NONCLASSICALITY AND ENTANGLEMENT IN CONTINUOUS VARIABLE SYSTEMS

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Jebathilagar Solomon Ivan

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Abstract

Nonclassicality and entanglement are two important features exhibited by continuous variable quantum states. This thesis is centered on the connection between nonclassicality and entanglement in the context of continuous variable quantum systems. Evidently, nonclassicality is a prerequisite for entanglement. The connection between the two has been well explored in the context of Gaussian states, namely in the context of squeezing nonclassicality. We study the connection in the context of other well known nonclassicalities, namely nonclassical photon number statistics and antibunching. By definition, every classical state is a convex sum of coherent states, and hence is separable. Nonclassicality does not imply entanglement, but every entangled state is nonclassical. Negativity under Partial Transpose (NPT) implies nonclassicality, but Positivity under Partial Transpose (PPT) by itself does not indicate that the state is classical or separable. A PPT state can be separable or entangled, can be classical or nonclassical.

Chapter 1 is primarily introductory in nature, bringing forth the various concepts involved in the theory of entanglement, both in the finite dimensional situation as well as in the infinite dimensional case of continuous variable systems. It is expository in nature and collects some of the techniques useful later in the thesis.

In Chapter 2 we bring forth a relationship between nonclassicality and entanglement. The problem of studying the interrelationship between nonclassicality and entanglement is tied to the fact that there is no simple test which can conclude in a definite manner if a given generic mixed state is classical or not, and there is no single test which can answer with certainty if a mixed state is entangled or separable. However, in very special or specific cases one can make definitive statements. For states of a single mode of radiation which are diagonal in the Fock basis, the issue of classicality/nonclassicality has been settled. This is possible thanks to the result of the classical Stieltjes moment problem [170]. We bring out the possibility of using such nonclassical (non-Gaussian) resources to generate useful entanglement. With a product state of the form $\hat{\rho}_{\rm in}^{(ab)} = \hat{\rho}^{(a)} \otimes |0\rangle_{bb}\langle 0|$ as input, the output two-mode state $\hat{\rho}_{\rm out}^{(ab)}$ of a beamsplitter is shown to be NPT whenever the photon number distribution (PND) statistics $\{p(n_a)\}$ associated with the mixed state $\hat{\rho}^{(a)}$ of the input a-mode is antibunched or otherwise nonclassical, i.e., if $\{p(n_a)\}$ fails to respect any one of an infinite sequence of necessary and sufficient classicality conditions. We establish the equivalence of classicality and PPT of $\hat{\rho}_{\rm out}^{(ab)}$ in this kind of situations. Thus NPT is a necessary and sufficient test of entanglement of $\hat{\rho}_{\rm out}^{(ab)}$. Furthermore $\hat{\rho}_{\rm out}^{(ab)}$ is shown to be distillable if $\hat{\rho}^{(a)}$ is antibunched or violates any one of an infinite sequence of three term classicality conditions. We also discuss the issue of distillability arising from an intrinsically higher order violation of classicality. This is the only second instance in

continuous variable entanglement theory where NPT has turned out to be a necessary and sufficient criterion for entanglement, the earlier instance being that of two-mode Gaussian states. A preliminary version of these results is found in [194]. We attempt to estimate the entanglement of formation (EOF) of entangled states generated in the above manner. We evaluate both upper and lower bounds on EOF for very special examples. Our principal tool in this scheme is the fact that average entanglement does not increase under local operations and classical communications (LOCC). The general idea used has been to project out the state into 2×2 subspaces, and then use Wootter's formula for the entanglement of formation of a two-qubit system to estimate the entanglement; such a process is clearly an LOCC. However, a drawback with such a scheme is the fact that one cannot estimate more than one ebit of entanglement even from a highly entangled state. For the simple example of an entangled state generated by passing through a 50:50 beamsplitter an arbitrary mixture of the ground state and n^{th} Fock state on Alice's side, with Bob's side in the ground state, we give a distillation procedure whereby we distill more entanglement than given by lower bound for EOF in [76]. We extend these ideas to entangled states generated from PND's which correspond to a very special superposition of coherent states, and we demonstrate distillation procedures which distill well above one ebit of entanglement. We also indicate the possibility of using the Terhal-Vollbrect formula [69,76] in estimating entanglement, in a more general context, using a truncation scheme.

The study undertaken in Chapter 2 is continued in Chapter 3 from a more general perspective. We describe a single test which, if successful, is able to simultaneously establish both the nonclassicality and NPT entanglement of a given two-mode state. We extend the notion of antibunching to two-mode systems through the Mandel matrix construct, and show that nonclassicality at this level naturally separates into two distinct kinds, Type I and Type II, depending on whether the sub Poissonian statistics is visible or not at a single-mode level. The "Type" of a nonclassical state is invariant under the action of every $U(2)$ beamsplitter. A state could go from separable to entangled under beamsplitter action, but its Type is invariant. Type II states are special in the sense that one may pass such states through any $U(2)$ beamsplitter, even then can never detect antibunching locally i.e., in a single-mode. We construct examples of both types. We introduce a beamsplitter invariant definition for the Mandel parameter, extended to the case of two-mode systems through the nonpositivity of the Mandel matrix. That we are able to do so is because the Mandel matrix transforms covariantly under beamsplitter action. However, we find that the two-mode Mandel parameter can take values less than -1, as compared to the Mandel parameter in the single-mode case. This feature seems to expose the limitation of the beamsplitter as an entangling device, as there are en-

tangled states that the beamsplitter cannot produce. The two-mode Mandel parameter is relevant only within the Type under consideration. We explore the production of bipartite entanglement from separable nonclassical states by beamsplitters, we trace back the entanglement to the nonclassicality involved, and we illustrate this aspect through several examples. We demonstrate distillable entanglement in this context. We extend these ideas to the case of generating tripartite entanglement through generalised beamsplitters, and examine their detection through simple moment-based tests which trace back the entanglement to a particular type of nonclassicality. We also demonstrate the possibility of generating genuine tripartite entanglement from two-mode Mandel type nonclassicality.

In Chapter 4 the EOF of an arbitrary two-mode Gaussian state is computed. In this context, we bring out the intimate connection between the two-mode squeeze parameter as a measure of the strength of nonclassicality and alternatively as a measure of entanglement. Apart from a conjecture, our analysis rests on two main ingredients. One of them is a four-parameter canonical form we develop for the covariance matrix, one of these parameters, the squeeze parameter, acting as a measure of EOF. The other is the generalisation of the EPR correlation used in the work of Giedke *et al* [70] to noncommuting variables. The conjecture is in respect of an extremal property of this correlation [327].

In Chapter 5 we study the compatibility conditions between the (global) spectrum and the spectra of the individual modes of a general n-mode Gaussian state. We present an elementary proof for the compatibility conditions, making optimal use of beamsplitter and two-mode squeezing transformations. An unexpected bye-product of our elementary approach is the result that every two-mode Gaussian state is uniquely determined, modulo local transformations, by its global spectrum and local spectra, a property shared not even by a pair of qubits [18].

In Chapter 6 we obtain the operator-sum representation of all the quantum limited single-mode Bosonic Gaussian channels. The analysis lends insight into how certain unphysical processes such as the transposition map, or scaling of the Weyl-ordered characteristic function, or a combination of both can be rendered physical through a threshold Gaussian noise. The motive here is to bring out this aspect in a transparent manner through the operator-sum representation. We have that the scaling of the diagonal weight function and scaling of the Husimi Q function correspond to physical processes. As will be seen in the following Chapter, the fact that scaling of the Q function is physical is of critical relevance when one defines a measure of non-Gaussianity for quantum states. This Chapter further explores the notion of nonclassicality breaking and the notion of entanglement breaking in light of the operator-sum representation.

Having brought out the connection between nonclassicality and entanglement, and having exposed nonclassicality as a resource, it is useful to understand this resource as being Gaussian and non-Gaussian. Chapters 2 and 3 primarily dealt with non-Gaussian states and the nonclassicality associated with them, but Chapters 4, 5, and 6, dealt with Gaussian states and issues regarding them. In Chapter 7 we bring out the essential difference between these two very different resources through the consideration of cumulants. Since the higher order cumulants defined through an s-ordered quasi-probability is independant of the ordering parameter s and hence is intrinsic to the state, every non vanishing cumulant of order greater than two serves as an indicator of non-Gaussianity. We introduce a new measure for non-Gaussianity based on the negentropy of the Q function. We show that our measure satisfies some of the requirements that a good non-Gaussianity measure should satisfy, especially the invariance of the measure under uniform scaling of the Q function. The scale invariance of the measure is demanded by the fact that scaling of the Q function is a valid physical transformation as shown in Chapter 6. The measure is well supported by the fact that the Marcinkiewicz theorem holds for phase space distributions too [358]. We analytically evaluate this non-Gaussianity measure for mixed entangled states generated by passing the photon-added thermal state through a $U(2)$ beamsplitter, the ancilla being in the ground state. We find for these examples that the non-Gaussianity as evaluated by our measure, is independent of temperature, which is a direct manifestation of scale invariance. That we are able to evaluate the non-Gaussianity for these mixed entangled states is because of the invariance of the measure under passive transformations. We also evaluate the measure for the phase averaged coherent state. In a recent work [361, 362], Genoni *et al* introduced distance based measures of non-Gaussianity of a state through the Hilbert-Schmidt distance and relative entropy defined at the density operator level. We compare their measure with ours for the simple example of the photon-added thermal state [216].

Finally we conclude with some remarks and discuss possible future directions of research, particularly in the context of the use of non-Gaussian resources in quantum information processing.

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1 *Generation of Distillable Entanglement From Nonclassical Photon Statistics* (In preparation), the preliminary version is found in: J. Solomon Ivan, N. Mukunda, and R. Simon, *Generation of NPT Entanglement from Nonclassical Photon Statistics*, quant-ph/0603255.

2 J. Solomon Ivan, S. Chaturvedi, E. Ercolessi, G. Marmo, G. Morandi, N. Mukunda, R. Simon *Entanglement and nonclassicality for multi-mode radiation field states*, arXiv:1009.6104. (Accepted in PRA)

3 J. Solomon Ivan and R. Simon, *Entanglement of Formation for Gaussian states*, quant-ph/0808.1658 (Submitted to PRL).

4 J. Solomon Ivan and R. Simon, *Compatability conditions on local and global spectra for n-mode Gaussian states*, quant-ph/0812.2805.

5 J. Solomon Ivan, Krishnakumar Sabapathy, and R. Simon, *Operator-sum representation for Bosonic Gaussian channels*, arXiv:1012.4266.

6 J. Solomon Ivan, M. Sanjay Kumar, and R. Simon, *A measure of non-Gaussianity of quantum states*, quant-ph/0812.2800 (Submitted to PRA).

1 Basic Ideas

1.1 Introduction

The state of a quantum mechanical system S is described by a density operator $\hat{\rho}$ acting on a Hilbert space $\mathcal{H}$. The system in consideration specifies the dimension of the Hilbert space. The operator $\hat{\rho}$, satisfies the following three defining properties :

$$\hat{\rho} = \hat{\rho}^{\dagger}, \quad \mathrm{Tr}\,\hat{\rho} = 1, \quad \hat{\rho} \geq 0. \tag{1.1}$$

A pure state is described by a normalised (unit) vector $|\Psi\rangle$ in the Hilbert space $\mathcal{H}$, and the density operator corresponding to $|\Psi\rangle$ is given by

$$\hat{\rho} = |\Psi\rangle\langle\Psi|. \tag{1.2}$$

Clearly, distinct vectors in the Hilbert space do not correspond to distinct states. All unit vectors in $\mathcal{H}$ which differ from one another by phase factors, represent one and the same state. In other words, states are represented by an equivalence class of unit vectors of the Hilbert space. It is clear that the state $\hat{\rho}$ in Eq. (1.2), satisfies the three defining requirements in Eq. (1.1).

The most general state of a quantum mechanical system S is described by a 'mixed' state $\hat{\rho}$, which is a convex combination of pure states, i.e.,

$$\hat{\rho} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|, \quad p_k > 0, \quad \sum_k p_k = 1. \tag{1.3}$$

The quantum state space thus forms a convex set in which pure states correspond to the case when all but one of the p_k's are zero. The pure states satisfy the additional

requirement

$$\hat{\rho}^2 = \hat{\rho}, \tag{1.4}$$

and correspond to the extremal points of the convex state space. They cannot be realised as nontrivial convex combinations of other states. While $\mathrm{Tr}\hat{\rho}^2 = 1$ for pure states, mixed states satisfy

$$\mathrm{Tr}\hat{\rho}^2 < 1, \tag{1.5}$$

and correspond to non extremal points of the convex state space.

The probabilities p_k's and the ensemble realisation in Eq. (1.3) are in general associated with a preparation procedure. The nontriviality of the ensemble realisation arises from the fact that the $|\Psi_k\rangle$'s need not be orthogonal, or even linearly independent, and that the set of ensembles realising a given mixed state $\hat{\rho}$ is a huge family [1]. A preparation procedure yields an ensemble realisation for $\hat{\rho}$, but given a $\hat{\rho}$ it is impossible to tell which preparation procedure it was derived from.

The natural objects of interest are the expectation values of observables. Observables in quantum theory are represented by hermitian operators $\hat{\Omega}$,

$$\hat{\Omega} = \sum_i \lambda_i |\Phi_i\rangle\langle\Phi_i| = \sum_i \lambda_i P_i, \quad \sum_i P_i = \mathbb{1}, \tag{1.6}$$

and P_i's are projection operators obeying $P_i P_j = \delta_{ij} P_i$. The $\{\lambda_i\}$ are interpreted as the outcome or eigenvalue of an experiment corresponding to the observable $\hat{\Omega}$, and $\{|\Phi_i\rangle\}$ as the corresponding eigenstates. The eigenvalues λ_i are real, but can be negative. The *expectation value* of an observable $\hat{\Omega}$ with respect to a pure state $|\Psi\rangle$ is given by $\langle\hat{\Omega}\rangle = \langle\Psi|\hat{\Omega}|\Psi\rangle$. In the case of a mixed state $\hat{\rho}$, the expectation value is given by

$$\langle\hat{\Omega}\rangle = \mathrm{Tr}(\hat{\Omega}\hat{\rho}) = \sum_k p_k \langle\Psi_k|\hat{\Omega}|\Psi_k\rangle. \tag{1.7}$$

The expectation value $\langle\hat{\Omega}\rangle$ is interpreted as the *average value* of the observable $\hat{\Omega}$ over repeated trials of the experiment, with the same state $\hat{\rho}$ prepared each time.

Though the average value of the outcome is calculated as in Eq. (1.7), a particular trial yields a particular eigenvalue λ_i as the outcome of the experiment. The probability of occurance q_i of the ith outcome corresponding to the eigenvalue λ_i is given by

$$q_i = \mathrm{Tr}(P_i\hat{\rho}) = \sum_k p_k \langle\Psi_k|P_i|\Psi_k\rangle. \tag{1.8}$$

Given a particular outcome i, the state of the system after measurement is no more represented by $\hat{\rho}$, but collapses to the corresponding eigenstate of the observed eigenvalue,

$$\hat{\rho}' = \frac{P_i \hat{\rho} P_i}{\mathrm{Tr}(P_i \hat{\rho})}. \tag{1.9}$$

This is the Von Neumann collapse postulate. Since the given state has collapsed into a particular eigenstate of the observable, it is no more useful in the study of the properties of the original state. Thus for a new trial, one has to repeat the preparation procedure to obtain the $\hat{\rho}$, and rerun the experiment. Such a measurement scheme is called as the Von Neumann projective measurement, and the probabilities q_i are called the Von Neumann projection valued measure.

To summarise, the density operator $\hat{\rho}$ completely specifies all the properties of the system. All expectation values of all possible experimental observables $\hat{\Omega}$ are captured by $\hat{\rho}$.

