TOPICS IN CONFORMAL FIELD THEORY AND STRING THEORY

By

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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Dedicated to

Maa, Baba, and Didi

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Contents

Summary

In this thesis, we deal with two problems. In the first problem (chapter 2), we focus on conformal field theories (CFTs) where we deal with a novel method to compute correlators in a certain class of two-dimensional CFTs — Wess-Zumino-Witten (WZW) models. In the second problem (chapter 3), we prove certain analyticity properties in closed superstring field theory (SFT). Both the problems are introduced and the motivations behind these studies are discussed in chapter 1.

In chapter 2, we consider the construction of genus zero correlators of $SU(N)_k$ WZW models involving two Kac-Moody primaries in the fundamental and two in the anti-fundamental representation from modular averaging of the contribution of the vacuum conformal block. We perform the averaging by two prescriptions — averaging over the stabiliser group associated with the correlator and averaging over the entire modular group. For the first method, in cases where we find the orbit of the vacuum conformal block to be finite, modular averaging reproduces the exact result for the correlators. In other cases, we perform the modular averaging numerically, the results are in agreement with the exact answers. Construction of correlators from averaging over whole of the modular group is more involved. Here, we find some examples where modular averaging does not reproduce the correlator. We find a close relationship between the modular averaging sums of the theories related by level-rank duality. We establish a one to one correspondence between elements of the orbits of the vacuum conformal blocks of dual theories. The contributions of paired terms to their respective correlators are simply re-

lated. One consequence of this is that the ratio between the OPE coefficients associated with dual correlators can be obtained analytically without performing the sums involved in the modular averagings. The pairing of terms in the modular averaging sums for dual theories suggests an interesting connection between level-rank duality and semi-classical holographic computations of the correlators in the theories.

In chapter 3, we consider the off-shell momentum space Green's functions in closed superstring field theory. Recently in [23], the off-shell Green's functions — after explicitly removing contributions of massless states — have been shown to be analytic on a domain (to be called the LES domain) in complex external momenta variables. Analyticity of off-shell Green's functions in local QFTs without massless states in the primitive domain is a well-known result. Use of complex Lorentz transformations and Bochner's tube theorem allow us to extend the LES domain to a larger subset of the primitive domain. For the 2-, 3- and 4-point functions, the full primitive domain is recovered. For the 5-point function, we are not able to obtain the full primitive domain analytically, only a large part of it is recovered. While this problem arises also for higher-point functions, it is expected to be only a technical issue.

We discuss the future directions in chapter 4.

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1 Introduction

The Standard Model is a remarkably successful theory. It is a quantum theory describing the electromagnetic, weak and the strong interactions in nature. Its biggest shortcoming lies in its inability to describe gravity in the quantum regime. Furthermore, the constituents of dark matter and their interactions remain poorly understood, making it clear that there is much physics beyond Standard Model. String theory has the promise to provide a quantum theory of gravity unifying all forces of nature [1–4]. It is a theory of interacting strings and other extended objects like D-branes. Its spectrum always contains a massless spin-2 state whose low energy interactions are as in general relativity. String perturbation theory is known to be ultraviolet finite. At present, the most promising direction to connect string theory with nature is by compactifying ten dimensional string theories on six dimensional Calabi-Yau manifolds. Much progress has taken place in this direction, and this remains an active area of research. Furthermore, the techniques of perturbative quantum field theory fail in the case of strong interactions (QCD) at low energies. However, a close cousin of QCD, i.e. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions at strong coupling can be understood via its string theory dual due to a mathematical correspondence [5, 6]. This has given hope that string theory can shed light on strong dynamics.

Conformal field theories (CFTs) are central to the research in string theory. String worldsheet dynamics is described by two-dimensional CFTs. Furthermore, the AdS/CFT correspondence enables the study of $(D + 1)$-dimensional gravitational dynamics via D-

dimensional CFTs. CFTs also describe the universality classes of statistical models at their critical point. CFTs are quantum field theories that have symmetry under conformal transformations (which include scaling) of space-time [7, 8]. In CFTs, the product of two operators located at distinct nearby points can be expanded as a convergent linear sum of operators located at a single point, known as operator product expansion (OPE). Schematically,

$$O_i(x)O_j(y) = \sum_k C^k_{ij}(x-y)O_k(y)\ , \tag{1.1}$$

where $C^k_{ij}(x-y)$ are complex-valued functions. Due to this, correlators in CFTs can be expressed as a sum over contributions of primary operators (which are characterized by their conformal dimensions) running in the intermediate channel. Each of these contributions is fixed (up to a coefficient) by conformal symmetry which is known as the conformal block corresponding to the primary operator. The coefficients are related to the three-point coefficients that encode interactions. The associativity of the operator algebra implies that the operators can be permuted inside a correlator which is known as the crossing symmetry. In the bootstrap approach, we invoke crossing symmetry to solve for CFT data i.e., operator dimensions, three-point coefficients. Particularly in two dimensions, CFTs have infinite local symmetries that enable us to bootstrap solvable models, e.g., rational CFTs which have finite number of primary operators. Classification of all two-dimensional CFTs and relating them to three-dimensional quantum gravity via AdS/CFT correspondence are active areas of research.

Besides the world-sheet formulation, string theory can also be formulated along the lines of a quantum field theory. This formulation is known as string field theory, which is a quantum field theory with countable infinite number of fields $\{\phi^\alpha(k)\}$ and non-local interaction vertices, whose action is directly written in momentum space [9–19]. The field contents and vertices are so chosen that it can reproduce perturbative amplitudes of string theory. Furthermore, using standard quantum field theory techniques like mass renormalization and shift of vacuum one can deal with the infrared divergences of string amplitudes (i.e., S-matrix elements) in string field theory. For closed superstring field theory (SFT), after the Lorentz covariant gauge fixing the action has the following general form in a background with D non-compact space-time dimensions [18].

$$
\begin{aligned}
S = & \int \frac{d^D k}{(2\pi)^D} \phi^\alpha(k) K_{\alpha\beta}(k) \phi^\beta(-k) \\
& + \sum_n \int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_n}{(2\pi)^D} (2\pi)^D \delta^{(D)}(k_1 + \cdots + k_n) \\
& \qquad \times V^{(n)}_{\alpha_1 \ldots \alpha_n}(k_1, \ldots, k_n) \times \phi^{\alpha_1}(k_n) \cdots \phi^{\alpha_n}(k_n) \,,
\end{aligned}
\tag{1.2}
$$

where $K_{\alpha\beta}(k)$ is the kinetic operator (typically quadratic in momenta), and the interaction vertices $V^{(n)}_{\alpha_1 \ldots \alpha_n}(k_1, \ldots, k_n)$ have non-local behaviour i.e., as one or few k_i approach infinity the dominant factor in a vertex takes the form $e^{-c_{ij} k_i \cdot k_j}$ for some matrix c_{ij} with large positive eigen values. This formulation also enables us to prove important properties, e.g., unitarity, crossing symmetry of superstring amplitudes [20–23].

Now let us discuss the problems addressed in this thesis in more detail. Chapter 2 deals with the first problem which is on CFT and a duality. The bootstrap [24, 25] serves as an extremely useful tool in the study of conformal field theories (see [26–29] for reviews). An interesting direction of study is its interplay with duality symmetries. For example, in [30] it was found that S-duality invariant points of $\mathcal{N} = 4$ supersymmetric Yang-Mills saturate the bootstrap bounds on the anomalous dimensions of low twist non-BPS operators, in [31] it was found that crossing has interesting implications for the structure of the S-matrix in Chern Simons theories with matter. Recently, a rather simple proposal has been put forward to generate crossing symmetric genus zero correlation functions in two dimensional conformal field theories [32]. We construct correlation functions in $SU(N)_k$ Wess-Zumino-Witten (WZW) models[1] using the proposal and examine level-rank duality of the models in this context. WZW models are 2d CFTs having currents that are conformal chiral primaries of dimension one. The non-abelian current (Kac-Moody) algebra

[1] We provide the necessary details of $SU(N)_k$ WZW models at the beginning of section 2.2 of chapter 2.

is an affine Lie algebra associated with a semisimple Lie group[2]. This algebra generates the spectrum of the theory which contains conformal Kac-Moody primaries transforming under that Lie group. WZW models have wide applications in various contexts including non-abelian bosonization in 2d [33], description of bosonic string theory on AdS_3 [34], description of 2d black holes in string theory [35].

In two dimensions, crossing together with modular invariance has provided strong constraints from the early days [36–45]. For some recent developments in 2D bootstrap see [46–66], and in particular [67–73] for work on theories with currents. The basic idea in [32] is to make use of transformation properties of conformal blocks under crossing to arrive at crossing symmetric candidate correlation functions. Correlation functions are generated by starting from a seed contribution (as given by the contributions of conformal blocks of some primaries of low dimension running in the intermediate channel) and summing over the orbit of the seed under crossing transformations to obtain a crossing symmetric candidate correlation function. In two dimensions, crossing symmetry acts as the modular group on conformal blocks. Thus the sum over the orbit of the seed contribution corresponds to "modular averaging" [3]. It was shown in [32] that modular averaging can be used to successfully compute genus zero four-point functions of minimal models. Modular averaging has appeared in the physics literature in the context of three-dimensional quantum gravity and is often referred to as Farey tail sums (see e.g. [75–81]). It was argued in [32] that terms that arise from the orbit of the seed contribution would arise naturally in a semiclassical holographic AdS_3 dual computation of the CFT correlator.

In chapter 2, we consider the construction of genus zero correlators of $SU(N)_k$ WZW models involving two Kac-Moody primaries in the fundamental and two in the anti-fundamental representation from modular averaging of the contribution of the vacuum conformal block. We perform the averaging by two prescriptions — averaging over the stabiliser group associated with the correlator and averaging over the entire modu-

[2] In our case the Lie group will be $SU(N)$.

[3] This is very similar in spirit to the proposal of [74] to compute partition functions from vacuum characters.

lar group. For the first method, in cases where we find the orbit of the vacuum conformal block to be finite, modular averaging reproduces the exact result for the correlators. In other cases, we perform the modular averaging numerically, the results are in agreement with the exact answers. Construction of correlators from averaging over whole of the modular group is more involved. Here, we find some examples where modular averaging does not reproduce the correlator. We find a close relationship between the modular averaging sums of the theories related by level-rank duality. We establish a one to one correspondence between elements of the orbits of the vacuum conformal blocks of dual theories. The contributions of paired terms to their respective correlators are simply related. One consequence of this is that the ratio between the OPE coefficients associated with dual correlators can be obtained analytically without performing the sums involved in the modular averagings. The pairing of terms in the modular averaging sums for dual theories suggests an interesting connection between level-rank duality and semi-classical holographic computations of the correlators in the theories.

The second problem that we deal with in this thesis is discussed in chapter 3, which is on certain analyticity properties in SFT. In [23] de Lacroix, Erbin and Sen (LES) showed that the infrared safe part[4] of the off-shell amputated n-point Green's function $G(p_1, \ldots, p_n)$ in SFT as a function of $(n-1)D$ complex variables (taking into account the momentum conservation for external ingoing D-momenta $p_1, \ldots, p_n$: $\sum_{a=1}^{n} p_a = 0$) is analytic on a domain (LES domain). On the other hand in local quantum field theories *without* massless particles, the off-shell amputated n-point Green's function $G(p_1, \ldots, p_n)$ is known to be analytic on a domain called the primitive domain [82–86]. This result follows from causality constraints on the position space Green's functions in a local QFT and representing the momentum space Green's functions as Fourier transforms of the position space correlators[5]. The primitive domain contains the LES domain as a proper subset. In

[4] We precisely define it at the beginning of chapter 3.

[5] But the lack of a position space description of closed superstring field theory forces us to work directly in the momentum space.

several complex variables, the analyticity domain cannot be arbitrary[6]. For example, the shape of the primitive domain allows the actual domain of holomorphy of $G(p_1, \ldots, p_n)$ to be larger than itself (e.g. [87, 88]). This can be used to prove various analyticity properties [89–96] of the S-matrix of QFTs, since the S-matrix is defined as the on-shell connected amputated n-point Green's function for $n \geq 3$. These properties are to be read as the artifact of the shape of the primitive domain, irrespective of the functional form of $G(p_1, \ldots, p_n)$ which is defined on it. In particular, the derivations [88–90][7] of certain analyticity properties [89, 90, 97, 98] of the S-matrix use the information of only the LES domain as a subregion of the primitive domain.

From [23] we already know that the infrared safe part of the off-shell amputated n-point Green's function in SFT is analytic on the LES domain. Hence [23] basically established that in superstring theory any possible departure from those analyticity properties of the full S-matrix that rely only on the LES domain is entirely due to the presence of massless states. This is also true for local QFTs that have massless states. Thus, with respect to the aforenamed analytic properties [89, 90, 97, 98], the S-matrix of the superstring theory displays similar behaviour to that of a standard local QFT with massless particles. In local QFTs *without* massless states, analyticity properties [91–96] of the S-matrix rely on the properties of the primitive domain that are not restricted to its LES subregion. We know that local QFTs *with* massless states could possibly deviate from these properties for their full S-matrix. However, any such departures are entirely due to the presence of massless states, i.e. the infrared safe parts of respective amplitudes in local QFTs with massless states must satisfy all these analyticity properties. At this stage, it is natural to ask that whether the infrared safe part of the S-matrix of the superstring theory (despite having non-local vertices) satisfies the last-mentioned analyticity properties, or not. They satisfy them, only if the relevant part of the off-shell amputated n-point Green's function in SFT

[6]General properties of the theory of functions of several complex variables have been briefly discussed in appendix B.1.

[7]For example, [88] recovered the JLD domain, [89] proved the crossing symmetry of the $2 \to 2$ scattering amplitudes.

can be shown to be analytic on the full primitive domain extending the LES domain.

In chapter 3, we show that the use of complex Lorentz transformations and Bochner's tube theorem allow us to extend the LES domain holomorphically to a larger subset of the primitive domain. For the 2-, 3- and 4-point functions, the full primitive domain is recovered. For the 5-point function, we are not able to obtain the full primitive domain analytically, only a large part of it is recovered. While this problem arises also for higher-point functions, it is expected to be only a technical issue.

We discuss the future directions of the two aforementioned problems in chapter 4. In appendices A and B we provide supplementary materials for the chapters 2 and 3 respectively.

2 Crossing, Modular Averages and $N \leftrightarrow k$ in WZW Models

Correlators of $SU(N)_k$ Wess-Zumino-Witten models have been well studied starting from the seminal work of [37], where four-point functions involving two Kac-Moody primaries in the fundamental and two in the anti-fundamental representation of $SU(N)$ were computed.

Recently, a novel proposal to compute correlators has been put forward in [32], by the method of "modular averaging" (we will review this in detail in section 2.1). In this chapter, we will analyse four-point WZW correlators involving two Kac-Moody primaries in the fundamental and two in the anti-fundamental representation of $SU(N)$ (same as those studied in [37]) using modular averaging. We carry out the averaging by two methods — averaging over the stabiliser subgroup of the correlator and over the entire modular group, mostly focussing on the first one (we review these prescriptions in section 2.1). For averaging performed using the stabilser group, we find that the correlators can be constructed from modular averaging of the contribution of the vacuum block in all the cases we examine. Primary examples of models where the sums can be done exactly are models with $N = k$ (the orbits for these models are finite). For models where we have not been able to show that the orbit is finite, we consider examples with specific values of N and k, and perform the averaging numerically. On the other hand, construction of correlators from averaging over whole of the modular group is more involved. Here, we find some examples where modular averaging does not reproduce the correlator.

An interesting feature of WZW models is level-rank duality [99]. Dual primary fields under $N \leftrightarrow k$ are related by transposition of the Young tableaux of their representations. The correlators considered in this chapter are the simplest related to each other by this duality. From the point of view of modular averaging, both N and k simply appear as parameters in the matrices associated with the action of the modular group on the conformal blocks. Thus modular averaging puts N and k on a more equal footing; one can hope that writing correlators as modular averages can reveal various aspects of level-rank duality. This expectation is borne out. We establish a one to one correspondence between elements of the orbits of the vacuum conformal blocks of dual theories. The contributions of paired terms to their respective correlators are simply related. This allows us to obtain the ratio between the OPE coefficients associated with dual correlators analytically without performing the sums involved in the modular averagings. The pairing of terms also indicates that holographic computations can make some properties of the level-rank duality manifest.

This chapter is organised as follows. In section 2.1, we briefly review some basic ingredients that will be necessary for our analysis. In section 2.2 (and appendix A.1) we obtain the transformation properties of the conformal blocks of the correlators under the action of the modular group. In section 2.3 (and appendices A.3, A.4) we compute correlators by modular averaging. In section 2.4, we examine level-rank duality.

2.1 Review

We start by recalling some basic facts about four-point functions in two dimensional conformal field theories. We then go on to describe the proposal of [32] to construct crossing symmetric correlation functions from modular averaging.

The four-point correlator of operators O_1, O_2, O_3 and O_4 in 2D CFTs on the Riemann sphere can be written as the product of a factor that determines its transformation properties under global conformal transformations and a function of a conformally invariant cross-ratio. It will be our convention to take

$$\langle O_1(z_1,\bar{z}_1)O_2(z_2,\bar{z}_2)O_3(z_3,\bar{z}_3)O_4(z_4,\bar{z}_4)\rangle = G_0(z_a,\bar{z}_a)G_{1234}(x,\bar{x}) \tag{2.1}$$

with

$$G_0(z_a,\bar{z}_a) = \prod_{a<b}(z_{ab}^{\mu_{ab}}\cdot\bar{z}_{ab}^{\bar{\mu}_{ab}}), \tag{2.2}$$

where $z_{ab} = z_a - z_b$ $(a,b = 1..4)$, $\mu_{ab} = (\frac{1}{3}\sum_{c=1}^{4} h_c) - h_a - h_b$ (h_i being the dimensions of the operators O_i) and the cross-ratio

$$x = \frac{z_{12}z_{34}}{z_{14}z_{32}}. \tag{2.3}$$

Conformal transformations can be used to set z_2 to 0 and z_3 to 1 and set z_4 to infinity, the coordinate z_1 then corresponds to the cross-ratio. Thus the cross-ratio space is the Riemann sphere with three punctures.

Correlators in two dimensional CFTs can be constructed from holomorphic and antiholomorophic conformal blocks. Although correlators need to be single valued functions of the cross-ratio space[1], there is no such requirement on the conformal blocks. Conformal blocks have monodromies in the cross-ratio space. Thus it is natural to consider conformal blocks as functions in the universal covering space of the cross-ratio space. This is $\mathbb{H}_+ = \{u + iv \mid v > 0 \text{ and } u, v \in \mathbb{R}\}$, the upper half plane[2]. The elliptic lambda function

$$\lambda(\tau) = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^4, \tag{2.4}$$

where $\tau = u + iv$ provides a surjective map ($x = \lambda(\tau)$) from $\mathbb{H}_+$ to the cross-ratio space

[1] We will be dealing with bosonic operators.

[2] The observation that conformal blocks should be single-valued on the upper half plane was made in [101], where an elliptic recursion representation was obtained for them.

[100]. $PSL(2,\mathbb{Z})$ action on the upper half plane has a close connection to the map. Under the action of the generators of the modular group

$$T : \tau \to \tau + 1 \;\text{ and }\; S : \tau \to -\frac{1}{\tau}, \tag{2.5}$$

images in the cross-ratio space have rather simple transformations

$$T \cdot x = \frac{x}{x-1} \;\text{ and }\; S \cdot x = 1 - x. \tag{2.6}$$

Furthermore, the function $\lambda(\tau)$ is invariant under the normal subgroup $\Gamma(2)$ of $PSL(2,\mathbb{Z})$:

$$\lambda(\gamma\tau) = \lambda(\tau), \forall \gamma \in \Gamma(2). \tag{2.7}$$

Thus, the condition that correlators have to be single valued in the cross-ratio space translates to invariance under $\Gamma(2)$ in $\mathbb{H}_+$.

At this stage, it is natural to seek for the interpretation of the action of the entire $PSL(2,\mathbb{Z})$ on the correlators in the CFT. For this, one has to look at crossing symmetry. For a general ordering of the operators, we define

$$\langle O_p(z_p,\bar{z}_p)O_q(z_q,\bar{z}_q)O_r(z_r,\bar{z}_r)O_s(z_s,\bar{z}_s)\rangle = G_0(z_a,\bar{z}_a)G_{pqrs}(x_{pqrs},\bar{x}_{pqrs}), \tag{2.8}$$

with G_0 as defined in (2.2) and

$$x_{pqrs} = \frac{z_{pq}z_{rs}}{z_{ps}z_{rq}}. \tag{2.9}$$

Note that with this we have $x = x_{1234}$, where x is the cross-ratio introduced in (2.3). Our choice of G_0 is invariant under permutations of the operators $\{O_a(z_a)\}$ inside the correlator thus crossing symmetry reduces to the statement that $G_{abcd}(x_{abcd})$ is invariant under action of the same permutation on $\{a,b,c,d\}$ in both the subscripts. Permutations that leave the cross ratio x invariant yield:

$$G_{1234}(x,\bar{x}) = G_{2143}(x,\bar{x}) = G_{3412}(x,\bar{x}) = G_{4321}(x,\bar{x}). \tag{2.10}$$

On the other hand, permutations which act non-trivially on the cross-ratio[3] give

$$\begin{aligned} G_{1234}(x,\bar{x}) &= G_{1243}(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}) = G_{3241}(\frac{1}{1-x},\frac{1}{1-\bar{x}}) = G_{3214}(\frac{1}{x},\frac{1}{\bar{x}}) \\ &= G_{4231}(1-x,1-\bar{x}) = G_{4213}(\frac{x-1}{x},\frac{\bar{x}-1}{\bar{x}}). \end{aligned} \tag{2.11}$$

The arguments of the functions in (2.11) can be related by the actions of S and T as given in (2.6). The actions are isomorphic to the anharmonic group, S_3. This is precisely equal to $PSL(2,\mathbb{Z})/\Gamma(2)$. Thus crossing symmetry and single valuedness[4] together specify the full $PSL(2,\mathbb{Z})$ action on the correlators. Combining (2.6),(2.10) and (2.11) they can be written in a very compact form [32]:

$$\vec{G}(\gamma\tau,\gamma\bar{\tau}) = \sigma(\gamma)\cdot\vec{G}(\tau,\bar{\tau}), \qquad \gamma\in PSL(2,\mathbb{Z}) \tag{2.12}$$

where

$$\vec{G} = (G_{1234}(\tau,\bar{\tau}), G_{2134}(\tau,\bar{\tau}), G_{4132}(\tau,\bar{\tau}), G_{1432}(\tau,\bar{\tau}), G_{2431}(\tau,\bar{\tau}), G_{4231}(\tau,\bar{\tau}))^t \tag{2.13}$$

and $\sigma(\gamma)$ are the six dimensional matrices associated with the linear representation of

[3] These relations differ from the ones in [32] since our choice for the cross-ratio x is different.
[4] Recall that correlators need to be invariant under $\Gamma(2)$ so that they are single valued.

$PSL(2,\mathbb{Z})/\Gamma(2) = S_3$ with

$$\sigma(S) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma(T) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.14}$$

We note that there is further simplification when all or some of the operators O_a are identical. For instance, in the case that all the four operators are identical $\vec{G}$ has only one independent component. Equation (2.12) requires it to be a modular invariant scalar.

Modular averaging can be used to obtain solutions of equations of the form of (2.12). The general structure of four-point functions in a CFT gives fiducial functions over which the averaging can be performed. Conformal invariance implies that the stripped correlators in (2.8) can be written as a sum over contributions associated with conformal primaries (ϕ_k):

$$G_{pqrs}(y,\bar{y}) = \sum_k C_{O_pO_q\phi_k} C_{O_rO_s\phi_k} \times y^{h_{\phi_k}-\frac{\mathfrak{H}}{3}} \bar{y}^{\bar{h}_{\phi_k}-\frac{\bar{\mathfrak{H}}}{3}} F^{\phi_k}_{pqrs}(y,\bar{y}), \tag{2.15}$$

where $C_{O_pO_q\phi_k}$, $C_{O_rO_s\phi_k}$ are three point structure constants, $\mathfrak{H} = (h_p + h_q + h_r + h_s)$ and $\bar{\mathfrak{H}} = (\bar{h}_p + \bar{h}_q + \bar{h}_r + \bar{h}_s)$. The functions $F^{\phi_k}_{pqrs}(y,\bar{y})$ are analytic at $y, \bar{y} = 0$ and $F^{\phi_k}_{pqrs}(0,0) = 1$. It will be our convention to call $\{y^{h_{\phi_k}-\frac{\mathfrak{H}}{3}} \bar{y}^{\bar{h}_{\phi_k}-\frac{\bar{\mathfrak{H}}}{3}} F^{\phi_k}_{pqrs}(y,\bar{y})\}$ as the conformal block corresponding to primary ϕ_k. These can be further factorized into holomorphic and anti-holomorchic conformal blocks for each ϕ_k. Given the form of (2.15), in the limit of $y \to 0$ the stripped correlator is well approximated by including contributions from the low lying primaries that appear in the sum i.e.

$$G_{pqrs}(y,\bar{y}) \approx G^{\text{light}}_{pqrs}(y,\bar{y}) = \sum_{k \le k_{\max}} C_{O_pO_q\phi_k} C_{O_rO_s\phi_k} \times y^{h_{\phi_k}-\frac{\mathfrak{H}}{3}} \bar{y}^{\bar{h}_{\phi_k}-\frac{\bar{\mathfrak{H}}}{3}} F^{\phi_k}_{pqrs}(y,\bar{y}) \quad \text{for} \quad y \to 0. \tag{2.16}$$

where the sum now runs over primaries which have weights less than or equal to $(h_{k_{\max}}, \bar{h}_{k_{\max}})$. The simplest approximation is to keep only the primary with the lowest weight. Reference [32] proposed that modular averaging of $\vec{G}^{\text{light}}$ can be used to construct candidate CFT correlators which satisfy the requirements single-valuedness and crossing.

$$\vec{G}^{\text{candidate}}(\tau,\bar{\tau}) = \mathcal{N}^{-1} \cdot \sum_{\gamma \in PSL(2,\mathbb{Z})} \sigma^{-1}(\gamma) \cdot \vec{G}^{\text{light}}(\gamma\tau, \gamma\bar{\tau}), \tag{2.17}$$

where $\mathcal{N}$ is a normalisation which can be determined from the $\tau \to i\infty$ ($y \to 0$) behaviour of $\vec{G}(\tau,\bar{\tau})$. In general, the sum in (2.17) is difficult to perform and might even need regularisation. The complications associated with dealing with a sum involving vector valued modular objects can be ameliorated for correlators with identical operators. As described earlier, in the presence of identical operators, various components of $\vec{G}$ (as defined in (2.13)) become related - the vector space effectively collapses to a lower dimensional one. As a result, the subgroup of $PSL(2,\mathbb{Z})$ that leaves any particular component of the vector inert under action of $\sigma(\gamma)$ is enhanced[5]. If the subgroup associated with the component G_a in the collapsed vector space is Γ_a, a natural candidate G_a can be constructed by defining

$$G_a^{\text{candidate}}(\tau,\bar{\tau}) = \mathcal{N}^{-1} \cdot \sum_{\gamma \in \Gamma_a} G_a^{\text{light}}(\gamma\tau, \gamma\bar{\tau}). \tag{2.18}$$

The above program to obtain CFT correlators was implemented for minimal models in [32]. It was found that for a large number of them, the candidate correlators did match with the exact ones by taking only the contribution of the Virasoro vacuum block while constructing G_a^{light} - the lightest block served the purpose.

