given by

$$\begin{aligned} (\gamma^+)^{\alpha\beta} &= \sqrt{2}\begin{pmatrix} 1_{8\times8} & 0_{8\times8} \\ \\ 0_{8\times8} & 0_{8\times8} \end{pmatrix} \qquad & (\gamma^+)_{\alpha\beta} &= \sqrt{2}\begin{pmatrix} 0_{8\times8} & 0_{8\times8} \\ \\ 0_{8\times8} & -1_{8\times8} \end{pmatrix} \\ (\gamma^-)^{\alpha\beta} &= \sqrt{2}\begin{pmatrix} 0_{8\times8} & 0_{8\times8} \\ \\ 0_{8\times8} & 1_{8\times8} \end{pmatrix} \qquad & (\gamma^-)_{\alpha\beta} &= \sqrt{2}\begin{pmatrix} -1_{8\times8} & 0_{8\times8} \\ \\ 0_{8\times8} & 0_{8\times8} \end{pmatrix} \end{aligned} \tag{B.0.30}$$

Appendix C

Motivating the Ansatz

C.0.2 The polynomial dependance of vertex operators on the pure spinor ghost field

In writing the most general form of the integrated vertex in equation (2.3.39), we assumed that it does not depend upon the $\bar{\lambda}\lambda$ factors. In this appendix, we justify this assumption. First we recall that the integrated vertex U can also be determined by integrating the b ghost around the unintegrated vertex V, i.e.,

$$U(z) = \oint \frac{dw}{2\pi i}\, b(w)V(z) \tag{C.0.1}$$

In the pure spinor formalism, the b ghost is a composite operator which involves different powers of $\bar{\lambda}\lambda$ in the denominator [55]. So, naively, one might expect that the integrated vertex will also involve different powers of $\bar{\lambda}\lambda$ in denominator. However, it is possible to work in a gauge in which the vertex operators are independent of the $\bar{\lambda}\lambda$ terms. To see this, we recall from the RNS formalism that the massive states also appear in the OPEs of the massless vertex operators. This allows us, in principle, to construct the massive vertex operators from the knowledge of the massless vertex operators. More specifically for open strings, this construction, pointed out to the authors by Nathan Berkovits, goes as follows. If V_1, V_2 are unintegrated and U_1, U_2 are integrated massless vertex operators respectively, then we have

$$QU_1 = \partial_{\mathbb{R}} V_1 \qquad \text{and} \qquad QU_2 = \partial_{\mathbb{R}} V_2 \tag{C.0.2}$$

We now take the contour integral of U_1 around the integrand of U_2 and define

$$U_3(z) \equiv \oint \frac{dw}{2\pi i} U_1(w) U_2(z) \tag{C.0.3}$$

Acting on this with the BRST operator Q and using (C.0.2), we obtain

$$QU_3 = \oint \frac{dw}{2\pi i} U_1(w) QU_2(z) = \oint \frac{dw}{2\pi i} U_1(w) \partial_z V_2(z) \equiv \partial_z V_3 \tag{C.0.4}$$

where,

$$V_3(z) \equiv \oint \frac{dw}{2\pi i} U_1(w) V_2(z) \tag{C.0.5}$$

and in the first equality in (C.0.4), we have used the fact that $\oint dw\, \partial_{\mathbb{R}} V_1(w)$ is zero.

Now, if we choose the momentum k_1 and k_2 of U_1 and U_2 to satisfy

$$(k_1 + k_2)^2 = 2k_1 \cdot k_2 \equiv (k_3)^2 = -m^2 = -\frac{n}{\alpha'} \tag{C.0.6}$$

then, by construction, the V_3 and U_3 will be unintegrated and integrated massive vertex operators respectively of open string states at mass level n.

One might ask how do we know that the U_3 and V_3 as defined in (C.0.3) and (C.0.5) do not vanish. To answer this question, we recall that the OPE of two massless vertex operators necessarily contain the massive vertex operators (this is necessary for the consistency of the theory and is well known from the RNS formalism). Now, the integrated vertices U_1 and U_2 have conformal weight one. Hence, by dimensional analysis, it is easy to see that the integrand involving the integrated massive vertex operator can only appear at the first order pole in (C.0.3) and hence its contour integral can't vanish. By a similar argument, we see that V_3 as defined in (C.0.5) can't vanish.

Since the massless vertices can be chosen to be independent of $\bar{\lambda}\lambda$ in denominator [7], this construction shows that the massive vertices can also be constructed without using the $\bar{\lambda}\lambda$ in the denominator. Moreover, since the massless vertices do not involve JJ and ∂J terms, the above construction also shows why JJ and ∂J terms do not appear in the massive vertices. In appendix C.0.3, we give another argument for this based on group theory.