1.2 Composite systems

Consider a bipartite system S which consists of subsystems A and B. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ denote the Hilbert spaces of the subsystems, then the Hilbert space of the total system is the tensor product $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$ of the Hilbert spaces of the subsystems. Let the dimension of $\mathcal{H}_A$ be m, and that of $\mathcal{H}_B$ be n. Let $\{|\psi_j\rangle\}$ form an ONB in $\mathcal{H}_A$, and $\{|\phi_\alpha\rangle\}$ an ONB in $\mathcal{H}_B$. Then any pure state $|\Psi\rangle$ of the combined system can be written as

$$|\Psi\rangle = \sum_{j,\alpha} c_{j\alpha} |\psi_j\rangle \otimes |\phi_\alpha\rangle. \tag{1.10}$$

A pure state $|\Psi\rangle$ of the bipartite system S is said to be a product state if and only if the expansion coefficients $c_{j\alpha}$ have the product form $c_{j\alpha} = x_j y_\alpha$, i.e., the $m \times n$ coefficient matrix c is the outer product of two vectors. Any state $|\Psi\rangle$ which cannot be written in the product form

$$|\Psi\rangle \neq |\psi\rangle \otimes |\phi\rangle, \tag{1.11}$$

is said to be *entangled*.

Theorem 1.1 *Given a state $|\Psi\rangle$ in the tensor product space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$, it can*

always be written in the form [2]

$$\begin{aligned} |\Psi\rangle &= \sum_{j=1}^{r} \sqrt{\lambda_j} |\psi'_j\rangle \otimes |\phi'_j\rangle, \quad \text{where} \\ r &\leq \min(m,n), \quad \lambda_j > 0, \quad \sum_j \lambda_j = 1, \end{aligned} \tag{1.12}$$

and $\{|\psi'_j\rangle\}$ *and* $\{|\phi'_j\rangle\}$ *are vectors from an ONB in* $\mathcal{H}_A$ *and* $\mathcal{H}_B$ *respectively.*

Proof: This can easily be seen from the singular value decomposition of the c matrix, i.e.,

$$c \to c' = VcW^T, \tag{1.13}$$

where V and W correspond to independent local unitary change of basis in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, i.e.,

$$|\psi'_k\rangle = \sum_j V^*_{jk} |\psi_j\rangle, \qquad |\phi'_\beta\rangle = \sum_\alpha W^*_{\alpha\beta} |\phi_\alpha\rangle. \tag{1.14}$$

V and W are chosen such that c' is diagonal.

The integer r is known as the Schmidt rank of $|\Psi\rangle$. The Schmidt rank r, and the Schmidt coefficients $\{\lambda_i\}$, constitute the local invariants of the state. It is clear that for product states the Schmidt rank is one. The Schmidt rank r, of a given bipartite pure state $|\Psi\rangle$, is thus an entanglement witness of the state.

The notion of entanglement for the case of mixed states is much more subtle. As we have already seen in Eq. (1.3), any mixed state of the bipartite system $A+B$ can be written as

$$\begin{aligned} \hat{\rho}_{AB} &= \sum_k p_k |\Psi_k\rangle\langle\Psi_k|, \\ p_k &> 0, \quad \sum_k p_k = 1. \end{aligned} \tag{1.15}$$

A state $\hat{\rho}_{AB}$ is said to be *separable*, if there exists an ensemble realisation $\{|\Psi_k\rangle,\ p_k\}$ of $\hat{\rho}_{AB}$, such that all the $|\Psi_k\rangle$'s are product states [3], i.e.,

$$\hat{\rho}_{AB} = \sum_k p_k \hat{\rho}_{Ak} \otimes \hat{\rho}_{Bk}, \tag{1.16}$$

where the p_k's are positive, and $\hat{\rho}_{Ak}$'s and $\hat{\rho}_{Bk}$'s are density operators of subsystems A and B respectively. Without loss of generality, these density operators can be chosen to correspond to pure states. Stated differently, any convex sum of product states is by definition a separable state, and separable states constitute a convex subset of the convex state space. Any state which cannot be written as a convex sum of product states is said to be *entangled*. The subtle part of the definition stems from the fact that one has to run, in principle, through all possible decompositions of a state to conclude if it is separable or not.

1.2.1 Partial trace

Consider a bipartite system in the state $\hat{\rho}_{AB}$. Suppose we are interested in the subsystem A alone, i.e., we are interested in measurement of a local observable $\hat{\Omega}_A$ which acts only on the Hilbert space $\mathcal{H}_A$. The action of such an observable is described by the operator $\hat{\Omega}_A \otimes \mathbb{1}_B$ on the extended Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. Using the resolution of identity in the B subsystem with the choice of an ONB $\{|\phi_\alpha\rangle\} \in \mathcal{H}_B$, the expectation value of the A subsystem observable $\hat{\Omega}_A$ is

$$\begin{aligned} \mathrm{Tr}(\hat{\Omega}_A \otimes \mathbb{1}_B \hat{\rho}_{AB}) &= \mathrm{Tr}_A \mathrm{Tr}_B(\hat{\Omega}_A \otimes \mathbb{1}_B \hat{\rho}_{AB}) \\ &= \sum_{k,\alpha} \langle\psi_k| \otimes \langle\phi_\alpha|\hat{\Omega}_A \otimes \mathbb{1}_B \hat{\rho}_{AB}|\psi_k\rangle \otimes |\phi_\alpha\rangle \\ &= \mathrm{Tr}_A(\hat{\Omega}_A \hat{\rho}_A), \quad \text{where} \\ \hat{\rho}_A &= \mathrm{Tr}_B \hat{\rho}_{AB} = \sum_\alpha \langle\phi_\alpha|\hat{\rho}_{AB}|\phi_\alpha\rangle. \end{aligned} \tag{1.17}$$

Clearly, $\hat{\rho}_A$ is an operator on the Hilbert space $\mathcal{H}_A$. The trace operation executed only on the B subsystem is called partial trace, and the resulting $\hat{\rho}_A$ is called the reduced density operator of subsystem A. It is clear that partial trace preserves the defining requirements on a density operator. The notions indicated above hold irrespective of whether $\hat{\rho}_{AB}$ was pure or mixed. Every observable of subsystem A sees the state $\hat{\rho}_{AB}$ as if it were the state $\hat{\rho}_A$. Clearly, partial trace of pure bipartite states leads to either pure or mixed states of the subsystem. If we begin with a bipartite pure entangled state, the partial traced state of the subsystem is definitely mixed, and the matrix rank of the reduced state is the Schmidt rank of the initial bipartite pure entangled state. Thus, partial trace can be viewed as an entanglement witness for bipartite pure entangled states. The concept of partial trace also renders another explanation to the origin of mixed states, i.e., through the process of considering evolutions in composite systems and then discarding one of the subsystems, we are able to generate mixed states of a subsystem. The notion of mixed

states was earlier considered through the notion of a preparation procedure.

1.2.2 Positive Operator Valued Measure

The notion of Positive Operator Valued Measure (POVM) is the generalisation of the Von Neumann measurement scheme, and is easily understood in the context of measurement in composite systems.

Consider a system A with Hilbert space $\mathcal{H}_A$, to be in the state $\hat{\rho}_A$, and an auxiliary system B with Hilbert space $\mathcal{H}_B$, to be in the state $\hat{\rho}_B$. Then the state of the combined system is

$$\begin{aligned} \hat{\rho}_S &= \hat{\rho}_A \otimes \hat{\rho}_B, \quad \text{where} \\ (\hat{\rho}_A \otimes \hat{\rho}_B)_{m\alpha,n\beta} &= (\hat{\rho}_A)_{mn}(\hat{\rho}_B)_{\alpha\beta}. \end{aligned} \tag{1.18}$$

A Von Neumann measurement on the combined system system is represented by projection operators

$$P_i P_j = \delta_{ij} P_j, \quad \sum_i P_i = \mathbb{1}. \tag{1.19}$$

The probability of the ith outcome of such a test given that the state of the combined system is in $\hat{\rho}_A \otimes \hat{\rho}_B$ is,

$$q_i = \mathrm{Tr}[P_i(\hat{\rho}_A \otimes \hat{\rho}_B)] = \sum_{m\alpha,n\beta} (P_j)_{m\alpha,n\beta}(\hat{\rho}_A)_{mn}(\hat{\rho}_B)_{\alpha\beta}. \tag{1.20}$$

This can be equivalently written as

$$\begin{aligned} q_i &= \mathrm{Tr}_A(M_i \hat{\rho}_A), \quad \text{where} \\ (M_i)_{mn} &= \sum_{\alpha\beta} (P_i)_{m\alpha,n\beta}(\hat{\rho}_B)_{\alpha\beta}, \end{aligned} \tag{1.21}$$

and $\{M_i\}$ are operators on the Hilbert space $\mathcal{H}_A$ of the A subsystem. The hermitian nonnegative operators M_i which need not commute clearly satisfy

$$\sum_i M_i = \mathbb{1}. \tag{1.22}$$

Each member of the set $\{M_i\}$ is called a positive operator valued measure (POVM) [4, 5], since each M_i is a positive operator by construction. The main difference between a Von Neumann type measurement and a POVM is that the M_i are not projection operators

and the number of outcomes is independent of the dimensionality of the Hilbert space $\mathcal{H}_A$. The probability of the ith outcome is now given by

$$q_i = \mathrm{Tr}(M_i \hat{\rho}), \tag{1.23}$$

as compared to Von Neumann $\mathrm{Tr}(P_i \hat{\rho})$. We have removed the subscript A to indicate the comparison at system level. The Von Neumann collapse postulate holds, except that the state of the subsystem after the measurement is the partial trace of the collapsed state of the composite system.

1.3 Quantum Dynamics

Consider a system with Hilbert space $\mathcal{H}$. The set of all density operators $\hat{\rho}$ acting on $\mathcal{H}$ is a subset of the set of all linear operators acting on $\mathcal{H}$. The set of all linear transformations on $\mathcal{H}$ forms a vector space. If $\mathcal{H}$ is n dimensional, then this new vector space is clearly n^2 dimensional. Quantum evolutions are linear transformations on the linear operators acting on $\mathcal{H}$. Linear transformations on this new vector space are called linear maps to distinguish them from linear operators on $\mathcal{H}$. They are sometimes called super-operators.

It would seem that any linear map acting on the density operator $\hat{\rho}$, and preserves the three defining properties of density operators in Eq. (1.1), is a valid quantum evolution. This is not so! Further conditions beyond (1.1) arise from looking at composite systems. Suppose we were dealing with only closed systems, then any map which preserves the defining properties of the density operators would have appeared physical. Since we may be a part of a larger system but observing only our system locally, it becomes imperative that the map under consideration takes valid density operators of the extended system to valid density operators. Thus one is lead to consider what are called completely positive maps.

1.3.1 Completely positive maps

Consider a bipartite system S with Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. A linear map Λ acting on the space of operators acting on $\mathcal{H}_A$ can be extended to act on the space of operators acting on $\mathcal{H}_S$ through the definition of the map $\Lambda \otimes \mathbb{1}_n$, where $\mathbb{1}_n$ is the identity map on subsystem B, i.e., the map Λ acts only on the A subsystem, but leaves the B subsystem as it is. Every possible choice of $\mathcal{H}_B$ gives us a possible extension of the above kind for Λ. A map Λ is said to be completely positive if it is positive for all such possible extensions. By positive we mean that it takes valid density operators acting on $\mathcal{H}_S$ to valid density operators acting on $\mathcal{H}_S$. The subtle part of the definition stems from the fact that one

has to in principle run over all possible extensions specified by the choices of $\mathcal{H}_B$, to conclude if a given positive map Λ is completely positive or not. We have the following important result :

Theorem 1.2 *The action of any trace-preserving completely positive map Λ on a density operator $\hat{\rho}$ can always be written in the following form :*

$$\Lambda(\hat{\rho}) = \sum_{\alpha} W_{\alpha}\hat{\rho}W_{\alpha}^{\dagger}, \quad \sum_{\alpha} W_{\alpha}^{\dagger}W_{\alpha} = \mathbb{1}. \tag{1.24}$$

We don t give the proof here, but a heuristic way to see this is as follows. Let $\{|\psi_j\rangle\}$ and $\{|\phi_\alpha\rangle\}$ be a set of ONB in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Suppose that the density operator of the bipartite system is initially in the state

$$\hat{\rho}_A \otimes |0\rangle_{BB}\langle 0|, \tag{1.25}$$

where $|0\rangle_B$ denotes a pure state in the B subsystem. Evolve the state unitarily in the combined system so that

$$\hat{\rho}_A \otimes |0\rangle_{BB}\langle 0| \rightarrow U(\hat{\rho}_A \otimes |0\rangle_{BB}\langle 0|)U^{\dagger}. \tag{1.26}$$

Now performing a partial trace over the B system Hilbert space yields

$$\begin{aligned} \hat{\rho}'_A &= \mathrm{Tr}_B(U(\hat{\rho}_A \otimes |0\rangle_{BB}\langle 0|)U^{\dagger} \\ &= \sum_{\alpha}\langle\phi_\alpha|U(\hat{\rho}_A \otimes |0\rangle_{BB}\langle 0|)U^{\dagger}|\phi_\alpha\rangle \\ &= \sum_{\alpha}\langle\phi_\alpha|U|0\rangle_B\, \hat{\rho}_A\, {}_B\langle 0|U^{\dagger}|\phi_\alpha\rangle. \end{aligned} \tag{1.27}$$

If we denote

$$W_\alpha = \langle\phi_\alpha|U|0\rangle_B, \tag{1.28}$$

then we can express $\hat{\rho}'_A$ as

$$\hat{\rho}'_A = \sum_{\alpha} W_\alpha \hat{\rho}_A W_{\alpha}^{\dagger}. \tag{1.29}$$

It follows from the unitarity of U that

$$\begin{aligned}\sum_\alpha W_\alpha^\dagger W_\alpha &= \sum_\alpha {}_B\langle 0|U^\dagger|\phi_\alpha\rangle\langle\phi_\alpha|U|0\rangle_B \\ &= {}_B\langle 0|U^\dagger U|0\rangle_B = \mathbb{1}_A. \end{aligned} \tag{1.30}$$

Thus we have two ways of picturing completely positive maps [6–9].

- Every trace preserving completely positive linear map Λ, has an operator sum representation as in Eq. (1.24).
- Every trace preserving completely positive linear map Λ, has an unitary representation as in Eq. (1.27).

What we have just demonstrated is that an unitary representation of a completely positive map can in fact be viewed as an operator sum representation. The nontrivial part of the theorems is that every trace-preserving completely positive map can be obtained in this manner. A simple way to understanding this aspect is by reasoning that any reasonable evolution should be accomplished as a unitary (hamiltonian) evolution on a larger system. It is useful to note that in the unitary realization one begins with a product state of the combined system, and a pure state for the B subsystem proves sufficient. It is clear from the definition and Eq. (1.24) that the set of all trace preserving completely positive maps form a convex set.

1.4 Detecting entanglement

One of the foremost problems of quantum information theory has been the development of tools for detection of entanglement. Since a given density operator $\hat{\rho}_{AB}$ of a bipartite system S, has infinitely many decompositions [1], and since we cannot possibly run through all of them to see if $\hat{\rho}_{AB}$ is separable, it is imperative that we devise efficient methods of detecting entanglement. Bell inequalities provide us with sufficient criteria for entanglement, and entropy based inequalities can also detect entanglement in suitable cases. These are scalar manifestations of entanglement, which has its roots at the density operator level. Since we only measure scalar quantities in the laboratory, the scalar manifestations of entanglement are crucial from an experimental point of view. From a theoretical perspective, the scalar manifestation of entanglement is intimately connected to the theory of positive maps which has a direct bearing on the concept of entanglement. We briefly discuss these ideas below.