[5]In the case that all the operators are distinct, this subgroup is $\Gamma(2)$ for all the components.

2.2 $SU(N)_k$ WZW Model: Conformal Blocks, Actions of S and T

As mentioned in the introduction, our focus will be on WZW correlators involving two Kac-Moody primaries in the fundamental and two in the anti-fundamental representation. In this section, we will obtain the transformation properties of the conformal blocks associated with the correlators under the action of crossing.

We begin by recalling some basic facts about the correlators (our discussion follows that of [7, 8, 37, 38]) and in the process set up our notation. The $SU(N)$ WZW model at level k on the two sphere is described by the action:

$$S_k^{\mathrm{WZW}}[g] = \frac{k}{16\pi}\int d^2z\, \mathrm{Tr}(\partial^\mu g^{-1}\partial_\mu g) - \frac{ik}{24\pi}\int_B d^3\vec{X}\, \epsilon_{\alpha\beta\gamma}\mathrm{Tr}(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g), \quad k = 1, 2, .. \tag{2.19}$$

where $g(z,\bar{z})$ is a matrix valued bosonic field which takes values in the group $SU(N)$. The second term is an integral over the three ball B, whose boundary is the two sphere. The pre-factors of the two terms in the action are chosen so that theory is conformal at the quantum level. The action enjoys an $SU(N)(z) \times SU(N)(\bar{z})$ invariance. The associated currents are

$$j(z) \equiv -k(\partial_z g)g^{-1}, \;\; \bar{j}(\bar{z}) \equiv kg^{-1}(\partial_{\bar{z}} g) \tag{2.20}$$

which can be expanded in terms of the generators of $SU(N)$ as

$$j(z) = \sum\nolimits_a j^a(z)t^a, \;\; \bar{j}(\bar{z}) = \sum\nolimits_a \bar{j}^a(\bar{z})t^a. \tag{2.21}$$

The Laurent series expansion coefficients of the currents together with the Virasoro generators generate two copies of the Kac-Moody algebra at level k.

Kac-Moody primaries serve as the highest weight states in the theory. For the (N, k) theory the spectrum of Kac-Moody primaries consists operators transforming in all representations of $SU(N)$ which have integrable Young tableaux i.e. those in which the number of columns is at most k. The conformal dimension of a Kac-Moody primary transforming in a representation R is

$$h_R = \frac{C(R)}{2(k+N)}, \tag{2.22}$$

where $C(R)$ is the quadratic Casimir of the representation.

We will follow the notation of [37] and denote a fundamental Kac-Moody primary by ${g_\alpha}^\beta(z,\bar{z})$, where α is a fundamental index of the $SU(N)$ left and β is a fundamental index of the $SU(N)$ right. On the other hand, an anti-fundamental will be denoted by ${g_\rho^{-1}}^\sigma$, where where ρ is an anti-fundamental index of the $SU(N)$ right and σ is an anti-fundamental index of the $SU(N)$ left. The conformal dimension of these fields can be easily obtained from (2.22)

$$h_g = h_{g^{-1}} = \frac{N^2-1}{2N(k+N)}. \tag{2.23}$$

For correlators involving two fundamentals and two anti-fundamentals, primaries that run in the intermediate channels will be as per the fusion rules

$$g \times g^{-1} = \mathbb{1} + \theta, \qquad g \times g = \xi + \chi, \qquad g^{-1} \times g^{-1} = \xi + \chi, \tag{2.24}$$

where $\mathbb{1}$ is the identity field, θ the adjoint, ξ the antisymmetric and χ the symmetric. The associated dimensions are

$$h_{\mathbb{1}} = 0, \quad h_\theta = \frac{N}{N+k}, \quad h_\xi = \frac{(N-2)(N+1)}{N(N+k)} \quad \text{and} \quad h_\chi = \frac{(N+2)(N-1)}{N(N+k)}. \tag{2.25}$$

Our main interest will be the correlator

$$\langle g g^{-1} g^{-1} g \rangle \equiv \langle {g_{\alpha_1}}^{\beta_1}(z_1,\bar{z}_1) \cdot {g_{\beta_2}^{-1}}^{\alpha_2}(z_2,\bar{z}_2) \cdot {g_{\beta_3}^{-1}}^{\alpha_3}(z_3,\bar{z}_3) \cdot {g_{\alpha_4}}^{\beta_4}(z_4,\bar{z}_4) \rangle \tag{2.26}$$

Recall that as per our conventions α_1, α_4 are $SU(N)$ left fundamental indices, α_2, α_3 are

$SU(N)$ left anti-fundamental indices, β_1,β_4 are $SU(N)$ right fundamental indices, β_2,β_3 are $SU(N)$ right anti-fundamental indices. We will be eventually interested in making choices for the indices such that the correlator contains two pairs of identical operators so that we can carry out modular averaging as per the prescription in (2.18). For this we need the conformal blocks associated with the correlator and their transformations under the modular group.

The correlator has been studied in detail in [37]. We briefly describe their analysis adopting the discussion to our conventions. First, we define the stripped correlator $G^{\beta_1\alpha_2\alpha_3\beta_4}_{\alpha_1\beta_2\beta_3\alpha_4}(x,\bar{x})$ as in (2.1)

$$\langle gg^{-1}g^{-1}g\rangle = (\prod_{a<b} z_{ab}^{\mu_{ab}}\bar{z}_{ab}^{\bar{\mu}_{ab}})G^{\beta_1\alpha_2\alpha_3\beta_4}_{\alpha_1\beta_2\beta_3\alpha_4}(x,\bar{x}), \tag{2.27}$$

where x is the cross-ratio defined in (2.3). Invariance of the correlator under $SU(N)$ left and right implies

$$G^{\beta_1\alpha_2\alpha_3\beta_4}_{\alpha_1\beta_2\beta_3\alpha_4}(x,\bar{x}) = \sum_{A,B=1,2}(I_A)(\bar{I}_B)G_{AB}(x,\bar{x}), \tag{2.28}$$

where

$$I_1 = \delta^{\alpha_2}_{\alpha_1}\delta^{\alpha_3}_{\alpha_4},\quad \bar{I}_1 = \delta^{\beta_1}_{\beta_2}\delta^{\beta_4}_{\beta_3},\quad I_2 = \delta^{\alpha_3}_{\alpha_1}\delta^{\alpha_2}_{\alpha_4}\ \text{ and }\ \bar{I}_2 = \delta^{\beta_1}_{\beta_3}\delta^{\beta_4}_{\beta_2}. \tag{2.29}$$

One then imposes the Knizhnik-Zamolodchikov (KZ) equations on the correlator. The KZ equations are a consequence of the Kac-Moody symmetries. For a correlator involving Kac-Moody primaries ϕ_i, transforming in the representations R_i they are

$$\left[\partial_{z_i} - \frac{1}{k+N}\sum_{j\neq i}\frac{\sum_a t^a_{R_i}\otimes t^a_{R_j}}{z_i - z_j}\right]\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle = 0,\ \forall\, i, \tag{2.30}$$

where $t^a_{R_i}$ are $SU(N)$ generators in the representation R_i. Similar set of equations hold in the anti-holomorphic coordinates. Imposing them on the correlator (2.26) yields the following equations for the matrix G_{AB} defined in (2.28).

$$\frac{\partial G}{\partial x} = \left[\frac{1}{x}P + \frac{1}{x-1}Q\right]G \ \text{ and }\ \frac{\partial G}{\partial\bar{x}} = G\left[\frac{1}{\bar{x}}P^t + \frac{1}{\bar{x}-1}Q^t\right], \tag{2.31}$$

where the matrices P and Q are given by

$$P = -\frac{1}{N(k+N)}\begin{pmatrix} \frac{2(N^2-1)}{3} & N \\ 0 & -\frac{N^2+2}{3} \end{pmatrix} \text{ and } Q = -\frac{1}{N(k+N)}\begin{pmatrix} -\frac{N^2+2}{3} & 0 \\ N & \frac{2(N^2-1)}{3} \end{pmatrix}. \tag{2.32}$$

The general solution to these equations takes the form

$$G_{AB}(x,\bar{x}) = X_{ij}F^i_A(x)F^j_B(\bar{x}), \tag{2.33}$$

where the indices i, j run over the primaries in the intermediate channel. These are the identity ($\mathbb{1}$) and the adjoint (θ) fields. $F^i_A(x)$ are the conformal blocks

$$\begin{aligned}
F^{\mathbb{1}}_1(x) &= x^{-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}}F\left(\frac{1}{\tilde{k}}, -\frac{1}{\tilde{k}}; 1-\frac{N}{\tilde{k}}; x\right), \\
F^{\mathbb{1}}_2(x) &= \frac{1}{k}x^{1-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}}F\left(1+\frac{1}{\tilde{k}}, 1-\frac{1}{\tilde{k}}; 2-\frac{N}{\tilde{k}}; x\right), \\
F^{\theta}_1(x) &= x^{h_\theta-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}}F\left(\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}}, \frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; 1+\frac{N}{\tilde{k}}; x\right), \\
F^{\theta}_2(x) &= -Nx^{h_\theta-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}}F\left(\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}}, \frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; \frac{N}{\tilde{k}}; x\right),
\end{aligned} \tag{2.34}$$

where $\tilde{k} = k + N$ and $F(a, b, c; x)$ is the Gauss hypergeometric function[6]. We define the holomorphic and the anti-holomorphic blocks:

$$\mathcal{F}^{\mathbb{1}}(x) = I_1F^{\mathbb{1}}_1(x) + I_2F^{\mathbb{1}}_2(x) \tag{2.35}$$

$$\bar{\mathcal{F}}^{\mathbb{1}}(\bar{x}) = \bar{I}_1F^{\mathbb{1}}_1(\bar{x}) + \bar{I}_2F^{\mathbb{1}}_2(\bar{x}) \tag{2.36}$$

$$\mathcal{F}^{\theta}(x) = I_1F^{\theta}_1(x) + I_2F^{\theta}_2(x) \tag{2.37}$$

$$\bar{\mathcal{F}}^{\theta}(\bar{x}) = \bar{I}_1F^{\theta}_1(\bar{x}) + \bar{I}_2F^{\theta}_2(\bar{x}). \tag{2.38}$$

With this, the correlator factorises into holomorphic and anti-holomorphic parts:

$$G^{\beta_1\alpha_2\alpha_3\beta_4}_{\alpha_1\beta_2\beta_3\alpha_4}(x,\bar{x}) = X_{ij}\mathcal{F}^i(x)\bar{\mathcal{F}}^j(\bar{x}). \tag{2.39}$$

[6]Our conventions for the definition of the Gauss hypergeometric function will be same as that of [102].

As discussed in section 2.1, general correlators transform as a six dimensional modular vector under the action of the modular group. Just as in the correlator described above, there are two holomorphic and two anti-holomorphic blocks associated with each correlator. This implies that the vector valued modular form requires 24 coefficients for its specification. This number is large even if one wants to carry out modular averaging as per (2.17) numerically. Luckily, one can simplify the computation by exploiting the fact that (2.39) implies that the X_{ij} are independent of the $SU(N)$ left and right tensor indices. We will make choices for these so that the correlator has two pairs of identical operators i.e. we will take $\alpha_1 = \alpha_4, \beta_1 = \beta_4, \alpha_2 = \alpha_3, \beta_2 = \beta_3$. With this we have

$$I_1 = I_2 \equiv I \qquad \text{and} \qquad \bar{I}_1 = \bar{I}_2 \equiv \bar{I}. \tag{2.40}$$

As a result, the six dimensional vector space collapses to a three dimensional one (after use of equation (2.10)):

$$\vec{G} = \left(G^{\beta_1\alpha_2\alpha_2\beta_1}_{\alpha_1\beta_2\beta_2\alpha_1}(\tau,\bar{\tau}), G^{\beta_1\alpha_2\alpha_1\beta_2}_{\alpha_1\beta_2\beta_1\alpha_2}(\tau,\bar{\tau}), G^{\beta_1\alpha_1\alpha_2\beta_2}_{\alpha_1\beta_1\beta_2\alpha_2}(\tau,\bar{\tau}) \right), \tag{2.41}$$

its transformations under the modular group as given by (2.12) reduces to

$$\begin{aligned} \vec{G}(T\cdot\tau, T\cdot\bar{\tau}) &= \sigma(T)\cdot\vec{G}(\tau,\bar{\tau}), \\ \vec{G}(S\cdot\tau, S\cdot\bar{\tau}) &= \sigma(S)\cdot\vec{G}(\tau,\bar{\tau}), \end{aligned} \tag{2.42}$$

where

$$\sigma(T) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \sigma(S) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.43}$$

We list the conformal blocks associated with the three correlators in (2.41) and their transformation properties under the modular group in appendix A.1.

We will primarily perform the modular averaging as per the algorithm in (2.18) (although also briefly consider averaging as per the prescription in (2.17) in appendix A.4). For the representation of $PSL(2,\mathbb{Z})$ generated by the matrices in (2.43), it is easy to see that the vector $(1,0,0)$ is left invariant by the subgroup generated by the actions of S and T^2. This is called the theta group [103]. This subgroup is an index 3 subgroup of $PSL(2,\mathbb{Z})$ which contains $\Gamma(2)$ as an index 2 normal subgroup. In order to carry out the modular averaging as per (2.18), we require the actions of the elements of this subgroup on the conformal blocks associated with the stripped correlator $G^{\beta_1\alpha_2\alpha_2\beta_1}_{\alpha_1\beta_2\beta_2\alpha_1}(\tau,\bar{\tau})$. These blocks are

$$
\begin{aligned}
\mathcal{H}^{\mathbb{1}}(x) &= IF^{\mathbb{1}}_1(x) + IF^{\mathbb{1}}_2(x) \\
\mathcal{H}^{\theta}(x) &= IF^{\theta}_1(x) + IF^{\theta}_2(x),
\end{aligned}
\tag{2.44}
$$

with I and $\bar{I}$ as defined in (2.40).

The transformation properties of these blocks under S and T^2 can be obtained from appendix A.1. The action of T^2 is given by

$$
\mathcal{H}^i\left(T^2.x\right) = \mathcal{H}^j(x)\, M_{ji}(T^2),
\tag{2.45}
$$

where

$$
M(T^2) = e^{-i4\pi(N^2-1)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi N/\tilde{k}} \end{pmatrix}.
\tag{2.46}
$$

The action of S is given by

$$
\mathcal{H}^i(S.x) = \mathcal{H}^j(x)\, M_{ji}(S),
\tag{2.47}
$$

where

$$
M(S) = \begin{pmatrix} -\frac{\tilde{k}\Gamma(N/\tilde{k})\Gamma(k/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(-1/\tilde{k})} & -\frac{N\Gamma^2(N/\tilde{k})}{\Gamma(N/\tilde{k}-1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})} \\ -\frac{\Gamma^2(k/\tilde{k})}{N\Gamma(k/\tilde{k}-1/\tilde{k})\Gamma(k/\tilde{k}+1/\tilde{k})} & \frac{\tilde{k}\Gamma(N/\tilde{k})\Gamma(k/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(-1/\tilde{k})} \end{pmatrix}.
\tag{2.48}
$$

Successive actions of $M(T^2)$ and $M(S)$ can be used to obtain the action of any element γ of the theta subgroup of the modular group on $\mathcal{H}^i(x)$, we shall denote the associated matrix by $M(\gamma)$. With the definitions in (2.44), the most general form of solutions to the KZ equations with two identical operators can be written as

$$G^{\beta_1\alpha_2\alpha_2\beta_1}_{\alpha_1\beta_2\beta_2\alpha_1}(x,\bar{x}) = X_{ij}\mathcal{H}^i(x)\bar{\mathcal{H}}^j(\bar{x}). \tag{2.49}$$

Under the action of an element γ of the theta subgroup, the matrix X transforms as

$$X \to M(\gamma)XM^\dagger(\gamma). \tag{2.50}$$

We note that under composition

$$M(\gamma_2.\gamma_1) = M(\gamma_1).M(\gamma_2). \tag{2.51}$$

2.3 Correlators from Modular Averaging

Having obtained the transformation properties of the conformal blocks we now turn to constructing correlators from modular averaging. In this section, we will carry out the modular averaging as per the prescription in (2.18). As described in the previous section, we will focus on the correlator (2.26) after making choices for $SU(N)$ left and right indices so that two pairs of operators are identical. G^{light} will be taken to be the contribution of the vacuum conformal block, as in [32] we will refer to this as the seed contribution. The transformation (2.50) of the matrix X implies that one can write the result of modular averaging as

$$X^{\text{av}} = \mathcal{N}^{-1} \cdot \sum_{\gamma \in \Gamma} M(\gamma) \cdot C_{\text{seed}} \cdot M(\gamma)^\dagger, \tag{2.52}$$

where we have used Γ to denote the theta subgroup and

$$C_{\text{seed}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.53}$$

The normalization constant $\mathcal{N}$ is determined by demanding $[X]_{11} = 1$, so that the $x \to 0$ behaviour of the correlator is correct. For comparison we record the (exact)result of [37]:

$$X^{\mathrm{KZ}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\Gamma(N/\tilde{k}-1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})\Gamma^2(1-N/\tilde{k})}{N^2\Gamma(1-N/\tilde{k}+1/\tilde{k})\Gamma(1-N/\tilde{k}-1/\tilde{k})\Gamma^2(N/\tilde{k})} \end{pmatrix}. \tag{2.54}$$

Before carrying out the sum in explicit examples, let us discuss some generalities. Any element of Γ can be expressed as

$$\gamma = T^{2n_1} S T^{2n_2} S \cdots S T^{2n_k}, \tag{2.55}$$

for some choice of integers n_i (see e.g. [7]). Since we are dealing with a normalised sum, the sum can be reduced to be over the orbit of C_{seed}. Given this, our interest shall be in γ whose action will generate distinct elements. In this context, note that for all (N, k) the action of $M(T^2)$ on C_{seed} is trivial. Also, in the representations under consideration (which are given in (2.46)), T^2 has finite order. Thus, all distinct $M(\gamma)$ can be generated by considering non-negative values of n_i upto the order of T^2. Furthermore, for $M(\gamma)$ of the form $e^{i\alpha}\mathbb{1}$, its action (2.50) on any X is trivial. We define $m(N, k)$ as the smallest positive integer such that

$$M(T^{2m(N,k)}) \propto \mathbb{1}. \tag{2.56}$$

With this, given the trivial actions described above, a list of γs whose actions contain the orbit of C_{seed} can be constructed by considering $\mathbb{1}$ and all elements of the form

$$\gamma = S T^{2r_1} S \cdots S T^{2r_\ell}, \tag{2.57}$$

with ℓ taking values over natural numbers, $r_i = 1 \cdots (m-1)$ for $i = 1 \cdots (\ell - 1)$ and $r_\ell = 0 \cdots (m-1)$. We define the length of an element in the list to be the value of ℓ associated with it (and denote it as $\ell(\gamma)$). $\mathbb{1}$ is defined to be the element of zero length.

The composition rule (2.51) implies

$$M(\gamma) = M(T^{2r_\ell})M(S)\cdots M(S)M(T^{2r_1})M(S). \tag{2.58}$$

If the stabilser of C_{seed} under the action $C_{\text{seed}} \to M(\gamma)\cdot C_{\text{seed}}\cdot M(\gamma)^\dagger$ has finite index, then the sum reduces to a finite number of terms. Otherwise, one has to deal with an infinite sum. We begin by discussing some models in which the stabiliser is of finite index.

Models with $N = k$ are particularly simple. For $N = k$, the actions of S and T as given by (2.48) and (2.46) can be written as

$$M(S) = \begin{pmatrix} \sin\frac{\pi}{2k} & -k\cos\frac{\pi}{2k} \\ -\frac{1}{k}\cos\frac{\pi}{2k} & -\sin\frac{\pi}{2k} \end{pmatrix}, \qquad M(T^2) = e^{-\frac{2\pi i}{3}\cdot\frac{(N^2-1)}{N^2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.59}$$

Note that $M(T^4) \propto \mathbb{1}$, thus the highest power of T that needs to be included while generating the matrices $M(\gamma)$ in the list in (2.57) is T^2. Let us start by discussing a particular example.

$\underline{N = 3, k = 3}$: For $N = 3, k = 3$, the matrices $M(S)$ and $M(T^2)$ are

$$M(S) = \begin{pmatrix} \frac{1}{2} & -\frac{3\sqrt{3}}{2} \\ -\frac{1}{2\sqrt{3}} & -\frac{1}{2} \end{pmatrix}, \qquad M(T^2) = e^{-\frac{16\pi i}{27}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.60}$$

The orbit of C_{seed} consists of three matrices. It is generated by the action of $\mathbb{1}, S$ and ST^2. We tabulate the results of these actions in Table 2.1. The normalised sum over the orbit (2.52) reproduces the KZ result.

γ	$M(\gamma)\cdot C_{\text{seed}}\cdot M(\gamma)^{\dagger}$
$\mathbb{1}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
S	$\begin{pmatrix} \frac{1}{4} & -\frac{1}{4\sqrt{3}} \\ -\frac{1}{4\sqrt{3}} & \frac{1}{12} \end{pmatrix}$
ST^2	$\begin{pmatrix} \frac{1}{4} & \frac{1}{4\sqrt{3}} \\ \frac{1}{4\sqrt{3}} & \frac{1}{12} \end{pmatrix}$
X^{av}	$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{pmatrix}$

Table 2.1: Orbit of the vacuum block for $N = 3, k = 3$

For general values N $(= k)$, one can show that the orbit of C_{seed} is finite by taking repeated products of the matrices $M(S)$ and $M(T^2)$. The orbit is the set

$$\left\{\begin{pmatrix} \sin^2\alpha & -\frac{1}{k}\sin\alpha\cos\alpha \\ -\frac{1}{k}\sin\alpha\cos\alpha & \frac{1}{k^2}\cos^2\alpha \end{pmatrix}\right\} \tag{2.61}$$

where $\alpha = \frac{\pi(2s+1)}{2k}$ with $s = 0\cdots(k-1)$ for k odd, and $\alpha = \frac{\pi s}{2k}$ with $s = 0\cdots(2k-1)$ for k even (we derive this in appendix A.2).

The sums over the orbits can be performed using the identities

$$\sum_{s=0}^{k-1}\sin^2\frac{\pi(2s+1)}{2k} = \frac{k}{2} = \sum_{s=0}^{k-1}\cos^2\frac{\pi(2s+1)}{2k}, \qquad \sum_{s=0}^{k-1}\sin\frac{\pi(2s+1)}{k} = 0$$

for k odd and

$$\sum_{s=0}^{2k-1}\sin^2\frac{\pi s}{2k} = k = \sum_{s=0}^{2k-1}\cos^2\frac{\pi s}{2k}, \qquad \sum_{s=0}^{2k-1}\sin\frac{\pi s}{k} = 0$$

for k even. Normalising the sum, one finds

$$X^{\text{av}} = \begin{pmatrix} 1 & 0 \\ 0 & 1/k^2 \end{pmatrix}, \tag{2.62}$$

which is in agreement with (2.54).

We now turn to models with $N \neq k$ models with finite orbits. For $k = 1$ and any finite N the actions of S and T^2 as given by (2.48) and (2.46) take the identity block to a multiple of itself. Thus the adjoint block decouples and upon modular averaging the correlator is given by $|\mathcal{F}^1_{\mathbb{1}}(\tau)|^2$, in keeping with [37]. Next, we discuss two models: $N = 4, k = 2$ and $N = 2, k = 4$. These examples will reappear in our discussion of the properties of modular averaging under interchange of N and k in section 2.4.

$\underline{N = 4, k = 2}$: For $N = 4, k = 2$ we note that $M(T^6) \propto \mathbb{1}$. The orbit of C_{seed} consists of four matrices. It is generated by the action of $\mathbb{1}$, S, ST^2 and ST^4. The normalised sum over the orbit (2.52) reproduces the KZ result which is $\frac{1}{16\sqrt[3]{2}}$.

$\underline{N = 2, k = 4}$: For $N = 2, k = 4$ we note that $M(T^6) \propto \mathbb{1}$. The orbit of C_{seed} consists of four matrices. It is generated by the action of $\mathbb{1}$, S, ST^2 and ST^4. The normalised sum over the orbit (2.52) reproduces the KZ result which is $\frac{1}{2\sqrt[3]{4}}$.

