SOME ASPECTS OF AMPLITUDES IN QUANTUM FIELD THEORIES

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A thesis submitted to the
Board of Studies in Physical Sciences
In partial fulfillment of requirements
for the Degree of
DOCTOR OF PHILOSOPHY
of
HOMI BHABHA NATIONAL INSTITUTE

July, 2018

Homi Bhabha National Institute[1]

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared Mritunjay Kumar Verma entitled "Some Aspects of Amplitudes in Quantum Field Theories" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. “Testing Subleading Multiple Soft Graviton Theorem for CHY Prescription”, Subhroneel Chakrabarti, Sitender Kashyap, Biswajit Sahoo, Ashoke Sen, Mritunjay Verma, *Journal of High Energy Physics*, **2018**, 01, 090

2. “Subleading Soft Theorem for Multiple Soft Gravitons”, Subhroneel Chakrabarti, Sitender Kashyap, Biswajit Sahoo, Ashoke Sen, Mritunjay Verma, *Journal of High Energy Physics*, **2017**, 12, 150

3. “Exploring Perturbative Conformal Field Theory in Mellin Space", Amin A. Nizami, Arnab Rudra, Sourav Sarkar, Mritunjay Verma, *Journal of High Energy Physics*, **2017**, 01, 102

Others (Not Included in the Thesis)

1. “Mellin Amplitudes for Fermionic Conformal Correlators”, Josua Faller, Sourav Sarkar, Mritunjay Verma, *Journal of High Energy Physics*, **2018**, 03, 106

2. “Integrated Massive Vertex Operator in Pure Spinor Formalism”, Subhroneel Chakrabarti, Sitender Kashyap, Mritunjay Verma, [Arxiv: 1802.04486/hep-th]

3. “Theta Expansion of First Massive Vertex Operator in Pure Spinor”, Subhroneel Chakrabarti, Sitender Kashyap, Mritunjay Verma, *Journal of High Energy Physics*, **2018**, 01, 019

4. “Implications of the AdS-CFT Correspondence on Space-Time and World-sheet OPE Coefficients", Sudip Ghosh, Sourav Sarkar, Mritunjay Verma, [Arxiv: 1703.06132 [hep-th]]

5. “Surviving in a Metastable de Sitter Space-Time”, Sitender Kashyap, Swapnomay Mondal, Ashoke Sen, Mritunjay Verma, *Journal of High Energy Physics*, **2015**, 09, 139

Review Article

1. “Superstring Field Theory and its Applications”, Corinne de Lacroix, Harold Erbin, Sitender Kashyap, Ashoke Sen, Mritunjay Verma , International Journal of Modern Physics A, **2017**, Vol. 32, No. 28n29, 1730021

Talks given in conferences

- “ *Soft Graviton Theorem in Generic Quantum Theory of Gravity*”, Discussion Meeting on String Field Theory and String Phenomenology, 11-15 February 2018, Harish Chandra Research Institute (HRI), Allahabad, India.

- "*Multiple Soft Graviton Theorem in Generic Quantum Theory of Gravity*", QCD Meets Gravity Workshop, 11-15 December 2017, University of California, Los Angeles (UCLA), USA.

- "*CFTs in Mellin Space*", National Strings Meeting, 5-10 December 2017, National Institute of Science Education and Reseach (NISER), Bhubaneshwar, India.

Mritunjay Kumar Verma

To my family

ACKNOWLEDGEMENTS

I have benefited enormously from my interactions with a number of people during my Ph.D. which is difficult to express in words. First of all, I am greatly indebted to my supervisor Rajesh Gopakumar for his academic as well as non academic support over the years. His influence in the working out of this thesis is inestimable and I shall always be thankful to him.

It is very difficult to express my thanks to Ashoke Sen. Not only he has taught physics in a marvelous way, he has also been a great mentor and a great source of inspiration as a wonderful human. His simple yet deep ways of looking at things and analyzing situations will always be something to look forward to.

Over the years, I learnt immensely by casual discussions with the HRI and ICTS members. I especially enjoyed academic and non academic discussions with Anirban Basu, Dileep Jatkar, R. Loganayagam, Anshuman Maharana, Sumathi Rao and R. Thangadurai.

I have learnt a lot from my discussions with my collaborators Subhroneel Chakrabarti, Josua Faller, Sudip Ghosh, Sitender Kashyap, Swapnomay Mondal, Amin Nizami, Arnab Rudra, Biswajit Sahoo and Sourav Sarkar all of whom were very good in finding mistakes in my logic and teaching me new ways to think about the situation.

My stay in HRI and ICTS was made easy by the presence of wonderful people who taught me many things about life in general. I am especially thankful to Sitender, Ruchi, Kasi, Pramod, Deepak, Teresa, Roji, Faroogh, Bhubanesh, Arnab and Victor for many wonderful memories.

Finally and most importantly, none of this would have been possible without the unconditional support of my family who valued education above all else and supported me in my decision of pursuing a very non traditional career despite all the difficulties.

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SYNOPSIS

In this work, we focus on three different aspects of the amplitudes in the perturbative quantum field theories. In the first line of work, we considered the conformal field theories in the mellin space. The second work was related to the amplitudes in the diffeomorphism invariant theories when some of the gravitons carry very low momenta. Finally, the third work was related to the so called CHY prescription for the tree amplitudes involving the massless particles. We describe each of the three works below.

1. Feynman Rules for CFTs in Mellin Space [1]

The first work is related to the correlation functions in the conformal field theories (CFTs). In the Wilsonian picture of quantum field theories, the CFTs are the fixed points of the renormalization group flow. Apart from their relevance to better understanding of the usual QFTs as the end points of the renormalization group flow, the CFTs living at these fixed points are relevant for many statistical mechanical systems near the critical points as well. The CFTs are also extremely important in string theory. In particular, the most well understood duality in the context of AdS CFT correspondence, namely the correspondence between the Type IIB string theory in the bulk and the $\mathcal{N} = 4$ SYM in the boundary involves a conformal field theory.

The aim of the first work was to understand the CFTs in the mellin space. The position to mellin space transformation of the CFTs is done with respect to the cross ratios of position space. Hence, given any amplitude, the number of independent mellin variables in it is same as the number of independent cross ratios. This is very convenient since the non trivial information about a CFT correlator is only a function of the cross ratios and not the individual space-time points. The mellin amplitude of a correlator involving only the scalar operators is defined as [2, 3]

$$A\left(\{x^i\}\right) = \prod_{1\leq i<j\leq n} \left(\int_{-i\infty}^{i\infty} \frac{ds^{ij}}{2\pi i} \, \Gamma\left(s^{ij}\right) \left(x^i - x^j\right)^{-2s^{ij}} \right) \prod_{i=1}^{n} \delta \left(\Delta^i - \sum_{j=1}^{n} s^{ij} \right) M\left(\{s^{ij}\}\right)$$

Here s^{ij} are the Mellin variables and $M\left(\{s^{ij}\}\right)$ is defined to be the Mellin amplitude. The variable Δ^i is the scaling dimension of the operator inserted at x^i.

One of the most important property of the mellin amplitude is that the spectrum of primary operators and their descendants become very transparent in this space. The mellin amplitude is a meromorphic function of its arguments and its poles in different channels correspond to twists of the operators exchanged in that channel in the intermediate state.

The goal of this work was to obtain Feynman rules for perturbative CFTs in the mellin space. We succeeded in deriving the tree Feynman rules in an arbitrary CFT in arbitrary dimensions involving

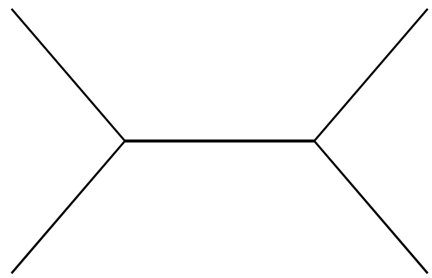

Figure 1: 4-point exchange diagram

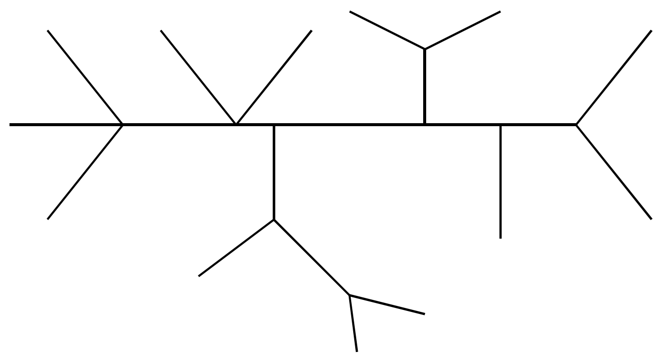

Figure 2: Example of an arbitrary tree diagram

the scalar operators [1]. In particular, we found that the propagator in the mellin space corresponds to a beta function. More specifically, if we consider a Feynamn diagram as shown in figure 3.12, its mellin amplitude is given by

$$M(s_{12}) = \frac{1}{2\Gamma(\gamma)}\beta\Big(\frac{\gamma - s_{12}}{2}, \frac{D}{2} - \gamma\Big)$$

where γ is the conformal dimension of the internal lines connecting the two vertices, D is the space-time dimension and s_{12} is the Mellin momentum flowing through the propagator. An advantage of our approach is that this expression is valid for an arbitrary number of external lines at two vertices provided we define the Mellin momenta s_{ab} appropriately.

The importance of this result is that one can now use this propagator to write the amplitude for an arbitrary tree diagram in a perturbative CFT. For an arbitrary tree diagram, such as the one shown in figure 2, the mellin amplitude is given by the product over all the propagator factors as

$$M(s_{ab}) = \prod \frac{1}{2\Gamma(\gamma_{ab})}\beta\left(\frac{\gamma_{ab} + k_{ab}^2}{2}, \frac{D}{2} - \gamma_{ab}\right)$$

where, the product is over all the internal lines. γ_{ab} is the conformal dimension of the internal line joining a^{th} and b^{th} vertices. k_{ab} is the total Mellin momentum flowing through the propagator joining

a^{th} and the b^{th} vertices.

From the expression of the amplitude, it is clear that its poles occur when the argument of the beta function becomes zero or a negative integer, i.e.

$$k_{ab}^2 + \gamma_{ab} = -2n \qquad , \qquad n = 0, 1, 2\cdots$$

The physical interpretation of these poles is that the primary ($n = 0$) fields with conformal dimension γ_{ab}, along with all of its descendants ($n > 0$) are propagating through the internal line. Thus, this representation of the amplitude in terms of the product over beta functions is very convenient in reading off the conformal family contributing to a given process.

2. Subleading Multiple Soft Graviton Theorem [4]

In the second work, we consider the field theories which have diffeomorphism invariance. One universal feature of the diffeomorphism invariant theories is the presence of a massless spin 2 particle [5]. These particles are the familiar gravitons and are responsible for the gravitational interaction. For theories containing spin 2 particles, it was shown by Weinberg that the amplitudes in these theories of processes containing soft gravitons can be related to the amplitudes without the soft gravitons [6]. More specifically, he showed that if we expand the S matrix of a theory involving arbitrary number of soft gravitons in the powers of the momenta of the soft gravitons, then the leading term in this expansion can be written as product of the amplitude without the soft gravitons multiplied by a factor which depends upon the momentum and polarizations of the soft gravitons. Moreover, this soft factor was universal and did not depend upon the details of the theory but only on the linear momenta carried by the particles.

Generally, a universal feature is related to some symmetries of the theory. Recently, it was found that the universal soft factorization in the gravitational theories are related to some asymptotic symmetries of space-time [7–9], called BMS symmetry. The ward identities associated with these BMS symmetries give rise to the Weinberg's soft theorem. Based on these motivations, the subleading term in the expansion of the S-matrix in terms of the soft momenta was also worked out. However, unlike leading order result, the extension of the subleading result for more than one goft graviton turned out to be more involved. The case of double soft graviton was investigated for specific theories in [10–17]. However, it was not clear whether these results are universal or dependent upon theory in hand.

In [18, 19], building on the ideas from the string field theory [20], a new approach based on Feynman diagrams was developed by Ashoke Sen to prove the factorization property of the amplitudes involving the soft gravitons. This was done for multiple gravitons at leading order and a single graviton at the subleading order.

Using the method of [18, 19, 21], in [4], we extended the soft graviton theorem to subleading order in the soft graviton momenta for an arbitrary number of soft gravitons. More specifically, we showed that given any theory having the diffeomorphism invariance, the amplitudes involving an arbitrary number of soft gravitons factorizes at the subleading order. The result does not depend upon the finite energy external states involved in the scattering. It only depends upon the linear and angular momenta carried by these states. Unlike the leading and the single subleading order results, the general result involves a term which has a purely quantum origin. This term, called contact term, arises at subleading order only when we have more than one soft graviton. A subsequent study by Alok Laddha and Ashoke Sen showed that this universal contact term drops out when we consider a classical limit [22].

Since our result at the subleading order does not depend upon the theory, any theory of quantum gravity in the low energy limit must satisfy this. Another feature of our result is that it is valid in arbitrary dimensions and only assumes the finiteness of the S-matrix. Whenever the S-matrix is not finite, our results are valid at tree level only. This happens in four and less number of dimensions. In 5 dimensions, naively, one encounters some IR enhancement in individual diagrams indicating possible breakdown of multiple soft theorem. However, it is expected that such IR enhanced contributions cancel after summing over all the diagrams.

3. Testing Soft Theorem Using CHY Formalism [23]

The third work was related to a new formalism developed by Freddy Cachazo, Song He and Ellis Ye Yuan for computing the tree level scattering amplitudes involving massless particles in quantum field theories [24–27]. This is known as the CHY formalism. This work was motivated by the desire to test our result on the general subleading soft graviton theorem in a specific theory. For this, we considered the Einstein's theory of gravity in the CHY formalism. Since our subleading result should be valid in any theory, it should also be valid for the purely Einstein's gravity containing only gravitons. Our goal was to derive the multiple subleading soft graviton theorem in Einstein's gravity using the CHY formalism and compare with our earlier result derived using Feynman diagram technique.

In the CHY formalism, an n-point tree amplitude of the massless particles is expressed as a sum over discrete set of points in the moduli space of an n-punctured Riemann sphere. The positions of the punctures are obtained by solving the so called scattering equations

$$\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b} = 0 \qquad\qquad \forall a \in \{1, 2, ..., n\}$$

where $\{\sigma_a\}$ are the holomorphic coordinates of the punctures and $\{p_a\}$ are momenta of the massless particles. Using the $SL(2,C)$ invariance of the moduli space of the sphere with n-punctures, we can fix the positions of three punctures and using $(n-3)$ number of independent scattering equations as constraint relations, the CHY formula is given as,

$$\mathcal{M}_n = \int \Big[\prod_{\substack{c=1\\ c\neq p,q,r}}^{n} d\sigma_c \Big] (\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki}) \Big[\prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b} \Big) \Big] I_n(\{p\},\{\epsilon\},\{\sigma\}) \, .$$

where $\sigma_{ab} \equiv \sigma_a - \sigma_b$.

The details of the theory is contained in the integrand I_n. It is determined by demanding some consistency requirements. For the purely Einstein's gravity, it is given by [27]

$$I_n(\{p\},\{\epsilon\},\{\sigma\}) = 4\,(-1)^n\,(\sigma_s - \sigma_t)^{-2}\,\det(\Psi^{st}_{st})$$

where Ψ is a $2n \times 2n$ anti-symmetric matrix and Ψ^{st}_{st} is obtained by removing s-th and t-th row from first n rows and removing s-th and t-th columns from first n columns of Ψ. The matrix Ψ has the form

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

where A, B, C are $n \times n$ matrices defined as,

$$A_{ab} = \begin{cases} \dfrac{p_a \cdot p_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases} \quad , \qquad B_{ab} = \begin{cases} \dfrac{\epsilon_a . \epsilon_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases}$$

$$C_{ab} = \begin{cases} \dfrac{\epsilon_a . p_b}{\sigma_a - \sigma_b} & a \neq b \\ -\displaystyle\sum_{c=1, c\neq a}^{n} \dfrac{\epsilon_a . p_c}{\sigma_a - \sigma_c} & a = b \, . \end{cases}$$

The polarizations $\varepsilon_{a,\mu\nu}$ of the gravitons are related to the space-time vectors ϵ_a by $\epsilon_{a,\mu}\epsilon_{a,\nu} \to \varepsilon_{a,\mu\nu}$.

The single and the double soft graviton result using the CHY formalism had been tried earlier in the literature [15, 16, 28, 29]. However, there were some issues with these results. Our aim was to resolve these issues and derive the most general tree level subleading soft graviton result in Einstein's gravity using this formalism. In doing this, we first needed to classify the various solutions of the scattering equations when there are an arbitrary number of the soft gravitons. We did this analysis in [23]. Using this result, the general case of the soft graviton theorem for an arbitrary number of the soft graviton was derived by performing the contour integrals in CHY formula successively. Our results for the Einstein's gravity matched perfectly with the general results derived in [4]. This not only showed the consistency with the known result but it was also a non trivial check for the validity of the CHY formalism itself given that the formalism has not yet been derived from first principles.

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Chapter 1

Introduction

Ever since Newton introduced the powerful language of mathematics [30] to describe the observable universe, we have been uncovering its secrets with ever increasing success. In the centuries following Newton, we discovered how the nature works at the scales which could be probed by our eyes. This included the regime in which the size of objects to be described were "large enough" and their speeds were "small enough". The laws which were uncovered during this period corresponded to the gravitational interaction which works at the scales of planets, stars and galaxies, the electromagnetic interactions which govern the behaviour of light, electricity and magnetism, and also the laws of thermodynamics which are needed when there is a large collection of particles. Using these laws expressed in mathematical language, one could very accurately predict the outcome of some event and then go and check whether the laws were applicable in real world or not.

Towards the dawn of 20th century, we started observing phenomenon which could not be described using these "classical laws". Moreover, with enough information in hand about the material objects, the time was ripe enough to ask questions about the nature of space time itself in which these objects reside. The developments along these lines led to the birth of quantum mechanics which is indispensable when we consider very small objects and special and general theories of relativity which are crucial for understanding the nature of space-time itself.

It was soon realised that a more accurate description of nature at sub atomic scales is provided by the quantum version of field theories, namely quantum field theories (QFTs). The progress along this direction led to a very accurate picture of nature which has been very rigorously verified in experiments upto nuclear scales. This description was called standard model and it describes the three known interactions, namely, strong, weak and electromagnetism in a unified manner. However, even though the standard model is very successful, it excludes the most obvious force visible to us, namely, gravity. The classical aspects of gravity is described by Einstein's general theory of relativity. Together with standard model of particle physics, the general relativity has very successfully described the evolution of the universe since the time its age was just fraction of seconds.

Despite the impressive success of the standard model and general relativity, they fail to describe some very important and interesting aspects of nature, e.g., the origin of the universe itself, the interior of black holes and questions about what happens at Plank scale. The main reason of the failure is that the classical theory of gravity is insufficient in situations where the gravity is very strong and the standard model is insufficient in situations when we want to probe very very small length scales.

The inadequacy of the standard model and the general relativity is also made clear from the Wilsonian point of view. The standard model and the naive quantisation of the gravity suffer from so called ultraviolet divergences. In the Wilsonian picture, this is a reflection of the fact that the theory is accurate only upto some length scale and breaks down beyond it. Some other theory takes over below this scale and the original theory remains just an effective description of this new theory above this scale. These issues imply that we need a theory which should replace the standard model and general relativity and provide a UV complete description of nature.

In constructing the standard model and general relativity, a number of guidance was provided by the requirement of mathematical consistency along with the experimental observations. In particular, the general relativity was constructed largely on aesthetic grounds using the tools of differential geometry. However, once constructed, it was used to make predictions which matched with the observations. Similarly, the standard model was based upon some prior experimental and clever use of mathematical tools. However, not all of its predictions were verified at the time of its construction. Some very verified much later.

The theories, which are needed when the standard model and general relativity fail, must reduce to these known theories when the latter are known to be applicable. However, they can be very different in the new regime. One of the goals of theoretical physics is to discover the theory which describes nature in these extreme regimes. Now, we don't have the direct access to the regimes in which known theories fail. This makes the search for new theories very difficult. In the absence of experimental inputs, one way to proceed is to use the mathematical consistency to go beyond the already known theories. To be sure, the ultimate validity of the theories constructed in this manner can only be decided by comparing their predictions with the actual observations. However, this may have to wait till we have enough technological advances to probe the regime where existing theories fail.

The string theory is an attempt in this direction. Its goal is to provide an UV complete theory incorporating all the interactions. In string theory, the fundamental particles are the one dimensional vibrating strings. The consistency with quantum mechanics demands that these one dimensional strings must propagate in 10 space-time dimensions. The connection with the observed 4 dimensional world is made by making the six of the dimensions to be compact and smaller than the length scales probed in experiments so far. One feature of string theory is that it automatically provides a consistent theory of quantum gravity since one of the vibration modes of strings is graviton which mediates gravity. The Einstein's equation also naturally emerge from string theory. Moreover, string theory is free from ultraviolet divergences making it a UV complete theory.

The formalism of string theory makes use of QFTs. In particular, a very special type of QFTs, namely, conformal field theories (CFTs) play a very important role in its formulation. In the Wilsonian picture, the CFTs are the fixed points of the renormalisation group flow. Some instances where CFTs appear in string theory are

1. A propagating string traces out a two dimensional surface in space-time. The theory living on this surface is a CFT. Demanding that this CFT be unbroken at quantum level gives rise to the Einstein's equation of motion in space-time.
2. In the context of AdS CFT correspondence, the boundary theory dual to the bulk string theory usually involves a conformal field theory.

Apart from their relevance in string theory, the CFTs are also important for better understanding of usual QFTs as the end points of the renormalisation group flow and describe many statistical systems near their critical points.

In this thesis, we focus on some aspects of conformal field theories and gravity as we describe now. The dynamical information of a conformal field theory is encoded in the conformal dimensions of the operators and the 3-point functions. The position and the momentum spaces are not ideal for making this fact transparent. E.g., the momentum space is useful when we have to extract information about the mass spectrum of the theory. However, due to the absence of single particle spectrum, the momentum representation of CFTs do not provide any simplification. However, it turns out that there is a mathematical representation which is more suited to CFTs. This is so called Mellin representation [2, 3]. This representation is sensitive to the scaling behaviour of the operators and hence is suitable for CFTs. It encodes the information about the scaling dimensions and the 3-point functions of the theory in the same way as momentum space encodes the information about the mass spectrum in usual QFTs.

In this work, we focus on the situations where CFT is perturbed by scalar marginal operators. In such situations, we can write a Feynman diagrammatic expansion for the correlation functions of the theory. This is the regime of weakly coupled CFT. Our goal in this work is to derive tree Feynman rules for such CFTs in the Mellin space for scalar operators. We find that the scalar propagators are given by the beta functions. We also give diagrammatic rules to write down the expression of an arbitrary loop Feynman diagram as an integration over the Schwinger parameters.

In the second work, we focus on some aspects of the gravitational interaction in the low energy infrared (IR) limit. In an scattering experiment, due to the limit on their resolution, any detector can't distinguish between the amplitudes with very low energy massless particles from the amplitudes without the low energy massless particles. It turns out that if these massless particles arise due to some gauge symmetries, then some special thing can happen. For a theory involving spin-2 massless gravitons, we have general coordinate invariance as gauge symmetry [5]. Using this, it was shown by Weinberg that in the expansion of an S-matrix (involving an arbitrary number of low energy massless gravitons) in powers of the momenta of the soft gravitons, the leading term can be written as product of the S-matrix without the soft gravitons multiplied by a factor which depends only upon the momenta and the polarisations of the gravitons [6]. Moreover, this soft factor turns out to be universal and does not depend upon the details of the theory.

Following a diagrammatic method developed by Ashoke sen [18, 19], we prove that the universal factorisation property shown by Weinberg for arbitrary number of gravitons for the leading term actually holds upto the subleading term in the expansion of S-matrix in terms of soft momenta. The diagrammatic method also shows that the universality is lost beyond the subleading term and the factorisation property itself is lost beyond the subsubleading order. The universal soft factor for the case of multiple soft gravitons contains a piece which has a purely quantum origin [22].

Finally, we derive the soft graviton theorem in the specific case of Einstein's gravity and match with our universal result. For this, we do not use the Feynman diagrammatic method but instead use the CHY (Cachazo-He- Yuan) prescription for computing the tree amplitudes involving massless particles [24–27]. The CHY prescription was developed by making a connection between the factorisation of the Riemann sphere with punctures with the factorisation property of the amplitudes. In this method, an n-point amplitude involving the massless particles is computed by doing a specific sum over the moduli space of Riemann sphere with n punctures. Unlike the Feynman diagram method, each term in this sum corresponding to some amplitude is gauge invariant by itself. We use this method to derive the multiple subleading soft graviton theorem in Einstein's gravity and find the result to be in perfect agreement with the universal result derived using the diagrammatic technique. In the process, we also fix some signs in the CHY description of the Einstein's gravity. Apart from confirming the multiple soft graviton theorem, our result also provides a non trivial test of the CHY prescription itself.

The rest of the thesis is organised as follows. In chapter 2, we derive the Feynman rules in mellin space for CFTs involving the scalar operators and give diagrammatic rules to write down the mellin amplitude of an arbitrary loop diagram. In chapter 3, we consider the multiple subleading soft graviton theorem and derive it for an arbitrary theory having general coordinate invariance using the Feynman diagrammatic method. Finally, in chapter 4, we consider the Einstein's gravity and derive the subleading multiple soft graviton theorem for it using the CHY prescription. In appendix A, we review the Mellin representation of the conformal field theories and discuss its properties. Appendix B is devoted to the review of the CHY formalism. Appendices C, D and E are devoted to the details of some calculations in chapters 2, 3 and 4 respectively.

Chapter 2

Feynman Rules for Tree Mellin Amplitudes of Scalar Fields

In this chapter, we shall be describing the tree Feynman rules in Mellin space for the conformal field theories (CFTs). It has been realized in the last few years, beginning with the pioneering work of Mack [2, 31] (see also [3]), that Mellin space provides the natural setting for the study of CFTs. The Mellin transform of a CFT correlator is a meromorphic function in the Mellin variables. In particular, for a four point function, the isolated simple poles locate the conformal twists of the operators in the spectrum whereas the residues at these poles contain information about the 3-point couplings. Thus the CFT data (operator dimensions and OPE coefficients) is at once made manifest in the Mellin space representation. Mellin amplitudes are also conformally invariant making conformal symmetry manifest in Mellin space.

Usually in quantum field theory, we Fourier transform the position space correlators to write Feynman rules in momentum space. The important advantage in doing so is that translation invariance leads to momentum conservation and the position space integrals are reduced to simple products in momentum space at tree level. In momentum space, conformal transformations have a non-linear action and as a result the conventional way of doing perturbative QFT in momentum space is not so advantageous for CFTs.

Various important features of QFT such as locality, causality and unitarity can be understood in terms of the analytic properties of momentum space amplitudes. The isolated poles of the momentum space propagator correspond to single-particle states and the branch cuts on the real axis give the multi-particle states (Kählen Lehmann spectral representation) and the amplitudes factorise on residues at the poles to lower point amplitudes. In a CFT, we do not have single particle states characterised by the masses since mass is a dimensionful parameter. Hence the propagators in momentum space have branch cuts extending to the origin. In the radial quantization of CFT, the dilatation operator acts as the Hamiltonian. The eigenvalues of this operator are discrete for $d > 2$. This discrete set of operators appear in the operator product expansion (OPE) as the exchanged primaries and descendants in an interacting CFT (in $d > 2$) . So it is desirable to have a representation for correlation functions in CFTs that makes this discrete spectrum manifest. As shown by Mack, it turns out that Mellin space provides such a representation.

The analogy of the Mellin space CFT correlators with scattering amplitudes is also striking. This has been explored in the context of the AdS/CFT correspondence. Following Mack, the application of the Mellin representation of conformal correlation functions was explored at strong coupling for large N CFTs using tree level Witten diagrams in AdS [32–38]. While at tree level, there seem to be a set of Feynman rules to write the Mellin amplitudes, the loop level seems to be significantly more involved. In the flat space limit of AdS/CFT, a relation between the bulk scattering amplitude and the CFT Mellin amplitudes was also suggested in [32, 35] and later put on a firm footing in [36, 39–41]. To be precise, the flat space S-matrix is expressed as an integral transform of the CFT Mellin amplitude and the Mellin variables, in the flat space limit, turn into flat space kinematic invariants (the Mandelstam variables). This scheme also relates the S-Matrix program in QFTs to the Bootstrap program in CFTs [42].

It was shown recently that it may be possible to bootstrap the full holographic correlator for the four point function of one-half BPS single trace operators in the context of IIB supergravity in $AdS_5 \times S^5$ [43, 44]. In the context of higher-spin holography, there have been efforts to understand the non-locality in the bulk interactions with Mellin amplitudes in the dual free CFT [45–47]. A new approach to the conformal bootstrap has also been developed in Mellin space [48,49] (see also [50–52]). In this method, conformal correlation functions are expanded in a manifestly crossing symmetric basis of functions provided by exchange Witten diagrams (in three channels). Demanding consistency with the Operator Product Expansion (OPE) one obtains constraints on operator dimensions and OPE coefficients.

The Mellin representation for tensor and fermionic operators and the factorization of Mellin amplitudes has also been studied recently [41, 53]. The Mellin representation has also been explored in the context of minimal model CFTs in [54] and for open string amplitudes in [55]. It was explored in the weak coupling regime in [56, 57] in the context of SYM and has also been used to calculate corrections beyond the planar limit to the 4-point function of a primary operator in $\mathcal{N} = 4$ SYM in [58] (see also [59–63]).

The goal of this work is to further explore the suitability of the Mellin representation for studying perturbative CFTs. We shall work with *weakly* coupled CFTs and attempt to formulate Feynman rules in Mellin space for perturbative field theory computations. We consider an exactly marginal perturbation around a free CFT and investigate whether it is possible to obtain a set of Feynman rules that can be used to calculated Mellin amplitudes. For simplicity, we shall restrict to scalar operators in this chapter. We present a complete derivation of the Feynman rules associated to tree level amplitudes in complete generality. For this purpose, we also develop a diagrammatic algorithm to write down the Mellin amplitude for any Feynman diagram (upto arbitrary loop order) as an integral over

Schwinger parameters corresponding to the internal propagators in the diagram. We further relax the conformality of the integrals, we consider, to study Mellin amplitudes in free CFTs with a generic perturbation. It turns out that when we consider integrals that enjoy a scale covariance only (as opposed to the full conformal covariance) the corresponding "Mellin amplitudes" can be interpreted as "off-shell" quantities that reduce to the "on-shell" conformal Mellin amplitudes under an LSZ like prescription.

This chapter is organized as follows. In section 2.1, we give a quick review of the Mellin amplitude for conformal field theories and also describe the situation when the Feynman rules developed in this chapter are useful. A more general introduction to the Mellin amplitudes is given in appendix A. In section 2.2, we consider some simple tree level Feynman diagrams involving only scalar fields and derive their Mellin amplitude. This is to introduce the general strategy that we follow for deriving the Mellin amplitude of a general tree level Feynman diagram. In section 2.3, we provide a general derivation for the Feynman rules for tree level diagrams. For this, we develop an algorithmic method for writing down the Mellin amplitude for an arbitrary Feynman diagram (tree as well as loops) as an integral over the Schwinger parameters for the internal propagators. In section 2.4, we consider the Mellin amplitudes for loop diagrams involving scalar fields and write an integral expression for the Mellin amplitude for such diagrams. In section 2.5, we extend the notion of Mellin amplitude to include generic scalar deformations of a free CFT which may break conformal invariance.

Throughout this chapter, the space-time Lorentz indices will be suppressed. We shall use the indices $\{i, j, \cdots\}$ for external vertices and the indices $\{a, b, \cdots\}$ for internal vertices. For convenience, we shall use the upstair indices for denoting the external vertices and the lower indices for denoting the internal vertices. This turns out to be useful for us mainly because of the fact that our analysis does not depend on how many external legs are attached to a given internal vertex. This will become clear when we consider explicit calculations. More details on the notations and conventions relevant for this chapter can be found in the appendix C.

2.1 Mellin amplitude and perturbative CFTs

The Mellin amplitude for an arbitrary n-point function is defined by the Mellin transformation of the position space correlation function [2, 31]

$$A\left(x^{i}\right)=\prod_{1\leq i<j\leq n}\left(\int_{-i\infty}^{i\infty}\frac{ds^{ij}}{2\pi i}\,\Gamma\left(s^{ij}\right)\left(x^{i}-x^{j}\right)^{-2s^{ij}}\right)\prod_{i=1}^{n}\delta\left(\Delta^{i}-\sum_{j=1}^{n}s^{ij}\right)M\left(s^{ij}\right)\tag{2.1}$$

Here s^{ij} are the Mellin variables and $M\left(s^{ij}\right)$ is defined to be the Mellin amplitude. The variable Δ^{i} is the scaling dimension of the operator inserted at x^{i}. One strips $M\left(s^{ij}\right)$ of the factors of $\Gamma(s^{ij})$ for convenience. This turns out to be particularly useful for large N gauge theories where these Gamma functions account for the poles corresponding to the multi trace operators whereas $M\left(s^{ij}\right)$ accounts for poles corresponding to the single particle states. Here, we summarize some important properties of the Mellin amplitude defined in (2.1). More details on the Mellin amplitude and its properties can be found in appendix A.

1. The delta function constraints in the definition of Mellin amplitude (2.1) ensure the covariance of $A(\{x_i\})$ under conformal transformations. More precisely, under inversion

$$\left(x^i - x^j\right)^2 \to \frac{\left(x^i - x^j\right)^2}{(x^i)^2 (x^j)^2}$$

The correlation function $A\left(\{x^i\}\right)$ transforms as

$$A\left(\{x^i\}\right) \to \left[\prod_{i=1}^{n} (x^i)^{-2\Delta^i}\right] A\left(\{x^i\}\right)$$

The delta function constraints $\sum\limits_{j\neq i} s^{ij} = \Delta^i$ ensure that both sides of (2.1) transform in the same way.

2. The Mellin amplitude $M(\{s^{ij}\})$ is manifestly conformally invariant. The conformal transformations act on the position space variables x^i. The x^i dependence of the expression (2.1) and the delta functions imposing constraints on the Mellin variables ensure that $A(\{x^i\})$ is conformally covariant.

3. The Mellin variables s^{ij} are symmetric in i and j. So the number of Mellin variables s^{ij} is $n(n-1)/2$. However, due to the n delta function constraints, the number of independent Mellin variables is only $\frac{n(n-3)}{2}$. This is also the number of independent cross-ratios for n points.

4. The delta function constraints can be solved in terms of the "dual Mellin momenta" [2] with $s^{ij} = k^i \cdot k^j$ and $(k^i)^2 = -\Delta^i$ and overall Mellin momentum conservation $\sum_i k^i = 0$. These are fictitious momenta associated with each x^i. We refer to $(k^i)^2 = -\Delta^i$ as the "on-shell" condition for Mellin momenta.

As mentioned above, we shall consider perturbative CFTs in this chapter. More precisely, we shall consider the situations in which a CFT is perturbed by marginal operators in an expansion in the coupling parameter. If the coupling parameters are exactly marginal, the amplitudes will be those of an exact CFT. This situation arises, e.g., when we perturb free $\mathcal{N} = 4$ SYM by interaction coupling g_{YM} (or $\lambda = g_{YM}^2 N$ in the large N limit).

Now, given an exact CFT such as free field CFT of scalar fields ϕ_i, we can consider perturbation by interaction vertices of the form

$$S_{int} = \frac{g_n}{n!} \int d^D x \, \phi_1 \phi_2 \cdots \phi_n \tag{2.2}$$

The dimensions of the operators are given by $[\phi_i] = \Delta_i$ and the condition of marginality is given by

$$\sum_{i=1}^{n} \Delta_i = D \tag{2.3}$$

where, D denotes the space-time dimension.

The interaction vertices of the form (2.2) will contribute to n-point function in perturbation theory. We shall see how these contributions can be evaluated using the Feynman diagram technique. For this, we shall make use of the fact that the free field 2-point function for an operator of dimension Δ is given by

$$\frac{1}{(x-y)^{\Delta}} \tag{2.4}$$

We shall not assume any other details of the theory except the existence of interaction vertices of the form given in (2.2).

2.2 Some Examples of Tree Diagrams

In this section we consider a few simple tree level examples which will illustrate the general strategy we shall follow for deriving the Mellin amplitude of Feynman diagrams involving only scalar fields. Specifically, we shall be looking at the contact interaction diagram, the diagram with one internal propagator and the diagram with two internal propagators. The Mellin amplitude for the contact interaction diagram and one propagator diagram were presented in [56]. We begin with these examples for pedagogy and completeness of our presentation.

2.2.1 Contact Interaction

The position space Feynman diagram for the contact interaction is shown in Figure 2.1. In this diagram, N external lines are meeting at the vertex u. We denote the scaling dimension of the field correponding to the external vertex x^i by Δ^i. As mentioned earlier, we choose to place the index upstairs to keep the notation compact when we discuss more complicated Feynman diagrams.

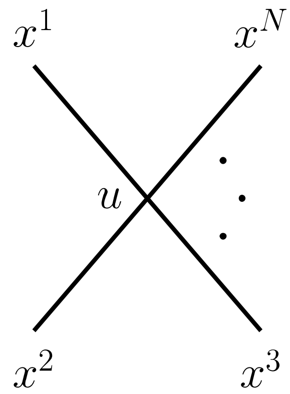

Figure 2.1: Contact Interaction Diagram

The position space correlation function corresponding to the contact interaction is given by

$$I = \int \frac{d^D u}{2(2\pi)^{D/2}} \left[\prod_{i=1}^{N} (x^i - u)^{-2\Delta^i} \Gamma(\Delta^i) \right] \tag{2.5}$$

The factors of $\Gamma(\Delta^i)$ and π have been included for the sake of convenience[1] later on. We follow these conventions throughout the chapter.

As mentioned in section 2.1, this expression is covariant under conformal transformations provided we impose the following 'conformality condition' on the conformal dimensions

$$\sum_{i=1}^{N} \Delta_i = D \tag{2.6}$$

We now introduce a Schwinger parameter for each propagator via the identity

$$\frac{1}{(x-y)^{2\Delta}} = \frac{1}{\Gamma(\Delta)} \int_0^\infty d\alpha \, \alpha^{\Delta - 1} \exp\left[-\alpha (x-y)^2\right] \tag{2.7}$$

Using this identity in (2.5) gives,

$$I = \prod_{i=1}^{N} \left[\int_0^\infty d\alpha^i (\alpha^i)^{\Delta^i - 1} \right] \int \frac{d^D u}{2(2\pi)^{D/2}} \exp\left[- \left(\sum_{i=1}^{N} \alpha^i (x^i - u)^2 \right) \right]$$

[1] In the rest of the chapter, we denote the measure as

$$\frac{d^D u}{2(2\pi)^{D/2}} \equiv \mathcal{D}u$$

The factors of $\Gamma(\Delta^i)$ present in (2.5) are cancelled by the corresponding factors in (2.7).

Performing the Gaussian integration over u, we obtain

$$I = \frac{1}{2}\prod_{i=1}^{N}\int_0^\infty d\alpha^i \left(\frac{(\alpha^i)^{\Delta^i-1}}{(\sum_i \alpha^i)^{\frac{D}{2}}}\right) \exp\left[-\frac{1}{\sum_i \alpha^i}\left(\sum_j \sum_{i<j} \alpha^i \alpha^j (x^{ij})^2\right)\right] \tag{2.8}$$

where, $x^{ij} \equiv x^i - x^j$.

We shall now render (2.8) particularly suitable for imposing the conformality conditions (2.6). For this, we insert the following partition of unity in (2.8)

$$1 = \int_0^\infty dv\, \delta\left(v - \sum_i \alpha^i\right)$$

We then rescale the Schwinger parameters, $\alpha^i \to \sqrt{v}\,\alpha^i$ and perform the integration over the auxiliary variable v using delta function. The end result is

$$I = \prod_{i=1}^{N}\left[\int_0^\infty d\alpha^i (\alpha^i)^{\Delta^i-1}\right]\left(\sum_i \alpha^i\right)^{\sum_i \Delta^i - D} \exp\left(-\sum_j \sum_{i<j} \alpha^i \alpha^j (x^{ij})^2\right) \tag{2.9}$$

We now impose the conformality condition (2.6) on equation (2.9) to obtain

$$I = \prod_{i=1}^{N}\left[\int_0^\infty d\alpha^i (\alpha^i)^{\Delta^i-1}\right] \exp\left(-\sum_j \sum_{i<j} \alpha^i \alpha^j (x^{ij})^2\right) \tag{2.10}$$

To proceed further, we now use the inverse Mellin transform representation of the exponential function

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \Gamma(s) x^{-s} \tag{2.11}$$

Here c is a real number greater than or equal to zero (if $c = 0$, then the contour of integration has a dent at the origin so as to put the pole at the origin on the left). The contour can be shifted to the right freely as all the poles are on negative real axis (note that the poles of $\Gamma(s)$ are at 0 and all negative integers). This freedom in shifting the contour (or equivalently, the freedom in the choice of c) turns out to be very crucial as we shall see below.

Using (2.11) for the exponential factor in (2.10), we obtain,

$$I = \prod_j \prod_{i<j}\left[\int_{c^{ij}-i\infty}^{c^{ij}+i\infty} [ds^{ij}](x^{ij})^{-2s^{ij}}\Gamma(s^{ij})\right]\prod_{i=1}^{N}\left[\int_0^\infty d\alpha^i (\alpha^i)^{\rho^i-1}\right] \tag{2.12}$$

where, the s^{ij} (corresponding to x^{ij}) are our Mellin variables and c^{ij} are real numbers greater than zero. Also,

$$\rho^i \equiv \Delta^i - \sum_{\substack{j=1\\ j\neq i}}^{N} s^{ij} \quad , \qquad\qquad [ds^{ij}] \equiv \frac{ds^{ij}}{2\pi i} \tag{2.13}$$

If ρ^i had any real part, the integration over the Schwinger parameters α^i in (2.12) will give divergent result. However, as explained in the appendix C.2, the integrals $\int_0^\infty d\alpha^i (\alpha^i)^{\rho^i - 1}$ behave as delta function inside the contour integration provided the real part of the exponent ρ^i is zero along the contour. More explicitly, as shown in the appendix C.2, we have the following result

$$\int_{c-i\infty}^{c+i\infty} [ds] f(s) \int_0^\infty dt\, t^{s_0 - s - 1} = \int_{c-i\infty}^{c+i\infty} [ds] f(s) \left(2\pi i \delta(s - s_o)\right) \tag{2.14}$$

if we choose $c = \text{Re}(s_0)$ (so that the real part of the exponent $s_0 - s$ is zero along the contour).

Thus, for the Schwinger parameter integrals in (2.12) to be well defined, we need to ensure that the real part of the exponents ρ^i vanish along the contour of integration. Using the expression of ρ^i given in (2.13), this means that we need to choose the set $\{c_{ij}\}$ in such a way that they satisfy

$$\sum_{\substack{j=1\\ j\neq i}}^{N} c^{ij} = \Delta^i \tag{2.15}$$

We observe from the discussion in section 2.1 that (2.15) is the same set of constraints that s^{ij} must satisfy if the expression in (2.12) has to transform correctly under the conformal transformations. Therefore we can infer that a solution to (2.15) exists (or else it would lead to a contradiction) and the Schwinger parameter integrals in (2.12) are well defined.

Using (2.14), the expression (2.12) becomes

$$I = \prod_j \prod_{i<j} \left[\int_{c^{ij}-i\infty}^{c^{ij}+i\infty} [ds^{ij}] (x^{ij})^{-2s^{ij}} \Gamma(s^{ij}) \right] \left[\prod_i 2\pi i \delta \left(\Delta^i - \sum_{j\neq i} s^{ij} \right) \right] \tag{2.16}$$

As discussed above, the constraints $\rho_i = 0$ (enforced by the delta functions) are precisely the constraints on the Mellin variables discussed in section 2.1. These constraints originate from the fact that the position space correlation function is covariant under conformal transformations. We also note that these constraints reduce the number of independent Mellin variables from $N(N-1)/2$ to $N(N-3)/2$. A careful look at (2.16) tells us that the N delta functions force the $(x^{ij})^{-2s^{ij}}$ terms to combine and form $\frac{N(N-3)}{2}$ cross ratios between the external vertices x^i and some extra factors that give appropriate transformation properties to the position space correlator.

We can now read off the Mellin ampltitude corresponding to the Feynman diagram in Figure 2.1. Comparing (2.16) with the defining expression of Mellin amplitude (2.1), we find that the Mellin amplitude for contact interaction is just 1.

2.2.2 Tree With One Internal Propagator

The next Feynman diagram that we consider (Figure 2.2) involves two internal vertices connected by an internal propagator. This example will give us the expression for the scalar propagator in Mellin space. γ denotes the scaling dimension of the internal propagator.

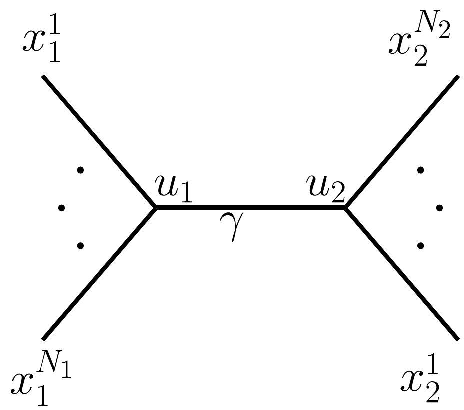

Figure 2.2: Two vertex

The position space expression for this diagram is given by

$$I = \int \mathcal{D}u_1 \, \mathcal{D}u_2 \left[\prod_{i \in 1} (x_1^i - u_1)^{-2\Delta_1^i} \Gamma(\Delta_1^i) \prod_{j \in 2} (x_2^j - u_2)^{-2\Delta_2^j} \Gamma(\Delta_2^j) (u_2 - u_1)^{-2\gamma} \right]$$

The conformality conditions for the two interaction vertices in this diagram are,

$$\sum_{i \in 1} \Delta_1^i + \gamma = D \quad , \qquad \sum_{i \in 2} \Delta_2^i + \gamma = D \tag{2.17}$$

We again use the identity (2.7) and introduce the Schwinger parameters for each propagator (internal as well as external)

$$\begin{aligned} I \;\; = \;\; & \left[\prod_{i \in 1} \int_0^\infty d\alpha_1^i (\alpha_1^i)^{\Delta_1^i - 1} \prod_{j \in 2} \int_0^\infty d\alpha_2^j (\alpha_2^j)^{\Delta_2^j - 1} \frac{1}{\Gamma(\gamma)} \int_0^\infty dt \; t^{\gamma - 1} \right] \int \mathcal{D}u_1 \, \mathcal{D}u_2 \\ & \exp\left(- \sum_{i \in 1} \alpha_1^i (x_1^i - u_1)^2 - \sum_{j \in 2} \alpha_2^j (x_2^j - u_2)^2 - t(u_2 - u_1)^2 \right) \end{aligned}$$

where t is the Schwinger parameter for the internal propagator.

Performing the u_1 integration, we obtain

$$I = \frac{1}{2\Gamma(\gamma)} \prod_{a=1}^{2} \prod_{i\in a} \Big(\int d\alpha_a^i (\alpha_a^i)^{\Delta_a^i - 1} \Big) \int_0^\infty dt\, t^{\gamma-1} \int \mathcal{D}u_2 \, \exp\Big(- \sum_{j\in 2} \alpha_2^j (x_2^j - u_2)^2 \Big)$$
$$\Big(\sum_{i\in 1} \alpha_1^i + t\Big)^{-D/2} \exp\Big(- \Big(\sum_{i\in 1} \alpha_1^i + t\Big)^{-1} \Big\{ \sum_{(i,j)\in 1} \alpha_1^i \alpha_1^j (x_{11}^{ij})^2 + t \sum_{i\in 1} \alpha_1^i (x_1^i - u_2)^2 \Big\} \Big) \tag{2.18}$$

Next, we insert the partition of unity

$$1 = \int_0^\infty dy\, \delta\Big(y - \sum_{i\in 1} \alpha_1^i - t \Big)$$

in the integral of (2.18), rescale the Schwinger parameters

$$\alpha_1^i \to \sqrt{y}\, \alpha_1^i \qquad , \qquad t \to \sqrt{y}\, t \tag{2.19}$$

and perform the integration over the variable y using the delta function. The result is

$$I = \frac{1}{\Gamma(\gamma)} \prod_{a=1}^{2} \prod_{i\in a} \Big[\int_0^\infty d\alpha_a^i (\alpha_a^i)^{\Delta_a^i - 1} \Big] \int_0^\infty dt\, t^{\gamma-1} \Big(\sum_{i\in 1} \alpha_1^i + t \Big)^{\sum_{i\in 1} \Delta_1^i + \gamma - D}$$
$$\exp\Big(- \sum_{(i,j)\in 1} \alpha_1^i \alpha_1^j (x_{11}^{ij})^2 \Big) \int \mathcal{D}u_2 \exp\Big(- \sum_{j\in 2} \alpha_2^j (x_2^j - u_2)^2 - t \sum_{i\in 1} \alpha_1^i (x_i - u_2)^2 \Big)$$

Next, we perform the u_2 integration, insert the following partition of unity in the integral

$$1 = \int_0^\infty dy\, \delta\Big(y - \sum_{i\in 2} \alpha_2^i - t \sum_{i\in 1} \alpha_1^i \Big) \ ,$$

carry out similar rescalings as in (2.19) (but this time, with the variables α_2^i and t) and perform the integration over the auxiliary variable y. This gives,

$$I = \frac{1}{\Gamma(\gamma)} \prod_{i\in 1} \Big[\int_0^\infty d\alpha_1^i (\alpha_1^i)^{\Delta_1^i - 1} \Big] \prod_{j\in 2} \Big[\int_0^\infty d\alpha_2^j (\alpha_2^j)^{\Delta_2^j - 1} \Big] \int_0^\infty dt\, t^{\gamma-1}$$
$$\exp\Big(-(1+t^2) \sum_{(i,j)\in 1} \alpha_1^i \alpha_1^j (x_{11}^{ij})^2 - \sum_{(i,j)\in 2} \alpha_2^i \alpha_2^j (x_{22}^{ij})^2 + t \sum_{i\in 1} \sum_{j\in 2} \alpha_1^i \alpha_2^j (x_{12}^{ij})^2 \Big)$$
$$\Big(\Big(1+t^2\Big) \sum_{i\in 1} \alpha_1^i + t \sum_{i\in 2} \alpha_2^i \Big)^{\sum_{i\in 1} \Delta_1^i + \gamma - D} \Big(\sum_{i\in 2} \alpha_2^i + t \sum_{i\in 1} \alpha_1^i \Big)^{\sum_{i\in 2} \Delta_2^i + \gamma - D}$$

We impose the conformality conditions (2.17), and then use the identity (2.11) for each exponential factor. After some rearrangement, we obtain,

$$I = \frac{1}{\Gamma(\gamma)} \prod_{(i,j)\in 1+2} \left(\int_{c_{ab}^{ij}-i\infty}^{c_{ab}^{ij}+i\infty} [ds_{ab}^{ij}] \Gamma(s_{ab}^{ij}) \Big((x_{ab}^{ij})^2 \Big)^{-s_{ab}^{ij}} \right) \prod_{i\in 1} \left(\int_0^\infty d\alpha_1^i (\alpha_1^i)^{\rho_1^i - 1} \right)$$
$$\prod_{j\in 2} \left(\int_0^\infty d\alpha_2^j (\alpha_2^j)^{\rho_2^j - 1} \right) \int_0^\infty dt\, t^{\gamma - s_{12} - 1} (1+t^2)^{-s_{11}} \qquad (2.20)$$

where,

$$\begin{aligned} \rho_1^i &\equiv \Delta_1^i - \sum_{j\in 1} s_{11}^{ij} - \sum_{j\in 2} s_{12}^{ij} \quad , \qquad (i \in 1) \\ \rho_2^i &\equiv \Delta_2^i - \sum_{j\in 2} s_{22}^{ij} - \sum_{j\in 1} s_{12}^{ji} \quad , \qquad (i \in 2) \\ s_{12} &\equiv \sum_{i\in 1}\sum_{j\in 2} s_{12}^{ij} \quad ; \qquad s_{aa} \equiv \sum_{1\le i<j\le N_a} s_{aa}^{ij} \ , \quad a = 1,2 \end{aligned}$$

Once again, we choose c_{ab}^{ij} appropriately so that the integrals over the Schwinger parameters α_1^i and α_2^i act as delta functions in the Mellin space (as explained in the previous section). This means that the Mellin variables satisfy the constraints

$$\rho_1^i = 0 = \rho_2^i \qquad \forall\, i \qquad (2.21)$$

These reduce the number of independent Mellin variables from $(N_1+N_2)(N_1+N_2-1)/2$ to $(N_1+N_2)(N_1+N_2-3)/2$. Summing over i and using the conformality conditions (2.17) gives useful relations between the Mellin variables

$$\sum_{i\in 1} \Delta_1^i = 2s_{11} + s_{12} = D - \gamma \quad , \qquad \sum_{i\in 2} \Delta_2^i = 2s_{22} + s_{12} = D - \gamma \qquad (2.22)$$

By comparing (2.20) with the definition of Mellin amplitude (2.1) (taking into account the constraints (2.21)), we can easily read off the Mellin amplitude to be

$$M(s_{12}) = \frac{1}{\Gamma(\gamma)} \int_0^\infty dt\, t^{\gamma - s_{12} - 1} (1+t^2)^{-s_{11}} = \frac{1}{2\Gamma(\gamma)} \beta\Big(\frac{\gamma - s_{12}}{2}, \frac{D}{2} - \gamma \Big) \qquad (2.23)$$

where we have used (2.22) to simplify the arguments of beta function.

The physical interpretation of the amplitude is clear. We can identify the beta function to be the Mellin space propagator. Moreover, the poles of the beta function have clear physical interpretation. At this stage, it is convenient to introduce dual Mellin momenta. If the Mellin momentum flowing

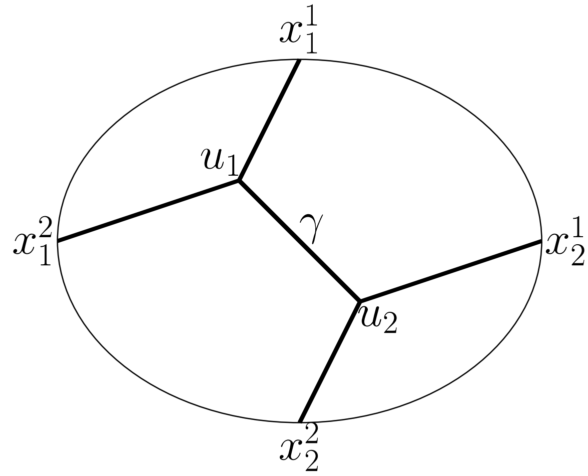

Figure 2.3: 4-point exchange Witten diagram

into the internal propagator through the external vertex x_a^i be k_a^i (where we have suppressed the dual spacetime index), then the full momentum propagating through the internal propagator is

$$k = \sum_{i \in 1} k_1^i = -\sum_{i \in 2} k_2^i$$

and the kinematical variable entering into the propagator is

$$s_{12} = \sum_{i \in 1} \sum_{j \in 2} s_{12}^{ij} = \left(\sum_{i \in 1} k_1^i \right) \cdot \left(\sum_{j \in 2} k_2^j \right) = -k^2$$

This means that the poles of the propagator appear at particular values of the k^2, namely

$$s_{12} = -k^2 = \gamma + 2n \qquad ; \quad n = 0, 1, 2, \cdots \tag{2.24}$$

These poles correspond to the primary and its leading twist descendants ($n = 0$) and the satellite ($n > 0$) propagating states[2].

It is instructive to compare the above result (2.23) at weak coupling with an analogous result at strong coupling obtained using Witten diagrams in the dual bulk theory in AdS in [32, 33, 35]. As an example, we look at the result obtained in [33] for the 4-point function exchange Witten diagram involving scalars as shown in Figure 2.3.

[2]As shown by Mack [2], the poles of the Mellin amplitude occur at $s = \gamma - \ell + 2n$, $(n = 0, 1, 2 \cdots)$. The first pole $(n = 0)$ corresponds to the exchanged primary operator and all its leading twist descendants (i.e. those operators in the conformal multiplet whose dimensions and spins keep $\gamma - \ell$ fixed). The higher poles $(n > 0)$ correspond to the satellite poles. The above result (2.24) is consistent with this expectation.

The Mellin amplitude for this diagram is given by (for a coupling constant g),

$$
\begin{aligned}
M(s_{12}) &= \frac{1}{2}\frac{g^2}{(s_{12}-\gamma)}\frac{\Gamma\left(\frac{\Delta_1^1+\Delta_1^2+\gamma-\frac{D}{2}}{2}\right)\Gamma\left(\frac{\Delta_2^1+\Delta_2^2+\gamma-\frac{D}{2}}{2}\right)}{\Gamma\left(1+\gamma-\frac{D}{2}\right)} \\
& {}_3F_2\left(\frac{2-\Delta_1^1-\Delta_1^2+\gamma}{2},\frac{2-\Delta_2^1-\Delta_2^2+\gamma}{2},\frac{\gamma-s_{12}}{2};\frac{2+\gamma-s_{12}}{2},1+\gamma-\frac{D}{2};1\right)
\end{aligned}
$$

This Mellin amplitude has the same analytic structure as our corresponding result (2.23) for the weak coupling case which is proportional to $\beta\left(\frac{\gamma-s_{12}}{2},\frac{D}{2}-\gamma\right)$.

It would be interesting to understand the extrapolation of the weak coupling results to the strong coupling results (in the particular example we have considered, how the beta function of the weakly coupled regime extrapolates to the ${}_3F_2$ hypergeometric function in the strong coupling regime). In the maximally supersymmetric case, it may be possible to use the integrability of the boundary field theory as well as the string theory in the bulk to understand this interpolation between the results at strong coupling and at weak coupling.

2.2.3 Tree With Two Internal Propagators

In order to check our interpretation of the result (2.23) as the propagator in Mellin space, we consider one more example before generalising to arbitrary tree level diagrams. We consider a Feynman diagram with two internal propagators (see figure 2.4).

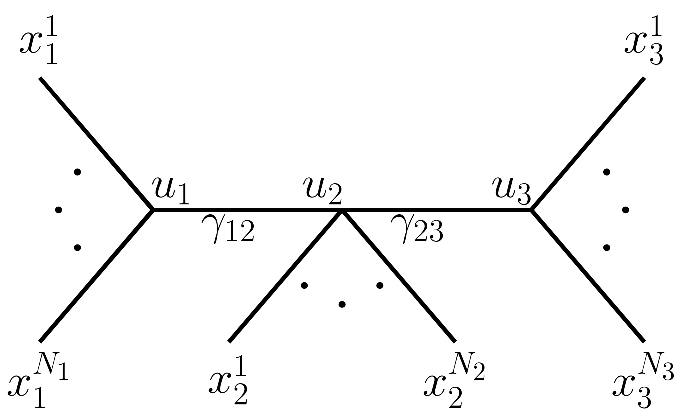

Figure 2.4: Three vertex tree

The position space expression for this is given by

$$I = \int \mathcal{D}u_1\mathcal{D}u_2\mathcal{D}u_3 \Bigg[\prod_{i\in 1}\Big\{(x_1^i - u_1)^{-2\Delta_1^i}\Gamma(\Delta_1^i)\Big\}\prod_{i\in 2}\Big\{(x_2^i - u_2)^{-2\Delta_2^i}\Gamma(\Delta_2^i)\Big\}$$
$$\times \prod_{i\in 3}\Big\{(x_3^i - u_3)^{-2\Delta_3^i}\Gamma(\Delta_3^i)\Big\}(u_2-u_1)^{-2\gamma_{12}}(u_2-u_3)^{-2\gamma_{23}}\Bigg] \qquad (2.25)$$

In this case, the conformality conditions are

$$\sum_{i\in a}\Delta_a^i = D - \gamma_{a,a+1} - \gamma_{a-1,a} \quad , \qquad 1 \leq a \leq 3$$

where, $\gamma_{01} = 0 = \gamma_{34}$.

For extracting the Mellin amplitude, we follow the same strategy as in the previous examples. However, now we have to choose an ordering of the vertices u_a for conducting the manipulations. All the choices lead to the same result eventually[3]. We follow the order $u_1 \to u_2 \to u_3$. The final result turns out to be

$$I = \prod_{a=1}^{3}\prod_{b=a}^{3}\Bigg(\prod_{(i,j)\in a+b}\int_{c_{ab}^{ij}-i\infty}^{c_{ab}^{ij}+i\infty}\left[ds_{ab}^{ij}\right]\Gamma(s_{ab}^{ij})\big(x_{ab}^{ij}\big)^{-2s_{ab}^{ij}}\Bigg)\prod_{a=1}^{3}\Bigg(\prod_{i\in a}\int_0^{\infty} d\alpha_a^i(\alpha_a^i)^{\rho_a^i-1}\Bigg)M(s_{ab})$$

where,

$$\rho_a^i \equiv \Delta_a^i - \sum_{j\in a}s_{aa}^{ij} - \sum_{\substack{b=1\\b\neq a}}^{3}\Bigg(\sum_{j\in b}s_{ab}^{ij}\Bigg) \quad , \qquad 1 \leq a \leq 3$$

Again, the integration over the variables α_a^i impose the constraints $\rho_a^i = 0$ which can be re-written as (using the conformality conditions)

$$\sum_{i\in a}\Delta_a^i = 2s_{aa} + \sum_{\substack{b=1\\b\neq a}}^{3}s_{ab} = D - \gamma_{a-1,a} - \gamma_{a,a+1} \quad , \qquad 1 \leq a \leq 3$$

Due to these constraints, the number of independent Mellin variables are only $N(N-3)/2$ (where

[3]The different choices for this ordering lead to integrals over the Schwinger parameters which are not manifestly equal. For diagrams with higher number of interaction vertices, it can often be difficult to show that these different integrals corresponding to the same Feynman diagram are all equal.

N is total number of external states). The Mellin amplitude is given by

$$\begin{aligned} M(s_{ab}) &= \frac{1}{\Gamma(\gamma_{12})\Gamma(\gamma_{23})}\int_0^\infty dt_{12}(t_{12})^{\gamma_{12}-s_{12}-s_{13}-1}\int_0^\infty dt_{23}(t_{23})^{\gamma_{23}-s_{13}-s_{23}-1} \\ &\quad \left(1+t_{12}^2(1+t_{23}^2)\right)^{-s_{11}}\left(1+t_{23}^2\right)^{-s_{22}-s_{12}} \\ &= \left[\frac{1}{2\Gamma(\gamma_{12})}\beta\Big(\frac{\gamma_{12}-s_{12}-s_{13}}{2},\frac{D}{2}-\gamma_{12}\Big)\right]\left[\frac{1}{2\Gamma(\gamma_{23})}\beta\Big(\frac{\gamma_{23}-s_{23}-s_{13}}{2},\frac{D}{2}-\gamma_{23}\Big)\right] \end{aligned} \tag{2.26}$$

This result, being a product of two beta functions with appropriate arguments, is consistent with our interpretation of the Mellin space propagator (2.23). Moreover, the poles of the propagator occur when the negative of the total Mellin momenta squared flowing through it is equal to the conformal dimension of the primary and descendants. This is easily seen by introducing the dual Mellin momenta and writing the arguments of beta functions in terms of these momenta.

2.3 General Tree Level Feynman Diagrams

We now consider general tree level Feynman diagrams. We shall show that the Mellin amplitude for an arbitrary tree diagram is given by the product over internal propagators. For each internal propagator, we obtain a factor of beta function with appropriate arguments consistent with the examples considered in the previous section.

In subsection 2.3.1, we present a diagrammatic algorithm to write down the Mellin amplitude (in terms of integrals over the Schwinger parameters) for any diagram involving only scalar operators. In subsection 2.3.2, we consider a tree diagram with n internal vertices such that one can go from one end of the diagram to the other end without encountering any branches (figure 2.11). Finally, in subsection 2.3.3, we consider a completely general tree diagram.

2.3.1 Diagrammatic Rules for Writing Mellin Amplitude

In this subsection, we present a diagrammatic technique which will be helpful in directly writing down the Mellin space amplitudes as integrals over the Schwinger parameters. These rules can be used for any tree as well as loop diagrams and will allow us to avoid going through all the algebraic manipulations, as described in the examples of the section 2.2.

For developing these rules, we shall use a simplified way to represent the Feynman diagrams. In our diagrammatic algorithm, the external lines in a Feynman diagram would not be playing any significant role. Hence, to simplify the diagrammatic representation, we represent the set of external lines attached to an interaction vertex by a small hollow circle at the vertex and the internal propagator by dashed lines. We call this the *skeleton of the Feynman diagram*. The skeleton for the single propagator Feynman diagram we considered earlier, is shown in Figure 2.5. Note that this way of representing a Feynman diagram is insensitive to the number of external legs attached to any given interaction vertex.

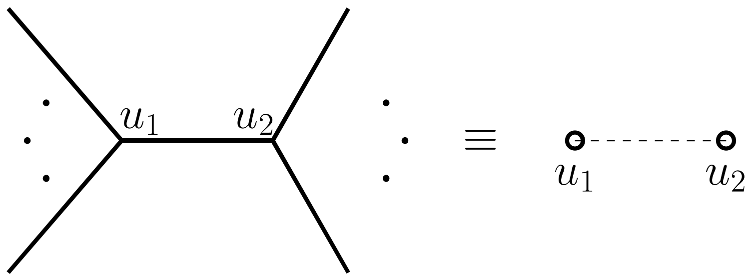

Figure 2.5: Skeleton of the single propagator diagram

Illustrating the rules

We now consider an explicit example of the two propagator case of section 2.2 to understand the diagrammatic rules. The skeleton of this Feynman diagram is shown in Figure 2.6. Considering this example serves a two-fold purpose. Apart from being an explicit (and simple) example of the application of the rules, it also helps us understand the origin of the rules. The essential idea is to represent the steps of derivation leading to the Mellin amplitude as a series of diagram. The components of the diagram are assigned some weight factors. The Mellin amplitude can be written in terms of the weight factors of the final diagram obtained after integrating over all the position space vertices.

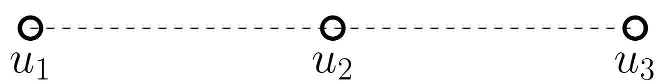

Figure 2.6: Tree level three vertex

We start with the position space expression for this diagram which is given in equation (2.25). For integrating over the interaction vertices, we choose the ordering $u_1 \to u_2 \to u_3$. We first consider the effect of integration over the u_1 variable. When u_1 is integrated over, we get terms of the form $\alpha_1^i \alpha_1^j (x_1^i - x_1^j)^2$ and $\alpha_1^i t_{12}(x_1^i - u_2)^2$ in the exponent. The factors of α^i eventually do not contribute to the Mellin amplitude (their role is in providing the delta function constraints on the Mellin variables as seen in examples of previous section). After using the Cahen Mellin identity for the exponentials, the coefficient of $\alpha_1^i \alpha_1^j (x_1^i - x_1^j)^2$ essentially becomes the part of Mellin amplitude. Keeping this in mind, we assign a weight 1 to the factor $(x_1^i - x_1^j)^2$. We also assign a weight of t_{12} with the factor $(x_1^i - u_2)^2$ (the reason for this will become clear shortly).

The statements made in the previous paragraph can be nicely captured by a diagrammatic means.

We take the diagram in figure 2.6 and replace the small hollow circle associated to the vertex u_1 with a bigger circle and the dashed lines connecting the adjacent un-integrated vertex with solid lines. We associate a weight 1 with this bigger circle and a weight t_{12} with the solid line (which is the Schwinger parameter associated with the line joining the vertices 1 and 2). This has been shown in Figure 2.7 (from now on, we won't write the vertex indices u_a)

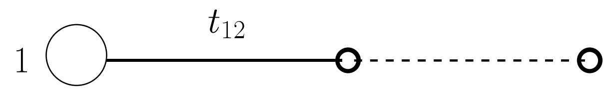

Figure 2.7: Diagrammatic representation of integration over vertex 1

Next, we perform the integration over the u_2 vertex. This gives rise to terms of the form $(1 + t_{12}^2)\alpha_1^i\alpha_1^j(x_1^i - x_1^j)^2$, $t_{12}\alpha_1^i\alpha_2^j(x_1^i - x_2^j)^2$, $\alpha_2^i\alpha_2^j(x_2^i - x_2^j)^2$, $\alpha_2^i t_{23}(x_2^i - u_3)^2$ and $\alpha_1^i t_{12}t_{23}(x_1^i - u_3)^2$ in the exponent. Keeping in mind that the coefficients of $\alpha_a^i\alpha_b^j(x_a^i - x_b^j)^2$ eventually become part of the Mellin amplitude, we diagrammatically represent this step by replacing the small circle around the u_2 vertex with a bigger circle, replace the dashed line connecting it with u_3 vertex by a solid line and making one more circle around the u_1 vertex. We also connect the vertices u_1 and u_3 by a different solid line. From the terms just mentioned, we see that we need to associate a weight 1 with the circle around u_2 vertex, a weight t_{23} with the solid line connecting u_2 and u_3 vertices, a weight $t_{12}t_{23}$ with the solid line connecting u_1 and u_3 and a weight t_{12}^2 with the new circle around u_1. This step, combined with the first step, can be represented as in Figure 2.8.