1.4.1 Entropic inequalities

Entropic inequalities originate from the observation that there is more information in an entangled state viewed as a whole than viewed as aggregate of information in the subsystems. A simple example such as a maximally entangled state in 2×2 dimensions illustrates this. The state when viewed in 2×2 dimensions as a whole is a pure state, but when we look at the state of either of the subsystems the state is a random mixture proportional to the identity operator. Thus from either of the subsystems we gain no knowledge of the state. Such a qualitative feature can be made quantitative through entropic inequalities such as

$$S(\hat{\rho}_A) \leq S(\hat{\rho}_{AB}), \quad S(\hat{\rho}_B) \leq S(\hat{\rho}_{AB}), \tag{1.31}$$

where $S(\hat{\rho}) = -\mathrm{Tr}(\hat{\rho} \log_2 \hat{\rho})$ is the Von Neumann entropy of a state $\hat{\rho}$. Any separable state obey the inequalities, but entangled states need not [10–13]. The idea is easily generalised to entropic inequalities such as those based on Renyi quantum entropies.

1.4.2 Majorisation

Majorisation is a technique that helps us compare two vector quantities. In the context of classical probability theory, it becomes useful when we compare two discrete probability distributions. Based on majorisation, we may be able to conclude if one probability distribution is more 'spread out' than the other, or in other words if one probability distribution is more 'disordered' than the other. We now state the definition [14, 15].

Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be two vector quantities, arranged in the nondecreasing order, i.e., $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots y_n$. Then we say $x \prec y$ (x is majorised by y) if and only if

$$\begin{aligned} \sum_{j=1}^{k} x_j &\geq \sum_{j=1}^{k} y_j, \ \forall \ k \leq n, \quad \text{and} \\ \sum_{j=1}^{n} x_j &= \sum_{j=1}^{n} y_j. \end{aligned} \tag{1.32}$$

That majorisation captures the disorderliness is seen through its implication on entropy. Let the vectors x and y denote two probability distributions and let $x \prec y$, then $H(\{x_i\}) \geq H(\{y_i\})$, where $H(.)$ is the Shannon entropy [14, 16]. The majorisation relation is more fundamental in capturing disorderliness, in the sense that the entropic inequality can be seen to follow as a consequence of the majorisation relation.

As already mentioned, entangled states have more information when seen as a whole rather than in their parts. This statement is made mathematically precise through the following majorisation relation :

Theorem 1.3 *If a bipartite mixed state $\hat{\rho}_{AB}$ is separable and $\hat{\rho}_A$ and $\hat{\rho}_B$ are the reduced density matrices of the subsystems A and B, then*

$$\lambda(\hat{\rho}_{AB}) \prec \lambda(\hat{\rho}_A), \text{ and } \lambda(\hat{\rho}_{AB}) \prec \lambda(\hat{\rho}_B). \tag{1.33}$$

Here, $\lambda(\hat{\rho}_{AB})$, $\lambda(\hat{\rho}_A)$, and $\lambda(\hat{\rho}_B)$, constitute the eigenvalues of $\hat{\rho}_{AB}$, $\hat{\rho}_A$, and $\hat{\rho}_B$, arranged in the nondecreasing order [17].

The above majorisation relation can be violated if the state $\hat{\rho}_{AB}$ is entangled. Clearly, entropic inequalities such as Eq. (1.31) are implied by the majorisation relation in Eq. (1.33).

Majorisation relations are not only useful in detecting entanglement, but appear in more general scenarios such as when we are dealing with quantum evolutions. The Schur-Horn lemma plays a pivotal role in this [14, 15]. We will discuss more of the majorisation relations in Chapter 5, where we study compatibility relations for Gaussian states [18].

1.4.3 Bell's inequalities

Bell inequalities arose initially from the study of quantum theory from the perspective of classical probability theory. One of the profound implications of quantum theory is that it gives rise to new possibilities in the correlation of distant events that cannot be explained by classical local models. Bell observed that there is an upper bound on the correlation of distant events as explained by a classical local model, and that quantum mechanics could violate it [19]. It was evident that any such violation was easily explained through entanglement. We now have the well established fact that violation of any of the Bell type inequalities is a clear manifestation of entanglement. Here we briefly discuss one such inequality, namely the Clauser-Horne- Shimony-Holt (CHSH) inequality [20].

The CHSH inequality refers to correlation experiments involving two dicotomic observables at two sites. The observed values of each of these observables can be taken to be ± 1. Let the observables in the Alice's side be denoted A_1 and A_2, and those on Bob's side B_1 and B_2. The outcomes of the experiment in each trial is denoted by a_1, a_2 and b_1, b_2 respectively. Define the correlation function between two observables A and B, A on Alice's side and B on Bob's side, as

$$E(A, B) = \langle ab \rangle, \tag{1.34}$$

which is the average value of the correlation over repeated trials. Then the CHSH inequality reads as

$$|\mathcal{B}| = |E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2)| \leq 2. \tag{1.35}$$

This can be easily seen from the fact that for a given trial $\mathcal{B}$ is ± 2, hence the average is always less than or equal to 2.

The quantum mechanical version of the CHSH inequality for 2×2 dimensional systems is easily stated through the definition of the Bell-CHSH observable

$$\hat{\mathcal{B}} = \hat{a}_1 \cdot \sigma \otimes (\hat{b}_1 + \hat{b}_2) \cdot \sigma + \hat{a}_2 \cdot \sigma \otimes (\hat{b}_1 - \hat{b}_2) \cdot \sigma. \tag{1.36}$$

Here $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$ and $\hat{b}_2$ are arbitrary unit vectors in R^3, $\hat{a} \cdot \sigma = \sum_{i=1}^{3} a_i \sigma_i$, and σ_i are the Pauli matrices. Any $\hat{a} \cdot \sigma$ corresponds to a spin observable with eigenvalues ± 1. With this definition the CHSH inequality for a bipartite 2×2 dimensional system reads as

$$\mathrm{Tr}(\hat{\rho}\hat{\mathcal{B}}) \leq 2. \tag{1.37}$$

The constraint imposed by the above equation is not generally obeyed by quantum mechanical systems. For instance, for the choice of the various unit vectors, each separated by angle of 22.5 degrees and with the singlet chosen as the state, $\mathrm{Tr}(\hat{\rho}\hat{\mathcal{B}}) = 2\sqrt{2}$ [5], clearly a violation of the CHSH bound implied by a local classical model. An upper bound on the maximum possible expectation value of the Bell-CHSH observable was obtained by Cirelson to be $2\sqrt{2}$ [21]. The remarkable aspect of the inequality is that experiments confirmed the violation [22], in complete agreement with quantum mechanical predictions, thus demonstrating entanglement. This inequality can be extended to more observables on each side [23] in a bipartite setup, and to more number of parties [24]. Violation in the latter case will indicate presence of multipartite entanglement. The Bell inequalities are only one of the ways to detect entanglement, and they are not very powerful in the sense that there are inseparable states obeying the Bell inequalities [3, 25]. That is, there is entanglement that is not detected through violation of Bell inequalities. See [26] for a review.

1.4.4 Positive maps

The theory of positive maps is an inescapable ingredient in the theory of entanglement. The first use of them was demonstrated by Peres in [27]. He observed that a separable state remains a state (positive) if subjected to partial transposition (PT). A crucial

observation by the Horodecki's that partial transpose is a positive map but not a completely positive one, led to the exploration of the intimate connection between the theory of entanglement and positive maps [28]. Earlier in Eq. (1.24), we had introduced the notion of completely positive map. We noted that a linear map Λ is completely positive if and only if the extended map $\Lambda \otimes \mathbb{1}_n$ is positive for all n. We now introduce the notion of positive but non completely positive maps. We say that a linear map Λ is positive but not completely positive if Λ takes density operators to density operators, but it has an extension which fails to do so.

Let the map Λ act on operators on Hilbert space $\mathcal{H}_A$, and let its extension act on operators on the Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. Suppose the extended map given by $\Lambda \otimes \mathbb{1}_n$, does not take positive operators on $\mathcal{H}_S$ to positive operators on $\mathcal{H}_S$, but the map Λ takes positive operators on $\mathcal{H}_A$ to positive operators on $\mathcal{H}_A$. Then the map Λ is said to be positive but not completely positive. Such a map could be tool a in detecting entanglement in bipartite systems specified by the Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. If such a map acts on a separable state $\hat{\rho}_{AB}$ on $\mathcal{H}_S$, we have the following to be true :

$$\begin{aligned}(\Lambda \otimes \mathbb{1}_n)(\hat{\rho}_{AB}) &= (\Lambda \otimes \mathbb{1}_n) \sum_k p_k \hat{\rho}_{Ak} \otimes \hat{\rho}_{Bk} \\ &= \sum_k p_k \Lambda(\hat{\rho}_{Ak}) \otimes \hat{\rho}_{Bk} \geq 0. \end{aligned} \tag{1.38}$$

This follows from the linearity and positivity of the map Λ. On the other hand, if the extended map acted on an entangled state, it could lead to the following possibility :

$$\begin{aligned}(\Lambda \otimes \mathbb{1}_n)(\hat{\rho}_{AB}) &= (\Lambda \otimes \mathbb{1}_n)(\sum_k p_k \hat{\rho}_{ABk}) \\ &= \sum_k p_k (\Lambda \otimes \mathbb{1}_n)(\hat{\rho}_{ABk}) \ngeq 0. \end{aligned} \tag{1.39}$$

This possibility arises because the map Λ is not completely positive. Thus a positive but not completely positive map helps us detect entanglement! We have the following important theorem [28] :

Theorem 1.4 *Let $\hat{\rho}_{AB}$ act on the Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. Then $\hat{\rho}_{AB}$ is separable if and only if for every positive map Λ on $\mathcal{H}_A$, the operator*

$$(\Lambda \otimes \mathbb{1}_n)(\hat{\rho}_{AB}) \tag{1.40}$$

is positive.

We now give a brief discussion on positive maps. It is convenient to go to an indexed

notation to describe them. The three defining properties of a density operator in this notation are

$$\rho_{rs} = \rho^*_{sr}\,, \quad \rho_{rs}x^*_r x_s \geq 0\,, \quad \rho_{rr} = 1. \tag{1.41}$$

Summation over repeated indices is implied as usual. Any linear positive map Λ takes valid density operators to valid density operators. Such a map can be expressed as

$$\begin{aligned} &\Lambda:\ \hat{\rho} \to \hat{\rho}' \\ &\rho'_{r's'} = \Lambda_{r's',rs}\,\rho_{rs}. \end{aligned} \tag{1.42}$$

The hermiticity requirement of $\hat{\rho}'$ demands that

$$\Lambda_{s'r',sr} = \Lambda^*_{r's',rs}, \tag{1.43}$$

trace preservation of $\hat{\rho}'$ demands that

$$\Lambda_{r'r',rs} = \delta_{rs}, \tag{1.44}$$

and preservation of positivity implies that

$$\rho'_{r's'}x^*_{r'}x_{s'} \geq 0 \quad \Rightarrow \quad \Lambda_{r's',rs}\rho_{rs}x^*_{r'}x_{s'} \geq 0. \tag{1.45}$$

From spectral resolution of $\hat{\rho}$, it is sufficient to consider positivity requirement on any of its projectors, hence we only require

$$\Lambda_{r's',rs}x^*_{r'}x_{s'}y^*_r y_s \geq 0. \tag{1.46}$$

Let us define a new matrix M by permuting the indices of Λ:

$$\Lambda_{r's',rs} = M_{r'r,s's}. \tag{1.47}$$

The hermiticity condition in Eq. (1.43) now reads

$$M_{r'r,s's} = M^*_{s's,r'r}, \tag{1.48}$$

i.e., the matrix M is hermitian. Hence we have the spectral resolution

$$M_{r'r,s's} = \sum_\alpha \lambda_\alpha \xi_{r'r}(\alpha)\xi^*_{s's}(\alpha). \tag{1.49}$$

Rewriting the positivity requirement of $\hat{\rho}'$ in Eq. (1.45) in terms of the matrix M, we have

$$x^*_{r'} y^*_r M_{r'r,s's} x_{s'} y_s, \tag{1.50}$$

which means that the matrix M is positive on product vectors. With this, any positive linear map reads as

$$\begin{aligned} \rho^{'}_{r's'} &= M_{r'r,s's}\rho_{rs} \\ &= \sum_\alpha \lambda_\alpha \xi_{r'r}(\alpha)\rho_{rs}\xi^*_{s's},(\alpha) \Rightarrow \\ \hat{\rho}' &= \sum_\alpha \lambda_\alpha \hat{\xi}(\alpha)\hat{\rho}\hat{\xi}^\dagger(\alpha). \end{aligned} \tag{1.51}$$

The eigenvalues $\{\lambda_\alpha\}$ need not be positive, since our only requirement was positivity M on product vectors. A completely positive map is a positive map as in Eq. (1.51) where all its eigenvalues, i.e., its $\{\lambda_\alpha\}$ are positive [29–31].

1.4.5 Entanglement Witness

The convex structure of the state space and the convex structure of the set of all separable states for a given bipartite system enable us employ the ideas known in the context of convex sets to distinguish entangled states from separable ones. The simple idea that a convex set and a point lying outside it can always be separated by a hyper plane is manifested in the context of state space as an entanglement witness. The following theorem due to Horodecki's captures this idea [28].

Theorem 1.5 *A state $\hat{\rho}_{AB}$ acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if*

$$\mathrm{Tr}(\hat{A}\hat{\rho}_{AB}) \geq 0 \tag{1.52}$$

for every hermitian operator $\hat{A}$ satisfying $\mathrm{Tr}(\hat{A}\hat{P}\otimes\hat{Q}) \geq 0$, where $\hat{P}$ and $\hat{Q}$ are projections operating on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

The implication of the theorem is that if a state $\hat{\rho}_{AB}$ satisfies the inequality

$$\mathrm{Tr}(\hat{A}\hat{\rho}_{AB}) < 0 \tag{1.53}$$

for such hermitian $\hat{A}$ which is positive on product vectors, we definitely know that the state $\hat{\rho}_{AB}$ was entangled. Any $\hat{A}$ which is positive semidefinite will not serve our purpose

in detecting entanglement, as such an operator would be positive on entangled states too. Thus the operator $\hat{A}$ though positive on product vectors, has to be necessarily indefinite to detect entanglement of some states. Thus, if such a witness $\hat{A}$ were to detect entanglement of a state $\hat{\rho}_{AB}$, the state $\hat{\rho}_{AB}$ is a point lying outside the convex set of separable states, and the numerical matrix determined by $\hat{A}$ is a hyper plane separating this point from the convex subset of separable states. The observation that any entanglement witness is a hermitian operator which is positive on product vectors but not positive definite, reminds us of the defining property of a positive map in Eq. (1.51). In other words, we have an isomorphism between positive but not completely positive maps and entanglement witnesses [32]. The first use of an entanglement witness was in [3], where the flip operator $\hat{V}$ served as an entanglement witness. It is easy to check for systems of 2×2 dimensions that the flip operator defined in the standard basis is the M matrix corresponding to partial transpose. It is useful to note that any Bell observable $\hat{\mathcal{B}}$ as in Eq. (1.36) can also be viewed as an entanglement witness.

