CLASSICAL AND QUANTUM SUBLEADING SOFT THEOREM IN FOUR SPACETIME DIMENSIONS

By

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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Biswajit Sahoo

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To my parents

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Summary

The thesis starts with studying one infrared property of quantum gravity S-matrices and ends with some prediction of gravitational memory for scattering of astrophysical objects which can be experimentally tested by space-based gravitational wave detector in near future. In this thesis we discuss the following developments in the classical and quantum soft photon and soft graviton theorem:

- We explain Sen's covariantization technique for deriving soft photon and soft graviton theorem in general spacetime dimensions. Then we discuss some infrared issues in four spacetime dimensions for deriving soft theorem from S-matrix analysis.
- We describe how to take the classical limit of soft theorem and relate it to the long-wavelength electromagnetic and gravitational waveform. From these understandings and some naive physical assumptions, we predict the subleading soft factor in four spacetime dimensions.
- By direct S-matrix analysis in D=4, using a particular kind of infrared regularisation, we derive subleading soft theorem for loop amplitudes which turns out to be logarithmic in the energy of the soft particle and in classical limit this result agrees with the earlier prediction.
- Finally by independent classical analysis we derive long-wavelength / large retarded time electromagnetic and gravitational waveform and prove it's relation with soft factors in four spacetime dimensions.

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1 Soft theorem in general spacetime dimensions

In a generic theory of quantum electrodynamics or quantum theory of gravity, soft photon/graviton theorem gives an amplitude with a set of finite energy external particles (hard particles) with arbitrary mass and spin and one or more low energy external photons/gravitons (soft photons/gravitons), in terms of the amplitude without the low energy photons/gravitons [1–18, 43–49, 100, 101]. Though most of the derivations are done for specific theories, here in this chapter we describe a prescription pioneered by Sen [100, 101], which do not assume any particular theory.

1.1 Sen's Covariantization prescription for one soft photon/graviton

In this section we describe the covariantization prescription developed by Sen [100, 101] and carried forward in [11, 15, 17] for deriving subleading soft theorem for one external soft particle (photon/ graviton) for a generic theory of quantum electrodynamics or quantum gravity. We combine soft photon and graviton within a single soft field and at the end from the final soft theorem result we can extract soft photon or soft graviton theorem unambiguously [17]. The steps for covariantization are as follows:

- Consider a generic theory of quantum gravity in D non-compact spacetime dimensions which can have $U(1)$ gauge symmetry and assumed to be UV complete and background independent. Suppose the theory is described in terms of $U(1)$ gauge invariant and general coordinate invariant 1PI effective action having arbitrary number of fields with arbitrary mass and spin.

- Expand the 1PI effective action in powers of all the fields of the theory including photon and graviton field about their vacuum expectation values so that one point function of the fluctuation of any field vanishes.

- Now gauge fix the action using Lorentz covariant gauge fixing condition. The propagators computed from this gauge fixed action have only simple poles at it's renormalised mass square values. Use this gauge fixed action to derive the Feynman rules for vertices associated with only finite energy particles (hard particles).

- The gauge fixed action is Lorentz invariant but not manifestly $U(1)$ gauge invariant or general coordinate invariant. Now we covariantize the action with respect to soft photon and soft graviton background[1]. Broadly this covariantization means replacing the flat metric by the metric associated with soft graviton background and replace the ordinary spacetime derivative by covariant derivative containing soft photon field, spin connection and/or Christoffel connection associated with soft graviton background.

- We need to add the action for soft photon and soft graviton fields which are needed for deriving propagators for soft particles and the vertices describing interactions between soft particles[2]. We also need to include generic non-minimal coupling terms in the action describing non-minimal interactions between soft photon/graviton

[1]If we consider superstring field theory as the quantum theory of gravity then covariantization with respect to soft graviton field means deforming the background target space metric used for constructing world-sheet CFT by the soft graviton mode (which is a marginal deformation). Now the background independence tells that these two different super-string field theories are related to each other by field redefinition [109].

[2]This would be important in proving multiple soft photon/graviton theorem, but not necessary for proving single soft photon/graviton theorem.

field and finite energy fields[3].

- Now we derive the Feynman rules for the vertices describing interactions between finite energy particles and soft photons or/and gravitons from the full covariantized action after adding the non-minimal contributions described above.

- Now after having all the ingredients we have to compute tree level amplitude for a scattering event associated with arbitrary number of finite energy particles and one soft photon/graviton and expand the amplitude in terms of the energy of the soft photon/graviton to relate it to the amplitude without soft photon/graviton. This gives the soft theorem to all loop order in perturbation theory.

We shall use the reduced Planck unit $8\pi G = c = \hbar = 1$ and the Minkowski metric is mostly positive in our convention. All our finite energy fields carry tangent space indices such that both bosons and fermions can be analysed at one go. We shall use $a, b, c, d, \cdots$ as tangent space indices, $\mu, \nu, \rho, \sigma, \cdots$ as curved space indices and all the indices will be lowered or raised by Minkwoski metric η. $\{\Phi^\alpha\}$ will represent the set of fields present in the theory (given as 1PI effective action) which belongs to some reducible representation of the Lorentz group SO(1, D-1)[4]. Here α index in the superscript represents different fields as well as spin and/or polarisation indices of any one of the fields. The external soft photon field will be denoted by $A_\mu(x)$ and external soft graviton field will be denoted by $S_{\mu\nu}(x)$. They have the following plane wave modes:

$$A_\mu(x) \quad = \quad \varepsilon_\mu(k)\, e^{ik.x} \,, \qquad k^\mu \varepsilon_\mu = 0 \,, \tag{1.1.1}$$

$$S_{\mu\nu}(x) \quad = \quad \varepsilon_{\mu\nu}(k)\, e^{ik.x} \,, \qquad \varepsilon_{\mu\nu}\, \eta^{\mu\nu} \;=\; 0 \,,\; k^\mu \varepsilon_{\mu\nu} \;=\; k^\nu \varepsilon_{\mu\nu} \;=\; 0. \tag{1.1.2}$$

[3]For example soft photon field can couple to finite energy fields via field strength and soft graviton field can couple to finite energy fields via Riemann tensor and its various combinations.

[4]Though super-string field theory (a candidate of quantum theory of gravity) has infinite number of fields but if we are interested in studying a scattering event at some energy scale we can always integrate out all the massive fields with masses greater than the energy scale as they can not be produced in the scattering event. So in that sense at any energy scale we always have finite number of fields in the 1PI effective action.

The background metric associated with one soft graviton mode is given by,

$$g_{\mu\nu} \;=\; \eta_{\mu\nu} \;+\; 2S_{\mu\nu}, \qquad\quad with \;\; S_{\mu\nu} = S_{\nu\mu} \;, \quad \eta^{\mu\nu}S_{\mu\nu} = 0 \tag{1.1.3}$$

Correspondingly the vielbeins are,

$$e^{a}_{\mu} \;=\; \delta^{a}_{\mu} \;+\; S^{\;a}_{\mu} \;+\cdots\;, \qquad E^{\mu}_{a} \;=\; \delta^{\mu}_{a} \;-\; S^{\;\mu}_{a} \;+\;\cdots\;. \tag{1.1.4}$$

Above we have chosen the soft graviton field as symmetric and traceless, so that $\sqrt{-det(g)} = 1$ in the covariantized action. For proving single soft theorem upto subleading order, we only need to keep terms linear in soft photon/graviton field and also terms upto one derivative on soft photon/graviton field. We combine soft photon and graviton field within a single soft field as $\xi_r \equiv (\varepsilon_\mu, \varepsilon_{\mu\nu})$.

Suppose under global $U(1)$ transformation some real component field Φ_α transforms as $\Phi_\alpha \;\rightarrow\; [exp(iQ\theta)]_\alpha^{\;\beta}\Phi_\beta$, where θ is the $U(1)$ global transformation parameter and Q is the generator of global $U(1)$ transformation on real component field Φ_α[5]. Then the covariantization procedure suggests to replace a set of ordinary derivatives operating on Φ_α by,

$$\partial_{a_1}\partial_{a_2}...\partial_{a_m}\Phi_\alpha \;\;\rightarrow\;\; E^{\mu_1}_{a_1}E^{\mu_2}_{a_2}\cdots E^{\mu_m}_{a_m}\; D_{\mu_1}D_{\mu_2}\cdots D_{\mu_m}\;\Phi_\alpha, \tag{1.1.5}$$

where

$$D_\mu\Phi_\alpha \;=\; \partial_\mu\Phi_\alpha \;-\; iQ_\alpha^{\;\;\beta}A_\mu\;\Phi_\beta \;+\; \frac{1}{2}\;\omega^{ab}_{\mu}\;(\Sigma_{ab})_\alpha^{\;\;\beta}\;\Phi_\beta\;. \tag{1.1.6}$$

Here Σ^{ab} is the spin angular momenta, normalised by specifying it's operation on a co-

[5]Usually complex fields have $U(1)$ global charge but since we are considering gravity and gauge theory together working in terms of real field components are easier. For example in place of a complex scalar field we will work with two real scalar fields considering them in a two component vector which rotates under SO(2) and Q is the generator of SO(2) transformation.

variant vector field ϕ_d as,

$$(\Sigma^{ab})_c{}^d\, \phi_d \;=\; (\delta^a{}_c \eta^{bd} \;-\; \delta^b{}_c \eta^{ad})\phi_d \tag{1.1.7}$$

and ω_μ^{ab} is the spin connection defined upto first order in $S_{\mu\nu}$ as,

$$\omega_\mu^{ab} \;=\; \partial^b S_\mu^{\;a} \;-\; \partial^a S_\mu^{\;b}\,. \tag{1.1.8}$$

Similarly two covariant derivatives operating on a field can be expressed upto terms linear in soft field and only one derivative on the soft field as,

$$\begin{aligned} D_\mu D_\nu \Phi_\alpha \;&=\; \partial_\mu(D_\nu\Phi_\alpha) \;-iQ_\alpha{}^\beta A_\mu(D_\nu\Phi_\beta) \;+\; \frac{1}{2}\omega_\mu{}^{ab}(\Sigma_{ab})_\alpha{}^\beta(D_\nu\Phi_\beta) \;-\Gamma^\rho_{\mu\nu}\partial_\rho\Phi_\alpha \\ &=\; \partial_\mu\partial_\nu\Phi_\alpha \;-\; iQ_\alpha{}^\beta\partial_\mu A_\nu\Phi_\beta \;-\; iQ_\alpha{}^\beta\,(A_\mu\partial_\nu\Phi_\beta + A_\nu\partial_\mu\Phi_\beta) \\ &\quad + \frac{1}{2}\omega_\mu{}^{ab}(\Sigma_{ab})_\alpha{}^\beta\partial_\nu\Phi_\beta \;+\; \frac{1}{2}\omega_\nu{}^{ab}(\Sigma_{ab})_\alpha{}^\beta\partial_\mu\Phi_\beta \;-\; \Gamma^\rho_{\mu\nu}\partial_\rho\phi_\alpha \;+\; \cdots \end{aligned} \tag{1.1.9}$$

where

$$\Gamma^\rho_{\mu\nu} \;=\; \Big(\,\partial_\mu S_\nu^\rho + \partial_\nu S_\mu^\rho - \partial^\rho S_{\mu\nu}\Big) \;+\; \text{terms quadratic in } S_{\mu\nu}\,. \tag{1.1.10}$$

The terms represented by "$\cdots$" in eq.(1.1.9) are either quadratic in the soft fields (would be important in proving double soft theorem) or contains more than one derivative on the soft field (would be important in proving soft theorem at sub-subleading order) would not play any role in proving subleading soft theorem for one soft particle. Also the last term containing Christoffel connection will not contribute to our analysis following the argument given in the appendix of [101].

We shall follow the convention that all the particles are ingoing and in all the Feynman diagrams the thick lines represent hard particles and thin line represents combined soft particle. To determine the vertices containing two finite energy particles and one soft particle we need to covariantize the following quadratic part of the 1PI effective action:

$$S^{(2)} \;=\; \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\,\frac{d^D q_2}{(2\pi)^D}\;\Phi_\alpha(q_1)\,\mathcal{K}^{\alpha\beta}(q_2)\,\Phi_\beta(q_2)\,(2\pi)^D\delta^{(D)}(q_1+q_2)\;. \tag{1.1.11}$$

Here $\Phi_\alpha(q)$ is the Fourier transform of the field $\Phi_\alpha(x)$ and if Φ_α is a Grassmann even field then the kinetic operator satisfies,

$$\mathcal{K}^{\alpha\beta}(q) \;=\; \mathcal{K}^{\beta\alpha}(-q)\;. \tag{1.1.12}$$

If Φ_α is a Grassmann odd field then the right hand side of (1.1.12) should contain an extra negative sign but final soft theorem result will be independent of this. The full renormalized propagator for a finite energy particle with momentum q^μ and renormalised mass M is given by,

$$D_F(q)_{\alpha\beta} \;\equiv\; i\,\mathcal{K}^{-1}_{\alpha\beta}(q) \;\equiv\; (q^2+M^2)^{-1}\,\Xi_{\alpha\beta}(q)\;, \tag{1.1.13}$$

From the invariance of the action $S^{(2)}$ under global $U(1)$ transformation $\Phi_\alpha \to [exp\,(iQ\theta)]_\alpha{}^\beta\,\Phi_\beta$, the kinetic operator needs to satisfy

$$Q_\gamma^{\ \alpha}\,\mathcal{K}^{\gamma\beta} \;+\; \mathcal{K}^{\alpha\gamma}\,Q_\gamma^{\ \beta} \;=\; 0. \tag{1.1.14}$$

In index free matrix notation this relation translates to,

$$Q^T\mathcal{K} \;+\; \mathcal{K}Q \;=0. \tag{1.1.15}$$

This imposes the following condition on the numerator of the propagator defined in (1.1.13),

$$Q\,\Xi \;+\; \Xi\,Q^T \;=\; 0. \tag{1.1.16}$$

The vertex describing the minimal interaction between two finite energy particles and one soft particle with polarisation and momentum $(\xi,\ k)$ upto linear order in soft momentum, can be read off from the following covariantized part of the action(1.1.11),

$$\begin{aligned} S^{(3)} &= \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k) \\ &\quad \Phi_\alpha(q_1)\Bigg[-\varepsilon_\mu \frac{\partial \mathcal{K}^{\alpha\gamma}(q_2)}{\partial q_{2\mu}}\,Q_\gamma^{\ \beta} - \frac{1}{4}\,(k_\mu\varepsilon_\nu + k_\nu\varepsilon_\mu)\,\frac{\partial^2\mathcal{K}^{\alpha\gamma}(q_2)}{\partial q_{2\mu}\partial q_{2\nu}}\,Q_\gamma^{\ \beta} \\ &\quad -\varepsilon_{\mu\nu}q_2^\nu\,\frac{\partial \mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu}} + \frac{1}{2}\,(k_b\varepsilon_{a\mu} - k_a\varepsilon_{b\mu})\,\frac{\partial \mathcal{K}^{\alpha\gamma}(q_2)}{\partial q_{2\mu}}\,(\Sigma^{ab})_\gamma^{\ \beta}\Bigg]\,\Phi_\beta(q_2)\,. \end{aligned} \tag{1.1.17}$$

First, third, and fourth terms within the square bracket above directly follows from covariantization of single derivative on Φ_β within $\mathcal{K}^{\alpha\beta}(q_2)$ following eq.(1.1.6) multiplied by inverse vielbein. The second term within the square bracket follows from covariantization of two derivative on Φ_β within $\mathcal{K}^{\alpha\beta}(q_2)$ due to the second term in the r.h.s. of eq.(1.1.9). With $S^{(3)}$ one needs to add non-minimal part of the action describing interaction between two finite energy particles and one soft photon coupled via field strength[6],

$$\overline{S}^{(3)} \equiv \frac{1}{2}\int \frac{d^D q_1}{(2\pi)^D}\frac{d^D q_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k)\,F_{\mu\nu}(k)\,\Phi_\alpha(q_1)\,C^{\alpha\beta,\mu\nu}(q_2)\,\Phi_\beta(q_2) \tag{1.1.18}$$

where

$$F_{\mu\nu}(k) = i\,[k_\mu\varepsilon_\nu(k) - k_\nu\varepsilon_\mu(k)]$$

and $C^{\alpha\beta,\mu\nu}(q)$ is a generic non-minimal coupling[7]. From the global $U(1)$ invariance and symmetry of the action, C has to satisfy the following properties for grassmann even field Φ_α ,

$$Q_\gamma^{\ \alpha}\,C^{\gamma\beta,\mu\nu}(q_2) + C^{\alpha\gamma,\mu\nu}(q_2)\,Q_\gamma^{\ \beta} = 0 \tag{1.1.19}$$

[6] We can add similar non-minimal action for soft graviton as well, but this kind of interaction will not affect single soft graviton theorem upto subleading order as it involved Riemann tensor of soft graviton field, which generates vertex quadratic in soft graviton momentum [11].

[7] An example of such kind of non-minimal coupling in spinor QED is $\bar{\psi}\gamma^{\mu\nu}F_{\mu\nu}\psi$ where $C^{\mu\nu} \sim \gamma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$

$$C^{\alpha\beta,\mu\nu}(q_2) = -C^{\alpha\beta,\nu\mu}(q_2) \tag{1.1.20}$$

$$C^{\alpha\beta,\mu\nu}(q_2) = C^{\beta\alpha,\mu\nu}(-q_1-k). \tag{1.1.21}$$

If Φ_α is a Grassmann odd field then the r.h.s. of the last equation above sould have an extra minus sign.

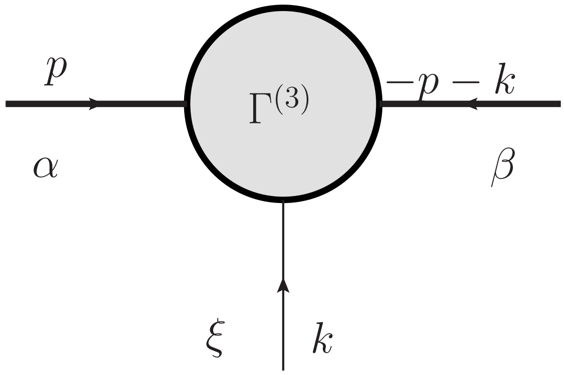

Figure 1.1: 1PI vertex describing interection between two finite energy particles and one soft particle. Figure adopted from [17].

The expression of the 1PI vertex for Fig.(1.1) is derived from (1.1.17) and (1.1.18) after symmetrization in α,β,

$$\begin{aligned}
&\Gamma^{(3)\alpha\beta}(\xi,k;\ p,-p-k)\\
&= +\frac{i}{2}\Bigg[\ \varepsilon_\mu(k)\frac{\partial\mathcal{K}^{\alpha\gamma}(-p-k)}{\partial p_\mu}\,\mathcal{Q}_\gamma^{\ \beta} - \ \varepsilon_\mu(k)\frac{\partial\mathcal{K}^{\beta\gamma}(p)}{\partial p_\mu}\,\mathcal{Q}_\gamma^{\ \alpha} - \frac{1}{4}\,(k_\mu\varepsilon_\nu+k_\nu\varepsilon_\mu)\frac{\partial^2\mathcal{K}^{\alpha\gamma}(-p-k)}{\partial p_\mu\partial p_\nu}\mathcal{Q}_\gamma^{\ \beta}\\
&-\frac{1}{4}\,(k_\mu\varepsilon_\nu+k_\nu\varepsilon_\mu)\frac{\partial^2\mathcal{K}^{\beta\gamma}(p)}{\partial p_\mu\partial p_\nu}\,\mathcal{Q}_\gamma^{\ \alpha} + \ i\,(k_\mu\varepsilon_\nu-k_\nu\varepsilon_\mu)\,C^{\alpha\beta,\mu\nu}(-p-k) \ + \ i\,(k_\mu\varepsilon_\nu-k_\nu\varepsilon_\mu)C^{\beta\alpha,\mu\nu}(p)\\
&-\varepsilon_{\mu\nu}(k)(p+k)^\nu\,\frac{\partial\mathcal{K}^{\alpha\beta}(-p-k)}{\partial p_\mu} \ - \ \varepsilon_{\mu\nu}(k)p^\nu\,\frac{\partial\mathcal{K}^{\beta\alpha}(p)}{\partial p_\mu}\\
&-\frac{1}{2}(k_b\varepsilon_{a\mu}-k_a\varepsilon_{b\mu})\,\frac{\partial\mathcal{K}^{\alpha\gamma}(-p-k)}{\partial p_\mu}(\Sigma^{ab})_\gamma^{\ \beta} \ + \ \frac{1}{2}\,(k_b\varepsilon_{a\mu}-k_a\varepsilon_{b\mu})\,\frac{\partial\mathcal{K}^{\beta\gamma}(p)}{\partial p_\mu}(\Sigma^{ab})_\gamma^{\ \alpha}\Bigg].
\end{aligned} \tag{1.1.22}$$

After using the relations (1.1.12),(1.1.1),(1.1.2), (1.1.21),(1.1.15) and doing Taylor series expansion in k^μ, upto linear order in k^μ we get the following index free form of $\Gamma^{(3)}$,

$$\begin{aligned}
&\Gamma^{(3)}(\xi, k;\ p, -p-k) \\
&= i\Bigg[+ \varepsilon_\mu \frac{\partial\mathcal{K}(-p)}{\partial p_\mu}\, Q \;+\; \frac{1}{2}\,\varepsilon_\mu k_\nu\, \frac{\partial^2\mathcal{K}(-p)}{\partial p_\mu \partial p_\nu}\, Q \;+\; i\,(k_\mu\varepsilon_\nu - k_\nu\varepsilon_\mu)\, C^{\mu\nu}(-p) \\
&- \varepsilon_{\mu\nu}p^\nu\, \frac{\partial\mathcal{K}(-p)}{\partial p_\mu} \;+\; \frac{1}{2}\,\varepsilon_{b\mu}k_a\, \frac{\partial\mathcal{K}(-p)}{\partial p_\mu}\,(\Sigma^{ab}) \;-\; \frac{1}{2}\,\varepsilon_{b\mu}k_a\,(\Sigma^{ab})^T\, \frac{\partial\mathcal{K}(-p)}{\partial p_\mu} \;-\; \frac{1}{2}\,\varepsilon_{\mu\nu}p^\nu k_\rho\, \frac{\partial^2\mathcal{K}(-p)}{\partial p_\mu \partial p_\rho}\Bigg] .
\end{aligned} \tag{1.1.23}$$

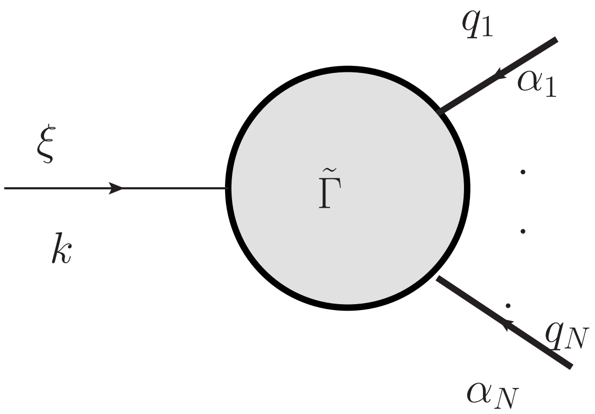

Figure 1.2: Amputated Greens function of N number of finite energy particles and one soft particle. This Feynman diagram represents sum over all the Feynman diagrams leaving the diagrams where the soft particle is attached to some external leg. Figure adopted from [17].

The amputated Greens function of N number of finite energy particles and one soft particle in Fig.(1.2) is evaluated by covariantizing the N particle amputated Greens function $\Gamma(q_1, q_2, \cdots, q_N)$ with respect to soft particle(photon/graviton) background. Since the diagram $\widetilde{\Gamma}$ does not have any scalar propagator which contributes inverse power in soft momentum, it starts contributing from subleading order in soft momentum expansion. So we need the leading contribution in soft momentum from this diagram,

$$\widetilde{\Gamma}(\xi, k;\ q_1, q_2, \cdots, q_N) \;=\; -\sum_{i=1}^{N} \Big\{ \prod_{\substack{j=1 \\ j\neq i}}^{N} \epsilon_{\alpha_j} \Big\}\, \epsilon_{\alpha_i} \left[Q^{\alpha_i}_{\beta_i}\, \varepsilon_\mu(k)\, \frac{\partial}{\partial q_{i\mu}} \;+\; \varepsilon_{\mu\nu}(k)\delta^{\alpha_i}_{\beta_i}\, q_i^\nu\, \frac{\partial}{\partial q_{i\mu}} \right] \Gamma^{\alpha_1 \cdots \beta_i \cdots \alpha_N}(q_1, ..., q_N) , \tag{1.1.24}$$

where $\epsilon_{\alpha_i}(q_i)$ denotes the polarization tensor of the i'th finite energy particle. Note that both $\widetilde{\Gamma}$ and Γ are distributions of momenta as in our notation $\widetilde{\Gamma}$ contains $(2\pi)^D\delta^{(D)}(q_1 + \cdots + q_N + k)$ and Γ contains $(2\pi)^D\delta^{(D)}(q_1 + \cdots + q_N)$.

1.2 Steps for evaluating Feynman diagrams

Here we describe five steps which one have to follow consecutively to evaluate the Feynman diagrams.

1. First write down the contribution of any Feynman diagram using the vertices and propagators derived in §1.1 and multiply the polarisation tensor of finite energy particles from the left.

2. Now use the following relations and their momentum derivatives to move (Σ^{ab}) and $(\Sigma^{ab})^T$ to the possible extreme right position,

$$(\Sigma^{ab})^T\,\mathcal{K}(-p) = -\mathcal{K}(-p)\,(\Sigma^{ab}) + p^a\frac{\partial\mathcal{K}(-p)}{\partial p_b} - p^b\frac{\partial\mathcal{K}(-p)}{\partial p_a}\,, \tag{1.2.25}$$

$$(\Sigma^{ab})\,\Xi(-p) = -\,\Xi(-p)\,(\Sigma^{ab})^T - p^a\frac{\partial\Xi(-p)}{\partial p_b} + p^b\frac{\partial\Xi(-p)}{\partial p_a}\,. \tag{1.2.26}$$

   The above relations follow from the Lorentz covariance of $\mathcal{K}$ and Ξ. Similarly move the $U(1)$ charge matrix Q to the possible extreme right using (1.1.15), (1.1.16), (1.1.20) and their momentum derivative relations.

3. Now Taylor expand $\mathcal{K}$, Ξ and $\Gamma^{\alpha_1,\cdots,\alpha_N}$ in powers of soft momenta k^μ and keep terms up to the required order.

4. Now move the momentum derivative from $\mathcal{K}$ to Ξ to the maximal possible extent using the following relations which directly follows from eq.(1.1.13),

$$\frac{\partial\mathcal{K}(-p)}{\partial p_\mu}\Xi(-p) = -\,\mathcal{K}(-p)\frac{\partial\Xi(-p)}{\partial p_\mu} + 2ip^\mu \tag{1.2.27}$$

$$\frac{\partial \Xi(-p)}{\partial p_\mu}\mathcal{K}(-p) = -\Xi(-p)\frac{\partial \mathcal{K}(-p)}{\partial p_\mu} + 2ip^\mu . \tag{1.2.28}$$

$$\frac{\partial^2 \mathcal{K}(-p)}{\partial p_\mu \partial p_\nu}\Xi(-p) = 2i\eta^{\mu\nu} - \frac{\partial \mathcal{K}(-p)}{\partial p_\mu}\frac{\partial \Xi(-p)}{\partial p_\nu} - \frac{\partial \mathcal{K}(-p)}{\partial p_\nu}\frac{\partial \Xi(-p)}{\partial p_\mu} - \mathcal{K}(-p)\frac{\partial^2 \Xi(-p)}{\partial p_\mu \partial p_\nu}. \tag{1.2.29}$$

5. In the final step use the on-shell condition for finite energy particle,

$$\epsilon^T(p)\mathcal{K}(-p) = 0. \tag{1.2.30}$$

1.3 Subleading soft theorem for one external soft photon/graviton

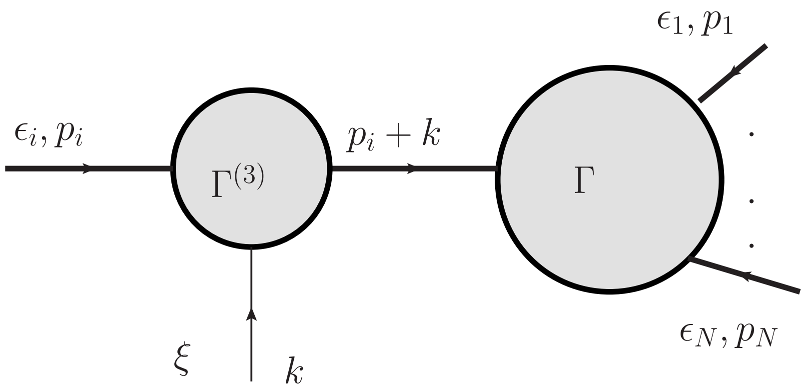

Figure 1.3: Diagram representing sum over all possible Feynman diagrams where the soft particle is attached to some external leg via $\Gamma^{(3)}$. Figure adopted from [17].

The diagram in Fig.1.3 starts contributing at leading order since it contains a nearly on-shell propagator which contributes inverse power in soft momentum. On the other hand the diagram in Fig.1.4 starts contributing at subleading order relative to the contribution from Fig.1.3. Following the steps described in §1.2 we can evaluate the diagrams. Here we are giving the final expressions, for intermediate steps the reader can look into the references [17,101]. The N-particle amputated Greens function after stripping out the i'th finite energy particle polarisation tensor is denoted by,

$$\Gamma^{(i)\alpha_i}(p_i) \equiv \left\{ \prod_{\substack{j=1 \\ j\neq i}}^{N} \epsilon_{j,\alpha_j}(p_j) \right\} \Gamma^{\alpha_1\alpha_2\cdots\alpha_N}(p_1, \cdots, p_N) \,. \tag{1.3.31}$$

The contribution of Fig.(1.3) turns out to be,

$$\begin{aligned} A_i \;=\; & \{p_i \cdot k\}^{-1} \Bigg[\varepsilon_\mu p_i^\mu \epsilon_i^T Q_i^T\, \Gamma^{(i)}(p_i) \,+\, \varepsilon_\mu p_i^\mu\, k_\rho\, \epsilon_i^T Q_i^T \frac{\partial \Gamma^{(i)}(p_i)}{\partial p_{i\rho}} \,+\, (\varepsilon_\mu k_\nu - \varepsilon_\nu k_\mu)\, \epsilon_i^T\, \mathcal{N}^{\mu\nu}_{(i)}(-p_i)\, \Gamma^{(i)}(p_i) \\ & + \varepsilon_{\mu\nu} p_i^\mu p_i^\nu\, \epsilon_i^T \Gamma^{(i)}(p_i) \,+\, \varepsilon_{\mu\nu} p_i^\mu p_i^\nu k_\rho\, \epsilon_i^T \frac{\partial \Gamma^{(i)}(p_i)}{\partial p_{i\rho}} \,+\, \varepsilon_{b\mu} p_i^\mu k_a\, \epsilon_i^T (\Sigma_i^{ab})^T\, \Gamma^{(i)}(p_i) \Bigg] \,. \end{aligned} \tag{1.3.32}$$

where,

$$\mathcal{N}^{\mu\nu}_{(i)}(-p_i) = \frac{i}{4} \frac{\partial \mathcal{K}_i(-p_i)}{\partial p_{i\nu}} \frac{\partial \Xi_i(-p_i)}{\partial p_{i\mu}}\, Q_i^T \,+\, \frac{1}{2}\, C^{\mu\nu}_{(i)}(-p_i) \Xi_i(-p_i) \,, \tag{1.3.33}$$

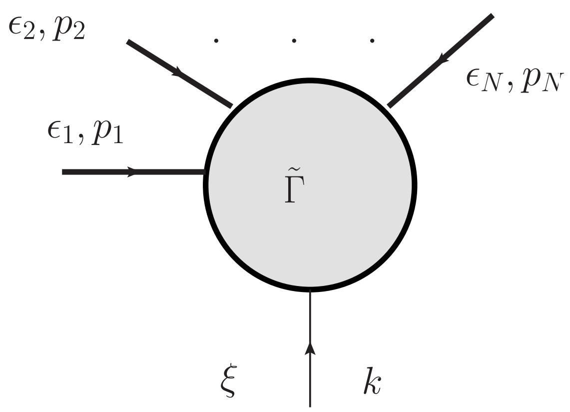

Figure 1.4: Diagram representing sum over all possible Feynman diagrams where the soft particle is not attached to any external leg. Figure adopted from [17].

Contribution of Fig.(1.4) directly follows from (1.1.24),

$$B = -\sum_{i=1}^{N} \left[\varepsilon_\mu(k)\, \epsilon_i^T\, Q_i^T \frac{\partial \Gamma^{(i)}(p_i)}{\partial p_{i\mu}} \,+\, \varepsilon_{\mu\nu}(k) p_i^\nu\, \epsilon_i^T \frac{\partial \Gamma^{(i)}(p_i)}{\partial p_{i\mu}} \right]. \tag{1.3.34}$$

Now $\sum_{i=1}^{N} A_i + B$ will give the full $(N+1)$ point amplitude in terms of N point amplitude.

Hence the subleading soft theorem for one soft particle turns out to be:

$$\begin{aligned}
&\Gamma^{(N+1)}(\xi, k;\ \{\epsilon_i, p_i\}) \\
&= \sum_{i=1}^{N} \{p_i \cdot k\}^{-1}\ \epsilon_i^T \left\{ \varepsilon_\mu p_i^\mu\ Q_i^T\ +\ \varepsilon_{\mu\nu} p_i^\mu p_i^\nu \right\} \Gamma^{(i)}(p_i) \\
&\ +\ \sum_{i=1}^{N} \{p_i \cdot k\}^{-1}\ \varepsilon_\mu k_\rho\ \epsilon_i^T\ Q_i^T \left\{ p_i^\mu \frac{\partial}{\partial p_{i\rho}}\ -\ p_i^\rho \frac{\partial}{\partial p_{i\mu}} \right\} \Gamma^{(i)}(p_i) \\
&\ +\sum_{i=1}^{N} \{p_i \cdot k\}^{-1}\ \varepsilon_{b\mu} p_i^\mu\ k_a\ \epsilon_i^T \left\{ p_i^b \frac{\partial}{\partial p_{ia}}\ -\ p_i^a \frac{\partial}{\partial p_{ib}}\ +\ (\Sigma_i^{ab})^T \right\} \Gamma^{(i)}(p_i) \\
&\ +\ \sum_{i=1}^{N} \{p_i \cdot k\}^{-1}\ (\varepsilon_\mu k_\nu - \varepsilon_\nu k_\mu)\ \epsilon_i^T\ \mathcal{N}_{(i)}^{\mu\nu}(-p_i)\ \Gamma^{(i)}(p_i)\ . \qquad (1.3.35)
\end{aligned}$$

From here, we can recover the single soft photon theorem upto subleading order after setting the polarization tensor for graviton to zero. If the external finite energy particles are charge eigenstates of $U(1)$ charge generator Q i.e. $Q\epsilon_i = q_i\epsilon_i$ then the subleading soft photon theorem takes the standard form of "Low's subleading soft photon theorem" [45],

$$\begin{aligned}
&\Gamma^{(N+1)}(\varepsilon, k;\ \{\epsilon_i, p_i\}) \\
&=\ \sum_{i=1}^{N} \left[q_i \frac{\varepsilon_\mu p_i^\mu}{p_i \cdot k}\ +\ q_i \frac{\varepsilon_\mu k_\rho}{p_i \cdot k} \left\{ p_i^\mu \frac{\partial}{\partial p_{i\rho}}\ -\ p_i^\rho \frac{\partial}{\partial p_{i\mu}} \right\} \right] \Gamma^{(N)}(\{\epsilon_i, p_i\}) \\
&\quad +\ \sum_{i=1}^{N} \frac{1}{p_i \cdot k}\ (\varepsilon_\mu k_\nu - \varepsilon_\nu k_\mu)\ \epsilon_i^T\ \mathcal{N}_{(i)}^{\mu\nu}(-p_i)\ \Gamma^{(i)}(p_i)\ . \qquad (1.3.36)
\end{aligned}$$

In the above equation second line of r.h.s is the general structure of non-universal term appearing in the subleading order of soft photon theorem. This non-universal term in soft photon theorem has been explored in [97] for some particular examples of non-minimal couplings in effective field theories of QED..

Similarly by setting the polarization vector of photon to be zero in expression(1.3.35) we recover "Cachazo-Strominger subleading soft graviton theorem" [7].

$$\Gamma^{(N+1)}(\varepsilon, k;\ \{\epsilon_i, p_i\})$$

$$= \sum_{i=1}^{N} \epsilon_i^T \left[\frac{\varepsilon_{\mu\nu} p_i^\mu p_i^\nu}{p_i \cdot k} + \frac{\varepsilon_{b\mu} p_i^\mu k_a}{p_i \cdot k} \left\{ p_i^b \frac{\partial}{\partial p_{ia}} - p_i^a \frac{\partial}{\partial p_{ib}} + (\Sigma_i^{ab})^T \right\} \right] \Gamma^{(i)}(p_i) \tag{1.3.37}$$

Above in the subleading soft factor the full expression within curly bracket represents transpose of total angular momentum of i'th finite energy particle,

$$j_i^{ab} \equiv p_i^b \frac{\partial}{\partial p_{ia}} - p_i^a \frac{\partial}{\partial p_{ib}} + \Sigma_i^{ab}. \tag{1.3.38}$$

1.4 Sub-subleading soft graviton theorem and subleading multiple soft graviton theorem

In this section we explain what extra ingredients we need to prove the sub-subleading soft graviton theorem and subleading multiple soft graviton theorem for a generic theory of quantum gravity and then directly state the results.

1.4.1 Sub-subleading soft theorem for one external soft graviton

For proving sub-subleading soft graviton theorem we have to compute the same two diagrams in Fig.1.3 and 1.4 but now to one higher order in soft momentum expansion. For this we need two extra ingredients [11] :

1. In the covariantization procedure of quadratic part of 1PI action, now we need to include contribution of two new terms with eq.(1.1.17), which are quadratic in soft momentum. One comes from covariantizing two derivatives operating on Φ_β: covariantization of $\partial_\mu \partial_\nu \Phi_\beta$ gives additional term $\frac{1}{2}\partial_{(\mu}\omega_{\nu)}^{ab}(\Sigma_{ab})_\beta{}^\gamma \Phi_\gamma$. Another new contribution comes from covariantizing three derivatives operating on Φ_β: covariantization of $\partial_\mu \partial_\nu \partial_\rho \Phi_\beta$ gives additional term $-\partial_{(\rho}\Gamma^\sigma_{\mu\nu)}\partial_\sigma \Phi_\beta$.

2. In the quadratic order in soft momentum we need to add a generic coupling term contribute to $\Gamma^{(3)}(\varepsilon, k; p, -p-k)$ describing non-minimal interaction between two finite energy particles and one soft graviton via Riemann tensor. We consider the general form of the non-minimal action with specific soft graviton momentum k,

$$\overline{S}^{(3)}_{gr} = \frac{1}{2}\int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(q_1+q_2+k)\,R_{\mu\nu\rho\sigma}(k)\;\Phi_\alpha(q_1)\mathcal{B}^{\alpha\beta,\mu\nu\rho\sigma}(q_2)\phi_\beta(q_2) \tag{1.4.39}$$

where $R_{\mu\nu\rho\sigma}(k)$ is the Fourier transform of linearised Riemann tensor given as,

$$R_{\mu\nu\rho\sigma}(k) = \varepsilon_{\mu\rho}k_\nu k_\sigma - \varepsilon_{\mu\sigma}k_\nu k_\rho - \varepsilon_{\nu\rho}k_\sigma k_\mu + \varepsilon_{\nu\sigma}k_\mu k_\rho \tag{1.4.40}$$

and $\mathcal{B}^{\alpha\beta,\mu\nu\rho\sigma}(q)$ is a generic function of momentum, spin of finite energy particles. Here $\mathcal{B}$ satisfies $\mathcal{B}^{\alpha\beta,\mu\nu\rho\sigma}(q_2) = \mathcal{B}^{\beta\alpha,\mu\nu\rho\sigma}(-q_2-k)$.

Including the above two new kind of contributions and following the steps described in §1.2, the sub-subleading contribution turns out to be [11]:

$$\begin{aligned}
&\Delta_{(sub)^2-leading}\,\Gamma^{(N+1)}(\varepsilon, k;\ \{\epsilon_i, p_i\})\\
=\ &\frac{1}{2}\sum_{i=1}^{N}\frac{\varepsilon_{ac}k_bk_d}{p_i\cdot k}\,\epsilon_i^T\Big[p_i^b\frac{\partial}{\partial p_{ia}} - p_i^a\frac{\partial}{\partial p_{ib}} + (\Sigma_i^{ab})^T\Big]\Big[p_i^d\frac{\partial}{\partial p_{ic}} - p_i^c\frac{\partial}{\partial p_{id}} + (\Sigma_i^{cd})^T\Big]\,\Gamma^{(i)}(p_i)\\
&+\frac{1}{2}\sum_{i=1}^{N}\frac{1}{p_i\cdot k}\,R_{\mu\rho\nu\sigma}(k)\,\epsilon_i^T\,\mathcal{N}^{\mu\rho\nu\sigma}_{(i)}(-p_i)\,\Gamma^{(i)}(p_i)
\end{aligned} \tag{1.4.41}$$

where

$$\begin{aligned}
\mathcal{N}^{\mu\rho\nu\sigma}_{(i)}(-p_i) =\ &\frac{i}{3}p_i^\nu\,\frac{\partial\mathcal{K}_i(-p_i)}{\partial p_{i\mu}}\,\frac{\partial^2\Xi_i(-p_i)}{\partial p_{i\rho}\partial p_{i\sigma}} - \frac{i}{6}p_i^\rho\,\frac{\partial^2\mathcal{K}_i(-p_i)}{\partial p_{i\mu}\partial p_{i\nu}}\,\frac{\partial\Xi_i(-p_i)}{\partial p_{i\sigma}}\\
&+\frac{i}{4}\,\frac{\partial\mathcal{K}_i(-p_i)}{\partial p_{i\mu}}\,\frac{\partial\Xi_i(-p_i)}{\partial p_{i\rho}}\,(\Sigma_i^{\nu\sigma})^T - \frac{1}{4}\,(\Sigma_i^{\mu\rho})^T\,(\Sigma_i^{\nu\sigma})^T + i\mathcal{B}^{\mu\rho\nu\sigma}_{(i)}(-p_i)\,\Xi_i(-p_i)
\end{aligned} \tag{1.4.42}$$

The second line of r.h.s in eq.(1.4.41) represents the non-universal term appearing in the sub-subleading order of soft graviton theorem. For various effective theory of quantum gravity with non-minimal couplings the non-universal part has been tested explicitly [11, 97].

1.4.2 Subleading multiple soft graviton theorem

For proving double soft graviton theorem we need two new kind of vertices besides the vertices available for proving single soft graviton theorem. It turns out that the ingredients needed for proving single and double soft graviton theorem is enough for proving multiple soft graviton theorem though the proof is much more involved [15]. The two new kind of vertices are

1. We need to evaluate a four point 1PI vertex $\Gamma^{(4)}$ describing interaction between two finite energy particles and two soft gravitons. This vertex can be derived from the covariantization of quadratic part of 1PI effective action $S^{(2)}$ with background metric $g_{\mu\nu} = \eta_{\mu\nu} + 2S_{\mu\nu} + 2S_{\mu\rho}S^{\rho}{}_{\nu}$, but now one has to keep terms upto quadratic order in soft graviton field $S_{\mu\nu}$.

2. There are possible Feynman diagrams containing three graviton vertex $V^{(3)}$ of which one graviton is virtual and connected to a finite energy leg by vertex $\Gamma^{(3)}$. This three graviton vertex along with graviton propagator can be worked out from the Einstein-Hilbert action for the soft graviton metric.

We have to evaluate four kinds of diagrams for extracting subleading multiple soft factor:

- Diagrams where all the soft gravitons are connected to finite energy particle lines by $\Gamma^{(3)}$ vertices. These diagrams start contributing from leading order in soft momenta expansion.

- Diagrams where one soft graviton is connected directly to the N-point amputated Greens function through $\widetilde{\Gamma}$ and the rest of the soft gravitons are connected to finite energy particle lines by $\Gamma^{(3)}$ vertices. These diagrams start contributing at the subleading order in soft momenta expansion.

- Diagrams where two soft gravitons are connected to a finite energy line via vertex $\Gamma^{(4)}$ and the rest of the soft gravitons are connected to finite energy particle lines by $\Gamma^{(3)}$ vertices. These diagrams start contributing at the subleading order in soft momenta expansion.

- Diagrams where two soft gravitons are connected to a scalar line via three graviton vertex $V^{(3)}$ - graviton propagator -vertex $\Gamma^{(3)}$ and the rest of the soft gravitons are connected to finite energy particle lines by $\Gamma^{(3)}$ vertices. These diagrams start contributing at the subleading order in soft momenta expansion.

For M number of soft gravitons with polarisations and momenta $\{\varepsilon_r(k_r), k_r\}$ the subleading multiple soft graviton theorem becomes [15]:

$$\begin{aligned}
&\Gamma^{(N+M)}(\{\varepsilon_r, k_r\}\,,\ \{\epsilon_i, p_i\}) \\
= &\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}\right\}\Bigg[\left\{\prod_{r=1}^{M}S_r^{(0)}\right\}\Gamma^{\alpha_1\cdots\alpha_N} + \sum_{s=1}^{M}\left\{\prod_{\substack{r=1\\ r\neq s}}^{M}S_r^{(0)}\right\}\left[S_s^{(1)}\Gamma\right]^{\alpha_1\cdots\alpha_N} \\
&+\sum_{\substack{r,u=1\\ r<u}}^{M}\left\{\prod_{\substack{s=1\\ s\neq r,u}}^{M}S_r^{(0)}\right\}\left\{\sum_{j=1}^{N}\frac{1}{p_j\cdot(k_r+k_u)}\,\mathcal{M}(p_j;\varepsilon_r,k_r;\varepsilon_u,k_u)\right\}\Gamma^{\alpha_1\cdots\alpha_N}\Bigg]
\end{aligned} \tag{1.4.43}$$

where $S_r^{(0)}$ is the leading soft factor for r'th graviton

$$S_r^{(0)} = \sum_{i=1}^{N}\frac{\varepsilon_{r,\mu\nu}p_i^\mu p_i^\nu}{p_i\cdot k_r}, \tag{1.4.44}$$

$S_r^{(1)}$ is the subleading soft factor for r'th graviton

$$\left[S_r^{(1)}\Gamma\right]^{\alpha_1\cdots\alpha_N} = \sum_{i=1}^{N}\frac{\varepsilon_{r,b\mu}p_i^\mu\ k_{ra}}{p_i\cdot k_r}\left\{p_i^b\frac{\partial\Gamma^{\alpha_1\cdots\alpha_N}}{\partial p_{ia}} - p_i^a\frac{\partial\Gamma^{\alpha_1\cdots\alpha_N}}{\partial p_{ib}} + (\Sigma_i^{ab})_{\beta_i}{}^{\alpha_i}\,\Gamma^{\alpha_1\cdots\beta_i\cdots\alpha_N}\right\}$$

(1.4.45)

and $\mathcal{M}(p_j;\varepsilon_r,k_r;\varepsilon_u,k_u)$ is the contact term receives contribution from the last two kind of diagrams described above and the expression of it is as follows:

$$
\begin{aligned}
&\mathcal{M}(p_j;\varepsilon_r,k_r;\varepsilon_u,k_u)\\
=\;& (p_j\cdot k_r)^{-1}\,(p_j\cdot k_s)^{-1}\Big[-(k_r\cdot k_s)\,(p_j\cdot\varepsilon_r\cdot p_j)\,(p_j\cdot\varepsilon_s\cdot p_j)\;+\;2(p_j\cdot k_s)\,(p_j\cdot\varepsilon_r\cdot p_j)\,(p_j\cdot\varepsilon_s\cdot k_r)\\
&+2(p_j\cdot k_r)\,(p_j\cdot\varepsilon_s\cdot p_j)\,(p_j\cdot\varepsilon_r\cdot k_s)\;-\;2(p_j\cdot k_r)\,(p_j\cdot k_s)\,(p_j\cdot\varepsilon_r\cdot\varepsilon_s\cdot p_j)\Big]\\
&+(k_r\cdot k_s)^{-1}\Big[-(k_s\cdot\varepsilon_r\cdot\varepsilon_s\cdot p_j)\,(k_s\cdot p_j)\;-\;(k_r\cdot\varepsilon_s\cdot\varepsilon_r\cdot p_j)\,(k_r\cdot p_j)\\
&+(k_s\cdot\varepsilon_r\cdot\varepsilon_s\cdot p_j)\,(k_r\cdot p_j)+(k_r\cdot\varepsilon_s\cdot\varepsilon_r\cdot p_j)\,(k_s\cdot p_j)\;-\;(\varepsilon_{r,\rho\sigma}\,\varepsilon_s^{\rho\sigma})(k_r\cdot p_j)(k_s\cdot p_j)\\
&-2(p_j\cdot\varepsilon_r\cdot k_s)(p_j\cdot\varepsilon_s\cdot k_r)\;+\;(p_j\cdot\varepsilon_s\cdot p_j)(k_s\cdot\varepsilon_r\cdot k_s)\;+\;(p_j\cdot\varepsilon_r\cdot p_j)\,(k_r\cdot\varepsilon_s\cdot k_r)\Big].
\end{aligned}
\tag{1.4.46}
$$

An explicit derivation of subleading multiple soft theorem for arbitrary number of external soft photons and gravitons are given in [17].

1.5 Infrared issues and validity of the assumptions

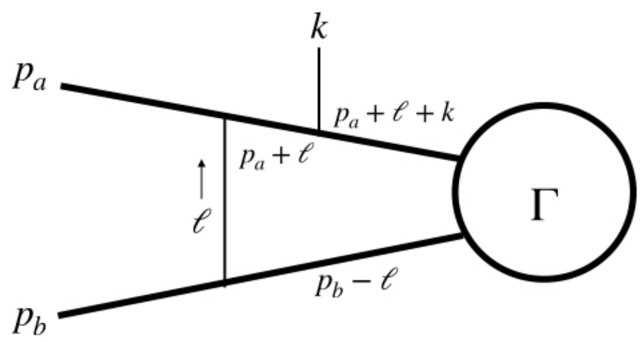

Figure 1.5: Possible one loop diagram contributing to amputated Green's function $\widetilde{\Gamma}$ can produce soft factor in the denominator. In the diagram thick lines represent finite energy particles and thin line represents photon/graviton.

Deriving soft theorem from Sen's covariantization procedure assumes that the soft momentum in the denominator comes only from propagators and not from 1PI vertices. Now here we will show that in $D = 4$ this assumption breaks down. Consider the one loop diagram in Fig.1.5 which is one of the diagrams contributing to amputated Green's function $\widetilde{\Gamma}$ in Fig.1.2. We have assumed that the contribution of $\widetilde{\Gamma}$ starts at the subleading order. But when the loop momentum in Fig.1.5 becomes of the same order as the external soft momentum, each of the scalar propagators with momenta $p_a + \ell$, $p_a + \ell + k$ and $p_b - \ell$ contributes one soft momentum in the denominator. There are also two powers of soft momentum in the denominator due to virtual soft particle propagator. Now in D non-compact spacetime dimensions the loop integration involved $\int d^D\ell$, so in the specified region of integration net power of soft momentum is $D - 5$. Hence in $D = 4$ Fig.1.5 contributes one power of soft momentum in the denominator, violating the assumption described in the beginning of the section. So our soft photon/graviton theorem upto subleading order is valid for loop amplitudes in $D \geq 5$ and only at tree level in D=4.

A similar kind of analysis has been performed in [11] and [15] which tell us that sub-subleading soft graviton theorem is valid for loop amplitudes in $D \geq 5$ and subleading multiple soft graviton theorem is valid for loop amplitudes in $D \geq 6$.

2 Classical limit of soft theorem

Although soft theorem is a relation between quantum scattering amplitudes – amplitudes with soft photon or graviton to amplitudes without soft photon or graviton – one can also relate soft theorem to classical scattering amplitudes. Ref. [19] produced a more direct relation between soft theorem and classical scattering in generic space-time dimensions by directly taking the classical limit of a quantum scattering amplitude. This relates various terms in soft theorem to appropriate terms in the radiative part of the electromagnetic and gravitational fields in classical scattering in generic space-time dimensions. Reversing the logic, one can use the classical scattering data to give an alternative definition of the soft factors.

We start with multiple soft graviton theorem and then demand that a large number of low energy gravitons should come out from the scattering process so that we can declare the coherent state of the large number of gravitons as gravity wave. This is known as the classical limit and in this limit, it turns out that masses of the scattered objects are large in the Planck unit. Here we also assume that the energy loss due to gravitational radiation is small compare to the energy of the objects participated in the scattering process. This leads to probe scatterer limit or large impact parameter limit for a two-body scattering and the wavelength of gravitational radiation has to be large compared to the characteristic length scale of scattering. After this analysis, we show that in general spacetime dimensions, the long wavelength gravitational waveform is proportional to the single soft graviton factor when we replace the asymptotic trajectory of the scattered object in the classical angular momentum part.

Since classical scattering is well defined even in four space-time dimensions, one can hope to use the classical definition of soft factors to understand soft theorem in four dimensions. Since in higher dimensions, the soft theorem expresses the low frequency radiative part of the electromagnetic and gravitational fields in terms of momenta and angular momenta of incoming and outgoing finite energy particles, the naive guess will be that the same formula will continue to hold in four dimensions. However in carrying out this procedure we encounter an obstacle [20, 33]. As we have seen in the last chapter, the subleading terms in the soft theorem contain a factor of angular momentum $j^{\mu\nu}$ of the individual particles involved in the scattering (e.g. look back to eq.(1.3.37)), with the classical orbital angular momentum given by $x^\mu p^\nu - x^\nu p^\mu$, where $x^\mu(\tau)$ and $p^\mu(\tau)$ label the asymptotic coordinates and momenta of the particle as a function of the proper time. In dimensions larger than four, p^μ approaches a constant and x^μ approaches the form $c^\mu + \alpha\, p^\mu\, \tau$ with constant c^μ and α as $\tau \to \infty$. Therefore $j^{\mu\nu}$ is independent of τ as $\tau \to \infty$ and we can use the asymptotic value of $j^{\mu\nu}$ computed this way to evaluate the soft factor. However in four space-time dimensions the long range gravitational and / or electromagnetic forces acting on the particles produce an additional term of the form $b^\mu \ln \tau$ in the expression for x^μ. This gives a logarithmically divergent term of the form $(b^\mu p^\nu - b^\nu p^\mu) \ln \tau$ in the expression for $j^{\mu\nu}$, making the subleading soft factor divergent.