Finally, we present some models whose orbits do not seem to be finite. We will analyse the models numerically. As described in our general discussion in the beginning of the section, a list of γs whose actions contain the orbit of C_{seed} can be obtained by considering elements of the form (2.57). To implement the numerics, we will organise the sum over the actions of the elements of the list in terms of the length of the elements. We define[7]

$$X^{\text{av}}(\ell_{\max}) = \mathcal{N}(\ell_{\max})^{-1} \cdot \sideset{}{'}\sum_{\ell(\gamma) \leq \ell_{\max}} M(\gamma) \cdot C_{\text{seed}} \cdot M(\gamma)^{\dagger}, \tag{2.63}$$

where the primed sum indicates that we include distinct elements of the orbit of C_{seed} in

[7] Our implementation of the numerics is similar to [32].

the sum. The normalisation constant $\mathcal{N}(\ell_{\max})$ is determined by requiring $X^{\rm av}_{11}(\ell_{\max}) = 1$, so that the $x \to 0$ behaviour of the correlator is correctly reproduced at every value of $\ell_{\max}$.

<u>$N = 2, k = 3$:</u> For $N = 2, k = 3$, we have performed sum in (2.63) upto $\ell_{\max} = 9$. This involves 429226 distinct contributions to the sum. We find $X^{\rm av}_{22}(9) = 0.29863$, which is in good agreement with the exact result (2.54), $X^{\rm KZ}_{22} \approx 0.29831$. The off diagonal entries of $X^{\rm av}(9)$ are of the order of 10^{-13}. Figure 2.1 shows our results for $X^{\rm av}_{22}(\ell_{\max})$ as a function of $\ell_{\max}$. Note that $X^{\rm av}_{22}(\ell_{\max})$ approaches the exact result in an oscillatory manner. Prior to normalisation of the sum, both the $(1,1)$-element as well as the $(2,2)$-element of the matrix have approximately linear growths (all terms in the sum make positive definite contributions to these elements). However, as exhibited by the plot, the ratio of the two quantities (which is $X^{\rm av}_{22}(\ell_{\max})$) tends to a constant. Off-diagonal entries are small as a result of phase cancellations.

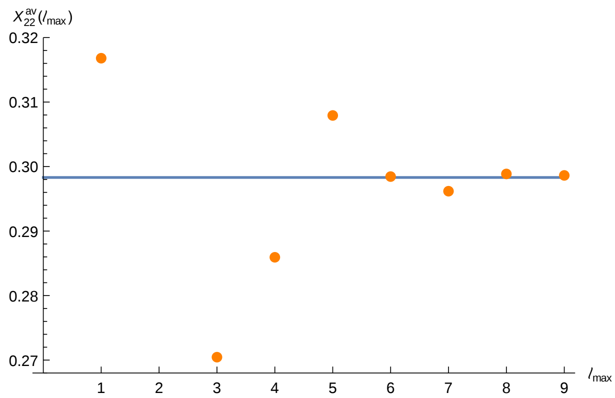

Figure 2.1: Orange dots show $X^{\rm av}_{22}(\ell_{\max})$ in the range $[0.268, 0.320]$ plotted against $\ell_{\max}$. Blue horizontal line at 0.29831 represents $X^{\rm KZ}_{22}$.

<u>$N = 3, k = 2$:</u> For $N = 3, k = 2$, we have performed sum in (2.63) upto $\ell_{\max} = 9$. This involves 429226 distinct contributions to the sum. We find $X^{\rm av}_{22}(9) = 0.0932166$, which is in good agreement with the exact result (2.54), $X^{\rm KZ}_{22} \approx 0.0931172$. The off diagonal entries of $X^{\rm av}(9)$ are of the order of 10^{-14}. Figure 2.2 shows our results for $X^{\rm av}_{22}(\ell_{\max})$ as a function of $\ell_{\max}$. As in the previous example, $X^{\rm av}_{22}(\ell_{\max})$ approaches the exact result in an

oscillatory manner. Other features of the numerics are also similar [8].

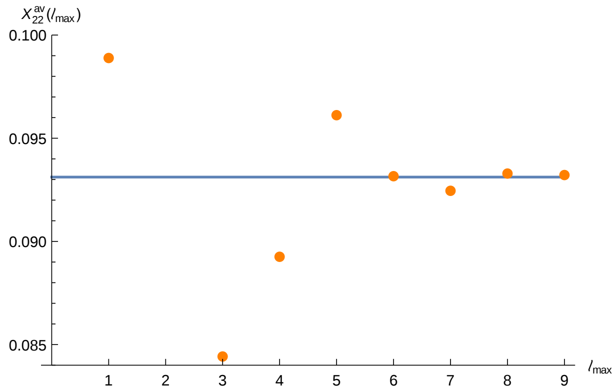

Figure 2.2: Orange dots show $X_{22}^{\rm av}(\ell_{\rm max})$ in the range $[0.084, 0.100]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.0931172 represents $X_{22}^{\rm KZ}$.

$N = 4, k = 3$: For $N = 4, k = 3$, we have performed sum in (2.63) upto $\ell_{\rm max} = 8$. This involves 2338785 distinct contributions to the sum. We find $X_{22}^{\rm av}(8) = 0.0592407$, which is in good agreement with the exact result (2.54), $X_{22}^{\rm KZ} \approx 0.0591147$. The off diagonal entries of $X^{\rm av}(8)$ are of the order of 10^{-14}. Figure 2.3 shows our results for $X_{22}^{\rm av}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$.

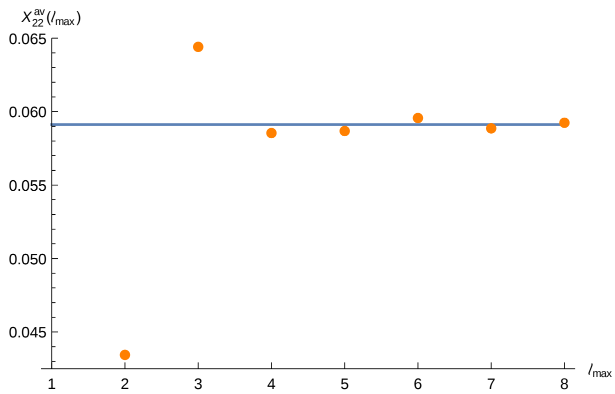

Figure 2.3: Orange dots show $X_{22}^{\rm av}(\ell_{\rm max})$ in the range $[0.0425, 0.0650]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.0591147 represents $X_{22}^{\rm KZ}$.

$N = 3, k = 4$: For $N = 3, k = 4$, we have performed sum in (2.63) upto $\ell_{\rm max} = 8$. This involves 2338785 distinct contributions to the sum. We find $X_{22}^{\rm av}(8) = 0.117725$, which is in good agreement with the exact result (2.54), $X_{22}^{\rm KZ} \approx 0.117474$. The off diagonal entries

[8] This is also true for all models that we study numerically.

of $X^{av}(8)$ are of the order of 10^{-14}. Figure 2.4 shows our results for $X^{av}_{22}(\ell_{max})$ as a function of ℓ_{max}.

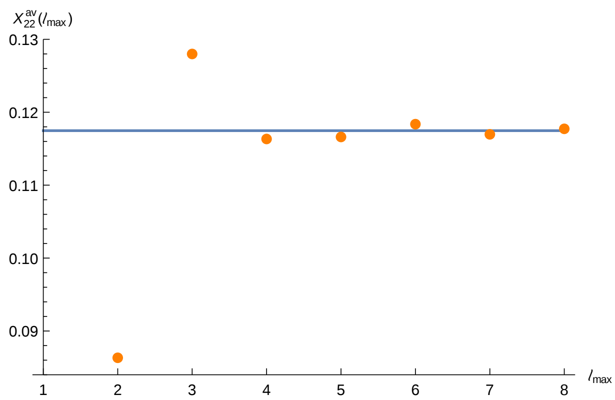

Figure 2.4: Orange dots show $X^{av}_{22}(\ell_{max})$ in the range $[0.084, 0.130]$ plotted against ℓ_{max}. Blue horizontal line at 0.117474 represents X^{KZ}_{22}.

It is interesting to ask whether it is possible to develop an understanding of the nature of the orbit associated with the (N, k) model and at what value of ℓ it terminates (if at all). We have developed a systematic algorithm for this purpose, we discuss this in appendix A.6.

As the values of N and k are increased the numerics can become quite involved. Getting accurate results might require large values of ℓ_{max}. Models with (N, k) equals to $(5, 6)$ and $(6, 5)$ provide examples of this. We discuss them in appendix A.3.

Large N: It is interesting to consider the large N limit of the system, this can be interesting from the point of view of holography. For finite k, the matrices $M(S)$ and $M(T^2)$ have following $\frac{1}{N}$ expansions upto order $\frac{1}{N^2}$.

$$M(S) = \begin{pmatrix} \frac{1}{k} + \frac{\pi^2(k^2-1)}{6kN^2} & -N + \frac{\pi^2}{6N} - \frac{k}{3N^2}\left(\pi^2 k + 3k\psi^{(2)}(1)\right) \\ (-1 + \frac{1}{k^2})\frac{1}{N} & -\frac{1}{k} - \frac{\pi^2(k^2-1)}{6kN^2} \end{pmatrix}, \tag{2.64}$$

$$M(T^2) = \begin{pmatrix} e^{\frac{2\pi i}{3}} - \frac{4\pi k}{3N} e^{\frac{\pi i}{6}} - \frac{4\pi}{9N^2}\{(2(-1)^{2/3}\pi - 3(-1)^{1/6})k^2 + 3(-1)^{1/6}\} & 0 \\ 0 & e^{\frac{2\pi i}{3}} + \frac{2\pi k}{3N} e^{\frac{\pi i}{6}} - \frac{2\pi}{9N^2}\{((-1)^{2/3}\pi + 3(-1)^{1/6})k^2 + 6(-1)^{1/6}\} \end{pmatrix}, \tag{2.65}$$

where $\psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z)$ is the Polygamma function. We have performed modular averaging using above matrices and obtained the associated correlators (it is not possible to carry out the sums analytically, we have performed them making specific choices of N and k with ($N \gg k$) using the numerical recipe described in the first part of this section). The agreement with the results of KZ is good even for low values of ℓ.

The results the for $k = 2,\ 3$ at $\ell_{\max} = 1,\ 2$ are summarised in figures 2.5, 2.6.

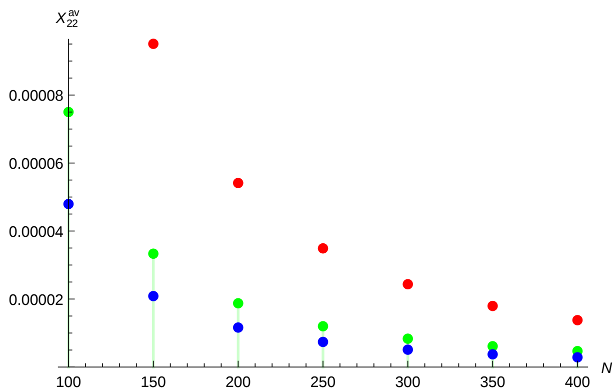

Figure 2.5: Plot for $k = 2$. Red dots show $X_{22}^{\text{av}}(1)$ while blue dots show $X_{22}^{\text{av}}(2)$ in the range $[0, 0.0000965]$ plotted against N. Green dots represent X_{22}^{KZ} against N.

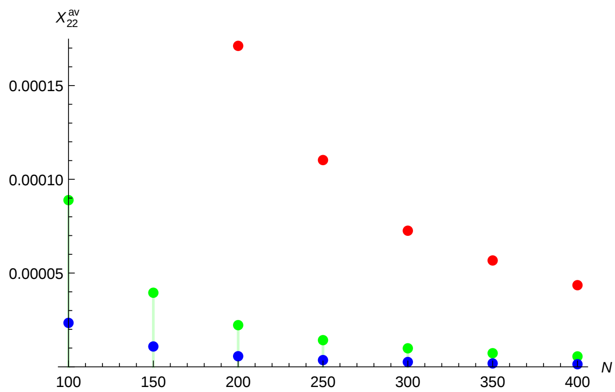

Figure 2.6: Plot for $k = 3$. Red dots show $X_{22}^{\text{av}}(1)$ while blue dots show $X_{22}^{\text{av}}(2)$ in the range $[0, 0.0001750]$ plotted against N. Green dots represent X_{22}^{KZ} against N.

The results indicate that one can obtain correlators by taking the large N limit of the matrices $M(S)$ and $M(T^2)$ (even working at low ℓ). This hints that low ℓ terms should be the most relevant in the context of semi-classical holography.

Finally, we have also considered the prescription for constructing correlators by averaging over the whole $PSL(2,\mathbb{Z})$ (2.17). This involves averaging over a vector and hence is more complicated. We briefly present our results on this in appendix A.4 and leave more detailed explorations for the future.

In summary, in all the cases that we have examined, modular averaging over the theta subgroup successfully reproduces the result of [37]. The correlators can be considered as extremal in the sense of [32]. For extremal correlators, modular averaging sums can be thought of as providing an alternate prescription for their computation. Next, we will examine the properties of these sums involved under interchange of N and k.

2.4 $N \leftrightarrow k$ in Modular Averages

As described in the introduction, an interesting property of WZW models is level-rank duality. In this section, we will show that there is a simple one to one correspondence between individual terms in the modular averaging sums for correlators in the (N,k) and (k,N) theories.

We will be simultaneously dealing with the (N,k) and (k,N) theories in this section, let us begin by introducing notation adapted for the purpose. We will include labels in the matrices (2.46) and (2.48) which generate the actions of S and T^2, to indicate the theory they belong to.

$$M_{N,k}(T^2) = e^{-i4\pi(N^2-1)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi N/\tilde{k}} \end{pmatrix} \equiv e^{i\alpha(N,k)} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi(N,k)} \end{pmatrix} \tag{2.66}$$

and

$$M_{N,k}(S) = \begin{pmatrix} -\frac{\tilde{k}\Gamma(N/\tilde{k})\Gamma(k/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(-1/\tilde{k})} & -\frac{N\Gamma^2(N/\tilde{k})}{\Gamma(N/\tilde{k}-1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})} \\ -\frac{\Gamma^2(k/\tilde{k})}{N\Gamma(k/\tilde{k}-1/\tilde{k})\Gamma(k/\tilde{k}+1/\tilde{k})} & \frac{\tilde{k}\Gamma(N/\tilde{k})\Gamma(k/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(-1/\tilde{k})} \end{pmatrix} \equiv \begin{pmatrix} a_s(N,k) & b_s(N,k) \\ c_s(N,k) & d_s(N,k) \end{pmatrix}. \quad (2.67)$$

We note that $d_s(N,k) = -a_s(N,k)$ and $b_s(N,k).c_s(N,k) = 1 + a_s(N,k).d_s(N,k)$. Also, $a_s(N,k)$ and the product $b_s(N,k).c_s(N,k)$ are symmetric under the interchange of N and k, i.e.

$$a_s(N,k) = a_s(k,N), \qquad d_s(N,k) = d_s(k,N), \qquad b_s(N,k).c_s(N,k) = b_s(k,N).c_s(k,N). \quad (2.68)$$

Recall that the matrices given in (2.58) provide a list whose actions contain the orbit of C_{seed}. We will denote the matrices in the list by

$$M^{\ell}_{N,k}(r_1, r_2\cdots, r_\ell) \equiv M^{\ell}_{N,k}(r_i) \equiv M_{N,k}(T^{2r_\ell})M_{N,k}(S)\cdots M_{N,k}(S)M_{N,k}(T^{2r_1})M_{N,k}(S). \quad (2.69)$$

Note that with this $M^{\ell}_{N,k}(r_i)$ is a function of $r_1, r_2\cdots r_l$; with $r_i = 1\cdots(m(N,k)-1)$ for $i = 1\cdots(\ell-1)$ and $r_\ell = 0\cdots(m(N,k)-1)$ with $m(N,k)$ as defined in (2.56). We define $M^0_{N,k}$ to be the identity matrix. We now introduce another set of matrices

$$\tilde{M}^{\ell}_{N,k}(p_1, p_2\cdots, p_\ell) \equiv \tilde{M}^{\ell}_{N,k}(p_i) \equiv M_{N,k}(T^{-2p_\ell})M_{N,k}(S)\cdots M_{N,k}(S)M_{N,k}(T^{-2p_1})M_{N,k}(S). \quad (2.70)$$

$\tilde{M}^{\ell}_{N,k}(p_i)$ is a function of $p_1, p_2\cdots p_l$; with $p_i = 1\cdots(m(N,k)-1)$ for $i = 1\cdots(\ell-1)$ and $p_\ell = 0\cdots(m(N,k)-1)$. We will define $\tilde{M}^0_{N,k}$ to be the identity matrix.

At any given length ℓ, the set of matrices generated from the action of $M^{\ell}_{N,k}(r_i)$ on C_{seed} is exactly same as the set generated from the action of $\tilde{M}^{\ell}_{N,k}(p_i)$ on C_{seed} i.e.

$$\begin{aligned} &\Big\{M^{\ell}_{N,k}(r_i)C_{\text{seed}}M^{\dagger\ell}_{N,k}(r_i); r_i = 1\cdots(m(N,k)-1) \text{ for } \quad i = 1\cdots(\ell-1), r_\ell = 0\cdots(m(N,k)-1)\Big\} \\ = &\Big\{\tilde{M}^{\ell}_{N,k}(p_i)C_{\text{seed}}\tilde{M}^{\dagger\ell}_{N,k}(p_i); p_i = 1\cdots(m(N,k)-1) \text{ for } \quad i = 1\cdots(\ell-1), p_\ell = 0\cdots(m(N,k)-1))\Big\}. \end{aligned}$$

$$(2.71)$$

This is a consequence of the fact that for any X following equality (between sets) holds

$$\left\{M_{N,k}(T^{2r})XM^{\dagger}_{N,k}(T^{2r}); r = 0\cdots(m(N,k)-1)\right\} = \left\{M_{N,k}(T^{-2p})XM^{\dagger}_{N,k}(T^{-2p}); p = 0\cdots(m(N,k)-1)\right\} \tag{2.72}$$

Given the equivalence in (2.71), while carrying out modular averaging, either set can be used to generate the sum over the orbit of C_{seed}. While establishing the relationship between the modular averages in the (N,k) and (k,N) theories, it will be useful to generate the orbit for the (N,k) theory using the $M^{\ell}_{N,k}$ matrices and for the (k,N) theory using $\tilde{M}^{\ell}_{k,N}$ matrices. The essential point will be to establish that the actions of the two matrices[9]

$$M^{\ell}_{N,k}(r_1, r_2\cdots r_\ell) \;\text{ and }\; \tilde{M}^{\ell}_{k,N}(r_1, r_2\cdots r_\ell) \tag{2.73}$$

on C_{seed} are closely related. Let us begin by looking at the general from of the matrices $M^{\ell}_{N,k}(r_1, r_2\cdots r_\ell)$ and $\tilde{M}^{\ell}_{N,k}(r_1, r_2\cdots r_\ell)$. As shown in appendix A.5, they can be written as

$$M^{\ell}_{N,k}(r_1,\cdots r_\ell) = \exp\left(i\alpha(N,k)(\sum r_i)\right)\begin{pmatrix} a^{\ell}_{N,k}(r_1,\cdots r_\ell) & b_s(N,k)b^{\ell}_{N,k}(r_1,\cdots r_\ell) \\ c_s(N,k)c^{\ell}_{N,k}(r_1,\cdots r_\ell) & d^{\ell}_{N,k}(r_1,\cdots r_\ell)\end{pmatrix} \tag{2.74}$$

$$\tilde{M}^{\ell}_{N,k}(r_1,\cdots r_\ell) = \exp\left(-i\alpha(N,k)(\sum r_i)\right)\begin{pmatrix} \tilde{a}^{\ell}_{N,k}(r_1,\cdots r_\ell) & b_s(N,k)\tilde{b}^{\ell}_{N,k}(r_1,\cdots r_\ell) \\ c_s(N,k)\tilde{c}^{\ell}_{N,k}(r_1,\cdots r_\ell) & \tilde{d}^{\ell}_{N,k}(r_1,\cdots r_\ell),\end{pmatrix} \tag{2.75}$$

with the functions appearing above obeying the relationships

$$\begin{aligned}\tilde{a}^{\ell}_{k,N}(r_1,\cdots r_\ell) &= a^{\ell}_{N,k}(r_1,\cdots r_\ell), \qquad \tilde{b}^{\ell}_{k,N}(r_1,\cdots r_\ell) = b^{\ell}_{N,k}(r_1,\cdots r_\ell),\\ \tilde{c}^{\ell}_{k,N}(r_1,\cdots r_\ell) &= c^{\ell}_{N,k}(r_1,\cdots r_\ell), \qquad \tilde{d}^{\ell}_{k,N}(r_1,\cdots r_\ell) = d^{\ell}_{N,k}(r_1,\cdots r_\ell).\end{aligned} \tag{2.76}$$

[9]Note since $\gcd(k+N,N) = \gcd(k,N) = \gcd(k+N,k)$, $m(N,k) = m(k,N)$. This implies that the arguments of $M^{\ell}_{N,k}$ and $\tilde{M}^{\ell}_{k,N}$ take the same values.

Now, let us discuss the implications of these relations for modular averages. As mentioned before, we will generate the orbit of the (N,k) theory using the matrices $M^{\ell}_{N,k}$ and the (k,N) theory using the $\tilde{M}^{\ell}_{k,N}$ matrices. Firstly, note that (2.74) and (2.75) imply that any duplications in the action of $M^{\ell}_{N,k}$ on C_{seed} implies a duplication in the action of $\tilde{M}^{\ell}_{k,N}$ on C_{seed} and vice versa[10] i.e.

$$M^{\ell}_{N,k}(r_i)C_{\text{seed}}M^{\dagger\ell}_{N,k}(r_i) = M^{\ell}_{N,k}(s_i)C_{\text{seed}}M^{\dagger\ell}_{N,k}(s_i) \Longleftrightarrow \tilde{M}^{\ell}_{k,N}(r_i)C_{\text{seed}}\tilde{M}^{\dagger\ell}_{k,N}(r_i) = \tilde{M}^{\ell}_{k,N}(s_i)C_{\text{seed}}\tilde{M}^{\dagger\ell}_{k,N}(s_i) \tag{2.77}$$

Furthermore, we have

$$M^{\ell}_{N,k}(r_i)C_{\text{seed}}M^{\dagger\ell}_{N,k}(r_i)\Big|_{11} = \tilde{M}^{\ell}_{k,N}(r_i)C_{\text{seed}}\tilde{M}^{\dagger\ell}_{k,N}(r_i)\Big|_{11} \tag{2.78}$$

and

$$c_s^2(k,N)M^{\ell}_{N,k}(r_i)C_{\text{seed}}M^{\dagger\ell}_{N,k}(r_i)\Big|_{22} = c_s^2(N,k)\tilde{M}^{\ell}_{k,N}(r_i)C_{\text{seed}}\tilde{M}^{\dagger\ell}_{k,N}(r_i)\Big|_{22}. \tag{2.79}$$

With this[11], it is natural to pair the matrix

$$M^{\ell}_{N,k}(r_i)C_{\text{seed}}M^{\dagger\ell}_{N,k}(r_i)$$

in the orbit of C_{seed} of the (N,k) theory with the matrix

$$\tilde{M}^{\ell}_{k,N}(r_i)C_{\text{seed}}\tilde{M}^{\dagger\ell}_{k,N}(r_i)$$

in the orbit of C_{seed} of the (k,N) theory. This establishes our one to one correspondence between the terms that appear in the modular averaging sums of the two theories. Note that (2.78) implies that the normalisations of both the sums are equal. With this, (2.79)

[10] This together with (2.71) explains why the number of duplicates for theories related under $N \leftrightarrow k$ were same in our numerical analysis in section 2.3.

[11] It is easy to check that these relationships hold for the (4,2) and (2,4) models (which have finite orbits). For other models we have checked them numerically.

implies that the all paired terms in the sums contribute to the sums with the ratio

$$\frac{c_s^2(N,k)}{c_s^2(k,N)}. \tag{2.80}$$

Of course, since the ratio is same for all the pairs, from the point of view of modular averaging one can trivially write the relation (even without performing the sums)

$$\frac{X^{\mathrm{av}}(N,k)\big|_{22}}{X^{\mathrm{av}}(k,N)\big|_{22}} = \frac{c_s^2(N,k)}{c_s^2(k,N)} = \frac{k^2\Gamma^4\left(k/\tilde{k}\right)\Gamma^2\left(N/\tilde{k}-1/\tilde{k}\right)\Gamma^2\left(N/\tilde{k}+1/\tilde{k}\right)}{N^2\Gamma^2\left(k/\tilde{k}-1/\tilde{k}\right)\Gamma^2\left(k/\tilde{k}+1/\tilde{k}\right)\Gamma^4\left(N/\tilde{k}\right)}. \tag{2.81}$$

One can check by making use of gamma function identities that this is indeed consistent with the KZ result (2.54). Thus, the one to one correspondence between the terms in the two sums has given us relations between OPE coefficients in the theories (as OPE coefficients can be obtained by taking the small cross-ratio limit of the expressions of the correlators in terms of conformal blocks).

It is natural to ask if the one to one correspondence between the terms in the modular averaging sums in the two theories has any physical interpretation. In this context, we note that it was argued in [32] that for "heavy operators" the modular averaging for genus zero correlators can be interpreted as a semiclassical AdS_3 dual computation. More specifically, if the operator dimensions are of the order of the central charge (c) of the theory but less than $c/12$ then the bulk path integral has saddles corresponding to geodesic propagation of heavy particles between the operator insertion points in the boundary [106–115]. Performing the sum over the saddles incorporating the back reaction of the heavy particle geodesics on the geometry and exchange of light primaries, yields the sum over modular channels. But, the operators considered in this chapter cannot be made heavy in the semi-classical limit, since $h_g/c \sim 1/Nk$. One possibility is that the situation is similar to [74] where the topological sectors for the saddle point sum was as given in the semi classical limit even in the quantum regime. In any case, a computation similar to ours for operators satisfying the heavy operator criterion should help reveal how level-rank duality works from a holographic point of view.

