UNCERTAINTY RELATIONS, QUANTUM COHERENCE AND QUANTUM MEASUREMENT

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DECLARATION

I, Gautam Sharma, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Gautam Sharma

List of Publications arising from the thesis

Journal

1. "Trade-off relations for coherence and disturbance", Gautam Sharma and Arun Kumar Pati, *Phys. Rev. A*, **2018**, *97*, 062308.

2. "Quantum uncertainty relation based on the mean deviation", Gautam Sharma, Chiranjib Mukhopadhyay, Sk Sazim and Arun Kumar Pati, *Phys. Rev. A*, **2018**, *98*, 032106.

3. "Quantum Coherence, Coherent Information and Information Gain Quantum Measurement", Gautam Sharma, Sk Sazim and Arun Kumar Pati, *EPL(EuroPhysics Letters)*, **2019**, *127*, 50004.

4. "Fine grained uncertainty determines preparation contextuality", Gautam Sharma, Shiltaditya Mal and Sk Sazim, *arXiv:1905.09695 [quant-ph]*, **2019**.(Preprint)

Conferences

1. December 2018: **Poster presentation** titled "Quantum Coherence, Information gain and Disturbance", QIPA 2018, Harish-Chandra Research Institute, Prayagraj(Allahabad), India.
2. May 2018: **Poster presentation** titled "Quantum uncertainty relation based on the mean deviation", International Conference on Quantum Frontiers and Fundamentals 2018, Raman Research Institute, Bengaluru, India.
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7 Summary and future directions

In this thesis we have investigated various aspects of quantum measurements with use of quantum coherence and uncertainty relations. As discussed before our results will not only help in enhancing the foundational understanding of the topic, it will also be of great signficance in various information processing tasks.

We have shown that quantum coherence has very important connection with the disturbance and information gain in a measurement. Earlier, it was known that the disturbance caused to a system is dependent on information gain, whose value is not a very accessible quantitatively. We were able to show relation between disturbance and initial coherence of the system, over which we have more control than the information gain. Hence, our results will be very useful in information processing tasks which require the initial system to not change much. Since, our trade-off relation are not tight for higher dimensional systems, one can try to find more tighter trade-off relations for coherence and disturbance. It might be possible that by using other measures of coherence and disturbance one can find tighter trade-off relations.

Another important contribution of this thesis in applications is that we were able to show that initial coherence of the system sets an upper bound on the information gain in a measurement. Previously, it was known that information gain in a measurement is upper bounded by entanglement developed between the system and the apparatus. Although, the entanglement that is created, is not in our control as compared to the initial coherence of the system. Our analysis, which also included noisy measurement scenarios, will definitely be of significance in information processing tasks which try to maximize the information gain. In future, one can do similar analysis for more generic measurement scenarios.

With regards to fundamental understanding of quantum uncertainties in measurement, we have shown that one may not necessarily always choose standard deviation to be a measure of deviation. Not only this, we have shown that mean deviation has wider applicability than the standard deviation and the most classical states are not necessarily the Gaussian states when we consider mean deviation as a measure of uncertainty. This is useful in the sense that there might be potentials which generate a probability distribution where standard deviation diverges, in that case mean deviation might turn out to be useful. In addition, we obtained better results for detecting EPR violation. In future, one should apply the mean deviation or the generalized deviation measures for various other tasks like formulating error disturbance relations. An important application would be in metrology, where new Cramer Rao bound could be useful. Just like standard deviation, one may try to find out classical and quantum part of the mean deviation uncertainty.

The last part of this thesis establishes an important connection between uncertainty relation and preparation contextuality. Previously a connection between uncertainty relations and non-locality has been shown, we have tried to generalize this by including local correlations also, which are responsible for contextuality. One should try to find a connection between uncertainty relations and measurement contextuality also. As we have better understanding of uncertainty relations than contextuality, the relation between the two will definitely be of significance in understanding contextuality.

Summary

The quantum measurement problem is one of the central problem associated with the understanding of quantum mechanics. How the measurement process affects the quantum state, is still an unsolved issue. The difficulty in understanding of quantum measurements is because of its difference with the well understood classical measurements. Two ways in which quantum measurements differ from classical measurements are: 1) They disturb the system irreversibly and 2) They only give a probability distribution of possible outcomes. In this thesis we have studied these aspects of quantum measurement theory through quantum coherence and uncertainty relations. The study covers the aspects of both foundational understanding of the topic along with application in quantum information processing.

Quantum coherence, which is considered a resource for various information processing tasks, has been shown to obey a trade-off relation with the disturbance caused to a quantum system in a quantum measurement. We demonstrate this relation through some examples in qubit and qutrit systems. Moreover, for a bipartite

system quantum coherence, quantum correlations (both entanglement and discord) and disturbance obey a trade-off relation among themselves. Next, we show that in a particular measurement process the maximum information that we can extract is upper bounded by the initial coherence of the system. The extracted information is equal to the coherent information, that the system can send to the apparatus. We also study the measurement scenario in presence of environment and obtain various constraints on initial coherence of the system and final entanglement between system and apparatus. From the perspective of apparatus, we found that by using a more robust apparatus one can gain more information.

Uncertainty relations capture the inherent imprecision associated with two or more incompatible observables. We provide a new mean-deviation based uncertainty measure and derive uncertainty relations using them. Our new measure has wider applicability than the standard deviation based uncertainty measure. We apply our uncertainty relation to demonstrate quantum steering and were able to do so for much lower efficiency than the standard deviation based uncertainty relations. In a separate work we have also shown a connection between the fine-grained uncertainty relation and the preparation contextuality.

1 Introduction

Quantum theory was founded in the early part of 20^{th} century to model the behavior of the microscopic world. Ever since its inception, it is known to have features which mark a dramatic departure from the classical world that we know. Inspite of that, the theory has survived the experimental tests till now. Among these features, the most startling features have been the *unpredictability* and *decoherence* phenomena associated with quantum measurements.

The work in this thesis is important in understanding both the foundational problems and the applications of quantum theory in information processing. The fundamental problems and applications of quantum theory go hand in hand. Without the understanding of the fundamental concepts, no progress is possible in applications.

Any measurement, classical or quantum, gives information about the system. While classical measurements can give information about the measurement outcome with certainty and without causing any changes in the system being measured. A quantum measurement only gives a probabilistic description of possible measurement outcomes and also causes irreversible disturbance to the measured system unless the system is already in one of the eigenstate of the observable measured. In fact, it was in this spirit of *disturbance caused to the system*, Heisenberg had given the uncertainty principle [2]. However, the uncertainty principle captures the notion of preparation uncertainty, which makes it impossible to create a state in which position and momentum are sharply defined.

There are two challenges associated with any measurement: (i) To cause minimum disturbance to the system and (ii) To extract maximum information from the system. We will first consider the work that has been done till now in understanding disturbance during a quantum measurement and how to minimize it. **Disturbance** caused to the system can be kept zero when we don't extract any information at all, but that amounts to a trivial case. Apart from this special case, in general, a measurement which extracts some fixed amount of information always changes the initial state of the system. Extensive amount of work has been done in this direction by deriving information vs disturbance trade-off relations, in various measurement setups and using different quantifiers of information and disturbance [3–15]. Also, we know that, without any disturbance or initial information of the system, one can't determine a single quantum state [16, 17]. For minimizing disturbance, researchers have also explored the idea of weak measurements and enhancing the information gain [18–22].

Next, we briefly discuss information gain during a quantum measurement. **Information gain** from a system happens by creating correlations between the system and apparatus. It can establish both classical and quantum correlations of which the quantum correlations is considered to be entanglement [23–26]. It is a well accepted fact that to extract more information from the system, one needs to establish strong entanglement between the system and apparatus [27–31].

In the first part of this thesis (Chapter 3 and 4) we look at what role does coherence play to determine disturbance caused to the system and information gain. We first look at how coherence can determine the disturbance caused to the system. The motivation to explore in this direction comes from the fact that the measurement process not only disturbs the state of the system it also leads to loss of coherence. It is also known that both the direct and indirect (when the ancilla is also there) measurements can cause decoherence in a system [32–34]. If the system does not get disturbed there will be no decoherence. Moreover, quantum coherence is a resource which can be used for various quantum processing tasks [35]. Quantum coherence obeys trade-off relations with path distinguishability, quantum entanglement, asymmetry and also with coherence measured in different basis. Therefore we ask the question: Can quantum coherence be the resource which gets consumed and appears in form of disturbance? Or is there a trade-off relations between the two quantities? We find that indeed there exists a trade-off relation for both single and bipartite systems, which we describe later.

Next, we consider the relation between quantum coherence and information gain. It was recently shown that quantum coherence is the resource that creates correlations [36, 37]. Although, information gain and information loss has been studied in the past [25, 38–41], but the studies are far from being complete. In fact it was shown, in [39] that the maximal information that we can extract is equal to the classical correlation developed between system and apparatus. This is very interesting because it hints that if we have more initial coherence in the system then we can develop more correlation between the system and apparatus and hence will be able to have more information gain. We therefore look for a connection between the initial coherence of the system and information gain from a measurement.

That quantum coherence can determine the information gain from a measurement is also indicated by the following observation. Suppose our initial system is in one of the eigenstate of the observable being measured. Then the measurement will always predict the state with certainty, hence there is no information gain. On the other hand if the initial state of the system is in a superposition (has non-zero coherence) of the eigenstates, the measurement outcome will be probabilistic which amounts to non-zero information. Hence, we see that coherence indeed is very intimately connected with information gain in a measurement. Later we will show a quantitative connection between the two quantities.

The 2nd part of thesis (Chapter 5 and 6) explores quantum uncertainty relations and its various applications in understanding the foundational aspects of quantum physics. Traditionally, quantum uncertainty relations were expressed using the standard deviation [2, 42, 43], as the measure of uncertainty. Despite Entropic uncertainty relations taking a center stage with fresh developments [44], the standard deviation based uncertainty is still preferred in the text books[45]. In recent years, standard deviation based uncertainty relations beyond the Robertson-Schrödinger inequality have been proved, for sum of deviations [46] and for multiple observables

[47]. However, we ask the question: What new information and advantages can we get if we use other forms of deviation measure to quantify uncertainty? In this thesis we use the **mean-deviation** as the measure of uncertainty and give new uncertainty relations present their applicability and applications.

The uncertainty relations based on deviations and entropic measures, capture only the incompatibility between the observable in a coarse way, i.e., they fail to capture the incompatibility between different outcomes of multiple observables. To address this issue, *fine-grained uncertainty relations* were proposed, which consist of the inequalities for all possible combination of outcomes [48]. For mutually unbiased and mutually biased observables, an upper bound has been found for these inequalities [49]. These uncertainty relations have also been applied to demonstrate the non-locality present in bipartite [48] and tripartite scenarios [50]. Motivated by this, we seek out to explore the connection between fine-grained uncertainty relations and quantum contextuality, of which non-locality is a special case. We indeed find a connection between uncertainty and preparation contextuality.

1.1 Outline of the Thesis

The organization of this thesis is as follows. In the remaining part of this introductory chapter we briefly present the quantum formalism and the mathematical structure of quantum measurement theory which will be necessary for the next chapters. In Chapter 3, we present a trade-off relation between quantum coherence and disturbance caused to a system by a quantum measurement. We also present various examples to demonstrate the tightness of the trade-off inequality. Moving ahead, in chapter 4 we show that the initial coherence of the system sets an upper bound on the information gain by doing a measurement. We study the information gain in noisy scenario and obtain conditions on the coherence and robustness of apparatus for maximizing information gain. Then we move into the 2nd half of the thesis, in which chapter 5 presents a new measure for deviation and its advantages and applications. Finally, in chapter 6 we show a connection between uncertainty relations and preparation contextuality.

2 Underlying Theory

In this thesis, we have studied various aspects of quantum measurement using quantum coherence and uncertainty relations. The study requires understanding of the underlying mathematical structure of quantum theory, in particular the quantum operations. We also use properties of Von-Neumann entropy and quantum coherence. Therefore, we will give a brief description of the necessary underlying theory in this chapter.

2.1 Quantum formalism

The formalism of quantum mechanics consists of basic postulates and the underlying mathematical structure. We briefly present the postulates and then describe the mathematical framework necessary for our purpose.

Postulate 1: At a given point of time the state of the physical system is denoted by a state vector $|\psi\rangle$, which is an element of Hilbert space $\mathcal{H}$.

Postulate 2: In quantum mechanics, the observables are represented by a linear

Hermitian operator $\mathcal{A}$, where $\mathcal{A}$ is a square matrix whose eigenstates form a complete and orthonormal basis.

Postulate 3: A quantum measurement on the system $|\psi\rangle$ enacted by $\mathcal{A}$, will yield one of the eigenvalues "a" of $\mathcal{A}$ as outcome with probability $p_a = |\langle a|\psi\rangle|^2$, which is also called the *Born rule*. After the measurement, if we renew the measurement, then the system collapses into one of the eigenstates $|a\rangle$.

Postulate 4: In absence of any measurement, the state of the quantum system evolves according to the $Schr\ddot{o}dinger's$ equation.

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle ,$$

where H is the Hamiltonian operator.

Postulate 5:For composite systems, the total Hilbert space is tensor product of individual Hilbert spaces, i.e., $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 ... \mathcal{H}_n$.

2.1.1 Quantum system

As mentioned in postulate 1, the state of a quantum system belongs to a Hilbert space. It is therefore important to understand what is a Hilbert space and for that we introduce the definition of an *inner product space*. A inner product space is a complex vector space which has an inner product (denoted by $\langle\psi|\phi\rangle$), which associates a complex number with any two elements $|\psi\rangle$ and $|\phi\rangle$ of the inner product space. For $|\psi\rangle, |\phi\rangle$ and $|\xi\rangle \in \mathcal{H}$, the inner product satisfies the following properties:

1. $\langle\psi|\psi\rangle \geq 0; \langle\psi|\psi\rangle = 0$, iff $|\psi\rangle$ is a null vector.

2. $\langle \alpha\psi + \beta\phi|\xi\rangle = \alpha \langle\psi|\xi\rangle + \beta \langle\phi|\xi\rangle$, where α, β are complex numbers.

3. $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^\dagger$.

The inner product defines a norm of the state $|\psi\rangle$, as $||\psi|| = \langle\psi|\psi\rangle$. An inner product space which is complete with respect to the norm is known as a **Hilbert space** [51, 52]. All the finite dimensional inner product spaces are complete and hence they are also Hilbert spaces. Only a limited number of infinite dimensional complex vector spaces are Hilbert spaces.

The states corresponding to Hilbert spaces are the pure states. But in the presence of noise, we have access only to a classical mixture of the pure states. To include such cases one needs to introduce the density matrix formalism. A density matrix ρ is a d-dimensional positive semi-definite Hermitian operator with $\mathrm{Tr}(\rho) = 1$, expressed as follows:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| \,.$$

The above expression implies that the system is in state $|\psi_i\rangle$ with probability p_i. However, it should be noted that there is no unique way to express a density matrix as a classical mixture of pure states. For pure states $\mathrm{Tr}[\rho^2] = 1$ and and the state corresponds to a projection operator, while $Tr[\rho^2] < 1$ for mixed states.

Another important property of density matrices that will be useful in this thesis is that, a $d-$dimensional density matrix can be expressed in terms d^2-1 parameters. The d^2-1 parameters form a Bloch vector which belong to an d^2-1 dimensional sphere. Using the Bloch vectors one can expand the density matrix in the following form [53, 54]

$$\rho = \frac{1}{d} I + \vec{b} \cdot \vec{\Gamma}, \tag{2.1}$$

where $\vec{b}$ are the Bloch vectors and $\vec{\Gamma}$ are the generalized Gell-mann matrices in d-dimension. The Bloch vectors lie within a sphere of radius $|\vec{b}| \leq \sqrt{\frac{d-1}{2d}}$. The generalized Gell-mann matrices have the property that they are traceless, i.e., $\mathrm{Tr}(\Gamma_i) = 0$ and mutually orthogonal, i.e., $\mathrm{Tr}(\Gamma_i \Gamma_j) = 2\delta_{ij}$. In the Bloch sphere the pure state correspond to the points on the surface of the sphere while the mixed states are inside. It should be noted that for $d = 2$ all the points in side the Bloch sphere are valid states, while for $d \geq 3$ not all the points inside the Bloch sphere correspond to valid quantum states [53, 54]. It is because of this fact we will see, that the fine-grained inequality derived in Chapter 6 is not tight for $d \geq 3$.

Please note that for $d = 2$, the following Bloch vector representation that is followed in most textbooks [55–58] differs from our representation in Eq.(2.1) only by a normalization factor.

$$\rho = \frac{1}{2}\big(I + \vec{t} \cdot \vec{\sigma_i}\big), \tag{2.2}$$

where for $d = 2$ the generalized Gellmann matrices are equivalent to Pauli matrices. In Eq.(2.2) the set of Bloch vectors are confined in a 3-dimensional Bloch sphere of radius $|\vec{t}| = 1$.

2.1.2 Quantum measurement

The idea of quantum measurement theory was first put in a formal way by Dirac in [59]. But a rigorous mathematical framework of quantum measurement was given by Von Neumann [60]. After which Lüders [61] made a small extension to Von Neumann's projective measurement postulate. Later, Davies and Kraus had generalized the quantum measurements for non-ideal scenarios [62, 63].

A quantum measurement is a non-unitary evolution of the quantum system which gives information about the physical quantity in which we are interested. As mentioned in Postulate 2.1, after the measurement, the state collapses into one of the eigenstates of the observable being measured. This kind of measurement can be described using the projection operators, $\Pi_a = |a\rangle\langle a|$, such that $\sum_a a\Pi_a = A$. The system on which measurement is being done is assumed to have the form $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. Using the Born rule we get the probability of obtaining outcome as a, as

$$\begin{aligned} p_a &= \sum_i p_i|\langle a|\psi_i\rangle|^2 \\ &= \mathrm{Tr}[\rho\Pi_a]. \end{aligned}$$

The above expression is a modification of Born rule for noisy states. However, just as states are noisy, the measurement process itself can be noisy. The noisy measurement process or the open system dynamics can be implemented using a set of Kraus operators K_j, such that $\sum_j K_j^\dagger K_j = 1$[55, 56]. Note that, for projective measurements $\mathrm{Tr}[K_i K_j] = \delta_{ij}$. Under the action of these operators the evolution of the state ρ is given by

$$\rho \to \mathcal{E}(\rho) = \sum_j K_j \rho K_j^\dagger.$$

We denote the above operation by Λ. It is also called a quantum operation or a quantum map or a quantum channel. For the final state $\mathcal{E}(\rho)$ to be a valid density matrix, it is necessary that the quantum operation Λ is convex linear on the set of density matrices and completely positive trace preserving (CPTP) map. These properties of a quantum map can be listed as below

- **Linearity**: A linear map Λ satisfies the following relation for all the operators in Hilbert space $\mathcal{H}$

$$\Lambda(\alpha\mathcal{O}_1 + \beta\mathcal{O}_2) = \alpha\Lambda(\mathcal{O}_1) + \beta\Lambda(\mathcal{O}_2),$$

  where α and β are complex numbers.

- **Trace Preservation**: A trace preserving map Λ, ensures that $\mathrm{Tr}[\mathcal{O}] = \mathrm{Tr}[\Lambda(\mathcal{O})]$, for any operator $\mathcal{O}$ in Hilbert space $\mathcal{H}$.

- **Complete positivity**: A map Λ is positive if $\Lambda(\mathcal{O})$ is positive semidefinite for all positive semidefinite operators $\mathcal{O}$. For complete positivity it is required that the map $I_R \otimes \Lambda$ acting on a operator $\mathcal{O}_R \otimes \mathcal{O}$, is also positive. The dimension R of the reference system is given by, $\dim(\mathcal{H}_R) \geq \dim(\mathcal{H}_S)$.

2.1.3 Examples of Quantum Channels

In this section, we list a few important quantum channels to be used in this thesis, for qubit and qutrit states.