1.4.6 Partial Transpose

The partial transpose test is a classic example of the application of the theory of positive maps in detecting entanglement. Partial transpose was initially introduced by Peres in [27] to detect entanglement. It is defined as follows. Consider the matrix elements of a state $\hat{\rho}_{AB}$ of a bipartite system specified by the Hilbert space $\mathcal{H}_S$ of $m \times n$ dimensions, i.e.,

$$\hat{\rho}_{i\alpha,j\beta} = \langle\psi_i| \otimes \langle\phi_\alpha|\hat{\rho}_{AB}|\psi_j\rangle \otimes |\phi_\beta\rangle. \tag{1.54}$$

The partial transposed matrix $\hat{\rho}_{AB}^{PT}$ is defined through its matrix elements thus :

$$\hat{\rho}^{\mathrm{PT}}_{i\alpha,j\beta} = \hat{\rho}_{i\beta,j,\alpha}. \tag{1.55}$$

Such an operation is easily seen in its matrix form. The state $\hat{\rho}_{AB}$ of the $m \times n$ system can be written as

$$\hat{\rho}_{AB} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ . & \cdots & . \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}, \tag{1.56}$$

i.e., as an $m \times m$ array of $n \times n$ matrices A_{ij} acting on the second Hilbert space $\mathcal{H}_B$. The whole matrix is defined by the matrix elements $\{A_{ij}\}_{\alpha\beta} = \rho_{i\alpha,j\beta}$. Then partial transpose

is simply the transposition of the $\{A_{ij}\}$ matrices.

$$\hat{\rho}_{AB}^{\mathrm{PT}} = \begin{pmatrix} A_{11}^{\mathrm{T}} & \cdots & A_{1m}^{\mathrm{T}} \\ . & \cdots & . \\ A_{m1}^{\mathrm{T}} & \cdots & A_{mm}^{\mathrm{T}} \end{pmatrix}. \tag{1.57}$$

Under partial transpose a separable state goes to a valid state, i.e.,

$$\hat{\rho}_{AB}^{\mathrm{PT}} = \sum_k p_k \hat{\rho}_{Ak} \otimes \hat{\rho}_{Bk}^{\mathrm{T}} = \sum_k p_k \hat{\rho}_{Ak} \otimes \hat{\rho}_{Bk}^{*} \geq 0. \tag{1.58}$$

Since every $\hat{\rho}_{Bk}^{*}$ is a valid state, $\hat{\rho}_{AB}^{\mathrm{PT}}$ is a valid state. An entangled state under partial transposition can result in a non positive operator. Thus partial transpose detects entanglement. Even though the operation is basis dependent, the conclusions are not. We have the following theorem for lower dimensional composite systems [28] :

Theorem 1.6 *A state $\hat{\rho}_{AB}$ of a $2\otimes 2$ or $2\otimes 3$ system is separable if and only if its partial transposition is a positive operator.*

1.4.7 Reduction Criterion

Yet another example of a positive map that is not completely positive is the reduction map. Consider the map

$$\Lambda(\hat{\rho}) = \mathbb{1} - \hat{\rho}. \tag{1.59}$$

This maps is clearly positive, since the eigenvalues of a density operator are positive and less than one. Now consider the extension of this map to composite systems. A separable state satisfies the following inequalities :

$$\mathbb{1} \otimes \hat{\rho}_B - \hat{\rho}_{AB} \geq 0, \quad \hat{\rho}_A \otimes \mathbb{1} - \hat{\rho}_{AB} \geq 0, \tag{1.60}$$

but entangled states need not. Here $\hat{\rho}_A$ and $\hat{\rho}_B$ are the partially traced versions of $\hat{\rho}_{AB}$ for Bob's and Alice's subsystems respectively. The above two conditions are jointly called the reduction criterion [33]. The reduction inequalities in Eq. (1.60) imply the majorisation inequalities in Eq. (1.33), and consequently imply the entropic inequalities in Eq. (1.31) [34]. It is known that the reduction criterion is weaker than partial transpose test in detecting entanglement, i.e., there are entangled states that partial transpose can detect, but reduction criterion cannot [33]. Thus, entanglement criteria defined through the reduction map, majorisation, and entropic inequalities, are all weaker than

partial transpose in detecting entanglement. It is also known that more general entropic inequalities other than that stated in Eq. (1.31) are implied by the reduction criteria [35].

We now digress to introduce the important notion of distillability before continuing the discussion on positive maps. This digression is useful, as both the partial transpose map and the reduction criterion are intimately connected to the concept of distillation.

1.5 Distillation

A central aspect of quantum information theory is transmission of quantum information, i.e., transmission of quantum states through quantum channels. The whole idea rests largely on the teleportation protocol [36] which consumes a maximally entangled state in the process of transmitting a qubit from Alice's side to Bob's side. Anything less than a maximally entangled state results in the loss of fidelity of the transmission. Thus a primary resource for achieving a perfect teleportation is a shared maximally entangled state. Thus arises the need for creating maximally entangled states starting with lesser entangled states. The aim of a distillation or purification procedure is to extract from a large ensemble of low fidelity EPR pairs a smaller sub-ensemble of high fidelity EPR pairs, using only local operations and classical communication (LOCC); these may then be used for faithful teleportation [37–39].

We begin by illustrating a purification procedure in the context of pure states, and then extend it to mixed states, before considering formal definitions. We stick to 2×2 dimensional systems for the sake of simplicity.

Suppose Alice and Bob share two copies of an entangled state $|\psi\rangle$ which is not maximally entangled, i.e.,

$$\begin{aligned} |\psi\rangle \otimes |\psi\rangle &= (\alpha|0\rangle_A|0\rangle_B + \sqrt{1-\alpha^2}|1\rangle_A|1\rangle_B) \otimes \\ &\quad (\alpha|0\rangle_{A'}|0\rangle_{B'} + \sqrt{1-\alpha^2}|1\rangle_{A'}|1\rangle_{B'}), \end{aligned} \tag{1.61}$$

where $|0\rangle$ and $1\rangle$ are the eigenstates of the σ_z operator, and A and A' refer to the Alice's side particles, and B and B' to Bob's. Expanding the above, we have

$$\begin{aligned} |\psi\rangle \otimes |\psi\rangle &= \alpha^2|0\rangle_A|0\rangle_B|0\rangle_{A'}|0\rangle_{B'} + (1-\alpha^2)|1\rangle_A|1\rangle_B|1\rangle_{A'}|1\rangle_{B'} + \\ &\quad \sqrt{2}\alpha\sqrt{1-\alpha^2}\left[\frac{|0\rangle_A|0\rangle_B|1\rangle_{A'}|1\rangle_{B'} + |1\rangle_A|1\rangle_B|0\rangle_{A'}|0\rangle_{B'}}{\sqrt{2}}\right] \end{aligned} \tag{1.62}$$

for the state of the two pairs. Now let Bob make a local measurement for the z-component of the spin. He can get either 2, –2, or zero. Suppose the result is zero, then Bob informs

Alice over phone and they decide by mutual consent to retain the state, else they decide to discard the state. The probability of such an outcome is $2\alpha^2(1-\alpha^2)$. The resultant state after the experiment is the term indicated in the square bracket :

$$|\psi_r\rangle = \frac{|0\rangle_A|0\rangle_B|1\rangle_{A'}|1\rangle_{B'} + |1\rangle_A|1\rangle_B|0\rangle_{A'}|0\rangle_{B'}}{\sqrt{2}}. \tag{1.63}$$

The subscript r denotes that the state is a result of local measurement. Suppose we do the following renaming, i.e., $|0'\rangle_A = |0\rangle_A|1\rangle_{A'}$, $|1'\rangle_A = |1\rangle_A|0\rangle_{A'}$, $|0'\rangle_B = |0\rangle_B|1\rangle'_B$, and $|1'\rangle_B = |1\rangle_B|0\rangle_{B'}$, the state that Alice and Bob share is then a singlet. Such procedure of generating the maximally entangled state is inefficient in the sense that we may end up losing part of the entanglement the parties initially shared through our very act of discarding some of the outcomes. But the method gets more and more efficient as Alice and Bob apply such a protocol collectively to more and more pairs. It was shown in [38] that Alice and Bob can obtain n singlets from k $|\psi\rangle$'s with the ratio approaching

$$\lim_{n,k\to\infty} \frac{n}{k} = E(|\psi\rangle) = -\alpha^2 \log_2 \alpha^2 - (1-\alpha^2)\log_2(1-\alpha^2). \tag{1.64}$$

$E(|\psi\rangle)$ is the entropy of entanglement and equals the Shannon entropy of the squares of the Schmidt coefficients of $|\psi\rangle$

Distillation procedures in the context of mixed states are more sophisticated. For the sake of simplicity we outline a procedure illustrated in [40]; it captures the essence of distillation in the context of mixed states. Suppose Alice and Bob share two copies of the mixed state

$$\hat{\rho}_{AB} = f|\phi^+\rangle_{AB}\langle\phi^+| + (1-f)|\psi^+\rangle_{AB}\langle\psi^+|, \tag{1.65}$$

where $|\phi^+\rangle_{AB}$ and $|\psi^+\rangle_{AB}$ are Bell states defined as

$$\begin{aligned} |\phi^+\rangle_{AB} &= \frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}, \\ |\psi^+\rangle_{AB} &= \frac{|01\rangle_{AB} + |10\rangle_{AB}}{\sqrt{2}}. \end{aligned} \tag{1.66}$$

Unless $f = \frac{1}{2}$, the state is inseparable. This can be seen for example through the partial transpose test. The aim here is to increase the fraction f of $|\phi^+\rangle_{AB}\langle\phi^+|$ through some local operations and decisions taken through mutual consent arrived at through classical communication. To this end, they perform a bilateral local CNOT operation, i.e., CNOT operations performed on pairs AA' and BB'. Such an operation is clearly local across

the Alice and Bob divide. The action of such an operation is summarised below.

$$\begin{aligned}
&|\phi^+\rangle_{AB}|\phi^+\rangle_{A'B'} \rightarrow |\phi^+\rangle_{AB}|\phi^+\rangle_{A'B'},\\
&|\psi^+\rangle_{AB}|\phi^+\rangle_{A'B'} \rightarrow |\psi^+\rangle_{AB}|\psi^+\rangle_{A'B'},\\
&|\phi^+\rangle_{AB}|\psi^+\rangle_{A'B'} \rightarrow |\phi^+\rangle_{AB}|\psi^+\rangle_{A'B'},\\
&|\psi^+\rangle_{AB}|\psi^+\rangle_{A'B'} \rightarrow |\psi^+\rangle_{AB}|\phi^+\rangle_{A'B'}.
\end{aligned} \tag{1.67}$$

After this local operation the state Alice and Bob share is

$$\begin{aligned}
\hat{\rho}_{ABA'B'} &= (f^2|\phi^+\rangle_{AB}\langle\phi^+| + (1-f)^2|\psi^+\rangle_{AB}\langle\psi^+|) \otimes |\phi^+\rangle_{A'B'}\langle\phi^+| + \\
&\quad f(1-f)(|\phi^+\rangle_{AB}\langle\phi^+| + |\psi^+\rangle_{AB}\langle\psi^+|) \otimes |\psi^+\rangle_{A'B'}\langle\psi^+|.
\end{aligned} \tag{1.68}$$

Now they perform local z-component measurements on the A' and B' particles. If they get the results to be correlated, they decide to retain the remaining pair, else they discard the remaining pair. The success probability is given by $f^2 + (1-f)^2$, and they share the state

$$\hat{\rho}'_{AB} = \frac{1}{f^2 + (1-f)^2}(f^2|\phi^+\rangle_{AB}\langle\phi^+| + (1-f)^2|\psi^+\rangle_{AB}\langle\psi^+|). \tag{1.69}$$

The fraction $f' = f^2/(f^2 + (1-f)^2) > f$ for $f > 1/2$. The procedure is clearly lossy as in the case of pure states in that we may lose entanglement in some of the outcomes. But given a sufficiently large initial ensemble we can, with a finite probability at every step, tend towards a higher fidelity of the desired pure maximally entangled state by iterating this procedure. Alice and Bob can, at the end of the procedure, *distill* a smaller ensemble of pairs with entanglement fidelity f arbitrarily close to unity [40]. These pairs can then be used for faithful teleportation. Distillation procedures in the context of mixed states were initially described in [37].

A careful analysis of a general distillation procedure tells us that there are three aspects to it, namely, local general measurements, classical communication, and post-selection.

Local general measurements : These are the most general possible measurements performed on the Alice's and Bob's side. They are described by two sets of operators A_i and B_j which satisfy the completeness relations

$$\sum_i A_i^\dagger A_i = \mathbb{1}, \quad \sum_i B_j^\dagger B_j = \mathbb{1}. \tag{1.70}$$

They can be realised by appending additional systems locally, evolving them together,

and then performing joint measurements locally. Under these actions, a shared state evolves as

$$\hat{\rho}_{AB} \rightarrow \sum_{ij} A_i \otimes B_j \hat{\rho}_{AB} A_i^\dagger \otimes B_j^\dagger. \tag{1.71}$$

Such a procedure is clearly local.

Classical communication : This simply means that while performing the local general measurements, Alice's and Bob's actions can be correlated through mutual consent arrived at through communication over the phone, in which case, the evolution is described by

$$\hat{\rho}_{AB} \rightarrow \sum_{i} A_i \otimes B_i \hat{\rho}_{AB} A_i^\dagger \otimes B_i^\dagger. \tag{1.72}$$

This describes a combination of both local general measurement and classical communication.

Post-selection : This is the procedure by which Alice and Bob choose to retain certain outcomes of the local general measurement subject to classical communication. Suppose they retained the state corresponding to the i^{th} and j^{th} local outcomes, the resultant state is

$$\frac{A_i \otimes B_j \hat{\rho}_{AB} A_i^\dagger \otimes B_j^\dagger}{\text{Tr}(A_i \otimes B_j \hat{\rho}_{AB} A_i^\dagger \otimes B_j^\dagger)}. \tag{1.73}$$

The denominator in Eq. (1.73) ensures the normalisation [41–44].

Any manipulation involving the above three procedures is succinctly called as *local operation and classical communication* (LOCC). Clearly any LOCC operation described by Eq. (1.71), is a completely positive map as in Eq. (1.49), but each of the W's are in tensor product form $A_i \otimes B_j$. Thus, any LOCC can be thought of as a separable super operator acting on $\hat{\rho}_{AB}$. It is known that every LOCC can be represented by a separable super operator, but not every separable super operator is an LOCC [44].

To summarise, a purification or distillation procedure is essentially one of extracting singlets from multiple copies of shared entangled states through LOCC. A state is said to be distillable if one can actually distill, using a pre-agreed protocol, pure singlets from multiple copies of the state. However such a definition may appear imprecise from the implementation point of view, since the set the of all LOCC available to the two parties is truely enormous.

It was shown in [45] that every entangled state of a 2×2 dimensional system is distil-

lable. The idea is that even if an entangled state in 2×2 dimensions had singlet fraction less than $\frac{1}{2}$, it can always be converted by LOCC, .i.e., a local filtering operation, to a state with singlet fraction greater than $\frac{1}{2}$ and, consequently, the recursion protocol described in [39], could be used for further distillation. A tangible criterion for distillability follows from this fact [45].

Theorem 1.7 *A state $\hat{\rho}_{AB}$ is distillable if and only if, for some two dimensional projectors P and Q, and for some n, the state $P \otimes Q \hat{\rho}_{AB}^{\otimes n} P \otimes Q$ is entangled.*

Thus, projecting out the given n copies of the state $\hat{\rho}_{AB}$ into a 2×2 dimensional subspace, and demonstrating entanglement in that subspace, amounts to showing the distillability of the state in contention.

An implication of this definition is that states which are positive under partial transpose (PPT) cannot be distilled by LOCC. This can for example be seen from the fact if a state is PPT, then n copies of the state is also PPT, and LOCC doesn't take PPT states outside the set of PPT states, i.e., LOCC is a PPT-preserving operation. Thus if

$$\begin{aligned} \hat{\rho}'_{AB} &= \sum_i A_i \otimes B_i \hat{\rho}_{AB} A_i^\dagger \otimes B_i^\dagger, \quad \text{then} \\ \hat{\rho}'^{PT}_{AB} &= \sum_i A_i \otimes B_i^T \hat{\rho}^{PT}_{AB} A_i^\dagger \otimes B_i^{\dagger T}. \end{aligned} \tag{1.74}$$

If $\hat{\rho}^{PT}_{AB}$ is positive, $\hat{\rho}'^{PT}_{AB}$ is also positive, since it is derived by the action of a completely positive map on $\hat{\rho}^{PT}_{AB}$. Thus, negativity under partial transpose (NPT) is a necessary condition for distillability. It is known that any state that violates the reduction criteria in Eq. (1.60) is distillable [33].