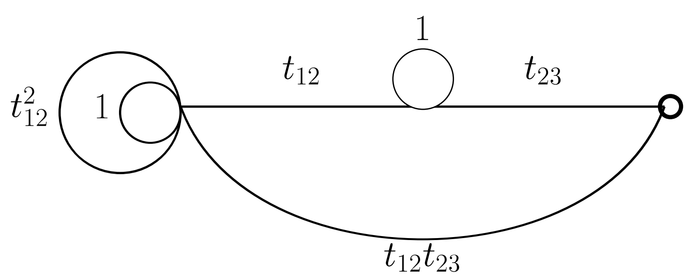

Figure 2.8: Diagrammatic representation of integration over second vertex

At this stage, we can state our strategy as follows: as mentioned above, the coefficients of $\alpha_a^i\alpha_b^j(x_a^i - x_b^j)^2$ in exponent in the final expression becomes the part of the Mellin amplitude. We have chosen to represent the coefficients for the case $a = b$ (i.e. $\alpha_a^i\alpha_a^j(x_a^i - x_a^j)^2$) by associating the weight factors to the circles drawn around the vertex u_a. On the other hand, the coefficients for the case $a \neq b$ (i.e. $\alpha_a^i\alpha_b^j(x_a^i - x_b^j)^2$) are represented by associating the weight factors to the solid line connecting the vertices u_a and u_b.

Finally, we integrate over the third vertex. The effect of this integration is represented by making the small circle around that vertex bigger. Drawing another circles around the first two vertices each and drawing another solid line connecting the first and second vertices. We associate a weight of 1 for the circle around the third vertex, a weight of t_{23}^2 for the new circle around the second vertex, a weight of $t_{12}^2 t_{23}^2$ for the new circle around the first vertex and a weight of $t_{12} t_{23}^2$ for new solid line connecting the first two vertices. This is shown in Figure 2.9.

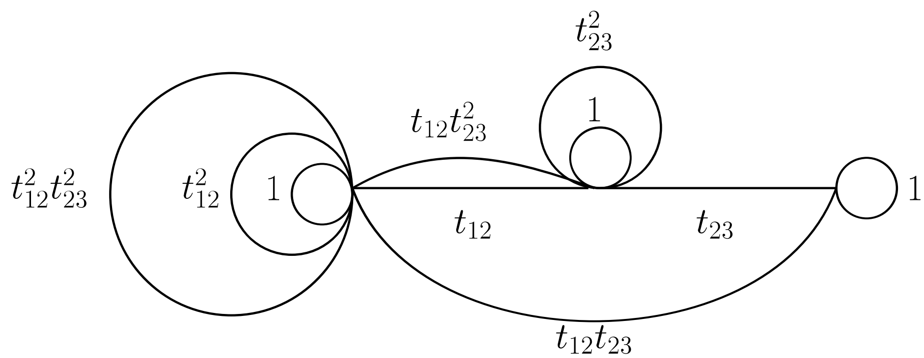

Figure 2.9: Diagrammatic representation of integration over third vertex

Next, we replace the two lines between the first two vertices by a single line and associate a weight which is sum of the weights of previous two lines. Similarly, we replace the multiple circles at each vertex by a single circle and associate a weight which is sum of the weights of all circles initially present. After combining multiple lines and circles, Figure 2.9 has been redrawn in Figure 2.10.

To write the Mellin amplitude,

1. For each initial dashed line between the vertices u_a and u_b, we associate an integral

$$\frac{1}{\Gamma(\gamma_{ab})} \int_0^\infty dt_{ab} \, (t_{ab})^{\gamma_{ab}-1}$$

2. For each solid line and circle in the final diagram, we include in the integrand, the corresponding weight factor raised to the power of s_{ab} where a and b are the two vertices associated with

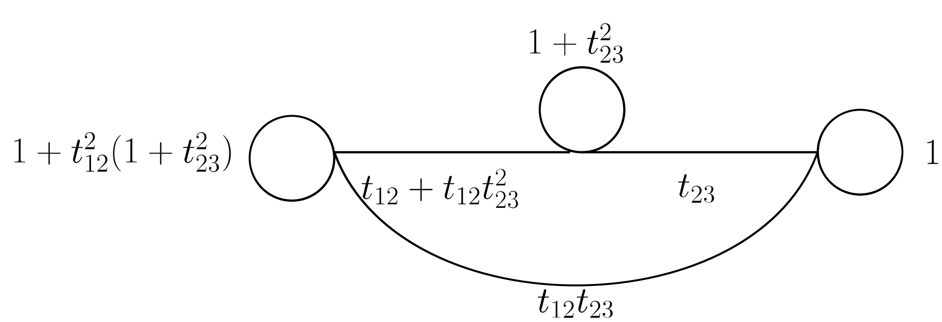

Figure 2.10: Final step for writing the Mellin amplitude

the line or the circle (in which case $a = b$).

Following the steps described above, the Mellin amplitude for the three vertex tree can be obtained to be

$$\begin{aligned} M(s_{ab}) &= \frac{1}{\Gamma(\gamma_{12})\Gamma(\gamma_{23})} \int_0^\infty dt_{12}(t_{12})^{\gamma_{12}-1} \int_0^\infty dt_{23}(t_{23})^{\gamma_{23}-1} \\ &\quad \Big[1 + t_{12}^2(1 + t_{23}^2)\Big]^{-s_{11}} \Big[1 + t_{23}^2\Big]^{-s_{22}} \Big[t_{12}(1 + t_{23}^2)\Big]^{-s_{12}} \Big[t_{12}t_{23}\Big]^{-s_{13}} \Big[t_{23}\Big]^{-s_{23}} \end{aligned}$$

This is same as the expression (2.26) given in the previous section.

General Rules

We shall now state the general rules for writing down the Mellin amplitude for any given Feynman diagram involving scalar fields. With a little thought, we can convince ourselves that the method described above works for any diagram. This is essentially due to the reason that for deriving the Mellin amplitude of any diagram, we need to integrate over the position space vertices and introduce the Mellin variables in the same manner. Hence, the steps of integration over the vertices, for any diagram, can be captured in a diagrammatic manner as described above for three vertex tree diagram.

We start with the skeleton and follow the steps given below for each interaction vertex, one at a time. For a general Feynman diagram there is a freedom to choose the order in which the different vertices are integrated over one by one. This procedure works for any chosen ordering.

Diagrammatic representation of integrating over an interaction vertex

At any interaction vertex on the skeleton (which has not been integrated yet), in general, there will be a small hollow circle denoting the external lines, and dashed and solid lines for the internal propagators. To represent the effect of integration over this vertex, we do the following:

1. Replace the small circle with a bigger circle and associate a weight 1.

2. If this vertex is connected by a solid line (with weight t) to another vertex which already has a circle with some weight, draw another circle at that vertex. Associate a weight t^2 to this new circle.

3. If this vertex (that is being integrated over) is connected to another vertex with a dashed line, we replace that dashed line with a solid line and associate a weight equal to the Schwinger parameter for this internal line.

4. After the third step, if this vertex (which is being integrated over) happens to be connected to two or more vertices $\{a\}$ by solid lines with weights $\{t_a\}$, then we join each pair of those vertices by a solid line as well. To the new line joining vertex a and b, we associate the weight $t_a t_b$.

5. If any two vertices are connected by multiple lines (or any vertex has multiple circles) with each line (or circle) being associated with some weight, we replace them with a single line (or a single circle) with a weight equal to the sum of the weights of the individual lines (or circles). The final diagram should have a single line between any two vertices and a single circle at each vertex.

6. If the vertex (which is being integrated over) is only connected with internal lines but no external lines (in other words, it does not have small circle), then we do not make a bigger circle around it. However, the steps 2-5 are still applicable.

Writing the Mellin amplitude

1. For each initial dashed line between the points a and b, we write an integral

$$\frac{1}{\Gamma(\gamma_{ab})}\int_0^\infty dt_{ab}\ (t_{ab})^{\gamma_{ab}-1}$$

2. For each solid line between the interaction vertices a and b (and a circle at a) in the final diagram, we include in the integrand a factor equal to the corresponding weight raised to the power s_{ab} (s_{aa} for the circle).

Although the output of this procedure is always the Mellin amplitude which is unique, the exact expression for the integrand (function of the Schwinger parameters for the internal lines) depends on the order we choose for integrating over the vertices. Also, it is not easy to show by direct evaluation of the integrals that these different integrals are in fact equal. For our purposes, we shall choose the order for integrating over the vertices that leads to the simplest integral.

2.3.2 n-Vertex Simple Tree

In this subsection, we consider a tree level diagram with any number of interaction vertices such that all the internal propagators are connected as a single chain. In other words, there are no branches on the skeleton as shown in Figure 2.11. We refer to this diagram as the *simple n-vertex tree*. The examples considered in section 2.2 are special cases of this.

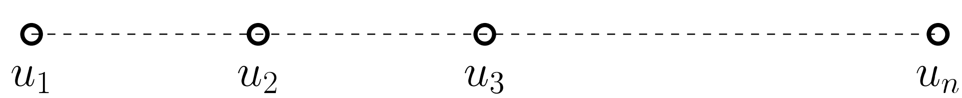

Figure 2.11: Simple tree with n vertices

The position space amplitude for this diagram is given by the following integral expression

$$I \;=\; \prod_{a=1}^{n}\Bigg[\int \mathcal{D}u_a\Bigg\{\prod_{i\in a}(x_a^i-u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\Bigg\}\Bigg\{(u_a-u_{a+1})^{-2\gamma_{a,a+1}}\Bigg\}\Bigg]$$

This expression can be brought to the standard form

$$I \;=\; \prod_{a=1}^{n}\prod_{b=a}^{n}\Bigg(\prod_{(i,j)\in a+b}\int_{c_{ab}^{ij}-i\infty}^{c_{ab}^{ij}+i\infty}\big[ds_{ab}^{ij}\big]\,\Gamma(s_{ab}^{ij})\big(x_{ab}^{ij}\big)^{-2s_{ab}^{ij}}\Bigg)\prod_{a=1}^{n}\prod_{i\in a}\Bigg(2\pi i\delta(\rho_a^i)\Bigg)M(s_{ab}) \quad (2.27)$$

$$\text{where,}\qquad \rho_a^i\equiv\Delta_a^i-\sum_{j\in a}s_{aa}^{ij}-\sum_{\substack{b=1\\ b\neq a}}^{n}\Bigg(\sum_{j\in b}s_{ab}^{ij}\Bigg)\quad,\qquad 1\le a\le n \quad (2.28)$$

For the standard order of integration over the position space vertices (namely, $u_1\to u_2\to\cdots\to u_n$), the Mellin amplitude $M(s_{ab})$ in (2.27) is given by

$$M(s_{ab}) \;=\; \frac{1}{\prod\limits_{a=1}^{n-1}\Gamma(\gamma_{a,a+1})}\prod_{a=1}^{n-1}\Bigg\{\int_0^\infty dt_{a,a+1}\big(t_{a,a+1}\big)^{R_a-1}\Big(G_a^n\Big)^{-Q_a}\Bigg\} \quad (2.29)$$

where,

$$R_a=\gamma_{a,a+1}-\sum_{b=a+1}^{n}\sum_{c=1}^{a}s_{cb}\quad,\qquad Q_a=\sum_{b=1}^{a}s_{ba}\quad,\qquad 1\le a\le n-1 \quad (2.30)$$

$$G_a^b=1+t_{a,a+1}^2(1+t_{a+1,a+2}^2(\cdots\cdots+t_{b-1,b}^2))\qquad 1\le a\le n-1,\qquad a\le b \quad (2.31)$$

To evaluate this integral, we start with the Schwinger variable $t_{n-1,n}$ and make a coordinate transformation and a rescaling simultaneously

$$1+t_{n-1,n}^2=y_{n-1}\quad,\qquad y_{n-1}t_{n-2,n-1}^2\to t_{n-2,n-1}^2$$

By making a further coordinate transformation $y_{n-1}-1\to y_{n-1}$, the integration over y_{n-1} can be recognised as a beta function and we obtain

$$M(s_{ab}) \;=\; \frac{1}{\prod\limits_{a=1}^{n-1}\Gamma(\gamma_{a,a+1})}\prod_{a=1}^{n-2}\Bigg\{\int_0^\infty dt_{a,a+1}\big(t_{a,a+1}\big)^{R_a-1}\Big(G_a^{n-1}\Big)^{-Q_a}\Bigg\}$$
$$\frac{1}{2}\beta\Big(\frac{R_{n-1}}{2},\frac{R_{n-2}-R_{n-1}}{2}+Q_{n-1}\Big)$$

We iteratively perform similar steps in order to integrate over the remaining Schwinger parameters and make necessary simplifications to obtain,

$$M(s_{ab}) \;=\; \frac{1}{\prod\limits_{a=1}^{n-1}\Gamma(\gamma_{a,a+1})}\prod_{a=1}^{n-1}\frac{1}{2}\beta\left(\frac{\gamma_{a,a+1}-\sum\limits_{b\in L_{a,a+1}}\sum\limits_{c\in R_{a,a+1}}s_{bc}}{2}\;,\;\frac{D-2\gamma_{a,a+1}}{2}\right) \tag{2.32}$$

where,

$$\sum_{b\in L_{a,a+1}}\sum_{c\in R_{a,a+1}}s_{bc}\equiv\sum_{b=1}^{a}\sum_{c=a+1}^{n}s_{bc}$$

The $L_{a,a+1}$ and $R_{a,a+1}$ appearing in above two equations stand for Left and Right respectively. If we cut the diagram 2.11 along the propagator $t_{a,a+1}$, the vertices will get divided in two sets. The set $L_{a,a+1}$ includes all the vertices which lie to to left of the cut and the set $R_{a,a+1}$ includes all the vertices which lie to the right of the cut. An example for $n = 4$ is given in Figure 2.12 in which the sets $L_{3,4}$ and $R_{3,4}$ have been shown.

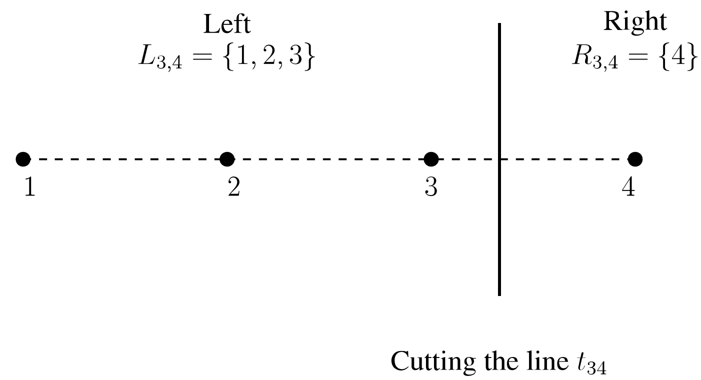

Figure 2.12: Left and Right of a cut line

The result (2.32) is consistent with the previous examples as the Mellin amplitude is a product over all the propagator factors (each of which is a beta function with appropriate arguments). We can again introduce dual Mellin momenta and replace the Mellin variables in the arguments of beta functions in favour of the total Mellin momenta flowing through the propagator. The propagators develop a pole when the negative of total Mellin momenta squared flowing through it becomes equal to the conformal dimension of a primary or descendant flowing through it.

2.3.3 General Tree

Finally, we consider a completely general tree Feynman diagram and show that the Mellin amplitude for it can be written in a simple form as product over all the internal propagator factors. The derivation

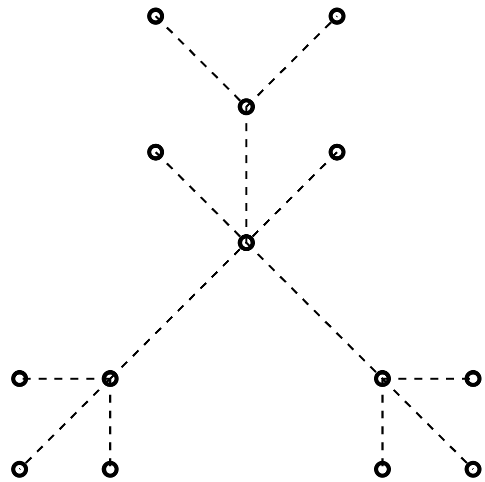

Figure 2.13: Example of a general tree level Feynman diagram (skeleton)

of Mellin amplitude for an arbitrary tree Feynman diagrams is a graph theoretic exercise and does not shed any light on the physical significance of the result itself. Hence, in this section, we shall only state the final result and discuss it's physical significance. The details of the derivation have been presented in the appendix C.4.1.

From the diagrammatic rules given in section 2.3.1, we know that the amplitude can be written in the following form

$$M\left(\{s_{ab}\}\right) = \prod \left[\int_0^\infty dt_{ab} \frac{(t_{ab})^{\gamma_{ab}-1}}{\Gamma(\gamma_{ab})}\right] F\Big(\{t_{ab}\}, \{s_{ab}\}\Big) \tag{2.33}$$

The product runs over all the internal lines[4]. F is function of the Schwinger parameters and the Mellin variables.

The function F depends on the order of integration of the position space vertices and, in general, is a very complicated function of the Schwinger parameters. It turns out that for the tree diagrams, it is possible to make a choice for the order in which the vertices are integrated over such that the integral (2.33) can be performed easily. This has been described in detail in the appendix C.4.1. With such a choice, the function F can be expressed as

$$F \;=\; \prod_{\text{all propagators}} (t_{ab})^{-P_{ab}} (A_{ab})^{-Q_{ab}}$$

[4]Since we are considering a general tree which may have brances, it is not necessary that neighbouring vertices will always be labelled with consecutive integers.

where,

$$P_{ab} = \sum_{c\in L_{ab}} \sum_{d\in R_{ab}} s_{cd} \qquad ; \qquad Q_{ab} = \sum_{\substack{c\in L_{ab}\\ c\neq a}} \sum_{\substack{d\in \widetilde{L}_{ab}\\ d\neq a}} s_{cd} \quad + \sum_{d\in L_{ab}} s_{ad}$$

The term involving the double sum in Q_{ab} is absent if there is no branching at the vertex a in the skeleton. The tilde in one of the L in this double sum denotes the fact that we should not include terms of the type s_{cd} where c and d are on the same branch in the set L_{ab}.

To define the function A_{ab}, we shall need a reference vertex which can be chosen freely from any one of the end vertices (a vertex with only one dashed line attached to it) on the skeleton. Let that vertex be $\mathcal{P}$ (see figure C.1 and the related discussion in appendix C.4.1), then

$$A_{ab} \equiv 1 + t_{ab}^2 \left(1 + t_{bc}^2 \left(1 + ...(1 + t_{o\mathcal{P}}^2)...\right)\right)$$

b,c,...,o are all on the shortest continuous route from a to the reference vertex $\mathcal{P}$.

The final result, after integrating over all the Schwinger parameters in (2.33), turns out to be a product of beta functions with one beta function for each internal propagator. The arguments of beta functions involve the Left and Right part of the propagator as in the case of simple tree in previous subsection. Since there may be branches in our tree, we need to specify what Left and Right of a cut line mean in this context. As a rule, we refer to the part of the diagram (after the cut) having the reference vertex $\mathcal{P}$ as the Right. With this, we can write the Mellin amplitude for a completely general tree as

$$M\left(\{s_{ab}\}\right) = \prod \frac{1}{2\Gamma(\gamma_{ab})} \beta \left(\frac{\gamma_{ab} - \sum\limits_{c\in L_{ab}} \sum\limits_{d\in R_{ab}} s_{cd}}{2} \,,\, \frac{D}{2} - \gamma_{ab} \right) \tag{2.34}$$

The product is over all the internal propagators of the diagram.

The physical interpretation of the above result becomes clear if we again consider the dual Mellin momenta. As before, we denote the Mellin momentum flowing into the diagram through the external vertex x_a^i by k_a^i. We first note that the constraints on the Mellin variables are automatically satisfied if the total Mellin momentum is conserved. The constraint satisfied by the Mellin variables is

$$\sum_b \left(\sum_{j\in b} s_{ab}^{ij} \right) = \Delta_a^i \qquad \Longrightarrow \qquad k_a^i \cdot \left(\sum_b \sum_{j\in b} k_b^j \right) = 0$$

where, we have used $\Delta_a^i = -(k_a^i)^2$.

The simplest way to satisfy the above equation is by demanding that the total Mellin momenta is conserved, namely $\sum\limits_b \sum\limits_{j\in b} k_b^j = 0$.

Now, the full Mellin momentum propagating through an internal propagator, joining the vertices a and b, is

$$k = \sum_{c \in L_{ab}} \sum_{i \in c} k_c^i = -\sum_{c \in R_{ab}} \sum_{i \in c} k_c^i$$

and the kinematical variable in the propagator is

$$\sum_{c \in L_{ab}} \sum_{d \in R_{ab}} s_{cd} = \sum_{c \in L_{ab}} \sum_{d \in R_{ab}} \sum_{i \in c} \sum_{j \in d} s_{cd}^{ij} = \left(\sum_{c \in L_{ab}} \sum_{i \in c} k_c^i \right) \cdot \left(\sum_{d \in R_{ab}} \sum_{j \in d} k_d^j \right) = -k^2$$

In terms of the Mellin momenta, the expression for the propagator can be written as

$$\frac{1}{2\Gamma(\gamma_{ab})} \beta \left(\frac{1}{2} \left(\gamma_{ab} + k^2 \right), \frac{D}{2} - \gamma_{ab} \right) = \frac{1}{\Gamma(\gamma_{ab})} \sum_{n=0}^{\infty} \frac{\Gamma\left(\gamma_{ab} - \frac{D}{2} + 1 + n \right)}{n! \Gamma\left(\gamma_{ab} - \frac{D}{2} + 1 \right)} \frac{1}{k^2 + \gamma_{ab} + 2n}$$

This shows that the total Mellin momentum (squared) flowing through the propagator has poles at $-\gamma_{ab} - 2n$. These correspond to the propagation of a primary field and the corresponding descendants (see footnote 4). The above sum representation of the propagator is analogous to the Kählén Lehmann spectral representation in ordinary quantum field theories.

The Feynman rules for tree level Feynman diagrams in perturbative CFT for scalar fields is now obvious. The propagator for any internal line is given by (2.34) and we simply multiply all the propagator factors of the diagram.

We would like to wrap up this discussion with a brief recapitulation of the most important points we have learnt so far. Mellin space provides a manifest conformally invariant representation for correlation functions in a CFT. At tree level, there exist a set of Mellin space Feynman rules that can be associated with Feynman diagrams involving scalar operators. Some linear combinations of the Mellin variables that appear in the propagators can be interpreted as Mandelstam variables constructed out of the (hypothetical) external Mellin momenta flowing into the diagram. The invariance of the amplitude under special conformal transformation allows for a statement of conservation of Mellin momentum. All the Mellin variables (or equivalently all the Mandelstam variables) are not independent and the number of independent Mellin variables is equal to the number of independent cross ratios between the external vertices in the diagram. Mellin space also allows a spectral representation for the correlation functions as any propagator in the diagram has a discrete infinite set of poles corresponding to the exchanged primary field and its descendants.

2.4 One-Loop Feynman Diagram

After deriving the Mellin space Feynman rules for tree level diagrams, the next step is to consider loop diagrams. We have not yet been able to derive the Feynman rules for loop diagrams. In this section, we shall content ourselves with the expression for the one-loop Mellin amplitude as an integral over the internal Schwinger parameters. It may be possible to derive the loop Feynman rules in Mellin space using an approach similar to the one presented in appendix C.4.2 which treats the n-vertex simple tree in a way different from what we have already seen in section 2.3.

The position space amplitude for the loop Feynman diagram in figure (2.14) is given by

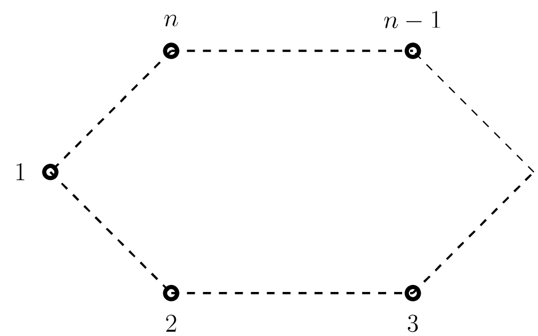

Figure 2.14: One loop diagram with n internal vertices

$$I = \prod_{a=1}^{n}\left[\int \mathcal{D}u_a \prod_{i\in a}\left\{(x_a^i - u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\right\}\right]\prod_{b=1}^{n}(u_b - u_{b+1})^{-2\gamma_{b,b+1}} \tag{2.35}$$

where $n+1 \equiv 1$.

2.4.1 One-Loop Mellin Amplitude

The Mellin amplitude for the n-vertex one-loop diagram can be derived using the position space amplitude (2.35) by following the same procedure as in the previous sections for tree diagrams. The Mellin amplitude is written as integral over the Schwinger parameters and the integrand depends upon the order in which we perform the integration over the interaction vertices in position space. For the cyclic order of integration ($u_1 \to u_2 \to \cdots \to u_n$), the Mellin amplitude turns out to be

$$M(s_{ab}) = \prod_{a=1}^{n}\left[\int_0^\infty dt_{a,a+1}\frac{(t_{a,a+1})^{\gamma_{a,a+1}-1}}{\Gamma(\gamma_{a,a+1})}\right]\prod_{a=1}^{n-1}\prod_{b=a}^{n}\Big(\widetilde{H}_a^b + \widetilde{K}_a\widetilde{K}_b\Big)^{-s_{ab}} \tag{2.36}$$

where,

$$
\begin{aligned}
\widetilde{H}_a^b &\equiv t_{a,a+1}\cdots t_{b-1,b}G_b^{n-1} && 1\le a<b\le n-1\\
\widetilde{H}_a^a &\equiv G_a^{n-1} && 1\le a\le n-1\\
\widetilde{H}_a^n &\equiv 0 && 1\le a\le n\\
\widetilde{K}_a &\equiv \Big(t_{1n}t_{12}\cdots t_{a-1,a}G_a^{n-1}+t_{a,a+1}\cdots t_{n-1,n}\Big) && 1\le a\le n-1\\
\widetilde{K}_n &= 1
\end{aligned}
$$

We have defined G_a^b in eq. C.1.

2.4.2 Special Case: Loop With 3 Internal Vertices

The conformal Mellin amplitude of one loop diagram with 3 internal vertices (“delta” diagram) can be exactly evaluated in terms of a tree amplitude (“star” tree diagram). This happens due to the standard “star-delta” relation in an analogy with a similar result in electrical circuits.

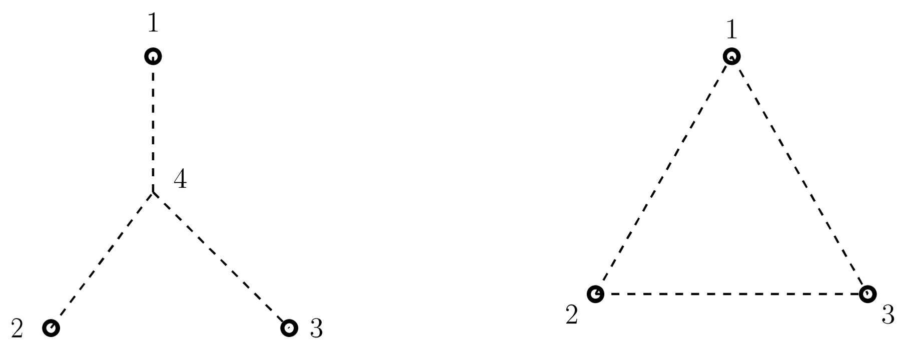

Figure 2.15: Skeleton of the “star” and the “delta”

The position space expression for the star diagram is

$$I_{star} \;=\; \int \mathcal{D}u_4 \prod_{a=1}^{3}\left[\int \mathcal{D}u_a \left\{\prod_{i\in a}(x_a^i-u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\right\}(u_a-u_4)^{-2\gamma'_{a,4}}\right]$$

To show the equivalence with the 3 vertex loop, we need to perform the integration over the central vertex u_4. For this vertex, we perform the standard algebraic steps as in section 2.2.1 and obtain

$$
\begin{aligned}
&I_{star} = \\
&\prod_{a=1}^{3}\left\{\int \mathcal{D}u_a \prod_{i\in a}(x_a^i - u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\right\}\prod_{a=1}^{3}\left(\int_0^\infty dt_{a,4}\,\frac{(t_{a,4})^{\gamma'_{a,4}-1}}{\Gamma(\gamma'_{a,4})}\right) \\
&\left(\prod_{a=1}^{3}\int [ds_a]\Gamma(s_a)\right)\left\{t_{14}t_{24}(u_1-u_2)\right\}^{-2s_3}\left\{t_{14}t_{34}(u_1-u_3)\right\}^{-2s_2}\left\{t_{24}t_{34}(u_2-u_3)\right\}^{-2s_1}
\end{aligned}
$$

Integration over the Schwinger parameters $t_{a,4}$ give 3 delta functions. We can thus perform the 3 integrals over s_a. The resulting expression is proportional to the 3 vertex loop amplitude in position space, i.e.

$$
\begin{aligned}
I_{star} &= \frac{\Gamma(\gamma_{12})\Gamma(\gamma_{23})\Gamma(\gamma_{13})}{\prod_{a=1}^{3}\Gamma(\gamma'_{a,4})}\prod_{a=1}^{3}\left\{\int \mathcal{D}u_a \prod_{i\in a}(x_a^i-u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\right\} \\
&\quad\times (u_1-u_2)^{-2\gamma_{12}}(u_1-u_3)^{-2\gamma_{13}}(u_2-u_3)^{-2\gamma_{23}}
\end{aligned}
$$

where,

$$
\gamma'_{14} = \gamma_{12}+\gamma_{13} \qquad , \qquad \gamma'_{24} = \gamma_{12}+\gamma_{23} \qquad , \qquad \gamma'_{34} = \gamma_{13}+\gamma_{23}
$$

Thus, if we represent the internal propagator of 3 vertex loop by γ_{ab} and those of the star diagram by γ'_{a4}, then the above relation says

$$
I_{star}(s_{ab},\gamma'_{ab}) = \frac{\Gamma(\gamma_{12})\Gamma(\gamma_{23})\Gamma(\gamma_{13})}{\Gamma(\gamma'_{14})\Gamma(\gamma'_{24})\Gamma(\gamma'_{34})} \quad\times\quad I_{delta}(s_{ab},\gamma_{ab})
$$

The star diagram is a tree diagram and its Mellin space amplitude can be easily written down using the Feynman rules given in the previous sections. Thus, we find the 3 vertex loop amplitude in Mellin space to be

$$
\begin{aligned}
I_{delta} =\ & \frac{1}{8\Gamma(\gamma_{12})\Gamma(\gamma_{23})\Gamma(\gamma_{13})}\beta\left(\frac{\gamma_{12}+\gamma_{13}-s_{12}-s_{13}}{2},\frac{D}{2}-\gamma_{12}-\gamma_{13}\right) \\
&\beta\left(\frac{\gamma_{12}+\gamma_{23}-s_{12}-s_{23}}{2},\frac{D}{2}-\gamma_{12}-\gamma_{23}\right)\beta\left(\frac{\gamma_{13}+\gamma_{23}-s_{13}-s_{23}}{2},\frac{D}{2}-\gamma_{13}-\gamma_{23}\right)
\end{aligned}
$$

2.5 Non-Conformal Mellin Amplitudes

In this section, we revisit some of the tree level Feynman diagrams we have been considering so far. However, this time we relax the conformality conditions imposed on them. The motivation for defining these "non-conformal Mellin amplitudes" comes from noting that exactly marginal deformations of CFTs are rare (generally arising only in some special supersymmteric gauge theories).

We have been considering CFTs whose Lagrangian descriptions are in terms of scalar fields. Since the couplings generically run with the energy scale, the beta function is non-zero and conformal invariance is broken. Thus, even a classically marginal perturbation generally breaks conformal invariance once quantum effects are included. We thus study these non-conformal Mellin amplitudes by considering a generic scalar perturbation around a free CFT that may not preserve any of the conformal or scale symmetry. At an operational level, we relax the conformality conditions that we have been imposing at each interaction vertex.

2.5.1 Some Examples

As a concrete example, we consider the simple n-vertex tree of figure 2.11. The conformal Mellin amplitude for this diagram was given in (2.29). If we do not impose the conformality conditions, then instead of (2.27), we obtain

$$I = \prod_{a=1}^{n}\prod_{b=a}^{n}\left(\prod_{(i,j)\in a+b}\int_{-i\infty}^{i\infty}\left[ds_{ab}^{ij}\right]\Gamma(s_{ab}^{ij})\left(x_{ab}^{ij}\right)^{-2s_{ab}^{ij}}\right)\widetilde{M}(s_{ab})$$

where,

$$\begin{aligned}\widetilde{M}(s_{ab}) &= \prod_{a=1}^{n}\left(\prod_{i\in a}\int_0^\infty d\alpha_a^i(\alpha_a^i)^{\rho_a^i-1}\right)\prod_{a=1}^{n-1}\left\{\int_0^\infty \frac{dt_{a,a+1}}{\Gamma(\gamma_{a,a+1})}\left(t_{a,a+1}\right)^{R_a-1}\left(G_a^n\right)^{-Q_a}\right\}\\ &\quad\times\prod_{a=1}^{n}\left\{\sum_{b=1}^{a-1}\left(H_b{}^a\sum_{i\in b}\alpha_b^i\right)+\sum_{b=a}^{n}\left(H_a{}^b\sum_{i\in b}\alpha_b^i\right)\right\}^{-\lambda_a}\\ &\equiv \delta\left(\sum_{a=1}^{n}(\lambda_a-\rho_a)\right)M(s_{ab}) \qquad (2.37)\end{aligned}$$

where ρ_a^i is defined in equation (2.28) and

$$\begin{aligned}G_a^c &= 1+t_{a,a+1}^2(1+t_{a+1,a+2}^2(\cdots\cdots+t_{c-1,c}^2)) \qquad 1\le a\le n-1\\ H_a{}^b &= t_{a,a+1}t_{a+1,a+2}\cdots t_{b-1,b}G_b^n \quad , \qquad H_a{}^a = G_a{}^n\\ \lambda_a &= D-\sum_{i\in a}\Delta_a^i-\gamma_{a-1,a}-\gamma_{a,a+1} \quad , \qquad 1\le a\le n\\ \rho_a &= \sum_{i\in a}\rho_a^i \quad , \qquad 1\le a\le n\end{aligned}$$

It is a simple exercise to extract the overall delta function from the expression of $\widetilde{M}(s_{ab})$ as we have done in equation (2.37). We shall take $M(s_{ab})$ in (2.37) to be the definition of the "non-conformal

Mellin amplitude" for n-vertex simple tree[5].

One crucial limitation of this treatment that should be noted here is that the delta function in (2.37) is graph dependent and consequently the definition of $M(s_{ab})$ is also graph dependent. Therefore, although we can calculate $M(s_{ab})$ for individual diagrams, that is not exactly equal to doing perturbation theory in Mellin space. The delta function emerges in this context from the fact that the position space integrals still scale in a given way, although this scaling property depends on the particular graph being considered and may not refer to any symmetry of the theory itself.

The conformality condition amounts to setting all λ_a equal to zero. However, we work with non zero λ_a. For concreteness, we consider the special cases $n = 1$ and $n = 2$ which give us some interesting results.

Contact Interaction

For a single vertex (i.e. $n = 1$), the expression (2.37) gives

$$\widetilde{M}(s^{ij}) = \prod_{i=1}^{N}\left(\int_0^\infty d\alpha^i (\alpha^i)^{\rho^i-1}\right)\Big(\sum_{i=1}^{N}\alpha^i\Big)^{\sum\limits_{i=1}^{N}\Delta^i - D}$$

All symbols have their usual meaning as used previously. In the conformal case, the expression above just gives delta function constraints on the Mellin variables. We now evaluate this in the non conformal case. For this, we insert the partition of unity

$$1 = \int_0^\infty dq \;\; \delta\left(q - \sum_{i=1}^{N}\alpha^i\right)$$

in the above integral, make the coordinate transformations $\alpha^i = q\, y^i$ and use the identity

$$\prod_{i=1}^{N}\int_0^\infty dx^i \; (x^i)^{\rho^i-1}\delta\left(1 - \sum_{i=1}^{N}x^i\right) = \frac{\prod\limits_{i=1}^{N}\Gamma(\rho^i)}{\Gamma\left(\sum\limits_{i=1}^{N}\rho^i\right)} \tag{2.38}$$

Using the expression for ρ_i, we finally obtain,

$$M(s^{ij}) \;\; = \;\; \frac{\prod\limits_{i=1}^{N}\Gamma\left(\Delta^i - \sum\limits_{\substack{j=1\\ j\neq i}}^{N}s^{ij}\right)}{\Gamma\left(D - \sum\limits_{i=1}^{N}\Delta^i\right)} \tag{2.39}$$

[5] We are using the same symbol $M(s_{ab})$ to denote both conformal as well as non-conformal Mellin amplitudes. However, the distinction should be clear from the context.

We have used the constraint arising from the overall delta function (see the definition (2.37)) to simplify the arguments of the Gamma function in denominator.

Tree With One Internal Propagator

We next consider the tree diagram with one internal propagator (i.e. $n = 2$). For $n = 2$, the expression (2.37) reduces to

$$\widetilde{M}(s_{ab}) = \frac{1}{\Gamma(\gamma)}\prod_{a=1}^{2}\prod_{i\in a}\left(\int_0^\infty d\alpha_a^i(\alpha_a^i)^{\rho_a^i-1}\right)\int_0^\infty dt\ (t)^{\gamma-s_{12}-1}\big(1+t^2\big)^{-s_{11}}$$
$$\left(\big(1+t^2\big)\sum_{i\in 1}\alpha_1^i + t\sum_{i\in 2}\alpha_2^i\right)^{-\lambda_1}\left(\sum_{i\in 2}\alpha_2^i + t\sum_{i\in 1}\alpha_1^i\right)^{-\lambda_2} \tag{2.40}$$

where, we have relabelled $t_{12}\to t$ and $\gamma_{12}\to\gamma$ to match with our notation in section 2.2. After some manipulations (see Appendix C.4.3), the non-conformal Mellin amplitude can be extracted to be,

$$M(s_{ab}) =$$
$${}_3F_2\left(\gamma-\frac{D}{2}+\lambda_1+\lambda_2\ ,\ \frac{R_1+\rho_1-\lambda_1}{2}\ ,\ \frac{R_1+\rho_2-\lambda_2}{2}\ ;\ \frac{R_1+\rho_1+\lambda_1}{2},\frac{R_1+\rho_2+\lambda_2}{2}\ ;\ 1\right)$$
$$\times\frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\left[\prod_{i\in a}\Gamma(\rho_a^i)\right]\left[\frac{\Gamma\left(\frac{R_1+\rho_1-\lambda_1}{2}\right)\Gamma\left(\frac{R_1+\rho_2-\lambda_2}{2}\right)}{\Gamma\left(\frac{R_1+\rho_1+\lambda_1}{2}\right)\Gamma\left(\frac{R_1+\rho_2+\lambda_2}{2}\right)}\right] \tag{2.41}$$

where, $R_1 = \gamma - s_{12}$ and $\lambda_a = D-\gamma-\sum_{i\in a}\Delta_a^i$.

We now look at the pole structure and the conformal limit of the expression (2.41).

Pole Structure

The poles of the amplitude (2.41) occur when the arguments of the gamma functions in the numerator in second line are zero or negative integers (${}_3F_2$ does not give rise to any pole). Thus, as a function of the Mellin variables, the poles of the amplitude lie at

$$\frac{R_1+\rho_1-\lambda_1}{2} = -n \qquad\Longrightarrow\qquad s_{11}+s_{12} = \sum_{i\in 1}\Delta_a^i+\gamma-\frac{D}{2}+2n$$

and,

$$\frac{R_1+\rho_2-\lambda_2}{2} = -n' \qquad\Longrightarrow\qquad s_{22}+s_{12} = \sum_{i\in 2}\Delta_a^i+\gamma-\frac{D}{2}+2n'$$

where, n, n' are zero or arbitrary positive integers, i.e. $0, 1, 2, \cdots$.

This shows that in the non-conformal case there are two sets of poles, which in the conformal limit $\lambda_a \to 0$, coalesce to give the one set of poles we had earlier.

Conformal Limit

In the conformal limit, we have $\lambda_1,\ \lambda_2 \to 0$. To impose this limit, we first take $\lambda_1 \to 0$ keeping λ_2 fixed and non-zero. In this limit, two of the arguments of the ${}_3F_2$ hypergeometric function in (2.41) will become identical and it will thus reduce to a ${}_2F_1$ hypergeometric function

$$\begin{aligned}\widetilde{M}(s_{ab}) \;=\; & \frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\left[\prod_{i\in a}\Gamma(\rho_a^i)\right]\left[\frac{\Gamma\left(\frac{R_1+\rho_2-\lambda_2}{2}\right)}{\Gamma\left(\frac{R_1+\rho_2+\lambda_2}{2}\right)}\right]\delta\Big(\rho_1+\rho_2-\lambda_2\Big)\\ & {}_2F_1\left(\gamma-\frac{D}{2}+\lambda_2\ ,\ \frac{R_1+\rho_2-\lambda_2}{2}\ ;\ \frac{R_1+\rho_2+\lambda_2}{2}\ ;\ 1\right)\end{aligned}$$

Now using the Gauss identity (C.11), we obtain after some simplification

$$\widetilde{M}(s_{ab}) \;=\; \frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\left[\prod_{i\in a}\Gamma(\rho_a^i)\right]\left[\frac{\delta\Big(\rho_1+\rho_2-\lambda_2\Big)}{\Gamma\left(\lambda_2\right)}\right]\beta\left(\frac{R_1+\rho_2-\lambda_2}{2}\ ,\ \frac{D}{2}-\gamma\right)$$

If we now take the limit $\lambda_2 \to 0$, we recover the Mellin amplitude (2.23) for the conformal case along with the delta function constraints (2.21) on the Mellin variables.

2.5.2 Scale Invariant Amplitudes and Off-Shell Interpretation

In usual QFTs, we often consider correlation functions in which the external legs are off-shell. One puts these external legs on-shell via the LSZ procedure. It turns out that we can define analogous "off-shell" objects for conformal field theories in Mellin space as well.

For this purpose, we consider position space correlation functions with scale covariance (as in a theory with scale symmetry) [6] although they need not have the full conformal covariance. We expect any physically interesting scale invariant theory to be conformally invariant as well (see [64] and the references therein). However, it is still interesting to consider this case, since as we shall argue below, the corresponding "Mellin amplitudes" seem to be "off-shell" quantities that reduce to the "on-shell" Mellin amplitudes[7] of conformally invariant theories through an LSZ like procedure.

We can imagine extending the definition of the Mellin amplitude to scale invariant theories in the following manner

$$A\left(x^i\right) = \prod_{i<j}\left(\int_{-i\infty}^{i\infty}\frac{ds^{ij}}{2\pi i}\Gamma(s^{ij})(x^i-x^j)^{-2s^{ij}}\right)\delta\left(\sum_i\Delta^i-\sum_{i\neq j}\sum_j s^{ij}\right)M\left(s^{ij}\right)$$

[6] In our notations, this would be equivalent to setting $\sum_a \lambda_a = 0$ although individual λ_a need not be equal to 0.

[7] $(k_a^i)^2 = -\Delta_a^i$ being the on-shell condition.

As opposed to the N (number of external lines) delta function constraints for the conformal amplitude (2.1), in this case we only have one overall constraint on the Mellin variables resulting from the covariance under scale transformations. Therefore, the number of Mellin variables in this case is only $N(N-1)/2-1$ which is also the correct number of independent kinematical variables in a scale invariant theory.

For this case also, we can introduce the dual Mellin momenta in exactly the same manner as before (see section 2.1). In terms of these dual momenta, the overall delta function constraint translates to the condition $\sum_i (k^i)^2 = -\sum_i \Delta^i$ which is weaker than the conformal case $(k^i)^2 = -\Delta^i$. However, we can still demand the conservation of these dual momenta, i.e. $\sum_i k^i = 0$[8].

The contact interaction diagram has only one interaction vertex and ensuring scale invariance automatically ensures conformal invariance as well. However, for more than one interaction vertices, there is a difference between the scale and conformally invariant Mellin amplitudes and the expressions for the former can be obtained by imposing $\sum_a \lambda_a = 0$ on the corresponding non-conformal Mellin amplitudes. For example, for the single propagator case, to obtain the scale invariant Mellin amplitude, we would need to impose $\lambda_1 + \lambda_2 = 0$ on (2.41).

From the examples given in subsection 2.5.1, we can see that each amplitude has a factor involving the product over Gamma functions, namely, $\prod_i \Gamma(\rho^i)$ (we have suppressed the label for the internal vertices). We also know that the conformal Mellin amplitudes involve product over delta functions with the same arguments, namely, $\prod_i \delta(\rho^i)$. In terms of dual Mellin momenta, we can write

$$\rho^i = \Delta^i - \sum_{j\neq i} s^{ij} = (k^i)^2 + \Delta^i$$

Thus, in the conformal case, the delta functions impose a set of constraints $(k^i)^2 + \Delta^i = 0$ which is the "on-shell" condition for the Mellin momenta. In contrast, for the scale invariant amplitudes, we have Gamma functions with the same arguments for each external leg. In the space of Mellin momenta, the "on-shell point" (or equivalently the conformal theory) lies at the pole of the Gamma function (where its argument vanishes). This motivates us to interpret these Gamma functions as external leg factors. It is in this sense that the scale invariant amplitudes are "off-shell" objects and imposing the conformality conditions is akin to an LSZ prescription in which the external leg factors are replaced by the corresponding delta functions.

[8] This is not possible when the scale covariance is also absent.

Chapter 3

Multiple Subleading Soft Graviton Theorem

In this chapter, our goal will be to derive the subleading multiple soft graviton theorem. We start by describing what we mean by the soft graviton theorem. The soft graviton theorem expresses the scattering amplitude of finite energy external states and low energy gravitons[1] in terms of the amplitude without the low energy gravitons [5, 6, 65, 66]. More specifically, suppose we have a quantum field theory describing some particles including the gravitons. We now consider an S-matrix element in this theory involving N external states of finite energy momenta $p_1, p_2, \cdots, p_N$ and M external gravitons of small momenta[2] $k_1, k_2, \cdots, k_M$. We can do a Laurent series expansion (i.e., we allow for the inverse powers of the momenta) of this S matrix in the powers of the soft momenta $k_1, k_2, \cdots, k_M$. This is not very surprising since we can always perform a Laurent series expansion for a well behaved function. However, in the situation described above, a non trivial thing happens. It turns out that the coefficients of the expansion are given in terms of the S matrix without the soft gravitons. This happens for the leading, subleading and the subsubleading orders in the expansion and, in general, breaks down beyond that. Moreover, the leading and the subleading terms in the expansion take a universal form whereas the subsubleading terms depend upon the theory. This property is called the soft graviton theorem.

One way to think about the soft graviton theorem is as follows. Suppose, the theory has no infrared divergences. Then, a scattering without the soft graviton is a localized phenomenon : particles come together, interact in a localized region and then go far from each other. We now add some low energy gravitons in the system. We can think of this situation as the same scattering but in a slowly varying gravitational background. Now, it is a property of the slowly varying gravitational fields that we can set the metric to be flat metric and its first derivative to be zero in a local region. However, we can't set the second derivative to be zero. Thus, if we are interested in the leading and the sublead-

[1]When we say low or finite energy, we mean it in the center of momentum frame, otherwise, this notion is ambiguous. E.g., we could take the whole system and boost it. In that case, the soft gravitons will also acquire the finite energy. A more Lorentz invariant notion of what we mean by the low energy can be given by saying that the quantity $\frac{p_i \cdot k}{p_j \cdot p_\ell}$ (where p_i and k denote the momenta of finite energy particles and soft gravitons respectively) is small.

[2]We shall consider the situation in which all the soft momenta become small at the same order. This is called the simultaneous soft limit.

ing behavior of the amplitude in soft momenta, we can relate the scattering in the presence of soft gravitons to the scattering in the absence of the soft gravitons in a universal manner.

We shall be proving the soft graviton theorem in a theory which has general coordinate invariance. We shall also assume that the quantum theory is free from anomalies and hence the general coordinate invariance is not broken by the quantum effects.

We shall not make any assumption about the finite energy external states. They can be massive or massless. In particular, they can be finite energy gravitons themselves. We shall also not make any assumption about their interactions. E.g., the finite energy sector can involve the strong and weak interactions. The external states can even be composite objects such as black holes or gold atoms. However, we shall assume that the S matrix of the theory is infrared finite. Now, in 4 and lower non compact dimensions, the S matrix is infrared divergent at loop order. This happens because the massless particles have long range interactions in 4 and lower non compact dimensions. Thus, the idea that scattering is a local event breaks down. In other words, at tree level, we can talk about the soft theorem in any dimensions. However, once we start including the loop effects, we need to be careful about the infrared effects[3]. Due to these IR divergences, our derivation will be valid for arbitrary loop amplitudes in higher than 4 non compact dimensions but only for tree amplitudes in 4 and lower non compact dimensions[4].

The soft graviton theorems have been investigated intensively during the last few years [15, 16, 28, 67–89] due to their connection to asymptotic symmetries [7–9, 90–97]. They have also been investigated in string theory [18, 19, 98–110]. In particular in specific quantum field theories and string theories, amplitudes with several finite energy external states and one soft graviton have been analyzed to subsubleading order, leading to the subsubleading soft graviton theorem in these theories. A general proof of the soft graviton theorem in a generic quantum theory of gravity was given in [18,19,21] for one external soft graviton and arbitrary number of other finite energy external states carrying arbitrary mass and spin. For specific theories, soft graviton amplitudes with two soft gravitons have also been investigated in [10–17]. Our goal in this chapter will be to derive, in a generic quantum theory of gravity, the form of the soft graviton theorem to the first subleading order in soft momentum for arbitrary number of soft gravitons and for arbitrary number of finite energy external states carrying arbitrary mass and spin.

The rest of the chapter is organized as follows. In section 3.1, we describe the general strategy followed here for proving the soft graviton theorem and give the final result. In section 3.2 we prove the subleading soft graviton theorem for two external soft gravitons and arbitrary number of

[3]Note that in quantum field theories we introduce the notion of inclusive cross section in which the infrared divergences cancel (E.g., even in 4 dimensions). However, the soft theorem we are interested in is a statement about the S matrix and not the inclusive cross section.

[4]In 5 non compact dimensions, naively, one encounters some IR enhancement in individual diagrams indicating possible breakdown of multiple soft theorem. However, it is expected that such IR enhanced contributions cancel after summing over all the diagrams. See the discussion in section 3.4.

external states of arbitrary mass and spin. In section 3.2.3 we carry out various consistency checks of this formula. These include test of gauge invariance and also comparison with existing results. We generalize the result to the case of multiple soft gravitons in section 3.3. The infrared issues are discussed in section 3.4.

3.1 General strategy and the main result

In this section, we describe the general strategy we shall be following for deriving the subleading soft graviton theorem. We shall be using a Feynman diagram technique for this purpose. The method used is summarized below.

1. Assume that the theory is described by a gauge invariant 1PI effective action. This is a reasonable assumption since given the Lagrangian of the theory, there is a well defined notion of the 1PI effective action. The use of 1PI effective action allows us to include the complicated interactions. E.g., we may want to include the QCD interactions (involving, e.g., protons and pions coupled to gravity) for which elementary Feynman diagrams do not exist. However, it can be easily taken into account in the 1PI effective action. We shall assume the most general form of the 1PI effective action and proceed from there.

2. To determine the Feynman rules, we expand the action in powers of all the fields representing the fluctuation over the background value. The most relevant is the expansion of the metric field

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu} + \cdots \tag{3.1}$$

   We are expanding the metric around the flat space which has Poincare invariance[5]. This means that we are assuming that the flat space is the solution to the equations of motion derived from the 1PI effective action. This is necessary since the notion of S matrix makes sense only in the flat space. If we have a small cosmological constant Λ, we can treat it as perturbation around the flat space. In our universe, we can talk about the S matrix by ignoring the effect of Λ. However, the value of Λ puts a restriction on the validity of the soft graviton theorem. In particular, the soft graviton theorem will break down when the soft momenta becomes comparable to the inverse of the size of cosmological horizon

$$k^\mu \sim \frac{1}{\text{size of cosmological horizon}}$$

   Thus, for the validity of the soft graviton theorem, the momenta of the soft gravitons should be

[5]Since we assume that the background is Poincare invariant, only the scalar fields can have non zero vacuum expectation value.

small compared to the momenta of the other particles. However, the soft momenta should be large compared to the scale set by the cosmological constant.

3. We now add manifestly Poincare invariant gauge fixing term to the 1PI effective action. E.g., the gauge fixing term can be $(\partial_\mu h^{\mu\nu})^2$. This will break the general coordinate invariance but not the Poincare invariance. We shall not need any explicit choice of the gauge fixing term since our manipulations will be formal. However, we shall use the fact that it is Poincare invariant.

4. After fixing the gauge, we can now derive the Feynman rules of the theory and compute the S matrix elements (note that the propagator is ill defined without gauge fixing). However, we want to prove the result for an arbitrary theory with diffeomorphism invariance. Hence, we don't want to work with any specific 1PI effective action. Due to this reason, we shall follow an alternative approach. We shall use the gauge fixed action to derive the Feynman rules for the finite energy particles but not the soft gravitons[6].

5. For the soft gravitons, we note that we are working with 1PI effective action. Hence, we only need to draw tree Feynman diagrams for computing the S matrix elements. Now, in a tree Feynman diagram, the momenta of each internal line is fixed by the momenta of the external lines. Thus, there is a clear separation between the "hard" and "soft" internal lines and we can pretend that the soft and finite energy gravitons are two different particles. This allows us to treat the soft gravitons differently[7].

6. Exploiting the above fact, for calculating the coupling (and hence Feynman rules) of the soft gravitons with the finite energy particles, we covariantize the gauge fixed effective action of step 4 with respect to the metric $\eta_{\mu\nu} + 2S_{\mu\nu}$ where $S_{\mu\nu}$ denotes the soft graviton. Given a Lorentz invariant action, we can always covariantize it (by replacing the flat space metric with curved space metric, the ordinary derivatives by covariant derivatives and introducing non minimal terms involving the Riemann tensor). This is the reason we allowed only the Lorentz invariant gauge fixing terms in step 4. Note that we are thinking of $h_{\mu\nu}$ in (3.1) as the field which describes only the finite energy gravitons. For describing the soft gravitons, we have introduced a new field $S_{\mu\nu}$ whose coupling with the other fields in the theory is determined by covariantizing the gauge fixed action with respect to $S_{\mu\nu}$.

   We could have expanded the metric in the original 1PI effective action in step 2 as $g_{\mu\nu} = \eta_{\mu\nu} + 2S_{\mu\nu} + 2h_{\mu\nu}$. This would be equivalent to the above procedure[8] if the gauge fixing term we choose in step 4 for the hard particles (including hard gravitons) does not break the

[6]The 1PI effective action is for all gravitons including the soft ones. Hence, one can definitely derive the Feynman rules for the soft gravitons also using the gauge fixed 1PI effective action itself.

[7]If we had worked with ordinary action, we would need to draw the loop diagrams as well. In loop diagrams, the loop momenta run over all possible values and hence the internal loop lines can't be identified as soft or finite energy lines.

[8]This equivalence is only upto the terms involving the Riemann tensor.

general coordinate invariance with respect to the soft gravitons. This would mean choosing the gauge fixing term to be $(D_\mu h^{\mu\nu})^2 = 0$ instead of $(\partial_\mu h^{\mu\nu})^2 = 0$ where D_μ denotes the covariant derivative with respect to the metric $\eta_{\mu\nu} + 2S_{\mu\nu}$. This is a particular choice of gauge for the hard particles. We can't choose the gauge fixing term to be general coordinate invariant. However, we can choose the gauge fixing term for the hard part to be general coordinate invariant under the soft gauge transformations. This procedure introduces interaction between the hard and soft gravitons. However, this is just a gauge artifact and hence the Feynman rules will be different than if we had a different choice of the gauge fixing term (the S matrix elements will be same for any gauge choice).

7. While covariantizing the action, we are allowed to add possible terms involving the Riemann tensor of the soft graviton field $S_{\mu\nu}$. However, the Riemann tensor has two derivatives of $S_{\mu\nu}$ and hence gives two powers of soft momenta in numerator. This means that the presence of the Riemann tensor will affect the soft graviton theorem only at the subsubleading order. Since we shall be interested only upto subleading order, the Riemann tensor ambiguity is irrelevant for our purpose.

8. After obtaining the Feynman rules, we can now draw all possible diagrams and compute them in the usual manner.

Following the procedure described above, our final result for an amplitude with N external finite energy particles carrying polarizations and momenta (ϵ_i, p_i) for $i = 1, \cdots, N$, and M soft gravitons carrying polarizations and momenta (ε_r, k_r) for $r = 1, \cdots, M$, takes the form

$$\begin{aligned} A \;=\; & \left\{\prod_{i=1}^{N} \epsilon_{i,\alpha_i}(p_i)\right\} \left[\left\{\prod_{r=1}^{M} S_r^{(0)}\right\} \Gamma^{\alpha_1\cdots\alpha_N} + \sum_{s=1}^{M} \left\{\prod_{\substack{r=1\\ r\neq s}}^{M} S_r^{(0)}\right\} \left[S_s^{(1)}\Gamma\right]^{\alpha_1\cdots\alpha_N} \right. \\ & \left. + \sum_{\substack{r,u=1\\ r<u}}^{M} \left\{\prod_{\substack{s=1\\ s\neq r,u}}^{M} S_s^{(0)}\right\} \left\{\sum_{j=1}^{N} \{p_j\cdot(k_r+k_u)\}^{-1} \;\mathcal{M}(p_j;\varepsilon_r,k_r,\varepsilon_u,k_u)\right\} \Gamma^{\alpha_1\cdots\alpha_N}\right], \end{aligned} \tag{3.2}$$

where

$$S_r^{(0)} = \sum_{\ell=1}^{N} (p_\ell\cdot k_r)^{-1}\, \varepsilon_{r,\mu\nu}\, p_\ell^\mu p_\ell^\nu\,, \tag{3.3}$$

$$[S_s^{(1)}\Gamma]^{\alpha_1\cdots\alpha_N} = \sum_{j=1}^{N} (p_j\cdot k_s)^{-1}\, \varepsilon_{s,b\mu}\, k_{sa}\, p_j^\mu \left[p_j^b \frac{\partial\Gamma^{\alpha_1\cdots\alpha_N}}{\partial p_{ja}} - p_j^a \frac{\partial\Gamma^{\alpha_1\cdots\alpha_N}}{\partial p_{jb}} + (J^{ab})_{\beta_j}^{\;\alpha_j}\, \Gamma^{\alpha_1\cdots\alpha_{j-1}\beta_j\alpha_{j+1}\cdots\alpha_N}\right] \tag{3.4}$$

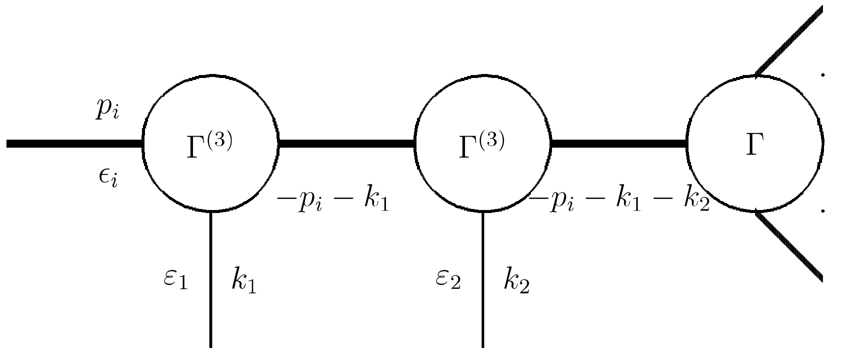

Figure 3.1: A leading contribution to the amplitude with two soft gravitons.

$$
\begin{aligned}
&\mathcal{M}(p_i;\varepsilon_1,k_1,\varepsilon_2,k_2) \\
=\;& (p_i\cdot k_1)^{-1}(p_i\cdot k_2)^{-1}\Bigg\{ -k_1\cdot k_2\, p_i\cdot\varepsilon_1\cdot p_i\, p_i\cdot\varepsilon_2\cdot p_i \\
&+2\,p_i\cdot k_2\, p_i\cdot\varepsilon_1\cdot p_i\, p_i\cdot\varepsilon_2\cdot k_1 + 2\,p_i\cdot k_1\, p_i\cdot\varepsilon_2\cdot p_i\, p_i\cdot\varepsilon_1\cdot k_2 - 2\,p_i\cdot k_1\, p_i\cdot k_2\, p_i\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i\Bigg\} \\
&+(k_1\cdot k_2)^{-1}\Bigg\{ -(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)(k_2\cdot p_i) - (k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)(k_1\cdot p_i) \\
&\quad +(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)(k_1\cdot p_i) + (k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)(k_2\cdot p_i) - \varepsilon_1^{\gamma\delta}\varepsilon_{2\gamma\delta}(k_1\cdot p_i)(k_2\cdot p_i) \\
&\quad -2(p_i\cdot\varepsilon_1\cdot k_2)(p_i\cdot\varepsilon_2\cdot k_1) + (p_i\cdot\varepsilon_2\cdot p_i)(k_2\cdot\varepsilon_1\cdot k_2) + (p_i\cdot\varepsilon_1\cdot p_i)(k_1\cdot\varepsilon_2\cdot k_1)\Bigg\},
\end{aligned}
\tag{3.5}
$$

and $\Gamma^{\alpha_1\cdots\alpha_N}$ is defined such that

$$\Gamma(\epsilon_1,p_1,\cdots,\epsilon_N,p_N) \equiv \left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}\right\}\Gamma^{\alpha_1\cdots\alpha_N}\,, \tag{3.6}$$

gives the amplitude without the soft gravitons, including the momentum conserving delta function. The indices $\alpha,\beta,\gamma,\delta$ run over all the fields of the theory and J^{ab} is the (reducible) representation of the spin angular momentum generator on the fields. The indices a,b as well as μ,ν,ρ are space-time coordinate / momentum labels. We shall use Einstein summation convention for the indices $\alpha,\beta,\cdots$ carried by the fields and also for the space-time coordinate labels $a,b\cdots$ and $\mu,\nu,\cdots$, but not for the indices $r,s,\cdots$ labelling the external soft gravitons and $i,j,\cdots$ labelling the external finite energy particles. For the signature of the space-time metric we shall use mostly + sign convention.

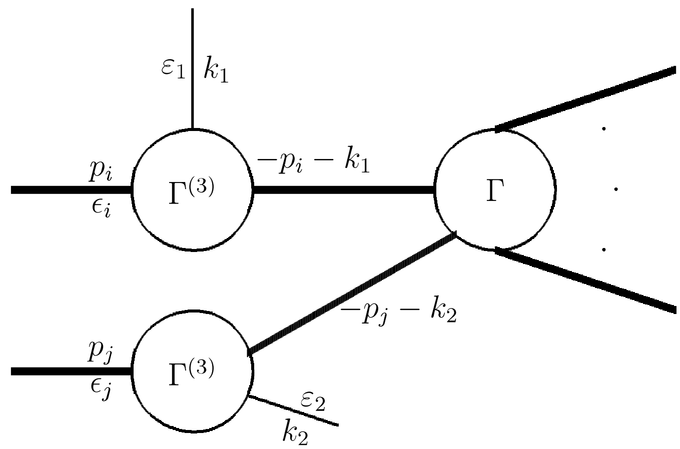

Figure 3.2: A leading contribution to the amplitude with two soft gravitons.

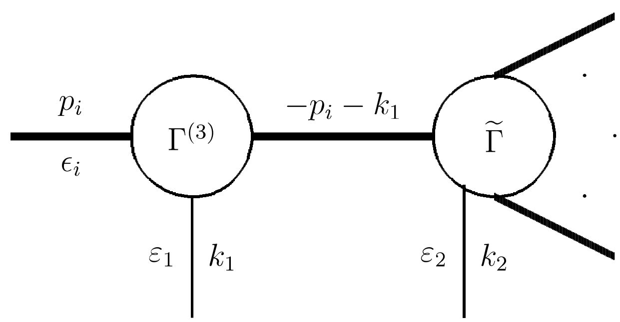

Figure 3.3: A subleading contribution to the amplitude with two soft gravitons. The subamplitude $\widetilde{\Gamma}$ excludes all diagrams where the soft particle carrying momentum k_2 gets attached to one of external lines of $\widetilde{\Gamma}$.

3.2 Amplitudes with two soft gravitons

In this section we shall analyze an amplitude with arbitrary number of finite energy external states and two soft gravitons in the limit when the momenta carried by the soft gravitons become soft at the same rate. The relevant diagrams are shown in Figs. 3.1-3.5. We use the convention that all external momenta are ingoing, thick lines represent finite energy propagators and thin lines represent soft propagators. ε_r, k_r for $r = 1, 2$ represent the polarizations and momenta carried by the soft gravitons subject to the constraint

$$\eta^{\mu\nu}\,\varepsilon_{r,\mu\nu} = 0, \quad k_r^\mu\,\varepsilon_{r,\mu\nu} = 0\,. \tag{3.7}$$

$\Gamma^{(3)}$ and $\Gamma^{(4)}$ denote one particle irreducible (1PI) three and four point functions and Γ denotes full amputated Green's function. In Fig. 3.3, $\widetilde{\Gamma}$ denotes sum of all amputated Feynman diagrams in which the soft graviton is not attached to an external leg via a 1PI three point function. The internal thick lines of the diagrams represent full quantum corrected propagators carrying finite momentum. For Figs. 3.1 and 3.3 we also have to consider diagrams where the two soft gravitons are exchanged.

Among these diagrams the contributions from Fig. 3.1 and Fig. 3.2 have two nearly on-shell propagators giving two powers of soft momentum in the denominators. For example in Fig. 3.1 the line carrying momentum $p_i + k_1$ is proportional to

$$\{(p_i + k_1)^2 + M_i^2\}^{-1} = (2p_i \cdot k_1)^{-1}\,, \tag{3.8}$$

using the on-shell condition $k_1^2 = 0$, $p_i^2 + M_i^2 = 0$ if the mass of the internal state is the same as the mass of the i-th external state. Therefore the contribution from these diagrams begins at the leading order. The rest of the diagrams have only one nearly on-shell propagator and therefore their contribution begins at the subleading order. The contribution from Fig. 3.5 is somewhat deceptive – it appears to have one nearly on-shell propagator carrying finite energy giving one power of soft momentum in the denominator and a soft internal propagator giving two powers of soft momentum in the denominator. However the three graviton vertex has two powers of soft momentum in the numerator. Therefore the contribution from this diagram begins with one inverse power of soft momentum and is subleading.

3.2.1 Feynman rules

For deriving the expressions of vertices, we follow the strategy developed in [18, 19, 21]. We begin with the 1PI effective action of the theory and use Lorentz covariant gauge fixing conditions such that the propagators computed from this gauge fixed action do not have double poles. We now find the coupling of the soft graviton to the rest of the fields by covariantizing this action. As in [19, 21] we shall assume that all the fields carry tangent space indices so that covariantization corresponds to

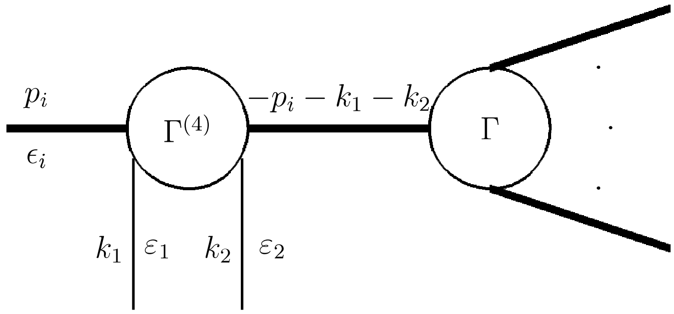

Figure 3.4: A subleading contribution to the amplitude with two soft gravitons.