Measurement Channel

A measurement channel is the simplest measurement which has kraus operators given by projective measurements, i.e., $K_i = \Pi_i = |\psi_i\rangle\langle\psi_i|$. The evolution of the state is represented as

$$\rho \longrightarrow \mathcal{E}(\rho) = \sum_i \Pi_i \rho \Pi_i = \rho^D = \sum_i p_i |\psi_i\rangle\langle\psi_i| \,,$$

where $p_i = \mathrm{Tr}[\rho \Pi_i]$.

Dephasing Channel

The evolution of a state under action of dephasing channel is given by

$$\rho \longrightarrow \mathcal{E}(\rho) = p\sigma_z \rho \sigma_z + (1-p)\rho.$$

where $\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is one of the Pauli matrices. The kraus operators of this channel are given by $K_1 = \sqrt{1-p}I, K_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $K_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This is equivalent to a measurement channel for $p = 1/2$.

Weak measurement channel

The implementation of the weak measurement channel by using projective measurement operators was given in Ref.[64, 65]. The advantage of this approach is that it can capture both strong and weak measurements. For this channel the Kraus operators have the following form
$K(x) = \sqrt{\frac{1-x}{2}}\Pi_0 + \sqrt{\frac{1+x}{2}}\Pi_1,$
$K(-x) = \sqrt{\frac{1+x}{2}}\Pi_0 + \sqrt{\frac{1-x}{2}}\Pi_1,$
where Π_0 and Π_1 are projectors. As required for the Kraus operators, for the weak measurement channel also, we have $K(x)^\dagger K(x) + K(-x)^\dagger K(-x) = \mathcal{I}$. The strength of the measurement in this case is captured by the parameter x which belongs to $[0, 1]$. The strength increases as x increases from 0 to 1. These operators satisfy the following properties: (i) For $x = 0$, we have, $K(x)$=$K(-x)$= $\frac{\mathcal{I}}{\sqrt{2}}$, i.e., no measurement takes place (ii) For $x = 1$, we have maximum strength measurement and the Kraus operators reduce to projectors, i.e., $K(x) = \Pi_1$ and $K(-x) = \Pi_0$ and (iii) $[K(x), K(-x)]$=0.

Depolarizing Channel

This channel captures the case when we lose the qubit state completely with probability p, i.e. the state transforms as

$$\rho \to \mathcal{E}(\rho) = pI/2 + (1-p)\rho$$

The Kraus operators for the qubit depolarizing channel are given by

$K_1 = \sqrt{1-\frac{3p}{4}} I_2, K_2 = \sqrt{\frac{p}{4}}\sigma_x, K_3 = \sqrt{\frac{p}{4}}\sigma_y, K_4 = \sqrt{\frac{p}{4}}\sigma_z,$

where σ_x, σ_y and σ_z are the Pauli matrices.

Depolarizing Channel for Qutrits : The Kraus operators for the single qutrit depolarizing channel [66] are given by

$K_1 = \sqrt{1-\frac{8p}{9}} I_3, K_2 = \sqrt{\frac{p}{9}} Y, K_3 = \sqrt{\frac{p}{9}} Z,$

$K_4 = \sqrt{\frac{p}{9}} YZ, K_5 = \sqrt{\frac{p}{9}} Y^2 Z, K_6 = \sqrt{\frac{p}{9}} Y Z^2,$

$K_7 = \sqrt{\frac{p}{9}} Y^2 Z^2, K_8 = \sqrt{\frac{p}{9}} Y^2, K_9 = \sqrt{\frac{p}{9}} Z^2.$

$$\text{where } Y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \text{ and } \omega = \exp^{\frac{2\pi i}{3}}.$$

Amplitude Damping Channel

The amplitude damping channel models the spontaneous emission of a photon from an atom. In spontaneous emission the atom gets decayed from excited to ground state by losing a photon. The emission happens even when the atom is in superposition of ground state and excited state. The Kraus operators for qubit Amplitude damping channel are given by

$K_1 = \sqrt{q} \left|0\right\rangle \left\langle 1\right|, K_2 = \left|0\right\rangle \left\langle 0\right| + \sqrt{1-q} \left|1\right\rangle \left\langle 1\right|.$

Amplitude Damping Channel for Qutrits: The Kraus operators for Amplitude damping channel are given by

$K_1 = \sqrt{q_0} \left|0\right\rangle \left\langle 1\right| + \sqrt{2q(1-q)} \left|1\right\rangle \left\langle 2\right|, K_2 = \sqrt{q} \left|0\right\rangle \left\langle 2\right|,$

$K_3 = |0\rangle\langle 0| + \sqrt{1-q}\,|1\rangle\langle 1| + \sqrt{1-q}\,|2\rangle\langle 2|$.

2.2 Quantum Entropy

The quantum entropy function of a density matrix ρ is defined as

$$S(\rho) =\equiv -\operatorname{Tr}[\rho \log \rho].$$

This is also known as the Von Neumann entropy of the state. The quantum entropy is a function of the eigenvalues of the density matrix, which can be seen by writing the density matrix in diagonal form as $\rho = \sum_x p_x |x\rangle\langle x|$. For such a spectral decomposition of the density matrix quantum entropy, $S(\rho) = -p_x \log p_x$, is equal to the classical entropy of the variable x. The interpretation of the entropy function $S(\rho)$ is that it captures classical uncertainty of the component states of the total density matrix. Upon having complete knowledge of the density matrix we gain $S(\rho)$ qubits of information. Quantum entropy or Von Neumann entropy has the following properties which will be useful for us in next chapters.

- **Non-Negativity**: $S(\rho) \geq 0$. This is necessary for a measure of information, since information should never be negative.

- **Minimum and maximum value**: For a density matrix of dimension d, the entropy is constrained by $0 \leq S(\rho) \leq \log d$. The minimum value occurs when ρ is a pure state and maximum is reached for a maximally mixed state, i.e., $\rho = \frac{I}{d}$.

- **Concavity**: The entropy is a concave function of the density operators, i.e., for $\rho = \sum_x p_x \rho_x$, $S(\rho) \geq \sum_x p_x S(\rho_x)$.

- **Isometric invariance**: The entropy of a state ρ doesn't change under the action of isometry or unitary operations, i.e., $S(\rho) = S(U\rho U^\dagger)$, where U is a unitary operator. This is so, because entropy is a function of eigenvalues and an isometric operation doesn't affect the eigenvalues.

- **Sub-additivity**: For a bipartite state ρ_{AB}, $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$. The equality is reached whenever $\rho_{AB} = \rho_A \otimes \rho_B$, i.e., the local states are independent.

- **Strong sub-additivity**: For a tripartite state ρ_{ABC}, $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$. It can be proved that strong-subadditivity implies sub-additivity.

Although there is a vast literature on quantum entropy and its properties, we limit ourselves to above only. The other essential quantities and properties, we will introduce when we need them in the later chapters.

3

A trade-off relation for coherence and disturbance

This chapter is based on our work done in paper titled "A trade-off relation for coherence and disturbance"[31]. In this chapter we explore the connection between the initial coherence of the system and the disturbance caused to the system by a measurement. We do this for both single party and bipartite systems. But before deriving a trade-off relation, we discuss how to quantify quantum coherence and disturbance caused to the system.

3.1 Quantum Coherence

Its important to first know the historical development made in the understanding of quantum coherence. Coherence is a property of the physical system in the quantum world that can be used to drive various non-classical phenomena. Hence, coherence can be viewed as a resource, which enables us to perform useful quantum information processing tasks. Much before the resource theory of coherence was developed [67–70], coherence was viewed as a resource similar to entanglement. In fact, similar to the entanglement swapping, the coherence swapping has been proposed that can create coherent superposition from two incoherent states [71]. After the development of the resources theory of coherence, this was shown to be complementary to the path distinguishability in an interferometer [72]. Similarly, a complementarity relation between quantum coherence and entanglement was proved in Ref.[73]. Also, coherence in two incompatible basis were shown to be complementary to each other by proving that they indeed satisfy an uncertainty relation [74]. Complementarity of coherence with mixedness and asymmetry was also investigated in Refs.[75–77].

Quantum coherence captures the superposition present in the system, in a given basis. Since, it is a basis dependent quantity hence it is necessary to first fix the reference basis in which we define a quantitative measure of coherence. An axiomatic approach to quantify quantum coherence was developed by Baumgratz et al. in Ref.[67] by characterizing incoherent states $\mathcal{I}$ and incoherent operations Λ. For a given reference basis $|i\rangle, (i = 0, 1, ...d-1)$, all incoherent states are of the form $\rho = \sum_i p_i |i\rangle \langle i|$ such that $\sum_i p_i = 1$. All incoherent operators are defined as CPTP maps, which map the set of incoherent states onto itself. A genuine measure of quantum coherence should fulfill the following requirements: (i) Non-negativity: $C(\rho) \geq 0$ in general. The equality is satisfied iff ρ is an incoherent state. (ii) Monotonicity: $C(\rho)$ does not increase under the action of incoherent operations, i.e., $C(\Lambda(\rho)) \leq C(\rho)$, where Λ is an incoherent operation. (iii) Strong monotonicity: $C(\rho)$ does not increase on average under selective incoherent operations, i.e., $\sum_i q_i C(\sigma_i) \leq C(\rho)$, where $q_i = \mathrm{Tr}[K_i \rho K^{\dagger}{}_i]$ are the probabilities, $\sigma_i = K_i \rho K^{\dagger}{}_i / q_i$ are post measurement states and K_i are the incoherent Kraus operators. (iv) Convexity: $C(\rho)$ is a convex function of the state, i.e., $\sum_i p_i C(\rho_i) \geq C(\sum_i p_i \rho_i)$. It can be noted that conditions (iii) and (iv) put together imply the condition (ii).

The measures that fulfill the above requirements are the l_1 norm of coherence and the relative entropy of coherence. In this chapter we will use the relative entropy of coherence which is given by

$$C_r(\rho) = S(\rho^D) - S(\rho), \tag{3.1}$$

where $S(\rho) = -\mathrm{Tr}(\rho \log_2(\rho))$ is the von Neumann Entropy of the density matrix ρ and ρ^D denotes the state obtained by deleting the off-diagonal elements of ρ in a given basis $\{|i\rangle\}$. For a d-dimensional state, $0 \leq C_r(\rho) \leq \log_2(d)$. Hence using the above definition we can define the maximally coherent state with $C_r(\rho) = \log_2(d)$, which is the case when $|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$. Another definition of coherence based on the matrix norm is the l_1 norm of coherence, which is given by $C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|$, where $\rho_{ij} = \langle i| \rho |j\rangle$. Also a geometric measure of coherence was defined in Ref.[36], as $C_g(\rho) = 1 - \max_{\sigma \epsilon I} F(\rho, \sigma)$, where I is the set of all incoherent states and the fidelity $F(\rho, \sigma) = ||\sqrt{\rho}\sqrt{\sigma}||_1^2$.

With this limited understanding of quantum coherence, which is sufficient for our purpose we move on to quantify the disturbance caused to a system.

3.2 Disturbance

In quantum mechanics, disturbance caused by a measurement process can be defined with respect to both the observable and the state. The disturbance for an observable due to measurement of another observable was defined in Ref.[78, 79] and for states in connection with the error-disturbance relations [80–82]. However, here we consider the disturbance caused to a state when the system is subjected to measurement process and CPTP maps and do not aim to formulate error-disturbance relations. We say that a system is disturbed when the initial and final state do not coincide. We define disturbance as an irreversible change in the state of the system, caused by CPTP evolution. It is thus required that the quantity D that measures disturbance should satisfy the following conditions[83]: (i) D should be a function of the initial state ρ and the CPTP map $\mathcal{E}$ only, i.e., $D = D(\rho, \mathcal{E})$, (ii) $D(\rho, \mathcal{E})$ should be null iff the CPTP map is invertible on the initial state ρ, because for invertible maps the change in state can be reversed hence the system is not disturbed by our definition, (iii) $D(\rho, \mathcal{E})$ should be monotonically non-decreasing under successive application of CPTP maps. This condition makes sure that the disturbance cannot be reversed by subsequent measurements, and (iv) $D(\rho, \mathcal{E})$ should be continuous for maps and initial states which do not differ too much.

Several definitions of disturbance have been proposed using the fidelity and the Bures distance between the initial and final state [3, 5, 84], but they fail to satisfy the irreversibility condition. Moreover, the fidelity based definition is non-zero for unitary transformations and disturb a system in a non-classical way [85], as we know that change in the quantum state due to unitary operations is reversible and hence do not cause any disturbance as per our definition. Also these definitions can be null for non-invertible maps and they are not monotonically non-decreasing under successive application of CPTP maps, therefore they fail to satisfy conditions (ii) and (iii). These are the main reasons why we have adopted the measure of disturbance given in Ref.[83]. For the sake of completeness, later in section 3.6 we also discuss the trade-off relation for the geometric measure of coherence and the fidelity based measure of disturbance. With the physically motivated conditions given in $(i) - (iv)$, it was shown by Maccone that all the above conditions are met by the following definition of disturbance [83]:

$$\begin{aligned} D(\rho, \mathcal{E}) &\equiv S(\rho) - I_c(\rho, \mathcal{E}) \\ &= S(\rho) - S(\mathcal{E}(\rho)) + S((\mathcal{E} \otimes I)(|\Psi\rangle_{SR}\langle\Psi|)) \end{aligned} \tag{3.2}$$

where $I_c = S(\mathcal{E}(\rho)) - S((\mathcal{E} \otimes I)(|\Psi\rangle_{SR}\langle\Psi|))$ is the coherent information [86, 87] of the system passing through a noisy channel, and $|\Psi\rangle_{SR}\langle\Psi|$ is a purification of ρ, such that $\rho = \rho_S = \mathrm{Tr}_R(|\Psi\rangle_{SR}\langle\Psi|)$. Since $I_c(\rho, \mathcal{E}) \leq S(\rho)$, we have $D(\rho, \mathcal{E}) \geq 0$. We know that a CPTP map is invertible iff the coherent information is equal to the Von Neumann entropy $S(\rho)$ of the state [86], and hence disturbance will always be null for all the invertible maps. The map $\mathcal{E} \otimes I$ acts on $|\Psi\rangle_{SR}$ with $\mathcal{E}$ acting on the system Hilbert space and I acts on the ancilla Hilbert space. The quantity I_c is non-increasing under successive application of CPTP maps, which makes the disturbance measure monotonically non-decreasing under CPTP maps. It is clear from the definition of $D(\rho, \mathcal{E})$ that for a d-dimensional density matrix ρ it satisfies, $0 \leq D \leq 2\log_2(d)$. With these basic definitions for the quantum coherence and disturbance now we present our main results.

3.3 Trade-off relations: Coherence, Entanglement, Quantum Correlations and Disturbance

In quantum information processing, the role of coherence, entanglement and quantum correlations cannot be condoned. However, when we send a quantum state through a noisy channel the system tends to loose these delicate quantum features. In practical scenarios, the action of noise and measurement cannot be evaded. In this section, we shall investigate how the initial coherence of the density matrix respect a trade-off relation with the disturbance caused by a quantum operation. Similarly, for a bipartite state we will explore how the quantum features like coherence, entanglement and quantum discord respect a trade-off relation with the disturbance caused by a CPTP map.

3.3.1 Coherence-Disturbance trade-off relation

Here, we prove that there exists, a trade-off relation between the amount of coherence contained in a quantum state and the disturbance caused to a system by a CPTP map. Consider a d-dimensional system with a density matrix ρ, initially the system and ancilla ρ_R share a pure bipartite state $|\Psi\rangle_{SR}$, with $\rho = \mathrm{Tr}_R(|\Psi\rangle_{SR}\langle\Psi|)$. When the system undergoes a quantum operation $\mathcal{E}$ the evolution is represented as $\rho \rightarrow \mathcal{E}(\rho) = \sum_i K_i \rho K_i{}^\dagger$, where K_i are the Kraus operator elements with $\sum_i K_i{}^\dagger K_i = I$. During the action of the CPTP map, the system undergoes disturbance as given in Eq.(3.2) ,i.e., $D(\rho, \mathcal{E}) = S(\rho) - I_c(\rho, \mathcal{E})$. For such a noisy evolution, we will prove that the trade-off relation between the coherence and the disturbance is given by

$$2C_r(\rho) + D(\rho, \mathcal{E}) \leq 2\log(d). \tag{3.3}$$

The proof is as follows :

$$\begin{aligned}
&2C_r(\rho) + D(\rho, \mathcal{E}) \\
&= 2S(\rho^D) - S(\rho) - S(\mathcal{E}(\rho)) + S(\mathcal{E} \otimes I(|\Psi\rangle_{SR} \langle\Psi|)) \\
&\leq 2S(\rho^D) - S(\rho) - S(\mathcal{E}(\rho)) + S(\mathcal{E}(\rho)) + S(\rho_R') \\
&= 2S(\rho^D) - S(\rho) + S(\rho_R) \\
&= 2S(\rho^D) \\
&\leq 2\log(d)
\end{aligned}$$

where ρ_R' is the final state of ancilla and the log has base 2. The first inequality is obtained by using the subadditivity of quantum entropy. The next inequality is obtained using the fact that there is no change in entropy of ancilla and the next equality follows using the fact that initial bipartite state is a pure state thus $S(\rho) = S(\rho_R)$. Final inequality comes from the maximum value of entropy of a state. Thus, for a given value of non-zero disturbance, the quantum coherence cannot reach its maximum value. There is a trade-off between these two quantum features. Also, note that in the proof no where we use the coherence measure in a particular basis. Therefore, the relation holds true in any basis we want to define the coherence and for all CPTP maps.

3.3.2 Coherence-Disturbance trade-off for the measurement channel

While the trade-off relation holds true for all quantum channels, the bound is tighter in the case of measurement channels. The quantum operation for the measurement channel is given by

$$\rho \longrightarrow \mathcal{E}(\rho) = \sum_k \Pi_k \rho \Pi_k = \rho^D = \sum_k \rho_k \left|k\right\rangle \left\langle k\right| ,$$

where Π_k are the projection operators. This is also known as dephasing channel. Now if we consider an environment state $\left|0\right\rangle_E$ so that $\left|\Psi\right\rangle_{SR} \otimes \left|0\right\rangle_E$ is also a pure state, then the evolution $(\mathcal{E} \otimes I)(\left|\Psi\right\rangle_{SR} \left\langle\Psi\right|)$ is equivalent to unitary evolution of the tripartite state (U acts on $\mathcal{H}_S \otimes \mathcal{H}_E$).

$$U \otimes \mathcal{I}(\left|\Psi\right\rangle_{SR} \otimes \left|0\right\rangle_E) \longrightarrow \left|\Psi'\right\rangle_{SRE} .$$

Since $\left|\Psi'\right\rangle_{SRE}$ is also a pure state, we have $S(\rho'_{SR}) = S(\rho'_E)$, where $\rho'_{SR} = (\mathcal{E} \otimes I)(\left|\Psi\right\rangle_{SR} \left\langle\Psi\right|) = \mathrm{Tr}[U(\left|\Psi\right\rangle_{SR} \left\langle\Psi\right| \otimes \left|0\right\rangle_E \left\langle 0\right|)U^\dagger]$. Then, using subadditivity of entropy one can obtain the following trade-off relation

$$C(\rho) + D(\rho, \mathcal{E}) \leq \log d_E, \tag{3.4}$$

where d_E =dim($\mathcal{H}_E$), is the dimension of the Hilbert space of the environment. In Eq.(3.4) both the quantities $C(\rho)$ and $D(\rho, \mathcal{E})$ are basis dependent. In the case of $dim(\mathcal{H}_S) = dim(\mathcal{H}_E) = d$, the trade-off relation given in Eq.(3.4) is tighter than the one given in Eq.(3.3).

3.3.3 Trade-off between coherence, entanglement and disturbance

In the previous section, we have proved the trade-off of coherence and disturbance for a single system. However, when we deal with a composite system it can have quantum coherence, entanglement and quantum correlation beyond entanglement such as the discord. Then, a natural question to ask here is if there exists any trade-off relation between the coherence, entanglement and disturbance caused by CPTP maps. Already, we know that for pure bipartite states there is a trade-off relation between the relative entropy of coherence and the bipartite entanglement, i.e., $C(\rho_A) + E(|\Psi\rangle_{AB}) \leq \log d$, where d is the dimension of the subsystem Hilbert space of A [73]. Even for mixed bipartite states ρ_{AB} one can prove a trade-off relation between coherence of one subsystem and entanglement of formation $E_f(\rho_{AB})$ [88].