Since we do not expect the radiative component of the metric or gauge fields to diverge in classical scattering in four space-time dimensions, this suggests that the divergence in the subleading soft factor is a breakdown of the power series expansion in the energy ω of the soft particle. Therefore the soft factor must contain non-analytic terms in ω. The natural guess is that the soft factor at the subleading order is given by replacing the factors of $\ln \tau$ in the naive expression by $\ln \omega^{-1}$. This has been tested in [20] by considering several examples of classical scattering in four space-time dimensions.[1] In this chapter

[1]The existence of various logarithmic terms in classical scattering has been known earlier [110–113]. Soft theorem provides a systematic procedure for computing the coefficient of the logarithmic term in the

we consider a general scattering process where all particles involved in the scattering have masses of the same order, and then determine the logarithmic terms in the classical soft factor using the $\ln \tau \to \ln \omega^{-1}$ replacement rule.

The organisation of this chapter is as follows. In §2.1 we describe how to take classical limit of leading multiple soft photon theorem and determine electromagnetic radiation from classical scattering of charged objects. Then we will explain how this analysis can be extended for deriving gravitational radiation from multiple soft graviton theorem. §2.2 describes the analysis of the logarithmic terms in D=4 in the subleading soft expansion for general classical scattering.

2.1 Classical electromagnetic and gravitational radiation from soft theorem

Consider a scattering event where N number of finite energy charged particles are involved with charges and momenta $\{q_a, p_a\}$ for $a = 1, \cdots N$, with the convention that momenta and charges are positive for ingoing particles and negative for outgoing particles. Suppose in this scattering event total M number of soft photons are coming out with polarisation and momenta $\{\varepsilon_r, k_r\}$ for $r = 1, \cdots, M$. Then up to leading order in soft momenta expansion the $(N + M)$ point amplitude is related to N point amplitude as [2, 17],

$$\Gamma^{(N+M)}(\{\varepsilon_r, k_r\}\ ,\ \{\epsilon_a, p_a, q_a\}) \;=\; \left\{\prod_{r=1}^{M} S^{(0)}_{em}(\varepsilon_r, k_r)\right\} \Gamma^{(N)}(\{\epsilon_a, p_a, q_a\}) \tag{2.1.1}$$

where

$$S^{(0)}_{em}(\varepsilon_r, k_r) \;=\; \sum_{a=1}^{N} q_a \frac{\varepsilon_{r,\mu} p_a^{\mu}}{p_a \cdot k_r} \tag{2.1.2}$$

For deriving long wavelength classical EM radiation from soft photon theorem we are closely following reference [19]. Now the differential cross-section for M number of soft

subleading soft factor without detailed knowledge of the forces responsible for the scattering.

photon emission all having polarisation ε and energy between ω to $\omega(1+\delta)$ within solid angle $\Delta\Omega$ is given by,

$$\Delta\sigma = \frac{A^M}{M!} \times \left|\Gamma^{(N)}\right|^2 \tag{2.1.3}$$

where

$$A = \frac{\omega^{D-2}\,\omega\delta\,\Delta\Omega}{(2\pi)^{D-1}} \frac{1}{2\omega} \left|S^{(0)}_{em}(\varepsilon, k)\right|^2 . \tag{2.1.4}$$

Now for a classical electromagnetic scattering, the scattered objects must have charges large in the unit of $\sqrt{\alpha}$, where α is the fine structure constant. Also, only if a large number of photons come out from the scattering event, we can declare the coherent state of photons as an electromagnetic wave. Let us suppose that $q_a \sim q$ is large, in that case $A \sim q^2$ is also large for fixed ω , δ and $\Delta\Omega$. So for a classical scattering with a large value of A we can maximize $\Delta\sigma$ w.r.t M to find out the number of soft photon emission at the peak of the distribution.

$$\begin{aligned} \frac{d(\ln\Delta\sigma)}{dM} &= 0 \\ \Rightarrow M &\simeq A \end{aligned} \tag{2.1.5}$$

Total energy carried by the soft photons in this bin,

$$M\omega = A\omega = \frac{\omega^{D-1}\,\delta\,\Delta\Omega}{2^D(\pi)^{D-1}} \left|S^{(0)}_{em}(\varepsilon, k)\right|^2 \tag{2.1.6}$$

Now independently for this classical scattering process, we can compute the electromagnetic energy from the long wave-length electromagnetic gauge field. Here we first show how the Fourier transform in time variable of the gauge field is directly proportional to soft photon factor. Then we compute the energy of EM radiation and show it's equivalence with the expression (2.1.6).

Let us consider Maxwell's equation in Lorentz gauge:

$$\Box A_\mu(x) = -j_\mu(x), \qquad \Box \equiv \eta^{\alpha\beta}\,\partial_\alpha\,\partial_\beta\,, \tag{2.1.7}$$

where $j_\mu(x)$ is the EM current density determined in terms of the trajectory of scattered objects. Then the retarded gauge field solution is given by

$$A_\mu(x) = -\int d^D y\, G_r(x,y)\, j_\mu(y)\,, \tag{2.1.8}$$

where $G_r(x,y)$ is the retarded Green's function:

$$G_r(x,y) = \int \frac{d^D\ell}{(2\pi)^D}\, e^{i\ell.(x-y)}\, \frac{1}{(\ell^0+i\epsilon)^2 - \vec{\ell}^2}\,. \tag{2.1.9}$$

Now performing Fourier transformation in time variable we get,

$$\begin{aligned}\widetilde{A}_\mu(\omega,\vec{x}) &\equiv \int dt\, e^{i\omega t}\, A_\mu(t,\vec{x}) \\ &= -\int d^D y\, j_\mu(y) \int \frac{d^{D-1}\ell}{(2\pi)^{D-1}}\, e^{i\omega y^0 + i\vec{\ell}.(\vec{x}-\vec{y})}\, \frac{1}{(\omega+i\epsilon)^2 - \vec{\ell}^2}\,.\end{aligned} \tag{2.1.10}$$

For large $|\vec{x}|$, we can evaluate this integral using a saddle point approximation as follows [19]. Defining $\vec{\ell}_\parallel$ and $\vec{\ell}_\perp$ as components of $\vec{\ell}$ along $\vec{x}-\vec{y}$ and transverse to $\vec{x}-\vec{y}$ respectively, we get

$$\widetilde{A}_\mu(\omega,\vec{x}) = -\int d^D y\, j_\mu(y) \int \frac{d^{D-2}\ell_\perp}{(2\pi)^{D-2}}\, \frac{d\ell_\parallel}{2\pi}\, e^{i\omega y^0 + i\ell_\parallel |\vec{x}-\vec{y}|}\, \frac{1}{(\omega+i\epsilon)^2 - \ell_\parallel^2 - \vec{\ell}_\perp^2}\,. \tag{2.1.11}$$

First consider the case $\omega > 0$. We now close the $\ell_\parallel$ integration contour in the upper half plane, picking up residue at the pole at $\sqrt{(\omega+i\epsilon)^2 - \vec{\ell}_\perp^2}$. This gives

$$\widetilde{A}_\mu(\omega,\vec{x}) = i\int d^D y\, j_\mu(y) \int \frac{d^{D-2}\ell_\perp}{(2\pi)^{D-2}}\, e^{i\omega y^0 + i|\vec{x}-\vec{y}|\,\sqrt{(\omega+i\epsilon)^2 - \vec{\ell}_\perp^2}}\, \frac{1}{2\sqrt{(\omega+i\epsilon)^2 - \vec{\ell}_\perp^2}}\,. \tag{2.1.12}$$

For large $|\vec{x}-\vec{y}|$ the exponent is a rapidly varying function of $\vec{\ell}_\perp$ and therefore we can carry out the integration over $\vec{\ell}_\perp$ using saddle point approximation. The saddle point is located at $\vec{\ell}_\perp = 0$. Expanding the exponent to order $\vec{\ell}_\perp^2$ and carrying out gaussian integration over $\vec{\ell}_\perp$ we get:

$$\widetilde{A}_\mu(\omega,\vec{x}) \;\simeq\; i\,\mathcal{N}\,e^{i\omega R}\int d^Dy\,e^{ik.y}\,j_\mu(y) \tag{2.1.13}$$

where we have made the approximation $|\vec{x}| >> |\vec{y}|$ and defined $k \equiv -\omega(1,\hat{n}),\quad \hat{n} \equiv \vec{x}/|\vec{x}|,\quad R \equiv |\vec{x}|$ for outgoing EM radiation[2]. Here $\mathcal{N}$ is given by,

$$\mathcal{N} \;=\; \left(\frac{\omega}{2\pi iR}\right)^{\frac{(D-2)}{2}}\frac{1}{2\omega} \tag{2.1.14}$$

At leading order the asymptotic trajectory of the ingoing/outgoing objects as proper time $\sigma \to \pm\infty$,

$$r_a^\mu(\sigma) \;=\; V_a^\mu\sigma \;+\cdots \tag{2.1.15}$$

where V_a^μ is the four velocity of a'th particle. The asymptotic value of EM current density is,

$$j^\mu(y) \;=\; \sum_{a\in in}\int_{-\infty}^{0}d\sigma\,q_aV_a^\mu\,\delta^{(D)}(y-V_a\sigma) \;+\; \sum_{a\in out}\int_{0}^{\infty}d\sigma\,q_aV_a^\mu\,\delta^{(D)}(y-V_a\sigma) \tag{2.1.16}$$

where q_a is the EM charge of a'th particle. Then substituting in eq.(2.1.13) and performing the σ integration we get,

$$\begin{aligned}\widetilde{A}_\mu(\omega,\vec{x}) \;&=\; i\mathcal{N}e^{i\omega R}\left[\sum_{a\in in}q_a\frac{V_{a\mu}}{ik\cdot V_a} \;-\; \sum_{a\in out}q_a\frac{V_{a\mu}}{ik\cdot V_a}\right]\\ &=\; \mathcal{N}e^{i\omega R}\sum_{a=1}^{N}q_a\,\frac{p_{a\mu}}{p_a\cdot k}\end{aligned} \tag{2.1.17}$$

[2]Since we are following the convention that momenta of outgoing particles carries an extra negative sign so momentum for outgoing EM radiation $k \equiv -\omega(1,\hat{n})$.

where $p_a^\mu = \eta_a m_a V_a^\mu$ with $\eta_a = \pm 1$ for incoming/outgoing particles and we are following the convention that charges carrie extra negative sign for outgoing particles. Hence comparing with the expression of the leading soft factor for one soft photon emission in eq.(2.1.2) we get,

$$\varepsilon^\mu \widetilde{A}_\mu(\omega, \vec{x}) \quad = \quad \mathcal{N} e^{i\omega R}\, S_{em}^{(0)}(\varepsilon, k) \tag{2.1.18}$$

In [19] the above relation has been extended and tested for a classical scattering process up to subleading order when energy loss due to EM radiation is small compared to the energy of each individual particles. In that limit the non-universal term appeared in the subleading soft photon theorem (the second line in the r.h.s. of eq.(1.3.36)) turns out to be small relative to the universal piece with the angular momenta operator replaced by classical angular momenta of finite energy particles.

Now we will use the expression (2.1.18) to compute energy of EM radiation along $\hat{n}$ direction around a solid angle $\Delta\Omega$ within energy range ω to $\omega(1+\delta)$. Since at large distance from the source the radiation looks like plane wave and we are only interested to find out the radiation with particular polarisation ε, we can treat $\varepsilon^\mu A_\mu(t, \vec{x})$ as a scalar field for $\vec{x} = R\hat{n}$. If we denote the radial coordinate along $\hat{n}$ at a large distance from scattering center as $z = \hat{n}\cdot\vec{x}$, then the relevant part of $\widetilde{\phi}(\omega, \vec{x})$ follows from eq.(2.1.18),

$$\widetilde{\phi}(\omega, \vec{x}) \;\equiv\; \mathcal{N} e^{i\omega z}\, S_{em}^{(0)}(\varepsilon, k) \tag{2.1.19}$$

where

$$\phi(t, \vec{x}) \;=\; \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \widetilde{\phi}(\omega, \vec{x}). \tag{2.1.20}$$

Energy-momentum tensor for the scalar field is:

$$T^{\mu\nu}(x) \;=\; \partial^\mu\phi\partial^\nu\phi \;-\frac{1}{2}\eta^{\mu\nu}\partial^\rho\phi\partial_\rho\phi \tag{2.1.21}$$

The energy radiated per unit area along $\hat{n}$ direction computed for the scalar field form (2.1.19) turns out to be,

$$\int dz\, T^{zt} \;=\; \int_0^\infty \frac{d\omega}{\pi}\, \omega^2\, |\mathcal{N}|^2 \left| S^{(0)}_{em}(\varepsilon, k) \right|^2 \tag{2.1.22}$$

Now the area corresponding to solid angle $\Delta\Omega$ is $R^{D-2}\Delta\Omega$. So the energy of EM radiation along $\hat{n}$ direction around a solid angle $\Delta\Omega$ within energy range ω to $\omega(1+\delta)$ and polarisation ε is,

$$\frac{\omega\delta}{\pi}\, \omega^2\, |\mathcal{N}|^2 \left| S^{(0)}_{em}(\varepsilon, k) \right|^2 \times R^{D-2}\Delta\Omega \tag{2.1.23}$$

Now substituting the expression of $\mathcal{N}$ from eq.(2.1.14) in the above expression, we get a perfect agreement with the expression in (2.1.6).

Analogously we start with subleading multiple soft graviton theorem expression in eq.(1.4.43) and take the classical limit to derive long wavelength gravitational waveform [19]. Here we will be brief and only discuss the key points in the analysis.

- Here classical limit corresponds to mass/energy of the objects participating in the scattering event being large in Planck mass unit. This implies a large number of gravitons come out from the scattering process and we can declare the coherent state of the large number of gravitons as gravitational wave.

- We also need to ensure that the total energy of gravitational radiation must be small compared to the energy of individual particles involved in the scattering process. This can be achieved for a $2 \to 2$ scattering process if we consider the impact parameter is large or consider the probe scatterer limit (one of the objects is heavy compare to other in the initial state).

- In the classical limit the angular momentum operator $j_a^{\mu\nu}$ in eq.(1.3.38) have to replace by $i\, \mathbf{J}_a^{\mu\nu}$, where $\mathbf{J}_a^{\mu\nu}$ is the classical angular momentum determined in terms of the trajectory and intrinsic spin of a'th particle.

$$j_a^{\mu\nu} \to i\,\mathbf{J}_a^{\mu\nu} = i(x_a^{\mu} p_a^{\nu} - x_a^{\nu} p_a^{\mu}) + \Sigma_a^{\mu\nu} \tag{2.1.24}$$

- In the classical and soft limit described above, the contact term containing $\mathcal{M}$ in eq.(1.4.43) turns out to be suppressed relative to the other subleading part containing classical angular momenta. So we will drop the contact term for our classical analysis. In this classical soft limit, the multiple soft factor for M number of soft graviton emission takes form,

$$\prod_{r=1}^{M} S_{gr}(\varepsilon_r, k_r) \tag{2.1.25}$$

where

$$S_{gr}(\varepsilon_r, k_r) = \sum_a \frac{\varepsilon_{r,\mu\nu} p_a^{\mu} p_a^{\nu}}{p_a \cdot k_r} + i \sum_a \frac{\varepsilon_{r,\mu\nu} p_a^{\nu} k_{r\rho} \mathbf{J}_a^{\rho\mu}}{p_a \cdot k_r}. \tag{2.1.26}$$

Now performing analogous study as like the soft photon emission, here we get the following relation for the trace reversed metric fluctuation [19],

$$\int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \varepsilon^{\mu\nu} e_{\mu\nu}(t, \vec{x}) = \mathcal{N}\, e^{i\omega R}\, S_{gr}(\varepsilon, k) \tag{2.1.27}$$

where the deviation of metric from Minkowski metric is defined by $h_{\mu\nu}(x) \equiv \frac{1}{2}(g_{\mu\nu}(x) - \eta_{\mu\nu})$ and the trace reversed metric is defined by $e_{\mu\nu}(x) \equiv h_{\mu\nu}(x) - \frac{1}{2} h_{\rho}^{\rho}(x) \eta_{\mu\nu}$.

2.2 Classical soft factor in D=4

The goal of this section will be to calculate the logarithmic terms in the soft factors in four space-time dimensions by examining them in the classical limit.

In dimensions larger than 4, the soft factors for photons and gravitons are given respectively by

$$S_{\rm em} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a + i \sum_a q_a \frac{\varepsilon_\mu k_\rho \mathbf{J}_a^{\rho\mu}}{p_a.k}\,, \tag{2.2.1}$$

and

$$S_{\rm gr} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + i \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho \mathbf{J}_a^{\rho\mu}}{p_a.k}\,. \tag{2.2.2}$$

Here the sum over a runs over all the incoming and outgoing particles, and q_a, p_a and $\mathbf{J}_a$ denote the charge, momentum and angular momentum of the a-th particle, counted with positive sign for an ingoing particle and negative sign for an outgoing particle. $S_{\rm em}$ may also contain a non-universal term at the subleading order. For S-matrix elements in quantum theory, $\mathbf{J}_a$ is a differential operator involving derivatives with respect to the external momenta. However in the classical limit in which the external finite energy states are macroscopic, $\mathbf{J}_a$ represents the classical angular momenta carried by the external particles. In this limit the soft factors describe the radiative part of the low frequency electromagnetic and gravitational fields during a classical scattering [19] as described in (2.1.18) and (2.1.27).

In applying (2.2.1), (2.2.2) to four dimensional theories, the complication arises from the contribution to $\mathbf{J}_a^{\mu\nu}$ from the orbital angular momentum. They are computed from the form of the asymptotic trajectories:

$$r_a^\mu(\sigma) = \eta_a \frac{1}{m_a} p_a^\mu \sigma + c_a^\mu \ln|\sigma| + \cdots\,, \tag{2.2.3}$$

where η_a is positive for incoming particles and negative for outgoing particles, m_a is the mass of the a-th particle and the proper time σ is large and negative for incoming particles and large and positive for outgoing particles. The term proportional to $\ln|\sigma|$ represents the effect of long range electromagnetic and/or gravitational interaction between the particles. This gives, for large $|\sigma|$,

$$\mathbf{J}_a^{\mu\nu} \simeq r_a^\mu(\sigma) p_a^\nu - r_a^\nu(\sigma) p_a^\mu + \text{spin} = (c_a^\mu p_a^\nu - c_a^\nu p_a^\mu) \ln|\sigma| + \cdots\,. \tag{2.2.4}$$

Here and in the following we shall use the convention that when a variable is followed by an argument (σ) it denotes the value of the variable at proper time σ, but when a variable is written without an argument, we take it to be its σ independent asymptotic value. Therefore in (2.2.3), (2.2.4) the p_a^μ's denote the asymptotic values of p_a^μ, reflecting the fact that the difference between $p_a^\mu(\sigma) = m_a \eta_a dr_a^\mu / d\sigma$ and p_a^μ approaches zero asymptotically.

Analysis of [20] indicates that if we substitute (2.2.4) into (2.2.1) and (2.2.2) and replace $\ln|\sigma|$ by $\ln\omega^{-1}$ – where $\omega = k_0$ is the frequency of the outgoing soft radiation – we can recover the logarithmic terms in the soft factors up to overall phases. This gives, up to overall phases:

$$S_{\text{em}} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a + i \ln\omega^{-1} \sum_a q_a \frac{\varepsilon_\mu k_\rho (c_a^\rho p_a^\mu - c_a^\mu p_a^\rho)}{p_a.k} \,, \tag{2.2.5}$$

and

$$S_{\text{gr}} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + i \ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho (c_a^\rho p_a^\mu - c_a^\mu p_a^\rho)}{p_a.k} \,. \tag{2.2.6}$$

Note that although S_{em} may contain a non-universal term at the subleading order, the term proportional to $\ln\omega^{-1}$ comes from orbital angular momentum and is universal.

Irrespective of what forces are operative during the scattering, the coefficient c_a^μ are determined only by the long range forces acting on the incoming and the outgoing particles. These will be taken to be electromagnetic and / or gravitational interaction. We shall now compute c_a^μ due to electromagnetic and gravitational interactions. We know from explicit comparison with known results that in the case of scattering via electromagnetic interactions there are no additional phases in the soft factor, but in the case of gravitational long range interaction there is an additional phase reflecting the effect of backscattering of the soft photon or soft graviton in the background gravitational field [110, 112, 113]. This phase will also be determined below.

2.2.1 Effect of electromagnetic interactions

We shall first study the effect of logarithmic correction to the trajectory due to long range electromagnetic interaction. For this we need to compute the gauge potential $A^{(b)}_\mu(x)$ at space-time point x due to particle b. We have

$$A^{(b)}_\mu(x) = \frac{1}{2\pi}\int d\sigma\, \eta_b\, q_b\, V_{b\mu}(\sigma)\, \delta_+(-(x-r_b(\sigma))^2), \qquad V^\mu_b(\sigma) \equiv \frac{dr_{b\mu}(\sigma)}{d\sigma} \simeq \eta_b\, \frac{p^\mu_b}{m_b}\,, \tag{2.2.7}$$

where δ_+ denotes the usual Dirac delta function with the understanding that we have to choose the zero of the argument for which $x^0 > r^0_b(\sigma)$. V_b denotes the asymptotic four velocity of the b-th particle. In evaluating (2.2.7) we shall ignore the logarithmic corrections to the trajectory and take $r_b(\sigma) \simeq V_b\,\sigma$. This gives, using $V^2_b = -1$,

$$\delta_+(-(x-r_b(\sigma))^2) = \delta_+(-x^2+2\,V_b.x\,\sigma+\sigma^2+\cdots) \simeq \frac{1}{2|V_b.x+\sigma|}\,\delta(\sigma+V_b.x+\sqrt{(V_b.x)^2+x^2})\,, \tag{2.2.8}$$

where the sign in front of the square root has been chosen to ensure that $x^0 > x^0_b(\sigma)$ at the solution. Substituting this into (2.2.7) we get

$$A^{(b)}_\mu(x) \simeq \frac{1}{4\pi}\,\frac{\eta_b\, q_b V_{b\mu}}{\sqrt{(V_b.x)^2+x^2}}\,. \tag{2.2.9}$$

From this we calculate

$$F^{(b)}_{\mu\nu}(x) = \partial_\mu A^{(b)}_\nu(x) - \partial_\nu A^{(b)}_\mu(x) \simeq -\frac{\eta_b\, q_b}{4\pi}\,\frac{x_\mu V_{b\nu} - x_\nu V_{b\mu}}{\{(V_b.x)^2+x^2\}^{3/2}}\,. \tag{2.2.10}$$

At the location $r_a = V_a\sigma = -V_a|\sigma|\eta_a$ of the a-th particle we get, using $V^2_a = -1$

$$F^{(b)}_{\mu\nu}(r_a(\sigma)) \simeq \eta_a\, \eta_b\, \frac{q_b}{4\pi\,\sigma^2}\,\frac{V_{a\mu}V_{b\nu} - V_{a\nu}V_{b\mu}}{\{(V_b.V_a)^2-1\}^{3/2}}\,. \tag{2.2.11}$$

Now the a-th particle will feel the field produced by the b-th particle if either both a-th and the b-th particle are outgoing or if both particles are ingoing. Therefore the equation of motion for the a-th particle takes the form

$$\frac{dp_{a\mu}(\sigma)}{d\sigma} = q_a \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} F^{(b)}_{\mu\nu}(r_a(\sigma))\, V^\nu_a(\sigma) \simeq \frac{1}{\sigma^2} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \eta_a\,\eta_b\, \frac{q_a q_b}{4\pi} \frac{V_a.V_b V_{a\mu} + V_{b\mu}}{\{(V_b.V_a)^2-1\}^{3/2}}\,. \tag{2.2.12}$$

On the other hand we have

$$\frac{dp_{a\mu}(\sigma)}{d\sigma} = \frac{m_a}{\eta_a}\frac{d^2 r_{a\mu}}{d\sigma^2} = -\frac{m_a}{\eta_a}\frac{c_{a\mu}}{\sigma^2}\,, \tag{2.2.13}$$

where in the last step we used (2.2.3). Comparing (2.2.12), (2.2.13) we get

$$c^\mu_a = -\frac{1}{m_a} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \eta_b\, \frac{q_a q_b}{4\pi} \frac{V_a.V_b V^\mu_a + V^\mu_b}{\{(V_b.V_a)^2-1\}^{3/2}} = -\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi} \frac{m_b^2\, p_a.p_b\, p^\mu_a + m_a^2 m_b^2\, p^\mu_b}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,, \tag{2.2.14}$$

and

$$c^\mu_a p^\nu_a - c^\nu_a p^\mu_a = -\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi} \frac{m_a^2 m_b^2\, \{p^\mu_b p^\nu_a - p^\nu_b p^\mu_a\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,. \tag{2.2.15}$$

Eqs.(2.2.5) and (2.2.6) now give[3]

$$S_{\rm em} = \sum_a \frac{\varepsilon_\mu p^\mu_a}{p_a.k}\, q_a - i\, \ln\omega^{-1} \sum_a \frac{q_a\, \varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi} \frac{m_a^2 m_b^2\, \{p^\rho_b p^\mu_a - p^\mu_b p^\rho_a\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,, \tag{2.2.16}$$

and

$$S_{\rm gr} = \sum_a \frac{\varepsilon_{\mu\nu} p^\mu_a p^\nu_a}{p_a.k} - i\, \ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p^\nu_a k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi} \frac{m_a^2 m_b^2\, \{p^\rho_b p^\mu_a - p^\mu_b p^\rho_a\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,. \tag{2.2.17}$$

[3] Note that even if we assume that the logarithmic corrections to the trajectories are generated predominantly by electromagnetic interaction, the resulting acceleration can generate logarithmic corrections to the gravitational radiation during the scattering.

2.2.2 Effect of gravitational interactions

Let us now suppose that the logarithmic correction to the trajectories arise due to gravitational interaction. We introduce the graviton field $h_{\mu\nu}$ and its trace reversed version $e_{\mu\nu}$ via the equations

$$h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu})/2, \qquad e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h_\rho^{\ \rho}\,. \tag{2.2.18}$$

Then the analog of (2.2.7) for the gravitational field produced at x due to the b-th particle is

$$e^{(b)}_{\mu\nu}(x) = \frac{1}{2\pi}\int d\sigma\, m_b\, V_{b\mu}(\sigma)\, V_{b\nu}(\sigma)\, \delta_+(-(x - r_b(\sigma))^2)\,. \tag{2.2.19}$$

Using $r_b(\sigma) = V_b\, \sigma + \cdots$ we get the analog of (2.2.9)

$$e^{(b)}_{\mu\nu}(x) \simeq \frac{1}{4\pi}\frac{m_b\, V_{b\mu}\, V_{b\nu}}{\sqrt{(V_b.x)^2 + x^2}}\,. \tag{2.2.20}$$

The associated Christoffel symbol is given by, in the weak field approximation,

$$\begin{aligned}\Gamma^{(b)\alpha}_{\rho\tau}(x) \;=\; & -\frac{m_b}{4\pi}\frac{1}{\{(V_b.x)^2 + x^2\}^{3/2}}\,\eta^{\alpha\mu}\Bigg[\left\{V_{b\mu}V_{b\tau} + \frac{1}{2}\eta_{\mu\tau}\right\}\left\{x_\rho + V_b.x\, V_{b\rho}\right\} \\ & + \left\{V_{b\mu}V_{b\rho} + \frac{1}{2}\eta_{\mu\rho}\right\}\{x_\tau + V_b.x\, V_{b\tau}\} - \left\{V_{b\rho}V_{b\tau} + \frac{1}{2}\eta_{\rho\tau}\right\}\left\{x_\mu + V_b.x\, V_{b\mu}\right\}\Bigg]\end{aligned} \tag{2.2.21}$$

From this we can write down the equation of motion of the a-th particle

$$\begin{aligned}\frac{d^2 r_a^\alpha(\sigma)}{d\sigma^2} \;&=\; -\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \Gamma^{(b)\alpha}_{\rho\tau}(r_a(\sigma))\, V_a^\rho(\sigma)\, V_a^\tau(\sigma) \\ &\simeq\; -\eta_a\,\frac{1}{4\pi\sigma^2}\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} m_b\,\frac{1}{\{(V_b.V_a)^2 - 1\}^{3/2}}\left[-\frac{1}{2}V_a^\alpha + \frac{1}{2}V_b^\alpha\left\{2(V_b.V_a)^3 - 3V_b.V_a\right\}\right].\end{aligned} \tag{2.2.22}$$

On the other hand using (2.2.3) the left hand side is given by $-c_a^\alpha/\sigma^2$. This gives

$$c_a^\alpha = \eta_a \frac{1}{4\pi} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} m_b \frac{1}{\{(V_b.V_a)^2-1\}^{3/2}} \left\{-\frac{1}{2}V_a^\alpha + \frac{1}{2}V_b^\alpha\left(2(V_b.V_a)^3 - 3V_b.V_a\right)\right\}, \quad (2.2.23)$$

and

$$\begin{aligned} c_a^\rho p_a^\mu - c_a^\mu p_a^\rho &= \frac{1}{8\pi\sigma^2} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} m_a\, m_b \frac{1}{\{(V_b.V_a)^2-1\}^{3/2}} (V_b^\rho V_a^\mu - V_b^\mu V_a^\rho)\left\{2(V_b.V_a)^3 - 3V_b.V_a\right\} \\ &= \frac{1}{8\pi} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\left\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\right\}. \end{aligned} \quad (2.2.24)$$

Substituting this into (2.2.5) and (2.2.6) we get,[4] up to overall phases:

$$\begin{aligned} S_{\rm em} &= \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a + \frac{i}{8\pi}\ln\omega^{-1} \sum_a \frac{q_a\,\varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho) \\ &\qquad \times\left\{2(p_b.p_a)^2 - 3m_a^2m_b^2\right\}, \end{aligned} \quad (2.2.25)$$

and

$$\begin{aligned} S_{\rm gr} &= \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + \frac{i}{8\pi}\ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho) \\ &\qquad \times\left\{2(p_b.p_a)^2 - 3m_a^2m_b^2\right\}. \end{aligned} \quad (2.2.26)$$

In this case we expect the wave-form of the gauge field / metric to also have an additional phase factor reflecting the effect of the gravitational drag on the soft particle due to the other particles. For this let us characterize the asymptotic trajectory of the soft particle as

$$x^\mu(\tau) = n^\mu\,\tau + m^\mu\,\ln|\tau|\,, \quad (2.2.27)$$

[4]Even if the logarithmic correction to the trajectory is generated by gravitational interaction, the particles can emit electromagnetic waves. This happens for example if we have a scattering of a charged particle and a neutral particle.

where τ is the affine parameter associated with the trajectory, $n = (1, \hat{n})$ is a null vector along the asymptotic direction of motion of the soft particle and m^μ is a four vector to be determined. Now substituting (2.2.27) into the equation of motion

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} , \tag{2.2.28}$$

and using the form (2.2.21) of $\Gamma^\mu_{\nu\rho}$, we get the following expression for m^μ by comparing the $1/\tau^2$ terms on the two sides of the equations of motion:

$$m^\alpha = -\frac{1}{4\pi} \sum_{\substack{b \\ \eta_b=-1}} \frac{m_b}{|n.V_b|^3} V_b^\alpha (V_b.n)^3 = \frac{1}{4\pi} \sum_{\substack{b \\ \eta_b=-1}} m_b V_b^\alpha = -\frac{1}{4\pi} \sum_{\substack{b \\ \eta_b=-1}} p_b^\alpha . \tag{2.2.29}$$

Now eliminating τ in terms of $t \equiv x^0$ using (2.2.27), we can express (2.2.27) as

$$x^i = n^i t + (m^i - n^i m^0) \ln |t| + \text{finite} . \tag{2.2.30}$$

Therefore if we denote by $k = (k^0, k) = -\omega(1, \hat{n})$ the four momentum of the soft particle, the overall $-$ sign reflecting the fact that it is an outgoing particle, the wave-function of the particle will be proportional to

$$\exp\left[-i\vec{k}.\left\{\vec{x} - \hat{n}t - (\vec{m} - \hat{n}\, m^0) \ln |t|\right\}\right] = \exp[-i\omega t + i\omega\hat{n}.\vec{x}] \, \exp[i(\vec{k}.\vec{m} + \omega\, m^0) \ln |t|] . \tag{2.2.31}$$

The second factor can be regarded as an additional infrared divergent contribution to the soft factor. Using $|t| \sim R$ where R is the distance of the soft particle from the scattering center, and eq.(2.2.29), we can express the second factor in (2.2.31) as

$$\exp[ik.m \, \ln R] = \exp\left[-\frac{i}{4\pi} \ln R \sum_{\substack{b \\ \eta_b=-1}} k.p_b\right] . \tag{2.2.32}$$

Since this is a pure phase it does not affect the flux. However it does produce observable effect on the electromagnetic / gravitational wave-form [33].

It follows from the analysis of [110,112,113] that the effect of gravitational backscattering of the soft photon / graviton actually converts $\ln R$ in (2.2.32) to $\ln(R\,\omega)$. This has been reviewed in [20]. It is natural to absorb this multiplicative factor in the wave-form into the definition of the soft factors. Expanding the exponential in a power series, picking up the term of order $\omega\ln(\omega R)$ in the expansion, and multiplying this by the leading soft factor, we get additional contributions to the soft photon and soft graviton factor at the subleading order

$$\frac{i}{4\pi}\left(\ln\omega^{-1}+\ln R^{-1}\right) S^{(0)}_{\rm em}\sum_{\substack{b\\ \eta_b=-1}} k.p_b, \qquad \text{and} \qquad \frac{i}{4\pi}\left(\ln\omega^{-1}+\ln R^{-1}\right) S^{(0)}_{\rm gr}\sum_{\substack{b\\ \eta_b=-1}} k.p_b\,. \tag{2.2.33}$$

Adding these to (2.2.25) and (2.2.26) we get the net soft factors to be

$$\begin{aligned} S_{\rm em} \;=\; & \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k}\,q_a + \frac{i}{4\pi}\left(\ln\omega^{-1}+\ln R^{-1}\right)\sum_{\substack{b\\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k}\,q_a \\ & + \frac{i}{8\pi}\ln\omega^{-1}\sum_a \frac{q_a\,\varepsilon_\mu k_\rho}{p_a.k}\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2-m_a^2m_b^2\}^{3/2}}\,(p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\left\{2(p_b.p_a)^2-3m_a^2m_b^2\right\}, \end{aligned} \tag{2.2.34}$$

and

$$\begin{aligned} S_{\rm gr} \;=\; & \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + \frac{i}{4\pi}\left(\ln\omega^{-1}+\ln R^{-1}\right)\sum_{\substack{b\\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \\ & + \frac{i}{8\pi}\ln\omega^{-1}\sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k}\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2-m_a^2m_b^2\}^{3/2}}\,(p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\left\{2(p_b.p_a)^2-3m_a^2m_b^2\right\}. \end{aligned} \tag{2.2.35}$$

2.2.3 Effect of electromagnetic and gravitational interactions

We now combine the results of last two subsections to write down the general expression for the soft factor when both gravitational interaction and electromagnetic interactions are responsible for the logarithmic corrections to the trajectory. The logarithmic terms get added up, yielding the results:

$$
\begin{aligned}
S_{\text{em}} \;=\;& \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k}\, q_a + \frac{i}{4\pi}\left(\ln\omega^{-1} + \ln R^{-1}\right) \sum_{\substack{b\\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k}\, q_a \\
& - i\,\ln\omega^{-1} \sum_a \frac{q_a\,\varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi}\, \frac{m_a^2 m_b^2\,\{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \\
& + \frac{i}{8\pi}\,\ln\omega^{-1} \sum_a \frac{q_a\,\varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,(p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\left\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\right\},
\end{aligned}
\tag{2.2.36}
$$

and

$$
\begin{aligned}
S_{\text{gr}} \;=\;& \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + \frac{i}{4\pi}\left(\ln\omega^{-1} + \ln R^{-1}\right) \sum_{\substack{b\\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \\
& - i\,\ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu}\, p_a^\nu\, k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi}\, \frac{m_a^2 m_b^2\,\{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \\
& + \frac{i}{8\pi}\,\ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,(p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\left\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\right\}.
\end{aligned}
\tag{2.2.37}
$$

Note that the soft factors given in (2.2.36) and (2.2.37) depend only on the charges and momenta carried by the external states. Therefore these can be reinterpreted as multiplicative soft factors in the full quantum theory – since there is no angular momentum there is no derivative with respect to the external momenta. In the next chapter we shall carry out some explicit quantum computations to examine to what extent this holds.

3 Subleading soft theorem in four spacetime dimensions from S-matrix analysis

Scattering amplitude in four spacetime dimensions for a theory with massless particles is infrared divergent. So from the perspective of soft theorem it was not *a priori* clear how to interpret a relation whose both sides are divergent [40,66,69,114]. Here in this chapter we analyse directly the quantum subleading soft factor by considering one loop scattering of charged scalar fields in the presence of gravitational and electromagnetic interaction. The difference with the previous analysis, *e.g.* in [66], is that we do not insist on a power series expansion in ω and calculating the coefficients of the power series expansion. Instead we allow for possible non-analytic terms of order $\ln \omega^{-1}$ in the soft expansion. This analysis yields results consistent with the classical results, although the quantum results contain additional real part which we interpret as the result of back reaction of the radiation on the motion of the particles.

In Chapter-1 we gave a general derivation of soft theorem including loop corrections as long as 1PI vertices do not generate soft factor in the denominator [11, 15, 17, 101]. Now one could ask to what extent we could derive the soft factors in D=4, using the result of [11, 15, 17, 101]. To this end we note that there are two distinct sources of logarithmic terms in the soft theorem. The first is the region of integration in which the loop momentum is large compared to the energy of the external soft particle. In this region we expect the arguments of [11,15,101] to be valid, and we find that the contribution from this region can indeed be obtained by applying the usual soft operator on the amplitude without the soft graviton. The other source is the region of integration in which the loop momentum is small compared to the external soft momentum. The contribution from this region cannot be derived using the usual soft theorem, and need to be computed explicitly.

The chapter is organized as follows. §3.1 describes some general strategy for dealing with the infrared divergent part of the S-matrix and extracting the quantum soft factor by making use of momentum conservation. §3.2 describes one loop quantum computation of the logarithmic terms in the soft photon theorem in scalar quantum electrodynamics (scalar QED). §3.3 describes a similar computation in the soft graviton theorem in a theory of charge neutral scalars interacting with the gravitational field. In §3.4 we consider charged scalars interacting via both gravitational and electromagnetic interaction, and determine the one loop contribution to the quantum soft graviton factor due to electromagnetic interaction and one loop contribution to the quantum soft photon factor due to gravitational interaction. In§3.5 we briefly sketch the derivation of multiple soft photon theorem up to subleading order in D=4. §3.6 contains a summary and a discussion of our results where we also discuss various special cases of our classical result.

3.1 How to treat momentum conservation and infrared divergences

In quantum theory, single soft theorem is expected to relate an amplitude $\Gamma^{(n,1)}$ with n finite energy external states carrying momenta $p_1, \cdots p_n$ and one soft particle of momentum k to an amplitude $\Gamma^{(n)}$ with just n finite energy external states carrying momenta $p_1, \cdots p_n$.

This relation takes the form

$$\Gamma^{(n,1)}(p_1,\cdots p_n,k) \simeq S(\varepsilon,k;\{p_a\})\,\Gamma^{(n)}(p_1,\cdots p_n)\,, \tag{3.1.1}$$

where $S(\varepsilon,k;\{p_a\})$ is the soft factor S_{em} or S_{gr}. There is however a potential problem. While the amplitude $\Gamma^{(n,1)}$ has momentum conservation $\sum_a p_a + k = 0$, the amplitude $\Gamma^{(n)}$ has momentum conservation $\sum_a p_a = 0$. Therefore we cannot keep the p_a's and k as independent variables in (3.1.1). Usually this problem is overcome by including the momentum conserving delta-functions in the definition of the amplitudes $\Gamma^{(n,1)}$ and $\Gamma^{(n)}$ and treating (3.1.1) as a relation between distributions. The soft factor $S(\varepsilon,k;\{p_a\})$ appearing in (3.1.1) is treated as a differential operator that also acts on the delta function and generates the Taylor series expansion of $\delta\left(\sum_a p_a + k\right)$ in power series of the momentum k of the soft particle. The subleading term in this expansion, given by $k^\mu\{\partial/\partial p_b^\mu\}\delta\left(\sum_a p_a\right)$ for any b, is included in the full subleading soft theorem in dimensions $D > 4$. However since in $D = 4$ we only analyze subleading terms containing $\ln\omega^{-1}$ factors, the term proportional to derivative of the delta function will not appear in our analysis.

In four space-time dimensions there are additional issues due to infrared divergence. Both the amplitudes $\Gamma^{(n,1)}$ and $\Gamma^{(n)}$ have infrared divergences which can be represented as overall multiplicative factors multiplying infrared finite amplitudes. For electromagnetic interactions these factors are common and can be factored out of the amplitudes but for gravity there is a residual infrared divergent factor in $\Gamma^{(n,1)}$ besides the ones that appear in $\Gamma^{(n)}$. In any case we shall denote by $\exp[K]$ the infrared divergent factor of $\Gamma^{(n)}$ and define regulated amplitudes via the relation:

$$\Gamma^{(n)} = \exp[K]\,\Gamma^{(n)}_{\text{reg}}, \quad \Gamma^{(n,1)} = \exp[K]\,\Gamma^{(n,1)}_{\text{reg}}\,. \tag{3.1.2}$$

K is in general a function of the momenta p_a of the finite energy particles. This makes $\Gamma^{(n)}_{\text{reg}}$ free from infrared divergences, but $\Gamma^{(n,1)}_{\text{reg}}$ still contains some residual infrared divergences for gravitational interaction. Eq.(3.1.1) is now replaced by[1]

$$\Gamma^{(n,1)}_{\rm reg}(p_1,\cdots p_n,k) \simeq S(\varepsilon,k;\{p_a\})\,\Gamma^{(n)}_{\rm reg}(p_1,\cdots p_n)\,. \tag{3.1.3}$$

The residual infrared divergences in $\Gamma^{(n,1)}_{\rm reg}$ will be reflected in the infrared divergent contributions to $S(\varepsilon,k;\{p_a\})$.

There is however a potential ambiguity in the definition of $\Gamma^{(n,1)}_{\rm reg}$ and hence of $S(\varepsilon,k;\{p_a\})$. This is due to the fact that in the definition of K we can add a term of the form $Q.\sum_a p_a$ for any vector Q (which could be a function of the p_a's) since by the momentum conserving delta function in $\Gamma^{(n)}$, $\sum_a p_a$ vanishes. However addition of such a term changes the definition of $\Gamma^{(n,1)}_{\rm reg}$ in (3.1.2) by a multiplicative factor of $\exp[k.Q]$ since the momentum conserving delta function in $\Gamma^{(n,1)}$ gives $k+\sum_a p_a=0$. This has the effect of multiplying $S(\varepsilon,k;\{p_a\})$ by $\exp[k.Q]$. Expanding $\exp(k.Q)$ as $(1+k.Q)$ we see that the the additional contribution appears at the subleading order, and has the form of $k.Q$ multiplying the leading soft factor. It does not affect the $\ln\omega^{-1}$ terms that we are after since the leading soft factor has no $\ln\omega^{-1}$ term and Q is ω independent. However this can affect the genuine infrared divergent terms proportional to $\ln R$ in the expression for $\Gamma^{(n,1)}_{\rm reg}$, since in the definition of Q we can include terms proportional to $\ln R$. Choosing $Q=-U\,\ln R$ for some vector U constructed from the p_a's amounts to having an additive contribution to $S^{(1)}$ of the form

$$-\ln R\, k.U\, S^{(0)}(\varepsilon,k;\{p_a\})\,. \tag{3.1.4}$$

3.2 Soft photon theorem in scalar QED

Consider a theory containing a U(1) gauge field A_μ and n scalars $\phi_1,\cdots\phi_n$ of masses $m_1,\cdots m_n$ and carrying U(1) charges $q_1,\cdots q_n$, satisfying $\sum_{a=1}^n q_a=0$. We further assume

[1]The situation here is somewhat different from the one in [66]. Since the logarithmic term in $S(\varepsilon,k;\{p_a\})$ that we are after is being represented as a multiplicative factor instead of a differential operator, the infrared divergent factor on the right hand side can be moved past S to the extreme left.

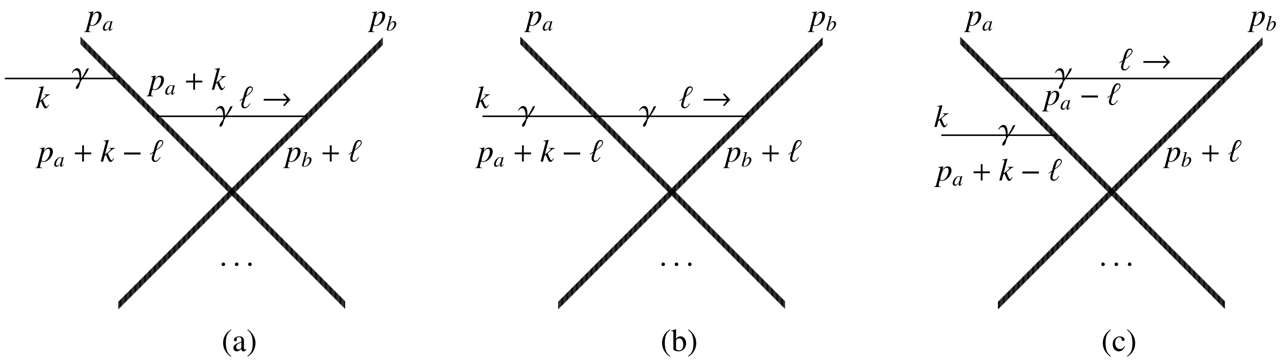

Figure 3.1: One loop contribution to $\Gamma^{(n,1)}$ involving internal photon line connecting two different legs. The thick lines represent scalar particles and the thin lines carrying the symbol γ represent photons. There are other diagrams related to this by permutations of the external scalar particles.

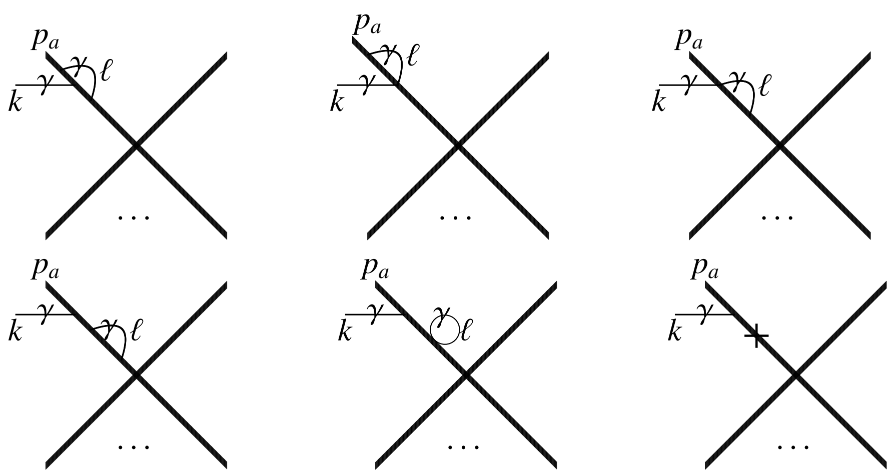

Figure 3.2: One loop contribution to $\Gamma^{(n,1)}$ involving internal photon line connecting two different points on the same leg. There are other diagrams related to this by permutations of the external scalar particles. In the last term the + on the scalar line represents a counterterm associated with mass renormalization that has to be adjusted to cancel the net contribution proportional to $1/(p_a.k)^2$.

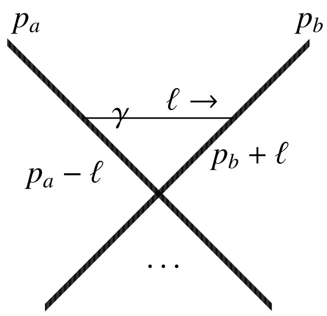

Figure 3.3: One loop contribution to $\Gamma^{(n)}$. There are other diagrams related to this by permutations of the external scalar particles.

that there is a non-derivative contact interaction between the n-scalars. Then the relevant part of the action takes the form

$$\int d^4x \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \sum_{a=1}^{n} \Big\{ (\partial_\mu \phi_a^* + i q_a A_\mu \phi_a^*)(\partial^\mu \phi_a - i q_a A^\mu \phi_a) + m_a^2 \phi_a^* \phi_a \Big\}$$
$$+ \lambda\, \phi_1 \cdots \phi_n + \lambda\, \phi_1^* \cdots \phi_n^* \Big]. \qquad (3.2.1)$$

We consider in this theory an amplitude with one external outgoing photon of momentum k and n external states corresponding to the fields $\phi_1, \cdots \phi_n$, carrying momenta $p_1, \cdots p_n$. All momenta are counted as positive if ingoing so that if the a-th particle is outgoing it will have negative p_a^0. Our goal will be to analyze this amplitude at one loop order, involving an internal photon connecting two matter lines. The relevant diagrams have been shown in Figs. 3.1 and 3.2. We denote by $\Gamma^{(n,1)}$ the sum over tree and one loop contribution to this amplitude. $\Gamma^{(n)}$ will denote the amplitude without the external soft photon to one loop order. One loop contribution to $\Gamma^{(n)}$ has been shown in Fig. 3.3.

In our analysis we shall ignore graphs with self energy insertions on external legs and assume that we follow on-shell renormalization so that the mass parameters appearing in the tree level propagators are the physical masses. The wave-function renormalization of the external scalars cancel between $\Gamma^{(n)}$ and $\Gamma^{(n,1)}$.

We shall use Feynman gauge and decompose the photon propagator of momentum ℓ, connecting the leg a to the leg b for $b \neq a$, with ℓ flowing from the a-th leg to the b-th leg, as [108]

$$-\,i\frac{\eta^{\mu\nu}}{\ell^2 - i\epsilon} = -\frac{i}{\ell^2 - i\epsilon}\left\{K^{\mu\nu}_{(ab)} + G^{\mu\nu}_{(ab)}\right\} \tag{3.2.2}$$

where,

$$K^{\mu\nu}_{(ab)} = \ell^\mu \ell^\nu \frac{(2p_a - \ell).(2p_b + \ell)}{(2p_a.\ell - \ell^2 + i\epsilon)(2p_b.\ell + \ell^2 - i\epsilon)}, \quad G^{\mu\nu}_{(ab)} = \eta^{\mu\nu} - K^{\mu\nu}_{(ab)}\,. \tag{3.2.3}$$

Note that p_a and p_b refer to the external momenta flowing into the legs a and b, and not necessarily the momenta of the lines to which the photon propagator attaches (which may have additional contribution from external soft momentum, *e.g.* in Figs. 3.1(a)). ℓ denotes the momentum flowing from leg a to leg b. For $a = b$ we do not carry out any decomposition.

Since the K-photon polarization is proportional to $\ell^\mu \ell^\nu$, it is pure gauge. This allows us to sum over K-photon insertions using Ward identities

$$\frac{-i}{p_c^2 + m_c^2}\,\ell^\mu\, i\, q_c\,(2p_{c\mu} + \ell_\mu)\,\frac{-i}{(p_c + \ell)^2 + m_c^2} = -q_c\left[\frac{-i}{(p_c + \ell)^2 + m_c^2} - \frac{-i}{p_c^2 + m_c^2}\right], \tag{3.2.4}$$

and

$$q_c\,[i\,q_c\,\varepsilon.(2p_c + 2\ell + k) - i\,q_c\,\varepsilon.(2p_c + k)] - 2\,i\,q_c^2\,\varepsilon.\ell = 0\,, \tag{3.2.5}$$

whose diagrammatic representations have been shown in Fig. 3.4. Sum over all insertions of the K-photons to either $\Gamma^{(n)}$ or $\Gamma^{(n,1)}$ produces an exponential factor [108]

$$\exp\left[i\sum_{a<b} q_a\, q_b \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2 - i\epsilon}\,\frac{(2p_a - \ell).(2p_b + \ell)}{(2p_a.\ell - \ell^2 + i\epsilon)(2p_b.\ell + \ell^2 - i\epsilon)}\right]. \tag{3.2.6}$$

Therefore we may write

$$\Gamma^{(n)} \;=\; \exp\left[K_{\rm em}\right]\left\{\Gamma^{(n)}_{\rm tree} + \Gamma^{(n)}_{\rm G}\right\}, \qquad \Gamma^{(n,1)} = \exp\left[K_{\rm em}\right]\left\{\Gamma^{(n,1)}_{\rm tree} + \Gamma^{(n,1)}_{\rm G} + \Gamma^{(n,1)}_{\rm self}\right\},$$

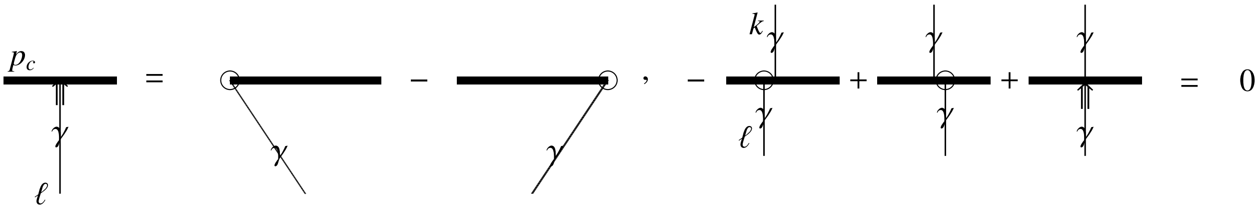

Figure 3.4: Diagrammatic representations of (3.2.4) and (3.2.5). The arrow on the photon line represents that the polarization of the photon is taken to be equal to the momentum entering the vertex. The circle denotes a simple vertex $-q_c$ with the polarization of the incoming photon stripped off.

$$K_{\text{em}} \equiv \frac{i}{2}\sum_{\substack{a,b\\ b\neq a}} q_a\, q_b \int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{(2p_a-\ell).(2p_b+\ell)}{(2p_a.\ell-\ell^2+i\epsilon)(2p_b.\ell+\ell^2-i\epsilon)}\,, \tag{3.2.7}$$

where $\Gamma^{(n)}_{\text{G}}$ and $\Gamma^{(n,1)}_{\text{G}}$ are computed by replacing the internal photons by the G-photons in Figs. 3.3 and 3.1 respectively and $\Gamma^{(n,1)}_{\text{self}}$ denotes the sum of diagrams in Fig. 3.2 for which we use the full photon propagator. Therefore a relation of the form $\Gamma^{(n,1)} = S_{\text{em}}\Gamma^{(n)}$ takes the form

$$\Gamma^{(n,1)}_{\text{tree}} + \Gamma^{(n,1)}_{\text{G}} + \Gamma^{(n,1)}_{\text{self}} = S_{\text{em}} \left\{ \Gamma^{(n)}_{\text{tree}} + \Gamma^{(n)}_{\text{G}} \right\}. \tag{3.2.8}$$

Now it is easy to see that Fig. 3.3 vanishes when we replace the internal photon by G-photon. Therefore $\Gamma^{(n)}_{\text{G}} = 0$, and we have:[2]

$$\Gamma^{(n)}_{\text{tree}} + \Gamma^{(n)}_{\text{G}} = \Gamma^{(n)}_{\text{tree}} = i\,\lambda\,. \tag{3.2.9}$$

If we write $S_{\text{em}} = S^{(0)}_{\text{em}} + S^{(1)}_{\text{em}}$ where $S^{(0)}_{\text{em}}$ is the leading soft factor $\sum_{a=1}^{n} q_a\, \varepsilon.p_a/k.p_a$ and $S^{(1)}_{\text{em}}$ is the subleading multiplicative factor containing logarithmic terms, then eq.(3.2.8) can be written as

$$\Gamma^{(n,1)}_{\text{tree}} + \Gamma^{(n,1)}_{\text{G}} + \Gamma^{(n,1)}_{\text{self}} = i\lambda \sum_{a=1}^{n} q_a \frac{\varepsilon.p_a}{k.p_a} + i\lambda\, S^{(1)}_{\text{em}}\,, \tag{3.2.10}$$

[2]Note that we are not explicitly writing the momentum conserving delta function, but are implicitly assuming that both sides of (3.2.8) are multiplied by the appropriate delta functions. We also implicitly assume that the delta function $\delta(\sum_a p_a + k)$ on the left hand side has been expanded in a power series in k.

to one loop order. Now $\Gamma^{(n,1)}_{\rm tree}$ is equal to the first term on the right hand side up to terms involving Taylor series expansion of the momentum conserving delta function in powers of k, but the latter are subleading contributions without any logarithmic terms and can be ignored in our analysis. Therefore (3.2.10) can be rewritten as:

$$\Gamma^{(n,1)}_{\rm self} + \Gamma^{(n,1)}_{\rm G} = i\lambda\, S^{(1)}_{\rm em}\,. \tag{3.2.11}$$

This is a simple algorithm for determination of $S^{(1)}_{\rm em}$.