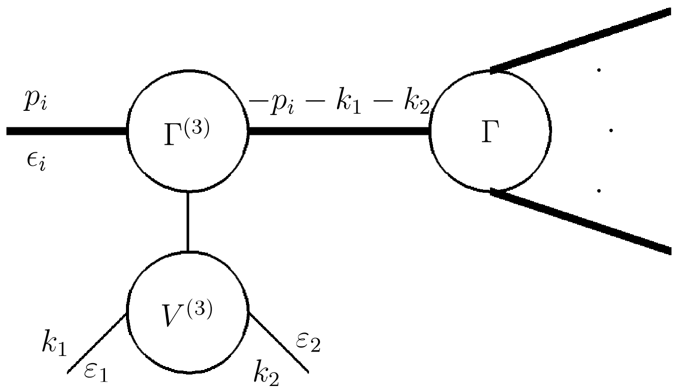

Figure 3.5: A subleading contribution to the amplitude with two soft gravitons.

replacing ordinary derivatives by covariant derivatives and then converting the tensor indices arising from derivatives to tangent space indices by contraction with inverse vielbeins. For simplicity we shall choose a gauge in which the metric associated with the external soft gravitons always has determinant -1 so that we do not need to worry about the multiplicative factor of $\sqrt{-\det g}$ while covariantizing the action. This is done by parametrizing the metric as

$$g_{\mu\nu} = \left(e^{2S\eta}\,\eta\right)_{\mu\nu} = \eta_{\mu\nu} + 2\,S_{\mu\nu} + 2\,S_{\mu\rho}S^{\rho}_{\ \nu} + \cdots\,, \quad S_{\mu\nu} = S_{\nu\mu}, \quad S_{\mu}^{\ \mu} = 0\,, \tag{3.9}$$

where all indices are raised and lowered by the flat metric η. We also introduce the vielbein fields

$$e_{\mu}^{\ a} = \left(e^{S\eta}\right)_{\mu}^{\ a} = \delta_{\mu}^{\ a} + S_{\mu}^{\ a} + \frac{1}{2}S_{\mu}^{\ b}S_{b}^{\ a} + \cdots\,, \quad E_{a}^{\ \mu} = \left(e^{-S\eta}\right)_{a}^{\ \mu} = \delta_{a}^{\ \mu} - S_{a}^{\ \mu} + \frac{1}{2}S_{a}^{\ b}S_{b}^{\ \mu} + \cdots \tag{3.10}$$

Covariantization of the action now involves the following step. Let $\{\phi_\alpha\}$ denote the set of all the fields of the theory. We replace a chain of ordinary derivatives $\partial_{a_1}\cdots\partial_{a_n}$ acting on a field ϕ_α by

$$E_{a_1}^{\ \mu_1}\cdots E_{a_n}^{\ \mu_n} D_{\mu_1}\cdots D_{\mu_n} \tag{3.11}$$

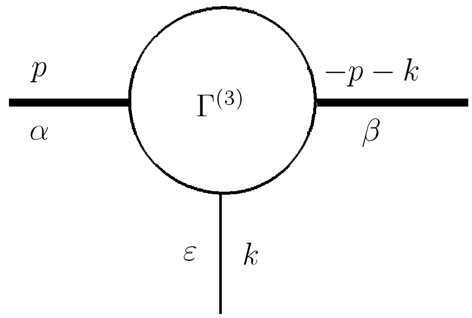

Figure 3.6: A 1PI vertex involving two finite energy particles and one soft particle.

where

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha + \frac{1}{2}\omega_\mu^{ab}(J_{ab})_\alpha^{\ \beta}\phi_\beta \,, \tag{3.12}$$

with $(J_{ab})_\alpha^{\ \beta}$ representing the action of spin angular momentum generator on all the fields, normalized so that acting on a covariant vector field ϕ_c, we have

$$(J^{ab})_c^{\ d} = \delta^a_{\ c}\eta^{bd} - \delta^b_{\ c}\eta^{ad} \,. \tag{3.13}$$

For our analysis we shall only need the expression for ω_μ^{ab} to first order in $S_{\mu\nu}$. This is given by[9]

$$\omega_\mu^{ab} = \partial^b S_\mu^{\ a} - \partial^a S_\mu^{\ b} \,. \tag{3.14}$$

For each pair of covariant derivatives acting on the field ϕ_α, we also have a contribution from the Christoffel symbol

$$D_\mu D_\nu \phi_\alpha = \cdots - \left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} D_\rho \,\phi_\alpha \,, \tag{3.15}$$

where

$$\left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} = \partial_\mu S_\nu^{\ \rho} + \partial_\nu S_\mu^{\ \rho} - \partial^\rho S_{\mu\nu} + \text{terms involving quadratic and higher powers of } S \,, \tag{3.16}$$

and $\cdots$ terms represent the usual derivatives and spin connection term. Since we shall compute subleading soft graviton amplitudes we shall only keep terms up to first order in the derivatives of soft gravitons. Also for amplitudes with two soft gravitons we only need to keep up to terms with two powers of soft graviton field $S_{\mu\nu}$. As we shall see, for specific vertices we can make further truncation of the action.

Let us now derive the form of the three point vertex involving one soft graviton and two finite

[9]Terms involving higher powers of S will give rise to vertices that have two or more soft gravitons, *and* a power of soft momentum. Such vertices will not contribute to the amplitude to subleading order in soft momentum.

energy fields, as shown in Fig. 3.6. For this we first express the quadratic part of the 1PI action as

$$\frac{1}{2}\int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2)\phi_\alpha(q_1)\,\mathcal{K}^{\alpha\beta}(q_2)\,\phi_\beta(q_2)\,, \tag{3.17}$$

where we take

$$\mathcal{K}^{\alpha\beta}(q)=\mathcal{K}^{\beta\alpha}(-q)\,. \tag{3.18}$$

For grassmann odd fields there will be an extra minus sign in this equation, but it does not affect the final results. If the soft graviton carries polarization ε and momentum k, then the coupling of single soft graviton to the fields ϕ_α, obtained by covariantizing (3.17), takes the form [21]

$$\begin{aligned} S^{(3)} &= \frac{1}{2}\int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k) \\ &\times\Phi_\alpha(q_1)\Bigg[-\varepsilon_{\mu\nu}q_2^\nu\frac{\partial}{\partial q_{2\mu}}\mathcal{K}^{\alpha\beta}(q_2)+\frac{1}{2}(k_b\,\varepsilon_{a\mu}-k_a\,\varepsilon_{b\mu})\frac{\partial}{\partial q_{2\mu}}\mathcal{K}^{\alpha\gamma}(q_2)\left(J^{ab}\right)_\gamma^{\ \beta} \\ &-\frac{1}{2}\frac{\partial^2\mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu}\partial q_{2\nu}}q_{2\rho}\left(k_\mu\varepsilon_\nu^{\ \rho}+k_\nu\varepsilon_\mu^{\ \rho}-k^\rho\varepsilon_{\mu\nu}\right)\Bigg]\Phi_\beta(q_2)\,. \end{aligned} \tag{3.19}$$

In this equation the first term inside the square bracket represents the effect of multiplication by $E_a^{\ \mu}=\delta_a^{\ \mu}-S_a^{\ \mu}$ in (3.11). The second term is the effect of the spin connection (3.14) appearing in the definition of the covariant derivative in (3.12) and the third term is the effect of the Christoffel symbol appearing in (3.15). From this we can derive an expression for the soft graviton vertex shown in Fig. 3.6 to order k:

$$\begin{aligned} &\Gamma^{(3)\alpha\beta}(\varepsilon,k;p,-p-k) \\ =\ &\frac{i}{2}\Bigg[-\varepsilon_{\mu\nu}(p+k)^\nu\frac{\partial}{\partial p_\mu}\mathcal{K}^{\alpha\beta}(-p-k)-\varepsilon_{\mu\nu}p^\nu\frac{\partial}{\partial p_\mu}\mathcal{K}^{\beta\alpha}(p) \\ &+\frac{1}{2}(k_a\,\varepsilon_{b\mu}-k_b\,\varepsilon_{a\mu})\frac{\partial}{\partial p_\mu}\mathcal{K}^{\alpha\gamma}(-p-k)\left(J^{ab}\right)_\gamma^{\ \beta}-\frac{1}{2}(k_a\,\varepsilon_{b\mu}-k_b\,\varepsilon_{a\mu})\frac{\partial}{\partial p_\mu}\mathcal{K}^{\beta\gamma}(p)\left(J^{ab}\right)_\gamma^{\ \alpha} \\ &-\frac{1}{2}\frac{\partial^2\mathcal{K}^{\alpha\beta}(-p-k)}{\partial p_\mu\partial p_\nu}(-p_\rho-k_\rho)\left(k_\mu\varepsilon_\nu^{\ \rho}+k_\nu\varepsilon_\mu^{\ \rho}-k^\rho\varepsilon_{\mu\nu}\right) \\ &-\frac{1}{2}\frac{\partial^2\mathcal{K}^{\beta\alpha}(p)}{\partial p_\mu\partial p_\nu}p_\rho\left(k_\mu\varepsilon_\nu^{\ \rho}+k_\nu\varepsilon_\mu^{\ \rho}-k^\rho\varepsilon_{\mu\nu}\right)\Bigg]\,. \end{aligned} \tag{3.20}$$

Using (3.18), (3.7), and expanding each term in Taylor series in the soft momentum k, we arrive at

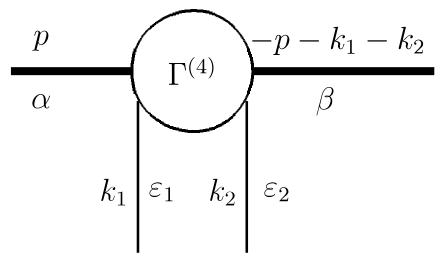

Figure 3.7: A 1PI vertex involving two finite energy particles and two soft particles.

the following expression for the vertex $\Gamma^{(3)}$ in Fig. 3.6 to order k:

$$\begin{aligned}&\Gamma^{(3)}(\varepsilon, k; p, p-k)\\ =\ & i\left[-\varepsilon_{\mu\nu}p^{\nu}\frac{\partial\mathcal{K}(-p)}{\partial p_{\mu}}-\frac{1}{2}\varepsilon_{\mu\nu}p^{\nu}k_{\rho}\frac{\partial^{2}\mathcal{K}(-p)}{\partial p_{\mu}\partial p_{\rho}}+\frac{1}{2}k_{a}\varepsilon_{b\mu}\frac{\partial\mathcal{K}(-p)}{\partial p_{\mu}}J^{ab}-\frac{1}{2}k_{a}\varepsilon_{b\mu}(J^{ab})^{T}\frac{\partial\mathcal{K}(-p)}{\partial p_{\mu}}\right],\end{aligned} \tag{3.21}$$

where we have used a matrix notation and $(J^{ab})^T$ denotes the transpose of J^{ab}, i.e. $((J^{ab})^T)^{\alpha}_{\ \gamma} = (J^{ab})_{\gamma}^{\ \alpha}$.

Next we consider the four point vertex containing two soft gravitons and two finite energy particles as shown in Fig. 3.7. Since this vertex appears in Fig. 3.4 which begins contributing at the subleading order, we need to evaluate this to leading power in the soft momentum. Therefore we can ignore the spin connection and Christoffel symbol terms in the expression for the covariant derivatives appearing in (3.11), and only focus on the contribution from the $E_a^{\ \mu}$ terms. Since we have two soft gravitons, we need to keep terms quadratic in the soft graviton field $S_{\mu\nu}$. These can come from two sources – either one power of S from two $E_a^{\ \mu}$'s or two powers of S from a single $E_a^{\ \mu}$. The resulting action is given by

$$\begin{aligned}&\frac{1}{2}\int\frac{d^{D}q_{1}}{(2\pi)^{D}}\frac{d^{D}q_{2}}{(2\pi)^{D}}\frac{d^{D}\ell_{1}}{(2\pi)^{D}}\frac{d^{D}\ell_{2}}{(2\pi)^{D}}(2\pi)^{D}\delta^{(D)}(q_{1}+q_{2}+\ell_{1}+\ell_{2})\Phi_{\alpha}(q_{1})\Phi_{\beta}(q_{2})\\ &\times\left[\frac{1}{2}S_{\mu\nu}(\ell_{1})S_{\rho\sigma}(\ell_{2})q_{2}^{\nu}q_{2}^{\sigma}\frac{\partial^{2}\mathcal{K}^{\alpha\beta}(q_{2})}{\partial q_{2\mu}\partial q_{2\rho}}+\frac{1}{2}S_{\mu}^{\ b}S_{b\nu}q_{2}^{\nu}\frac{\partial\mathcal{K}^{\alpha\beta}(q_{2})}{\partial q_{2\mu}}\right].\end{aligned} \tag{3.22}$$

Using this and the symmetry (3.18), we get the following form of the vertex shown in Fig. 3.7 to leading order in soft momenta, written in the matrix notation:

$$\begin{aligned}&\Gamma^{(4)}(\varepsilon_{1}, k_{1}, \varepsilon_{2}, k_{2}; p, -p-k_{1}-k_{2})\\ =\ & i\left[\varepsilon_{1,\mu\nu}\varepsilon_{2,\rho\sigma}p^{\nu}p^{\sigma}\frac{\partial^{2}\mathcal{K}(-p)}{\partial p_{\mu}\partial p_{\rho}}+\frac{1}{2}\left(\varepsilon_{1,\mu}^{\ \ b}\,\varepsilon_{2,b\nu}+\varepsilon_{2,\mu}^{\ \ b}\,\varepsilon_{1,b\nu}\right)p^{\nu}\frac{\partial\mathcal{K}(-p)}{\partial p_{\mu}}\right].\end{aligned} \tag{3.23}$$

Next let us consider the contribution from the amplitude in Fig. 3.8 for off-shell external momenta

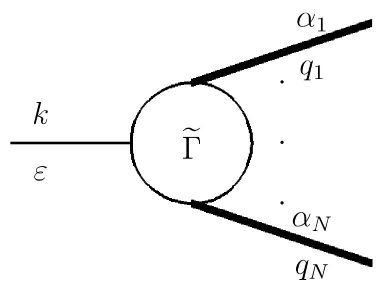

Figure 3.8: An amputated amplitude with one external soft particle and many external finite energy particles. We exclude from this any diagram where the soft particle gets attached to one of the external lines.

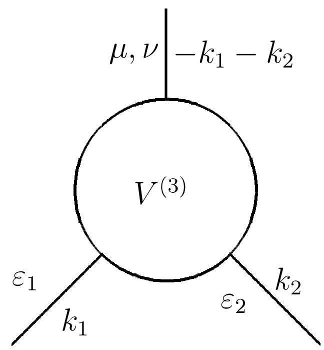

Figure 3.9: A 1PI vertex involving three soft gravitons.

$q_1, \cdots, q_N$. This can be obtained by covariantizing the truncated Green's function $\Gamma^{\alpha_1 \cdots \alpha_N}(q_1, \cdots q_N)$ without the soft graviton. Since this amplitude appears inside Fig. 3.3 which begins contributing at the subleading order, we only need the leading contribution from this amplitude. This is easily computed using the covariantization procedure, giving the result [19]

$$\widetilde{\Gamma}^{\alpha_1 \cdots \alpha_N}(\varepsilon, k; q_1, \cdots q_N) = - \sum_{i=1}^{N} \varepsilon_{\mu\nu} q_i^{\mu} \frac{\partial}{\partial q_{i\nu}} \Gamma^{\alpha_1 \cdots \alpha_N}(q_1, \cdots, q_N), \tag{3.24}$$

reflecting the effect of having to multiply every factor of momentum (derivative with respect to space-time coordinates) by inverse vielbeins as in (3.11).

The next vertex to be evaluated is the three point vertex of three soft gravitons as shown in Fig. 3.9, involving external on-shell soft gravitons carrying momenta k_1, k_2 and polarizations $\varepsilon_1, \varepsilon_2$ respectively and internal soft graviton carrying momenta $-k_1 - k_2$ and polarization labelled by the pair of indices (μ, ν). This vertex appears in Fig. 3.5 which begins contributing at the subleading order. Therefore we need to evaluate this vertex to leading order in soft momenta – given by the Einstein-Hilbert action. This is best done by regarding the external soft gravitons as background field $S_{\mu\nu}$ so that the vertex can be regarded as the one point function of the internal graviton in the presence of soft graviton background. This is proportional to $R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$ computed from the soft graviton metric. Evaluating this to quadratic order in $S_{\rho\sigma}$ we can read out the vertex. Using standard results on the expansion of connection and curvature in powers of fluctuations in the metric (see *e.g.* [111, 112]) we find that the vertex takes the form:

$$
\begin{aligned}
& V^{(3)}_{\mu\nu}(\varepsilon_1, k_1, \varepsilon_2, k_2) \\
= \;& \frac{i}{2}\,\varepsilon_{1,ab}\varepsilon_{2,cd}\Bigg[\Bigg\{\eta_{\mu\nu}\eta^{ac}\eta^{bd}k_1^{\rho}k_{2\rho} - 2\eta^{ad}\eta^{c}{}_{\nu}k_2^{b}k_{2\mu} - 2\eta^{cb}\eta^{a}{}_{\nu}k_1^{d}k_{1\mu} + 2\eta^{ad}\eta^{c}{}_{\nu}k_{1\mu}k_2^{b} \\
& +2\eta^{cb}\eta^{a}{}_{\mu}k_1^{d}k_{2\nu} - 2\eta^{ac}\eta^{bd}k_{1\mu}k_{2\nu} - 4\eta^{a}{}_{\nu}\eta^{c}{}_{\mu}k_1^{d}k_2^{b} + 2\eta^{c}{}_{\mu}\eta^{d}{}_{\nu}k_2^{b}k_2^{a} + 2\eta^{a}{}_{\mu}\eta^{b}{}_{\nu}k_1^{d}k_1^{c}\Bigg\} \\
& +\{\mu\leftrightarrow\nu\}\Bigg]\,. \qquad (3.25)
\end{aligned}
$$

We now turn to the computation of the propagators. In the normalization in which the three point vertex of Fig. 3.9 is given by (3.25), the soft graviton propagator in the de Donder gauge takes the form:

$$
G_{\mu\nu,\rho\sigma}(k) = -\frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\frac{i}{k^2}\,, \qquad (3.26)
$$

where μ, ν are the indices carried by one of the gravitons and ρ, σ are the indices carried by the other graviton.

The final ingredient is the propagator for an internal finite energy line carrying momentum q. This is given by $i\,\mathcal{K}^{-1}_{\alpha\beta}(q)$. We define

$$
\Xi^{j}_{\alpha\beta}(q) = i\,\mathcal{K}^{-1}_{\alpha\beta}(q)\,(q^2 + M_j^2)\,, \qquad (3.27)
$$

where M_j is the mass of the j-th external state. Then the propagator can be expressed as

$$
\Delta(q) = (q^2 + M_j^2)^{-1}\,\Xi^{j}(q)\,, \qquad (3.28)
$$

where we have adopted the matrix notation dropping the indices α, β.

Now from (3.27) we have

$$
\mathcal{K}(q)\,\Xi^{i}(q) = i\,(q^2 + M_i^2)\,. \qquad (3.29)
$$

Taking derivatives of this with respect to momenta we arrive at the following relations:

$$
\begin{aligned}
\frac{\partial\mathcal{K}(-p)}{\partial p_\mu}\Xi^{i}(-p) \;&=\; -\mathcal{K}(-p)\frac{\partial\Xi^{i}(-p)}{\partial p_\mu} + 2\,i\,p^{\mu}\,, \\
\frac{\partial^2\mathcal{K}(-p)}{\partial p_\mu\partial p_\nu}\Xi^{i}(-p) &= -\frac{\partial\mathcal{K}(-p)}{\partial p_\mu}\frac{\partial\Xi^{i}(-p)}{\partial p_\nu} - \frac{\partial\mathcal{K}(-p)}{\partial p_\nu}\frac{\partial\Xi^{i}(-p)}{\partial p_\mu} - \mathcal{K}(-p)\frac{\partial^2\Xi^{i}(-p)}{\partial p_\mu\partial p_\nu} + 2\,i\,\eta^{\mu\nu}\,,
\end{aligned} \qquad (3.30)
$$

Finally rotational invariant of $\mathcal{K}$ implies the following relations:

$$
\begin{aligned}
(J^{ab})^T \mathcal{K}(-p) &= -\mathcal{K}(-p) J^{ab} + p^a \frac{\partial \mathcal{K}(-p)}{\partial p_b} - p^b \frac{\partial \mathcal{K}(-p)}{\partial p_a}\,, \\
J^{ab}\, \Xi^i(-p) &= -\Xi^i(-p)(J^{ab})^T - p^a \frac{\partial \Xi^i(-p)}{\partial p_b} + p^b \frac{\partial \Xi^i(-p)}{\partial p_a}\,.
\end{aligned} \tag{3.31}
$$

3.2.2 Evaluation of the diagrams

We begin with the evaluation of Fig. 3.1. Even though we can use the form (3.28) for the internal propagator for any j, (3.28) being independent of j due to (3.27), we shall use the form (3.28) with $j = i$ when the soft gravitons attach to the i-th external line. In this case the propagator carrying momentum $-p_i - k$ for some soft momentum k takes the form

$$
\Delta(-p_i - k) = \{(p_i + k)^2 + M_i^2\}^{-1}\, \Xi^i(-p_i - k) = (2p_i \cdot k + k^2)^{-1}\, \Xi^i(-p_i - k)\,. \tag{3.32}
$$

We now define

$$
\Gamma^{\alpha_i}_{(i)}(p_i) = \left\{ \prod_{\substack{j=1 \\ j\neq i}}^{N} \epsilon_{j,\alpha_j} \right\} \Gamma^{\alpha_1\cdots\alpha_N}(p_1, \cdots, p_N)\,, \tag{3.33}
$$

with the understanding that $\Gamma^{\alpha_i}_{(i)}(p_i)$ also implicitly depends on the p_j's and ϵ_j's for $j \neq i$. Using this we can express the contribution from Fig. 3.1 as

$$
\begin{aligned}
A_1 \;\equiv\; & \sum_{i=1}^{N} (2p_i \cdot k_1)^{-1} (2p_i \cdot (k_1 + k_2) + 2k_1 \cdot k_2)^{-1}\, \epsilon_i^T\, \Gamma^{(3)}(\varepsilon_1, k_1; p_i, -p_i - k_1)\, \Xi^i(-p_i - k_1) \\
& \Gamma^{(3)}(\varepsilon_2, k_2; p_i + k_1, -p_i - k_1 - k_2)\, \Xi^i(-p_i - k_1 - k_2)\, \Gamma_{(i)}(p_i + k_1 + k_2)\,,
\end{aligned} \tag{3.34}
$$

where we have summed over soft graviton insertion on different external legs. We now use the expression (3.21) for $\Gamma^{(3)}$ and manipulate this expression as follows:

1. Take all the J^{ab} factors to the extreme right using (3.31) and their derivatives with respect to p^μ.

2. Expand $\mathcal{K}$, Ξ^i and $\Gamma_{(i)}$ in Taylor series expansion in k_1, k_2, and keep up to the first subleading terms in soft momenta.

3. Use the relations (3.30) to move all momentum derivatives to the extreme right to the extent possible.

4. Finally use the on-shell condition

$$
\epsilon_i^T \mathcal{K}(-p) = 0\,, \tag{3.35}
$$

to set all terms in which the left-most $\mathcal{K}$ does not have a derivative acting on it to zero.

While these steps are sufficient to arrive at the final result given in (3.39), for the analysis of section 3.3 we shall need some of the results that appear in the intermediate stages. For example, Taylor series expansion in k, together with the use of (3.30), (3.31) leads to the result

$$\Gamma^{(3)}(\varepsilon, k; p, -p-k)\ \Xi^i(-p-k) \;=\; \Big[2\,\varepsilon^{\mu\nu} p_\mu p_\nu + i\,\varepsilon_{\mu\nu} p^\nu \mathcal{K}(-p)\frac{\partial \Xi^i(-p)}{\partial p_\mu} + 2\,\varepsilon_{b\mu} k_a p^\mu (J^{ab})^T + \mathcal{K}(-p)\ \mathcal{Q}(p,k)\Big] \tag{3.36}$$

to subleading order. Here

$$\mathcal{Q}(p,k) \equiv \frac{i}{2}\, k\cdot p\, \varepsilon_{b\mu}\, \frac{\partial^2 \Xi(-p)}{\partial p_\mu \partial p_b} + i\,\varepsilon_{b\mu}\, k_a\, \frac{\partial \Xi(-p)}{\partial p_\mu}\, (J^{ab})^T\,, \tag{3.37}$$

denotes a term that receives contribution from subleading order in soft momentum. We shall see that its contribution to the amplitude vanishes due to (3.35). Using (3.36) we can express the amplitude (3.34) as

$$\begin{aligned} A_1 \;=\; & \sum_{i=1}^N (2p_i\cdot k_1)^{-1}(2p_i\cdot(k_1+k_2)+2k_1\cdot k_2)^{-1} \\ & \epsilon_i^T\Big[2\,\varepsilon_1^{\mu\nu} p_{i\mu}p_{i\nu} + i\,\varepsilon_{1,\mu\nu} p_i^\nu \mathcal{K}(-p_i)\frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} + 2\,\varepsilon_{1,b\mu} k_{1a} p_i^\mu (J^{ab})^T + \mathcal{K}(-p_i)\ \mathcal{Q}(p_i,k_1)\Big] \\ & \Big[2\,\varepsilon_2^{\rho\sigma}(p_{i\rho}+k_{1\rho})(p_{i\sigma}+k_{1\sigma}) + i\,\varepsilon_{2,\rho\sigma}(p_i^\sigma + k_1^\sigma)\mathcal{K}(-p_i-k_1)\frac{\partial \Xi^i(-p_i-k_1)}{\partial p_{i\rho}} \\ & +2\,\varepsilon_{2,d\rho}\, k_{2c}\,(p_i^\rho + k_1^\rho)\,(J^{cd})^T + \mathcal{K}(-p_i)\ \mathcal{Q}(p_i,k_2)\Big]\Gamma_{(i)}(p_i+k_1+k_2)\,, \end{aligned} \tag{3.38}$$

to first subleading order. Expanding the terms inside the second square bracket and $\Gamma_{(i)}$ in a Taylor series expansion in k_1 and k_2, and using (3.35), (3.31), we get, up to subleading order,

$$\begin{aligned} A_1 \;=\; & \sum_{i=1}^N (p_i\cdot k_1)^{-1}\{p_i\cdot(k_1+k_2)+k_1\cdot k_2\}^{-1}\epsilon_i^T\Bigg[\varepsilon_1^{\sigma\tau} p_{i\sigma}p_{i\tau}\ \varepsilon_{2\mu\nu} p_i^\nu\, p_i^\mu\ \Gamma_{(i)}(p_i) \\ & +2\varepsilon_1^{\sigma\tau} p_{i\sigma}p_{i\tau}\ \varepsilon_{2,\mu\nu} k_1^\nu\, p_i^\mu\ \Gamma_{(i)}(p_i) + \varepsilon_1^{\sigma\tau} p_{i\sigma}p_{i\tau} k_{2a}\varepsilon_{2,b\mu} p_i^\mu (J^{ab})^T\Gamma_{(i)}(p_i) \\ & +\varepsilon_{1,b\sigma} k_{1a} p_i^\sigma\ \varepsilon_{2,\mu\nu} p_i^\nu\, p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \varepsilon_1^{\sigma\tau} p_{i\sigma}p_{i\tau}\ \varepsilon_{2,\mu\nu} p_i^\nu\, p_i^\mu\ (k_1+k_2)_\rho\ \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{i\rho}} \\ & +\frac{1}{2} i\,(k_1\cdot p_i)\,\varepsilon_{1,\mu\sigma}\,\varepsilon_{2,\rho\nu}\, p_i^\sigma\, p_i^\nu\ \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}}\ \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}}\ \Gamma_{(i)}(p_i)\Bigg]\,. \end{aligned} \tag{3.39}$$

To this, we also need to add an expression in which we interchange $(k_1, \varepsilon_1) \leftrightarrow (k_2, \varepsilon_2)$. This gives the amplitude

$$
\begin{aligned}
A'_1 \;=\; & \sum_{i=1}^{N} (p_i \cdot k_2)^{-1} (p_i \cdot (k_2 + k_1) + 2k_2 \cdot k_1)^{-1} \epsilon_i^T \Bigg[\varepsilon_2^{\sigma\tau} p_{i\sigma} p_{i\tau}\, \varepsilon_{1,\mu\nu} p_i^{\nu}\, p_i^{\mu}\, \Gamma_{(i)}(p_i) \\
& + 2\varepsilon_2^{\sigma\tau} p_{i\sigma} p_{i\tau}\, \varepsilon_{1,\mu\nu} k_2^{\nu}\, p_i^{\mu}\, \Gamma_{(i)}(p_i) + \varepsilon_2^{\sigma\tau} p_{i\sigma} p_{i\tau} k_{1a} \varepsilon_{1,b\mu} p_i^{\mu} (J^{ab})^T \Gamma_{(i)}(p_i) \\
& + \varepsilon_{2,b\sigma} k_{2a} p_i^{\sigma}\, \varepsilon_{1,\mu\nu} p_i^{\nu}\, p_i^{\mu} (J^{ab})^T \Gamma_{(i)}(p_i) + \varepsilon_2^{\sigma\tau} p_{i\sigma} p_{i\tau}\, \varepsilon_{1,\mu\nu} p_i^{\nu}\, p_i^{\mu}\, (k_2 + k_1)_{\rho}\, \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{i\rho}} \\
& + \frac{1}{2} i\, (k_2 \cdot p_i)\, \varepsilon_{2,\mu\sigma} \varepsilon_{1,\rho\nu} p_i^{\sigma} p_i^{\nu} \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \Gamma_{(i)}(p_i) \Bigg] . \qquad (3.40)
\end{aligned}
$$

The contribution from Fig. 3.2 can be evaluated by knowing the result for single soft graviton insertion since the two parts of the diagram on which the two soft gravitons are inserted can be evaluated independently. We shall express this as

$$
\begin{aligned}
& (2p_i \cdot k_1)^{-1}\, (2p_j \cdot k_2)^{-1} \left\{ \varepsilon_i^T\, \Gamma^{(3)}(\varepsilon_1, k_1; p_i, -p_i - k_1)\, \Xi^i(-p_i - k_1) \right\} \\
& \otimes \left\{ \varepsilon_j^T\, \Gamma^{(3)}(\varepsilon_2, k_2; p_j, -p_j - k_2)\, \Xi^j(-p_j - k_2) \right\} \Gamma_{(i,j)}(p_i + k_1, p_j + k_2)\, , \qquad (3.41)
\end{aligned}
$$

where $\Gamma_{(i,j)}^{\alpha_i\alpha_j}$ is defined in the same way as $\Gamma_{(i)}$ except that we now strip off both the polarization tensors of the i-th and the j-th leg:

$$
\Gamma_{(i,j)}^{\alpha_i\alpha_j}(p_i, p_j) \equiv \left\{ \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{N} \epsilon_{\ell,\alpha_\ell} \right\} \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \cdots, p_N)\, . \qquad (3.42)
$$

It is understood that in (3.41) the terms inside the first curly bracket contracts with the first index α_i of $\Gamma_{(i,j)}$ and the terms inside the second bracket contracts with the second index α_j of $\Gamma_{(i,j)}$. By manipulating the matrices acting on the i-th and the j-th leg independently in the same way as before, using the results

$$
\epsilon_{i,\alpha} \Gamma_{(i,j)}^{\alpha\beta}(p_i, p_j) = \Gamma_{(j)}^{\beta}(p_j), \quad \epsilon_{j,\beta} \Gamma_{(i,j)}^{\alpha\beta}(p_i, p_j) = \Gamma_{(i)}^{\alpha}(p_i), \qquad (3.43)
$$

and summing over insertions on all external legs, we arrive at the following result for the amplitude up to first subleading order:

$$
\begin{aligned}
A_2 \;=\;& \sum_{\substack{i,j=1\\ i\neq j}}^{N} (p_i\cdot k_1)^{-1}\,(p_j\cdot k_2)^{-1}\,\varepsilon_{1,\mu\nu}\,p_i^\mu p_i^\nu\,\varepsilon_{2,\rho\sigma}\,p_j^\rho p_j^\sigma\;\Gamma(\varepsilon_1,k_1,\varepsilon_2,k_2;\epsilon_1,p_1,\cdots,\epsilon_N,p_N)\\
&+\sum_{\substack{i,j=1\\ i\neq j}}^{N} (p_i\cdot k_1)^{-1}\,(p_j\cdot k_2)^{-1}\,\varepsilon_{2,\rho\sigma}\,p_j^\rho p_j^\sigma\,\epsilon_i^T\;\left[\varepsilon_{1,\mu\nu}\,p_i^\mu p_i^\nu k_{1\tau}\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{i\tau}}+k_{1a}\,\varepsilon_{1,b\mu}\,p_i^\mu\,(J^{ab})^T\Gamma_{(i)}(p_i)\right]\\
&+\sum_{\substack{i,j=1\\ i\neq j}}^{N} (p_i\cdot k_2)^{-1}\,(p_j\cdot k_1)^{-1}\,\varepsilon_{1,\rho\sigma}\,p_j^\rho p_j^\sigma\,\epsilon_i^T\;\left[\varepsilon_{2,\mu\nu}\,p_i^\mu p_i^\nu k_{2\tau}\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{i\tau}}+k_{2a}\,\varepsilon_{2,b\mu}\,p_i^\mu\,(J^{ab})^T\Gamma_{(i)}(p_i)\right].
\end{aligned}
\tag{3.44}
$$

Next we consider the contribution from Fig. 3.3. The contribution from this term has at most one pole in the soft momentum and therefore begins at subleading order. Therefore we only need the leading contribution from this diagram. For this we use the result (3.24) for the off-shell amplitude shown in Fig. 3.8. This gives the following expression for the contribution from Fig. 3.3:

$$
A_3 = -\sum_{i=1}^{N}(2p_i\cdot k_1)^{-1}\epsilon_i^T\Gamma^{(3)}(\varepsilon_1,k_1;p_i,-p_i-k_1)\,\Xi^i(-p_1-k_1)\sum_{j=1}^{N}\epsilon_j^T\varepsilon_2^{\mu\nu}p_{j\mu}\frac{\partial}{\partial p_j^\nu}\Gamma_{(i,j)}(p_i,p_j)\,, \tag{3.45}
$$

where again we have summed over the insertion of the first soft graviton on all external finite energy states. We can now manipulate this using the form of $\Gamma^{(3)}$ given earlier. This leads to

$$
A_3 \;=\; -\sum_{i=1}^{N}(p_i\cdot k_1)^{-1}\,\varepsilon_1^{\rho\sigma}p_{i\rho}p_{i\sigma}\,\sum_{j=1}^{N}\epsilon_j^T\varepsilon_2^{\mu\nu}p_{j\mu}\frac{\partial}{\partial p_j^\nu}\Gamma_{(j)}\,. \tag{3.46}
$$

The diagram obtained by interchanging $(k_1,\varepsilon_1)\leftrightarrow(k_2,\varepsilon_2)$ gives

$$
A_3' \;=\; -\sum_{i=1}^{N}(p_i\cdot k_2)^{-1}\,\varepsilon_2^{\rho\sigma}p_{i\rho}p_{i\sigma}\,\sum_{j=1}^{N}\epsilon_j^T\varepsilon_1^{\mu\nu}p_{j\mu}\frac{\partial}{\partial p_j^\nu}\Gamma_{(j)}\,. \tag{3.47}
$$

Fig. 3.4 also begins contributing at the subleading order. Therefore we only need its leading contribution, which is given by

$$
A_4 = \sum_{i=1}^{N}\{2p_i\cdot(k_1+k_2)\}^{-1}\epsilon_i^T\Gamma^{(4)}(\varepsilon_1,k_1,\varepsilon_2,k_2;p_i,-p_i-k_1-k_2)\,\Xi^i(-p_i-k_1-k_2)\,\Gamma_{(i)}(p_i)\,. \tag{3.48}
$$

This can be evaluated using the expression (3.23) for the vertex $\Gamma^{(4)}$ shown in Fig. 3.7 and manipulating the resulting expression in the same way as the previous diagrams. The result is

$$A_4 = \sum_{i=1}^{N}\{p_i\cdot(k_1+k_2)\}^{-1}\epsilon_i^T\Bigg[-2\varepsilon_{1,\mu}{}^{\nu}\varepsilon_{2,\nu\rho}p_i^\rho p_i^\mu - \frac{i}{2}\Big(\varepsilon_{1,\mu\sigma}\varepsilon_{2,\rho\nu}p_i^\sigma p_i^\nu+\varepsilon_{1,\rho\sigma}\varepsilon_{2,\mu\nu}p_i^\sigma p_i^\nu\Big)\frac{\partial\mathcal{K}(-p_i)}{\partial p_{i\mu}}\frac{\partial\Xi^i(-p_i)}{\partial p_{i\rho}}\Bigg]\Gamma_{(i)}(p_i)\,. \qquad (3.49)$$

Finally we turn to the computation of the diagram shown in Fig. 3.5. Its contribution is given by

$$A_5 = V^{(3)\mu\nu}(\varepsilon_1,k_1,\varepsilon_2,k_2)G_{\mu\nu,\rho\sigma}(k_1+k_2)\ \sum_{i=1}^{N}\epsilon_i^T\,\Gamma^{(3)(\rho\sigma)}\left(k_1+k_2;p_i,-p_i-k_1-k_2\right)\ \Gamma_{(i)}(p_i)\,, \qquad (3.50)$$

where $V^{(3)}$ and $G_{\mu\nu,\rho\sigma}$ have been defined in (3.25) and (3.26) respectively, and $\Gamma^{(3)(\rho\sigma)}$ is defined via the equation

$$\Gamma^{(3)}\left(\varepsilon,k;p,-p-k\right)=\varepsilon_{\rho\sigma}\Gamma^{(3)(\rho\sigma)}\left(k;p,-p-k\right)\,. \qquad (3.51)$$

Using the leading order expression for $\Gamma^{(3)}$ given in (3.21), and the relations (3.30), (3.31), (3.35) this can be brought to the form

$$\begin{aligned}A_5 = & \sum_{i=1}^{N}\{p_i\cdot(k_1+k_2)\}^{-1}(k_1\cdot k_2)^{-1}\epsilon_i^T\Big[-(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)(k_2\cdot p_i)-(k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)(k_1\cdot p_i)\\ &+(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)(k_1\cdot p_i)+(k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)(k_2\cdot p_i)-\varepsilon_1^{cd}\varepsilon_{2,cd}\left(k_1\cdot p_i\right)\left(k_2\cdot p_i\right)\\ &-2(p_i\cdot\varepsilon_1\cdot k_2)(p_i\cdot\varepsilon_2\cdot k_1)+(p_i\cdot\varepsilon_2\cdot p_i)(k_2\cdot\varepsilon_1\cdot k_2)+(p_i\cdot\varepsilon_1\cdot p_i)(k_1\cdot\varepsilon_2\cdot k_1)\Big]\Gamma_{(i)}(p_i)\end{aligned} \qquad (3.52)$$

The full amplitude is given by

$$\begin{aligned}A = & A_1+A_1'+A_2+A_3+A_3'+A_4+A_5\\ = & \left\{\sum_{i=1}^{N}(p_i\cdot k_1)^{-1}\,\varepsilon_{1,\mu\nu}\,p_i^\mu p_i^\nu\right\}\left\{\sum_{j=1}^{N}(p_j\cdot k_2)^{-1}\,\varepsilon_{2,\rho\sigma}\,p_j^\rho p_j^\sigma\right\}\Gamma(\epsilon_1,p_1,\cdots,\epsilon_N,p_N)\\ &+\left\{\sum_{j=1}^{N}(p_j\cdot k_2)^{-1}\,\varepsilon_{2,\rho\sigma}\,p_j^\rho p_j^\sigma\right\}\ \sum_{i=1}^{N}(p_i\cdot k_1)^{-1}\,\varepsilon_{1,b\mu}\,k_{1a}\,p_i^\mu\,\epsilon_i^T\ \left[p_i^b\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{ia}}-p_i^a\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{ib}}+(J^{ab})^T\Gamma_{(i)}(p_i)\right]\\ &+\left\{\sum_{j=1}^{N}(p_j\cdot k_1)^{-1}\,\varepsilon_{1,\rho\sigma}\,p_j^\rho p_j^\sigma\right\}\ \sum_{i=1}^{N}(p_i\cdot k_2)^{-1}\,\varepsilon_{2,b\mu}\,k_{2a}\,p_i^\mu\,\epsilon_i^T\ \left[p_i^b\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{ia}}-p_i^a\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{ib}}+(J^{ab})^T\Gamma_{(i)}(p_i)\right]\\ &+\left\{\sum_{i=1}^{N}\{p_i\cdot(k_1+k_2)\}^{-1}\,\mathcal{M}(p_i;\varepsilon_1,k_1,\varepsilon_2,k_2)\right\}\Gamma(\epsilon_1,p_1,\cdots,\epsilon_N,p_N)\,,\end{aligned} \qquad (3.53)$$

where

$$
\begin{aligned}
\mathcal{M}(p_i;\varepsilon_1,k_1,\varepsilon_2,k_2) \;=\;& (p_i\cdot k_1)^{-1}(p_i\cdot k_2)^{-1}\Bigg\{-(k_1\cdot k_2)\,(p_i\cdot\varepsilon_1\cdot p_i)\,(p_i\cdot\varepsilon_2\cdot p_i) \\
&+2\,(p_i\cdot k_2)\,(p_i\cdot\varepsilon_1\cdot p_i)\,(p_i\cdot\varepsilon_2\cdot k_1)+2\,(p_i\cdot k_1)\,(p_i\cdot\varepsilon_2\cdot p_i)\,(p_i\cdot\varepsilon_1\cdot k_2) \\
&-2\,(p_i\cdot k_1)\,(p_i\cdot k_2)\,(p_i\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)\Bigg\} \\
+(k_1\cdot k_2)^{-1}\Bigg\{&-(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)\,(k_2\cdot p_i)-(k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)\,(k_1\cdot p_i) \\
&+(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)\,(k_1\cdot p_i)+(k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)\,(k_2\cdot p_i)-\varepsilon_1^{cd}\varepsilon_{2,cd}\,(k_1\cdot p_i)\,(k_2\cdot p_i) \\
&-2(p_i\cdot\varepsilon_1\cdot k_2)\,(p_i\cdot\varepsilon_2\cdot k_1)+(p_i\cdot\varepsilon_2\cdot p_i)\,(k_2\cdot\varepsilon_1\cdot k_2)+(p_i\cdot\varepsilon_1\cdot p_i)\,(k_1\cdot\varepsilon_2\cdot k_1)\Bigg\}\,.
\end{aligned}
\tag{3.54}
$$

Here we have used the shorthand notation $p_i\cdot\varepsilon_1\cdot p_i\equiv\varepsilon_{1,\mu\nu}p_i^\mu p_i^\nu$ etc. $\mathcal{M}$ receives contributions from the first two terms in (3.39) and (3.40) and also from (3.49) and (3.52).

3.2.3 Consistency checks

In this section we shall carry out various consistency checks on our result. First we shall check the internal consistency of our result. Then we shall compare our results with the previous results derived for specific theories.

Internal consistency

The first internal consistency check of our result comes from the requirement of gauge invariance. This means that if we make the transformation

$$
\varepsilon_{r,\mu\nu}\to k_{r\mu}\,\xi_{r\nu}+k_{r\nu}\,\xi_{r\mu}\,,\qquad r=1,2\,,
\tag{3.55}
$$

for any vector ξ_r satisfying $k_r\cdot\xi_r=0$, the result (3.53) does not change. Checking this involves tedious but straightforward algebra, and needs use of the equations

$$
\sum_{i=1}^{N} p_{i\mu}\,\Gamma^{\alpha_1\cdots\alpha_N}=0\,,
\tag{3.56}
$$

and

$$
\sum_{i=1}^{N}\left[p_i^b\,\frac{\partial\Gamma^{\alpha_1\cdots\alpha_N}}{\partial p_{ia}}-p_i^a\,\frac{\partial\Gamma^{\alpha_1\cdots\alpha_N}}{\partial p_{ib}}+(J^{ab})_{\beta_i}{}^{\alpha_i}\,\Gamma^{\alpha_1\cdots\alpha_{i-1}\beta_i\alpha_{i+1}\cdots\alpha_N}\right]=0\,,
\tag{3.57}
$$

reflecting respectively translational and rotational invariance of the amplitude without the soft graviton. While making this analysis we also need to be careful to ensure that while passing $p_{i\mu}$ through $\partial/\partial p_{j\nu}$ in order to make use of (3.56), we have to take into account the extra terms proportional to $\delta_{ij}\delta^\nu_\mu$. For this reason the terms in the third and fourth lines of (3.53) are not gauge invariant by themselves – their gauge variation cancels against the variation of the term in the last line of (3.53). More specifically if we denote by δ_r the gauge variation:

$$\delta_r \ : \qquad \varepsilon_{r,\mu\nu} \to \varepsilon_{r,\mu\nu} + k_{r\mu}\xi_{r\nu} + k_{r\nu}\xi_{r\mu} \, , \tag{3.58}$$

for some vector ξ_r satisfying $k_r \cdot \xi_r = 0$, then under δ_1 the term in the third line of (3.53) remains unchanged, but the term in the fourth line changes by

$$-2 \sum_{i=1}^{N} (p_i \cdot k_2)^{-1} \, \varepsilon_{2,b\mu} \, p_i^b p_i^\mu \; k_2 \cdot \xi_1 \; \epsilon_i^T \Gamma_{(i)} \, . \tag{3.59}$$

On the other hand we get, after using momentum conservation equation $\sum_{j=1}^N p_j \Gamma_{(i)} = 0$,

$$\sum_{i=1}^{N} \{p_i \cdot (k_1 + k_2)\}^{-1} \, \delta_1 \mathcal{M}(p_i; \varepsilon_1, k_1, \varepsilon_2, k_2) \; \epsilon_i^T \Gamma_{(i)} = 2 \sum_{i=1}^{N} (p_i \cdot k_2)^{-1} \, \varepsilon_{2,b\mu} \, p_i^b p_i^\mu \; k_2 \cdot \xi_1 \; \epsilon_i^T \Gamma_{(i)} \tag{3.60}$$

Using this one can easily verify that the δ_1 variation of the fourth line and the last line of (3.53) cancel. A similar analysis shows that the δ_2 variation of the third and the last lines of (3.53) cancel, and that the fourth line of (3.53) is invariant under δ_2.

The second consistency requirement arises from the fact that individual terms in (3.53) depend on the off-shell data on $\Gamma^{\alpha_1 \cdots \alpha_N}$ while the actual result should be insensitive to such off-shell extension. For example if we add to $\Gamma^{\alpha_1 \cdots \alpha_N}$ any term proportional to $p_i^2 + M_i^2$, it does not affect the on-shell amplitude without the soft gravitons since it vanishes on-shell. However $\partial \Gamma^{\alpha_1 \cdots \alpha_N}/\partial p_{i_\mu}$ receives a contribution proportional to p_i^μ that does not vanish on-shell. We note however that in (3.53) the derivatives of $\Gamma^{\alpha_1 \cdots \alpha_N}$ come in a very special combination that vanishes under addition of any term to $\partial\Gamma/\partial p_{i\mu}$ proportional to p_i^μ. Therefore (3.53) is not sensitive to such additional terms in Γ.

More generally we can add to $\Gamma^{\alpha_1 \cdots \alpha_N}$ any term proportional to $\mathcal{K}^{\alpha_i \beta}(-p_i) \mathcal{G}_\beta^{\alpha_1 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_N}$ for any function $\mathcal{G}$, since its contribution to on-shell amplitudes without the soft gravitons vanishes due to (3.35). Using (3.35) and the rotational invariance of $\mathcal{K}$ described in (3.31), is easy to see however that the addition of such terms to Γ does not affect (3.53).

Comparison with known results

In order to compare the amplitude with known results, it is convenient to rewrite the amplitude (3.53) as a sum of two terms $\mathcal{A}_1 + \mathcal{A}_2$ by adding and subtracting a specific term given in the last two lines of (3.61):

$$\begin{aligned}
\mathcal{A}_1 &= \left\{ \sum_{i=1}^{N}(p_i\cdot k_1)^{-1}\,\varepsilon_{1,\mu\nu}\,p_i^\mu p_i^\nu \right\} \left\{ \sum_{j=1}^{N}(p_j\cdot k_2)^{-1}\,\varepsilon_{2,\rho\sigma}\,p_j^\rho p_j^\sigma \right\} \Gamma(\epsilon_1,p_1,\cdots,\epsilon_N,p_N) \\
&+\left\{ \sum_{j=1}^{N}(p_j\cdot k_2)^{-1}\,\varepsilon_{2,\rho\sigma}\,p_j^\rho p_j^\sigma \right\} \sum_{i=1}^{N}(p_i\cdot k_1)^{-1}\,\varepsilon_{1,b\mu}\,k_{1a}\,p_i^\mu\,\epsilon_i^T\,\left[p_i^b\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{ia}}-p_i^a\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{ib}}+(J^{ab})^T\Gamma_{(i)}(p_i)\right] \\
&+\left\{ \sum_{j=1}^{N}(p_j\cdot k_1)^{-1}\,\varepsilon_{1,\rho\sigma}\,p_j^\rho p_j^\sigma \right\} \sum_{i=1}^{N}(p_i\cdot k_2)^{-1}\,\varepsilon_{2,b\mu}\,k_{2a}\,p_i^\mu\,\epsilon_i^T\,\left[p_i^b\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{ia}}-p_i^a\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{ib}}+(J^{ab})^T\Gamma_{(i)}(p_i)\right] \\
&+(k_1\cdot k_2)^{-1}\sum_{i=1}^{N}\,(p_i\cdot k_1)^{-1}\,(p_i\cdot k_2)^{-1}\,\Big\{(k_1\cdot\varepsilon_2\cdot k_1)(p_i\cdot\varepsilon_1\cdot p_i)\,(p_i\cdot k_2) \\
&\qquad +(k_2\cdot\varepsilon_1\cdot k_2)\,(p_i\cdot\varepsilon_2\cdot p_i)\,(p_i\cdot k_1)\Big\}\,\epsilon_i^T\Gamma_{(i)}(p_i)\,, \qquad (3.61)
\end{aligned}$$

$$\mathcal{A}_2 = \left\{ \sum_{i=1}^{N}\mathcal{N}(p_i;\varepsilon_1,k_1,\varepsilon_2,k_2)\right\}\Gamma(\epsilon_1,p_1,\cdots,\epsilon_N,p_N)\,, \qquad (3.62)$$

where

$$\begin{aligned}
\mathcal{N}(p_i;\varepsilon_1,k_1,\varepsilon_2,k_2) &= \{p_i\cdot(k_1+k_2)\}^{-1}\,\mathcal{M}(p_i;\varepsilon_1,k_1,\varepsilon_2,k_2) \\
&-(k_1\cdot k_2)^{-1}\,(p_i\cdot k_1)^{-1}\,(p_i\cdot k_2)^{-1} \\
&\times\Big\{(k_1\cdot\varepsilon_2\cdot k_1)\,(p_i\cdot\varepsilon_1\cdot p_i)\,(p_i\cdot k_2)+(k_2\cdot\varepsilon_1\cdot k_2)\,(p_i\cdot\varepsilon_2\cdot p_i)\,(p_i\cdot k_1)\Big\}\,,
\end{aligned} \qquad (3.63)$$

$\mathcal{M}$ being given in (3.54). With this definition $\mathcal{A}_1$ and $\mathcal{A}_2$ can be shown to be separately gauge invariant.

Refs. [15, 16] computed the double soft limit for scattering of gravitons in Einstein gravity using CHY scattering equations [24–27, 113]. Since our result is valid for general finite energy external states in any theory, it must also be valid for scattering of gravitons. Therefore we can compare the two results. The contribution in [15,16] comes from two separate terms, the degenerate solutions and non-degenerate solutions. The contribution from the degenerate solutions agrees with our amplitude $\mathcal{A}_2$ given in (3.62) up to a sign after using momentum conservation rules (3.56). The contribution from the non-degenerate solutions were evaluated in [16] to give only the first three lines of (3.61). However the analysis was carried out in a gauge in which $k_1\cdot\varepsilon_2=0$ and $k_2\cdot\varepsilon_1=0$. For this choice of gauge the contribution from the last two lines of (3.61) vanishes. However, the results of [15, 16] differs from ours by a sign. But, apart from the issue of sign, there is agreement between our results and the results in pure gravity derived from CHY equations in [15, 16], with (3.61) giving the full gauge invariant version of the contribution from non-degenerate solutions of CHY equations. As we shall show in the next chapter by reanalyzing the double soft limit of the CHY formula for the scattering amplitudes, the result obtained from the CHY formula actually agrees with ours including the sign [23].

Ref. [11] computed the double soft limit of graviton scattering amplitude in four space-time dimensions using BCFW recursion relations [114]. This analysis was also carried out in the gauge $k_1 \cdot \varepsilon_2 = 0$ and $k_2 \cdot \varepsilon_1 = 0$. In this gauge the subleading contribution to $\mathcal{A}_1$ comes only from the second and the third lines which, written in the spinor helicity notation, has the standard form involving derivatives with respect to the spinor helicity variables, called 'non-contact terms' in [11]. Therefore we focus on the $\mathcal{A}_2$ term. Ref. [16] showed that the contribution from the degenerate solution to the CHY equations agrees with the 'contact terms' computed in [11] using BCFW recursion relations. Therefore our result for $\mathcal{A}_2$ agrees with the contact terms of [11] up to the sign factor discussed earlier. We have also verified this independently by noting that in the gauge $k_1 \cdot \varepsilon_2 = 0$ and $k_2 \cdot \varepsilon_1 = 0$ many of the terms in $\mathcal{A}_2$ vanish and the remaining terms take the form

$$\sum_{i=1}^{N} \{p_i \cdot (k_1 + k_2)\}^{-1} \Bigg[- \ (p_i \cdot k_1)^{-1} (p_i \cdot k_2)^{-1} (k_1 \cdot k_2) \ (p_i \cdot \varepsilon_1 \cdot p_i) \ (p_i \cdot \varepsilon_2 \cdot p_i) \\ - 2 \ (p_i \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_i) - \ (k_1 \cdot k_2)^{-1} \, \varepsilon_1^{cd} \varepsilon_{2,cd} \ (k_1 \cdot p_i) \ (k_2 \cdot p_i) \Bigg] \Gamma(\epsilon_1, p_1, \cdots, \epsilon_N, p_N) \ . \tag{3.64}$$

By expressing this in the spinor helicity notation we find that when the two soft gravitons carry the same helicity (3.64) vanishes. This is in agreement with the result of [11]. On the other hand when the two soft gravitons carry opposite helicities, $\mathcal{A}_2$ gives a non-zero result that agrees with the 'contact terms' of [11] up to a sign. We have not tried to resolve this discrepancy in sign between our results and that of [11]. However given that we have now verified that the CHY result for contact terms actually comes with a sign opposite to that found in [15, 16] and agrees with our amplitude $\mathcal{A}_2$ [23], it seems that the difference in sign between our results and the BCFW results may be due to some differences in convention, *e.g.* the difference in the choice of sign of the graviton polarization tensor.[10]

3.3 Amplitudes with arbitrary number of soft gravitons

The method described in the earlier sections can now be generalized to derive the expression for the amplitude with multiple soft gravitons when the momenta carried by all the soft gravitons become small at the same rate. As mentioned towards the begining of this chapter, we claim that the subleading soft graviton amplitude with M soft gravitons carrying momenta $k_1, \cdots, k_M$ and polarizations

[10] We have used the convention that the graviton polarization tensors in four dimensions are given by squares of the gauge field polarization tensors without any extra sign.

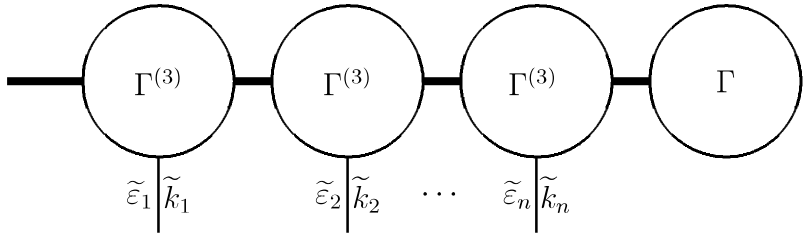

Figure 3.10: A leading contribution to the amplitude with multiple soft gravitons.

$\varepsilon_1, \cdots, \varepsilon_M$ and N finite energy particles carrying momenta $p_1, \cdots, p_N$ and polarizations $\epsilon_1, \cdots, \epsilon_N$ is given by

$$\begin{aligned}
A \;=\; & \prod_{r=1}^{M} \left\{ \sum_{i=1}^{N} (p_i \cdot k_r)^{-1} \, \varepsilon_{r,\mu\nu} \, p_i^\mu p_i^\nu \right\} \Gamma(\epsilon_1, p_1, \cdots, \epsilon_N, p_N) \\
&+ \sum_{s=1}^{M} \sum_{j=1}^{N} (p_j \cdot k_s)^{-1} \, \varepsilon_{s,b\mu} \, k_{sa} \, p_j^\mu \, \epsilon_j^T \left[p_j^b \frac{\partial \Gamma_{(j)}(p_j)}{\partial p_{ja}} - p_j^a \frac{\partial \Gamma_{(j)}(p_j)}{\partial p_{jb}} + (J^{ab})^T \Gamma_{(j)}(p_j) \right] \\
&\qquad \times \prod_{\substack{r=1 \\ r \neq s}}^{M} \left\{ \sum_{i=1}^{N} (p_i \cdot k_r)^{-1} \, \varepsilon_{r,\mu\nu} \, p_i^\mu p_i^\nu \right\} \\
&+ \sum_{\substack{r,u=1 \\ r<u}}^{M} \left\{ \sum_{j=1}^{N} \{p_j \cdot (k_r + k_u)\}^{-1} \, \mathcal{M}(p_j; \varepsilon_r, k_r, \varepsilon_u, k_u) \, \epsilon_j^T \, \Gamma_{(j)}(p_j) \right\} \prod_{\substack{s=1 \\ s \neq r,u}}^{M} \left\{ \sum_{i=1}^{N} (p_i \cdot k_s)^{-1} \, \varepsilon_{s,\mu\nu} \, p_i^\mu p_i^\nu \right\},
\end{aligned} \tag{3.65}$$

where $\mathcal{M}(p_j; \varepsilon_r, k_r, \varepsilon_u, k_u)$ has been defined in (3.54). Independently of the general argument given below, we have used Cadabra [115, 116] and Mathematica [117] to check (3.65) explicitly for amplitudes with three soft gravitons.

We begin by reviewing the derivation of the leading term given in the first line of (3.65). For this note that this term may be rearranged as

$$\sum_{\substack{A_1, \cdots A_N;\; A_i \subset \{1, \cdots, M\} \\ A_i \cap A_j = \emptyset \text{ for } i \neq j;\; A_1 \cup A_2 \cup \cdots \cup A_N = \{1, \cdots M\}}} \prod_{i=1}^{N} \left\{ \prod_{r \in A_i} \varepsilon_{r,\mu\nu} \, p_i^\mu p_i^\nu \right\} \left\{ \prod_{r \in A_i} (p_i \cdot k_r)^{-1} \right\} \Gamma(\epsilon_1, p_1, \cdots, \epsilon_N, p_N) \tag{3.66}$$

Physically the i-th term in the product represents the contribution from the soft gravitons in the set A_i attached to the i-th finite energy external line. To see how we get this factor, let us denote the momenta of the soft gravitons attached from the outermost end to the innermost end of the i-th line in a given graph by $\widetilde{k}_1, \cdots \widetilde{k}_n$. The corresponding polarizations are denoted by $\widetilde{\varepsilon}_1, \cdots \widetilde{\varepsilon}_n$. This is shown in Fig. 3.10. The unordered set $\{\widetilde{k}_1, \cdots, \widetilde{k}_n\}$ coincides with the set $\{k_s; s \in A_i\}$. A similar statement holds for the polarizations. The leading contribution from the products of three point vertices and propagators associated with the i-th line of the graph may be computed using (3.36), (3.35) and is given by

$$\left\{\prod_{r=1}^{n}\widetilde{\varepsilon}_{r,\mu\nu}\, p_i^\mu p_i^\nu\right\}\ \ \{p_i\cdot\widetilde{k}_1\}^{-1}\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)\}^{-1}\cdots\{p_i\cdot(\widetilde{k}_1+\cdots+\widetilde{k}_n)\}^{-1}\,. \tag{3.67}$$

The total contribution obtained after summing over all permutations of the momenta $\widetilde{k}_1,\cdots,\widetilde{k}_n$ using (D.1) is given by

$$\begin{aligned}
&\left\{\prod_{r=1}^{n}\widetilde{\varepsilon}_{r,\mu\nu}\, p_i^\mu p_i^\nu\right\}\sum_{\text{permutations of }\widetilde{k}_1,\cdots\widetilde{k}_n}\{p_i\cdot\widetilde{k}_1\}^{-1}\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)\}^{-1}\cdots\{p_i\cdot(\widetilde{k}_1+\cdots+\widetilde{k}_n)\}^{-1}\\
&=\ \left\{\prod_{r=1}^{n}\widetilde{\varepsilon}_{r,\mu\nu}\, p_i^\mu p_i^\nu\right\}\left\{\prod_{s=1}^{n}(p_i\cdot\widetilde{k}_s)^{-1}\right\}=\left\{\prod_{r\in A_i}\varepsilon_{r,\mu\nu}\, p_i^\mu p_i^\nu\right\}\left\{\prod_{s\in A_i}(p_i\cdot k_s)^{-1}\right\}.
\end{aligned}\tag{3.68}$$

This reproduces (3.66).