This is given by

$$C_r(\rho_A) + E_f(\rho_{AB}) \leq \log d. \tag{3.5}$$

The proof follows from the Carlen-Lieb inequality [89] . This inequality says that $E_f(\rho_{AB}) \leq min\{S(\rho_A), S(\rho_B)\}$. Assuming that $S(\rho_A)$ is the minimum one, we have $E_f(\rho_{AB}) \leq S(\rho_A)$ which is equivalent to Eq.(3.5). Below, we prove that there is indeed a trade-off relation for the coherence, relative entropy of entanglement and disturbance caused by measurement or a CPTP map on the bipartite state. Suppose, we have the bipartite state ρ_{AB} with purification $|\Psi\rangle_{ABR}$, such that $\rho_{AB} = \text{Tr}_R(|\Psi\rangle_{ABR}\langle\Psi|)$. The relative entropy of entanglement was defined in Ref. [90, 91], as

$$E_R(\rho_{AB}) = min_{\sigma_{AB}} S(\rho_{AB}||\sigma_{AB}).$$

where σ_{AB} belongs to the set of all separable states. Note that, a mixed state is called separable if it can be written in the form $\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k$, where ρ_A^k and ρ_B^k are states of the subsystems with probability p_k. Now suppose that $\rho_{AB} \to \mathcal{E}(\rho_{AB})$, then, the disturbance for the bipartite state under a quantum channel is defined as

$$\begin{aligned} D(\rho_{AB}, \mathcal{E}) &= S(\rho_{AB}) - I_c((\rho_{AB})) \\ &= S(\rho_{AB}) - S(\mathcal{E}(\rho_{AB}) + S(\mathcal{E} \otimes \mathcal{I}(|\Psi\rangle_{ABR}\langle\Psi|). \end{aligned} \tag{3.6}$$

Here, the relative entropy of quantum coherence for the bipartite state can be defined as

$$C(\rho_{AB}) = S(\rho_{AB}^D) - S(\rho_{AB}). \tag{3.7}$$

where ρ_{AB}^D is the diagonal part of ρ_{AB} in the basis $\{|i\rangle \otimes |\mu\rangle\} \in \mathcal{H}_{AB}$. Using the above definitions of coherence, entanglement and disturbance for the bipartite state ρ_{AB}, we can get a trade-off relation of the following form

$$C(\rho_{AB}) + E_R(\rho_{AB}) + D(\rho_{AB}, \mathcal{E}) \leq 2\log(d_{AB}). \tag{3.8}$$

The proof of the relation is as follows:

$$\begin{aligned}
&C(\rho_{AB}) + E_R(\rho_{AB}) + D(\rho_{AB}, \mathcal{E}) \\
&= S(\rho_{AB}^D) + min_{\sigma_{AB}} S(\rho_{AB}||\sigma_{AB}) - S(\mathcal{E}(\rho_{AB}) + S(\mathcal{E} \otimes \mathcal{I}(|\Psi\rangle_{ABR}\langle\Psi|) \\
&\leq S(\rho_{AB}^D) + S(\rho_{AB}||\rho_A \otimes \rho_B) + S(\rho_{AB}) \\
&= S(\rho_{AB}^D) + S(\rho_A) + S(\rho_B) \\
&\leq 2\log(d_{AB}).
\end{aligned}$$

where d_{AB} is the dimension of the state ρ_{AB}. The first inequality is obtained using subadditivity of $S(\mathcal{E} \otimes \mathcal{I}(|\Psi\rangle_{ABR}\langle\Psi|)$ and the fact that, $min_\sigma S(\rho_{AB}||\sigma) \leq S(\rho_{AB}||\rho_A \otimes \rho_B)$. The final inequality follows from maximum value of the entropy of the states, i.e., $S(\rho_A) \leq \log(d_A)$, $S(\rho_B) \leq \log(d_B)$ and $S(\rho_{AB}) \leq \log(d_{AB})$.

The trade-off relation in Eq.(3.8) suggests that the sum of quantumness such as the coherence and entanglement cannot be large if the disturbance is also large. Also, for a fixed coherence $C(\rho)$, there is a trade-off between entanglement and disturbance caused to the quantum system. For separable states, $E_R(\rho_{AB}) = 0$, and we have $C(\rho_{AB}) + D(\rho_{AB}, \mathcal{E}) \leq 2\log(d_{AB})$.

3.3.4 Trade-off between Coherence, Quantum Discord and Disturbance

In the last section we proved a trade-off relation for coherence, entanglement and disturbance caused by a CPTP map on a bipartite system. Similarly, one can ask if other quantum correlations like quantum discord satisfies a similar trade-off relation. It was shown in Ref.[37] that for multipartite states, creation of quantum discord with multipartite incoherent operations is bounded by the amount of quantum coherence consumed in its subsystems during the process. This interplay between coherence and quantum discord suggests that coherence, quantum discord and disturbance of a bipartite system may satisfy a trade-off relation. We will now prove that, they also satisfy a trade-off relation. Quantum discord of a bipartite state is defined in Ref.[24] as

$$Q_D(\rho_{AB}) = min_{\Pi_i^B}[I(\rho_{AB}) - J(\rho_{AB})_{\Pi_i^B}].$$

where $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ is the mutual information between the subsystems A and B and $J(\rho_{AB})_{\Pi_i^B} = S(\rho_A) - S(A|\Pi_i^B)$, represents the amount of information gained about the subsystem A by measuring the subsystem B. Here,

Π_i^B are the measurement operators corresponding to Von-Neumann measurement on the subsystem B and $S(A|\Pi_i^B)$ is the conditional entropy after measurement has been performed on the subsystem B. Using the definitions of disturbance and coherence given in Eq.(3.7) and Eq.(3.6) respectively, for a bipartite state, we get a trade-off relation of the following form

$$C(\rho_{AB}) + Q_D(\rho_{AB}) + D(\rho_{AB}, \mathcal{E}) \leq 2\log(d_{AB}). \tag{3.9}$$

The proof of the relation is as follows:

$$\begin{aligned}
& C(\rho_{AB}) + Q_D(\rho_{AB}) + D(\rho_{AB}, \mathcal{E}) \\
& = S(\rho_{AB}^D) + min_{\Pi_i^B}[I(\rho_{AB}) - J(\rho_{AB})_{\Pi_i^B}] - S(\mathcal{E}(\rho_{AB})) + \quad S(\mathcal{E} \otimes \mathcal{I}(|\Psi\rangle_{ABR} \langle\Psi|)) \\
& \leq S(\rho_{AB}^D) + I(\rho_{AB}) + S(\rho_{AB}) \\
& = S(\rho_{AB}^D) + S(\rho_A) + S(\rho_B) \\
& \leq 2\log(d_{AB}).
\end{aligned}$$

where the proof is similar to the proof of trade-off relation of entanglement, coherence and disturbance of bipartite state. Thus, for separable states the coherence and discord cannot be arbitrarily large for a given disturbance $D(\rho_{AB}, \mathcal{E})$ caused to the quantum system. For classical-classical states such as $\rho_{AB} = \sum p_k |k\rangle\langle k| \otimes |k\rangle\langle k|$, one has $Q_D(\rho_{AB}) = 0$ and in that case one has $C(\rho_{AB}) + D(\rho_{AB}, \mathcal{E}) \leq 2\log(d_{AB})$.

3.4 Examples

To gain some physical insight, in this section, we analyze the coherence disturbance trade-off relation for different quantum channels for a single qubit and later for the single qutrit density matrix. The trade-off relations can be neatly presented for few channels. Let us consider a two qubit pure composite state of system and ancilla in $\{|+\rangle, |-\rangle\}$ basis.

$$|\Psi\rangle_{SR} = \sqrt{\lambda_0}\,|+\rangle_S\,|+\rangle_R + \sqrt{\lambda_1}\,|-\rangle_S\,|-\rangle_R\,.$$

For this composite state, the density matrix of system in computational basis is given by

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & \lambda_0 - \lambda_1 \\ \lambda_0 - \lambda_1 & 1 \end{bmatrix}.$$

In this basis, it has non-zero coherence which is given by

$$\begin{aligned} C_r(\rho) &= -\mathrm{Tr}[\rho^D \log_2 \rho^D] + \mathrm{Tr}[\rho \log_2 \rho] \\ &= 1 + \lambda_0 \log_2 \lambda_0 + \lambda_1 \log_2 \lambda_1. \end{aligned} \tag{3.10}$$

Disturbance of a state depends on both the state density matrix and the quantum channel. We give the expressions of the disturbance and present the trade-off relations, for few channels as examples.

3.4.1 Weak measurement channel

We use the weak measurement channel presented in the previous chapter. Under the weak measurement channel the state changes as

$$\rho \to \mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{(1-x^2)}(\lambda_0-\lambda_1) \\ \sqrt{(1-x^2)}(\lambda_0-\lambda_1) & 1 \end{bmatrix}. \tag{3.11}$$

Disturbance for the weak measurement channel is given by

$$\begin{aligned} D(\rho,\mathcal{E}) &= -\mathrm{Tr}\left[\rho\log_2\rho\right] + \mathrm{Tr}[\mathcal{E}(\rho)\log_2\mathcal{E}(\rho)] - \mathrm{Tr}[\mathcal{E}\otimes I(|\Psi\rangle\langle\Psi|)\log_2\mathcal{E}\otimes I(|\Psi\rangle\langle\Psi|] \\ &= -\lambda_0\log_2\lambda_0 - \lambda_1\log_2\lambda_1 \\ &- \frac{1-\sqrt{1-4\lambda_0\lambda_1x^2}}{2}\log_2\frac{1-\sqrt{1-4\lambda_0\lambda_1x^2}}{2} - \frac{1+\sqrt{1-4\lambda_0\lambda_1x^2}}{2}\log_2\frac{1+\sqrt{1-4\lambda_0\lambda_1x^2}}{2} \\ &+ \left(\frac{1}{2}-\frac{\lambda_0-\lambda_1}{2}\sqrt{1-x^2}\right)\log_2\left(\frac{1}{2}-\frac{\lambda_0-\lambda_1}{2}\sqrt{1-x^2}\right) \\ &+ \left(\frac{1}{2}+\frac{\lambda_0-\lambda_1}{2}\sqrt{1-x^2}\right)\log_2\left(\frac{1}{2}+\frac{\lambda_0-\lambda_1}{2}\sqrt{1-x^2}\right). \end{aligned} \tag{3.12}$$

It can be checked that $D(\rho,\mathcal{E})$ increases monotonically as x is increased from 0 to 1. By using Eq.(3.10) and Eq.(3.12) we indeed see that the relation $C(\rho)+D(\rho,\mathcal{E}) \leq$

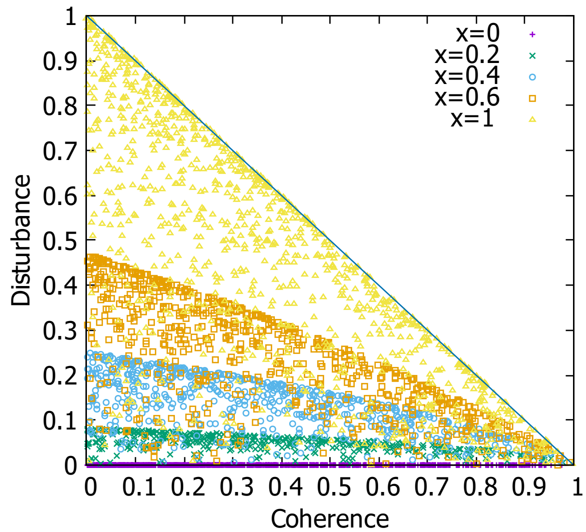

Figure 3.1: The trade-off between Coherence $C(\rho)$ and Disturbance $D(\rho, \mathcal{E})$ for the weak measurement channel. The figure shows coherence along the X-axis and disturbance along Y-axis. Random states were generated and coherence and entropy were calculated, using the Matlab package [1].

1 holds. This is depicted in Figure.3.1. This relation is tighter than our original relation given in Eq.(3.3). The same trade-off relation is also obtained for the bit-flip, phase flip and bit-phase flip channels for a single qubit system.

3.4.2 Depolarizing Channel

By using the Kraus operators defined for depolarizing channel in previous chapter, the state changes as

$$\rho \to \mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} 1 & (1-p)(\lambda_0-\lambda_1) \\ (1-p)(\lambda_0-\lambda_1) & 1 \end{bmatrix}.$$

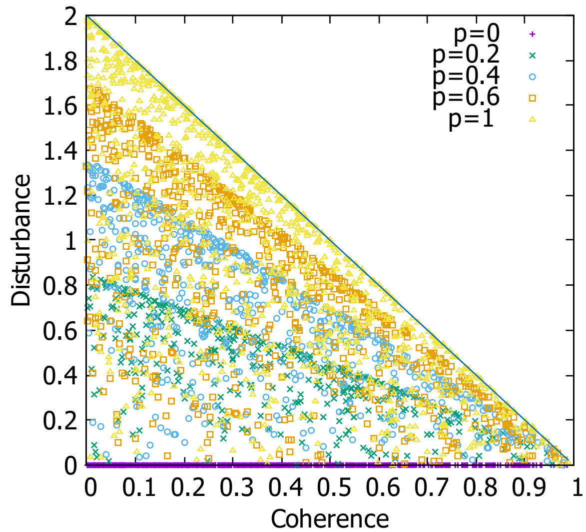

Figure 3.2: The trade-off between Coherence $C(\rho)$ and Disturbance $D(\rho, \mathcal{E})$ for the depolarising channel. The figure shows coherence along the X-axis and disturbance along Y-axis. Random states were generated and coherence and entropy were calculated, using the Matlab package [1].

The disturbance for the depolarizing channel is given by

$$\begin{aligned}
D(\rho,\mathcal{E}) = &-\lambda_0\log_2\lambda_0 - \lambda_1\log_2\lambda_1 + \left(\frac{1}{2}-\frac{\lambda_0-\lambda_1}{2}(1-p)\right)\log_2\left(\frac{1}{2}-\frac{\lambda_0-\lambda_1}{2}(1-p)\right)\\
&+\left(\frac{1}{2}+\frac{\lambda_0-\lambda_1}{2}(1-p)\right)\log_2\left(\frac{1}{2}+\frac{\lambda_0-\lambda_1}{2}(1-p)\right) - \frac{p\lambda_0}{2}\log_2\frac{p\lambda_0}{2} - \frac{p\lambda_1}{2}\log_2\frac{p\lambda_1}{2}\\
&-\left(\frac{(1-\frac{p}{2})+\sqrt{(1-\frac{p}{2})^2-4\lambda_0\lambda_1(p-\frac{3p^2}{4})}}{2}\right)\log_2\left(\frac{(1-\frac{p}{2})+\sqrt{(1-\frac{p}{2})^2-4\lambda_0\lambda_1(p-\frac{3p^2}{4})}}{2}\right)\\
&-\left(\frac{(1-\frac{p}{2})-\sqrt{(1-\frac{p}{2})^2-4\lambda_0\lambda_1(p-\frac{3p^2}{4})}}{2}\right)\log_2\left(\frac{(1-\frac{p}{2})-\sqrt{(1-\frac{p}{2})^2-4\lambda_0\lambda_1(p-\frac{3p^2}{4})}}{2}\right).
\end{aligned} \tag{3.13}$$

Again it is easy to check that $D(\rho,\mathcal{E})$ increases monotonically with p. Moreover, using Eq.(3.10) and Eq.(3.13) we get $2C(\rho)+D(\rho,\mathcal{E})\leq 2$, which is same as Eq.(3.3) for a qubit, and is depicted in Figure.3.2.

3.4.3 Amplitude Damping Channel

Under the amplitude channel the state transforms as

$$\rho \to \mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} 1+q & \sqrt{(1-q)}(\lambda_0-\lambda_1) \\ (\sqrt{(1-q)}+q)(\lambda_0-\lambda_1) & 1-q \end{bmatrix}.$$

The disturbance of the amplitude damping channel is given by

$$
\begin{aligned}
D(\rho,\mathcal{E}) = & -\frac{1}{2}(1+\lambda_0-\lambda_1)\log_2\frac{1}{2}(1+\lambda_0-\lambda_1) \\
& -\frac{1}{2}(1-\lambda_0+\lambda_1)\log_2\frac{1}{2}(1-\lambda_0+\lambda_1) - (1-q\lambda_1)\log_2(1-q\lambda_1) - q\lambda_1\log_2 q\lambda_1 \\
& +\frac{1}{2}\left(1-\sqrt{q^2+(\lambda_0-\lambda_1)^2(1-q)}\right)\log_2\frac{1}{2}\left(1-\sqrt{q^2+(\lambda_0-\lambda_1)^2(1-q)}\right) \\
& +\frac{1}{2}\left(1+\sqrt{q^2+(\lambda_0-\lambda_1)^2(1-q)}\right)\log_2\left(1+\sqrt{q^2+(\lambda_0-\lambda_1)^2(1-q)}\right).
\end{aligned}
\tag{3.14}
$$

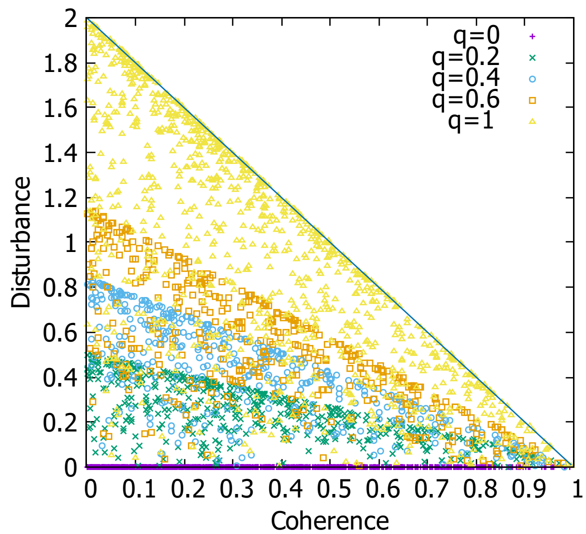

Figure 3.3: The trade-off between Coherence $C(\rho)$ and Disturbance $D(\rho,\mathcal{E})$ for the amplitude damping channel. The figure shows coherence along the X-axis and disturbance along Y-axis. Random states were generated and coherence and entropy were calculated, using the Matlab package [1].

For the amplitude damping channel also $D(\rho,\mathcal{E})$ and $C(\rho)$ follow the original relation Eq.(3.3). The trade-off relations derived above can be seen as given in

Figure.3.3.

Numerical data shows that the trade-off relation for the coherence and disturbance given in Eq.(3.3) is satisfied for all the above channels for single qubit systems. The amount of disturbance reduces as the measurement strength is decreased which is expected in the case of all the channels. It can be also seen that, the trade-off between coherence and disturbance is channel dependent. The trade-off relation obeyed for a single qubit state in case of weak measurement channel is tighter than Eq.(3.3). While the amplitude damping and depolarising channels follow the original relation for a single qubit state.