Having introduced the useful concept of distillation or purification, and having shown that negativity under partial transposition is a necessary condition for distillability, an immediate question that arises is the following. Are there entangled states that are nondistillable? A immediate way of answering this question is by answering the following simpler question. Are there entangled states that are PPT? The answer to this was given by the Horodecki's in [45, 46]. There they constructed states that are entangled but PPT. Such states were called *bound entangled*, meaning one cannot distill any entanglement from them. To detect entanglement in states that are PPT is a nontrivial task, since we need to devise methods that are 'stronger' that partial transpose. This has led the search for stronger criteria in detecting entanglement. We discuss some of them below.

1.5.1 Range Criterion

The range criterion is a useful tool to detect entanglement, particularly in cases where partial transpose fails. The statement of the criterion is summarised in the following theorem.

Theorem 1.8 *If a state $\hat{\rho}_{AB}$ acting on the space $\mathcal{H}_{AB}$ is separable, then there exists a family of product vectors $|\psi_j\rangle \otimes |\phi_\alpha\rangle$ such that,*

(i) they span the range of $\hat{\rho}_{AB}$,

(ii) the vectors $\{|\psi_j\rangle \otimes |\phi_\alpha^*\rangle\}_{i=1}^k$ span the range of $\hat{\rho}_{AB}^{PT}$ (where $*$ denotes complex conjugation in the basis in which partial transposition was performed).

In [46], the range criterion was applied to detect entanglement in a PPT state. Consequently, it led to the elegant construction of UPB's (Unextendible Product Basis) in [47]. These are a set of product orthogonal vectors in $\mathcal{H}_{AB}$ that has fewer elements than the dimension of the space, but there does not exist any product vector orthogonal to all of them. Thus, a UPB is a partial basis that cannot be completed into a product basis. A simple example of such a UPB for a $3 \otimes 3$ dimensional system is

$$\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle - |1\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle \otimes (|1\rangle - |2\rangle), \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |2\rangle, \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \otimes |0\rangle, \\
|\psi_4\rangle &= \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle) \otimes (|0\rangle + |1\rangle + |2\rangle).
\end{aligned} \tag{1.75}$$

Given a UPB, it is easy to construct bound entangled states. Consider the projector

$$P_{UPB} = \sum_{i=0}^{4} |\psi_i\rangle\langle\psi_i|, \tag{1.76}$$

now construct the state

$$\hat{\rho}_{AB} = \frac{1}{4}(\mathbb{1} - P_{UPB}). \tag{1.77}$$

The range of $\hat{\rho}_{AB}$ contains no product vectors, otherwise one would have been able to extend the product basis. Clearly $\hat{\rho}_{AB}^{PT}$ is positive, i.e., $\hat{\rho}_{AB}$ is PPT. The state $\hat{\rho}_{AB}$ is thus PPT entangled, and hence bound entangled. The range criteria as a tool was successfully extended to the case of continuous variables in [48], to detect bound entanglement.

1.5.2 Non-decomposable maps

The presence of PPT bound entanglement necessitates the definition of decomposable and non-decomposable positive maps. A positive map is said to be decomposable, if it can be represented in the form

$$\Lambda = \Lambda^1_{CP} + \Lambda^2_{CP} \circ T, \tag{1.78}$$

where Λ^1_{CP} and Λ^2_{CP} are completely positive maps, $\circ$ denotes composition, and T denotes the transposition map. Clearly, a decomposable map acting on a composite system is no stronger than partial transpose in detecting entanglement. It turns out that in $2 \otimes 2$ and 2×3 dimensional systems, every positive map is decomposable [49, 50]. Thus positivity under partial transpose turns out to be a necessary and sufficient condition for entanglement in these systems, as indicated earlier. However, in higher dimensional systems there are positive maps that cannot be decomposed as above. Any map that cannot be written as in Eq. (1.78), is said to be a non-decomposable map. Clearly, non-decomposable maps can potentially detect entanglement that partial transpose cannot. Thus the study of non-decomposable maps has turned out to be an integral part of entanglement theory. Examples of such maps can be found in [6, 51–54]. In [51], it was demonstrated that given a UPB one could construct a non-decomposable map.

There seems to be no simple universal way of showing a positive map to be non-decomposable. One possible way is to construct a PPT state, and then show that the map detects entanglement in the PPT state, thus establishing that the map is indecomposable. This particular route to demonstrating non-decomposability has led to search for systematically characterising PPT states [55, 56].

1.5.3 Uncertainty relations

The technique of detecting entanglement through uncertainty relations is based on the fact that separable states, in addition to obeying the general uncertainty principle which arise from non-commutativity of operators, have to obey additional constraints simply arising from the fact that they are separable. Such a technique, though initially introduced in the context of continuous variable systems [57, 58], has been successfully used to detect entanglement in finite dimensional systems [59–61]. Such a technique is powerful as it detects even bound entanglement [60, 61]. We will discuss these ideas later when we deal with continuous variable entanglement.

1.6 Measures of Entanglement

Since entanglement is a fundamental resource in quantum information science, it is important that one is able to quantify it. Any measure of entanglement has to meet some desirable physical requirements. Some reasonable requirements are enumerated below [37, 41].

(i) For any separable state as in Eq. (1.16), which does not have any entanglement, the measure of entanglement E should be zero :

$$E(\hat{\rho}_{AB}^{\text{sep}}) = 0. \tag{1.79}$$

(ii) The second requirement concerns the invariance of the entanglement measure under simple local unitary transformations, since such a transformation amounts to a change of basis locally and hence can be undone locally in a deterministic manner. Any local unitary operation should not be able to change the quantity of entanglement shared :

$$E(\hat{\rho}_{AB}) = E(U_A \otimes U_B \hat{\rho}_{AB} U_A^\dagger \otimes U_B^\dagger). \tag{1.80}$$

(iii) The third requirement states that the average entanglement should not increase under local operations, classical communications, and post-selection (Eq. (1.72)), since if the converse was possible, it would mount to creating entanglement from less entangled states through local operations alone. It is true that through appropriate post-selection, we are able to extract pure state singlets by LOCC as shown in Eq. (1.73), but we are able to do so only at the cost of discarding other outcomes. Thus, on the average entanglement does not increase. Suppose given the state $\hat{\rho}_{AB}$ we get the post-selected states $\hat{\rho}_{AB}^i$ with probability p_i, we then require

$$E(\hat{\rho}_{AB}) \geq \sum_i p_i E(\hat{\rho}_{AB}^i). \tag{1.81}$$

(iv) Finally one would require that the measure of entanglement is additive. Given two entangled bipartite states $\hat{\rho}_1$ and $\hat{\rho}_2$ with the combined system in $\hat{\rho}_1 \otimes \hat{\rho}_2$ we would like to have

$$E(\hat{\rho}_1 \otimes \hat{\rho}_2) = E(\hat{\rho}_1) + E(\hat{\rho}_2). \tag{1.82}$$

For bipartite pure states there is a unique measure of entanglement. Given a bipartite

pure state in the Schmidt form

$$|\Psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}|\psi_i\rangle_A \otimes |\phi_i\rangle_B, \tag{1.83}$$

the unique measure of its entanglement is

$$E(|\Psi\rangle_{AB}) = -\sum_i \lambda_i \log_2 \lambda_i, \tag{1.84}$$

the Shannon entropy of the squares of the Schmidt coefficients. It is the Von Neumann entropy of the reduced density matrix of either subsystem. The uniqueness of the measure follows from the fact that given k copies of the state $|\Psi\rangle$ one can distill n copies of singlets and, conversely, given n copies of singlets one can produce k copies of the given state. However, such a reversible transformation is to be understood in the sense of the limit $k, n \to \infty$ [38]. With this interconvertibility available, we have

$$nE(|\Psi\rangle_{AB}) = kE(|S\rangle_{AB}), \tag{1.85}$$

where $|S\rangle_{AB}$ is the singlet state. The interconvertibility implies that

$$E(|\Psi\rangle_{AB}) = \lim_{n,k\to\infty} \frac{k}{n} E(|S\rangle_{AB}) = -\sum_i \lambda_i \log_2 \lambda_i. \tag{1.86}$$

By convention, one takes $E(|S\rangle_{AB})$ to be one [37, 62]. This measure which is called the entropy of entanglement or simply entanglement, ranges from zero for a product state to $\log_2 d$ for maximally entangled states in $d \otimes d$ dimensions.

1.6.1 Entanglement of Formation

Having defined a unique measure of entanglement for bipartite pure states, it is now possible to extend this measure to define a good measure for the case of bipartite mixed states. As indicated earlier, the set of ensembles $\varepsilon = \{p_i, \hat{\rho}^i_{AB}\}$ realising a given mixed state $\hat{\rho}_{AB}$ is an infinitely huge family [1]. Keeping this fact in mind, the definition for the entanglement measure for the case of bipartite mixed states is defined as

$$E(\hat{\rho}_{AB}) = \min_{\varepsilon} \sum_i p_i E(\hat{\rho}^i_{AB}). \tag{1.87}$$

The minimisation has to be carried out with respect to the possible ensemble decompositions. Clearly, any separable state has an entanglement measure zero by definition. The above measure is called the *Entanglement of Formation* (EOF). The EOF satisfies the

first three requirements required of a good measure. The regularised version of the EOF is defined as [63, 64]

$$E_c(\hat{\rho}_{AB}) = \lim_{n\to\infty} \frac{E(\hat{\rho}_{AB}^{\otimes n})}{n}, \tag{1.88}$$

the entanglement cost of $\hat{\rho}_{AB}$. By definition the entanglement cost of a state is expected to be lesser than its EOF, since a minimisation over a much larger ensemble set is to be carried out. In case the EOF of a state is additive, then its entanglement cost E_c is equal to its EOF. It is now known that EOF is in general not additive [65, 66].

The EOF of a state is in general very hard to evaluate. It is thus remarkable that Wootters was able to provide a closed form expression for the EOF of an arbitrary two qubit state [67, 68]. The other cases where closed form expressions have been found are those states with a high degree of symmetry [69–72]. We will come back to this measure when we discuss the EOF for two-mode Gaussian states.

1.6.2 Distillable entanglement

This is the amount of pure state entanglement that can be distilled from multiple copies of a given entangled mixed state through LOCC. There are no hard and fast rules as to how one goes about it. Distillable entanglement, as a measure of entanglement, is protocol dependent. Given n copies of the shared entangled mixed state $\hat{\rho}_{AB}$, where n is very large, if one is able to extract m copies of pure state singlets by LOCC, the distillable entanglement is defined as

$$D(\hat{\rho}_{AB}) = \lim_{n\to\infty} \frac{m}{n}. \tag{1.89}$$

Fundamental prototypes of various protocols were initially introduced in [37]. As indicated earlier, not all entangled states are distillable, thus for bound entangled states distillable entanglement is zero. Distillable entanglement as a measure of entanglement is however useful in the sense that it actually gives an estimate of the useful entanglement one can harvest from multiple copies of a given state through LOCC. This measure being protocol dependant, is not unique. As an example in the case of two qubits, the one way hashing protocol gives a finite yield with

$$\frac{m}{n} \approx 1 - S(\hat{\rho}_{AB}), \tag{1.90}$$

where $S(\hat{\rho}_{AB})$ is the Von Neumann entropy of the state $\hat{\rho}_{AB}$. In comparison, the recursion protocol similar to the one discussed in Section 1.5, gives almost a zero yield [37]. It

is known that any distillable entanglement is a lower bound on the entanglement of formation [37].

1.6.3 Distance based measures

This category of measures arise from the convex structure of state space. We know that the set all states in a given bipartite system form a convex set, so does the set of all separable states of that system. And clearly, the set of all separable states form a subset in the state space. Thus one can talk of *distance* between the given entangled state and a separable state. Let $\mathcal{S}$ be the set of all separable states. The measure of entanglement for a given state $\hat{\sigma}$ is defined as

$$E(\hat{\sigma}) = \min_{\hat{\rho}\in\mathcal{S}} D(\hat{\sigma}||\hat{\rho}), \tag{1.91}$$

where D is any measure of *distance* (not necessarily a metric) between the two density operators $\hat{\sigma}$ and $\hat{\rho}$, such that it satisfies the requirements put forth above. One measure which satisfies the first three requirements is based on the quantum relative entropy defined as

$$D(\hat{\sigma}||\hat{\rho}) = S(\hat{\sigma}||\hat{\rho}) = \mathrm{Tr}[\hat{\hat{\sigma}}(\ln\hat{\sigma} - \ln\hat{\rho})]. \tag{1.92}$$

The quantum relative entropy is not symmetric and does not satisfy the triangle inequality required of a metric. One could have defined the measure with $\hat{\sigma}$ and $\hat{\rho}$ interchanged, however in such a case, pure entangled states have infinite measure of entanglement, which is not desirable. It is known that for pure bipartite states, the relative entropy of entanglement is the Von Neumann reduced entropy [41, 42]. The relative entropy of entanglement has been evaluated for Bell diagonal states, however a closed form expression is not known in the case of general two qubit mixed states. It is known that the relative entropy of entanglement is not additive [71], that it is always less than the entanglement of formation [42], and that it is an upper bound on the distillable entanglement, thus a lower bound on the entanglement of formation [42, 73]. While in the case of pure states one can distill as much entanglement as is present, the mixed state scenario is different. One cannot distill all the entanglement that is present. The existence of bound entanglement is a manifestation of this aspect. This has led to consideration of thermodynamical analogies in the context of mixed state entanglement [42, 62, 74]. Other distance measures such as the Bures metric have also been analogously extended to define entanglement measures [42].

1.6.4 Negativity

Most of the measures of entanglement discussed above are hard to compute for an arbitrary mixed entangled state. Especially, the entanglement of formation and the relative entropy of entanglement require an optimisation procedure, which is in general not easy. Thus it is useful to have a practical measure which is easily calculable. The negativity or the logarithmic negativity is one such measure. The negativity for a bipartite state $\hat{\rho}_{AB}$ is defined as

$$\mathcal{N}(\hat{\rho}_{AB}) = \frac{||\hat{\rho}_{AB}^{PT}||_1 - 1}{2}, \tag{1.93}$$

where $||.||_1$ denotes the trace norm. The trace norm for a hermitian operator is the sum of the absolute values of its eigenvalues. Thus $\mathcal{N}(\hat{\rho}_{AB})$ is simply the sum of negative eigenvalues of $\hat{\rho}_{AB}^{PT}$. Given negativity, we can define yet another measure of entanglement, the logarithmic negativity which is defined as

$$E_{\mathcal{N}}(\hat{\rho}_{AB}) = \log_2 ||\hat{\rho}_{AB}^{PT}||_1. \tag{1.94}$$

The logarithmic negativity satisfies the last three of the requirements required of a good measure of entanglement. It is local unitary invariant, does not increase under LOCC, and is additive. However, since it is based on partial transpose, it fails to measure entanglement in bound entangled states [75]. In a similar fashion, one can define calculable measures of entanglement based on negativity based on other positive maps [76, 77].

1.7 Continuous variables

In the earlier Sections, our concern was systems described by finite dimensional Hilbert spaces. We now consider extension of some of these ideas to the case of continuous variable systems. Such a study is neccesitated by the very nature of the current available technology. One approach towards practical implementation of the fundamental ideas of quantum information processing has been through the currently available quantum optical technology. Many fundamental aspects of quantum information theory have already been demonstrated in various labs, essentially through quantum optical technology [78–111]. Entanglement or EPR correlation was initially demonstrated in [78, 79]. Teleportation in the continuous variable context was initially discussed in [82] and then demonstrated in [83]. It was subsequently reported in other experiments [85, 86, 88–90, 92, 93]. Continuous variable dense coding has been reported in [96, 103–105]. Entanglement of the polarisation degree of freedom of photons was established in [84–87].