We now turn to the analysis of the subleading terms. For this let us first analyze the contribution from the products of the propagators and vertices in Fig. 3.10 to subleading order. Using (3.36) this may be expressed as

$$\begin{aligned}
&\{2p_i\cdot\widetilde{k}_1\}^{-1}\{2p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)+2\,\widetilde{k}_1\cdot\widetilde{k}_2\}^{-1}\cdots\left\{2p_i\cdot(\widetilde{k}_1+\cdots+\widetilde{k}_n)+2\sum_{\substack{r,u=1\\ r<u}}^{n}\widetilde{k}_r\cdot\widetilde{k}_u\right\}^{-1}\\
&\epsilon_i^T\left[2\,\widetilde{\varepsilon}_1^{\mu\nu}\ p_{i\mu}p_{i\nu}+i\,\widetilde{\varepsilon}_{1,\mu\nu}p_i^\nu\mathcal{K}(-p_i)\frac{\partial\Xi^i(-p_i)}{\partial p_{i\mu}}+2\,\widetilde{\varepsilon}_{1,b\mu}\widetilde{k}_{1a}p_i^\mu(J^{ab})^T+\mathcal{K}(-p_i)\ \mathcal{Q}(p_i,\widetilde{k}_1)\right]\\
&\Big[2\,\widetilde{\varepsilon}_2^{\mu\nu}\{p_{i\mu}+\widetilde{k}_{1\mu}\}\{p_{i\nu}+\widetilde{k}_{1\nu}\}+i\,\widetilde{\varepsilon}_{2,\mu\nu}(p_i^\nu+\widetilde{k}_1^\nu)\mathcal{K}(-p_i-\widetilde{k}_1)\frac{\partial\Xi^i(-p_i-\widetilde{k}_1)}{\partial p_{i\mu}}\\
&\qquad\qquad+2\,\widetilde{\varepsilon}_{2,b\mu}\widetilde{k}_{2a}p_i^\mu(J^{ab})^T+\mathcal{K}(-p_i)\ \mathcal{Q}(p_i,\widetilde{k}_2)\Big]\\
&\cdots\Big[2\,\widetilde{\varepsilon}_n^{\mu\nu}\{p_{i\mu}+\widetilde{k}_{1\mu}+\cdots+\widetilde{k}_{(n-1)\mu}\}\{p_{i\nu}+\widetilde{k}_{1\nu}+\cdots+\widetilde{k}_{(n-1)\nu}\}\\
&+i\,\widetilde{\varepsilon}_{n,\mu\nu}(p_i^\nu+\widetilde{k}_1^\nu+\cdots+\widetilde{k}_{n-1}^\nu)\,\mathcal{K}(-p_i-\widetilde{k}_1-\widetilde{k}_2-\cdots\widetilde{k}_{n-1})\frac{\partial\Xi^i(-p_i-\widetilde{k}_1-\widetilde{k}_2-\cdots-\widetilde{k}_{n-1})}{\partial p_{i\mu}}\\
&\qquad\qquad+2\,\widetilde{\varepsilon}_{n,b\mu}\widetilde{k}_{na}p_i^\mu(J^{ab})^T+\mathcal{K}(-p_i)\ \mathcal{Q}(p_i,\widetilde{k}_n)\Big]\,\Gamma_{(i)}(p_i+\widetilde{k}_1+\cdots+\widetilde{k}_n)\,.
\end{aligned}\tag{3.69}$$

First let us analyze the contribution from the $\widetilde{k}_r\cdot\widetilde{k}_u$ terms in the denominator. Since this is subleading, we need to expand one of the denominators to first order in $\widetilde{k}_r\cdot\widetilde{k}_u$, set $\widetilde{k}_r\cdot\widetilde{k}_u=0$ in the rest of the denominators, and pick the leading contribution from all other factors. This leads to

$$-\left\{\sum_{m=2}^{n}\sum_{\substack{r,u=1\\ r<u}}^{m}\frac{\widetilde{k}_r\cdot\widetilde{k}_u}{p_i\cdot(\widetilde{k}_1+\cdots+\widetilde{k}_m)}\right\}\left\{\prod_{\ell=1}^{n}\frac{1}{p_i\cdot(\widetilde{k}_1+\cdots+\widetilde{k}_\ell)}\right\}\left\{\prod_{s=1}^{n}\widetilde{\varepsilon}_{s,\mu\nu}p_i^\mu p_i^\nu\right\}\epsilon_i^T\ \Gamma_{(i)}(p_i)\,. \tag{3.70}$$

After performing the sum over all permutations of $\widetilde{k}_1, \cdots, \widetilde{k}_n$ using (D.3) this gives

$$-\prod_{s=1}^{n} \left\{ (p_i \cdot \widetilde{k}_s)^{-1} \, \widetilde{\varepsilon}_{s,\mu\nu} \, p_i^\mu p_i^\nu \right\} \sum_{\substack{r,u=1 \\ r<u}}^{n} \widetilde{k}_r \cdot \widetilde{k}_u \, \{p_i \cdot (\widetilde{k}_r + \widetilde{k}_u)\}^{-1} \, . \tag{3.71}$$

Next we consider the terms involving the contraction of $\widetilde{\varepsilon}_u$ with $\widetilde{k}_r$ for $r < u$, coming from the first term inside each square bracket in (3.69). Since this term is subleading, once we pick one of these factors we must pick the leading terms from all the other factors. Again using (3.35) we can express the sum of all such contributions as

$$2 \sum_{\substack{r,u=1 \\ r<u}}^{n} \left\{ \prod_{\substack{s=1 \\ s\neq u}}^{n} \widetilde{\varepsilon}_{s,\mu\nu} p_i^\mu p_i^\nu \right\} \widetilde{\varepsilon}_{u,\mu\nu} \, p_i^\mu \, \widetilde{k}_r^\nu \left\{ \prod_{m=1}^{n} \frac{1}{p_i \cdot (\widetilde{k}_1 + \cdots + \widetilde{k}_m)} \right\} \epsilon_i^T \, \Gamma_{(i)}(p_i) \, . \tag{3.72}$$

After summing over all permutations of $(\widetilde{k}_1, \widetilde{\varepsilon}_1), \cdots, (\widetilde{k}_n, \widetilde{\varepsilon}_n)$ using (D.2) this gives

$$\begin{aligned} &2 \left\{ \prod_{s=1}^{n} (p_i \cdot \widetilde{k}_s)^{-1} \right\} \sum_{\substack{r,u=1 \\ r<u}}^{n} \{p_i \cdot (\widetilde{k}_r + \widetilde{k}_u)\}^{-1} \left\{ \prod_{\substack{s=1 \\ s\neq r,u}}^{n} \widetilde{\varepsilon}_{s,\mu\nu} p_i^\mu p_i^\nu \right\} \\ &\quad \left\{ (p_i \cdot \widetilde{k}_u) \, (p_i \cdot \widetilde{\varepsilon}_r \cdot p_i) \, (p_i \cdot \widetilde{\varepsilon}_u \cdot \widetilde{k}_r) + (p_i \cdot \widetilde{k}_r) \, (p_i \cdot \widetilde{\varepsilon}_u \cdot p_i) \, (p_i \cdot \widetilde{\varepsilon}_r \cdot \widetilde{k}_u) \right\} \epsilon_i^T \, \Gamma_{(i)}(p_i) \, . \end{aligned} \tag{3.73}$$

We now turn to the rest of the contribution from (3.69) in which we drop the $\widetilde{k}_r \cdot \widetilde{k}_u$ factors in the denominator and also the terms involving contraction of $\widetilde{k}_r$ with $\widetilde{\varepsilon}_u$ in the first term inside each square bracket. Our first task will be to expand the factors of $\mathcal{K}$ and Ξ^i in Taylor series expansion in powers of the soft momenta. It is easy to see however that to the first subleading order, the order $\widetilde{k}^\mu$ terms in the expansion of Ξ^i do not contribute to the amplitude. This is due to the fact that once we have picked a subleading term proportional to $\widetilde{k}_s^\rho \partial^2 \Xi^i / \partial p_i^\mu \partial p_i^\rho$, we must replace the argument of $\mathcal{K}$ by $-p_i$ in the accompanying factor and in all other factors we must pick the leading term. In this case repeated use of (3.35) shows that the corresponding contribution vanishes. Therefore we can replace all factors of $\partial \Xi^i(-p_i - \widetilde{k}_1 - \cdots)/\partial p_i^\mu$ by $\partial \Xi^i(-p_i)/\partial p_i^\mu$. Similar argument shows that all the $\mathcal{K}(-p_i)\mathcal{Q}$ terms, and the terms involving contraction of $\widetilde{\varepsilon}_u$ with $\widetilde{k}_r$ in the second term inside each square bracket in (3.69), give vanishing contribution at the subleading order. This allows us to express the rest of the contribution from (3.69) as

$$
\begin{aligned}
&(2p_i\cdot\widetilde{k}_1)^{-1}\{2p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)\}^{-1}\cdots\{2p_i\cdot(\widetilde{k}_1+\cdots+\widetilde{k}_n)\}^{-1} \\
&\epsilon_i^T\left[2\,\mathcal{E}_1+2\,\mathcal{L}_1+2\,\widetilde{\varepsilon}_{1,b\mu}\widetilde{k}_{1a}p_i^\mu(J^{ab})^T\right] \\
&\left[2\,\mathcal{E}_2+2\,\mathcal{L}_2+i\,\widetilde{\varepsilon}_{2,\mu\nu}p_i^\nu\widetilde{k}_{1\rho}\frac{\partial\mathcal{K}(-p_i)}{\partial p_{i\rho}}\frac{\partial\Xi^i(-p_i)}{\partial p_{i\mu}}+2\,\widetilde{\varepsilon}_{2,b\mu}\widetilde{k}_{2a}p_i^\mu(J^{ab})^T\right]\cdots \\
&\left[2\,\mathcal{E}_n+2\,\mathcal{L}_n+i\,\widetilde{\varepsilon}_{n,\mu\nu}p_i^\nu\,(\widetilde{k}_{1\rho}+\cdots+\widetilde{k}_{n-1,\rho})\frac{\partial\mathcal{K}(-p_i)}{\partial p_{i\rho}}\,\frac{\partial\Xi^i(-p_i)}{\partial p_{i\mu}}+2\,\widetilde{\varepsilon}_{n,b\mu}\widetilde{k}_{na}p_i^\mu(J^{ab})^T\right] \\
&\Gamma_{(i)}(p_i+\widetilde{k}_1+\cdots+\widetilde{k}_n)\,,
\end{aligned}
\tag{3.74}
$$

where,

$$
\mathcal{E}_s=\widetilde{\varepsilon}_s^{\mu\nu}p_{i\mu}p_{i\nu},\qquad \mathcal{L}_s=\frac{i}{2}\widetilde{\varepsilon}_s^{\mu\nu}p_{i\nu}\mathcal{K}(-p_i)\,\frac{\partial\Xi^i(-p_i)}{\partial p_{i\mu}}\,. \tag{3.75}
$$

We now expand (3.74) in powers of soft momenta. Even though $\mathcal{L}_s$ is leading order, its contribution to the amplitude vanishes by (3.35) unless there is some other matrix sitting between ϵ_i^T and $\mathcal{L}_s$. The possible terms come from picking up either the term proportional to $\partial\mathcal{K}/\partial p_{i\rho}\,\partial\Xi/\partial p_{i\mu}$ or $(J^{ab})^T$ from one of the factors. Both these terms are subleading and therefore we can pick at most one such term in the product, with the other factors being given by $\mathcal{E}_s+\mathcal{L}_s$. Therefore if we expand (3.74) and pick the subleading factor from the r-th term in the product, then in the product of $\mathcal{E}_s+\mathcal{L}_s$, we can drop all factors of $\mathcal{L}_s$ for $s<r$ since they sit to the left of the subleading factor and will vanish due to (3.35). This gives the following expression for the subleading contribution to (3.74):

$$
\begin{aligned}
&(p_i\cdot\widetilde{k}_1)^{-1}\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)\}^{-1}\cdots\{p_i\cdot(\widetilde{k}_1+\cdots+\widetilde{k}_n)\}^{-1} \\
&\epsilon_i^T\left[\sum_{r=1}^n\left\{\prod_{s=1}^{r-1}\mathcal{E}_s\right\}\,\left[\frac{i}{2}\,\widetilde{\varepsilon}_{r,\mu\nu}p_i^\nu\,(\widetilde{k}_{1\rho}+\cdots+\widetilde{k}_{r-1,\rho})\frac{\partial\mathcal{K}(-p_i)}{\partial p_{i\rho}}\,\frac{\partial\Xi^i(-p_i)}{\partial p_{i\mu}}+\,\widetilde{\varepsilon}_{r,b\mu}\widetilde{k}_{ra}p_i^\mu(J^{ab})^T\right]\right. \\
&\left.\left\{\prod_{s=r+1}^n(\mathcal{E}_s+\mathcal{L}_s)\right\}\Gamma_{(i)}(p_i)\right. \\
&+(p_i\cdot\widetilde{k}_1)^{-1}\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)\}^{-1}\cdots\{p_i\cdot(\widetilde{k}_1+\cdots+\widetilde{k}_n)\}^{-1}\left\{\prod_{s=1}^n\mathcal{E}_s\right\}\,\sum_{r=1}^n\widetilde{k}_{r\rho}\,\frac{\partial\Gamma_{(i)}(p_i)}{\partial p_{i\rho}}\,.
\end{aligned}
\tag{3.76}
$$

The last term comes from the Taylor series expansion of $\Gamma_{(i)}$ in powers of soft momenta. In the product the $(\mathcal{E}_s+\mathcal{L}_s)$'s are ordered from left to right in the order of increasing s. We now manipulate the product $\prod_{s=r+1}^n(\mathcal{E}_s+\mathcal{L}_s)$ as follows. If the subleading factor is the one proportional to $\partial\mathcal{K}/\partial p_{i\rho}\,\partial\Xi/\partial p_{i\mu}$ then we leave the product of the factors $(\mathcal{E}_s+\mathcal{L}_s)$ for $s>r$ unchanged. However if the subleading factor is the one proportional to $(J^{ab})^T$, then we expand the product of the factors

$(\mathcal{E}_s + \mathcal{L}_s)$ for $s > r$ as

$$(\mathcal{E}_{r+1} + \mathcal{L}_{r+1}) \cdots (\mathcal{E}_n + \mathcal{L}_n) = \mathcal{E}_{r+1} \cdots \mathcal{E}_n + \sum_{u=r+1}^{n} \mathcal{E}_{r+1} \cdots \mathcal{E}_{u-1} \, \mathcal{L}_u \, (\mathcal{E}_{u+1} + \mathcal{L}_{u+1}) \cdots (\mathcal{E}_n + \mathcal{L}_n) \tag{3.77}$$

Using this, and combining the contribution from the first term on the right hand side of (3.77) with the last term in (3.76), we can express (3.76) as

$$\begin{aligned}
&(p_i \cdot \widetilde{k}_1)^{-1} \{p_i \cdot (\widetilde{k}_1 + \widetilde{k}_2)\}^{-1} \cdots \{p_i \cdot (\widetilde{k}_1 + \cdots + \widetilde{k}_n)\}^{-1} \\
&\epsilon_i^T \left[\sum_{r=1}^{n} \Bigg\{ \prod_{\substack{s=1 \\ s \neq r}}^{n} \mathcal{E}_s \Bigg\} \Bigg\{ \widetilde{\varepsilon}_{r,b\mu} \widetilde{k}_{ra} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \widetilde{\varepsilon}_r^{\mu\nu} p_{i\mu} p_{i\nu} \, \widetilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}}{\partial p_{i\rho}} \Bigg\} \right. \\
&+ \frac{i}{2} \sum_{r=1}^{n} \Bigg\{ \prod_{s=1}^{r-1} \mathcal{E}_s \Bigg\} \widetilde{\varepsilon}_{r,\mu\nu} p_i^\nu \, (\widetilde{k}_{1\rho} + \cdots + \widetilde{k}_{r-1,\rho}) \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\rho}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} \Bigg\{ \prod_{s=r+1}^{n} (\mathcal{E}_s + \mathcal{L}_s) \Bigg\} \Gamma_{(i)} \\
&\left. + \frac{i}{2} \sum_{r=1}^{n} \sum_{u=r+1}^{n} \Bigg\{ \prod_{\substack{s=1 \\ s \neq r}}^{u-1} \mathcal{E}_s \Bigg\} \widetilde{\varepsilon}_{r,b\mu} \widetilde{k}_{ra} p_i^\mu (J^{ab})^T \, \widetilde{\varepsilon}_{u,\rho\sigma} p_i^\sigma \mathcal{K}(-p_i) \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \Bigg\{ \prod_{s=u+1}^{n} (\mathcal{E}_s + \mathcal{L}_s) \Bigg\} \Gamma_{(i)} \right] .
\end{aligned} \tag{3.78}$$

We now use (3.31) to move the $\mathcal{K}(-p_i)$ factor in the last term to the left of $(J^{ab})^T$ and use (3.35). This allows us to express (3.78) as

$$\begin{aligned}
&(p_i \cdot \widetilde{k}_1)^{-1} \{p_i \cdot (\widetilde{k}_1 + \widetilde{k}_2)\}^{-1} \cdots \{p_i \cdot (\widetilde{k}_1 + \cdots + \widetilde{k}_n)\}^{-1} \\
&\epsilon_i^T \left[\sum_{r=1}^{n} \Bigg\{ \prod_{\substack{s=1 \\ s \neq r}}^{n} \mathcal{E}_s \Bigg\} \Bigg\{ \widetilde{\varepsilon}_{r,b\mu} \widetilde{k}_{ra} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \widetilde{\varepsilon}_r^{\mu\nu} p_{i\mu} p_{i\nu} \widetilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}}{\partial p_{i\rho}} \Bigg\} \right. \\
&+ \frac{i}{2} \sum_{r=1}^{n} \Bigg\{ \prod_{s=1}^{r-1} \mathcal{E}_s \Bigg\} \widetilde{\varepsilon}_{r,\mu\nu} p_i^\nu \, (\widetilde{k}_{1\rho} + \cdots + \widetilde{k}_{r-1,\rho}) \frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\rho}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\mu}} \Bigg\{ \prod_{s=r+1}^{n} (\mathcal{E}_s + \mathcal{L}_s) \Bigg\} \Gamma_{(i)} \\
&+ \frac{i}{2} \sum_{\substack{r,u=1 \\ r<u}}^{n} \Bigg\{ \prod_{\substack{s=1 \\ s \neq r}}^{u-1} \mathcal{E}_s \Bigg\} \widetilde{\varepsilon}_{r,b\mu} \widetilde{k}_{ra} p_i^\mu \, \widetilde{\varepsilon}_{u,\rho\sigma} p_i^\sigma \left(p_i^a \frac{\partial \mathcal{K}(-p_i)}{\partial p_{ib}} - p_i^b \frac{\partial \mathcal{K}(-p_i)}{\partial p_{ia}} \right) \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \\
&\qquad\qquad \left. \Bigg\{ \prod_{s=u+1}^{n} (\mathcal{E}_s + \mathcal{L}_s) \Bigg\} \Gamma_{(i)} \right] .
\end{aligned} \tag{3.79}$$

It is easy to see that terms proportional to $p_i^b \partial\mathcal{K}/\partial p_{ia}$ in the fourth line of (3.79) cancels the terms in the third line of (3.79). Therefore we are left with

$$
\begin{aligned}
&(p_i \cdot \widetilde{k}_1)^{-1}\{p_i \cdot (\widetilde{k}_1 + \widetilde{k}_2)\}^{-1} \cdots \{p_i \cdot (\widetilde{k}_1 + \cdots + \widetilde{k}_n)\}^{-1} \\
&\epsilon_i^T \left[\sum_{r=1}^{n} \left\{ \prod_{\substack{s=1 \\ s\neq r}}^{n} \mathcal{E}_s \right\} \left\{ \widetilde{\varepsilon}_{r,b\mu} \widetilde{k}_{ra} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \widetilde{\varepsilon}_r^{\mu\nu} p_{i\mu} p_{i\nu} \widetilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}}{\partial p_{i\rho}} \right\} \right. \\
&\left. + \frac{i}{2} \sum_{\substack{r,u=1 \\ r<u}}^{n} \left\{ \prod_{\substack{s=1 \\ s\neq r}}^{u-1} \mathcal{E}_s \right\} p_i \cdot \widetilde{k}_r \, \widetilde{\varepsilon}_{r,b\mu} \, p_i^\mu \, \widetilde{\varepsilon}_{u,\rho\sigma} \, p_i^\sigma \, \frac{\partial \mathcal{K}(-p_i)}{\partial p_{ib}} \frac{\partial \Xi^i(-p_i)}{\partial p_{i\rho}} \left\{ \prod_{s=u+1}^{n} (\mathcal{E}_s + \mathcal{L}_s) \right\} \Gamma_{(i)} \right].
\end{aligned}
\tag{3.80}
$$

First consider the term in the second line of (3.80). We sum over all permutations of $(\widetilde{\varepsilon}_1, \widetilde{k}_1), \cdots, (\widetilde{\varepsilon}_n, \widetilde{k}_n)$. After the sum over r is performed, this expression is already invariant under the permutations of the soft gravitons inserted on the i-th line. Therefore we simply have to sum the expression in the first line over all permutations using (D.1), producing the result:

$$
\left\{ \prod_{s=1}^{n} (p_i \cdot \widetilde{k}_s)^{-1} \right\} \epsilon_i^T \left[\sum_{r=1}^{n} \prod_{\substack{s=1 \\ s\neq r}}^{n} \{ \widetilde{\varepsilon}_s^{\mu\nu} p_{i\mu} p_{i\nu} \} \left\{ \widetilde{\varepsilon}_{r,b\mu} \widetilde{k}_{ra} p_i^\mu (J^{ab})^T \Gamma_{(i)}(p_i) + \widetilde{\varepsilon}_r^{\mu\nu} p_{i\mu} p_{i\nu} \widetilde{k}_{r\rho} \frac{\partial \Gamma_{(i)}}{\partial p_{i\rho}} \right\} \right. . \tag{3.81}
$$

Since this is already subleading, we have to pick the leading contribution from all the other external legs, producing factors of $\prod_{s\in A_j} \left\{ (p_j \cdot k_s)^{-1} \varepsilon_{s,\mu\nu} p_j^\mu p_j^\nu \right\}$ after summing over permutations of the soft gravitons. Finally we sum over all ways of distributing the soft gravitons on the external lines. The net contribution from these terms is given by

$$
\sum_{r=1}^{M} \sum_{i=1}^{N} (p_i \cdot k_r)^{-1} \, \varepsilon_{r,b\rho} \, k_{ra} \, p_i^\rho \, \epsilon_i^T \, \left[p_i^b \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ia}} + (J^{ab})^T \Gamma_{(i)}(p_i) \right] \prod_{\substack{s=1 \\ s\neq r}}^{M} \left\{ \sum_{j=1}^{N} (p_j \cdot k_s)^{-1} \, \varepsilon_{s,\mu\nu} \, p_j^\mu p_j^\nu \right\} \tag{3.82}
$$

We now combine this with the contribution from the sum of graphs where one soft graviton attaches to the amplitude via the vertex $\widetilde{\Gamma}$ shown in Fig. 3.8 and the other soft gravitons attach to the external lines. Using (3.24) we get the contribution from these graphs to be

$$
- \sum_{r=1}^{M} \left[\prod_{\substack{s=1 \\ s\neq r}}^{M} \left\{ \sum_{j=1}^{N} (p_j \cdot k_s)^{-1} \, \varepsilon_{s,\mu\nu} \, p_j^\mu p_j^\nu \right\} \right] \sum_{i=1}^{N} \varepsilon_{r,ab} \, p_i^a \, \epsilon_i^T \, \frac{\partial \Gamma_{(i)}(p_i)}{\partial p_{ib}} \,. \tag{3.83}
$$

The sum of (3.82) and (3.83) reproduces the terms in the second and third line of (3.65).

Let us now turn to the contribution from the last line of (3.80). We express $p_i \cdot \widetilde{k}_r$ factor as

$$
p_i \cdot (\widetilde{k}_1 + \cdots + \widetilde{k}_r) - p_i \cdot (\widetilde{k}_1 + \cdots \widetilde{k}_{r-1}) \tag{3.84}
$$

so that each term in (3.84) cancels one of the denominator factors in the first line of (3.80). Now we are supposed to sum over all permutations of the soft gravitons carrying the labels $1, \cdots n$. However instead of summing over all permutations of $\widetilde{k}_1, \cdots, \widetilde{k}_n$ in one step, let us first fix the positions of all soft gravitons except the one carrying momentum $\widetilde{k}_r$, and sum over all insertions of the soft graviton carrying momentum $\widetilde{k}_r$ to the left of the one carrying momentum $\widetilde{k}_u$. Using (3.84) at each step, it is easy to see that the contributions from the terms cancel pairwise. For example for three soft gravitons, with 1 fixed to the left of 3, and the position of 2 summed over on all positions to the left of 3, we have

$$\begin{aligned} & \{p_i\cdot\widetilde{k}_1\}^{-1}\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)\}^{-1}\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2+\widetilde{k}_3)\}^{-1}\,\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)-p_i\cdot\widetilde{k}_1\} \\ & +\{p_i\cdot\widetilde{k}_2\}^{-1}\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2)\}^{-1}\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2+\widetilde{k}_3)\}^{-1}\{p_i\cdot\widetilde{k}_2\} \\ = \; & \{p_i\cdot\widetilde{k}_1\}^{-1}\,\{p_i\cdot(\widetilde{k}_1+\widetilde{k}_2+\widetilde{k}_3)\}^{-1}\,. \end{aligned} \tag{3.85}$$

As a result of this pairwise cancellation, at the end we are left with only one term arising from the insertion of $\widetilde{k}_r$ just to the left of $\widetilde{k}_u$. In order to express the result in a convenient form we relabel the gravitons attached to the i-th line from left to right, other than the one carrying momentum $\widetilde{k}_r$, as

$$(\widehat{\varepsilon}_1,\widehat{k}_1),\cdots,(\widehat{\varepsilon}_{u-2},\widehat{k}_{u-2}),(\widetilde{\varepsilon}_u,\widetilde{k}_u),(\widehat{\varepsilon}_{u+1},\widehat{k}_{u+1}),\cdots,(\widehat{\varepsilon}_n,\widehat{k}_n)\,. \tag{3.86}$$

and sum over all insertions of the graviton carrying the quantum numbers $(\widetilde{\varepsilon}_r,\widetilde{k}_r)$ to the left of $(\widetilde{\varepsilon}_u,\widetilde{k}_u)$. Then for fixed r, s the result is given by

$$\begin{aligned} & (p_i\cdot\widehat{k}_1)^{-1}\{p_i\cdot(\widehat{k}_1+\widehat{k}_2)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2})\}^{-1} \\ & \{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u+\widehat{k}_{u+1}+\cdots+\widehat{k}_n)\}^{-1} \\ & \epsilon_i^T\left[\frac{i}{2}\left\{\prod_{s=1}^{u-2}\widehat{\mathcal{E}}_s\right\}\widetilde{\varepsilon}_{r,b\mu}\,p_i^\mu\,\widetilde{\varepsilon}_{u,\rho\sigma}\,p_i^\sigma\,\frac{\partial\mathcal{K}(-p_i)}{\partial p_{ib}}\frac{\partial\Xi^i(-p_i)}{\partial p_{i\rho}}\left\{\prod_{s=u+1}^{n}(\widehat{\mathcal{E}}_s+\widehat{\mathcal{L}}_s)\right\}\right]\Gamma_{(i)}(p_i)\,, \end{aligned} \tag{3.87}$$

where

$$\widehat{\mathcal{E}}_s=\widehat{\varepsilon}_s^{\mu\nu}p_{i\mu}p_{i\nu},\qquad \widehat{\mathcal{L}}_s=\frac{i}{2}\widehat{\varepsilon}_s^{\mu\nu}p_{i\nu}\mathcal{K}(-p_i)\,\frac{\partial\Xi^i(-p_i)}{\partial p_{i\mu}}\,. \tag{3.88}$$

Next we add to this a term obtained by exchanging the positions of r and u. This is equivalent to exchanging the ρ and b indices in $(\partial\mathcal{K}/\partial p_{ib})(\partial\Xi/\partial p_{i\rho})$ and gives

$$\begin{aligned} & (p_i\cdot\widehat{k}_1)^{-1}\{p_i\cdot(\widehat{k}_1+\widehat{k}_2)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2})\}^{-1} \\ & \{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u+\widehat{k}_{u+1}+\cdots+\widehat{k}_n)\}^{-1} \\ & \epsilon_i^T\left[\frac{i}{2}\left\{\prod_{s=1}^{u-2}\widehat{\mathcal{E}}_s\right\}\widetilde{\varepsilon}_{r,b\mu}\,p_i^\mu\,\widetilde{\varepsilon}_{u,\rho\sigma}\,p_i^\sigma\,\frac{\partial\mathcal{K}(-p_i)}{\partial p_{i\rho}}\frac{\partial\Xi^i(-p_i)}{\partial p_{ib}}\left\{\prod_{s=u+1}^{n}(\widehat{\mathcal{E}}_s+\widehat{\mathcal{L}}_s)\right\}\right]\Gamma_{(i)}(p_i)\,. \end{aligned} \tag{3.89}$$

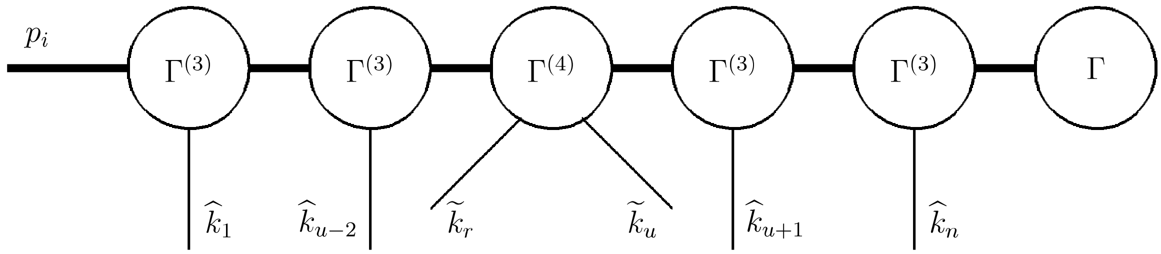

Figure 3.11: A subleading contribution to the amplitude with multiple soft gravitons.

Summing over the remaining permutations corresponds to treating the r-th and u-th graviton as one unit sitting together, and summing over all permutations of the $n-1$ objects generated this way. However it will be more convenient for us to first add to this the contribution from the diagrams shown in in Fig. 3.11 and Fig. 3.12. Since these diagrams are subleading, we need to pick the leading contribution from all the vertices and propagators. For the product of any of the $\Gamma^{(3)}$ vertices and the propagator to the right of this vertex, we can use (3.36) to generate factors of $(\widehat{\mathcal{E}}_s + \widehat{\mathcal{L}}_s)$ in the numerator. Using (3.35) we can argue that in all such factors to the left of the vertex, where momentum $\widetilde{k}_r + \widetilde{k}_u$ enters the finite energy line, we can drop the $\widehat{\mathcal{L}}_s$ factors. Following analysis similar to the ones leading to (3.49) and (3.52), we arrive at the following results for Figs.3.11 and 3.12 respectively,

$$
\begin{aligned}
&(p_i\cdot \widehat{k}_1)^{-1}\{p_i\cdot(\widehat{k}_1+\widehat{k}_2)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2})\}^{-1} \\
&\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u+\widehat{k}_{u+1}+\cdots+\widehat{k}_n)\}^{-1} \\
&\epsilon_i^T\left[\left\{\prod_{s=1}^{u-2}\widehat{\mathcal{E}}_s\right\}\left\{-2\varepsilon_{r\mu}{}^{\nu}\varepsilon_{u,\nu\rho}p_i^\rho p_i^\mu-\frac{i}{2}\Big(\varepsilon_{r,\mu\sigma}\varepsilon_{u,\rho\nu}p_i^\sigma p_i^\nu+\varepsilon_{r,\rho\sigma}\varepsilon_{u,\mu\nu}p_i^\sigma p_i^\nu\Big)\frac{\partial\mathcal{K}(-p_i)}{\partial p_{i\mu}}\frac{\partial\Xi^i(-p_i)}{\partial p_{i\rho}}\right\}\right. \\
&\qquad\qquad\left.\left\{\prod_{s=u+1}^{n}(\widehat{\mathcal{E}}_s+\widehat{\mathcal{L}}_s)\right\}\right]\Gamma_{(i)}(p_i)\,, \qquad (3.90)
\end{aligned}
$$

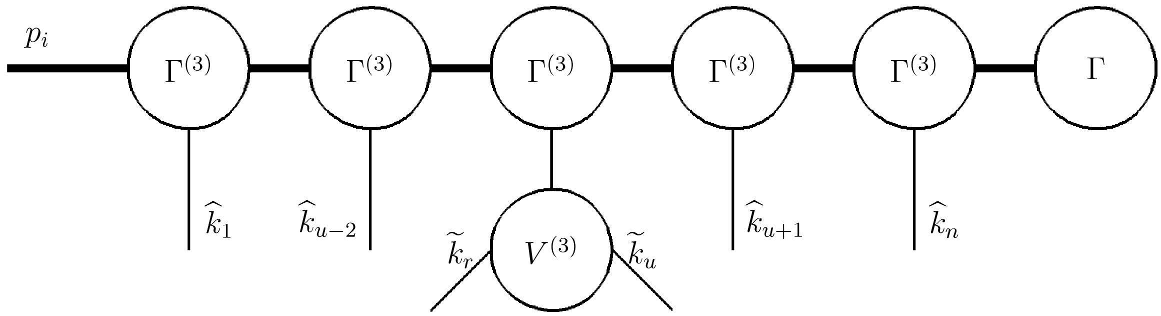

Figure 3.12: Another subleading contribution to the amplitude with multiple soft gravitons.

and[11]

$$\begin{aligned}
&(\widetilde{k}_r\cdot\widetilde{k}_u)^{-1}\,(p_i\cdot\widehat{k}_1)^{-1}\{p_i\cdot(\widehat{k}_1+\widehat{k}_2)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2})\}^{-1}\\
&\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u+\widehat{k}_{u+1}+\cdots+\widehat{k}_n)\}^{-1}\\
&\epsilon_i^T\left[\left\{\prod_{s=1}^{u-2}\widehat{\mathcal{E}}_s\right\}\Big\{-(\widetilde{k}_u\cdot\widetilde{\varepsilon}_r\cdot\widetilde{\varepsilon}_u\cdot p_i)(\widetilde{k}_u\cdot p_i)-(\widetilde{k}_r\cdot\widetilde{\varepsilon}_u\cdot\widetilde{\varepsilon}_r\cdot p_i)(\widetilde{k}_r\cdot p_i)\right.\\
&+(\widetilde{k}_u\cdot\widetilde{\varepsilon}_r\cdot\widetilde{\varepsilon}_u\cdot p_i)(\widetilde{k}_r\cdot p_i)+(\widetilde{k}_r\cdot\widetilde{\varepsilon}_u\cdot\widetilde{\varepsilon}_r\cdot p_i)(\widetilde{k}_u\cdot p_i)-\widetilde{\varepsilon}_r^{cd}\varepsilon_{u,cd}(\widetilde{k}_r\cdot p_i)(\widetilde{k}_u\cdot p_i)\\
&-2(p_i\cdot\widetilde{\varepsilon}_r\cdot\widetilde{k}_u)(p_i\cdot\widetilde{\varepsilon}_u\cdot\widetilde{k}_r)+(p_i\cdot\widetilde{\varepsilon}_u\cdot p_i)(\widetilde{k}_u\cdot\widetilde{\varepsilon}_r\cdot\widetilde{k}_u)+(p_i\cdot\widetilde{\varepsilon}_r\cdot p_i)(\widetilde{k}_r\cdot\widetilde{\varepsilon}_u\cdot\widetilde{k}_r)\Big\}\\
&\qquad\left.\left\{\prod_{s=u+1}^{n}(\widehat{\mathcal{E}}_s+\widehat{\mathcal{L}}_s)\right\}\right]\Gamma_{(i)}(p_i)\,. \qquad (3.91)
\end{aligned}$$

After adding these to (3.87), (3.89) the terms involving derivatives of $\mathcal{K}$ and Ξ get canceled. Once these terms cancel, we can drop the terms proportional to $\widehat{\mathcal{L}}_s$. The result takes the form

$$\begin{aligned}
&(p_i\cdot\widehat{k}_1)^{-1}\{p_i\cdot(\widehat{k}_1+\widehat{k}_2)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2})\}^{-1}\\
&\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u)\}^{-1}\cdots\{p_i\cdot(\widehat{k}_1+\cdots+\widehat{k}_{u-2}+\widetilde{k}_r+\widetilde{k}_u+\widehat{k}_{u+1}+\cdots+\widehat{k}_n)\}^{-1}\\
&(\widetilde{k}_r\cdot\widetilde{k}_u)^{-1}\,\epsilon_i^T\left[\left\{\prod_{s=1}^{u-2}\widehat{\mathcal{E}}_s\right\}\left\{\prod_{s=u+1}^{n}\widehat{\mathcal{E}}_s\right\}\Big\{-2\,\widetilde{k}_r\cdot\widetilde{k}_u\;\varepsilon_{r\mu}{}^{\nu}\varepsilon_{u,\nu\rho}p_i^\rho p_i^\mu-(\widetilde{k}_u\cdot\widetilde{\varepsilon}_r\cdot\widetilde{\varepsilon}_u\cdot p_i)(\widetilde{k}_u\cdot p_i)\right.\\
&-(\widetilde{k}_r\cdot\widetilde{\varepsilon}_u\cdot\widetilde{\varepsilon}_r\cdot p_i)(\widetilde{k}_r\cdot p_i)+(\widetilde{k}_u\cdot\widetilde{\varepsilon}_r\cdot\widetilde{\varepsilon}_u\cdot p_i)(\widetilde{k}_r\cdot p_i)+(\widetilde{k}_r\cdot\widetilde{\varepsilon}_u\cdot\widetilde{\varepsilon}_r\cdot p_i)(\widetilde{k}_u\cdot p_i)\\
&-\widetilde{\varepsilon}_r^{cd}\widetilde{\varepsilon}_{u,cd}(\widetilde{k}_r\cdot p_i)(\widetilde{k}_u\cdot p_i)-2(p_i\cdot\widetilde{\varepsilon}_r\cdot\widetilde{k}_u)(p_i\cdot\widetilde{\varepsilon}_u\cdot\widetilde{k}_r)+(p_i\cdot\widetilde{\varepsilon}_u\cdot p_i)(\widetilde{k}_u\cdot\widetilde{\varepsilon}_r\cdot\widetilde{k}_u)\\
&\left.+(p_i\cdot\widetilde{\varepsilon}_r\cdot p_i)(\widetilde{k}_r\cdot\widetilde{\varepsilon}_u\cdot\widetilde{k}_r)\Big\}\right]\Gamma_{(i)}(p_i)\,. \qquad (3.92)
\end{aligned}$$

We can now sum over all permutations of the soft gravitons carrying momenta $\widehat{k}_1,\cdots,\widehat{k}_{u-2}$, $\widehat{k}_{u+1},\cdots,\widehat{k}_n$ and the relative position of the unit carrying momentum $\widetilde{k}_r+\widetilde{k}_u$ among these. The only factors that differ for different permutations are the factors in the first two lines of (3.92). Sum over

[11] We could have dropped the $\widehat{\mathcal{L}}_s$ factors from (3.91) using (3.35), but will postpone this till the next step.

permutations using (D.1) converts these to

$$\left\{\prod_{s=1}^{u-1}(p_i\cdot\widehat{k}_s)^{-1}\right\}\left\{\prod_{s=u+1}^{n}(p_i\cdot\widehat{k}_s)^{-1}\right\}\{p_i\cdot(\widetilde{k}_r+\widetilde{k}_u)\}^{-1}=\{p_i\cdot(\widetilde{k}_r+\widetilde{k}_u)\}^{-1}\left\{\prod_{\substack{s=1\\ s\neq r,u}}^{n}(p_i\cdot\widetilde{k}_s)^{-1}\right\} \tag{3.93}$$

where we have used the fact that the unordered set $\{\widetilde{k}_r,\widetilde{k}_u,\widehat{k}_1,\cdots,\widehat{k}_{u-2},\widehat{k}_{u+1},\cdots\widehat{k}_n\}$ corresponds to the set $\{\widetilde{k}_1,\cdots,\widetilde{k}_n\}$. Using a similar relation for the polarizations we can express the product of $\widehat{\mathcal{E}}_s$ factors in (3.92) as $\prod_{s\neq r,u}\mathcal{E}_s$. We now sum over all possible choices of r,u from the set $\{1,\cdots,n\}$, and add to this the contribution (3.71), (3.73). This gives

$$\begin{aligned}&\sum_{\substack{r,u=1\\ r<u}}^{n}\left\{\prod_{\substack{s=1\\ s\neq r,u}}^{n}(\widetilde{\varepsilon}_s^{\mu\nu}p_{i\mu}p_{i\nu})\right\}\left\{\prod_{\substack{s=1\\ s\neq r,u}}^{n}(p_i\cdot\widetilde{k}_s)^{-1}\right\}\{p_i\cdot(\widetilde{k}_r+\widetilde{k}_u)\}^{-1}\mathcal{M}(p_i;\widetilde{\varepsilon}_r,\widetilde{k}_r,\widetilde{\varepsilon}_u,\widetilde{k}_u)\,\epsilon_i^T\,\Gamma_{(i)}(p_i)\\ &=\sum_{\substack{r,u\in A_i\\ r<u}}\left\{\prod_{\substack{s\in A_i\\ s\neq r,u}}(\varepsilon_s^{\mu\nu}p_{i\mu}p_{i\nu})\right\}\left\{\prod_{\substack{s\in A_i\\ s\neq r,u}}(p_i\cdot k_s)^{-1}\right\}\{p_i\cdot(k_r+k_u)\}^{-1}\mathcal{M}(p_i;\varepsilon_r,k_r,\varepsilon_u,k_u)\,\epsilon_i^T\,\Gamma_{(i)}(p_i)\,,\end{aligned} \tag{3.94}$$

where we have used the fact that the set $\{\widetilde{k}_1,\cdots,\widetilde{k}_n\}$ corresponds to the set $\{k_a;a\in A_i\}$, and that a similar relation exists also for the polarization tensors.

Summing over all insertions of all other soft gravitons on other legs we now get the result

$$\begin{aligned}\sum_{\substack{A_1,\cdots A_N;\ A_i\subset\{1,\cdots,M\}\\ A_i\cap A_j=\emptyset\text{ for }i\neq j;\ A_1\cup A_2\cup\cdots\cup A_N=\{1,\cdots M\}}}\ &\sum_{i=1}^{N}\left[\prod_{\substack{j=1\\ j\neq i}}^{N}\prod_{q\in A_j}\left\{(p_j\cdot k_q)^{-1}\varepsilon_{q,\mu\nu}\,p_j^\mu p_j^\nu\right\}\right]\\ &\sum_{\substack{r,u\in A_i\\ r<u}}\Bigg[\{p_i\cdot(k_r+k_u)\}^{-1}\prod_{\substack{s\in A_i\\ s\neq r,u}}\left\{(p_i\cdot k_s)^{-1}\,\varepsilon_{s,\mu\nu}\,p_i^\mu p_i^\nu\right\}\\ &\mathcal{M}(p_i;\varepsilon_r,k_r,\varepsilon_u,k_u)\,\Gamma(\epsilon_1,p_1,\cdots,\epsilon_N,p_N)\Bigg]\,.\end{aligned} \tag{3.95}$$

After rearrangement of the sums and products, this reproduces the terms on the last line of (3.65). This completes our proof that amplitudes with multiple soft gravitons are given by (3.65).

3.3.1 Consistency check

We now briefly discuss the gauge invariance of the general multiple subleading soft graviton result (3.65). For this it will be useful to use the compact notation for the amplitude A as given in eq.(3.2). Let us suppose that we transform ε_p by the gauge transformation δ_p defined in (3.58). Then the non-vanishing contribution to $\delta_p\, A$ is given by

$$\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}\right\}\sum_{\substack{s=1\\ s\neq p}}^{M}\left\{\prod_{\substack{r=1\\ r\neq s,p}}^{M} S_r^{(0)}\right\}\delta_p S_p^{(0)}\;[S_s^{(1)}\Gamma]^{\alpha_1\cdots\alpha_p}$$
$$+\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}\right\}\sum_{\substack{r,u=1\\ r<u}}^{M}\left\{\prod_{\substack{s=1\\ s\neq r,u}}^{M} S_s^{(0)}\right\}\left\{\sum_{j=1}^{N}\{p_j\cdot(k_r+k_u)\}^{-1}\;\delta_p\,\mathcal{M}(p_j;\varepsilon_r,k_r,\varepsilon_u,k_u)\right\}\Gamma^{\alpha_1\cdots\alpha_N}\,. \tag{3.96}$$

The first line of (3.96) can be evaluated using (3.59), and yields the result

$$-2\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}\right\}\sum_{\substack{s=1\\ s\neq p}}^{M}\left\{\prod_{\substack{r=1\\ r\neq s,p}}^{M} S_r^{(0)}\right\}S_s^{(0)}\;k_s\cdot\xi_p\;\Gamma^{\alpha_1\cdots\alpha_N}\,. \tag{3.97}$$

The second line of (3.96) receives contribution from the choices $r=p$ or $u=p$. Since $\mathcal{M}(p_i;\varepsilon_r,k_r,\varepsilon_u,k_u)$ is symmetric under the exchange of r and u, we can take $u=p$ and replace the $r<u$ constraint in the sum by $r\neq p$. Therefore the second line of (3.96) takes the form

$$\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}\right\}\sum_{\substack{r=1\\ r\neq p}}^{M}\left\{\prod_{\substack{s=1\\ s\neq r,p}}^{M} S_s^{(0)}\right\}\left\{\sum_{j=1}^{N}\{p_j\cdot(k_r+k_p)\}^{-1}\;\delta_p\,\mathcal{M}(p_j;\varepsilon_r,k_r,\varepsilon_p,k_p)\right\}\Gamma^{\alpha_1\cdots\alpha_N}\,. \tag{3.98}$$

Using (3.60) we can now express this as

$$2\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}\right\}\sum_{\substack{r=1\\ r\neq p}}^{M}\left\{\prod_{\substack{s=1\\ s\neq r,p}}^{M} S_s^{(0)}\right\}S_r^{(0)}\;k_r\cdot\xi_p\;\;\Gamma^{\alpha_1\cdots\alpha_N}\,. \tag{3.99}$$

This precisely cancels (3.97), establishing gauge invariance of the amplitude.

3.4 Infrared issues

In our analysis we have assumed that possible soft factors in the denominator arise from propagators but not from the 1PI vertices. This holds when the number of non-compact space-time dimensions D is sufficiently high. However we shall now show that for $D\leq 5$, individual contributions violate this condition due to infrared effects in the loop. Let us consider for example the diagram shown in Fig. 3.13. In the 1PI effective field theory, this corresponds to a graph similar to one shown in Fig. 3.3, but with both soft gravitons connected to the vertex $\widetilde{\Gamma}$. If there is no inverse power of soft momenta from $\widetilde{\Gamma}$ then this contribution is subsubleading and can be ignored. However let us consider the limit in which the loop momentum ℓ in Fig. 3.13 becomes soft – of the same order as the external soft momenta. In this limit each of the propagators carrying momenta $p_i+\ell$, $p_i+\ell+k_2$, $p_j-\ell$

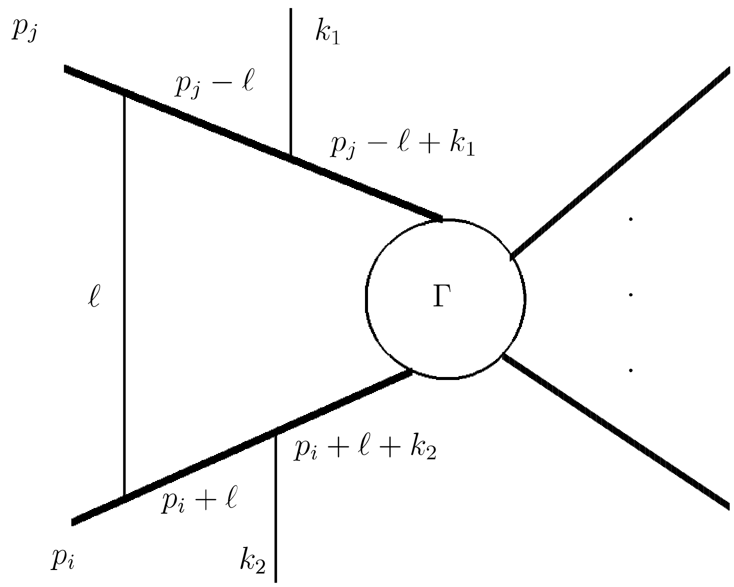

Figure 3.13: A possible subleading contribution in five non-compact dimensions.

and $p_j - \ell + k_1$ gives one power of soft momentum in the denominator and the soft propagator carrying momentum ℓ gives two powers of soft momentum in the denominator. On the other hand in D non-compact space-time dimensions the loop momentum integration measure goes as D powers of soft momentum. Therefore the net power of soft momentum that we get from this graph for soft ℓ is $D - 6$, and in $D = 5$ this integral can give a term containing one power of soft momentum in the denominator, giving a subleading contribution. Since we have not included these diagrams in our analysis we conclude that for loop amplitudes our result is valid for $D \geq 6$. It is easy to see by simple power counting that higher loop amplitudes do not lead to any additional enhancement from the infrared region of loop momenta.

Similar analysis can be carried out for multiple soft graviton amplitudes described in section 3.3. As we connect each external soft graviton to an internal nearly on-shell line carrying finite energy, the number of powers of soft momentum in the denominator goes up by one. However the required number of powers of soft momentum in the denominator of the subleading contribution also goes up by one. Therefore the result of section 3.3 continues to be valid for loop amplitudes for $D \geq 6$, irrespective of the number of external soft gravitons.

Even though this analysis shows that individual diagrams can give contributions beyond what we have included in our analysis for $D \leq 5$, we expect that for $D = 5$ such contributions will cancel when we sum over all diagrams. This expectation arises out of standard results on factorization of soft loops [118, 119] that tells us that after summing over graphs, the contribution from the region of soft loop momentum takes the form of a product of an amplitude without soft loop and a soft factor that arises from graphs like Fig. 3.13 without the external soft lines. Since the graphs like Fig. 3.13 without soft external lines do not receive large contribution from the small ℓ region, and are furthermore independent of the external soft momenta, their contribution may be absorbed into the definition of the amplitude without the soft gravitons. Therefore we conclude that the contribution from the loop momentum integration region for small ℓ in graphs like Fig. 3.13 must cancel in the sum over graphs. Nevertheless since our general analysis relies on the analysis of individual contributions of different graphs of the type shown in Fig. 3.1-3.5, and since the coefficients of Taylor series expansion of these individual contributions as well as those not included in Fig. 3.1-3.5 (like Fig. 3.13) do receive large contribution from small loop momentum region, we cannot give a fool-proof argument that our general result is not affected by infrared contributions of the type described above.

Note that similar infrared enhancement also occurs for amplitudes with single soft graviton, but by analyzing the tensor structure of these contributions it was argued in [21] that gauge invariance prevents corrections to the soft theorem from such effects to subsubleading order for $D \geq 5$. Similar argument has not been developed for multiple soft graviton amplitudes.

This problem of course does not arise for tree amplitudes where the vertices are always polynomial in momenta. Therefore for tree amplitudes our results hold in all dimensions.

Chapter 4

Soft Graviton Theorem Using CHY Prescription

In the last chapter, we derived the subleading multiple soft graviton theorem using the Feynman diagram technique for an arbitrary theory which has diffeomorphism invariance. In this chapter, we continue with this aspect of the scattering amplitudes. More specifically, our goal in this chapter will be two fold. First of all, we want to test our general result (3.2) for some specific theory. However, we shall not be using the Feynman diagram technique applied to a particular theory for testing this result. We shall be doing this in a more non trivial setting by using the Cachazo-He-Yuan (CHY) prescription for the tree amplitude calculation [24–27, 29, 113, 120]. In the CHY proposal, instead of drawing Feynman diagrams and evaluating them using the Feynman rules, an n-point tree amplitude is given by doing a specific sum over a specific set of discrete points in the moduli space of an n-punctured Riemann sphere $\mathcal{M}_{0,n}$. We shall be using this proposal and apply to Einstein's gravity at tree level in arbitrary dimensions. Along with testing the subleading soft graviton result, as our second goal, it will also provide a non trivial test of the CHY proposal itself.

We start by recalling the multiple soft graviton theorem (3.2) proved in the last chapter and express it in a form which is more suitable for our purposes. Let us suppose that we have n external finite energy particles carrying polarizations and momenta (ε_a, p_a) for $a = 1, \cdots, n$, and m soft gravitons carrying polarizations and momenta $(\varepsilon_{n+r}, \tau\, k_{n+r})$ for $r = 1, \cdots, m$. ε_a for $1 \leq a \leq n$ can be any tensor or spinor representation of the Lorentz group. Then for small τ the amplitude $\mathbf{M}_{n+m}$

may be expressed in terms of the amplitude $\mathbf{M}_n$ without the soft gravitons as follows:[1]

$$\begin{aligned} \mathbf{M}_{n+m} &= \tau^{-m}\left\{\prod_{r=1}^{m} S^{(0)}_{n+r}\right\}\mathbf{M}_n + \tau^{-m+1}\sum_{s=1}^{m}\left\{\prod_{\substack{r=1\\ r\neq s}}^{m} S^{(0)}_{n+r}\right\}\left[S^{(1)}_{n+s}\mathbf{M}_n\right] \\ &+\tau^{-m+1}\sum_{\substack{r,u=1\\ r<u}}^{m}\left\{\prod_{\substack{s=1\\ s\neq r,u}}^{m} S^{(0)}_{n+s}\right\}\left\{\sum_{a=1}^{n}\{p_a\cdot(k_{n+r}+k_{n+u})\}^{-1}\ \mathcal{M}(p_a;\varepsilon_{n+r},k_{n+r},\varepsilon_{n+u},k_{n+u})\right\}\mathbf{M}_n\,, \\ &+O(\tau^{-m+2}) \end{aligned} \tag{4.1}$$

where

$$S^{(0)}_{n+r} = \sum_{a=1}^{n}(p_a\cdot k_{n+r})^{-1}\,\varepsilon_{(n+r),\mu\nu}\,p_a^{\mu}p_a^{\nu}\,, \tag{4.2}$$

$$[S^{(1)}_{n+s}\mathbf{M}_n] = \sum_{a=1}^{n}(p_a\cdot k_{n+s})^{-1}\,\varepsilon_{n+s,\sigma\mu}\,k_{n+s,\rho}\,p_a^{\mu}\left[p_a^{\sigma}\frac{\partial}{\partial p_{a\rho}} - p_a^{\rho}\frac{\partial}{\partial p_{a\sigma}} + J^{\rho\sigma}\right]\mathbf{M}_n\,, \tag{4.3}$$

and $\mathcal{M}$ has been defined in (3.5). Our goal in this chapter will be to prove this result for Einstein's gravity using the CHY prescription.

This chapter is organized as follows. In section 4.1, we summarize the results related to the CHY proposal for the Einstein's gravity which will be needed in our analysis. A more general review of the CHY prescription is given in the appendix B. In section 4.2, we review the derivation of the single soft graviton theorem at subleading order [28,69,77,78]. In the process, we also fix the normalization of the CHY prescription for the Einstein's gravity which have been ignored in earlier works. This is crucial for fixing the sign issue mentioned in section 3.2.3. In section 4.3, we consider the double soft graviton result using the CHY prescription and derive the results of [15,16] in a more systematic manner. Finally, we consider the multiple soft gravitons in section 4.4 and show that the result derived using the CHY method agrees with the above result (4.1). In all our analysis, we work in a general gauge without making any specific gauge choice[2].

Before proceeding further, we again mention that the references [15,16] derived the double soft graviton theorem from CHY scattering equation[3]. The result derived in the last chapter differs from these references by a sign. In this chapter, we show by careful analysis that the sign that we get

[1] In [4] the amplitudes $\mathbf{M}_n$ included the momentum conserving delta-functions in their definition. In this chapter, we shall use the CHY formula for $\mathbf{M}_n$ and will not include the momentum conserving delta functions in the definition of $\mathbf{M}_n$. As has been discussed below (4.14), the soft graviton theorem takes the same form as in [4] for amplitudes without delta functions.

[2] The only exception is that we do impose $\varepsilon^{\mu}_{\mu} = 0$ gauge condition on the polarization tensor ε of the soft gravitons. Using on-shell condition this also implies $k^{\mu}\varepsilon_{\mu\nu} = 0$ where k is the momentum of the soft graviton.

[3] Double and multiple soft theorem for other soft particles from CHY scattering equations has been studied in [10,29,121]

from the double soft limit of the CHY scattering formula agrees with our result (4.1) provided we normalize the amplitude so that the single soft theorem comes with the correct sign. We also find few extra terms in the intermediate stages of analysis that were missed in the analysis of [15, 16], but they cancel in the final result.

4.1 Cachazo-He-Yuan prescription for Einstein's gravity

We start by summarizing the CHY proposal for Einstein's gravity [25–27, 29, 113, 120]. According to this proposal, an n-point tree amplitude involving massless particles can be derived from a sum over discrete set of points in the moduli space of an n-punctured Riemann sphere $\mathcal{M}_{0,n}$. The position of the punctures corresponding to these points in $\mathcal{M}_{0,n}$ are determined from the solutions of so called scattering equations:

$$\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b} = 0 \qquad\qquad \forall a \in \{1, 2, ..., n\}\,, \tag{4.4}$$

where $\{\sigma_a\}$ are the holomorphic coordinates of the punctures and $\{p_a\}$ are momenta of massless particles. Using the $SL(2, C)$ invariance of $\mathcal{M}_{0,n}$ we can fix the positions of three punctures and use any $(n-3)$ of the n scattering equations as constraint relations. The CHY formula is given as,

$$\mathrm{M}_n = \int \Big[\prod_{\substack{c=1\\ c\neq p,q,r}}^{n} d\sigma_c \Big] (\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki}) \Big[\prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b} \Big) \Big] \mathcal{I}_n\,, \tag{4.5}$$

where $\sigma_{ab} \equiv \sigma_a - \sigma_b$. Here we used SL(2,C) invariance to fix three punctures p, q, r, and made use of the linear dependence of the scattering equations to remove three scattering equations for i, j, k particles. The SL(2,C) invariance of (4.5) requires

$$\mathcal{I}_n \to \mathcal{I}_n \prod_{a=1}^{n} (c\,\sigma_a + d)^4 \quad \text{under} \quad \sigma_a \to (a\,\sigma_a + b)\,(c\,\sigma_a + d)^{-1}\,. \tag{4.6}$$

Using this requirement, together with multi-linearity in all the polarization vectors of n-gravitons and gauge invariance of M_n, CHY gave the following form of the integrand $\mathcal{I}_n$ for Einstein gravity [25]:

$$\mathcal{I}_n = 4\,(-1)^n\,(\sigma_s - \sigma_t)^{-2}\,\det(\Psi^{st}_{st}) \tag{4.7}$$

where Ψ is a $2n \times 2n$ anti-symmetric matrix defined below and Ψ^{st}_{st} is obtained by removing s-th and t-th row from first n rows and removing s-th and t-th columns from first n columns of Ψ. The $(-1)^n$ factor was not present in the original formula, but we have included it in order to get soft graviton theorems with conventional signs for the leading soft graviton theorem. This factor has also appeared recently in [122]. The overall n independent normalization and sign of the amplitude will be irrelevant for our analysis. Ψ has the form:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \tag{4.8}$$

where A, B, C are $n \times n$ matrices defined as,

$$A_{ab} = \begin{cases} \dfrac{p_a \cdot p_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases} , \qquad B_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases}$$

$$C_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot p_b}{\sigma_a - \sigma_b} & a \neq b \\ -\sum\limits_{c=1, c \neq a}^{n} \dfrac{\epsilon_a \cdot p_c}{\sigma_a - \sigma_c} & a = b \end{cases} \tag{4.9}$$

Here for each a, ϵ_a is a space-time vector. In the expression for $\mathcal{I}_n$, each ϵ_a appears quadratically. In the final formula, we replace $\epsilon_{a,\mu}\epsilon_{a,\nu}$ by $\varepsilon_{a,\mu\nu}$ and identify ε_a as the polarization tensor for the a-th graviton.

Since the total number of independent scattering equations is equal to the total number of independent variables, the integration over σ_a's reduce to a sum over discrete set of points, it can be seen using simple counting that the total number of solutions to the scattering equation is $(n-3)!$ [24].

In the rest of the chapter, we shall use the CHY formula (4.5) to study soft limits – limits in which one or more external momenta become small. Our general strategy will be as follows. We first represent the integration over σ_a's as contour integrals using (B.23), taking the contours to lie close to the solutions of the scattering equations. In this region we can make appropriate approximation on the integrand using the soft limit. Once we make this approximation, we now deform the contours away from the solutions of the scattering equation, possibly through regions where the original approximation fails, but Cauchy's theorem guarantees that the value of the integral is unchanged. As we shall see, this gives an effective method for approximating the CHY formula in the soft limit.

4.2 Single soft-graviton theorem

In this section, we review the derivation of subleading single soft graviton theorem from CHY prescription given in [28, 69, 77, 78], taking special care about the signs. We shall see that the $(-1)^n$ factor in (4.7) is necessary for getting the conventional signs. We also work with general polarizations of external gravitons without making any gauge choice (other than the one given in footnote 2) as in [78] – this will be necessary for generalizing the analysis to multiple soft graviton theorem where special gauge choices of the type used *e.g.* in [28] is not possible. Single soft graviton theorem for general gauge choice has been derived in [78].

4.2.1 Statement of the theorem

We shall begin by stating the subleading single soft graviton theorem in a form which is convenient for our purpose. We consider the scattering amplitude of $(n+1)$ massless gravitons with $(n+1)$'th particle momentum soft and expand the amplitude about the soft momentum. Let $p_1, \cdots p_n$ denote the momenta of the finite energy particles, and $p_{n+1} = \tau\, k_{n+1}$ denote the momentum of the soft particle. We shall take the soft limit by taking $\tau \to 0$ limit at fixed k_{n+1}. We also denote by ε_i the polarization tensor of the i-th graviton. Then the subleading soft theorem takes the form

$$\mathbf{M}_{n+1}(p_1, p_2, ..., p_n, \tau k_{n+1}) = \Big[\frac{1}{\tau} S^{(0)}_{n+1} + S^{(1)}_{n+1} + O(\tau)\Big] \mathbf{M}_n(p_1, p_2, ..., p_n) \tag{4.10}$$

where,

$$S^{(0)}_{n+1} = \sum_{a=1}^{n} \frac{\varepsilon_{(n+1),\mu\nu}\, p_a^\mu\, p_a^\nu}{k_{n+1}.p_a}\,, \tag{4.11}$$

$$S^{(1)}_{n+1} = \sum_{a=1}^{n} \frac{\varepsilon_{(n+1),\mu\nu}\, p_a^\mu\, k_{(n+1),\rho}\, \widehat{J}_a^{\rho\nu}}{k_{n+1}.p_a}\,. \tag{4.12}$$

Here, $\widehat{J}_a^{\mu\nu}$ is the total angular momentum acting on the a-th state, defined as,

$$\widehat{J}_a^{\mu\nu} = p_a^\nu \frac{\partial}{\partial p_{a,\mu}} - p_a^\mu \frac{\partial}{\partial p_{a,\nu}} + J_a^{\mu\nu} \tag{4.13}$$

where $J_a^{\mu\nu}$ is the spin angular momentum of the a-th hard graviton, defined as

$$(J^{\mu\nu}\,\varepsilon)_{\rho\sigma} \equiv (J^{\mu\nu})_{\rho\sigma}{}^{\alpha\beta}\varepsilon_{\alpha\beta} = \delta^\mu_\rho \varepsilon^\nu{}_\sigma - \delta^\nu_\rho \varepsilon^\mu{}_\sigma + \delta^\mu_\sigma \varepsilon_\rho{}^\nu - \delta^\nu_\sigma \varepsilon_\rho{}^\mu\,. \tag{4.14}$$

There is one subtle point that must be mentioned here. The full amplitude contains a momentum conserving delta function, and the derivation of the multiple soft graviton theorem derived in last chapter included this momentum conserving delta function in the definition of the amplitude. In that case the external momenta can be taken as independent and the derivatives with respect to momenta, present in the definition of $J^{\mu\nu}$, are well defined. On the other hand, the CHY formula given in (4.5) does not have this delta function. It is however easy to see that as long as (4.5) satisfies the soft graviton theorem, the full amplitude including the delta function also satisfies the soft graviton theorem [68]. To see this, we multiply both sides of (4.10) by $\delta^{(D)}(p_1 + \cdots + p_n + \tau\, k_{n+1})$, and manipulate the right hand side as follows:

$$
\begin{aligned}
& \tau^{-1} \sum_{a=1}^{n} \frac{\varepsilon_{(n+1),\mu\nu}\, p_a^\mu\, p_a^\nu}{k_{n+1}\cdot p_a} \mathbf{M}_n \left(\delta^{(D)}(p_1+\cdots+p_n) + \tau\, k_{n+1,\rho} \frac{\partial}{\partial p_{a,\rho}} \delta^{(D)}(p_1+\cdots+p_n) \right) \\
& + \sum_{a=1}^{n} \frac{\varepsilon_{(n+1),\mu\nu}\, p_a^\mu\, k_{(n+1),\rho}}{k_{n+1}\cdot p_a} \left\{ p_a^\nu \frac{\partial}{\partial p_{a,\rho}} - p_a^\rho \frac{\partial}{\partial p_{a,\nu}} + J_a^{\rho\nu} \right\} \mathbf{M}_n\, \delta^{(D)}(p_1+\cdots+p_n) + O(\tau) \\
= \; & \tau^{-1} \sum_{a=1}^{n} \frac{\varepsilon_{(n+1),\mu\nu}\, p_a^\mu\, p_a^\nu}{k_{n+1}\cdot p_a} \mathbf{M}_n\, \delta^{(D)}(p_1+\cdots+p_n) \\
& + \sum_{a=1}^{n} \frac{\varepsilon_{(n+1),\mu\nu}\, p_a^\mu\, k_{(n+1),\rho}}{k_{n+1}\cdot p_a} \left\{ p_a^\nu \frac{\partial}{\partial p_{a,\rho}} - p_a^\rho \frac{\partial}{\partial p_{a,\nu}} + J_a^{\rho\nu} \right\} \left\{ \mathbf{M}_n\, \delta^{(D)}(p_1+\cdots+p_n) \right\} \\
& + O(\tau)\,,
\end{aligned}
\tag{4.15}
$$

where we have used the fact that

$$
\sum_{a=1}^{n} \frac{\varepsilon_{(n+1),\mu\nu}\, p_a^\mu\, k_{(n+1),\rho}}{k_{n+1}\cdot p_a} p_a^\rho \frac{\partial}{\partial p_{a,\nu}} \delta^{(D)}(p_1+\cdots+p_n) = 0\,, \tag{4.16}
$$

as a consequence of the condition $\varepsilon^\mu_\mu = 0$ that we impose on the polarization tensors of the soft gravitons. Since the right hand side of (4.15) now takes the form of the right hand side of the soft graviton theorem with the momentum conserving delta function included in the definition of the amplitude, we see that once we prove eq.(4.10) for the CHY amplitudes, it also holds for the full amplitude including the momentum conserving delta functions. A similar result holds also for the multiple soft graviton theorem (4.1).

While proving (4.10), or more generally for the multiple gravitons for the CHY amplitude (4.5), we shall treat all the external momenta as independent while computing the derivatives of the amplitude with respect to the external momenta. However while verifying the equality of the two sides of the equation we shall make use of the conservation of total momenta, since the final identity we are interested in proving is not (4.10) or its generalization to multiple soft graviton case, but the ones obtained from these by multiplying both sides of the equation by momentum conserving delta functions.

4.2.2 Single soft limit of CHY formula

We now begin the analysis of the single soft limit of CHY formula. Let us introduce some compact notations,[4]

$$f_a^n \equiv \sum_{\substack{b=1\\b\neq a}}^{n} \frac{p_a{\cdot}p_b}{\sigma_a-\sigma_b} \ \text{ for } 1\leq a\leq n, \quad f_{n+r}^n \equiv \sum_{b=1}^{n}\frac{k_{n+r}{\cdot}p_b}{\sigma_{n+r}-\sigma_b}\,, \tag{4.17}$$

$$\int D\sigma \equiv \int \Big[\prod_{\substack{c=1\\c\neq p,q,r}}^{n} d\sigma_c\Big](\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki})\,, \tag{4.18}$$

and rewrite the CHY formula for the $(n+1)$-point scattering amplitude as,

$$\begin{aligned}&\mathbf{M}_{n+1}(p_1,p_2,...,p_n,\tau\, k_{n+1})\\ &= \int D\sigma\int d\sigma_{n+1}\Big[\prod_{\substack{a=1\\a\neq i,j,k}}^{n}\delta\Big(\sum_{\substack{b=1\\b\neq a}}^{n}\frac{p_a{\cdot}p_b}{\sigma_{ab}}+\tau\,\frac{p_a.k_{n+1}}{\sigma_{a(n+1)}}\Big)\Big]\delta\Big(\tau\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\sigma_{(n+1)b}}\Big)\, I_{n+1}\,.\end{aligned} \tag{4.19}$$

While carrying out various manipulations below, we shall not be careful about orders of the delta functions. However it should be understood that while doing the final integration over the σ_a's, the delta functions must be brought to the same order in which they were arranged initially. With this understanding we expand the product of delta functions inside the integral (4.19) in a Taylor series in τ as:

$$\begin{aligned}&\prod_{\substack{a=1\\a\neq i,j,k}}^{n}\delta\Big(\sum_{\substack{b=1\\b\neq a}}^{n}\frac{p_a{\cdot}p_b}{\sigma_{ab}}+\tau\,\frac{p_a.k_{n+1}}{\sigma_{a(n+1)}}\Big)\\ &= \prod_{\substack{a=1\\a\neq i,j,k}}^{n}\delta\Big(\sum_{\substack{b=1\\b\neq a}}^{n}\frac{p_a{\cdot}p_b}{\sigma_{ab}}\Big)+\tau\sum_{l=1}^{n}\frac{p_l.k_{n+1}}{\sigma_{l(n+1)}}\delta'\Big(\sum_{\substack{c=1\\c\neq l}}^{n}\frac{p_l.p_c}{\sigma_{lc}}\Big)\prod_{\substack{a=1\\a\neq i,j,k,l}}^{n}\delta\Big(\sum_{\substack{b=1\\b\neq a}}^{n}\frac{p_a{\cdot}p_b}{\sigma_{ab}}\Big)+O(\tau^2)\\ &\equiv \delta^{(0)}+\tau\,\delta^{(1)}+O(\tau^2)\,.\end{aligned} \tag{4.20}$$

The integrand can also be expanded in powers of τ as,

$$I_{n+1} = I_{n+1}\Big|_{\tau=0}+\tau\,\frac{\partial I_{n+1}}{\partial\tau}\Big|_{\tau=0}+O(\tau^2)\equiv I_{n+1}^{(0)}+\tau\, I_{n+1}^{(1)}+O(\tau^2)\,. \tag{4.21}$$

[4] For single soft gravitons we only need f_{n+1}^n, but later we shall make use of the definition of f_{n+r}^n for general r.

Now substituting (4.20) and (4.21) in (4.19) and keeping terms up to subleading order in τ, we obtain

$$\begin{aligned}
\mathbf{M}_{n+1}(\{p_a\},\tau\,k_{n+1}) &= \frac{1}{\tau}\int D\sigma\int d\sigma_{n+1}\Big[\delta^{(0)}+\tau\,\delta^{(1)}\Big]\,\delta(f^n_{n+1})\,(I^{(0)}_{n+1}+\tau\,I^{(1)}_{n+1}) \\
&= \frac{1}{\tau}\int D\sigma\,\delta^{(0)}\int d\sigma_{n+1}\,\delta(f^n_{n+1})\,I^{(0)}_{n+1}+\int D\sigma\int d\sigma_{n+1}\,\delta^{(1)}\,\delta(f^n_{n+1})\,I^{(0)}_{n+1} \\
&\quad+\int D\sigma\,\delta^{(0)}\int d\sigma_{n+1}\,\delta(f^n_{n+1})\,I^{(1)}_{n+1}\,.
\end{aligned} \tag{4.22}$$

Now we shall analyze the three terms appearing in equation (4.22) one by one.

First term

Using the expressions (4.7), (4.8) and (4.9), we get, on putting $\tau=0$,

$$I^{(0)}_{n+1} = -C^2_{n+1,n+1}I_n = -\Big(\sum_{a=1}^{n}\frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\Big)^2 I_n\,. \tag{4.23}$$

Here, the negative sign comes from the $(-1)^n$ factor in the definition of I_n. Using this, the first term on the right hand side of (4.22) can be written as

$$\begin{aligned}
\mathcal{A}_1 &\equiv \frac{1}{\tau}\int D\sigma\,\delta^{(0)}\int d\sigma_{n+1}\,\delta(f^n_{n+1})\,I^{(0)}_{n+1} \\
&= -\frac{1}{\tau}\int D\sigma\prod_{\substack{a=1\\ a\neq i,j,k}}^{n}\delta\Big(\sum_{b=1,b\neq a}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)\int d\sigma_{n+1}\delta\Big(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\sigma_{(n+1)b}}\Big)\Big(\sum_{a=1}^{n}\frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\Big)^2 I_n \\
&= -\frac{1}{\tau}\int D\sigma\prod_{\substack{a=1\\ a\neq i,j,k}}^{n}\delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)I_n\oint d\sigma_{n+1}\Big(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\Big)^2.
\end{aligned} \tag{4.24}$$

In going to the last line, we have used the contour integral representation of the delta function as described in (B.23), with the σ_{n+1} integration contour wrapping the solutions of the scattering equation of the $(n+1)$-th particle. We now deform the contour and do the contour integration about the poles of rest of the integrand. The deformed contour will wrap the other poles in clockwise direction, and therefore the residue theorem will have an extra minus sign. We shall also need to take care of poles at infinity if they exist. If we denote collectively by $\{R_i\}$ all these poles, the first term becomes

$$\mathcal{A}_1 = \frac{1}{\tau}\int D\sigma\prod_{\substack{a=1\\ a\neq i,j,k}}^{n}\delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)I_n\oint_{\{R_i\}}d\sigma_{n+1}\frac{1}{\Big(\sum\limits_{b=1}^{n}\frac{k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b}\Big)}\Big(\sum_{a=1}^{n}\frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\Big)^2. \tag{4.25}$$

Using momentum conservation $\sum_{a=1}^{n}p_a = -\tau\,k_{n+1}$ and the on-shell conditions $k^2_{n+1}=0$, $\epsilon_{n+1}.k_{n+1}=0$, we can see that for large σ_{n+1}, the last term in (4.25) falls off as $(\sigma_{n+1})^{-4}$ and the last but one term grows as $(\sigma_{n+1})^2$. Therefore there is no pole at ∞. The rest of the poles inside the deformed contour are the simple poles at $\sigma_{n+1} = \sigma_a$ coming from the combination of the last two terms in (4.25). Computing the residue at each such pole, we obtain the result

$$\mathcal{A}_1 \;=\; \frac{1}{\tau}\sum_{a=1}^{n} \frac{(\epsilon_{n+1}.p_a)^2}{k_{n+1}.p_a}\,\mathbf{M}_n\,. \tag{4.26}$$

Replacing $\epsilon_{n+1,\mu}\epsilon_{n+1,\nu}$ by $\varepsilon_{n+1,\mu\nu}$ in the above expression, we see that it agrees with the leading soft theorem (4.10) including the sign.

Second Term

The second term on the right hand side of (4.22) is given by

$$\begin{aligned}\mathcal{A}_2 \;&\equiv\; \int D\sigma \int d\sigma_{n+1}\,\delta^{(1)}\,\delta(f_{n+1}^{n})\,I_{n+1}^{(0)}\\ &=\; -\int D\sigma \int d\sigma_{n+1} \sum_{l=1}^{n}\frac{p_l.k_{n+1}}{\sigma_{l(n+1)}}\delta'\Big(\sum_{\substack{c=1\\c\neq l}}^{n}\frac{p_l.p_c}{\sigma_{lc}}\Big)\Big[\prod_{\substack{a=1\\a\neq i,j,k,l}}^{n}\delta\Big(\sum_{\substack{b=1\\b\neq a}}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)\Big]\\ &\qquad \delta\Big(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\sigma_{(n+1)b}}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+1}.p_d}{\sigma_{n+1}-\sigma_d}\Big)^2 I_n\,.\end{aligned} \tag{4.27}$$

Let us focus on the σ_{n+1} integration. The relevant integral is

$$\int d\sigma_{n+1}\frac{p_l.k_{n+1}}{\sigma_l-\sigma_{n+1}}\delta\Big(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+1}.p_d}{\sigma_{n+1}-\sigma_d}\Big)^2. \tag{4.28}$$

We can analyze this by deforming the σ_{n+1} integration contour in the same way as before. The only extra complication is that along with the simple poles at $\sigma_{n+1} = \sigma_a$ $(a \neq \ell)$, we now also have a double pole at σ_ℓ and we have to be a little more careful in evaluating the residue. This can be done, yielding the result

$$\sum_{\substack{b=1\\b\neq l}}^{n}\frac{1}{\sigma_l-\sigma_b}\Big[2\big(\epsilon_{n+1}.p_l\big)\big(\epsilon_{n+1}.p_b\big)-\big(\epsilon_{n+1}.p_l\big)^2\frac{k_{n+1}.p_b}{k_{n+1}.p_l}-\big(\epsilon_{n+1}.p_b\big)^2\frac{k_{n+1}.p_l}{k_{n+1}.p_b}\Big]\,. \tag{4.29}$$

Substituting this into (4.27) we get

$$\begin{aligned}\mathcal{A}_2 \;&=\; -\int D\sigma\, I_n \sum_{l=1}^{n}\Big[\prod_{\substack{a=1\\a\neq i,j,k,l}}^{n}\delta\Big(\sum_{\substack{b=1\\b\neq a}}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)\Big]\delta'\Big(\sum_{\substack{c=1\\c\neq l}}^{n}\frac{p_l.p_c}{\sigma_{lc}}\Big)\sum_{\substack{b=1\\b\neq l}}^{n}\frac{1}{\sigma_l-\sigma_b}\\ &\qquad\Big[2\big(\epsilon_{n+1}.p_l\big)\big(\epsilon_{n+1}.p_b\big)-\big(\epsilon_{n+1}.p_l\big)^2\frac{k_{n+1}.p_b}{k_{n+1}.p_l}-\big(\epsilon_{n+1}.p_b\big)^2\frac{k_{n+1}.p_l}{k_{n+1}.p_b}\Big]\,.\end{aligned} \tag{4.30}$$

The derivative of the delta function can be obtained by using the representation (B.23) of the delta function. Instead of contour integral of $1/f$ we have contour integral of $-1/f^2$. We shall see, however, that we do not need to make use of this explicit representation.