3.5 Examples from the qutrit system

Consider a two qutrit pure composite state of system and ancilla in the $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle\}$ basis.

$$|\Psi\rangle_{SR} = \sqrt{\lambda_0}\,|\alpha\rangle_S\,|\alpha\rangle_R + \sqrt{\lambda_1}\,|\beta\rangle_S\,|\beta\rangle_R + \sqrt{\lambda_2}\,|\gamma\rangle_S\,|\gamma\rangle_R\,,$$

where $|\alpha\rangle = \frac{|0\rangle-|1\rangle+|2\rangle}{\sqrt{3}}$, $|\beta\rangle = \frac{|0\rangle+|1\rangle-|2\rangle}{\sqrt{3}}$ and $|\gamma\rangle = \frac{-|0\rangle+|1\rangle+|2\rangle}{\sqrt{3}}$. For this composite state, the density matrix of system in computational basis is given by

$$\rho = \frac{1}{3}\begin{bmatrix} 1 & 2\lambda_0 - 1 & 2\lambda_1 - 1 \\ 2\lambda_0 - 1 & 1 & 2\lambda_2 - 1 \\ 2\lambda_1 - 1 & 2\lambda_2 - 1 & 1 \end{bmatrix}.$$

In this basis it has non-zero coherence which is given by

$$C_r(\rho) = \log_2 3 + \lambda_0 \log_2 \lambda_0 + \lambda_1 \log_2 \lambda_1 + \lambda_2 \log_2 \lambda_2. \tag{3.15}$$

Depolarizing Channel

Under the depolarising channel the qutrit state changes as

$$\rho \to \mathcal{E}(\rho) = \begin{bmatrix} 1-\frac{2p}{3} & \frac{2\lambda_0-1}{3}p & \frac{2\lambda_1-1}{3}p \\ \frac{2\lambda_0-1}{3}p & 1-\frac{2p}{3} & \frac{2\lambda_2-1}{3}p \\ \frac{2\lambda_1-1}{3}p & \frac{2\lambda_2-1}{3}p & 1-\frac{2p}{3} \end{bmatrix}.$$

The disturbance for the depolarizing channel is given by

$$\begin{aligned} D(\rho,\mathcal{E}) = {} & -\lambda_0 \log_2 \lambda_0 - \lambda_1 \log_2 \lambda_1 - \lambda_2 \log_2 \lambda_2 + (\frac{p}{3} + (1-p)\lambda_0) \log_2(\frac{p}{3} + (1-p)\lambda_0) \\ & + (\frac{p}{3} + (1-p)\lambda_1) \log_2(\frac{p}{3} + (1-p)\lambda_1) + (\frac{p}{3} + (1-p)\lambda_2) \log_2(\frac{p}{3} + (1-p)\lambda_2) \\ & - 2\frac{p\lambda_0}{3} \log_2 \frac{p\lambda_0}{3} - (\lambda_0 - \frac{8p\lambda_0}{9}) \log_2(\lambda_0 - \frac{8p\lambda_0}{9}) - 2\frac{p\lambda_1}{3} \log_2 \frac{p\lambda_1}{3} \\ & - (\lambda_1 - \frac{8p\lambda_1}{9}) \log_2(\lambda_1 - \frac{8p\lambda_1}{9}) - 2\frac{p\lambda_2}{3} \log_2 \frac{p\lambda_2}{3} - (\lambda_2 - \frac{8p\lambda_2}{9}) \log_2(\lambda_2 - \frac{8p\lambda_2}{9}). \end{aligned} \tag{3.16}$$

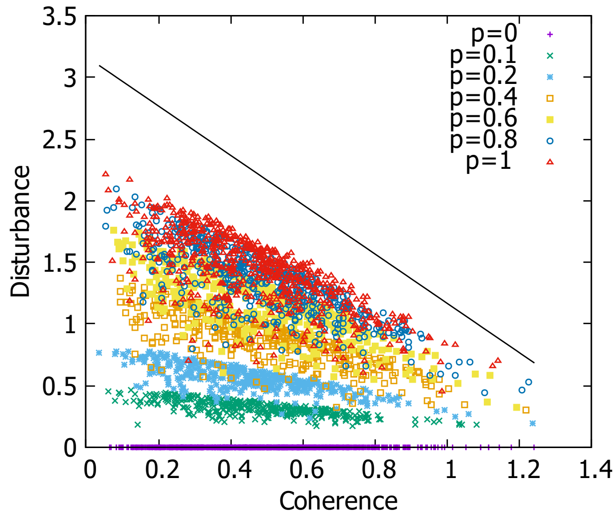

Figure 3.4: The trade-off between Coherence $C(\rho)$ and Disturbance $D(\rho, \mathcal{E})$ for the depolarizing channel applied on a qutrit state. The figure shows coherence along the X-axis and disturbance along Y-axis. The straight line corresponds to $2C_r(\rho) + D(\rho, \mathcal{E}) = 2\log_2 3$. Random states were generated and coherence and entropy were calculated, using the Matlab package [1].

From Eq.(3.15) and Eq.(3.16) we plot Coherence and Disturbance in Figure.4. We find that in this case the trade-off relation is stronger than the our original relation Eq.(3.3).

Amplitude Damping Channel

Under the amplitude channel the qutrit state transforms as

$$\rho \to \mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} \frac{1+q}{3} & \frac{\sqrt{1-q}}{3}(2\lambda_0 + 2q - 1) & \frac{\sqrt{1-q}}{3}(2\lambda_1 - 1) \\ \frac{\sqrt{1-q}}{3}(2\lambda_0 + 2q - 1) & \frac{1}{3}(1 + 2q(1-q)) & \frac{1-q}{3}(2\lambda_1 - 1) \\ \frac{\sqrt{1-q}}{3}(2\lambda_1 - 1) & \frac{1-q}{3}(2\lambda_1 - 1) & \frac{1-q}{3} \end{bmatrix}.$$

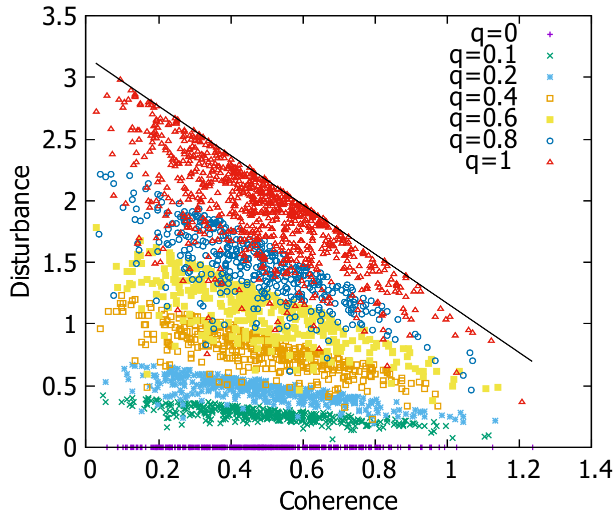

Figure 3.5: The trade-off between Coherence $C(\rho)$ and Disturbance $D(\rho, \mathcal{E})$ for the amplitude damping channel applied on a qutrit state. The figure shows coherence along the X-axis and disturbance along Y-axis. Random states were generated and coherence and entropy were calculated, using the Matlab package [1].

The disturbance for the amplitude damping channel is given by

$$
\begin{aligned}
D(\rho, \mathcal{E}) = & -2\lambda_0 \log_2 \lambda_0 - \lambda_1 \log_2 \lambda_1 - \lambda_2 \log_2 \lambda_2 + (\lambda_0 + q\lambda_1) \log_2(\lambda_0 + q\lambda_1) \\
& + ((1-q)\lambda_1 + (3-2q)q\lambda_2) \log_2((1-q)\lambda_1 + (3-2q)q\lambda_2) \\
& - q\lambda_1 \log_2 q\lambda_1 - ((1-q)\lambda_1) \log_2((1-q)\lambda_1) - (3-2q)q\lambda_2 \log_2 (3-2q)q\lambda_2.
\end{aligned}
\tag{3.17}
$$

For the amplitude damping channel $D(\rho, \mathcal{E})$ and $C(\rho)$ follow the original relation Eq.(3.3). The trade-off relations derived above can be verified with the given Figure.3.5.

3.6 Trade-off relation between fidelity based disturbance and geometric coherence

The choice of disturbance measure as given in Eq.(3.2) was motivated through a set of axioms and also was justified as that helps us to study how the quantum coherence may be degraded by a noisy quantum channel. However, one may ask if there exists a trade-off between other definitions of coherence and disturbance. Indeed, we find that the geometric measure of coherence defined in Ref.[36] obeys a trade-off with the fidelity based disturbance measure given in Ref.[3, 5, 84]. For a state evolving under measurement as $\rho \to \mathcal{E}(\rho)$, disturbance can be defined using fidelity between initial and final state as

$$D(\rho, \mathcal{E}) = 1 - F(\rho, \mathcal{E}(\rho)), \tag{3.18}$$

where $F(\rho, \mathcal{E}(\rho)) = [Tr\sqrt{\rho^{1/2}\mathcal{E}(\rho)\rho^{1/2}}]$. It should be noted the disturbance in Eq.(3.18) is reversible and is non-zero even for unitary evolution of a state. The geometric measure of coherence is given by

$$C_g(\rho) = 1 - \max_{\delta \epsilon I} F(\rho, \delta) \tag{3.19}$$

where I is the set of all incoherent states. If the measurement process always leads to incoherent state,i.e., $\mathcal{E}(\rho)\epsilon I$, then using equations Eq.(3.18) and Eq.(3.19) we get the following relation

$$C_g(\rho) \leq D(\rho, \mathcal{E}) \tag{3.20}$$

The above relation tells us that if the final state is an incoherent state, then the disturbance caused to the state will be at least equal to the amount of coherence in the system. If the system is in the eigenstate or the dephased state, then the coherence will be zero and the disturbance will also be zero as the state will no longer be disturbed. We present the relation in Equation.(3.20) using an example of the measurement channel acting on a single qubit state. The Kraus operators for measurement channel are projection operators Π_0 and Π_1 in the computational basis. Under the measurement channel, the system evolves as $\rho \to \mathcal{E}(\rho) = \Pi_0\rho\Pi_0 + \Pi_1\rho\Pi_1$.

By using Bloch vector representation the states ρ and $\mathcal{E}(\rho)$ can be represented as

$$\rho = \frac{\mathcal{I} + \vec{r}.\sigma}{2}, \mathcal{E}(\rho) = \frac{\mathcal{I} + \vec{s}.\sigma}{2}.$$

Then the Fidelity between ρ and $\mathcal{E}(\rho)$ has the form

$$F(\rho, \mathcal{E}(\rho)) = \frac{1}{2}[1 + \vec{r}.\vec{s} + \sqrt{(1 - |\vec{r}|^2)(1 - |\vec{s}|^2)}].$$

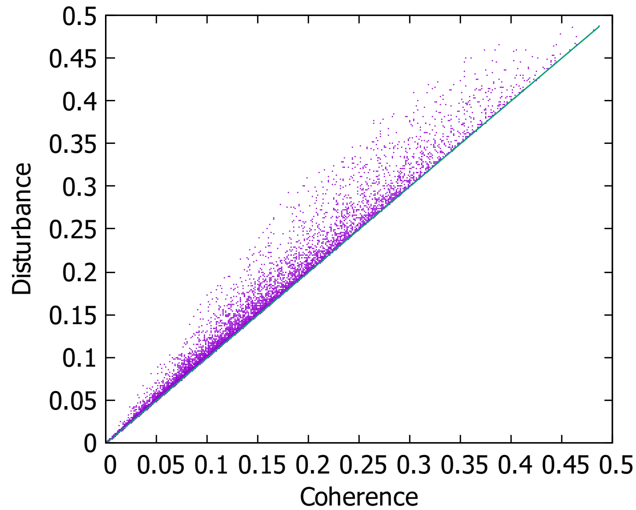

Figure 3.6: The trade-off between geometric measure of coherence and fidelity based disturbance measure is plotted obeying the relation in Eq In Chapter 3, we present a trade-off relation between quantum coherence and disturbance caused to a system by a quantum measurement. We also present3.20. Random states were generated using the Matlab package [1].

Therefore, we can write the disturbance using the Bloch vectors as

$$D(\rho, \mathcal{E}) = \frac{1}{2}[1 - \vec{r}.\vec{s} - \sqrt{(1 - |\vec{r}|^2)(1 - |\vec{s}|^2)}]. \tag{3.21}$$

The maximum value of Fidelity of a qubit with an incoherent state is given by [92]

$$\max_{\delta \epsilon I} F(\rho, \delta) = \frac{1}{2}[1 + \sqrt{1 - r_x^2 - r_y^2}]$$

Now, we can also express the geometric coherence using the Bloch vectors of the state ρ as

$$C_g(\rho) = \frac{1}{2}[1 - \sqrt{1 - r_x^2 - r_y^2}]. \tag{3.22}$$

In Figure.(3.6), we see that the trade-off relation is respected.

3.7 Conclusion

To summarize this chapter, we have shown that there exists a trade-off relation between the coherence of a state and disturbance caused by a CPTP map or a measurement channel on a quantum system. For measurement channel we find a tighter trade-off relation. Moreover, we obtain a trade-off relation for the quantum coherence, relative entropy of entanglement and disturbance for a bipartite system. Similar relation is also obtained for the quantum coherence, quantum discord and disturbance for a bipartite state. The trade-off relation for the coherence and disturbance has been illustrated for weak measurement channel and other quantum channels. Our results capture the intuition that coherence, entanglement and quantum discord for a quantum system should respect a trade-off relation with disturbance. Our results provide a deep physical meaning about the relation between quantum coherence and disturbance which can be widely applied in various contexts. We hope that these results will find interesting applications where we send single or composite systems under noisy channels that tend to loose quantum coherence and entanglement. If we wish to maintain coherence or entanglement or both, then we need to send the quantum states through a channel that does not disturb the system to a greater extent. In future it will be interesting to see if other measures of coherence and entanglement respects the trade-off relation with disturbance.

4 Quantum Coherence, Coherent Information and information gain

This chapter is based on our work titled "Quantum Coherence, Coherent Information and information gain" [93]. In the previous chapter we have shown that quantum coherence plays an important role in determining the disturbance caused to the system by a measurement. However, if a measurement only disturbs the system without giving any information, there is no meaning to doing it. It is known that extracting information can't happen without disturbing the system. The information gain from a system provides a lower bound to disturbance caused to the system[5, 83]. Therefore, in this chapter we ask the question how quantum coherence affects the amount of information that we can extract from the system. For that we first discuss how to quantify the information gain from a measurement.

4.1 Information gain

Suppose we have a random classical variable X, which takes values x with probability p_x. The information content of such a variable is given by the Shannon Entropy funtion $H(X) = -\sum_x p_x \log p_x$. But now instead of a classical variable, we have a quantum ensemble $\rho = \{p_x, \rho_x\}$. The information content of this ensemble is given by the von Neumann Entropy(quantum entropy), $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$ [56]. If we can write ρ as an ensemble of pure states, i.e., $\rho_x = |x\rangle\langle x|$, quantum entropy is equal to classical entropy, i.e., $S(\rho) = -\sum_x p_x \log p_x$. The quantum entropy $S(\rho)$ ought to be different than $H(X)$, because it captures both classical and inherent quantum uncertainty associated with the system, while $H(X)$ captures only the classical uncertainty associated with the random variable.

However, in a quantum measurement there is a restriction on the maximum information that we can extract from an ensemble. For the ensemble $\rho = \sum_x p_x \rho_x$ with classical index X, we do a measurement using POVM Λ_y then the amount of information that we can extract is equal to the mutual information $I(X;Y)$, with Y being the random variable we obtain as outcome. The maximum value of the mutual information, $\max_{\Lambda_y} I(X;Y)$, that we can access from the ensemble $\rho = \sum_x p_x \rho_x$, is given by Holevo quantity [55, 56] which is defined as

$$\chi(\{p_x, \rho_x\}) = \sum_x p_x S(\rho_x || \rho).$$

It should be noted that classical correlations [25] and the Holevo quantity are related concepts [94]. A bipartite state will have classical correlations if after application of rank one Projective Operator Valued Measurement (POVM) (i.e., entanglement breaking operation) on one of the parties will transform the state to either classical-quantum (CQ) or quantum classical (QC) states [94], i.e., $\rho_{CQ} = \sum_i p_i |i\rangle\langle i|_C \otimes \rho_Q^i$ or $\rho_{QC} = \sum_i p_i \rho_Q^i \otimes |i\rangle\langle i|_C$, where ρ^i are not orthogonal. The classical correlations [25] of a quantum state ρ_{SA} is given by $J = \sup_{\pi_j} [S(\rho_S) - \sum_j p_j S(\rho_{S|\pi_j^A})]$, where $\rho_{S|\pi_j^A}$ are the post measurement states with probability p_i due to the application of projective measurements ($\{\pi_j\}$) on the part A of ρ_{SA}. Then, the average post measurement state will be of the form of QC state. Therefore, the classical correlations of the state ρ_{SA} is equivalent to the maximum possible mutual information of ρ_{QC}, i.e., $J = I(\rho_{QC}) = S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i)$, hence, the classical correlations is nothing but the Holevo quantity [94].

4.2 Quantum coherence of system and Information Gain

Here, we ask whether the quantum coherence of the system plays a role in determining the information gain during a measurement process.

To answer the above query, we study the quantum measurement process in which the apparatus is initially in a mixed state. We adopt and analyze the measurement process described in Ref.[39], where the initial system is in a pure state $|\Psi\rangle_S = \sum_i s_i |s_i\rangle$ and the initial state of the apparatus is a mixed state as given by $\rho_A = \sum_\ell a_\ell |a_\ell\rangle\langle a_\ell|$, with a_ℓ as eigenvalues of ρ_A and $\{|a_\ell\rangle\}$ as the eigenbasis. The measurement process can be described by a unitary operator, U, which acts on the system and apparatus together, such that

$$U(|s_i\rangle\langle s_j| \otimes \rho_A)U^\dagger = |s_i\rangle\langle s_j| \otimes \rho_A^{(ij)}, \tag{4.1}$$

where $\rho_A^{(ij)}$ is an operator acting on $\mathcal{H}_A$. When the apparatus interacts with the system, it builds up correlation which is encoded in ρ_A^{ii}. Here, we have assumed that the state $|s_i\rangle\,|a_\ell\rangle$ of system and apparatus is transformed into, $|s_i\rangle\,|\tilde{a}_{i\ell}\rangle$. In an ideal measurement process, the final states of apparatus corresponding to different system states are orthogonal to each other and can be distinguished perfectly. But more often, this is not the case and the information extracted is not maximum. As we are trying to extract maximum information of the system from the apparatus, we assume the apparatus states $|\tilde{a}_{i\ell}\rangle$ corresponding to different system states to be orthogonal to each other, i.e., $\langle\tilde{a}_{i\ell}\rangle\,\tilde{a}_{j\ell} = \delta_{ij}$. This unitary evolution allows us to develop correlation between the system and the apparatus which is given by

$$|\Psi\rangle_S\,\langle\Psi| \otimes \rho_A \rightarrow U(|\Psi\rangle_S\,\langle\Psi| \otimes \rho_A)U^\dagger = \rho'_{SA}.$$

The final joint state in general is an entangled state as given by [39]

$$\rho'_{SA} = \sum_i |s_i|^2\,|s_i\rangle\,\langle s_i| \otimes \rho_A^{(ii)} + \sum_{i\neq j} s_i s_j^*\,|s_i\rangle\,\langle s_j| \otimes \rho_A^{(ij)}.$$

Note that the first term on the right carries the extractable information due to measurement. To extract the maximum possible information from the system, we should be able to distinguish between the different post measurement apparatus states $\rho_A^{(ii)}$ precisely. The maximum amount of accessible information from the apparatus is given by the Holevo quantity,

$$I_m = S\left(\sum_i |s_i|^2 \rho_A^{(ii)}\right) - \sum_i |s_i|^2 S(\rho_A^{(ii)}). \tag{4.2}$$

This quantity can also be identified with the classical correlations of the state $\rho'^{c}_{SA} = \sum_i |s_i|^2 |s_i\rangle \langle s_i| \otimes \rho_A^{(ii)}$. Note also that the state ρ'^{c}_{SA} is only classically correlated although the final state ρ'_{SA} has non-zero entanglement. It was argued that the maximum possible information gain during the process may not be identified by the entanglement developed in the system and apparatus. This can be understood by the following inequality [39]

$$E_r(\rho'_{SA}) \geq I_m. \tag{4.3}$$

where $E_r(\rho_{AB}) = \min_{\sigma_{AB}} S(\rho_{AB}||\sigma_{AB})$, is the relative entropy of entanglement [90, 91] and σ_{AB} is a separable state.