Therefore we need to focus on the evaluation of the one loop contribution to $\Gamma^{(n,1)}_{\rm G}$ and $\Gamma^{(n,1)}_{\rm self}$ by summing the diagrams in Figs. 3.1 and 3.2, with the internal photon replaced by G-photon in Fig. 3.1. We first consider the diagrams in Fig. 3.1. It is easy to see that the G-photon contribution to Fig. 3.1(c) vanishes. Therefore we need to focus on Figs, 3.1(a) and (b). The contribution from Fig. 3.1(a) is given by

$$\begin{aligned}\mathcal{I}_1 &= \lambda\, q_a^2\, q_b\, \frac{\epsilon.p_a}{k.p_a} \int \frac{d^4\ell}{(2\pi)^4} \left[2k.(2p_b+\ell) - \frac{2k.\ell\,(2p_a-\ell).(2p_b+\ell)}{(2p_a.\ell-\ell^2+i\epsilon)}\right] \\ &\qquad \times \frac{1}{\ell^2-i\epsilon}\, \frac{1}{2p_a.(k-\ell)+(k-\ell)^2-i\epsilon}\, \frac{1}{2p_b.\ell+\ell^2-i\epsilon}\end{aligned} \tag{3.2.12}$$

and the contribution from Fig. 3.1(b) is given by

$$\begin{aligned}\mathcal{I}_2 &= -\lambda\, q_a^2\, q_b \int \frac{d^4\ell}{(2\pi)^4} \left[2\epsilon.(2p_b+\ell) - \frac{2\epsilon.\ell\,(2p_a-\ell).(2p_b+\ell)}{(2p_a.\ell-\ell^2+i\epsilon)}\right] \\ &\qquad \times \frac{1}{\ell^2-i\epsilon}\, \frac{1}{2p_a.(k-\ell)+(k-\ell)^2-i\epsilon}\, \frac{1}{2p_b.\ell+\ell^2-i\epsilon}\end{aligned} \tag{3.2.13}$$

Both $\mathcal{I}_1$ and $\mathcal{I}_2$ are infrared finite since for small ℓ the integrands diverge as $1/\ell^3$. The terms involving logarithm of k come from the region of ℓ integration where the components $|\ell^\mu|$ are large compared to $\omega \equiv k_0$ but small compared to the p_a's. In this range we can approximate $\mathcal{I}_1$ and $\mathcal{I}_2$ as

$$\mathcal{I}_1 \simeq -\lambda\, q_a^2\, q_b\, \frac{\epsilon.p_a}{k.p_a} \int_{\rm reg} \frac{d^4\ell}{(2\pi)^4} \left[k.p_b - \frac{k.\ell\; p_a.p_b}{p_a.\ell+i\epsilon}\right] \frac{1}{\ell^2-i\epsilon}\, \frac{1}{p_a.\ell+i\epsilon}\, \frac{1}{p_b.\ell-i\epsilon}$$

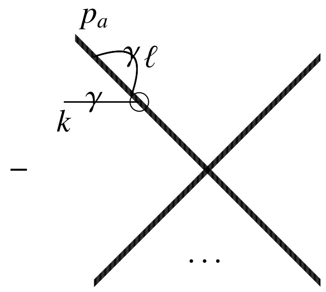

Figure 3.5: Sum of the first four diagrams in Fig. 3.2 with ε replaced by k.

$$= \quad -\lambda\, q_a^2 q_b\, \frac{\epsilon.p_a}{k.p_a}\left[\, k.p_b + p_a.p_b\, k^\mu \frac{\partial}{\partial p_a^\mu}\right] \int_{\rm reg} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon}\,, \tag{3.2.14}$$

and

$$\begin{aligned} \mathcal{I}_2 \quad &\simeq \quad \lambda\, q_a\, q_a q_b \int_{\rm reg} \frac{d^4\ell}{(2\pi)^4}\left[\epsilon.p_b - \frac{\epsilon.\ell\; p_a.p_b}{p_a.\ell + i\epsilon}\right] \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon} \\ &= \quad \lambda\, q_a^2 q_b \left[\epsilon.p_b \;+\; p_a.p_b\, \epsilon^\mu \frac{\partial}{\partial p_a^\mu}\right] \int_{\rm reg} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon}, \end{aligned} \tag{3.2.15}$$

where the subscript reg indicates that the integration needs to be carried out over the region where $|\ell^\mu|$ is large compared to ω but small compared to the energies of the finite energy particles. Adding $\mathcal{I}_1$ and $\mathcal{I}_2$ and summing over a, b we get the total contribution to $\Gamma^{(n,1)}_{\rm G}$ to one loop order:

$$\begin{aligned} \Gamma^{(n,1)}_G \quad &= \quad -\lambda \sum_{\substack{a,b \\ b\neq a}} (q_a)^2 q_b \left[\frac{\epsilon.p_a}{k.p_a}\; k.p_b + \frac{\epsilon.p_a}{k.p_a}\; p_a.p_b\, k^\mu \frac{\partial}{\partial p_a^\mu} - \epsilon.p_b \;-\; p_a.p_b\, \epsilon^\mu \frac{\partial}{\partial p_a^\mu}\right] \\ &\qquad \int_{\rm reg} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon} \\ &= \quad -\lambda \sum_{\substack{a,b \\ b\neq a}} (q_a)^2\, q_b \frac{\varepsilon_\mu k_\nu}{p_a.k}\left\{p_a^\mu \frac{\partial}{\partial p_{a\nu}} - p_a^\nu \frac{\partial}{\partial p_{a\mu}}\right\} \int_{\rm reg} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{p_a.p_b}{(p_a.\ell + i\epsilon)\,(p_b.\ell - i\epsilon)}\,. \end{aligned} \tag{3.2.16}$$

The contribution to $\Gamma^{(n,1)}_{\rm self}$ from Fig. 3.2 can be analyzed using the following indirect method. First of all we note that the net dependence on ε and k from the first four diagrams must be of the form $\varepsilon.p_a\, f(p_a.k)$ for some function f. To determine f, we can set $\varepsilon = k$ and sum over all insertions of the external photon using the Ward identities shown in Fig. 3.4. The final result, given in Fig. 3.5, has the form:

$$\frac{C_1}{p_a.k}\,, \tag{3.2.17}$$

for some constant C_1. Therefore we get

$$p_a.k\, f(p_a.k) = \frac{C_1}{p_a.k} \quad \Rightarrow \quad f(p_a.k) = \frac{C_1}{(p_a.k)^2}\,. \tag{3.2.18}$$

The fifth and sixth diagrams also have the form

$$\frac{C_2}{(p_a.k)^2} \quad \text{and} \quad \frac{C_3}{(p_a.k)^2}\,, \tag{3.2.19}$$

for appropriate constants C_2 and C_3. Now since we are using on-shell renormalization the counterterm proportional to C_3 is to be adjusted precisely so that the net contribution proportional to $1/(p_a.k)^2$ vanishes. Therefore we must choose $C_3 = -C_1 - C_2$, and the total contribution to $\Gamma^{(n,1)}_{\text{self}}$ from all the diagrams in Fig. 3.2 vanishes. We have verified this by explicitly computing the Feynman diagrams in Fig. 3.2.

From (3.2.11) we now see that the net contribution to the logarithmic terms in $S^{(1)}_{\text{em}}$ is obtained by dividing $\Gamma^{(n,1)}_{\text{G}}$ given in (3.2.16) by $i\,\lambda$. This can be written as

$$S^{(1)}_{\text{em}} \;=\; \sum_c q_c \frac{\varepsilon_\mu k_\nu}{p_c.k} \left\{ p_c^\mu \frac{\partial}{\partial p_{c\nu}} - p_c^\nu \frac{\partial}{\partial p_{c\mu}} \right\} K^{\text{reg}}_{\text{em}}\,, \tag{3.2.20}$$

where $K^{\text{reg}}_{\text{em}}$ is the factor K_{em} defined in (3.2.7) with the understanding that integration over the loop momentum ℓ will run over the range where $|\ell^\mu|$ is larger than ω but small compared to the momenta of the finite energy external states:

$$K^{\text{reg}}_{\text{em}} \equiv \frac{i}{2} \sum_{\substack{a,b\\ b\neq a}} q_a\, q_b \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{(2p_a - \ell).(2p_b + \ell)}{(2p_a.\ell - \ell^2 + i\epsilon)(2p_b.\ell + \ell^2 - i\epsilon)}\,. \tag{3.2.21}$$

So essentially we need to evaluate $K_{\rm em}^{\rm reg}$. For this we need to evaluate the integral:[3]

$$\begin{aligned}\mathcal{I}_{ab} &\equiv \int_{\rm reg}\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\frac{1}{p_b.\ell-i\epsilon} \\ &= -\frac{1}{E_aE_b}\int_{\rm reg}\frac{d^3\ell}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\ell^0}{2\pi}\frac{1}{(\ell^0-|\vec{\ell}|+i\epsilon)(\ell^0+|\vec{\ell}|-i\epsilon)}\frac{1}{\ell^0-\vec{v}_a.\vec{\ell}-i\epsilon}\frac{1}{\ell^0-\vec{v}_b.\vec{\ell}+i\epsilon},\end{aligned} \tag{3.2.22}$$

where $E_a = p_a^0$, $E_b = p_b^0$, $\vec{v}_a = \vec{p}_a/E_a$ and $\vec{v}_b = \vec{p}_b/E_b$. In writing down the above equation we have assumed that E_a and E_b are positive, i.e. both lines represent incoming states. The integrand has simple poles at,

$$\ell^0 = (|\vec{\ell}|-i\epsilon)\,,\ -(|\vec{\ell}|-i\epsilon)\,,\ (\vec{v}_a.\vec{\ell}+i\epsilon)\,,\ (\vec{v}_b.\vec{\ell}-i\epsilon)\,. \tag{3.2.23}$$

So now if we close the contour in the lower half plane we have to take the pole contributions from $\ell^0 = (|\vec{\ell}|-i\epsilon)$ and $\ell^0 = (\vec{v}_b.\vec{\ell}-i\epsilon)$. This gives

$$\begin{aligned}\mathcal{I}_{ab} &= \frac{i}{E_aE_b}\int_{\rm reg}\frac{d^3\vec{\ell}}{(2\pi)^3}\frac{1}{2|\vec{\ell}|}\frac{1}{|\vec{\ell}|-\vec{v}_a.\vec{\ell}}\frac{1}{|\vec{\ell}|-\vec{v}_b.\vec{\ell}} \\ &\quad+\frac{i}{E_aE_b}\int_{\rm reg}\frac{d^3\vec{\ell}}{(2\pi)^3}\frac{1}{(\vec{v}_b.\vec{\ell})^2-|\vec{\ell}|^2}\frac{1}{(\vec{v}_b-\vec{v}_a).\vec{\ell}-i\epsilon}\,.\end{aligned} \tag{3.2.24}$$

Note that we have removed the $i\epsilon$'s from the denominators that never vanish.

Let us first analyze the second term. Since the result should be Lorentz invariant, it should not depend on the chosen frame. For simplicity choose a frame in which $\vec{v}_b$ and $\vec{v}_a$ are along the positive z-axis with $|\vec{v}_b| > |\vec{v}_a|$. Denoting by θ the angle between $\vec{\ell}$ and the z-axis, we can express the second term in (3.2.24) as

$$\mathcal{I}'_{ab} = \frac{i}{E_aE_b(2\pi)^2}\frac{1}{|\vec{v}_a-\vec{v}_b|}\int_{\rm reg}\frac{d|\vec{\ell}|}{|\vec{\ell}|}\int_{-1}^{1}d(cos\theta)\frac{1}{|\vec{v}_b|^2cos^2\theta-1}\frac{1}{cos\theta-i\epsilon} \tag{3.2.25}$$

Without the $i\epsilon$ piece of the last term the integral vanishes since the integrand is an odd

[3]Since the ℓ^μ integration runs over a limited range, one might wonder why we are choosing the ℓ^0 integration range from $-\infty$ to ∞. To this end, note that once we have imposed the range restriction on $|\vec{\ell}|$, we can let the ℓ^0 integral in (3.2.22) run over the entire real axis since the regions outside the allowed range do not generate any logarithmic contribution.

function of $\cos\theta$. However the imaginary part of the last term makes the integral non-vanishing. Using $1/(x-i\epsilon) = i\pi\delta(x) + P(1/x)$ in the integral, and using the fact that the value of $|\vec{\ell}|$ for which our approximation of the integrand is valid ranges from ω to some finite energy, we get,

$$\mathcal{I}'_{ab} \simeq \frac{1}{4\pi\, E_a E_b}\ln\omega^{-1}\frac{1}{|\vec{v}_a - \vec{v}_b|} = \frac{1}{4\pi}\ln\omega^{-1}\frac{1}{\sqrt{(p_a.p_b)^2 - m_a^2 m_b^2}}\,, \tag{3.2.26}$$

where in the intermediate stage we used $|\vec{p}_a||\vec{p}_b| = |\vec{p}_a.\vec{p}_b|$, since $\vec{p}_a$ and $\vec{p}_b$ are parallel.

If both the legs a and b are outgoing instead of ingoing, then E_a and E_b are negative and the signs of the $i\epsilon$ in the last two terms in (3.2.22) are reversed. But this can be brought back to the form given in (3.2.22) by making a change of variables $\ell^\mu \to -\ell^\mu$. Therefore the net result for the residue at $\ell^0 = \vec{v}_b.\vec{\ell} - i\epsilon$ will continue to be described by (3.2.26). Finally if one of the momenta is outgoing and the other is ingoing, then both the $i\epsilon$'s in the last two terms of (3.2.22) come with the same sign. By changing variables from ℓ^μ to $-\ell^\mu$ if necessary, we can ensure that both the poles are in the upper half plane and close the contour to the lower half plane. In this case there will be no analog of the contribution given in (3.2.26).

We now turn to the contribution from the first term on the right hand side of (3.2.24), which we will call $\mathcal{I}''_{ab}$. We will again evaluate this integral in the frame in which $\vec{v}_a$ and $\vec{v}_b$ are parallel to the z-axis with $|\vec{v}_b| > |\vec{v}_a|$. We get

$$\begin{aligned}
\mathcal{I}''_{ab} &= \frac{i}{E_a E_b}\int_{\rm reg}\frac{d^3\vec{l}}{(2\pi)^3}\frac{1}{2|\vec{l}|}\frac{1}{|\vec{l}| - \vec{v}_a.\vec{l}}\frac{1}{|\vec{l}| - \vec{v}_b.\vec{l}} \\
&= \frac{i}{8\pi^2 E_a E_b}\ln\omega^{-1}\int_{-1}^{1} d(cos\theta)\,\frac{1}{v_b - v_a}\left[\frac{v_b}{1 - v_b\cos\theta} - \frac{v_a}{1 - v_a\cos\theta}\right] \\
&= \frac{i}{8\pi^2}\ln\omega^{-1}\frac{1}{|\vec{p}_b|E_a - |\vec{p}_a|E_b}\ln\left[\frac{(E_a - |\vec{p}_a|)(E_b + |\vec{p}_b|)}{(E_a + |\vec{p}_a|)(E_b - |\vec{p}_b|)}\right] \\
&= -\frac{i}{8\pi^2}\ln\omega^{-1}\frac{1}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}\ln\left[\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}\right].
\end{aligned} \tag{3.2.27}$$

It is easy to check that the form of the contribution remains unchanged even when both legs are outgoing or one leg is incoming and the other leg is outgoing.

Combining these results we get

$$K_{\rm em}^{\rm reg} = \frac{i}{2}\sum_{\substack{a,b\\ b\neq a}} q_a\, q_b\, \frac{1}{4\pi}\, \ln\omega^{-1}\, \frac{p_a.p_b}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \left\{\delta_{\eta_a\eta_b,1} - \frac{i}{2\pi}\ln\left(\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2p_b^2}}\right)\right\}. \tag{3.2.28}$$

Using (3.2.20) we can now write down the expression for the logarithmic term in the subleading soft factor $S_{\rm em}^{(1)}$

$$\begin{aligned} &-\frac{i}{4\pi}\ \ln\omega^{-1} \sum_{a=1}^{n} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} q_a^2\, q_b\, \frac{\epsilon_\mu\, k_\rho}{p_a.k}\, \frac{m_a^2 m_b^2 [p_a^\mu p_b^\rho - p_b^\mu p_a^\rho]}{[(p_a.p_b)^2 - m_a^2 m_b^2]^{3/2}} \\ &-\frac{1}{8\pi^2}\ \ln\omega^{-1} \sum_{\substack{a,b\\ b\neq a}} q_a^2\, q_b\, \ln\left[\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2p_b^2}}\right] \frac{p_a^2 p_b^2}{\{(p_a.p_b)^2 - p_a^2p_b^2\}^{3/2}} \left\{-\epsilon.p_b + \frac{\epsilon.p_a}{k.p_a} k.p_b\right\} \\ &+\frac{1}{4\pi^2}\ \ln\omega^{-1} \sum_{\substack{a,b\\ b\neq a}} q_a^2 q_b\, \frac{p_a.p_b}{(p_a.p_b)^2 - p_a^2 p_b^2} \left\{-\epsilon.p_b + \frac{\epsilon.p_a}{k.p_a} k.p_b\right\}. \end{aligned} \tag{3.2.29}$$

The term in the first line agrees with the classical expression for $S_{\rm em}^{(1)}$ given by the second term of (2.2.16). The rest of the contribution is extra.

We have also checked that (3.2.29) holds if instead of scalars we have interacting fermions. This confirms that the logarithmic correction to the soft factor is independent of the spin of the particle.

We end this section by making some observation on the results derived above:

1. Suppose we assume the validity of the naive version of the subleading soft photon theorem:[4]

$$\Gamma^{(n,1)} = \{S_{\rm em}^{(0)} + \widehat{S}_{\rm em}^{(1)}\}\,\Gamma^{(n)}, \tag{3.2.30}$$

[4] Since the presence of the logarithmic term makes the finite part ambiguous, we consider only the logarithmic terms in the subleading factor.

where the 'hat' on $S^{(1)}$ denotes that we are using the differential operator form that arises in the quantum theory:

$$S^{(0)}_{\rm em} = \sum_a q_a \frac{\varepsilon.p_a}{p_a.k}, \qquad \widehat{S}^{(1)}_{\rm em} = \sum_a q_a \frac{\varepsilon_\mu k_\nu}{p_a.k}\left\{p^\mu_a \frac{\partial}{\partial p_{a\nu}} - p^\nu_a \frac{\partial}{\partial p_{a\mu}}\right\}. \tag{3.2.31}$$

Then using (3.2.7) and the fact that $\Gamma^{(n)}_{\rm G}$ vanishes at one loop order, we get

$$\Gamma^{(n,1)}_{\rm tree} + \Gamma^{(n,1)}_{\rm self} + \Gamma^{(n,1)}_{\rm G} = S^{(0)}_{\rm em}\Gamma^{(n)}_{\rm tree} + \{\widehat{S}^{(1)}_{\rm em} K_{\rm em}\}\,\Gamma^{(n)}_{\rm tree} + \widehat{S}^{(1)}_{\rm em}\Gamma^{(n)}_{\rm tree}\,. \tag{3.2.32}$$

Using $\Gamma^{(n)}_{\rm tree} = i\,\lambda$, using (3.2.10) to replace the left hand side, and throwing away terms like $\widehat{S}^{(1)}_{\rm em}\Gamma^{(n)}_{\rm tree}$ which vanishes, we get

$$S^{(1)}_{\rm em} = \widehat{S}^{(1)}_{\rm em}\, K_{\rm em}\,. \tag{3.2.33}$$

In the definition of $K_{\rm em}$ the integration over loop momentum runs over all range and we have an infrared divergence from the region of small ℓ. However if we make an ad hoc restriction that the loop momentum integral will run in the range much larger than the energy ω of the external soft photon, then $K_{\rm em}$ reduces to $K^{\rm reg}_{\rm em}$ defined in (3.2.21) and we recover the correct logarithmic terms in $S^{(1)}_{\rm em}$ as given in (3.2.20). This suggests an ad hoc rule for computing the logarithmic terms in the soft expansion in quantum theory – begin with the usual soft expansion and explicitly evaluate the action of the differential operator on the amplitude, restricting the region of loop momentum integration to lie in a range larger than the soft momenta but smaller than the momenta of the finite energy particles. With hindsight, this prescription can be justified by noting that the general arguments of [11, 101], that assumes existence of 1PI effective action with no powers of soft momenta coming from the vertices, breaks down for the contribution where the loop momentum is smaller than the external soft momenta. On the other hand we do not expect any large contribution from the region of integration where the loop momentum is of the order of the external momenta or larger.

This argument also suggests that although we have carried out the explicit calculation only at one loop order, the result may be valid to all orders in perturbation theory, since $K_{\rm em}$ is known to be valid to all orders in perturbation theory [108].

2. The second observation concerns the relation between the classical and the quantum results. As already noted, compared to the classical result that agrees with the first line of (3.2.29), the quantum result found here has an extra term given in the second and third line of (3.2.29). If however we replace in (3.2.22) the Feynman propagator for the photon by the retarded propagator, we get only the contribution from the first line of (3.2.29), since the contribution from the pole at $\ell^2 = 0$ can then be avoided by appropriate choice of contour. Therefore at least for the soft photon theorem in quantum electrodynamics, the rule for relating the quantum and the classical result seems to be to replace the Feynman propagator of the photon in the loop in the quantum result by retarded propagator.

We shall now write down the results for the other cases and test if the generalization of observation 1 works. We shall also explore if the results satisfy the generalization of observation 2.

3.3 Soft graviton theorem in gravitational scattering

We now turn to the analysis of the soft graviton theorem in the scattering of scalar particles, interacting via gravity, to one loop order. The action is taken to be

$$\int d^4x\,\sqrt{-\det g}\left[\frac{1}{16\pi G}R-\sum_{a=1}^{n}\left\{g^{\mu\nu}\,\partial_\mu\phi_a^*\,\partial_\nu\phi_a+m_a^2\phi_a^*\phi_a\right\}+\lambda\,\phi_1\cdots\phi_n+\lambda\,\phi_1^*\cdots\phi_n^*\right]. \tag{3.3.1}$$

Even though in this case we could take the scalar fields to be real, we have kept them complex in order to extend the analysis to the case where the scalars have both electromagnetic and gravitational interaction. As in §3.2, we shall postulate a relation of the form

$$\Gamma^{(n,1)} = \left\{ S^{(0)}_{\rm gr} + S^{(1)}_{\rm gr} \right\} \Gamma^{(n)}\,, \tag{3.3.2}$$

and try to determine the logarithmic terms in $S^{(1)}_{\rm gr}$ by comparing the two sides up to one loop order.

We shall carry out our computation in the de Donder gauge in which the propagator of a graviton of momentum ℓ is given by:

$$-\frac{i}{\ell^2 - i\epsilon}\,\frac{1}{2}\,(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma})\,. \tag{3.3.3}$$

For our analysis we also need the vertices involving the graviton. The scalar-scalar-graviton vertex, with the scalars carrying ingoing momenta p_1, p_2 and the graviton carrying ingoing momentum $-p_1 - p_2$ and Lorentz index $(\mu\nu)$, is given by

$$-\,i\,\kappa\,[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - \eta_{\mu\nu}(p_1.p_2 - m^2)]\,, \tag{3.3.4}$$

where $\kappa = \sqrt{8\pi G} = 1$ in our convention. The vertex involving two scalars carrying ingoing momenta p_1, p_2, and two gravitons carrying ingoing momenta k_1, k_2 and Lorentz indices $(\alpha\beta)$ and $(\mu\nu)$ is given by[5]

$$\begin{aligned} 2\,i\,\kappa^2 \Big[-\eta_{\alpha\mu}\,\eta_{\beta\nu}\,p_1.p_2 \;+\; \frac{1}{2}\eta_{\alpha\beta}\,\eta_{\mu\nu}\,p_1.p_2 \;-\; \eta_{\alpha\beta}\,p_{1\mu}p_{2\nu} \;-\; \eta_{\mu\nu}\,p_{1\alpha}p_{2\beta} \\ +\,2\,\eta_{\alpha\mu}\left\{p_{1\beta}p_{2\nu} + p_{2\beta}p_{1\nu}\right\} \;+\; m^2\,(\eta_{\mu\alpha}\eta_{\nu\beta} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta})\Big]\,. \end{aligned} \tag{3.3.5}$$

If we label the ingoing graviton momenta by k_1, k_2 and $k_3 = -k_1 - k_2$ and the Lorentz indices carried by them by $(\mu\alpha)$, $(\nu\beta)$ and $(\sigma\gamma)$ respectively, then the 3-graviton vertex

[5] In writing this and other vertices we already include the symmetry factor related to exchange of identical particles. Therefore if we were to use this vertex to compute tree level two graviton, two scalar amplitude, no further symmetry factor is necessary.

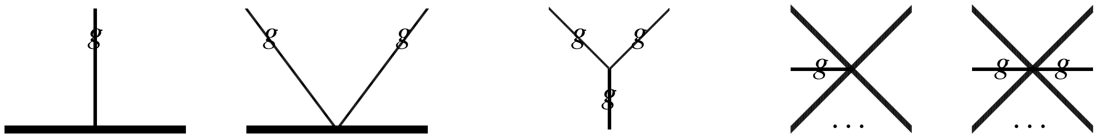

Figure 3.6: This diagram shows various vertices induced from the action (3.3.1) that are needed for our computation. Here the thinner lines carrying the symbol g denote gravitons and the thicker lines denote scalars.

takes the form:

$$
\begin{aligned}
i\kappa \quad \Big[\quad & (k_1.k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma} + k_2.k_1\eta_{\nu\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma} + k_1.k_3\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma} \\
+ \quad & k_3.k_1\eta_{\sigma\gamma}\eta_{\mu\nu}\eta_{\alpha\beta} + k_2.k_3\eta_{\nu\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma} + k_3.k_2\eta_{\sigma\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\
- \quad & 2\,(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta} + k_{2\mu}k_{3\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma} + k_{3\nu}k_{1\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma}) \\
- \quad & 4(k_1.k_2 + k_2.k_3 + k_3.k_1)\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu} \\
+ \quad & (k_1.k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma} + k_2.k_3\eta_{\nu\sigma}\eta_{\beta\gamma}\eta_{\mu\alpha} + k_3.k_1\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\nu\beta}) \\
+ \quad & 2(k_{1\sigma}k_{2\mu}\eta_{\alpha\nu}\eta_{\beta\gamma} + k_{2\mu}k_{3\nu}\eta_{\sigma\alpha}\eta_{\gamma\beta} + k_{3\nu}k_{1\sigma}\eta_{\mu\beta}\eta_{\alpha\gamma} \\
+ \quad & k_{2\sigma}k_{1\nu}\eta_{\mu\beta}\eta_{\alpha\gamma} + k_{3\mu}k_{2\sigma}\eta_{\nu\gamma}\eta_{\beta\alpha} + k_{1\nu}k_{3\mu}\eta_{\sigma\beta}\eta_{\gamma\alpha}) \\
- \quad & \frac{1}{2}(k_1.k_2 + k_2.k_3 + k_3.k_1)\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\Big]\,.
\end{aligned} \tag{3.3.6}
$$

In (3.3.5) and (3.3.6) it is understood that the vertices need to be symmetrized under the exchange of the pair of Lorentz indices carried by each external graviton, *e.g.* $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$ in (3.3.5) and $\mu \leftrightarrow \alpha$, $\nu \leftrightarrow \beta$ and $\sigma \leftrightarrow \gamma$ in (3.3.6). Even though (3.3.6) has a complicated form, we shall need the form of the vertex when one of the external momenta (say k_3) is small compared to the others. In this limit it simplifies.

The vertex where a graviton carrying Lorentz index $(\mu\nu)$ attaches to n scalar fields is given by:

$$
i\kappa\lambda\,\eta_{\mu\nu}\,. \tag{3.3.7}
$$

The vertex where two gravitons carrying Lorentz index $(\mu\nu)$ and $(\rho\sigma)$ attach to n scalar

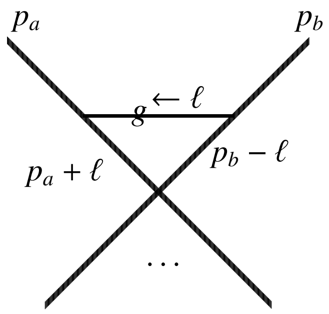

Figure 3.7: Diagram contributing to $\Gamma^{(n)}$.

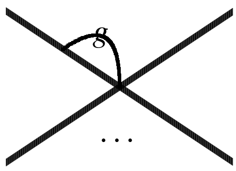

Figure 3.8: Another diagram contributing to $\Gamma^{(n)}$. We can also have a diagram where both ends of the internal graviton are attached to the n-scalar vertex, but this vanishes in dimensional regularization and so we have not displayed them.

fields is given by:

$$-\, i\kappa^2\lambda \left(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right) . \tag{3.3.8}$$

We also need the vertex containing two scalars and three gravitons for evaluating the fifth diagram of Fig. 3.10. However even without knowing the form of this vertex one can see that this diagram does not generate contributions proportional to $\ln\omega^{-1}$. Therefore we have not written down the expression for this vertex.

We can use these vertices to compute one loop contribution to the n scalar amplitude $\Gamma^{(n)}$ and n-scalar and one soft graviton amplitude $\Gamma^{(n,1)}$. At one loop order $\Gamma^{(n)}$ receives contribution from diagrams shown in Fig. 3.7 that are analogous to Fig. 3.3 with the internal photon replaced by a graviton. There are also some additional diagrams shown in Fig. 3.8.

The relevant diagrams for $\Gamma^{(n,1)}$ include the analogs Figs. 3.1 and 3.2 with all photons replaced by gravitons. This have been shown in Figs. 3.9 and 3.10. However there are also some extra diagrams that we shall list below:

1. There are diagrams where the external graviton couples to the internal graviton via

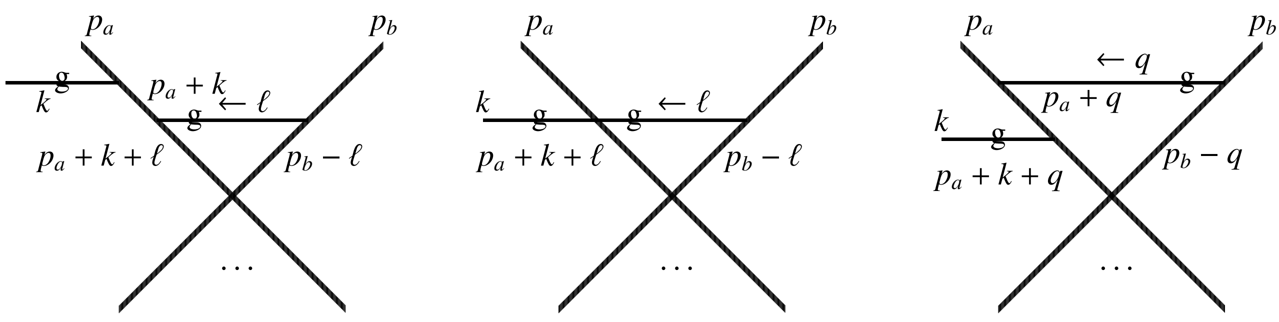

Figure 3.9: One loop contribution to $\Gamma^{(n,1)}$ involving internal graviton line connecting two different legs. The thicker lines represent scalar particles and the thinner lines represent gravitons.

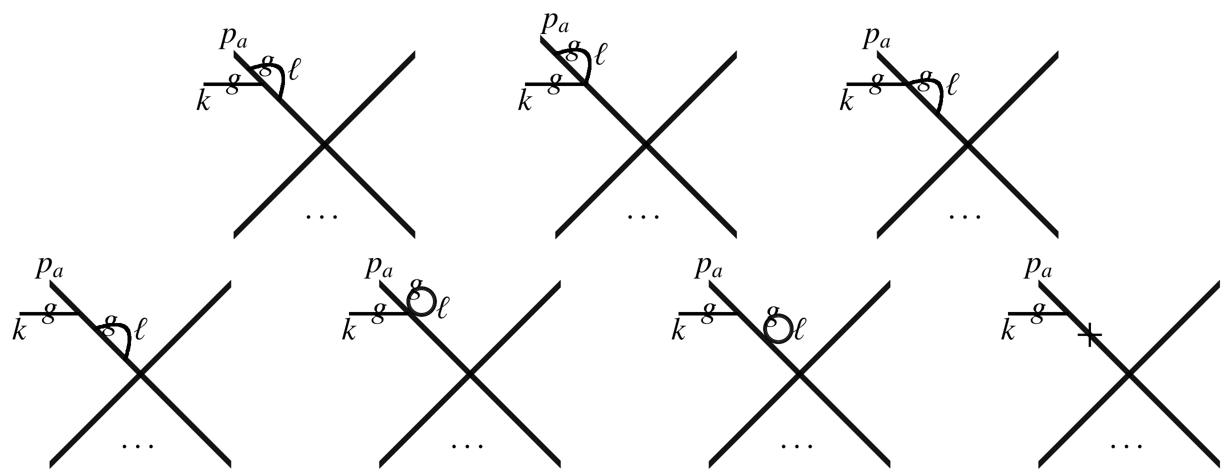

Figure 3.10: One loop contribution to $\Gamma^{(n,1)}$ involving internal graviton line connecting two different points on the same leg.

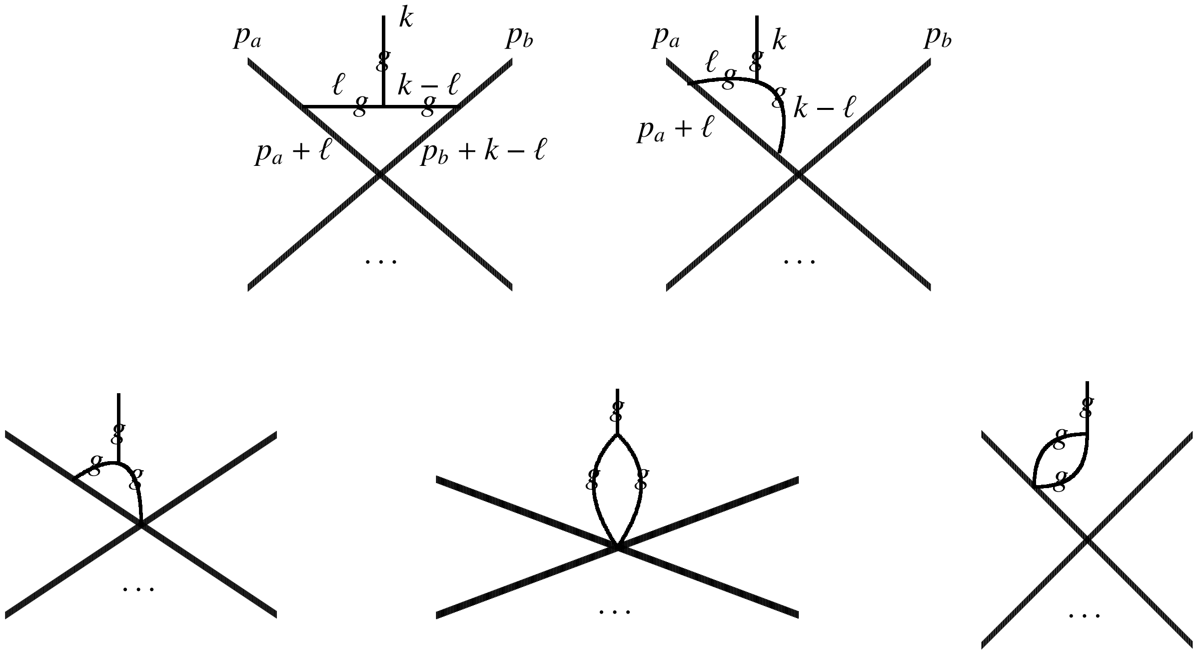

Figure 3.11: Diagrams involving 3-graviton vertex.

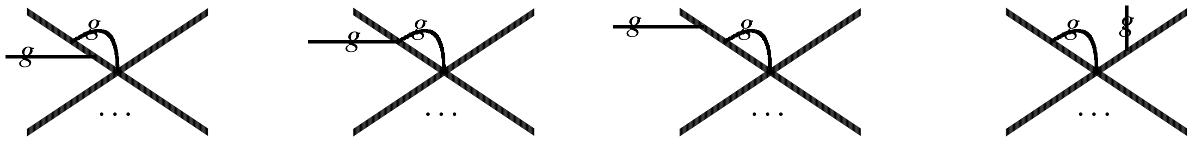

Figure 3.12: Diagrams where the internal graviton attaches to the n-point vertex.

the cubic coupling (3.3.6). Examples of these are shown in Fig. 3.11.

2. There are diagrams where one end of the internal graviton attaches to the n-scalar vertex via the coupling (3.3.7). These have been shown in Figs. 3.12.

3. There are diagrams where the external graviton attaches to the scalar n-point vertex via the coupling (3.3.7) or (3.3.8). These have been shown in Fig. 3.13. The first diagram can be made to vanish by taking the external graviton polarization to be traceless: $\varepsilon_\rho^{\ \rho} = 0$. The second diagram has no logarithmic terms. Therefore we shall ignore these diagrams in subsequent discussions.

4. There are diagrams of the type shown in Fig. 3.14 where two ends of the internal graviton attach to the n-scalar vertex. In dimensional regularization these diagrams vanish. Therefore we shall ignore these diagrams in our analysis.

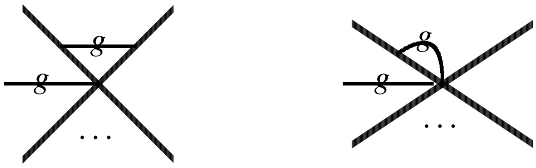

Figure 3.13: Diagrams where the external graviton attaches to the n-point vertex. The first diagram vanishes if we take the external graviton polarization to be traceless. The second diagram has no logarithmic terms.

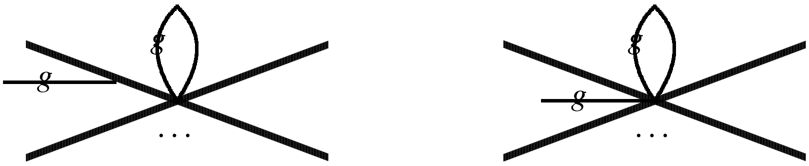

Figure 3.14: Diagrams where both ends of the internal graviton attach to the n-point vertex. In dimensional regularization these diagrams vanish. Even if we use momentum cut-off, these diagrams cannot have any contribution proportional to $\ln \omega^{-1}$ since the soft momentum k does not flow through any loop.

Our analysis of these diagrams will proceed as in §3.2, but there will be some important differences that we shall point out below. For an internal graviton of momentum ℓ, whose two ends are attached to two scalar lines a and b with ℓ flowing from the leg b towards the leg a, as in Figs. 3.7, 3.9, the analog of Grammer-Yennie decomposition of the graviton propagator will be taken to be

$$G^{\mu\nu,\rho\sigma}_{(ab)}(\ell, p_a, p_b) = (\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}) - K^{\mu\nu,\rho\sigma}_{(ab)}(\ell, p_a, p_b), \tag{3.3.9}$$

$$K^{\mu\nu,\rho\sigma}_{(ab)}(\ell, p_a, p_b) = C(\ell, p_a, p_b)\,[(p_a+\ell)^\mu \ell^\nu + (p_a+\ell)^\nu \ell^\mu]\,[(p_b-\ell)^\rho \ell^\sigma + (p_b-\ell)^\sigma \ell^\rho],$$

where

$$C(\ell, p_a, p_b) = \frac{(-1)}{\{p_a.(p_a+\ell) - i\epsilon\}\,\{p_b.(p_b-\ell) - i\epsilon\}\{\,\ell.(\ell+2p_a) - i\epsilon\}\,\{\ell.(\ell-2p_b) - i\epsilon\}} \Big[2(p_a.p_b)^2 - p_a^2 p_b^2 - \ell^2(p_a.p_b) - 2(p_a.p_b)(p_a.\ell) + 2(p_a.p_b)(p_b.\ell)\Big]. \tag{3.3.10}$$

If one end of an internal graviton is attached to the n-scalar vertex and the other end is attached to the a'th scalar leg as in Figs. 3.8, 3.12, with ℓ flowing from the vertex towards the a'th leg, we express the propagator as:

$$- \frac{i}{\ell^2 - i\epsilon} \frac{1}{2} \left\{ G_{(a)}^{\mu\nu,\rho\sigma}(\ell, p_a) + K_{(a)}^{\mu\nu,\rho\sigma}(\ell, p_a) \right\} , \tag{3.3.11}$$

where

$$G_{(a)}^{\mu\nu,\rho\sigma}(\ell, p_a) = (\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}) - K_{(a)}^{\mu\nu,\rho\sigma}(\ell, p_a) , \tag{3.3.12}$$

$$K_{(a)}^{\mu\nu,\rho\sigma}(\ell, p_a) = \widetilde{C}(\ell, p_a) \left[(p_a + \ell)^\mu \ell^\nu + (p_a + \ell)^\nu \ell^\mu \right] \eta^{\rho\sigma} , \tag{3.3.13}$$

and

$$\widetilde{C}(\ell, p_a) = - \frac{2\, p_a.(p_a - \ell)}{\{p_a.(p_a + \ell) - i\epsilon\} \{ \ell.(\ell + 2p_a) - i\epsilon\}} . \tag{3.3.14}$$

For internal gravitons whose one end is attached to a 3-graviton vertex instead of a scalar, as in Fig 3.11, we do not carry out any Grammer-Yennie decomposition.

The decomposition into G and K-gravitons is not arbitrary but has been chosen to ensure two properties:

1. The K-graviton polarization, being proportional to ℓ, is pure gauge and allows us to sum over K-graviton insertions using Ward identities. The relevant Ward identities have been shown in Fig. 3.15, with the quantity $A(p, k, \ell, \xi, \zeta)$ is given by

$$\begin{aligned} A(p, k, \ell, \xi, \zeta) \;=\; & 2\, i\, \xi.p\, \zeta^{\mu\nu} \Big[2\, (2\, p_\mu + \ell_\mu)\, k_\nu + 2\, k_\mu k_\nu - \eta_{\mu\nu} \Big\{ k.(2p + \ell) + k^2 \Big\} \Big] \\ & + 2\, i\, \xi.(k + \ell)\, \zeta^{\mu\nu} \Big[-2\, p_\mu\, (p + \ell)_\nu + \eta_{\mu\nu}\, \{ p.(p + \ell) + m^2 \} \Big] \\ & + 2\, i\, (\xi^\alpha k^\beta + \xi^\beta k^\alpha)\, \zeta^{\mu\nu} \Big[\eta_{\alpha\mu}\, \eta_{\beta\nu}\, p.(p + k + \ell) - \frac{1}{2}\, \eta_{\alpha\beta}\, \eta_{\mu\nu}\, p.(p + k + \ell) \\ & + \eta_{\alpha\beta}\, p_\mu\, (p + k + \ell)_\nu + \eta_{\mu\nu}\, p_\alpha\, (p + k + \ell)_\beta - 2\, \eta_{\alpha\mu}\, p_\beta (p + k + \ell)_\nu \\ & - 2\, \eta_{\alpha\mu}\, p_\nu (p + k + \ell)_\beta + m^2 \Big(\eta_{\mu\alpha}\, \eta_{\nu\beta} - \frac{1}{2}\, \eta_{\mu\nu}\, \eta_{\alpha\beta} \Big) \Big] . \end{aligned} \tag{3.3.15}$$

Due to this additional term, the sum over K-gravitons will leave behind some resid-

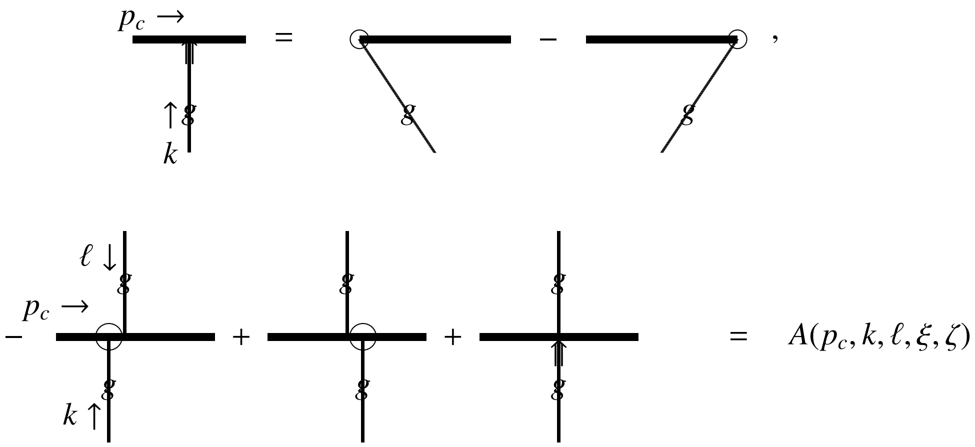

Figure 3.15: Analog of Fig. 3.4 for gravity. The arrow on the graviton line represents that the polarization of the graviton carrying momentum k is taken to be equal to $\xi_\mu k_\nu + \xi_\nu k_\mu$. The polarization of the graviton carrying momentum k is taken to be $\zeta_{\rho\sigma}$. In the first diagram the circle on the left denotes a vertex $-2\,\xi.(p_c + k)$ while the circle on the right denotes a vertex $-2\,\xi.p_c$. $A(p_c, k, \ell, \xi, \zeta)$ appearing on the right hand side of the second diagram is given in eq.(3.3.15).

ual terms that will be discussed below.

2. In any one loop diagram contributing to the amplitude $\Gamma^{(n)}$ without external soft graviton, the result vanishes if we replace the internal graviton by G-graviton.

With this convention the K-graviton contribution to Fig. 3.7 for gravity can be computed as in §3.2, leading to a contribution of the form $i\lambda K_{\rm gr}$ to $\Gamma^{(n)}$, where $K_{\rm gr}$ is the gravitational counterpart of $K_{\rm em}$. The relevant part of the expression for $K_{\rm gr}$ will be described later. The K-graviton contribution to Fig. 3.8 can be carried out similarly, leading to an expression of the form $i\lambda \widetilde{K}_{\rm gr}$. $\widetilde{K}_{\rm gr}$ has no infrared divergence and we shall not write down its expression explicitly although it is straightforward to do so. The G-graviton contributions to Fig. 3.7 and 3.8 vanish by construction. Therefore the net contribution to $\Gamma^{(n)}$ to one loop order may be written as $i\lambda\, \exp[K_{\rm gr} + \widetilde{K}_{\rm gr}]$.

The K-graviton contributions to Figs. 3.9 and 3.12 may be evaluated similarly, with the factorized term giving $i\lambda\, S^{(0)}_{\rm gr}\, \exp[K_{\rm gr} + \widetilde{K}_{\rm gr}]$. There are however some left-over terms arising as follows:

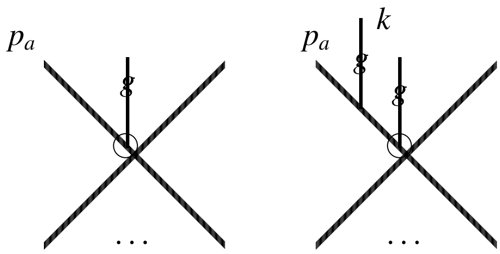

Figure 3.16: Figure illustrating the difference in the factorized K-graviton contribution to $\Gamma^{(n)}$ and $\Gamma^{(n+1)}$.

1. As shown in Fig. 3.15, in the sum over K-graviton insertions in $\Gamma^{(n,1)}$ there is a residual contribution A that comes from lack of complete cancellation among terms where a K-graviton is inserted to the two sides of a scalar-scalar-graviton vertex and into the scalar-scalar-graviton vertex.

2. As explained in the caption of Fig. 3.15, the circled vertices are momentum dependent. Therefore the two circled vertices shown in Fig. 3.16 are not the same, one carries a factor of $\xi.p_a$ while the other carries a factor of $\xi.(p_a + k)$. The left hand figure is relevant for $\Gamma^{(n)}$ while the right-hand figure is relevant for $\Gamma^{(n,1)}$. Therefore, even after factoring out $\exp[K_{\rm gr} + \widetilde{K}_{\rm em}]$ factor multiplying $\Gamma^{(n)}$, we are left with an additional contribution to $\Gamma^{(n,1)}$ from sum over K-gravitons that must be accounted for.

We shall denote the sum of these two types of residual contributions as $\Gamma^{(n,1)}_{\rm residual}$. The G-graviton contributions to Figs. 3.9 and 3.12 will be denoted by $\Gamma^{(n,1)}_{G}$ and the net contribution from Fig. 3.10 will be called $\Gamma^{(n,1)}_{\rm self}$. Finally the contribution to the diagrams in Fig. 3.11 involving 3-graviton coupling will be denoted by $\Gamma^{(n,1)}_{3-{\rm graviton}}$. In principle we should also include the contributions from Fig. 3.13 and Fig. 3.14, but we ignore them since they do not generate logarithmic terms. In this case the analog of (3.2.11) takes the form:

$$\Gamma^{(n,1)}_{\rm self} + \Gamma^{(n,1)}_{\rm G} + \Gamma^{(n,1)}_{3-{\rm graviton}} + \Gamma^{(n,1)}_{\rm residual} = i\lambda\, S^{(1)}_{\rm gr}\,. \tag{3.3.16}$$

We shall now briefly describe how we evaluate these contributions and then give the final result. First let us consider $\Gamma^{(n,1)}_{\rm residual}$. This receives contribution from Fig. 3.9 and Fig. 3.12. As explained above, there are two kinds of terms: one due to the right hand side of the second figure of Fig. 3.15, and the other due to the momentum dependence of the circled vertices in Fig. 3.15. It turns out that the residual part of the K-graviton contribution from Fig. 3.12 does not have any logarithmic term. On the other hand the residual part of the K-graviton contribution from Fig. 3.9 receives logarithmic contribution only from the region where the loop momentum is large compared to ω. The result takes the form:

$$\Gamma^{(n,1)}_{\rm residual} = -(i\lambda)\,\frac{i}{2}\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n}\,[2(p_a.p_b)^2-p_a^2p_b^2]\,\frac{p_a.\varepsilon.p_a}{p_a^2}\int_{\rm reg}\frac{d^4l}{(2\pi)^4}\,\frac{1}{[p_a.l-i\epsilon]\,[p_b.l+i\epsilon]\,[l^2-i\epsilon]}\,. \tag{3.3.17}$$

This contribution may be evaluated following a procedure similar to the one used in §3.2.

Contribution to $\Gamma^{(n,1)}_{3-\text{graviton}}$ arises from the five diagrams in Fig. 3.11, but only the first two give terms proportional to $\ln\omega^{-1}$. Individually these diagrams suffer from collinear divergence from region of integration where the momenta of the internal gravitons become parallel to that of the external graviton, but these divergences cancel in the sum over such graphs after using momentum conservation. Therefore we always work with sum of these diagrams. The net contribution from these diagrams receive logarithmic contribution from two regions – one where the loop momentum is large compared to ω and the other where the loop momentum is small compared to ω. We shall analyze the contribution from the region of small loop momentum later. Contribution from the region where the loop momentum is large compared to ω may be approximated as

$$\begin{aligned}
&-(i\lambda)\,\frac{i}{4}\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n}\int_{\rm reg}\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{[p_a.\ell-i\epsilon]\,[p_b.\ell+i\epsilon]\,[\ell^2-i\epsilon]}\\
&\left[-8\,(p_a.\varepsilon.p_b)\,(p_a.p_b)\;+\;2\,(p_a.\varepsilon.p_a)\,p_b^2\;+\;2\,(p_b.\varepsilon.p_b)\,p_a^2-2\,\{2(p_a.p_b)^2-p_a^2p_b^2\}\,\frac{\ell.\varepsilon.\ell}{\ell^2-i\epsilon}\right]\\
&-(i\lambda)\,\frac{i}{2}\sum_{a=1}^{n}\int_{\rm reg}\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{[p_a.\ell-i\epsilon]^2}\,\frac{1}{[\ell^2-i\epsilon]}\left[-2\,p_a^2\,(p_a.\varepsilon.p_a)\;+\;\frac{(p_a^2)^2}{p_a.\ell-i\epsilon}\,(p_a.\varepsilon.\ell)\right].
\end{aligned} \tag{3.3.18}$$

In arriving at this result we have used integration by parts and also conservation of total momentum $\sum_{a=1}^{n} p_a = 0$. We have also used the fact that in the expression for the graviton propagator carrying momentum $(k-\ell)$ in the second diagram of Fig .3.11, we can use the identity

$$\frac{1}{(k-\ell)^2 - i\epsilon} = \frac{2\ell.k}{\{(k-\ell)^2 - i\epsilon\}\{\ell^2 - i\epsilon\}} + \frac{1}{\ell^2 - i\epsilon}, \tag{3.3.19}$$

and ignore the contribution from the $(\ell^2 - i\epsilon)^{-1}$ term, since the expression for the amplitude involving this term has no k-dependent denominator and therefore cannot have a $\ln\omega^{-1}$ term.[6] Similar manipulations will be used in other terms as well.

Contribution to $\Gamma^{(n,1)}_{\text{self}}$ given in Fig. 3.10 may be analyzed following the argument given below (3.2.16). We assume a general form $\varepsilon_{\mu\nu} p_a^\mu p_a^\nu f(p_a.k)$ for this amplitude based on Lorentz invariance and replace $\varepsilon_{\mu\nu}$ by $\xi_\mu k_\nu + \xi_\nu k_\mu$ for an arbitrary vector ξ satisfying $k.\xi = 0$. Then the amplitude reduces to $2\, p_a.\xi\, p_a.k\, f(p_a.k)$. On the other hand the diagrams in Fig. 3.10 for this choice of polarization may be evaluated using the Ward identity given in Fig. 3.15. Due to the presence of the non-vanishing right-hand side in Fig. 3.15, the result does not vanish. Comparing this with the expected result $2\, p_a.\xi\, p_a.k\, f(p_a.k)$, we can compute $f(p_a.k)$ and hence $\Gamma^{(n,1)}_{\text{self}}$. It turns out that it receives logarithmic contribution from region of integration where the loop momentum is large compared to ω. The result is:

$$\Gamma^{(n,1)}_{\text{self}} = -(i\lambda)\,\frac{i}{2}\sum_{a=1}^{n} p_a^2\, p_a.\varepsilon.p_a \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{[p_a.\ell - i\epsilon]^2\,[\ell^2 - i\epsilon]}. \tag{3.3.20}$$

This cancels the term in the last line of (3.3.18).