Third term

We now turn to the evaluation of the third term on the right hand side of (4.22)

$$\mathcal{A}_3 = \int D\sigma\, \delta^{(0)} \int d\sigma_{n+1}\, \delta(f^n_{n+1})\, I^{(1)}_{n+1}\,. \tag{4.31}$$

We begin by introducing some notations. We denote by $\widetilde{\Psi}$ the matrix Ψ^{st}_{st} defined in (4.8) and the paragraph above it for n finite energy external states, with some fixed choice of s and t in the range $1 \le s, t \le n$. We shall denote by $\widehat{\Psi}$ the same matrix for $n+1$ external states with the $(n+1)$-th state describing a soft graviton carrying momentum $\tau\, k_{n+1}$. $\widetilde{\Psi}$ is a $(2n-2)\times(2n-2)$ matrix and $\widehat{\Psi}$ is a $2n \times 2n$ matrix. We shall denote by $\widetilde{P}$ the Pfaffian of $\widetilde{\Psi}$ and by $\widehat{P}$ the Pfaffian of $\widehat{\Psi}$:

$$\widetilde{P} = \frac{1}{2^{n-1}\,(n-1)!}\, \epsilon^{\alpha_1\cdots\alpha_{2n-2}} \widetilde{\Psi}_{\alpha_1\alpha_2} \cdots \widetilde{\Psi}_{\alpha_{2n-3}\alpha_{2n-2}}\,, \qquad \widehat{P} = \frac{1}{2^n\, n!} \epsilon^{\alpha_1\cdots\alpha_{2n}} \widehat{\Psi}_{\alpha_1\alpha_2} \cdots \widehat{\Psi}_{\alpha_{2n-1}\alpha_{2n}}\,. \tag{4.32}$$

Furthermore we shall denote by $\widehat{P}_{\alpha\beta} = -\widehat{P}_{\beta\alpha}$ for $\alpha < \beta$ the Pfaffian of the matrix obtained by eliminating from $\widehat{P}$ the α-th row and column and β-th row and column. Similarly $\widehat{P}_{\alpha\beta\gamma\delta}$ is defined to be totally anti-symmetric in $\alpha, \beta, \gamma, \delta$, which, for $\alpha < \beta < \gamma < \delta$, is given by the Pfaffian of the matrix obtained by eliminating the $\alpha, \beta, \gamma, \delta$-th rows and columns of the matrix $\widehat{\Psi}$. Similar definitions and properties hold for $\widetilde{P}_{\alpha\beta}$ and $\widetilde{P}_{\alpha\beta\gamma\delta}$. Then we have

$$I_{n+1} = 4(-1)^{n+1}\,(\sigma_s - \sigma_t)^{-2}\, \det \widehat{\Psi} = 4(-1)^{n+1}\,(\sigma_s - \sigma_t)^{-2}\, \widehat{P}^2\,, \tag{4.33}$$

and therefore

$$I^{(1)}_{n+1} \equiv \left.\frac{\partial I_{n+1}}{\partial \tau}\right|_{\tau=0} = 8(-1)^{n+1}\,(\sigma_s - \sigma_t)^{-2}\; \widehat{P}\, \left.\frac{\partial \widehat{P}}{\partial \tau}\right|_{\tau=0}\,. \tag{4.34}$$

Now from the definition (4.32) of the Pfaffian it follows that

$$\frac{\partial \widehat{P}}{\partial \tau} = \sum_{\substack{\alpha,\beta=1\\ \alpha<\beta,\alpha,\beta\neq s,t}}^{2n+2} (-1)^{\alpha-\beta+1} \left.\frac{\partial \widehat{\Psi}_{\alpha\beta}}{\partial \tau} \widehat{P}_{\alpha\beta}\right|_{\tau=0}\,. \tag{4.35}$$

Here we have adopted the notation that in $\widehat{\Psi}_{\alpha\beta}$ and $\widehat{P}_{\alpha\beta}$ we shall let the indices run over $1 \le \alpha, \beta \le (n+1)$, $\alpha, \beta \ne s, t$, and $(n+2) \le \alpha, \beta \le 2n+2$. Therefore the last rows and columns will be called $(2n+2)$-th rows and columns, even though they are actually $2n$-th rows and columns. We have also implicitly assumed that the integers s and t are consecutive – otherwise there will be extra minus signs when one of α or β falls between s and t. Our final result will be valid even when s and t are not consecutive.

Now using the explicit form of the $(n+1)\times(n+1)$ matrices A, B, C given in (B.42) for $n+1$ gravitons, with $p_{n+1} = \tau\, k_{n+1}$, we get,

$$\begin{aligned}
&\frac{\partial A_{ab}}{\partial\tau} = 0 \ \text{ for } 1\le a,b\le n, \quad \frac{\partial A_{a(n+1)}}{\partial\tau} = \frac{p_a.k_{n+1}}{\sigma_a-\sigma_{n+1}} \ \text{ for } 1\le a\le n,\\
&\frac{\partial B_{ab}}{\partial\tau} = 0 \ \text{ for } 1\le a,b\le n+1\,,\\
&\frac{\partial C_{ab}}{\partial\tau} = 0 \ \text{ for } 1\le a,b\le n,\, a\ne b, \quad \frac{\partial C_{a(n+1)}}{\partial\tau} = \frac{\epsilon_a.k_{n+1}}{\sigma_a-\sigma_{n+1}} \ \text{ for } 1\le a\le n\,,\\
&\frac{\partial C_{(n+1)b}}{\partial\tau} = 0 \ \text{ for } 1\le b\le n\,, \quad \frac{\partial C_{aa}}{\partial\tau} = -\frac{\epsilon_a.k_{n+1}}{\sigma_a-\sigma_{n+1}} \ \text{ for } 1\le a\le n, \quad \frac{\partial C_{(n+1)(n+1)}}{\partial\tau} = 0\,.
\end{aligned} \tag{4.36}$$

On the other hand using (4.35) and (4.8), we get

$$\begin{aligned}
\frac{\partial\widehat{P}}{\partial\tau} &= \sum_{\substack{a,b=1\\ a<b,a,b\ne s,t}}^{n+1} (-1)^{a-b+1}\frac{\partial A_{ab}}{\partial\tau}\widehat{P}_{ab} + \sum_{\substack{a,b=1\\ a<b}}^{n+1}(-1)^{a-b+1}\frac{\partial B_{ab}}{\partial\tau}\widehat{P}_{(n+1+a)(n+1+b)}\\
&\quad + \sum_{\substack{a,b=1\\ a\ne s,t}}^{n+1}(-1)^{a-n-b}\frac{\partial(-C^T)_{ab}}{\partial\tau}\widehat{P}_{a(n+1+b)}\,.
\end{aligned} \tag{4.37}$$

Using (4.36), (4.37) we can rewrite (4.34) as,

$$\begin{aligned}
I^{(1)}_{n+1} &= 8\,(-1)^{n+1}\,(\sigma_s-\sigma_t)^{-2}\,\widehat{P}\Bigg\{\sum_{\substack{a=1\\ a\ne s,t}}^{n}(-1)^{a-n}\frac{\partial A_{a(n+1)}}{\partial\tau}\widehat{P}_{a(n+1)}\\
&\quad + \sum_{a=1}^{n}(-1)^{a}\frac{\partial C_{a(n+1)}}{\partial\tau}\widehat{P}_{(n+1)(n+1+a)} + \sum_{\substack{a=1\\ a\ne s,t}}^{n}(-1)^{n+1}\frac{\partial C_{aa}}{\partial\tau}\widehat{P}_{a(n+1+a)}\Bigg\}\Bigg|_{\tau=0}\,.
\end{aligned} \tag{4.38}$$

We shall now evaluate the various components $\widehat{P}_{\alpha\beta}$ that appear in the expression (4.38). For this it will be useful to express $\widehat{\Psi}$ in the matrix form:

$$\widehat{\Psi} = \begin{pmatrix}
A_{ab} & \frac{\tau p_a.k_{n+1}}{\sigma_a-\sigma_{n+1}} & -C^T_{ad} & \frac{p_a.\epsilon_{n+1}}{\sigma_a-\sigma_{n+1}}\\
\frac{\tau k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} & 0 & \frac{\tau k_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d} & -C_{(n+1)(n+1)}\\
C_{cb} & \frac{\tau\epsilon_c.k_{n+1}}{\sigma_c-\sigma_{n+1}} & B_{cd} & \frac{\epsilon_c.\epsilon_{n+1}}{\sigma_c-\sigma_{n+1}}\\
\frac{\epsilon_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} & C_{(n+1)(n+1)} & \frac{\epsilon_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d} & 0
\end{pmatrix}, \tag{4.39}$$

where the indices a, b, c, d for the matrices A, B, C run from 1 to n (with $a, b \ne s, t$), and the $(n+1)$-

th components of these matrices have been written down explicitly. Also useful will be the explicit form of $C_{(n+1)(n+1)}$:

$$C_{(n+1)(n+1)} = -\sum_{c=1}^{n} \frac{\epsilon_{n+1}\cdot p_c}{\sigma_{n+1}-\sigma_c}\,. \tag{4.40}$$

We begin with $\widehat{P}_{a(n+1)}$. This is the Pfaffian of the matrix (4.39) without the a-th and $(n+1)$-th rows and columns. Expanding this about the last column we get

$$\widehat{P}_{a(n+1)} = \sum_{\substack{e=1\\ e\neq s,t,a}}^{n} (-1)^e \frac{\epsilon_{n+1}\cdot p_e}{\sigma_e-\sigma_{n+1}} \widehat{P}_{ae(n+1)(2n+2)} + \sum_{d=1}^{n} (-1)^{n+d} \frac{\epsilon_{n+1}\cdot \epsilon_d}{\sigma_d-\sigma_{n+1}} \widehat{P}_{a(n+1)(n+1+d)(2n+2)}\,. \tag{4.41}$$

We now notice that for $\tau=0$, the matrix obtained by eliminating the $(n+1)$-th and $(2n+2)$-th rows and columns of $\widehat{\Psi}$, is in fact the matrix $\widetilde{\Psi}$. Therefore we can rewrite (4.41) as

$$\widehat{P}_{a(n+1)}|_{\tau=0} = \sum_{\substack{e=1\\ e\neq s,t,a}}^{n} (-1)^{e+1} \frac{\epsilon_{n+1}\cdot p_e}{\sigma_{n+1}-\sigma_e} \widetilde{P}_{ae} + \sum_{d=1}^{n} (-1)^{n+d+1} \frac{\epsilon_{n+1}\cdot \epsilon_d}{\sigma_{n+1}-\sigma_d} \widetilde{P}_{a(n+d)}\,. \tag{4.42}$$

Similarly we have

$$\widehat{P}_{(n+1)(n+1+a)}|_{\tau=0} = \sum_{\substack{e=1\\ e\neq s,t}}^{n} (-1)^{e} \frac{\epsilon_{n+1}\cdot p_e}{\sigma_{n+1}-\sigma_e} \widetilde{P}_{e(n+a)} + \sum_{\substack{d=1\\ d\neq a}}^{n} (-1)^{n-d+1} \frac{\epsilon_{n+1}\cdot \epsilon_d}{\sigma_{n+1}-\sigma_d} \widetilde{P}_{(n+a)(n+d)}\,. \tag{4.43}$$

To analyze $\widehat{P}_{a(n+1+a)}$ we expand it in the $(n+1)$-th row. For $\tau=0$ only the $(2n+2)$-th column contributes and gives

$$\begin{aligned}\widehat{P}_{a(n+1+a)}|_{\tau=0} &= -(-1)^{n-1} C_{(n+1)(n+1)} \widehat{P}_{a(n+1)(n+1+a)(2n+2)}|_{\tau=0} \\ &= (-1)^{n-1}\left(\sum_{c=1}^{n} \frac{\epsilon_{n+1}\cdot p_c}{\sigma_{n+1}-\sigma_c}\right) \widetilde{P}_{a(n+a)}\,. \end{aligned} \tag{4.44}$$

In order to evaluate the right hand side of (4.38) we also need $\widehat{P}|_{\tau=0}$. This is evaluated by expanding the Pfaffian of the matrix $\widehat{\Psi}$ given in (4.39) in the $(n+1)$-th row. The only contribution comes from the last element in the row, giving the result

$$\widehat{P}|_{\tau=0} = -(-1)^n C_{(n+1)(n+1)}|_{\tau=0}\,\widetilde{P} = (-1)^n \sum_{c=1}^{n} \frac{\epsilon_{n+1}\cdot p_c}{\sigma_{n+1}-\sigma_c}\,\widetilde{P}\,. \tag{4.45}$$

Substituting these results in (4.38) and using (4.36) we get the result for $I^{(1)}_{n+1}$ which can be substituted into (4.31). Using the definition of f^n_{n+1} given in (4.17) and the definition (B.23) of the delta function, we now get

$$\begin{aligned}\mathcal{A}_3 \;=\; & -8\int D\sigma\,\delta^{(0)}\oint d\sigma_{n+1}\left(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b}\right)^{-1}(\sigma_s-\sigma_t)^{-2}\,\widetilde{P}\,\left(\sum_{g=1}^{n}\frac{\epsilon_{n+1}.p_g}{\sigma_{n+1}-\sigma_g}\right)\\
&\Bigg[\sum_{\substack{a=1\\a\neq s,t}}^{n}(-1)^{a-n}\frac{p_a.k_{n+1}}{\sigma_a-\sigma_{n+1}}\Bigg\{\sum_{\substack{e=1\\e\neq s,t,a}}^{n}(-1)^{e+1}\frac{\epsilon_{n+1}.p_e}{\sigma_{n+1}-\sigma_e}\widetilde{P}_{ae}+\sum_{d=1}^{n}(-1)^{n+1+d}\frac{\epsilon_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d}\widetilde{P}_{a(n+d)}\Bigg\}\\
&+\sum_{a=1}^{n}(-1)^{a}\frac{\epsilon_a.k_{n+1}}{\sigma_a-\sigma_{n+1}}\Bigg\{\sum_{\substack{e=1\\e\neq s,t}}^{n}(-1)^{e}\frac{\epsilon_{n+1}.p_e}{\sigma_{n+1}-\sigma_e}\widetilde{P}_{e(n+a)}+\sum_{\substack{d=1\\d\neq a}}^{n}(-1)^{n-d+1}\frac{\epsilon_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d}\widetilde{P}_{(n+a)(n+d)}\Bigg\}\\
&\qquad+\sum_{\substack{a=1\\a\neq s,t}}^{n}\left(\frac{\epsilon_a.k_{n+1}}{\sigma_{n+1}-\sigma_a}\right)\left(\sum_{c=1}^{n}\frac{\epsilon_{n+1}.p_c}{\sigma_{n+1}-\sigma_c}\right)\widetilde{P}_{a(n+a)}\Bigg]. \qquad (4.46)\end{aligned}$$

The σ_{n+1} contour winds anti-clockwise around the zeroes of $\left(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b}\right)$. We can now deform the contour towards infinity, picking up residues from $\sigma_{n+1}=\sigma_a$, – it is easy to see that there are no poles at ∞. For the coefficients of $\widetilde{P}_{ab}$, $\widetilde{P}_{(n+a)(n+b)}$ and $\widetilde{P}_{a(n+b)}$ for $a\neq b$, the integrand has single poles at $\sigma_{n+1}=\sigma_a$ and $\sigma_{n+1}=\sigma_b$, and the result can be computed easily by picking up the residues. Finally for the coefficient of $\widetilde{P}_{a(n+a)}$ the integrand has a double pole at $\sigma_{n+1}=\sigma_a$, therefore to evaluate the residue we have to expand it in a Laurent series around the pole and pick the coefficient of the $(\sigma_{n+1}-\sigma_a)^{-1}$ term. After some algebra, one finds:

$$\begin{aligned}&\mathcal{A}_3 = 8\int D\sigma\delta^{(0)}\,(\sigma_s-\sigma_t)^{-2}\,\widetilde{P}\\
&\Bigg[\sum_{\substack{a,b=1\\a\neq b;a,b\neq s,t}}^{n}(-1)^{a+b-n}\,(\sigma_a-\sigma_b)^{-1}\left\{\frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}-\frac{\epsilon_{n+1}.p_b}{k_{n+1}.p_b}\right\}k_{n+1}.p_a\,\epsilon_{n+1}.p_b\,\widetilde{P}_{ab}\\
&+\sum_{\substack{a,b=1\\a\neq b}}^{n}(-1)^{a+b-n}\,(\sigma_a-\sigma_b)^{-1}\left\{\frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}-\frac{\epsilon_{n+1}.p_b}{k_{n+1}.p_b}\right\}k_{n+1}.\epsilon_a\,\epsilon_{n+1}.\epsilon_b\,\widetilde{P}_{(n+a)(n+b)}\\
+&\sum_{\substack{a,b=1\\a\neq b;a\neq s,t}}^{n}(-1)^{a+b}\,(\sigma_a-\sigma_b)^{-1}\left\{\frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}-\frac{\epsilon_{n+1}.p_b}{k_{n+1}.p_b}\right\}\left\{k_{n+1}.p_a\,\epsilon_{n+1}.\epsilon_b-k_{n+1}.\epsilon_b\,\epsilon_{n+1}.p_a\right\}\widetilde{P}_{a(n+b)}\\
&+\sum_{\substack{a=1\\a\neq s,t}}^{n}\sum_{\substack{c=1\\c\neq a}}^{n}(\sigma_a-\sigma_c)^{-1}\left\{\frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}-\frac{\epsilon_{n+1}.p_c}{k_{n+1}.p_c}\right\}\left\{k_{n+1}.\epsilon_a\,\epsilon_{n+1}.p_c-k_{n+1}.p_c\,\epsilon_{n+1}.\epsilon_a\right\}\,\widetilde{P}_{a(n+a)}\Bigg].\end{aligned}$$

(4.47)

4.2.3 Comparison with soft graviton theorem

The first term $\mathcal{A}_1$ given in (4.26) exactly matches the leading soft graviton result given by the first term on the right hand side of (4.10) once we make the identification $\varepsilon_{a,\mu\nu} = \epsilon_{a,\mu}\,\epsilon_{a,\nu}$. To check that the second and the third terms $\mathcal{A}_2$ and $\mathcal{A}_3$ given respectively in (4.30) and (4.47) agree with the known subleading soft graviton theorem, we first express (4.12) as

$$S^{(1)}_{n+1} = S^{(1)}_{\rm orbital} + S^{(1)}_{\rm spin} \tag{4.48}$$

where

$$\begin{aligned}
S^{(1)}_{\rm orbital} &= \sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}\Big[\big(\epsilon_{n+1}.p_a\big)k_{n+1,\rho}\frac{\partial}{\partial p_{a\rho}} - \big(k_{n+1}.p_a\big)\epsilon_{(n+1),\nu}\frac{\partial}{\partial p_{a\nu}}\Big], \\
S^{(1)}_{\rm spin} &= \sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}\epsilon_{(n+1),\nu}k_{(n+1),\mu}J^{\mu\nu}_a\,.
\end{aligned} \tag{4.49}$$

Therefore we have, from (4.5)

$$\begin{aligned}
S^{(1)}_{n+1}\,\mathbf{M}_n &= \{S^{(1)}_{\rm orbital} + S^{(1)}_{\rm spin}\}\int D\sigma \prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)I_n \\
&= \int D\sigma \sum_{l=1}^{n}\left\{S^{(1)}_{\rm orbital}\,\delta\Big(\sum_{\substack{c=1\\ c\neq l}}^{n}\frac{p_l.p_c}{\sigma_l-\sigma_c}\Big)\right\}\prod_{\substack{a=1\\ a\neq i,j,k,l}}^{n}\delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)\,I_n \\
&\quad + \int D\sigma \prod_{\substack{a=1\\ a\neq i,j,k}}^{n}\delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)\,\{S^{(1)}_{\rm orbital} + S^{(1)}_{\rm spin}\}I_n\,.
\end{aligned} \tag{4.50}$$

First let us compute the operation of $S^{(1)}_{\rm orbital}$ on the delta function,

$$\begin{aligned}
&S^{(1)}_{\rm orbital}\delta\Big(\sum_{\substack{c=1\\ c\neq l}}^{n}\frac{p_l.p_c}{\sigma_l-\sigma_c}\Big) \\
&= \delta'\Big(\sum_{\substack{c=1\\ c\neq l}}^{n}\frac{p_l.p_c}{\sigma_l-\sigma_c}\Big)\sum_{a=1}^{n}\frac{(\epsilon_{n+1}.p_a)}{k_{n+1}.p_a}\Big[\big(\epsilon_{n+1}.p_a\big)k_{n+1,\rho}\frac{\partial}{\partial p_{a\rho}} - \big(k_{n+1}.p_a\big)\epsilon_{n+1,\rho}\frac{\partial}{\partial p_{a\rho}}\Big]\sum_{\substack{c=1\\ c\neq l}}^{n}\frac{p_l.p_c}{\sigma_l-\sigma_c} \\
&= \delta'\Big(\sum_{\substack{c=1\\ c\neq l}}^{n}\frac{p_l.p_c}{\sigma_l-\sigma_c}\Big)\sum_{\substack{a=1\\ a\neq l}}^{n}\frac{1}{\sigma_l-\sigma_a}\Big[-2\big(\epsilon_{n+1}.p_l\big)\big(\epsilon_{n+1}.p_a\big) + \big(\epsilon_{n+1}.p_l\big)^2\frac{k_{n+1}.p_a}{k_{n+1}.p_l} \\
&\quad + \big(\epsilon_{n+1}.p_a\big)^2\frac{k_{n+1}.p_l}{k_{n+1}.p_a}\Big]\,.
\end{aligned} \tag{4.51}$$

With this, the first term on the right hand side of equation (4.50) becomes,

$$-\int D\sigma \sum_{l=1}^{n} \prod_{\substack{a=1\\ a\neq i,j,k,l}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a\cdot p_b}{\sigma_{ab}}\Big)\delta'\Big(\sum_{\substack{c=1\\ c\neq l}}^{n} \frac{p_l\cdot p_c}{\sigma_{lc}}\Big) \sum_{\substack{a=1\\ a\neq l}}^{n} \frac{1}{\sigma_l-\sigma_a}$$
$$\Big[2\big(\epsilon_{n+1}\cdot p_l\big)\big(\epsilon_{n+1}\cdot p_a\big) - \big(\epsilon_{n+1}\cdot p_l\big)^2 \frac{k_{n+1}\cdot p_a}{k_{n+1}\cdot p_l} - \big(\epsilon_{n+1}\cdot p_a\big)^2 \frac{k_{n+1}\cdot p_l}{k_{n+1}\cdot p_a}\Big]\; I_n\,. \qquad (4.52)$$

This exactly agrees with $\mathcal{A}_2$ given in equation (4.30).

It remains to show that the second term on the right hand side of equation (4.50) agrees with $\mathcal{A}_3$. Using the definition of I_n given in (4.7) and that $\det(\Psi^{st}_{st}) = \det\widetilde{\Psi} = \widetilde{P}^2$, we get

$$I_n = 4\,(-1)^n\,(\sigma_s-\sigma_t)^{-2}\,\widetilde{P}^2\,. \qquad (4.53)$$

In order to calculate the action of $S^{(1)}$ on this, it will be useful to record how $S^{(1)}_{\text{spin}}$ defined in (4.49) acts on the 'square root' ϵ of the graviton polarization tensor ε. This is done by identifying the action of $J^{\mu\nu}$ on ϵ as

$$\big(J^{\mu\nu}\,\epsilon\big)_\rho \equiv \big(J^{\mu\nu}\big)_\rho{}^\alpha\epsilon_\alpha = \delta^\mu_\rho\epsilon^\nu - \delta^\nu_\rho\epsilon^\mu\,. \qquad (4.54)$$

This allows us to express $S^{(1)}_{n+1}$ as

$$S^{(1)}_{n+1} = \sum_{c=1}^{n} \frac{\epsilon_{n+1}\cdot p_c}{k_{n+1}\cdot p_c}\;\epsilon^\mu_{n+1}\;k^\nu_{n+1}\;\left(p_{c\mu}\frac{\partial}{\partial p_{c\nu}} - p_{c\nu}\frac{\partial}{\partial p_{c\mu}} + \epsilon_{c\mu}\frac{\partial}{\partial \epsilon_{c\nu}} - \epsilon_{c\nu}\frac{\partial}{\partial \epsilon_{c\mu}}\right)\,, \qquad (4.55)$$

and therefore

$$S^{(1)}_{n+1}\,I_n = 8\,(-1)^n\,(\sigma_s-\sigma_t)^{-2}\,\widetilde{P}\,S^{(1)}_{n+1}\widetilde{P}\,. \qquad (4.56)$$

Using (4.55) and (B.42) we get

$$\begin{aligned}
S^{(1)}_{n+1}A_{ab} &= \frac{1}{\sigma_a-\sigma_b}\left[\frac{\epsilon_{n+1}\cdot p_a}{k_{n+1}\cdot p_a} - \frac{\epsilon_{n+1}\cdot p_b}{k_{n+1}\cdot p_b}\right]\;\epsilon^\mu_{n+1}\;k^\nu_{n+1}\,\big[p_{a\mu}p_{b\nu} - p_{a\nu}p_{b\mu}\big]\,,\\
S^{(1)}_{n+1}B_{ab} &= \frac{1}{\sigma_a-\sigma_b}\left[\frac{\epsilon_{n+1}\cdot p_a}{k_{n+1}\cdot p_a} - \frac{\epsilon_{n+1}\cdot p_b}{k_{n+1}\cdot p_b}\right]\;\epsilon^\mu_{n+1}\;k^\nu_{n+1}\,\big[\epsilon_{a\mu}\epsilon_{b\nu} - \epsilon_{a\nu}\epsilon_{b\mu}\big]\,,\\
S^{(1)}_{n+1}C_{ab} &= \frac{1}{\sigma_a-\sigma_b}\;\epsilon^\mu_{n+1}\;k^\nu_{n+1}\left[\frac{\epsilon_{n+1}\cdot p_a}{k_{n+1}\cdot p_a} - \frac{\epsilon_{n+1}\cdot p_b}{k_{n+1}\cdot p_b}\right]\big[\epsilon_{a\mu}\,p_{b\nu} - \epsilon_{a\nu}p_{b\mu}\big] \quad \text{for } a\neq b\,,\\
S^{(1)}_{n+1}C_{aa} &= -\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{1}{\sigma_a-\sigma_b}\;\epsilon^\mu_{n+1}\;k^\nu_{n+1}\left[\frac{\epsilon_{n+1}\cdot p_a}{k_{n+1}\cdot p_a} - \frac{\epsilon_{n+1}\cdot p_b}{k_{n+1}\cdot p_b}\right]\big[\epsilon_{a\mu}\,p_{b\nu} - \epsilon_{a\nu}p_{b\mu}\big]\,.
\end{aligned} \qquad (4.57)$$

We now use a formula similar to (4.37) with $(n+1)$ replaced by n and ∂_τ replaced by $S^{(1)}_{n+1}$:

$$S^{(1)}_{n+1}\widetilde{P} = \sum_{\substack{a,b=1\\ a<b,a,b\neq s,t}}^{n} (-1)^{a-b+1}(S^{(1)}_{n+1}A_{ab})\,\widetilde{P}_{ab} + \sum_{\substack{a,b=1\\ a<b}}^{n}(-1)^{a-b+1}(S^{(1)}_{n+1}B_{ab})\,\widetilde{P}_{(n+a)(n+b)} + \sum_{\substack{a,b=1\\ a\neq s,t}}^{n}(-1)^{a-n-b+1}(-S^{(1)}_{n+1}C_{ba})\,\widetilde{P}_{a(n+b)}\,, \tag{4.58}$$

and use (4.56), (4.57) to express the second term on the right hand side of eq.(4.50) as

$$\begin{aligned}
&\int D\sigma \prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{p_a\cdot p_b}{\sigma_{ab}}\Big)\, 8\,(-1)^n\,(\sigma_s-\sigma_t)^{-2}\,\widetilde{P}\\
&\Bigg[\sum_{\substack{a,b=1\\ a\neq b,a,b\neq s,t}}^{n} (-1)^{a-b}\,\widetilde{P}_{ab}\,(\sigma_a-\sigma_b)^{-1}\left\{\frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}-\frac{\epsilon_{n+1}.p_b}{k_{n+1}.p_b}\right\}\,\epsilon_{n+1}.p_b\,k_{n+1}.p_a\\
&+\sum_{\substack{a,b=1\\ a\neq b}}^{n}(-1)^{a-b}\,\widetilde{P}_{(n+a)(n+b)}\,(\sigma_a-\sigma_b)^{-1}\left\{\frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}-\frac{\epsilon_{n+1}.p_b}{k_{n+1}.p_b}\right\}\,\epsilon_{n+1}.\epsilon_b\,k_{n+1}.\epsilon_a\\
&+\sum_{\substack{a,b=1\\ a\neq s,t;a\neq b}}^{n}(-1)^{a-n-b}\,\widetilde{P}_{b(n+a)}\frac{1}{\sigma_a-\sigma_b}\left\{\frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}-\frac{\epsilon_{n+1}.p_b}{k_{n+1}.p_b}\right\}\{\epsilon_{n+1}.\epsilon_a\,k_{n+1}.p_b-\epsilon_{n+1}.p_b\,k_{n+1}.\epsilon_a\}\\
&-\sum_{\substack{a=1\\ a\neq s,t}}^{n}(-1)^n\,\widetilde{P}_{a(n+a)}\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{1}{\sigma_a-\sigma_b}\left\{\frac{\epsilon_{n+1}.p_a}{k_{n+1}.p_a}-\frac{\epsilon_{n+1}.p_b}{k_{n+1}.p_b}\right\}\{\epsilon_{n+1}.\epsilon_a\,k_{n+1}.p_b-\epsilon_{n+1}.p_b\,k_{n+1}.\epsilon_a\}\Bigg].
\end{aligned} \tag{4.59}$$

This is identical to (4.47).

This finishes the proof that the CHY prescription for graviton scattering amplitude is consistent with the subleading single soft graviton theorem.

4.3 Double soft graviton theorem

In this section we shall consider the limit of the graviton scattering amplitude when two of the gravitons carry soft momenta. We shall reproduce the result of [15, 16] with opposite sign and in general gauge. During this analysis we also find some additional terms that were left out in the analysis of [15, 16], which nevertheless cancel at the end.

4.3.1 Double soft limit

We shall first follow [10, 29] to analyze the solutions to the scattering equations of $(n+2)$ particles with momenta $\{p_1, p_2, ..., p_n, \tau\, k_{n+1}, \tau\, k_{n+2}\}$ in the double soft limit $\tau \to 0$ keeping k_{n+1}, k_{n+2} fixed. The scattering equations of first n particles, $(n+1)$-th particle and $(n+2)$-th particle are given respectively by:

$$\sum_{\substack{b=1\\b\neq a}}^{n} \frac{p_a.p_b}{\sigma_a-\sigma_b} + \frac{\tau\, p_a.k_{n+1}}{\sigma_a-\sigma_{n+1}} + \frac{\tau\, p_a.k_{n+2}}{\sigma_a-\sigma_{n+2}} = 0 \qquad \forall a \in \{1,2,...,n\}\,, \tag{4.60}$$

$$\sum_{b=1}^{n} \frac{k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} + \frac{\tau\, k_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}} = 0\,, \tag{4.61}$$

$$\sum_{b=1}^{n} \frac{k_{n+2}.p_b}{\sigma_{n+2}-\sigma_b} + \frac{\tau\, k_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} = 0\,, \tag{4.62}$$

where we have removed overall factors of τ from the last two equations. We also have momentum conservation relation,

$$p_1^\mu + p_2^\mu + ... + p_n^\mu + \tau\,(k_{n+1}^\mu + k_{n+2}^\mu) = 0\,. \tag{4.63}$$

The solution to the scattering equations can be divided into two classes [10,29]: degenerate solutions where $\sigma_{n+1}-\sigma_{n+2} \sim \tau$ and non-degenerate solutions for which $\sigma_{n+1}-\sigma_{n+2} \sim 1$ in the $\tau \to 0$ limit. For both classes of solutions σ_a's for $1 \leq a \leq n$ remain finite distance away from each other and from σ_{n+1}, σ_{n+2} as $\tau \to 0$. In order to show that there are no other types of solutions we shall now show that the total number of degenerate and non-degenerate solutions add up to $(n-1)!$ – the actual number of solutions for scattering equation of $(n+2)$ particles [25].

At $\tau = 0$, eq.(4.60) describes the scattering equations for n particles which have $(n-3)!$ solutions for $\{\sigma_1, \sigma_2, ..., \sigma_n\}$. Generically these solutions are non-degenerate, with non-coincident σ_a's. Therefore to count the number of non-degenerate and degenerate solutions in the $\tau \to 0$ limit, we have to multiply $(n-3)!$ by the number of solutions for σ_{n+1} and σ_{n+2} for fixed $\sigma_1, \cdots, \sigma_n$.

Let us first count the number of non-degenerate solutions. For fixed $\sigma_1, \cdots, \sigma_n$, we can ignore the last term in (4.61) in $\tau \to 0$ limit and express this as a polynomial equation for σ_{n+1}:

$$\sum_{a=1}^{n} k_{k+1}.p_a \prod_{\substack{b=1\\b\neq a}}^{n} (\sigma_{n+1}-\sigma_b) = 0\,. \tag{4.64}$$

Naively this is of degree $(n-1)$, but momentum conservation (4.63) makes the coefficient of the $(\sigma_{n+1})^{n-1}$ term $-\tau\, k_{n+1}.k_{n+2}$ which vanish in the $\tau \to 0$ limit. Therefore in this limit this is a polynomial equation of degree $(n-2)$ and gives $(n-2)$ solutions for σ_{n+1}. Similarly eq.(4.62) gives

$(n-2)$ solutions for σ_{n+2}. Therefore the total number of non-degenerate solutions is given by

$$(n-3)! \times (n-2)^2 = (n-2)! \times (n-2)\,. \tag{4.65}$$

For counting the number of degenerate solutions, we first define ρ, ξ through [10, 29],

$$\sigma_{n+1} = \rho - \frac{\xi}{2}\,, \qquad \sigma_{n+2} = \rho + \frac{\xi}{2}\,, \tag{4.66}$$

and add and subtract (4.61) and (4.62) to write them as,

$$\sum_{b=1}^{n} \frac{k_{n+1}.p_b}{\rho - \frac{\xi}{2} - \sigma_b} + \sum_{b=1}^{n} \frac{k_{n+2}.p_b}{\rho + \frac{\xi}{2} - \sigma_b} = 0\,, \tag{4.67}$$

$$\sum_{b=1}^{n} \frac{k_{n+1}.p_b}{\rho - \frac{\xi}{2} - \sigma_b} - \sum_{b=1}^{n} \frac{k_{n+2}.p_b}{\rho + \frac{\xi}{2} - \sigma_b} - \frac{2\tau\, k_{n+2}.k_{n+1}}{\xi} = 0\,. \tag{4.68}$$

Expanding ξ as $\xi = \tau\,\xi_1 + \tau^2\,\xi_2 + O(\tau^3)$, the second equation may be written as,

$$\frac{1}{\xi_1} = \frac{1}{k_{n+1}.k_{n+2}} \sum_{b=1}^{n} \frac{k_{n+1}.p_b}{\rho - \sigma_b} = -\frac{1}{k_{n+1}.k_{n+2}} \sum_{b=1}^{n} \frac{k_{n+2}.p_b}{\rho - \sigma_b}\,. \tag{4.69}$$

On the other hand (4.67) in the $\tau \to 0$ limit gives

$$\sum_{b=1}^{n} \frac{(k_{n+1} + k_{n+2}).p_b}{\rho - \sigma_b} = 0$$

$$\Rightarrow \sum_{b=1}^{n} (k_{n+1} + k_{n+2}).p_b \prod_{a=1, a\neq b}^{n} (\rho - \sigma_b) = 0 \tag{4.70}$$

This is a polynomial equation in ρ of degree $(n-2)$, since the ρ^{n-1} coefficient vanishes by momentum conservation (4.63) as $\tau \to 0$. Hence it gives $(n-2)$ solutions for ρ for a given set of $\{\sigma_1, \sigma_2, ..., \sigma_n\}$. For each such solution, ξ_1 is fixed uniquely from (4.69). Therefore we will get total of

$$(n-3)!(n-2) = (n-2)! \tag{4.71}$$

degenerate solutions. Adding this to (4.65) we get $(n-1)!$ solutions which is the expected total number of solutions of the scattering equations for $(n+2)$ particles. This shows that we have not missed any solution.

4.3.2 Contribution from non-degenerate solutions

The contribution from the non-degenerate solutions to subleading order in τ may be written as,

$$\int D\sigma \int d\sigma_{n+1} d\sigma_{n+2} \Big[\prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a.p_b}{\sigma_{ab}} + \tau \frac{p_a.k_{n+1}}{\sigma_{a(n+1)}} + \tau \frac{p_a.k_{n+2}}{\sigma_{a(n+2)}} \Big) \Big]$$
$$\delta\Big(\tau \sum_{b=1}^{n} \frac{k_{n+1}.p_b}{\sigma_{(n+1)b}} + \tau^2 \frac{k_{n+1}.k_{n+2}}{\sigma_{(n+1)(n+2)}} \Big) \delta\Big(\tau \sum_{b=1}^{n} \frac{k_{n+2}.p_b}{\sigma_{(n+2)b}} + \tau^2 \frac{k_{n+2}.k_{n+1}}{\sigma_{(n+2)(n+1)}} \Big) I_{n+2}\,, \qquad (4.72)$$

where it is understood that we pick contributions from those zeroes of the δ-functions for which $\sigma_{n+1} - \sigma_{n+2} \sim 1$ as $\tau \to 0$. The product of first $(n-3)$ delta functions can be expanded in powers of τ as:

$$\begin{aligned} & \prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a.p_b}{\sigma_{ab}} + \tau \frac{p_a.k_{n+1}}{\sigma_{a(n+1)}} + \tau \frac{p_a.k_{n+2}}{\sigma_{a(n+2)}} \Big) \\ = & \prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a.p_b}{\sigma_{ab}} \Big) + \tau \sum_{l=1}^{n} \Big[\frac{p_l.k_{n+1}}{\sigma_{l(n+1)}} + \frac{p_l.k_{n+2}}{\sigma_{l(n+2)}} \Big] \delta'\Big(\sum_{\substack{c=1\\ c\neq l}}^{n} \frac{p_l.p_c}{\sigma_{lc}} \Big) \prod_{\substack{a=1\\ a\neq i,j,k,l}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a.p_b}{\sigma_{ab}} \Big) \\ & \equiv \delta^{(0)} + \tau\, \delta^{(1)}\,. \end{aligned} \qquad (4.73)$$

On the other hand the product of last two delta functions have the form:

$$\delta\Big(\tau \sum_{b=1}^{n} \frac{k_{n+1}.p_b}{\sigma_{(n+1)b}} + \tau^2 \frac{k_{n+1}.k_{n+2}}{\sigma_{(n+1)(n+2)}} \Big) = \frac{1}{\tau} \Big[\delta(f^n_{n+1}) + \tau \frac{k_{n+1}.k_{n+2}}{\sigma_{(n+1)(n+2)}} \delta'(f^n_{n+1}) + O(\tau^2) \Big]\,, \qquad (4.74)$$

$$\delta\Big(\tau \sum_{b=1}^{n} \frac{k_{n+2}.p_b}{\sigma_{(n+2)b}} + \tau^2 \frac{k_{n+2}.k_{n+1}}{\sigma_{(n+2)(n+1)}} \Big) = \frac{1}{\tau} \Big[\delta(f^n_{n+2}) + \tau \frac{k_{n+2}.k_{n+1}}{\sigma_{(n+2)(n+1)}} \delta'(f^n_{n+2}) + O(\tau^2) \Big]\,, \qquad (4.75)$$

where f^n_{n+r} has been defined in (4.17). Finally the integrand I_{n+2} may be expanded as,

$$I_{n+2} \;=\; I_{n+2}|_{\tau=0} + \tau \frac{\partial I_{n+2}}{\partial \tau}|_{\tau=0} + O(\tau^2) \equiv I^{(0)}_{n+2} + \tau\, I^{(1)}_{n+2} + O(\tau^2)\,. \qquad (4.76)$$

Substituting all these expansions in the $(n+2)$-point amplitude (4.72), we get

$$\begin{aligned}
&\frac{1}{\tau^2}\int D\sigma \int d\sigma_{n+1}\, d\sigma_{n+2}\big[\delta^{(0)}+\tau\,\delta^{(1)}\big]\Big[\delta(f^n_{n+1})+\tau\,\frac{k_{n+1}.k_{n+2}}{\sigma_{(n+1)(n+2)}}\,\delta'(f^n_{n+1})\Big]\\
&\Big[\delta(f^n_{n+2})+\tau\,\frac{k_{n+2}.k_{n+1}}{\sigma_{(n+2)(n+1)}}\,\delta'(f^n_{n+2})\Big]\,\big[I^{(0)}_{n+2}+\tau\,I^{(1)}_{n+2}\big]\\
=\;&\frac{1}{\tau^2}\int D\sigma\,\delta^{(0)}\int d\sigma_{n+1}\,d\sigma_{n+2}\,\delta(f^n_{n+1})\,\delta(f^n_{n+2})\,I^{(0)}_{n+2}\\
&+\frac{1}{\tau}\int D\sigma\int d\sigma_{n+1}\,d\sigma_{n+2}\,\delta^{(1)}\,\delta(f^n_{n+1})\,\delta(f^n_{n+2})\,I^{(0)}_{n+2}\\
&+\frac{1}{\tau}\int D\sigma\,\delta^{(0)}\int d\sigma_{n+1}\,d\sigma_{n+2}\,\delta(f^n_{n+1})\,\delta(f^n_{n+2})\,I^{(1)}_{n+2}\\
&+\frac{1}{\tau}\int D\sigma\,\delta^{(0)}\int d\sigma_{n+1}\,d\sigma_{n+2}\,\frac{k_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}}\,\delta'(f^n_{n+1})\,\delta(f^n_{n+2})\,I^{(0)}_{n+2}\\
&+\frac{1}{\tau}\int D\sigma\,\delta^{(0)}\int d\sigma_{n+1}\,d\sigma_{n+2}\,\delta(f^n_{n+1})\frac{k_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}}\,\delta'(f^n_{n+2})\,I^{(0)}_{n+2}\,.
\end{aligned}\tag{4.77}$$

For evaluating this, it will be convenient to introduce two soft parameters τ_1 and τ_2 instead of a single parameter τ and take the external momenta to be $p_1,\cdots p_n,\tau_1\,k_{n+1},\tau_2\,k_{n+2}$. I_{n+2} is then given by

$$I_{n+2}=4\,(-1)^{n+2}\,(\sigma_s-\sigma_t)^{-2}\,\widehat{P}^2 \tag{4.78}$$

where $\widehat{P}$ is the Pfaffian of the matrix

$$\widehat{\Psi}=\begin{pmatrix}
A_{ab} & \frac{\tau_1 p_a.k_{n+1}}{\sigma_a-\sigma_{n+1}} & \frac{\tau_2 p_a.k_{n+2}}{\sigma_a-\sigma_{n+2}} & -C^T_{ad} & \frac{p_a.\epsilon_{n+1}}{\sigma_a-\sigma_{n+1}} & \frac{p_a.\epsilon_{n+2}}{\sigma_a-\sigma_{n+2}}\\
\frac{\tau_1 k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} & 0 & 0 & \frac{\tau_1 k_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d} & -C_{(n+1)(n+1)} & -\frac{\tau_1\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}}\\
\frac{\tau_2 k_{n+2}.p_b}{\sigma_{n+2}-\sigma_b} & 0 & 0 & \frac{\tau_2 k_{n+2}.\epsilon_d}{\sigma_{n+2}-\sigma_d} & -\frac{\tau_2\epsilon_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}} & -C_{(n+2)(n+2)}\\
C_{cb} & \frac{\tau_1\epsilon_c.k_{n+1}}{\sigma_c-\sigma_{n+1}} & \frac{\tau_2\epsilon_c.k_{n+2}}{\sigma_c-\sigma_{n+2}} & B_{cd} & \frac{\epsilon_c.\epsilon_{n+1}}{\sigma_c-\sigma_{n+1}} & \frac{\epsilon_c.\epsilon_{n+2}}{\sigma_c-\sigma_{n+2}}\\
\frac{\epsilon_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} & C_{(n+1)(n+1)} & \frac{\tau_2\epsilon_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}} & \frac{\epsilon_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d} & 0 & \frac{\epsilon_{n+1}.\epsilon_{n+2}}{\sigma_{n+1}-\sigma_{n+2}}\\
\frac{\epsilon_{n+2}.p_b}{\sigma_{n+2}-\sigma_b} & \frac{\tau_1\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} & C_{(n+2)(n+2)} & \frac{\epsilon_{n+2}.\epsilon_d}{\sigma_{n+2}-\sigma_d} & \frac{\epsilon_{n+2}.\epsilon_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} & 0
\end{pmatrix}. \tag{4.79}$$

In (4.79) we have set terms with two or more powers of τ to be zero. The values of $C_{(n+1)(n+1)}$ and $C_{(n+2)(n+2)}$ to linear order in τ are:

$$C_{(n+1)(n+1)}=-\sum_{a=1}^{n}\frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}-\frac{\tau_2\,\epsilon_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}}\,,\quad C_{(n+2)(n+2)}=-\sum_{a=1}^{n}\frac{\epsilon_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}-\frac{\tau_1\,\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}}\,. \tag{4.80}$$

At the end of the computation we shall take the limit $\tau_1,\tau_2\to\tau\to 0$.

We shall now analyze the different terms on the right hand side of (4.77). However, before we proceed to evaluate the integrals, we would like to remind the reader of the general strategy for analyzing these integrals that was described in the last paragraph of section 4.1. The expansion

(4.77) of the amplitude in powers of τ is valid when the integration contours wrap around the non-degenerate solutions for which $\sigma_{n+1} - \sigma_{n+2} \sim 1$. Once we have made this approximation we can deform the contours of σ_{n+1} and σ_{n+2} even to regions where $|\sigma_{n+1} - \sigma_{n+2}| << 1$. Treating (4.77) as the integrand still gives the correct result by Cauchy's theorem even though the original integrand is no longer approximated by (4.77). A similar remark will hold for the contribution from degenerate solutions – we shall make our approximation in the region where the integration contour is close to the degenerate solution of the scattering equation, and once the approximation is made, we shall be free to deform the contour away from the region where the approximation is valid.

First two terms

In (4.77), the first two terms on the right hand side may be analyzed as in the case of single soft graviton by carrying out the integration over σ_{n+2} and σ_{n+1} independently. For this we note from (4.79) that

$$\widehat{P}|_{\tau_1=\tau_2=0} = -C_{(n+1)(n+1)}\, C_{(n+2)(n+2)}\, \widetilde{P} = -\left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)\left(\sum_{a=1}^{n} \frac{\epsilon_{n+2}\cdot p_a}{\sigma_{n+2}-\sigma_a}\right)\widetilde{P}\,, \quad (4.81)$$

where $\widetilde{P}$ is the Pfaffian of the matrix $\widetilde{\Psi}$ for n-particle scattering amplitude without soft graviton. This gives

$$I^{(0)}_{n+2} = \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)^2 \left(\sum_{a=1}^{n} \frac{\epsilon_{n+2}\cdot p_a}{\sigma_{n+2}-\sigma_a}\right)^2 I_n\,. \quad (4.82)$$

We can now proceed to evaluate the first two terms on the right hand side of (4.77) as in sections 4.2.2 and 4.2.2. The first term is given by

$$\begin{aligned}\mathcal{B}_0 \;\equiv\; & \frac{1}{\tau^2}\int D\sigma\, \delta^{(0)}\, I_n \oint_{\{A_i\}} d\sigma_{n+1} \left(\sum_{a=1}^{n} \frac{k_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1} \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)^2 \\ & \oint_{\{B_i\}} d\sigma_{n+2} \left(\sum_{a=1}^{n} \frac{k_{n+2}\cdot p_a}{\sigma_{n+2}-\sigma_a}\right)^{-1} \left(\sum_{a=1}^{n} \frac{\epsilon_{n+2}\cdot p_a}{\sigma_{n+2}-\sigma_a}\right)^2, \end{aligned} \quad (4.83)$$

where $\{A_i\}$ and $\{B_i\}$ are defined as the set of points satisfying

$$\sum_{a=1}^{n} \frac{k_{n+1}\cdot p_a}{\rho-\sigma_a} = 0 \quad \text{at} \quad \rho = A_i\,, \qquad \sum_{a=1}^{n} \frac{k_{n+2}\cdot p_a}{\rho-\sigma_a} = 0 \quad \text{at} \quad \rho = B_i\,. \quad (4.84)$$

We can now carry out integration over σ_{n+1} and σ_{n+2} independently by deforming the contours to ∞. The contribution comes from residues at the poles at σ_a – which can be evaluated following the procedure described in section 4.2.2 – and the contribution from ∞ can be evaluated after using momentum conservation. The result is

$$\begin{aligned}\mathcal{B}_0 &\equiv \frac{1}{\tau^2}\int D\sigma\,\delta^{(0)}\,I_n\left[\sum_{a=1}^{n}\frac{(\epsilon_{n+1}.p_a)^2}{k_{n+1}.p_a}+\tau\,(\epsilon_{n+1}.k_{n+2})^2\,(k_{n+1}.k_{n+2})^{-1}\right]\\ &\qquad\left[\sum_{a=1}^{n}\frac{(\epsilon_{n+2}.p_a)^2}{k_{n+2}.p_a}+\tau\,(\epsilon_{n+2}.k_{n+1})^2\,(k_{n+1}.k_{n+2})^{-1}\right]. \end{aligned}\tag{4.85}$$

Using $\mathbf{M}_n=\int D\sigma\,\delta^{(0)}\,I_n$, this may be rewritten as

$$\begin{aligned}\mathcal{B}_0 &= \tau^{-2}\,S^{(0)}_{n+1}\,S^{(0)}_{n+2}\,\mathbf{M}_n+\tau^{-1}\Big[S^{(0)}_{n+1}\,(\epsilon_{n+2}.k_{n+1})^2\,(k_{n+1}.k_{n+2})^{-1}\\ &\qquad +S^{(0)}_{n+2}\,(\epsilon_{n+1}.k_{n+2})^2\,(k_{n+1}.k_{n+2})^{-1}\Big]\mathbf{M}_n+O(\tau^0)\,,\end{aligned}\tag{4.86}$$

where we define

$$S^{(0)}_r\equiv\sum_{a=1}^{n}\frac{(\epsilon_r.p_a)^2}{p_a.k_r}\,,\qquad S^{(1)}_r\equiv\sum_{a=1}^{n}\frac{\epsilon_{r,\mu}\,\epsilon_{r,\nu}\,p_a^\mu\,k_{r,\rho}\,\widehat{J}_a^{\rho\nu}}{k_r.p_a}\,.\tag{4.87}$$

For evaluating the second term on the right hand side of (4.77) we note that $\delta^{(1)}$ defined in (4.73) contains sum of two sets of terms – one set involving $1/\sigma_{l(n+1)}$ and the other set involving $1/\sigma_{l(n+2)}$. First consider the set involving $1/\sigma_{l(n+1)}$. In this case we can carry out the integration over σ_{n+2} first following the procedure of section 4.2.2 and arrive at the product of $S^{(0)}_{n+2}$ times an integral of the form given in (4.27). The contribution from the pole at $\sigma_{n+2}=\infty$ can be ignored since that will produce an extra factor of τ and give a subsubleading contribution. The integration over σ_{n+1} is proportional to (4.27) and can be analyzed as in section 4.2.2, leading to the result (4.30) multiplied by $S^{(0)}_{n+2}$. Using the equality between (4.30), (4.52) and the first term on the right hand side of (4.50), this may be expressed as

$$\tau^{-1}\,S^{(0)}_{n+2}\int D\sigma\,(S^{(1)}_{n+1}\,\delta^{(0)})\,I_n=\tau^{-1}\,S^{(0)}_{n+2}S^{(1)}_{n+1}\mathbf{M}_n-\tau^{-1}\,S^{(0)}_{n+2}\int D\sigma\delta^{(0)}\,S^{(1)}_{n+1}I_n\,,\tag{4.88}$$

where in the left hand side of this equation it is understood that only the orbital part of $S^{(1)}_{n+1}$ acts on $\delta^{(0)}$. The contribution from the second term in $\delta^{(1)}$ introduced in (4.73) is given by an expression similar to that in (4.88) but with $(n+1)$ and $(n+2)$ interchanged. Therefore the total contribution from the second term on the right hand side of (4.77) may be expressed as

$$\mathcal{B}_1=\frac{1}{\tau}\left\{S^{(0)}_{n+1}S^{(1)}_{n+2}+S^{(0)}_{n+2}S^{(1)}_{n+1}\right\}\,\mathbf{M}_n-\frac{1}{\tau}\,S^{(0)}_{n+2}\int D\sigma\delta^{(0)}\,S^{(1)}_{n+1}I_n-\frac{1}{\tau}\,S^{(0)}_{n+1}\int D\sigma\delta^{(0)}\,S^{(1)}_{n+2}I_n\,.\tag{4.89}$$

We shall now turn to the analysis of the last three terms in (4.77).

Third Term:

We now consider the third term on the right hand side of (4.77):

$$\frac{1}{\tau}\int D\sigma\,\delta^{(0)}\int d\sigma_{n+1}\,d\sigma_{n+2}\,\delta(f^n_{n+1})\,\delta(f^n_{n+2})\,I^{(1)}_{n+2}\,. \tag{4.90}$$

To evaluate this we have to first evaluate $I^{(1)}_{n+2}$. We have the analog of (4.34):

$$I^{(1)}_{n+2}\equiv\left(\frac{\partial I_{n+2}}{\partial\tau_1}+\frac{\partial I_{n+2}}{\partial\tau_2}\right)\Big|_{\tau_1=\tau_2=0}=8(-1)^{n+2}\,(\sigma_s-\sigma_t)^{-2}\;\widehat{P}\,\left[\frac{\partial\widehat{P}}{\partial\tau_1}+\frac{\partial\widehat{P}}{\partial\tau_2}\right]\Big|_{\tau_1=\tau_2=0}, \tag{4.91}$$

where $\widehat{P}$ is the Pfaffian of the matrix $\widehat{\Psi}$ given in (4.79).

Let us now examine the τ_1 derivative of $\widehat{P}$. Since we have to set $\tau_2=0$ at the end, we can make this replacement even before taking the τ_1 derivative. In this case the Pfaffian $\widehat{P}$ of the matrix (4.79) reduces to

$$(-1)^n\,C_{(n+2)(n+2)}\,Pf\begin{pmatrix} A_{ab} & \frac{\tau_1 p_a.k_{n+1}}{\sigma_a-\sigma_{n+1}} & -C^T_{ad} & \frac{p_a.\epsilon_{n+1}}{\sigma_a-\sigma_{n+1}}\\ \frac{\tau_1 k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} & 0 & \frac{\tau_1 k_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d} & -C_{(n+1)(n+1)}\\ C_{cb} & \frac{\tau_1\epsilon_c.k_{n+1}}{\sigma_c-\sigma_{n+1}} & B_{cd} & \frac{\epsilon_c.\epsilon_{n+1}}{\sigma_c-\sigma_{n+1}}\\ \frac{\epsilon_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} & C_{(n+1)(n+1)} & \frac{\epsilon_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d} & 0\end{pmatrix}. \tag{4.92}$$

One can now recognize this matrix as the same matrix that appears in (4.39). Therefore when the τ_1 derivative acts on the Pfaffian of this matrix, then at $\tau_1=0$ it will have the same form as that in (4.47) after σ_{n+1} integration. The extra factor of $C_{(n+2)(n+2)}$ at $\tau_1=0$, multiplied by a similar factor coming from the $\widehat{P}$ factor in (4.91), will generate a factor of $S^{(0)}_{n+2}$ after integration over σ_{n+2}. There is an additional contribution from $\sigma_{n+2}=\infty$, but this is subsubleading and may be ignored. Using the equality of (4.47), (4.59) and second term on the right hand side of (4.50), the net contribution of this term to the right hand side of (4.77) can be shown to have the form:

$$S^{(0)}_{n+2}\int D\sigma\,\delta^{(0)}\,S^{(1)}_{n+1}\,I_n\,. \tag{4.93}$$

A similar term with $(n+1)$ and $(n+2)$ exchanged comes from the $\partial\widehat{P}/\partial\tau_2$ term in (4.91). The sum of these two give

$$\mathcal{B}_2\;\equiv\;\frac{1}{\tau}S^{(0)}_{n+2}\int D\sigma\,\delta^{(0)}S^{(1)}_{n+1}I_n+\frac{1}{\tau}S^{(0)}_{n+1}\int D\sigma\,\delta^{(0)}S^{(1)}_{n+2}I_n\,. \tag{4.94}$$

This cancels the last two terms in (4.89).

This however is not the complete contribution from the third term in (4.77), since we still have to include the contribution where τ_1 derivative acts on the $C_{(n+2)(n+2)}$ factor in (4.92), and a similar contribution with $(n+1)$ and $(n+2)$ exchanged. Now from (4.80) we have

$$\frac{\partial C_{(n+2)(n+2)}}{\partial \tau_1} = -\frac{\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} \,. \tag{4.95}$$

Therefore at $\tau_1 = \tau_2 = 0$, the contribution to $\partial \widehat{P}/\partial \tau_1$, with the derivative acting on the first term in (4.92), reduces to

$$\frac{\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} C_{(n+1)(n+1)}|_{\tau_1=\tau_2=0}\, \widetilde{P} = -\frac{\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} \sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\, \widetilde{P}\,, \tag{4.96}$$

where $\widetilde{P}$ is the Pfaffian of the matrix $\begin{pmatrix} A_{ab} & -(C^T)_{ad} \\ C_{cb} & B_{cd} \end{pmatrix}$. Substituting this into (4.91) and setting $\tau_1 = \tau_2 = 0$, using (4.81), and adding also the extra contribution from the $\partial \widehat{P}/\partial \tau_2$ term in (4.91), we get the net extra contribution to $I^{(1)}_{n+2}$ to be

$$\begin{aligned} & 8\,(-1)^n\,(\sigma_s-\sigma_t)^{-2} \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right) \left(\sum_{a=1}^{n} \frac{\epsilon_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\right) \widetilde{P}^2 \\ & \left[\frac{\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} \sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a} + \frac{\epsilon_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}} \sum_{a=1}^{n} \frac{\epsilon_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\right] . \end{aligned} \tag{4.97}$$

Noting that $4\,(-1)^n\,(\sigma_s-\sigma_t)^{-2}\,\widetilde{P}^2 = I_n$, the contribution of (4.97) to (4.90) is given by

$$\begin{aligned} & \frac{2}{\tau} \int D\sigma\, \delta^{(0)}\, I_n \int d\sigma_{n+1}\, d\sigma_{n+2}\, \delta(f^n_{n+1})\, \delta(f^n_{n+2}) \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right) \left(\sum_{a=1}^{n} \frac{\epsilon_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\right) \\ & \left[\frac{\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} \sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a} + \frac{\epsilon_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}} \sum_{a=1}^{n} \frac{\epsilon_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\right] . \end{aligned} \tag{4.98}$$

We now regard the integrals over σ_{n+1} and σ_{n+2} as contour integrals as in (B.23). Let us examine the first term in the square bracket. It takes the form

$$\begin{aligned} & \frac{2}{\tau} \int D\sigma\, \delta^{(0)}\, I_n \oint_{\{A_i\}} d\sigma_{n+1} \oint_{\{B_i\}} d\sigma_{n+2} \left(\sum_{a=1}^{n} \frac{k_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1} \left(\sum_{a=1}^{n} \frac{k_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\right)^{-1} \\ & \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right) \left(\sum_{a=1}^{n} \frac{\epsilon_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\right) \left(\frac{\epsilon_{n+2}.k_{n+1}}{\sigma_{n+2}-\sigma_{n+1}}\right) \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right) , \end{aligned} \tag{4.99}$$

where, $\{A_i\}$ and $\{B_i\}$ have been defined in (4.84) and accordingly the σ_{n+1} and σ_{n+2} contours run anti-clockwise around the poles of the first and second factors of the integrand respectively. We shall now deform the σ_{n+2} contour away from the pole towards infinity. The only pole is at $\sigma_{n+2} = \sigma_{n+1}$ and a possible contribution from $\sigma_{n+2} = \infty$. Let us first examine the contribution from the pole at σ_{n+1}. This is given by

$$\begin{aligned}\mathcal{B}_3 \equiv & -\frac{2}{\tau} \int D\sigma\, \delta^{(0)}\, I_n \oint_{\{A_i\}} d\sigma_{n+1} \left(\sum_{a=1}^{n} \frac{k_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1} \left(\sum_{a=1}^{n} \frac{k_{n+2}.p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1} \\ & \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right)^{2} \left(\sum_{a=1}^{n} \frac{\epsilon_{n+2}.p_a}{\sigma_{n+1}-\sigma_a}\right) (\epsilon_{n+2}.k_{n+1}) \,. \end{aligned} \tag{4.100}$$

Next we analyze the contribution from $\sigma_{n+2} = \infty$. Using momentum conservation equation $\sum_{a=1}^{n} p_a = -\tau(k_{n+1} + k_{n+2})$ this is given by

$$\begin{aligned}& \frac{2}{\tau} \int D\sigma\, \delta^{(0)}\, I_n \oint_{\{A_i\}} d\sigma_{n+1} \left(\sum_{a=1}^{n} \frac{k_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1} \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\right)^{2} \\ & \qquad \times (\epsilon_{n+2}.k_{n+1})^2\, (k_{n+1}.k_{n+2})^{-1} \,. \end{aligned} \tag{4.101}$$

Now we deform the σ_{n+1} contour to ∞, picking up residues at σ_a. The result is

$$\mathcal{B}_4 \equiv -\frac{2}{\tau}\, \mathbf{M}_n\, S^{(0)}_{n+1}\, (\epsilon_{n+2}.k_{n+1})^2\, (k_{n+1}.k_{n+2})^{-1} \,. \tag{4.102}$$

The residue at $\sigma_{n+1} = \infty$ has an additional factor of τ. Therefore its contribution is subsubleading.

The contribution from the second term in the square bracket in (4.98) can be obtained from (4.100), (4.102) by exchange of $(n+1)$ and $(n+2)$ and is given by $\mathcal{B}_5 + \mathcal{B}_6$, where

$$\begin{aligned}\mathcal{B}_5 \equiv & -\frac{2}{\tau} \int D\sigma\, \delta^{(0)}\, I_n \oint_{\{B_i\}} d\sigma_{n+2} \left(\sum_{a=1}^{n} \frac{k_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\right)^{-1} \left(\sum_{a=1}^{n} \frac{k_{n+1}.p_a}{\sigma_{n+2}-\sigma_a}\right)^{-1} \\ & \left(\sum_{a=1}^{n} \frac{\epsilon_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\right)^{2} \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}.p_a}{\sigma_{n+2}-\sigma_a}\right) (\epsilon_{n+1}.k_{n+2}) \,, \end{aligned} \tag{4.103}$$

and

$$\mathcal{B}_6 \equiv -\frac{2}{\tau}\, \mathbf{M}_n\, S^{(0)}_{n+2}\, (\epsilon_{n+1}.k_{n+2})^2\, (k_{n+1}.k_{n+2})^{-1} \,. \tag{4.104}$$

Fourth and fifth terms

The fourth term on the right hand side of (4.77) is given by:

$$\begin{aligned}
&\frac{1}{\tau}\int D\sigma\delta^{(0)}\int d\sigma_{n+1}d\sigma_{n+2}\frac{k_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}}\delta'(f^n_{n+1})\delta(f^n_{n+2})I^{(0)}_{n+2}\\
= &-\frac{1}{\tau}\int D\sigma\delta^{(0)}\oint_{\{A_i\}}d\sigma_{n+1}\oint_{\{B_i\}}d\sigma_{n+2}\frac{k_{n+1}.k_{n+2}}{\sigma_{n+1}-\sigma_{n+2}}\Big(\sum_{c=1}^{n}\frac{k_{n+1}.p_c}{\sigma_{n+1}-\sigma_c}\Big)^{-2}\Big(\sum_{a=1}^{n}\frac{k_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\Big)^{-1}\\
&\Big(\sum_{b=1}^{n}\frac{\epsilon_{n+1}.p_b}{\sigma_{n+1}-\sigma_b}\Big)^2\Big(\sum_{b=1}^{n}\frac{\epsilon_{n+2}.p_b}{\sigma_{n+2}-\sigma_b}\Big)^2 I_n\,. \qquad (4.105)
\end{aligned}$$

We now deform the σ_{n+1} contour away from the poles $\{A_i\}$ towards infinity and pick up residues at $\sigma_{n+1}=\sigma_{n+2}$ as well as the contribution from infinity. There are no poles at $\sigma_{n+1}=\sigma_a$. The contribution from the pole at σ_{n+2} is given by

$$\begin{aligned}
\mathcal{B}_7 \;\equiv\; &\frac{1}{\tau}\int D\sigma\,\delta^{(0)}\,I_n\oint_{\{B_i\}}d\sigma_{n+2}\Big(\sum_{c=1}^{n}\frac{k_{n+1}.p_c}{\sigma_{n+2}-\sigma_c}\Big)^{-2}\Big(\sum_{a=1}^{n}\frac{k_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\Big)^{-1}(k_{n+1}.k_{n+2})\\
&\Big(\sum_{b=1}^{n}\frac{\epsilon_{n+2}.p_b}{\sigma_{n+2}-\sigma_b}\Big)^2\Big(\sum_{b=1}^{n}\frac{\epsilon_{n+1}.p_b}{\sigma_{n+2}-\sigma_b}\Big)^2. \qquad (4.106)
\end{aligned}$$

On the other hand the contribution from infinity is given by, after using momentum conservation,

$$\begin{aligned}
\mathcal{B}_8 \;\equiv\; &-\frac{1}{\tau}\int D\sigma\delta^{(0)}\oint_{\{B_i\}}d\sigma_{n+2}\;\frac{(\epsilon_{n+1}.k_{n+2})^2}{k_{n+1}.k_{n+2}}\Big(\sum_{a=1}^{n}\frac{k_{n+2}.p_a}{\sigma_{n+2}-\sigma_a}\Big)^{-1}\Big(\sum_{b=1}^{n}\frac{\epsilon_{n+2}.p_b}{\sigma_{n+2}-\sigma_b}\Big)^2 I_n\\
= \;&\frac{1}{\tau}\mathbf{M}_n\,S^{(0)}_{n+2}\,(k_{n+1}.k_{n+2})^{-1}\,(\epsilon_{n+1}.k_{n+2})^2\,, \qquad (4.107)
\end{aligned}$$

where in the second step we have performed integration over σ_{n+2}, picking up residues at σ_b. In this case, as $\sigma_{n+2}\to\infty$, the contribution to the integrand goes as τ, and therefore this does not contribute at the subleading order.

The contribution from the fifth term on the right hand side of (4.77) can be evaluated in a similar manner, giving contributions similar to (4.106) and (4.107), with $n+1$ and $n+2$ exchanged:

$$\begin{aligned}
\mathcal{B}_9 \;\equiv\; &\frac{1}{\tau}\int D\sigma\,\delta^{(0)}\,I_n\oint_{\{A_i\}}d\sigma_{n+1}\Big(\sum_{c=1}^{n}\frac{k_{n+2}.p_c}{\sigma_{n+1}-\sigma_c}\Big)^{-2}\Big(\sum_{a=1}^{n}\frac{k_{n+1}.p_a}{\sigma_{n+1}-\sigma_a}\Big)^{-1}(k_{n+1}.k_{n+2})\\
&\Big(\sum_{b=1}^{n}\frac{\epsilon_{n+1}.p_b}{\sigma_{n+1}-\sigma_b}\Big)^2\Big(\sum_{b=1}^{n}\frac{\epsilon_{n+2}.p_b}{\sigma_{n+1}-\sigma_b}\Big)^2, \qquad (4.108)
\end{aligned}$$

and

$$\mathcal{B}_{10}\equiv\frac{1}{\tau}\mathbf{M}_n\,S^{(0)}_{n+1}\,(k_{n+1}.k_{n+2})^{-1}\,(\epsilon_{n+2}.k_{n+1})^2\,. \qquad (4.109)$$

Adding the contributions $\mathcal{B}_0$ to $\mathcal{B}_{10}$ we get the total contribution from non-degenerate solutions:

$$
\begin{aligned}
\mathcal{B} \;=\; & \{\tau^{-2} S^{(0)}_{n+1}\, S^{(0)}_{n+2}\mathbf{M}_n + \tau^{-1} S^{(0)}_{n+1}\, S^{(1)}_{n+2}\mathbf{M}_n + \tau^{-1} S^{(0)}_{n+2}\, S^{(1)}_{n+1}\mathbf{M}_n\} \\
& -\frac{2}{\tau}\int D\sigma\,\delta^{(0)}\, I_n \oint_{\{A_i\}} d\rho \left(\sum_{a=1}^n \frac{k_{n+1}.p_a}{\rho-\sigma_a}\right)^{-1} \left(\sum_{a=1}^n \frac{k_{n+2}.p_a}{\rho-\sigma_a}\right)^{-1} \left(\sum_{a=1}^n \frac{\epsilon_{n+1}.p_a}{\rho-\sigma_a}\right)^{2} \left(\sum_{a=1}^n \frac{\epsilon_{n+2}.p_a}{\rho-\sigma_a}\right)(\epsilon_{n+2}.k_{n+1}) \\
& -\frac{2}{\tau}\int D\sigma\,\delta^{(0)}\, I_n \oint_{\{B_i\}} d\rho \left(\sum_{a=1}^n \frac{k_{n+2}.p_a}{\rho-\sigma_a}\right)^{-1} \left(\sum_{a=1}^n \frac{k_{n+1}.p_a}{\rho-\sigma_a}\right)^{-1} \left(\sum_{a=1}^n \frac{\epsilon_{n+2}.p_a}{\rho-\sigma_a}\right)^{2} \left(\sum_{a=1}^n \frac{\epsilon_{n+1}.p_a}{\rho-\sigma_a}\right)(\epsilon_{n+1}.k_{n+2}) \\
& +\frac{1}{\tau}\int D\sigma\,\delta^{(0)}\, I_n \oint_{\{B_i\}} d\rho \left(\sum_{c=1}^n \frac{k_{n+1}.p_c}{\rho-\sigma_c}\right)^{-2} \left(\sum_{a=1}^n \frac{k_{n+2}.p_a}{\rho-\sigma_a}\right)^{-1} (k_{n+1}.k_{n+2}) \left(\sum_{b=1}^n \frac{\epsilon_{n+2}.p_b}{\rho-\sigma_b}\right)^{2} \left(\sum_{b=1}^n \frac{\epsilon_{n+1}.p_b}{\rho-\sigma_b}\right)^{2} \\
& +\frac{1}{\tau}\int D\sigma\,\delta^{(0)}\, I_n \oint_{\{A_i\}} d\rho \left(\sum_{c=1}^n \frac{k_{n+2}.p_c}{\rho-\sigma_c}\right)^{-2} \left(\sum_{a=1}^n \frac{k_{n+1}.p_a}{\rho-\sigma_a}\right)^{-1} (k_{n+1}.k_{n+2}) \left(\sum_{b=1}^n \frac{\epsilon_{n+1}.p_b}{\rho-\sigma_b}\right)^{2} \left(\sum_{b=1}^n \frac{\epsilon_{n+2}.p_b}{\rho-\sigma_b}\right)^{2} .
\end{aligned}
\tag{4.110}
$$

The term in the first line was derived in [16]. The terms in the second and the third line vanish in the gauge chosen in [16]. The terms in the fourth and fifth lines were missed in the analysis of [16]. We shall see that these terms cancel similar terms arising in the analysis of the degenerate solutions. Using the gauge transformation laws of $S^{(0)}_{n+1}\, S^{(1)}_{n+2}\mathbf{M}_n$ and $S^{(0)}_{n+2}\, S^{(1)}_{n+1}\mathbf{M}_n$ analyzed in [4], one can verify that (4.110) is invariant under the gauge transformations of the soft graviton polarizations.