Now, we will show that during a quantum measurement the information gain is actually equal to the coherent information for the system and the apparatus state

ρ'_{AS}. Using Eq.(4.2), we can write the information gain as

$$\begin{aligned} I_m &= S\left(\sum_i |s_i|^2 \rho_A^{(ii)}\right) - \sum_i |s_i|^2 S(\rho_A^{(ii)}) \\ &= S(\rho'_A) - S(\rho_A) \\ &= S(\rho'_A) - S(\rho'_{SA}) \\ &= I_c(S'\rangle A'). \end{aligned} \tag{4.4}$$

Note that using Eq.(4.1) we have $\sum_i |s_i|^2 S(\rho_A^{(ii)}) = \sum_i |s_i|^2 S(\rho_A) = S(\rho_A)$. The quantity, $I_c(S'\rangle A') = S(\rho'_A) - S(\rho'_{SA})$ is the coherent information of the final state from the system to the final state of the apparatus [56–58, 86, 95]. Hence, the extractable information is exactly equal to the amount of distinct quantum information, one may send from the system to the apparatus via measurement. Therefore, this finding suggest that the extractable information is actually of quantum origin even though it is captured by the classical correlations. Next, we will prove a trade-off relation for the information gain, the coherence of the apparatus states and the mixedness of the initial apparatus state. If we fix a basis for the apparatus, then $C_r(\rho'_A) = S(\rho'^D_A) - S(\rho'_A)$. Using Eq.(4.4), we find that

$$I_m + C_r(\rho'_A) + S(\rho_A) \leq \log N, \tag{4.5}$$

where $S(\rho_A)$ denotes mixedness of the apparatus and N is its dimension. This relation tells us that to maximize information gain the coherence in the final apparatus as well as the mixedness of the initial state of the apparatus should be as minimum as possible. This relation is stronger than the one given in Ref.[39], which reads as

$$I_m + S(\rho_A) \leq \log N, \tag{4.6}$$

i.e., the more mixedness in the apparatus state, the more difficult will be to extract the information. However, one may wonder whether $C_r(\rho'_A) \neq 0$. We note that after the interaction, we have $\rho'_{SA} = \sum_{ij} s_i s_j^* \left|s_i\right\rangle \left\langle s_j\right| \otimes \rho_A^{(ij)}$, which yields

$$\rho'_A = \sum_i |s_i|^2 \rho_A^{(ii)} = \sum_\ell a_\ell \sum_i |s_i|^2 \left|a_{i\ell}\right\rangle \left\langle a_{i\ell}\right|,$$

where $\left\langle a_{i\ell}\right\rangle a_{j\ell} = \delta_{ij}$, $\forall\ \ell$. Since, $\rho_A^{(ii)}$ are not orthogonal, $\mathrm{C}(\rho'_A)$ has non zero coherence. Alternatively, the above equation tells us that $\sum_i |s_i|^2 \left|a_{i\ell}\right\rangle \left\langle a_{i\ell}\right|$ is diagonal in different basis for different ℓ. The state, ρ'_A can be cast into a classical-quantum state where index ℓ becomes the flag, and coherence of such state can be evaluated to be nonzero in any local bases for the apparatus [96, 97]. Hence, the coherence of the state ρ'_A is not zero in a particular basis.

Now, we ask how does the initial coherence of the system govern the information gain? We will actually prove that the initial coherence of the system puts an upper bound on the information gain during the measurement. We can write the initial coherence of the system state in the basis $\{\left|s_i\right\rangle\}$ as,

$$C_r(\left|\Psi\right\rangle \left\langle\Psi\right|) = -\sum_i |s_i|^2 \log |s_i|^2,$$

which is exactly same as the maximum information possible in the measurement.

This quantity is always greater than the information gain, i.e., $C_r(|\Psi\rangle\langle\Psi|) \geq I_m$. The above inequality comes from the fact that $S(\sum_i |s_i|^2 \rho_A^{(ii)}) \leq -\sum_i |s_i|^2 \log |s_i|^2 + \sum_i |s_i|^2 S(\rho_A^{(ii)})$ [55]. Hence, we get

$$C_r(|\Psi\rangle\langle\Psi|) \geq I_m = I_c(S'\rangle A'). \tag{4.7}$$

Therefore, the extractable information is upper bounded by the coherence of the initial state of the system. The reason why the measurement process cannot extract maximum information, $(C_r(|\Psi\rangle\langle\Psi|))$, is that the apparatus is mixed. This can be understood by the complementarity relation between the information gain and mixedness of the system given in Eq.(4.6).

As the coherence of the initial systems can be better control than the entanglement in the final state ρ'_{AS}, our relation given in Eq.(4.7) may be more useful operationally than Eq.(4.3). In the limiting case of pure apparatus, the maximum information gain is equal to the initial coherence of the system and also to the entanglement developed between the system and the apparatus [39].

4.3 Coherence, Entanglement and Information Gain in presence of environment

We can generalize the above measurement scenario to a more realistic one by considering the environment also. Note that the environment here will not cause decoherence to the system rather we will use it to purify the mixed state of the apparatus, and hence, the initial joint state of the apparatus and environment can be expressed as $|\Psi_{AE}\rangle = \sum_\ell \sqrt{a_\ell}\,|a_\ell\rangle\,|e_\ell\rangle$. The transformation of the complete state is now given by

$$\sum_i s_i\,|s_i\rangle \otimes |\Psi_{AE}\rangle \longrightarrow \sum_i s_i\,|s_i\rangle\,|\Psi^i_{AE}\rangle\,,$$

where $|\Psi^i_{AE}\rangle = \sum_\ell \sqrt{a_\ell}\,|\tilde{a}_{i\ell}\rangle\,|e_\ell\rangle$. One can easily see that tracing out the environment, gives the same post-measurement state of the apparatus, which was obtained using only the mixed apparatus. Now, the final joint state of system and apparatus is obtained by the tracing out the environment, i.e.,

$$\rho_{S'A'} = \sum_{ij} s_i s_j^* (\sum_k \langle e_k|\,|\Psi^i_{AE}\rangle\,\langle\Psi^j_{AE}|\,|e_k\rangle) \otimes |s_i\rangle\,\langle s_j|\,.$$

On tracing out the environment we should get the the same post measurement joint state of system and apparatus. Also, tracing the environment should yield the final apparatus state, i.e., $\sum_k \langle e_k|\,|\Psi^i_{AE}\rangle\,\langle\Psi^j_{AE}|\,|e_k\rangle = \rho_A^{(ij)}$. Note that during the whole process, the state of the environment remains unaffected. For a better understanding of the Eq.(4.6), we can look at the initial joint state $|\Psi_{AE}\rangle$ as a bigger pure apparatus which can be used to extract the maximum information from the system. However, as we have no access to the environment, we loose out some information.

In this section, we will prove that the information gain by the apparatus can never exceed the entropy exchange between the apparatus and the system during the measurement process. Also, we can prove a new complementarity relation for the information gain and the coherence content of the final state of the system after measurement.

Note that when the system and the apparatus interact unitarily, the apparatus undergoes a noisy quantum evolution (Φ) as given by

$$\rho_A \longrightarrow \Phi(\rho_A) = \sum_\mu A_\mu \rho A_\mu^\dagger$$
$$= \mathrm{Tr}_{SE}[U(|\Psi_S\rangle\langle\Psi_S| \otimes |\Psi_{AE}\rangle\langle\Psi_{AE}|)U^\dagger] = \sum_i |a_i|^2 \rho_A^{(ii)}.$$

Now, the coherent information for the state ρ_A and channel Φ is $I_c(A\rangle E) = S(\Phi(\rho_A)) - S(\Phi \otimes \mathcal{I} \, |\Psi\rangle_{AE} \langle\Psi|) = S(\rho'_A) - S(\rho'_{AE})$. This quantity represents how much entanglement is retained by the apparatus and the environment after the apparatus interacts with the system. Since $|\Psi_{SAE}\rangle$ evolves unitarily, we have $S(\rho'_{AE}) = S(\rho'_S)$. Using the inequality $I_c(A\rangle E) \leq S(\rho_A)$, we have $S(\rho'_A) - S(\rho'_{AE}) \leq S(\rho_A)$ and hence

$$I_m = S(\rho'_A) - S(\rho_A) \leq S(\rho'_{AE}).$$

Since $S(\rho'_{AE}) = S(\rho'_S) = S_e$ is the entropy exchange between the apparatus and the system [98, 99], we have

$$I_m \leq S_e. \tag{4.8}$$

The entropy exchange S_e is intrinsic property of the apparatus and the dynamical map Φ that the apparatus undergoes. Also, this represents the entropy increase of the system state if it is initially in the pure state. Therefore, one can say that the information gain by the apparatus can never exceed the entropy exchange between the apparatus and the system during the measurement. Next, one may ask is there any trade-off relation between the information gain and the coherence of the final state of the system in case of non-ideal measurement. Using the above relation, we find that indeed they satisfy a complementarity relation which is given by

$$I_m + C_r(\rho'_S) \leq \log M, \tag{4.9}$$

where M is the dimension of the system. One may wonder whether the evolved state of the system will have non-zero coherence in the basis $\{|s_i\rangle\}$. However, we note that after the interaction, we have $\rho'_{SA} = \sum_{ij} s_i s_j^* |s_i\rangle\langle s_j| \otimes \rho_A^{(ij)}$, which yields

$$\begin{aligned}\rho'_S &= \sum_{ij} s_i s_j^* \mathrm{Tr}[\rho_A^{(ij)}] |s_i\rangle\langle s_j| \\ &= \sum_i |s_i|^2 |s_i\rangle\langle s_i| + \sum_{i\neq j} s_i s_j^* \mathrm{Tr}[\rho_A^{(ij)}] |s_i\rangle\langle s_j|,\end{aligned}$$

as $\mathrm{Tr}(\rho_A^{(ii)}) = 1$. Clearly, ρ'_S is not diagonal in $\{|s_i\rangle\}$ basis unless $\mathrm{Tr}(\rho_A^{(ij)}) = \delta_{ij}$ and therefore, has a non-zero coherence, i.e., $C_r(\rho'_S) \neq 0$.

Now, we will show the trade off relations between the entanglement, coherence, and information gain by the apparatus in the presence of environment. It was shown by Vedral [39], that the entanglement between apparatus and environment and the information from the measurement obey the following the complementarity relation

$$E_r(\rho'_{AE}) + I_m \leq S(\rho'_A), \tag{4.10}$$

where ρ'_A is the final state of the apparatus. It was argued that for extracting larger information from the system, the final entanglement between the apparatus and the environment should be less. From Eq.(4.10), one can obtain the following complementarity relation involving the final entanglement between apparatus and environment, extractable information, and final coherence of the apparatus.

$$E_r(\rho'_{AE}) + I_m + C_r(\rho'_A) \leq \log N. \tag{4.11}$$

This equation tells that for extracting larger information, we want both the final coherence of the apparatus and the final entanglement between apparatus and environment to be small. However, prima facie, this is not clear whether at the same time the final coherence of the apparatus and the final entanglement between apparatus and environment will be small. We will use a recently introduced complementarity relation between coherence and entanglement to provide better intuition in this regard.

For the bipartite system $|\Psi_{AE}\rangle$ we have $C_r(\rho_A) + E(\rho_{AE}) \leq \log N$ [73], where $E(\cdot)$ may be any bona-fide measure of entanglement. Although this relation is basis independent, but it does not properly reveal much information about the dual nature of the two resources. However, without loss of generality, if one can choose a preferred basis in which the coherence of any sub-system is maximum (cf.,

[100]) then, the relation becomes more relevant and informative. Therefore, the correct interpretation of the complementarity relation is as follows: If we have large "maximum" coherence of a subsystem then the entanglement of the bipartite system will be small and vice versa.

Note that the joint state of apparatus and environment undergoes a local operation on its subsystem(apparatus), hence the entanglement $E_r(\rho_{AE})$ can never increase. Therefore, we also have the complementarity relation of the form $C_r(\rho_A) + E_r(\rho'_{AE}) \leq \log N$. From this equation, we conclude that we should have large initial coherence of the apparatus, to keep the final entanglement between apparatus and environment small. Now, we also have a complementarity relation between final states of system and apparatus, $C_r(\rho'_A) + E_r(\rho'_{AS}) \leq \log N$. This relation tells us that the more the final entanglement between system and apparatus is, the less will be the final coherence of the apparatus.

Therefore, it is clear from the above discussion and Eq.(4.11), that to extract maximum information from a measurement process, the final entanglement between the system and apparatus should be maximum while the initial coherence of the apparatus should be as large as possible.

4.4 Disturbance in the apparatus and information gain

It was shown in Ref.[39] that the more information about a degree of freedom of a system one can gain during a measurement, the more will be the disturbance in the system. Here, we ask the opposite question: To maximize the information gain, how robust the apparatus should be? When we treat the measuring apparatus quantum mechanically, not only the apparatus disturbs the system, but also the apparatus is disturbed by the system. To answer this, we will introduce a legitimate measure of disturbance for a quantum system discussed in Ref.[4, 13, 83, 101, 102]. For a quantum state ρ_A evolving under a CPTP (Completely Positive and Trace Preserving) map Φ [103], the disturbance caused to the apparatus is given by (See Eq.(3.2))

$$D(\rho_A, \Phi) = S(\rho_A) - I_c(\rho_A, \Phi),$$

where $I_c(\rho_A, \Phi) = \Phi(\rho_A) - S(\Phi \otimes \mathbb{I}(|\Psi\rangle\langle\Psi|_{AE})$ is the coherent information and $|\Psi\rangle_{AE}$ is the purification of ρ_A such that $\rho_A = \mathrm{Tr}_E[|\Psi\rangle\langle\Psi|_{AE}]$.

Here, to quantify the disturbance to the apparatus during the measurement process in presence of environment, we use the above quantifier. For this particular scenario, the disturbance to the apparatus caused during quantum measurement is given by

$$D(\rho_A, \Phi) = S(\rho_A) - [S(\rho'_A) - S(\rho'_{AE})].$$

Noticing the fact that $I_m = S(\rho'_A) - S(\rho_A)$ and $S(\rho'_{AE}) \leq S(\rho'_A) + S(\rho_E)$, we finally have

$$D(\rho_A, \Phi) + I_m + C_r(\rho'_A) \leq 2 \log N \tag{4.12}$$

where N is the dimension of the apparatus and the environment. This relation is tight in the sense that the disturbance itself is bounded by $0 \leq D(\rho_A, \Phi) \leq 2 \log N$. The Eq.(4.12) tells a very interesting feature of the apparatus itself. For a fixed amount of coherence of the apparatus final state, in order to gain more information, apparatus should be disturbed less. This is in agreement with the intuition that for maximal information gain, apparatus should be more robust during interaction with the system.

4.5 Conclusion

Quantum measurement process plays a fundamental role in physics and continues to hurl us with new insights. We have studied the role of quantum coherence of the system and the apparatus in a quantum measurement process. We consider a measurement procedure, where the extracted information from a pure state using a mixed apparatus, is upper bounded by the initial coherence of the system. Since, we have better control over the initial coherence of the system compared to the final entanglement between apparatus and system, our result provides a realistic estimate of the extractable information. In addition, we show that, the extractable information is exactly equal to the coherent information from the final joint state of the system to the apparatus. This provides a new meaning to the information gain as the amount of the distinct quantum information that is being sent from the system to the apparatus. This finding shows us that the extractable information is rather quantum in nature than classical. We also show that the information gain by the apparatus is bounded by the entropy exchange. Further, we prove trade off relation between the information gain, disturbance, and coherence of the apparatus. To give holistic description of the measurement, we include the environment to purify the measurement apparatus and we find that to extract more information from the measurement, the apparatus should have large initial coherence and we should be able to develop maximum entanglement between the system and apparatus. The measurement procedure described here can be extended to more general measurement scenario in which the evolved apparatus states are not strictly orthogonal to each other. We hope that these findings will provide new insights to the role of coherence and coherent information in quantum measurement.

5 Mean deviation based uncertainty relations and their applications

This chapter is based on our work done in paper titled "Quantum uncertainty relation based on the mean deviation" [104].

5.1 Mean deviation based uncertainty relations

We first define the mean deviation based uncertainty for an observable with respect to a quantum state.

Definition (Mean Deviation(MD) based uncertainty) - For any physical observable $A = \sum_a a \left|a\right\rangle \left\langle a\right|$, the mean deviation based uncertainty of an observable A on the state $|\Psi\rangle$ is defined as

$$\Delta_M A = \sum_a |a - \langle A\rangle| \, |\langle\Psi| \, |a\rangle \, |^2$$
$$= \langle\Psi| \, A'^2 \, |\Psi\rangle \, . \tag{5.1}$$

Note that $\Delta_M A$ is always non-negative and vanishes only when $|\Psi\rangle$ is an eigenstate of the observable A. Let us define a positive, Hermitian operator $A' = \sum_a \sqrt{|a - \langle a\rangle\,|} \, |a\rangle \langle a|$, and hence $A'^2 = \sum_a |a - \langle A\rangle| \, |a\rangle \langle a|$. Similarly for the operator $B = \sum_b b \, |b\rangle \langle b|$, we define B' and write down the uncertainty of B on the state $|\Psi\rangle$ as

$$\Delta_M B = \sum_b |b - \langle B\rangle| \, |\langle\Psi| \, |b\rangle \, |^2$$
$$= \langle\Psi| \, B'^2 \, |\Psi\rangle \, . \tag{5.2}$$

Let us now define two vectors $|\Psi_1\rangle = A' \, |\Psi\rangle$, and $|\Psi_2\rangle = B' \, |\Psi\rangle$, then we have $||\Psi_1||^2 = \langle\Psi|A'^2|\Psi\rangle = \Delta_M A$ and $||\Psi_2||^2 = \langle\Psi|B'^2|\Psi\rangle = \Delta_M B$. Now, the product of $\Delta_M A$ and $\Delta_M B$ on the state $|\Psi\rangle$ respects the following inequality

$$\Delta_M A \Delta_M B \geq \frac{1}{4} | \, \langle\Psi| \, [A', B'] \, |\Psi\rangle \, |^2. \tag{5.3}$$

Here the inequality follows from the Cauchy-Schwarz inequality for two unnormalized vectors $|\Psi_1\rangle$ and $|\Psi_2\rangle$. This is the Robertson form of mean deviation uncertainty relation for products of uncertainties. It is also possible to cast the uncertainty relation in sum form instead of the product form above. For incompatible observers, the triviality of the Standard Deviation(SD) based uncertainty relations was removed rather recently [46]. Below we present a similar uncertainty relation in terms of the mean deviation.

(MD based uncertainty relation for incompatible observables) Theorem : For observables A and B, system state $|\Psi\rangle$ and any state $|\Psi^{\perp}\rangle$ orthogonal to the system state, the following uncertainty relation holds -

$$\Delta_M A + \Delta_M B \geq \pm i \langle [A', B'] \rangle + | \langle \Psi | A' \pm i B' | \Psi^{\perp} \rangle |^2. \tag{5.4}$$

Here we choose the sign outside the commutator in such a way that the first term in the RHS remains positive.

Proof- We write $\Delta_M A = || A' |\Psi\rangle ||$ and $\Delta_M B = || iB' |\Psi\rangle ||$ to obtain

$$||(A' \mp iB')\Psi||^2 = \Delta_M A + \Delta_M B \mp i \langle [A', B'] \rangle .$$

Now the LHS of this expression can be lower bounded using the Cauchy-Schwartz inequality as $||(A' \mp iB')\Psi||^2 \geq | \langle \Psi | A' \pm iB' | \Psi^{\perp} \rangle |^2$, for every $|\Psi^{\perp}\rangle$ orthogonal to $|\Psi\rangle$. This completes the proof. □

Similar to the Robertson-Schrödinger uncertainty relation, the MD based uncertainty relation given in (5.3) can be trivial when $|\Psi\rangle$ is an eigenstate of either A or B. However, the lower bound in (5.4) is non-trivial for every $|\Psi^{\perp}\rangle$, barring the case when $|\Psi\rangle$ is a common eigenstate of both A' and B'.