One loop contribution from the diagrams involving G-gravitons in Figs. 3.9 and 3.12 may be evaluated following the procedure described in §3.2. We find that the G-graviton contribution to Fig. 3.12 has no logarithmic contribution. Therefore we are left with the G-graviton contributions to Fig. 3.9. These diagrams have the same structure as in scalar QED and can be evaluated similarly. As in the case of scalar QED, these diagrams receive

[6] This manipulation can be carried out only for terms containing at least two powers of ℓ in the numerator so that each of the terms in (3.3.19) generates infrared finite integral.

significant contribution only from the region where the loop momentum is large compared to ω and small compared to the momenta of finite energy particles. The net logarithmic contributions from these diagrams is given by

$$\begin{aligned}
\Gamma_G^{(n,1)} &= -(i\lambda)\,\frac{i}{2}\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n}\int\frac{d^4l}{(2\pi)^4}\,\frac{1}{[p_a.l-i\epsilon]\,[p_b.l+i\epsilon]\,[l^2-i\epsilon]}\\
&\qquad\left[8(p_a.p_b)\,(p_a.\varepsilon.p_b)\;-\;2p_b^2\,(p_a.\varepsilon.p_a)-\;[2(p_a.p_b)^2-p_a^2p_b^2]\left(\frac{p_a.\varepsilon.p_a}{p_a^2}\;+\;2\,\frac{p_a.\varepsilon.l}{p_a.l}\right)\right]\\
&\qquad+(i\lambda)\,\frac{i}{2}\left(\frac{p_a.\varepsilon.p_a}{p_a.k}\right)\int\frac{d^4l}{(2\pi)^4}\,\frac{1}{[p_a.l-i\epsilon]\,[p_b.l+i\epsilon]\,[l^2-i\epsilon]}\\
&\qquad\qquad\left[4(p_a.p_b)\,(p_b.k)-[2(p_a.p_b)^2-p_a^2p_b^2]\,\frac{k.l}{p_a.l}\right].
\end{aligned}\tag{3.3.21}$$

The total logarithmic terms in $\Gamma_{\rm G}^{(n,1)}$, $\Gamma_{\rm self}^{(n,1)}$, $\Gamma_{\rm residual}^{(n,1)}$ and $\Gamma_{\rm 3-graviton}^{(n,1)}$ from the region of integration where the loop momentum is large compared to ω, can be expressed as[7]

$$(i\lambda)\,\widehat{S}_{\rm gr}^{(1)}\,K_{\rm gr}^{\rm reg}\,,\tag{3.3.22}$$

where $\widehat{S}_{\rm gr}^{(1)}$ is the quantum subleading soft graviton operator

$$\widehat{S}_{\rm gr}^{(1)}=\sum_a\frac{\varepsilon_{\mu\rho}p_a^\rho k_\nu}{p_a.k}\left\{p_a^\mu\frac{\partial}{\partial p_{a\nu}}-p_a^\nu\frac{\partial}{\partial p_{a\mu}}\right\},\tag{3.3.23}$$

and

$$K_{\rm gr}^{\rm reg}\equiv\frac{i}{2}\sum_{\substack{a,b\\ b\neq a}}\left\{(p_a.p_b)^2-\frac{1}{2}p_a^2p_b^2\right\}\int_{\rm reg}\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2-i\epsilon}\,\frac{1}{(p_a.\ell-i\epsilon)\,(p_b.\ell+i\epsilon)}\,.\tag{3.3.24}$$

$K_{\rm gr}^{\rm reg}$ is the analog of $K_{\rm em}^{\rm reg}$ for gravitational scattering, namely it is the factor that appears in the exponent of the soft factor in the scattering of n scalars, with the understanding that the integration over loop momentum is restricted to the region larger than ω. We

[7] It is natural to conjecture that this pattern continues to hold also for subsubleading soft graviton theorem, i.e. the universal part of the subsubleading contribution is given by the action of the subsubleading soft graviton operator $\widehat{S}_{\rm gr}^{(2)}$ acting on $\exp[K_{\rm gr}^{\rm reg}]$. But we have not verified this by explicit computation.

note however that the full expression for $K_{\rm gr}$ has more terms – (3.3.24) already involves an approximation that the loop momentum is small compared to the energies of external lines since this is the region that generates $\ln\omega^{-1}$ terms. Explicit evaluation gives the following expression for the terms involving $\ln\omega^{-1}$:

$$K_{\rm gr}^{\rm reg} = \frac{i}{2}\sum_{\substack{a,b\\ b\neq a}} \frac{1}{4\pi}\ \ln\omega^{-1}\ \frac{\left\{(p_a.p_b)^2 - \frac{1}{2}p_a^2 p_b^2\right\}}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \left\{\delta_{\eta_a\eta_b,1} - \frac{i}{2\pi}\ln\left(\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}\right)\right\}. \tag{3.3.25}$$

At this stage the only remaining terms are the contributions to $\Gamma^{(n,1)}_{3-\text{graviton}}$ from regions of loop momentum integration where the loop momentum is small compared to ω. These come from the first two diagrams in Fig. 3.11. In the first diagram there are two relevant regions: when ℓ is small and when $k-\ell$ is small, but they are related to each other by $\ell \to k-\ell$ and $a \leftrightarrow b$ symmetry. In the second diagram the relevant region is when ℓ is small. The net contribution from these regions may be approximated by

$$\lambda \sum_{a=1}^{n}\sum_{b=1}^{n}\int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{[2k.\ell - \ell^2 + i\epsilon]\,[p_a.\ell - i\epsilon]\,[\ell^2 - i\epsilon]}\left[2(p_a.\varepsilon.p_b)\,(p_a.k) - 2(p_b.\varepsilon.p_b)\,\frac{(p_a.k)^2}{p_b.k}\right], \tag{3.3.26}$$

with the understanding that the integration over ℓ runs in the region where the components of ℓ are small compared to ω. The result may be expressed as

$$i\lambda\,(\ln\omega^{-1} + \ln R^{-1})\left[\frac{i}{4\pi}\sum_{\substack{b\\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k} + \frac{1}{8\pi^2}\sum_a \frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}\sum_b p_b.k\,\ln\frac{m_b^2}{(p_b.\hat{k})^2}\right], \tag{3.3.27}$$

where $1/R$ is an infrared lower cut-off on momentum integration and $\hat{k} = -k/\omega = (1,\hat{n})$.

Adding (3.3.22) to (3.3.27) and dividing by $i\lambda$ we get the terms involving $\ln\omega^{-1}$ and $\ln R$ in $S^{(1)}_{\rm gr}$:

$$S^{(1)}_{\rm gr} \;=\; \widehat{S}^{(1)}_{gr}\, K^{\rm reg}_{\rm gr}$$

$$+\frac{1}{4\pi}(\ln\omega^{-1}+\ln R^{-1})\left[i\sum_{\substack{b\\ \eta_b=-1}}k.p_b\sum_a\frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}+\frac{1}{2\pi}\sum_a\frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}\sum_b p_b.k\,\ln\frac{m_b^2}{(p_b.\hat{k})^2}\right]. \tag{3.3.28}$$

3.4 Generalizations

In this section we shall consider the case where the scalars interact via both electromagnetic and gravitational interaction via the action:

$$\int d^4x\,\sqrt{-\det g}\bigg[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{16\pi G}R-\sum_{a=1}^{n}\Big\{g^{\mu\nu}(\partial_\mu\phi_a^*+iq_aA_\mu\phi_a^*)(\partial_\nu\phi_a-iq_aA_\nu\phi_a)$$
$$+m_a^2\phi_a^*\phi_a\Big\}+\lambda\,\phi_1\cdots\phi_n+\lambda\,\phi_1^*\cdots\phi_n^*\bigg]. \tag{3.4.1}$$

For this analysis we need two new vertices, the graviton-photon-photon vertex and the graviton-photon-scalar-scalar vertex. If the graviton carries an ingoing momentum q and Lorentz index $(\rho\sigma)$, and the two photons carry ingoing momenta k_1 and k_2 and Lorentz indices μ and ν respectively, then the graviton-photon-photon vertex is given by:

$$-i\kappa\Big[\eta_{\rho\sigma}\Big(-k_1.k_2\,\eta_{\mu\nu}+\,k_{1\nu}k_{2\mu}\Big)+\,\eta_{\mu\nu}\Big(k_{1\rho}k_{2\sigma}\,+\,k_{2\rho}k_{1\sigma}\Big)\,+\,k_1.k_2\Big(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}\Big)$$
$$-\Big(k_{1\sigma}k_{2\mu}\eta_{\rho\nu}+k_{2\sigma}k_{1\nu}\eta_{\rho\mu}+k_{1\rho}k_{2\mu}\eta_{\sigma\nu}+k_{2\rho}k_{1\nu}\eta_{\sigma\mu}\Big)\Big]. \tag{3.4.2}$$

On the other hand the vertex with a pair of scalars carrying charges q, $-q$ and momenta p_1 and p_2, a graviton carrying Lorentz indices $(\mu\nu)$ and momentum k_1 and a photon carrying Lorentz index ρ and momentum k_2, all counted ingoing, is given by

$$-i\kappa\,q\Big[\eta_{\mu\rho}(p_1-p_2)_\nu+\eta_{\nu\rho}(p_1-p_2)_\mu-\eta_{\mu\nu}(p_1-p_2)_\rho\Big]. \tag{3.4.3}$$

In this theory we shall analyze the extra terms in both the soft graviton theorem and the soft photon theorem.

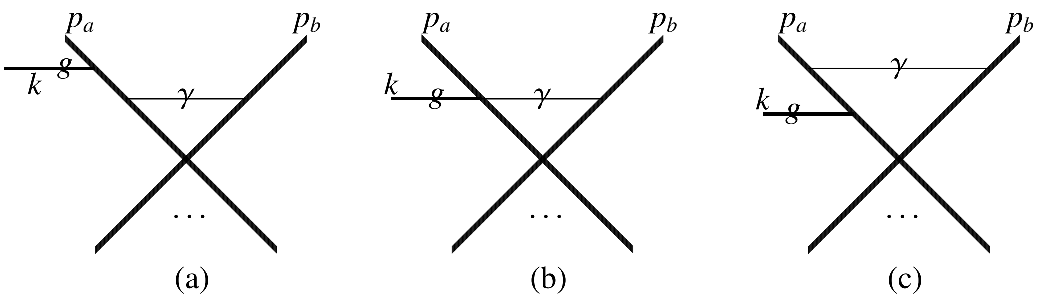

Figure 3.17: Contribution to soft graviton amplitude due to internal photon whose two ends are connected to two different scalar lines. Here the thickest lines denote scalars, lines of medium thickness carrying the symbol g denote gravitons and the thin lines carrying the symbol γ denote photons.

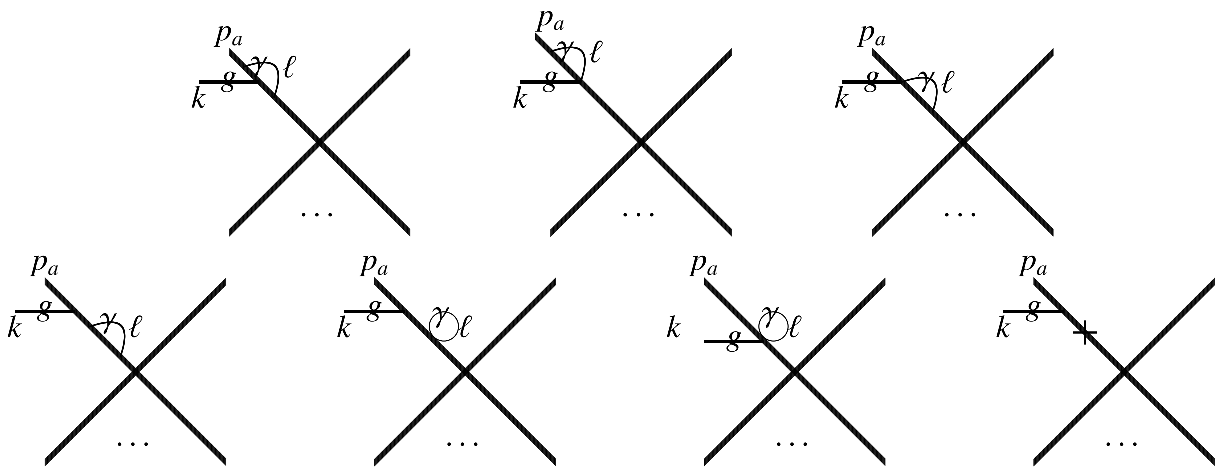

Figure 3.18: One loop contribution to soft graviton amplitude involving internal photon line connecting two points on the same leg.

There are two other vertices that are needed for our analysis. For example the sixth diagram of Fig. 3.18 needs the vertex containing two scalars, two photons and one graviton, whereas the sixth diagram of Fig. 3.23 requires the two scalar, two graviton and one photon vertex. However even without knowing the form of these vertices one can see that these diagrams do not generate contributions proportional to $\ln \omega^{-1}$. Therefore we have not written down the expressions for these vertices.

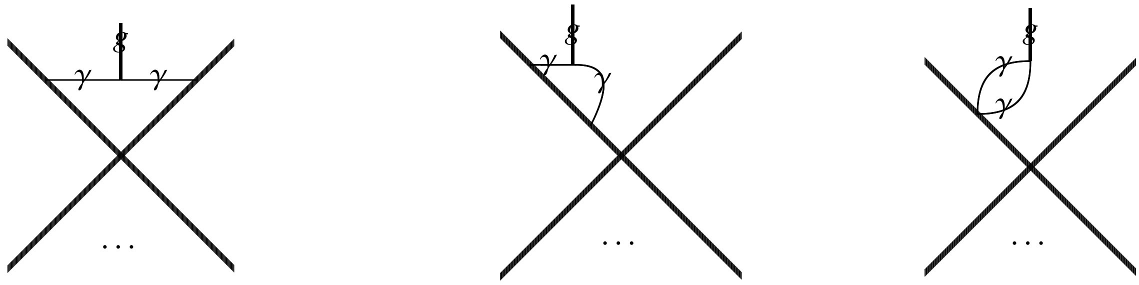

Figure 3.19: Diagrams containing graviton-photon-photon vertex that contribute to the soft photon contribution to the soft graviton theorem.

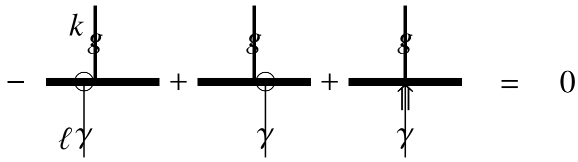

Figure 3.20: The Ward identity for the photon in the presence of a graviton-scalar-scalar vertex.

3.4.1 Soft graviton theorem

We first consider the soft graviton theorem. In this case besides the contributions analyzed in §3.3, we also have the diagrams of Fig. 3.17 and Fig. 3.18, obtained by replacing, in the diagrams in §3.2, the external photon by a graviton but keeping the internal line as a photon. We also have an additional set of diagrams shown in Fig. 3.19 where the external graviton connects to the internal photon. Diagrams in which the external graviton attaches to the n-scalar vertex vanish for $\varepsilon_\rho^{\ \rho} = 0$ and have not been displayed. We carry out Grammer-Yennie decomposition for the internal photons in Fig. 3.17 following (3.2.3), but not for diagrams of the form shown in Fig. 3.18 and Fig. 3.19. The sum over K-photons factorize as in §3.2 and gives the factor of $\exp[K_{\rm em}]$ that cancels between $\Gamma^{(n)}$ and $\Gamma^{(n,1)}$. In this case there is no residual contribution since the analog of Fig. 3.4 holds with the upper photon in the second identity replaced by a graviton (see Fig. 3.20). This leads to the analog of (3.2.11) with an additional contribution to the left hand side given by diagrams of the form shown in Fig. 3.19. Denoting this contribution by $\Gamma^{(n,1)}_{\gamma\gamma g}$ we arrive at the relation

$$\Gamma^{(n,1)}_{\rm self} + \Gamma^{(n,1)}_{\rm G} + \Gamma^{(n,1)}_{\gamma\gamma g} = i\,\lambda\, S^{(1)}_{\rm gr}\,, \tag{3.4.4}$$

with the understanding that both sides represent contributions in addition to what already appear in (3.3.16). None of the terms have any infrared divergence, and therefore there are no logarithmic terms from the region of integration in which the loop momentum is small compared to ω. We shall describe below the organization of the various terms and then state the final result:

1. One can analyze $\Gamma^{(n,1)}_{\rm self}$ represented by the graphs in Fig. 3.18 by following the procedure described below (3.2.16). We replace the external graviton polarization by a pure gauge form $(\xi_\mu k_\nu + \xi_\nu k_\mu)$ and apply Ward identity to evaluate the sum over the graphs in Fig. 3.18. In this case the Ward identity has an additional contribution as shown in the right hand side of the second diagram in Fig. 3.21. It turns out however that its contribution to the amplitude does not have any logarithmic term. Therefore $\Gamma^{(n,1)}_{\rm self}$ does not generate any logarithmic contribution.

2. $\Gamma^{(n,1)}_{\gamma\gamma g}$ receives contribution proportional to $\ln\omega^{-1}$ from the first two diagrams of Fig. 3.19, from the region where the loop momentum is large compared to ω.

3. Finally, the G-photon contribution $\Gamma^{(n,1)}_{\rm G}$ from the first two diagrams in Fig. 3.17 also has terms proportional to $\ln\omega^{-1}$ from the region where the loop momentum is large compared to ω.

The net logarithmic contribution from $\Gamma^{(n,1)}_{\gamma\gamma g}$ and $\Gamma^{(n,1)}_{\rm G}$ is given by:

$$(i\lambda)\,\widehat{S}^{(1)}_{\rm gr}\;K^{\rm reg}_{\rm em}\,. \tag{3.4.5}$$

After removing the $i\,\lambda$ factor, we have to add this to (3.3.28) to get the total logarithmic

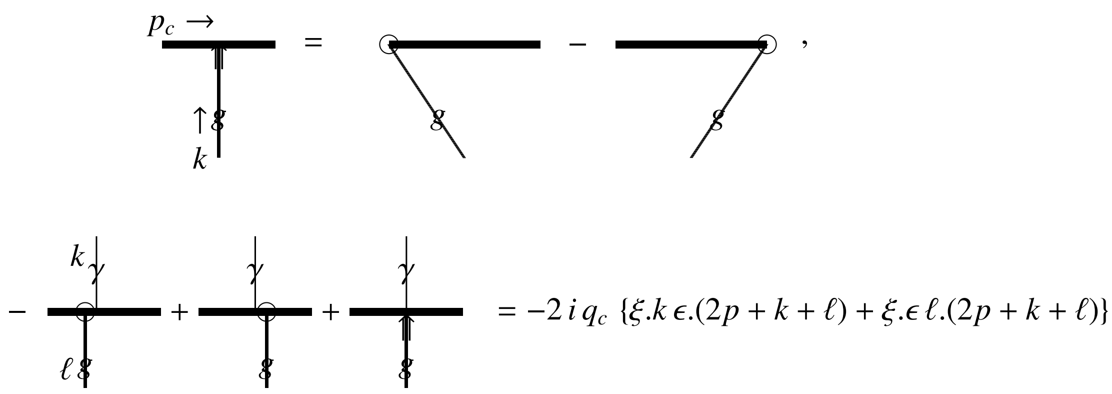

Figure 3.21: Analog of Fig. 3.4 for graviton in the presence of a photon. The graviton carries a polarization $(\xi_\mu \ell_\nu + \xi_\nu \ell_\mu)$ and the photon carries a polarization ϵ. The circled vertex has been explained in the caption of Fig. 3.15.

contribution to $S^{(1)}_{\rm gr}$:

$$
\begin{aligned}
S^{(1)}_{\rm gr} \;&=\; \widehat{S}^{(1)}_{gr}\left(K^{\rm reg}_{\rm em} + K^{\rm reg}_{\rm gr}\right) \\
&+\frac{1}{4\pi}(\ln\omega^{-1} + \ln R^{-1})\left[i \sum_{\substack{b\\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + \frac{1}{2\pi}\sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \sum_b p_b.k \,\ln\frac{m_b^2}{(p_b.\hat{k})^2}\right].
\end{aligned}
\tag{3.4.6}
$$

This reproduces terms proportional to $\ln\omega^{-1}$ in the sum of (3.6.6) and (3.6.7) after using (3.2.28) and (3.3.25).

3.4.2 Soft photon theorem

Next we shall consider the soft photon theorem. In this case we have all the diagrams considered in §3.2, but also extra diagrams where the internal photon of Figs. 3.1 and 3.2 is replaced by an internal graviton, as shown in Figs. 3.22 and 3.23, and two additional sets of diagrams: one where one end of the internal graviton connects to the external photon as in Fig. 3.24 and the other where one end of the internal graviton is attached to the n-scalar vertex as in Fig. 3.25. There is also an additional diagram obtained by

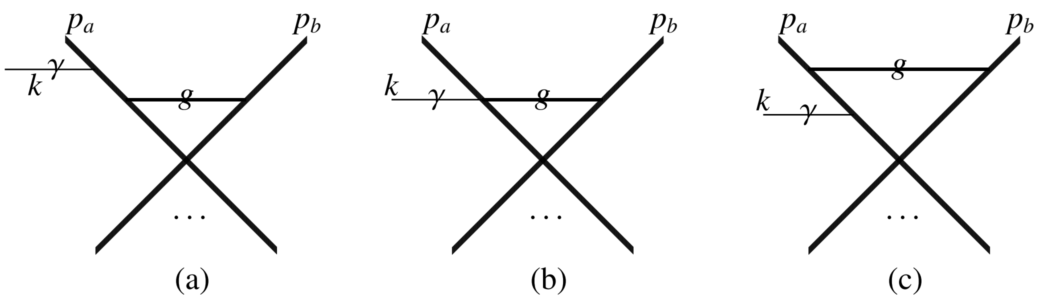

Figure 3.22: One loop contribution to soft photon amplitude involving internal graviton line connecting two different legs.

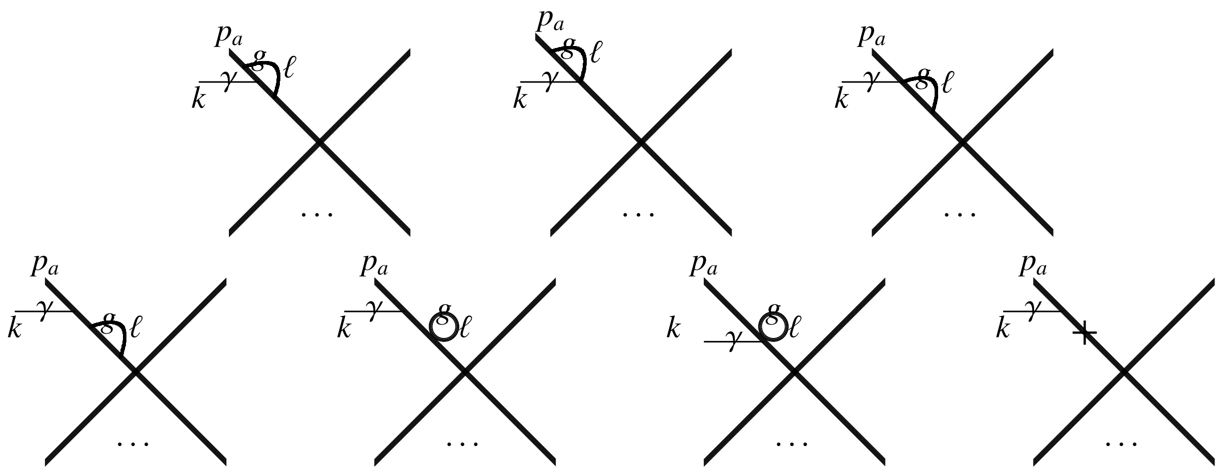

Figure 3.23: Diagrams in which the external photon and both ends of the internal graviton attach to the same scalar leg.

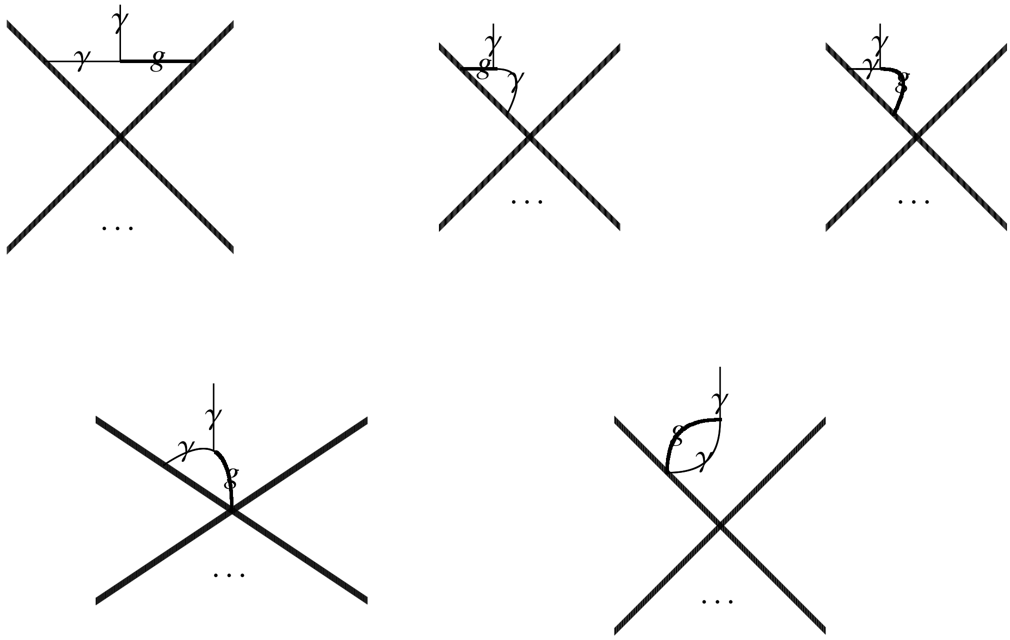

Figure 3.24: Diagrams involving graviton-photon-photon vertex that need to be included in computing the soft graviton contribution to the soft photon theorem.

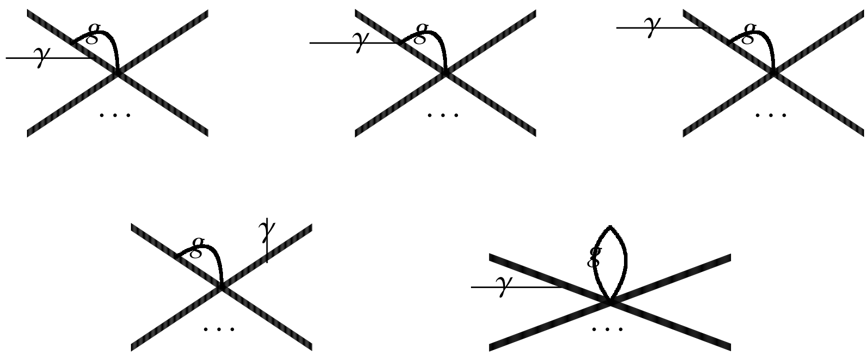

Figure 3.25: Diagrams with external soft photon and an internal graviton where the internal graviton attaches to the n-point vertex.

replacing in the first diagram of Fig. 3.14 the external graviton by the external photon, but this vanishes in dimensional regularization.

We shall analyze the diagrams in Figs. 3.22 and 3.25 using Grammer-Yennie decomposition for the internal graviton following the rules described in (3.3.9)-(3.3.14). The result of summing over K-gravitons in $\Gamma^{(n,1)}$ will generate the factor of $\exp[K_{\text{gr}} + \widetilde{K}_{\text{gr}}]$ which cancels a similar factor in the expression of $\Gamma^{(n)}$. However there will be residual part that will be left over due to non-cancellation of the sum over K-graviton insertions reflected in the right-hand side of Fig. 3.21. Another residual contribution arises due to the momentum dependence of the circled vertices; as illustrated in Fig, 3.16, the factorized contribution of K-gravitons for $\Gamma^{(n)}$ and $\Gamma^{(n,1)}$ differ. The only difference in the present case is that the external graviton carrying momentum k in Fig. 3.16 is replaced by an external photon. As in §3.3, we shall denote these residual contributions in the sum over K-gravitons by $\Gamma^{(n,1)}_{\text{residual}}$. The G-graviton contribution to Figs. 3.22 and 3.25 will be denoted by $\Gamma^{(n,1)}_{\text{G}}$. The contribution from diagrams involving the coupling of graviton to photon, as shown in Fig. 3.24, will be denoted by $\Gamma^{(n,1)}_{\gamma\gamma g}$, and the contributions from Fig. 3.23 will be denoted by $\Gamma^{(n,1)}_{\text{self}}$. Then the generalization of (3.2.11) takes the form:

$$\Gamma^{(n,1)}_{\text{self}} + \Gamma^{(n,1)}_{\text{G}} + \Gamma^{(n,1)}_{\gamma\gamma g} + \Gamma^{(n,1)}_{\text{residual}} = i\,\lambda\, S^{(1)}_{\text{gr}}\,, \tag{3.4.7}$$

again with the understanding that both sides represent additional contribution besides those described in §3.2.

Analysis of various terms on the left hand side of (3.4.7) goes as follows:

1. $\Gamma^{(n,1)}_{\text{self}}$ can be shown to vanish using the same argument given below (3.2.16). In this case the relevant Ward identities given in Figs. 3.4 and 3.20 do not have any left-over extra contributions.

2. It turns out that $\Gamma^{(n,1)}_{\text{residual}}$, given by the left-over contribution after summing over K-graviton insertions in Figs. 3.22 and 3.25, does not receive any logarithmic terms

either from the region of loop momentum integration small compared to ω or from regions of loop momentum integration large compared to ω.

3. $\Gamma_{\rm G}^{(n,1)}$ receives contributions proportional to $\ln\omega^{-1}$ only from the G-graviton contribution to Fig. 3.22, from region of integration where the loop momentum is larger than ω.

4. The individual diagrams contributing to $\Gamma_{\gamma\gamma g}^{(n,1)}$ have collinear divergence from the region where the momenta of the internal graviton and photon are parallel to the momentum of the external photon. This cancels in the sum over all diagrams in Fig. 3.24. The second and third diagrams of Fig. 3.24 each has contribution proportional to $\ln\omega^{-1}$ from the region of integration where the loop momentum is large compared to ω, but the sum of these contributions vanishes. Finally, $\Gamma_{\gamma\gamma g}^{(n,1)}$ receives contributions proportional to $\ln\omega^{-1}$ from the first two diagrams in Fig. 3.24, from the region where the momentum of the internal graviton is smaller than ω.

The net logarithmic contribution from the region of integration where the loop momentum is larger than ω is given by

$$i\,\lambda\,\widehat{S}^{(1)}_{\rm em}\,K_{\rm gr}^{\rm reg}\,. \tag{3.4.8}$$

On the other hand the contribution to $\Gamma_{\gamma\gamma g}^{(n,1)}$ from the small loop momentum region is given by:

$$i\,\lambda\,(\ln\omega^{-1}+\ln R^{-1})\left[\frac{i}{4\pi}\sum_{\substack{b\\ \eta_b=-1}} k.p_b\sum_a\frac{\varepsilon_\mu p_a^\mu}{p_a.k}\,q_a+\frac{1}{8\pi^2}\sum_{a=1}^n\frac{q_a\varepsilon_\mu p_a^\mu}{p_a.k}\sum_{b=1}^n(p_b.k)\,\ln\left(\frac{-p_b^2}{(p_b.\hat{k})^2}\right)\right]. \tag{3.4.9}$$

One difference from the previous diagrams of this type, *e.g.* the ones shown in Fig. 3.11, is that the divergent contribution comes only from the region where the internal graviton momentum becomes small, and not when the internal photon momentum becomes small. This reflects the fact that while photons feel the long range gravitational force due to other particles, the graviton, being charge neutral, does not feel any long range Coulomb force.

After removing the $i\lambda$ factors from (3.4.8) and (3.4.9), we have to add them to (3.2.20) to get the total soft factor $S^{(1)}_{\rm em}$. This gives

$$
\begin{aligned}
S^{(1)}_{\rm em} &= \widehat{S}^{(1)}_{\rm em}\left(K^{\rm reg}_{\rm em}+K^{\rm reg}_{\rm gr}\right)\\
+(\ln\omega^{-1}+\ln R^{-1})&\left[\frac{i}{4\pi}\sum_{\substack{b\\ \eta_b=-1}} k.p_b\sum_a\frac{\varepsilon_\mu p_a^\mu}{p_a.k}q_a+\frac{1}{8\pi^2}\sum_{a=1}^n\frac{q_a\varepsilon_\mu p_a^\mu}{p_a.k}\sum_{b=1}^n(p_b.k)\,\ln\left(\frac{-p_b^2}{(p_b.\hat{k})^2}\right)\right].
\end{aligned}
\tag{3.4.10}
$$

This reproduces terms proportional to $\ln\omega^{-1}$ in the sum of (3.6.4) and (3.6.5) after using the explicit forms of $K^{\rm reg}_{\rm em}$ and $K^{\rm reg}_{\rm gr}$ given in (3.2.28) and (3.3.25).

3.5 Steps to derive subleading multiple soft photon theorem

In this section summarising our learnings from §3.2 we will derive double soft photon theorem. Then we will see that we can very easily extend our analysis to multiple soft photon case and directly write down the result of multiple soft photon theorem up to subleading order. Sum over the tree and one loop diagrams with two external photons with momenta k_1 and k_2 will be denoted by $\Gamma^{(n,2)}$ and $\Gamma^{(n)}$ will denote the amplitude without any soft photon up to one loop order. We will consider the momenta k_1 and k_2 are softer in the same rate as denoted by $k_1^\mu=\omega n_1^\mu$ and $k_2^\mu=\tau\omega n_2^\mu$, where τ is an $O(1)$ dimensionless parameter and n_1^μ, n_2^μ are unit null vectors.

- The KG decomposition performed in eq.(3.2.3) also works for diagrams containing two external photons attached with a scalar line with two three point vertices and analysis directly follows from Fig.3.4. One can also verify that the same KG decomposition works when both the external photons are attached to a scalar line via two scalar, two photon vertex. This is diagrammatically shown in Fig.3.26.

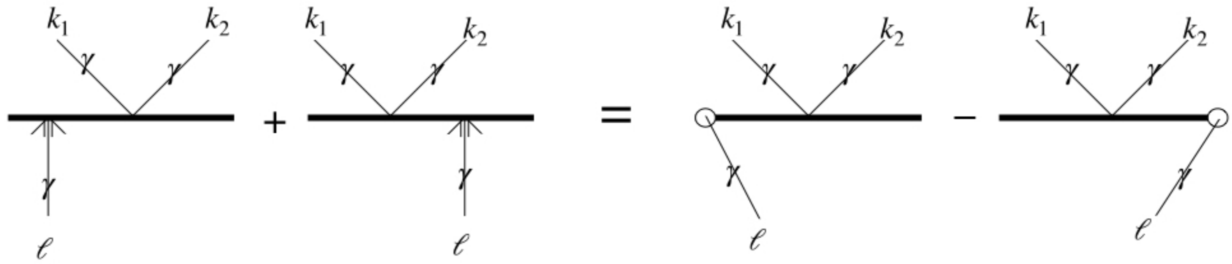

Figure 3.26: The diagram shows that even in presence of the vertex with two external photon connecting to a scalar line K-G decomposition works similar to Figure(3.4). Here the photon line with arrow represents K-photon and the vertex with circle is the same as described in Figure(3.4).

Therefore we can write,

$$\Gamma^{(n)} = \exp[K_{em}]\left\{\Gamma^{(n)}_{tree} + \Gamma^{(n)}_{G}\right\}, \qquad \Gamma^{(n,2)} = \exp[K_{em}]\left\{\Gamma^{(n,2)}_{tree} + \Gamma^{(n,2)}_{G} + \Gamma^{(n,2)}_{self}\right\} \tag{3.5.11}$$

where $\Gamma^{(n)}_{tree}$ represents n-scalar tree level amplitude, $\Gamma^{(n)}_{G}$ represents one loop n-scalar amplitude with virtual G-photon, $\Gamma^{(n,2)}_{tree}$ represents n-scalar, two photon tree-level amplitude, $\Gamma^{(n,2)}_{G}$ represents n-scalar, two photon one loop amplitude with virtual G-photon and $\Gamma^{(n,2)}_{self}$ represents all set of one loop diagrams where both ends of the virtual photon are connected to same scalar line.

- The diagrams contributing to $\Gamma^{(n,2)}_{self}$ have to be computed with full photon propagator. But following the logic of §3.2 they can not contribute to the subleading order[8].

- In §3.2 we have seen that contribution from the diagrams with G-photon propagator connecting scalars with momenta p_a and p_b vanishes. Which implies one loop n-scalar amplitude $\Gamma^{(n)}_{G}$ vanishes and the contribution from the diagrams in Fig.3.27 also vanishes.

[8]In double soft photon theorem subleading order is of order $\frac{ln\omega}{\omega}$.

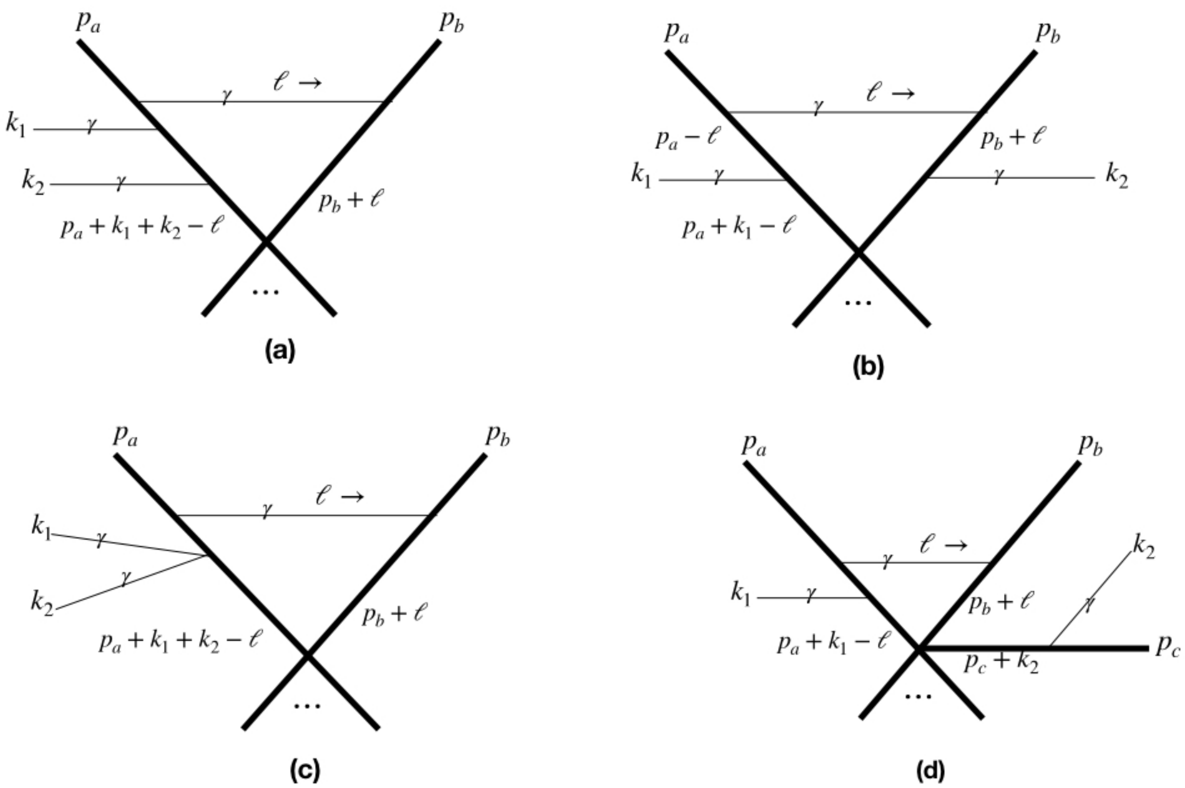

Figure 3.27: One loop contribution to $\Gamma_G^{(n,2)}$ involving internal G-photon line connecting two different legs and for diagram-(a),(b),(c): both the external photons are inside the virtual photon propagator and for diagram-(d): one photon is inside the virtual photon propagator and the other is connected to some different leg. The thick lines represent scalar particles and the thin lines carrying symbol γ represent photons. There are other diagrams related to this by permutaions of scalar lines and exchange of $k_1 \leftrightarrow k_2$.

- Now for tree amplitude leading double soft theorem takes form,

$$\Gamma_{tree}^{(n,2)} = (i\lambda)\left(\sum_{a=1}^{n} q_a \frac{\varepsilon_1 \cdot p_a}{p_a.k_1}\right)\left(\sum_{b=1}^{n} q_b \frac{\varepsilon_2 \cdot p_b}{p_b.k_2}\right). \tag{3.5.12}$$

Substituting this result in eq.(3.5.11) and following the arguments above the algorithm for determining subleading double soft photon factor $S^{(1)}_{double}$,

$$\Gamma_G^{(n,2)} = i\lambda\, S^{(1)}_{double} \tag{3.5.13}$$

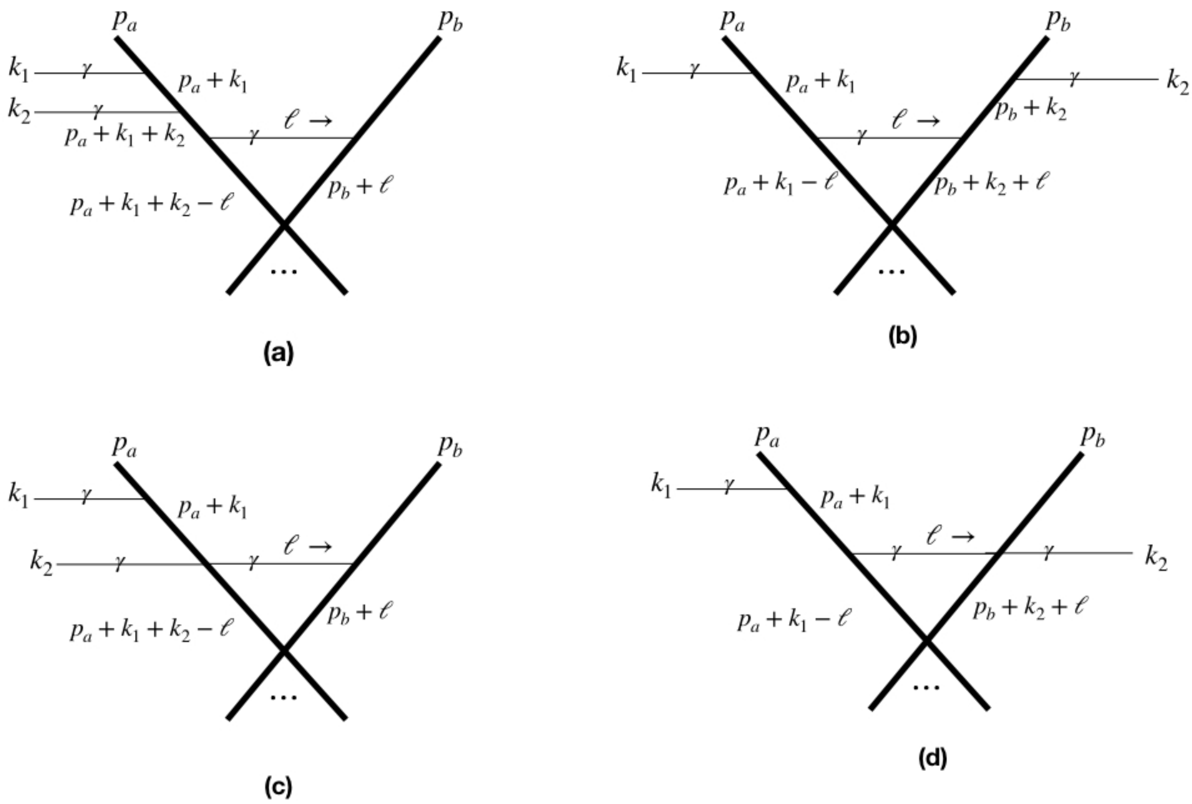

Figure 3.28: One loop contribution to $\Gamma_G^{(n,2)}$ involving internal G-photon line connecting two different legs and no external photon is inside the virtual photon propagator. The thick lines represent scalar particles and the thin lines carrying symbol γ represent photons. There are other diagrams related to this by permutaions of scalar lines and exchange of $k_1 \leftrightarrow k_2$. For diagram-(b) one don't need to include diagrams with $k_1 \leftrightarrow k_2$ exchange.

where in $\Gamma_G^{(n,2)}$ we only need to analyse the contribution from diagrams in Fig.3.28,3.29,3.30,3.31 and extract order $\frac{\ln\omega}{\omega}$ contribution.

All the diagrams in Fig.3.28 with G-photon propagator are IR finite but contributes to $\mathcal{O}(\omega^{-1}\ln\omega)$ order in the integration region $|\omega| << |\ell^\mu| << |p_a^\mu|, |p_b^\mu|$. Contribution from Fig.3.28(a) in this regulated region of integration,

$$
\begin{aligned}
(Ia) \quad \simeq \quad & -\lambda q_a^3 q_b \frac{\varepsilon_1.p_a}{p_a.k_1} \frac{\varepsilon_2.p_a}{p_a.(k_1+k_2)} \int_{reg} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{1}{p_a.\ell + i\epsilon} \frac{1}{p_b.\ell - i\epsilon} \\
& \left[p_b.(k_1+k_2) - \ell.(k_1+k_2) \frac{p_a.p_b}{p_a.\ell + i\epsilon} \right] \qquad (3.5.14)
\end{aligned}
$$

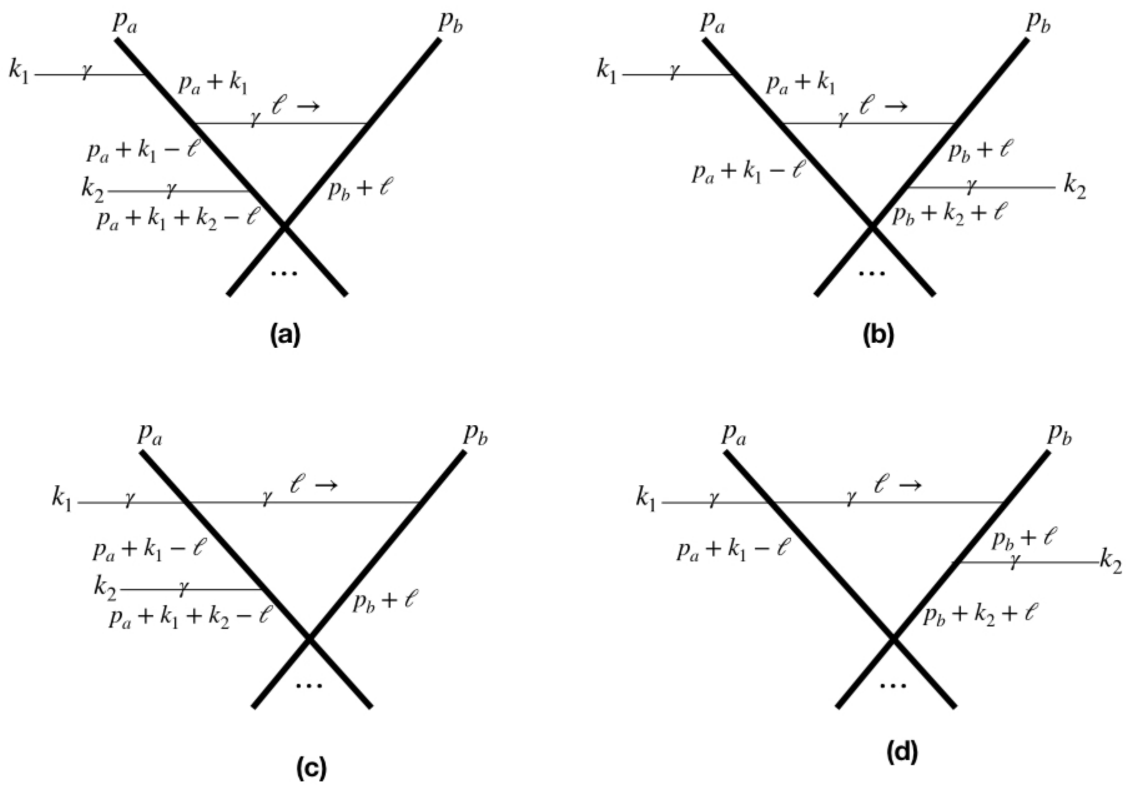

Figure 3.29: One loop contribution to $\Gamma_G^{(n,2)}$ involving internal G-photon line connecting two different legs and one external photon is inside the virtual photon propagator. The thick lines represent scalar particles and the thin lines carrying symbol γ represent photons. There are other diagrams related to this by permutaions of scalar lines and exchange of $k_1 \leftrightarrow k_2$.

Contribution from Fig.3.28(b) in the regulated region of integration,

$$\begin{aligned}(Ib) \quad &\simeq \quad -\lambda q_a^2 q_b^2 \frac{\varepsilon_1.p_a}{p_a.k_1}\frac{\varepsilon_2.p_b}{p_b.k_2}\int_{reg}\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2 - i\epsilon}\frac{1}{p_a.\ell + i\epsilon}\frac{1}{p_b.\ell - i\epsilon}\\ &\quad \left[p_b.k_1 + p_a.k_2 - p_a.p_b\frac{\ell.k_1}{p_a.\ell + i\epsilon} - p_a.p_b\frac{\ell.k_2}{p_b.\ell - i\epsilon}\right] \qquad (3.5.15)\end{aligned}$$

Contribution from Fig.3.28(c) in the regulated region of integration,

$$(Ic) \quad \simeq \quad \lambda q_a^2 q_b^2 \frac{\varepsilon_1.p_a}{p_a.k_1}\int_{reg}\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2 - i\epsilon}\frac{1}{p_a.\ell + i\epsilon}\frac{1}{p_b.\ell - i\epsilon}$$

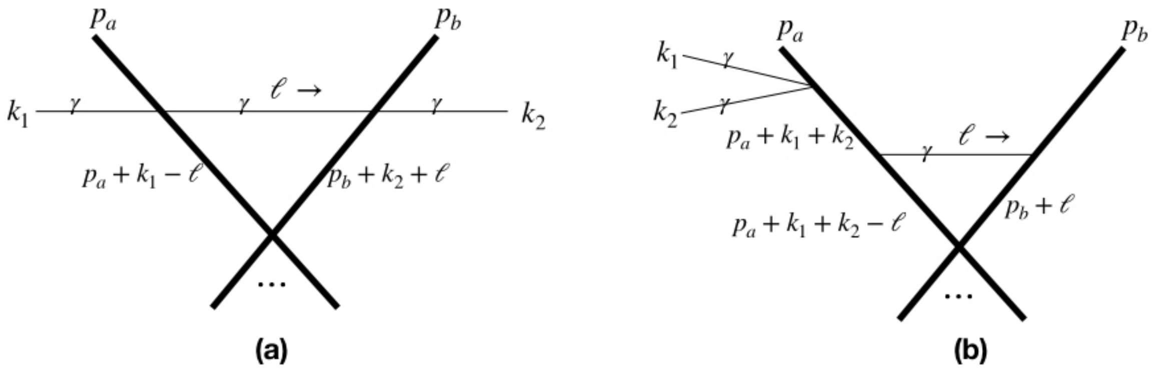

Figure 3.30: One loop contribution to $\Gamma_G^{(n,2)}$ involving internal G-photon line connecting two different legs and (a)both the external photons attached to the virtual photon propagator with two scalar-two photon vertices, (b)both the external photons connected to a scalar line by two scalar-two photon vertex outside the virtual G-photon propagator. There are other diagrams related to this by permutaions of scalar lines.