4.3.3 Contribution from degenerate solutions

When the integration contours of σ_{n+1} and σ_{n+2} wrap around the degenerate solutions for which $|\sigma_{n+1}-\sigma_{n+2}| \sim \tau$, we will carry out the integration in ρ and ξ variables introduced in (4.66). In terms of these variables the integration measure takes the form,

$$
\int d\sigma_{n+1}\, d\sigma_{n+2}\, \delta(f^n_{n+1})\, \delta(f^n_{n+2}) \cdots = -2 \int d\rho\, d\xi\, \delta(f^n_{n+1}+f^n_{n+2})\, \delta(f^n_{n+1}-f^n_{n+2}) \cdots \tag{4.111}
$$

with the understanding that ρ integration is done using the first delta function and ξ integration is done using the second delta function. Therefore the contribution to the $(n+2)$-point amplitude from the degenerate solutions becomes,

$$
\begin{aligned}
& -\frac{2}{\tau^2}\int D\sigma\,\delta^{(0)} \int d\rho\, d\xi\, \delta\Big(\sum_{b=1}^n \frac{k_{n+1}.p_b}{\rho-\frac{\xi}{2}-\sigma_b} + \sum_{b=1}^n \frac{k_{n+2}.p_b}{\rho+\frac{\xi}{2}-\sigma_b}\Big) \\
& \qquad \delta\Big(\sum_{b=1}^n \frac{k_{n+1}.p_b}{\rho-\frac{\xi}{2}-\sigma_b} - \sum_{b=1}^n \frac{k_{n+2}.p_b}{\rho+\frac{\xi}{2}-\sigma_b} - \frac{2\,\tau\, k_{n+1}.k_{n+2}}{\xi}\Big)\, I_{n+2}\,,
\end{aligned}
\tag{4.112}
$$

where it is understood that we sum over contributions from those zeroes of the second δ-function for which $\xi \sim \tau$. By carrying out the integration over ξ using the second delta function, this integral can be approximated as

$$-\frac{2}{\tau}\int D\sigma\,\delta^{(0)}\int d\rho\,\delta\Big(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\rho-\sigma_b}+\sum_{b=1}^{n}\frac{k_{n+2}.p_b}{\rho-\sigma_b}\Big)\,\frac{\xi_1^2}{2\,k_{n+1}.k_{n+2}}\,I_{n+2}|_{\xi=\tau\,\xi_1}+O(\tau^0)\,, \quad (4.113)$$

where

$$\xi_1 = 2\,k_{n+1}.k_{n+2}\,\left(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\rho-\sigma_b}-\sum_{b=1}^{n}\frac{k_{n+2}.p_b}{\rho-\sigma_b}\right)^{-1}\,. \quad (4.114)$$

Now in the degeneration limit we can evaluate I_{n+2} by regarding this as the square of the Pfaffian of the matrix $\widehat{\Psi}$ which now takes the form

$$\widehat{\Psi}\simeq\begin{pmatrix} A_{ab} & \frac{\tau\,p_a.k_{n+1}}{\sigma_a-\rho} & \frac{\tau\,p_a.k_{n+2}}{\sigma_a-\rho} & -C^T_{ad} & \frac{p_a.\epsilon_{n+1}}{\sigma_a-\rho} & \frac{p_a.\epsilon_{n+2}}{\sigma_a-\rho} \\ \frac{\tau\,k_{n+1}.p_b}{\rho-\sigma_b} & 0 & -\frac{\tau\,k_{n+1}.k_{n+2}}{\xi_1} & \frac{\tau\,k_{n+1}.\epsilon_d}{\rho-\sigma_d} & -C_{(n+1)(n+1)} & -\frac{\epsilon_{n+2}.k_{n+1}}{\xi_1} \\ \frac{\tau\,k_{n+2}.p_b}{\rho-\sigma_b} & \frac{\tau\,k_{n+1}.k_{n+2}}{\xi_1} & 0 & \frac{\tau\,k_{n+2}.\epsilon_d}{\rho-\sigma_d} & \frac{\epsilon_{n+1}.k_{n+2}}{\xi_1} & -C_{(n+2)(n+2)} \\ C_{cb} & \frac{\tau\,\epsilon_c.k_{n+1}}{\sigma_c-\rho} & \frac{\tau\,\epsilon_c.k_{n+2}}{\sigma_c-\rho} & B_{cd} & \frac{\epsilon_c.\epsilon_{n+1}}{\sigma_c-\rho} & \frac{\epsilon_c.\epsilon_{n+2}}{\sigma_c-\rho} \\ \frac{\epsilon_{n+1}.p_b}{\rho-\sigma_b} & C_{(n+1)(n+1)} & -\frac{\epsilon_{n+1}.k_{n+2}}{\xi_1} & \frac{\epsilon_{n+1}.\epsilon_d}{\rho-\sigma_d} & 0 & -\frac{\epsilon_{n+1}.\epsilon_{n+2}}{\tau\,\xi_1} \\ \frac{\epsilon_{n+2}.p_b}{\rho-\sigma_b} & \frac{\epsilon_{n+2}.k_{n+1}}{\xi_1} & C_{(n+2)(n+2)} & \frac{\epsilon_{n+2}.\epsilon_d}{\rho-\sigma_d} & \frac{\epsilon_{n+2}.\epsilon_{n+1}}{\tau\,\xi_1} & 0 \end{pmatrix}\,, \quad (4.115)$$

where we now have

$$C_{(n+1)(n+1)}\simeq-\sum_{a=1}^{n}\frac{\epsilon_{n+1}.p_a}{\rho-\sigma_a}+\frac{\epsilon_{n+1}.k_{n+2}}{\xi_1}\,,\qquad C_{(n+2)(n+2)}\simeq-\sum_{a=1}^{n}\frac{\epsilon_{n+2}.p_a}{\rho-\sigma_a}-\frac{\epsilon_{n+2}.k_{n+1}}{\xi_1}\,. \quad (4.116)$$

Given the appearance of $1/\tau$ in the $(2n+3)$-$(2n+4)$-th element of the matrix, one may wonder whether the Pfaffian may give a leading contribution of order $1/\tau$, thereby upsetting the counting of powers of τ. It is easy to see however that the coefficient of this element in the expansion of the Pfaffian, given by the Pfaffian obtained by eliminating the $(2n+3)$ and $(2n+4)$-th rows and columns of this matrix, actually goes as τ and therefore does not upset the counting of powers of τ. With this understanding we can now compute the Pfaffian of $\widehat{\Psi}$, arriving at the result [29]:

$$
\begin{aligned}
I_{n+2} &= \Big[(\xi_1)^{-2}\epsilon_{n+1}.\epsilon_{n+2}\, k_{n+1}.k_{n+2} - (\xi_1)^{-2}\epsilon_{n+1}.k_{n+2}\, \epsilon_{n+2}.k_{n+1} \\
&\qquad\qquad -C_{(n+1)(n+1)}\, C_{(n+2)(n+2)}\Big]^2 I_n + \mathcal{O}(\tau) \\
&= \Big[(\xi_1)^{-2}\epsilon_{n+1}.\epsilon_{n+2}\, k_{n+1}.k_{n+2} - (\xi_1)^{-2}\epsilon_{n+1}.k_{n+2}\, \epsilon_{n+2}.k_{n+1} \\
&\quad -\Big\{\sum_{c=1}^{n} \frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c} - (\xi_1)^{-1}\,\epsilon_{n+1}.k_{n+2}\Big\}\ \Big\{\sum_{c=1}^{n} \frac{\epsilon_{n+2}.p_c}{\rho-\sigma_c} + (\xi_1)^{-1}\epsilon_{n+2}.k_{n+1}\Big\}\Big]^2 I_n \\
&\quad +\mathcal{O}(\tau)\,.
\end{aligned}
\tag{4.117}
$$

Substituting this into (4.112) we get

$$
\begin{aligned}
&-\frac{4k_{n+1}.k_{n+2}}{\tau}\int D\sigma\,\delta^{(0)}\, I_n \int d\rho\,\delta\Big(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\rho-\sigma_b} + \sum_{b=1}^{n}\frac{k_{n+2}.p_b}{\rho-\sigma_b}\Big)\Big(\sum_{a=1}^{n}\frac{(k_{n+1}-k_{n+2}).p_a}{\rho-\sigma_a}\Big)^{-2} \\
&\Big[\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+2}.p_d}{\rho-\sigma_d}\Big) - \epsilon_{n+1}.\epsilon_{n+2}\,\frac{1}{4\,k_{n+1}.k_{n+2}}\Big(\sum_{a=1}^{n}\frac{(k_{n+1}-k_{n+2}).p_a}{\rho-\sigma_a}\Big)^2 \\
&+\Big(\epsilon_{n+2}.k_{n+1}\sum_{c=1}^{n}\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c} - \epsilon_{n+1}.k_{n+2}\,\sum_{c=1}^{n}\frac{\epsilon_{n+2}.p_c}{\rho-\sigma_c}\Big)\frac{1}{2k_{n+1}.k_{n+2}}\,\left(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\rho-\sigma_b} - \sum_{b=1}^{n}\frac{k_{n+2}.p_b}{\rho-\sigma_b}\right)\Big]^2 .
\end{aligned}
\tag{4.118}
$$

By making use of the δ-function, we can rewrite this as

$$\frac{k_{n+1}.k_{n+2}}{\tau}\int D\sigma\,\delta^{(0)}\,I_n\int d\rho\,\delta\Big(\sum_{b=1}^n\frac{k_{n+1}.p_b}{\rho-\sigma_b}+\sum_{b=1}^n\frac{k_{n+2}.p_b}{\rho-\sigma_b}\Big)\Big(\sum_{a=1}^n\frac{k_{n+1}.p_a}{\rho-\sigma_a}\Big)^{-1}\Big(\sum_{a=1}^n\frac{k_{n+2}.p_a}{\rho-\sigma_a}\Big)^{-1}$$
$$\Bigg[\Bigg\{\Big(\sum_{c=1}^n\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^n\frac{\epsilon_{n+2}.p_d}{\rho-\sigma_d}\Big)+\epsilon_{n+1}.\epsilon_{n+2}\,\frac{1}{k_{n+1}.k_{n+2}}\Big(\sum_{a=1}^n\frac{k_{n+1}.p_a}{\rho-\sigma_a}\Big)\Big(\sum_{a=1}^n\frac{k_{n+2}.p_a}{\rho-\sigma_a}\Big)\Bigg\}^2$$
$$-2\,\Bigg\{\Big(\sum_{c=1}^n\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^n\frac{\epsilon_{n+2}.p_d}{\rho-\sigma_d}\Big)\Bigg\}(k_{n+1}.k_{n+2})^{-1}$$
$$\Big(\epsilon_{n+2}.k_{n+1}\sum_{c=1}^n\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\sum_{b=1}^n\frac{k_{n+2}.p_b}{\rho-\sigma_b}+\epsilon_{n+1}.k_{n+2}\,\sum_{c=1}^n\frac{\epsilon_{n+2}.p_c}{\rho-\sigma_c}\,\sum_{b}\frac{k_{n+1}.p_b}{\rho-\sigma_b}\Big)$$
$$+\epsilon_{n+1}.\epsilon_{n+2}\,(k_{n+1}.k_{n+2})^{-2}\,\Big(\sum_{a=1}^n\frac{k_{n+1}.p_a}{\rho-\sigma_a}\Big)\Big(\sum_{a=1}^n\frac{k_{n+2}.p_a}{\rho-\sigma_a}\Big)$$
$$\Big(\epsilon_{n+2}.k_{n+1}\sum_{c=1}^n\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}-\epsilon_{n+1}.k_{n+2}\,\sum_{c=1}^n\frac{\epsilon_{n+2}.p_c}{\rho-\sigma_c}\Big)\,\left(\sum_{b=1}^n\frac{k_{n+1}.p_b}{\rho-\sigma_b}-\sum_{b=1}^n\frac{k_{n+2}.p_b}{\rho-\sigma_b}\right)$$
$$-(k_{n+1}.k_{n+2})^{-2}\Big(\epsilon_{n+2}.k_{n+1}\sum_{c=1}^n\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}-\epsilon_{n+1}.k_{n+2}\,\sum_{c=1}^n\frac{\epsilon_{n+2}.p_c}{\rho-\sigma_c}\Big)^2\sum_{b=1}^n\frac{k_{n+1}.p_b}{\rho-\sigma_b}\sum_{b=1}^n\frac{k_{n+2}.p_b}{\rho-\sigma_b}\Bigg]. \tag{4.119}$$

Note that there are many other ways of writing this expression – by replacing one or more factors of $\sum_{b=1}^n\frac{k_{n+1}.p_b}{\rho-\sigma_b}$ by $-\sum_{b=1}^n\frac{k_{n+2}.p_b}{\rho-\sigma_b}$ and vice versa. We have chosen the one that will be most convenient for our analysis, but the final result does not depend on this choice.

We now represent the delta function as a contour integration as in (B.23)

$$\int d\rho\,\delta\Big(\sum_{b=1}^n\frac{k_{n+1}.p_b}{\rho-\sigma_b}+\sum_{b=1}^n\frac{k_{n+2}.p_b}{\rho-\sigma_b}\Big)\,\cdots=\oint d\rho\,\Big(\sum_{b=1}^n\frac{k_{n+1}.p_b}{\rho-\sigma_b}+\sum_{b=1}^n\frac{k_{n+2}.p_b}{\rho-\sigma_b}\Big)^{-1}\,\cdots \tag{4.120}$$

and deform the ρ integration contour away from the zeroes of the argument of the delta function. This will generate three kinds of terms, the residues at $\rho=\sigma_a$ for $1\le a\le n$, residues at the zeroes of $\Big(\sum\limits_{a=1}^n\frac{k_{n+1}.p_a}{\rho-\sigma_a}\Big)$ and $\Big(\sum\limits_{a=1}^n\frac{k_{n+2}.p_a}{\rho-\sigma_a}\Big)$ – called $\{A_i\}$ and $\{B_i\}$ in (4.84) – and residue at ∞. We shall now analyze these three kinds of terms one by one.

Residue at σ_a: This gives

$$\begin{aligned}\mathcal{D}_1 \;\equiv\; & -\tau^{-1}\,\mathbf{M}_n\,\sum_{a=1}^{n}\{p_a.(k_{n+1}+k_{n+2})\}^{-1}(p_a.k_{n+1})^{-1}(p_a.k_{n+2})^{-1}\\ & \Bigg[(k_{n+1}.k_{n+2})\,\Big\{\epsilon_{n+1}.p_a\;\epsilon_{n+2}.p_a+(k_{n+1}.k_{n+2})^{-1}\epsilon_{n+1}.\epsilon_{n+2}\;k_{n+1}.p_a\;k_{n+2}.p_a\Big\}^2\\ & -2\;\epsilon_{n+1}.p_a\;\epsilon_{n+2}.p_a\Big\{\epsilon_{n+2}.k_{n+1}\;\epsilon_{n+1}.p_a\;k_{n+2}.p_a+\epsilon_{n+1}.k_{n+2}\;\epsilon_{n+2}.p_a\;k_{n+1}.p_a\Big\}\\ & -(k_{n+1}.k_{n+2})^{-1}\epsilon_{n+1}.\epsilon_{n+2}\;k_{n+1}.p_a\;k_{n+2}.p_a\\ & \Big\{\epsilon_{n+2}.k_{n+1}\;\epsilon_{n+1}.p_a\;k_{n+2}.p_a-\epsilon_{n+2}.k_{n+1}\;\epsilon_{n+1}.p_a\;k_{n+1}.p_a\\ & +\epsilon_{n+1}.k_{n+2}\;\epsilon_{n+2}.p_a\;k_{n+1}.p_a-\epsilon_{n+1}.k_{n+2}\;\epsilon_{n+2}.p_a\;k_{n+2}.p_a\Big\}\\ & -(k_{n+1}.k_{n+2})^{-1}\;k_{n+1}.p_a\;k_{n+2}.p_a\;\{\epsilon_{n+2}.k_{n+1}\;\epsilon_{n+1}.p_a-\epsilon_{n+1}.k_{n+2}\;\epsilon_{n+2}.p_a\}^2\Bigg]\end{aligned} \tag{4.121}$$

where the overall minus sign comes from the reversal of the integration contour. Expanding the square and using the convention

$$\varepsilon_{n+1}=\epsilon_{n+1}\otimes\epsilon_{n+1},\qquad \varepsilon_{n+2}=\epsilon_{n+2}\otimes\epsilon_{n+2} \tag{4.122}$$

we get

$$\mathcal{D}_1 \;=\; \tau^{-1}\,\mathbf{M}_n\,\sum_{a=1}^{n}\{p_a.(k_{n+1}+k_{n+2})\}^{-1}\,\mathcal{M}(p_a;\varepsilon_{n+1},k_{n+1},\varepsilon_{n+2},k_{n+2})\,, \tag{4.123}$$

where

$$\begin{aligned}& \mathcal{M}(p_a;\varepsilon_{n+1},k_{n+1},\varepsilon_{n+2},k_{n+2})\\ =\; & -(p_a.k_{n+1})^{-1}(p_a.k_{n+2})^{-1}\Bigg[k_{n+1}.k_{n+2}\;p_a.\varepsilon_{n+1}.p_a\;p_a.\varepsilon_{n+2}.p_a+2\;p_a.\varepsilon_{n+1}.\varepsilon_{n+2}.p_a\;k_{n+1}.p_a\;k_{n+2}.p_a\\ & -2\;p_a.\varepsilon_{n+2}.k_{n+1}\;p_a.\varepsilon_{n+1}.p_a\;k_{n+2}.p_a-2\;p_a.\varepsilon_{n+1}.k_{n+2}\;p_a.\varepsilon_{n+2}.p_a\;k_{n+1}.p_a\\ & +(k_{n+1}.k_{n+2})^{-1}\;(k_{n+1}.p_a)\;(k_{n+2}.p_a)\Big\{\varepsilon_{n+1}.\varepsilon_{n+2}\;k_{n+1}.p_a\;k_{n+2}.p_a\\ & -\;p_a.\varepsilon_{n+1}.\varepsilon_{n+2}.k_{n+1}\;k_{n+2}.p_a+\;p_a.\varepsilon_{n+1}.\varepsilon_{n+2}.k_{n+1}\;k_{n+1}.p_a\\ & -\;p_a.\varepsilon_{n+2}.\varepsilon_{n+1}.k_{n+2}\;k_{n+1}.p_a+\;p_a.\varepsilon_{n+2}.\varepsilon_{n+1}.k_{n+2}\;k_{n+2}.p_a\\ & -k_{n+1}.\varepsilon_{n+2}.k_{n+1}\;p_a.\varepsilon_{n+1}.p_a-k_{n+2}.\varepsilon_{n+1}.k_{n+2}\;p_a.\varepsilon_{n+2}.p_a+2\;k_{n+2}.\varepsilon_{n+1}.p_a\;k_{n+1}.\varepsilon_{n+2}.p_a\;\Big\}\Bigg]\,.\end{aligned} \tag{4.124}$$

Residue at $\{A_i\}$ **and** $\{B_i\}$: The contribution to (4.119) from the residues at A_i and B_i defined in

(4.84) are given by

$$\begin{aligned}
\mathcal{D}_2 \;\equiv\; & -\frac{k_{n+1}.k_{n+2}}{\tau}\int D\sigma\,\delta^{(0)}\,I_n \oint_{\{A_i\}} d\rho\,\Big(\sum_{b=1}^{n}\frac{k_{n+2}.p_b}{\rho-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{k_{n+1}.p_a}{\rho-\sigma_a}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{k_{n+2}.p_a}{\rho-\sigma_a}\Big)^{-1} \\
& \Bigg[\Bigg\{\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+2}.p_d}{\rho-\sigma_d}\Big)\Bigg\}^2 \\
& -2\Bigg\{\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+2}.p_d}{\rho-\sigma_d}\Big)\Bigg\}(k_{n+1}.k_{n+2})^{-1}\epsilon_{n+2}.k_{n+1}\sum_{c=1}^{n}\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\sum_{b=1}^{n}\frac{k_{n+2}.p_b}{\rho-\sigma_b}\Bigg] \\
& -\frac{k_{n+1}.k_{n+2}}{\tau}\int D\sigma\,\delta^{(0)}\,I_n \oint_{\{B_i\}} d\rho\,\Big(\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\rho-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{k_{n+1}.p_a}{\rho-\sigma_a}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{k_{n+2}.p_a}{\rho-\sigma_a}\Big)^{-1} \\
& \Bigg[\Bigg\{\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+2}.p_d}{\rho-\sigma_d}\Big)\Bigg\}^2 \\
& -2\Bigg\{\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+1}.p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+2}.p_d}{\rho-\sigma_d}\Big)\Bigg\}(k_{n+1}.k_{n+2})^{-1}\epsilon_{n+1}.k_{n+2}\sum_{c=1}^{n}\frac{\epsilon_{n+2}.p_c}{\rho-\sigma_c}\sum_{b=1}^{n}\frac{k_{n+1}.p_b}{\rho-\sigma_b}\Bigg].
\end{aligned} \tag{4.125}$$

These terms were missed in the analysis of [15]. They cancel similar contributions (4.110) coming from non-degenerate solutions.

Residue at ∞: Finally we can examine the residue of the integral (4.119) at $\rho=\infty$. This is given by

$$\begin{aligned}
& \frac{k_{n+1}.k_{n+2}}{\tau}\,\mathbf{M}_n\Big(\sum_{b=1}^{n}(k_{n+1}+k_{n+2}).p_b\Big)^{-1}\Big(\sum_{a=1}^{n}k_{n+1}.p_a\Big)^{-1}\Big(\sum_{a=1}^{n}k_{n+2}.p_a\Big)^{-1} \\
& \Bigg[\Bigg\{\Big(\sum_{c=1}^{n}\epsilon_{n+1}.p_c\Big)\Big(\sum_{d=1}^{n}\epsilon_{n+2}.p_d\Big)+\epsilon_{n+1}.\epsilon_{n+2}\,\frac{1}{k_{n+1}.k_{n+2}}\Big(\sum_{a=1}^{n}k_{n+1}.p_a\Big)\Big(\sum_{a=1}^{n}k_{n+2}.p_a\Big)\Bigg\}^2 \\
& -2\Bigg\{\Big(\sum_{c=1}^{n}\epsilon_{n+1}.p_c\Big)\Big(\sum_{d=1}^{n}\epsilon_{n+2}.p_d\Big)\Bigg\}(k_{n+1}.k_{n+2})^{-1} \\
& \Big(\epsilon_{n+2}.k_{n+1}\sum_{c=1}^{n}\epsilon_{n+1}.p_c\sum_{b=1}^{n}k_{n+2}.p_b+\epsilon_{n+1}.k_{n+2}\,\sum_{c=1}^{n}\epsilon_{n+2}.p_c\,\sum_{b}k_{n+1}.p_b\Big) \\
& +\epsilon_{n+1}.\epsilon_{n+2}\,(k_{n+1}.k_{n+2})^{-2}\,\Big(\sum_{a=1}^{n}k_{n+1}.p_a\Big)\Big(\sum_{a=1}^{n}k_{n+2}.p_a\Big) \\
& \Big(\epsilon_{n+2}.k_{n+1}\sum_{c=1}^{n}\epsilon_{n+1}.p_c-\epsilon_{n+1}.k_{n+2}\,\sum_{c=1}^{n}\epsilon_{n+2}.p_c\Big)\,\Bigg(\sum_{b=1}^{n}k_{n+1}.p_b-\sum_{b=1}^{n}k_{n+2}.p_b\Bigg) \\
& -(k_{n+1}.k_{n+2})^{-2}\Big(\epsilon_{n+2}.k_{n+1}\sum_{c=1}^{n}\epsilon_{n+1}.p_c-\epsilon_{n+1}.k_{n+2}\,\sum_{c=1}^{n}\epsilon_{n+2}.p_c\Big)^2\sum_{b=1}^{n}k_{n+1}.p_b\sum_{b=1}^{n}k_{n+2}.p_b\Bigg].
\end{aligned} \tag{4.126}$$

Using momentum conservation law $\sum_{a=1}^{n} p_a = -\tau\,(k_{n+1} + k_{n+2})$ we can see that this is of order τ. Therefore this contribution is subsubleading and can be ignored.

4.3.4 Total contribution

The full amplitude corresponding to two soft gravitons is obtained by adding the contributions (4.110) from the non-degenerate solutions and the contributions (4.123) and (4.125) from the degenerate solutions. This is given by

$$\begin{aligned} \mathbf{M}_{n+2} &= \tau^{-2}\, S^{(0)}_{n+1}\, S^{(0)}_{n+2}\mathbf{M}_n + \tau^{-1}\, S^{(0)}_{n+1}\, S^{(1)}_{n+2}\mathbf{M}_n + \tau^{-1}\, S^{(0)}_{n+2}\, S^{(1)}_{n+1}\mathbf{M}_n \\ &\quad + \tau^{-1}\,\mathbf{M}_n \sum_{a=1}^{n} \{p_a.(k_{n+1}+k_{n+2})\}^{-1} \mathcal{M}(p_a;\varepsilon_{n+1},k_{n+1},\varepsilon_{n+2},k_{n+2})\,. \end{aligned} \tag{4.127}$$

This agrees with the result given in (4.1) for two soft gravitons. As shown in last chapter, (4.127) is invariant under the gauge transformations of the soft graviton polarizations.

Note that (4.125) cancels part of the contribution from (4.110). This suggests that there may be a better way of organising the calculation instead of representing it as a sum of contributions from degenerate and non-degenerate solutions.

4.4 Multiple soft graviton theorem

In this section we shall generalize the analysis of the previous section to the case where arbitrary number of gravitons become soft.

4.4.1 Degenerate and non-degenerate solutions

We assume that there are $n+m$ number of gravitons and m of them become soft. We parametrize the m soft momenta as

$$p_a^\mu = \tau\, k_a^\mu \quad , \qquad a = n+1,\cdots,n+m\,, \tag{4.128}$$

and take the soft limit by taking $\tau \to 0$ at fixed k_a. The momentum conservation now takes the form

$$p_1^\mu + \cdots + p_n^\mu + \tau\,(k_{n+1}^\mu + \cdots + k_{n+m}^\mu) = 0\,. \tag{4.129}$$

In this case the full solution space of the scattering equations gets divided into different sectors. These different sectors correspond to the case when a group of r_1 punctures associated with soft gravitons come within a distance of order τ of each other, another group of r_2 punctures associated with soft gravitons come within a distance of order τ of each other and so on. A detailed analysis of the number of solutions of each type can be found in appendix E. In this section our goal will be to prove that for the subleading multiple soft graviton amplitude, only two sectors contribute – non-degenerate solutions where all the punctures are finite distance away from each other and degenerate solutions where two of the punctures come within a distance τ of each other and all other punctures are finite distance away from each other.

To prove this, we note that the CHY formula for the amplitude is given by

$$\mathbf{M}_{n+m} = \int D\sigma \prod_{q=1}^{m} d\sigma_{n+q} \Big[\prod_{a\neq i,j,k} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n+m} \frac{p_a.p_b}{\sigma_a - \sigma_b} \Big) \Big] \, I_{n+m} \,. \tag{4.130}$$

We now analyze the product of delta functions and focus on the last m of them corresponding to the soft gravitons. These m delta functions are given by

$$\delta\Big(\tau \sum_{b=1}^{n} \frac{k_{n+q}.p_b}{\sigma_{n+q} - \sigma_b} + \tau^2 \sum_{\substack{u=1\\ u\neq q}}^{m} \frac{k_{n+q}.k_{n+u}}{\sigma_{n+q} - \sigma_{n+u}} \Big) \,, \qquad 1 \leq q \leq m \,. \tag{4.131}$$

Each of these delta functions gives a factor of $\frac{1}{\tau}$ to the amplitude irrespective of how many of the m σ_a's come within a distance of order τ of each other. These are the only source of singular τ dependence coming from the delta functions. Therefore we get a net factor of τ^{-m} from the delta functions.

Next, we consider the τ factors coming from the measure when r of the m σ_a's associated with the soft gravitons come within a distance of order τ of each other. Without loss of generality we can label these as $\sigma_{n+m-r+q}$ for $1 \leq q \leq r$. We now make following redefinitions of the coordinates

$$\begin{aligned} \sigma_a &= \sigma'_a \quad \text{for } 1 \leq a \leq n+m-r+1 \,, \\ \sigma_{n+m-r+q} &= \sigma'_{n+m-r+1} + \tau\, \xi_q \,, \qquad (q = 2, \cdots, r) \,. \end{aligned} \tag{4.132}$$

For the above coordinate transformation, we have

$$\prod_{c=1}^{n+m} d\sigma_c = \tau^{r-1} \left(\prod_{c=1}^{n+m-r+1} d\sigma'_c \right) \left(\prod_{q=2}^{r} d\,\xi_q \right) \,. \tag{4.133}$$

We shall prove shortly that I_{n+m} does not give rise to any singular behavior in the $\tau \to 0$ limit for finite σ'_c, ξ_q. Assuming this to be the case, we see that this contribution is leading only for $r = 1$ and can receive subleading contribution only for $r = 1, 2$. In other words, for the subleading soft graviton amplitude, the contributions come only from those solutions for which none of the punctures go close to each other (non degenerate solutions) or two of the punctures go close to each other (degenerate solutions).

Let us now prove that I_{n+m} has a finite limit as $\tau \to 0$ with σ'_c, ξ_q fixed (i.e., when r punctures go close to each other). We define

$$p \equiv n + m - r \,, \tag{4.134}$$

and express I_{n+m} as

$$I_{n+m} = 4\,(-1)^{n+m}\,(\sigma_s - \sigma_t)^{-2}\,\widehat{P}^2 \,, \tag{4.135}$$

where $\widehat{P}$ is the Pfaffian of the matrix

$$\left[\begin{array}{c|c|c|c|c||c|c|c}
(A_p)_{ab} & \frac{\tau p_a.k_{p+1}}{\sigma_a-\sigma_{p+1}} & \cdots & \frac{\tau p_a.k_{p+r}}{\sigma_a-\sigma_{p+r}} & -(C_p^T)_{ad} & -\frac{\epsilon_{p+1}.p_a}{\sigma_{p+1}-\sigma_a} & \cdots & -\frac{\epsilon_{p+r}.p_a}{\sigma_{p+r}-\sigma_a} \\
\hline
\frac{\tau k_{p+1}.p_b}{\sigma_{p+1}-\sigma_b} & 0 & \cdots & \frac{\tau^2 k_{p+1}.k_{p+r}}{\sigma_{p+1}-\sigma_{p+r}} & -\frac{\tau\epsilon_d.k_{p+1}}{\sigma_d-\sigma_{p+1}} & -C_{p+1,p+1} & \cdots & -\frac{\tau\epsilon_{p+r}.k_{p+1}}{\sigma_{p+r}-\sigma_{p+1}} \\
\hline
\vdots & \vdots & \cdots\vdots\cdots & \vdots & \vdots & \vdots & \cdots\vdots\cdots & \vdots \\
\hline
\frac{\tau k_{p+r}.p_b}{\sigma_{p+r}-\sigma_b} & \frac{\tau^2 k_{p+r}.k_{p+1}}{\sigma_{p+r}-\sigma_{p+1}} & \cdots & 0 & -\frac{\tau\epsilon_d.k_{p+r}}{\sigma_d-\sigma_{p+r}} & -\frac{\tau\epsilon_{p+1}.k_{p+r}}{\sigma_{p+1}-\sigma_{p+r}} & \cdots\vdots\cdots & -C_{p+r,p+r} \\
(C_p)_{cb} & \frac{\tau\epsilon_c.k_{p+1}}{\sigma_c-\sigma_{p+1}} & \cdots & \frac{\tau\epsilon_c.k_{p+r}}{\sigma_c-\sigma_{p+r}} & (B_p)_{cd} & \frac{\epsilon_c.\epsilon_{p+1}}{\sigma_c-\sigma_{p+1}} & \cdots & \frac{\epsilon_c.\epsilon_{p+r}}{\sigma_c-\sigma_{p+r}} \\
\hline\hline
\frac{\epsilon_{p+1}.p_b}{\sigma_{p+1}-\sigma_b} & C_{p+1,p+1} & \cdots & \frac{\tau\epsilon_{p+1}.k_{p+r}}{\sigma_{p+1}-\sigma_{p+r}} & \frac{\epsilon_{p+1}.\epsilon_d}{\sigma_{p+1}-\sigma_d} & 0 & \cdots & \frac{\epsilon_{p+1}.\epsilon_{p+r}}{\sigma_{p+1}-\sigma_{p+r}} \\
\hline
\vdots & \vdots & \cdots\vdots\cdots & \vdots & \vdots & \vdots & \cdots\vdots\cdots & \vdots \\
\hline
\frac{\epsilon_{p+r}.p_b}{\sigma_{p+r}-\sigma_b} & \frac{\tau\epsilon_{p+r}.k_{p+1}}{\sigma_{p+r}-\sigma_{p+1}} & \cdots & C_{p+r,p+r} & \frac{\epsilon_{p+r}.\epsilon_d}{\sigma_{p+r}-\sigma_d} & \frac{\epsilon_{p+r}.\epsilon_{p+1}}{\sigma_{p+r}-\sigma_{p+1}} & \cdots\vdots\cdots & 0
\end{array}\right] \tag{4.136}$$

where the matrices A, B and C are defined as in (B.42) for $m+n=p+r$ particles and A_p, B_p and C_p denote the first $p \times p$ blocks of these matrices. Using the parametrization (4.132), the matrix (4.136) may be expressed as

$$\left[\begin{array}{c|c|c|c|c||c|c|c}
(A_p)_{ab} & \frac{\tau p_a.k_{p+1}}{\sigma_a-\sigma_{p+1}} & \cdots & \frac{\tau p_a.k_{p+r}}{\sigma_a-\sigma_{p+r}} & -(C_p^T)_{ad} & -\frac{\epsilon_{p+1}.p_a}{\sigma_{p+1}-\sigma_a} & \cdots & -\frac{\epsilon_{p+r}.p_a}{\sigma_{p+r}-\sigma_a} \\
\hline
\frac{\tau k_{p+1}.p_b}{\sigma_{p+1}-\sigma_b} & 0 & \cdots & -\frac{\tau k_{p+1}.k_{p+r}}{\xi_r} & -\frac{\tau\epsilon_d.k_{p+1}}{\sigma_d-\sigma_{p+1}} & -C_{p+1,p+1} & \cdots & -\frac{\epsilon_{p+r}.k_{p+1}}{\xi_r} \\
\hline
\vdots & \vdots & \cdots\vdots\cdots & \vdots & \vdots & \vdots & \cdots\vdots\cdots & \vdots \\
\hline
\frac{\tau k_{p+r}.p_b}{\sigma_{p+r}-\sigma_b} & \frac{\tau k_{p+r}.k_{p+1}}{\xi_r} & \cdots & 0 & -\frac{\tau\epsilon_d.k_{p+r}}{\sigma_d-\sigma_{p+r}} & \frac{\epsilon_{p+1}.k_{p+r}}{\xi_r} & \cdots\vdots\cdots & -C_{p+r,p+r} \\
(C_p)_{cb} & \frac{\tau\epsilon_c.k_{p+1}}{\sigma_c-\sigma_{p+1}} & \cdots & \frac{\tau\epsilon_c.k_{p+r}}{\sigma_c-\sigma_{p+r}} & (B_p)_{cd} & \frac{\epsilon_c.\epsilon_{p+1}}{\sigma_c-\sigma_{p+1}} & \cdots & \frac{\epsilon_c.\epsilon_{p+r}}{\sigma_c-\sigma_{p+r}} \\
\hline\hline
\frac{\epsilon_{p+1}.p_b}{\sigma_{p+1}-\sigma_b} & C_{p+1,p+1} & \cdots & -\frac{\epsilon_{p+1}.k_{p+r}}{\xi_r} & \frac{\epsilon_{p+1}.\epsilon_d}{\sigma_{p+1}-\sigma_d} & 0 & \cdots & -\frac{\epsilon_{p+1}.\epsilon_{p+r}}{\tau\xi_r} \\
\hline
\vdots & \vdots & \cdots\vdots\cdots & \vdots & \vdots & \vdots & \cdots\vdots\cdots & \vdots \\
\hline
\frac{\epsilon_{p+r}.p_b}{\sigma_{p+r}-\sigma_b} & \frac{\epsilon_{p+r}.k_{p+1}}{\xi_r} & \cdots & C_{p+r,p+r} & \frac{\epsilon_{p+r}.\epsilon_d}{\sigma_{p+r}-\sigma_d} & \frac{\epsilon_{p+r}.\epsilon_{p+1}}{\tau\xi_r} & \cdots\vdots\cdots & 0
\end{array}\right] \tag{4.137}$$

We now note the following features of the matrix shown in (4.137):

1. $(A_p)_{ab}$, $(B_p)_{ab}$ and $(C_p)_{ab}$ have finite $\tau \to 0$ limit.

2. There are four blocks of the matrix formed by the vertical double line and the horizontal double line. We note that

(a) The upper left block has r rows and r columns that are proportional to τ.

(b) The upper right and lower left blocks have finite $\tau \to 0$ limit.

(c) In the lower right block, the diagonal matrix elements vanish, whereas the off-diagonal matrix elements are proportional to $1/\tau$.

We now note that the inverse powers of τ appear only in the elements of the lower right block of size $r \times r$. Let us suppose first that r is even. Then we get a maximum contribution of $\tau^{-r/2}$ from the Pfaffian of the lower right block, and the coefficient of this term is proportional to the Pfaffian of the upper left block of (4.137). This matrix has r rows and columns with every element proportional to τ, and therefore its Pfaffian will be proportional to $\tau^{r/2}$, cancelling the $\tau^{-r/2}$ factor. Therefore this term is finite as $\tau \to 0$.

If r is odd, then the singular terms in the lower right block can give a maximum contribution of $\tau^{-(r-1)/2}$. This is given by a sum of terms, one of which is proportional to the Pfaffian of the matrix given by the lower right block of size $(r-1) \times (r-1)$, multiplied by the Pfaffian of the matrix obtained by eliminating the last $(r-1)$ rows and columns of $\widehat{\Psi}$. The other terms are related to this one by rearrangement of the last r rows and columns and can be analyzed similarly. It is easy to see that the matrix obtained by eliminating the last $(r-1)$ rows and columns of (4.137) has r rows and columns proportional to each other in the $\tau \to 0$ limit, and therefore its Pfaffian gives a factor of $\tau^{(r-1)/2}$. This cancels the $\tau^{-(r-1)/2}$ factor, giving a finite $\tau \to 0$ limit.

Next we need to consider the possibility that we may not choose the maximally singular terms from the lower right block. One such term corresponds to choosing the Pfaffian of the matrix given by the last $k \times k$ block for some even integer $k < r$, multiplied by the Pfaffian of the matrix obtained by eliminating the last k rows and columns, but from the latter we do not pick any term that has $1/\tau$ factor. The Pfaffian of the $k \times k$ matrix goes as $\tau^{-k/2}$, whereas the matrix obtained by eliminating the last k rows and columns has r rows (and columns) given by linear combinations of $(r-k)$ independent vectors in the $\tau \to 0$ limit. Therefore its Pfaffian goes as $\tau^{k/2}$. This again shows that the product has a finite $\tau \to 0$ limit. The other terms of this kind are related to the one discussed above by rearrangement of the last r rows and columns and are therefore also finite as $\tau \to 0$.

This finishes our proof that I_{n+m} remains finite in the $\tau \to 0$ limit.

4.4.2 Contribution from non-degenerate solutions

In this section we shall compute the contribution to the amplitude from non-degenerate solutions to the scattering equations. The amplitude is given by

$$
\begin{aligned}
& \mathbf{M}_{n+m}(p_1, ..., p_n, \tau\, k_{n+1}, \cdots, \tau\, k_{n+m}) \\
= & \int D\sigma \int \Big[\prod_{q=1}^{m} d\sigma_{n+q}\Big]\Big[\prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a{\cdot}p_b}{\sigma_{ab}} + \tau \sum_{v=1}^{m} \frac{p_a.k_{n+v}}{\sigma_{a(n+v)}}\Big)\Big] \\
& \prod_{r=1}^{m} \delta\Big(\tau \sum_{b=1}^{n} \frac{k_{n+r}{\cdot}p_b}{\sigma_{(n+r)b}} + \tau^2 \sum_{\substack{u=1\\ u\neq r}}^{m} \frac{k_{n+r}.k_{n+u}}{\sigma_{(n+r)(n+u)}}\Big)\, I_{n+m}\,. \qquad (4.138)
\end{aligned}
$$

We now expand the various factors in this expression up to order τ assuming that $(\sigma_a - \sigma_b) \sim 1$ for all pairs a, b with $a \neq b$, $1 \leq a, b \leq n+m$. Expansion of the first $(n-3)$ delta functions in (4.138) takes the form:

$$
\begin{aligned}
& \prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a{\cdot}p_b}{\sigma_{ab}} + \tau \sum_{v=1}^{m} \frac{p_a.k_{n+v}}{\sigma_{a(n+v)}}\Big) \\
= & \prod_{\substack{a=1\\ a\neq i,j,k}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a{\cdot}p_b}{\sigma_{ab}}\Big) + \tau \sum_{l=1}^{n}\sum_{v=1}^{m} \frac{p_l.k_{n+v}}{\sigma_{l(n+v)}} \delta'\Big(\sum_{\substack{c=1\\ c\neq l}}^{n} \frac{p_l{\cdot}p_c}{\sigma_{lc}}\Big) \prod_{\substack{a=1\\ a\neq i,j,k,l}}^{n} \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a{\cdot}p_b}{\sigma_{ab}}\Big) + O(\tau^2) \\
\equiv & \ \delta^{(0)} + \tau\, \delta^{(1)} + O(\tau^2)\,. \qquad (4.139)
\end{aligned}
$$

Expansion of the last m delta functions in (4.138) takes the form:

$$
\begin{aligned}
& \prod_{r=1}^{m} \delta\Big(\tau \sum_{b=1}^{n} \frac{k_{n+r}{\cdot}p_b}{\sigma_{(n+r)b}} + \tau^2 \sum_{\substack{u=1\\ u\neq r}}^{m} \frac{k_{n+r}.k_{n+u}}{\sigma_{(n+r)(n+u)}}\Big) \\
= & \ \tau^{-m} \prod_{r=1}^{m} \Big[\delta(f^n_{n+r}) + \tau \sum_{\substack{u=1\\ u\neq r}}^{m} \frac{k_{n+r}.k_{n+u}}{\sigma_{(n+r)(n+u)}} \delta'(f^n_{n+r}) + O(\tau^2)\Big] \\
= & \ \tau^{-m} \left[\prod_{r=1}^{m} \delta(f^n_{n+r}) + \tau \sum_{v=1}^{m}\sum_{\substack{u=1\\ u\neq v}}^{m} \frac{k_{n+v}.k_{n+u}}{\sigma_{(n+v)(n+u)}} \delta'(f^n_{n+v}) \prod_{\substack{r=1\\ r\neq v}}^{m} \delta(f^n_{n+r}) + O(\tau^2)\right], \qquad (4.140)
\end{aligned}
$$

where f^n_{n+r} has been defined in (4.17). Finally, expansion of the integrand I_{n+m} gives,

$$
I_{n+m} \;=\; I_{n+m}\Big|_{\tau=0} + \tau \frac{\partial I_{n+m}}{\partial \tau}\Big|_{\tau=0} + O(\tau^2) \;\equiv\; I^{(0)}_{n+m} + \tau\, I^{(1)}_{n+m} + O(\tau^2)\,. \qquad (4.141)
$$

Then, up to subleading order, the non degenerate contribution becomes

$$
\begin{aligned}
& \frac{1}{\tau^m}\int D\sigma \int \Big[\prod_{q=1}^{m} d\sigma_{n+q}\Big]\big[\delta^{(0)}+\tau\,\delta^{(1)}\big]\,\Big[I^{(0)}_{n+m}+\tau\, I^{(1)}_{n+m}\Big] \\
& \times\left[\prod_{r=1}^{m}\delta(f^n_{n+r})+\tau\sum_{v=1}^{m}\sum_{\substack{u=1\\u\neq v}}^{m}\frac{k_{n+v}.k_{n+u}}{\sigma_{(n+v)(n+u)}}\delta'(f^n_{n+v})\prod_{\substack{r=1\\r\neq v}}^{m}\delta(f^n_{n+r})\right] \\
= \;& \frac{1}{\tau^m}\int D\sigma\, \delta^{(0)}\int \Big[\prod_{q=1}^{m} d\sigma_{n+q}\Big]\prod_{r=1}^{m}\delta(f^n_{n+r})\; I^{(0)}_{n+m} \\
& +\frac{1}{\tau^{m-1}}\int D\sigma\int\Big[\prod_{q=1}^{m} d\sigma_{n+q}\Big]\;\delta^{(1)}\prod_{r=1}^{m}\delta(f^n_{n+r})\, I^{(0)}_{n+m} \\
& +\frac{1}{\tau^{m-1}}\int D\sigma\;\delta^{(0)}\int\Big[\prod_{q=1}^{m} d\sigma_{n+q}\Big]\prod_{r=1}^{m}\delta(f^n_{n+r})\, I^{(1)}_{n+m} \\
& +\frac{1}{\tau^{m-1}}\int D\sigma\;\delta^{(0)}\int\Big[\prod_{q=1}^{m} d\sigma_{n+q}\Big]\sum_{v=1}^{m}\sum_{\substack{u=1\\u\neq v}}^{m}\frac{k_{n+v}.k_{n+u}}{\sigma_{(n+v)(n+u)}}\delta'(f^n_{n+v})\prod_{\substack{r=1\\r\neq v}}^{m}\delta(f^n_{n+r})\, I^{(0)}_{n+m}\,.
\end{aligned}
\tag{4.142}
$$

For evaluating $I^{(0)}_{n+m}$ and $I^{(1)}_{n+m}$ it will be convenient to introduce m soft parameters $\tau_1,\tau_2,...,\tau_m$ generalizing the approach followed in (4.79), and label the momenta of $(n+m)$ particles as $p_1,p_2,...,p_n,\tau_1\, k_{n+1},\tau_2\, k$ Then to linear order in τ_i, I_{n+m} is given by

$$
I_{n+m}=4(-1)^{n+m}(\sigma_s-\sigma_t)^{-2}\widehat{P}^2 \tag{4.143}
$$

where $\widehat{P}$ is the Pfaffian of the matrix

$$
\widehat{\Psi}=\left[\begin{array}{cccc|cccc}
A_{ab} & \frac{\tau_1 p_a.k_{n+1}}{\sigma_a-\sigma_{n+1}} & \dots & \frac{\tau_m p_a.k_{n+m}}{\sigma_a-\sigma_{n+m}} & -C^T_{ad} & \frac{p_a.\epsilon_{n+1}}{\sigma_a-\sigma_{n+1}} & \dots & \frac{p_a.\epsilon_{n+m}}{\sigma_a-\sigma_{n+m}} \\
\frac{\tau_1 k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} & 0 & \dots & 0 & \frac{\tau_1 k_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d} & -C_{(n+1)(n+1)} & \dots & -\frac{\tau_1\epsilon_{n+m}.k_{n+1}}{\sigma_{n+m}-\sigma_{n+1}} \\
. & . & \dots & . & . & . & \dots & . \\
. & . & \dots & . & . & . & \dots & . \\
\frac{\tau_m k_{n+m}.p_b}{\sigma_{n+m}-\sigma_b} & 0 & \dots & 0 & \frac{\tau_m k_{n+m}.\epsilon_d}{\sigma_{n+m}-\sigma_d} & -\frac{\tau_m\epsilon_{n+1}.k_{n+m}}{\sigma_{n+1}-\sigma_{n+m}} & \dots & -C_{(n+m)(n+m)} \\
\hline
C_{cb} & \frac{\tau_1\epsilon_c.k_{n+1}}{\sigma_c-\sigma_{n+1}} & \dots & \frac{\tau_m\epsilon_c.k_{n+m}}{\sigma_c-\sigma_{n+m}} & B_{cd} & \frac{\epsilon_c.\epsilon_{n+1}}{\sigma_c-\sigma_{n+1}} & \dots & \frac{\epsilon_c.\epsilon_{n+m}}{\sigma_c-\sigma_{n+m}} \\
\frac{\epsilon_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} & C_{(n+1)(n+1)} & \dots & \frac{\tau_m\epsilon_{n+1}.k_{n+m}}{\sigma_{n+1}-\sigma_{n+m}} & \frac{\epsilon_{n+1}.\epsilon_d}{\sigma_{n+1}-\sigma_d} & 0 & \dots & \frac{\epsilon_{n+1}.\epsilon_{n+m}}{\sigma_{n+1}-\sigma_{n+m}} \\
. & . & \dots & . & . & . & \dots & . \\
. & . & \dots & . & . & . & \dots & . \\
\frac{\epsilon_{n+m}.p_b}{\sigma_{n+m}-\sigma_b} & \frac{\tau_1\epsilon_{n+m}.k_{n+1}}{\sigma_{n+m}-\sigma_{n+1}} & \dots & C_{(n+m)(n+m)} & \frac{\epsilon_{n+m}.\epsilon_d}{\sigma_{n+m}-\sigma_d} & \frac{\epsilon_{n+m}.\epsilon_{n+1}}{\sigma_{n+m}-\sigma_{n+1}} & \dots & 0
\end{array}\right].
\tag{4.144}
$$

The values of $C_{(n+r)(n+r)}$ for $r = 1, 2, ..., m$ are:

$$C_{(n+r)(n+r)} = -\sum_{a=1}^{n} \frac{\epsilon_{n+r}.p_a}{\sigma_{n+r} - \sigma_a} - \sum_{\substack{q=1\\q\neq r}}^{m} \tau_q \frac{\epsilon_{n+r}.k_{n+q}}{\sigma_{n+r} - \sigma_{n+q}} \tag{4.145}$$

With this, (4.144) gives

$$\widehat{P}|_{\tau_i=0} = (-1)^{mn+m(m+1)/2}\, \widetilde{P} \prod_{q=1}^{m} \Big(\sum_{b=1}^{n} \frac{\epsilon_{n+q}.p_b}{\sigma_{n+q} - \sigma_b} \Big), \tag{4.146}$$

where $\widetilde{P}$ is the Pfaffian of (4.144) for $m = 0$. Therefore we have

$$I^{(0)}_{n+m} = (-1)^m \prod_{q=1}^{m} \Big(\sum_{b=1}^{n} \frac{\epsilon_{n+q}.p_b}{\sigma_{n+q} - \sigma_b} \Big)^2 I_n\,. \tag{4.147}$$

To evaluate $I^{(1)}_{n+m}$ we use the analogue of (4.91):

$$I^{(1)}_{n+m} \equiv \sum_{r=1}^{m} \frac{\partial I_{n+m}}{\partial \tau_r}\Big|_{\tau_1=\tau_2=...=\tau_m=0} = 8(-1)^{n+m}(\sigma_s - \sigma_t)^{-2} \widehat{P} \sum_{r=1}^{m} \frac{\partial \widehat{P}}{\partial \tau_r}\Big|_{\tau_1=\tau_2=...=\tau_m=0}. \tag{4.148}$$

The strategy for evaluating $\widehat{P}\frac{\partial \widehat{P}}{\partial \tau_r}\Big|_{\{\tau_q=0\}}$ will be to first set $\tau_q = 0$ for $q \neq r$ in the matrix (4.144) and then expand the Pfaffian successively about the rows $(n + 1)$ to $(n + m)$ except the $(n + r)$-th row. This gives

$$\widehat{P} = \pm \Big(\prod_{\substack{q=1\\q\neq r}}^{m} C_{(n+q)(n+q)} \Big) \widehat{P}^{(n)}_r, \tag{4.149}$$

where $\widehat{P}^{(n)}_r$ is the Pfaffian of the matrix $\widehat{\Psi}$ for $(n + 1)$ graviton scattering with the first n gravitons carrying momenta $p_1, \cdots, p_n$ and polarizations $\varepsilon_1, \cdots \varepsilon_n$ and the last one carrying soft momentum $\tau_r k_{n+r}$ and polarization ε_{n+r}. The overall sign in (4.149) will not be needed for our analysis. Using

(4.145) and (4.149) we get

$$
\begin{aligned}
\widehat{P}\frac{\partial \widehat{P}}{\partial \tau_r}\Big|_{\tau_1=\ldots=\tau_m=0} &= \Big(\prod_{\substack{q=1\\q\neq r}}^{m} C_{(n+q)(n+q)}\Big)\Big|_{\tau_1=\ldots=\tau_m=0}\widehat{P}_r^{(n)}\frac{\partial}{\partial \tau_r}\Big[\Big(\prod_{\substack{q=1\\q\neq r}}^{m} C_{(n+q)(n+q)}\Big)\widehat{P}_r^{(n)}\Big]\Big|_{\tau_1=\ldots=\tau_m=0} \\
&= \prod_{\substack{q=1\\q\neq r}}^{m}\Big(C_{(n+q)(n+q)}\Big)^2\widehat{P}_r^{(n)}\frac{\partial \widehat{P}_r^{(n)}}{\partial \tau_r}\Big|_{\tau_1=\ldots=\tau_m=0} \\
&\quad +\widetilde{P}^2\sum_{\substack{u=1\\u\neq r}}^{m}\Big[\frac{\epsilon_{n+u}.k_{n+r}}{\sigma_{n+u}-\sigma_{n+r}}\Big]\Big[\sum_{a=1}^{n}\frac{\epsilon_{n+u}.p_a}{\sigma_{n+u}-\sigma_a}\Big]\prod_{\substack{q=1\\q\neq u}}^{m}\Big(C_{(n+q)(n+q)}\Big)^2\Big|_{\tau_1=\ldots=\tau_m=0}.
\end{aligned}
\tag{4.150}
$$

Hence the expression for $I_{n+m}^{(1)}$ becomes,

$$
\begin{aligned}
I_{n+m}^{(1)} &= 8\,(-1)^{n+m}(\sigma_s-\sigma_t)^{-2}\sum_{r=1}^{m}\prod_{\substack{q=1\\q\neq r}}^{m}\Big(C_{(n+q)(n+q)}\Big)^2\widehat{P}_r^{(n)}\frac{\partial \widehat{P}_r^{(n)}}{\partial \tau_r}\Big|_{\tau_1=\ldots=\tau_m=0} \\
&\quad +2\,(-1)^m I_n\sum_{r=1}^{m}\sum_{\substack{u=1\\u\neq r}}^{m}\Big[\frac{\epsilon_{n+u}.k_{n+r}}{\sigma_{n+u}-\sigma_{n+r}}\Big]\Big[\sum_{a=1}^{n}\frac{\epsilon_{n+u}.p_a}{\sigma_{n+u}-\sigma_a}\Big]\prod_{\substack{q=1\\q\neq u}}^{m}\Big(C_{(n+q)(n+q)}\Big)^2\Big|_{\tau_1=\ldots=\tau_m=0}.
\end{aligned}
\tag{4.151}
$$

We shall now proceed to evaluate the right hand side of (4.142).

First term:

The first term on the right hand side of (4.142) may be expressed as

$$
\begin{aligned}
\mathcal{F}_0 &\equiv \frac{1}{\tau^m}\int D\sigma\delta^{(0)}\int\Big[\prod_{q=1}^{m}d\sigma_{n+q}\Big]\prod_{r=1}^{m}\delta(f_{n+r}^{n})\;I_{n+m}^{(0)} \\
&= \frac{(-1)^m}{\tau^m}\int D\sigma\delta^{(0)}\;I_n\prod_{r=1}^{m}\Big\{\oint_{\{A_{r,i}\}}d\sigma_{n+r}\Big(\sum_{b=1}^{n}\frac{k_{n+r}.p_b}{\sigma_{n+r}-\sigma_b}\Big)^{-1}\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+r}.p_d}{\sigma_{n+r}-\sigma_d}\Big)^2\Big\}\,,
\end{aligned}
\tag{4.152}
$$

where for fixed r, $\{A_{r,i}\}$ are defined as the set of points satisfying

$$
\sum_{b=1}^{n}\frac{k_{n+r}.p_b}{\sigma_{n+r}-\sigma_b}=0\qquad at\quad \sigma_{n+r}=A_{r,i}\,. \tag{4.153}
$$

We can now carry out integration over all $\{\sigma_{n+r}\}$ independently by deforming the contours to ∞. The contribution comes from residues at the poles at σ_a – which can be evaluated following the procedure described in section 4.2.2 – and the contribution from ∞ can be evaluated after using momentum conservation. The result up to subleading order is

$$\begin{aligned}\mathcal{F}_0 &= \frac{1}{\tau^m}\int D\sigma\delta^{(0)}\, I_n\,\prod_{r=1}^{m}\Big[\sum_{a=1}^{n}\frac{\big(\epsilon_{n+r}.p_a\big)^2}{k_{n+r}.p_a}+\tau\Big(\epsilon_{n+r}.\sum_{\substack{u=1\\u\neq r}}^{m}k_{n+u}\Big)^2\Big(k_{n+r}.\sum_{\substack{u=1\\u\neq r}}^{m}k_{n+u}\Big)^{-1}\Big]\\ &= \frac{1}{\tau^m}\Big[\prod_{r=1}^{m}S^{(0)}_{n+r}\Big]\mathbf{M}_n+\frac{1}{\tau^{m-1}}\sum_{v=1}^{m}\Big(\epsilon_{n+v}.\sum_{\substack{u=1\\u\neq v}}^{m}k_{n+u}\Big)^2\Big(k_{n+v}.\sum_{\substack{u=1\\u\neq v}}^{m}k_{n+u}\Big)^{-1}\Big[\prod_{\substack{r=1\\r\neq v}}^{m}S^{(0)}_{n+r}\Big]\mathbf{M}_n\,,\end{aligned}\tag{4.154}$$

where $S^{(0)}_{n+r}$ has been defined in (4.87).

Second term

The second term on the right hand side of (4.142) is given by

$$\begin{aligned}\mathcal{F}_1 &\equiv \frac{(-1)^m}{\tau^{m-1}}\int D\sigma\, I_n\,\sum_{l=1}^{n}\delta'\Big(\sum_{\substack{c=1\\c\neq l}}^{n}\frac{p_l.p_c}{\sigma_{lc}}\Big)\prod_{\substack{a=1\\a\neq i,j,k,l}}^{n}\delta\Big(\sum_{\substack{b=1\\b\neq a}}^{n}\frac{p_a.p_b}{\sigma_{ab}}\Big)\sum_{v=1}^{m}\int\Big[\prod_{q=1}^{m}d\sigma_{n+q}\Big]\frac{p_l.k_{n+v}}{\sigma_{l(n+v)}}\\ &\Big[\prod_{r=1}^{m}\delta\Big(\sum_{e=1}^{n}\frac{k_{n+r}.p_e}{\sigma_{n+r}-\sigma_e}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+r}.p_d}{\sigma_{n+r}-\sigma_d}\Big)^2\Big]\,.\end{aligned}\tag{4.155}$$

The integrals over $\{\sigma_{n+q}\}$ for $q\neq v$ are of the form (4.24) and can be analyzed as in section 4.2.2 to produce factors of $-S^{(0)}_{n+q}$. The contribution from the pole at $\sigma_{n+q}=\infty$ is subsubleading and can be ignored. The remaining integration over σ_{n+v} has the form given in (4.27) and can be analyzed as in section 4.2.2. The final result, given in (4.30), can in turn be related, using the equality of (4.30), (4.52) and the first term on the right hand side of (4.50), to

$$S^{(1)}_{n+v}\,\mathbf{M}_n-\int D\sigma\,\delta^{(0)}\,S^{(1)}_{n+v}\,I^{(0)}\,,\tag{4.156}$$

where $S^{(1)}_{n+v}$ has been defined in (4.87). Then the result up to subleading order is

$$\begin{aligned}\mathcal{F}_1 &= \frac{1}{\tau^{m-1}}\sum_{v=1}^{m}\Big[\,\Big[\prod_{\substack{r=1\\r\neq v}}^{m}S^{(0)}_{n+r}\Big]\Big(S^{(1)}_{n+v}\,\mathbf{M}_n-\int D\sigma\,\delta^{(0)}\,S^{(1)}_{n+v}\,I_n\Big)\Big]\\ &= \frac{1}{\tau^{m-1}}\sum_{v=1}^{m}\Big[\prod_{\substack{r=1\\r\neq v}}^{m}S^{(0)}_{n+r}\Big]\Big(S^{(1)}_{n+v}\,\mathbf{M}_n\Big)-\frac{1}{\tau^{m-1}}\sum_{v=1}^{m}\Big[\prod_{\substack{r=1\\r\neq v}}^{m}S^{(0)}_{n+r}\Big]\int D\sigma\,\delta^{(0)}\,S^{(1)}_{n+v}\,I_n\,.\end{aligned}\tag{4.157}$$

Third term:

We shall now consider the third term in the right hand side of (4.142):

$$\frac{1}{\tau^{m-1}}\int D\sigma\,\delta^{(0)}\int\Big[\prod_{q=1}^{m}d\sigma_{n+q}\Big]\prod_{r=1}^{m}\delta(f^{n}_{n+r})\,I^{(1)}_{n+m}\,. \tag{4.158}$$

Substituting the first term of $I^{(1)}_{n+m}$ given in (4.151) into (4.158) and taking the sum over r out of the integration we get,

$$\frac{1}{\tau^{m-1}}\sum_{r=1}^{m}\int D\sigma\,\delta^{(0)})\left\{\prod_{\substack{q=1\\q\neq r}}^{m}\int dq_{n+q}\,\delta(f^{n}_{n+q})\,(C_{(n+q)(n+q)})^{2}\right\}$$
$$\int d\sigma_{n+r}\,\delta(f^{n}_{n+r})\,8\,(-1)^{m+n}\,(\sigma_s-\sigma_t)^{-2}\,\widehat{P}^{(n)}_{r}\,\frac{\partial\widehat{P}^{(n)}_{r}}{\partial\tau_r}\bigg|_{\tau_1=\cdots=\tau_m=0}\,. \tag{4.159}$$

We can easily see that the integration over each σ_{n+q} for $q\neq r$ generates a factor of $-S^{(0)}_{n+q}$. On the other hand integration over σ_{n+r} has exactly the structure of third term (4.31) of single soft graviton case with $I^{(1)}_{n+1}$ given in (4.34). Using the equality of (4.31), (4.47), (4.59) and the second term on the right hand side of (4.50), the expression (4.159) can be written as:

$$\mathcal{F}_2\equiv\frac{1}{\tau^{m-1}}\sum_{r=1}^{m}\Big(\prod_{\substack{q=1\\q\neq r}}^{m}S^{(0)}_{n+q}\Big)\int D\sigma\,\delta^{(0)}\,S^{(1)}_{n+r}\,I_n\,. \tag{4.160}$$

On the other hand, substituting the second term on the right hand side of (4.151) into (4.158) we get:

$$2\frac{(-1)^m}{\tau^{m-1}}\int D\sigma\delta^{(0)}I_n\sum_{r=1}^{m}\sum_{\substack{u=1\\u\neq r}}^{m}\prod_{\substack{q=1\\q\neq u}}^{m}\oint_{\{A_{q,i}\}}d\sigma_{n+q}\Big(\sum_{c=1}^{n}\frac{k_{n+q}\cdot p_c}{\sigma_{n+q}-\sigma_c}\Big)^{-1}C^{2}_{(n+q)(n+q)}$$
$$\oint_{\{A_{u,i}\}}d\sigma_{n+u}\Big(\sum_{b=1}^{n}\frac{k_{n+u}\cdot p_b}{\sigma_{n+u}-\sigma_b}\Big)^{-1}\Big(\frac{\epsilon_{n+u}\cdot k_{n+r}}{\sigma_{n+u}-\sigma_{n+r}}\Big)\Big(\sum_{a=1}^{n}\frac{\epsilon_{n+u}\cdot p_a}{\sigma_{n+u}-\sigma_a}\Big)\,. \tag{4.161}$$

We shall evaluate the integration over σ_{n+u} by deforming it to ∞. During this deformation we shall encounter poles at $\sigma_{n+u}=\sigma_{n+r}$ for $1\leq r\leq m$, $r\neq u$, and at $\sigma_{n+u}=\infty$ but there are no poles at $\{\sigma_a\}$ for $1\leq a\leq n$. The contribution to (4.161) from the residue at the pole at $\{\sigma_{n+r}:1\leq r\leq m\}$

is given by:

$$\begin{aligned}\mathcal{F}_3 &\equiv 2\frac{(-1)^{m-1}}{\tau^{m-1}}\int D\sigma\delta^{(0)} I_n \sum_{r=1}^{m}\sum_{\substack{u=1\\u\neq r}}^{m}(\epsilon_{n+u}.k_{n+r})\prod_{\substack{q=1\\q\neq u}}^{m}\oint_{\{A_{q,i}\}} d\sigma_{n+q}\Big(\sum_{c=1}^{n}\frac{k_{n+q}.p_c}{\sigma_{n+q}-\sigma_c}\Big)^{-1}\\ &\quad \Big(\sum_{d=1}^{n}\frac{\epsilon_{n+q}.p_d}{\sigma_{n+q}-\sigma_d}\Big)^{2}\Big(\sum_{b=1}^{n}\frac{k_{n+u}.p_b}{\sigma_{n+r}-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{\epsilon_{n+u}.p_a}{\sigma_{n+r}-\sigma_a}\Big)\\ &= -\frac{2}{\tau^{m-1}}\sum_{r=1}^{m}\sum_{\substack{u=1\\u\neq r}}^{m}(\epsilon_{n+u}.k_{n+r})\Big[\prod_{\substack{q=1\\q\neq r,u}}^{m} S^{(0)}_{n+q}\Big]\int D\sigma\delta^{(0)} I_n \oint_{\{A_{r,i}\}} d\sigma_{n+r}\Big(\sum_{c=1}^{n}\frac{k_{n+r}.p_c}{\sigma_{n+r}-\sigma_c}\Big)^{-1}\\ &\quad \Big(\sum_{d=1}^{n}\frac{\epsilon_{n+r}.p_d}{\sigma_{n+r}-\sigma_d}\Big)^{2}\Big(\sum_{b=1}^{n}\frac{k_{n+u}.p_b}{\sigma_{n+r}-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{\epsilon_{n+u}.p_a}{\sigma_{n+r}-\sigma_a}\Big). \end{aligned} \tag{4.162}$$

On the other hand the contribution to (4.161) from the residue at the pole at $\sigma_{n+u}=\infty$ is given by:

$$\mathcal{F}_4 \equiv -\frac{2}{\tau^{m-1}}\mathbf{M}_n\sum_{u=1}^{m}\Big(\prod_{\substack{q=1\\q\neq u}}^{m} S^{(0)}_{n+q}\Big)\Big(\epsilon_{n+u}.\sum_{\substack{r=1\\r\neq u}}^{m}k_{n+r}\Big)^{2}\Big(k_{n+u}.\sum_{\substack{v=1\\v\neq u}}^{m}k_{n+v}\Big)^{-1}. \tag{4.163}$$

Fourth term:

The fourth term on the right hand side of (4.142) is given by:

$$\begin{aligned}&\frac{1}{\tau^{m-1}}\int D\sigma\,\delta^{(0)}\int\Big[\prod_{q=1}^{m}d\sigma_{n+q}\Big]\sum_{v=1}^{m}\sum_{\substack{u=1\\u\neq v}}^{m}\frac{k_{n+v}.k_{n+u}}{\sigma_{(n+v)(n+u)}}\delta'(f^n_{n+v})\prod_{\substack{r=1\\r\neq v}}^{m}\delta(f^n_{n+r})I^{(0)}_{n+m}\\ &= \frac{(-1)^{m-1}}{\tau^{m-1}}\int D\sigma\delta^{(0)}\, I_n\sum_{v=1}^{m}\sum_{\substack{u=1\\u\neq v}}^{m}\prod_{\substack{r=1\\r\neq v}}^{m}\oint_{\{A_{r,i}\}} d\sigma_{n+r}\Big(\sum_{b=1}^{n}\frac{k_{n+r}.p_b}{\sigma_{n+r}-\sigma_b}\Big)^{-1}\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+r}.p_d}{\sigma_{n+r}-\sigma_d}\Big)^{2}\\ &\quad\Big[\oint_{\{A_{v,i}\}} d\sigma_{n+v}\frac{k_{n+v}.k_{n+u}}{\sigma_{n+v}-\sigma_{n+u}}\Big(\sum_{a=1}^{n}\frac{k_{n+v}.p_a}{\sigma_{n+v}-\sigma_a}\Big)^{-2}\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+v}.p_c}{\sigma_{n+v}-\sigma_c}\Big)^{2}\Big],\end{aligned} \tag{4.164}$$

where in the last line we have used the definition of derivative of delta function inside the contour integration as $\delta'(f)=-\frac{1}{f^2}$. For evaluating the integral over σ_{n+v}, we shall deform the contour away from $\{A_{v,i}\}$ towards the infinity and pick the residues at $\sigma_{n+v}=\sigma_{n+u}$ as well as the contribution from infinity. There is no pole at $\sigma_{n+v}=\sigma_a$. The contribution from the pole at σ_{n+u} is given by

$$\begin{aligned}
\mathcal{F}_5 &\equiv \frac{(-1)^m}{\tau^{m-1}}\int D\sigma\delta^{(0)}\, I_n\, \sum_{v=1}^{m}\sum_{\substack{u=1\\u\neq v}}^{m}\prod_{\substack{r=1\\r\neq v}}^{m}\oint_{\{A_{r,i}\}} d\sigma_{n+r}\,\Big(\sum_{b=1}^{n}\frac{k_{n+r}.p_b}{\sigma_{n+r}-\sigma_b}\Big)^{-1}\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+r}.p_d}{\sigma_{n+r}-\sigma_d}\Big)^2 \\
&\quad \Big(\sum_{a=1}^{n}\frac{k_{n+v}.p_a}{\sigma_{n+u}-\sigma_a}\Big)^{-2}\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+v}.p_c}{\sigma_{n+u}-\sigma_c}\Big)^2 k_{n+v}.k_{n+u} \\
&= \frac{1}{\tau^{m-1}}\sum_{v=1}^{m}\sum_{\substack{u=1\\u\neq v}}^{m}\Big[\prod_{\substack{r=1\\r\neq u,v}}^{m} S^{(0)}_{n+r}\Big]\int D\sigma\delta^{(0)} I_n\,\oint_{\{A_{u,i}\}} d\sigma_{n+u}\Big(\sum_{b=1}^{n}\frac{k_{n+u}.p_b}{\sigma_{n+u}-\sigma_b}\Big)^{-1}\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+u}.p_d}{\sigma_{n+u}-\sigma_d}\Big)^2 \\
&\quad \Big(\sum_{a=1}^{n}\frac{k_{n+v}.p_a}{\sigma_{n+u}-\sigma_a}\Big)^{-2}\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+v}.p_c}{\sigma_{n+u}-\sigma_c}\Big)^2 k_{n+v}.k_{n+u}\,. \qquad (4.165)
\end{aligned}$$

On the other hand the contribution from infinity is given by, after using momentum conservation,

$$\begin{aligned}
\mathcal{F}_6 &\equiv \frac{(-1)^{m-1}}{\tau^{m-1}}\int D\sigma\,\delta^{(0)}\, I_n\,\sum_{v=1}^{m}\prod_{\substack{r=1\\r\neq v}}^{m}\oint_{\{A_{r,i}\}} d\sigma_{n+r}\,\Big(\sum_{b=1}^{n}\frac{k_{n+r}.p_b}{\sigma_{n+r}-\sigma_b}\Big)^{-1}\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+r}.p_d}{\sigma_{n+r}-\sigma_d}\Big)^2 \\
&\quad \big(k_{n+v}.\sum_{\substack{u=1\\u\neq v}}^{m}k_{n+u}\big)^{-1}\big(\epsilon_{n+v}.\sum_{\substack{u=1\\u\neq v}}^{m}k_{n+u}\big)^2 \\
&= \frac{1}{\tau^{m-1}}\sum_{v=1}^{m}\big(k_{n+v}.\sum_{\substack{u=1\\u\neq v}}^{m}k_{n+u}\big)^{-1}\big(\epsilon_{n+v}.\sum_{\substack{u=1\\u\neq v}}^{m}k_{n+u}\big)^2\prod_{\substack{r=1\\r\neq v}}^{m} S^{(0)}_{n+r}\,\mathbf{M}_n\,. \qquad (4.166)
\end{aligned}$$

Adding the contributions from $\mathcal{F}_0$ to $\mathcal{F}_6$ we get the total contribution from non-degenerate solutions

$$\begin{aligned}
\mathcal{F} &= \tau^{-m}\Big[\prod_{r=1}^{m} S^{(0)}_{n+r}\Big]\mathbf{M}_n + \tau^{-(m-1)}\sum_{v=1}^{m}\Big[\prod_{\substack{r=1\\r\neq v}}^{m} S^{(0)}_{n+r}\Big]\Big(S^{(1)}_{n+v}\mathbf{M}_n\Big) \\
&\quad -\frac{2}{\tau^{m-1}}\sum_{r=1}^{m}\sum_{\substack{u=1\\u\neq r}}^{m}(\epsilon_{n+u}.k_{n+r})\Big[\prod_{\substack{q=1\\q\neq r,u}}^{m} S^{(0)}_{n+q}\Big]\int D\sigma\delta^{(0)} I_n\,\oint_{\{A_{r,i}\}} d\sigma_{n+r}\Big(\sum_{c=1}^{n}\frac{k_{n+r}.p_c}{\sigma_{n+r}-\sigma_c}\Big)^{-1} \\
&\quad \Big(\sum_{d=1}^{n}\frac{\epsilon_{n+r}.p_d}{\sigma_{n+r}-\sigma_d}\Big)^2\Big(\sum_{b=1}^{n}\frac{k_{n+u}.p_b}{\sigma_{n+r}-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{\epsilon_{n+u}.p_a}{\sigma_{n+r}-\sigma_a}\Big) \\
&\quad +\frac{1}{\tau^{m-1}}\sum_{r=1}^{m}\sum_{\substack{u=1\\u\neq r}}^{m}(k_{n+r}.k_{n+u})\Big[\prod_{\substack{q=1\\q\neq u,r}}^{m} S^{(0)}_{n+q}\Big]\int D\sigma\delta^{(0)} I_n \\
&\quad \oint_{\{A_{u,i}\}} d\sigma_{n+u}\Big(\sum_{b=1}^{n}\frac{k_{n+u}.p_b}{\sigma_{n+u}-\sigma_b}\Big)^{-1}\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+u}.p_d}{\sigma_{n+u}-\sigma_d}\Big)^2\Big(\sum_{a=1}^{n}\frac{k_{n+r}.p_a}{\sigma_{n+u}-\sigma_a}\Big)^{-2}\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+r}.p_c}{\sigma_{n+u}-\sigma_c}\Big)^2, \\
&\qquad (4.167)
\end{aligned}$$

where in the last line we have relabelled the dummy indices from what they were in (4.165).