Case for generalized deviation measure

The obvious generalization of the mean deviation based uncertainty relation derived above would be to consider the situation for arbitrary exponent α, which would subsume the mean deviation based uncertainty measure and the usual variance based uncertainty measure as special cases, viz., when $\alpha = 1$ or $\alpha = 2$ respectively. More concretely, similar to Eq.(5.1), we seek to define generalized deviations as

$$\begin{aligned}\Delta_M^\alpha A &= \sum_a |\langle\Psi|\,|a\rangle|^2 |a - \langle A\rangle|^\alpha \\ &= \langle\Psi| {A'_\alpha}^2 |\Psi\rangle . \end{aligned} \tag{5.5}$$

where $\{a\}$ is the set of eigenvalues of the observable A and $\{|a\rangle\}$ are the corresponding eigenvectors. As done earlier let us define a positive semi-definite operator $A'_\alpha = \sum_a \sqrt{|a - \langle a\rangle|^\alpha}\,|a\rangle\langle a|$. Hence, ${A'_\alpha}^2 = \sum_a |a - \langle A\rangle|^\alpha |a\rangle\langle a|$. For the operator $B = \sum_b b\,|b\rangle\langle b|$, we similarly define B'_α and write down the uncertainty of B on the state $|\Psi\rangle$ as

$$\begin{aligned}\Delta_M B &= \sum_b |b - \langle B\rangle|^\alpha |\langle\Psi|\,|b\rangle|^2 \\ &= \langle\Psi| {B'_\alpha}^2 |\Psi\rangle . \end{aligned} \tag{5.6}$$

The resulting product and sum uncertainty relations are now expressed as

$$\Delta_M^\alpha A \Delta_M^\alpha B \geq \frac{1}{4} |\langle\Psi| [A'_\alpha, B'_\alpha] |\Psi\rangle|^2, \tag{5.7}$$

and

$$\Delta_M^\alpha A + \Delta_M^\alpha B \geq \pm i \langle [A'_\alpha, B'_\alpha] \rangle + | \langle \Psi | A'_\alpha \pm i B'_\alpha | \Psi^\perp \rangle |^2, \tag{5.8}$$

respectively.

5.1.1 Intelligent states

It is natural to wonder which quantum states are the most 'classical' in the sense of incurring the least amount of uncertainty for incompatible observables. A canonical example is that of the coherent states for a quantum harmonic oscillator [45]. These states have been given the moniker of 'intelligent' states in the literature and studied for the SD based uncertainty relations [105]. It is well known that a Gaussian wave-function saturates the uncertainty bound of standard deviation based uncertainty relation. Specifically, for the position and momentum operators the lower bound is given by

$$\Delta X \Delta P = \frac{\hbar}{2}.$$

The SD for the position observable is defined as $\Delta X = \sqrt{\langle \Psi | X^2 | \Psi \rangle - (\langle \Psi | X | \Psi \rangle)^2}$ and SD of momentum is defined in a similar way. However, if we move away from the SD based approach, the situation is less clear. For median based uncertainty relations, it was numerically shown [106] that the wave function corresponding to the Cauchy probability distribution is, in fact, more 'intelligent' than the Gaussian wave function, which is not reflected in the SD based uncertainty relations, owing to the fact that the SD (or indeed even the mean) does not exist in general for the Cauchy type probability distribution. One can easily calculate the product of mean deviation uncertainties in position and momentum for the Gaussian wavefunction and the product is given by ($\hbar = 1$)

$$\Delta_M X \Delta_M P = \frac{1}{\pi}.$$

The only difference in this case as compared to the SD is the factor π in the denominator.

In the quest for finding intelligent states in the MD case, let us now digress a bit. The differential entropy was introduced by Shannon himself in a bid to generalize the Shannon entropy for continuous settings. It is defined as follows-

Definition (differential entropy) - If X is a random variable with a probability density function p whose support is the set $\mathbb{X}$, then the differential entropy $H(X)$ is defined as

$$H(X) = -\int_{\mathbb{X}} p(x) \ln p(x)\ dx. \tag{5.9}$$

It can be shown that the probability distribution that maximizes the differential entropy given a fixed SD is a Gaussian. We now prove the analog of this result for the MD case.

(Probability density function which maximizes differential entropy for a fixed value of mean deviation) Theorem:- The Laplace distribution maximizes the differential entropy if the mean deviation is fixed (say μ) and the mean is set to zero.

Proof. We have to maximize $H(x)$ subject to the following constraints

1. $\mu = \int p(x)|x|dx$,
2. $\int p(x)dx = 1$.

Now introducing the Lagrange multipliers λ and γ, the functional derivative of the following quantity $\int[-p(x)\ln p(x) + \lambda|x|p(x) + \gamma p(x)]dx$ must vanish for maximization, which immediately leads to the result

$$-1 - \ln p(x) + \lambda|x| + \gamma = 0. \tag{5.10}$$

Utilizing the constraints to eliminate the Lagrange multipliers, we end up with the following Laplace probability density function, i.e.,

$$p(x) = \frac{1}{2\mu}\exp\left(-\frac{|x|}{\mu}\right).$$

□

This constrained optimization procedure bears a strong resemblance to the way one singles out the Gibbs distribution by fixing the average energy and maximizing the von Neumann entropy. Here we begin by fixing the mean deviation, which is a measure of dispersion, unlike the average energy. However, the above result immediately spawns the question - are the wave functions giving rise to the Laplace probability distribution as 'intelligent' as the Gaussian wave function as far as the

MD based uncertainty relation is concerned ? We answer this question in the affirmative through the following proposition.

(States as intelligent as Gaussian states in the context of the MD uncertainty relation) Proposition : States with wave function generating the Laplace probability distribution are as intelligent as Gaussian states in the context of the MD uncertainty relation.

Proof- We assume the following position space wave function ($\hbar = 1$)

$$\psi(x) = \frac{1}{\sqrt{2\mu}} \exp\left(-\frac{|x|}{2\mu}\right). \tag{5.11}$$

The mean deviation corresponding to the above wavefunction is given by $\Delta_M X = \mu$. The momentum space wave function, i.e., the Fourier transform of the position space wave function is a Cauchy distribution

$$\tilde{\psi}(p) = \frac{2\sqrt{\mu}}{\sqrt{\pi}(1+4\mu^2p^2)}. \tag{5.12}$$

The mean deviation for momentum is, therefore, given by $\Delta_M P = \frac{1}{\pi\mu}$. Hence, the product of MD based uncertainties in position and momentum reads as

$$\Delta_M X \Delta_M P = \frac{1}{\pi}. \tag{5.13}$$

This is exactly the expression for Gaussian wave function given earlier, thus completing our proof. $\square$

We note in passing that the above wave function arises naturally as a solution of the

Schrödinger equation for the one-dimensional Dirac delta potential [107]. However, the corresponding state is not as 'intelligent' as the Gaussian state for the product of SD based uncertainties as it satisfies $\Delta X \Delta P = \frac{1}{\sqrt{2}}$, whereas in the Gaussian case, $\Delta X \Delta P = \frac{1}{2}$. One is tempted to ask the question - are these the most intelligent states in the MD scenario ? The answer to this question is quite tricky, as has been pointed out in an online forum [108] by Frederic Grosshans. He showed, if one uses the entropic uncertainty relation for position and momentum in conjunction with the fact that the differential entropy is related to the mean deviation, one can get a bound on the product of the MD uncertainties, which is somewhat lower than $\frac{1}{\pi}$. This bound is saturated in the case that both the position space and momentum space wave functions generate Laplace probability distributions - which is not possible since the Fourier transform of a Laplace distribution is not another Laplace distribution. Thus, we leave the problem of finding more intelligent states than the ones discussed in this section for future work.

5.1.2 State independent MD based uncertainty relation

One of the nice features of entropic uncertainty relations is the fact that they are state independent and consequently bring out the incompatibility of pairs of observables without having to worry about states for which the uncertainty relation becomes trivial. It is thus a natural question to ask whether we can have state independent uncertainty relations for other measures of uncertainty. For SD, this was addressed by Huang [109]. In this subsection, we provide a state independent MD based uncertainty relation for the sum of an arbitrary number of observables.

Setting- Suppose we construct m number of bases $\{\mathcal{B}_i\}_{i=1...m}$ for an n-dimensional Hilbert space. Now let $|a_i^j\rangle$ be the j-th basis element for the i-th basis. We now assume m Hermitian operators of the form $\mathcal{O}_i = \sum_j a_i^j |a_i^j\rangle\langle a_i^j|$. We consider the probabilities corresponding to the measurement outcome of observables as $p_i^j = |\langle a_i^j|\Psi\rangle|^2$, where $|\Psi\rangle$ is the corresponding state. The aim is to provide a state independent lower bound for the sum of the MD based uncertainties of these observables. To this end, we formulate the following uncertainty relation

(state independent MD based uncertainty relation) Theorem : The following MD based sum uncertainty relation holds for multiple observables $\{\mathcal{O}_i\}$ and any $\alpha \in \mathbb{R}$

$$\alpha \sum_i \Delta_M(\mathcal{O}_i) \geq C - \sum_i \ln \max_{\min_k a_i^k \leq \beta_i \leq \max_k a_i^k} \sum_j e^{-\alpha|a_i^j - \beta_i|}. \tag{5.14}$$

Here, C in the lower bound is the logarithm of the maximum overlap [110] between operator spectra, and consequently is state independent. To prove this result, we first consider the following lemma.

Lemma - For $\alpha \in \mathbb{R}$,

$$\alpha\Delta_M(\mathcal{O}_i) \geq H(\mathcal{O}_i) - \ln \sum_j e^{-\alpha|a_i^j - \langle \mathcal{O}_i\rangle|}. \tag{5.15}$$

where $H(\mathcal{O}_i) = -\sum_j p_i^j \log_2 p_i^j$ is the Shannon entropy of the observable $\mathcal{O}_i$and $\langle \mathcal{O}_i\rangle$ is its mean.

Proof- Using the inequality $e^x \geq 1 + x$ in conjunction with $x = -\alpha|a_i^k - \langle \mathcal{O}_i\rangle| -$

$\ln p_i^k \sum_j e^{-\alpha|a_i^j - \langle \mathcal{O}_i \rangle|}$, we find that

$$\begin{aligned}
1 &= \sum_k p_i^k \frac{e^{-\alpha|a_i^k - \langle \mathcal{O}_i \rangle|}}{p_i^k \sum_j e^{-\alpha|a_i^j - \langle \mathcal{O}_i \rangle|}}, \\
&\geq \sum_k p_i^k (1 - \alpha|a_i^k - \langle \mathcal{O}_i \rangle| - \ln p_i^k \sum_j e^{-\alpha|a_i^j - \langle \mathcal{O}_i \rangle|}), \\
&= \sum_k p_i^k - \alpha \sum_k p_i^k |a_i^k - \langle \mathcal{O}_i \rangle| - \sum_k p_i^k \ln p_i^k \\
&\quad - \sum_k p_i^k \ln \sum_j e^{-\alpha|a_i^j - \langle \mathcal{O}_i \rangle|}, \\
&= 1 - \alpha \Delta_M(\mathcal{O}_i) + H(\mathcal{O}_i) - \ln \sum_j e^{-\alpha|a_i^j - \langle \mathcal{O}_i \rangle|}.
\end{aligned} \tag{5.16}$$

This completes the proof of the lemma. □

Now summing over i in (5.15), we get the state dependent MD based sum uncertainty relation.

$$\begin{aligned}
\alpha \sum_i \Delta_M(\mathcal{O}_i) &\geq \sum_i H(\mathcal{O}_i) - \sum_i \ln \sum_j e^{-\alpha|a_i^j - \langle \mathcal{O}_i \rangle|}, \\
&= C - \sum_i \ln \sum_j e^{-\alpha|a_i^j - \langle \mathcal{O}_i \rangle|}.
\end{aligned} \tag{5.17}$$

Now, noticing that $\min_k a_i^k \leq \langle \mathcal{O}_i \rangle \leq \max_k a_i^k$, we find

$$\alpha \sum_i \Delta_M(\mathcal{O}_i) \geq C - \sum_i \ln \max_{\min_k a_i^k \leq \beta_i \leq \max_k a_i^k} \sum_j e^{-\alpha|a_i^j - \beta_i|}. \tag{5.18}$$

This accomplishes the goal of finding a MD based state independent uncertainty relation for multiple observables.

5.2 New Uncertainty Relation: Examples

The constraints placed on quantum wave functions are less severe than restricting the possible solutions of the Schrödinger equation only to functions for which the standard deviation does not blow up at any point. This gives rise to perfectly licit solutions of the Schrödinger equation, for which the standard deviation based uncertainty relations are inapplicable. One way of dealing with this problem is to resort to the *semi interquartile range (SIQR)* as a measure of spread [106]. However, analytical expressions for SIQR of arbitrary distributions are notoriously hard to calculate. As we argue below, the MD based uncertainty relation (5.3) derived above is algebraically more tractable and as such, an excellent candidate to fill this lacuna. These relations only demand that the mean be well-defined, which is a less stringent condition than requiring an well-behaved standard deviation. In this section, we illustrate two different scenarios where, in some regimes, the SD based uncertainty relations are inapplicable, but the new uncertainty relations hold true. We illustrate this using two examples, one being the F-distribution, the other being the Pareto distribution.

5.2.1 F-distribution

Let us now consider the probability distribution function known as F-distribution, whose expression is given by

$$f(x;d_1,d_2) = \frac{1}{\beta(\frac{d_1}{2},\frac{d_2}{2})}\left(\frac{d_1}{d_2}\right)^{\frac{d_1}{2}} x^{\frac{d_1}{2}-1}\left(1+\frac{d_1}{d_2}x\right)^{-\frac{d_1+d_2}{2}}. \tag{5.19}$$

where $x \geq 0$, $\beta(a,b)$ being the two-parameter beta function family and the parameters d_1 and d_2 being positive integers. The mean, given by $\frac{d_2}{d_2-2}$, exists for all $d_2 > 2$. The standard deviation $\sigma = \sqrt{\frac{2{d_2}^2(d_1+d_2-2)}{d_1(d_2-2)^2(d_2-4)}}$, exists however only for $d_2 > 4$. Thus, we note that the standard deviation for this distribution does not exist for $d_2 = \{3,4\}$ even though the mean exists.

It can be shown that this distribution arises as a solution to the Schrödinger equation for the following form of potential (V_0 being a constant energy shift parameter).

$$\begin{aligned} V(x) = & V_0 - \frac{\hbar^2}{32m}[(2d_1-2)(2d_1-6)x^{-2} - 2\frac{d_1}{d_2}(2d_1-2)(d_1+d_2)x^{-1}\left(1+\frac{d_1}{d_2}x\right)^{-1} \\ & + \frac{{d_1}^2}{d_2}(d_1+d_2)(d_1+d_2+4)\left(1+\frac{d_1}{d_2}x\right)^{-2}]. \end{aligned} \tag{5.20}$$

It is clear that in the regime $d_2 \in (2,4]$, the SD-based uncertainty relation is meaningless. However, the mean deviation for F-distribution is perfectly defined in that regime(see Fig.5.1) and is, in general, for $d_2 > 2$, given by

$$\Delta_M X = \frac{1}{\mathcal{N}}[2\left(\frac{d_2}{d_2-2}\right)^{\frac{2+d_1}{2}}\left(\frac{{}_2F_1(\frac{d_1}{2},\frac{d_1+d_2}{2},\frac{d_1+2}{2},-\frac{d_1}{d_2-2})}{d_1}-\frac{{}_2F_1(\frac{d_1+2}{2},\frac{d_1+d_2}{2},\frac{d_1+4}{2},-\frac{d_1}{d_2-2})}{d_1+2}\right)$$
$$+\frac{2\left(\frac{d_2-2}{d_1}\right)^{\frac{d_2}{2}}\left(\frac{d_1}{d_2}\right)^{\frac{-d_1}{2}}\left((d_2)_2F_1(\frac{d_2-2}{2},\frac{d_1+d_2}{2},\frac{d_2}{2},\frac{2-d_2}{d_1})-(d_2-2)_2F_1(\frac{d_2}{2},\frac{d_1+d_2}{2},\frac{d_1+2}{2},\frac{2-d_2}{d_1})\right)}{(d_2-2)^2}].$$

where $\mathcal{N} = \frac{1}{B(\frac{d_1}{2},\frac{d_2}{2})^{\frac{1}{2}}}\left(\frac{d_1}{d_2}\right)^{\frac{-d_1}{4}}$ and ${}_2F_1(a,b,c,z)$ is the hyper-geometric function. We acknowledge that this is not a potential that one comes across very often in literature. However, this is a possible physical potential, and may turn out to be relevant for future works.

5.2.2 Pareto distribution

As another example of a physical situation where the mean deviation based uncertainty relation is meaningful in contradistinction with SD based uncertainty relations, let us assume a solution of the Schrödinger equation of the form

$$\psi(x) = \begin{cases} f(x), & \text{if } x \geq \lambda \\ \phi(x), & \text{otherwise} \end{cases} \tag{5.21}$$

where $f(x)$ arises from the Pareto distribution and is defined as

$$f(x) = \sqrt{p}\sqrt{\frac{\alpha\lambda^{\alpha}}{x^{\alpha+1}}}. \tag{5.22}$$

where $p \in (0,1)$ and $\lambda \geq 0$. In order to ensure the continuity of the wave function,

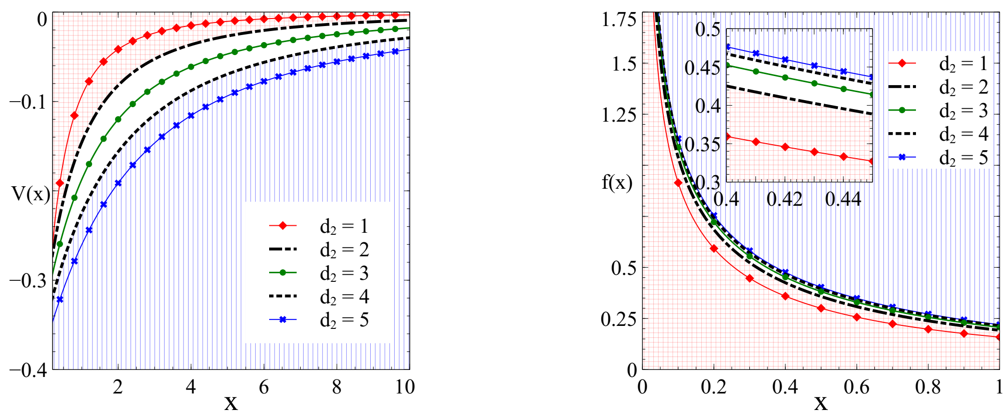

Figure 5.1: Graphical depiction of the efficacy of MD based uncertainty relations. The figure on left is the plot of potential $V(x)$ as a function of position x, while the right hand side figure plots the probability distribution function. The blue striped zone, with an example depicted by the blue line corresponding to $d_2 = 5$, is where both SD based uncertainty relations and MD based uncertainty relations are applicable. The white zone, with an example depicted by the green line corresponding to $d_2 = 3$, is where the MD based uncertainty relations apply but SD based ones do not. The red squared zone, with an example furnished by the red line corresponding to $d_2 = 1$, is where both the MD and SD based uncertainty relations fail to apply. The lines corresponding to $d_2 = 2$ (dot-dashed) and $d_2 = 4$ (dashed) set the boundaries between these zones. We set $d_1 = 1$ throughout.

the constraints on $\phi(x)$ are given by

- i) $\phi(\lambda) = \sqrt{\frac{\alpha}{2\lambda}}$,
- ii) $\phi'(\lambda) = -\sqrt{\frac{\alpha}{2\lambda}}\frac{\alpha+1}{2\lambda}$,
- iii) $\int_{-\infty}^{\lambda} |\phi(x)|^2 dx = 1 - p$.

Obviously, one can construct families of functions satisfying these properties. For each such function, the corresponding physical potential can be found. Now, it is easy to see that the fluctuation in position for this wave function $\psi(x)$ blows up, yet the mean is well defined in the regime $\alpha \in (1,2]$ - thus the standard deviation based uncertainty relation becomes meaningless in this regime, yet the mean deviation based uncertainty relations are perfectly meaningful even in this scenario. Mean position $\beta = \langle X \rangle$ can now be easily seen to be finite for legitimate wave functions $\phi(x)$. Therefore, $\Delta_M X = \int_{-\infty}^{\infty} |\psi(x)|^2 |x - \langle X \rangle| dx = \int_{-\infty}^{\lambda} |\phi(x)|^2 |x - \beta| dx + \int_{\lambda}^{\infty} |f(x)|^2 |x - \beta| dx$. Now, both the terms are finite, therefore the mean deviation is finite in this case.