In [85] two qubit entanglement was demonstrated, and in [87] three qubit entanglement. In [86] teleportation of a single photon's polarisation state was reported. Coherent state based key distribution was reported in [97–100]. Other experiments that report continuous variable entanglement are found in [80, 81, 91, 94, 95, 101, 102, 107, 108, 111]. In [94] atomic ensemble entanglement was reported, and in [95, 101, 111] multipartite entanglement was reported. In [109, 110] teleportation between light and matter was reported.

A fundamental attribute to most of the above mentioned experiments has been the use of nonclassical resources. It is well known that states of radiation, naturally divide into classical and nonclassical types [112], nonclassicality being more fundamental than entanglement in the sense that it is a prerequisite for entanglement [113–115]. A primary reason for the possibility of many experiments in the context of quantum information theory has been the availability of nonclassical resources, primarily in the form of quadrature squeezing [116–135]. It is well known that this kind of nonclassicality, when processed appropriately with the help of passive devices like beamsplitters, can create entanglement, particularly in Gaussian states [113–115, 136–140]. Squeezing has been, and continues to be studied as a resource in the context of continuous variable entanglement.

Bearing in mind this technological scenario, a study of entanglement from the perspective of quantum states of radiation becomes desirable. Any such study requires one to deal with a system of quantum harmonic oscillators, systems whose Hilbert spaces are infinite dimensional. The potential role of such systems in quantum information theory have been explored in [141–163]. In [141] the issue of encoding a qubit in an oscillator was discussed, and the possibility of using coherent states to carry out quantum computational tasks was discussed in [164]. Such a study showed the possibility of using continuous variable set up to do quantum information tasks carried out on finite dimensional systems. In [142] the possibility of key distribution using squeezed states was discussed. In [143, 147, 148] quantum cryptography with Gaussian states was analysed. Multipartite entanglement and its potential use in quantum networks were considered in [144, 145, 150, 154–156]. In [151] universal quantum computation based on continuous variable cluster states using linear optics and homodyne measurement was explored. In [152] the problem of quantum state engineering was considered. In [146, 149, 157–159] continuous variable Gaussian channels were studied. In [161, 162, 165, 166] the possibility of generating entanglement in nanomechanical oscillators was explored; such systems are also modelled as a system of quantum harmonic oscillators. Thus it is natural to undertake the study of entanglement in the context of continuous variables.

The simplest study of entanglement in the context of bipartite continuous variable systems is the study of two-oscillator systems. For this bipartite continuous variable

system $S = A + B$, subsystems A and B consists of a single quantum harmonic oscillator each. The total system S is specified by respective annihilation and creation operator pairs $\hat{a}^\dagger$, $\hat{a}$, and $\hat{b}^\dagger$, $\hat{b}$ acting on the Hilbert spaces $\mathcal{H}^a$ and $\mathcal{H}^b$. Their only non-vanishing commutators are

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = \mathbb{1}. \tag{1.95}$$

The Fock or photon number states for the two modes provide a natural set of ONB's for $\mathcal{H}^a$ and $\mathcal{H}^b$ respectively. They are given as,

$$\begin{aligned} |n\rangle &= (n!)^{-1/2}(\hat{a}^\dagger)^n|0\rangle_a\,, \\ |m\rangle &= (m!)^{-1/2}(\hat{b}^\dagger)^m|0\rangle_b\,, \\ & n, m = 0, 1, 2, \ldots \end{aligned} \tag{1.96}$$

Then the product states $|n, m\rangle \equiv |n\rangle_a \otimes |m\rangle_b$ form an ONB for $\mathcal{H}^a \otimes \mathcal{H}^b$ [167]. This description is easily extended to multipartite continuous variable systems through introduction of additional creation and annihilation operator pairs.

1.7.1 Quasi-probabilities

When dealing with quantum mechanics of continuous variables, i.e., radiation field modes, it is useful to go over to their description in the language of quasi-probabilities. A state of a quantum mechanical system described by a density operator $\hat{\rho}$ can be equivalently described in the language of their s-ordered quasi-probabilities in a complete manner. The s-ordered quasi-probabilities capture all the statistical information present in the density operator $\hat{\rho}$. In this set up a density operator $\hat{\rho}$ is mapped into a quasi-probability, and the observables to corresponding ordered functions in phase space. Considering a single-mode for simplicity, the s-ordered quasi-probability corresponding to a state $\hat{\rho}$ is defined as

$$\begin{aligned} W_\rho(\alpha, s) &= \mathrm{Tr}(\hat{\rho}\hat{T}(\alpha, s)), \quad \text{where,} \\ \hat{T}(\alpha, s) &= \frac{1}{\pi}\int D(\xi, s)\exp(\alpha\xi^* - \alpha^*\xi)d^2\xi, \quad \text{and} \\ D(\xi, s) &= \exp(\hat{a}^\dagger\xi^* - \hat{a}\xi + \frac{1}{2}s|\xi|^2). \end{aligned} \tag{1.97}$$

$D(\xi, s)$ is the displacement operator corresponding to a given s, and α is the phase space variable which is denoted as $\alpha = \frac{q+ip}{\sqrt{2}}$. The parameter s takes the values $-1 \leq s \leq 1$. The three defining properties of a density operator given in Eq. (1.1) transcribe into the

following requirements on the s-ordered quasi-probability :

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}) = 1 \quad &\Leftrightarrow \quad \int W_\rho(\alpha, s) d^2\alpha = 1, \\ \hat{\rho}^\dagger = \hat{\rho} \quad &\Leftrightarrow \quad W_\rho(\alpha, s) = W_\rho^*(\alpha, s), \\ \hat{\rho} \geq 0 \quad &\Leftrightarrow \quad \mathrm{Tr}(\hat{\rho}\hat{\rho}') \geq 0 \Leftrightarrow \int W_\rho(\alpha, s) W_{\rho'}(\alpha, -s) d^2\alpha \geq 0. \end{aligned} \tag{1.98}$$

The first requirement demands that the s-ordered quasi-probability is normalised to one with respect to integration over the phase space, while the second demands that it be real. For the third requirement to be satisfied one has to in principle check for positivity of the trace inner product of the given density operator $\hat{\rho}$ with respect to all valid density operators $\hat{\rho}'$, which amounts to checking for positivity of the overlap integral of the given s-ordered quasi-probability with respect to all valid $(-s)$-ordered quasi-probabilities. The defining requirements at the density operator level doesn't enforce pointwise positivity of the s-ordered quasi-probability, further, a valid s-ordered quasi-probability can be highly singular.

Objects of interest are the expectation values of observables with respect to the given density operator $\hat{\rho}$. Given an observable $\hat{\Omega}$, assume that it possesses an s-ordered power series expansion

$$\hat{\Omega} = \sum_{n,m=0}^{\infty} \Omega_{n,m}(s) \{\hat{a}^{\dagger n} \hat{a}^m\}_s, \tag{1.99}$$

where $\{\hat{a}^{\dagger n} \hat{a}^m\}_s$ is the s-ordered product given by

$$\{\hat{a}^{\dagger n} \hat{a}^m\}_s = \frac{1}{\pi} \int \hat{T}(\alpha, s) \alpha^{*n} \alpha^m d^2\alpha, \tag{1.100}$$

then the expectation value $\langle \hat{\Omega} \rangle$ is given by

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}\hat{\Omega}) \quad &= \quad \int \Omega(\alpha, -s) W_\rho(\alpha, s) d^2\alpha, \quad \text{where} \\ \Omega(\alpha, -s) \quad &= \quad \sum_{n,m=0}^{\infty} \Omega_{n,m}(-s) \alpha^{*n} \alpha^m. \end{aligned} \tag{1.101}$$

Evaluating expectation values of s-ordered operators with respect to a density operator $\hat{\rho}$ corresponds to evaluating the function Ω which is obtained from $\hat{\Omega}$ simply by replacing $\{\hat{a}^{\dagger n} \hat{a}^m\}_s$ by $\alpha^{*n} \alpha^m$, and then evaluating its overlap integral with the s-ordered quasi-probability $W_\rho(\alpha, s)$ corresponding to the state $\hat{\rho}$. Conversely, any observable $\hat{\Omega}$ can be

written as

$$\hat{\Omega} = \frac{1}{\pi} \int \Omega(\alpha, s) \hat{T}(\alpha, s) d^2\alpha. \tag{1.102}$$

The s-ordered quasi-probabilities enables us to recast quantum mechanics in the language of statistical mechanics. Given an observable $\hat{\Omega}$, we have an s-ordered weight function $\Omega(\alpha, -s)$, which captures the content of $\hat{\Omega}$ for each $-1 \leq s \leq 1$.

The more familiar s-ordered quasi-probabilities are Sudarshan's ϕ function corresponding to $s = 1$, the Wigner function corresponding to $s = 0$, and the Husimi or Q function corresponding to $s = -1$. The notion of s-ordered quasi-probability can be easily extended to the multi-mode case as well [167–169].

1.7.2 Sudarshan's ϕ function

It turns out that any density operator corresponding to a single mode of radiation field can always be expanded as

$$\hat{\rho} = \int \frac{d^2z}{\pi} \phi_\rho(z) |z\rangle\langle z|, \tag{1.103}$$

where $\phi_\rho(z) = W_\rho(z, 1)$, and $|z\rangle$ is the coherent state. This is called as the diagonal representation. It is remarkable that every density operator can be expanded diagonally in the coherent state basis. This has been possible because of the over completeness of the coherent state basis [112]. This representation enables one to evaluate ensemble averages of normally ordered operators. Normal ordering corresponds to shifting all the creation operators to the left and all the annihilation operators to the right. The useful concept that arises out of this representation is the notions of classicality and nonclassicality. If $\phi_\rho(z)$ corresponding to a density operator $\hat{\rho}$ is pointwise nonnegative in the complex plane, then the density operator is a convex combination of coherent states. Since the coherent states are the most elementary of all quantum mechanical states exhibiting classical behaviour, any state that can be written as a convex combination of these elementary states is deemed classical. We have,

$$\phi_\rho(z) \geq 0 \ \text{ for all } \ z \in \mathcal{C} \Rightarrow \hat{\rho} \text{ is classical.} \tag{1.104}$$

Any state which cannot be written so is deemed nonclassical. This definition is readily extended to the multi-mode case. Not all states are classical, and in fact the ϕ function can be highly singular. For the Fock states $|n\rangle$, the diagonal function $\phi_\rho(z)$ turns out to be n^{th} derivative of the delta function. We will make extensive use of nonclassicality in

the context of entanglement in bipartite continuous variable systems.

1.7.3 Nonclassicality

One of the primary concerns in quantum optics over the years has been nonclassicality. This notion has been explored in various ways and its manifestations detailed [170–192]. Physical manifestations such as photon number oscillations have been explored in [170, 175, 178, 181, 183, 193]. More recently, nonclassicality has been viewed as a resource for entanglement [113–115, 137–139, 171, 194].

An important concern in the study of nonclassicality is in respect of the methods employed to detect nonclassicality. We briefly describe one of the techniques. Suppose we have the operator

$$F(\hat{a}^\dagger, \hat{a}) = \sum_{ij} c_{ij} \hat{a}^{\dagger j} \hat{a}^i. \tag{1.105}$$

Define the positive operator $F^\dagger F$ and normal order it to obtain $: F^\dagger F :$. Then the expectation value of this normal ordered operator in a state $\hat{\rho}$ is

$$\langle : F^\dagger F : \rangle = \mathrm{Tr}(\hat{\rho} : F^\dagger F :) = \int \frac{d^2 z}{\pi} \phi_\rho(z) |F(z^*, z)|^2. \tag{1.106}$$

Calculating $\langle : F^\dagger F : \rangle$ is equivalent to evaluating the phase space average of a pointwise positive function $|F(z^*, z)|^2$ with respect to $\phi_\rho(z)$. Thus if the expectation value of a positive normal ordered operator with respect to a state turns out to be negative, then we can surely conclude that the state was nonclassical. However if the expectation value turns out to be positive, we cannot conclude that the state was classical. In principle, one has to run over all possible positive functions over the phase space to test if a state is classical or not.

The method outlined above has been discussed and used in [170, 182, 186, 189–191]. The idea can be further extended through the use of positive polynomials which are not sum of squares of other polynomials [191]. It is useful to note that there is no conclusive test for nonclassicality, of an arbitrary mixed state. There are however two exceptions, namely single-mode Gaussians [185] and single-mode phase invariant states [170].

Another important aspect in the study of nonclassicality has been its quantification. Measures of nonclassicality have been discussed in [171, 173, 174, 177, 188, 195–198]. Distance based measures were discussed in [173, 188, 195, 196], where the measure of nonclassicality was defined as the least 'distance' of a nonclassical state with respect to the set of all classical states. The distance based measures are in general hard to evaluate

because of the inherent minimisation procedure involved in the definition. In [174, 177] a measure based on the smoothening of the ϕ function was defined. One can go from the ϕ function to any s-ordered quasi-probability through a Gaussian convolution parametrised by τ. The point τ_c at which the resultant quasi-probability is just about a true probability gives us a measure of nonclassicality. In [197, 198] a measure of nonclassicality based on Hudson's theorem was defined. Hudson's theorem says that the only pure state whose Wigner function is positive is a Gaussian state [199]. This fact indicates that any measure that captures the negativity of the Wigner function is a measure of nonclassicality of the state. More recently, in [171] a measure of nonclassicality inspired from an entanglement point of view was defined. Nonclassicality of a single-mode state was defined as the maximal bipartite entanglement it can produce when coupled with additional classical resources and passed through a 50:50 beamsplitter.

1.7.4 The Wigner function

The Wigner function is the s-ordered quasi-probability corresponding to $s = 0$. It is obtained by convoluting the ϕ with a particular Gaussian weight function. Thus, in principle, it is not as singular as the ϕ function, but nevertheless it can be pointwise non-positive in phase space. In this description, density operators are put in correspondence with real valued functions over the phase space through the rule

$$W_\rho(q,p) = \frac{1}{\pi}\int dq' \langle q - q'|\hat{\rho}|q + q'\rangle \exp(2iq'p). \tag{1.107}$$

We recall that $\alpha = \frac{q+ip}{\sqrt{2}}$. We may write $W_\rho(q,p)$ as $W_\rho(\xi)$ for convenience [200, 201], where ξ is the pair $\{q,p\}$. The Wigner description offers several advantages, from the theoretical and experimental perspectives. It turns out to be convenient in the context of unitary evolution of states under the action quadratic hamiltonians. Given the unitary operator $U(S)$ corresponding to a quadratic hamiltonian which is captured by a symplectic group element $S \in Sp(2n,R)$, a state evolves as

$$\hat{\rho} \to \hat{\rho}' = U(S)\hat{\rho}U(S)^\dagger. \tag{1.108}$$

This transformation takes a very simple form in the Wigner description :

$$S:\ \hat{\rho} \to \hat{\rho}' = U(S)\hat{\rho}U(S)^\dagger \Leftrightarrow W_\rho(\xi) \to W_{\rho'}(\xi) = W_\rho(S^{-1}\xi). \tag{1.109}$$

That is, $W_{\rho'}(S\xi) = W_\rho(\xi)$ for every canonical transformation $S \in Sp(2n,R)$. That is, the Wigner function transforms as a $Sp(2n,R)$ scalar field [202]. This is also true

of inhomogeneous linear canonical transformations involving phase space translations as well.