2.5 Conclusions

In this chapter, we have analysed correlators involving two fundamentals and two anti-fundamentals in $SU(N)_k$ WZW theories using modular averaging. After determining the transformations of the conformal blocks under S and T transformations, correlators were expressed as sum of the action of the elements of the theta subgroup of $PSL(2, \mathbb{Z})$ on the vacuum block. We found that for all models with $N = k$ the orbit of the vacuum block is finite and modular averaging reproduces the correlators correctly. In models where we were unable to characterise the orbit we performed the sums numerically; modular averaging successfully reproduced the correlators, providing strong evidence that the correlators examined in this chapter are extremal in the sense of [32]. We also considered construction of correlators from averaging over the entire modular group. This is more involved. Here we have found examples where the averaging does not reproduce the correlator (see appendix A.4). Interestingly, [32] argues that it is the modular averaging over the theta subgroup that has a direct interpretation in the holographic context.

We have found a close relationship between modular averaging for correlators involving fundamentals and anti-fundamentals in the (N, k) and (k, N) theories. In section 2.4, we established a one to one correspondence between the orbits of the vacuum conformal blocks of the two theories. The contributions of the paired terms to their respective sums were given by a ratio of elements of braids matrices in the theories. This allowed us to obtain a simple relationship between OPE coefficients.

3 Analyticity of Off-shell Green's Functions in Superstring Field Theory

The study of analytic properties of amplitudes has led to the understanding of various aspects of QFTs. There is much less literature on the analytic properties of amplitudes in string theory. The focus of this chapter will be to obtain the primitive domain (defined in section 3.1.1) of analyticity of the off-shell amputated Green's functions in closed superstring field theory (SFT). The analyticity on the primitive domain is useful to derive analytic properties of superstring amplitudes, similar to the case of a local QFT.

Off-shell momentum space amputated n-point Green's function[1] in SFT is defined by usual momentum space Feynman rules [18, 20][2]. It can be computed by summing over connected Feynman diagrams with n amputated external legs carrying ingoing D-momenta $p_1, \ldots, p_n$. SFT contains massless states. We consider the off-shell n-point Green's function $G(p_1, \ldots, p_n)$ in SFT after explicitly removing contributions coming from one or more massless internal propagators (for more details, see [23]). That is, the relevant part of the perturbative expansion of the off-shell n-point Green's function keeps only those Feynman diagrams that do not contain any internal line corresponding to a massless particle. We call this part of the off-shell Green's function as the infrared safe part[3]. In [23]

[1]Hereafter Green's functions will always refer to momentum space amputated Green's functions.

[2]The general form of its action has been given in (1.2).

[3]Hereafter off-shell Green's functions in SFT will refer to this part of respective off-shell Green's functions in SFT, if not explicitly stated. This part of the off-shell Green's functions — when all the external particles are massless — precisely gives the vertices of the Wilsonian effective field theory of massless fields, obtained by integrating out the massive fields in superstring theory [18, 117].

de Lacroix, Erbin and Sen (LES) showed that the infrared safe part of the off-shell n-point Green's function $G(p_1, \ldots, p_n)$ in SFT as a function of $(n-1)D$ complex variables (taking into account the momentum conservation $\sum_{a=1}^{n} p_a = 0$ for external momenta) is analytic on a domain which we call as the LES domain (reviewed in section 3.1.2). At the heart of this result, it has been proven that each of the relevant Feynman diagrams $\mathrm{F}(p_1, \ldots, p_n)$ has an integral representation in terms of loop integrals as presented below, whenever the external momenta lie on the LES domain.

$$\mathrm{F}(p_1, \ldots, p_n) = \int \prod_{r=1}^{L} \frac{d^D k_r}{(2\pi)^D} \frac{f(k_1, \ldots, k_L; p_1, \ldots, p_n)}{\prod_{s=1}^{\mathcal{I}} \left(\left(\ell_s(\{k_r\}; \{p_a\})\right)^2 + m_s^2 \right)} \,. \tag{3.1}$$

The above analytic function F multiplied by a factor of $(2\pi)^D \delta^{(D)}(p_1 + \cdots + p_n)$ gives the usual Feynman diagram. Equation (3.1) represents an n-legged L-loop graph with $\mathcal{I}$ internal lines, where k_r is a loop momentum, ℓ_s is the momentum of the internal line with mass m_s $(\neq 0)$[4], and f is a regular function whenever its arguments take finite complex values. The function f contains the product of the vertex factors associated with the vertices of the graph, as well as possible momentum dependence from the numerators of the internal propagators. The momentum ℓ_s of an internal line is usually a linear combination of the loop momenta and the external momenta. Due to certain non-local properties of the vertices in SFT, the graph is manifestly UV finite as long as for each r, k_r^0 integration contour ends at $\pm i\infty$, and k_r^i, $i = 1, \ldots, (D-1)$ integration contours end at $\pm\infty$. The prescription for the loop integration contours has been given as follows [20]. At origin, i.e. $p_a = 0$ $\forall a = 1, \ldots, n$ each loop energy integral is to be taken along the imaginary axis from $-i\infty$ to $i\infty$ and each spatial component of loop momenta is to be taken along the real axis from $-\infty$ to ∞. With this, F(0,…,0) has been shown to be finite as all the poles of the integrand in any complex k_r^μ plane are at a finite distance away from the loop integration contour. As we vary the external momenta from the origin to other complex values if some of such poles approach the k_r^μ contour, the contour has to be bent away from those

[4]In our notation, $\ell_s^2 = -(\ell_s^0)^2 + \sum_{i=1}^{D-1} (\ell_s^i)^2$ for each $s = 1, \ldots, \mathcal{I}$.

poles keeping its ends fixed at $\pm i\infty$ for loop energies and $\pm\infty$ for loop momenta. It has been shown that there exists a path inside the LES domain connecting the origin to any other point $p \equiv (p_1, \ldots, p_n)$ of the LES domain such that when we vary external momenta from origin to that point p along that path, the loop integration contours in any graph can be deformed away avoiding poles of the integrand which approach them [23]. Hence the integral representation (3.1) for $\mathrm{F}(p_1, \ldots, p_n)$ when the external momenta lie on the LES domain is well defined where the poles of the integrand are at a finite distance away from the (deformed)loop integration contours.

In this chapter, we aim to generalize the result of [23] by showing that the infrared safe part of the off-shell n-point Green's function in SFT is analytic on a larger domain than the LES domain. As will be reviewed in section 3.1.2, the analyticity property of the off-shell Green's functions in SFT is invariant under the action of a D-dimensional complex Lorentz transformation on all the external momenta. Thus the off-shell Green's functions in SFT are also analytic at points that are obtained by the action of such transformations on points in the LES domain [23]. We consider LES domain adjoining these new points. The primitive domain essentially contains the union of a certain family of convex tube domains. We call such tubes as the primitive tubes. Within each such primitive tube, we identify a connected tube which is also contained in the LES domain and its shape allows us to carry out a holomorphic extension to its convex hull inside the corresponding primitive tube, due to a classic theorem by Bochner [116]. The domain thus found may still be smaller. We explicitly work out the cases of the three-, four- and five-point functions to determine whether such convex hulls fully obtain respective primitive tubes, or not. The extension to the primitive domain is trivial for the two-point function. In the case of three-point function, indeed such extensions yield all the 6 possible primitive tubes. Also for the four-point function, by such extensions all the 32 possible primitive tubes are obtained. Whereas for the five-point function, out of 370 possible primitive tubes, for 350 of them we are able to show that such extensions obtain each of them fully. The technique that we employ for aforesaid checks seems difficult to implement analytically for the remaining 20 primitive tubes whose shapes are complicated. This technical difficulty is a feature of the higher point functions as well. However in any case, our work establishes that based on a geometric consideration only, the LES domain is holomorphically extended inside all the primitive tubes. This extension does not depend on the details of the Green's functions. Thus with respect to all the analyticity properties of the S-matrix which can be obtained from this extended domain[5], superstring theory behaves like a standard local QFT that has massless states.

We organize the chapter as follows. In section 3.1, we briefly review certain properties of the primitive domain and the LES domain which are useful for our purpose. In section 3.2, we start with the general scheme to extend the LES domain holomorphically. We apply this scheme to the case of the three-point function in section 3.2.1. In section 3.2.2 (and appendix B.6) we deal with the case of the four-point function, and in section 3.2.3 the case of five-point function. We discuss certain real limits within each of the primitive tubes in section 3.3. As our analysis uses properties of the theory of functions of several complex variables, we discuss them in appendix B.1.

3.1 Review

In this section we review certain properties of the two domains, namely the primitive domain and the LES domain which will be useful for our analysis. Both the domains are domains in the complex manifold $\mathbb{C}^{(n-1)D}$ given by $p_1 + \cdots + p_n = 0$. The origin of $\mathbb{C}^{(n-1)D}$ which is given by $p_a = 0\ \forall a = 1, \ldots, n$ will be denoted by O. We count p_a as positive if ingoing, negative otherwise. We shall use Minkowski metric with mostly plus signature.

[5]Note that for the 2-, 3- and 4-point functions, the extended domain is equal to the primitive domain.

3.1.1 The primitive domain

The primitive domain $\mathcal{D}$ is given by

$$\begin{aligned}\mathcal{D} = \ \Big\{ p \equiv (p_1, \ldots, p_n) :\ & \textstyle\sum_{a=1}^{n} p_a = 0 \text{ and for each } I \in \wp^*(X) \\ & \text{either, Im } P_I \neq 0,\ (\text{Im } P_I)^2 \leq 0 \quad \text{or, Im } P_I = 0,\ -P_I^2 < M_I^2 \Big\}, \end{aligned} \tag{3.2}$$

where $X = \{1, \ldots, n\}$ is the set of first n natural numbers. $\wp^*(X) = \{I \subsetneq X, \text{ except } \emptyset\}$ is the collection of all non-empty proper subsets of X. P_I is defined to be equal to $\sum_{a \in I} p_a$. M_I is the threshold of production of any (multi-particle) state in a channel containing the external states in the set $\{p_a,\ a \in I\}$, i.e. M_I is the invariant threshold mass for producing any set of intermediate states in the collision of particles carrying total momentum P_I.

Clearly, $O \in \mathcal{D}$. Primitive domain is star-shaped with respect to O, i.e. the straight line segment connecting O and any point $p \in \mathcal{D}$ which is given by $\{tp :\ t \in [0, 1]\}$ lie entirely inside $\mathcal{D}$. Hence the primitive domain is path-connected as any two points $p^{(1)},\ p^{(2)} \in \mathcal{D}$ can be connected via the straight line segments $p^{(1)}O$ and $Op^{(2)}$. Furthermore primitive domain is simply connected, i.e. any closed curve within $\mathcal{D}$ can be continuously shrunk to the point O, which is a property of a star-shaped domain.

Primitive domain essentially contains the union of a family of mutually disjoint tube[6] domains denoted by $\{\mathcal{T}_\lambda,\ \lambda \in \Lambda^{(n)}\}$ [85, 87, 118, 119]. Any member $\mathcal{T}_\lambda$ of this family will be called a primitive tube. To describe this family of tubes the following definitions are needed. We consider the space $\mathbb{R}^{n-1}$ of n real variables $s_1, \ldots, s_n$ linked by the relation $s_1 + \cdots + s_n = 0$. We define $S_I = \sum_{a \in I} s_a$ for each $I \in \wp^*(X)$. The family of planes $\{S_I = 0,\ I \in \wp^*(X)\}$ (where the planes $S_I = 0$ and $S_{X \setminus I} = 0$ are identical) divides the above space $\mathbb{R}^{n-1}$ into open convex cones[7] with common apex at the origin. Any such

[6] A subset of $\mathbb{C}^m$ is called a tube if it is equal to $\mathbb{R}^m + iA$ for some subset A of $\mathbb{R}^m$ where m is a given natural number. A is called the base of the tube.

[7] A subset A of $\mathbb{R}^m$ is called a cone if any point $p \in A \implies \alpha p \in A\ \forall \alpha > 0$.

cone will be called a cell, γ_λ. Within a cell each S_I is of definite sign $\lambda(I)$. Thus a cell can be written as

$$\gamma_\lambda = \left\{ s \in \mathbb{R}^{n-1} : \ \lambda(I) S_I > 0 \quad \forall I \in \wp^*(X) \right\}, \tag{3.3}$$

where $\lambda : \wp^*(X) \to \{-1, 1\}$ is a sign-valued map with the following properties.

$$\begin{aligned}
&\text{(i) } \forall I \in \wp^*(X) \quad \lambda(I) = -\lambda(X \setminus I), \\
&\text{(ii) } \forall I, J \in \wp^*(X) \text{ with } I \cap J = \emptyset \text{ and } \lambda(I) = \lambda(J) \quad \lambda(I \cup J) = \lambda(I) = \lambda(J)\,.
\end{aligned} \tag{3.4}$$

The first property is compatible with $s_1 + \cdots + s_n = 0$ and the second property is compatible with $S_{I \cup J} = S_I + S_J$ whenever $I \cap J = \emptyset$. $\Lambda^{(n)}$ denotes the collection of all possible maps λ satisfying above properties. Corresponding to each cell γ_λ, now we associate an open convex tube domain $\mathcal{T}_\lambda$ (primitive tube) given by[8]

$$\begin{aligned}
\mathcal{T}_\lambda &= \left\{ p \equiv (p_1, \ldots, p_n) : \ \textstyle\sum_{a=1}^n p_a = 0, \ \lambda(I) \mathrm{Im}\, P_I \in V^+ \quad \forall I \in \wp^*(X) \right\} \\
&= \mathbb{R}^{(n-1)D} \ + \ i C_\lambda \,, \\
C_\lambda &= \left\{ \mathrm{Im}\, p \in \mathbb{R}^{(n-1)D} : \ \lambda(I) \mathrm{Im}\, P_I \in V^+ \quad \forall I \in \wp^*(X) \right\},
\end{aligned} \tag{3.5}$$

where V^+ is the open forward lightcone in $\mathbb{R}^D$. C_λ is the conical base of the tube $\mathcal{T}_\lambda$. Although the primitive domain is non-convex the primitive tubes $\mathcal{T}_\lambda$ are convex (see appendix B.2). Hence each tube $\mathcal{T}_\lambda$ is path-connected as the entire straight line segment $p^{(1)} p^{(2)}$ connecting any two points $p^{(1)},\ p^{(2)} \in \mathcal{T}_\lambda$ is contained in the tube $\mathcal{T}_\lambda$.

[8]With $i = \sqrt{-1}$.

3.1.2 The LES domain

The LES domain is given by

$$\mathcal{D}' = \Big\{ p \equiv (p_1, \dots, p_n) : \ \forall a \ \mathrm{Im}\, p_a^\mu = 0,\ \mu \neq 0,1;\ \sum\nolimits_{a=1}^n p_a = 0 \text{ and for each } I \in \wp^*(X)$$
$$\text{either, } \mathrm{Im}\, P_I \neq 0,\ (\mathrm{Im}\, P_I)^2 \leq 0 \quad \text{or, } \mathrm{Im}\, P_I = 0,\ -P_I^2 < M_I^2 \Big\}\,, \tag{3.6}$$

where all the Im p_a thereby Im P_I are allowed to lie only on the two dimensional Lorentzian plane[9] $p^0 - p^1$. Clearly $O \in \mathcal{D}'$ and the LES domain $\mathcal{D}'$ is contained in the primitive domain $\mathcal{D}$.

In [23] it has been argued that the domain of holomorphy of the n-point Green's function $G(p_1, \dots, p_n)$ in SFT is a connected region in $\mathbb{C}^{(n-1)D}$ containing the origin, and it is invariant under the action of Lorentz transformations $\tilde{\Lambda}$ with complex parameters, i.e. $\tilde{\Lambda}$ is any complex matrix satisfying $\tilde{\Lambda}^T \eta \tilde{\Lambda} = \eta$ for η being the Minkowski metric in $\mathbb{R}^D$. We call the set of such matrices the complex Lorentz group, $\mathfrak{L}$. In general, the action of a complex Lorentz transformation $\tilde{\Lambda}$ is defined on the complex manifold $\mathbb{C}^{(n-1)D}$ taking a point to another point of the same manifold given by

$$(p_1, \dots, p_n) \longmapsto (\tilde{\Lambda} p_1, \dots, \tilde{\Lambda} p_n)\,, \tag{3.7}$$

which we abbreviate as $p \mapsto \tilde{\Lambda} p$. Note that the same $\tilde{\Lambda}$ acts on all p_a.

As as consequence, the result of [23] automatically generalizes to a larger domain than the LES domain $\mathcal{D}'$, i.e. $G(p)$ is analytic on the domain $\tilde{\mathcal{D}}'$ given by

$$\tilde{\mathcal{D}}' = \Big\{ \tilde{\Lambda} p :\ p \in \mathcal{D}',\ \tilde{\Lambda} \in \mathfrak{L} \Big\}\,. \tag{3.8}$$

Clearly $\tilde{\mathcal{D}}' \supset \mathcal{D}'$, since $\mathfrak{L}$ contains the identity matrix. Hereafter we refer $\tilde{\mathcal{D}}'$ as the LES

[9] By a two dimensional Lorentzian plane we refer to any two dimensional plane in $\mathbb{R}^D$ which contains the p^0-axis.

domain.

3.2 Extension of the LES domain $\tilde{\mathcal{D}}'$

We identify a family of tubes lying inside the primitive tube $\mathcal{T}_\lambda$ which is a convex tube domain (described by equation (3.5) in section 3.1.1) such that any member of the family is also contained in the LES domain $\tilde{\mathcal{D}}'$ (described in section 3.1.2). Any member of this family is convex and it can be characterized by a set of $D-2$ angles $\vec{\theta}$ which specifies a two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}}$[10] with $\vec{\theta} = 0$ specifying the $p^0 - p^1$ plane. Hence we denote a member by $\mathcal{T}_\lambda^{\vec{\theta}}$. The convex tube $\mathcal{T}_\lambda^{\vec{\theta}}$ is given by

$$
\begin{aligned}
\mathcal{T}_\lambda^{\vec{\theta}} &= \Big\{ p \equiv (p_1, \ldots, p_n) : \ \forall a \ \mathrm{Im}\, p_a \in \ p^0 - p^{\vec{\theta}} \ \text{plane}; \ \sum\nolimits_{a=1}^{n} p_a = 0, \\
&\qquad\qquad \text{and } \lambda(I) \mathrm{Im}\, P_I \in V^+ \quad \forall I \in \wp^*(X) \Big\} \\
&= \mathbb{R}^{(n-1)D} \ + \ i C_\lambda^{\vec{\theta}} \,,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
C_\lambda^{\vec{\theta}} &= \Big\{ (\mathrm{Im}\, p_1, \ldots, \mathrm{Im}\, p_n) \text{ on manifold } \sum\nolimits_{a=1}^{n} \mathrm{Im}\, p_a = 0 \text{ such that} \\
&\qquad \forall a \ \mathrm{Im}\, p_a \in \ p^0 - p^{\vec{\theta}} \text{ plane and } \lambda(I) \mathrm{Im}\, P_I \in V^+ \quad \forall I \in \wp^*(X) \Big\} \,,
\end{aligned}
$$

where the base $C_\lambda^{\vec{\theta}}$ is a subset of $\mathbb{R}^{(n-1)D}$. Any such tube $\mathcal{T}_\lambda^{\vec{\theta}}$ can be obtained by acting a real rotation on the tube $\mathcal{T}_\lambda^{\vec{\theta}=0}$.

Clearly $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ lies inside the primitive tube $\mathcal{T}_\lambda$ as well as it is contained in the LES domain $\tilde{\mathcal{D}}'$. Now $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is a tube given by $\mathbb{R}^{(n-1)D} + i(\bigcup_{\vec{\theta}} C_\lambda^{\vec{\theta}})$. Although each tube $\mathcal{T}_\lambda^{\vec{\theta}}$ is convex

[10]Here the axis $p^{\vec{\theta}}$ lies in the subspace $\mathbb{R}^{D-1}$ of points $(p^1, \ldots, p^{D-1})$. It can be specified by a point on the unit sphere S^{D-2} in $\mathbb{R}^{D-1}$. With a set of $D-2$ angles $(\theta_1, \ldots, \theta_{D-2}) \equiv \vec{\theta}$ where $0 \leq \theta_1, \ldots, \theta_{D-3} \leq \pi$ and $0 \leq \theta_{D-2} < 2\pi$, the axis $p^{\vec{\theta}}$ can be explicitly written as

$$
p^{\vec{\theta}} = (\cos\theta_1, \ \sin\theta_1 \cos\theta_2, \ \sin\theta_1 \sin\theta_2 \cos\theta_3, \ \ldots, \ \sin\theta_1 \cdots \sin\theta_{D-3} \cos\theta_{D-2}) \,.
$$

(see appendix B.2) the tube $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is non-convex[11] (see appendix B.3). However the tube $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is path-connected (see appendix B.4).

We apply Bochner's tube theorem [116, 120][12] on the connected tube $(\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}})$[13]. The theorem states that any open connected tube $\mathbb{R}^m + iA$ has a holomorphic extension[14] to the domain $\mathbb{R}^m + i\mathrm{Ch}(A)$ where $\mathrm{Ch}(A)$ is the smallest convex set containing the set A, called the convex hull of A.

By the above application, since $\bigcup_{\vec{\theta}} C_\lambda^{\vec{\theta}}$ is non-convex we always have a holomorphic extension of $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ to a domain given by $\mathbb{R}^{(n-1)D} + i\mathrm{Ch}(\bigcup_{\vec{\theta}} C_\lambda^{\vec{\theta}})$. Furthermore C_λ contains $\mathrm{Ch}(\bigcup_{\vec{\theta}} C_\lambda^{\vec{\theta}})$ since C_λ is a convex cone containing $\bigcup_{\vec{\theta}} C_\lambda^{\vec{\theta}}$. In subsequent subsections, we deal with the explicit cases of the three-, four- and five-point functions, where up to the four-point function we obtain that for each C_λ such extension yields the full of C_λ, i.e. $\mathrm{Ch}(\bigcup_{\vec{\theta}} C_\lambda^{\vec{\theta}}) = C_\lambda$, and for five-point function we obtain subcases in which we are able to prove this equality.

For the two-point function (i.e. $n = 2$), $\mathcal{T}_\lambda = \bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$. That is, whenever $p \in \mathcal{T}_\lambda$, $\mathrm{Im}\, p_1$ $(= -\mathrm{Im}\, p_2)$ lies on some two dimensional Lorentzian plane.

Remarks

The properties of the tube $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ as a domain in several complex variables have been used here to extend it holomorphically. From the work of [23], we know that all the relevant Feynman diagrams (those which do not have any internal line of a massless particle) in the perturbative expansion of the n-point Green's function are analytic in the common tube $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$. Hence our extension of $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is valid to all orders in perturbation theory.

[11] $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is non-convex when $n > 2$.
[12] In [120] it has been stated as the 'convex tube theorem' at the end of its third section. This version of the theorem is suitable for our purpose.
[13] This tube can be thickened in order to make it open (see appendix B.5).
[14] By a holomorphic extension Ω' of a domain Ω in $\mathbb{C}^m$ we mean any larger domain Ω' containing Ω, with the property that all the functions which are holomorphic on Ω are also holomorphic on Ω'.

A proper application of Bochner's tube theorem requires us to thicken the connected tube $\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}$ in order to make it open. But the thickened tubes (as in appendix B.5) are not identical for all the Feynman diagrams. However the intersection of all these thickened tubes corresponding to distinct diagrams certainly contains $\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}$. We consider any relevant Feynman diagram. The corresponding thickened tube can be holomorphically extended to its convex hull due to Bochner's tube theorem. This extended domain contains the tube $\mathrm{Ch}(\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}) = \mathbb{R}^{(n-1)D} + i\mathrm{Ch}(\bigcup_{\vec{\theta}} C_{\lambda}^{\vec{\theta}})$. Clearly, $\mathrm{Ch}(\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}})$ lies in the intersection of all such extensions corresponding to distinct diagrams since $\mathrm{Ch}(\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}})$ only includes convex combinations of points from $\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}$. Hence $\mathrm{Ch}(\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}})$ is the domain where all the relevant Feynman diagrams (at all orders in perturbation theory) are analytic.

3.2.1 Three-point function

For three-point function, we have $n = 3$ and the sign-valued maps $\lambda(I)$ (described by equation (3.4)) can be given explicitly as follows. In this case, the primitive domain $\mathcal{D}$ essentially contains the union of 6 mutually disjoint tubes denoted by $\{\mathcal{T}_a^{(3)\pm},\ a = 1, 2, 3\}$ and these primitive tubes are given by[15]

$$\mathcal{T}_a^{(3)\pm} = \left\{ p \in \mathbb{C}^{2D} :\ \mathrm{Im}\, p \in C_a^{(3)\pm} \right\}, \tag{3.10}$$

where $p = (p_1, p_2, p_3)$ is linked by $p_1 + p_2 + p_3 = 0$. Their conical bases are defined by

$$C_a^{(3)+} = -C_a^{(3)-} = \left\{ \mathrm{Im}\, p :\ \mathrm{Im}\, p_b,\ \mathrm{Im}\, p_c \in V^+ \right\}, \tag{3.11}$$

where (abc) = permutation of (123). In order to define each of the above cones $C_a^{(3)+}$ ($C_a^{(3)-}$), we require a certain pair of imaginary external momenta which (or their negative) are specified to be in the open forward lightcone V^+. For a given conical base, this in turn

[15]Primitive tubes are generally defined in (3.5). Here, an additional superscript $^{(3)}$ in the notations for the tubes stands for the 3-point function.

fixes all other Im P_I to be in specific lightcone[16].