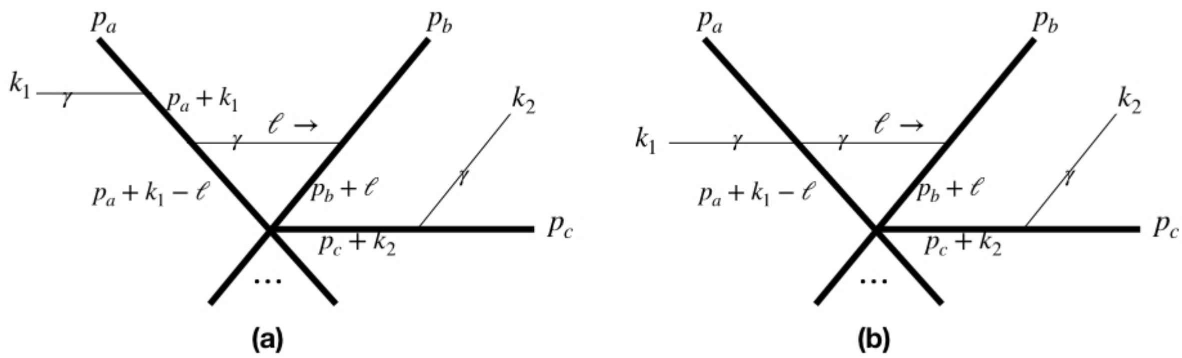

Figure 3.31: One loop contribution to $\Gamma_G^{(n,2)}$ involving internal G-photon line connecting two different legs and one of the external photon connecting to some different leg where the virtual photon propagator is not connected. The thick lines represent scalar particles and the thin lines carrying symbol γ represent photons. There are other diagrams related to this by permutaions of scalar lines and exchange of $k_1 \leftrightarrow k_2$.

$$\left[\varepsilon_2.p_b - \varepsilon_2.\ell \frac{p_a.p_b}{p_a.\ell + i\epsilon}\right] \tag{3.5.16}$$

Contribution from Fig.3.28(d) in the regulated region of integration,

$$(Id) \quad \simeq \quad \lambda q_a^2 q_b^2 \frac{\varepsilon_1.p_a}{p_a.k_1} \int_{reg} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{1}{p_a.\ell + i\epsilon} \frac{1}{p_b.\ell - i\epsilon}$$

$$\left[\varepsilon_2.p_a - \varepsilon_2.\ell \frac{p_a.p_b}{p_b.\ell - i\epsilon}\right] \tag{3.5.17}$$

Now summing over pair a, b for the contributions (Ia), (Ic) with permutations of $a \leftrightarrow b$ and $(\varepsilon_1, k_1) \leftrightarrow (\varepsilon_2, k_2)$, we get

$$\begin{aligned}
\mathcal{I} \quad = \quad & -\lambda \sum_{a=1}^{n} \sum_{\substack{b=1 \\ b\neq a}}^{n} \left[q_a \frac{\varepsilon_2.p_a}{p_a.k_2}\, q_a \frac{\varepsilon_{1\mu} k_{1\nu}}{p_a.k_1} \left(p_a^{\mu} \frac{\partial}{\partial p_{a\nu}} - p_a^{\nu} \frac{\partial}{\partial p_{a\mu}} \right) + q_a \frac{\varepsilon_1.p_a}{p_a.k_1}\, q_a \frac{\varepsilon_{2\mu} k_{2\nu}}{p_a.k_2} \left(p_a^{\mu} \frac{\partial}{\partial p_{a\nu}} - p_a^{\nu} \frac{\partial}{\partial p_{a\mu}} \right) \right] \\
& \times q_a q_b \int_{reg} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{1}{p_a.\ell + i\epsilon} \frac{1}{p_b.\ell - i\epsilon}\, p_a \cdot p_b
\end{aligned} \tag{3.5.18}$$

Summing over pair a, b for the contributions (Ib) with permutation of $a \leftrightarrow b$ and (Id) with permutations of $a \leftrightarrow b$ and $(\varepsilon_1, k_1) \leftrightarrow (\varepsilon_2, k_2)$, we get

$$\begin{aligned}
\mathcal{I}' \quad = \quad & -\lambda \sum_{a=1}^{n} \sum_{\substack{b=1 \\ b\neq a}}^{n} \left[q_b \frac{\varepsilon_2.p_b}{p_b.k_2}\, q_a \frac{\varepsilon_{1\mu} k_{1\nu}}{p_a.k_1} \left(p_a^{\mu} \frac{\partial}{\partial p_{a\nu}} - p_a^{\nu} \frac{\partial}{\partial p_{a\mu}} \right) + q_a \frac{\varepsilon_1.p_a}{p_a.k_1}\, q_b \frac{\varepsilon_{2\mu} k_{2\nu}}{p_b.k_2} \left(p_b^{\mu} \frac{\partial}{\partial p_{b\nu}} - p_b^{\nu} \frac{\partial}{\partial p_{b\mu}} \right) \right] \\
& \times q_a q_b \int_{reg} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{1}{p_a.\ell + i\epsilon} \frac{1}{p_b.\ell - i\epsilon}\, p_a \cdot p_b
\end{aligned} \tag{3.5.19}$$

In the regulated region of integration $|\omega| << |\ell^{\mu}| << |p_a^{\mu}|, |p_b^{\mu}|$, contributions from the diagrams in Fig.3.29 appears in order ω^{-1}. So for extracting order $\frac{\ln\omega}{\omega}$ contribution we can ignore the contributions from this diagrams. For example diagram-3.29(a) contribution in the regulated region of integration turns out:

$$\begin{aligned}
(IIa) \quad \simeq \quad & \lambda q_a^3 q_b\, (\varepsilon_2.p_a)\, \frac{\varepsilon_1.p_a}{p_a.k_1} \int_{reg} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{1}{(p_a.\ell + i\epsilon)^2} \frac{1}{p_b.\ell - i\epsilon} \\
& \times \left[p_b.k_1 \; - \; k_1.\ell \frac{p_a.p_b}{p_a.\ell + i\epsilon} \right] \\
\sim \quad & O(\omega^{-1})
\end{aligned}$$

Similarly we can ignore the contribution from the diagrams in Fig.3.30 as they contribute in order $O(\ln\omega)$ in the regulated region of integration. As argued above contribution from diagrams in Fig.3.27 vanishes when we evaluated them with G-photon propagator. The contribution from diagrams in Fig.3.31 can be easily read off from single soft photon theorem analysis following §3.2 and they contribute in $\frac{\ln\omega}{\omega}$ order. Contribution from diagrams-3.31(a) and 3.31(b) after summing over a, b, c with all possible permutations of (a, b, c) and $(\varepsilon_1, k_1) \leftrightarrow (\varepsilon_2, k_2)$ turns out:

$$\begin{aligned} \mathcal{I}'' \quad &\simeq \quad -\lambda \sum_{c=1}^{n} \sum_{\substack{a=1\\a\neq c}}^{n} \sum_{\substack{b=1\\b\neq c,a}}^{n} \left[q_c \frac{\varepsilon_2.p_c}{p_c.k_2}\, q_a \frac{\varepsilon_{1\mu}k_{1\nu}}{p_a.k_1}\left(p_a^{\mu}\frac{\partial}{\partial p_{a\nu}} - p_a^{\nu}\frac{\partial}{\partial p_{a\mu}}\right) + q_c \frac{\varepsilon_1.p_c}{p_c.k_1}\, q_a \frac{\varepsilon_{2\mu}k_{2\nu}}{p_a.k_2}\left(p_a^{\mu}\frac{\partial}{\partial p_{a\nu}} - p_a^{\nu}\frac{\partial}{\partial p_{a\mu}}\right)\right] \\ &\times q_a q_b \int_{reg} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon}\, p_a \cdot p_b \end{aligned} \tag{3.5.20}$$

Now summing $\mathcal{I}$, $\mathcal{I}'$ and $\mathcal{I}''$ we get the $O(\omega^{-1}\ln\omega)$ contribution of $\Gamma_G^{(n,2)}$,

$$\begin{aligned} \Gamma_G^{(n,2)} \quad &= \quad -\lambda \sum_{c=1}^{n} \sum_{a=1}^{n} \sum_{\substack{b=1\\b\neq a}}^{n} \left[q_c \frac{\varepsilon_2.p_c}{p_c.k_2}\, q_a \frac{\varepsilon_{1\mu}k_{1\nu}}{p_a.k_1}\left(p_a^{\mu}\frac{\partial}{\partial p_{a\nu}} - p_a^{\nu}\frac{\partial}{\partial p_{a\mu}}\right) + q_c \frac{\varepsilon_1.p_c}{p_c.k_1}\, q_a \frac{\varepsilon_{2\mu}k_{2\nu}}{p_a.k_2}\left(p_a^{\mu}\frac{\partial}{\partial p_{a\nu}} - p_a^{\nu}\frac{\partial}{\partial p_{a\mu}}\right)\right] \\ &\times q_a q_b \int_{reg} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon}\, p_a \cdot p_b \end{aligned} \tag{3.5.21}$$

Hence from eq.(3.5.13) one can read off the form of subleading double soft photon factor,

$$S^{(1)}_{double} \quad = \quad \sum_{c=1}^{n} \sum_{a=1}^{n} \left[q_c \frac{\varepsilon_2.p_c}{p_c.k_2}\, q_a \frac{\varepsilon_{1\mu}k_{1\nu}}{p_a.k_1}\left(p_a^{\mu}\frac{\partial}{\partial p_{a\nu}} - p_a^{\nu}\frac{\partial}{\partial p_{a\mu}}\right) + q_c \frac{\varepsilon_1.p_c}{p_c.k_1}\, q_a \frac{\varepsilon_{2\mu}k_{2\nu}}{p_a.k_2}\left(p_a^{\mu}\frac{\partial}{\partial p_{a\nu}} - p_a^{\nu}\frac{\partial}{\partial p_{a\mu}}\right)\right] K_{em}^{reg} \tag{3.5.22}$$

where K_{em}^{reg} expression is given in eq.(3.2.28).

From our experience of the derivation of double soft photon theorem now we can easily guess the form of multiple soft photon theorem. One observation is that for m number of soft photons the one loop diagrams having one or more soft photons attached inside the virtual G-photon propagator will not contribute to subleading order. This observation suggests that we can derive subleading multiple soft photon theorem following the strategy of [15, 17]. For m number of soft photons, the subleading multiple soft photon factor turns out to be:

$$S^{(1)}_{multiple} = \sum_{i=1}^{m} \Big\{ \prod_{\substack{j=1\\ j\neq i}}^{m} \sum_{c=1}^{n} q_c \frac{\varepsilon_j.p_c}{p_c.k_j} \Big\} \sum_{a=1}^{n} q_a \frac{\varepsilon_{i\mu} k_{i\nu}}{p_a.k_i} \left(p_a^{\mu} \frac{\partial}{\partial p_{a\nu}} - p_a^{\nu} \frac{\partial}{\partial p_{a\mu}} \right) K_{em}^{reg} \quad (3.5.23)$$

Now after substituting K_{em}^{reg} from eq.(3.2.28) in the above equation we find that the subleading multiple soft photon theorem becomes of order $O(\omega^{-m+1} \ln \omega)$.

3.6 Summary and analysis of the results

In this section we shall summarize the results of this chapter and then discuss various aspects of the results. Finally we shall consider some special limits and compare with known results. We shall use $\hbar = c = 8\pi G = 1$ units.

3.6.1 Summary of the results

In order to give a uniform treatment of the classical soft photon and soft graviton theorem, we shall denote by $\phi(\vec{x}, t)$ the radiative part of the metric or electromagnetic field at a point $\vec{x}$ at time t for a scattering event around the origin. For electromagnetic field, ϕ can be directly identified with the gauge field. For the gravitational field we define

$$h_{\mu\nu} = (g_{\mu\nu} - \eta_{\mu\nu})/2, \qquad e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h_\rho^{\ \rho}\,, \quad (3.6.1)$$

and take ϕ to be $e_{\mu\nu}$. For both electromagnetism and gravity we define classical soft factor $S(\varepsilon, k)$ in D space-time dimensions via the relation:

$$\begin{aligned}\int dt\, e^{i\omega t}\, \varepsilon.\phi(\vec{x}, t) &= e^{i\omega R} \left(\frac{\omega}{2\pi i R}\right)^{(D-2)/2} \frac{1}{2\omega} S(\varepsilon, k) \\ &= -\frac{i}{4\pi R} e^{i\omega R} S(\varepsilon, k) \quad \text{for} \quad D = 4\,, \end{aligned} \quad (3.6.2)$$

where ε is the polarization tensor of the soft particle so that $\varepsilon.\phi = \varepsilon^\mu A_\mu$ for gauge fields and $\varepsilon^{\mu\nu} e_{\mu\nu}$ for gravity, and

$$k = -\omega(1, \hat{n}), \quad \hat{n} \equiv \vec{x}/|\vec{x}|, \quad R = |\vec{x}|\,. \tag{3.6.3}$$

On the other hand the quantum soft factor $S(\varepsilon, k)$ is the ratio of an amplitude with an outgoing soft photon or graviton with momentum k and polarization ε and an amplitude without such a soft particle. It was shown in [19] that in the classical limit the quantum soft factor reduces to the classical soft factor for $D > 4$. Our interest will be in analyzing the situation in $D = 4$.

We consider the scattering of n particles carrying electric charges $\{q_a\}$ and momenta $\{p_a\}$ for $a = 1, \cdots n$. In our convention the momenta / charges carry extra minus sign if they are outgoing. The particles are taken to interact via electromagnetic and gravitational interactions besides other short range interactions whose nature we need not know. The symbol η_a takes value $+1$ (-1) if the a-th particle is ingoing (outgoing). Then the classical result for the soft photon factor $S_{\rm em}(\varepsilon, k)$, containing terms of order ω^{-1} and $\ln\, \omega^{-1}$, is[9]

$$\begin{aligned}
S_{\rm em} \;=\;& \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k}\, q_a - i\, \ln\omega^{-1} \sum_a \frac{q_a\, \varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi}\, \frac{m_a^2 m_b^2 \{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \\
& + \frac{i}{4\pi}\,(\ln\omega^{-1} + \ln R^{-1}) \sum_{\substack{b\\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k}\, q_a \\
& + \frac{i}{8\pi}\, \ln\omega^{-1} \sum_a \frac{q_a\, \varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\, (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho) \left\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\right\}.
\end{aligned} \tag{3.6.4}$$

Since for real polarization the subleading contribution is purely imaginary, it does not af-

[9] In this and subsequent expressions R arises as an infrared cut-off. For the classical result the $\ln R$ terms arise due to long range gravitational force on the soft photon or graviton during its journey from the scattering center to the detector over a distance R. For the quantum part, the natural infrared cut-off is provided by the resolution of the detector. For a detector placed at a distance R from the scattering center, the best energy resolution possible is of order $1/R$. Therefore it is again natural to take R as the infrared regulator.

fect the flux to this order. However the flux for circular polarization and / or the wave-form of the electromagnetic field do receive subleading contribution. An identical situation prevails for gravity.

The quantum result for $S_{\rm em}$ has additional terms:[10]

$$
\begin{aligned}
\Delta S_{\rm em} = {} & \frac{1}{16\pi^2}\, \ln\omega^{-1} \sum_a q_a \frac{\varepsilon_\mu k_\nu}{p_a.k} \left\{ p_a^\mu \frac{\partial}{\partial p_{a\nu}} - p_a^\nu \frac{\partial}{\partial p_{a\mu}} \right\} \\
& \sum_{b\neq a} \left[\frac{\left\{ 2\, q_a q_b p_a.p_b + 2\,(p_a.p_b)^2 - p_a^2 p_b^2 \right\}}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \ln\left(\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \right) \right] \\
& + \frac{1}{8\pi^2} (\ln\omega^{-1} + \ln R^{-1}) \sum_a \frac{q_a \varepsilon_\mu p_a^\mu}{p_a.k} \sum_b (p_b.k)\, \ln\left(\frac{m_b^2}{(p_b.\hat{k})^2} \right) .
\end{aligned} \tag{3.6.5}
$$

The classical results are universal, independent of the theory and the nature of external particles. We expect that the quantum results are also universal, but we have derived them by working with one loop amplitudes in scalar QED coupled to gravity. It is easy to check that (3.6.4), (3.6.5) are invariant under gauge transformation $\varepsilon_\mu \to \varepsilon_\mu + \xi\, k_\mu$ for any constant ξ.

As will be discussed in §3.6.2, the quantum correction (3.6.5) should not be directly added to (3.6.4) and substituted into (3.6.2) to compute the radiative component of the classical electromagnetic field. Rather, when the contribution (3.6.5) is small compared to (3.6.4), we can substitute (3.6.4) into (3.6.2) to compute the classical electromagnetic field produced by a scattering event.

As discussed in §3.1, the quantum results are ambiguous and are defined up to addition of a term to $S_{\rm em}$ of the form $\ln R^{-1}\, k.U\, S^{(0)}_{\rm em}$ where $S^{(0)}_{\rm em}$ is the leading soft factor given by the first term on the right hand side of (3.6.4) and U is a vector constructed out of the p_a's. By choosing $U = (8\pi^2)^{-1} \sum_b p_b\, \ln(m_b^2/\mu^2)$, we can replace the $\ln m_b^2$ term in the coefficient of

[10]Note however that when we express the results in terms of the frequency / wavelength of the soft photon / graviton and momenta of the finite energy particles, neither the classical nor the quantum result has any power of $\hbar$. We shall discuss later the conditions under which we expect the quantum results to be small compared to the classical results.

$\ln R^{-1}$ in the last line of (3.6.5) by $\ln \mu^2$ for any mass parameter μ. This makes manifest the fact that the coefficient is not divergent in the $m_b \to 0$ limit. The coefficient of $\ln \omega^{-1}$ cannot be changed this way, but in this case the finiteness of $m_b \to 0$ limit follows as a consequence of cancellation between the second and third line of (3.6.5) and momentum conservation.

If we want to consider the situation where we ignore the effect of gravity, then we need to set the terms proportional to $\ln \omega^{-1}$ that are linear in q_c's to zero. On the other hand if we want to consider the situation where we ignore the effect of electromagnetic interaction between the particles during scattering (but still use electromagnetic interaction to compute soft photon emission process), we have to set the terms proportional to $\ln \omega^{-1}$ that are cubic in the q_c's to zero.

The classical result for soft graviton factor takes the form

$$\begin{aligned}
S_{\rm gr} \;=\; & \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} - i\,\ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi} \frac{m_a^2 m_b^2 \,\{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \\
& + \frac{i}{4\pi}\,(\ln\omega^{-1} + \ln R^{-1}) \sum_{\substack{b\\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \\
& + \frac{i}{8\pi}\,\ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \,(p_b^\rho p_a^\mu - p_b^\mu p_a^\rho) \left\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\right\} .
\end{aligned} \tag{3.6.6}$$

The quantum result has additional terms

$$\begin{aligned}
\Delta S_{\rm gr} \;=\; & \frac{1}{16\pi^2}\,\ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\rho} p_a^\rho k_\nu}{p_a.k} \left\{ p_a^\mu \frac{\partial}{\partial p_{a\nu}} - p_a^\nu \frac{\partial}{\partial p_{a\mu}} \right\} \\
& \sum_{b\neq a} \left[\frac{\left\{2\, q_a q_b p_a.p_b + 2\,(p_a.p_b)^2 - p_a^2 p_b^2\right\}}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \ln\left(\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \right) \right] \\
& + \frac{1}{8\pi^2}\,(\ln\omega^{-1} + \ln R^{-1}) \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \sum_b p_b.k\, \ln \frac{m_b^2}{(p_b.\hat{k})^2}\,,
\end{aligned} \tag{3.6.7}$$

where $\hat{k} = -k/\omega = (1, \hat{n})$. Again the classical results are valid universally. The quantum results are obtained from one loop calculation in scalar QED coupled to gravity, but we expect them to be universal. As in the case of (3.6.5), the $\ln m_b^2$ term in the coefficient of $\ln R^{-1}$ in the last line of (3.6.7) can be replaced by $\ln \mu^2$ by exploiting the ambiguity in the definition of the soft factor discussed in §3.1. One can check that (3.6.6), (3.6.7) are invariant under gauge transformations $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \xi_\mu k_\nu + \xi_\nu k_\mu$ for any constant vector ξ_μ.

If we want to consider the situation where we ignore the effect of electromagnetic interactions, then we need to set the terms proportional to $\ln \omega^{-1}$ that are quadratic in q_c's to zero. On the other hand if we want to consider the situation where we ignore the effect of gravitational interaction between the particles during scattering (but still use gravitational interaction to compute soft graviton emission process), we have to set the q_c independent terms in the coefficient of $\ln \omega^{-1}$ to zero.

3.6.2 Discussion of results

First we shall briefly outline how these results are derived. The classical results (3.6.4) and (3.6.6) are the result of direct application of classical soft theorem to subleading order. As described in [20], the soft factor involves orbital angular momenta of initial and final particles and these diverge logarithmically in the elapsed time τ in four dimensions due to the long range gravitational / electromagnetic force on the incoming and outgoing particles that generates a term proportional to $\ln|\tau|$ in the trajectory. We follow the prescription of [20] of replacing $\ln|\tau|$ by $\ln \omega^{-1}$ to arrive at the first and third lines of the classical results (3.6.4), (3.6.6) in the last chapter. The second lines of (3.6.4) and (3.6.6) arise from additional phases that are not directly determined by soft theorem. They represent the effect of long range gravitational force on the outgoing soft photon or graviton which causes the soft particle to slow down and also backscatter.

Quantum results are the result of direct one loop computation in a field theory of multiple charged scalars, coupled to electromagnetic and gravitational fields. We simply evaluate the order ω^{-1} and $\ln\omega^{-1}$ terms in the scattering amplitude of multiple finite energy scalars and an outgoing soft photon or graviton of energy ω, and express this as the product of the amplitude without the soft photon or graviton and a multiplicative factor that we call the soft factor. The latter is given by the sum of (3.6.4) and (3.6.5) for soft photon and the sum of (3.6.6) and (3.6.7) for the soft graviton. Even though the S-matrix elements with and without the soft particle are infrared divergent, much of this cancels when we take the ratio of the two. The remaining infrared divergent part is regulated by the infra-red length cut-off R and is responsible for the terms proportional to $\ln R$ in these expressions. This is related to the quantity σ'_n introduced in [66].

The different terms proportional to $\ln\omega^{-1}$ in (3.6.4), (3.6.5) and in (3.6.6), (3.6.7) have different origin. We shall explain them in the context of the soft graviton factor, but the case of soft photon factor is very similar.

1. We begin with the classical result (3.6.6). The term proportional to $q_a q_b$ in the first line represents the effect of late time gravitational radiation due to the late time acceleration of the particles via long range electromagnetic interaction. The term in the last line of (3.6.6) represents the effect of late time gravitational radiation due to the late time acceleration of the particles via long range gravitational interaction. We expect the scale of these logarithms to be set by the largest length scale involved in the classical scattering process, *e.g.* the typical distance of closest approach between the particles involved in the scattering. This is taken to be larger than or of the order of the Schwarzschild radii of the particles and much larger than the Compton wave-lengths of the particles involved in the scattering. In the quantum one loop computation both these terms arise from the region of loop momentum integration where the loop momentum is large compared to ω but small compared to the energies of the other particles. In this case the scale of these logarithms is again set by the largest length scale involved in the quantum scattering which is

the inverse of the typical energy carried by the finite energy external states. For one loop result to be reliable, this needs to be taken to be large compared to the Schwarzschild radii of these particles.

2. The term in the second line of (3.6.6) proportional to $(\ln\omega^{-1}+\ln R^{-1})$ represents the effect of gravitational drag on the soft graviton due to the other finite energy particles in the final state. This has the effect of causing a time delay, represented by the $\ln R^{-1}$ term, for the soft graviton to travel to a distance R. This also has the effect of inducing backscattering of the soft graviton, represented by the $\ln\omega^{-1}$ term. In the quantum computation these terms arise from region of loop momentum integration where the loop momentum is smaller than ω and larger than the infrared cut-off R^{-1}. This term has appeared *e.g.* in [110, 112, 113]. As mentioned in footnote 9, the scale of these logarithms is set by the effective infrared cut-off, *e.g.* the distance R to the detector for the classical scattering and the resolution of the detector for the quantum scattering. The latter in turn has a lower limit set by R^{-1} since we cannot measure the energy of the outgoing particle with an accuracy better than R^{-1} if the detector is placed at a distance R from the scattering center.

3. We emphasize that the classical results are obtained by replacing in the classical soft theorem the logarithmically divergent terms by $\ln\omega^{-1}$ and not by direct calculation of electromagnetic and gravitational radiation during classical scattering. In special cases the equivalence of these two procedures was tested in [20] by direct classical computation. In principle similar tests can be done for the general formulae (3.6.4) and (3.6.6), but we have not done this.

4. We now turn to the additional terms (3.6.7) that arise in the quantum computation. First note that both these terms are real for real polarizations unlike the classical result where the coefficients of $\ln\omega^{-1}$ terms are imaginary for real polarizations. The terms in the first two lines come from regions of loop momentum integration where the loop momentum is large compared to ω but small compared to the ener-

gies of the other particles, while the term in the third line arise from region of loop momentum integration where the loop momentum is small compared to ω and large compared to the infrared cut-off R^{-1}.

5. In the quantum computation the terms that arise from loop momenta large compared to ω, namely the terms in the first and third line of (3.6.6) and the first two lines of (3.6.7), can be generated using a simple algorithm. As discussed earlier, the amplitude without the soft graviton has an infrared divergent factor multiplying it. Let us call this the IR factor. If in the integration over loop momenta of this IR factor we restrict the loop momentum integration to be large compared to ω and apply the usual subleading soft differential operator that arises in higher dimensions to this IR factor, we recover precisely the results given in the first and third line of (3.6.6) and the first two lines of (3.6.7). The rest of the contribution that arises from integration region where the loop momentum is small compared to ω cannot be recovered this way. This indicates that the general argument of [11, 101], based on general coordinate invariance of 1PI effective action and power counting assuming that loops do not generate inverse power of soft momentum, remain valid in four dimensions as well as long as the loop momentum is large compared to the external soft momentum.

Since the real infrared divergent part of the amplitude reflects the effect of real graviton emission, our interpretation of the extra contributions (3.6.7) in the quantum theory is that they reflect the effect of backreaction of soft radiation on the classical trajectories. To this end note that the validity of the classical limit described in [19] requires that the total energy carried by soft radiation should remain small compared to the energies of the finite energy objects taking part in the scattering. Here 'soft radiation' represents those particles which are not included in the sum over a in (3.6.6). Therefore we should expect that the extra terms arising in the quantum theory should be small in the limit when the total energy carried by the soft radiation is small.

In order to test this hypothesis we need to consider a scattering where the energy carried away by soft radiation remains small compared to the energies of finite energy objects. One way to achieve this is to consider scattering at large impact parameter so that each incoming particle gets deflected by a small amount and the energy radiated during this process remains small. In this case the momenta $\{p_a\}$ come in approximately equal and opposite pairs – the incoming and the corresponding outgoing particle. Now in eq.(3.6.7) the last term changes sign under $p_b \to -p_b$ and also under $p_a \to -p_a$. This shows that it is small for small deflection scattering. The first term on the right hand side of (3.6.7) changes sign under $(p_b, q_b) \to -(p_b, q_b)$ and also under $(p_a, q_a) \to -(p_a, q_a)$, due to the argument of the log getting inverted under each of these operations. This shows that the terms approximately cancel making the result small. There is one exception to this that arises when $q_b = -q_a$, $p_b \simeq -p_a$, i.e. the pairs (a, b) represent the incoming and the corresponding outgoing particle. In this case there is no other term that cancels this since the sum does not include the $b = a$ term, and we need to explicitly evaluate this and show that it vanishes. This can be checked explicitly by first evaluating the derivatives in the second line of (3.6.7), then setting $p_b = -p_a + \epsilon$ and then carefully evaluating the result in the $\epsilon \to 0$ limit. Even though individual terms diverge in the $\epsilon \to 0$ limit, a careful analysis shows that the result vanishes. This confirms that quantum corrections are small in this limit.

Another situation discussed in [19], where the radiated energy remains small compared to the energies of the hard particles, is the probe limit in which one of the particles has a large mass M and the other particles are lighter carrying energy small compared to M. We shall now verify that in this case too the quantum corrections (3.6.7) are small compared to the classical result (3.6.6). For this we shall work in a frame in which the heavy particle is initially at rest, and using gauge invariance choose the polarization tensor ε to have only spatial components. After the scattering the heavy particle acquires a momentum but it is small compared to M. In this case the dominant contribution to (3.6.6), of order M, comes from choosing a to be one of the light particles and b to be the heavy particle in the second and third line of (3.6.6). However in the quantum correction (3.6.7) similar contribution cancels between the choice of b as the initial state heavy particle and the final state heavy particle, and we do not get any contribution proportional to M. This again shows that quantum corrections are small compared to the classical result in this limit.

We must emphasize however that the quantum analysis is carried out for single soft graviton emission. If we want to relate the quantum result to the radiative component of the classical gravitational field as in [19], then we need to first consider multiple soft graviton emission and then take the classical limit. The analysis of [19] relied on the fact that the soft factors associated with different bins in the phase space are independent of each other, i.e. the probability of emitting certain number of soft particles in one bin does not depend on how many soft particles are emitted in the other bin. This independence breaks down when the total energy carried by the soft particles becomes comparable to the energies of the hard particles – precisely when the quantum correction (3.6.7) becomes comparable to the classical result (3.6.6). Therefore we should not use (3.6.7) to modify the classical result (3.6.6). Instead we should use the smallness of (3.6.7) as a test of when the classical result (3.6.6) is valid. An identical discussion holds for electromagnetism[11].

3.6.3 Special cases

As a special case we can consider the situation described in [33] where a neutral massive object of mass M at rest decays into a heavy object of mass $M_0 \simeq M$ and a set of neutral light objects carrying mass $m_a << M$ and momentum $p_a = -e_a(1, \vec{\beta}_a)$ with $e_a << M$ for $a = 1, \cdots N$. Our goal will be to write down the classical soft graviton factor for this case. We shall take the polarization tensor of the soft graviton to have components only along the spatial direction, since the result for the other components may be found by using invariance under the gauge transformation $\varepsilon_{\mu\nu} \rightarrow \varepsilon_{\mu\nu} + \xi_\mu k_\nu + \xi_\nu k_\mu$ for any vector ξ. If we denote the momentum carried by final state heavy object of mass M_0 by p_{N+1}, then we

[11]For subleading multiple soft photon theorem we tried to give a sketch of proof in §3.5.

have $p^0_{N+1} \simeq -M_0$ and $|p^i_{N+1}| << M_0$. Examining (3.6.6) with $q_a = q_b = 0$ we see that dominant term proportional to $\ln\omega^{-1}$ comes from the terms where we choose $b = N + 1$ and a labels any of the N finite energy states. Using the relation $e_a^2 = m_a^2/(1-\vec{\beta}_a^2)$, the net contribution takes the form:

$$\begin{aligned}
& \frac{i}{4\pi}\ln\omega^{-1}\, M_0 \sum_{a=1}^{N} e_a \frac{\varepsilon^{ij}\beta_{ai}\beta_{aj}}{1-\hat{n}.\vec{\beta}_a} + \frac{i}{8\pi}\ln\omega^{-1}\, M_0 \sum_{a=1}^{N} e_a \frac{\varepsilon^{ij}\beta_{ai}\beta_{aj}}{1-\hat{n}.\vec{\beta}_a}\frac{(-e_a)(2e_a^2-3m_a^2)}{(e_a^2-m_a^2)^{3/2}} \\
= \; & \frac{i}{8\pi}\ln\omega^{-1}\, M_0 \sum_{a=1}^{N} e_a \frac{\varepsilon^{ij}\beta_{ai}\beta_{aj}}{1-\hat{n}.\vec{\beta}_a}\frac{2\vec{\beta}_a^3+1-3\vec{\beta}_a^2}{|\vec{\beta}_a|^3} + \cdots\,, \qquad (3.6.8)
\end{aligned}$$

where $\cdots$ contain terms without a factor of M_0 and are therefore smaller in the limit of large M_0. This agrees with the results of [33]. As discussed in [33], this produces a late time tail in the gravitational wave-form that falls off as inverse power of time.

Note that when all the final state light particles are massless, so that $|\vec{\beta}_a| = 1$ for $1 \le a \le N$, the expression (3.6.8) vanishes. This would be the situation during binary black hole merger when the final state particles are only gravitons. However since in such processes the radiation carries away an appreciable fraction of the mass of the parent system, the $\cdots$ terms in (3.6.8) could be significant even though their contribution will be suppressed by the ratio of the total energy carried away by radiation to the mass of the parent system. We shall now evaluate the result without making any approximation. In this case in the sum over a and b in (3.6.6), either a or b (or both) represents a massless particle. Recalling that when p_a and p_b are both outgoing then $p_a.p_b$ is negative, we can express the terms in (3.6.6) proportional to $\ln\omega^{-1}$ as

$$\frac{i}{4\pi}\ln\omega^{-1}\sum_{a=1}^{N+1}\varepsilon^{ij}p_{ai}p_{aj} + \frac{i}{4\pi}\ln\omega^{-1}\sum_{a=1}^{N+1}\varepsilon^{ij}p_{ai}\sum_{\substack{b=1\\ b\neq a}}^{N+1}p_{bj} = 0\,, \qquad (3.6.9)$$

where in the last step we have used conservation of spatial momentum $\sum_{b=1}^{N+1} p_{bj} = 0$. Therefore we see that even without making any approximation, the coefficient of the $\ln\omega^{-1}$ term in the classical soft graviton factor continues to vanish.

Another special case we can consider is when a charge neutral object of mass M at rest breaks apart into two charge neutral objects of masses m_1 and m_2, spatial momenta $\vec{p}$ and $-\vec{p}$ and energies $e_1 = \sqrt{m_1^2 + \vec{p}^2}$ and $e_2 = \sqrt{\vec{p}^2 + m_2^2}$. In this case if we take the polarization tensor of the soft graviton to have components only along the spatial direction, then the contribution from the initial state to (3.6.6) vanishes and we need to only compute the contribution from a pair of final states. This can be easily evaluated and the terms proportional to $\ln\omega^{-1}$ take the form

$$\frac{i}{8\pi}\ln\omega^{-1}\,\varepsilon_{ij}p^i p^j\,(e_1+e_2)\left\{\frac{1}{e_1-\hat{n}.\vec{p}}+\frac{1}{e_2+\hat{n}.\vec{p}}\right\} \times\left[\frac{e_1e_2+\vec{p}^2}{\{(e_1e_2+\vec{p}^2)^2-m_1^2m_2^2\}^{3/2}}\left\{2(e_1e_2+\vec{p}^2)^2-3m_1^2m_2^2\right\}-2\right]. \quad (3.6.10)$$

Next special case we shall analyze is that of scattering of massless particles, again focussing on the classical result (3.6.6). Defining

$$P \equiv \sum_{\eta_a=1} p_a = -\sum_{\eta_a=-1} p_a\,, \quad (3.6.11)$$

and the fact that $p_a.p_b$ is negative for $\eta_a\eta_b = 1$, we can express the term proportional to $\ln\omega^{-1}$ in (3.6.6) for massless particles as

$$-\frac{i}{2\pi}\ln\omega^{-1}\,k.P\sum_{\substack{a\\ \eta_a=1}}\frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}+\frac{i}{2\pi}\ln\omega^{-1}\,\varepsilon_{\mu\nu}P^\mu P^\nu\,. \quad (3.6.12)$$

Note that this involves only the momenta of the initial state particles and is insensitive to the momenta of the final state particles. This asymmetry is related to the fact that in our analysis we are considering soft particle only in the final state and not in the initial state.

More generally one can show that for a general scattering process involving both massive and massless particles, the terms proportional to $\ln\omega^{-1}$ in the classical formula (3.6.6) is not sensitive to the details of the final state massless particles except through overall momentum conservation. To see this let us first consider terms that could involve a final state massless particle momenta and the initial state momenta. These come from choosing a to be an initial state and b to be a final state massless state in the term in the second line of (3.6.6). The net contribution from such terms is given by

$$\frac{i}{4\pi}\ln\omega^{-1}\sum_{\substack{b\ \mathrm{massless}\\ \mathrm{J}_b=-1}} k.p_b \sum_{\substack{a\\ \eta_a=1}} \frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{k.p_a} = -\frac{i}{4\pi}\ln\omega^{-1}\ k.(P-P_{\mathrm{massive}}) \sum_{\substack{a\\ \eta_a=1}} \frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{k.p_a}\,, \quad (3.6.13)$$

where $-P$ denotes total outgoing momentum as defined in (3.6.11) and $-P_{\mathrm{massive}}$ denotes the total outgoing momentum carried by the massive particles. Therefore this does not depend explicitly on the momenta of the outgoing massless states except through momentum conservation.

Next we consider terms that involve a pair of final state momenta at least one of which is massless. This term receives contribution from all three lines on the right hand side of (3.6.6) with the restriction $\eta_a = 1$, $\eta_b = 1$, and either m_a or m_b or both zero. Therefore the term proportional to $q_a q_b$ vanishes. Also the coefficient of $\ln\omega^{-1}$ in the summand in the last two lines simplifies to

$$\frac{i}{4\pi}\frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}\,p_b.k - \frac{i}{4\pi}\frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}\,p_b.k + \frac{i}{4\pi}\,\varepsilon_{\mu\nu}\,p_a^\mu p_b^\nu\,. \quad (3.6.14)$$

In the first term the sum over a and b includes the term where $b = a$, but in the second and the third term the sum excludes the $b = a$ term. Therefore the first two terms almost cancel, leaving behind a contribution where we set $b = a$. This left over contribution $\frac{i}{4\pi}\,\varepsilon_{\mu\nu}p_a^\mu p_a^\nu$ can now be added to the last term to include in the sum over a or b also the contribution where $b = a$. The net contribution from the terms where either a or b or both represent massless state is then

$$\frac{i}{4\pi}\ln\omega^{-1}\sum_{\substack{a,b;\eta_a=\eta_b=-1\\ \text{either a or b massless}}} \varepsilon_{\mu\nu}\,p_a^\mu p_b^\nu\,. \quad (3.6.15)$$

This can be rewritten as

$$\frac{i}{4\pi}\ln\omega^{-1}\varepsilon_{\mu\nu}\left(\sum_{a,b;\eta_a=\eta_b=-1} p_a^\mu p_b^\nu - \sum_{\substack{a,b;\eta_a=\eta_b=-1\\ \text{a and b massive}}} p_a^\mu p_b^\nu\right) = \frac{i}{4\pi}\ln\omega^{-1}\varepsilon_{\mu\nu}\left(P^\mu P^\nu - P^\mu_{\text{massive}}P^\nu_{\text{massive}}\right). \tag{3.6.16}$$

This also does not depend on the details of the momenta of massless final state particles except for the total momentum carried by these particles.

4 Classical soft theorem in four spacetime dimensions

Classical soft graviton theorem in four space-time dimensions determines the gravitational wave-form emitted during a scattering process at late and early retarded time, in terms of the four momenta of the ingoing and outgoing objects. As discussed in the last two chapters, this result was conjectured earlier by taking the classical limit of the quantum soft graviton theorem, and making some assumption about how to deal with the infrared divergences of the soft factor.

Our goal in this chapter will be to prove the classical soft graviton theorem in four space-time dimensions directly in the classical theory. In particular we prove the following result. Let us consider a scattering process in which a set of m objects carrying four momenta $p'_1, \cdots, p'_m$ come together, scatter via some (unknown) interactions and disperse as n objects carrying momenta $p_1, \cdots, p_n$. The special case $m = 1$ will describe an explosion in which a single bound system fragments into many objects, including radiation. We shall choose the origin of the space-time coordinate system so that the scattering takes place within a finite neighbourhood of the origin. Let us also suppose that we have a gravitational wave detector placed at a faraway point $\vec{x}$, and define

$$R = |\vec{x}|, \qquad \hat{n} = \frac{\vec{x}}{R}, \qquad n = (1, \hat{n})\,. \tag{4.0.1}$$

We shall consider the limit of large R and analyze only the terms of order $1/R$ in the gravitational wave-form. We define the retarded time at the detector:

$$u \equiv t - R + 2\,G\,\ln R \sum_{b=1}^{n} p_b.n\,. \tag{4.0.2}$$

Here $t - R$ is the usual retarded time and the $2\,G\,\ln R \sum_{b=1}^{n} p_b.n$ takes into account the effect of the long range gravitational force on the gravitational wave as it travels from the scattering center to the detector. G denotes the Newton's constant. We have used units in which the velocity of light c has been set equal to 1, – this is the unit we shall use throughout the chapter. We also define the deviation of the metric $g_{\mu\nu}$ from flat metric via:

$$h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu})/2, \qquad e_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,h_{\rho\sigma}\,. \tag{4.0.3}$$

Let us first assume that the objects do not carry charge so that gravity is the only long range force acting on the objects at late and early time, although during the scattering they may undergo complicated interactions. Then at late and early retarded time, our result for the gravitational wave-form at the detector is given by:

$$\begin{aligned}
e^{\mu\nu}(t,R,\hat{n}) &= \frac{2\,G}{R}\left[-\sum_{a=1}^{n} p_a^{\mu}\,p_a^{\nu}\,\frac{1}{n.p_a} + \sum_{a=1}^{m} p_a'^{\mu}\,p_a'^{\nu}\,\frac{1}{n.p_a'}\right] \\
&- \frac{4\,G^2}{R\,u}\left[\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a.p_b}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}}\left\{\frac{3}{2}p_a^2 p_b^2 - (p_a.p_b)^2\right\}\frac{n_\rho p_a^{\mu}}{n.p_a}\,(p_b^{\rho}p_a^{\nu} - p_b^{\nu}p_a^{\rho})\right. \\
&\quad \left. - \sum_{b=1}^{n} p_b.n\left\{\sum_{a=1}^{n}\frac{1}{p_a.n}\,p_a^{\mu}p_a^{\nu} - \sum_{a=1}^{m}\frac{1}{p_a'.n}\,p_a'^{\mu}\,p_a'^{\nu}\right\}\right] + O(u^{-2})\,, \quad \text{as } u\to\infty \\
e^{\mu\nu}(t,R,\hat{n}) &= \frac{4\,G^2}{R\,u}\left[\sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m} \frac{p_a'.p_b'}{\{(p_a'.p_b')^2 - p_a'^2 p_b'^2\}^{3/2}}\left\{\frac{3}{2}p_a'^2\,p_b'^2 - (p_a'.p_b')^2\right\}\right. \\
&\quad \left. \times \frac{n_\rho p_a'^{\mu}}{n.p_a'}\,(p_b'^{\rho}\,p_a'^{\nu} - p_b'^{\nu}\,p_a'^{\rho})\right] + O(u^{-2})\,, \quad \text{as } u\to-\infty,
\end{aligned} \tag{4.0.4}$$

where $O(u^{-2})$ includes terms of order $u^{-2}\ln|u|$. The term on the right hand side of the first line represents a constant jump in $h_{\mu\nu}$ during the passage of the gravitational wave, and is known as the memory effect [23–29, 115–117]. This is related to the leading soft theorem [30]. The terms of order $1/u$ are related to logarithmic corrections to the subleading soft theorem. These have been verified in various examples via explicit calculations [110, 118, 119]. The sum over a in (4.0.4) also includes the contribution from finite frequency radiation emitted during the scattering.

As already discussed in [16,33], in case of decay ($m = 1$), if at most one of the final objects is massive and the rest are massless, including radiation, then the terms proportional to $1/u$ in the expression for $e^{\mu\nu}$ cancel. This will be the case for binary black hole merger where the initial state is a single bound system, and the final state consists of a single massive black hole and gravitational radiation. Therefore absence of $1/u$ tails in such decays can be taken as a test of general theory of relativity.

If the objects participating in the scattering process are charged, with the incoming objects carrying charges $q'_1, \cdots, q'_m$ and outgoing objects carrying charges $q_1, \cdots, q_n$, then there are further corrections to (4.0.4) due to long range electromagnetic forces between the incoming and the outgoing objects. These corrections have been given in (4.3.30).

A similar result can be given for the profile of the electromagnetic vector potential a_μ at the detector at late and early retarded time. The results are given in (4.3.23), (4.3.24).

Although these results derived in this chapter independently, they have been conjectured earlier from soft graviton theorem following the chain of arguments given in the last two chapters. Emboldened by the success of these arguments, we describe in §4.4 a new conjecture for terms of order $u^{-2} \ln |u|$ at late and early retarded time. These have been given in (4.4.7), (4.4.8) and (4.4.9). We have done a numerical estimate of this new tail memory in §4.5 for various astrophysical scattering processes.

4.1 Some useful results

In this section we shall review some simple mathematical results that will be useful for our analysis.

4.1.1 Different Fourier transforms

We shall deal with functions of four variables $x \equiv (t, \vec{x}) \equiv (x^0, x^1, x^2, x^3)$ describing the space-time coordinates. Given any such function $F(x)$, we shall introduce the following different kinds of Fourier transforms:

$$\widehat{F}(k) \equiv \int d^4x\, e^{-ik.x}\, F(t,\vec{x}), \quad \bar{F}(t,\vec{k}) \equiv \int d^3x\, e^{-i\vec{k}.\vec{x}}\, F(t,\vec{x}), \quad \widetilde{F}(\omega,\vec{x}) \equiv \int dt\, e^{i\omega t}\, F(t,\vec{x})\,. \tag{4.1.1}$$

The inverse relations are

$$F(t,\vec{x}) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik.x}\, \widehat{F}(k), \quad F(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}.\vec{x}}\, \bar{F}(t,\vec{k}), \quad F(t,\vec{x}) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \widetilde{F}(\omega,\vec{x})\,. \tag{4.1.2}$$

Note that we are using the convention $k.x \equiv \eta_{\mu\nu} k^\mu x^\nu = -k^0 x^0 + \vec{k}.\vec{x}$.

4.1.2 Radiative field at large distance

Let us consider a differential equation of the form:

$$\Box F(x) = -j(x), \qquad \Box \equiv \eta^{\alpha\beta}\, \partial_\alpha\, \partial_\beta\,, \tag{4.1.3}$$

where $j(x)$ is some given function. The retarded solution to this equation is given by

$$F(x) = -\int d^4y\, G_r(x,y)\, j(y)\,, \tag{4.1.4}$$

where $G_r(x,y)$ is the retarded Green's function:

$$G_r(x,y) = \int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.(x-y)}\, \frac{1}{(\ell^0+i\epsilon)^2 - \vec{\ell}^2}\,. \tag{4.1.5}$$

Using (4.1.1) we get

$$\widetilde{F}(\omega,\vec{x}) = -\int d^4y\, j(y) \int \frac{d^3\ell}{(2\pi)^3}\, e^{i\omega y^0 + i\vec{\ell}.(\vec{x}-\vec{y})}\, \frac{1}{(\omega+i\epsilon)^2 - \vec{\ell}^2}\,. \tag{4.1.6}$$

For large $|\vec{x}|$, we can evaluate this integral using a saddle point approximation as follows [19]. Defining $\vec{\ell}_{||}$ and $\vec{\ell}_\perp$ as components of $\vec{\ell}$ along $\vec{x}-\vec{y}$ and transverse to $\vec{x}-\vec{y}$ respectively, we get

$$\widetilde{F}(\omega,\vec{x}) = -\int d^4y\, j(y) \int \frac{d^2\ell_\perp}{(2\pi)^2}\,\frac{d\ell_{||}}{2\pi}\, e^{i\omega y^0 + i\ell_{||}\,|\vec{x}-\vec{y}|}\, \frac{1}{(\omega+i\epsilon)^2 - \ell_{||}^2 - \vec{\ell}_\perp^2}\,. \tag{4.1.7}$$

First consider the case $\omega > 0$. We now close the $\ell_{||}$ integration contour in the upper half plane, picking up residue at the pole at $\sqrt{(\omega+i\epsilon)^2 - \vec{\ell}_\perp^2}$. This gives

$$\widetilde{F}(\omega,\vec{x}) = i\int d^4y\, j(y) \int \frac{d^2\ell_\perp}{(2\pi)^2}\, e^{i\omega y^0 + i\,|\vec{x}-\vec{y}|\,\sqrt{(\omega+i\epsilon)^2-\vec{\ell}_\perp^2}}\, \frac{1}{2\sqrt{(\omega+i\epsilon)^2 - \vec{\ell}_\perp^2}}\,. \tag{4.1.8}$$

For large $|\vec{x}-\vec{y}|$ the exponent is a rapidly varying function of $\vec{\ell}_\perp$ and therefore we can carry out the integration over $\vec{\ell}_\perp$ using saddle point approximation. The saddle point is located at $\vec{\ell}_\perp = 0$. Expanding the exponent to order $\vec{\ell}_\perp^2$ and carrying out gaussian integration over $\vec{\ell}_\perp$ we get:

$$\widetilde{F}(\omega,\vec{x}) = i\int d^4y\, j(y)\, e^{i\omega y^0 + i\,(\omega+i\epsilon)\,|\vec{x}-\vec{y}|}\, \frac{\omega+i\epsilon}{2\,\pi\, i\,|\vec{x}-\vec{y}|}\, \frac{1}{2(\omega+i\epsilon)} \simeq \frac{1}{4\pi R}\, e^{i\omega R} \int d^4y\, e^{-ik.y}\, j(y)\,, \tag{4.1.9}$$

where we have made the approximation $|\vec{x}| >> |\vec{y}|$, and,

$$k \equiv \omega(1,\hat{n}), \quad \hat{n} \equiv \vec{x}/|\vec{x}|, \quad R \equiv |\vec{x}|\,. \tag{4.1.10}$$

A similar analysis can be carried out for $\omega < 0$, leading to the same final expression. Using (4.1.1), eq.(4.1.9) may be written as

$$\widetilde{F}(\omega, \vec{x}) \simeq \frac{1}{4\pi R}\, e^{i\omega R}\, \widehat{j}(k)\,. \tag{4.1.11}$$

This is a known formula (see *e.g.* [120]), but the derivation given above also gives its limitations. In arriving at the right hand side of (4.1.9) we used the approximation $|\vec{x}| >> |\vec{y}|$. Therefore in the integration over $\vec{y}$ there is a natural infrared cut-off given by $|\vec{x}| = R$. If the y integral is convergent then there is no need of such a cut-off, but in case the y integral diverges from the large y region, we need to explicitly impose the cut-off. We can implement the cut-off by putting a cut-off on y^0, since typically the source $j(y)$ will have support inside the light-cone $|\vec{y}| \leq |y^0|$ for large y. For example, for positive y^0 we can implement the infrared cut-off by adding to k^0 an imaginary part $i\,\Lambda\, R^{-1}$ for some fixed number Λ. In that case for $y^0 << R/\Lambda$ this additional factor has no effect on (4.1.9), but for $y^0 >> R/\Lambda$ there is an exponential suppression factor that cuts off the integration over y. For negative y^0 the corresponding modification of k corresponds to adding an imaginary part $-i\,\Lambda\, R^{-1}$ to k^0.

4.1.3 Late and early time behaviour from Fourier transformation

In our analysis we shall encounter functions $\widetilde{F}(\omega, \vec{x})$ that are non-analytic as $\omega \to 0$, – having singularities either of the form $1/\omega$ or of the form $\ln\omega$. On general grounds we expect these singular small ω behaviour to be related to the behaviour of $F(t, \vec{x})$ as $t \to \pm\infty$. We shall now determine the precise correspondence between the small ω behaviour of $\widetilde{F}(\omega, \vec{x})$ and large $|t|$ behaviour of $F(t, \vec{x})$. Since the analysis will be carried out at fixed $\vec{x}$, we shall not display the $\vec{x}$ dependence of various quantities in subsequent discussions.

First we shall consider singularities of the form $1/\omega$ for small ω. For this consider a function of the form:

$$\widetilde{F}(\omega) = C\, e^{i\omega\phi}\, \frac{1}{\omega}\, f(\omega)\,. \tag{4.1.12}$$

Here C and ϕ are constants that could depend on $\vec{x}$. $f(\omega)$ is a function of ω that is smooth at $\omega = 0$ with $f(0) = 1$ and falls off sufficiently fast as $\omega \to \infty$ so as to make the Fourier integral over ω well defined. Our final result will not depend on $f(\omega)$, but for definiteness we shall choose

$$f(\omega) = \frac{1}{\omega^2 + 1}\,. \tag{4.1.13}$$

This gives

$$F(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \widetilde{F}(\omega) = C \int \frac{d\omega}{2\pi}\, e^{-i\omega u} \frac{1}{\omega}\, f(\omega), \qquad u \equiv t - \phi\,. \tag{4.1.14}$$

In order to define the integral around $\omega = 0$, we need to choose an appropriate $i\epsilon$ prescription. However since $1/(\omega + i\epsilon)$ and $1/(\omega - i\epsilon)$ differ by a term proportional to $\delta(\omega)$, whose Fourier transform is a u independent constant, the difference will not be of interest to us. For definiteness , we shall work with $1/(\omega + i\epsilon)$. Then we have

$$F(t) = C\, \frac{1}{2\pi} \int d\omega\, e^{-i\omega u} \frac{1}{\omega + i\epsilon} f(\omega) = -i\, C\, H(u) + O(e^{-u})\,, \tag{4.1.15}$$

where H is the Heaviside step function. This result is obtained by closing the contour in the lower (upper) half plane for positive (negative) u, and picking up the residues at the poles. The order e^{-u} contribution comes from the residues at the poles of $f(\omega)$. The step function $H(u)$ gives a jump in $e_{\mu\nu}$ between $u \to -\infty$ and $u \to \infty$, leading to the memory effect [23–26].

Let us now turn to the Fourier transform of the logarithmic terms. We consider functions of the form:

$$\widetilde{F}(\omega) = C\, e^{i\omega\phi}\, \ln\omega\, f(\omega)\,. \tag{4.1.16}$$

Again we need to consider the different $i\epsilon$ prescriptions, and this time the difference between the two choices is not trivial. Therefore we consider[1]

$$F_\pm(t) = C \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, e^{i\omega\phi}\, \ln(\omega \pm i\epsilon)\, f(\omega) = C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \ln(\omega \pm i\epsilon)\, f(\omega)\,. \quad (4.1.17)$$

For $u > 0$ we can close the contour in the lower half plane. In this case F_- gets contribution only from the poles of $f(\omega)$ and therefore is suppressed by factors of e^{-u}. Similarly for $u < 0$, F_+ is suppressed by powers of e^{-u}. Furthermore, using $\ln(\omega + i\epsilon) = \ln(\omega - i\epsilon) + 2\,\pi\, i\, H(-\omega)$, we have

$$F_+ - F_- = i\,C \int_{-\infty}^{0} d\omega\, e^{-i\omega u} f(\omega) \simeq -\frac{C}{u}, \quad \text{for} \quad u \to \pm\infty\,. \quad (4.1.18)$$

Using these results we get

$$F_+ \equiv C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \ln(\omega + i\epsilon)\, f(\omega) \to \begin{cases} -\dfrac{C}{u} & \text{for} \quad u \to \infty, \\ 0 & \text{for} \quad u \to -\infty, \end{cases}$$

$$F_- \equiv C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \ln(\omega - i\epsilon)\, f(\omega) \to \begin{cases} 0 & \text{for} \quad u \to \infty, \\ \dfrac{C}{u} & \text{for} \quad u \to -\infty. \end{cases} \quad (4.1.19)$$

Next we shall consider the integrals:

$$G_\pm \equiv C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \omega\, \{\ln(\omega \pm i\epsilon)\}^2\, f(\omega)\,. \quad (4.1.20)$$

As before, G_+ vanishes for large negative u and G_- vanishes for large positive u up to

[1] If $F(t)$ is real, we must have from (4.1.1) $\widehat{F}(\omega) = \widehat{F}(-\omega)^*$. Now since $\ln(-\omega + i\epsilon)^* = \ln(-\omega - i\epsilon) = \ln(\omega + i\epsilon) - i\pi$, we see that $\ln(\omega + i\epsilon)$ is not a good candidate for $\widetilde{F}(\omega)$. This can be rectified by averaging over $\ln(\omega + i\epsilon)$ and $\ln(-\omega - i\epsilon)$. However since the two differ by a constant, whose Fourier transform, being proportional to $\delta(u)$, does not affect the behaviour at large $|u|$, we shall ignore this complication. A similar remark holds for $\ln(\omega - i\epsilon)$.

exponentially suppressed corrections. Furthermore we have

$$
\begin{aligned}
G_+ - G_- &= 4\pi\, i\, C \int_{-\infty}^{0} \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \omega\, \{\ln(\omega - i\epsilon) + i\pi\}\, f(\omega) \\
&= -4\pi\, C\, \frac{d}{du} \int_{-\infty}^{0} \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \{\ln(\omega - i\epsilon) + i\pi\}\, f(\omega)\,. \qquad (4.1.21)
\end{aligned}
$$

Changing integration variable to $v = \omega\, u$ we can express this as

$$
\begin{aligned}
G_+ - G_- &= -2\, C\, \frac{d}{du}\left[u^{-1} \int_{-\infty\times \mathrm{sign}\, u}^{0} dv\, e^{-iv}\, \{\ln(v - i\epsilon) - \ln u + i\pi\}\, f(v/u)\right] \\
&= -2\, C\, \frac{d}{du}\left[-i\, u^{-1}\, \ln u + O(u^{-1})\right] = -2\, i\, C\, u^{-2}\, \ln|u| + O(u^{-2})\,. \quad (4.1.22)
\end{aligned}
$$

This gives

$$
G_+ \equiv C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \omega\{\ln(\omega + i\epsilon)\}^2 f(\omega) \to \begin{cases} -\, 2\, i\, C\, u^{-2}\, \ln|u| \quad \text{for} \quad u \to \infty, \\ \\ 0 \quad \text{for} \quad u \to -\infty, \end{cases}
$$

$$
G_- \equiv C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \omega\{\ln(\omega - i\epsilon)\}^2 f(\omega) \to \begin{cases} 0 \quad \text{for} \quad u \to \infty, \\ \\ 2\, i\, C\, u^{-2}\, \ln|u| \quad \text{for} \quad u \to -\infty\,, \end{cases} \qquad (4.1.23)
$$

up to corrections of order u^{-2}.