5.3 Some applications of the mean-deviation based uncertainty relations

Apart from being one of the cornerstones of quantum theory, uncertainty relations can be applied to provide new insights on future quantum technologies. Uncertainty relations have successfully been utilized, among other applications, as (non-linear) entanglement witnesses [111, 112], in determining the speed limit of evolution of quantum states [113–115], in determining the purity of states [116], detecting EPR steering [117] and in determining the degree of non-locality of proposed physical theories through retrieval games [48]. Till now, most of the tasks mentioned above have been performed using either the SD-based form or the entropic form of the uncertainty relations. Thus, it is natural to wonder how our mean deviation based uncertainty relations fare over standard deviation based uncertainty relations in these applications. In the present work, we seek to provide an answer in two such situations. First, we consider the problem of detecting EPR-steering. Finally, we analyze the utility of mean deviation based uncertainty relations in entanglement detection.

5.3.1 Detection of EPR violation

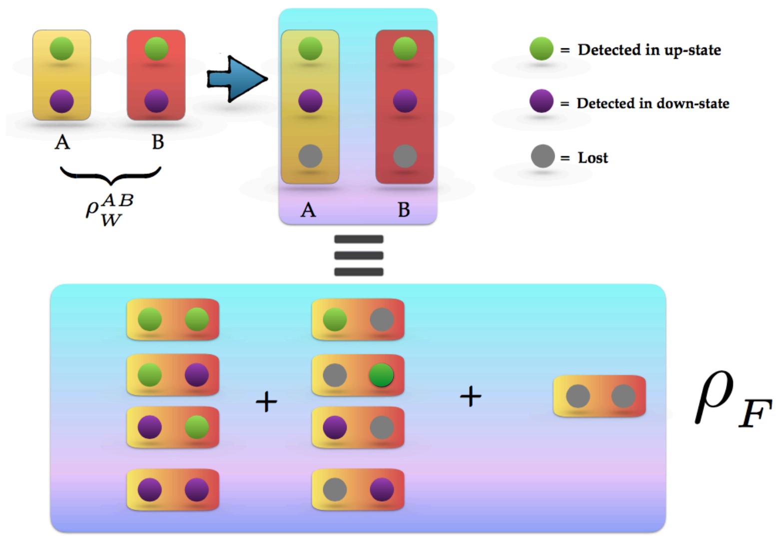

Figure 5.2: A schematic diagram for the lossy detector scenario. The two-qubit initial Werner state is written in terms of a 9×9 density matrix ρ_F taking into account particle loss before measurement.

One of the oldest philosophical objections to the quantum theory is the EPR argument [118, 119] of local realism. Typically, constraining the theory to satisfy local realism at one or more subsystems results in certain inequalities [119, 120], the violation of which for some quantum state implies the untenability of the EPR assumption. Here we consider the MD based local uncertainty relations for a subsystem A of a bipartite state $|\Psi\rangle_{AB}$. For a set of observables $\{\mathcal{O}_i\}$, we define inferred mean deviations as $\Delta_{M_{\inf}}\mathcal{O}_i = \sum_{\mathcal{O}^B} P(\mathcal{O}^B)\Delta_M(\mathcal{O}_i|\mathcal{O}^B)$ and inferred mean $|\langle C\rangle_{\inf}| = \sum_{\mathcal{O}^B} P(\mathcal{O}^B)|\langle C|\mathcal{O}^B\rangle|$ and apply them for product uncertainty relations of the form $\Delta_M\mathcal{O}_1\Delta_M\mathcal{O}_2 \geq \frac{1}{4}|\langle\Psi_A|[\mathcal{O}_1', \mathcal{O}_2']|\Psi_A\rangle|^2 = |\langle C\rangle|^2$ and corresponding sum uncertainty relations in the Robertson-like form.

Theorem (EPR violation)- If we replace the mean deviations by the inferred mean deviations $\Delta_{M_{\inf}}\mathcal{O}_i$, and the mean $|\langle C\rangle|$ by the inferred mean $|\langle C\rangle_{\inf}|$ in an uncertainty relation of the form $\Delta_M\mathcal{O}_1\Delta_M\mathcal{O}_2 \geq \frac{1}{4}|\langle\Psi|[\mathcal{O}_1', \mathcal{O}_2']|\Psi\rangle|^2 = |\langle C\rangle|^2$ - the violation of the resulting inequality is a manifestation of the EPR paradox.

Proof - To demonstrate the EPR phenomena we assume using the local realism argument that there exists an element of reality $\lambda_{\mathcal{O}_i}$ which probabilistically predetermines the result for the measurement of the observable $\mathcal{O}_i$ performed at A. For two different elements of reality we can have a joint probability distribution of the form $P(\lambda_1, \lambda_2) = P(\lambda)$. Now for the product of two mean deviations

$$\begin{aligned}
\Delta_{M_{\inf}}\mathcal{O}_1\Delta_{M_{\inf}}\mathcal{O}_2 &= \sum_{\mathcal{O}_1^B} P(\mathcal{O}_1^B)\Delta_M(\mathcal{O}_1|\mathcal{O}_1^B)\sum_{\mathcal{O}_2^B} P(\mathcal{O}_2^B)\Delta_M(\mathcal{O}_2|\mathcal{O}_2^B) \\
&\geq \sum_{\lambda} P(\lambda)\Delta_M(\mathcal{O}_1|\lambda)\Delta_M(\mathcal{O}_2|\lambda) \\
&\geq \sum_{\lambda} P(\lambda)|\langle C|\lambda\rangle|^2 \\
&\geq |\langle C\rangle_{\inf}|^2.
\end{aligned}$$

The EPR inequality for sum of mean deviations can be proved in a similar way as

$$\Delta_{M_{inf}}\mathcal{O}_1 + \Delta_{M_{inf}}\mathcal{O}_2 = \sum_i \sum_{\mathcal{O}_i^B} P(\mathcal{O}_i^B)\Delta_M(\mathcal{O}_i|\mathcal{O}_i^B)$$
$$= \sum_\lambda [\Delta_M(\mathcal{O}_1|\lambda) + \Delta_M(\mathcal{O}_2|\lambda)] \geq 2|\langle D\rangle_{\text{inf}}|. \tag{5.23}$$

Here $\langle D\rangle$ corresponds to the first term in the rhs of the sum uncertainty relation (5.4). This completes the proof of the EPR inequality in terms of (inferred) mean deviation. □

In the next part of this subsection, we concentrate on applying the above result in an experimental setup.

Detection of EPR violation in lossy scenario through MD uncertainty relation

Armed with the inequality (5.23) above, we now consider an experimental scenario for observing EPR violation upon measurement on one of the parties in a two-qubit setting. Specifically, we work with the set-up proposed in Ref. [121], which we outline below for the sake of completeness. See Fig. 5.2 for an illustration.

Scenario- Consider two spatially separated particles at locations A and B respectively, each of which can either be in spin-up (+1) or in spin-down (-1) configuration. Now assume, they share a singlet state $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|1\rangle_A |{-1}\rangle_B - |{-1}\rangle_A |1\rangle_B)$ along with a white noise. Their shared state is thus described in general by the two qubit Werner family of states $\rho_W^{AB} = p\,|\Psi\rangle_{AB}\langle\Psi| + \frac{1-p}{4}\mathbb{I}_4$. Now, let us assume that a detector observes the spin of the each of the particles. However, this detector is inefficient in the sense that sometimes it may fail to conclusively detect a particle in either spin-up or spin-down configuration due to loss of that particle before measurement. This is parametrized by introducing an overall detection efficiency η. The detection space for each of the spins is now that of a qutrit with possible outcomes being

- spin-up (+1),
- spin-down (-1),
- lost particle (0).

The full bipartite density matrix can now be represented as $\rho_W \rho_{vac}$, where $\rho_{vac} = |0\rangle \langle 0|$ is the multi mode vacuum state in which the undetected particles are collected. We follow the Schwinger representation to write $|1\rangle$ as $|1,0\rangle$ and $|-1\rangle$ as $|0,1\rangle$, where the $\{i,j\}$ in $|i,j\rangle$ denote the number of particles in spin-up and spin-down configurations respectively. The creation operators at sites A and B are $a_{\pm}^{\dagger}$ and $b_{\pm}^{\dagger}$ respectively, with '+' for statistics of spin-up and '-' for statistics of spin-down particles. Likewise, we denote the creation operators for the vacuum state at sites A and B as $a_{\pm,vac}^{\dagger}$ and $b_{\pm,vac}^{\dagger}$ respectively. The detection mechanism is described as follows. The particles are led through a beam splitter which couples the field and vacuum modes, after which the modes are transformed as $a_{\pm} \rightarrow \sqrt{\eta} a_{\pm} + \sqrt{1-\eta} a_{\pm,vac}$ and $b_{\pm} \rightarrow \sqrt{\eta} b_{\pm} + \sqrt{1-\eta} b_{\pm,vac}$. The final two-qutrit density matrix ρ_F is now derived by tracing over the lost photon modes. There are a total of nine basis states

- $|u_{1-4}\rangle = |\pm 1\rangle_A |\pm 1\rangle_B$;
- $|u_{5,6}\rangle = |\pm 1\rangle_A |0\rangle_B$;

- $|u_{7,8}\rangle = |0\rangle_A |\pm1\rangle_B$ and,

- $|u_9\rangle = |0\rangle_A |0\rangle_B$.

The final form of the density matrix ρ_F is now given in block-diagonal form by

$$\rho_F = \begin{bmatrix} \eta^2\rho_W & 0 & 0 \\ 0 & \frac{\eta}{2}(1-\eta)\mathbb{I}_4 & 0 \\ 0 & 0 & (1-\eta)^2 \end{bmatrix}.$$

To find the condition for detecting EPR violation, we use the MD based sum uncertainty relation for the spin operators. Working in the Schwinger representation, we express the spin operators in terms of the particle creation and destruction operators. The spin operators at location A are $J_x^A = (a_+^\dagger a_- + a_-^\dagger a_+)/2$, $J_y^A = i(a_-^\dagger a_+ - a_+^\dagger a_-)/2$, $J_z^A = (a_+^\dagger a_+ - a_-^\dagger a_-)/2$, and the number operator $N^A = a_+^\dagger a_+ + a_-^\dagger a_-$. Similarly at location B, the operators J_x^B, J_y^B, J_z^B and N^B are defined using $b_\pm$. For detection of at most a single particle per mode, it can be shown that for a measurement at A, the following inequality holds.

$$\Delta_M J_x^A + \Delta_M J_y^A + \Delta_M J_z^A \geq \frac{3}{2}\langle N^A\rangle - \frac{\langle N^A\rangle^2}{2}. \tag{5.24}$$

To prove the above inequality we have used the inequality $\langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2 \leq \frac{\langle N\rangle^2}{4}$. The EPR paradox is manifested if the above inequality is violated for inferred mean

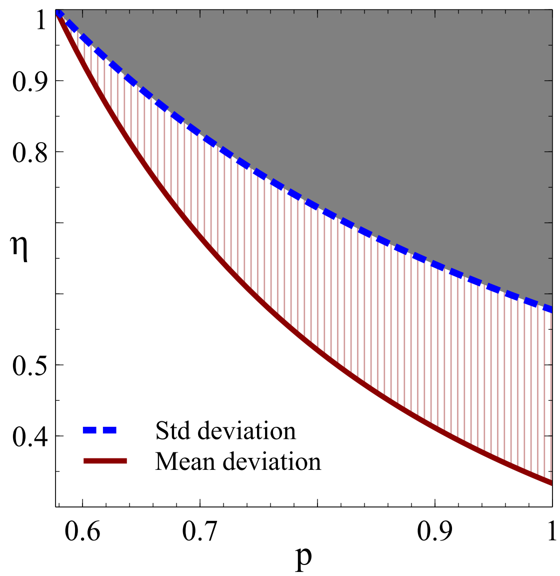

Figure 5.3: Plots of MD and SD based uncertainty relation in detecting EPR violation. SD based uncertainty relations cannot detect EPR violation if the detector efficiency η plotted along y-axis (with respect to the Werner noise parameter p) is below the dashed blue curve. However, up to the limit of the solid maroon curve, i.e., in the maroon striped zone, the MD based uncertainty relation can still detect such EPR violation.

deviations, i.e.,

$$\Delta_{M_{\mathrm{inf}}} J_x^A + \Delta_{M_{\mathrm{inf}}} J_y^A + \Delta_{M_{\mathrm{inf}}} J_z^A < \frac{3}{2} \langle N^A \rangle - \frac{\langle N^A \rangle^2}{2} \tag{5.25}$$

The inferred mean deviations $\Delta_{M_{\mathrm{inf}}} J_x^A, \Delta_{M_{\mathrm{inf}}} J_y^A$ and $\Delta_{M_{\mathrm{inf}}} J_z^A$ are the average errors corresponding to the elements of reality that exist for J_x^A, J_y^A and J_z^A respectively. For the loss-included Werner state ρ_F, we compute the inferred mean deviations $\Delta_{M_{\mathrm{inf}}} J_x^A = \Delta_{M_{\mathrm{inf}}} J_y^A = \Delta_{M_{\mathrm{inf}}} J_z^A = \frac{\eta}{2}(1 - \eta^2 p^2)$ and $\langle N^A \rangle = \eta$. Putting these values in (5.25), we get $\eta p^2 > \frac{1}{3}$. We compare this result with the corresponding result obtained in [121] utilizing the SD based uncertainty relation in Fig. 5.3

to note that for the same value of noise parameter p, the MD based uncertainty can detect EPR violation with a less efficient detector than the SD based uncertainty. However, for maximum detector efficiency, i.e., for $\eta = 1$, we can detect steerability of werner states for $p > \frac{1}{\sqrt{3}}$, which is exactly the same bound derived using standard deviation uncertainty [122].

Is the mean deviation optimal for detection of EPR violation ? -We ask at this point, whether any other measure of deviation defined in the same way as the standard deviation or mean deviation, may allow us to detect EPR violation with detectors with even less efficiency. Perhaps, the reader may wonder, it is even possible to detect EPR violation with an extremely inefficient detector, so long as the deviation measure is carefully chosen. In this subsection, we show that such optimism is not correct and mean deviation based bound for EPR violation can not be bettered through choosing a suitable exponent for the measure of deviation, when that exponent is less than unity.
For an arbitrary $\alpha \leq 1$, the generalized α-deviation is defined as

$$\Delta_M^\alpha J = \sum_a |\langle\Psi|\,|a\rangle\,|^2 |a - \langle J\rangle\,|^\alpha. \tag{5.26}$$

where a are the eigenvalues of the observable J and $|a\rangle$ are the corresponding eigenvectors.

When we use $\Delta_M^\alpha J$ as the measure of uncertainty for the spin operators they satisfy the following uncertainty relation.

$$\sum_{i=x,y,z} \Delta_M^{\alpha} J_i \geq \frac{3\eta}{2^{\alpha}} + \sum_{m=1}^{\infty} \frac{\eta^{2m}\alpha(\alpha-1)...(\alpha-2m-2)}{2^{\alpha}(2m-1)!} \left[\frac{\eta(\alpha-2m-1)}{2m} - 1\right]. \tag{5.27}$$

Now if the inferred generalized mean deviations violate the inequality (5.27), we say that the given state exhibits EPR violation. The sum of inferred generalized mean deviations are now given as following

$$\sum_{i=x,y,z} \Delta_{M_{inf}}^{\alpha} J_i = \frac{3\eta}{2^{\alpha}} + \sum_{m=1}^{\infty} \frac{3\eta^{2m+1}p^{2m}\alpha(\alpha-1)...(\alpha-2m-2)}{2^{\alpha}(2m-1)!} \left[\frac{\eta(\alpha-2m-1)}{2m} - 1\right]. \tag{5.28}$$

where p is the noise parameter of Werner state.

From Eq.(5.27) and Eq.(5.28) we note that, on comparing each term of the series (so that the uncertainty inequality is saturated), we get, from the m-th term of the series a relation of the form $\eta \geq \frac{1}{3p^{2m}}$. Therefore the lowest efficiency that we can have is for $m = 1$, $\eta = \frac{1}{3p^2}$. This is equal to the lowest efficiency that we get using the mean deviation uncertainty relations.

5.3.2 Entanglement detection

Quantum entanglement is the key resource behind many quantum technologies. Coupled with the fact that the complexity of complete state tomography grows exponentially with the dimension, this renders the problem of detection of entanglement in a quantum state via non-tomographic, e.g. witness based methods extremely important. However, the linear witnesses guaranteed to exist vide the Hahn-Banach theorem are often not as strong as desired. Thus considering non-linear witnesses is quite natural. One such family of non-linear witness is furnished by local uncertainty relations, the violation of which implies entanglement in the global state [111]. This method was refined further to derive a necessary criteria for separability in finite-dimensional systems based on inequalities for variances of observables [112]. More concretely, it was proven that for an entangled two qubit state $|\Psi_1\rangle = a\,|00\rangle + b\,|11\rangle$, with $a > b$ there exist observables $\{\mathcal{O}_i\}$ such that $\sum_i \Delta^2(\mathcal{O}_i)_{|\Psi_1\rangle\langle\Psi_1|} = 0$,and the following inequality is obeyed for separable states.

$$\sum_i \Delta^2(\mathcal{O}_i) \geq 2a^2b^2, \tag{5.29}$$

where $\mathcal{O}_i = |\Psi_i\rangle\langle\Psi_i|$, $i = 1, ..., 4$ with $|\Psi_2\rangle = a\,|01\rangle + b\,|10\rangle$, $|\Psi_3\rangle = b\,|01\rangle - a\,|10\rangle$ and $|\Psi_4\rangle = b\,|00\rangle - a\,|11\rangle$. Using the above inequality, we can detect the entanglement in members of the Werner family of states $\rho_W^{AB} = p\,|\Psi_1\rangle\langle\Psi_1| + \frac{1-p}{4}\mathbb{I}$ for $p > \sqrt{1 - \frac{8a^2b^2}{3}}$. If we choose $a = b = \frac{1}{\sqrt{2}}$, we can detect entanglement for $p > \frac{1}{\sqrt{3}}$. Using the MD uncertainty relation, we similarly obtain the result $\sum_i \Delta_M(\mathcal{O}_i)_{|\Psi_1\rangle\langle\Psi_1|} = 0$. For a pure separable state, we note the following inequality being obeyed, i.e.,

$$\sum_i \Delta_M \mathcal{O}_i \geq 4a^2b^2, \tag{5.30}$$

where $\Delta_M \mathcal{O}_i$ is the mean deviation uncertainty of the observable $\mathcal{O}_i$, for any separable state. Now using (5.30), it is straightforward to check that we can detect the entanglement of Werner states for $p > \sqrt{1 - \frac{8a^2b^2}{3}}$. Thus, we note that for the Werner family of states, the MD based uncertainty relation is as good a tool as the SD based uncertainty when it comes to entanglement detection. This construction can detect all the bipartite pure entangled states, and, for two qudit systems, many bound entangled states as well [1].

We note that the scheme of detection of entanglement is quite different than the setup we considered for the detection of EPR violation earlier. However, in lossy scenario, it may be shown that one can detect entanglement for any value of detector efficiency using the mean deviation based uncertainty measure considered here. This is consistent with the result obtained in [121] which assumes the standard deviation as the measure of uncertainty. EPR steering is, in general, a strictly stronger form of quantum correlation than quantum entanglement [120, 123, 124] - thus detection of EPR steering tends to be more demanding than detection of entanglement alone.