The cones (3.11) reside on the manifold Im p_1 + Im p_2 + Im $p_3 = 0$. In order to assign coordinates to the points in $C_a^{(3)+}$, let us choose $\{\text{Im } p_b,\ \text{Im } p_c\}$ as our set of basis vectors. On the other hand, for $C_a^{(3)-}$, let us choose $\{-\text{Im } p_b, -\text{Im } p_c\}$ as our basis. With this, any of the above cones is contained in a $\mathbb{R}^{2D}$ and is of the following common form

$$C^{(3)} = \left\{\vec{Q} = (P_\alpha, P_\beta) : \ P_\alpha,\ P_\beta \in V^+\right\}, \tag{3.12}$$

where any $\vec{Q} \in C^{(3)}$ can be written as a $D \times 2$ matrix given by

$$\vec{Q} = \begin{pmatrix} P_\alpha^0 & P_\beta^0 \\ P_\alpha^1 & P_\beta^1 \\ \vdots & \vdots \\ P_\alpha^{D-1} & P_\beta^{D-1} \end{pmatrix}, \tag{3.13}$$

with conditions $P_r^0 > +\sqrt{\sum_{i=1}^{D-1}(P_r^i)^2} \quad \forall r = \alpha, \beta$ ensuring that both the columns belong to the forward lightcone V^+. Hence given a $\vec{Q}$, the quantities $P_r^0 - \sqrt{\sum_i (P_r^i)^2},\ r = \alpha, \beta$ are a pair of positive numbers. Furthermore, the two columns of $\vec{Q}$ in general do not lie on a same two dimensional Lorentzian plane.

Now we consider cones $C^{(3)\vec{\theta}}$ containing points $\tilde{\vec{Q}}$ where both the columns not only belong to V^+ but also lie on a same two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}}$ characterized by a $\vec{\theta}$. We shall show that taking points from these cones for various $\vec{\theta}$, a convex combination of them represents given $\vec{Q}$ in (3.13). This will complete the proof of $\text{Ch}(\bigcup_{\vec{\theta}} C^{(3)\vec{\theta}}) = C^{(3)}$, since we already have $C^{(3)} \supset \text{Ch}(\bigcup_{\vec{\theta}} C^{(3)\vec{\theta}})$ as discussed right above the remarks in section 3.2.

[16]For $n = 3$, the total number of possible $P_I = 2^3 - 2 = 6$.

We consider two points $\tilde{\vec{Q}}_1 \in C^{(3)\vec{\theta}_1}$ and $\tilde{\vec{Q}}_2 \in C^{(3)\vec{\theta}_2}$ given by

$$\tilde{\vec{Q}}_1 = 2\begin{pmatrix} P^0_\alpha - \epsilon & \epsilon \\ P^1_\alpha & 0 \\ \vdots & \vdots \\ P^{D-1}_\alpha & 0 \end{pmatrix}, \quad \tilde{\vec{Q}}_2 = 2\begin{pmatrix} \epsilon & P^0_\beta - \epsilon \\ 0 & P^1_\beta \\ \vdots & \vdots \\ 0 & P^{D-1}_\beta \end{pmatrix}, \tag{3.14}$$

where we take any ϵ satisfying $0 < \epsilon < \min\{P^0_r - \sqrt{\sum_{i=1}^{D-1}(P^i_r)^2},\ r = \alpha, \beta\}$ so that for each $r = \alpha, \beta$ we have $P^0_r - \epsilon > +\sqrt{\sum_{i=1}^{D-1}(P^i_r)^2}$. Both the columns of $\tilde{\vec{Q}}_1$ lie on a same two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}_1}$ where also the first column P_α of $\vec{Q}$ in (3.13) lies. Similarly, both the columns of $\tilde{\vec{Q}}_2$ lie on the two dimensional Lorentzian plane where P_β lies. Now it is easy to check that the following relation holds

$$\frac{\tilde{\vec{Q}}_1}{2} + \frac{\tilde{\vec{Q}}_2}{2} = \vec{Q}\,. \tag{3.15}$$

Equation (3.15) establishes that each of the 6 primitive tubes given in (3.10) can be obtained as holomorphic extension of a tube where the latter is contained in the LES domain $\tilde{\mathcal{D}}'$.

3.2.2 Four-point function

For four-point function, we have $n = 4$ and the maps $\lambda(I)$ (described by equation (3.4)) can be given explicitly as follows. In this case, the primitive domain $\mathcal{D}$ essentially contains the union of 32 mutually disjoint tubes denoted by $\{\mathcal{T}^{(4)\pm}_a,\ \mathcal{T}^{(4)\pm}_{ab},\ 1 \le a, b \le 4,\ a \ne b\}$[17] and these primitive tubes are given by [85, 87]

$$\mathcal{T}^{(4)\pm}_a = \left\{p \in \mathbb{C}^{3D} :\ \mathrm{Im}\ p \in C^{(4)\pm}_a\right\}, \quad \mathcal{T}^{(4)\pm}_{ab} = \left\{p \in \mathbb{C}^{3D} :\ \mathrm{Im}\ p \in C^{(4)\pm}_{ab}\right\}, \tag{3.16}$$

[17] Here a superscript $^{(4)}$ in the notations for the tubes stands for the 4-point function.

where $p = (p_1, \ldots, p_4)$ is linked by $p_1 + \cdots + p_4 = 0$. Their conical bases are defined by

$$\begin{aligned} C_a^{(4)+} &= -C_a^{(4)-} = \left\{ \text{Im}\, p : \; \text{Im}\, p_b, \; \text{Im}\, p_c, \; \text{Im}\, p_d \in V^+ \right\}, \\ C_{ab}^{(4)+} &= -C_{ab}^{(4)-} = \left\{ \text{Im}\, p : \; -\text{Im}\, p_b, \; \text{Im}\, (p_b + p_c), \; \text{Im}\, (p_b + p_d) \in V^+ \right\}, \end{aligned} \tag{3.17}$$

where $(abcd)$ = permutation of (1234). Note that in order to describe each of the above cones we require a certain set of three Im P_I each of which (or its negative) is specified to be in the open forward lightcone V^+. For a given conical base, this in turn fixes all other Im P_I to be in specific lightcone[18] (see appendix B.6).

The cones (3.17) reside on the manifold Im $p_1 + \cdots +$ Im $p_4 = 0$. Due to this link we can choose any three linear combinations of Im $p_1, \ldots,$ Im p_4 which are linearly independent as our set of basis vectors, to describe a given cone. In particular as our basis, we choose $\{\text{Im}\, p_b, \; \text{Im}\, p_c, \; \text{Im}\, p_d\}$ for the cones $C_a^{(4)+}$, whereas we choose $\{-\text{Im}\, p_b, -\text{Im}\, p_c, -\text{Im}\, p_d\}$ for the cones $C_a^{(4)-}$. Besides, as our basis, we choose $\{-\text{Im}\, p_b, \; \text{Im}\, (p_b + p_c), \; \text{Im}\, (p_b + p_d)\}$ for the cones $C_{ab}^{(4)+}$[19], whereas we choose $\{\text{Im}\, p_b, -\text{Im}\, (p_b + p_c), -\text{Im}\, (p_b + p_d)\}$ for the cones $C_{ab}^{(4)-}$. With this, any of the above cones is contained in a $\mathbb{R}^{3D}$ and is of the following common form

$$C^{(4)} = \left\{ \vec{Q} = (P_\alpha, P_\beta, P_\gamma) : \; P_\alpha, P_\beta, P_\gamma \in V^+ \right\}, \tag{3.18}$$

where any $\vec{Q} \in C^{(4)}$ can be written as a $D \times 3$ matrix given by

$$\vec{Q} = \begin{pmatrix} P_\alpha^0 & P_\beta^0 & P_\gamma^0 \\ P_\alpha^1 & P_\beta^1 & P_\gamma^1 \\ \vdots & \vdots & \vdots \\ P_\alpha^{D-1} & P_\beta^{D-1} & P_\gamma^{D-1} \end{pmatrix}, \tag{3.19}$$

with conditions $P_r^0 > +\sqrt{\sum_{i=1}^{D-1} (P_r^i)^2} \quad \forall r = \alpha, \beta, \gamma$ ensuring that each of the columns

[18]For $n = 4$, the total number of possible $P_I = 2^4 - 2 = 14$.

[19]Instead, one can choose $\{\text{Im}\, p_b, \; \text{Im}\, p_c, \; \text{Im}\, p_d\}$ as the basis to describe points in any of the cones $C_{ab}^{(4)+}$. This change of basis is a linear invertible transformation and the work of this section 3.2.2 can be recast in this new basis (e.g., see appendix B.6).

belong to the forward lightcone V^+. Hence given a $\vec{Q}$ the quantities $P_r^0 - \sqrt{\sum_i (P_r^i)^2}$, $r = \alpha, \beta, \gamma$ are three positive numbers. Furthermore the columns of $\vec{Q}$ in general do not lie on a same two dimensional Lorentzian plane.

Now we consider cones $C^{(4)\vec{\theta}}$ containing points $\vec{\tilde{Q}}$ where all the three columns not only belong to V^+ but also lie on the same two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}}$ characterized by a $\vec{\theta}$. We shall show that taking points from these cones for various $\vec{\theta}$, a convex combination of them represents given $\vec{Q}$ in (3.19). This will complete the proof of $\mathrm{Ch}(\bigcup_{\vec{\theta}} C^{(4)\vec{\theta}}) = C^{(4)}$, since we already have $C^{(4)} \supset \mathrm{Ch}(\bigcup_{\vec{\theta}} C^{(4)\vec{\theta}})$ as discussed right above the remarks in section 3.2.

We consider three points $\vec{\tilde{Q}}_1 \in C^{(4)\vec{\theta}_1}$, $\vec{\tilde{Q}}_2 \in C^{(4)\vec{\theta}_2}$ and $\vec{\tilde{Q}}_3 \in C^{(4)\vec{\theta}_3}$ given by

$$\vec{\tilde{Q}}_1 = 3\begin{pmatrix} P_\alpha^0 - \epsilon & \epsilon/2 & \epsilon/2 \\ P_\alpha^1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ P_\alpha^{D-1} & 0 & 0 \end{pmatrix}, \quad \vec{\tilde{Q}}_2 = 3\begin{pmatrix} \epsilon/2 & P_\beta^0 - \epsilon & \epsilon/2 \\ 0 & P_\beta^1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & P_\beta^{D-1} & 0 \end{pmatrix},$$
$$\vec{\tilde{Q}}_3 = 3\begin{pmatrix} \epsilon/2 & \epsilon/2 & P_\gamma^0 - \epsilon \\ 0 & 0 & P_\gamma^1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & P_\gamma^{D-1} \end{pmatrix}, \tag{3.20}$$

where we take any ϵ satisfying $0 < \epsilon < \min\{P_r^0 - \sqrt{\sum_{i=1}^{D-1}(P_r^i)^2},\ r = \alpha, \beta, \gamma\}$ so that for each $r = \alpha, \beta, \gamma$ we have $P_r^0 - \epsilon > +\sqrt{\sum_{i=1}^{D-1}(P_r^i)^2}$. All the three columns of $\vec{\tilde{Q}}_1$ lie on a same two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}_1}$ where also the first column P_α of $\vec{Q}$ in (3.19) lies. Similarly, all the columns of $\vec{\tilde{Q}}_2$ lie on the two dimensional Lorentzian plane where P_β lies, and all the columns of $\vec{\tilde{Q}}_3$ lie on the two dimensional Lorentzian plane where P_γ lies. Now it is easy to check that the following relation holds

$$\frac{\vec{\tilde{Q}}_1}{3} + \frac{\vec{\tilde{Q}}_2}{3} + \frac{\vec{\tilde{Q}}_3}{3} = \vec{Q}\,. \tag{3.21}$$

Equation (3.21) establishes that each of the 32 primitive tubes given in (3.16) can be obtained as holomorphic extension of a tube where the latter is contained in the LES domain $\tilde{\mathcal{D}}'$.

3.2.3 Five-point function

For five-point function, we have $n = 5$ and in this case the primitive domain $\mathcal{D}$ essentially contains the union of 370 mutually disjoint tubes whose conical bases are given by [85][20]

$$\begin{aligned}
C_a^{(5)+} = -C_a^{(5)-} \quad &= \Big\{ \text{Im}\, p : \ \text{Im}\, p_b,\ \text{Im}\, p_c,\ \text{Im}\, p_d,\ \text{Im}\, p_e \in V^+ \Big\},\\
C_{ab}^{(5)+} = -C_{ab}^{(5)-} \quad &= \Big\{ \text{Im}\, p : \ -\text{Im}\, p_b,\ \text{Im}\,(p_b + p_c),\ \text{Im}\,(p_b + p_d),\ \text{Im}\,(p_b + p_e) \in V^+ \Big\},\\
C_{ab}'^{(5)+} = -C_{ab}'^{(5)-} \quad &= \Big\{ \text{Im}\, p : \ \text{Im}\,(p_a + p_c),\ \text{Im}\,(p_a + p_d),\ \text{Im}\,(p_a + p_e),\\
&\qquad\qquad \text{Im}\,(p_b + p_c),\ \text{Im}\,(p_b + p_d),\ \text{Im}\,(p_b + p_e) \in V^+ \Big\},\\
C_{ab,c}^{(5)+} = -C_{ab,c}^{(5)-} \quad &= \Big\{ \text{Im}\, p : \ \text{Im}\, p_c,\ -\text{Im}\,(p_b + p_c),\ \text{Im}\,(p_b + p_d),\ \text{Im}\,(p_b + p_e) \in V^+ \Big\},\\
C_{ab,c}'^{(5)+} = -C_{ab,c}'^{(5)-} \quad &= \Big\{ \text{Im}\, p : \ -\text{Im}\,(p_b + p_c),\ \text{Im}\,(p_a + p_c),\ \text{Im}\,(p_b + p_d),\ \text{Im}\,(p_b + p_e) \in V^+ \Big\},\\
C_{a,bc}^{(5)+} = -C_{a,bc}^{(5)-} \quad &= \Big\{ \text{Im}\, p : \ \text{Im}\, p_d,\ \text{Im}\, p_e,\ \text{Im}\,(p_a + p_b),\ \text{Im}\,(p_a + p_c) \in V^+ \Big\}\,,
\end{aligned} \tag{3.22}$$

where $(abcde)$ = permutation of (12345) and $\text{Im}\, p = (\text{Im}\, p_1, \ldots, \text{Im}\, p_5)$ is linked by the relation $\text{Im}\, p_1 + \cdots + \text{Im}\, p_5 = 0$. Due to this link we can choose any four linear combinations of $\text{Im}\, p_1, \ldots, \text{Im}\, p_5$ which are linearly independent as our set of basis vectors, to describe a cone which is given from the above list (3.22). Hence any of these cones is contained in a $\mathbb{R}^{4D}$ with a choice for a basis.

[20] Here a superscript $^{(5)}$ in the notations for the cones stands for the 5-point function. And following [85] we use a prime only to distinguish between two classes of cones having same indices (ab) or (ab, c).

We note that in order to describe each of the cones in (3.22) except the cones $C'^{(5)+}_{ab}$, $C'^{(5)-}_{ab}$, we require a certain set of four Im P_I each of which (or its negative) is specified to be in the open forward lightcone V^+. This in turn fixes all other Im P_I to be in specific lightcone[21]. Now we confine ourselves to these cones which are 350 in numbers[22]. To describe any of these cones we choose the corresponding certain set of four Im P_I as our basis (in a similar manner to the cases of the three-point and four-point functions, as demonstrated in detail in the sections 3.2.1, and 3.2.2 respectively). With this, any of these cones is of the following common form

$$C^{(5)} = \left\{\vec{Q} = (P_\alpha, P_\beta, P_\gamma, P_\delta) : \; P_\alpha, P_\beta, P_\gamma, P_\delta \in V^+\right\}, \tag{3.23}$$

where any $\vec{Q} \in C^{(5)}$ can be written as a $D \times 4$ matrix given by

$$\vec{Q} = \begin{pmatrix} P^0_\alpha & P^0_\beta & P^0_\gamma & P^0_\delta \\ P^1_\alpha & P^1_\beta & P^1_\gamma & P^1_\delta \\ \vdots & \vdots & \vdots & \vdots \\ P^{D-1}_\alpha & P^{D-1}_\beta & P^{D-1}_\gamma & P^{D-1}_\delta \end{pmatrix}, \tag{3.24}$$

with conditions $P^0_r > +\sqrt{\sum_{i=1}^{D-1}(P^i_r)^2} \quad \forall r = \alpha, \beta, \gamma, \delta$. Given a $\vec{Q}$ as in (3.24) it can now be represented as the following convex combination.

$$\frac{\vec{\tilde{Q}}_1}{4} + \frac{\vec{\tilde{Q}}_2}{4} + \frac{\vec{\tilde{Q}}_3}{4} + \frac{\vec{\tilde{Q}}_4}{4} = \vec{Q}\,, \tag{3.25}$$

[21] For $n = 5$, the total number of possible $P_I = 2^5 - 2 = 30$.

[22] Each of $C'^{(5)+}_{ab}$ and $C'^{(5)-}_{ab}$ is symmetric under the interchange of a, b which is evident from (3.22). Hence they are 20 in total.

where $\vec{\tilde{Q}}_r,\ r = 1,\ldots,4$ are given by

$$
\begin{aligned}
\vec{\tilde{Q}}_1 &= 4\begin{pmatrix} P^0_\alpha - \epsilon & \epsilon/3 & \epsilon/3 & \epsilon/3 \\ P^1_\alpha & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ P^{D-1}_\alpha & 0 & 0 & 0 \end{pmatrix}, \quad
\vec{\tilde{Q}}_2 = 4\begin{pmatrix} \epsilon/3 & P^0_\beta - \epsilon & \epsilon/3 & \epsilon/3 \\ 0 & P^1_\beta & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & P^{D-1}_\beta & 0 & 0 \end{pmatrix}, \\
\vec{\tilde{Q}}_3 &= 4\begin{pmatrix} \epsilon/3 & \epsilon/3 & P^0_\gamma - \epsilon & \epsilon/3 \\ 0 & 0 & P^1_\gamma & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & P^{D-1}_\gamma & 0 \end{pmatrix}, \quad
\vec{\tilde{Q}}_4 = 4\begin{pmatrix} \epsilon/3 & \epsilon/3 & \epsilon/3 & P^0_\delta - \epsilon \\ 0 & 0 & 0 & P^1_\delta \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & P^{D-1}_\delta \end{pmatrix},
\end{aligned}
\tag{3.26}
$$

where we take any ϵ satisfying the condition: $0 < \epsilon < \min\{P^0_r - \sqrt{\sum_{i=1}^{D-1}(P^i_r)^2},\ r = \alpha,\beta,\gamma,\delta\}$.

Equation (3.25) establishes that each of the primitive tubes described by (3.22) except the ones whose conical bases are $C'^{(5)+}_{ab}$, $C'^{(5)-}_{ab}$ can be obtained as holomorphic extension of a tube where the latter is contained in the LES domain $\tilde{\mathcal{D}}'$.

The above technique has limitations. Following difficulty arrises when we consider the remaining 20 cones which are given by $C'^{(5)+}_{ab}$, $C'^{(5)-}_{ab}$. To describe points in $C'^{(5)+}_{ab}$ let us choose the set

$$
\Big\{\mathrm{Im}\,(p_a + p_c),\ \mathrm{Im}\,(p_a + p_d),\ \mathrm{Im}\,(p_a + p_e),\ \mathrm{Im}\,(p_b + p_c)\Big\}
$$

as our basis, and to describe points in $\tilde{C}'^{(5)-}_{ab}$ let us choose the set

$$
\Big\{-\mathrm{Im}\,(p_a + p_c),\ -\mathrm{Im}\,(p_a + p_d),\ -\mathrm{Im}\,(p_a + p_e),\ -\mathrm{Im}\,(p_b + p_c)\Big\}
$$

as our basis. With this any of the cones $C'^{(5)+}_{ab}$, $C'^{(5)-}_{ab}$ is of the following common form

$$C'^{(5)} = \left\{ \vec{Q} = (P_\alpha, P_\beta, P_\gamma, P_\delta) : \; P_\alpha,\; P_\beta,\; P_\gamma,\; P_\delta,\; (P_\beta + P_\delta - P_\alpha) \text{ and } (P_\gamma + P_\delta - P_\alpha) \in V^+ \right\}. \tag{3.27}$$

Due to additional constraints on the linear combinations $(P_\beta + P_\delta - P_\alpha)$ and $(P_\gamma + P_\delta - P_\alpha)$ the technique which we have employed in earlier cases seems difficult to implement here analytically, in order to check the validity of $\mathrm{Ch}(\bigcup_{\vec{\theta}} C'^{(5)\vec{\theta}}) = C'^{(5)}$. Here each cone $C'^{(5)\vec{\theta}}$ is to be obtained from $C'^{(5)}$ by putting further restrictions on its points $\vec{Q} = (P_\alpha, P_\beta, P_\gamma, P_\delta)$ so that $\forall r = \alpha, \beta, \gamma, \delta \;\; P_r$ lies on the two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}}$. That is, it is difficult to find a set of points $\vec{\tilde{Q}}_r,\; r = 1, \ldots, m$ for some m[23], each of which has four columns satisfying the six conditions as stated in (3.27) and furthermore all the four columns lie on a two dimensional Lorentzian plane, in such a way that a convex combination of these m points produce a general point $\vec{Q}$ in (3.27). As an illustration we work with one of these 20 problematic cones in appendix B.7 (in which case, as a trial we take $m = 4$).

3.3 Limits within $\mathcal{T}_\lambda$

In section 3.2, we have shown that for an n-point Green's function and given any λ from the possible set $\Lambda^{(n)}$, the tube $\bigcup_{\vec{\theta}} \mathcal{T}^{\vec{\theta}}_\lambda$ has holomorphic extension inside the primitive tube $\mathcal{T}_\lambda$ where the former tube is contained in the LES domain $\tilde{\mathcal{D}}'$.

As per the equations (3.6) and (3.8), if we take the limit $\mathrm{Im}\, P_I \to 0,\; \mathrm{Im}\, P_I \in \mathcal{T}^{\vec{\theta}}_\lambda$ for a collection of subsets $\{I\} \subset \wp^*(X)$, the n-point Green's function $G(p)$ in SFT is finite whenever we restrict their real parts by $-P_I^2 < M_I^2$ for each I belonging to that collection $\{I\}$. Here $\mathrm{Re}\, P_J$ are kept arbitrary for all $J \in \wp^*(X) \setminus \{I\}$. In fact, for a given collection $\{I\}$ by taking such limits within $\mathcal{T}^{\vec{\theta}}_\lambda$ for any $\vec{\theta}$ and restricting corresponding real parts, we

[23]Here $m \leq 4D + 1$, due to Carathéodory's theorem: if A is a non-empty subset of $\mathbb{R}^q$, then any point of the convex hull of A is representable as a convex combination of at most $q + 1$ points of A.

reach to same value $G(p)$ for all $\vec{\theta}$. Now that $\bigcup_{\vec{\theta}}\mathcal{T}_{\lambda}^{\vec{\theta}}$ has an unique holomorphic extension given by $\mathrm{Ch}(\bigcup_{\vec{\theta}}\mathcal{T}_{\lambda}^{\vec{\theta}}) \subset \mathcal{T}_{\lambda}$, we reach to above value $G(p)$ in the limit $\mathrm{Im}\, P_I \to 0,\ \mathrm{Im}\, P_I \in \mathrm{Ch}(\bigcup_{\vec{\theta}}\mathcal{T}_{\lambda}^{\vec{\theta}})$ with above constraints on the real parts.

Evidently the family of holomorphic functions $\{G_{\lambda}(p),\ \lambda \in \Lambda^{(n)}\}$[24] defined on the family of mutually disjoint tubes $\{\mathrm{Ch}(\bigcup_{\vec{\theta}}\mathcal{T}_{\lambda}^{\vec{\theta}}),\ \lambda \in \Lambda^{(n)}\}$ coincide on a real domain $\mathcal{R}$ given by

$$\mathcal{R} = \left\{ p \equiv (p_1, \ldots, p_n) : \ \sum\nolimits_{a=1}^{n} p_a = 0 \ \text{ and } \ \forall I \in \wp^*(X)\ \ -P_I^2 < M_I^2 \right\}. \tag{3.28}$$

3.4 Conclusions

In this chapter, we have shown that for any n-point Green's function in superstring field theory, the LES domain $\tilde{\mathcal{D}}'$ due to its shape always admits a holomorphic extension within the primitive domain $\mathcal{D}$ where the latter is basically the union of the convex primitive tubes. In the process we have found that the LES domain $\tilde{\mathcal{D}}'$ contains a non-convex connected tube within each convex primitive tube. The former tube being non-convex allows to include all the new points from its convex hull which is the set of all convex combinations of points in that tube. The convex tube thus obtained is a holomorphic extension of the former non-convex tube due to a classic theorem by Bochner, and lies inside the corresponding primitive tube.

Up to the four-point function such extension yields the full of the primitive domain. We have proved this result, in section 3.2.1 for the three-point function obtaining all the 6 primitive tubes, and in section 3.2.2 for the four-point function obtaining all the 32 primitive tubes. The appropriate real limits within those tubes in both cases are also attained (as discussed in section 3.3).

In section 3.2.3, we are able to show that for the five-point function such extension yields the full of 350 primitive tubes out of 370 primitive tubes which are possible in this case.

[24] Here $G_{\lambda}(p)$ denotes the analytic continuation of $G(p)$ defined on $\bigcup_{\vec{\theta}}\mathcal{T}_{\lambda}^{\vec{\theta}}$ to the domain $\mathrm{Ch}(\bigcup_{\vec{\theta}}\mathcal{T}_{\lambda}^{\vec{\theta}})$.

The technique employed in this section can not be applied (as it is) for the remaining 20 primitive tubes, their shape being complicated. However within all these 370 extensions inside respective primitive tubes (obtaining 350 of them fully) the appropriate real limits are attained (as discussed in section 3.3). The difficulty arising for the 20 primitive tubes has been demonstrated in appendix B.7 which is a generic feature for all higher-point functions as well. We expect it to be only a technical issue.