4.4.3 Contribution from degenerate solutions

We now evaluate the contribution to the amplitude from the degenerate solutions of the scattering equation. As shown in section 4.4.1, we only have to consider the case where two of the punctures come close to each other, and that this contribution begins at the subleading order. We can choose the two punctures out of m punctures in $\binom{m}{2}$ ways. We shall consider the contribution of two generic punctures σ_{n+p} and σ_{n+u} coming close to each other and then sum over all possible $\binom{m}{2}$ terms.

When $|\sigma_{n+p} - \sigma_{n+u}| \sim \tau$, it is convenient to go to the (ρ, ξ) coordinate system as in the double soft graviton case:

$$\sigma_{n+p} = \rho - \frac{\xi}{2}, \qquad \sigma_{n+u} = \rho + \frac{\xi}{2}\,. \tag{4.168}$$

In terms of these variables, we have

$$\int d\sigma_{n+p}\, d\sigma_{n+u}\, \delta(f^n_{n+p})\, \delta(f^n_{n+u}) = -2 \int d\rho\, d\xi\, \delta(f^n_{n+p} + f^n_{n+u})\, \delta(f^n_{n+p} - f^n_{n+u})\,, \tag{4.169}$$

with the understanding that ρ integration needs to be done using the first delta function and ξ integration using the second delta function. Therefore, the contribution of this solution to the amplitude becomes, to subleading order in τ:

$$\begin{aligned}
\mathbf{M}^{(p,u)}_{n+m} &= -\frac{2}{\tau^m} \int D\sigma\, \delta^{(0)} \prod_{\substack{r=1\\ r\neq p,u}}^{m} d\sigma_{n+r}\, \delta(f^n_{n+r}) \int d\rho\, d\xi\, \delta\Bigg(\sum_{b=1}^{n} \frac{k_{n+p}.p_b}{\rho - \frac{\xi}{2} - \sigma_b} + \sum_{b=1}^{n} \frac{k_{n+u}.p_b}{\rho + \frac{\xi}{2} - \sigma_b} \Bigg) \\
&\quad \delta\Bigg(\sum_{b=1}^{n} \frac{k_{n+p}.p_b}{\rho - \frac{\xi}{2} - \sigma_b} - \sum_{b=1}^{n} \frac{k_{n+u}.p_b}{\rho + \frac{\xi}{2} - \sigma_b} - \frac{2\tau k_{n+p}.k_{n+u}}{\xi} \Bigg)\, I_{n+m} \\
&\simeq -\frac{2}{\tau^{m-1}} \int D\sigma\, \delta^{(0)} \prod_{\substack{r=1\\ r\neq p,u}}^{m} d\sigma_{n+r}\, \delta(f^n_{n+r}) \int d\rho\, \delta\Bigg(\sum_{b=1}^{n} \frac{k_{n+p}.p_b}{\rho - \frac{\xi}{2} - \sigma_b} + \sum_{b=1}^{n} \frac{k_{n+u}.p_b}{\rho + \frac{\xi}{2} - \sigma_b} \Bigg) \\
&\quad \frac{\xi_1^2}{2\, k_{n+p}.k_{n+u}}\, I_{n+m}\,,
\end{aligned} \tag{4.170}$$

where in the second step we have explicitly performed the ξ integration using the second delta function, and

$$\xi_1 \equiv 2k_{n+p}.k_{n+u} \left(\sum_{b=1}^{n} \frac{k_{n+p}.p_b}{\rho - \sigma_b} - \sum_{b=1}^{n} \frac{k_{n+u}.p_b}{\rho - \sigma_b} \right)^{-1}. \tag{4.171}$$

Now in the degeneration limit we can evaluate I_{n+m} by regarding this as the square of the Pfaffian of the matrix $\widehat{\Psi}$ given in (4.136). In computing the Pfaffian, it is convenient to first shift the $(n+p)^{th}$ and

$(n+u)^{th}$ rows and columns to the $(n+m-1)^{th}$ and $(n+m)^{th}$ positions and also the $(2n+m+p)^{th}$ and $(2n+m+u)^{th}$ rows and columns to the $(2n+2m-1)^{th}$ and $(2n+2m)^{th}$ positions, and then evaluate the Pfaffian. Now, if we exchange two rows (say ℓ and k) and the corresponding columns (ℓ and k) , then the value of Pfaffian changes by a sign. However one can easily see that the combined effect of all the movements is to not generate any sign. Now the problem effectively reduces to that of computing I_{n+m} for two soft gravitons, and using (4.117) we get,

$$\begin{aligned} I_{n+m} &= \Bigg[(\xi_1)^{-2}\epsilon_{n+p}.\epsilon_{n+u}\, k_{n+p}.k_{n+u} - (\xi_1)^{-2}\epsilon_{n+p}.k_{n+u}\, \epsilon_{n+u}.k_{n+p} \\ &\qquad -C_{n+p,n+p}\, C_{n+u,n+u}\Bigg]^2 I_{n+m-2} + \mathcal{O}(\tau) \\ &= \Bigg[(\xi_1)^{-2}\epsilon_{n+p}.\epsilon_{n+u}\, k_{n+p}.k_{n+u} - (\xi_1)^{-2}\epsilon_{n+p}.k_{n+u}\, \epsilon_{n+u}.k_{n+p} \\ &\quad -\Big\{\sum_{c=1}^{n}\frac{\epsilon_{n+p}.p_c}{\rho-\sigma_c} - (\xi_1)^{-1}\,\epsilon_{n+p}.k_{n+u}\Big\}\Big\{\sum_{c=1}^{n}\frac{\epsilon_{n+u}.p_c}{\rho-\sigma_c} + (\xi_1)^{-1}\epsilon_{n+u}.k_{n+p}\Big\}\Bigg]^2 I_{n+m-2} \\ &\quad +\mathcal{O}(\tau)\,, \end{aligned} \tag{4.172}$$

where I_{n+m-2} is the integrand for the scattering amplitude of $(n+m-2)$ gravitons in which we have removed the s-th and the u-th gravitons from the original set. Therefore, to the desired order of expansion, the contribution to the $(n+m)$-point amplitude from the degenerate solution becomes,

$$\begin{aligned} &\mathbf{M}_{n+m}^{(p,u)} \\ &= -\frac{4k_{n+p}.k_{n+u}}{\tau^{m-1}}\int D\sigma\,\delta^{(0)}\prod_{\substack{r=1\\ r\neq p,u}}^{m} d\sigma_{n+r}\delta(f_{n+r}^{n})I_{n+m-2}\int d\rho\,\delta\Big(\sum_{b=1}^{n}\frac{k_{n+p}.p_b}{\rho-\sigma_b}+\sum_{b=1}^{n}\frac{k_{n+u}.p_b}{\rho-\sigma_b}\Big) \\ &\Big(\sum_{a=1}^{n}\frac{(k_{n+p}-k_{n+u}).p_a}{\rho-\sigma_a}\Big)^{-2}\Bigg[\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+p}.p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+u}.p_d}{\rho-\sigma_d}\Big) - \frac{\epsilon_{n+p}.\epsilon_{n+u}}{4\,k_{n+p}.k_{n+u}}\Big(\sum_{a=1}^{n}\frac{(k_{n+p}-k_{n+u}).p_a}{\rho-\sigma_a}\Big)^2 \\ &+\Big(\epsilon_{n+u}.k_{n+p}\sum_{c=1}^{n}\frac{\epsilon_{n+p}.p_c}{\rho-\sigma_c} - \epsilon_{n+p}.k_{n+u}\sum_{c=1}^{n}\frac{\epsilon_{n+u}.p_c}{\rho-\sigma_c}\Big)\frac{1}{2k_{n+p}.k_{n+u}}\Big(\sum_{b=1}^{n}\frac{k_{n+p}.p_b}{\rho-\sigma_b}-\sum_{b=1}^{n}\frac{k_{n+u}.p_b}{\rho-\sigma_b}\Big)\Bigg]^2. \end{aligned} \tag{4.173}$$

In the above expression, we are only interested in the τ independent contribution of I_{n+m-2}. Hence, we can evaluate it at $\tau=0$. In terms of I_n, it is given by

$$I_{n+m-2} = (-1)^{m-2}\prod_{\substack{r=1\\ r\neq p,u}}^{m}\Big(\sum_{b=1}^{n}\frac{\epsilon_{n+r}.p_b}{\sigma_{n+r}-\sigma_b}\Big)^2 I_n\ + O(\tau)\,. \tag{4.174}$$

The integration over σ_{n+r} for $r\neq p,u$ in (4.173) can now be performed by the standard contour deformation, producing a factor of $(-1)^{m-2}\prod_{r=1,r\neq p,u}^{m} S^{(0)}_{n+r}$. The remaining integral over ρ has exactly the same form as (4.118) and can be analyzed as in section 4.3.3. Using (4.123), (4.125) we see that the final result for (4.173) is given by a sum of two terms:

$$\mathbf{M}^{(p,u)(1)}_{n+m} = \tau^{-m+1}\prod_{\substack{r=1\\ r\neq p,u}}^{m} S^{(0)}_{n+r}\sum_{a=1}^{n}\{p_a.(k_{n+p}+k_{n+u})\}^{-1}\,\mathcal{M}(p_a;\varepsilon_{n+p},k_{n+p},\varepsilon_{n+u},k_{n+u})\,\mathbf{M}_n\,, \tag{4.175}$$

and

$$\begin{aligned}
&\mathbf{M}^{(p,u)(2)}_{n+m}\\
&= -\frac{k_{n+p}.k_{n+u}}{\tau^{m-1}}\int D\sigma\delta^{(0)}\Big(\prod_{\substack{r=1\\ r\neq p,u}}^{m} S^{(0)}_{n+r}\Big) I_n \oint_{\{A_{s,i}\}} d\rho\,\Big(\sum_{b=1}^{n}\frac{k_{n+u}\cdot p_b}{\rho-\sigma_b}\Big)^{-2}\Big(\sum_{a=1}^{n}\frac{k_{n+p}\cdot p_a}{\rho-\sigma_a}\Big)^{-1}\\
&\qquad\Big\{\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+p}\cdot p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+u}\cdot p_d}{\rho-\sigma_d}\Big)\Big\}^2\\
&-\frac{k_{n+p}.k_{n+u}}{\tau^{m-1}}\int D\sigma\delta^{(0)}\Big(\prod_{\substack{r=1\\ r\neq p,u}}^{m} S^{(0)}_{n+r}\Big) I_n \oint_{\{A_{u,i}\}} d\rho\,\Big(\sum_{b=1}^{n}\frac{k_{n+u}\cdot p_b}{\rho-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{k_{n+p}\cdot p_a}{\rho-\sigma_a}\Big)^{-2}\\
&\qquad\Big\{\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+p}\cdot p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+u}\cdot p_d}{\rho-\sigma_d}\Big)\Big\}^2\\
&+\frac{2}{\tau^{m-1}}\int D\sigma\delta^{(0)}\Big(\prod_{\substack{r=1\\ r\neq p,u}}^{m} S^{(0)}_{n+r}\Big) I_n \oint_{\{A_{s,i}\}} d\rho\,\Big(\sum_{b=1}^{n}\frac{k_{n+u}\cdot p_b}{\rho-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{k_{n+p}\cdot p_a}{\rho-\sigma_a}\Big)^{-1}\\
&\qquad\Big\{\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+p}\cdot p_c}{\rho-\sigma_c}\Big)^2\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+u}\cdot p_d}{\rho-\sigma_d}\Big)\Big\}\epsilon_{n+u}.k_{n+p}\\
&+\frac{2}{\tau^{m-1}}\int D\sigma\delta^{(0)}\Big(\prod_{\substack{r=1\\ r\neq p,u}}^{m} S^{(0)}_{n+r}\Big) I_n \oint_{\{A_{u,i}\}} d\rho\,\Big(\sum_{b=1}^{n}\frac{k_{n+u}\cdot p_b}{\rho-\sigma_b}\Big)^{-1}\Big(\sum_{a=1}^{n}\frac{k_{n+p}\cdot p_a}{\rho-\sigma_a}\Big)^{-1}\\
&\qquad\Big\{\Big(\sum_{c=1}^{n}\frac{\epsilon_{n+p}\cdot p_c}{\rho-\sigma_c}\Big)\Big(\sum_{d=1}^{n}\frac{\epsilon_{n+u}\cdot p_d}{\rho-\sigma_d}\Big)^2\Big\}\epsilon_{n+p}.k_{n+u}\,.
\end{aligned} \tag{4.176}$$

$\mathcal{M}$ is the same function as defined in (4.124).

4.4.4 Total contribution

We can now add the expressions given in (4.175) and (4.176), sum over all possible choices p, u in the range $1 \le p < u \le m$, and add this to (4.167) to get the final result. The result is

$$\begin{aligned}\mathbf{M}_{n+m} \;&=\; \tau^{-m}\Big[\prod_{r=1}^{m} S^{(0)}_{n+r}\Big]\mathbf{M}_n + \tau^{-(m-1)}\sum_{v=1}^{m}\,\big[\prod_{\substack{r=1\\ r\neq v}}^{m} S^{(0)}_{n+r}\big]\Big(S^{(1)}_{n+v}\mathbf{M}_n\Big)\\ &+\tau^{-(m-1)}\sum_{\substack{p,u=1\\ p<u}}^{m}\Bigg(\prod_{\substack{r=1\\ r\neq p,u}}^{m} S^{(0)}_{n+r}\Bigg)\sum_{\substack{a=1\\ a\neq p,u}}^{n+m}\{p_a.(k_{n+p}+k_{n+u})\}^{-1}\,\mathcal{M}(p_a;\varepsilon_{n+p},k_{n+p},\varepsilon_{n+u},k_{n+u})\,\mathbf{M}_n\,.\end{aligned} \tag{4.177}$$

This agrees with the result (4.1) as claimed. This finishes the proof that the CHY prescription gives the same result for the subleading multiple soft graviton theorem as the Feynman diagram technique described in the previous chapter.

Appendix A

Mellin Representation of Conformal Field Theory

In this appendix, we review some basics of Mellin representation of the conformal field theories following [2, 32, 44, 123, 124].

A.1 Motivation for Mellin Representation

In usual quantum field theories (which have Poincare invariance), the amplitudes are easier to analyze in the momentum space as opposed to position space. One reason for this is that we are usually interested in perturbation around wave like solutions of the theory. For this purpose, the decomposition of solution in terms of sine and cosine modes is most appropriate if we are using the cartesian coordinates. The Fourier transform to momentum space naturally picks out these different modes. Some of the nice features of the momentum space amplitudes are

1. Tree level amplitudes in momentum space are given by products of propagator and vertices. No integration is needed to be performed. In position space, the two point function $G(x, y)$ is given by an integral kernel. However, in momentum space $G(p)$ is just a rational function.

2. The momentum space amplitudes have nice analytic properties as functions of external momenta. These analytic properties have nice physical interpretations. In particular, the poles in the amplitude correspond to stable or unstable single particle states[1] and the branch cuts on real axis correspond to the multi particle states. Moreover, the residues of the poles factorize on lower point amplitudes.

3. The physical requirements of locality, causality and unitarity are also translated into some analytic behaviour of the amplitude in momentum space. The locality is ensured by some kind

[1] The unstable particle is characterized by a non zero value of its on shell self energy diagram.

of polynomial boundedness of amplitudes for theories involving finite number of derivatives. The causality is related to the properties of analytic continuation of the amplitudes. Finally, the unitarity is encoded in various dispersion relations and the cutting rules.

All these properties are difficult to handle in the position space since the analytic behaviour of position space amplitudes is not very transparent. Now, we are interested in the conformal field theories. Naively, one might expect that since CFTs are also quantum field theories, we should also be able to analyze them in the momentum space. However, it turns out that the momentum space is not very useful for CFTs.

To see this, we recall that, in general, the non trivial interacting CFTs have a continuous spectrum (going to zero) since there is no mass (gap) scale in the theory. Due to the absence of isolated single particle states in the spectrum, the momentum space amplitudes do not have simple poles. More specifically, the role of Hamiltonian in CFTs in radial quantization on $S^{d-1} \times R$ is played by the dilatation operator since it generates the "time" translation. The spectrum of the theory corresponds to the eigenvalues $\{\Delta_i\}$ of this operator. This spectrum is discrete for $d > 2$. Now, the two point function of scalars in CFTs behaves as

$$\langle \phi(x_1)\phi(x_2)\rangle \;=\; \frac{1}{|x_1 - x_2|^{2\Delta}} \tag{A.1}$$

Δ is the scaling dimension of the scalars and it is not necessarily an integer in non trivial CFTs. The Fourier transform of this two point function is proportional to $p^{-2(d-\Delta)}$ which has a branch cut starting at $p^2 = 0$ for a generic non integer Δ. Thus, even for the simplest correlators, the momentum space behaviour is very complicated.

Another way to see the problem is to note that on the "cylinder" $S^{d-1} \times R$, the good quantities are not the momenta but the spherical harmonics. Moreover, CFTs have additional global symmetries, namely, scale invariance and the special conformal invariance apart from Poincare symmetry. Now, the special conformal transformations acts non trivially on the Mandelstam invariants $p_i.p_j$. This is unlike the Poincare transformations which keep the Mandelstam variables invariant.

Due to the above reasons, the momentum space does not provide a very useful representation for analyzing CFTs. Hence, correlators in CFTs are usually analyzed in the position space itself which again has all the disadvantages mentioned earlier. It prompts us to ask the question if there is representation which is more suitable for CFTs. One property we would like to have for this representation is that the amplitudes in this representation should possesses poles associated with the spectrum of the theory just as momentum space does for Poincare invariant theories. It turns out that there is indeed such a representation, namely, Mellin representation [2]. The Mellin representation makes use of the fact that the non trivial information of a CFT correlator depends only upon the so called cross ratios. Hence, instead of transforming each position space coordinate (just as we do when going from position to momentum space), it is more useful to make a transformation with respect to the cross ratios. We shall describe below this transformation and properties of CFT correlators in Mellin space. For this, we start by reviewing some of the properties of the CFT correlators in position space.

A.2 CFT Correlators in Position Space

A general n-point correlator in position space is given by (suppresing the indices on operators)

$$\mathcal{A}(x_i) \;=\; \Big\langle \mathcal{O}_{\Delta_1,\ell_1}(x_1)\mathcal{O}_{\Delta_2,\ell_2}(x_2)\cdots\mathcal{O}_{\Delta_n,\ell_n}(x_n)\Big\rangle \tag{A.2}$$

The Δ_i, ℓ_i denote the dimension and spin of the i-th operator. We shall focus on the case when $\mathcal{O}_i$ are primary operators. we shall also restrict to the case $\ell_i = 0$ for simplicity. We shall now analyze the consequence of the conformal symmetry on this correlator.

The Poincare invariance implies that $\mathcal{A}(x_i)$ can depend only upon[2] $x_{ij}^2 = (x_i - x_j)^2$. The scale invariance implies

$$\mathcal{O}_{\Delta_i}(\lambda x_i) \;=\; \lambda^{-\Delta_i}\mathcal{O}_{\Delta_i}(x_i) \qquad \Longrightarrow \qquad \mathcal{A}(\lambda x_i) \;=\; \lambda^{-\sum_{i=1}^{n}\Delta_i}\mathcal{A}(x_i) \tag{A.3}$$

Now, under scale transformation, we have $x_{ij}^2 \to \lambda^2 x_{ij}^2$. This means that the ratios $\frac{x_{ij}^2}{x_{kl}^2}$ will be scale invariant. Thus, if a theory had just the Poincare and scale invariance, the amplitude could be written in the form

$$\mathcal{A}(x_i) = \frac{1}{\prod\limits_{i<j}(x_{ij})^{2\Delta_{ij}}}\widetilde{\mathcal{A}}(r_{ij}) \tag{A.4}$$

where r_{ij} are a set of $\frac{n(n-1)}{2} - 1$ independent cross ratios which can be chosen, e.g., to be

$$r_{ij} = \frac{x_{ij}^2}{x_{12}^2} \qquad ; \qquad (i,j) \neq (1,2) \tag{A.5}$$

and Δ_{ij} are chosen to satisfy

$$2\sum_{i<j}\Delta_{ij} = \sum_{i=1}^{n}\Delta_i \tag{A.6}$$

Equations (A.4),(A.5) and (A.6) ensure that equation (A.3) holds.

Finally, we consider the consequence of the covariance under the special conformal transforma-

[2]For $\ell_i \neq 0$, there can be more tensor structures.

tion (SCT). To implement SCT, we note that SCT can be expressed as

$$SCT = \text{Inversion} \times \text{Translation} \times \text{Inversion} \tag{A.7}$$

Thus, if we can make our amplitude covariant under inversion transformation, then it will automatically be covariant under SCT. Now, under inversion, we have $x^\mu \to x^\mu/x^2$ and

$$\mathcal{O}_{\Delta_i}\left(\frac{x_i^\mu}{x_i^2}\right) = (x_i)^{2\Delta_i}\mathcal{O}_{\Delta_i}(x_i) \qquad \Longrightarrow \qquad \mathcal{A}\left(\frac{x_i^\mu}{x_i^2}\right) = \left[\prod_{i=1}^{n}(x_i)^{2\Delta_i}\right]\mathcal{A}(x_i) \tag{A.8}$$

Noting that $x_{ij}^2 \to x_{ij}^2/x_i^2x_j^2$ under inversion, it is easy to check that the "cross ratios" $\frac{x_{ij}^2x_{k\ell}^2}{x_{ik}^2x_{j\ell}^2}$ are invariant under the inversion. Total number of independent cross ratios is $\frac{n(n-3)}{2}$ for n points. We can prove this recursively. Suppose, we have a set of independent cross ratios for $(n-1)$ points $(x_1, \cdots, x_{n-1})$. We now add another point x_n. Using this, we can construct $n-2$ new cross ratios as

$$\frac{x_{1n}^2x_{23}^2}{x_{13}^2x_{2n}^2} \quad , \quad \frac{x_{12}^2x_{an}^2}{x_{1a}^2x_{2n}^2} \qquad \text{where} \quad a = 3, 4, \cdots, (n-1) \tag{A.9}$$

It can be easily verified that any other cross ratio built from the n points can be expressed in terms of the cross ratios of $n-1$ points and the cross ratios of equation (A.9). Now, if S_m is the number of independent cross ratios for m points, then we have

$$S_{n-1} + n - 2 = S_n \tag{A.10}$$

We also know that for 4 points, number of independent cross ratios are 2, i.e.,

$$S_4 = 2 \tag{A.11}$$

Solving the above two equations recursively, we find $S_n = \frac{n(n-3)}{2}$ which proves the desired result. It turns out that this result is valid as long as the number of space-time dimensions d is greater than the number of points n. If n is large compared to d, there are non trivial relations between the $\frac{n(n-3)}{2}$ cross ratios. More precisely, for a d-dimensional space-time, the number of conformal cross ratios is given by (see, e.g., [44])

$$\begin{aligned} S_n &= \frac{n(n-3)}{2} \qquad ; \qquad d+1 > n \\ S_n &= nd - \frac{1}{2}(d+1)(d+2) \qquad ; \qquad d+1 \leq n \end{aligned} \tag{A.12}$$

Returning to the covariance under SCT, to take into account (A.8), the correlator can be expressed in the form

$$\mathcal{A}(x_i) = \frac{1}{\prod\limits_{i<j}(x_{ij})^{2\Delta_{ij}}}\widetilde{\mathcal{A}}(u_n) \tag{A.13}$$

where $\{u_n\}$ are the independent cross ratios and Δ_{ij} must satisfy

$$\Delta_{ij} = \Delta_{ji} \quad , \qquad \sum_{\substack{j=1 \\ j\neq i}}^{n} \Delta_{ij} = \Delta_i \qquad ; \quad i = 1, 2, \cdots, n \tag{A.14}$$

For 4 point function, an explicit solution for Δ_{ij} is given by

$$\begin{aligned} \Delta_{14} &= -\frac{\Delta_2+\Delta_3}{2} \quad , \qquad \Delta_{24} = \frac{\Delta_2+\Delta_4}{2} \quad , \qquad \Delta_{34} = \frac{\Delta_3+\Delta_4}{2} \\ \Delta_{12} &= \frac{\Delta_1+\Delta_2}{2} \quad , \qquad \Delta_{13} = \frac{\Delta_1+\Delta_3}{2} \quad , \qquad \Delta_{23} = -\frac{\Delta_1+\Delta_4}{2} \end{aligned} \tag{A.15}$$

A.3 Mellin Representation of CFT Correlators

As mentioned earlier, in the Poincare invariant QFTs, the Fourier transform to momentum space is useful since it naturally picks out the different harmonic modes. However, this is not suitable for the CFTs since CFT correlators more naturally admit a power law decomposition (and not harmonic decomposition) as in equation (A.13). It turns out that the Mellin transform of a function does precisely this, i.e., it decomposes a function in terms of functions with definite scaling behaviour. More precisely, the Mellin transform $\widetilde{f}(s)$ of a function $f(x)$ is defined as

$$\widetilde{f}(s) = \int_0^\infty dx\, x^{s-1}\, f(x) \qquad \Longrightarrow \qquad f(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\, x^{-s}\, \widetilde{f}(s) \tag{A.16}$$

s is the complex Mellin variable. The contour in above equation runs along the imaginary axis. However, sometimes we need to deform it to pick up appropriate poles.

From (A.16), we see that the decomposition of $f(x)$ is in terms of power law behaviour. More precisely, the components of $f(x)$ with different power law behaviour are associated with the poles of $\widetilde{f}(s)$. In other words, if $f(x)$ has contributions from different scaling dimensions, the $\widetilde{f}(s)$ would be sensitive to these scaling dimensions. As an example, we consider a simple Mellin function

$$\widetilde{f}(s) \;=\; \frac{1}{s-\Delta} \qquad \text{(where } \Delta \text{ is real)} \tag{A.17}$$

For $x > 1$, we close the contour in (A.16) from right and obtain

$$f(x) \;=\; \operatorname*{Res}_{s\to\Delta}\left(\frac{x^{-s}}{s-\Delta}\right) \;=\; \frac{1}{x^{\Delta}} \tag{A.18}$$

This property of the Mellin transform is essentially the reason we consider the Mellin representation of CFT correlators. We now follow [2] to obtain the Mellin representation of 4-point CFT correlator. For details, please refer to [2].

We consider the 4-point correlator involving the scalar operators in position space. Restricting to $n = 4$, equation (A.13) gives

$$M(x_i) = \Big\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\Big\rangle = \prod_{i<j}(x_{ij})^{-2\Delta_{ij}} f(u,v) \tag{A.19}$$

where Δ_{ij} satisfy the constraint (A.14) and u, v are the conformal cross ratios given by

$$u \;\equiv\; \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad ; \qquad v \;\equiv\; \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{A.20}$$

We now make a Mellin transform with respect to the cross ratios u and v. This gives,

$$\begin{aligned} M(x_i) &= \prod_{i<j}(x_{ij})^{-2\Delta_{ij}} f(u,v) \\ &= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\beta_1 \int_{-i\infty}^{i\infty} d\beta_2\, u^{-\beta_1} v^{-\beta_2}\, \widetilde{f}(\beta_1,\beta_2) \prod_{i<j}(x_{ij})^{-2\Delta_{ij}} \\ &= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\beta_1 \int_{-i\infty}^{i\infty} d\beta_2\, \widetilde{f}(\beta_1,\beta_2) \prod_{i<j}(x_{ij})^{-2s_{ij}} \end{aligned} \tag{A.21}$$

In going to the 3rd line, we have used (A.20) and defined $s_{ij} \equiv \Delta_{ij} + \beta_{ij}$, where β_{ij} are linear combinations of β_1 and β_2 as

$$\beta_{12} = \beta_{34} = \beta_1 \qquad , \qquad \beta_{14} = \beta_{23} = \beta_2 \qquad , \quad \beta_{13} = \beta_{24} = -\beta_1 - \beta_2 \tag{A.22}$$

The β_{ij} and s_{ij} satisfy

$$\sum_{\substack{j=1\\ j\neq i}}^{4} \beta_{ij} \;=\; 0 \qquad \Longrightarrow \qquad \sum_{\substack{j=1\\ j\neq i}}^{4} s_{ij} \;=\; \Delta_i \quad , \qquad s_{ij} = s_{ji} \tag{A.23}$$

The conventional definition of Mellin amplitude includes a product of Gamma functions in the measure as

$$M(x_i) \;=\; \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} d^2s\; M(s_{ij})\prod_{i<j}\Gamma(s_{ij})(x_{ij})^{-2s_{ij}} \tag{A.24}$$

The integration in above equation is over two independent Mellin variables (say s_{12} and s_{13}). The factors of $\Gamma(s_{ij})$ are a matter of convention. However, these Gamma functions turn out to be very useful for large N CFTs as we shall see.

The above steps can be repeated for an arbitrary n-point function. In that case, the Mellin amplitude can be defined as

$$M\left(x_i\right) = \prod_{1\le i<j\le n}\left(\int_{-i\infty}^{i\infty}\frac{ds_{ij}}{2\pi i}\,\Gamma\left(s_{ij}\right)\left(x_{ij}\right)^{-2s_{ij}}\right)\prod_{i=1}^{n}\delta\left(\Delta_i-\sum_{\substack{j=1\\ j\neq i}}^{n}s_{ij}\right)M\left(s_{ij}\right) \tag{A.25}$$

We have imposed the analog of the constraint (A.23) for n point function by including explicit delta functions in the measure. These constraints take care of the covariance property (A.8) of the amplitude. Due to this, the total number of independent Mellin variables is $\frac{1}{2}n(n-3)$ which is same as the number of independent conformal cross ratios.

The constraints on the Mellin variables can be solved by introducing a set of fictitious momenta k_i^μ and identifying s_{ij} with some kind of Mandelstam variables as

$$s_{ij} \;=\; k_i.k_j \qquad , \qquad \sum_{i=1}^{n}k_i^\mu \;=\; 0 \quad , \qquad k_i^2 \;=\; -\Delta_i \tag{A.26}$$

The s_{ij} defined in this way automatically satisfy the delta function constraints. One motivation for this representation of the Mellin variables is that in the context of AdS CFT in the large radius limit, the s_{ij} behave as the flat space Mandelstam variables [32, 39, 40].

It is instructive to solve the constraints on Mellin variables explicitly and express the amplitude (A.25) as integral over just independent variables for $n = 4$. In this case, the four constraint equations are

$$s_{12}+s_{13}+s_{14} \;=\; \Delta_1 \qquad , \qquad s_{12}+s_{23}+s_{24} \;=\; \Delta_2$$

$$s_{13}+s_{23}+s_{34} \;=\; \Delta_3 \qquad , \qquad s_{14}+s_{24}+s_{34} \;=\; \Delta_4$$

Due to these constraints, there are only two independent Mellin variables. We shall choose them to be s_{12} and s_{13}. It turns out that instead of using s_{12} and s_{13}, it is more convenient to use the representation (A.26) and define shifted variables s and t as

$$s = -(k_1+k_2)^2 = \Delta_1+\Delta_2-2s_{12} \qquad \Longrightarrow \qquad s_{12} = \frac{1}{2}\big(\Delta_1+\Delta_2-s\big)$$

$$t = -(k_1+k_3)^2 = \Delta_1+\Delta_3-2s_{13} \qquad \Longrightarrow \qquad s_{13} = \frac{1}{2}\big(\Delta_1+\Delta_3-t\big)$$

In terms of s and t, the other Mellin variables can be solved as

$$s_{14} = \frac{s+t-\Delta_2-\Delta_3}{2} \quad , \quad s_{23} = \frac{s+t-\Delta_1-\Delta_4}{2}$$

$$s_{24} = \frac{\Delta_2+\Delta_4-t}{2} \quad , \quad s_{34} = \frac{\Delta_3+\Delta_4-s}{2}$$

Using s and t, the 4 point correlator can be expressed as

$$\mathcal{A}(x_i) = \frac{1}{\prod\limits_{i<j}(x_{ij})^{2\Delta_{ij}}}\widetilde{\mathcal{A}}(u,v) \tag{A.27}$$

where Δ_{ij} are given in (A.15) and

$$\begin{aligned}\widetilde{\mathcal{A}}(u,v) \;=\; & \int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\; M(s,t)\; u^{\frac{s}{2}}\; v^{-\frac{s+t}{2}}\;\Gamma\left[\frac{\Delta_1+\Delta_2-s}{2}\right]\Gamma\left[\frac{\Delta_1+\Delta_3-t}{2}\right] \\ & \Gamma\left[\frac{\Delta_3+\Delta_4-s}{2}\right]\Gamma\left[\frac{\Delta_2+\Delta_4-t}{2}\right]\Gamma\left[\frac{s+t-\Delta_2-\Delta_3}{2}\right]\Gamma\left[\frac{s+t-\Delta_1-\Delta_4}{2}\right]\end{aligned} \tag{A.28}$$

A.4 Properties of Mellin Amplitudes

The Mellin amplitude $M(s_{ij})$ have very nice properties which are not manifest in the position space amplitude. We first list some of these properties

1. The Mellin amplitudes are meromorphic functions of the independent Mellin variables[3], i.e., they only have poles but no branch cuts unlike momentum space amplitudes. The branch cuts in momentum space come from absence of a mass gap. However, the Mellin amplitudes which are sensitive to the scaling behaviour of operators do not have branch cuts since in most cases of interest, we only have discrete spectrum of dimensions of operators.

2. The poles in different channels correspond to the twists $\tau = \Delta - \ell + 2m \quad (m = 0, 1, 2, \cdots)$ of the operators exchanged in that channel in the intermediate state. The first pole $(m = 0)$

[3]This is true as long as we can ignore the multi trace operators. We shall consider the presence of the multi trace operators towards the end of this section.

corresponds to the exchanged primary operator and all its leading twist descendants (i.e. those operators in the conformal multiplet whose dimensions and spins keep $\Delta-\ell$ fixed). The higher poles ($m>0$) correspond to the satellite poles.

3. The Mellin amplitude factorizes in terms of the lower point amplitudes when the intermediate particle goes on-shell. In other words, the residues at poles of the correlators are related to the lower point correlators. This property of factorization is also present for usual QFTs in the momentum space.

4. The channel dualities (i.e. ability to evaluate the amplitude in different OPE channel) are simply manifest as the exchange of corresponding Mellin variables.

5. For large N CFTs where multi trace correlators factorize, the additional Gamma functions present in the definition of the Mellin amplitude take care of the contributions of the multi trace correlators. This means that the Mellin amplitude $M(s_{ij})$ contains information only about the single trace operators.

6. In the large radius limit of AdS, the Mellin amplitudes become flat space scattering amplitude and the Mellin variables s_{ij} become the Mandelstam variables of flat space (upto some proportionality factor) [32, 39, 40].

The first four properties are universal properties and essentially follow from the OPEs. We shall illustrate these for the case of 4-point function of scalar primary operators. The 5th property requires the large N CFT whereas the 6th property needs large N and large t' hooft coupling. We now elaborate on each of these property below.

Meromorphicity of Mellin Amplitudes

We start by considering the 4-point function

$$\mathcal{A}(x_i)=\Big\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\Big\rangle \tag{A.29}$$

The OPE in the s channel is given by

$$\begin{aligned}\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) &= \sum_{\Delta,n} c_{12}^{\Delta,n}\frac{(x_{12})_{\mu_1}\cdots(x_{12})_{\mu_\ell}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta+\ell-2n}}\,\partial^{2n}\mathcal{O}_{\Delta}^{\mu_1\cdots\mu_\ell}(x_2)\;+\;\cdots\\ \mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4) &= \sum_{\Delta,n} c_{34}^{\Delta,n}\frac{(x_{34})_{\mu_1}\cdots(x_{34})_{\mu_\ell}}{|x_{34}|^{\Delta_3+\Delta_4-\Delta+\ell-2n}}\,\partial^{2n}\mathcal{O}_{\Delta}^{\mu_1\cdots\mu_\ell}(x_4)\;+\;\cdots\end{aligned} \tag{A.30}$$

The $\cdots$ terms in (A.30) denote the contributions of the multitrace operators such $\mathcal{O}_{\Delta_i}\partial^{2n}\mathcal{O}_{\Delta_j}$. In the above OPE of two scalar operators, we have the contributions from spin ℓ operators $\mathcal{O}_{\Delta}^{\mu_1\cdots\mu_\ell}$.

This happens because the scalar operators are not coincident. Hence, when we expand the OPE in terms of operators at a single point, these operators can have non zero orbital angular momentum spins. The $c_{12}^{\Delta,n}$ are dimensionless coefficients and are known as OPE coefficients or 3-point function coefficients. The factor $|x_{12}|^{\Delta_1+\Delta_2-\Delta+\ell-2n}$ in RHS of above OPE is necessary for both sides to have the same conformal transformation properties (or equivalently the mass dimension).

Now, the correlator in (A.29) can be evaluated by doing an s channel OPE expansion which converts the correlator into a sum over two point functions. The leading behaviour of the amplitude in the s channel expansion is dictated by (A.30). Thus, the leading behaviour in s channel of the correlator (A.29) in $|x_{12}|$ variable goes as power law $|x_{12}|^{-(\Delta_1+\Delta_2-\Delta+\ell-2n)}$. We want to reproduce this behaviour from our definition of the Mellin amplitude given in (A.27) and (A.28). For this, we note that x_{12} enters in (A.27) through the prefactor involving Δ_{12} and through the definition of u in $\widetilde{\mathcal{A}}(u,v)$. Now, the factor $\Delta_1 + \Delta_2$ in the power law behaviour is taken care by the prefactor in (A.27) through the term involving Δ_{12}. For the remaining terms in power law, we note that $\widetilde{\mathcal{A}}(u,v)$ involves integration over $u^{s/2}$ (note that u involves x_{12}^2). This means that $M(s,t)$ must have poles at $s = \Delta - \ell + 2n$. This will ensure the expected behaviour of the amplitude in the s channel expansion.

The total Mellin amplitude $M(s,t)$ is a sum of contributions from each of these poles. Thus, for the s channel expansion, we can write

$$M(s,t) \sim \sum_{\Delta,n} \frac{M_n(t)}{s-(\Delta-\ell+2n)} \tag{A.31}$$

Thus, we have an infinite set of descendant poles (labelled by n) associated with each primary operator twist ($\tau = \Delta - \ell$). For each operator $\mathcal{O}_{\Delta,n} = \partial^{2n}\mathcal{O}_{\Delta}^{\mu_1\cdots\mu_\ell}$ contributing to the OPE in (A.30), we have a pole in (A.31). Due to the discreteness of dimensions of the primary operators, the sum in (A.30) is a discrete sum and hence the sum in (A.31) is also discrete. This implies that $M(s,t)$ has no branch cuts but is a meromorphic function.

An important point to note about the Mellin amplitude $M(s,t)$ is that it can only have simple poles. The higher order poles will not be consistent with the OPE structure (A.30). To see this, we note that if $M(s,t)$ had a double pole in s, then the s integral in (A.28) will go as

$$\int ds\, \frac{u^{s/2}}{(s-a)^2}\, f(s) \tag{A.32}$$

where a is a constant and $f(s)$ is an analytic function in s. The evaluation of this integral will give an expression proportional to $\log(u)$ which will be in contradiction with the OPE behaviour (A.30).

Factorization of Residue

We next consider the factorization of the residue. To evaluate the correlator in (A.29), we need to replace $\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)$ and $\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)$ by their OPE expansions (A.30). Now, we can choose our basis of operators so that the two point function becomes diagonal and the non zero contribution only occurs when same operators appear in both the OPEs. After doing this, the 4-point function reduces to a sum over 2-point functions as

$$\mathcal{A}_i(x_i) \;=\; \sum_{\Delta,n,m} c_{12}^{\Delta,n} c_{34}^{\Delta,m}\Big[\frac{C(x_{12},\partial_y)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta+\ell-2n}}\frac{C(x_{34},\partial_z)}{|x_{34}|^{\Delta_3+\Delta_4-\Delta+\ell-2m}}\langle\mathcal{O}_\Delta(y)\mathcal{O}_\Delta(z)\rangle\Big] \qquad \text{(A.33)}$$

Since OPE is consistent with the conformal symmetry, the above expression can also be expressed in terms of so called conformal blocks as

$$\mathcal{A}(x_i) = \frac{1}{\prod\limits_{i<j}(x_{ij})^{2\Delta_{ij}}}\sum_{\Delta} c_{12}^{\Delta} c_{34}^{\Delta}\;\; G_{\Delta,\ell}(u,v) \qquad \text{(A.34)}$$

The functions $G_{\Delta,\ell}(u,v)$ are the conformal blocks in s channel. They represent the kinematic data determined by the conformal symmetry. The sum in the above expression is only over Δ. The sum over n and m present in OPE (A.30) which correspond to the contribution from descendants is already included in the definition of the conformal block. It turns out that the conformal block $G_{\Delta,\ell}$ admit an expansion in powers of u as

$$G_{\Delta,\ell} = u^{\tau/2}\sum_{n=0}^{\infty} u^n g_n(v) \qquad \text{(A.35)}$$

where $g_n(v)$ admits a power series expansion in $1-v$. E.g., the first term is given by

$$g_0(v) = \left(\frac{v-1}{2}\right)^{\ell} {}_2F_1\left(\frac{\Delta+\ell-\Delta_1+\Delta_2}{2}, \frac{\Delta+\ell+\Delta_3-\Delta_4}{2};\; \Delta+\ell;\; 1-v\right) \qquad \text{(A.36)}$$

The expressions (A.33) and (A.34) represent the factorization of the amplitude in terms of two sets of 3-point functions (each set determined by c_{12}^{Δ} or c_{34}^{Δ}). For this to be consistent with Mellin amplitude, we must have the following expansion for the Mellin amplitude

$$M(s,t) \;\sim\; \sum_{\Delta,n}\frac{c_{12}^{\Delta}\, c_{34}^{\Delta}\; \mathcal{Q}_{\ell,n}(t)}{s-(\tau_\Delta+2n)} \qquad , \qquad \tau_\Delta = \Delta-\ell \qquad \text{(A.37)}$$

The $\mathcal{Q}_{\ell,n}(t)$ are a set of orthogonal polynomials of degree ℓ which are usually parametrized in terms of another polynomial $Q_{\ell,n}$ as

$$\mathcal{Q}_{\ell,n}(t) \;=\; -\frac{2\Gamma(\Delta+\ell)(\Delta-1)_\ell}{4^\ell\Gamma\left(\frac{\Delta+\ell+\Delta_1-\Delta_2}{2}\right)\Gamma\left(\frac{\Delta+\ell-\Delta_1+\Delta_2}{2}\right)\Gamma\left(\frac{\Delta+\ell+\Delta_3-\Delta_4}{2}\right)\Gamma\left(\frac{\Delta+\ell-\Delta_3+\Delta_4}{2}\right)} \frac{Q_{\ell,n}(t)}{n!(\Delta-\frac{d}{2}+1)_n\Gamma\left(\frac{-\Delta+\ell+\Delta_1+\Delta_2}{2}-n\right)\Gamma\left(\frac{-\Delta+\ell+\Delta_3+\Delta_4}{2}-n\right)}$$

$(a)_m$ denote the Pochhammer symbol

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} \tag{A.38}$$

and the polynomial $Q_{\ell,n}$ behave as

$$Q_{\ell,n}(t) \;=\; t^\ell \;+\; O(t^{\ell-1}) \tag{A.39}$$

Channel Duality

In the above analysis, we used the OPE in s channel. However, we could have used the OPE in the t channel as well ($x_1 \to x_3$). The operators in the intermediate state $\mathcal{O}_{\Delta,\ell}$ would then correspond to the operators appearing in the OPE of $\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_3}(x_3)$. Since the final result should be independent of the OPE channel used for calculation, the Mellin amplitude must have a set of poles similar to (A.31) and (A.37) in t variables as well (i.e. at $t = \tau + 2n$). For identical scalars, there should be a symmetry under exchange of the s and t channels. This is manifest in the Mellin amplitude which is symmetric in s and t variables. This is channel duality.

Large N Limit

In general CFTs in dimensions $d > 2$, given any two primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$ which have twists τ_1 and τ_2 respectively, the CFT contains an infinite family of double twists operators with arbitrarily increasing spin ℓ. Moreover, in the limit $\ell \to \infty$, their twists are given by $\tau_1 + \tau_2 + 2m$ where m is a positive integer. This makes the Mellin amplitude non meromorphic since these infinite sequence of poles start getting accumulated at these asymptotic values of the twists.

It turns out that in the large N limit, the above situation simplifies considerably. This happens because in the large N limit, the spin ℓ double trace operators of the form $\mathcal{O}_1\partial^{2n}\partial^{\mu_1\cdots\mu_\ell}\mathcal{O}_2$ have the twists $\tau = \Delta_1 + \Delta_2 + 2n + O(1/N^2)$. Similarly, the double trace operators $\mathcal{O}_3\partial^{2n}\partial^{\mu_1\cdots\mu_\ell}\mathcal{O}_4$ have the twists $\tau = \Delta_3 + \Delta_4 + 2n + O(1/N^2)$. Now, from expression (A.28), we see that the additional Gamma functions present in the definition of the Mellin amplitude have poles at $s = \Delta_1 + \Delta_2 + 2n$ and $s = \Delta_3 + \Delta_4 + 2n'$ (for positive integers n and n'). These poles are precisely the poles we expect due to the contributions of double trace operators. Thus, we don't need to include these poles into the Mellin amplitude $M(s,t)$. This implies that $M(s,t)$ only captures the single trace piece of the

OPE.

This is very convenient from the AdS CFT point of view. The single trace operators of the boundary CFT correspond to the single particle states in the bulk. Similarly, the multi trace operators correspond to the multi particles states. Now, suppose we are interested in the single particle scattering in bulk string theory. In the dual boundary theory of large N CFT, this will correspond to correlators of single trace operators. In the Mellin space, this information will be encoded in $M(s_{ij})$ which does not care about the multi trace operators. Thus, in the large N limit, if we are working with Mellin amplitudes, we don't need to worry about contamination by multi particle states if we consider the single particle scattering (i.e. in perturbative string theory). In other words, the Mellin amplitude gives a nice way to disentangle the contributions of the multi particle intermediate states in single particle scatterings.

Flat Space Limit

In the context of AdS CFT, we can consider the Mellin representation of the Witten diagrams. These Mellin amplitudes of the Witten diagrams again share all the universal properties of the Mellin space. They depend on the "radius" of the AdS and are meromorphic functions. Now, we can consider the flat space limit of AdS by sending the radius to infinity. Equivalently, we can consider particles which are much more energetic than the AdS length scale and hence they are insensitive to the AdS curvature. In effect, this requires a large t' hooft coupling. This is due to the fact that high energy particles in the AdS correspond to CFT operators with large scaling dimensions and hence the resulting CFT theory should be strongly coupled charecterized by a large t' hooft coupling.

Now, the location of the poles in Mellin amplitude depend upon the CFT coupling (or equivalently the radius of AdS). In taking the flat space limit, we make the coupling (or radius) very large. Due to this, the poles of the Mellin amplitudes start coming together and merge to form a branch cut. It turns out that in this situation, we can relate the AdS Mellin amplitude to the flat space scattering matrix. An explicit relation between the massless flat space scattering amplitude $\mathcal{T}(S_{ij})$ and the Mellin amplitude $M(s_{ij})$ is given by [32, 39, 40]

$$\begin{aligned}\mathcal{T}(S_{ij}) &= \lim_{R\to\infty}\frac{1}{\mathcal{N}}\delta^{d+1}\left(\sum_i k_i\right)\int_{-i\infty}^{i\infty} d\alpha\, e^{\alpha}\,\alpha^{\frac{1}{2}(d-\sum\Delta_i)}M\left(s_{ij}=-\frac{R^2 S_{ij}}{4\alpha}\right)\\ M(s_{ij}) &= \mathcal{N}\int_0^{\infty}\frac{d\beta}{\beta}\,e^{-\beta}\,\beta^{\frac{1}{2}(\sum\Delta_i-d)}\mathcal{T}\left(S_{ij}=-4\beta\frac{s_{ij}}{R^2}\right) \qquad , \quad s_{ij}>>1 \qquad \text{(A.40)}\end{aligned}$$

In the above expressions, $\mathcal{N}$ is a normalization factor which behaves as $\mathcal{N}\sim R^{n(1-d)/2+d+1}$, the $\{S_{ij}=k_i.k_j\}$ are the Mandelstam variables of the flat space scattering process, the $\{s_{ij}\}$ are the Mellin variables and the Δ_i are the dimensions of the external operators.

Appendix B

CHY Prescription for Tree Amplitude Calculation

In this appendix, we review the CHY prescription for computing the tree amplitudes involving the massless particles in arbitrary dimensions [24–27] (for review, see [120]).

B.1 Kinematic Space, Scattering Equations and CHY Formula

We start by recalling some facts about the singularity structures of the amplitudes. Since we are interested in the tree level amplitudes, the singularities correspond to the poles of the amplitudes and they occur when a subset of the Mandelstam variables become zero. If the momenta of the n particles involved in the scattering are denoted by $k_1, \cdots, k_n$, the Mandelstam variables can be defined by

$$2s_{a_1,a_2} \;=\; (k_{a_1} + k_{a_2})^2 \;=\; 2k_{a_1}.k_{a_2} \tag{B.1}$$

The total number of independent Mandelstam variables is $n(n-3)/2$. The codimension one singularity in the Mandelstam space is obtained by setting one of the Mandelstam variables to zero. Similarly, the codimension two singularity in the Mandelstam space is obtained by setting two of the Mandelstam variables to zero and so on. The highest codimension of a singularity in the Mandelstam space can be $n-3$. In general, studying the singularity subspaces in the Mandelstam space is a complicated problem. In the CHY formalism, we consider another space which encodes the information about the singularities of the Mandelstam space in a more transparent manner.

For this purpose, we consider a Riemann sphere with n punctures. If z is a choice of complex coordinate on the Riemann sphere, doing an $SL(2,\mathbb{C})$ transformation gives an equivalent coordinate system

$$z \to w \;\equiv\; \frac{az+b}{cz+d} \qquad a,b,c,d \;\in \mathbb{C} \quad \text{and} \quad ad-bc=1 \tag{B.2}$$

Now, the n punctured Riemann sphere can be parametrized by n holomorphic variables $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ together with the $SL(2,\mathbb{C})$ redundancy. Due to the $SL(2,\mathbb{C})$ freedom, any three points on the Riemann sphere can be fixed. The remaining $n-3$ points can be anywhere on the sphere. The different choice of locations of these $n-3$ points correspond to a different Riemann sphere (such that one can't go from one Riemann sphere to other by a holomorphic change of coordinates). The set of all Riemann spheres with n marked points having $SL(2,\mathbb{C})$ redundancy is called the moduli space of n punctured Riemann spheres and is denoted by $\mathcal{M}_{0,n}$. This means that the moduli space $\mathcal{M}_{0,n}$ is an $n-3$ dimensional complex space.

The moduli space $\mathcal{M}_{0,n}$ has boundaries. These boundaries correspond to the situation when some of the punctures come very close to each other. In this situation, the sphere degenerates. Our goal is to find a mapping between the singularities in the Mandelstam space and the degeneration limit of the n punctured Riemann sphere.

For concreteness, we consider the case of $n = 4$. The Mandelstam variables for this case are given by

$$\begin{aligned} s &= 2s_{12} = 2s_{34} = -(k_1+k_2)^2 \\ t &= 2s_{23} = 2s_{14} = -(k_2+k_3)^2 \\ u &= 2s_{13} = 2s_{24} = -(k_1+k_3)^2 \end{aligned} \tag{B.3}$$

which satisfy $s+t+u=0$.

Next, we consider the moduli space of 4 punctured Riemann sphere. We fix the location of three punctures using the $SL(2,\mathbb{C})$ invariance at $\sigma_1 = 0, \sigma_3 = 1$ and $\sigma_4 = \infty$. The coordinate σ_2 is free to vary and parametrizes the moduli space. The degeneration limits of the Riemann sphere correspond to the situations when σ_2 is very close to any one of the fixed punctures.

The mapping between the Mandelstam space and the $\mathcal{M}_{0,4}$ requires finding a relation between σ_2 and the Mandelstam variables s, t, u

$$\sigma_2 \;=\; f(s_{12}, s_{13}, s_{23}) \tag{B.4}$$

with the following demand

$$\begin{aligned} \sigma_2 &\to 0 \qquad \text{when} \qquad s_{12} \to 0 \\ \sigma_2 &\to \infty \qquad \text{when} \qquad s_{13} \to 0 \\ \sigma_2 &\to 1 \qquad \text{when} \qquad s_{14} \to 0 \end{aligned}$$

A relation satisfying these conditions is given by

$$\sigma_2 \;=\; -\frac{s_{12}}{s_{13}} \tag{B.5}$$

For the purpose of generalizing this to the case of n points, we rewrite this relation in terms of arbitrary value of all the 4 coordinates as

$$\frac{s_{12}}{\sigma_1-\sigma_2}+\frac{s_{13}}{\sigma_1-\sigma_3}+\frac{s_{14}}{\sigma_1-\sigma_4}=0 \tag{B.6}$$

This equation is covariant under the $SL(2,\mathbb{C})$ transformation after using the momentum conservation[1]. Moreover, by fixing the 3 coordinates as $\sigma_1 = 0, \sigma_3 = 1$ and $\sigma_4 = \infty$, we recover equation (B.5).

The generalization of equation (B.6) to the case of n points is given by the so called scattering equation as

$$\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a-\sigma_b} \;=\; 0 \qquad \forall a \tag{B.8}$$

Even though there are n equations, only $n-3$ of them are linearly independent. This is expected since there are only $n-3$ independent variables after taking into account the $SL(2,\mathbb{C})$ redundancy. This is reflected in the fact that three specific linear combinations of the scattering equations are zero, namely

$$\sum_{a=1}^{n}\sigma_a^m\left(\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{s_{ab}}{\sigma_a-\sigma_b}\right) \;=\; 0 \qquad \text{for} \quad m=0,1,2 \tag{B.9}$$

After making the connection between the Mandelstam variables and the location of punctures on the Riemann sphere, we propose following formula for any tree level amplitude involving the massless states

$$\mathcal{A}_n^{tree} \;=\; \int_{\mathcal{M}_{0,n}} \frac{\prod_{a=1}^{n} d\sigma_a}{\text{Vol } SL(2,\mathbb{C})} \prod_{b=1}^{n} \delta\left(\sum_{\substack{a=1\\ a\neq b}}^{n}\frac{s_{ab}}{\sigma_a-\sigma_b}\right) \mathcal{I}_n \tag{B.10}$$

The delta functions ensure that the integral receive contributions only from the points in the moduli

[1]To check this, we note that under the $SL(2,\mathbb{C})$ transformation (B.2), we have

$$\sigma_p-\sigma_q \;\rightarrow\; \frac{\sigma_p-\sigma_q}{(c\sigma_p+d)(c\sigma_q+d)} \tag{B.7}$$

space which are related to the Mandelstam variables through the scattering equations (B.8). The $\mathcal{I}_n$ is an unknown function which needs to be determined for a given theory.

Due to the fact that only $n-3$ of the scattering equations are independent, the above proposed equation (B.10) has a serious problem. If we find a solution for $n-3$ variables and insert into the scattering equations, the remaining 3 will get automatically satisfied. Thus, the integrand of (B.10) has a product of three delta functions with zero argument, namely $\delta(0)\delta(0)\delta(0)$, which will make the whole expression divergent. Due to this, we need to omit three delta functions. Suppose, we choose $a = p, q, r$ and omit the delta functions corresponding to these values. We want to do this in a covariant manner. It turns out that the following combination is independent of the choice of the three punctures

$$\sigma_{pq}\sigma_{qr}\sigma_{rp} \prod_{\substack{a=1\\a\neq p,q,r}}^{n} \delta\left(\sum_{\substack{b=1\\b\neq a}}^{n} \frac{s_{ba}}{\sigma_b - \sigma_a}\right) \tag{B.11}$$

where, $\sigma_{ab} \equiv \sigma_a - \sigma_b$.

Similarly, since we have fixed the locations of the three punctures (say $\sigma_\ell, \sigma_m, \sigma_n$), we also have

$$\frac{\prod_{a=1}^{n} d\sigma_a}{\text{Vol } SL(2,\mathbb{C})} = \sigma_{\ell m}\sigma_{mn}\sigma_{n\ell} \prod_{\substack{a=1\\a\neq \ell,m,n}}^{n} d\sigma_a \tag{B.12}$$

Thus, our proposed relation is modified to

$$\mathcal{A}_n^{tree} = \int_{\mathcal{M}_{0,n}} \prod_{\substack{a=1\\a\neq \ell,m,n}}^{n} d\sigma_a \, (\sigma_{\ell m}\sigma_{mn}\sigma_{n\ell})(\sigma_{pq}\sigma_{qr}\sigma_{rp}) \prod_{\substack{b=1\\b\neq p,q,r}}^{n} \delta\left(\sum_{\substack{c=1\\c\neq b}}^{n} \frac{s_{cb}}{\sigma_c - \sigma_b}\right) \mathcal{I}_n \tag{B.13}$$

The delta functions in the above equation should be treated with care since they correspond to Holomorphic variables. For proving soft graviton theorems, it is more convenient to use a contour integral representation for these delta functions. We discuss these issues in the next section.

The integrand in the above equation should be $SL(2,\mathbb{C})$ invariant. This imposes some restriction on the function $\mathcal{I}_n$ as we describe now. We denote the measure in (B.10) by $d\mu_n$, i.e.

$$d\mu_n \equiv \prod_{\substack{a=1\\a\neq \ell,m,n}}^{n} d\sigma_a \, (\sigma_{\ell m}\sigma_{mn}\sigma_{n\ell})(\sigma_{pq}\sigma_{qr}\sigma_{rp}) \prod_{\substack{b=1\\b\neq p,q,r}}^{n} \delta\left(\sum_{\substack{c=1\\c\neq b}}^{n} \frac{s_{cb}}{\sigma_c - \sigma_b}\right) \tag{B.14}$$

Under $SL(2,\mathbb{C})$ transformation (B.2), this transforms as

$$d\mu_n \;\to\; d\mu_n \prod_{a=1}^{n} (c\sigma_a + d)^{-4} \tag{B.15}$$

Thus, for the integrand in (B.13) to be $SL(2,\mathbb{C})$ invariant, we must have

$$\mathcal{I}_n \;\to\; \mathcal{I}_n \prod_{a=1}^{n} (c\sigma_a + d)^{4} \tag{B.16}$$

Our next job is to construct the functions $\mathcal{I}_n$ taking into account the above requirement for different theories. However, before doing this, we first turn to finding the number of independent solutions of the scattering equations.

Suppose, we are interested in finding the number of solutions $\mathcal{N}_n$ of the scattering equations for the case of n particles. We shall obtain this in a recursive manner. For this, we focus on the n-th scattering equation and write the scattering equations as

$$\sum_{\substack{b=1\\ b\neq a}}^{n-1} \frac{k_a.k_b}{\sigma_a - \sigma_b} + \frac{k_a.k_n}{\sigma_a - \sigma_n} \;=\; 0 \qquad \text{for} \quad a = 1, 2, \cdots, n-1 \tag{B.17}$$

$$\sum_{b=1}^{n-1} \frac{k_n.k_b}{\sigma_n - \sigma_b} \;=\; 0 \tag{B.18}$$

Now, we consider the limit in the kinematic space where the n-th particle is becoming soft, namely

$$k_n = \tau\, q \qquad \tau \to 0 \tag{B.19}$$

Now, in this limit, the number of solutions of the scattering equations should remain same unless we hit the soft limit $\tau = 0$. Upto leading order, the term involving $k_a.k_n$ in equation (B.17) can be ignored. The (B.17) then becomes the set of scattering equations for the $n-1$ particles which have $\mathcal{N}_{n-1}$ solutions. We denote these solutions as $\sigma_a^{(I)}$ where $a = 1, 2, \cdots, n-1$ and $I = 1, 2, \cdots, \mathcal{N}_{n-1}$. Substituting these solutions in (B.18), we find

$$\sum_{b=1}^{n-1} \frac{q.k_b}{\sigma_n - \sigma_b^{(I)}} \;=\; 0 \tag{B.20}$$

We now need to find the number of solutions of σ_n. For this, we note that the above equation is a polynomial equation for σ_n. Naively, the degree of this polynomial equation is $n-2$. However, it is easy to check that the coefficient of the $(\sigma_n)^{n-2}$ in this polynomial equation is $\sum_{a=1}^{n-1} k_a.q$ which is zero by momentum conservation. Thus, the degree of the polynomial equation is $n-3$ and hence the number of solutions of σ_n are $n-3$. Now, for $n=4$, we have already seen that the number of solutions is just one. Thus, we have following recursive equations

$$\mathcal{N}_4 = 1 \qquad , \qquad \mathcal{N}_n = (n-3)\mathcal{N}_{n-1} \tag{B.21}$$

The unique solution of these recursive equations is $\mathcal{N}_n = (n-3)!$. Thus, the total number of solutions of the scattering equations for n particles is $(n-3)!$.

B.2 Delta functions in CHY formula

In writing down the CHY formula for scattering amplitudes (B.13), we made use of the delta function involving the holomorphic variables. The delta functions in (B.13) are abstract quantities which are defined via the equation

$$\int d\sigma \; \delta(f(\sigma)) \; F(\sigma) = \sum_{(\alpha)} (f'(\sigma_{(\alpha)}))^{-1} F(\sigma_{(\alpha)}) \tag{B.22}$$

where f and F are arbitrary functions and $\sigma_{(\alpha)}$ are the zeroes of the function $f(\sigma)$. In contrast to the usual delta function, the weight factor is $(f'(\sigma_{(\alpha)}))^{-1}$ instead of $(|f'(\sigma_{(\alpha)})|)^{-1}$. Therefore we can also represent this as a contour integral

$$\sum_{\alpha} \oint_{\sigma_{(\alpha)}} d\sigma \; \frac{F(\sigma)}{f(\sigma)} \,, \tag{B.23}$$

where $\oint_{\sigma_{(\alpha)}}$ denotes an anti-clockwise contour around $\sigma_{(\alpha)}$, including the $(2\pi i)^{-1}$ factor.

To see the equivalence between (B.22) and (B.23), we just need to Taylor expand $f(\sigma)$ around $\sigma_{(\alpha)}$

$$f(\sigma) = f(\sigma_{(\alpha)}) \; + \; (\sigma - \sigma_{(\alpha)}) f'(\sigma_{(\alpha)}) \; + \cdots \tag{B.24}$$

By residue theorem, we shall only get $f'(\sigma_{(\alpha)})$ in the denominator and $F(\sigma_{(\alpha)})$ in the numerator after performing the contour integral in (B.23) showing the equivalence between the two expressions.