The Wigner function is also important from an experimental perspective, as it can be directly measured in a lab. The Wigner function by definition, is the expectation value of the displaced parity operator with respect to the given density operator[169], and thus can be observed in a laboratory. Indeed, the Wigner function of various quantum states of radiation have been reported [203–215].

1.7.5 The Q function

The Q function corresponds to the quasi-probability with order parameter $s = -1$ For a state $\hat{\rho}$ of a single-mode of radiation field it is defined as

$$Q_\rho(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle. \tag{1.110}$$

It is thus pointwise nonnegative in the complex plane, being the expectation value of a coherent state on a positive semidefinite density operator $\hat{\rho}$. Thus it is a true probability distribution. However it should be noted that not all valid probability distributions are valid Q functions. The quantum mechanical uncertainty principle places severe restrictions on probability distributions which qualify to be valid Q functions. The advantage of working with them is that one will deal only with true probabilities. We will consider them in more detail in Chapter 7 where we discuss non-Gaussianity [216].

1.8 Detection of Entanglement

Detecting entanglement in continuous variables has been an important pursuit, as methods employed in the case of finite dimensions cannot always be extended in a naive manner. Among tests based on positive maps, only the partial transpose test and reduction criterion have been extended [57, 217]. However, inseparability criteria (inequalities) defined through EPR-like operators based on the quadratic moments, initially introduced in [58], are effective in detecting entanglement in continuous variables. These inequalities are derived from first principles as consequences of separability. The uncertainty principle places certain restrictions on the moments, and all states have to obey the uncertainty principle [218]. However, separable states have to obey further constraints, which can be taken as inseparability criteria. Simple examples of such constraints are the Bell type inequalities, violation of which have been observed in entangled Gaussian states [219, 220].

1.8.1 Bell's inequalities

The Bell type inequalities as in Eq. (1.36) are extended to the continuous variable case through the definition of the parity operator. The parity operator takes the role of the spin observable, and the role of 'direction' of the spin observable is taken by displacement in phase space. The parity operator $\mathbf{\Pi}(\xi)$ for N modes is given as

$$\mathbf{\Pi}(\xi) = \otimes_{i=1}^{N} \Pi_i(\xi_i) = \otimes_{i=1}^{N} D_i(\xi_i)(-1)^{\hat{n}_i} D_i^{\dagger}(\xi_i), \tag{1.111}$$

where $D_i(\xi_i)$ is the phase space displacement operator of the i$^{\text{th}}$ mode, defined in Eq. (1.97). It happens that the expectation value of the parity operator $\mathbf{\Pi}(\xi)$ with a given state $\hat{\rho}$ gives its Wigner function [168, 169] :

$$W_{\rho}(\xi) = \mathrm{Tr}(\hat{\rho}\mathbf{\Pi}(\xi)). \tag{1.112}$$

Hence, the expectation value of the parity operator on simple states such as Gaussian states is easily evaluated. In [219] the Bell operator

$$\mathcal{B} = \Pi(0,0) + \Pi(0,\beta) + \Pi(\alpha,0) - \Pi(\alpha,\beta) \tag{1.113}$$

was considered, where α, and β are phase space displacements on the two modes. Separable continuous variable states have to satisfy the inequality $|\mathcal{B}| \leq 2$. It was demonstrated in [219] that the two-mode squeezed vacuum clearly violated this inequality. This idea was further generalised to the multi-mode case in [221].

Another equivalent approach towards Bell inequalities in continuous variable systems was given in [220]. This was done through the extension of the spin operators to infinite dimensions through a direct sum of the spin-$\frac{1}{2}$ Pauli operators. In this set up, the Pauli spin operators $\hat{\sigma}_z$, $\hat{\sigma}^{+}$, and $\hat{\sigma}^{-}$ are given by

$$\hat{\sigma}_z = (-1)^{\hat{n}}, \quad \hat{\sigma}^{+} = \hat{\sigma}^{-\dagger} = \sum_{n=0}^{\infty} |2n\rangle\langle 2n+1|. \tag{1.114}$$

It is easy to see that the pseudo-spin operators defined above, satisfy the $SU(2)$ algebra required of spin-$\frac{1}{2}$ operators. It is now easy to extend the Bell-CHSH inequalities in Eq. (1.36) to the continuous variable case. In [220] such an extension was done, and the entanglement in two-mode squeezed vacuum was demonstrated. The role of Bell's inequalities in detecting entanglement is also discussed in [222–225].

1.8.2 Partial transpose

We have already seen in Eq. (1.109) that the Wigner function transforms as a scalar field under the action of symplectic group elements corresponding to evolution under quadratic hamiltonians. The transpose map and the partial transpose map also take a geometric form in the Wigner description. For a single mode of radiation field, it follows from the definition of the Wigner function that transpose operation on the density operator is equivalent to complex conjugation of the elements of the density matrix in position representation, which transcribes faithfully into momentum reversal operation in the Wigner description :

$$\begin{aligned} \mathrm{T}: \ W_\rho(q,p) &\rightarrow W'_\rho(q,p) = W_\rho(q,-p) = W_\rho(\Lambda\xi), \\ \Lambda &= \mathrm{diag}(1,-1) = \sigma_3. \end{aligned} \tag{1.115}$$

This amounts to a mirror reflection which inverts the p coordinate, leaving the q coordinate unchanged. The transpose map takes density operators to density operators, but is unphysical as seen in Section 1.4.6. For a bipartite system of two modes of radiation field, partial transposition amounts to inverting the p coordinate for one of the subsystems. Its action on the Wigner function is given by

$$\begin{aligned} \mathrm{PT}: \ W_\rho(q_a,p_a,q_b,p_b) &\rightarrow W'_\rho(q_a,p_a,q_b,p_b) \\ &= W_\rho(q_a,p_a,q_b,-p_b). \end{aligned} \tag{1.116}$$

Though the partial transpose, as a map, is given by an elegant transformation with regard to the Wigner function, its usefulness as an entanglement witness is manifest only when we are able to test a phase space distribution for its Wigner quality. That is, one has to answer the question as to when is a function in phase space a Wigner function. Such a question was initially raised in the limited context of Gaussian Wigner functions in [201], and fully answered in [226] in that context. This was possible because of the Williamson's theorem [226]. This consequently led to the demonstration of partial transpose as an effective entanglement criterion in the case of two-mode Gaussians [57]. In general, one can test for entanglement in a limited manner through the manifestation of partial transpose on moments.

Given a state $\hat{\rho}^{(ab)}$ of a bipartite system of two modes of radiation field, a test for entanglement in $\hat{\rho}^{(ab)}$ through partial transpose would be to test for negativity of $\hat{\rho}^{(ab)PT}$. A direct approach would be to test for violation of positivity of the diagonals of $\hat{\rho}^{(ab)PT}$ in some orthogonal basis pertaining to the composite system Hilbert space $\mathcal{H}^a \otimes \mathcal{H}^b$. Such an effort may be tedious while one is dealing with continuous variable systems. A simpler

practical approach, but limited in scope, would be to test for violation of positivity of $\hat{\rho}^{(ab)PT}$ by taking its expectation value with a positive operator. Such a test may not in principle capture the negativity of $\hat{\rho}^{(ab)PT}$ in its entirety, but may prove to be useful in certain contexts. An immediate requirement of the procedure demands the systematic construction of positive operators acting on $\mathcal{H}^a \otimes \mathcal{H}^b$. Given an operator

$$\eta = \sum_{jklm} c_{jklm} \hat{a}^{\dagger j} \hat{a}^k \hat{b}^{\dagger l} \hat{b}^m, \tag{1.117}$$

the operator $\eta^\dagger \eta$ is positive by construction. A simple test for violation of positivity of $\hat{\rho}^{(ab)PT}$ is to then check if

$$\mathrm{Tr}(\hat{\rho}^{(ab)PT}(\eta^\dagger \eta)) = \mathrm{Tr}(\hat{\rho}^{(ab)}(\eta^\dagger \eta)^{PT}) < 0. \tag{1.118}$$

To this end, we need to know how partial transpose acts on an ordered monomial $(\hat{a}^{\dagger j} \hat{a}^k \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger l} \hat{b}^m \hat{b}^{\dagger r} \hat{b}^s)$. Since the creation and annihilation operators $\hat{b}^\dagger$ and $\hat{b}$ are real, transposition on Bob's side alone amounts to hermitian conjugation of operators on Bob's side. We have

$$(\hat{a}^{\dagger j} \hat{a}^k \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger l} \hat{b}^m \hat{b}^{\dagger r} \hat{b}^s)^{PT} = (\hat{a}^{\dagger j} \hat{a}^k \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger s} \hat{b}^r \hat{b}^{\dagger m} \hat{b}^l). \tag{1.119}$$

As a simple example demonstrating the effectiveness of the procedure occurs in the case of two-mode Gaussians. The violation of partial transpose at the level of variance matrix corresponding to $\eta^\dagger \eta$ being quadratic in the annihilation and creation operators of the two modes, turns out to be a both necessary and sufficient test for entanglement [57]. We shall see later how this procedure can be effective in more general contexts [227–229].

1.8.3 Inseparability criteria through uncertainty relations

This method of detecting entanglement in continuous variable systems is based on the fact that expectation values of nonlocal operators pertaining to the composite system have to obey additional constraints for separable states in addition to the usual uncertainty principle. As a simple example, consider the pair of EPR like operators $\hat{x}_a - \hat{x}_b$ and $\hat{p}_a + \hat{p}_b$. The sum of the variances of these two operators goes to zero for maximally entangled states. Such a state may appear unphysical though, but for the two-mode squeezed vacuum the total variance rapidly tends to zero with increasing degree of squeezing. And this state approaches the maximally entangled state as the squeeze parameter grows.

However for a separable state, assuming that the first moments were zero, we have

$$\begin{aligned}
&\langle(\hat{x}_a - \hat{x}_b)^2\rangle + \langle(\hat{p}_a + \hat{p}_b)^2\rangle = \\
&\sum_i p_i\langle\hat{x}_a^2\rangle_i + \sum_i p_i\langle\hat{x}_b^2\rangle_i - 2\sum_i p_i\langle\hat{x}_a\rangle_i\langle\hat{x}_b\rangle_i \\
&+ \sum_i p_i\langle\hat{p}_a^2\rangle_i + \sum_i p_i\langle\hat{p}_b^2\rangle_i + 2\sum_i p_i\langle\hat{p}_a\rangle_i\langle\hat{p}_b\rangle_i \\
&= \sum_i p_i(\langle\hat{x}_a^2\rangle_i - \langle\hat{x}_a\rangle_i^2) + \sum_i p_i(\langle\hat{p}_a^2\rangle_i - \langle\hat{p}_a\rangle_i^2) \\
&+ \sum_i p_i(\langle\hat{x}_b^2\rangle_i - \langle\hat{x}_b\rangle_i^2) + \sum_i p_i(\langle\hat{p}_b^2\rangle_i - \langle\hat{p}_b\rangle_i^2) + \\
&\sum_i p_i(\langle\hat{x}_a\rangle_i - \langle\hat{x}_b\rangle_i)^2 + \sum_i p_i(\langle\hat{p}_a\rangle_i + \langle\hat{p}_b\rangle_i)^2 \geq 2. \qquad (1.120)
\end{aligned}$$

Such a criterion was effectively used in [58, 230] to demonstrate entanglement in two-mode Gaussian states. The EPR uncertainty relation in Eq. (1.120) also becomes useful in evaluating the EOF of two-mode symmetric Gaussian states. This method was extended to the multi-mode scenario in [231]. Similar techniques, based on the uncertainties on moments were used in the case of finite dimensional systems to detect entanglement [59–61], and later extended to continuous variable systems in [232].

Positive maps on matrix of moments: This technique was recently introduced in [233]. It enables us to apply the positive maps familiar from the context of finite dimensional systems to detect entanglement in continuous variable systems through the matrix of moments. Any moment matrix generated by tensoring operators belonging to the individual systems is separable on separable states, thus inseparability of the moment matrix implies entanglement for the state. As a simple example, for two pairs of operators $\hat{f}_1$, $\hat{f}_2$ on Alice's side, and $\hat{g}_1$, $\hat{g}_2$ on Bob's side, the matrix of moments formed by the tensored set $(\hat{f}_1\hat{g}_1, \hat{f}_1\hat{g}_2, \hat{f}_2\hat{g}_1, \hat{f}_2\hat{g}_2)$, i.e.,

$$M = \begin{pmatrix}
\langle\hat{f}_1^2\hat{g}_1^2\rangle & \langle\hat{f}_1^2\hat{g}_1\hat{g}_2\rangle & \langle\hat{f}_1\hat{f}_2\hat{g}_1^2\rangle & \langle\hat{f}_1\hat{f}_2\hat{g}_1\hat{g}_2\rangle \\
\langle\hat{f}_1^2\hat{g}_2\hat{g}_1\rangle & \langle\hat{f}_1^2\hat{g}_2^2\rangle & \langle\hat{f}_1\hat{f}_2\hat{g}_2\hat{g}_1\rangle & \langle\hat{f}_1\hat{f}_2\hat{g}_2^2\rangle \\
\langle\hat{f}_2\hat{f}_1\hat{g}_1^2\rangle & \langle\hat{f}_2\hat{f}_1\hat{g}_1\hat{g}_2\rangle & \langle\hat{f}_2^2\hat{g}_1^2\rangle & \langle\hat{f}_2^2\hat{g}_1\hat{g}_2\rangle \\
\langle\hat{f}_2\hat{f}_1\hat{g}_2\hat{g}_1\rangle & \langle\hat{f}_2\hat{f}_1\hat{g}_2^2\rangle & \langle\hat{f}_2^2\hat{g}_2\hat{g}_1\rangle & \langle\hat{f}_2^2\hat{g}_2^2\rangle
\end{pmatrix}, \qquad (1.121)$$

is separable for separable states. This method enables us to indirectly (directly) use the theory of positive maps to detect entanglement in continuous variable systems.

1.9 Gaussian states

Gaussian states form an integral part of quantum information theory in the context of continuous variables. Initially they were studied in the context of radiation fields in quantum optics [185, 201, 202, 226, 234–240]. Much of the study was devoted to characterising them through the variance matrix and the exploration of nonclassicality of these states which was primarily in the form of squeezing. But with the development of quantum information theory the focus has shifted to the study of entanglement in these states [57, 58, 70, 136, 138, 140, 143, 144, 147–150, 156–159, 217, 241–271]. Teleporation, the fundamental protocol of quantum information theory has been achieved using them [83, 88–90, 96, 97, 103]. This has lead to the exploration of Gaussian states in a major way.

We review, very briefly, some of these developments. The potential role of Gaussian states in quantum information processes was initially realised in [82] in the form of teleportation. The entanglement involved in this protocol was first pointed out in [105]. Subsequently, the detection of entanglement of these states was completely characterised in [58, 272]. The effective use of the variance matrix formalism in the context of bipartite entanglement was initially carried out in [57], and this was possible because Gaussian states are completely specified by their variance matrix. The primary tool is the effective use of uncertainty principle in characterising them [192, 202, 218, 226]. The Williamson theorem has a fundamental role to play in this context [192, 202, 226, 272, 273].

In [241] a family of bound entangled Gaussian states was constructed, and in [217] the issue of distillability was solved: it was shown that every NPT Gaussian state is distillable. Purification of Gaussian states has been discussed in [259, 268, 271]. In [242], the issue of bipartite separability in the multi-mode case was completely solved. The various possible situations that could arise with regard to separability in tripartite systems was discussed in [243]. Entanglement in multipartite Gaussians was studied in [249, 250]. It was found that in special cases Gaussian multipartite entanglement can be reduced to two-mode Gaussian entanglements using local operations alone.