As a consequence, our result shows that with respect to all the analyticity properties of the S-matrix which can be obtained relying on above extended domain inside the primitive domain, the infrared safe part of the S-matrix of superstring theory has similar behaviour to that of a standard local QFT. Any non-analyticity of the full S-matrix of SFT is entirely due to the presence of massless states — which is also the case for a standard local QFT. The current approach is perturbative (because it uses Feynman diagrams) whereas the original proof of primitive analyticity for local QFTs is non-perturbative. Thus superstring amplitudes might also have potential singularities on the primitive domain arising from non-perturbative effects. Local QFTs are free from those. Furthermore, in local QFTs the following estimate holds on a primitive tube $\mathcal{T}_\lambda$ for each truncated cone $K^r \subset C_\lambda \cup \{0\}$[25].

$$\left|G\left(\mathrm{Re}\; p + i\mathrm{Im}\; p\right)\right| \;\leq\; A\frac{(1+ \parallel \mathrm{Re}\; p \parallel)^m}{\parallel \mathrm{Im}\; p \parallel^l} \qquad \forall\; \mathrm{Re}\; p \in \mathbb{R}^{(n-1)D},\; \mathrm{Im}\; p \in K^r \setminus \{0\}\;, \quad (3.29)$$

where the numbers $A, m, l > 0$ depend on K^r [82][26]. This is guaranteed as the off-shell n-point Green's function in a local QFT is equal to the Fourier-Laplace transform of some generalized function (more precisely, a tempered distribution) which is a position space correlator, and the analyticity of the off-shell Green's function on a primitive tube follows from causality constraints on the position space correlators. Equation (3.29) is a place

[25] A cone has been generally defined in footnote 7. Given a cone K and a positive number r, a truncated cone is defined as (using $\parallel v \parallel$ to denote the Euclidean norm of v):

$$K^r = \{v \in K : \parallel v \parallel \leq r\}\;.$$

[26] $|z|$ denotes the modulus of a complex number z.

where superstring field theory could differ since typically its non-local vertices prevent us from defining position space correlators.

4 Outlook

The work described in this thesis opens up several novel directions for research. We begin by listing some specific directions that we feel are interesting:

- In chapter 2, our focus was on correlators in $SU(N)_k$ WZW models involving two fundamentals and two anti-fundamentals only. We have proven that the modular averagings for these correlators in the (N, k) and (k, N) theories are related. A prescription relating general correlators of WZW models under level-rank duality has been given in [99]. The braid matrices of the theories for general correlators have been related in [104, 105]. It will be interesting to study the implications of these relations for modular averaging in more general correlators.

- As discussed in the later part of section 2.4 of chapter 2, we believe that our results give a strong hint that holographic computations can make various aspects of level-rank duality in WZW models manifest. A first step in this direction can be to consider correlators of heavy operators in the theories and analyse their conformal blocks in the semi-classical limit.

- In appendix A.4, the construction of correlators by averaging over the whole $PSL(2, \mathbb{Z})$ following (2.17) involves averaging over a vector. Here we have scanned correlators for low values of (N, k) and have found examples for which the modular averaging despite being crossing symmetric does not reproduce the correlator. Increasing N and k makes the numerics quite involved, we leave this for future work. Answering that whether the crossing symmetric modular averagings correspond to any CFT correlators or not would also be interesting.

– Chapter 3 proves the analyticity of the infrared safe part of off-shell 2-, 3- and 4-point Green's functions in SFT on the full primitive domain, at all orders in perturbation theory. We expect it to be true for higher-point functions as well. Obtaining the envelope of holomorphy of the primitive domain (as posed in [88]) still remains a challenging open question. Answering this potentially may lead to new dispersion relations in local QFTs, as well as SFT owing to our results.

– The difficulty arising for the twenty primitive tubes in the case for the five-point function has been demonstrated in appendix B.7. This is a generic feature that arises for all higher-point functions, in the course of determining whether or not the application of Bochner's tube theorem yields certain primitive tubes fully. Solving this may require numerical analysis. We leave this for future work.

– Whether the momentum space results in [23, 123] can be inverted to extract information on position space Green's functions in SFT remains an interesting question.

Following up on these directions will certainly strengthen our understanding of CFTs and string field theory. This in turn can have an impact on the more challenging goals such as completion of the AdS/CFT dictionary and a non-perturbative understanding of string theory. Of course, the ultimate goal is to connect these ideas to observations.

Appendix A.

A.1 Conformal Blocks and Their Transformations:

In this appendix, we list the conformal blocks associate with the following three correlators[1]

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar{z}_1)\cdot g^{-1}_{\beta_2}{}^{\alpha_2}(z_2,\bar{z}_2)\cdot g^{-1}_{\beta_3}{}^{\alpha_3}(z_3,\bar{z}_3)\cdot g_{\alpha_4}{}^{\beta_4}(z_4,\bar{z}_4)\rangle \tag{A.1}$$

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar{z}_1)\cdot g^{-1}_{\beta_2}{}^{\alpha_2}(z_2,\bar{z}_2)\cdot g_{\alpha_4}{}^{\beta_4}(z_3,\bar{z}_3)\cdot g^{-1}_{\beta_3}{}^{\alpha_3}(z_4,\bar{z}_4)\rangle \tag{A.2}$$

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar{z}_1)\cdot g_{\alpha_4}{}^{\beta_4}(z_2,\bar{z}_2)\cdot g^{-1}_{\beta_2}{}^{\alpha_2}(z_3,\bar{z}_3)\cdot g^{-1}_{\beta_3}{}^{\alpha_3}(z_4,\bar{z}_4)\rangle \tag{A.3}$$

and their transformation properties under the modular tranformations (after the identification (2.40) described in section 2.2). We will refer to the correlators listed above as the first, second and third correlators. Blocks and their transformation matrices will be given subscripts to indicate the correlator they belong to.

For the first correlator

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar{z}_1)\cdot g^{-1}_{\beta_2}{}^{\alpha_2}(z_2,\bar{z}_2)\cdot g^{-1}_{\beta_3}{}^{\alpha_3}(z_3,\bar{z}_3)\cdot g_{\alpha_4}{}^{\beta_4}(z_4,\bar{z}_4)\rangle$$

[1] The other three independent correlators in (2.13) are related to these by the interchange $I_1 \leftrightarrow I_2$. Thus they can be easily obtained from the data in this appendix.

the holomorphic conformal blocks[2] are

$$\begin{aligned}\mathcal{F}^{\mathbb{1}}_{(1)}(x) &= I_1 F^{\mathbb{1}}_{(1)1}(x) + I_2 F^{\mathbb{1}}_{(1)2}(x),\\ \mathcal{F}^{\theta}_{(1)}(x) &= I_1 F^{\theta}_{(1)1}(x) + I_2 F^{\theta}_{(1)2}(x),\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}F^{\mathbb{1}}_{(1)1}(x) &= x^{-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}} F\left(\frac{1}{\tilde{k}}, -\frac{1}{\tilde{k}}; 1-\frac{N}{\tilde{k}}; x\right),\\ F^{\mathbb{1}}_{(1)2}(x) &= \frac{1}{k} x^{1-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}} F\left(1+\frac{1}{\tilde{k}}, 1-\frac{1}{\tilde{k}}; 2-\frac{N}{\tilde{k}}; x\right),\\ F^{\theta}_{(1)1}(x) &= x^{h_\theta-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}} F\left(\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}}, \frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; 1+\frac{N}{\tilde{k}}; x\right),\\ F^{\theta}_{(1)2}(x) &= -N x^{h_\theta-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}} F\left(\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}}, \frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; \frac{N}{\tilde{k}}; x\right).\end{aligned} \tag{A.5}$$

The holomorphic blocks for the correlator

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar{z}_1)\cdot g^{-1\alpha_2}_{\beta_2}(z_2,\bar{z}_2)\cdot g_{\alpha_4}{}^{\beta_4}(z_3,\bar{z}_3)\cdot g^{-1\alpha_3}_{\beta_3}(z_4,\bar{z}_4)\rangle$$

are

$$\begin{aligned}\mathcal{F}^{\mathbb{1}}_{(2)}(x) &= I_1 F^{\mathbb{1}}_{(2)1}(x) + I_2 F^{\mathbb{1}}_{(2)2}(x),\\ \mathcal{F}^{\theta}_{(2)}(x) &= I_1 F^{\theta}_{(2)1}(x) + I_2 F^{\theta}_{(2)2}(x),\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}F^{\mathbb{1}}_{(2)1}(x) &= x^{-\frac{4h_g}{3}}(1-x)^{h_\chi-\frac{4h_g}{3}} F\left(\frac{1}{\tilde{k}}, 1-\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; 1-\frac{N}{\tilde{k}}; x\right),\\ F^{\mathbb{1}}_{(2)2}(x) &= -\frac{1}{k} x^{1-\frac{4h_g}{3}}(1-x)^{h_\chi-\frac{4h_g}{3}} F\left(1+\frac{1}{\tilde{k}}, 1-\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; 2-\frac{N}{\tilde{k}}; x\right),\\ F^{\theta}_{(2)1}(x) &= x^{h_{\hat{\theta}}-\frac{4h_g}{3}}(1-x)^{h_\chi-\frac{4h_g}{3}} F\left(1+\frac{1}{\tilde{k}}, \frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; 1+\frac{N}{\tilde{k}}; x\right),\end{aligned}$$

[2]The blocks for this correlator have already been discussed in the main text. We rewrite them here with the subscript convention discussed above, so as to have a consistent notation for this appendix.

$$F^{\theta}_{(2)2}(x) = -Nx^{h_{\hat\theta}-\frac{4h_g}{3}}(1-x)^{h_\chi-\frac{4h_g}{3}}F\left(\frac{1}{\tilde k},\frac{N}{\tilde k}+\frac{1}{\tilde k};\frac{N}{\tilde k};x\right). \tag{A.7}$$

The holomorphic blocks for the correlator

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar z_1)\cdot g_{\alpha_4}{}^{\beta_4}(z_2,\bar z_2)\cdot g^{-1}_{\beta_2}{}^{\alpha_2}(z_3,\bar z_3)\cdot g^{-1}_{\beta_3}{}^{\alpha_3}(z_4,\bar z_4)\rangle$$

are

$$\begin{aligned}\mathcal{F}^{\xi}_{(3)}(x) &= I_1F^{\xi}_{(3)1}(x)+I_2F^{\xi}_{(3)2}(x),\\ \mathcal{F}^{\chi}_{(3)}(x) &= I_1F^{\chi}_{(3)1}(x)+I_2F^{\chi}_{(3)2}(x),\end{aligned} \tag{A.8}$$

where

$$\begin{aligned}F^{\xi}_{(3)1}(x) &= x^{h_\xi-\frac{4h_g}{3}}(1-x)^{h_{\hat\theta}-\frac{4h_g}{3}}F\left(1-\frac{1}{\tilde k},\frac{N}{\tilde k}-\frac{1}{\tilde k};1-\frac{2}{\tilde k};x\right),\\ F^{\xi}_{(3)2}(x) &= -x^{h_\xi-\frac{4h_g}{3}}(1-x)^{h_{\hat\theta}-\frac{4h_g}{3}}F\left(-\frac{1}{\tilde k},\frac{N}{\tilde k}-\frac{1}{\tilde k};1-\frac{2}{\tilde k};x\right),\\ F^{\chi}_{(3)1}(x) &= x^{h_\chi-\frac{4h_g}{3}}(1-x)^{h_{\hat\theta}-\frac{4h_g}{3}}F\left(1+\frac{1}{\tilde k},\frac{N}{\tilde k}+\frac{1}{\tilde k};1+\frac{2}{\tilde k};x\right),\\ F^{\chi}_{(3)2}(x) &= x^{h_\chi-\frac{4h_g}{3}}(1-x)^{h_{\hat\theta}-\frac{4h_g}{3}}F\left(\frac{1}{\tilde k},\frac{N}{\tilde k}+\frac{1}{\tilde k};1+\frac{2}{\tilde k};x\right).\end{aligned} \tag{A.9}$$

With the choices for tensor indices as in (2.40), we will denote the holomorphic blocks of the three correlators by $\mathcal{H}^{i}_{(q)}(x)$ with $q=1,2,3$ i.e.

$$\begin{aligned}\mathcal{H}^{\mathbb{1}}_{(1)}(x) &= IF^{\mathbb{1}}_{(1)1}(x)+IF^{\mathbb{1}}_{(1)2}(x),\\ \mathcal{H}^{\theta}_{(1)}(x) &= IF^{\theta}_{(1)1}(x)+IF^{\theta}_{(1)2}(x),\\ \mathcal{H}^{\mathbb{1}}_{(2)}(x) &= IF^{\mathbb{1}}_{(2)1}(x)+IF^{\mathbb{1}}_{(2)2}(x),\\ \mathcal{H}^{\theta}_{(2)}(x) &= IF^{\theta}_{(2)1}(x)+IF^{\theta}_{(2)2}(x),\\ \mathcal{H}^{\xi}_{(3)}(x) &= IF^{\xi}_{(3)1}(x)+IF^{\xi}_{(3)2}(x),\\ \mathcal{H}^{\chi}_{(3)}(x) &= IF^{\chi}_{(3)1}(x)+IF^{\chi}_{(3)2}(x).\end{aligned} \tag{A.10}$$

We note that with $I_1 = I_2$ the three correlators are equal to those in (2.41).

The actions of T and S on these can be computed using the following identities of hypergeometric functions [102].

$$\begin{aligned} F(a,b;c;z) =& (1-z)^{c-a-b}F(c-a,c-b;c;z), \\ F(a,b;c;\frac{z}{z-1}) =& (1-z)^{a}F(a,c-b;c;z) = (1-z)^{b}F(c-a,b;c;z), \\ F(a,b;c;1-z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z^{c-a-b}F(c-a,c-b;c-a-b+1;z). \end{aligned} \tag{A.11}$$

$$\begin{aligned} F(a,b;c;1-z) =& \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z^{c-a-b}(1-z)^{1-c}F(1-b,1-a;1+c-a-b,z) \\ &+ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}(1-z)^{1-c}F(1+b-c,1+a-c;1+a+b-c;z) \end{aligned} \tag{A.12}$$

<u>Action of T</u>: The action of T on the blocks $\mathcal{H}^i_{(1)}(x)$ are given by

$$\mathcal{H}^i_{(1)}(T.x) = \mathcal{H}^j_{(2)}(x)\, M_{(1)ji}(T), \tag{A.13}$$

where

$$M_{(1)}(T) = (-1)^{-2(N^2-1)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{N/\tilde{k}} \end{pmatrix}. \tag{A.14}$$

The action of T on the blocks $\mathcal{H}^i_{(2)}(x)$ are given by

$$\mathcal{H}^i_{(2)}(T.x) = \mathcal{H}^j_{(1)}(x)\, M_{(2)ji}(T), \tag{A.15}$$

where

$$M_{(2)}(T) = (-1)^{-2(N^2-1)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{N/\tilde{k}} \end{pmatrix}. \tag{A.16}$$

The action of T on the blocks $\mathcal{H}^i_{(3)}(x)$ are given by

$$\mathcal{H}^i_{(3)}\,(T.x) = \mathcal{H}^j_{(3)}\,(x)\,M_{(3)ji}(T), \tag{A.17}$$

where

$$M_{(3)}(T) = -(-1)^{(N^2-3N-4)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & -(-1)^{2/\tilde{k}} \end{pmatrix}. \tag{A.18}$$

<u>Action of S</u>: The action of S on the blocks $\mathcal{H}^i_{(1)}(x)$ are given by

$$\mathcal{H}^i_{(1)}\,(S.x) = \mathcal{H}^j_{(1)}\,(x)\,M_{(1)ji}(S), \tag{A.19}$$

where

$$M_{(1)}(S) = \begin{pmatrix} -\frac{\tilde{k}\Gamma(N/\tilde{k})\Gamma(k/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(-1/\tilde{k})} & -\frac{N\Gamma^2(N/\tilde{k})}{\Gamma(N/\tilde{k}-1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})} \\ -\frac{\Gamma^2(k/\tilde{k})}{N\Gamma(k/\tilde{k}-1/\tilde{k})\Gamma(k/\tilde{k}+1/\tilde{k})} & \frac{\tilde{k}\Gamma(N/\tilde{k})\Gamma(k/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(-1/\tilde{k})} \end{pmatrix}. \tag{A.20}$$

The action of S on the blocks $\mathcal{H}^i_{(2)}(x)$ are given by

$$\mathcal{H}^i_{(2)}\,(S.x) = \mathcal{H}^j_{(3)}\,(x)\,M_{(2)ji}(S), \tag{A.21}$$

where

$$M_{(2)}(S) = \begin{pmatrix} \frac{\Gamma(k/\tilde{k})\Gamma(2/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(k/\tilde{k}+1/\tilde{k})} & \frac{N\Gamma(N/\tilde{k})\Gamma(2/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})} \\ \frac{\Gamma(k/\tilde{k})\Gamma(-2/\tilde{k})}{\Gamma(k/\tilde{k}-1/\tilde{k})\Gamma(-1/\tilde{k})} & -\frac{N\Gamma(N/\tilde{k})\Gamma(-2/\tilde{k})}{\Gamma(N/\tilde{k}-1/\tilde{k})\Gamma(-1/\tilde{k})} \end{pmatrix}. \tag{A.22}$$

The action of S on the blocks $\mathcal{H}^i_{(3)}(x)$ are given by

$$\mathcal{H}^i_{(3)}\,(S.x) = \mathcal{H}^j_{(2)}\,(x)\,M_{(3)ji}(S), \tag{A.23}$$

where

$$M_{(3)}(S) = \begin{pmatrix} \frac{2\Gamma(-2/\tilde{k})\Gamma(N/\tilde{k})}{\Gamma(-1/\tilde{k})\Gamma(N/\tilde{k}-1/\tilde{k})} & \frac{2\Gamma(2/\tilde{k})\Gamma(N/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})} \\ \frac{\Gamma(1-2/\tilde{k})\Gamma(-N/\tilde{k})}{\Gamma(-1/\tilde{k})\Gamma(k/\tilde{k}-1/\tilde{k})} & \frac{\Gamma(1+2/\tilde{k})\Gamma(-N/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(k/\tilde{k}+1/\tilde{k})} \end{pmatrix}. \tag{A.24}$$

A.2 Generators of the orbit for $N = k$ theories

In this appendix, we show that for general values of $N(= k)$ the orbit of C_{seed} is as given in (2.61). We will do this by showing that the orbit can in effect be generated by considering the action of matrices of the form

$$\begin{pmatrix} \sin\alpha & -k\cos\alpha \\ -\frac{1}{k}\cos\alpha & -\sin\alpha \end{pmatrix}, \tag{A.25}$$

on C_{seed}, where $\alpha = \frac{\pi(2s+1)}{2k}$ with $s = 0\cdots(k-1)$ for k odd, and $\alpha = \frac{\pi s}{2k}$ with $s = 0\cdots(2k-1)$ for k even. It is easy to check that the actions of these matrices on C_{seed} indeed generates the orbits described in (2.61). We begin by noting that for $M(\gamma)$ of the form

$$M(\gamma) \equiv \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix},$$

its action on C_{seed} yields

$$\begin{pmatrix} |a_\gamma|^2 & a_\gamma c_\gamma^* \\ a_\gamma^* c_\gamma & |c_\gamma|^2 \end{pmatrix}. \tag{A.26}$$

Thus, the result of the action only depends on a_γ and c_γ (and is independent of b_γ and d_γ). Furthermore, since (A.26) is quadratic in a_γ and c_γ, elements of the orbit are only sensitive to their relative sign. Thus deformations of $M(\gamma)s$ which modify b_γ, d_γ and the relative sign between a_γ, c_γ keep their actions on C_{seed} unchanged. We will use such deformations to show that the orbit is in effect generated by the matrices given in (A.25). Let us start by considering the first few matrices in the list (2.58) of $M(\gamma)$ (for theories with $N = k$). In what follows, we will use the symbol '$\sim$' to denote a deformation of a matrix $M(\gamma)$ which keeps its action on C_{seed} unchanged.

$$M(\mathbb{1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \sin\frac{\pi k}{2k} & -k\cos\frac{\pi k}{2k} \\ -\frac{1}{k}\cos\frac{\pi k}{2k} & -\sin\frac{\pi k}{2k} \end{pmatrix};$$

$$M(S) = \begin{pmatrix} \sin\frac{\pi}{2k} & -k\cos\frac{\pi}{2k} \\ -\frac{1}{k}\cos\frac{\pi}{2k} & -\sin\frac{\pi}{2k} \end{pmatrix};$$

$$M(ST^2) = \begin{pmatrix} \sin\frac{\pi}{2k} & -k\cos\frac{\pi}{2k} \\ \frac{1}{k}\cos\frac{\pi}{2k} & \sin\frac{\pi}{2k} \end{pmatrix} \sim \begin{pmatrix} \sin\frac{\pi(2k-1)}{2k} & -k\cos\frac{\pi(2k-1)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(2k-1)}{2k} & -\sin\frac{\pi(2k-1)}{2k} \end{pmatrix};$$

$$M(ST^2S) = \begin{pmatrix} \sin\frac{\pi(2-k)}{2k} & -k\cos\frac{\pi(2-k)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(2-k)}{2k} & -\sin\frac{\pi(2-k)}{2k} \end{pmatrix} \sim \begin{pmatrix} \sin\frac{\pi(2+k)}{2k} & -k\cos\frac{\pi(2+k)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(2+k)}{2k} & -\sin\frac{\pi(2+k)}{2k} \end{pmatrix};$$

$$\begin{aligned} M(ST^2ST^2) &= \begin{pmatrix} -\cos\frac{2\pi}{2k} & -k\sin\frac{2\pi}{2k} \\ \frac{1}{k}\sin\frac{2\pi}{2k} & -\cos\frac{2\pi}{2k} \end{pmatrix} \sim \begin{pmatrix} \sin\frac{\pi(3k-2)}{2k} & -k\cos\frac{\pi(3k-2)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(3k-2)}{2k} & -\sin\frac{\pi(3k-2)}{2k} \end{pmatrix} \\ &\sim \begin{pmatrix} \sin\frac{\pi(k-2)}{2k} & -k\cos\frac{\pi(k-2)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(k-2)}{2k} & -\sin\frac{\pi(k-2)}{2k} \end{pmatrix}; \end{aligned}$$

$$M(ST^2ST^2S) = \begin{pmatrix} -\sin\frac{3\pi}{2k} & k\cos\frac{3\pi}{2k} \\ \frac{1}{k}\cos\frac{3\pi}{2k} & \sin\frac{3\pi}{2k} \end{pmatrix} \sim \begin{pmatrix} \sin\frac{3\pi}{2k} & -k\cos\frac{3\pi}{2k} \\ -\frac{1}{k}\cos\frac{3\pi}{2k} & -\sin\frac{3\pi}{2k} \end{pmatrix}.$$

Proceeding as above, all the $M(\gamma)$ can be brought to the form in (A.25) by making use of the identities

$$\begin{pmatrix} \sin\beta & -k\cos\beta \\ -\frac{1}{k}\cos\beta & -\sin\beta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \sin\alpha & -k\cos\alpha \\ -\frac{1}{k}\cos\alpha & -\sin\alpha \end{pmatrix} = \begin{pmatrix} \sin(\alpha+\beta-\frac{\pi}{2}) & -k\cos(\alpha+\beta-\frac{\pi}{2}) \\ -\frac{1}{k}\cos(\alpha+\beta-\frac{\pi}{2}) & -\sin(\alpha+\beta-\frac{\pi}{2}) \end{pmatrix}$$

and

$$\begin{pmatrix} \sin\alpha & -k\cos\alpha \\ -\frac{1}{k}\cos\alpha & -\sin\alpha \end{pmatrix} \sim \begin{pmatrix} \sin(\alpha+\pi) & -k\cos(\alpha+\pi) \\ -\frac{1}{k}\cos(\alpha+\pi) & -\sin(\alpha+\pi) \end{pmatrix}$$

for any angle α and β.

For completeness, we provide the orbit the $N(=k) = 2$ theory. It can easily be checked that this is same as that given by the matrices in (2.61). For $N = 2, k = 2$ the matrices $M(S)$ and $M(T^2)$ are

$$M(S) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\sqrt{2} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad M(T^2) = e^{-\frac{i\pi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.27}$$

The orbit of C_{seed} consists of four matrices. It is generated by the action of $\mathbb{1}, S, ST^2$ and ST^2S. We tabulate the results of these actions in Table A.1. The normalised sum over the orbit (2.52) reproduces the KZ result.

γ	$M(\gamma) \cdot C_{\text{seed}} \cdot M(\gamma)^\dagger$
$\mathbb{1}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
S	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} \end{pmatrix}$
ST^2	$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} \end{pmatrix}$
ST^2S	$\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$
X^{av}	$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

Table A.1: Orbit of the vacuum block for $N = 2, k = 2$

A.3 Further numerical examples

Here we provide a couple of examples where the numerics are quite involved as discussed at the end of section 2.3.

$\underline{N = 5, k = 6}$: For $N = 5, k = 6$, the value of $m(5,6)$ as defined in (2.56) is 11. Thus with each increment in $\ell_{\max}$ by 1, there is approximately a tenfold increase in the number of new terms added to the sum (2.63). With the available computing resources we have performed the sum upto $\ell_{\max} = 6$. This involves 1193006 distinct contributions to the sum. We find $X^{\text{av}}_{22}(6) = 0.026177$, alongside we note the exact result (2.54), $X^{\text{KZ}}_{22} \approx 0.0405346$. The off diagonal entries of $X^{\text{av}}(6)$ are of the order of 10^{-14}. Figure A.1 shows our results for $X^{\text{av}}_{22}(\ell_{\max})$ as a function of $\ell_{\max}$, all qualitative features of the numerics are same as those in the examples discussed in section 2.3.