Finally we consider the integral:

$$
H \equiv C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \omega\, \ln(\omega + i\epsilon)\, \ln(\omega - i\epsilon)\, f(\omega)\,. \qquad (4.1.24)
$$

For evaluating this we use the result:

$$
G_+ + G_- - 2\, H \;=\; C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \omega\, \{\ln(\omega + i\epsilon) - \ln(\omega - i\epsilon)\}^2\, f(\omega)
$$

$$= \quad -2\pi C \int_{-\infty}^{0} d\omega\, e^{-i\omega u}\, \omega\; f(\omega) = O(u^{-2})\,. \tag{4.1.25}$$

Using (4.1.23) we now get:

$$H \equiv C \int \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \omega\, \ln(\omega + i\epsilon)\, \ln(\omega - i\epsilon)\, f(\omega) \rightarrow \begin{cases} -\,i\, C\, u^{-2}\, \ln|u| & \text{for} \quad u \to \infty\,, \\ \\ i\, C\, u^{-2}\, \ln|u| & \text{for} \quad u \to -\infty\,. \end{cases} \tag{4.1.26}$$

4.2 Proof of classical soft graviton theorem

We consider a scattering event in asymptotically flat space-time in which m objects carrying masses $\{m'_a\}$, four velocities $\{v'_a\}$ and four momenta $\{p'_a = m'_a\, v'_a\}$ for $1 \leq a \leq m$ come close, undergo complicated interactions, and disperse as n objects carrying masses $\{m_a\}$, four velocities $\{v_a\}$ and four momenta $\{p_a\}$ for $1 \leq a \leq n$. We do not assume that the interactions are weak, and they could involve exchange of energy and other quantum numbers, fusion and splitting. Our goal will be to compute the gravitational wave-form emitted during this scattering event at early and late retarded time. As discussed in §4.1.3, this is related to the behaviour of the Fourier transform of the wave-form in the low frequency limit.

Since we shall be interested in the long wavelength gravitational waves emitted by the system, we can represent the leading contribution to the energy momentum tensor of the incoming and outgoing objects by the energy momentum tensor of point particles, and include the effect of internal structure of the objects by adding subleading contributions involving higher derivative terms [121–129]. In fact, to the order at which we shall be working, it will be sufficient to keep just the leading term. For this reason, we shall henceforth refer to the incoming and outgoing objects as particles.

The strategy we shall follow will be to iteratively solve the coupled equations of motion of matter and gravity using Feynman diagram like techniques. This method has been widely used in recent years [112, 113, 130–132], most notably in [120, 133–135]. However the main difference between our approach and the earlier ones is in setting up the boundary conditions. In the usual approach we set the initial condition and evolve the system using the equations of motion, computing both the trajectories and the emitted radiation during this process. In our approach we take the initial and final momenta as given, but allow the interactions during the scattering to be arbitrary. Therefore while solving the equations we need to evolve the initial particle trajectories forward in time and the final particle trajectories backward in time, and compute the net gravitational wave emitted during the scattering.

For simplicity, in this section we shall consider the situation where the particles are uncharged so that there are no long range electromagnetic interactions between the asymptotic particles. The effect of such interactions will be incorporated in §4.3.4.

4.2.1 General set-up

We choose the origin of the space-time coordinate system to be somewhere within the region where the scattering takes place and denote by $\mathcal{R}$ a large but finite region of space-time so that the non-trivial part of the scattering occurs within the region $\mathcal{R}$. In particular we shall choose $\mathcal{R}$ to be sufficiently large so that outside the region $\mathcal{R}$ the only interaction that exists between the particles is the long range gravitational interaction. This has been shown in Fig. 4.1. We shall denote by L the linear size of $\mathcal{R}$ and analyze gravitational radiation at retarded time u for $|u| >> L$.

We define:

$$h_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} - \eta_{\mu\nu}), \quad e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma} \;\Leftrightarrow\; h_{\mu\nu} = e_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}e_{\rho\sigma}\,. \tag{4.2.1}$$

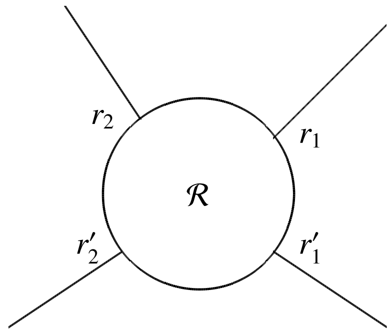

Figure 4.1: A scattering process in which the particles interact strongly inside the region $\mathcal{R}$ via some unspecified forces, but outside the region $\mathcal{R}$ the only force operative between the particles is the long range gravitational force.

We denote by $X_a(\sigma)$ for $1 \le a \le n$ the outgoing particle trajectories parametrized by the proper time[2] σ in the range $0 \le \sigma < \infty$, with $\sigma = 0$ labelling the point where the trajectory exits the region $\mathcal{R}$. Similarly $X'_a(\sigma)$ for $1 \le a \le m$ will denote the incoming particle trajectories parametrized by the proper time σ in the range $-\infty < \sigma \le 0$, with $\sigma = 0$ labelling the point where the trajectory enters the region $\mathcal{R}$. We now consider the Einstein's action coupled to these particles:

$$\begin{aligned} S \;=\; & \frac{1}{16\pi G}\int d^4x\,\sqrt{-\det g}\,R - \sum_{a=1}^{n}\int_0^\infty d\sigma\, m_a \left\{-g_{\mu\nu}(X(\sigma))\,\frac{dX_a^\mu}{d\sigma}\,\frac{dX_a^\nu}{d\sigma}\right\}^{1/2} \\ & -\sum_{a=1}^{m}\int_{-\infty}^{0} d\sigma\, m'_a \left\{-g_{\mu\nu}(X'(\sigma))\,\frac{dX_a'^\mu}{d\sigma}\,\frac{dX_a'^\nu}{d\sigma}\right\}^{1/2} . \end{aligned} \tag{4.2.2}$$

Note that we have included in the action the contribution only from part of the particle trajectories that lie outside the region $\mathcal{R}$. We shall argue later that this action is sufficient for determining the gravitational wave-form at late and early time. We now derive the equations of motion for $e_{\mu\nu}$ by extremizing the action (4.2.2) with respect to $e_{\mu\nu}$. This takes the form:

$$\sqrt{-\det g}\left(R^{\mu\nu} - \frac{1}{2}g^{\rho\sigma}R_{\rho\sigma}\,g^{\mu\nu}\right) = 8\,\pi\,G\,T^{X\mu\nu}\,, \tag{4.2.3}$$

[2]More precisely, σ is a parameter labelling the trajectory, that is set equal to the proper time after deriving the equations of motion.

where,

$$T^{X\mu\nu} \equiv \sum_{a=1}^{n} m_a \int_0^\infty d\sigma\, \delta^{(4)}(x - X_a(\sigma))\, \frac{dX_a^\mu}{d\sigma}\frac{dX_a^\nu}{d\sigma} + \sum_{a=1}^{m} m'_a \int_{-\infty}^0 d\sigma\, \delta^{(4)}(x - X'_a(\sigma))\, \frac{dX_a'^\mu}{d\sigma}\frac{dX_a'^\nu}{d\sigma}\,. \tag{4.2.4}$$

Note the factor of $\sqrt{-\det g}$ and the raised indices on the left hand side of (4.2.3) – this makes the right hand side independent of the metric. After imposing the de Donder gauge:

$$\eta^{\mu\nu}\partial_\mu h_{\nu\lambda} - \frac{1}{2}\partial_\lambda\left(\eta^{\rho\sigma}h_{\rho\sigma}\right) = 0 \quad \Leftrightarrow \quad \eta^{\mu\nu}\,\partial_\mu\, e_{\nu\lambda} = 0\,, \tag{4.2.5}$$

and expanding the left hand side of (4.2.3) in power series in $h_{\mu\nu}$, we can express the equations of motion of the metric as:

$$\eta^{\alpha\mu}\,\eta^{\beta\nu}\,\eta^{\rho\sigma}\partial_\rho\partial_\sigma e_{\alpha\beta} = -8\,\pi\, G\, T^{\mu\nu}(x), \qquad T^{\mu\nu} \equiv T^{X\mu\nu} + T^{h\mu\nu}\,, \tag{4.2.6}$$

where $T^{h\mu\nu}$ denotes the gravitational stress tensor, defined as what we obtain by taking all $e_{\alpha\beta}$ dependent terms on the left hand side of (4.2.3), except the terms linear in $e_{\alpha\beta}$, to the right hand side and dividing it by $8\,\pi\, G$. In all subsequent equations, the indices will be raised and lowered by $\eta_{\mu\nu}$.

Our goal is to compute $e_{\mu\nu}(t, \vec{x})$ at a point far away from the scattering center. We shall label $\vec{x}$ as $R\,\hat{n}$ where $\hat{n}$ is a unit vector and $R \equiv |\vec{x}|$. It follows from (4.1.4) and (4.1.11) that the retarded solution to (4.2.6) is given by [19][3]

$$\tilde{e}_{\mu\nu}(\omega, R, \hat{n}) = \frac{2\,G}{R}\, e^{i\omega R}\, \widehat{T}_{\mu\nu}(k) + O(R^{-2})\,, \tag{4.2.7}$$

where

$$\widehat{T}_{\mu\nu}(k) \equiv \int d^4x\, e^{-ik.x}\, T_{\mu\nu}(x)\,, \tag{4.2.8}$$

[3](4.2.7) can also be written as

$$e_{\mu\nu}(t, R, \hat{n}) = \frac{2\,G}{R}\, \tilde{T}_{\mu\nu}(t - R, \vec{k}) + O(R^{-2})\,.$$

is the Fourier transform of $T_{\mu\nu}(x)$ in all the variables and $k = \omega(1, \hat{n})$ as defined in (4.1.10). Therefore we need to compute $\widehat{T}_{\mu\nu}(k)$. Furthermore, it follows from the analysis of §4.1.3 that to extract the late and early time behaviour of $e_{\mu\nu}(t, \vec{x})$ we need to examine the non-analytic part of $\tilde{e}_{\mu\nu}(\omega, R, \hat{n})$ as a function of ω – in particular terms of order $1/\omega$ and $\ln\omega$. For this, we can restrict the integration over x in (4.2.8) to outside the region $\mathcal{R}$, since integration over a finite region of space-time will give an infrared finite contribution and cannot generate a singularity as $\omega \to 0$. This justifies the omission of the contribution to the action (4.2.2) from particle trajectories inside the region $\mathcal{R}$.

We shall compute $\widehat{T}_{\mu\nu}$ by solving the following equations iteratively:

$$
\begin{aligned}
T^{\mu\nu}(x) &= T^{X\mu\nu}(x) + T^{h\mu\nu}(x), \\
T^{X\mu\nu}(x) &\equiv \sum_{a=1}^{n} m_a \int_0^\infty d\sigma\, \delta^{(4)}(x - X_a(\sigma))\, \frac{dX_a^\mu}{d\sigma}\frac{dX_a^\nu}{d\sigma} \\
&\quad + \sum_{a=1}^{m} m'_a \int_{-\infty}^0 d\sigma\, \delta^{(4)}(x - X'_a(\sigma))\, \frac{dX_a'^\mu}{d\sigma}\frac{dX_a'^\nu}{d\sigma}, \\
\Box\, e_{\mu\nu} &= -8\,\pi\, G\, T_{\mu\nu} \equiv -8\,\pi\, G\, \eta_{\mu\alpha}\,\eta_{\nu\beta}\, T^{\alpha\beta},
\end{aligned}
$$

$$
\frac{d^2X_a^\mu}{d\sigma^2} = -\Gamma^\mu_{\nu\rho}(X(\sigma))\,\frac{dX_a^\nu}{d\sigma}\frac{dX_a^\rho}{d\sigma}, \qquad \frac{d^2X_a'^\mu}{d\sigma^2} = -\Gamma^\mu_{\nu\rho}(X'(\sigma))\,\frac{dX_a'^\nu}{d\sigma}\frac{dX_a'^\rho}{d\sigma}, \tag{4.2.9}
$$

with boundary conditions:

$$
X_a^\mu(\sigma=0) = r_a^\mu, \quad \lim_{\sigma\to\infty}\frac{dX_a^\mu}{d\sigma} = v_a^\mu = \frac{1}{m_a}\, p_a^\mu, \quad X_a'^\mu(\sigma=0) = r_a'^\mu, \quad \lim_{\sigma\to-\infty}\frac{dX_a'^\mu}{d\sigma} = v_a'^\mu = \frac{1}{m'_a}\, p_a'^\mu. \tag{4.2.10}
$$

Here $\Gamma^\mu_{\nu\rho}$ denotes the Christoffel symbol constructed from the metric $\eta_{\mu\nu}+2\,h_{\mu\nu}$. r_a denotes the point where the trajectory of the a-th outgoing particle intersects the boundary of $\mathcal{R}$ and r'_a denotes the point where the trajectory of the a-th incoming particle intersects the boundary of $\mathcal{R}$. T^h is the stress tensor of gravity, as defined below (4.2.6). $h_{\mu\nu}$ and hence $e_{\mu\nu}$ is required to satisfy retarded boundary condition. The starting solution for the iteration is taken to be

$$e_{\mu\nu} = 0, \quad X_a^\mu(\sigma) = r_a^\mu + v_a^\mu\,\sigma = r_a^\mu + \frac{1}{m_a}\,p_a^\mu\,\sigma, \quad X_a'^\mu(\sigma) = r_a'^\mu + v_a'^\mu\,\sigma = r_a'^\mu + \frac{1}{m_a'}\,p_a'^\mu\,\sigma\,. \tag{4.2.11}$$

We can give a uniform treatment of the incoming and the outgoing particles by defining:

$$X_{a+n}^\mu(\sigma) = X_a'^\mu(-\sigma), \quad m_{a+n} = m_a', \quad v_{a+n}^\mu = -v_a'^\mu, \quad r_{a+n}^\mu = r_a'^\mu, \quad p_{a+n}^\mu = -p_a'^\mu, \quad \text{for } 1 \le a \le m\,. \tag{4.2.12}$$

In this case we can express (4.2.9) and (4.2.10) as:

$$\begin{aligned}
&T^{\mu\nu}(x) = T^{X\mu\nu}(x) + T^{h\mu\nu}(x),\\
&T^{X\mu\nu}(x) \equiv \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\,\delta^{(4)}(x - X_a(\sigma))\,\frac{dX_a^\mu}{d\sigma}\,\frac{dX_a^\nu}{d\sigma}\\
&\Box e_{\mu\nu} = -8\,\pi\,G\,T_{\mu\nu}, \qquad \frac{d^2X_a^\mu}{d\sigma^2} = -\Gamma^\mu_{\nu\rho}(X(\sigma))\,\frac{dX_a^\nu}{d\sigma}\,\frac{dX_a^\rho}{d\sigma}\,, \quad \text{for } 1 \le a \le m+n\,,
\end{aligned} \tag{4.2.13}$$

and

$$X_a^\mu(\sigma = 0) = r_a^\mu, \quad \lim_{\sigma\to\infty} \frac{dX_a^\mu}{d\sigma} = v_a^\mu = \frac{1}{m_a}\,p_a^\mu, \quad \text{for } 1 \le a \le m+n\,. \tag{4.2.14}$$

Also the starting solution (4.2.11) for iteration may be written as

$$e_{\mu\nu} = 0, \quad X_a^\mu(\sigma) = r_a^\mu + v_a^\mu\,\sigma = r_a^\mu + \frac{1}{m_a}\,p_a^\mu\,\sigma, \quad \text{for } 1 \le a \le m+n\,. \tag{4.2.15}$$

From now on we shall follow this convention, with the understanding that the sum over a always runs from 1 to $(m+n)$ unless stated otherwise.

4.2.2 Leading order contribution

At the leading order in the expansion in powers of G, $T^{h\mu\nu}$ vanishes, and we have:

$$\begin{aligned}\widehat{T}^{\mu\nu}(k) = \widehat{T}^{X\mu\nu}(k) &= \int d^4x\, e^{-ik.x} \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, \delta^{(4)}(x - X_a(\sigma)) \frac{dX_a^\mu}{d\sigma} \frac{dX_a^\nu}{d\sigma} \\ &= \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, e^{-ik.X(\sigma)} \frac{dX_a^\mu}{d\sigma} \frac{dX_a^\nu}{d\sigma}, \qquad (4.2.16)\end{aligned}$$

where, as mentioned earlier, we have restricted the region of integration over x to outside the region $\mathcal{R}$. Using the leading order solution (4.2.15) we get

$$\begin{aligned}\widehat{T}^{\mu\nu}(k) &= \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, e^{-ik.(v_a\sigma + r_a)}\, v_a^\mu v_a^\nu = \sum_{a=1}^{m+n} m_a \frac{1}{i(k.v_a - i\epsilon)} e^{-ik.r_a}\, v_a^\mu v_a^\nu \\ &= \sum_{a=1}^{m+n} p_a^\mu p_a^\nu e^{-ik.r_a} \frac{1}{i(k.p_a - i\epsilon)}. \qquad (4.2.17)\end{aligned}$$

The $i\epsilon$ prescription is obtained by noting that addition of a small negative imaginary part to $k.v_a$ makes the σ integrals convergent. Therefore the poles must be in the upper half $k.v_a$ plane.

Since we are looking for terms that are singular at $\omega \to 0$, i.e. $k^\mu \to 0$, we can replace the $e^{-ik.r_a}$ factors by 1. This gives the leading soft factor associated with the memory effect.

4.2.3 First order correction to the gravitational field

We now turn to the next order contribution. We first solve for $e_{\mu\nu}$ satisfying the third equation in (4.2.13) as

$$\begin{aligned}\widehat{e}_{\mu\nu}(k) &= -8\pi G\, G_r(k)\, \widehat{T}_{\mu\nu}(k) = -8\pi G \sum_{a=1}^{m+n} p_{a\mu}\, p_{a\nu}\, e^{-ik.r_a} G_r(k) \frac{1}{i(k.p_a - i\epsilon)}, \\ G_r(k) &\equiv \frac{1}{(k^0 + i\epsilon)^2 - \vec{k}^2}. \qquad (4.2.18)\end{aligned}$$

One comment is in order here. The expression (4.2.16) for $\widehat{T}^{\mu\nu}(k)$, which we are using in (4.2.18), ignores the contribution from the region of integration $\mathcal{R}$. This was justified earlier since we were computing the singular part of $\widehat{T}_{\mu\nu}$. However, now we need the contribution to $\widehat{e}_{\mu\nu}$ from the full $\widehat{T}_{\mu\nu}$ since our goal will be to use this to compute $T^h_{\mu\nu}$, and also to compute the corrections to the particle trajectories, which, in turn, give corrections to $T^X_{\mu\nu}$. Once we compute these, we use (4.2.7) to compute $\widetilde{e}_{\mu\nu}$. At this stage, we can again restrict the integration region to outside $\mathcal{R}$ while taking the Fourier transform to compute the corrected $\widehat{T}_{\mu\nu}$. To address this issue, we first analyze the possible correction $\delta T^X_{\mu\nu}$ to $T^X_{\mu\nu}$ due to gravitational fields generated from inside $\mathcal{R}$. Since in four space-time dimensions the retarded Green's function has support on the future light-cone, the field sourced by energy momentum tensor inside $\mathcal{R}$ will have support on the future light-cone emerging from points inside $\mathcal{R}$. These intersect the time-like trajectories of the outgoing (or incoming) particles emerging from $\mathcal{R}$ only within a distance of order L – the size of $\mathcal{R}$. Therefore $\delta T^X_{\mu\nu}$ is affected only in this region. Since integration over this region will not produce a singular contribution to $\widehat{T}^X_{\mu\nu}(k)$ in the $\omega \to 0$ limit, this effect may be ignored. However the gravitational field produced from the sources inside $\mathcal{R}$ could give significant contribution to T^h, since we are not assuming the interactions inside $\mathcal{R}$ to be weak. We take this into account by regarding the contribution to $\widehat{e}_{\mu\nu}(k) = -8\pi G\, G_r(k)\widehat{T}_{\mu\nu}(k)$ from inside the region $\mathcal{R}$ as a flux of finite wavelength gravitational waves produced by $T_{\mu\nu}(x)$ inside $\mathcal{R}$, and include this in the sum over a. Therefore the outgoing momenta $\{p_a\}$ not only will include finite mass particles, but also the finite wave-length 'massless gravitons' emitted during the scattering process.

Using (4.2.18) we can calculate, at the next order,

$$
\begin{aligned}
e^{(b)}_{\mu\nu}(x) &= -8\pi G \int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.(x-r_b)} G_r(\ell)\, p_{b\mu}\, p_{b\nu}\, \frac{1}{i(\ell.p_b - i\epsilon)}, \\
h^{(b)}_{\mu\nu}(x) &= -8\pi G \int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.(x-r_b)} G_r(\ell) \left\{ p_{b\mu}\, p_{b\nu} - \frac{1}{2}\, p_b^2\, \eta_{\mu\nu} \right\} \frac{1}{i(\ell.p_b - i\epsilon)}, \qquad (4.2.19)
\end{aligned}
$$

where $e^{(b)}_{\mu\nu}$ is the gravitational field due to the b-th particle. This gives

$$\begin{aligned}
\Gamma^{(b)\mu}_{\nu\rho}(x) &= \eta^{\mu\alpha}\left\{\partial_\nu h^{(b)}_{\alpha\rho} + \partial_\rho h^{(b)}_{\alpha\nu} - \partial_\alpha h^{(b)}_{\nu\rho}\right\} \\
&= -8\pi G \int \frac{d^4\ell}{(2\pi)^4} e^{i\ell.(x-r_b)}\, G_r(\ell)\, \frac{1}{(\ell.p_b - i\epsilon)}\Big[\left\{\ell_\nu p^\mu_b p_{b\rho} + \ell_\rho p^\mu_b p_{b\nu} - \ell^\mu p_{b\nu} p_{b\rho}\right\} \\
&\quad -\frac{1}{2}\, p_b^2 \left\{\ell_\nu \delta^\mu_\rho + \ell_\rho \delta^\mu_\nu - \ell^\mu \eta_{\nu\rho}\right\}\Big]. \qquad (4.2.20)
\end{aligned}$$

These results will be used for two purposes. We shall substitute (4.2.20) into the last equation in (4.2.13) to compute the correction to the outgoing particle trajectories and hence to $T^X_{\mu\nu}$. We shall also use (4.2.19) to compute the leading contribution to $T^h_{\mu\nu}$.

Note that $e^{(b)}_{\mu\nu}(x)$ given in (4.2.19) satisfies:

$$\partial_\mu e^{\mu\nu} = \sum_{b=1}^{m+n} \partial_\mu\, e^{(b)\mu\nu}(x) = -8\pi G \sum_{b=1}^{m+n} \int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.(x-r_b)} G_r(\ell)\; p^\nu_b\,. \qquad (4.2.21)$$

As long as we restrict the integration range of ℓ to values for which $\ell.(r_c - r_a)$ is small for every pair a, c, we can take $e^{-i\ell.r_b}$ to be approximately independent of b, and the right hand side of (4.2.21) vanishes due to momentum conservation law $\sum_{b=1}^{m+n} p^\mu_b = 0$. Therefore $e^{\mu\nu}$ at this order satisfies the de Donder gauge condition:

$$\partial^\mu e_{\mu\nu} = 0\,. \qquad (4.2.22)$$

At the next order there is apparent violation of this condition due to the $\ell.r_b$ factors coming from the expansion of the exponential factor. This can be compensated by some boundary terms on $\partial\mathcal{R}$ coming from integration inside the region $\mathcal{R}$ [21], but since these terms will not contribute to the singular terms that are of interest to us, we shall ignore them.

In the next two subsections we shall compute the correction to $\widehat{T}^X$ and $\widehat{T}^h$ using these results. It is also possible to argue that in order to calculate the logarithmic terms of interest, we can stop at this order. The natural dimensionless expansion parameter is $GM\omega$ where M denotes the typical energy of the incoming / outgoing particles. Since the leading term (4.2.17) is of order $1/\omega$, the subleading corrections that we shall compute will be of order ω^0 multiplied by powers of $\ln\omega$. Higher order terms will involve higher powers of ω and will not be needed for our analysis.

4.2.4 Subleading contribution to the matter stress tensor

We begin by computing correction to the particle trajectory (4.2.15). Let Y_a^μ denote the correction:

$$X_a^\mu(\sigma) = v_a^\mu\,\sigma + r_a^\mu + Y_a^\mu(\sigma)\,. \tag{4.2.23}$$

Then Y_a^μ satisfies the differential equation and boundary conditions:

$$\frac{d^2Y_a^\mu}{d\sigma^2} = -\Gamma^\mu_{\nu\rho}(v_a\,\sigma + r_a)\,v_a^\nu\,v_a^\rho, \quad Y_a^\mu \to 0 \text{ as } \sigma \to 0, \quad \frac{dY_a^\mu}{d\sigma} \to 0 \text{ as } \sigma \to \infty\,, \tag{4.2.24}$$

where[4]

$$\Gamma^\mu_{\nu\rho} = \sum_{\substack{b=1\\ b\neq a}}^{m+n} \Gamma^{(b)\mu}_{\nu\rho}\,, \tag{4.2.25}$$

captures the effect of the gravitational field produced by all particles other than a. Some of these terms must vanish, *e.g.* the gravitational field produced by an outgoing particle should not affect an incoming particle. This however will follow automatically from the equations that we shall derive, and need not be imposed externally. Integrating (4.2.24) we get

$$\frac{dY_a^\mu(\sigma)}{d\sigma} = \int_\sigma^\infty d\sigma'\,\Gamma^\mu_{\nu\rho}(v_a\,\sigma' + r_a)\,v_a^\nu\,v_a^\rho\,, \tag{4.2.26}$$

and

$$Y_a^\mu(\sigma) = \int_0^\sigma d\sigma' \int_{\sigma'}^\infty d\sigma''\,\Gamma^\mu_{\nu\rho}(v_a\,\sigma'' + r_a)\,v_a^\nu\,v_a^\rho\,. \tag{4.2.27}$$

[4]The self-force effects [136] will not be important at this order.

Substituting (4.2.23) into (4.2.16) we get $\widehat{T}^{X\mu\nu}$ to subleading order:

$$\begin{aligned}\widehat{T}^{X\mu\nu}(k) &= \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, e^{-ik.(v_a\,\sigma+r_a)}\,\{1-ik.Y_a(\sigma)\}\left\{v_a^\mu+\frac{dY_a^\mu}{d\sigma}\right\}\left\{v_a^\nu+\frac{dY_a^\nu}{d\sigma}\right\}\\ &= \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, e^{-ik.(v_a\,\sigma+r_a)}\left[v_a^\mu v_a^\nu - ik.Y_a(\sigma)\,v_a^\mu v_a^\nu+\frac{dY_a^\mu}{d\sigma}\,v_a^\nu+v_a^\mu\,\frac{dY_a^\nu}{d\sigma}\right].\end{aligned} \tag{4.2.28}$$

Using (4.2.26), (4.2.27), we can express this as,

$$\begin{aligned}\widehat{T}^{X\mu\nu}(k) = \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, e^{-ik.(v_a\,\sigma+r_a)}\Big[v_a^\mu v_a^\nu - ik_\rho \int_0^\sigma d\sigma' \int_{\sigma'}^\infty d\sigma''\, \Gamma^\rho_{\alpha\beta}(v_a\,\sigma''+r_a) v_a^\alpha v_a^\beta\, v_a^\nu\, v_a^\mu\\ + \int_\sigma^\infty d\sigma'\,\Gamma^\mu_{\alpha\beta}(v_a\,\sigma'+r_a)\,v_a^\alpha v_a^\beta\, v_a^\nu + \int_\sigma^\infty d\sigma'\,\Gamma^\nu_{\alpha\beta}(v_a\,\sigma'+r_a)\,v_a^\alpha v_a^\beta\, v_a^\mu\Big].\end{aligned} \tag{4.2.29}$$

Substituting (4.2.20) and (4.2.25) into (4.2.29), and dropping the leading term given in (4.2.17), we get the first order correction to $\widehat{T}^X$:

$$\begin{aligned}\Delta\widehat{T}^{X\mu\nu}(k) &= -8\,\pi\, G \sum_{a=1}^{m+n}\sum_{b\neq a} m_a \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell.p_b-i\epsilon}\,G_r(\ell)\Big[\int_0^\infty d\sigma\int_0^\sigma d\sigma'\int_{\sigma'}^\infty d\sigma''\\ &\quad e^{-i\,k.v_a\sigma}\,e^{i\ell.v_a\sigma''}\Big\{-i\,v_a.p_b\,(2k.p_b\,v_a.\ell-k.\ell\,v_a.p_b)+\frac{i}{2}\,p_b^2(2k.v_a\,v_a.\ell-k.\ell\,v_a^2)\Big\}\,v_a^\nu\,v_a^\mu\\ &\quad + \int_0^\infty d\sigma\int_\sigma^\infty d\sigma'\,e^{-i\,k.v_a\sigma}\,e^{i\ell.v_a\sigma'}\\ &\quad \Big\{2\,\ell.v_a\,v_a.p_b\left(v_a^\nu p_b^\mu+v_a^\mu p_b^\nu\right)-(v_a.p_b)^2\left(\ell^\mu v_a^\nu+\ell^\nu v_a^\mu\right)\\ &\quad -2\,\ell.v_a\,p_b^2\,v_a^\mu\,v_a^\nu+\frac{1}{2}\,v_a^2 p_b^2\left(\ell^\mu v_a^\nu+\ell^\nu v_a^\mu\right)\Big\}\Big]e^{-ik.r_a-i\ell.(r_b-r_a)}\,.\end{aligned} \tag{4.2.30}$$

After carrying out the integrations over σ,σ',σ'', and using $p_a^\mu = m_a v_a^\mu$, we get

$$\begin{aligned}\Delta\widehat{T}^{X\mu\nu}(k) &= -8\,\pi\, G\sum_{a=1}^{m+n}\sum_{b\neq a}\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell.p_b-i\epsilon}\,G_r(\ell)\,e^{-ik.r_a-i\ell.(r_b-r_a)}\\ &\Big[\Big(2\,p_a.p_b\,k.p_b\,p_a.\ell-k.\ell\,(p_a.p_b)^2-p_b^2\,p_a.k\,p_a.\ell+\frac{1}{2}k.\ell\,p_a^2\,p_b^2\Big)\,p_a^\nu\,p_a^\mu\,\frac{1}{\ell.p_a}\,\frac{1}{k.p_a}\,\frac{1}{(\ell-k).p_a}\\ &-\Big\{2\,p_a.p_b\,\ell.p_a\left(p_a^\nu p_b^\mu+p_a^\mu p_b^\nu\right)-(p_a.p_b)^2\left(\ell^\mu p_a^\nu+\ell^\nu p_a^\mu\right)-2\,p_b^2\,\ell.p_a\,p_a^\mu\,p_a^\nu\\ &\qquad +\frac{1}{2}\,p_a^2\,p_b^2\left(\ell^\mu p_a^\nu+\ell^\nu p_a^\mu\right)\Big\}\times\frac{1}{\ell.p_a}\,\frac{1}{(\ell-k).p_a}\Big].\end{aligned} \tag{4.2.31}$$

For $|r_a^\mu - r_b^\mu| \sim L$, the ultraviolet divergence in the integration over ℓ is cut-off at L^{-1} due to the oscillatory phase factor $e^{-i\ell.(r_b-r_a)}$.

In order to evaluate the integral, we need to determine the $i\epsilon$ prescription for the poles in (4.2.31). The $i\epsilon$ prescription for the $1/\ell.p_b$ term has already been determined before. Similarly, since the $1/\ell.p_a$ factor comes from an integral in (4.2.30) of the form $\int_{\sigma'}^\infty d\sigma''\, e^{i\ell.v_a\sigma''}$ or $\int_\sigma^\infty d\sigma'\, e^{i\ell.v_a\sigma'}$, the $i\epsilon$ prescription will be to replace $1/\ell.p_a$ by $1/(\ell.p_a + i\epsilon)$. The $1/k.p_a$ factor comes from an integral of the form $\int_0^\infty d\sigma\, e^{-ik.v_a\sigma}$, and the $i\epsilon$ prescription will be to replace $1/k.p_a$ by $1/(k.p_a - i\epsilon)$. Finally, the $1/(\ell-k).p_a$ factor in (4.2.31) arises from an integral of the form $\int_0^\infty d\sigma\, e^{i(\ell-k).v_a\sigma}$, and the correct $i\epsilon$ prescription for this term is $1/((\ell-k).p_a + i\epsilon)$. Therefore, (4.2.31) should be written as

$$\begin{aligned}
\Delta\widehat{T}^{X\mu\nu}(k) \;=\; & -8\pi G \sum_{a=1}^{m+n}\sum_{b\neq a}\int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell.p_b - i\epsilon}\, G_r(\ell)\, e^{-ik.r_a - i\ell.(r_b-r_a)} \\
& \Bigg[\Big(2\, p_a.p_b\, k.p_b\, p_a.\ell - k.\ell\,(p_a.p_b)^2 - p_b^2\, p_a.k\, p_a.\ell + \frac{1}{2}k.\ell\, p_a^2\, p_b^2\Big)\, p_a^\nu\, p_a^\mu \\
& \qquad \times \frac{1}{\ell.p_a + i\epsilon}\,\frac{1}{k.p_a - i\epsilon}\,\frac{1}{(\ell-k).p_a + i\epsilon} \\
& -\Big\{2\, p_a.p_b\, \ell.p_a\left(p_a^\nu p_b^\mu + p_a^\mu p_b^\nu\right) - (p_a.p_b)^2\left(\ell^\mu p_a^\nu + \ell^\nu p_a^\mu\right) - 2\, p_b^2\, \ell.p_a\, p_a^\mu\, p_a^\nu + \frac{1}{2}\, p_a^2\, p_b^2\left(\ell^\mu p_a^\nu + \ell^\nu p_a^\mu\right)\Big\} \\
& \qquad \times \frac{1}{\ell.p_a + i\epsilon}\,\frac{1}{(\ell-k).p_a + i\epsilon}\Bigg].
\end{aligned} \tag{4.2.32}$$

Since we are interested in the singular term proportional to $\ln\omega$, we can simplify the analysis of the integral as follows. Since the expression is Lorentz covariant, we could evaluate it in a special frame in which p_a and p_b have only third component of spatial momenta. Let us denote by $\ell_\perp = (\ell^1, \ell^2)$ the transverse component of ℓ. Now since $p_a.\ell$ and $p_b.\ell$ are both linear in ℓ^0 and ℓ^3, we can use $p_a.\ell$ and $p_b.\ell$ as independent variables instead of ℓ^0 and ℓ^3. Then, if we ignore the poles of $G_r(\ell)$, we see that we have one pole in the $p_b.\ell$ plane and two poles on the same side of the real axis in the $p_a.\ell$ plane. Therefore we can deform the $p_a.\ell$ and $p_b.\ell$ integration contours away from the poles. However due to the presence of the $G_r(\ell)$ factor there are also poles at

$$(\ell^0 + i\epsilon + \ell^3)(\ell^0 + i\epsilon - \ell^3) = \ell_\perp^2 \,. \tag{4.2.33}$$

Therefore, for small but fixed $\ell_\perp$, if we deform the $(\ell^0 + \ell^3)$ contour to a distance of order $|\ell_\perp|$ away from the origin, a pole will approach the origin within a distance of order $|\ell_\perp|$ in the complex $(\ell^0 - \ell^3)$ plane. The integration contour could then be pinched between this pole and one of the poles of the $(\ell.p_a + i\epsilon)^{-1}\{(\ell - k).p_a + i\epsilon\}^{-1}(\ell.p_b - i\epsilon)^{-1}$ factor. However it is clear that in the complex ℓ^0 and complex ℓ^3 plane, the integration contour can be deformed so that the contour maintains a minimum distance of order $|\ell_\perp|$ from all the poles, which themselves are situated within a distance of order $|\ell_\perp|$ of the origin. This shows that while estimating the integrand to examine possible sources of singularity of the integral, we can take all the components of ℓ to be of order $\ell_\perp$ and need not worry about the regions where one or more components are smaller than the others. Since for $\ell^\mu \sim \ell_\perp$ the integration measure gives a factor of $|\ell_\perp|^4$, we see that in order to get a logarithmic correction, the integrand must be of order $|\ell_\perp|^{-4}$.

We now note that in both terms the integrand of (4.2.32) grow as $|\ell_\perp|^{-3}$ for $|\ell^\mu| \sim |\ell_\perp| << \omega$ and therefore there are no logarithmic corrections from this region. For $|r_b^\mu - r_a^\mu|^{-1} \sim L^{-1} >> |\ell^\mu| >> \omega$ we can replace $(k - \ell).p_a$ by $-\ell.p_a$, and drop the $e^{-i\ell.(r_b - r_a)}$ factor. In this case the integrand is of order $|\ell_\perp|^{-4}$ and the integral could have logarithmic contributions. To compute this, we note that in this region of integration the integral may be approximated as

$$\begin{aligned}
\Delta\widehat{T}^{X\mu\nu}(k) \quad &\simeq \quad -8\pi G \sum_a \sum_{b\neq a} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell.p_b - i\epsilon} G_r(\ell)\, e^{-ik.r_a} \\
&\Bigg[\Big(2\, p_a.p_b\, k.p_b\, p_a.\ell - k.\ell\,(p_a.p_b)^2 - p_b^2\, p_a.k\, p_a.\ell + \frac{1}{2}k.\ell\, p_a^2\, p_b^2\Big)\, p_a^\nu\, p_a^\mu\, \frac{1}{(\ell.p_a + i\epsilon)^2}\, \frac{1}{k.p_a - i\epsilon} \\
&-\Big\{2\, p_a.p_b\, \ell.p_a\left(p_a^\nu p_b^\mu + p_a^\mu p_b^\nu\right) - (p_a.p_b)^2\Big(\ell^\mu p_a^\nu + \ell^\nu p_a^\mu\Big) - 2\, p_b^2\, \ell.p_a\, p_a^\mu\, p_a^\nu \\
&\qquad + \frac{1}{2}\, p_a^2\, p_b^2\Big(\ell^\mu p_a^\nu + \ell^\nu p_a^\mu\Big)\Big\} \times \frac{1}{(\ell.p_a + i\epsilon)^2}\Bigg].
\end{aligned} \tag{4.2.34}$$

It will be understood that in this integral the integration over $\vec{\ell}_\perp$ is restricted to the region $L^{-1} >> |\vec{\ell}_\perp| >> \omega$. Since for fixed $\vec{\ell}_\perp$, the integration over ℓ^0 and ℓ^3 are finite, we do not need to impose separate cut-off on the ℓ^0 and ℓ^3 integrals. All the terms in (4.2.34) can be expressed in terms of the basic integral

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell.p_b - i\epsilon} G_r(\ell) \frac{1}{(\ell.p_a + i\epsilon)^2} \ell_\alpha = -\frac{\partial}{\partial p_a^\alpha} J_{ab} , \tag{4.2.35}$$

where

$$J_{ab} = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell.p_b - i\epsilon} G_r(\ell) \frac{1}{\ell.p_a + i\epsilon} . \tag{4.2.36}$$

It has been shown in appendix 4.6.1 that J_{ab} vanishes when a represents an incoming particle and b represents an outgoing particle or vice versa. On the other hand when a and b are both ingoing particles or both outgoing particles, we have, from (4.6.6),

$$J_{ab} = \frac{1}{4\pi} \ln\{L(\omega + i\epsilon\eta_a)\} \frac{1}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} , \tag{4.2.37}$$

where η_a is a number that takes value 1 for outgoing particles ($1 \le a \le n$) and -1 for incoming particles ($n + 1 \le a \le m + n$).[5] Using (4.2.37) we can express (4.2.35) as

$$\begin{aligned}
&\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell.p_b - i\epsilon} G_r(\ell) \frac{1}{(\ell.p_a + i\epsilon)^2} \ell_\alpha \\
= \; & -\frac{1}{4\pi} \ln\{L(\omega + i\epsilon\eta_a)\} \frac{\partial}{\partial p_a^\alpha} \frac{1}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} = -\frac{1}{4\pi} \ln\{L(\omega + i\epsilon\eta_a)\} \frac{p_b^2\, p_{a\alpha} - p_a.p_b\, p_{b\alpha}}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}} .
\end{aligned} \tag{4.2.38}$$

We now use this to evaluate the right hand side of (4.2.34). We can also replace $e^{ik.r_a}$ by 1 since the difference is higher order in the small ω limit. This gives

$$\begin{aligned}
\Delta\widehat{T}^{X\mu\nu}(k) \;=\; & 2G \sum_{a=1}^{m+n} \sum_{\substack{b\neq a \\ \eta_a\eta_b=1}} \frac{\ln\{L(\omega + i\epsilon\eta_a)\}}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}} \Bigg[\frac{k.p_b}{k.p_a} p_a^\mu p_a^\nu\, p_a.p_b \left\{ \frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2 \right\} \\
& + \frac{1}{2} p_a^\mu p_a^\nu\, p_a^2\, (p_b^2)^2 - \{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu\}\, p_a.p_b \left\{ \frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2 \right\} \Bigg] .
\end{aligned} \tag{4.2.39}$$

[5] This is opposite to the convention used in the earlier chapters.

The constraint $\eta_a\eta_b = 1$ means that the sum over b runs over incoming particles if a represents an incoming particle and runs over outgoing particles if a represents an outgoing particle.

4.2.5 Subleading contribution from the gravitational stress tensor

Let us now turn to the computation of $T^{h\mu\nu}$ defined via (4.2.6). A detailed calculation shows that to quadratic order in $h_{\mu\nu}$, it has the form:

$$\begin{aligned} 8\pi G\, T^{h\mu\nu} \;=\; & -2\Big[\frac{1}{2}\partial^\mu h_{\alpha\beta}\partial^\nu h^{\alpha\beta} + h^{\alpha\beta}\partial^\mu\partial^\nu h_{\alpha\beta} - h^{\alpha\beta}\partial^\nu\partial_\beta h_\alpha^{\ \mu} - h^{\alpha\beta}\partial^\mu\partial_\beta h_\alpha^{\ \nu} + h^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu} \\ & + \partial^\beta h^{\nu\alpha}\partial_\beta h_\alpha^{\ \mu} - \partial^\beta h^{\alpha\nu}\partial_\alpha h_\beta^{\ \mu}\Big] + h^{\mu\nu}\partial_\rho\partial^\rho h - 2\, h^\mu_{\ \rho}\partial^\sigma\partial_\sigma h^{\nu\rho} - 2\, h^\nu_{\ \rho}\partial^\sigma\partial_\sigma h^{\mu\rho} \\ & + \eta^{\mu\nu}\Big[\frac{3}{2}\partial^\rho h_{\alpha\beta}\partial_\rho h^{\alpha\beta} + 2h^{\alpha\beta}\partial^\rho\partial_\rho h_{\alpha\beta} - \partial^\beta h^{\alpha\rho}\partial_\alpha h_{\beta\rho}\Big] + h\Big[\partial^\rho\partial_\rho h^{\mu\nu} - \frac{1}{2}\partial^\rho\partial_\rho h\, \eta^{\mu\nu}\Big], \end{aligned} \tag{4.2.40}$$

where we have used de Donder gauge condition to simplify the expression. To the order that we shall be working, this is allowed due to the observation made below (4.2.21). This expression differs from some of the more standard expressions given *e.g.* in [137], since we have defined $8\pi G\, T^{h\mu\nu}$ as the collection of the quadratic terms in the expansion of $-\sqrt{-\det g}\,(R^{\mu\nu} - g^{\mu\nu}R/2)$. As already mentioned, all indices in (4.2.40) are raised and lowered using the flat metric η.

We shall manipulate (4.2.40) by expressing $h_{\alpha\beta}$ in the momentum space as given in (4.2.19). This gives a general expression of the form:

$$\begin{aligned} \widehat{T}^{h\mu\nu}(k) \;=\; & -8\pi G \sum_{a,b} e^{-ik.r_a} \int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.(r_a-r_b)}\, G_r(k-\ell)G_r(\ell)\, \frac{1}{p_b.\ell - i\epsilon}\, \frac{1}{p_a.(k-\ell) - i\epsilon} \\ & \times\Big\{p_{b\alpha}p_{b\beta} - \frac{1}{2}p_b^2\eta_{\alpha\beta}\Big\}\, \mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\, \Big\{p_{a\rho}p_{a\sigma} - \frac{1}{2}p_a^2\eta_{\rho\sigma}\Big\}, \end{aligned} \tag{4.2.41}$$

where,

$$
\begin{aligned}
&\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell) \\
= \quad & 2\Big[\frac{1}{2}\ell^{\mu}(k-\ell)^{\nu}\eta^{\rho\alpha}\eta^{\sigma\beta} + (k-\ell)^{\mu}(k-\ell)^{\nu}\eta^{\rho\alpha}\eta^{\sigma\beta} - (k-\ell)^{\nu}(k-\ell)^{\beta}\eta^{\rho\alpha}\eta^{\sigma\mu} \\
& -(k-\ell)^{\mu}(k-\ell)^{\beta}\eta^{\rho\alpha}\eta^{\sigma\nu} + (k-\ell)^{\alpha}(k-\ell)^{\beta}\eta^{\rho\mu}\eta^{\sigma\nu} + (k-\ell).\ell\,\eta^{\beta\nu}\eta^{\alpha\rho}\eta^{\sigma\mu} \\
& -\ell^{\rho}(k-\ell)^{\alpha}\eta^{\beta\nu}\eta^{\sigma\mu} - \frac{1}{2}(k-\ell)^{2}\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\sigma} \ + \eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\nu\sigma}(k-\ell)^{2} \ + \ \eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\mu\sigma}(k-\ell)^{2}\Big] \\
& -\eta^{\mu\nu}\Big[\frac{3}{2}(k-\ell).\ell\,\eta^{\rho\alpha}\eta^{\sigma\beta} + 2(k-\ell)^{2}\eta^{\rho\alpha}\eta^{\sigma\beta} - \ell^{\sigma}(k-\ell)^{\alpha}\eta^{\rho\beta}\Big] \\
& -\eta^{\alpha\beta}(k-\ell)^{2}\eta^{\rho\mu}\eta^{\sigma\nu} + \frac{1}{2}\eta^{\alpha\beta}(k-\ell)^{2}\eta^{\rho\sigma}\eta^{\mu\nu}\,. \qquad (4.2.42)
\end{aligned}
$$

In the $\ell^{\mu} \to 0$ limit the integrand diverges as $|\ell^{\mu}|^{-4}$ and therefore the integral has logarithmic infrared divergence. As discussed below (4.1.11), the lower cut-off on the ℓ^{μ} integral in this case is provided by R^{-1} where R is the distance to the detector (measured in flat metric). Formally, this can be achieved by adding to $k^0 = \omega$ a small imaginary part proportional to R^{-1}.

Now in (4.2.41) the $G_r(\ell)\, G_r(k-\ell)$ factor takes the form:

$$
G_r(\ell)\, G_r(k-\ell) = \frac{1}{(\ell^0 + i\epsilon)^2 - \vec{\ell}^2}\,\frac{1}{(k^0 - \ell^0 + i\epsilon)^2 - (\vec{k} - \vec{\ell})^2}\,. \qquad (4.2.43)
$$

As a result the poles of the two denominators in the ℓ^0 plane are on the opposite sides of the integration contour lying along the real axis. We shall express this as:

$$
G_r(\ell)^{*} G_r(k-\ell) - 2\,i\,\pi\,\delta(\ell^2)\Big\{H(\ell^0) - H(-\ell^0)\Big\}\, G_r(k-\ell)\,, \qquad (4.2.44)
$$

where H is the Heaviside step function. In this case in the first term the poles in both factors are in the upper half ℓ^0 plane. This allows us to deform the ℓ^0 contour away from these poles till we hit the zeros of the other denominators. In particular, following the argument given in the paragraph containing (4.2.33), one can argue that for $|\ell_\perp| > \omega$, we can deform the contours such that it maintains a distance of order $\ell_\perp$ from all the poles. We shall show in appendix 4.6.2 that the contribution from the terms proportional to $\delta(\ell^2)$ in (4.2.44) represents the contribution to $\widehat{T}^{\mu\nu}$ from the gravitational radiation (real gravitons) emitted during the scattering. Since this contribution has already been included by including the radiation contribution in the sum over a, we shall not discuss them any further in this section.

We shall now analyze possible logarithmic contribution to (4.2.41) with $G_r(\ell)$ replaced by $G_r(\ell)^*$. These can arise from three regions: $R^{-1} << |k^\mu - \ell^\mu| << \omega$, $R^{-1} << |\ell^\mu| << \omega$ and $L^{-1} >> |\ell^\mu| >> \omega$. Since each term in (4.2.42) has at least one power of $(k - \ell)$, one finds by simple power counting that there is no logarithmic contribution from the region $R^{-1} << |k^\mu - \ell^\mu| << \omega$. For $R^{-1} << |\ell^\mu| << \omega$ the integrand has four powers of ℓ in the denominator and could give logarithmic contribution. In this region we can replace the integrand by its leading term in the $\ell \to 0$ limit. In particular $\mathcal{F}^{\mu\nu}_{\alpha\beta;\rho\sigma}(k,\ell)$ may be approximated as

$$\mathcal{F}^{\mu\nu}_{\alpha\beta;\rho\sigma}(k,\ell) \simeq 2\,k^\mu\,k^\nu\,\eta_{\alpha\rho}\,\eta_{\beta\sigma} - 2\,k^\nu\,k_\beta\,\eta_{\rho\alpha}\,\delta^\mu_\sigma - 2\,k^\mu\,k_\beta\,\eta_{\rho\alpha}\,\delta^\nu_\sigma + 2k_\alpha\,k_\beta\,\delta^\mu_\rho\delta^\nu_\sigma\,, \tag{4.2.45}$$

where we have used $k^\rho k_\rho = 0$. A further simplification is possible by noting that eventually we shall use the $\widehat{T}^{h\mu\nu}(k)$ computed from (4.2.41) to calculate its contribution to subleading correction to asymptotic $\widetilde{e}_{\mu\nu}$ via (4.2.7). Since $\widetilde{e}_{\mu\nu}$ is determined only up to a gauge transformation

$$\widetilde{e}_{\mu\nu} \to \widetilde{e}_{\mu\nu} + k_\mu\xi_\nu + k_\nu\xi_\mu - k.\xi\,\eta_{\mu\nu}\,, \tag{4.2.46}$$

for any vector ξ, addition of a similar term to $\widehat{T}^h_{\mu\nu}$ and hence to $\mathcal{F}^{\mu\nu}_{\alpha\beta;\rho\sigma}(k,\ell)$ will not have any effect of $\widetilde{e}_{\mu\nu}$. Using this we can simplify (4.2.45) to:

$$\mathcal{F}^{\mu\nu}_{\alpha\beta;\rho\sigma}(k,\ell) \simeq -2\,k_\sigma\,k_\beta\,\eta_{\rho\alpha}\,\eta^{\mu\nu} + 2\,k_\alpha\,k_\beta\,\delta^\mu_\rho\delta^\nu_\sigma\,. \tag{4.2.47}$$

We can also make the approximations:

$$\frac{1}{p_a.(k-\ell)-i\epsilon} \simeq \frac{1}{p_a.k-i\epsilon}, \quad G_r(\ell)^* = \frac{1}{(\ell^0-i\epsilon)^2-\vec{\ell}^2},$$
$$G_r(k-\ell) = \frac{1}{(k^0-\ell^0+i\epsilon)^2-(\vec{k}-\vec{\ell})^2} \simeq \frac{1}{2(k.\ell+i\epsilon\omega)} \simeq \frac{1}{2(k.\ell+i\epsilon)}. \quad (4.2.48)$$

Substituting these into (4.2.41), we get the logarithmic contribution from the $R^{-1} << |\ell^\mu| << \omega$ region, denoted by $\widehat{T}^{(1)\mu\nu}(k)$:

$$\widehat{T}^{(1)\mu\nu}(k) = 8\pi G \sum_{a,b=1}^{m+n} \frac{1}{p_a.k-i\epsilon} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{k.\ell+i\epsilon} \frac{1}{p_b.\ell-i\epsilon} \frac{1}{(\ell^0-i\epsilon)^2-\vec{\ell}^2}$$
$$\left\{ p_a.p_b\, k.p_a\, k.p_b\, \eta^{\mu\nu} - \frac{1}{2} p_b^2\, (k.p_a)^2\, \eta^{\mu\nu} - (k.p_b)^2\, p_a^\mu\, p_a^\nu \right\}. \quad (4.2.49)$$

This integral, called K_b' in (4.6.10), has been evaluated in (4.6.11), and gives:

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{k.\ell+i\epsilon} \frac{1}{p_b.\ell-i\epsilon} \frac{1}{(\ell^0-i\epsilon)^2-\vec{\ell}^2} = \frac{1}{4\pi}\, \delta_{\eta_b,1}\, \ln\{(\omega+i\epsilon)R\}\, \frac{1}{k.p_b}. \quad (4.2.50)$$

Using this, we get:

$$\widehat{T}^{(1)\mu\nu}(k) = 2\,G\, \ln\{(\omega+i\epsilon)R\} \sum_{a=1}^{m+n} \sum_{b=1}^{n} \frac{1}{p_a.k-i\epsilon} \frac{1}{p_b.k-i\epsilon}$$
$$\left\{ p_a.p_b\, k.p_a\, k.p_b\, \eta^{\mu\nu} - \frac{1}{2} p_b^2\, (k.p_a)^2\, \eta^{\mu\nu} - (k.p_b)^2\, p_a^\mu\, p_a^\nu \right\}. \quad (4.2.51)$$

The terms proportional to $p_a.k\, p_b.k$ and $(k.p_a)^2$ inside the curly bracket cancel the denominator factor of $k.p_a$, and the result vanishes by momentum conservation after summing over a. Therefore we have

$$\widehat{T}^{(1)\mu\nu}(k) = -2\,G\, \ln\{(\omega+i\epsilon)R\} \sum_{a=1}^{m+n} \sum_{b=1}^{n} \frac{p_b.k}{p_a.k-i\epsilon}\, p_a^\mu p_a^\nu. \quad (4.2.52)$$

Next we turn to the contribution from the region $L^{-1} >> |\ell^\mu| >> \omega$. Simple power counting shows that the integrand goes as $|\ell|^{-4}$ in this region. Therefore, in order to extract the logarithmic term, we need to keep only the leading term in the integrand for large ℓ^μ. In particular, in the expression for $\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)$, we need to keep only quadratic terms in ℓ. Therefore we need to evaluate the integral:

$$\mathcal{I}^{\alpha\beta}_{ab} \equiv \int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.(r_a-r_b)}\, G_r(k-\ell)G_r(\ell)^*\, \frac{1}{p_b.\ell - i\epsilon}\, \frac{1}{p_a.(\ell-k)+i\epsilon}\ell^\alpha\, \ell^\beta\,. \tag{4.2.53}$$

Using $L^{-1} >> |\ell^\mu| >> \omega$, we can further approximate (4.2.53) by

$$\begin{aligned}
\mathcal{I}^{\alpha\beta}_{ab} &\simeq \int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\{(\ell^0-i\epsilon)^2-\vec{\ell}^2\}^2}\, \frac{1}{p_b.\ell-i\epsilon}\, \frac{1}{p_a.\ell+i\epsilon}\, \ell^\alpha\ell^\beta \\
&= \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, \frac{\partial}{\partial\ell_\alpha}\left\{\frac{1}{(\ell^0-i\epsilon)^2-\vec{\ell}^2}\right\}\, \frac{1}{p_b.\ell-i\epsilon}\, \frac{1}{p_a.\ell+i\epsilon}\, \ell^\beta \\
&= -\frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{(\ell^0-i\epsilon)^2-\vec{\ell}^2}\, \frac{1}{p_b.\ell-i\epsilon}\, \frac{1}{p_a.\ell+i\epsilon}\left\{\eta^{\alpha\beta} - \frac{p_a^\alpha\ell^\beta}{p_a.\ell+i\epsilon} - \frac{p_b^\alpha\ell^\beta}{p_b.\ell-i\epsilon}\right\} \\
&= -\frac{1}{2}\left\{\eta^{\alpha\beta} + p_a^\alpha\frac{\partial}{p_{a\beta}} + p_b^\alpha\frac{\partial}{\partial p_{b\beta}}\right\}\int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{(\ell^0-i\epsilon)^2-\vec{\ell}^2}\, \frac{1}{p_a.\ell+i\epsilon}\, \frac{1}{p_b.\ell-i\epsilon}
\end{aligned} \tag{4.2.54}$$

In the third step we have carried out an integration by parts.[6] We can now evaluate the integral using the result for J_{ab} given in (4.6.1). The result vanishes when a represents an ingoing particle and b represents an outgoing particle or vice versa. When both particles are ingoing or both particles are outgoing, the result is given in (4.6.6). This gives

$$\begin{aligned}
\mathcal{I}^{\alpha\beta}_{ab} &\simeq -\frac{1}{8\pi}\,\ln\{L(\omega+i\epsilon\eta_a)\}\left\{\eta^{\alpha\beta} + p_a^\alpha\frac{\partial}{p_{a\beta}} + p_b^\alpha\frac{\partial}{\partial p_{b\beta}}\right\}\frac{1}{\sqrt{(p_a.p_b)^2 - p_a^2p_b^2}} \\
&= -\frac{1}{8\pi}\,\ln\{L(\omega+i\epsilon\eta_a)\}\,\frac{1}{\left\{(p_a.p_b)^2-p_a^2p_b^2\right\}^{3/2}} \\
&\quad \times\left[\eta^{\alpha\beta}\{(p_a.p_b)^2 - p_a^2p_b^2\} + p_a^2\,p_b^\alpha\,p_b^\beta + p_b^2\,p_a^\alpha\,p_a^\beta - p_a.p_b\,(p_a^\alpha\,p_b^\beta + p_a^\beta\,p_b^\alpha)\right]
\end{aligned} \tag{4.2.55}$$

Note that the result diverges for $a=b$. This can be traced to the fact that if we replace the $p_a.(\ell-k)+i\epsilon$ factor in the denominator of (4.2.53) by $p_a.\ell+i\epsilon$ from the beginning,

[6]This can be justified as follows. First, following arguments similar to the one given below (4.2.32), we can consider the integration region to be $\omega << |\ell_\perp| << L^{-1}$, without any restriction on ℓ^0 and ℓ^3. Integration by parts will then give boundary contributions from $|\ell_\perp| = \omega$ and $|\ell_\perp| = L^{-1}$. These involve angular integration and do not generate any logarithmic terms.

then for $b = a$ the contour is pinched by the poles from both sides with separation of order ϵ, and we shall get a divergence in the $\epsilon \to 0$ limit. This shows that for $a = b$ we have to be more careful in evaluating the integral. We proceed by working with (4.2.53) without making any approximation at the beginning. If we work in the rest frame of p_a, then we can evaluate the ℓ^0 integral by closing the contour in the lower half plane, picking the residue at $\ell^0 = 0$ for outgoing $p_a = p_b$ and at $\ell^0 = k^0$ for incoming $p_a = p_b$. Let us for definiteness consider the case where the particle is outgoing, so that we pick up the residue from the pole at $\ell^0 = 0$. This reduces the integral to

$$I_{aa}^{\alpha\beta} = -2\pi i \frac{1}{p_a^0} \frac{1}{p_a.k - i\epsilon} \int \frac{d^3\ell}{(2\pi)^4} \frac{1}{\vec{\ell}^2} \frac{1}{\vec{\ell}^2 - 2\vec{k}.\vec{\ell} - i\epsilon} \ell^\alpha \ell^\beta . \tag{4.2.56}$$

Since this is potentially linearly divergent from the region of large $|\vec{\ell}|$, we expand the integrand in power series expansion in inverse powers of ℓ, keeping up to the first subleading term:

$$I_{aa}^{\alpha\beta} \simeq -2\pi i \frac{1}{p_a^0} \frac{1}{p_a.k - i\epsilon} \int \frac{d^3\ell}{(2\pi)^4} \frac{1}{\vec{\ell}^2} \left[\frac{1}{\vec{\ell}^2} + \frac{2\vec{k}.\vec{\ell}}{(\vec{\ell}^2)^2} \right] \ell^\alpha \ell^\beta . \tag{4.2.57}$$

The leading linearly divergent term, where we pick the $\ell^\alpha \ell^\beta$ term from the numerator, represents the usual infinite self energy of a classical point particle, and is regulated by the intrinsic size of the particle. In any case, this does not lead to any logarithmic terms. The potentially logarithmically divergent subleading contribution actually vanishes by $\vec{\ell} \to -\vec{\ell}$ symmetry since it has to be evaluated at $\ell^0 = 0$. Therefore we conclude that $I_{aa}^{\alpha\beta}$ does not have any logarithmic correction. A similar analysis can be carried out for the incoming particles, leading to the same conclusion.