C.0.3 General form of the Superfields

In this appendix, we give the method for writing down the ansatz (2.3.61) and (2.3.62) for the massive superfields which appear in the vertex operators and Lagrange multipliers. The same method can be very easily generalized for the construction of any massive vertex operator in the pure spinor formalism.

We start by arguing that the superfields appearing in the integrated vertex operator must be expressible in terms of the basic superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$. This follows because as shown in [9,74], the superfields appearing in the full set of superspace equations of motion can be expressed solely in terms of any of the basic superfields $\Psi_{m\alpha}, B_{mnp}$ or G_{mn}. Thus, the vertex operators should be expressible entirely in terms of any of these basic superfields[1]. For the unintegrated vertex operator (2.1.21), this can be seen using equations (2.1.22) and (2.1.24)-(2.1.27). From this, it is also

[1]This is similar to the case of the massless vertices. The massless vertices are also expressed entirely in terms of the superfields which appear in the $\mathcal{N} = 1$ super Yang Mills equations of motion in 10 dimensions.

clear that if we want to express the entire vertex operator in terms of only one or two basic superfield, we need to use the supercovariant derivative. However, if we use all the 3 superfields, then we can avoid the use of supercovariant derivatives (since the supercovariant derivative of the basic superfields can be expressed in terms of the basic superfields without supercovariant derivative using equations (2.1.24)-(2.1.27)).

We shall make use of all the 3 basic superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$. Thus, due to equations (2.1.24)-(2.1.27), the relation between the superfields in integrated vertex operator and these basic superfields can be expressed without using the super covariant derivative. Moreover, whatever be the functional form of these superfields, the Lorentz invariance implies that they can only involve 3 basic superfields, the momentum vector, the space-time metric η^{mn} and the gamma matrices. Thus, the functional dependence of all the superfields in the momentum space is

$$\text{Superfields in } U = f(B_{mnp}, G_{mn}, \Psi_{m\alpha}, k_m, \eta^{mn}, \text{Gamma Matrices})$$

Our goal now is to determine these functions. This can be done by making use of the group representation theory. To see this, we note that the physical degrees of freedom (encoded in the fields $\psi_{m\alpha}, b_{mnp}$ and g_{mn}) should match on both sides at each order in the theta expansion of the above equation. Moreover, since the right hand side does not involve supercovariant derivative, it follows that we can equate the coefficients in the theta expansion of the superfield in the left hand side at a given order with the coefficient at the same order in the theta expansion of the right hand side[2]. Since the right hand side involve only the basic superfields $\Psi_{m\alpha}, B_{mnp}$ and G_{mn}, it follows that any given order theta component of the superfield in the left side is related to the same order theta component of the basic superfields $\Psi_{m\alpha}, B_{mnp}$ and G_{mn}. We now focus on the theta independent component.

[2]Note that if we have a superspace equation of the form $S_\alpha = D_\alpha T$, then the ℓ^{th} order component of the superfield S_α will be related to $(\ell-1)^{th}$ and $(\ell+1)^{th}$ order components of the superfield T. However, if we have an equation of the form $S_\alpha = R_\alpha$, then the ℓ^{th} order component of S_α will be related to ℓ^{th} order component of R_α.

Using above argument, it follows that the theta independent components of the superfields in the left hand side must be expressible in terms of only the theta independent components of the basic superfields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$, namely b_{mnp}, g_{mn} and $\psi_{m\alpha}$[3].

Thus, for the theta independent components of the superfields in the integrated vertex, our problem has reduced to finding the correct physical degrees of freedom and to express them in terms of b_{mnp}, g_{mn} and $\psi_{m\alpha}$. The covariant expression for the full superfield can then be obtained by replacing b_{mnp}, g_{mn} and $\psi_{m\alpha}$ by B_{mnp}, G_{mn} and $\Psi_{m\alpha}$ respectively. The validity of this procedure can be justified by the fact that it gives an operator U which satisfies the correct BRST equation $QU = \partial_{\mathbb{R}} V$. Once we have an operator U which satisfies this equation, we are guaranteed that it is the correct integrated vertex irrespective of how we arrive at it.

Now, the correct degrees of freedom in the theta independent components of the superfields can be obtained by looking at their index structure and using the group theory. In the rest frame, the physical fields b_{mnp}, g_{mn} and $\psi_{m\alpha}$ form the **84, 44** and **128** representations of the little group SO(9). Thus, to determine the correct physical degrees of freedom in the theta independent components of the superfields, we need to find the number of **84, 44** and **128** representations of SO(9) in their theta independent components in the rest frame.