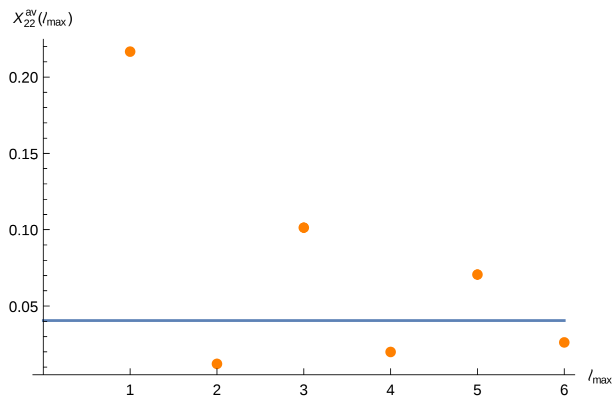

Figure A.1: Orange dots show $X^{\text{av}}_{22}(\ell_{\max})$ in the range $[0.005, 0.225]$ plotted against $\ell_{\max}$. Blue horizontal line at 0.0405346 represents X^{KZ}_{22}.

$\underline{N = 6, k = 5}$: For $N = 6, k = 5$, the value of $m(6,5)$ as defined in (2.56) is 11. Thus similarly, with each increment in $\ell_{\max}$ by 1, there is approximately a tenfold increase in the number of new terms added to the sum (2.63). With the available computing resources we have performed the sum upto $\ell_{\max} = 6$. This involves 1193006 distinct contributions to the sum. We find $X^{\text{av}}_{22}(6) = 0.0177022$, alongside we note the exact result (2.54), $X^{\text{KZ}}_{22} \approx 0.0274114$. The off diagonal entries of $X^{\text{av}}(6)$ are of the order of 10^{-14}. Figure A.2 shows our results for $X^{\text{av}}_{22}(\ell_{\max})$ as a function of $\ell_{\max}$. All the features of the numerics are similar to the previous example.

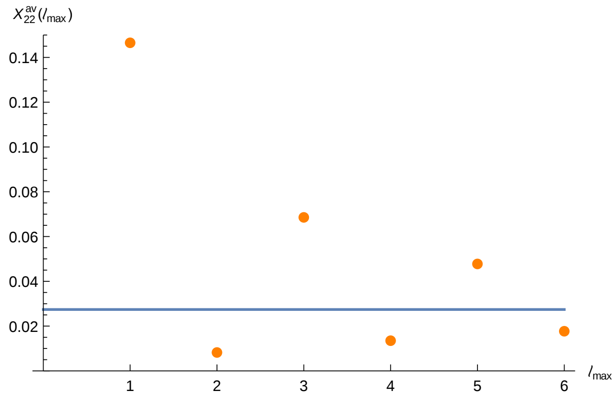

Figure A.2: Orange dots show $X_{22}^{\text{av}}(\ell_{\text{max}})$ in the range $[0.000, 0.150]$ plotted against ℓ_{max}. Blue horizontal line at 0.0274114 represents X_{22}^{KZ}.

A.4 Averaging over all of $PSL(2, \mathbb{Z})$

In this appendix, we briefly discuss the construction of correlator from averaging over the full modular group. To implement the prescription (2.17), the six holomorphic blocks in (A.10) of the three correlators in (2.41) can be put in a six dimensional row:

$$\vec{\mathcal{H}}(\tau) = \left(\mathcal{H}_{(1)}^{\mathbb{1}}(\tau), \mathcal{H}_{(1)}^{\theta}(\tau), \mathcal{H}_{(2)}^{\mathbb{1}}(\tau), \mathcal{H}_{(2)}^{\theta}(\tau), \mathcal{H}_{(3)}^{\xi}(\tau), \mathcal{H}_{(3)}^{\chi}(\tau)\right). \tag{A.28}$$

On this, T and S act as

$$\mathcal{H}^i(T.\tau) = \mathcal{H}^j(\tau)\mathcal{M}_{ji}(T) \quad \text{and} \quad \mathcal{H}^i(S.\tau) = \mathcal{H}^j(\tau)\mathcal{M}_{ji}(S) \tag{A.29}$$

with

$$\mathcal{M}(T) = \begin{pmatrix} 0 & M_{(1)}(T) & 0 \\ M_{(2)}(T) & 0 & 0 \\ 0 & 0 & M_{(3)}(T) \end{pmatrix} \quad \text{and} \quad \mathcal{M}(S) = \begin{pmatrix} M_{(1)}(S) & 0 & 0 \\ 0 & 0 & M_{(2)}(S) \\ 0 & M_{(3)}(S) & 0 \end{pmatrix}, \tag{A.30}$$

where the two dimensional matrices ($M_{(i)}(T)$ and $M_{(i)}(S)$) are as defined in appendix A.1. The light contribution as defined in (2.16) can be taken as

$$G_B^{\text{light}}(\tau,\bar{\tau}) = C^B_{i(B)j(B)}\mathcal{H}^{i(B)}(\tau)\bar{\mathcal{H}}^{j(B)}(\bar{\tau}), \quad B = 1,2,3\,, \tag{A.31}$$

where repeated indices are summed over with $i(1), j(1) \in \{1,2\}$, $i(2), j(2) \in \{3,4\}$ and $i(3), j(3) \in \{5,6\}$,

$$C^B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = 1,2,3\,. \tag{A.32}$$

Under the action $\gamma \in PSL(2,\mathbb{Z})$,

$$C^B_{i(B)j(B)}\mathcal{H}^{i(B)}(\tau)\bar{\mathcal{H}}^{j(B)}(\bar{\tau}) \to \mathcal{M}(\gamma)_{ki(B)}C^B_{i(B)j(B)}\mathcal{M}(\gamma)^\dagger_{j(B)l}\mathcal{H}^k(\tau)\bar{\mathcal{H}}^l(\bar{\tau})\,. \tag{A.33}$$

For each γ we arrange the three 6×6 matrices

$$\sigma^{-1}(\gamma)_{AB}\mathcal{M}(\gamma)_{ki(B)}C^B_{i(B)j(B)}\mathcal{M}(\gamma)^\dagger_{j(B)l}\quad,\quad A = 1,2,3\,, \tag{A.34}$$

in a three dimensional column $\vec{X}(\gamma)$. The sum (2.17) then reads

$$\vec{X}^{\text{av}} = \mathcal{N}^{-1}\cdot \sum_{\gamma\in PSL(2,\mathbb{Z})} \vec{X}(\gamma)\,, \tag{A.35}$$

where the normalisation $\mathcal{N}$ is the $(1,1)$ element of $[\,\sum_\gamma \vec{X}(\gamma)]^1$. Hence the candidate for the vector-valued modular function (2.41) is given by

$$[\vec{X}^{\text{av}}]^A_{kl}\mathcal{H}^k(\tau)\bar{\mathcal{H}}^l(\bar{\tau}), \quad A = 1,2,3\,. \tag{A.36}$$

To incorporate the distinct contributions $\vec{X}(\gamma)$ to the sum (A.35), elements γ are arranged in a list similar to (2.57) where we replace all T^{2r_i} by T^{r_i}, and m denotes the smallest positive integer such that

$$\mathcal{M}(T^m) \propto \mathbb{1}\,.$$

We perform the sum (A.35) taking distinct contributions of elements γ of all lengths upto a maximum value $\ell_{\max}$:

$$\vec{X}^{\text{av}}(\ell_{\max}) = \mathcal{N}(\ell_{\max})^{-1} \cdot \sum_{\ell(\gamma)\leq\ell_{\max}}{}^{'} \vec{X}(\gamma)\,, \tag{A.37}$$

where the primed sum indicates that distinct elements are added. Our results are as follows

<u>$N = 2, k = 2$:</u> For $N = 2, k = 2$, the sum (A.37) is finite and consists of six distinct contributions, reproducing the KZ result, $[\vec{X}^{\text{av}}]^1_{22} = \frac{1}{4}$.

<u>$N = 2, k = 4$:</u> For $N = 2, k = 4$, the sum (A.37) is finite and consists of four distinct contributions, reproducing the KZ result, $[\vec{X}^{\text{av}}]^1_{22} = \frac{1}{2\sqrt[3]{4}}$.

<u>$N = 2, k = 3$:</u> For $N = 2, k = 3$, the sum (A.37) seems to be infinite. We have performed the sum upto $\ell_{\max} = 6$. This invloves 83651 distinct contributions to the sum. We find $[\vec{X}^{\text{av}}]^1_{22}(6) = 0.296026$, which is in good agreement with the KZ result. Figure A.3 shows our results for $[\vec{X}^{\text{av}}]^1_{22}(\ell_{\max})$ as a function of $\ell_{\max}$.

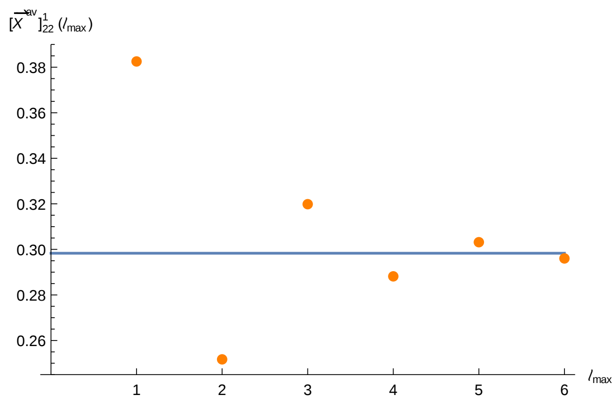

Figure A.3: Orange dots show $[\vec{X}^{\text{av}}]^1_{22}(\ell_{\max})$ in the range $[0.245, 0.390]$ plotted against $\ell_{\max}$. Blue horizontal line at 0.29831 represents the KZ result.

Finally, let us discuss some examples where modular averaging does not yield the correlator.

<u>$N = 3, k = 2$:</u> For $N = 3, k = 2$, the sum (A.37) seems to be infinite. We have performed the sum upto $\ell_{\max} = 6$. This invloves 664111 distinct contributions to the sum. We find

$[\vec{X}^{\rm av}]^1_{22}(6) = 0.151496$, which is not in agreement with the KZ result, although crossing symmetric. Figure A.4 shows our results for $[\vec{X}^{\rm av}]^1_{22}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$.

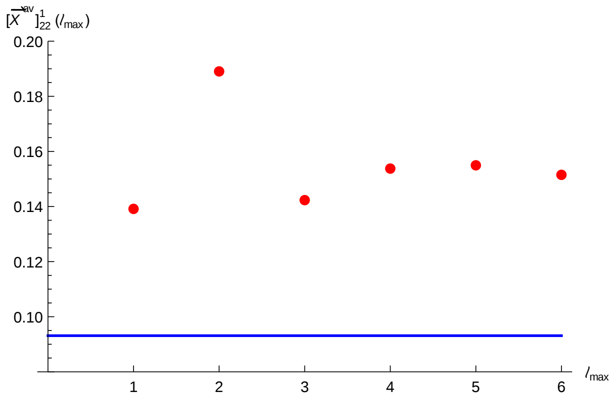

Figure A.4: Red dots show $[\vec{X}^{\rm av}]^1_{22}(\ell_{\rm max})$ in the range $[0.08, 0.20]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.0931172 represents the KZ result.

$N = 4, k = 2$: For $N = 4, k = 2$, the sum (A.37) seems to be infinite. We have performed the sum upto $\ell_{\rm max} = 8$. This invloves 69219 distinct contributions to the sum. We find $[\vec{X}^{\rm av}]^1_{22}(8) = 0.111064$, which is not in agreement with the KZ result, although crossing symmetric. Figure A.5 shows our results for $[\vec{X}^{\rm av}]^1_{22}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$.

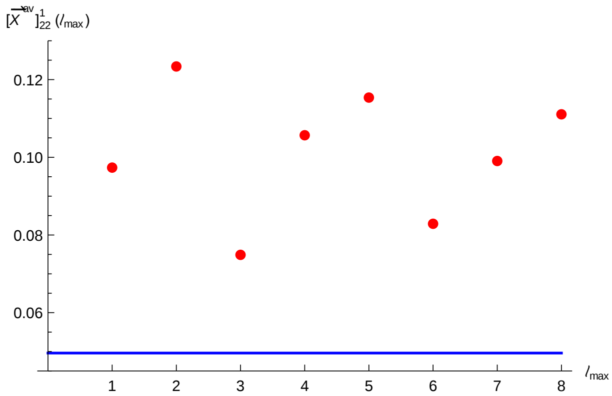

Figure A.5: Red dots show $[\vec{X}^{\rm av}]^1_{22}(\ell_{\rm max})$ in the range $[0.045, 0.130]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.0496063 represents the KZ result.

Thus while summing over the entire modular group we have found examples where the averaging does not reproduce the correlator. We note that, [32] argues that it is the modular averaging over the theta subgroup that has a direct interpretation in the holographic context.

Increasing N and k makes the numerics quite involved, we leave this for future work.

A.5 The matrices $M^{\ell}_{N,k}$ and $\tilde{M}^{\ell}_{N,k}$

In this appendix, we obtain the general form of the matrices $M^{\ell}_{N,k}$ and $\tilde{M}^{\ell}_{N,k}$. We then use these to derive the relations given in (2.76). The elements of matrices $M^{\ell}_{N,k}$ can be computed recursively in ℓ using their defining equation in (2.69)

$$M^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = M(T^{2r_{\ell+1}})M(S)M^{\ell}_{N,k}(r_1,\cdots r_\ell). \tag{A.38}$$

This gives the following relations for the functions that appear in (2.74)

$$\begin{aligned}
a^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) &= a_s(N,k)a^{\ell}_{N,k}(r_1\cdots r_\ell) + b_s(N,k)c_s(N,k)c^{\ell}_{N,k}(r_1\cdots r_\ell)\\
b^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) &= a_s(N,k)b^{\ell}_{N,k}(r_1\cdots r_\ell) + d^{\ell}_{N,k}(r_1\cdots r_\ell)\\
c^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) &= e^{ir_{\ell+1}\phi(N,k)}\left(d_s(N,k)c^{\ell}_{N,k}(r_1\cdots r_\ell) + a^{\ell}_{N,k}(r_1\cdots r_\ell)\right)\\
d^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) &= e^{ir_{\ell+1}\phi(N,k)}\left(d_s(N,k)d^{\ell}_{N,k}(r_1\cdots r_\ell) + b_s(N,k)c_s(N,k)b^{\ell}_{N,k}(r_1\cdots r_\ell)\right)
\end{aligned}$$

Similarly, the matrices $\tilde{M}^{\ell}_{N,k}$ can be computed recursively in ℓ using their defining equation in (2.70)

$$\tilde{M}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = M(T^{-2r_{\ell+1}})M(S)\tilde{M}^{\ell}_{N,k}(r_1,\cdots r_\ell). \tag{A.39}$$

This gives following relations for the functions that appear in (2.75)

$$\begin{aligned}
\tilde{a}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) &= a_s(N,k)\tilde{a}^{\ell}_{N,k}(r_1\cdots r_\ell) + b_s(N,k)c_s(N,k)\tilde{c}^{\ell}_{N,k}(r_1\cdots r_\ell)\\
\tilde{b}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) &= a_s(N,k)\tilde{b}^{\ell}_{N,k}(r_1\cdots r_\ell) + \tilde{d}^{\ell}_{N,k}(r_1\cdots r_\ell)\\
\tilde{c}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) &= e^{-ir_{\ell+1}\phi(N,k)}\left(d_s(N,k)\tilde{c}^{\ell}_{N,k}(r_1\cdots r_\ell) + \tilde{a}^{\ell}_{N,k}(r_1\cdots r_\ell)\right)\\
\tilde{d}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) &= e^{-ir_{\ell+1}\phi(N,k)}\left(d_s(N,k)\tilde{d}^{\ell}_{N,k}(r_1\cdots r_\ell) + b_s(N,k)c_s(N,k)\tilde{b}^{\ell}_{N,k}(r_1\cdots r_\ell)\right).
\end{aligned}$$

Now, making use of relations in (2.68) and the fact that[3]

$$e^{ir\phi(N,k)} = e^{-ir\phi(k,N)} \text{ for any integer r,} \tag{A.40}$$

it is easy to see that $\tilde{a}^{\ell}_{k,N}(r_i), \tilde{b}^{\ell}_{k,N}(r_i), \tilde{c}^{\ell}_{k,N}(r_i), \tilde{d}^{\ell}_{k,N}(r_i)$ have exactly the same recurrence relations as $a^{\ell}_{N,k}(r_i), b^{\ell}_{N,k}(r_i), c^{\ell}_{N,k}(r_i), d^{\ell}_{N,k}(r_i)$. Given that they have same initial values, hence the equalities in (2.76).

A.6 Truncation of Sums

The sum in (2.63) terminates after a (lowest) value $\ell^0_{\max}$ if the actions of γs in the list (2.57) with $\ell(\gamma) > \ell^0_{\max}$ do not generate new elements of the orbit of C_{seed}, i.e. the orbit is finite. Note that if there is an ℓ such that no new terms are generated, higher values of ℓ also do not generate new terms in the orbit (with this the value of the sum in (2.63) at higher $\ell_{\max}$ does not change). Thus comparison of the terms generated at a certain ℓ with the ones at lower ℓ can be used to determine the cases with finite orbit. It is possible to implement this consideration at each point in the (N, k) lattice (of course the non-trivial cases are for $N, k \geq 2$). Before discussing the details, we summarise our results. Truncations start from $\ell^0_{\max} = 1$. Here, it is possible to determine analytically the values of (N, k) for which the truncations occur - $(3, 3), (2, 4)$ and $(4, 2)$ are the only points where the modular sum truncates at $\ell^0_{\max} = 1$. For higher values of ℓ, except for cases with $N = k$ we have not been able to carry out a general analysis so as to determine the points in the (N, k) lattice for which truncations occur (the results for $N = k$ are given in section 2.3, recall that all these models exhibit truncation). For $N \neq k$ we have implemented the above algorithm numerically, and found that upto $\ell = 5$, for points in the (N, k) lattice with $N, k \leq 6$ (and $N \neq k$) there are no truncations.

The details of the analysis are as follows. We recall (A.26). The action of $\mathbb{1}$ on C_{seed} is

[3] Recall that $\phi(N, k) = \frac{2\pi N}{k+N}$.

given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{A.41}$$

The actions of γs of $\ell(\gamma) = 1$ on C_{seed} are given by

$$\begin{pmatrix} a_S^2 & a_S c_S e^{-\frac{i2\pi N r_1}{\tilde{k}}} \\ a_S c_S e^{\frac{i2\pi N r_1}{\tilde{k}}} & c_S^2 \end{pmatrix}, \tag{A.42}$$

for $r_1 = 0, \cdots, (m(N,k) - 1)$. Here a_S is the 1-1 entry of the matrix $M(S)$ (see for e.g (2.67)). Clearly the phases at the off-diagonal entries are the $m(N,k)$-th roots of unity, hence all distinct and add up to zero. The actions of γs of $\ell(\gamma) = 2$ on C_{seed} are given by[4]

$$\begin{pmatrix} 1 - 2a_S^2 + 2a_S^4 - 2a_S^2(a_S^2 - 1)\cos(\frac{2\pi N r_1}{\tilde{k}}) & a_S c_S e^{-\frac{i2\pi N r_2}{\tilde{k}}}(1 - e^{-\frac{i2\pi N r_1}{\tilde{k}}})(a_S^2 - (a_S^2 - 1)e^{\frac{i2\pi N r_1}{\tilde{k}}}) \\ a_S c_S e^{\frac{i2\pi N r_2}{\tilde{k}}}(1 - e^{\frac{i2\pi N r_1}{\tilde{k}}})(a_S^2 - (a_S^2 - 1)e^{-\frac{i2\pi N r_1}{\tilde{k}}}) & 2a_S^2 c_S^2(1 - \cos(\frac{2\pi N r_1}{\tilde{k}})) \end{pmatrix}, \tag{A.43}$$

for $r_1 = 1, \cdots, (m(N,k) - 1)$ and $r_2 = 0, \cdots, (m(N,k) - 1)$. Comparing with the structure of the terms generated at length zero (A.41) and one (A.42), we see that truncation requires that the following equality necessarily holds for all $r_1 = 1 \cdots (m(N,k) - 1)$.

$$\cos(\frac{2\pi N r_1}{\tilde{k}}) = \frac{2a_s^2 - 1}{a_S^2} \,. \tag{A.44}$$

Hence necessarily $a_S^2 \geq \frac{1}{4}$, which holds only when (N,k) lies on the line $N = 2$ or $k = 2$ or at the point $(3,3)$. Furthermore at any (N,k) the r.h.s. of (A.44) is fixed which restricts the number of values r_1 can take. This in turn gives a necessary condition for the possible values for $m(N,k)$: it must be 2 or 3. Hence $(2,2), (3,3), (2,4)$ and $(4,2)$ models are the only ones which satisfy this criterion. Going through each of these possibilities case by case one finds that truncation and $\ell^0_{\text{max}} = 1$ occurs for $(3,3), (2,4)$ and $(4,2)$. Similar considerations necessary to determine truncations at higher ℓ are more involved (except for the cases with $N = k$); we have implemented them numerically and found for points

[4] After using $b_S c_S = 1 + a_S d_S$ and $d_S = -a_S$.

in the (N, k) lattice with $N, k \leq 6$ (and $N \neq k$), there are no truncations upto $\ell = 5$.

Appendix B.

B.1 Several complex variables

The theory of functions of several complex variables (SCV) has key differences (known as Hartogs's phenomena) than the more familiar case of a single complex variable. For a systematic exposition to the methods of SCV and their various applications to QFT, see [82, 121]. Here we discuss one key difference that lies at the heart of our analysis in chapter 3.

A domain Ω in m complex variables $(z_1, \ldots, z_m)$ is an open and connected subset of $\mathbb{C}^m$[1]. A function is said to be holomorphic (or analytic) on Ω if it is holomorphic with respect to each variable z_j separately, with the other variables kept fixed inside Ω. The set of all functions which are holomorphic on a given Ω will be denoted by $\mathcal{H}(\Omega)$. By a holomorphic extension Ω' of a given Ω we mean any larger domain Ω' containing Ω, with the property that any $f \in \mathcal{H}(\Omega)$ can be analytically continued to Ω'. The existence of a holomorphic extension is a property of a given domain irrespective of the functions that are analytic on it. The largest possible holomorphic extension of a given domain is called the envelope of holomorphy. Note that a few but not all of the functions in $\mathcal{H}(\Omega)$ may be analytically continued to an even larger domain than the envelope of holomorphy. Given a particular function f we sometimes ask what is the largest domain where f can

[1]However in chapter 3, the word 'domain' has been used in slightly general sense and whenever necessary we have explicitly used the phrase 'open and connected' instead of using 'domain'.

be continued analytically. This largest domain is known as the domain of holomorphy of f.

In one complex variable, every domain is a domain of holomorphy of some function i.e., there always exists a function on the domain which cannot be continued outside it. However, in several complex variables this is not true. For example, any function f which is analytic on the following domain in two complex variables

$$\Omega_{(2)} = \left\{(z_1, z_2) \ : \ |z_1| < 1, \ \frac{1}{2} < |z_2| < 1\right\} \cup \left\{(z_1, z_2) \ : \ \frac{1}{2} < |z_1| < 1, \ |z_2| < 1\right\} \tag{B.1}$$

can be analytically continued to the domain

$$\Omega'_{(2)} = \left\{(z_1, z_2) \ : \ |z_1| < 1, \ |z_2| < 1\right\}. \tag{B.2}$$

The analytic continuation can be explicitly given as

$$F(z_1, z_2) = \frac{1}{2\pi i} \oint_{C_1} d\xi_1 \, \frac{1}{2\pi i} \oint_{C_2} d\xi_2 \, \frac{f(\xi_1, \xi_2)}{(\xi_1 - z_1)(\xi_2 - z_2)} , \tag{B.3}$$

where $C_1 = \{\xi_1 : \ |\xi_1| = 1 - \delta_1\}$, $C_2 = \{\xi_2 : \ |\xi_2| = 1 - \delta_2\}$ are anticlockwise contours and δ_1, δ_2 are two arbitrarily small positive numbers. By definition $f(\xi_1, \xi_2)$ is analytic in each variable when the other one is held fixed so that $(\xi_1, \xi_2) \in \Omega_{(2)}$. Therefore two successive applications of Cauchy integral formula yield that $F(z_1, z_2) = f(z_1, z_2)$ whenever $(z_1, z_2) \in \Omega_{(2)}$, proving the claim that F is the analytic continuation of f.

One such non-trivial extension theorem for SCV that has immense application in QFT is Bogoliubov's edge-of-the-wedge theorem (see, [82, 120, 122]). For our purpose, we use Bochner's tube theorem [116] which states that any domain of form $\mathbb{R}^m + iA, \ A \subset \mathbb{R}^m$ (called a tube) has a holomorphic extension to $\mathbb{R}^m + i\mathrm{Ch}(A)$ where $\mathrm{Ch}(A)$ is the smallest convex set containing the set A (called the convex hull of A). That is, we need to adjoin all possible finite convex combinations of points taken from the tube. In general, a finite convex combination of points x_i taken from a set A i.e., a point of the form $\sum_{i=1}^r t_i x_i,\ t_i \geq 0,\ \sum_i t_i = 1$ does not belong to the same set A.

B.2 Convexity of $\mathcal{T}_\lambda$, $\mathcal{T}_\lambda^{\vec{\theta}}$

The tube $\mathcal{T}_\lambda$ (described by equation (3.5)) is convex [91]. We derive it as follows. We take any m points $p^{(1)}, \ldots, p^{(m)} \in \mathcal{T}_\lambda$ and consider the convex combination $q = \sum_{r=1}^m t_r p^{(r)}$ where each $t_r \geq 0$ and $\sum_{r=1}^m t_r = 1$. Now if $q \in \mathcal{T}_\lambda$ then $\mathcal{T}_\lambda$ is convex.

Clearly $\sum_{a=1}^n q_a = 0$. We define $Q_I = \sum_{a \in I} q_a$ for each $I \in \wp^*(X)$. Hence we get $Q_I = \sum_{r=1}^m t_r P_I^{(r)}$ where for each r we have $P_I^{(r)} = \sum_{a \in I} p_a^{(r)}$. We also have $\lambda(I)\text{Im } P_I^{(r)} \in V^+$ since $p^{(1)}, \ldots, p^{(m)} \in \mathcal{T}_\lambda$. Therefore $\lambda(I)\text{Im } Q_I = \sum_{r=1}^m t_r \lambda(I)\text{Im } P_I^{(r)} \in V^+$. Hence $q \in \mathcal{T}_\lambda$. This proves the convexity of $\mathcal{T}_\lambda$.