Substituting (4.2.55) into (4.2.41) for $a \neq b$, we get the logarithmic contribution to $\widehat{T}^{h\mu\nu}(k)$ from the region $\omega << |\ell^\mu| << L^{-1}$, which we shall denote by $\widehat{T}^{(2)\mu\nu}(k)$:

$$\begin{aligned} \widehat{T}^{(2)\mu\nu}(k) \quad = \quad & G \sum_{a=1}^{m+n} \ln\{L(\omega + i\epsilon\eta_a)\} \sum_{\substack{b=1 \\ b\neq a, \eta_a\eta_b=1}}^{m+n} \frac{1}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}} \Big[- p_b^\mu p_b^\nu (p_a^2)^2 (p_b^2) \\ & + \{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu\} p_a.p_b \left\{ \frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2 \right\} \Big] . \end{aligned} \tag{4.2.58}$$

4.2.6 Gravitational wave-form at early and late time

Adding (4.2.17), (4.2.39), (4.2.52) and (4.2.58) we get the net logarithmic contribution to $\widehat{T}^{\mu\nu}(k)$ to the subleading order in the small ω expansion:[7]

$$\begin{aligned}
&\widehat{T}^{\mu\nu}(k) \\
= \; & \sum_{a=1}^{n} p_a^\mu\, p_a^\nu\, \frac{1}{i(k.p_a - i\epsilon)} - \sum_{a=1}^{m} p_a'^\mu\, p_a'^\nu\, \frac{1}{i(k.p_a' + i\epsilon)} \\
+ \; & 2\,G\, \ln\{L(\omega + i\epsilon)\} \sum_{a=1}^{n} \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a.p_b}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}} \left\{\frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2\right\} \frac{k_\rho p_a^\mu}{k.p_a} (p_b^\rho p_a^\nu - p_b^\nu p_a^\rho) \\
+ & \, 2\,G\, \ln\{L(\omega - i\epsilon)\} \sum_{a=1}^{m} \sum_{\substack{b=1\\ b\neq a}}^{m} \frac{p_a'.p_b'}{\{(p_a'.p_b')^2 - p_a'^2 p_b'^2\}^{3/2}} \left\{\frac{3}{2} p_a'^2 p_b'^2 - (p_a'.p_b')^2\right\} \frac{k_\rho p_a'^\mu}{k.p_a'} (p_b'^\rho p_a'^\nu - p_b'^\nu p_a'^\rho) \\
- \; & 2\,G\, \ln\{(\omega + i\epsilon)R\} \sum_{b=1}^{n} p_b.k \left[\sum_{a=1}^{n} \frac{1}{p_a.k - i\epsilon} p_a^\mu p_a^\nu - \sum_{a=1}^{m} \frac{1}{p_a'.k - i\epsilon} p_a'^\mu p_a'^\nu\right].
\end{aligned} \tag{4.2.59}$$

In (4.2.59) we can replace k by $\omega\, n$ with $n = (1, \hat{n})$. Comparing the second and last line of (4.2.59) we see that the term proportional to $\ln R$ exponentiates to a multiplicative factor of

$$\exp\left[-2\, i\, \omega\, G\, \ln R \sum_{b=1}^{n} p_b.n\right]. \tag{4.2.60}$$

Using (4.2.7), (4.1.15) and (4.1.19) we get the late and early time behaviour of the gravitational wave-form:

$$\begin{aligned}
e^{\mu\nu}(t, R, \hat{n}) \;=\; & \frac{2\,G}{R} \left[-\sum_{a=1}^{n} p_a^\mu\, p_a^\nu\, \frac{1}{n.p_a} + \sum_{a=1}^{m} p_a'^\mu\, p_a'^\nu\, \frac{1}{n.p_a'}\right] \\
& - \frac{4\,G^2}{R\,u} \Bigg[\sum_{a=1}^{n} \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a.p_b}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}} \left\{\frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2\right\} \frac{n_\rho p_a^\mu}{n.p_a} (p_b^\rho p_a^\nu - p_b^\nu p_a^\rho) \\
& \qquad - \sum_{b=1}^{n} p_b.n \left\{\sum_{a=1}^{n} \frac{1}{p_a.n} p_a^\mu p_a^\nu - \sum_{a=1}^{m} \frac{1}{p_a'.n} p_a'^\mu p_a'^\nu\right\}\Bigg], \quad \text{as } u \to \infty\,, \\
e^{\mu\nu}(t, R, \hat{n}) \;=\; & \frac{4\,G^2}{R\,u} \Bigg[\sum_{a=1}^{m} \sum_{\substack{b=1\\ b\neq a}}^{m} \frac{p_a'.p_b'}{\{(p_a'.p_b')^2 - p_a'^2 p_b'^2\}^{3/2}} \left\{\frac{3}{2} p_a'^2 p_b'^2 - (p_a'.p_b')^2\right\}
\end{aligned}$$

[7] The quantum computation of [16] gave rise to additional terms in the soft factor, but they do not seem to play any role in the classical gravitational wave-form found here.

$$\times \frac{n_\rho p_a'^\mu}{n.p_a'} (p_b'^\rho p_a'^\nu - p_b'^\nu p_a'^\rho)\Bigg], \qquad \text{as } u \to -\infty\,, \quad (4.2.61)$$

where, from (4.1.14) and (4.2.7), (4.2.60),

$$u = t - R + 2\,G\,\ln R \sum_{b=1}^{n} p_b.n\,. \quad (4.2.62)$$

In (4.2.61) we have adjusted the overall additive constant in the expression for $e^{\mu\nu}$ such that it vanishes in the far past.

4.3 Generalizations

In this section we shall derive the classical soft photon theorem. We shall also generalize the soft graviton theorem to include the effect of electromagnetic interactions among the incoming and the outgoing particles. In order to simplify our formulæ we shall drop the regulator factors of $e^{i\ell.(r_a-r_b)}$, $e^{ik.r_a}$ etc., with the understanding that momentum integrals have an upper cut-off L^{-1} and a lower cut-off R^{-1}.

4.3.1 Soft photon theorem with electromagnetic interactions

As in §4.2, we consider a scattering event in asymptotically flat space-time in which m particles carrying masses $\{m_a'; 1 \le a \le m\}$, four velocities $\{v_a'\}$, four momenta $\{p_a' = m_a'\, v_a'\}$ and charges $\{q_a'\}$ come close, undergo interactions, and disperse as n particles carrying masses $\{m_a; 1 \le a \le n\}$, four velocities $\{v_a\}$, four momenta $\{p_a\}$ and charges $\{q_a\}$. Our goal will be to compute the early and late time electromagnetic wave-form emitted during this scattering event. In this section we shall proceed by ignoring the gravitational interaction between the particles, but this will be included in §4.3.2. Since the analysis proceeds as in §4.2, we shall be brief, pointing out only the main differences. In particular, as in §4.2, we can treat the incoming particles as outgoing particles with four velocities $\{-v_a'\}$, four momenta $\{-p'_a\}$ and charges $\{-q'_a\}$. This allows us to drop the sum over incoming particles by extending the sum over a from 1 to $m+n$.

In the Lorentz gauge $\eta^{\alpha\beta}\partial_\alpha a_\beta = 0$, the equations replacing (4.2.13) are:

$$j^\mu(x) = \sum_a q_a \int d\sigma\, \delta^{(4)}(x - X_a(\sigma))\, \frac{dX_a^\mu}{d\sigma}, \qquad \Box\, a_\mu = -j_\mu, \qquad m_a \frac{d^2 X_a^\mu}{d\sigma^2} = q_a F^\mu{}_\nu(X_a(\sigma))\, \frac{dX_a^\nu}{d\sigma}\,. \tag{4.3.1}$$

We introduce the Fourier transforms via:

$$a_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik.x}\, \widehat{a}_\mu(k), \quad j_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik.x}\, \widehat{j}_\mu(k)\,. \tag{4.3.2}$$

This gives:

$$\widehat{j}^\mu(k) = \int d^4x\, e^{-ik.x}\, j^\mu(x) = \sum_a q_a \int d\sigma\, e^{-ik.X(\sigma)}\, \frac{dX_a^\mu}{d\sigma}\,. \tag{4.3.3}$$

The generalization of (4.2.7) for the asymptotic electromagnetic field is:

$$\tilde{a}^\mu(\omega, R, \hat{n}) = \frac{1}{4\pi R}\, e^{i\omega R}\, \widehat{j}^\mu(k) + O(R^{-2})\,. \tag{4.3.4}$$

We proceed to find iterative solutions to (4.3.1) in a power series expansion in the charges, beginning with the leading order solution for X^μ given in (4.2.11). Substituting this into (4.3.3) we find the leading order expression for $\widehat{j}^\mu(k)$:

$$\widehat{j}^\mu(k) = \sum_{a=1}^{m+n} q_a\, p_a^\mu\, \frac{1}{i(k.p_a - i\epsilon)}\,. \tag{4.3.5}$$

This is the leading soft factor. Using this we can get the analogs of (4.2.19) and (4.2.20):

$$a_\mu^{(b)}(x) = -\int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.x}\, G_r(\ell) q_b\, p_{b\mu}\, \frac{1}{i(\ell.p_b - i\epsilon)}\,, \tag{4.3.6}$$

$$F_{\nu\rho}^{(b)}(x) = \partial_\nu a_\rho^{(b)} - \partial_\rho a_\nu^{(b)} = -\int \frac{d^4\ell}{(2\pi)^4} e^{i\ell.x}\, G_r(\ell)\, \frac{q_b}{(\ell.p_b - i\epsilon)} (\ell_\nu p_{b\rho} - \ell_\rho p_{b\nu})\,, \tag{4.3.7}$$

where $a_\mu^{(b)}$ and $F_{\mu\nu}^{(b)}$ denote the gauge field and field strength produced by the b-th particle.

The analogs of (4.2.26), (4.2.27) take the form:

$$\frac{dY_a^\mu(\sigma)}{d\sigma} = -\frac{q_a}{m_a}\int_\sigma^\infty d\sigma'\, F^\mu{}_\nu(v_a\,\sigma' + r_a)\, v_a^\nu\,, \tag{4.3.8}$$

and

$$Y_a^\mu(\sigma) = -\frac{q_a}{m_a}\int_0^\sigma d\sigma'\int_{\sigma'}^\infty d\sigma''\, F^\mu{}_\nu(v_a\,\sigma'' + r_a)\, v_a^\nu\,. \tag{4.3.9}$$

Using these results we can proceed as in §4.2.4 to compute the next order correction $\Delta\widehat{j^\mu}(k)$ to $\widehat{j^\mu}(k)$. Since the analysis is identical to those in §4.2.4 we only quote the analog of (4.2.32):

$$\begin{aligned}\Delta\widehat{j^\mu}(k) \;=\; & \sum_{a=1}^{m+n}\sum_{\substack{b=1\\ b\neq a}}^{m+n} q_a^2 q_b \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell.p_b - i\epsilon}\,G_r(\ell)\bigg[k_\rho\,\frac{1}{(k-\ell).p_a - i\epsilon}\,\frac{1}{k.p_a - i\epsilon}\,\frac{1}{\ell.p_a + i\epsilon} \\ & \qquad\qquad\times(\ell_\nu p_{b\rho} - \ell_\rho p_{b\nu})\,p_a^\nu\,p_a^\mu \\ & -\frac{1}{(k-\ell).p_a - i\epsilon}\,\frac{1}{\ell.p_a + i\epsilon}(\ell_\nu p_b^\mu - \ell^\mu p_{b\nu})p_a^\nu\bigg]\,.\end{aligned} \tag{4.3.10}$$

This can be evaluated exactly as in §4.2.4, leading to the final result analogous to (4.2.39):

$$\begin{aligned}\Delta\widehat{j^\mu}(k) \;=\; & \frac{1}{4\pi}\ln(\omega + i\epsilon)\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n} q_a^2 q_b\, p_a^2\, p_b^2\,\frac{1}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}}\,\frac{k^\rho}{k.p_a}\left\{p_{b\rho}p_a^\mu - p_{a\rho}p_b^\mu\right\} \\ + & \frac{1}{4\pi}\ln(\omega - i\epsilon)\sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m} q_a'^2 q_b'\, p_a'^2\, p_b'^2\,\frac{1}{\{(p_a'.p_b')^2 - p_a'^2 p_b'^2\}^{3/2}}\,\frac{k^\rho}{k.p_a'}\left\{p'_{b\rho}p_a'^\mu - p'_{a\rho}p_b'^\mu\right\}\,.\end{aligned} \tag{4.3.11}$$

4.3.2 Gravitational contribution to the soft photon theorem

We shall now study the effect of gravitational interaction on the soft photon theorem. This modifies the last two equations in (4.3.1) as follows. First of all the equation for a_μ get modified to:

$$\partial_\nu\left(\sqrt{-\det g}\, g^{\nu\rho} g^{\mu\sigma} F_{\rho\sigma}\right) = -j^\mu\,. \tag{4.3.12}$$

Using Lorentz gauge condition $\eta^{\rho\sigma}\partial_\rho a_\sigma = 0$, this may be written as

$$\eta^{\mu\nu}\,\eta^{\rho\sigma}\,\partial_\rho\,\partial_\sigma\,a_\nu = -j^\mu - j_h^\mu\,, \tag{4.3.13}$$

where

$$j_h^\mu \equiv \partial_\nu\left\{\eta^{\alpha\beta}\,h_{\alpha\beta}\,\eta^{\nu\rho}\eta^{\mu\sigma}F_{\rho\sigma} - 2\,\left(h^{\nu\rho}\eta^{\mu\sigma} + \eta^{\nu\rho}h^{\mu\sigma}\right)F_{\rho\sigma}\right\} + \text{higher order terms}\,. \tag{4.3.14}$$

The equation for X^μ is modified to:

$$m_a\,\frac{d^2X_a^\mu}{d\sigma^2} = q_a F^\mu{}_\nu(X_a(\sigma))\,\frac{dX_a^\nu}{d\sigma} - m_a\,\Gamma^\mu_{\nu\rho}(X_a(\sigma))\,\frac{dX_a^\nu}{d\sigma}\,\frac{dX_a^\rho}{d\sigma}\,. \tag{4.3.15}$$

We shall now expand the above equations in powers of $h_{\alpha\beta}$ and then raise and lower all indices by the flat metric η. We begin with the analysis of (4.3.15). To the order that we are working, we can study the effect of the two terms on the right hand side of (4.3.15) separately. The effect of the first term on $\widehat{j}^\mu$ has already been analyzed in §4.3.1. The effect of the second term on $\widehat{T}^X_{\mu\nu}$ has been studied in §4.2.4, but this can be easily extended to $\widehat{j}^\mu$. The additional contribution to $\widehat{j}^\mu$ is given by:

$$\begin{aligned}
\Delta^{(1)}\widehat{j}^\mu(k) \;\; = \;\; & -8\pi G\sum_{a=1}^{m+n}\sum_{\substack{b=1\\ b\neq a}}^{m+n} q_a\int\frac{d^4\ell}{(2\pi)^4}\,G_r(\ell)\,\frac{1}{\ell\cdot p_b - i\epsilon}\\
& \Bigg[-\frac{\left\{2k\cdot p_b\;\ell\cdot p_a\;p_a\cdot p_b - k\cdot\ell\,(p_a\cdot p_b)^2 - \frac{1}{2}\left(2k\cdot p_a\;\ell\cdot p_a - k\cdot\ell\;p_a^2\right)p_b^2\right\}p_a^\mu}{[k\cdot p_a - i\epsilon]\,[(k-\ell)\cdot p_a - i\epsilon]\,[\ell\cdot p_a + i\epsilon]}\\
& +\frac{2p_b^\mu\ell\cdot p_a\;p_a\cdot p_b - \ell^\mu\,(p_a\cdot p_b)^2 - \frac{1}{2}\left(2p_a^\mu\,\ell\cdot p_a - \ell^\mu\,p_a^2\right)p_b^2}{[(k-\ell)\cdot p_a - i\epsilon]\,[\ell\cdot p_a + i\epsilon]}\Bigg]\,.
\end{aligned} \tag{4.3.16}$$

This integral can be evaluated as in §4.2.4 and yields the result:

$\Delta^{(1)}\widehat{j}^\mu(k)$

$$
\begin{aligned}
= & -G\,\log(\omega+i\epsilon)\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n} q_a\frac{k_\rho}{k\cdot p_a}\frac{p_a\cdot p_b}{\{(p_a\cdot p_b)^2-p_a^2p_b^2\}^{\frac{3}{2}}}\left(p_a^\mu p_b^\rho-p_a^\rho p_b^\mu\right)\{2\,(p_a\cdot p_b)^2-3p_a^2p_b^2\} \\
- & \;G\,\log(\omega-i\epsilon)\sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m} q_a'\frac{k_\rho}{k\cdot p_a'}\frac{p_a'\cdot p_b'}{\left\{\left(p_a'\cdot p_b'\right)^2-p_a'^2p_b'^2\right\}^{\frac{3}{2}}}\left(p_a'^\mu p_b'^\rho-p_a'^\rho p_b'^\mu\right)\{2\,(p_a'\cdot p_b')^2-3p_a'^2p_b'^2\}\,.
\end{aligned}
\tag{4.3.17}
$$

We now turn to the evaluation of j_h^μ given in (4.3.14). Using the expressions for $h_{\mu\nu}$ and $F_{\mu\nu}$ given in (4.2.19) and (4.3.7), we get:

$$
\widehat{j}_h^\mu(k)=8\pi G\sum_{a,b=1}^{m+n} q_b\int\frac{d^4\ell}{(2\pi)^4}G_r(\ell)\,G_r(k-\ell)\,\frac{1}{[\ell\cdot p_a-i\epsilon]\,[(k-\ell)\cdot p_b-i\epsilon]}\mathcal{F}^\mu(k,\ell)\,,
\tag{4.3.18}
$$

where

$$
\begin{aligned}
\mathcal{F}^\mu \;\equiv\; & p_b^\mu\left\{p_a^2\,k.(k-\ell)-2\,p_a.(k-\ell)\,p_a.k\right\}+(k^\mu-\ell^\mu)\,\left\{2\,k.p_a\,p_a.p_b-p_a^2\,k.p_b\right\} \\
& +p_a^\mu\,\{2\,p_a.(k-\ell)\,k.p_b-2\,k.(k-\ell)\,p_a.p_b\}\,.
\end{aligned}
\tag{4.3.19}
$$

We shall analyze this by expressing $G_r(\ell)\,G_r(k-\ell)$ as in (4.2.44). The term proportional to $\delta(\ell^2)$ can be analyzed as in appendix 4.6.2, and one can show that in this case there is no contribution from this term. This gives

$$
\widehat{j}_h^\mu(k)=8\pi G\sum_{a,b=1}^{m+n} q_b\int\frac{d^4\ell}{(2\pi)^4}G_r(\ell)^*\,G_r(k-\ell)\,\frac{1}{[\ell\cdot p_a-i\epsilon]\,[(k-\ell)\cdot p_b-i\epsilon]}\mathcal{F}^\mu(k,\ell)\,.
\tag{4.3.20}
$$

This integral could give logarithmic contributions from three regions: $R^{-1} << |\ell^\mu| << \omega$, $R^{-1} << |k^\mu-\ell^\mu| << \omega$ and $\omega << |\ell^\mu| << L^{-1}$. However, since $\mathcal{F}^\nu$ vanishes as $k-\ell\to 0$, there is no logarithmic divergence from the $R^{-1} << |k^\mu-\ell^\mu| << \omega$ region. Furthermore, since $\mathcal{F}^\nu$ does not have any quadratic term in ℓ, this rules out logarithmic contribution from the region $\omega << |\ell^\mu| << L^{-1}$. Therefore the only possible source of logarithmic divergence is the region $R^{-1} << |\ell^\mu| << \omega$. In this region,

$$\mathcal{F}^\nu \simeq -2\left\{(k\cdot p_a)^2\, p_b^\nu - k\cdot p_a\, k\cdot p_b\, p_a^\nu\right\}. \tag{4.3.21}$$

We have ignored the terms proportional to k^ν because such terms can be removed by gauge transformation $a_\mu \to a_\mu + \partial_\mu\phi$ for appropriate function ϕ. We can now evaluate the logarithmic contribution to the integral using the method described below (4.2.47), and the result is

$$\widehat{j}^\mu_h(k) \;=\; -2\,G\,\ln(\omega+i\epsilon)\sum_{a=1}^{n} k.p_a\left[\sum_{b=1}^{n} q_b\, p_b^\mu\,\frac{1}{k.p_b} - \sum_{b=1}^{m} q'_b\, p_b'^\mu\,\frac{1}{k.p'_b}\right], \tag{4.3.22}$$

after using charge conservation $\sum_{b=1}^{m+n} q_b = 0$.

4.3.3 Electromagnetic wave-form at early and late time

Adding (4.3.5), (4.3.11), (4.3.17) and (4.3.22), and using $k = \omega\, n$ and (4.3.4), (4.1.15), (4.1.19) we get

$$\begin{aligned}
4\,\pi\,R\,a^\mu(t,R,\hat{n}) \;\simeq\;& -\sum_{a=1}^{n} q_a\, p_a^\mu\frac{1}{n.p_a} + \sum_{a=1}^{m} q'_a\, p_a'^\mu\frac{1}{n.p'_a} \\
&+\frac{1}{u}\Bigg[-\frac{1}{4\pi}\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n} q_a^2 q_b\, p_a^2\, p_b^2\,\frac{1}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}}\frac{n^\rho}{n.p_a}\left\{p_{b\rho}p_a^\mu - p_{a\rho}p_b^\mu\right\} \\
&+ G\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n} q_a\frac{n_\rho}{n\cdot p_a}\frac{p_a\cdot p_b}{\{(p_a\cdot p_b)^2 - p_a^2 p_b^2\}^{\frac{3}{2}}}\left(p_a^\mu p_b^\rho - p_a^\rho p_b^\mu\right)\{2\,(p_a\cdot p_b)^2 - 3p_a^2 p_b^2\} \\
&+ 2\,G\sum_{a=1}^{n} n.p_a\left\{\sum_{b=1}^{n} q_b\, p_b^\mu\,\frac{1}{n.p_b} - \sum_{b=1}^{m} q'_b\, p_b'^\mu\,\frac{1}{n.p'_b}\right\}\Bigg], \qquad \text{as } u\to\infty\,,
\end{aligned} \tag{4.3.23}$$

and

$$4\,\pi\,R\,a^\mu(t,R,\hat{n}) \;\simeq\; \frac{1}{u}\left[\frac{1}{4\pi}\sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m} q_a'^2 q'_b\, p_a'^2\, p_b'^2\,\frac{1}{\{(p'_a.p'_b)^2 - p_a'^2 p_b'^2\}^{3/2}}\frac{n^\rho}{n.p'_a}\left\{p'_{b\rho}p_a'^\mu - p'_{a\rho}p_b'^\mu\right\}\right.$$

$$
-G\sum_{a=1}^{m}\sum_{\substack{b=1\\b\neq a}}^{m} q'_a \frac{n_\rho}{n\cdot p'_a}\frac{p'_a\cdot p'_b}{\{\left(p'_a\cdot p'_b\right)^2-p'^2_a p'^2_b\}^{\frac{3}{2}}}\left(p'^\mu_a p'^\rho_b - p'^\rho_a p'^\mu_b\right)\{2\left(p'_a\cdot p'_b\right)^2 - 3p'^2_a p'^2_b\}\Big],
$$
$$
\text{as } u\to-\infty\,. \qquad (4.3.24)
$$

This gives the wave-form of the electromagnetic field at early and late time. The term on the right hand side of the first line gives the constant shift in the vector potential, and is responsible for electromagnetic memory [50–52]. The rest of the terms are tail terms.

4.3.4 Electromagnetic contribution to the soft graviton theorem

We shall now analyze the effect of electromagnetic interaction on the soft graviton theorem. This affects our earlier analysis of soft graviton theorem in two ways. First of all, the Lorentz force on the outgoing and incoming particles changes the particle trajectories, producing an additional contribution to $\widehat{T}^{X\mu\nu}$. Analysis of this follows the same procedure that led to (4.2.32), (4.3.16), and the final result is given by:

$$
\begin{aligned}
\Delta^{(1)}\widehat{T}^{\mu\nu}(k) \;=\; & \sum_{a=1}^{m+n}\sum_{\substack{b=1\\b\neq a}}^{m+n} q_a q_b \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell.p_b - i\epsilon}G_r(\ell) \\
& \Big[\frac{1}{(k-\ell).p_a - i\epsilon}\frac{1}{k.p_a - i\epsilon}\frac{1}{\ell.p_a + i\epsilon}\left(p_a.\ell\, k.p_b - k.\ell\, p_a.p_b\right) p^\mu_a p^\nu_a \\
& -\frac{1}{(k-\ell).p_a - i\epsilon}\frac{1}{\ell.p_a + i\epsilon}\left(p_a.\ell\, p^\mu_b - \ell^\mu\, p_a.p_b\right) p^\nu_a \\
& -\frac{1}{(k-\ell).p_a - i\epsilon}\frac{1}{\ell.p_a + i\epsilon}\left(p_a.\ell\, p^\nu_b - \ell^\nu\, p_a.p_b\right) p^\mu_a\Big].
\end{aligned} \qquad (4.3.25)
$$

Evaluation of this using the method described below (4.2.32), gives

$$
\begin{aligned}
\Delta^{(1)}\widehat{T}^{\mu\nu}(k) \;=\; & \frac{1}{4\pi}\ln\{L(\omega+i\epsilon)\}\sum_{a=1}^{n}\sum_{\substack{b=1\\b\neq a}}^{n} q_a q_b \frac{1}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}}\Big[\frac{k.p_b}{k.p_a}p_a^2\, p_b^2\, p^\mu_a\, p^\nu_a \\
& + p_b^2\, p_a.p_b\, p^\mu_a\, p^\nu_a - p_a^2\, p_b^2\,(p^\mu_a p^\nu_b + p^\nu_a p^\mu_b)\Big] \\
& + \frac{1}{4\pi}\ln\{L(\omega-i\epsilon)\}\sum_{a=1}^{m}\sum_{\substack{b=1\\b\neq a}}^{m} q'_a q'_b \frac{1}{\{(p'_a.p'_b)^2 - p'^2_a p'^2_b\}^{3/2}}\Big[\frac{k.p'_b}{k.p'_a}p'^2_a\, p'^2_b\, p'^\mu_a\, p'^\nu_a
\end{aligned}
$$

$$+ p_b'^2\, p_a'.p_b'\, p_a'^\mu\, p_a'^\nu - p_a'^2\, p_b'^2\, (p_a'^\mu p_b'^\nu + p_a'^\nu p_b'^\mu)\Big]. \tag{4.3.26}$$

Second, there is an additional contribution to the stress tensor due to the electromagnetic field. Using the form of the electromagnetic field produced by the charged particle as given in (4.3.7), this additional contribution takes the form:

$$\begin{aligned}
\Delta^{(2)}\widehat{T}^{\mu\nu}(k) &= \int d^4x\, e^{-ik.x}\Big[\eta^{\mu\rho}\,\eta^{\nu\sigma}\,\eta^{\alpha\beta}\,F_{\rho\alpha}\,F_{\sigma\beta} - \frac{1}{4}\,\eta^{\mu\nu}\,\eta^{\rho\sigma}\,\eta^{\alpha\beta}\,F_{\rho\alpha}\,F_{\sigma\beta}\Big] + \text{higher order terms} \\
&\simeq \sum_{a=1}^{m+n}\sum_{b=1}^{m+n} q_a q_b \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell.p_b - i\epsilon}\,\frac{1}{(k-\ell).p_a - i\epsilon}\,G_r(\ell)\,G_r(k-\ell) \\
&\quad \Big[\ell^\mu (k-\ell)^\nu\, p_a.p_b - \ell^\mu p_a^\nu\, p_b.(k-\ell) - p_b^\mu\,(k-\ell)^\nu\,\ell.p_a + p_b^\mu p_a^\nu\,\ell.(k-\ell) \\
&\quad -\frac{1}{2}\eta^{\mu\nu}\,\{\ell.(k-\ell)\,p_a.p_b - \ell.p_a\,(k-\ell).p_b\}\Big].
\end{aligned} \tag{4.3.27}$$

We shall analyze this by expressing $G_r(\ell)\,G_r(k-\ell)$ as in (4.2.44). The term proportional to $\delta(\ell^2)$ can be analyzed as in appendix 4.6.2, and one can show that the contribution from this term can be interpreted as the soft graviton emission from electromagnetic wave produced during scattering. Since this is included in the sum over a in the soft factor, we do not need to include its contribution. This allows us to replace $G_r(\ell)$ by $G_r(\ell)^*$ in (4.3.27). Since each term in the numerator of the integrand carries a factor of ℓ and a factor of $(k-\ell)$, there is no logarithmic contribution from the $|\ell^\mu| << \omega$ and $|k^\mu - \ell^\mu| << \omega$ regions. Therefore we focus on the $|\ell^\mu| >> \omega$ region, and analyze the contribution using (4.2.53), (4.2.55). The final result is:

$$\begin{aligned}
\Delta^{(2)}\widehat{T}^{\mu\nu}(k) &= -\frac{1}{4\pi}\ln(\omega + i\epsilon)\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n} q_a q_b\, p_a.p_b\,\frac{1}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}}\,p_a^\mu(p_b^2\,p_a^\nu - p_a.p_b\,p_b^\nu) \\
&-\frac{1}{4\pi}\ln(\omega - i\epsilon)\sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m} q_a' q_b'\, p_a'.p_b'\,\frac{1}{\{(p_a'.p_b')^2 - p_a'^2 p_b'^2\}^{3/2}}\,p_a'^\mu(p_b'^2\,p_a'^\nu - p_a'.p_b'\,p_b'^\nu) \\
&-\frac{1}{4\pi}\ln(\omega + i\epsilon)\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n} q_a q_b\,\frac{1}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{1/2}}\,p_b^\mu\,p_a^\nu
\end{aligned}$$

$$-\frac{1}{4\pi}\ln(\omega-i\epsilon)\sum_{a=1}^{m}\sum_{\substack{b=1\\b\neq a}}^{m} q'_a q'_b \frac{1}{\{(p'_a.p'_b)^2-p'^2_a p'^2_b\}^{1/2}}\, p'^{\mu}_b\, p'^{\nu}_a\,. \tag{4.3.28}$$

Adding (4.3.26) and (4.3.28) we get the net electromagnetic contribution to the soft graviton theorem:

$$\begin{aligned}
&\Delta^{(1)}\widehat{T}^{\mu\nu}(k)+\Delta^{(2)}\widehat{T}^{\mu\nu}(k)\\
=\;&\frac{1}{4\pi}\ln(\omega+i\epsilon)\sum_{a=1}^{n}\sum_{\substack{b=1\\b\neq a}}^{n} q_a q_b \frac{p_a^2\, p_b^2}{\{(p_a.p_b)^2-p_a^2 p_b^2\}^{3/2}}\left[\frac{k.p_b}{k.p_a}\, p_a^{\mu}\, p_a^{\nu}-p_a^{\mu} p_b^{\nu}\right]\\
&+\frac{1}{4\pi}\ln(\omega-i\epsilon)\sum_{a=1}^{m}\sum_{\substack{b=1\\b\neq a}}^{m} q'_a q'_b \frac{p'^2_a\, p'^2_b}{\{(p'_a.p'_b)^2-p'^2_a p'^2_b\}^{3/2}}\left[\frac{k.p'_b}{k.p'_a}\, p'^{\mu}_a\, p'^{\nu}_a-p'^{\mu}_a p'^{\nu}_b\right].
\end{aligned} \tag{4.3.29}$$

From (4.2.7), (4.1.19), we can read out the additional contribution to the gravitational wave-form at early and late retarded time due to electromagnetic interactions:

$$\begin{aligned}
\Delta_{\rm em}\, e^{\mu\nu} \;\to\;& -\frac{G}{2\pi R\, u}\sum_{a=1}^{n}\sum_{\substack{b=1\\b\neq a}}^{n} q_a q_b \frac{1}{\{(p_a.p_b)^2-p_a^2 p_b^2\}^{3/2}}\left[\frac{k.p_b}{k.p_a}\, p_a^2\, p_b^2\, p_a^{\mu}\, p_a^{\nu}-p_a^2\, p_b^2\, p_a^{\mu} p_b^{\nu}\right]\\
&\qquad\qquad \text{as } u\to\infty,\\
\to\;& \frac{G}{2\pi R\, u}\sum_{a=1}^{m}\sum_{\substack{b=1\\b\neq a}}^{m} q'_a q'_b \frac{1}{\{(p'_a.p'_b)^2-p'^2_a p'^2_b\}^{3/2}}\left[\frac{k.p'_b}{k.p'_a}\, p'^2_a\, p'^2_b\, p'^{\mu}_a\, p'^{\nu}_a-p'^2_a\, p'^2_b\, p'^{\mu}_a p'^{\nu}_b\right]\\
&\qquad\qquad \text{as } u\to-\infty.
\end{aligned} \tag{4.3.30}$$

4.4 New conjectures at the subsubleading order

Emboldened by the success of soft theorem in correctly predicting the tail of the gravitational wave-form at the subleading order, we shall now propose new conjectures at the subsubleading order. It is known that in quantum gravity, the subsubleading soft factors are not universal. Nevertheless there are some universal terms that we could utilize [11]. These are terms that are quadratic in the orbital angular momenta. Our goal will be to make use of these universal terms to arrive at new conjectures on the late and early time tail of gravitational radiation. The non-universal terms do not involve orbital angular momenta and therefore do not have logarithmic divergences. Hence they will not affect our analysis.

Using the relation between quantum soft factors and classical gravitational wave-forms derived in [19], and ignoring the non-universal terms, we can write down the following form of the gravitational wave-form to subsubleading order:[8]

$$\begin{aligned} \widetilde{e}^{\mu\nu}(\omega,\vec{x}) \;=\;& \frac{2\,G}{R}\, e^{i\omega R} \exp\left[-2\,i\,G\,\ln\{R(\omega+i\epsilon)\}\sum_{b=1}^{n} p_b.k\right] \\ & \times \sum_{a=1}^{m+n}\left[-i\,\frac{p_a^\mu p_a^\nu}{p_a.k} - \frac{1}{p_a.k} J_a^{\rho(\nu} p_a^{\mu)} k_\rho + \frac{i}{2}\,\frac{1}{p_a.k}\, k_\rho\, k_\sigma\, J_a^{\mu\rho}\, J_a^{\nu\sigma}\right], \end{aligned} \tag{4.4.1}$$

where k has been defined in (4.1.10) and $J_a^{\rho\sigma}$ is the sum of the orbital and spin angular momenta of the a-th external particle:

$$J_a^{\rho\sigma} = X_a^\rho\, p_a^\sigma - X_a^\sigma\, p_a^\rho + \Sigma_a^{\rho\sigma}\,. \tag{4.4.2}$$

The second term in the last line of (4.4.1) differs by a sign from the expressions used *e.g.* in [19]. This can be traced to the fact that in [19] we treated the charges / momenta / angular momenta carried by ingoing particles as positive and of the outgoing particles as negative, whereas here we are following the opposite convention. Following (4.2.12), the spin $\Sigma_a'^{\mu\nu}$ for incoming particles are given by:

$$\Sigma_a'^{\mu\nu} = -\Sigma_{a+n}^{\mu\nu} \quad \text{for } 1 \le a \le m\,. \tag{4.4.3}$$

The phase factor $\exp\left[-2\,i\,G\,\ln\{R(\omega+i\epsilon)\}\sum_{b=1}^{n} p_b.k\right]$ in (4.4.1) is not determined by soft theorem, but is determined by independent computation [110–112], and is consistent with the term in the last line of (4.2.59). Due to the long range gravitational force between the outgoing / incoming particles, X_a^ρ has logarithmic corrections at late / early time [20],

[8] A non-trivial test of this formula for the scattering of massless particles can be found in [118].

leading to [16]

$$X_a^\rho\, p_a^\sigma - X_a^\sigma\, p_a^\rho \;=\; -G \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \ln|\sigma_a|\, \frac{p_b.p_a}{\{(p_b.p_a)^2 - p_a^2 p_b^2\}^{3/2}}\, (p_b^\rho p_a^\sigma - p_b^\sigma p_a^\rho)\, \left\{2(p_b.p_a)^2 - 3p_a^2 p_b^2\right\} + (r_a^\rho p_a^\sigma - r_a^\sigma p_a^\rho)\,, \tag{4.4.4}$$

where σ_a denotes the proper time of the a-th particle, and r_a is the constant that appeared in (4.2.23). The contribution proportional to $\ln|\sigma_a|$ arises from the correction term Y_a in (4.2.23). The conjecture of [16,20] was that the $\ln|\sigma_a|$ factor should be replaced by $\ln\omega^{-1}$ in (4.4.1). After including the $i\epsilon$ prescription described in this chapter, this conjecture translates to the rule that in (4.4.1), $J_a^{\rho\sigma}$ should be replaced by:

$$J_a^{\rho\sigma} \;=\; G \sum_{\substack{b\neq a\\ \eta_a\eta_b=1}} \ln(\omega + i\epsilon\eta_a) \frac{p_b.p_a}{\{(p_b.p_a)^2 - p_a^2 p_b^2\}^{3/2}}\, (p_b^\rho p_a^\sigma - p_b^\sigma p_a^\rho)\, \left\{2(p_b.p_a)^2 - 3p_a^2 p_b^2\right\} + (r_a^\rho p_a^\sigma - r_a^\sigma p_a^\rho) + \Sigma_a^{\rho\sigma}\,. \tag{4.4.5}$$

We now substitute (4.4.5) into (4.4.1) and expand the expression in powers of ω, including the $\exp\left[-2\, i\, G\, \ln\{R(\omega + i\epsilon)\} \sum_{b=1}^{n} p_b.k\right]$ term. Terms proportional to $\ln(\omega \pm i\epsilon)$ reproduce correctly (4.2.59). We shall focus on terms proportional to $\omega(\ln\omega)^2$. In the ω space these terms are subdominant compared to the order ω^0 terms that we have left out from the subleading terms in the gravitational wave-form. However after Fourier transformation, polynomials in ω produce local terms in time, while terms involving $\ln\omega$ produce tail terms that survive at late and early retarded time. Therefore the corrections to (4.2.61) at late and early time will be dominated by the terms proportional to $\omega(\ln\omega)^2$ in the expression for $\widetilde{e}^{\mu\nu}$. The order ω^0 terms may have other observational signature, *e.g.* the spin memory discussed in [32].

Expanding (4.4.1) in powers of ω, with $J_a^{\rho\sigma}$ given by (4.4.5), we get the corrections proportional to $\omega(\ln\omega)^2$. These take the form:

$$
\begin{aligned}
&\Delta_{\rm subsubleading}\,\widetilde{e}^{\mu\nu} \\
= \ & e^{i\omega R - 2\,i\,\omega\,G\,\ln R\,\sum_{d=1}^{n} p_d.n}\, i\,\frac{G^3}{R}\,\omega\Bigg[4\,\{\ln(\omega+i\epsilon)\}^2\sum_{b=1}^{n} p_b.n\sum_{c=1}^{n} p_c.n\sum_{a=1}^{m+n}\frac{p_a^\mu p_a^\nu}{p_a.n} \\
& +4\,\ln(\omega+i\epsilon)\sum_{c=1}^{n} p_c.n\sum_{a=1}^{m+n}\sum_{\substack{b=1\\ b\neq a,\eta_a\eta_b=1}}^{m+n}\ln(\omega+i\epsilon\eta_a)\frac{1}{p_a.n}\frac{p_a.p_b}{\{(p_a.p_b)^2-p_a^2p_b^2\}^{3/2}} \\
& \qquad \{2(p_a.p_b)^2-3p_a^2p_b^2\}\{n.p_b\,p_a^\mu\,p_a^\nu - n.p_a\,p_a^\mu\,p_b^\nu\} \\
& +\sum_{a=1}^{m+n}\sum_{\substack{b=1\\ b\neq a,\eta_a\eta_b=1}}^{m+n}\sum_{\substack{c=1\\ c\neq a,\eta_a\eta_c=1}}^{m+n}\{\ln(\omega+i\epsilon\eta_a)\}^2\frac{1}{p_a.n}\frac{p_a.p_b}{\{(p_a.p_b)^2-p_a^2p_b^2\}^{3/2}} \\
& \qquad \{2(p_a.p_b)^2-3p_a^2p_b^2\}\frac{p_a.p_c}{\{(p_a.p_c)^2-p_a^2p_c^2\}^{3/2}}\{2(p_a.p_c)^2-3p_a^2p_c^2\} \\
& \qquad \{n.p_b\,p_a^\mu - n.p_a\,p_b^\mu\}\,\{n.p_c\,p_a^\nu - n.p_a\,p_c^\nu\}\Bigg]. \qquad (4.4.6)
\end{aligned}
$$

Using (4.1.23), (4.1.26), we now get,

$$
\Delta_{\rm subsubleading}\,e_{\mu\nu}\rightarrow\begin{cases} u^{-2}\,\ln|u|\,F_{\mu\nu} & \text{as} \quad u\to\infty \\ u^{-2}\,\ln|u|\,G_{\mu\nu} & \text{as} \quad u\to-\infty\end{cases}, \qquad (4.4.7)
$$

where

$$
\begin{aligned}
F^{\mu\nu} \ = \ & 2\,\frac{G^3}{R}\Bigg[4\sum_{b=1}^{n} p_b.n\sum_{c=1}^{n} p_c.n\left\{\sum_{a=1}^{n}\frac{p_a^\mu p_a^\nu}{p_a.n}-\sum_{a=1}^{m}\frac{p_a'^\mu p_a'^\nu}{p_a'.n}\right\} \\
& +4\sum_{c=1}^{n} p_c.n\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{1}{p_a.n}\frac{p_a.p_b}{\{(p_a.p_b)^2-p_a^2p_b^2\}^{3/2}}\{2(p_a.p_b)^2-3p_a^2p_b^2\}\{n.p_b\,p_a^\mu\,p_a^\nu-n.p_a\,p_a^\mu\,p_b^\nu\} \\
& +2\sum_{c=1}^{n} p_c.n\sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m}\frac{1}{p_a'.n}\frac{p_a'.p_b'}{\{(p_a'.p_b')^2-p_a'^2p_b'^2\}^{3/2}}\{2(p_a'.p_b')^2-3p_a'^2p_b'^2\}\{n.p_b'\,p_a'^\mu\,p_a'^\nu-n.p_a'\,p_a'^\mu\,p_b'^\nu\} \\
& +\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n}\sum_{\substack{c=1\\ c\neq a}}^{n}\frac{1}{p_a.n}\frac{p_a.p_b}{\{(p_a.p_b)^2-p_a^2p_b^2\}^{3/2}}\{2(p_a.p_b)^2-3p_a^2p_b^2\}\frac{p_a.p_c}{\{(p_a.p_c)^2-p_a^2p_c^2\}^{3/2}} \\
& \qquad \{2(p_a.p_c)^2-3p_a^2p_c^2\}\{n.p_b\,p_a^\mu-n.p_a\,p_b^\mu\}\,\{n.p_c\,p_a^\nu-n.p_a\,p_c^\nu\}\Bigg], \qquad (4.4.8)
\end{aligned}
$$

and

$$\begin{aligned}
G^{\mu\nu} &= -2\frac{G^3}{R}\Bigg[2\sum_{c=1}^{n} p_c.n \sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m} \frac{1}{p'_a.n}\frac{p'_a.p'_b}{\{(p'_a.p'_b)^2 - p'^2_a p'^2_b\}^{3/2}}\{2(p'_a.p'_b)^2 - 3p'^2_a p'^2_b\} \\
&\qquad \{n.p'_b\, p'^{\mu}_a\, p'^{\nu}_a - n.p'_a\, p'^{\mu}_a\, p'^{\nu}_b\} \\
&- \sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m}\sum_{\substack{c=1\\ c\neq a}}^{m} \frac{1}{p'_a.n}\frac{p'_a.p'_b}{\{(p'_a.p'_b)^2 - p'^2_a p'^2_b\}^{3/2}}\{2(p'_a.p'_b)^2 - 3p'^2_a p'^2_b\}\frac{p'_a.p'_c}{\{(p'_a.p'_c)^2 - p'^2_a p'^2_c\}^{3/2}} \\
&\qquad \{2(p'_a.p'_c)^2 - 3p'^2_a p'^2_c\}\{n.p'_b\, p'^{\mu}_a - n.p'_a\, p'^{\mu}_b\}\{n.p'_c\, p'^{\nu}_a - n.p'_a\, p'^{\nu}_c\}\Bigg]. \qquad (4.4.9)
\end{aligned}$$

One can check that in case of binary black hole merger, regarded as a process in which a single massive object decays into a massive object and many massless particles (gravitational waves), $F^{\mu\nu}$ and $G^{\mu\nu}$ vanish.

One could attempt to prove these results following the same procedure employed in this chapter. For this we need to iteratively solve the equations of motion to one order higher than what has been done in this chapter. Terms of order $\omega \ln \omega$ would also receive contributions from the expansion of the factors of $e^{ik.r_a}$ in various expressions in this chapter, *e.g.* in (4.2.32), to first order in k, and will therefore depend on the additional data $\{r_a\}$. However the $\omega(\ln \omega)^2$ terms given in (4.4.6) do not suffer from any such ambiguity [138].

It may be possible to find higher order generalization of these results using the exponentiated soft factor discussed in [61, 103, 104, 129, 139].

4.5 Numerical estimate

Before concluding the chapter, we shall give estimates of the coefficient of the $1/u$ term in (4.0.4) in some actual physical processes.

1. Hypervelocity stars: When a binary star system comes close to the supermassive black hole at the center of the milky way, often one of them gets captured by the black hole and the other escapes with a high velocity, producing a hypervelocity star

[140]. This can be taken as a two body decay of a single bound object. Using the mass of the central black hole to be of order $10^5\, M_\odot$, the mass of the hypervelocity star to be of order $M_\odot$, its velocity to be of order $3 \times 10^{-3} c$, and the distance of the earth from the galactic center to be of order 25000 light-years, one can estimate the coefficient of the $1/u$ term in (4.0.4) to be of order 10^{-22} days. The minimum value of u needed for (4.0.4) to hold – namely when the kinetic energy dominates the gravitational potential energy [33] – in this case is about a day. This gives a strain of order 10^{-22}, which is at the edge of the detection sensitivity of the future space-based gravitational wave detectors.

2. Core collapse supernova: This case was already discussed in [33]. During this process the residual neutron star often gets a high velocity kick which could be of the order of 1000 km/sec, balanced by ejected matter in opposite direction at a speed up to 5000 km/sec. [141]. Taking the neutron star to have a mass of order $M_\odot$ and the supernova to be in our galaxy so that its distance from the earth is of the order of 10^5 light years, the coefficient of the $1/u$ term was computed to be of order 10^{-22} sec. The minimum value of u for which the asymptotic formula holds was found to be of order 1 sec. Therefore the strain at this time will be of order 10^{-22}, which is at the edge of the detection limit of the current gravitational wave detectors.

3. Binary black hole merger: As already pointed out, for binary black hole merger, the coefficient of the $1/u$ term vanishes due to cancellation between various terms. However the individual terms in (4.0.4) have the same order of magnitude as the memory effect when the asymptotic formula can be trusted, which is of the order of the light crossing time of the horizon. Therefore the observation of the memory effect without observation of the $1/u$ tail is a prediction of general theory of relativity that can be tested in future gravitational wave experiments.

4. Bullet cluster: The bullet cluster [142] consists of a pair of galaxy clusters, each

with mass of about $10^{14} M_\odot$ [143], passing through each other at a speed of about $10^{-2}\, c$. The system is situated at a distance of about 4×10^9 light-years from the earth. Using this data we get the coefficient of the $1/u$ term in (4.0.4) to be of the order of 10^{-6} year. The retarded time u for this system, – the time that has elapsed since the centres of the two clusters passed each other, – is about 1.5×10^8 years. This gives the current value of the strain produced by the bullet cluster in our neighbourhood to be about 10^{-14}. While this is much larger than the sensitivity of the current gravitational wave detectors, what the latter detect is not the strain but the change in the strain – more precisely its second u derivative that enters the expression for the Riemann tensor. For the bullet cluster this is too small an effect to be observed by the conventional gravitational wave detectors.

4.6 Appendices

4.6.1 Evaluation of some integrals

In this appendix we shall review the evaluation of several integrals following [16].

We begin with the integral

$$J_{ab} = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell.p_b - i\epsilon} G_r(\ell) \frac{1}{(\ell-k).p_a + i\epsilon} \simeq \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell.p_b - i\epsilon} \frac{1}{(\ell^0+i\epsilon)^2 - \vec{\ell}^2} \frac{1}{\ell.p_a + i\epsilon}, \tag{4.6.1}$$

with the understanding that the integration over ℓ is restricted to the region $L^{-1} >> |\vec{\ell}_\perp| >> \omega$. Simple power counting, together with the contour deformation arguments given in the paragraph containing (4.2.33), then shows that the logarithmic contribution can come only from the region $|\ell^\mu| \sim |\ell_\perp|$ for all μ. However, since the ℓ^0 and ℓ^3 integrals converge for fixed $\ell_\perp$, we shall take the range of these integrals to be unrestricted.

First consider the case where a represents an incoming particle and b represents an outgoing particle. In this case $p_a^0 = -p'^0_{a-n} < 0$ and $p_b^0 > 0$. In the ℓ^0 plane the poles of $G_r(\ell)$ are in the lower half plane, and the zeroes of $\ell.p_a + i\epsilon$ and $\ell.p_b - i\epsilon$ are also in the lower half plane. Therefore we can close the ℓ^0 integration contour in the upper half plane and the integral vanishes.