We shall illustrate this method by some examples now. First, consider the superfield C_{mn} in (2.3.39). Since, it is anti symmetric in its indices m and n, its only non zero components in the rest frame can be C_{0a} and C_{ab}. These can only form **9** and **36** representation of SO(9) and hence can't contain the physical massive fields. Thus, C_{mn} must be zero. Similarly, all the superfields whose theta independent components cannot form the **84, 44** or **128** representations of SO(9), must be zero. Next, we consider the superfield G^{β}_{mn}. Since, it is also anti symmetric in its vector indices m and n,

[3]This is not true for the massless states. One reason for this is that given a differential equation of the form $D_\alpha S = B_\alpha$ (where S and B_α are some superfields encoding the information about the massless states), one can't invert this to write an expression for S in terms of some differential operator acting on B_α since $k^2 = 0$ for the massless states. This is unlike the massive states where we can always invert this kind of equations.

going to the rest frame, we find that its non zero components can only be G_{0a}^{β} and G_{ab}^{β}. Our goal is to look for representations of SO(9) corresponding to the physical states. Now, the index structure of G_{0a}^{β} implies that its theta independent component forms the product representation **16** $\times$ **9** which contains one **128**. Similarly, G_{ab}^{β} contains one **128**. This means that the theta independent component of G_{mn}^{β} should contain two representations of **128** and hence there should be two terms involving $\Psi_{m\alpha}$ in the expansion of G_{mn}^{β} in terms of the basic fields B_{mnp}, G_{mn} and $\Psi_{m\alpha}$. After finding the correct number of terms, the next step is to write down the form of G_{mn}^{β} so that it has two terms involving $\Psi_{m\alpha}$. Taking into account the on shell conditions (2.1.27), we find

$$G_{pq}^{\beta} \;=\; g_2 \gamma_{[p}^{\beta\sigma} \Psi_{q]\sigma} + g_3 k^r \gamma_r^{\beta\sigma} k_{[p} \Psi_{q]\sigma} \tag{C.0.7}$$

where g_2 and g_3 are some unknown coefficients which need to be determined.

Appendix D

Decay Rate for Equation of State $p = w\,\rho$

In this appendix we shall compute the growth of decay rate with time for a general equation of state of the form $p = w\,\rho$ with $w > -1$. In this case the ρ and a are related as

$$\rho = \rho_0\, a^{-3(w+1)}\,, \tag{D.0.1}$$

for some constant ρ_0. As a result the dependence of a on t and τ are determined by the equations

$$\frac{1}{a}\frac{da}{dt} = \sqrt{\frac{8\pi G}{3}\rho} = C\, a^{-3(w+1)/2}, \qquad C \equiv \sqrt{\frac{8\pi G}{3}\rho_0}\,, \tag{D.0.2}$$

and

$$\frac{1}{a^2}\frac{da}{d\tau} = C\, a^{-3(w+1)/2}\,. \tag{D.0.3}$$

The solutions to these equations are

$$t = \frac{2}{3C(w+1)} a^{3(w+1)/2}, \quad \tau = \frac{2}{(3w+1)C} a^{(3w+1)/2}\,. \tag{D.0.4}$$

As a ranges from 0 to ∞, both t and τ also range from 0 to ∞.

We can now compute the decay rate using the first equation of (3.4.13):

$$\begin{aligned} D(t) &\equiv -\frac{1}{a}\frac{d}{d\tau}\ln P_0(\tau) \\ &= \frac{1}{a}\,4\pi K\int_0^{\tau} d\sigma\, a(\sigma)^4\,(\tau-\sigma)^2\,, \end{aligned} \tag{D.0.5}$$

where $a(\sigma)$ denotes the scale factor at conformal time σ. Changing integration variable to $b = a(\sigma)$, which due to (D.0.4) corresponds to

$$\sigma = \frac{2}{(3w+1)C} b^{(3w+1)/2}\,, \tag{D.0.6}$$

we can express (D.0.5) as

$$\begin{aligned} D(t) &= 4\pi K\, a^{-1}\left(\frac{2}{(3w+1)C}\right)^3 \frac{3w+1}{2}\int_0^a db\, b^4\, b^{(3w-1)/2}\,\left(a^{(3w+1)/2)} - b^{(3w+1)/2}\right)^2 \\ &= \frac{32\pi K}{3\,C^3}\, a^{9(w+1)/2}\,\frac{1}{(w+3)(3w+5)(9w+11)}\,. \end{aligned} \tag{D.0.7}$$

Using the relation between a and t given in (D.0.4), this can be expressed as

$$D(t) = 36\,\pi\, K\,\frac{(w+1)^3}{(w+3)(3w+5)(9w+11)}\, t^3\,. \tag{D.0.8}$$

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