5.4 Conclusion

In this work, we have provided an alternative formulation of state dependent as well as state independent quantum uncertainty relations in terms of the mean deviation rather than the usual standard deviation based approach. Furthermore, using F-distribution and Pareto distribution based wave functions, we showed that a definite quantification of quantum uncertainties can be given through our approach which is

[1]We also checked that the MD uncertainty relation based entanglement detection performs as well as the SD uncertainty relation based ones for two qubit Gisin family of states.

not possible for standard deviation based approaches in some cases. We have applied the new uncertainty relations in detecting EPR violation in a lossy scenario and in entanglement detection schemes. Of course, applications of mean deviation based uncertainty relations are not confined to these examples. For example, in future, one can formulate new error disturbance relations for successive measurements in terms of mean deviations [125–128]. Another interesting problem would be to find the analog of the Wigner Yanase skew information [129] for the quantum part of the mean deviation based uncertainty [130] and to study properties thereof, for example, whether this quantity is a true coherence monotone [67] unlike the WYSI [131]. For quantum metrological purposes, a mean deviation based formulation of the Cramer Rao bound [132] may turn out to be useful. Theorists working on quantum gravity have conjectured deformed uncertainty relations [133, 134]. It also remains an interesting direction to explore whether the search for signatures of such deformations, such as the existence of a minimum length scale, can be facilitated by the present work. Another possible direction of work is to explore the identity of intelligent states with respect to arbitrary deviation measures.

6 Demonstrating preparation contextuality via uncertainty relations

This chapter is based on our work done in arXiv preprint titled "Fine grained uncertainty determines preparation contextuality"[135]. In this chapter we show a relation between uncertainty relations and contextual nature of quantum theory. Unlike non-locatily and quantum steering, quantum contextuality capture more generic form of correlations which are not limited to only non-local measurements.

6.1 Preparation Noncontextuality from Parity Oblivious Random Access Codes

Preparation non-contextuality associated with an operational theory was first introduced in [136]. An operational theory provides the probabilities $p(k|P, M)$ of getting an outcome k given the preparation procedure P, and the measurement

M. Quantum theory is also an operational theory in which a preparation procedure P is represented by ρ_P and a measurement is represented by a positive operator valued measure (POVM), $\Lambda_{M,k}$. The probability of getting an outcome k is $p(k|P, M) = \mathrm{Tr}(\rho_{\mathrm{P}}\Lambda_{\mathrm{M,k}})$.

An operational theory is said to be preparation non-contextual if two preparations yield the same measurement statistics for all possible measurements, implies probability associated with two different preparations at the hidden variable level (λ) is also same, i.e,

$$\forall M\ \forall k;\ p(k|P, M) = p(k|P', M) \implies p(\lambda|P) = p(\lambda|P') \tag{6.1}$$

where λ is a hidden variable and P and P' denote two preparation procedures.

Preparation contextuality was demonstrated using parity oblivious communication games [137, 138]. In the game, Alice sends an N-bit string $x \in \{0, 1, ..., d-1\}^N$ to Bob, chosen uniformly. Whereas, Bob's task is to guess the y^{th} bit of the string x, using his measurement outcome b as shown in Fig.6.1. There is a cryptographic constraint that Alice can encode her message under the parity obliviousness condition that no information about the parity of x can be revealed to Bob. If $s \in Par$ where $Par \equiv \{s | s \in \{0, 1, .., d-1\}^N, \zeta \leq d-2\}$, with ζ denoting the number of zeroes appearing in a particular s, then no information about $x \cdot s = \oplus_i x_i s_i (\mod d) = l$, $\forall l \leq d-1$ should be revealed to Bob. We refer to this task as $N \to 1$ d-Parity oblivious random access codes (d-PORACs). The parity obliviousness condition can be cast down in the form of following equality

$$\forall s, b, l, l', y;\ \frac{1}{p(l)} \sum_{x\cdot s=l} p(b|x,y) = \frac{1}{p(l')} \sum_{x\cdot s=l'} p(b|x,y),$$

where $p(l) = \sum_{x\cdot s=l} p(x)$. As for all l parity strings x_l, we have d^{N-1} uniform choices, $p(l) = p(l')$. Thus, the above obliviousness condition reduces to

$$\forall s, b, l, l', y;\ \sum_{x\cdot s=l} p(b|x,y) = \sum_{x\cdot s=l'} p(b|x,y). \tag{6.2}$$

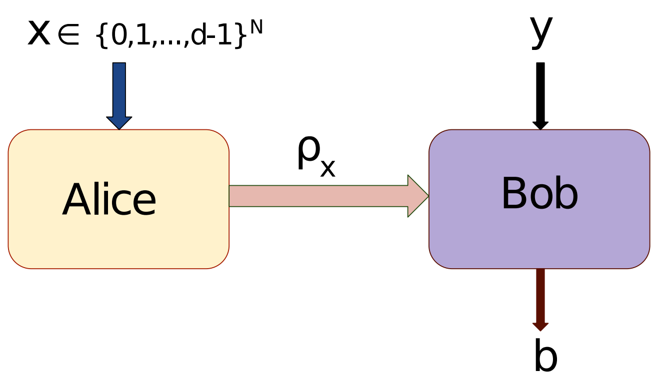

Figure 6.1: In this communication game, Alice encodes the classical string $x \in \{0, 1, ..., d-1\}^N$ in state ρ_x. On receiving the state ρ_x Bob's performs a measurement X_i chosen uniformly from a set of N observables, and tries to guess the y^{th} bit of x using his measurement outcome b.

Given the obliviousness constraint Bob's task is to maximize the average success probability of reporting the correct output $b = x_y$. The average probability of guessing the correct bit is given by

$$p(b = x_y) = \frac{1}{d^N N} \sum_{x \in \{0,1,...d-1\}^N} \sum_{y \in \{1,...,N\}} p(b|x,y),$$

Different operational theories provide different maximal success probability of the game. It was shown in [138] that an operational theory which admits a preparation non-contextual hidden variable model, the probability of success for $N \to 1$ d-PORAC is bounded by the following inequality,

$$\frac{1}{d^N N} \sum_{x \in \{0,1,...d-1\}^N} \sum_{y \in \{1,...,N\}} p(b|x,y) \leq \frac{N+d-1}{dN}. \tag{6.3}$$

Any operational theory which violates this inequality is contextual. We will show that quantum theory violates this inequality and that fine-grained uncertainty is just another way to express that.

6.2 Fine-Grained Uncertainty relations

Suppose, we want to measure N different observables X_i, where $i \in \{1, N\}$, and outcomes $x_i \in \{0, ..., d-1\}$. One can quantify the uncertainty associated with the measurements using entropic uncertainty relations as following

$$\sum_{i=1}^{N} H(X_i)_\rho \geq \beta,$$

where β depends on the compatibility between different observables. However, the entropy is a coarse way of measuring the uncertainty and incompatibility of a set of measurements. It does not reflect the uncertainty inherent in obtaining a particular combination of outcomes x_i for different measurements X_i. To circumvent this issue, fine-grained uncertainty relation was proposed in Ref.[48]. The uncertainty relation is a set of d^N inequalities of the following form

$$P^{cert}(\rho, x) = \sum_{i=1}^{N} p(X_i)p(x_i|X_i)_\rho \leq C_x(\mathcal{O}, \mathcal{P}), \tag{6.4}$$

where $C_x(\mathcal{O}, \mathcal{P})$ depends on the particular combination of measurement outcomes from set of observables $\mathcal{O} = \{X_i\}$ and chosen with distribution function $\mathcal{P} = \{p(X_i)\}$. The quantity $C_x(\mathcal{O}, \mathcal{P})$ captures the amount of uncertainty allowed in a particular physical theory. If $C_x(\mathcal{O}, \mathcal{P}) < 1$ for any x, one cannot obtain any outcome with certainty. Later, in Ref.[139] FUR were generalized for MUB, MUM and MBB for d dimensional systems. For a set of N mutual unbiased bases(MUBs) chosen with equal probability, the inequalities takes the following form [139]

$$\frac{1}{N} \sum_{i=1}^{N} p(x_i|X_i)_\rho \leq \frac{1}{d}\left(1 + \frac{d-1}{\sqrt{N}}\right). \tag{6.5}$$

Now, we will present FUR for a set of N arbitrary d-level observables, which also reproduces fine-grained upper bound for a set of MUBs.

Result 1 : *For a set of N arbitrary observables in dimension d, the FUR has* *the following form.*

$$\frac{1}{N}\sum_{i=1}^{N} p(x_i|X_i) \leq \frac{1}{d}\left(1 + \frac{(d-1)\sqrt{N + 2\sum_{j>k=1}^{N}\cos(\theta_{jk})}}{N}\right), \tag{6.6}$$

where $\cos(\theta_{jk})$ *is the angle between the Bloch vectors corresponding to eigenvectors* $|x_j\rangle$ *and* $|x_k\rangle$*.*

Proof. To prove this, we need to find the state $\rho_{\max}$ which maximize the left hand side of Eq.(6.6). The eigenvectors $|x_i\rangle$ corresponding to eigenvalues x_i and the state $\rho_{\max}$ can be expressed using Bloch vector representation as [54]

$$\rho_{x_i} = \frac{1}{d}I + \vec{x_i}\cdot\vec{\Gamma} \text{ and } \rho_{\max} = \frac{1}{d}I + \vec{b}\cdot\vec{\Gamma},$$

where $\vec{x_i}$ and $\vec{b}$ are the respective Bloch vectors and $\{\Gamma_i; i \in (0, ..., d-1)\}$ are the generalized Gell-mann matrices in dimension d. The length of the Bloch vector in d dimension should be less than $\sqrt{(d-1)/2d}$, where the maximum length indicate pure states. The generalized Gell-mann matrices are traceless, i.e. $\mathrm{Tr}(\Gamma_i) = 0$ and orthogonal, i.e. $\mathrm{Tr}(\Gamma_i\Gamma_j) = 2\delta_{ij}$ [54].

Now, using the Bloch vector representation, we find that

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N} p(x_i|X_i) &= \frac{1}{N}\sum_{i=1}^{N} \mathrm{Tr}[|x_i\rangle\langle x_i|\rho_{\max}] \\
&= \frac{1}{N}\sum_{i=1}^{N} \mathrm{Tr}\left[\left(\frac{1}{d}I + \vec{x_i}.\vec{\Gamma}\right)\left(\frac{1}{d}I + \vec{b}\cdot\vec{\Gamma}\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{d} + 2\vec{x_i}\cdot\vec{b}\right) \\
&= \frac{1}{d} + \frac{2}{N}\left(\sum_{i=1}^{N}\vec{x_i}\right)\cdot\vec{b}.
\end{aligned}$$

It is straightforward to see that the quantity $\left(\sum_{i=1}^{N}\vec{x_i}\right)\cdot\vec{b}$ is maximum when $\vec{b}$ is collinear with $\sum_{i=1}^{N}\vec{x_i}$, i.e., $\vec{b} = \eta\sum_{i=1}^{N}\vec{x_i}$, where η is scaling factor. For maximization, we have to find the appropriate value of η such that $|\vec{b}| = \sqrt{\frac{d-1}{2d}}$, which implies that $\rho_{\max}$ must be a pure state. Since, $|\sum_{i=1}^{N}\vec{x_i}| = \sqrt{N'}\sqrt{\frac{d-1}{2d}}$, which yields $\eta = \frac{1}{\sqrt{N'}}$, where $N' = N + 2\sum_{j>k=1}^{N}\cos(\theta_{jk})$. Thus, by substituting η, we find the Bloch vector, $\vec{b} = \frac{1}{\sqrt{N'}}\sum_{i=1}^{N}\vec{x_i}$ and which gives upper bound for the considered FUR. However, it should be noted that such a Bloch vector might not always represent a valid density matrix, since not all the points inside the d^2-1 dimensional Bloch sphere correspond to a valid density matrix, since it is not always positive-semidefinite [53]. This inequality is always tight for $d = 2$ systems, but not so for $d \geq 3$. $\square$

As a corollary of our derivation fine-grained upper bound for MUBs can be reproduced using the following lemma.

Lemma 1 : *The Bloch vectors belonging to d dimensional mutually unbiased* *bases are orthogonal to each other.*

Proof. We notice that the overlap between two mutually unbiased state vectors is

$$\frac{1}{d} = \mathrm{Tr}[\left(\frac{1}{d}I + \vec{x_i} \cdot \vec{\Gamma}\right)\left(\frac{1}{d}I + \vec{x_j} \cdot \vec{\Gamma}\right)] = \frac{1}{d} + 2\vec{x_i} \cdot \vec{x_j},$$

where we have used the tracelessness and orthogonality of the generalized Gell-mann matrices. Therefore, we get $\vec{x_i} \cdot \vec{x_j} = 0$. □

Using the Lemma 1 in Eq.(6.6), for any pair of mutually unbiased bases, $\cos(\theta_{jk}) = 0$ which gives the Eq.(6.5).

An example of the above inequality in qubit case, for measurements σ_x and σ_z, is given by [48]

$$\frac{1}{2}p(x_{\sigma_x}|\sigma_x) + \frac{1}{2}p(x_{\sigma_z}|\sigma_z) \leq \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right).$$

The above inequality is saturated for all 4 possible vectors $\vec{x} \in \{x_{\sigma_x}, x_{\sigma_z}\}$ and the maximally certain states are given by the eigenstates of $\frac{\sigma_x \pm \sigma_z}{\sqrt{2}}$.

6.3 Violating non-contextuality inequality with Fine-grained uncertainty

In this section, we show how FUR determines preparation contextuality of quantum theory.

As previously stated, there exist d^N such inequalities for N mutually unbiased observables Eq.(6.5). If we take the average over all such inequalities, we obtain

$$\frac{1}{d^N N} \sum_{x\in\{0,1,\ldots,d-1\}^N} \sum_{i=1}^{N} p(x_i|X_i)_\rho \leq \frac{1}{d}\left(1+\frac{d-1}{\sqrt{N}}\right), \tag{6.7}$$

where $x_i \in \{0,1,...,d-1\}$ are the measurement outcomes corresponding to observable X_i. If Alice encodes the classical string x by preparing ρ_x, and sends to Bob, who measures X_i to guess the i^{th} bit of x, then L.H.S of inequality 6.7 becomes the success probability of $N \to 1$ d PORAC game. Now, R.H.S of inequality (6.7) gives the quantum upper bound for success probability of the communication game. Later we also show that such encoding and decoding scheme also respects the parity obliviousness condition. Now, we state our result in terms of a theorem when Bob performs measurement with MUBs.

Theorem 2: *If Alice encodes the classical string x in maximally certain state and Bob measures with corresponding MUBs, then preparation contextuality of quantum theory can be revealed.*

Proof. The maximum success probability of the $N \to 1$ d-PORAC in quantum theory is exactly the R.H.S of the Eq.(6.7). On comparing the upper bound of $N \to 1$ d PORAC game with that of FUR, we find that $\frac{1}{d}\left(1+\frac{d-1}{\sqrt{N}}\right) \geq \frac{N+d-1}{dN}$. Therefore, we have obtained a probability which is greater than the maximum success probability obtained in a theory which is preparation non-contextual. □

If the above encoding and decoding scheme also respects parity obliviousness, then the preparation contextual nature of quantum theory can be revealed.

6.4 Illustrations

Example-1 First we present the simplest example of a $2 \to 1$, 2-PORAC. Although this has been presented earlier [137], we only highlight how the fine-grained uncertainty relations comes in the picture. The classical signal {00,01,10,11} are encoded in the states with Bloch vectors $\left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, because for σ_x and σ_y these states saturate the fine grained uncertainty relation. To decode the signal Bob measures with σ_x to measure the first bit and with σ_y to measure the second bit. Using this method he detects the correct signal with probability $\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) = 0.8535553 \geq \frac{n+1}{2n} = \frac{3}{4}$, and thus violates the inequality in Eq.(6.3). The parity obliviousness condition is also respected, since the parity 0 and 1 states are represented by the same density matrix operator ,i.e., $\frac{1}{2}\rho_{00} + \frac{1}{2}\rho_{11} = \frac{1}{2}\rho_{10} + \frac{1}{2}\rho_{01} = \frac{I}{2}$. Thus, by using the fine grained uncertainty relation we obtain a violation of preparation non-contextuality.

Example-2 Next, we show the example of $3 \to 1$ 2-PORAC. If Alice encodes the classical signal {000,001,010,011,100,101,110,111} in the states with Bloch vectors $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$, they saturate the fine-grained uncertainty for 3 observables σ_x, σ_y and σ_z with mutually unbiased bases. Bob employs σ_x,σ_y and σ_z operators to detect the first, second and third bit respectively and obtains correct signal with probability $\frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right) = 0.788675 \geq \frac{n+1}{2n} = \frac{2}{3}$. It has been shown that this is the optimal success probability Refs.[140, 141].

6.5 Conclusion

Optimal success probability of certain communication games reveal the fundamental limitations of different operational theories. Quantum advantage of random access code game with the additional constraint of parity obliviousness asserts that quantum theory is preparation contextual. Here we have shown that the degree of contextuality in quantum theory is limited by the amount of uncertainty allowed. To show this, we have derived a fine-grained uncertainty relations of N arbitrary observables of dimension d. Subsequently, we find analytically the quantum violation of the preparation contextuality inequality. Some partial results of optimal violations were known upto a few dimension with the help of numerical method i.e., semidefinite programming [142]. Our result is derived under the condition that dimension of the resource states corresponding to d-PORAC game is also d in classical or quantum theory.

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Thesis Highlight

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Thesis Title: Uncertainty Relations, Quantum Coherence and Quantum Measurement

Discipline: **Physical Sciences** **Sub-Area of Discipline:** **Quantum Information**

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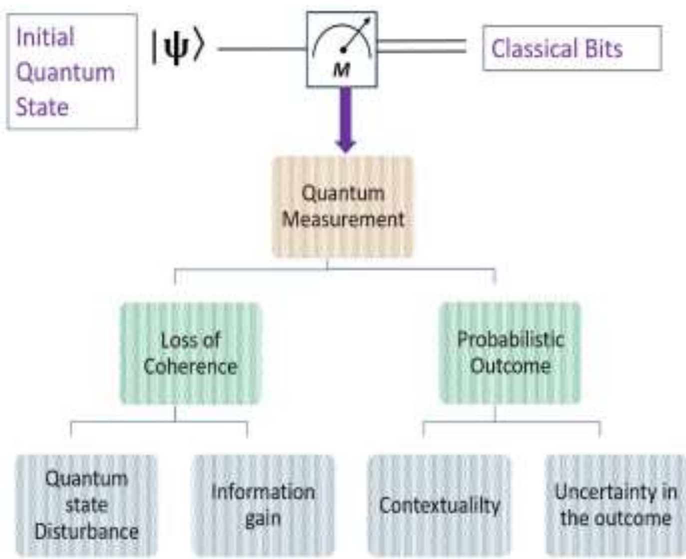

Measurement of quantum systems is remarkably different from doing measurement of classical systems. A quantum measurement disturbs a system irreversibly and gives only a probabilistic distribution of possible outcomes. These two aspects of quantum measurements are associated with loss of coherence of the system and uncertainty in the final outcome. This thesis uses quantum coherence and uncertainty relations to understand various phenomenon associated with quantum measurements.

It has been shown that quantum states with large initial coherence undergo less disturbance during a quantum measurement. This has been demonstrated for qubit and qutrit states. We discovered that for specific type of measurement processes quantum Coherence sets the upper bound on the maximum information that we can extract. Moreover, we proved that a more robust apparatus will be able to extract more information from the system.

An important contribution of this work has been the introduction of a new kind of uncertainty measure called “Mean deviation uncertainty” and new uncertainty relations based on them. This measure of uncertainty has wider applicability than the more commonly used standard deviation uncertainty. Moreover, our work raises the question of whether Gaussian states are the most classical states. We show that for mean deviation, based uncertainty relations Laplace states are as classical as the Gaussian states. We have used the new measure of uncertainty to detect Quantum Steering and Quantum Entanglement.

Another important contribution of this work is to establish a connection between uncertainty relations and Preparation Contextuality. Moreover, this work also established an upper bound on fine-grained uncertainty inequalities for arbitrary set of observables in all dimensions. This finding can lead to studying preparation contextual behavior of quantum theory via uncertainty relations.

Quantum measurement problem is one of the most significant problem associated with quantum theory, which has stayed since the origin of quantum theory itself. In future more work is needed in understanding and solving the problem thoroughly.