If a represents an outgoing particle and b represents an incoming particle, then the zeroes of $\ell.p_a + i\epsilon$ and $\ell.p_b - i\epsilon$ are in the upper half plane. Therefore the ℓ^0 integral does not vanish automatically. We can evaluate this by choosing a special frame in which p_a and $-p_b$ both carry spatial momenta along the third direction, with velocities β_a and β_b respectively. Then the integral takes the form:

$$\begin{aligned}
&\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{p_a^0 p_b^0} \frac{1}{\ell^0 - \beta_a\ell^3 - i\epsilon_1} \frac{1}{\ell^0 - \beta_b\,\ell^3 - i\epsilon_2} \frac{1}{(\ell^0 + i\epsilon)^2 - \vec{\ell}^2} \qquad (4.6.2)\\
= \ &i \int \frac{d^3\ell}{(2\pi)^3} \frac{1}{p_a^0 p_b^0} \frac{1}{(\beta_a - \beta_b)\ell^3 + i(\epsilon_1 - \epsilon_2)} \left[-\frac{1}{(1-\beta_a^2)(\ell^3)^2 + \vec{\ell}_\perp^2} + \frac{1}{(1-\beta_b^2)(\ell^3)^2 + \vec{\ell}_\perp^2} \right],
\end{aligned}$$

where $\ell_\perp \equiv (\ell^1, \ell^2)$. In the second step we have evaluated the ℓ^0 integral by closing its integration contour in the upper half plane. Since $\beta_a, \beta_b \le 1$, the denominators of the terms inside the square bracket never vanish and we have dropped the $i\epsilon$ factors in that term. If we now express $\{(\beta_a - \beta_b)\,\ell^3 + i(\epsilon_1 - \epsilon_2)\}^{-1}$ as a sum of its principal value and a term proportional to $\delta((\beta_a - \beta_b)\,\ell^3)$, then the contribution to the integral from the principle value term vanishes due to $\ell^3 \to -\ell^3$ symmetry. The term proportional to $\delta((\beta_a - \beta_b)\,\ell^3)$ forces ℓ^3 to vanish, in which case the two terms inside the square bracket cancel. Therefore J_{ab} vanishes also in this case.

If a and b both refer to outgoing particles, then the zero of $\ell.p_a + i\epsilon$ is in the upper half plane and the zero of $\ell.p_b - i\epsilon$ is in the lower half plane. Therefore if we close the contour in the upper half plane so as to avoid contribution from the residues at the poles of $G_r(\ell)$, we only pick up the residue at $\ell.p_a + i\epsilon = 0$. If we choose a frame in which $p_a = p_a^0(1, 0, 0, \beta_a)$ and $p_b = p_b^0(1, 0, 0, \beta_b)$, then the pole is at $\ell^0 = \beta_a\ell^3 + i\epsilon$, and the resulting integrand takes the form

$$-i\int \frac{d^3\ell}{(2\pi)^3}\,\frac{1}{p_a^0\,p_b^0}\,\frac{1}{(\beta_a-\beta_b)\,\ell^3+i\epsilon}\,\frac{1}{(1-\beta_a^2)(\ell^3)^2+\ell_\perp^2}\,. \tag{4.6.3}$$

If we express $((\beta_a-\beta_b)\,\ell^3+i\epsilon)^{-1}$ term as a sum of its principal value and $-i\pi\delta((\beta_a-\beta_b)\,\ell^3)$, then the contribution from the principal value part vanishes due to $\ell^3\to-\ell^3$ symmetry. The term proportional to $\delta((\beta_a-\beta_b)\,\ell^3)$ forces ℓ^3 to vanish. Integration over $\ell_\perp\equiv(\ell^1,\ell^2)$ in the range $\omega << |\ell_\perp| << L^{-1}$ now generates a factor of $2\pi\ln(\omega L)^{-1}$. This gives

$$J_{ab}=\frac{1}{4\pi}\ln(\omega\,L)\,\frac{1}{p_a^0p_b^0}\,\frac{1}{|\beta_a-\beta_b|}=\frac{1}{4\pi}\ln(\omega\,L)\,\frac{1}{\sqrt{(p_a.p_b)^2-p_a^2p_b^2}}\,, \tag{4.6.4}$$

where in the last step we have reexpressed the result in the covariant form.

If a and b both refer to incoming particles, then the zero of $\ell.p_a+i\epsilon$ is in the lower half plane and the zero of $\ell.p_b-i\epsilon$ is in the upper half plane. Therefore if we close the contour in the upper half plane, we only pick up the residue at $\ell.p_b-i\epsilon=0$. The integral can be evaluated similarly using the same frame as used above and yields the same result (4.6.4).

We can also determine the $i\epsilon$ prescription for the $\ln\omega$ term by noting that in (4.6.1), the factor

$$\{(k-\ell).p_a-i\epsilon\}^{-1}=\{-p_a^0\omega+\vec{p}_a.\vec{k}-\ell.p_a-i\epsilon\}^{-1} \tag{4.6.5}$$

preserves the $i\epsilon$ prescription under addition of a positive (negative) imaginary part to ω for positive (negative) p_a^0. Therefore the singularity in the complex ω plane must be located in the lower (upper) half plane for positive (negative) p_a^0. This shows that $\ln\omega$ in (4.6.4) stands for $\ln(\omega+i\epsilon\eta_a)$ where $\eta_a=1$ for outgoing particles and $\eta_a=-1$ for incoming particles. The final result may be written as:

$$J_{ab}=\frac{1}{4\pi}\,\delta_{\eta_a,\eta_b}\,\ln\{(\omega+i\,\epsilon\,\eta_a)L\}\,\frac{1}{\sqrt{(p_a.p_b)^2-p_a^2p_b^2}}\,. \tag{4.6.6}$$

Next we consider the integral:

$$K_b \equiv 2\int \frac{d^4\ell}{(2\pi)^4}\, G_r(k-\ell)\, \frac{1}{p_b.\ell - i\epsilon}\, G_r(\ell) \simeq \int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{k.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon}\, \frac{1}{(\ell^0+i\epsilon)^2 - \vec{\ell}^2}\,, \tag{4.6.7}$$

with ω providing the upper cut-off to the integral and R^{-1} providing the lower cut-off. The last expression is obtained by making the approximation $|\ell^\mu| << \omega$ since the logarithmic contribution arises from this region. This integral has the same structure as (4.6.1) with p_a replaced by k and can be evaluated similarly. There are however a few differences:

1. Due to the changes in the cut-off, $\ln(\omega L)$ factor in (4.6.4) will be replaced by $-\ln(\omega R)$.

2. The $i\epsilon$ prescription for the integral can be determined by noting that in the expression for $G_r(k-\ell) = \{(k^0 - \ell^0 + i\epsilon)^2 - (\vec{k} - \vec{\ell})^2\}^{-1}$ in (4.6.7), if we add a positive imaginary part to $k^0 = \omega$ then it does not change the $i\epsilon$ prescription for the poles, but adding a negative imaginary part will change the $i\epsilon$ prescription. Therefore the factors of $\ln\omega$ will correspond to $\ln(\omega + i\epsilon)$.

3. Since k represents an outgoing momentum, it follows from the arguments given below (4.6.2) that in order for the integral in (4.6.7) to be non-vanishing, p_b must also represent an outgoing momentum.

4. Since $k^2 = 0$, the denominator factor in (4.6.6) simplifies to

$$\sqrt{(k.p_b)^2 - k^2\, p_b^2} = -k.p_b\,, \tag{4.6.8}$$

   with the minus sign arising from the fact that when k and p_b both represent outgoing momenta, $k.p_b$ is negative.

With these ingredients we can express the final result for K_b as:

$$K_b = \frac{1}{4\pi}\, \delta_{\eta_b,1}\, \ln\{(\omega + i\epsilon)\, R\}\, \frac{1}{k.p_b}\,. \tag{4.6.9}$$

Finally we shall analyze the integral

$$K'_b \equiv 2\int \frac{d^4\ell}{(2\pi)^4}\, G_r(k-\ell)\,\frac{1}{p_b.\ell - i\epsilon}\,G_r(\ell)^* \simeq \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{k.\ell + i\epsilon}\,\frac{1}{p_b.\ell - i\epsilon}\,\frac{1}{(\ell^0 - i\epsilon)^2 - \vec{\ell}^2}\,, \tag{4.6.10}$$

with ω providing the upper cut-off to the integral and R^{-1} providing the lower cut-off. To evaluate this integral, note that $(K'_b)^*$ is formally equal to K_b with (k, p_b) replaced by $(-k, -p_b)$. The latter result can be read out for those of J_{ab} with incoming momenta. This gives

$$K'_b = \frac{1}{4\pi}\,\delta_{\eta_b,1}\,\ln\{(\omega + i\epsilon)\,R\}\,\frac{1}{k.p_b}\,. \tag{4.6.11}$$

4.6.2 Contribution from real gravitons

In the analysis in §4.2.5, we had left out the contribution of the second term of (4.2.44) in (4.2.41). This is given by:

$$\begin{aligned}\widehat{T}^{\mu\nu}_{\text{extra}}(k) \;=\;& 16\, i\,\pi^2 G \sum_{a,b}\int \frac{d^4\ell}{(2\pi)^4}\, G_r(k-\ell)\delta(\ell^2)\left\{H(\ell^0) - H(-\ell^0)\right\}\\ & \frac{1}{p_b.\ell - i\epsilon}\,\frac{1}{p_a.(k-\ell) - i\epsilon}\,\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\left\{p_{b\alpha}p_{b\beta} - \frac{1}{2}p_b^2\eta_{\alpha\beta}\right\}\left\{p_{a\rho}p_{a\sigma} - \frac{1}{2}p_a^2\eta_{\rho\sigma}\right\},\end{aligned} \tag{4.6.1}$$

where $\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)$ has been defined in (4.2.42), and it is understood that the integration over the momenta ℓ^μ is restricted to the range much below the cut-off L^{-1}, so that we can drop the exponential factors of $e^{-ik.r_a}$ and $e^{i\ell.(r_a - r_b)}$ that regulate the ultraviolet divergence in (4.2.41). We shall now analyze possible logarithmic contributions to this term from different regions of integration.

First of all, since each term in $\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)$ defined in (4.2.42) has a factor of $(k-\ell)$, a simple power counting shows that there are no logarithmic contributions from the region $|k^\mu - \ell^\mu| << \omega$. Therefore we need to analyze contributions from the regions $R^{-1} << |\ell^\mu| << \omega$ and $\omega << |\ell^\mu| << L^{-1}$. Power counting shows that in order to analyze logarithmic contribution from the region $R^{-1} << |\ell^\mu| << \omega$, we can replace the numerator by its $\ell \to 0$ limit. Therefore we need to analyze an integral of the form

$$\mathcal{E}_0 = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{p_b.\ell - i\epsilon} \frac{1}{p_a.(k-\ell) - i\epsilon} G_r(k-\ell)\delta(\ell^2)\left\{H(\ell^0) - H(-\ell^0)\right\} . \tag{4.6.2}$$

This can be reexpressed as:

$$\mathcal{E}_0 = \frac{1}{2\pi i} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{p_b.\ell - i\epsilon} \frac{1}{p_a.(k-\ell) - i\epsilon} G_r(k-\ell)\left\{G_r(\ell)^* - G_r(\ell)\right\} . \tag{4.6.3}$$

For $|\ell^\mu| << \omega$ the contribution reduces to one of the integrals defined in (4.6.7) or (4.6.10) and can be evaluated using (4.6.9) or (4.6.11). The result vanishes due to the cancellation between the contributions coming from the $G_r(\ell)$ and $G_r(\ell)^*$ terms. Therefore there is no logarithmic contribution from the $|\ell^\mu| << \omega$ region.

We now focus on the region $|\ell^\mu| >> \omega$. Power counting shows that the integral has linear divergence in this region. So we have to evaluate it carefully by keeping also the subleading terms in this limit. First let us consider the subleading contribution arising from the terms in $\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)$ that are linear in ℓ. These involve integrals of the form:

$$\mathcal{E}_1 = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{p_b.\ell - i\epsilon} \frac{1}{p_a.(k-\ell) - i\epsilon} G_r(k-\ell)\delta(\ell^2)\left\{H(\ell^0) - H(-\ell^0)\right\} \ell^\kappa . \tag{4.6.4}$$

In the region $|\ell^\mu| >> \omega$, we can approximate the integral as:

$$\mathcal{E}_1 \simeq -\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{p_b.\ell - i\epsilon} \frac{1}{p_a.\ell + i\epsilon} \frac{1}{2k.\ell - i\epsilon\ell^0} \delta(\ell^2)\left\{H(\ell^0) - H(-\ell^0)\right\} \ell^\kappa . \tag{4.6.5}$$

Now, since $\delta(\ell^2)$ factor puts the momentum ℓ on-shell, $p_a.\ell$ and $p_b.\ell$ never vanish in the integration region of interest and therefore we can drop the $i\epsilon$ factors.[9] $k.\ell$ can vanish only when ℓ is parallel to k, but by examining the numerator factor (4.2.42) we find that

[9]The only exception is when p_a and / or p_b represents a massless particle and ℓ becomes parallel to p_a and / or p_b producing a collinear divergence; but such divergences are known to cancel in gravitational theories [144].

there are always additional suppression factors in this limit that kill potential singularity at $k.\ell = 0$. Therefore the $i\epsilon\ell^0$ factor can be dropped from this term as well. For example the presence of a $p_a.k$ or $p_a.\ell$ factor in the numerator will mean that the ratio $p_a.k/p_a.\ell$ or $p_a.\ell/p_a.\ell$ becomes a independent in the limit when ℓ is parallel to k, and the result then vanishes after summing over a using momentum conservation $\sum_a p_a = 0$. A similar result holds for terms proportional to $p_b.k$ or $p_b.\ell$. Also, a combination of terms of the form $k^\mu \xi^\nu + \xi^\nu k^\mu - k.\xi\, \eta^{\mu\nu}$ will produce a term in the gravitational wave-form that is pure gauge and therefore can be removed. Therefore we can remove such terms appearing at the level of the integrand itself. Once the $i\epsilon$ factors are removed from all the denominators, the integrand of (4.6.5) becomes an odd function of ℓ and therefore vanishes after integration over ℓ.

We now turn to the contribution from terms in $\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)$ that are quadratic in ℓ. The corresponding integrals take the form:

$$\mathcal{E}_2 = \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{p_b.\ell - i\epsilon}\,\frac{1}{p_a.(k-\ell) - i\epsilon}\,G_r(k-\ell)\,\delta(\ell^2)\left\{H(\ell^0) - H(-\ell^0)\right\}\,\ell^\kappa \ell^\tau\,. \tag{4.6.6}$$

This has potential linear divergence from the region $|\ell^\mu| >> \omega$. Therefore we need to expand the $(p_a.(k-\ell) - i\epsilon)^{-1}$ factor in powers of $p_a.k$ to the first subleading order:

$$\frac{1}{p_a.(k-\ell) - i\epsilon} = -\frac{1}{p_a.\ell + i\epsilon} - \frac{p_a.k}{(p_a.\ell + i\epsilon)^2}\,. \tag{4.6.7}$$

We can argue as before that due to the presence of the $\delta(\ell^2)$ factor we can drop all the $i\epsilon$ factors in the denominator. In this case the contribution from the last term in (4.6.7) to the integral (4.6.6) vanishes by $\ell \to -\ell$ symmetry. On the other hand, when we substitute the first term on the right hand side of (4.6.7) into (4.6.6), the integrand is an even function of ℓ. In this case terms proportional to $H(\ell^0)$ and $-H(-\ell^0)$ give identical contributions, and we get:

$$\mathcal{E}_2 \simeq -\int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{p_b.\ell - i\epsilon}\,\frac{1}{p_a.\ell + i\epsilon}\,\frac{1}{k.\ell - i\epsilon}\,\delta(\ell^2)\,H(\ell^0)\,\ell^\kappa\ell^\tau\,. \tag{4.6.8}$$

Note that we have kept the $i\epsilon$ factors even though the presence of $\delta(\ell^2)$ makes them irrelevant.

Now from (4.6.1) and (4.2.42) we see that the indices κ and τ must either be free indices μ, ν, or be contracted with the index of p_a or p_b, or be contracted with each other. If they are contracted with each other then we have a factor of ℓ^2 and the contribution vanishes due to the $\delta(\ell^2)$ factor. If any one of them is contracted with p_b, then we have a factor of $p_b.\ell$ in the numerator that kills the denominator factor of $p_b.\ell - i\epsilon$. After summing over b and using momentum conservation law $\sum_b p_b = 0$, this contribution also vanishes. A similar argument can be given for terms where either ℓ^κ or ℓ^τ is contracted with p_a. The only term that survives is where (κ, τ) take values (μ, ν). Using this we can bring the contribution to (4.6.1) to the form

$$\widehat{T}^{\mu\nu}_{\text{extra}} = \frac{G}{\pi^2} \int \left\{d^4\ell\, \delta(\ell^2) H(\ell^0)\right\} \left\{\sum_{a,b=1}^{m+n} \frac{1}{(p_a.\ell - i\epsilon)\,(p_b.\ell + i\epsilon)}\right\} \left\{(p_a.p_b)^2 - \frac{1}{2} p_a^2 p_b^2\right\} \frac{\ell^\mu \ell^\nu}{i(k.\ell - i\epsilon)} \,. \tag{4.6.9}$$

We shall now show that this contribution can be interpreted as the effect of soft emission from the gravitational radiation produced during the scattering, and is therefore already accounted for when we include in the sum over a in the soft factor the contribution from the gravitational radiation produced during the scattering. For this we note that the flux of radiation in a phase space volume $\delta(\ell^2)\, H(\ell^0)\, d^4\ell$ carrying polarization $\varepsilon_{\mu\nu}$ is given by

$$\frac{G}{\pi^2} \left\{d^4\ell\, \delta(\ell^2) H(\ell^0)\right\} \left\{\sum_{a=1}^{m+n} \frac{p_a^\rho p_a^\sigma}{p_a.\ell - i\epsilon}\right\} \left\{\sum_{b=1}^{m+n} \frac{p_b^\kappa p_b^\tau}{p_b.\ell + i\epsilon}\right\} (\varepsilon_{\kappa\tau})^* \varepsilon_{\rho\sigma} \,. \tag{4.6.10}$$

This equation can be derived by using the relation between the leading soft factor (4.2.17) and the flux of radiation [19]. In this case the first factor inside the curly bracket gives the phase space volume, and the rest of the factors gives the flux of radiation produced in the scattering. Since we shall be interested in only the total flux, we can sum over polarizations using the formula

$$\sum_{\varepsilon}(\varepsilon_{\kappa\tau})^{*}\varepsilon_{\rho\sigma} = \frac{1}{2}\left(\eta_{\kappa\rho}\eta_{\tau\sigma} + \eta_{\kappa\sigma}\eta_{\tau\rho} - \eta_{\kappa\tau}\eta_{\rho\sigma}\right), \tag{4.6.11}$$

yielding the standard result for the total flux of gravitational radiation given *e.g.* in eq.(10.4.22) of [137]:

$$\frac{G}{\pi^2}\left\{d^4\ell\,\delta(\ell^2)H(\ell^0)\right\}\left\{\sum_{a=1}^{m+n}\frac{p_a^{\rho}p_a^{\sigma}}{p_a.\ell - i\epsilon}\right\}\left\{\sum_{b=1}^{m+n}\frac{p_b^{\kappa}p_b^{\tau}}{p_b.\ell + i\epsilon}\right\}\frac{1}{2}\left(\eta_{\kappa\rho}\eta_{\tau\sigma} + \eta_{\kappa\sigma}\eta_{\tau\rho} - \eta_{\kappa\tau}\eta_{\rho\sigma}\right). \tag{4.6.12}$$

The leading soft theorem (4.2.17), applied to this radiation flux, now shows that the contribution to the $\widehat{T}^{\mu\nu}$ due to the radiation is obtained by multiplying (4.6.12) by $-i/(\ell.k - i\epsilon)$ and integrating over ℓ. This gives the net leading contribution to the soft factor due to radiation to be

$$\widehat{T}^{R\mu\nu} = \frac{G}{\pi^2}\int\left\{d^4\ell\,\delta(\ell^2)H(\ell^0)\right\}\left\{\sum_{a,b=1}^{m+n}\frac{1}{(p_a.\ell - i\epsilon)\,(p_b.\ell + i\epsilon)}\right\}\left\{(p_a.p_b)^2 - \frac{1}{2}p_a^2p_b^2\right\}\frac{\ell^{\mu}\ell^{\nu}}{i(k.\ell - i\epsilon)}. \tag{4.6.13}$$

This agrees with (4.6.9), showing that the extra contribution (4.6.1) is already accounted for by including in the sum over a in the soft factor the contribution due to radiation.[10]

4.6.3 Position space analysis of $\widehat{T}^{X\mu\nu}$

In §4.2, §4.3 we have carried out our analysis in momentum space. This has the advantage that the expressions we obtain are similar to the ones that appear in the evaluation of Feynman diagrams, and various general techniques developed for computing amplitudes in quantum field theory may find applications here. Nevertheless it is instructive to see how some of these computations can also be performed directly in position space. In this appendix we shall show how to carry out the analysis of sections §4.2.3 and §4.2.4

[10] As in [19], this can also be expressed as angular integrals over appropriate functions of the radiative gravitational field and its derivatives, but we shall not describe this here.

directly in position space.

Our first task will be to compute the gravitational fields produced by the incoming and outgoing particles during a scattering, and study their effect on the motion of the other particles. At the leading order, the incoming and outgoing particle trajectories are given by (4.2.11), or equivalently (4.2.15). Using retarded Green's function in flat space-time, we get the following expression for the gravitational field produced by the b-th particle on the forward light-cone of the trajectory of the particle [16]:

$$e^{(b)}_{\mu\nu}(x) = 2\,G\,m_b\,\frac{v_{b\mu}v_{b\nu}}{\sqrt{(v_b.x)^2 + x^2}}, \qquad h^{(b)}_{\mu\nu} = e^{(b)}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,e^{(b)\rho}_{\ \ \rho}\,. \tag{4.6.1}$$

The associated Christoffel symbol is given by, in the weak field approximation,

$$\begin{aligned}\Gamma^{(b)\alpha}_{\rho\tau}(x) \;=\; & -2\,G\,m_b\,\frac{1}{\{(v_b.x)^2 + x^2\}^{3/2}}\,\eta^{\alpha\mu}\left[\left\{v_{b\mu}v_{b\tau} + \frac{1}{2}\eta_{\mu\tau}\right\}\left\{x_\rho + v_b.x\,v_{b\rho}\right\}\right.\\ & \left.+\left\{v_{b\mu}v_{b\rho} + \frac{1}{2}\eta_{\mu\rho}\right\}\{x_\tau + v_b.x\,v_{b\tau}\} - \left\{v_{b\rho}v_{b\tau} + \frac{1}{2}\eta_{\rho\tau}\right\}\left\{x_\mu + v_b.x\,v_{b\mu}\right\}\right].\end{aligned} \tag{4.6.2}$$

Since the field has support on the forward light-cone of the trajectory, it follows that in sufficiently far future and far past of the scattering event, the outgoing particles are affected by the gravitational field of the outgoing particles and the incoming particles are affected by the gravitational field of the incoming particles.

Let Y^μ_a denote the correction to the particle trajectory (4.2.15) due to the gravitational field produced by the other particles:

$$X^\mu_a(\sigma) = v^\mu_a\,\sigma + r^\mu_a + Y^\mu_a(\sigma)\,. \tag{4.6.3}$$

We shall use the compact notation described in (4.2.12), and define η_a to be a number that takes value 1 for outgoing particles ($1 \leq a \leq n$) and -1 for incoming particles

$(n+1 \le a \le m+n)$. Then Y_a^μ satisfies the differential equation and boundary conditions:

$$\frac{d^2 Y_a^\mu}{d\sigma^2} = -\Gamma^\mu_{\nu\rho}(v_a\,\sigma + r_a)\, v_a^\mu\, v_a^\nu, \quad Y_a^\mu \to 0 \text{ as } \sigma \to 0, \quad \frac{dY_a^\mu}{d\sigma} \to 0 \text{ as } \sigma \to \infty\,, \tag{4.6.4}$$

where

$$\Gamma^\mu_{\nu\rho} = \sum_{\substack{b=1\\ b\neq a, \eta_a\eta_b=1}}^{m+n} \Gamma^{(b)\mu}_{\nu\rho}\,. \tag{4.6.5}$$

The constraint $\eta_a\eta_b = 1$ reflects that the outgoing particles are affected by the gravitational field of the outgoing particles and the incoming particles are affected by the gravitational field of the incoming particles. Using (4.6.2), (4.6.4) and (4.6.5) we get, for $\sigma >> |r_a| \sim L$:

$$\frac{d^2 Y_a^\alpha(\sigma)}{d\sigma^2} \simeq \frac{2\,G}{\sigma^2} \sum_{\substack{b=1\\ b\neq a, \eta_a\eta_b=1}}^{m+n} m_b \frac{1}{\{(v_b.v_a)^2 - 1\}^{3/2}} \left[-\frac{1}{2} v_a^\alpha + \frac{1}{2} v_b^\alpha \left\{ 2(v_b.v_a)^3 - 3 v_b.v_a \right\} \right]. \tag{4.6.6}$$

This gives

$$\frac{dY_a^\alpha(\sigma)}{d\sigma} \simeq -\frac{2\,G}{\sigma} \sum_{\substack{b=1\\ b\neq a, \eta_a\eta_b=1}}^{m+n} m_b \frac{1}{\{(v_b.v_a)^2 - 1\}^{3/2}} \left[-\frac{1}{2} v_a^\alpha + \frac{1}{2} v_b^\alpha \left\{ 2(v_b.v_a)^3 - 3 v_b.v_a \right\} \right]. \tag{4.6.7}$$

Now in (4.2.28) we have the expression for $\widehat{T}^X_{\mu\nu}$ to subleading order:

$$\widehat{T}^{X\mu\nu}(k) = \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, e^{-ik.(v_a\,\sigma + r_a)} \left[v_a^\mu v_a^\nu - ik.Y_a(\sigma)\, v_a^\mu v_a^\nu + \frac{dY_a^\mu}{d\sigma}\, v_a^\nu + v_a^\mu \frac{dY_a^\nu}{d\sigma} \right]. \tag{4.6.8}$$

As discussed below (4.2.17), the integration over σ is made well defined by replacing ω by $\omega + i\epsilon$ for outgoing particles and by $\omega - i\epsilon$ for incoming particles. We now manipulate the second term by writing

$$e^{-ik.(v_a\,\sigma + r_a)} = \frac{i}{k.v_a} \frac{d}{d\sigma} e^{-ik.(v_a\,\sigma + r_a)} \tag{4.6.9}$$

and integrating over σ by parts. The boundary term at infinity vanishes due to the replacement of ω by $\omega + i\epsilon\eta_a$, while the boundary term at $\sigma = 0$ gives a finite contribution in the

$\omega \to 0$ limit and is not of interest to us. With this (4.6.8) can be expressed as

$$\widehat{T}^{X\mu\nu}(k) = \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, e^{-ik.(v_a\sigma + r_a)} \left[v_a^\mu v_a^\nu - \frac{1}{k.v_a}\, k.\frac{dY_a}{d\sigma}\, v_a^\mu v_a^\nu + \frac{dY_a^\mu}{d\sigma}\, v_a^\nu + v_a^\mu\, \frac{dY_a^\nu}{d\sigma} \right]. \tag{4.6.10}$$

After integration over σ the first term gives the leading term. In the other terms we can substitute the expression (4.6.7) for $dY_a/d\sigma$. Since the integrand is proportional to $1/\sigma$ in the range $L << \sigma << \omega^{-1}$, we get contribution proportional to $\ln((\omega + i\epsilon\eta_a)^{-1}/L)$. Therefore, with the help of (4.6.7), the logarithmic correction to $\widehat{T}^X$, given by the last three terms in (4.6.10), takes the form:

$$\begin{aligned}\Delta\widehat{T}^{X\mu\nu}(k) = 2G \sum_{a=1}^{m+n} m_a \ln\{L(\omega + i\epsilon\eta_a)\} \sum_{\substack{b=1\\ b\neq a, \eta_a\eta_b=1}}^{m+n} m_b \frac{1}{\{(v_b.v_a)^2 - 1\}^{3/2}} \\ \left[-\frac{v_a^\mu v_a^\nu}{k.v_a}\, k_\alpha \left\{ -\frac{1}{2} v_a^\alpha + \frac{1}{2} v_b^\alpha \left\{2(v_b.v_a)^3 - 3v_b.v_a\right\} \right\} + \left\{ -\frac{1}{2} v_a^\mu + \frac{1}{2} v_b^\mu \left\{2(v_b.v_a)^3 - 3v_b.v_a\right\} \right\} v_a^\nu \right. \\ \left. + \left\{ -\frac{1}{2} v_a^\nu + \frac{1}{2} v_b^\nu \left\{2(v_b.v_a)^3 - 3v_b.v_a\right\} \right\} v_a^\mu \right].\end{aligned} \tag{4.6.11}$$

After using the relations $p_a = m_a v_a$ and some simplification we get:

$$\begin{aligned}\Delta\widehat{T}^{X\mu\nu}(k) &= 2G \sum_{a=1}^{m+n} \ln\{L(\omega + i\epsilon\eta_a)\} \sum_{\substack{b=1\\ b\neq a, \eta_a\eta_b=1}}^{m+n} \frac{1}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}} \\ &\quad \times \left[\frac{k.p_b}{k.p_a}\, p_a^\mu p_a^\nu\, p_a.p_b \left\{ \frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2 \right\} \right. \\ &\quad \left. + \frac{1}{2} p_a^\mu p_a^\nu\, p_a^2\, (p_b^2)^2 - \{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu\}\, p_a.p_b \left\{ \frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2 \right\} \right]\end{aligned} \tag{4.6.12}$$

This is in perfect agreement with (4.2.39).

5 Outlook and open questions

In this thesis, we tried to develop a comprehensive understanding of soft photon and soft graviton theorem from the study of quantum S-matrix as well as from classical electromagnetic and gravitational waveform analysis. We started the thesis by describing a general prescription for deriving soft photon/graviton theorem for a generic theory of QED/quantum gravity. Then we show that some of the assumptions of this general prescription break down in D=4. Also, the S-matrix is IR divergent in four spacetime dimensions. Due to all these subtleties, before deriving the soft factor in D=4 from the study of S-matrix, we tried to develop an understanding of the soft factor for the classical scattering process. Studying the classical limit of soft photon and soft graviton theorem in generic spacetime dimensions, we are able to relate long-wavelength electromagnetic and gravitational waveform to the soft factors derived from S-matrix. Assuming the validity of these relations in D=4 and studying asymptotic trajectories of scattered particles, we conjectured the form of subleading soft factors in four spacetime dimensions. Then by direct analysis of one loop S-matrix by some specific way of removing IR divergences, we reproduced the classical soft factors conjectured earlier with some additional quantum terms. Finally, we have given an independent derivation of the long-wavelength electromagnetic and gravitational waveform for a classical scattering process in D=4, which predicts some new kind of electromagnetic and gravitational tail memory. We also gave some numerical estimation of gravitational tail memory following from the soft graviton theorem which could be tested in the near future. Here in this final chapter, our main intention would be to point out some open directions which one needs to pursue to get a complete understanding of this subject. We describe these future directions in the following three sections.

5.1 Soft theorem from S-matrix analysis

- We can ask the question whether Sen's covariantization procedure can be generalizable in non-abelian gauge field background. It seems that the generalization is pretty straight forward like the $U(1)$ gauge field case and we can derive soft gluon theorem up to subleading order for one external soft gluon [9, 64, 65, 72, 73, 145] in generic spacetime dimensions (for loop amplitude in $D > 4$). But if we try to derive multiple soft gluon theorem, it turns out that the ingredients one has to prove double soft gluon theorem [13] is not enough for proving multiple soft gluon theorem even at leading order in contrast with the proof of multiple soft graviton theorem given in §1.4.2. The key difference is that three gluon vertex is linear in gluon momenta and four gluon vertex is independent of gluon momenta while three, four,... graviton vertices are quadratic in graviton momenta. As an example, all the diagrams in Fig.5.1 contribute to the leading soft gluon theorem for four external gluons[1]. It seems that for large number of external gluons the diagrams contributing to leading soft gluon theorem is exponentially large and we don't have a proper understanding of how to analyze compactly and get a closed-form expression. But possibly in the classical limit all the contact terms appearing from three and four gluon interaction vertices drop out and we can make sense of classical soft gluon theorem [146].

[1]If we try to prove multiple soft gluon theorem in pure YM theory with external finite energy particles are also gluons, then colour ordering would be important. For a particular colour ordering, the appearence of contact terms depends on whether the soft gluons are in adjacent legs or non-adjacent legs.

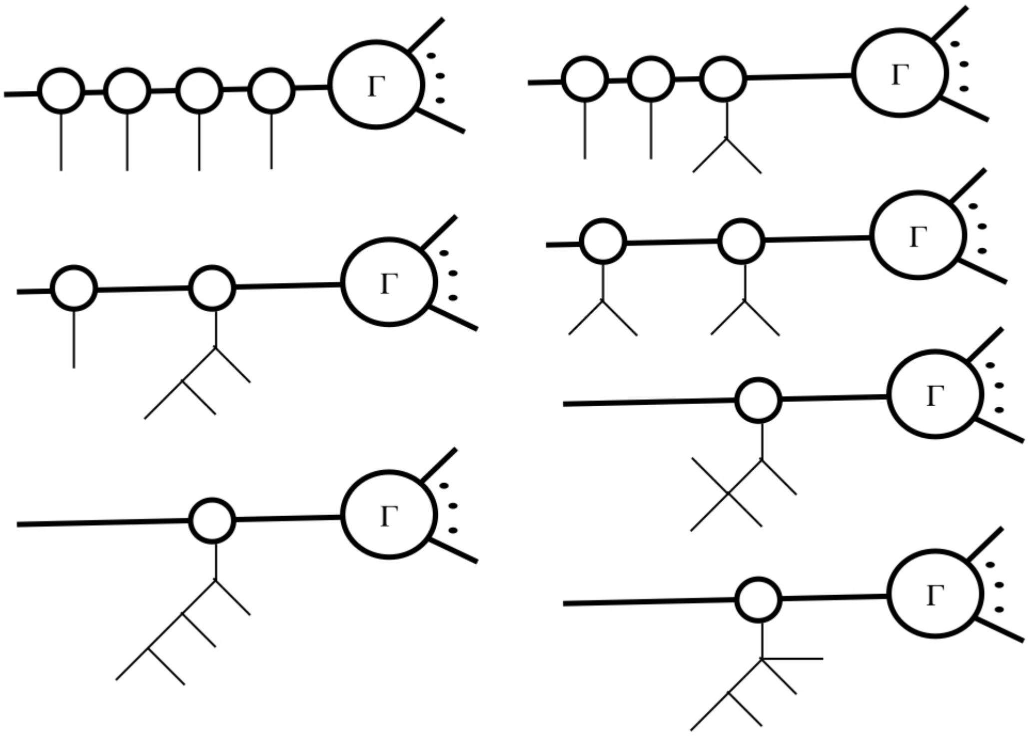

Figure 5.1: Possible set of diagrams contributing to leading soft gluon theorem for four external gluons. There are also diagrams where different gluons connected to different finite energy particle legs. In the diagrams, thick lines represent finite energy particles and thin lines represent soft gluons.

- Can we use the Grammer-Yennie decomposition in QCD [147] for studying soft gluon theorem for loop amplitudes in D=4, following Chapter-3 [16]? Here our naive understanding is that at one loop, the leading soft gluon factor will be loop corrected and produce $O(ln\omega/\omega)$ term at leading order. On the other hand, the soft graviton theorem receives loop correction in subleading order and produce $O(ln\omega)$ term [16]. With this understanding, it will be interesting to test the double copy relation in the soft limit at one-loop level [148]. Also performing an analogous classical analysis for Yang-Mills theory, as we performed for gravity [22], we can test the classical double copy relation from the classical soft expansion perspective in four spacetime dimensions[2].

[2]At $D > 4$ this has been tested in [146].

- In §4.4 we have conjectured new gravitational tail memory, where the gravitational waveform goes like $u^{-2}\ln u$ at large retarded time $u \to \pm\infty$. From the S-matrix analysis using Grammer-Yennie treatment, we expect the corresponding soft factor can be derived performing two loop calculations and extracting the coefficient of $\omega(\ln\omega)^2$. Some preliminary analysis suggests that in the regions where two loop integrals contribute to $\omega(\ln\omega)^2$, the integrals break into a product of two one loop integrals. So generalization of Chapter-3 at two loop order may be enough to extract this part of sub-subleading soft graviton factor.

- As we already explained, in four spacetime dimensions the S-matrix turns out to be IR divergent when massless particles run in the loop. Instead of making each of the S-matrices IR finite in the soft theorem analysis, we remove the same divergent factors from both the S-matrices and then extract the soft factors [16]. But we can ask whether we can study soft theorem in terms of IR finite S-matrices. There are some constructions of IR finite S-matrices [149–151], which could be helpful along this directions. For the Faddeev-Kulish kind of S-matrices without any cutoff, the leading soft theorem vanishes but from the physical point of view we know the existence of leading memory. So it is not yet clear how this kind of IR finite S-matrices can help to address the problem of deriving soft theorem from loop amplitudes.

5.2 Classical limit of soft theorem in D=4

Here we try to point out one problem in our understanding of the classical limit of the soft theorem in four spacetime dimensions. For simplicity, we are considering subleading multiple soft photon theorem given in eq.(3.5.23). In the classical limit, the multiple soft photon factor for M number of external soft photons takes form,

$$\prod_{r=1}^{M}\left\{\frac{1}{\omega}\widehat{S}^{(0)}(\varepsilon_r,n_r)+i\ \ln\omega\,\widehat{S}^{(ln)}_{cl}(\varepsilon_r,n_r)\ +\ \ln\omega\,\widehat{S}^{(ln)}_{qm}(\varepsilon_r,n_r)\right\} \tag{5.2.1}$$

where

$$\begin{aligned}
\widehat{S}^{(0)}(\varepsilon_r,n_r) &= \sum_a \frac{\varepsilon_{r,\mu}p_a^\mu}{p_a\cdot n_r}\,q_a \\
\widehat{S}^{(ln)}_{cl}(\varepsilon_r,n_r) &= \sum_a \frac{q_a\,\varepsilon_{r,\mu}n_{r\rho}}{p_a\cdot n_r}\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}}\frac{q_aq_b}{4\pi}\,\frac{m_a^2m_b^2\,\{p_b^\rho p_a^\mu-p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2-m_a^2m_b^2\}^{3/2}} \\
\widehat{S}^{(ln)}_{qm}(\varepsilon_r,n_r) &= -\frac{1}{16\pi^2}\sum_a q_a\,\frac{\varepsilon_{r,\mu}n_{r\nu}}{p_a\cdot n_r}\left\{p_a^\mu\frac{\partial}{\partial p_{a\nu}}-p_a^\nu\frac{\partial}{\partial p_{a\mu}}\right\} \\
&\quad \sum_{b\neq a}\left[\frac{\{2\,q_aq_bp_a.p_b\}}{\sqrt{(p_a.p_b)^2-p_a^2p_b^2}}\ \ln\left(\frac{p_a.p_b+\sqrt{(p_a.p_b)^2-p_a^2p_b^2}}{p_a.p_b-\sqrt{(p_a.p_b)^2-p_a^2p_b^2}}\right)\right]
\end{aligned} \tag{5.2.2}$$

Here $k_r^\mu\equiv\omega n_r^\mu$ and all the soft factors given above are real for real polarisation of soft photons. $\widehat{S}^{(0)}$ represents the leading soft factor, $\widehat{S}^{(ln)}_{cl}$ represents the classical subleading soft factor and $\widehat{S}^{(ln)}_{qm}$ represents the quantum subleading soft factor after stripping out the ω dependence. Now if following §2.1, we compute energy for a large number of photon radiation from soft theorem and compare it with the energy computed from the electromagnetic waveform, we get:

$$\left\{\varepsilon^\mu\widetilde{A}_\mu(\omega,\vec{x})\right\}\left\{\varepsilon^\nu\widetilde{A}_\nu(-\omega,\vec{x})\right\} = \left(\frac{1}{4\pi R}\right)^2\left|\frac{1}{\omega}\widehat{S}^{(0)}(\varepsilon,n)+i\ \ln\omega\,\widehat{S}^{(ln)}_{cl}(\varepsilon,n)\ +\ \ln\omega\,\widehat{S}^{(ln)}_{qm}(\varepsilon,n)\right|^2 \tag{5.2.3}$$

On the other hand by direct classical calculations of electromagnetic waveform in §4.3.1 we get,

$$\varepsilon^\mu\widetilde{A}_\mu(\omega,\vec{x}) = \frac{1}{4\pi R}\,e^{i\omega R}\left[\frac{1}{\omega}\widehat{S}^{(0)}(\varepsilon,n)\ +\ i\ \ln\omega\,\widehat{S}^{(ln)}_{cl}(\varepsilon,n)\right] \tag{5.2.4}$$

where $R \equiv \hat{n} \cdot \vec{x}$ and $k^\mu = \omega(1, \hat{n})$. Now only when $\widehat{S}^{(ln)}_{qm}(\varepsilon, n) = 0$, the equations (5.2.3) and (5.2.4) agrees. We have verified that in the limit of low momentum transfer in a $2 \to 2$ scattering $\widehat{S}^{(ln)}_{qm}(\varepsilon, n)$ is suppressed with respect to $\widehat{S}^{(ln)}_{cl}(\varepsilon, n)$. So we can make sense of the equivalence of this two equations for these kinds of scattering process. But if we consider a hard scattering process where momentum transfer is large, then eq.(5.2.3) and eq.(5.2.4) are not consistent with each other. But it may happen that the assumptions made in §2.1 break down for hard scattering process. Till now we don't have a better understanding to resolve this discrepency.

5.3 Extending the derivation of gravitational tail memory in higher order

In §4.4 we have conjectured sub-subleading gravitational tail memory, where the gravitational waveform goes like $u^{-2} \ln u$ at large retarded time $u \to \pm\infty$. In principle we can verify this by calculating the corrected trajectory of the scattered objects to one higher order and finding the corresponding gravitational waveform at $\mathcal{O}(G^3)$ by extending the analysis of Chapter-4. If the objects involved in the scattering have intrinsic spin then the effect of their spins will appear at $\mathcal{O}(u^{-2})$ tail memory. It would be interesting to work out the spin-dependent memory[3]. We can also ask whether we can generalize our prescription of deriving classical soft theorem to arbitrary $(sub)^n$-leading order to extract $\mathcal{O}(u^{-n}(\ln u)^{n-1})$ gravitational memory. The brute-force strategy could be the following [138]:

Suppose that the asymptotic trajectory of a'th particle is given by,

$$X^\mu_a(\sigma) = r^\mu_a + v^\mu_a \sigma + Y^\mu_a(\sigma) \tag{5.3.5}$$

[3]Though the full $\mathcal{O}(u^{-2})$ tail memory is a little ambiguous but possibly we can compute the spin-dependent part unambiguously.

with correction to straight line trajectory expanded as:

$$Y^\mu_a(\sigma) \quad = \quad \mathit{\Delta}_{(sub)^0} Y^\mu_a(\sigma) \;+\; \mathit{\Delta}_{(sub)} Y^\mu_a(\sigma) + \;\mathit{\Delta}_{(sub)^2} Y^\mu_a(\sigma) + \cdots \tag{5.3.6}$$

Then with this correction of trajectory the Fourier transform of the total (matter + gravitational) energy-momentum tensor can be expanded as:

$$\widehat{T}_{\mu\nu}(k) \quad = \quad \mathit{\Delta}_{(sub)^0}\widehat{T}_{\mu\nu}(k) \;+\; \mathit{\Delta}_{(sub)}\widehat{T}_{\mu\nu}(k) \;+\; \mathit{\Delta}_{(sub)^2}\widehat{T}_{\mu\nu}(k) \;+\cdots \tag{5.3.7}$$

And corresponding trace reversed metric fluctuation expansion:

$$e_{\mu\nu}(x) \quad = \quad \mathit{\Delta}_{(sub)^0} e_{\mu\nu}(x) \;+\; \mathit{\Delta}_{(sub)} e_{\mu\nu}(x) \;+\; \mathit{\Delta}_{(sub)^2} e_{\mu\nu}(x) + \cdots \tag{5.3.8}$$

where

$$\mathit{\Delta}_{(sub)^r} e_{\mu\nu}(x) \quad = \quad -\,8\pi G \int \frac{d^4\ell}{(2\pi)^4}\, G_r(\ell)\, e^{i\ell\cdot x}\, \mathit{\Delta}_{(sub)^r}\widehat{T}_{\mu\nu}(\ell) \tag{5.3.9}$$

for $r = 0, 1, 2, \cdots$. Now to derive $\mathit{\Delta}_{(sub)^r} Y^\mu_a(\sigma)$ we need to solve geodesic equation in the background metric $\sum\limits_{s=0}^{r-1} \mathit{\Delta}_{(sub)^s} e_{\mu\nu}$. Then knowing the trajectory of the particles upto $(sub)^r$ order and background metric upto $(sub)^{r-1}$ order we can compute $\mathit{\Delta}_{(sub)^r}\widehat{T}_{\mu\nu}(k)$. Finally substituting in eq.(5.3.9) and performing the integration, in principle we can evaluate $\mathit{\Delta}_{(sub)^r} e_{\mu\nu}(x)$ at large distance from scattering center and large retarded time.

In §4.4 and §4.5 we have seen that for a scattering process with at most one massive particle in the final state (e.g. black hole merger) the subleading and sub-subleading gravitational tail memory vanishes. It would be a useful prediction (if true) if we can show that all order gravitational tail memory vanishes for such kinds of scattering processes.

Introduction

Soft theorem is the infrared property of scattering amplitude with low energy photons and gravitons in external states. In a quantum theory of gravity, soft graviton theorem gives an amplitude with a set of finite energy external particles (hard particles) and one or more low energy external gravitons (soft gravitons), in terms of the amplitude without the low energy gravitons [1–18]. However when we take the classical limit, there is a different manifestation of the same theorem – it determines the low frequency component of the gravitational wave-form produced during a scattering process in terms of the momenta and spin of the incoming and outgoing objects, without any reference to the interactions responsible for the scattering [19–22], which is also related to the classical gravitational memory [23–33]. Recently people have interpreted soft graviton theorem as a Ward identity of the symmetry of asymptotically flat spacetime [6, 30, 34–42]. The inter-connection between these three subjects made the infrared structure of gravitational physics much more interesting. An analogous inter-connection has been established between soft photon theorem [1–4, 43–49], electromagnetic memory [50–53] and Ward identity for large gauge symmetry [54–63].

In last few years a large group of people have derived soft theorem from direct analysis of amplitudes in field theory and string theory [5, 7–9, 11–15, 64–105]. There are general arguments establishing their validity in any space-time dimensions in any theory as long as one maintains the relevant gauge symmetries – general coordinate invariance for soft graviton theorem and U(1) gauge invariance for soft photon theorem. What makes it a theorem is that the soft factors are universal (independent of the theory) up to some specific order of expansion in soft momenta. For example, soft photon theorem is only universal in leading order i.e. the leading soft factor only depends on the charges and momenta of external finite energy particles and no other information of the theory. At subleading order soft photon factor has a universal part along with a theory dependent piece due to non-minimal coupling of field strength with finite energy particles [17]. Soft graviton theorem is universal up to subleading order and non-universal terms appear at sub-subleading order due to non-minimal coupling of finite energy particles with Riemann tensor [11].

Very recently Sen has developed a covariantization prescription [101] for deriving single soft graviton theorem for a generic theory of quantum gravity which is valid to all orders of perturbation theory for non-compact spacetime dimensions greater than four and valid in tree level for four-dimensional spacetime. The basic idea is that we have to start with a general coordinate invariant one particle irreducible(1PI) effective action of a generic theory of quantum gravity(assumed to be UV complete and background independent). Now the full quantum corrected S-matrix of any scattering process can be obtained just by computing tree-level diagrams using Feynman rules derived from the 1PI effective action. To get the Feynman rules associated with the vertices involving soft gravitons we need to covariantize the 1PI effective action with respect to soft graviton background and include possible all non-minimal couplings associated with soft graviton with other particles.

In [15] we derived subleading multiple soft graviton theorem for a generic theory of quantum gravity generalizing Sen's covariantization prescription when all the external gravitons are soft at the same rate (simultaneous limit). Here it turns out that the ingredients needed to prove double soft graviton theorem is enough for deriving multiple soft graviton theorem up to subleading order though the derivation is much more involved. Multiple soft factors can be divided into two parts. For M number of soft graviton scattering the first part can be written as a multiplication of $(M-1)$ number of leading soft factor and one subleading soft factor which are fixed from single soft graviton theorem. The second part we call contact term which comes from the diagrams involving three graviton self-interaction and two gravitons two finite energy particles interaction vertices and the structure of this part is fixed from double soft graviton theorem. In [106], using Cachazo-He-Yuan (CHY) prescription [107] we derived multiple soft graviton theorem for Einstein gravity at tree level which agrees with our more general result of [15].

In [17] we generalized Sen's covariantization prescription in a background involving soft photon and soft graviton simultaneously and derived multiple soft photon-graviton theorem up to subleading order. Here we fixed the structure of non-universal term in the subleading order of single soft photon theorem for a theory with photon field coupled to any arbitrary field with arbitrary spin in all possible minimal and non-minimal ways. Considering some specific kind of non-minimal coupling terms appeared in effective field theory literature we determined the form of the non-universal piece of subleading soft factor and compared it with earlier results. The interesting feature in our multiple soft photon-graviton theorem result is that once we fix the non-universal term in the single soft photon theorem it determines all the terms in multiple soft photon-graviton theorem result and no other new kind of non-universal term appears. Here the contact term receives contribution from the process of soft graviton splitting into two soft photons and this contribution turns out to be spacetime dimension dependent.

In four spacetime dimensions, the S-matrix of a theory involving massless particles is IR divergent. So studying soft photon or graviton theorem in terms of IR divergent S-matrix is ambiguous [66]. But since soft theorem relates two S-matrices, we don't need to make individual S-matrices IR-finite but factored out the same IR divergent pieces from both the S-matrices and cancel it from both sides of soft theorem expression. In the case of spinor QED, Grammer and Yennie proposed a prescription for factoring out the IR-divergent part of the S-matrix by dividing the photon propagator into two pieces called G-photon and K-photon propagators [108]. The division is made in such a way that any loop diagram computed with K-photon propagator contains the full IR-divergent part and the same diagram computed with G-photon propagator is IR-finite.

In [16] we used Grammer-Yennie treatment to derive loop corrected subleading soft photon and graviton theorem. Evaluating loop amplitudes we found that soft factor in subleading order is logarithmic in the energy of external soft photon/graviton. This logarithmic soft factor is universal and independent of the spin of external finite energy particles and up to subleading order in soft momenta, the logarithmic term is one loop exact. In [16] we divided the logarithmic terms of soft photon theorem into two parts according to whether the contribution comes from massive particle propagator or massless photon propagator while evaluating loop integrals. The first part is called classical logarithmic term which takes care of the electromagnetic radiation coming out from a scattering process due to early and late time acceleration of charged particles as an effect of long-range electromagnetic force produced by other charged particles. We have also verified this from independent classical calculations. The other part is called quantum logarithmic term, interpreted as the effect of back-reaction due to soft radiation energy loss on the classical trajectories of the charged particles. This quantum term is suppressed relative to the classical term when the total energy loss in electromagnetic radiation is small.

Generalizing Grammer-Yennie technique to perturbative gravity in [16] we derived the logarithmic terms in subleading soft graviton theorem for loop amplitudes. Here one part of the classical logarithmic term represents gravitational radiation for deceleration or acceleration of hard particles due to long-range gravitational attraction by other particles at late time. Another part of the classical logarithmic term represents the backscattering effect of the external soft graviton with the background geometry produced by finite energy particles. Here the backscattering term in loop amplitude arises from the extra infrared divergent part in $(N+1)$ particle amplitude relative to N particle amplitude. This is evaluated using IR-cutoff R given by the distance of scattering center from detector position. Here we also have a quantum part of logarithmic soft factor interpreted as the effect of backreaction on the trajectory of finite energy particles due to soft gravitational radiation energy loss.

In [19] Laddha and Sen extracted large wavelength gravitational waveform studying the classical limit of soft graviton theorem. Here classical limit means a large number of gravitons should come out from a scattering process so that the coherent state of a large number of gravitons can be declared as gravity wave. This requires that the masses of the scattered objects have to be large in Planck mass unit. Here the soft limit means that the energy loss due to gravitational radiation has to be small compared to the energy of the objects that participated in the scattering process. This leads to probe scatterer limit or large impact parameter limit for a two-body scattering and the wavelength of gravitational radiation has to be large compared to the characteristic length scale of scattering.

In four spacetime dimensions performing Fourier transformation of the soft factor in soft energy variable one extracts gravitational memory. The leading soft graviton factor predicts a DC Christodoulou memory effect corresponding to the permanent displacement between the mirrors of a gravitational wave detector. Our logarithmic soft factor predicts a tail memory [22, 33], which tells that if we wait for a long time after the peak of gravitational waveform passes, we will not only get a permanent shift but it will have larger shift initially and with increasing time the mirrors will settle to their permanent shifted position predicted by Christodoulou. For an asymmetric explosion of type-II supernova (1987A), we have estimated the order of magnitude of displacement between the mirrors divided by the initial distance between the mirrors, which turns out in the order of 10^{-22} – which is in the edge of LIGO resolution but expected to be observed in eLISA. We also determined order of magnitude of strain in GW detector for bullet cluster scattering and hyper velocity star production [22].

Recently in [22] we have given a general proof of the relationship between long-wavelength gravitational waveform and the soft graviton factor [16] in four spacetime dimensions analogous to the result proved in higher spacetime dimensions [21]. To show the equivalence, we have to iteratively solve liniarized Einstein equation to derive background metric and geodesic equation to get asymptotic trajectory. At this stage, we observed a mapping between classical waveform calculation with the one-loop scattering matrix calculations and we expect that our prescription is quite simple to generalize for computing higher order memory effects. Emboldened by the success of soft theorem in correctly predicting the tail of the gravitational wave-form at the subleading order, we conjectured a new tail memory at the sub-subleading order [22].

The organization of the thesis is as followed,

- In Chapter 1 we describe a general prescription of deriving soft photon and soft graviton theorem for a generic theory of quantum electrodynamics and quantum gravity. Using this prescription we derive single soft photon and soft graviton theorem up to subleading order and will state the result of sub-subleading soft graviton theorem and subleading multiple soft graviton theorem. Then we show how some of the assumptions in this prescription break down in four spacetime dimensions.

- In Chapter 2 we explain how to take the classical limit of multiple soft photon and soft graviton theorem. From the classical limit of the soft theorem, we predict long wavelength electromagnetic and gravitational waveform. From the understanding of soft theorem in D>4 and asymptotic trajectory of particles in D=4 here we give the classical subleading soft factor in D=4 which is logarithmic in soft energy.

- We devote Chapter 3 for the derivation of subleading soft photon and soft graviton theorem for loop amplitudes in four spacetime dimensions. Then we briefly sketch the strategy for deriving subleading multiple soft photon theorem in D=4.

- Chapter 4 is solely devoted for deriving low frequency electromagnetic and gravitational waveform for a general classical scattering process. Using this we determine the electromagnetic and gravitational wave-form emitted during a scattering process at late and early retarded time, in terms of the charges and four momenta of the ingoing and outgoing objects. As an amusement here we have performed some numerical estimation of gravitational memory for various astrophysical scattering processes.

- In chapter 5 we will conclude the thesis with some future directions.

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