The absence of absolute value in the right hand side of (B.22) can some time cause confusion when we have multiple integration. Consider for example the integral

$$\int dx \, dy \, \delta(x-y) \, \delta(x+y) \, F(x,y) \,. \tag{B.25}$$

Let us do the y integration first. If we use the second delta function to do this integral then the result is

$$\int dx\, \delta(2x)\, F(x, -x) = \frac{1}{2} F(0, 0)\,. \tag{B.26}$$

On the other hand if we use the first delta function to carry out the y integral first then the result is

$$-\int dx \delta(2x) F(x, x) = -\frac{1}{2} F(0, 0)\,. \tag{B.27}$$

Therefore the two results do not agree. Thus, for holomorphic delta functions, it matters which delta function we use to perform which integral and hence we need to be careful when multiple integrals involving multiple holomorphic delta functions are present.

This ambiguity can be resolved if we regard the holomorphic delta functions as grassmann odd objects so that exchanging their positions costs a sign. We also regard the integration measure as wedge products so that changing the order of doing the integration also costs a sign. For a given order of integration and delta functions we shall follow the convention that the last integration will be done first using the last delta function, and successive integrations will follow the same order. If we want to use a different delta function for the last integration, we need to first bring that particular delta function to the end picking up appropriate minus sign and then carry out the integration using (B.22) or (B.23). In terms of the contour integral representation (B.23), it means that if we have multiple contour integrals, then the 1st integral should be done using the poles of the arguments of first delta function, second integral using the poles of the arguments of the second delta function and so on.

It is easy to see that with this convention, under a change of variables the product of delta functions pick up the inverse of the Jacobian without absolute value

$$\prod_{i=1}^{n} \delta\left(\sum_j A_{ij}\sigma_j\right) = (\det A)^{-1} \prod_{i=1}^{n} \delta(\sigma_i)\,, \tag{B.28}$$

where on both sides the delta functions in the product are arranged from left to right in the order of increasing i.

B.3 Construction of $\mathcal{I}_n$

We now turn to the construction of $\mathcal{I}_n$. We shall do this for the scalar theory with cubic interactions, Yang-Mills theory and Einstein's gravity theory. These examples will illustrate the general strategy. For the construction of $\mathcal{I}_n$ for other theories, please refer to the original literature. Before we can describe the construction of $\mathcal{I}_n$, we take a small detour to explain the concept of colour ordered amplitude.

When a theory has some internal symmetries or gauge redundancies, the particles in the theory carry the flavour and the colour indices respectively. We consider the case when these groups are $U(N)$. In that case, the structure constant of the group can be written in terms of the trace of the generators, namely

$$f^{abc} = -\frac{i}{\sqrt{2}}\, tr\big(T^a[T^b, T^c]\big) \tag{B.29}$$

The vertex factors and the propagators of the theory contain the explicit factors of these structure constants. This implies that the scattering amplitudes carry these flavour/colour indices. It turns out that for a theory containing a single type of particle, the tree level scattering amplitudes can be written in the form[2]

$$\mathcal{A}_n^{tree} = \sum_{a\in S_n/\mathbb{Z}_n} tr\big(T^{a_1}T^{a_2}\cdots T^{a_n}\big) M_n(\{p_{a_i}\},\{\epsilon_{a_i}\}) \tag{B.30}$$

The notation $S_n/\mathbb{Z}_n$ means that we need to sum over all the permutations S_n of the flavour/colour labels with the quotient by $\mathbb{Z}_n$ since trace is invariant under the cyclic permutation of the matrices T^a. The function $M_n(\{p_{a_i}\},\{\epsilon_{a_i}\})$ is called the colour ordered partial amplitude. The CHY formalism is very convenient in taking into account these partial amplitudes as we shall see below.

Now, we return to the construction of $\mathcal{I}_n$. For any theory, it must satisfy the condition (B.16). For specific theories, there might be more restrictions on the form of $\mathcal{I}_n$. We first try to construct a function which satisfies (B.16). For this, we consider the so called Park Taylor factor defined by

$$C_n[a] \;\equiv\; \frac{1}{(\sigma_{a_1}-\sigma_{a_2})(\sigma_{a_2}-\sigma_{a_3})\cdots(\sigma_{a_{n-1}}-\sigma_{a_1})} \tag{B.31}$$

This has the property that under the $SL(2,\mathbb{C})$ transformation (B.2), it transforms as

$$C_n[a] \to C_n[a]\prod_{a=1}^{n}(c\sigma_a + d)^2 \tag{B.32}$$

Thus, the product of two Park Taylor factors, namely $C_n[a]C_n[b]$ will satisfy the property (B.16). Now, the Park Taylor factors involve a specific ordering $(a_1, a_2, \cdots, a_n)$. Thus, the product of the Park Taylor factors is natural if we are interested in the partial amplitudes. For the full amplitude, we also need to multiply this by the trace over product of generators and sum over inequivalent

[2]For theories containing several type of particles, more complicated expressions may hold.

permutations. Thus, we define

$$\mathcal{C}_n \equiv \sum_{a\in S_n/\mathbb{Z}_n} tr\big(T^{a_1}T^{a_2}\cdots T^{a_n}\big)C_n[a] \tag{B.33}$$

The product $\mathcal{C}_n\mathcal{C}_n$ is a legitimate choice of the function $\mathcal{I}_n$.

We have succeeded in constructing an example of function $\mathcal{I}_n$. However, we still don't know the theory whose amplitude will correspond to this function. A hint is obtained by noting that the product $C_n[a]C_n[b]$ involves two orderings. Thus, the particles of the theory should carry two indices. Now, the function $C_n[a]C_n[b]$ does not involve any polarization tensor. Thus, we expect that this function should represent only the amplitudes (if it represent amplitudes at all) involving the scalar particles. A natural guess is that this function should represent a theory of scalar particles having a flavour group $U(N)\times U(N)$. Explicit calculations show that this is indeed the case and the amplitudes computed using the CHY formula with the function $\mathcal{C}_n\mathcal{C}_n$ correspond to so called bi adjoint scalar theory with cubic interaction described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi_{I\tilde{I}}\partial^\mu\Phi_{I\tilde{I}} + \frac{1}{3!}g_{IJK}\widetilde{g}_{\tilde{I}\tilde{J}\tilde{K}}\Phi_{I\tilde{I}}\Phi_{J\tilde{J}}\Phi_{K\tilde{K}} \tag{B.34}$$

Now, if we are interested in the ϕ^3 theory with no flavour index, we just need to forget about the flavour indices which come from the factor of trace over the product of generators in the expression of $\mathcal{C}_n$. Since $C_n^2[a]$ correspond to summation of diagrams which have specific orderings, to produce results corresponding to diagrams with no orderings, we need to sum over all possible orderings

$$\mathcal{I}_n^{\phi^3} = \frac{1}{2^{n-2}}\sum_{a\in S_n} C_n^2[a] \tag{B.35}$$

The overall numerical factor is necessary since there is an over counting.

We now turn to the Yang Mills theory which has a single colour group $U(N)$. We first focus on the partial amplitudes. To take into account the single colour group, the function $\mathcal{I}_n$ must contain one Park Taylor factor $C_n[a]$. Thus, the $\mathcal{I}_n$ for the YM theory for the partial amplitudes should be of the form $\mathcal{I}_n^{YM} = C_n[a]\mathcal{P}$ where $\mathcal{P}$ is an unknown function. The function $\mathcal{P}$ should satisfy the following properties

1. Under the $SL(2,\mathbb{C})$ transformation (B.2), it should transform as

   $$\mathcal{P} \to \mathcal{P}\prod_{a=1}^{n}(c\sigma_a + d)^2 \tag{B.36}$$

   so that $\mathcal{I}_n$ satisfy (B.16).

2. The YM amplitude is a function of the polarization vectors ϵ_a^μ and the momenta p_a^μ. Since rest

of the factors in the CHY formula do not involve ϵ_a^μ, they must enter through $\mathcal{P}$. In other words, $\mathcal{P}$ should be a function of the kinematic variables $(\epsilon_a^\mu, p_a^\mu)$ along with the puncture locations σ_a.

3. We shall demand that $\mathcal{P}$ is a polynomial in the kinematic variables $(\epsilon_a^\mu, p_a^\mu)$. In other words, $\mathcal{P}$ should not have any pole in the kinematic variables. This is demanded because we want to generate all the singularities of the amplitude by the scattering equations which connect the moduli space $\mathcal{M}_{0,n}$ with the Mandelstam space. We don't want to put these singularities of the amplitudes in the CHY formula by hand.

4. For the amplitude to be gauge invariant, we must have

$$\mathcal{P}(\epsilon_a^\mu + \alpha p_a^\mu) \;=\; \mathcal{P}(\epsilon_a^\mu) \qquad \Longrightarrow \qquad \mathcal{P}(\epsilon_a^\mu = p_a^\mu) \;=\; 0 \tag{B.37}$$

We demand that this should hold only when the scattering equations are satisfied. We shall also demand that the above equation is satisfied for each solution of the scattering equation independently[3].

5. From the usual QFT, we know that the Feynman amplitudes should be expressible as

$$\mathcal{A}_n(\epsilon_a^\mu, p_a^\mu) = \epsilon_1^{\mu_1}\epsilon_2^{\mu_2}\cdots\epsilon_n^{\mu_n}\widetilde{\mathcal{A}}_{\mu_1\mu_2\cdots\mu_n}(p_a^\mu) \tag{B.38}$$

For this to happen, the function $\mathcal{P}$ must also satisfy an identical relation and hence it must be linear in each polarization vector. In other words, we must have

$$\mathcal{P}(\lambda\epsilon_a^\mu) \;=\; \lambda\,\mathcal{P}(\epsilon_a^\mu) \tag{B.39}$$

We now give the form of the function $\mathcal{P}$ and explain how this satisfy the above requirements. We claim that the function $\mathcal{P}$ is given by

$$\mathcal{P}(\{k_a^\mu\},\{\epsilon_a^\mu\},\{\sigma_a\}) = \mathrm{Pf}'\Psi \;=\; -\frac{(-1)^{s+t}}{\sigma_s-\sigma_t}\,\mathrm{Pf}(\Psi_{st}^{st}) \tag{B.40}$$

$\mathrm{Pf}(\Psi)$ denotes the Pfaffian of a $2n\times 2n$ anti-symmetric matrix. The matrix Ψ_{st}^{st} is obtained by removing s-th and t-th row from first n rows and removing s-th and t-th columns from first n columns of Ψ. The matrix Ψ has the form

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \tag{B.41}$$

[3]In this respect, the CHY prescription differs from the Feynman diagram technique. To check the gauge invariance of amplitudes using the Feynman diagram technique, we need to sum over all the Feynman diagrams. However, in the CHY prescription, the contribution to the amplitude from each solution is gauge invariant by itself.

where A, B, C are $n \times n$ matrices defined as,

$$A_{ab} = \begin{cases} \dfrac{p_a.p_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases} \quad , \qquad B_{ab} = \begin{cases} \dfrac{\epsilon_a.\epsilon_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases}$$

$$C_{ab} = \begin{cases} \dfrac{\epsilon_a.p_b}{\sigma_a - \sigma_b} & a \neq b \\ -\displaystyle\sum_{c=1, c\neq a}^{n} \dfrac{\epsilon_a.p_c}{\sigma_a - \sigma_c} & a = b \end{cases}$$

The quantity $\text{Pf}'\Psi$ which corresponds to the Pfaffian of the matrix Ψ after removing the s-th and t-th rows and columns is called the reduced Pfaffian. The reason we need to remove two rows and two columns from the matrix Ψ is that it has two null eigenvectors $(\sigma_1^m, \sigma_2^m, \cdots, \sigma_n^m; 0, 0, \cdots, 0)$ for $m = 0, 1$. Due to this, the determinant and hence the Pfaffian of Ψ vanish trivially.

We now explain why reduced Pfaffian is a natural choice for our case. Using the momentum conservation and transversality condition $\epsilon_a.p_a = 0$, the equation (B.36) can be explicitly checked. The 2nd and 3rd conditions are trivially true. For the 4th condition, we note that if we replace ϵ_a^μ by p_a^μ in the $\text{Pf}'\Psi$, then the rows and columns labeled by a and $n + a$ become proportional to each other and $\text{Pf}'\Psi$ and hence the amplitude vanishes. Note that this happens only when the scattering equations are imposed and C_{aa} present in these rows and columns vanish.

Another motivation for the choice of $\mathcal{P}$ comes from a conjecture given in [125]. Suppose we have a function of the form $\frac{P}{Q}$, where P and Q are polynomials, with the following properties

1. It has $n - 3$ poles consistent with trivalent Feynman diagrams. In other words, the poles of the function are similar to the one which can arise in an arbitrary Feynman diagram in ϕ^3 theory.

2. The numerator of the function has $n - 2$ powers of the momenta.

3. The function satisfies the gauge invariance.

According to the conjecture, if these conditions are satisfied, then the function is unique[4]. Now, the $\text{Pf}'\Psi$ is a rational function with $n - 2$ powers of the momenta in the numerator. The poles of this function are consistent with the trivalent Feynman diagrams. Moreover, this function is gauge invariant. Thus, according to the conjecture, the function given in (B.40) should be unique.

The final form of the function $\mathcal{I}_n$ for an arbitrary amplitude (not colour ordered) in the YM theory is, thus, given by

$$\mathcal{I}_n^{\text{YM}} = \mathcal{C}_n \, \text{Pf}'(\Psi) \tag{B.42}$$

[4]There is also a stronger version of this conjecture which relaxes the condition that the poles should be consistent with the trivalent Feynman diagrams [125]. In other words, we allow for any arbitrary pole structure.

where, $\mathcal{C}_n$ and $\text{Pf}'(\Psi)$ are defined in (B.33) and (B.40) respectively.

Finally, we consider the Einstein's gravity theory. It has no flavour/colour group. Hence, the function $\mathcal{I}_n$ for it should not carry any Park Taylor factor. Moreover, it should be linear in polarization tensor $\epsilon^{\mu\nu}$ of each graviton which is symmetric and traceless. Now, if we consider a function $(\text{Pf}'\ \Psi)^2$, then it will contain product of polarization tensors in the form $\epsilon_1^{\mu_1}\epsilon_1^{\nu_1}\epsilon_2^{\mu_2}\epsilon_2^{\nu_2}\cdots\epsilon_n^{\mu_n}\epsilon_n^{\nu_n}$. If we identity $\epsilon^{\mu}\epsilon^{\nu}\equiv\epsilon^{\mu\nu}$ as the graviton polarization, then this function will have the desired property of describing the graviton scattering processes in CHY formalism. The correct factor for the gravity amplitudes is given by

$$\mathcal{I}_n^{\text{Gravity}} = 4(-1)^n(\text{Pf}'\ \Psi)^2 \ = \ \frac{4(-1)^n}{(\sigma_s-\sigma_t)^2}(\text{Pf}\ \Psi^{st}_{st})^2 = \frac{4(-1)^n}{(\sigma_s-\sigma_t)^2}\det(\Psi^{st}_{st}) \tag{B.43}$$

Appendix C

Appendix for chapter 2

In this appendix, we summarize our notations, conventions and useful identities used in chapter 2. We shall also give details of some calculations.

C.1 Notations and conventions

Convention for indices

1. The vertices are labeled by indices $a, b, \cdots$
2. The co-ordinate of the vertices are $u_a, u_b, \cdots$
3. The co-ordinate of all external points attached to the a^{th} vertex are x_a^i (we suppress the space-time Lorentz index)
4. The conformal dimension of the operator inserted at x_a^i is Δ_a^i
5. The squared distance between two points x_a^i and x_b^j is

$$(x_a^i - x_b^j)^2 \equiv (x_{ab}^{ij})^2 = (x_{ba}^{ji})^2$$

6. The Mellin variable dual to x_{ab}^{ij} is denoted by s_{ab}^{ij} which satisfies

$$s_{ab}^{ij} = s_{ba}^{ji} \qquad \text{and} \qquad s_{aa}^{ii} \equiv 0$$

Convention for summations and products

1. If there are N_a external lines meeting at the a^{th} vertex, then we denote

$$\sum_{i\in a} \equiv \sum_{i=1}^{N_a} = \text{ sum over all \textit{external} lines connected to the vertex } u_a$$

$$\prod_{i\in a} \equiv \prod_{i=1}^{N_a} = \text{ product over all \textit{external} lines connected to the vertex } u_a$$

2. For the double summation and products (which avoid over counting), we use the notations

$$\sum_{1\leq i<j\leq N_a} \sum \equiv \sum_{(i,j)\in a} \qquad ; \qquad \prod_{1\leq i<j\leq N_a} \prod \equiv \prod_{(i,j)\in a}$$

3. If the upper index is *not* mentioned then it implies that upper indices have been summed over all possible values. e.g. for the Mellin variables, we shall use

$$s_{aa} \equiv \sum_{(i,j)\in a} s_{aa}^{ij} \quad ; \quad s_{ab} = s_{ba} \equiv \sum_{i\in a} \sum_{j\in b} s_{ab}^{ij} \quad , \quad (a \neq b)$$

Some other conventions

1. Mellin measure $\quad [ds_{ij}] \equiv \frac{ds_{ij}}{2\pi i}$

2. Position space measure $\quad \mathcal{D}u \equiv \frac{d^D u}{2(2\pi)^{-D/2}}$

3. Mellin space delta function $\quad \delta_M(s - s_0) \equiv 2\pi i\, \delta(s - s_0)$

Shorthand notations

The Schwinger parameters in the integral expression of Mellin amplitude, for n-vertex simple tree and one loop diagrams appear in a nice structure. It is useful to introduce a short hand notations for these functions of Schwinger parameters. These notations turn out to be especially convenient for various manipulations. Along with these, we shall also introduce some functions of the Mellin variables.

Set 1 : G_a^c

$$\begin{aligned} G_a^c &\equiv 1 + t_{a,a+1}^2(1 + t_{a+1,a+2}^2(\cdots\cdots + t_{c-1,c}^2)) \qquad 1 \leq a \leq n-1 \\ G_a^a &\equiv 1 \qquad\qquad 1 \leq a \leq n-1 \end{aligned} \tag{C.1}$$

The value of the upstair index c will be greater than or equal to the lower index.

Set 2 : $\widetilde{H}_a^b$

$$\begin{aligned} \widetilde{H}_a^b &\equiv t_{a,a+1}\cdots t_{b-1,b}G_b^{n-1} && 1 \leq a < b \leq n-1 \\ \widetilde{H}_a^a &\equiv G_a^{n-1} && 1 \leq a \leq n-1 \\ \widetilde{H}_a^n &\equiv 0 && 1 \leq a \leq n \end{aligned}$$

$\widetilde{H}_a^b$ is not symmetric in its indices. A good mnemonic worth remembering is that upstairs index is always larger than or equal to the downstairs index.

Set 3 : K_a **and** $\widetilde{K}_a$

$$\begin{aligned} K_a &\equiv t_{a,a+1}\cdots t_{n-1,n} && 1 \leq a \leq n-1 \\ K_n &\equiv 1 \\ \widetilde{K}_a &\equiv \left(t_{1n}t_{12}\cdots t_{a-1,a}G_a^{n-1} + K_a\right) && 1 \leq a \leq n-1 \\ \widetilde{K}_n &= 1 \end{aligned}$$

Set 4

$$R_a \equiv \gamma_{a,a+1} - \sum_{c=a+1}^{n}\sum_{b=1}^{a}(s_{bc}) \qquad ; \qquad 1 \leq a \leq n-1$$

C.2 Mellin space delta function

In this section we want to show that

$$I = \int_{c-i\infty}^{c+i\infty}[ds]f(s)\int_0^\infty dt\; t^{s_0-s-1} = \int_{c-i\infty}^{c+i\infty}[ds]f(s)\left(2\pi i\delta(s-s_o)\right) \tag{C.2}$$

where $c = \text{Re}(s_0)$.

The above identity essentially shows that inside the contour integration, the real integral $\int_0^\infty dt\; t^{s_0-s-1}$ behaves as the delta function as long as the real part of $(s-s_0)$ is zero along the contour. In order to prove our claim, we first perform a change of variable $t = e^x$ to get

$$I = \int_{c-i\infty}^{c+i\infty}[ds]f(s)\int_{-\infty}^{\infty} dx\; e^{(s_0-s)x}$$

Now, since the argument $(s_0 - s)$ of the exponential function is purely imaginary along the contour of integration, the x integral is an integration over an oscillating function. Using the delta function representation

$$\delta(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{iyp}$$

we obtain

$$I = \int_{c-i\infty}^{c+i\infty} [ds] f(s)\,(2\pi i \delta(s_0 - s)) = f(s_0)$$

This proves the desired result (C.2).

C.3 Some Useful Identities

Some useful identities used in our derivations are as follows [126, 127]

1. The following Mellin-Barnes representation turns out to be very useful

$$\frac{1}{(1+z)^a} = \frac{1}{\Gamma(a)}\int_{-i\infty}^{i\infty} [ds]\; z^{-s}\Gamma(a-s)\Gamma(s) \tag{C.3}$$

2. The first Barnes lemma is

$$\int_{c-i\infty}^{c+i\infty} [ds]\; \beta(a+s, b-s)\beta(c+s, d-s) = \beta(a+d, b+c) \tag{C.4}$$

3. An useful rearrangement identity involving the product of beta functions is

$$\beta(a-u, u)\beta(d-u, k) = \beta(d-u, u)\beta(d, k) \qquad ; \quad \text{provided} \quad a = d + k \tag{C.5}$$

4. The recursive integral form of product of beta functions is

$$\prod_{a=1}^{m-1}\int_0^{\infty} dt_{a,a+1}\big(t_{a,a+1}\big)^{N_a - 1}\Big(G_a^m\Big)^{-L_a} = \prod_{a=1}^{m-1}\frac{1}{2}\beta\Big(\frac{N_a}{2}, L_a + \frac{N_{a-1} - N_a}{2}\Big) \tag{C.6}$$

where $N_0 \equiv 0$ and G_a^m is defined in appendix C.1.

5. The definition of the Hypergeometric function is

$$ {}_pF_q\Big(a_1,\cdots,a_p\ ;\ b_1,\cdots,b_q\ ;\ x\Big) \;=\; \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\cdots\Gamma(a_p+n)}{\Gamma(b_1+n)\cdots\Gamma(b_q+n)}x^n \tag{C.7} $$

with $|x|<1$.

6. The recursive integration formula for hypergeometric functions is

$$ \begin{aligned} &\beta\left(a_{p+1},b_{q+1}-a_{p+1}\right){}_{p+1}F_{q+1}\Big(a_1,\cdots,a_{p+1}\ ;\ b_1,\cdots,b_{q+1}\ ;\ x\Big) \\ &\qquad = \int_0^1 dt\ (t)^{a_{p+1}-1}(1-t)^{b_{q+1}-a_{p+1}-1}{}_pF_q\Big(a_1,\cdots,a_p\ ;\ b_1,\cdots,b_q\ ;\ tx\Big) \end{aligned} \tag{C.8} $$

7. Following identities relate two ${}_3F_2$ hypergeometric functions with different arguments

$$ \begin{aligned} {}_3F_2(a_1,a_2,a_3;b_1,b_2;1) \;=\; & {}_3F_2(a_1,b_2-a_2,b_2-a_3;b_2,b_1+b_2-a_1-a_3;1) \\ & \times\frac{\Gamma(b_1)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_1-a_1)\Gamma(b_1+b_2-a_2-a_3)} \end{aligned} \tag{C.9} $$

and,

$$ \begin{aligned} &{}_3F_2(a_1,a_2,a_3;b_1,b_2;1) \\ &= {}_3F_2(b_1-a_1,b_2-a_1,b_1+b_2-a_1-a_2-a_3;b_1+b_2-a_1-a_2,b_1+b_2-a_1-a_3;1) \\ &\quad\times\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(a_1)\Gamma(b_1+b_2-a_1-a_2)\Gamma(b_1+b_2-a_1-a_3)} \end{aligned} \tag{C.10} $$

8. The Gauss identity is

$$ {}_2F_1(a_1,a_2;b_1;1)=\frac{\Gamma(b_1)\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)} \tag{C.11} $$

9. An integral representation of ${}_2F_1$ is

$$ {}_2F_1(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{C.12} $$

10. Following identity relates two ${}_2F_1$ Hypergeometric function with different arguments

$$ {}_2F_1(a,b;c;z) \;=\; (1-z)^{-a}{}_2F_1\Big(a,c-b;c;\frac{z}{z-1}\Big) \tag{C.13} $$

C.4 Details of some calculations

C.4.1 Mellin amplitude of a completely general tree

In this appendix, we carry out the derivation of Mellin amplitude for an arbitrary tree Feynman diagram and show that the amplitude is given by product of beta functions with one beta function for each internal propagator.

From the diagrammatic rules described in section 2.3.1, we know that a general Mellin amplitude takes the form

$$M\left(\{s_{ab}\}\right) = \prod \left[\int_0^\infty dt_{ab} \frac{(t_{ab})^{\gamma_{ab}-1}}{\Gamma(\gamma_{ab})}\right] F\Big(\{t_{ab}\}, \{s_{ab}\}\Big) \tag{C.14}$$

The t_{ab} is the Schwinger parameter for the propagator joining the internal vertices a and b and the product is over all the internal propagators of the Feynman diagram. The function F is, in general, an arbitrary function of the Schwinger parameters t_{ab} and the Mellin variables s_{ab}.

As mentioned earlier, the function F, for any given graph, depends on the order of integration over the position space vertices. For a straight chain of propagators (as in section 2.3.2), the natural choice is to go from one end to the other without any jumps. This results in the simplest expression for F. For a tree with branches, this option is absent. We shall, however, prescribe an order that gives an expression for F such that the integration over the Schwinger parameters can be performed easily.

To specify the ordering, we first choose any one of the vertices, with only one edge attached, to be the reference vertex $\mathcal{P}$ (see figure C.1) on the skeleton diagram. In our prescription, the integration over this vertex will be carried out after performing integration over all the other vertices. For all the other vertices, the order is indicated by some arrows on the lines. For drawing these arrows, one needs to follow two rules. The first rule is that, among all the lines meeting at a vertex, there should be only one line with an outgoing arrow. All the other lines attached to that vertex should have ingoing arrows. The second rule is that any given vertex is integrated only after all the other vertices connected to it by (lines with) ingoing arrows have been integrated over. Thus, $\mathcal{P}$ is the only vertex with a single line which has an ingoing arrow and according to our prescription, it is integrated in the end. Figure C.1 gives an example of a compatible ordering using arrows. In this example, an order allowed by the above rules is $1 \to 3 \to 2 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10$. This, however, is not the only ordering consistent with our rules and all such allowed orderings are equally good for our purpose.

We can now write down the function F for this prescribed order of integration for any Feynman diagram using the diagrammatic rules described in section 2.3.1. We state the result here in a graph independent manner. From the diagramatic rules, we know that each pair of vertices a, b (which may be the same point also) of the Feynman diagram contributes a multiplicative factor raised to $-s_{ab}$ to

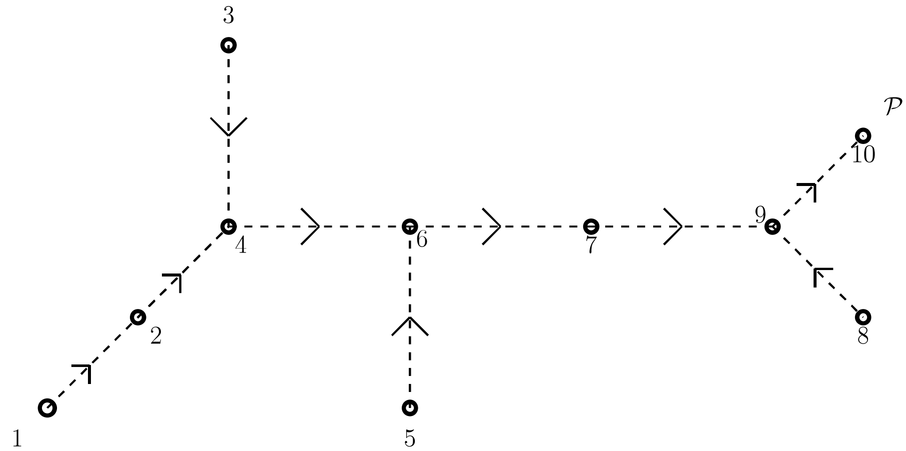

Figure C.1: Order of integration over the vertices depicted by the numbers (in increasing order)

F. In other words, if we denote this factor by $\mathcal{F}_{ab}$, then the function F is given by,

$$F = \prod_{a=1}^{N} \prod_{b=1}^{N} (\mathcal{F}_{ab})^{-s_{ab}} \tag{C.15}$$

where N is the total number of internal vertices.

We now describe the functional dependence of $\mathcal{F}$ on Schwinger parameters. For this, we draw the shortest continuous line (without raising the pen) between the two vertices a and b on the skeleton diagram (via the vertices that come on the way). We refer to the set of vertices that we cross as $\mathcal{A}_{ab}$. Since we are considering a tree, there will exist a vertex in this set that is *nearest* to the reference vertex $\mathcal{P}$. The set of vertices on the continuous route from this vertex to $\mathcal{P}$ is denoted by $\mathcal{B}_{ab}$.

To see an example, let us consider the pair of points (3,5) in figure C.1. Here, $\mathcal{A}_{35} = \{3, 4, 6, 5\}$. The vertex in the set $\mathcal{A}_{35}$ nearest to $\mathcal{P}$ is 6 and $\mathcal{B}_{35} = \{6, 7, 9, 10\}$.

Since the vertices in the sets $\mathcal{A}_{ab}$ and $\mathcal{B}_{ab}$ form individual chains, there is always a line on the skeleton diagram connecting the consecutive vertices in each of these sets. Below, we shall make use of the Schwinger parameters corresponding to the propagators in these lines.

The functional dependence of $\mathcal{F}_{ab}$ on the Schwinger parameters can now be easily written down by following the diagrammatic rules of section 2.3.1 and is given by

$$\mathcal{F}_{ab} = \left[\prod_{i=1}^{|\mathcal{A}_{ab}|-1} t_{\mathcal{A}_{ab}(i)\mathcal{A}_{ab}(i+1)} \right] K_{ab} \tag{C.16}$$

where $|\mathcal{A}_{ab}|$ denotes the number of vertices in the set $\mathcal{A}_{ab}$ and the product in the bracket is over those propagators which lie along the shortest continuous line joining the two vertices a and b. The function K_{ab} is given by

$$K_{ab} \equiv 1 + t^2_{\mathcal{B}_{ab}(i)\mathcal{B}_{ab}(i+1)} K_{\mathcal{B}_{ab}(i+1)\mathcal{B}_{ab}(i+2)} \qquad ; \qquad K_{\mathcal{B}_{ab}(|\mathcal{B}_{ab}|-1)\mathcal{B}_{ab}(|\mathcal{B}_{ab}|)} \equiv 1$$

It should be emphasized that the above form of $\mathcal{F}_{ab}$ is true only for the chosen order of integration and will be different if we change the order.

As an example, for the Feynman diagram in figure C.1, the factor $\mathcal{F}_{3,5}$ is given by

$$\mathcal{F}_{3,5} = t_{34} t_{46} t_{65} \left(1 + t^2_{67}\left(1 + t^2_{79}\left(1 + t^2_{9,10}\right)\right)\right)$$

By using (C.16) in (C.15) and rearranging the terms, we can write a simplified expression for the function F as

$$F \;=\; \prod_{\text{all propagators}} (t_{ab})^{-P_{ab}} (A_{ab})^{-Q_{ab}} \tag{C.17}$$

where,

$$A_{ab} \equiv 1 + t^2_{ab}\left(1 + t^2_{bc}\left(1 + ...(1 + t^2_{op})...\right)\right)$$

b,c,...,o are all on the shortest continuous route from a to the reference vertex $\mathcal{P}$. In other words, for each propagator on the skeleton graph, we draw the shortest continuous line connecting it with the propagator containing the reference vertex $\mathcal{P}$. $t_{ab}, t_{bc}, \cdots, t_{op}$ are the Schwinger parameters of the successive propagators on this line.

The functions P_{ab} and Q_{ab} in (C.17) are given by

$$P_{ab} = \sum_{c\in L_{ab}} \sum_{d\in R_{ab}} s_{cd} \qquad ; \qquad Q_{ab} = \sum_{\substack{c\in L_{ab} \\ c\neq a}} \sum_{\substack{d\in \widetilde{L}_{ab} \\ d\neq a}} s_{cd} \quad + \sum_{d\in L_{ab}} s_{ad}$$

The L_{ab} (R_{ab}) in the above definition refer to the set of vertices which lie to the left (right) of the propagator joining the vertex a and b. By convention, we call the set of vertices which include the reference vertex $\mathcal{P}$ as R_{ab}. The term involving the double sum in Q_{ab} is absent if not more than two lines meet at the vertex a in the skeleton graph. The tilde in one of the L in this double sum denotes the fact that we should not include terms of the type s_{cd} where c and d are on the same branch in the set L_{ab}.

Finally, we now need to integrate over the Schwinger parameters in (C.14). Since the integrals are not factorised, we shall have to carry out one integral at a time and we shall do that in an order compatible with the arrows on the skeleton (integrating over the line involving the reference vertex

$\mathcal{P}$ in the end). For example, for the figure C.1, a compatible order is $t_{12} \to t_{24} \to t_{34} \to t_{46} \to t_{56} \to t_{67} \to t_{89} \to t_{79} \to t_{9,10}$. Carrying out these integrals in exactly the same way as in the simple tree case in section 2.3.2 and using the conformality conditions, we obtain the desired result (2.34).

C.4.2 Mellin-Barnes approach to n-vertex tree

In this appendix, we present an alternative derivation of the factorization of n vertex tree diagram. This derivation makes use of the Barnes' first identity and shows the usefulness of the Barnes integrals for Mellin space.

We start by noting that the Mellin amplitude of n vertex simple tree of section (2.3.2) can be written in a form which closely resembles the loop amplitude (2.36)

$$M_n(s_{ab}) \;=\; \prod_{a=1}^{n-1}\left[\int_0^{\infty} dt_{a,a+1}(t_{a,a+1})^{\gamma_{a,a+1}-1}\right]\prod_{b=c}^{n}\prod_{c=1}^{n-1}\Big(\widetilde{H}_c^b + K_cK_b\Big)^{-s_{cb}}$$

where, the functions $\widetilde{H}_a^b$ and K_a are defined in the appendix (C.1).

The basic idea is to use the identity (C.3) to convert the terms involving sum as products of Mellin Barnes integrals. We then perform the integration over Schwinger parameters. The Mellin Barnes integrations are performed in the end by making repeated use of Barnes first lemma.

Using the identity (C.3) and noting that $\widetilde{H}_a^n \equiv 0$ (which means that we only need to introduce $n(n-1)/2$ Barnes variables w_{ab}), we obtain after making use of the definitions of $\widetilde{H}_a^b$ and K_a

$$\begin{aligned} M(s_{ab}) \;=\; & \prod_{b=1}^{n-1}\prod_{c=b}^{n-1}\left(\int_{-i\infty}^{i\infty}[dw_{bc}]\beta(s_{bc}-w_{bc},w_{bc})\right)\int_0^{\infty} dt_{n-1,n}\big(t_{n-1,n}\big)^{\mathcal{R}_{n-1}-1} \\ & \prod_{a=1}^{n-2}\left(\int_0^{\infty} dt_{a,a+1}(t_{a,a+1})^{\mathcal{R}_a-1}(G_a^{n-1})^{-\mathcal{Q}_a}\right) \end{aligned}$$

where,

$$\begin{aligned} \mathcal{R}_a &\equiv R_a - 2\sum_{b=1}^{a} w_{bb} - 2\sum_{b=1}^{a}\sum_{c=1}^{b-1} w_{bc} \qquad ; \qquad 1 \le a \le n-1 \\ \mathcal{Q}_a &\equiv \sum_{b=1}^{a}\Big(s_{ab}-w_{ab}\Big) \qquad ; \qquad 1 \le a \le n-1 \end{aligned}$$

$t_{n-1,n}$ integral is straightforward and it gives a Mellin space delta function. For integration over other Schwinger parameters, we use the identity (C.6). After simplifying the expressions by making use of the conformal conditions, we obtain

$$M(s_{ab}) \;=\; \prod_{b=1}^{n-1}\left\{\prod_{c=b}^{n-1}\int_{-i\infty}^{i\infty}[dw_{bc}]\,\beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=1}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)\delta_M\Big(\mathcal{R}_{n-1}\Big) \tag{C.18}$$

where, $\gamma_{01}\equiv 0\equiv\mathcal{R}_0$.

We note that we have obtained beta functions for only $n-2$ propagators. The missing propagator is hidden in the Mellin-Barnes integrals and the Mellin space delta function as we shall show below.

To perform the integration over the w_{ab} variables, we rewrite (C.18) as

$$\begin{aligned} M(s_{ab}) \;=\; & \prod_{b=1}^{n-2}\left\{\prod_{c=b}^{n-2}\int_{-i\infty}^{i\infty}[dw_{bc}]\,\beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=1}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big) \\ & \prod_{b=1}^{n-1}\left\{\int_{-i\infty}^{i\infty}[dw_{b,n-1}]\,\beta(s_{b,n-1}-w_{b,n-1},w_{b,n-1})\right\}\delta_M\Big(\mathcal{R}_{n-1}\Big) \end{aligned} \tag{C.19}$$

The first line does not involve the $w_{a,n-1}$ variables (note that for n vertex case, only $\mathcal{R}_{n-1}$ involves the $w_{a,n-1}$ variables). This helps in performing the integration over $w_{a,n-1}$ variables. We first use the delta function to get rid of integration over $w_{n-1,n-1}$ and then use the identity (C.4) to perform integration over other $w_{a,n-1}$ variables. After carefully keeping track of various terms, we obtain

$$\begin{aligned} M(s_{ab}) \;=\; & \frac{1}{2}\prod_{b=1}^{n-2}\left\{\prod_{c=b}^{n-2}\int_{-i\infty}^{i\infty}[dw_{bc}]\,\beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=1}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big) \\ & \beta\left(\sum_{a=1}^{n-1}s_{a,n-1}-\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\;\;,\;\;\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\right) \end{aligned} \tag{C.20}$$

The factor of $\frac{1}{2}$ in front arises due to the delta function in (C.19).

To proceed further, we note that the conformal condition at vertex 1 can be expressed in following way

$$s_{11}=\frac{R_1}{2}+\frac{D}{2}-\gamma_{12}$$

This allows us to use the rearrangement identity(C.5) as

$$\beta\left(s_{11}-w_{11},w_{11}\right)\beta\left(\frac{\mathcal{R}_1}{2},\frac{D-2\gamma_{12}}{2}\right)=\beta\left(\frac{R_1}{2},\frac{D-2\gamma_{12}}{2}\right)\beta\left(\frac{R_1}{2}-w_{11},w_{11}\right)$$

Using this, we can rewrite (C.20) as

$$\begin{aligned} M(s_{ab}) \;=\;& \frac{1}{2^2}\beta\Big(\frac{R_1}{2},\frac{D-2\gamma_{12}}{2}\Big)\int_{-i\infty}^{i\infty}[dw_{11}]\beta\left(\frac{R_1}{2}-w_{11},w_{11}\right) \\ & \int_{-i\infty}^{i\infty}[dw_{12}]\beta\left(s_{12}-w_{12},w_{12}\right)\int_{-i\infty}^{i\infty}[dw_{22}]\beta\left(s_{22}-w_{22},w_{22}\right) \\ & \prod_{b}^{n-2}\left\{\prod_{c}^{n-2}\int_{-i\infty}^{i\infty}[dw_{bc}]\;\beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=2}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big) \\ & \beta\left(\sum_{a=1}^{n-1}s_{a,n-1}-\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\;,\;\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\right) \end{aligned} \tag{C.21}$$

The integrals in the third line do not include the integrations over the (w_{11},w_{12},w_{22}) variables. Moreover, these variables appear only in the form of sum (i.e. $w_{11}+w_{12}+w_{22}$) in the beta functions of last two lines. Hence, it is useful to use a new set of coordinates as follows

$$\{w_{11},w_{12},w_{22}\}\to\{w_{11},w_{22},u_1\}\qquad;\quad u_1\equiv w_{11}+w_{12}+w_{22}$$

After this coordinate change, the last two lines of (C.21) do not include w_{12} or w_{22} variables anywhere. They just appear in the first two lines. Performing the integration over these variables using Barnes first lemma gives

$$\begin{aligned} M \;=\;& \frac{1}{2^3}\beta\Big(\frac{R_1}{2},\frac{D-2\gamma_{12}}{2}\Big)\int_{-i\infty}^{i\infty}[du_1]\beta\left(\frac{R_1}{2}+s_{12}+s_{22}-u_1,u_1\right)\beta\Big(\frac{R_2}{2}-u_1,\frac{D-2\gamma_{23}}{2}\Big) \\ & \prod_{b}^{n-2}\left\{\prod_{c}^{n-2}\int_{-i\infty}^{i\infty}[dw_{bc}]\;\beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=3}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big) \\ & \beta\left(\sum_{a=1}^{n-1}s_{a,n-1}-\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\;,\;\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\right) \end{aligned}$$

In the next step we use the conformal condition at the vertex 2 to use the rearrangement identity (C.5) for the two beta functions inside the integration in the first line of the above expression. After this, we note that the u_1 variable appears in the combination $u_1+w_{13}+w_{23}+w_{33}$ in all but one of the beta function. We exploit this by trading the u_1 variable for a new variable u_2 defined as $u_2 = u_1+w_{13}+w_{23}+w_{33}$. This allows us to perform the integrations over w_{13}, w_{23} and w_{33}

variables using the Barnes' first lemma. The end result after this step is

$$
\begin{aligned}
M \;=\;& \frac{1}{2^3}\beta\Big(\frac{R_1}{2},\frac{D-2\gamma_{12}}{2}\Big)\beta\Big(\frac{R_2}{2},\frac{D-2\gamma_{12}}{2}\Big)\int_{-i\infty}^{i\infty}[du_2]\beta\left(\frac{R_2}{2}+s_{13}+s_{23}+s_{33}-u_2,u_2\right)\\
& \prod_b^{n-2}\left\{\prod_c^{n-2}\int_{-i\infty}^{i\infty}[dw_{bc}]\;\beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=3}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)\\
& \beta\left(\sum_{a=1}^{n-1}s_{a,n-1}-\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\;\;,\;\;\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\right)
\end{aligned}
$$

Continuing this process iteratively, i.e. combining the beta functions and then making a change of coordinate (such that only two beta functions involve the appropriate w_{ab} variables), we obtain the desired result. We need to make repeated use of the identity (conformal condition at a^{th} vertex)

$$
\frac{R_a}{2}+\sum_{b=1}^{a+1}s_{b,a+1}=\frac{R_{a+1}}{2}+\frac{D}{2}-\gamma_{a+1,a+2} \tag{C.22}
$$

In the end step, we obtain the desired result

$$
\begin{aligned}
M(s_{ab}) \;=\;& \frac{1}{2}\left[\prod_{a=1}^{n-2}\frac{1}{2}\beta\Big(\frac{R_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)\right]\int_{-i\infty}^{i\infty}[du_{n-3}]\beta\Big(\frac{R_{n-2}}{2}-u_{n-3}\;,\;u_{n-3}\Big)\\
& \times\beta\left(\sum_{a=1}^{n-1}s_{a,n-1}-\frac{R_{n-1}}{2}+u_{n-3}\;\;,\;\;\frac{R_{n-1}}{2}-u_{n-3}\right)\\
=\;& \prod_{a=1}^{n-1}\frac{1}{2}\beta\Big(\frac{R_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)
\end{aligned}
$$

where, we have used the identity (C.22) for $a=n-1$ after performing the u_{n-3} integration.

C.4.3 Details of calculation in section 2.5.1

In this appendix, we present the details of the calculation leading to the equation (2.41). Our starting expression is (2.40). We insert a partition of unity in this expression in the form

$$
1=\int dq_a\;\;\delta\left(q_a-\sum_{i\in a}\Delta_a^i\right)\qquad;\qquad a=1,2
$$

and make the coordinate transformations $\alpha_a^i = q_a \; y_a^i \;\; (a=1,2)$ and then use the identity (2.38) to obtain

$$\widetilde{M}(s_{ab}) \;=\; \frac{1}{\Gamma(\gamma)} \prod_{a=1}^{2} \left[\frac{\prod_{i\in a} \Gamma(\rho_a^i)}{\Gamma\left(\rho_a\right)} \right] \int_0^\infty dt \; (t)^{R_1-1} \big(1+t^2\big)^{-s_{11}} \int_0^\infty dq_1 \; (q_1)^{\rho_1-1}$$
$$\int_0^\infty dq_2 \; (q_2)^{\rho_2-1} \left(q_1\big(1+t^2\big)+tq_2\right)^{-\lambda_1} \Big(t\; q_1 + q_2\Big)^{-\lambda_2}$$

where,

$$R_1 \equiv \gamma - s_{12} \qquad \text{and} \quad \rho_a \equiv \sum_{i\in a} \rho_a^i \qquad ; \quad a=1,2$$

To proceed further, we rescale $q_2 \to tq_1q_2$ and obtain,

$$\widetilde{M}(s_{ab}) \;=\; \frac{1}{\Gamma(\gamma)} \prod_{a=1}^{2} \left[\frac{\prod_{i\in a} \Gamma(\rho_a^i)}{\Gamma\left(\rho_a\right)} \right] \int_0^\infty dt \; (t)^{R_1+\rho_2-\lambda_2-1} \big(1+t^2\big)^{-s_{11}} \int_0^\infty dq_1 dq_2$$
$$(q_1)^{\rho_1+\rho_2-\lambda_1-\lambda_2-1} \; (q_2)^{\rho_2-1} \Big(1+t^2+t^2 q_2\Big)^{-\lambda_1} \Big(1+q_2\Big)^{-\lambda_2}$$

Integration over q_1 gives a delta function. We now take out the factor of $1+t^2$ from the term containing the single power of λ_1, rescale $t^2 \to t$ and make a further change of coordinates

$$q_2 = \frac{u}{1-u} \qquad , \quad t = \frac{v}{1-v}$$

This gives,

$$\widetilde{M}(s_{ab}) \;=\; \frac{1}{2\Gamma(\gamma)} \prod_{a=1}^{2} \left[\frac{\prod_{i\in a} \Gamma(\rho_a^i)}{\Gamma\left(\rho_a\right)} \right] \delta\Big(\rho_1+\rho_2-\lambda_1-\lambda_2\Big) \int_0^1 du \int_0^1 dv \; (v)^{\frac{R_1+\rho_2-\lambda_2}{2}-1}$$
$$(u)^{\rho_2-1} \big(1-u\big)^{\lambda_2-\rho_2-1} \big(1-v\big)^{s_{11}+\lambda_1-\frac{1}{2}(R_1+\rho_2-\lambda_2)-1} \Big(1+\frac{uv}{1-u}\Big)^{-\lambda_1}$$

Now, using the identities (C.12) and (C.13), we obtain

$$\widetilde{M}(s_{ab}) \;=\; \frac{1}{2\Gamma(\gamma)} \prod_{a=1}^{2} \left[\frac{\prod_{i\in a} \Gamma(\rho_a^i)}{\Gamma\left(\rho_a\right)} \right] \delta\Big(\rho_1+\rho_2-\lambda_1-\lambda_2\Big) \beta\left(\frac{R_1}{2}+\frac{\rho_2-\lambda_2}{2} \; , \; \frac{D}{2}-\gamma\right)$$
$$\int_0^1 du \; (u)^{\rho_2-1} \big(1-u\big)^{\rho_1-1} {}_2F_1\left(\lambda_1 \; , \; \frac{D}{2}-\gamma \; ; \; s_{11}+\lambda_1 \; ; \; u\right)$$

We have also used

$$s_{11} + \lambda_1 - \frac{1}{2}(R_1 + \rho_2 - \lambda_2) = \frac{D}{2} - \gamma$$

Next, we use the identity (C.8) for $a = 2, b = 1$ and the identity (C.9) to obtain

$$\begin{aligned}\widetilde{M}(s_{ab}) \;\; = \;\; & \frac{1}{2\Gamma(\gamma)} \prod_{a=1}^{2} \left[\frac{\prod_{i \in a} \Gamma(\rho_a^i)}{\Gamma(\rho_a)} \right] \beta\left(\frac{R_1}{2} + \frac{\rho_1 - \lambda_1}{2} \, , \, \frac{D}{2} - \gamma \right) \beta\Big(\rho_1, \rho_2\Big) \\ & {}_3F_2\left(\frac{D}{2} - \gamma \, , \, \lambda_2 \, , \, \rho_1 \, ; \, s_{11} + \rho_1, \lambda_1 + \lambda_2 \, ; \, 1 \right) \delta\Big(\rho_1 + \rho_2 - \lambda_1 - \lambda_2\Big)\end{aligned}$$

In writing the above expression, we have used the fact that ${}_3F_2$ is symmetric in first set of three indices and the second set of two indices.

The above expression is not symmetric in the two vertices. We can put it in a symmetric form by using the identity (C.10). After some simplification, we obtain the desired result (2.41).

Appendix D

Summation Identities for chapter 3

In this appendix, we list three summation identities that are used in the derivation of the multiple soft graviton theorem in section 3.3.

$$\sum_{\text{all permutations of subscripts } 1,\cdots,n} \prod_{m=1}^{n} (a_1 + a_2 + \cdots + a_m)^{-1} = \prod_{m=1}^{n} (a_m)^{-1} \ . \tag{D.1}$$

$$\begin{aligned} &\sum_{\text{all permutations of subscripts } 1,\cdots,n} \sum_{\substack{r,u=1 \\ r<u}}^{n} c_{ur} \prod_{\ell=1}^{n} (a_1 + \cdots + a_\ell)^{-1} \\ &= \prod_{m=1}^{n} (a_m)^{-1} \sum_{\substack{r,u=1 \\ r<u}}^{n} (a_r + a_u)^{-1} \left(a_u \, c_{ur} + a_r \, c_{ru}\right) . \end{aligned} \tag{D.2}$$

$$\begin{aligned} &\sum_{\text{all permutations of subscripts } 1,\cdots,n} \sum_{m=2}^{n} \sum_{\substack{r,u=1 \\ r<u}}^{m} b_{ru} \, (a_1 + \cdots + a_m)^{-1} \prod_{\ell=1}^{n} (a_1 + \cdots + a_\ell)^{-1} \\ &= \prod_{m=1}^{n} (a_m)^{-1} \sum_{\substack{r,u=1 \\ r<u}}^{n} b_{ru} \, (a_r + a_u)^{-1} \qquad \text{for } b_{rs} = b_{sr} \text{ for } 1 \le r < s \le n \, . \end{aligned} \tag{D.3}$$

The proof of these identities are given below.

First Identity

To prove the first identity, we note that the summand on the left hand side of (D.1) may be expressed as

$$\int_0^\infty ds_1 e^{-s_1 a_1} \int_0^\infty ds_2 e^{-s_2(a_1+a_2)} \cdots \int_0^\infty ds_n e^{-s_n(a_1+\cdots+a_n)} \,. \tag{D.4}$$

Defining new variables

$$t_1 = s_1 + s_2 + \cdots s_n, \quad t_2 = s_2 + \cdots + s_n, \cdots, \quad t_n = s_n \,, \tag{D.5}$$

we may express (D.4) as

$$\int_R dt_1 \, dt_2 \, \cdots dt_n \, e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n} \tag{D.6}$$

where the integration range R is

$$\infty > t_1 \geq t_2 \geq \cdots \geq t_{n-1} \geq t_n \geq 0 \,. \tag{D.7}$$

Summing over all permutations of the subscripts $1, \cdots, n$ can now be implemented by summing over permutations of $t_1, \cdots t_n$. This has the effect of making the integration range unrestricted, with each t_i running from 0 to ∞. The corresponding integral generates the right hand side of (D.1).

Second Identity

The proof of the second identity (D.2) follows from a simple variation of the first identity. For this note that the coefficient of the c_{ur} term on the left hand side of (D.2) for $r < u$ is given by a sum over permutations with the same summand as in (D.1), but with the restriction that we sum over those permutations in which r comes before u. Translated to (D.7) this means that after summing over permutations the restriction $t_r > t_u$ is still maintained. Therefore the result is

$$\int_{t_r \geq t_u} dt_1 \, dt_2 \, \cdots dt_n \, e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n} \,. \tag{D.8}$$

This integral can be easily evaluated to give

$$(a_1 \cdots a_n)^{-1} \, a_u \, (a_r + a_u)^{-1} \,. \tag{D.9}$$

This is precisely the coefficient of c_{ur} on the right hand side of (D.2). Similarly in the computation of the coefficient of c_{ru} for $r < u$ we only sum over those permutations for which u comes before r. This has the effect the changing the constraint $t_r \geq t_u$ to $t_r \leq t_u$ in (D.8) and reproduces correctly the coefficient on c_{ru} on the right hand side of (D.2).

Third Identity

Finally we consider (D.3). We begin with a different sum

$$\sum_{\text{all permutations of subscripts } 1,\cdots,n} \prod_{\ell=1}^{n} \Bigg(a_1 + a_2 + \cdots + a_\ell - \sum_{\substack{r,u=1\\ r<u}}^{\ell} b_{ru} \Bigg)^{-1} , \tag{D.10}$$

and note that the first subleading term in a Taylor series expansion of (D.10) in powers of b_{mn}'s give the left hand side of (D.3). We now manipulate this as before, arriving at the analog of (D.4):

$$\int_0^\infty ds_1 e^{-s_1 a_1} \int_0^\infty ds_2 e^{-s_2(a_1+a_2-b_{12})} \cdots \int_0^\infty ds_n e^{-s_n(a_1+\cdots+a_n-\sum_{\substack{r,u=1\\ r<u}}^{n} b_{ru})} . \tag{D.11}$$

The change of variables given in (D.5) converts this to

$$\begin{aligned} & \int_R dt_1\, dt_2\, \cdots dt_n\, e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n} \\ & \exp\Bigg[(t_2 - t_3) b_{12} + (t_3 - t_4)(b_{12} + b_{23} + b_{13}) + \cdots + (t_{n-1} - t_n) \sum_{\substack{r,u=1\\ r<u}}^{n-1} b_{ru} + t_n \sum_{\substack{r,u=1\\ r<u}}^{n} b_{ru} \Bigg] . \\ = \; & \int_R dt_1\, dt_2\, \cdots dt_n\, e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n} \exp\Bigg[\sum_{\substack{r,u=1\\ r<u}}^{n} b_{ru}\, t_u \Bigg] . \end{aligned} \tag{D.12}$$

We now expand the last factor of (D.12) in a Taylor series expansion and pick the coefficient of the b_{ru} term. This has the effect of multiplying the integrand by t_u and restrict the sum over permutations to those for which r remains to the left of u. However as b_{ru} is symmetric in r, u, there is also another term related to this one under the exchange of the subscripts r and u. Therefore the integral is given by

$$\int_{t_r > t_u} dt_1\, dt_2\, \cdots dt_n\, e^{-t_1 a_1 - t_2 a_2 - \cdots - t_n a_n}\, t_u + (r \leftrightarrow u) \; . \tag{D.13}$$

Evaluation of this integral gives

$$(a_1 \cdots a_n)^{-1} \left\{ \frac{a_u}{(a_r + a_u)^2} + \frac{a_r}{(a_r + a_u)^2} \right\} = (a_1 \cdots a_n)^{-1} \, (a_r + a_u)^{-1} . \tag{D.14}$$

This is precisely the coefficient of b_{ru} on the right hand side of (D.3).

We can also give recursive proof of all the identities without using the integral representations. Let us begin with the identity (D.1). Let us suppose that it holds for $(n-1)$ objects. We now organise the sum over permutations of all subscripts $1, \cdots, n$ in (D.1) by first fixing the last element to be some integer i, and summing over all permutations of the subscripts other than i. This gives, using (D.1) for $(n-1)$ objects,

$$(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)^{-1} (a_1 + \cdots a_n)^{-1} . \tag{D.15}$$

We now sum over all possible choices of i. This gives

$$\sum_{i=1}^{n} (a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)^{-1} (a_1 + \cdots a_n)^{-1} . \tag{D.16}$$

This can be written as

$$(a_1 \cdots a_n)^{-1} (a_1 + \cdots + a_n)^{-1} \sum_{i=1}^{n} a_i = (a_1 \cdots a_n)^{-1} , \tag{D.17}$$

reproducing the right hand side of (D.1).

A recursive proof of (D.2) can be given as follows. Let us again assume that the identity is valid for $(n-1)$ objects. Now for $u > r$, the coefficient of c_{ur} on the left hand side involves a sum over permutations of the subscripts $1, \cdots, n$, with the same summand as in identity (D.1), but with the restriction that r always appears to the left of u in the permutation. We now organise the sum as follows. First we fix the last element and sum over permutations of the first $(n-1)$ elements. If the last element is i with $i \neq u$, then the result, using (D.3) for $(n-1)$ objects, is given by

$$\left\{ \prod_{\substack{m=1 \\ m \neq i}}^{n} (a_m)^{-1} \right\} a_u \, (a_u + a_r)^{-1} \, (a_1 + \cdots a_n)^{-1} . \tag{D.18}$$

Note that i cannot be r since that will violate the rule that the r always appears to the left of u. On the other hand if the last element is u then the sum over permutations over the first $(n-1)$ elements becomes unrestricted and we can apply (D.1) to get

$$\left\{ \prod_{\substack{m=1 \\ m \neq u}}^{n} (a_m)^{-1} \right\} (a_1 + \cdots + a_n)^{-1} . \tag{D.19}$$

Therefore the total answer, obtained by summing over all possible choices of the last element (other than r), is

$$\sum_{i \neq r,u} \left\{ \prod_{\substack{m=1 \\ m \neq i}}^{n} (a_m)^{-1} \right\} a_u \, (a_u + a_r)^{-1} \, (a_1 + \cdots a_n)^{-1} + \left\{ \prod_{\substack{m=1 \\ m \neq u}}^{n} (a_m)^{-1} \right\} (a_1 + \cdots + a_n)^{-1} . \tag{D.20}$$

Elementary algebra reduces this to

$$(a_1 \cdots a_n)^{-1}\, a_u\, (a_r + a_u)^{-1}\,, \tag{D.21}$$

which is the coefficient of c_{ur} on the right hand side of (D.2). The analysis for the case $r > u$ is identical, with the roles of r and u interchanged.

Finally we turn to the proof of (D.3). By collecting the coefficients of b_{ru} on both sides and using the symmetry of b_{ru}, we can write this identity as

$$\begin{aligned} &\sum_{\text{all permutations of subscripts } 1,\cdots,n} \sum_{\substack{m=2\\ m\geq r,u}}^{n} (a_1+\cdots+a_m)^{-1} \prod_{\ell=1}^{n} (a_1+\cdots+a_\ell)^{-1} \\ = \;& (a_r+a_u)^{-1} \prod_{m=1}^{n} (a_m)^{-1}\,. \end{aligned} \tag{D.22}$$

As before, we shall proceed by assuming this to be valid for $(n-1)$ objects and then prove this for n objects. Let us first consider the contribution from the $m = n$ term in the sum on the left hand side of (D.22). The contribution of this term is given by

$$(a_1+\cdots+a_n)^{-2} \sum_{\text{all permutations of subscripts } 1,\cdots,n} \prod_{\ell=1}^{n-1} (a_1+\cdots+a_\ell)^{-1}\,. \tag{D.23}$$

We now perform the sum over all permutations by fixing the last element to be some fixed number i, sum over permutations of the rest for which we can use (D.1), and then sum over all choices of i. This gives

$$(a_1+\cdots+a_n)^{-2} \sum_{i=1}^{n} \left\{ \prod_{\substack{m=1\\ m\neq i}}^{n} (a_m)^{-1} \right\} = (a_1+\cdots+a_n)^{-1} \left\{ \prod_{m=1}^{n} (a_m)^{-1} \right\}. \tag{D.24}$$

Next we consider the contribution to the sum in the left hand side of (D.22) for $m \leq (n-1)$. This is given by

$$(a_1+\cdots+a_n)^{-1} \sum_{\text{all permutations of subscripts } 1,\cdots,n} \sum_{\substack{m=2\\ m\geq r,u}}^{n-1} (a_1+\cdots+a_m)^{-1} \prod_{\ell=1}^{n-1} (a_1+\cdots+a_\ell)^{-1} \tag{D.25}$$

We again perform the sum over permutations by fixing the last element to be some fixed number i, summing over permutation of the rest of the objects, and then summing over i. Note however that now i cannot be either r or u since then we cannot satisfy the constraint $m \geq r, u$. The sum over permutations can now be performed using (D.22) for $n-1$ objects and gives

$$
\begin{aligned}
&(a_1+\cdots+a_n)^{-1}\sum_{\substack{i=1\\ i\neq r,s}}^{n}\left\{\prod_{\substack{m=1\\ m\neq i}}^{n}(a_m)^{-1}\right\}(a_r+a_u)^{-1}\\
=\;&(a_1+\cdots+a_n)^{-1}(a_r+a_u)^{-1}\left\{\prod_{m=1}^{n}(a_m)^{-1}\right\}(a_1+\cdots+a_n-a_r-a_u)\,. \qquad \text{(D.26)}
\end{aligned}
$$

Adding this to (D.24) we get

$$
(a_r+a_u)^{-1}\left\{\prod_{m=1}^{n}(a_m)^{-1}\right\}, \qquad \text{(D.27)}
$$

which is precisely the right hand side of (D.22).

Appendix E

Counting different solutions of Scattering Equations for m soft gravitons

In this appendix, we shall analyze the solutions of the scattering equations for the case when there are m number of soft gravitons. For multiple soft gravitons, the solutions to the scattering equation fall into different classes. A given class corresponds to the case when a group of r_1 punctures carrying soft momenta come within a distance of order τ of each other, another group of r_2 punctures carrying soft momenta come within a distance of order τ of each other and so on. We shall now derive the number of solutions of the scattering equations for this situation. The scattering equations for the first n gravitons (which are finite energy gravitons) is given by

$$\sum_{\substack{b=1\\b\neq a}}^{n} \frac{p_a.p_b}{\sigma_a-\sigma_b} + \tau \sum_{v=1}^{m} \frac{p_a.k_{n+v}}{\sigma_a-\sigma_{n+v}} = 0 \qquad (a=1,\cdots,n)\,. \tag{E.1}$$

The scattering equations for the gravitons in the i-th group $\mathcal{S}_i$ containing r_i punctures that come with a distance of order τ can be written as

$$\sum_{b=1}^{n} \frac{k_{n+u}.p_b}{\sigma_{n+u}-\sigma_b} + \tau \sum_{\substack{v\in\mathcal{S}_i\\v\neq u}} \frac{k_{n+u}.k_{n+v}}{\sigma_{n+u}-\sigma_{n+v}} + \tau \sum_{\substack{v=1\\v\notin\mathcal{S}_i}}^{m} \frac{k_{n+u}.k_{n+v}}{\sigma_{n+u}-\sigma_{n+v}} = 0\,, \qquad \forall u\in\mathcal{S}_i\,, \tag{E.2}$$

where we have removed an overall factor of τ from the equation.

In the $\tau\to 0$ limit, we can ignore the second term in the left hand side of (E.1). The equation (E.1) then reduces to the scattering equations of n finite energy particles and hence the number of solutions for the set $(\sigma_1,\cdots,\sigma_n)$ is $(n-3)!$. To obtain the number of solutions for the rest of the punctures, we note that the first two terms in (E.2) are of order unity since the denominators of the second term is of the order τ, while the third term is of order τ. Hence, we can ignore the third term. With this, the scattering equation for the punctures in each group $\mathcal{S}_i$ decouple from each other and we can focus on solutions of the punctures of each group separately given the solution for the set

$(\sigma_1, \cdots, \sigma_n)$.

For ease of notation, we consider the case of first set $\mathcal{S}_1$ and assume that its punctures are labelled as $(\sigma_{n+1}, \cdots, \sigma_{n+r_1})$. We redefine the variables as

$$\sigma_{n+a} = \sigma_{n+1} + \tau\, \xi_a\,, \qquad a = 2, \cdots, r_1\,. \tag{E.3}$$

In the $\tau \to 0$ limit, all the r_1 punctures $(\sigma_{n+1}, \cdots, \sigma_{n+r_1})$ come within a distance of order τ of each other, and therefore $\xi_a \sim 1$. Our goal now is to find out how many solutions exist for the variables $(\sigma_{n+1}, \xi_2, \cdots, \xi_{r_1})$. We shall now prove that the number of solutions for this set is $(n-2)(r_1-1)!$. To do this, we write the scattering equations for the group $\mathcal{S}_1$ in terms of the variables in (E.3) to leading order in τ:

$$\sum_{b=1}^{n} \frac{k_{n+1}.p_b}{\sigma_{n+1}-\sigma_b} - \frac{k_{n+1}.k_{n+2}}{\xi_2} - \cdots - \frac{k_{n+1}.k_{n+r_1}}{\xi_{r_1}} = 0 \tag{E.4}$$

$$\sum_{b=1}^{n} \frac{k_{n+2}.p_b}{\sigma_{n+1}-\sigma_b} + \frac{k_{n+2}.k_{n+1}}{\xi_2} + \frac{k_{n+2}.k_{n+3}}{\xi_2-\xi_3} + \cdots + \frac{k_{n+2}.k_{n+r_1}}{\xi_2-\xi_{r_1}} = 0 \tag{E.5}$$

$$\vdots$$

$$\sum_{b=1}^{n} \frac{k_{n+r_1-1}.p_b}{\sigma_{n+1}-\sigma_b} + \frac{k_{n+r_1-1}.k_{n+1}}{\xi_{r_1-1}} + \cdots + \frac{k_{n+r_1-1}.k_{n+r_1-2}}{\xi_{r_1-1}-\xi_{r_1-2}} + \frac{k_{n+r_1-1}.k_{n+r_1}}{\xi_{r_1-1}-\xi_{r_1}} = 0 \tag{E.6}$$

$$\sum_{b=1}^{n} \frac{k_{n+r_1}.p_b}{\sigma_{n+1}-\sigma_b} + \frac{k_{n+r_1}.k_{n+1}}{\xi_{r_1}} + \cdots + \frac{k_{n+r_1}.k_{n+r_1-1}}{\xi_{r_1}-\xi_{r_1-1}} = 0 \tag{E.7}$$

Adding all the equations and using momentum conservation, we find that all the ξ_a's drop out of the equation and we get a polynomial equation of degree $n-2$ for σ_{n+1}. Hence it has $n-2$ solutions. We can now recursively prove that the number of solutions for the set $(\xi_2, \cdots, \xi_{r_1})$ is $(r_1-1)!$. We consider a situation in which the last momentum k_{n+r_1} is softer than the other r_1-1 soft momenta. We then replace k_{n+r_1} by $\omega\, \ell_{n+r_1}$ for a new soft parameter ω and fixed ℓ_{n+r_1}, and take the limit $\omega \to 0$.[1] In this limit, the terms involving k_{n+r_1} in (E.4)-(E.6) can be ignored and therefore they reduce to the analog of eqs.(E.4)-(E.7) for a cluster of (r_1-1) punctures. These by assumption have $(r_1-2)!$ solutions for the variables $(\xi_2, \cdots, \xi_{r_1-1})$. On the other hand, after we factor out an overall factor of ω from the last equation (E.7), it reduces to a polynomial equation for ξ_{r_1} of degree r_1-1 for a given solution of $(\sigma_1, \cdots, \sigma_{n+1}, \xi_2, \cdots, \xi_{r-1})$. Therefore ξ_{r_1} has (r_1-1) solutions. Thus, there are a total of $(r_1-1)!$ solutions for the set $(\xi_2, \cdots, \xi_r)$. For finite ω the solutions will change, but we expect their number to remain the same. Since for fixed $\sigma_1, \cdots \sigma_n$, σ_{n+1} has $(n-2)$ solutions, this

[1] This trick was used in [24, 128] for counting the number of solutions to the scattering equations.

proves our claim that the number of solutions for the set $(\sigma_{n+1}, \cdots, \sigma_{n+r_1})$ is $(n-2)(r_1-1)!$.

We can repeat the analysis for the remaining group of punctures with the result that the number of solutions for the punctures in the group $\mathcal{S}_i$ is $(n-2)(r_i-1)!$. The total number of solutions is, thus, given by

$$(n-3)!\prod_i [(r_i-1)!(n-2)] \tag{E.8}$$

where the product runs over all clusters containing punctures carrying soft momentum.

To see that it gives the correct number of total solutions $(n+m-3)!$ for the $n+m$ number of gravitons, we need to sum over all the possibilities. This is given by,

$$\mathcal{N} \equiv (n-3)! \sum_{\substack{\{m_r\} \\ \sum_{r=1}^{m} r\, m_r = m}} \left\{ \prod_{r=1}^{m} \big[(r-1)!(n-2)\big]^{m_r} \right\} \frac{m!}{\prod_{r=1}^{m}(r!)^{m_r} m_r!} \tag{E.9}$$

where m_r is the multiplicity for the r punctures coming together. The factor $\frac{m!}{\prod_{r=1}^{m}(r!)^{m_r} m_r!}$ is the number of ways of dividing m objects into groups of r objects with multiplicity m_r. To evaluate the constrained sum in (E.9), it is convenient to define the following unconstrained sum (generating function)

$$\begin{aligned}
\mathcal{N}(y) &\equiv (n-3)! \prod_{r=1}^{\infty} \sum_{m_r=0}^{\infty} \left\{ \big[(r-1)!(n-2)\big]^{m_r} \frac{m!}{(r!)^{m_r} m_r!} y^{r m_r} \right\} \\
&= m!(n-3)! \prod_{r=1}^{\infty} \exp\left[\frac{(n-2)}{r} y^r\right] \\
&= m!(n-3)!(1-y)^{-(n-2)} \\
&= m!(n-3)! \sum_{k=0}^{\infty} \frac{(n+k-3)!}{k!(n-3)!} y^k \,.
\end{aligned} \tag{E.10}$$

The coefficient of y^m in (E.10) is the quantity in (E.9). This coefficient is precisely $(n+m-3)!$ – the expected number of solutions of the scattering equations for $n+m$ particles. This shows that we have not left out any solutions.

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