Taking any m points $p^{(1)}, \ldots, p^{(m)} \in \mathcal{T}_\lambda^{\vec{\theta}}$ (described by equation (3.9)), now we show that the convex combination $q = \sum_{r=1}^m t_r p^{(r)}$ belongs to $\mathcal{T}_\lambda^{\vec{\theta}}$. In present case, since for each $r = 1, \ldots, m$ the Im $p_a^{(r)}$ $\forall a$, in turn all Im $P_I^{(r)}$ lie on the two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}}$, therefore we get that the Im Q_I for all I including singletons lie on the same plane $p^0 - p^{\vec{\theta}}$. Hence $\mathcal{T}_\lambda^{\vec{\theta}}$ is convex following similar steps to the above case.

B.3 Nonconvexity of $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$

We take a point $p^{(1)} = (p_1^{(1)}, \ldots, p_n^{(1)}) \in \mathcal{T}_\lambda^{\vec{\theta}=0}$. Hence Im $p_a^{(1)}$ $\forall a = 1, \ldots, n$ lie on the two dimensional Lorentzian plane $p^0 - p^1$. Now suppose the point $p^{(2)}$ is obtained by acting a real rotation[2] on the point $p^{(1)}$ so that Im $p_a^{(2)}$ $\forall a = 1, \ldots, n$ now lie on the two dimensional Lorentzian plane $p^0 - p^2$. Hence $p^{(2)} \in \mathcal{T}_\lambda^{\vec{\theta}_2}$ where $\vec{\theta}_2$ characterizes the two dimensional Lorentzian plane $p^0 - p^2$. Now for the point $q = \frac{1}{2}p^{(1)} + \frac{1}{2}p^{(2)}$ which is

[2]In equation (3.7), we have defined actions of complex Lorentz transformations which include real rotations.

the mid-point of the straight line segment connecting the two points $p^{(1)}$, $p^{(2)}$ we have $\mathrm{Im}\, q_a\ \forall a = 1, \dots, n$ lying on the two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}_3}$ where the $p^{\vec{\theta}_3}$-axis lies on the two dimensional plane $p^1 - p^2$ making an angle 45^0 with the positive p^2-axis. Hence the point $q \in \mathcal{T}_\lambda^{\vec{\theta}_3}$. However we can change $\mathrm{Im}\, p_1^{(2)}$ little bit, keeping all real parts and $\mathrm{Im}\, p_a^{(2)}$, $a = 2, \dots, n$ unchanged to obtain a new point $\tilde{p}^{(2)}$ such that $\tilde{p}^{(2)}$ still belongs to $\mathcal{T}_\lambda^{\vec{\theta}_2}$[3]. Consequently we get two points $p^{(1)} \in \mathcal{T}_\lambda^{\vec{\theta}=0}$ and $\tilde{p}^{(2)} \in \mathcal{T}_\lambda^{\vec{\theta}_2}$, for which the mid-point of the straight line segment connecting them is given by $\tilde{q} = \frac{1}{2}p^{(1)} + \frac{1}{2}\tilde{p}^{(2)}$. Although $\mathrm{Im}\, \tilde{q}_a\ \forall a = 2, \dots, n$ lie on the two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}_3}$, the D-momenta $\mathrm{Im}\, \tilde{q}_1 = \frac{1}{2}p_1^{(1)} + \frac{1}{2}\tilde{p}_1^{(2)}$ does not lie on the two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}_3}$ anymore. Hence $\tilde{q} \notin \bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$. In other words, $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ is non-convex.

To see this, let us take $\{p_1, \dots, p_{n-1}\}$ as our basis to describe points on the complex manifold $p_1 + \cdots + p_n = 0$. A generic point $p = (p_1, \dots, p_{n-1})$ on this manifold can be represented by an unique $D \times (n-1)$ matrix where the a-th column represent the D-momenta p_a. Since each $\mathcal{T}_\lambda^{\vec{\theta}}$ thereby $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$ reside on this manifold now the imaginary parts of the points $p^{(1)}$, $p^{(2)}$, $\tilde{p}^{(2)}$, q and $\tilde{q}$ can be represented in terms of $D \times (n-1)$

[3] In support of this see appendix B.5.

matrices as follows[4].

$$p^{(1)} = \begin{pmatrix} p_1^0 & p_2^0 & \cdots & p_{n-1}^0 \\ p_1^1 & p_2^1 & \cdots & p_{n-1}^1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},\ p^{(2)} = \begin{pmatrix} p_1^0 & p_2^0 & \cdots & p_{n-1}^0 \\ 0 & 0 & \cdots & 0 \\ p_1^1 & p_2^1 & \cdots & p_{n-1}^1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},\ \tilde{p}^{(2)} = \begin{pmatrix} p_1^0 & p_2^0 & \cdots & p_{n-1}^0 \\ 0 & 0 & \cdots & 0 \\ \tilde{p}_1^1 & p_2^1 & \cdots & p_{n-1}^1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$q = \begin{pmatrix} p_1^0 & p_2^0 & \cdots & p_{n-1}^0 \\ \frac{1}{2}p_1^1 & \frac{1}{2}p_2^1 & \cdots & \frac{1}{2}p_{n-1}^1 \\ \frac{1}{2}p_1^1 & \frac{1}{2}p_2^1 & \cdots & \frac{1}{2}p_{n-1}^1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},\ \tilde{q} = \begin{pmatrix} p_1^0 & p_2^0 & \cdots & p_{n-1}^0 \\ \frac{1}{2}p_1^1 & \frac{1}{2}p_2^1 & \cdots & \frac{1}{2}p_{n-1}^1 \\ \frac{1}{2}\tilde{p}_1^1 & \frac{1}{2}p_2^1 & \cdots & \frac{1}{2}p_{n-1}^1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{B.4}$$

Clearly all the columns of $\mathrm{Im}\, q$ lie on the two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}_3}$ where the $p^{\vec{\theta}_3}$-axis lies on the two dimensional plane $p^1 - p^2$ making an angle 45^0 with the positive p^2-axis. It is also evident that the first column of $\mathrm{Im}\, \tilde{q}$ does not lie on the two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}_3}$ although all the other columns of $\mathrm{Im}\, \tilde{q}$ lie on it.

B.4 Path-connectedness of $\bigcup_{\vec{\theta}}\mathcal{T}_\lambda^{\vec{\theta}}$

We take any two points $p^{(1)},\ p^{(2)} \in \bigcup_{\vec{\theta}}\mathcal{T}_\lambda^{\vec{\theta}}$. To show that the tube $\bigcup_{\vec{\theta}}\mathcal{T}_\lambda^{\vec{\theta}}$ is path-connected it is sufficient to find a path connecting the points $p^{(1)},\ p^{(2)}$ staying inside the tube $\bigcup_{\vec{\theta}}\mathcal{T}_\lambda^{\vec{\theta}}$. Let $p^{(1)} \in \mathcal{T}_\lambda^{\vec{\theta}_1}$ and $p^{(2)} \in \mathcal{T}_\lambda^{\vec{\theta}_2}$ for some $\vec{\theta}_1, \vec{\theta}_2$. Now if $\vec{\theta}_1 = \vec{\theta}_2$ $(= \vec{\theta},$ say) then $p^{(1)},\ p^{(2)}$

[4]For notational simplicity, we omit the prefix 'Im' in all the entries in the equation (B.4) however the entries should be understood as respective imaginary parts.

belong to the same tube $\mathcal{T}_\lambda^{\vec{\theta}}$. Since the tube $\mathcal{T}_\lambda^{\vec{\theta}}$ is convex (see appendix B.2) the straight line segment connecting $p^{(1)},\ p^{(2)}$ lies entirely inside $\mathcal{T}_\lambda^{\vec{\theta}}$. Now we consider the case when $\vec{\theta}_1 \neq \vec{\theta}_2$. In this case $p^{(1)} = (p_1^{(1)}, \dots, p_n^{(1)})$, where all the Im $p_a^{(1)}$ thereby Im $P_I^{(1)}$ lie on the two dimensional Lorentzian plane $p^0 - p^{\vec{\theta}_1}$. We consider the point $\tilde{p}^{(1)} = (\tilde{p}_1^{(1)}, \dots, \tilde{p}_n^{(1)}) \in \mathbb{C}^{(n-1)D}$ given by

$$\begin{aligned} \forall a \quad & \text{Im } \tilde{p}_a^{(1)0} = \text{Im } p_a^{(1)0}; \quad \text{Im } \tilde{p}_a^{(1)i} = 0,\ i = 1, \dots, D-1; \\ & \text{Re } \tilde{p}_a^{(1)\mu} = \text{Re } p_a^{(1)\mu},\ \mu = 0, \dots, D-1\ , \end{aligned} \tag{B.5}$$

where we switch off only the Im $p_a^{(1)i},\ i = 1, \dots, D-1$ components $\forall a$ in $p^{(1)}$ to obtain the point $\tilde{p}^{(1)}$. As the Im $p_a^{(1)0}$ component remains unaltered $\forall a$ the point $\tilde{p}^{(1)} \in \mathcal{T}_\lambda^{\vec{\theta}}$ for all $\vec{\theta}$. In particular $\tilde{p}^{(1)}$ belongs to both the tubes $\mathcal{T}_\lambda^{\vec{\theta}_1}$ and $\mathcal{T}_\lambda^{\vec{\theta}_2}$. Hence $\mathcal{T}_\lambda^{\vec{\theta}_1}$ being a convex tube the straight line segment $p^{(1)}\tilde{p}^{(1)}$ connecting $p^{(1)},\ \tilde{p}^{(1)}$ lies entirely inside $\mathcal{T}_\lambda^{\vec{\theta}_1}$, and $\mathcal{T}_\lambda^{\vec{\theta}_2}$ being a convex tube the straight line segment $\tilde{p}^{(1)}p^{(2)}$ connecting $\tilde{p}^{(1)},\ p^{(2)}$ lies entirely inside $\mathcal{T}_\lambda^{\vec{\theta}_2}$. Joining these two segments we get a path connecting $p^{(1)},\ p^{(2)}$ staying inside the tube $\bigcup_{\vec{\theta}} \mathcal{T}_\lambda^{\vec{\theta}}$. This completes the proof.

To see that $\tilde{p}^{(1)} \in \mathcal{T}_\lambda^{\vec{\theta}}$ for all $\vec{\theta}$, let us consider $\tilde{P}_I^{(1)} = \sum_{a\in I} \tilde{p}_a^{(1)}$ for an arbitrary non-empty proper subset I of X. Therefore we have Im $\tilde{P}_I^{(1)} = \sum_{a\in I}$Im $\tilde{p}_a^{(1)}$ and it is timelike because

$$\begin{aligned} (\text{Im } \tilde{P}_I^{(1)})^2 &= -(\sum\nolimits_{a\in I} \text{Im } \tilde{p}_a^{(1)0})^2 + \sum\nolimits_{i=1}^{D-1} (\sum\nolimits_{a\in I} \text{Im } \tilde{p}_a^{(1)i})^2 \\ &= -(\sum\nolimits_{a\in I} \text{Im } p_a^{(1)0})^2 < 0\ . \end{aligned} \tag{B.6}$$

Furthermore Im $\tilde{P}_I^{(1)}$ and Im $P_I^{(1)}$ belong to the same lightcone because of the following

$$\text{Im } \tilde{P}_I^{(1)0} = \sum\nolimits_{a\in I} \text{Im } \tilde{p}_a^{(1)0} = \sum\nolimits_{a\in I} \text{Im } p_a^{(1)0} = \text{Im } P_I^{(1)0} \implies \text{sgn}(\text{Im } \tilde{P}_I^{(1)0}) = \text{sgn}(\text{Im } P_I^{(1)0})\ . \tag{B.7}$$

B.5 Thickening of $\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}$

As mentioned at the beginning of chapter 3, the work of [23] showed that at any point p belonging to the LES domain $\mathcal{D}'$ all the relevant Feynman diagrams[5] in the perturbative expansion of an n-point Green's function in SFT are analytic. Any such Feynman diagram at the point p has an integral representation in terms of loop integrals where the poles of the integrand are at finite distance away from any of the loop integration contours. As per the discussion around equation (3.8), the above statement also holds for any point p belonging to the LES domain $\tilde{\mathcal{D}}'$.

Hence for any $p \in \bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}$ (thereby $\mathrm{Im}\, p \in \bigcup_{\vec{\theta}} C_{\lambda}^{\vec{\theta}} \subset \mathbb{R}^{(n-1)D}$) we can allow a small open ball $\mathcal{B}_{\mathrm{Im}\, p}$ in $\mathbb{R}^{(n-1)D}$ centered at $\mathrm{Im}\, p$ such that for any point $p' \in \mathcal{B}_{\mathrm{Im}\, p}$ the aforementioned poles of the integrand are still at a finite distance away from the loop integration contours in a given Feynman diagram [23]. Consequently, the same integral representation in terms of loop integrals holds at any of the new points p'. Therefore, by allowing such open balls for each $p \in \bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}$ we can make $\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}$ open, in which the given Feynman diagram still remains analytic. In this way, $\bigcup_{\vec{\theta}} \mathcal{T}_{\lambda}^{\vec{\theta}}$ can be thickened individually for all the relevant Feynman diagrams (at all orders in perturbation theory).

B.6 The cone $C_{12}^{(4)+}$

The cone $C_{12}^{(4)+}$ taken from the list (3.17) can be written as

$$C_{12}^{(4)+} = \left\{ \mathrm{Im}\, p : \ -\mathrm{Im}\, p_2,\ \mathrm{Im}\, (p_2 + p_3),\ \mathrm{Im}\, (p_2 + p_4) \in V^+ \right\}. \tag{B.8}$$

[5] As per the discussion at the beginning of chapter 3, diagrams that do not have any massless internal propagator are only relevant.

$C_{12}^{(4)+}$ resides on the manifold $\text{Im } p_1 + \cdots + \text{Im } p_4 = 0$. Here we show how specifying $\text{Im } p_2$ in V^-[6], and $\text{Im } (p_2 + p_3)$ and $\text{Im } (p_2 + p_4)$ in V^+ in turn determine the sign-valued map $\lambda(I)$ (described by equation (3.4)) uniquely. Now we consider the following set of seven $\text{Im } P_I$

$$\Big\{\text{Im } p_2,\ \text{Im } p_3,\ \text{Im } p_4,\ \text{Im } (p_2+p_3),\ \text{Im } (p_2+p_4),\ \text{Im } (p_3+p_4),\ \text{Im } (p_2+p_3+p_4)\Big\}\,. \quad \text{(B.9)}$$

We want to see that for any $\text{Im } p$ inside the cone $C_{12}^{(4)+}$ in which lightcone each element of the above set lies. Knowing this, similar information for any other possible $\text{Im } P_I$ can be determined using the relation $\text{Im } p_1 + \cdots + \text{Im } p_4 = 0$. Now the following information can be obtained since for any $\text{Im } p \in C_{12}^{(4)+}$ we have $-\text{Im } p_2,\ \text{Im } (p_2 + p_3),\ \text{Im } (p_2 + p_4) \in V^+$.

$$\begin{aligned}
&\text{Im } p_3 = -\text{Im } p_2 + \text{Im } (p_2 + p_3) \in V^+, \\
&\text{Im } p_4 = -\text{Im } p_2 + \text{Im } (p_2 + p_4) \in V^+, \\
&\text{Im } (p_3 + p_4) = \text{Im } p_3 + \text{Im } p_4 \in V^+, \\
&\text{Im } (p_2 + p_3 + p_4) = -\text{Im } p_2 + \text{Im } (p_2 + p_3) + \text{Im } (p_2 + p_4) \in V^+ \,.
\end{aligned} \qquad \text{(B.10)}$$

Hence the signs $\lambda(I)$ corresponding to the cone $C_{12}^{(4)+}$ is now known for any non-empty proper subset I of $\{1, \ldots, 4\}$.

In section 3.2.2, with $\{-\text{Im } p_2,\ \text{Im } (p_2 + p_3),\ \text{Im } (p_2 + p_4)\}$ as our basis, points in the cone $C_{12}^{(4)+}$ have been described. However any point in $C_{12}^{(4)+}$ can uniquely be written in a new basis given by $\{\text{Im } p_2,\ \text{Im } p_3,\ \text{Im } p_4\}$ since such a change of basis is a linear transformation $\mathcal{L}$ with $\det(\mathcal{L}) = -1$. This transformation law can be stated as: any point $\vec{Q} = (P_\alpha, P_\beta, P_\gamma)$ written in the basis $\{-\text{Im } p_2,\ \text{Im } (p_2 + p_3),\ \text{Im } (p_2 + p_4)\}$ can be written as $\mathcal{L}\vec{Q} = (-P_\alpha,\ P_\alpha + P_\beta,\ P_\alpha + P_\beta)$ in basis $\{\text{Im } p_2,\ \text{Im } p_3,\ \text{Im } p_4\}$.

[6] $V^-(= -V^+)$ is the open backward lightcone in $\mathbb{R}^D$.

Hence the point $\vec{Q}$ in the cone $C_{12}^{(4)+}$ as given in (3.19) in the new basis reads as

$$\mathcal{L}\vec{Q} = \begin{pmatrix} -P_\alpha^0 & P_\alpha^0 + P_\beta^0 & P_\alpha^0 + P_\gamma^0 \\ -P_\alpha^1 & P_\alpha^1 + P_\beta^1 & P_\alpha^1 + P_\gamma^1 \\ \vdots & \vdots & \vdots \\ -P_\alpha^{D-1} & P_\alpha^{D-1} + P_\beta^{D-1} & P_\alpha^{D-1} + P_\gamma^{D-1} \end{pmatrix}, \tag{B.11}$$

with conditions $P_r^0 > +\sqrt{\sum_{i=1}^{D-1}(P_r^i)^2} \quad \forall r = \alpha, \beta, \gamma$. And the points in (3.20) in the new basis read as

$$\mathcal{L}\vec{\tilde{Q}}_1 = 3\begin{pmatrix} -P_\alpha^0 + \epsilon & P_\alpha^0 - \epsilon/2 & P_\alpha^0 - \epsilon/2 \\ -P_\alpha^1 & P_\alpha^1 & P_\alpha^1 \\ \vdots & \vdots & \vdots \\ -P_\alpha^{D-1} & P_\alpha^{D-1} & P_\alpha^{D-1} \end{pmatrix}, \quad \mathcal{L}\vec{\tilde{Q}}_2 = 3\begin{pmatrix} -\epsilon/2 & P_\beta^0 - \epsilon/2 & \epsilon \\ 0 & P_\beta^1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & P_\beta^{D-1} & 0 \end{pmatrix},$$
$$\mathcal{L}\vec{\tilde{Q}}_3 = 3\begin{pmatrix} -\epsilon/2 & \epsilon & P_\gamma^0 - \epsilon/2 \\ 0 & 0 & P_\gamma^1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & P_\gamma^{D-1} \end{pmatrix}, \tag{B.12}$$

where ϵ satisfies the condition: $0 < \epsilon < \min\{P_r^0 - \sqrt{\sum_{i=1}^{D-1}(P_r^i)^2},\ r = \alpha, \beta, \gamma\}$. Clearly above points $\mathcal{L}\vec{Q}$, $\mathcal{L}\vec{\tilde{Q}}_1$, $\mathcal{L}\vec{\tilde{Q}}_2$ and $\mathcal{L}\vec{\tilde{Q}}_3$ written in the basis $\{\mathrm{Im}\ p_2,\ \mathrm{Im}\ p_3,\ \mathrm{Im}\ p_4\}$ are consistent with (B.10). Furthermore each columns of $\mathcal{L}\vec{\tilde{Q}}_1$ lie on the same two dimensional Lorentzian plane where P_α lies. Similarly all the columns of $\mathcal{L}\vec{\tilde{Q}}_2$ lie on the two dimensional Lorentzian plane where P_β lies, and all the columns of $\mathcal{L}\vec{\tilde{Q}}_3$ lie on the two dimensional Lorentzian plane where P_γ lies. Now it is easy to check that the following relation holds

$$\frac{\mathcal{L}\vec{\tilde{Q}}_1}{3} + \frac{\mathcal{L}\vec{\tilde{Q}}_2}{3} + \frac{\mathcal{L}\vec{\tilde{Q}}_3}{3} = \mathcal{L}\vec{Q}\,. \tag{B.13}$$

Which depicts nothing but the linearity of $\mathcal{L}$ on (3.21).

B.7 Difficulty arising for $C'^{(5)+}_{12}$

Consider a 5-point function, i.e. $n = 5$. We take the following conditions which describe one of the 20 problematic cones, $C'^{(5)+}_{12}$.

$$\begin{aligned} &\text{Im}\,(p_1 + p_3),\ \text{Im}\,(p_1 + p_4),\ \text{Im}\,(p_2 + p_3),\ \text{Im}\,(p_2 + p_4) \in V^+, \\ &-\text{Im}\,(p_1 + p_3 + p_4),\ -\text{Im}\,(p_2 + p_3 + p_4) \in V^+\ . \end{aligned} \tag{B.14}$$

Above conditions in turn imply that Im p_1, Im $p_2 \in V^+$ and Im p_3, Im $p_4 \in V^-$.

In this case, we choose $\{\text{Im}\ p_1,\ \text{Im}\ p_2, -\text{Im}\ p_3, -\text{Im}\ p_4\}$ as our basis to assign coordinates to the points in the cone $C'^{(5)+}_{12}$. The goal is to find $P_1, P_2, P_3, P_4 \in V^+$, in terms of which we can decompose a generic point $(\text{Im}\ p_1,\ \text{Im}\ p_2, -\text{Im}\ p_3, -\text{Im}\ p_4)$ in the cone in a convex combination of several points. Each of the terms in the decomposition should be in $\bigcup_{\vec{\theta}} C'^{(5)+,\vec{\theta}}_{12}$.

Consider the following decomposition in a sum of four terms,

$$\begin{aligned} (\text{Im}\ p_1,\ \text{Im}\ p_2, -\text{Im}\ p_3, -\text{Im}\ p_4) = &\ (\alpha_1 P_1,\ \alpha_2 P_1,\ \alpha_3 P_1,\ \alpha_4 P_1) \\ &+ (\beta_1 P_2,\ \beta_2 P_2,\ \beta_3 P_2,\ \beta_4 P_2) + (\gamma_1 P_3,\ \gamma_2 P_3,\ \gamma_3 P_3,\ \gamma_4 P_3) + (\delta_1 P_4,\ \delta_2 P_4,\ \delta_3 P_4,\ \delta_4 P_4)\ , \end{aligned} \tag{B.15}$$

where $\alpha_r, \beta_r, \gamma_r, \delta_r,\ r = 1, \ldots, 4$ all are real and non-negative. The conditions imposed by (B.14) imply for α_r,

$$\alpha_1 \geq \alpha_3,\ \alpha_4\ ; \quad \alpha_1 \leq \alpha_3 + \alpha_4\ ; \quad \alpha_2 \geq \alpha_3,\ \alpha_4\ ; \quad \alpha_2 \leq \alpha_3 + \alpha_4\ . \tag{B.16}$$

Exactly the same conditions should hold for β_r, γ_r and δ_r as well. Note that in case of equality we can stay within the cone using ϵ prescription for the p^0 components. For example, suppose we have,

$$\alpha_1 = \alpha_3\ ; \quad \beta_1 > \beta_3\ ; \quad \gamma_1 > \gamma_3\ ; \quad \delta_1 > \delta_3\ , \tag{B.17}$$

and all other inequalities (strictly) as in (B.16), without loss of generality. Then we can write the r.h.s. of (B.15) as

$$\begin{aligned}&(\alpha_1 P_1 + \tau\bar{\epsilon},\ \alpha_2 P_1,\ \alpha_3 P_1,\ \alpha_4 P_1) + (\beta_1 P_2 - \tau\bar{\epsilon},\ \beta_2 P_2,\ \beta_3 P_2,\ \beta_4 P_2)\\ &\quad + (\gamma_1 P_3,\ \gamma_2 P_3,\ \gamma_3 P_3,\ \gamma_4 P_3) + (\delta_1 P_4,\ \delta_2 P_4,\ \delta_3 P_4,\ \delta_4 P_4)\ ,\end{aligned} \tag{B.18}$$

where

$$0 < \tau < \min\{\beta_1 - \beta_3,\ \beta_1 - \beta_4\}\ ;$$

$$\bar{\epsilon} \equiv \begin{pmatrix}\epsilon\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix},\quad 0 < \epsilon < \min\{P_r^0 - |\vec{P}_r|,\ r = 1, \ldots, 4\}\ . \tag{B.19}$$

Clearly $P_2 - \frac{\tau}{\beta_1}\bar{\epsilon},\ P_2 - \frac{\tau}{\beta_1-\beta_3}\bar{\epsilon},\ P_2 - \frac{\tau}{\beta_1-\beta_4}\bar{\epsilon} \in V^+$ since $\frac{\tau}{\beta_1},\ \frac{\tau}{\beta_1-\beta_3},\ \frac{\tau}{\beta_1-\beta_4} < 1$. Hence for each term in (B.18), we remain inside the cone $\tilde{C}'^{+}_{12}$.

Now for each term in the decomposition (B.15), evidently all the four columns lie on a two dimensional Lorentzian plane. We need to solve for P_1, P_2, P_3, P_4 by inverting the following matrix equation,

$$\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3\\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4\end{pmatrix}\begin{pmatrix}P_1\\ P_2\\ P_3\\ P_4\end{pmatrix} = \begin{pmatrix}\mathrm{Im}\ p_1\\ \mathrm{Im}\ p_2\\ -\mathrm{Im}\ p_3\\ -\mathrm{Im}\ p_4\end{pmatrix}. \tag{B.20}$$

If we find that all the P_r are in V^+, then the proof is done and we can say that $C'^{(5)+}_{12} = \mathrm{Ch}(\bigcup_{\vec{\theta}} C'^{(5)+,\vec{\theta}}_{12})$. But subject to conditions (B.16), solving (B.20) seems to be difficult analytically.

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