The role of squeezing in generating entanglement in these states was discussed. in [136–138, 140], and some of the special properties of Gaussian states were discussed in [244, 246, 252, 253]. In [244, 246] it was found that Gaussian states cannot be distilled with Gaussian operations alone, and in [252] it was shown that the optimal cloning of Gaussian states required additional non-Gaussian resources. Some of the extremal properties of Gaussian states were discussed in [253]. One particular such extremal property helped solve the problem of determining the entanglement of formation in symmetric Gaussian states [70]. We will have more to say on this later in the thesis. Lower bounds

on the entanglement of formation of general bipartite Gaussian states was obtained in [262].

The determination of entanglement in bipartite Gaussian states through purity measurements was discussed in [248, 261, 263]. Such a study brought out the possibility of characterising Gaussian states through both global and local purities. Properties such as the loss of entanglement in evolution was discussed in [137, 260, 274].

Gaussian channels have been discussed in [149, 157, 269]. A definition for the most general Gaussian channel was given in [157]. The quantum channel capacity defined through coherent information was evaluated for a certain class of Gaussian channels in [158, 159].

The possibility of assessing the entanglement in two-mode Gaussian states using local parity measurements and classical communication was discussed in [247]. It was shown that given sufficiently large number of copies of a Gaussian state, its entanglement and the state itself can be characterised completely by LOCC!

The concept of entanglement monogamy [275, 276] was extended to the Gaussian case in [254, 255], and monogamy relations were established. In [251] the variance matrix set up was discussed from the perspective of convex sets, and numerical routines were set up to generate entanglement witnesses. More detailed reviews on Gaussian states can be found in [256–258].

We now describe the basic formalism for handling Gaussian states and their transformation, with particular attention to nonclassicality and entanglement.

1.9.1 Nonclassicality in Gaussians

Nonclassicality in Gaussian states primarily occurs in the form of quadrature squeezing. This quantum optical concept was initially explored in [185, 192, 237–240, 277, 278]. An elegant definition for squeezing through the variance matrix formalism was given in [192]. We briefly review this work.

Since zero-mean Gaussian states are completely specified by their variance matrix, we begin by giving the basic setup of variance matrix, first from the perspective of nonclassicality, and later we give the description from the entanglement perspective.

Consider an n mode quantum system with annihilation operators $\hat{a}_j$, $\hat{a}_j^\dagger$, $j = 1, 2, \ldots, n$, obeying the standard boson commutation relations

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\,, \quad [\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0, \tag{1.122}$$

or equivalently

$$\begin{aligned} [\hat{q}_j, \hat{p}_k] &= i\delta_{jk}\,, \quad [\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0, \text{ where} \\ \hat{a}_j &= \frac{\hat{q}_j + i\hat{p}_j}{\sqrt{2}}, \quad \hat{a}_j^\dagger = \frac{\hat{q}_j - i\hat{p}_j}{\sqrt{2}}. \end{aligned} \tag{1.123}$$

It is convenient to arrange the hermitian $\hat{q}_j$, $\hat{p}_j$ and the $\hat{a}_j$, $\hat{a}_j^\dagger$ in $2n$ component column vector forms :

$$\hat{\xi}^{(r)} = \begin{pmatrix} \hat{q}_1 \\ . \\ . \\ . \\ \hat{q}_n \\ \hat{p}_1 \\ . \\ . \\ . \\ \hat{p}_n \end{pmatrix}, \quad \hat{\xi}^{(c)} = \begin{pmatrix} \hat{a}_1 \\ . \\ . \\ . \\ \hat{a}_n \\ \hat{a}_1^\dagger \\ . \\ . \\ . \\ \hat{a}_n^\dagger \end{pmatrix}. \tag{1.124}$$

The 'vectors' $\hat{\xi}^{(c)}$ and $\hat{\xi}^{(r)}$ are related by a fixed numerical unitary matrix Γ

$$\begin{aligned} \hat{\xi}^{(c)} &= \Gamma\hat{\xi}^{(r)}, \quad \text{where} \\ \Gamma &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & i\mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{pmatrix}. \end{aligned} \tag{1.125}$$

The canonical commutation relations among the mode operators can now be succinctly written as

$$\begin{aligned} [\hat{\xi}_j^{(r)}, \hat{\xi}_k^{(r)}] &= i\beta_{jk}, \quad [\hat{\xi}_j^{(c)}, \hat{\xi}_k^{(c)}] = i\Sigma_{jk}, \\ i, j &= 1, 2, \ldots, 2n, \end{aligned} \tag{1.126}$$

where the $2n \times 2n$ dimensional matrices β and Σ are given in block form by

$$\beta = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{1.127}$$

We define the $2n \times 2n$ real variance matrix $V^{(r)}$ for a state $\hat{\rho}$ by

$$V^{(r)}_{jk} = \frac{1}{2}\mathrm{Tr}(\hat{\rho}\{\hat{\xi}^{(r)}_j \hat{\xi}^{(r)T}_k\}), \tag{1.128}$$

where $\{.,.\}$ is the anticommutator. We can alternatively arrive at the variance matrix by taking the expectation value of the positive operator $\hat{\xi}^{(r)}\hat{\xi}^{(r)T}$, i.e.,

$$\langle \hat{\xi}^{(r)}\hat{\xi}^{(r)T}\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{\xi}^{(r)}\hat{\xi}^{(r)T}) = V^{(r)} + \frac{i}{2}\beta.$$

We can write the variance matrix in an $n \times n$ block form as

$$\begin{aligned} V^{(r)} &= \begin{pmatrix} V_1 & V_2 \\ V_2^T & V_3 \end{pmatrix}, \\ (V_1)_{jk} &= \langle \hat{q}_j \hat{q}_k\rangle,\ (V_2)_{jk} = \frac{1}{2}\langle\{\hat{q}_j, \hat{p}_k\}\rangle,\ (V_3)_{jk} = \langle \hat{p}_j \hat{p}_k\rangle, \\ j,k &= 1,2,\ldots,n\,. \end{aligned} \tag{1.129}$$

The matrix V_1 gives the correlations among the $\hat{q}$'s, V_3 gives those among the $\hat{p}$'s, and V_2 the correlations between the $\hat{q}$'s and $\hat{p}$'s. For a state $\hat{\rho}$ with non zero mean $(\langle\hat{\xi}^{(r)}\rangle) \neq 0$, the variance matrix is defined by simply replacing $\hat{\xi}^{(r)}$ by $\Delta\hat{\xi}^{(r)} = \hat{\xi}^{(r)} - \langle\hat{\xi}^{(r)}\rangle$. Such a replacement corresponds to a rigid translation in phase space by amount $-\langle\hat{\xi}^{(r)}\rangle$, implemented by the displacement operator $D(-\langle\hat{\xi}^{(r)}\rangle)$.

The complex form of the variance matrix $V^{(c)}$ is generated by taking the expectation value of the positive operator $\hat{\xi}^{(c)}\hat{\xi}^{(c)\dagger}$, i.e.,

$$\begin{aligned} \langle \hat{\xi}^{(c)}\hat{\xi}^{(c)\dagger}\rangle &= V^{(c)} + \frac{1}{2}\Sigma, \quad \text{where} \\ V^{(c)} &= \Gamma V^{(r)} \Gamma^\dagger. \end{aligned} \tag{1.130}$$

Writing in $n \times n$ block form, we have

$$\begin{aligned} V^{(c)} &= \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}, \\ A_{jk} &= A^*_{kj} = \frac{1}{2}\langle\{\hat{a}_j, \hat{a}^\dagger_k\}\rangle,\ B_{jk} = B_{kj} = \langle \hat{a}_j \hat{a}_k\rangle. \end{aligned} \tag{1.131}$$

By definition, $V^{(r)}$ is symmetric positive definite, this also implies that $V^{(c)}$ is complex hermitian positive definite. It also implies that A is hermitian, and B is symmetric. We have the following relations between the blocks of the variance matrix in its real and

complex forms :

$$
\begin{aligned}
A &= \frac{1}{2}[V_1 + V_3 + i(V_2^T - V_2)], \\
B &= \frac{1}{2}[V_1 - V_3 + i(V_2^T + V_2)]; \\
V_1 &= \frac{1}{2}(A + A^* + B + B^*), \\
V_2 &= \frac{i}{2}(A - A^* - B + B^*), \\
V_3 &= \frac{1}{2}(A + A^* - B - B^*).
\end{aligned} \tag{1.132}
$$

Uncertainty principle : Given a real symmetric positive definite $2n \times 2n$ matrix $V^{(r)}$, it has to satisfy additional requirements to qualify as a variance matrix derived from a quantum mechanical state. The additional requirements are succinctly captured by the uncertainty principle [192] :

$$
V^{(r)} + \frac{i}{2}\beta \geq 0. \tag{1.133}
$$

The above condition is both necessary and sufficient to validate a given symmetric positive definite matrix $V^{(r)}$ as the variance matrix of some quantum state. The necessity of the condition follows from definition but the sufficiency part, which is nontrivial, follows from the use of Williamson theorem [272, 273]. In the case of Gaussian probability distributions this uncertainty principle on the variance matrix is both necessary and sufficient condition to qualify the probability as a Wigner distribution[226], since in this case the variance matrix completely specifies the quantum state. This is no longer true in more general contexts [279].

Unitary evolution : Unitary evolution of the mode operators under a quadratic hamiltonian corresponds to a symplectic transformation on the column vector $\hat{\xi}^{(r)}$:

$$
\begin{aligned}
U &= \exp(-iH) \Rightarrow U^\dagger \hat{\xi}^{(r)} U = S_{\rm H}^{(r)} \hat{\xi}^{(r)}, \quad \text{where} \\
H &= \sum_{j,k} h_{jk}^{(r)} \hat{\xi}_j^{(r)} \hat{\xi}_k^{(r)}, \quad \text{and} \\
S_{\rm H}^{(r)} &\in Sp(2n, R), \text{ i.e., } \quad S_{\rm H}^{(r)} \beta S_{\rm H}^{(r)T} = \beta.
\end{aligned} \tag{1.134}
$$

Under such a unitary evolution, the vector $\hat{\xi}^{(c)}$ transforms as

$$
\hat{\xi}^{(c)} \rightarrow \hat{\xi}'^{(c)} = S_{\rm H}^{(c)} \hat{\xi}^{(c)}, \quad S_{\rm H}^{(c)} = \Gamma S_{\rm H}^{(r)} \Gamma^\dagger. \tag{1.135}
$$

Given a state $\hat{\rho}$ with variance matrix $V^{(r)}$, under time evolution by a quadratic hamiltonian the variance matrix evolves through a congruence by the symplectic transformation corresponding to the hamiltonian :

$$\begin{aligned} \hat{\rho} &\rightarrow \hat{\rho}' = U(S^{(r)})\hat{\rho}U(S^{(r)})^{\dagger} \Rightarrow \\ V^{(r)} &\rightarrow V'^{(r)} = S^{(r)}V^{(r)}S^{(r)T}, \quad S^{(r)} \in Sp(2n,R). \end{aligned} \tag{1.136}$$

We have removed the subscript H for brevity. Similarly, the complex form of the variance matrix evolves as

$$V^{(c)} \rightarrow V'^{(c)} = S^{(c)}V^{(c)}S^{(c)\dagger}, \quad S^{(c)} = \Gamma S^{(r)}\Gamma^{\dagger}. \tag{1.137}$$

The symplectic group : The defining property of matrices comprising the symplectic group $Sp(2n,R)$ is

$$S^{(r)} \in Sp(2n,R) \Leftrightarrow S^{(r)}\beta S^{(r)T} = \beta. \tag{1.138}$$

The group is characterised by $n(2n+1)$ generators, of which n^2 generators are compact generators and total photon number. The remaining $n(n+1)$ generators are noncompact, and do not conserve the total photon number. The hermitian quadratic hamiltonians which correspond to the compact generators can be taken to be

$$\begin{aligned} &\frac{1}{4}(\hat{a}_j^{\dagger}\hat{a}_j + \hat{a}_j\hat{a}_j^{\dagger}), \quad j = 1,2,\ldots,n; \\ &\frac{1}{4}(\hat{a}_j^{\dagger}\hat{a}_k + \hat{a}_k^{\dagger}\hat{a}_j), \\ &\frac{i}{4}(\hat{a}_j^{\dagger}\hat{a}_k - \hat{a}_k^{\dagger}\hat{a}_j), \quad j < k = 2,\ldots,n. \end{aligned} \tag{1.139}$$

They commute with the total photon number

$$\hat{N} = \sum_{j=1}^{n} \hat{a}_j^{\dagger}\hat{a}_j, \tag{1.140}$$

and the unitary operators generated by them correspond to passive systems which preserve classicality. The remaining $n(n+1)$ linearly independent hermitian quadratic noncompact generators can be given by

$$\begin{aligned} &\frac{1}{4}(\hat{a}_j^{\dagger}\hat{a}_k^{\dagger} + \hat{a}_k\hat{a}_j), \\ &\frac{i}{4}(\hat{a}_j^{\dagger}\hat{a}_k^{\dagger} - \hat{a}_k\hat{a}_j), \quad j \leq k = 1,2,\ldots,n. \end{aligned} \tag{1.141}$$

The maximal compact subgroup of $Sp(2n,R)$ denoted by $K(n)$ is generated by the n^2 compact generators, and is given by

$$S^{(r)}(X,Y) = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}, \tag{1.142}$$

where X and Y are real $n \times n$ matrices obeying

$$XX^T + YY^T = \mathbb{1}, \quad XY^T = YX^T. \tag{1.143}$$

It is easy to see that

$$S^{(r)}(X,Y)S^{(r)T}(X,Y) = \mathbb{1}, \quad S^{(r)}(X,Y)\beta S^{(r)T}(X,Y) = \beta. \tag{1.144}$$

Such a matrix is both orthogonal and symplectic in $2n$ dimensions. Going over to the complex form, we have

$$\begin{aligned} S^{(c)}(X,Y) &= \Gamma S^{(r)}(X,Y)\Gamma^\dagger = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} = S^{(c)}(U), \\ U &= X - iY, \quad UU^\dagger = \mathbb{1}, \end{aligned} \tag{1.145}$$

That is $X - iY$ (and hence $X + iY$) is a unitary matrix. Thus the maximal compact subgroup $K(n)$ of $Sp(2n,R)$ is isomorphic to $U(n)$. In other words,

$$K(n) = SO(2n) \cap Sp(2n,R) = U(n). \tag{1.146}$$

An interesting property of $K(n)$ is that it acts transitively on the phase space unit sphere $S^{(2n-1)}$.

Theorem 1.9 *A variance matrix is squeezed if and only if*

$$l(V^{(r)}) < \frac{1}{2}, \tag{1.147}$$

where $l(V^{(r)})$ is the least eigenvalue of the variance matrix $V^{(r)}$ [192].

One is able make this statement because of the transitive action of $K(n)$ on the unit sphere $S^{(2n-1)}$, though one cannot in general diagonalise a given $V^{(r)}$ by $K(n)$ rotations [192].

1.9.2 Entanglement in Gaussians

We now revise our notation into one which is useful in describing entanglement in multimode Gaussian states. The earlier notation was useful from the perspective of passive devices, and hence nonclassicality as the action of any passive device is compactly described as a unitary transformation on the annihilation operators. With entanglement in perspective, it is useful to make a modewise grouping of the canonical variables.

Consider a bipartite system consisting of n modes, with m modes in possession of Alice and the remaining $n-m$ modes in Bob's possession. We introduce the following notation :

$$\begin{aligned} \hat{\xi}^{(a)} &= (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_m, \hat{p}_m); \\ \hat{\xi}^{(b)} &= (\hat{q}_{m+1}, \hat{p}_{m+1}, \hat{q}_{m+2}, \hat{p}_{m+2}, \ldots, \hat{q}_n, \hat{p}_n); \\ \hat{\xi} &= (\hat{\xi}^{(a)}, \hat{\xi}^{(b)}). \end{aligned} \tag{1.148}$$

The commutation relations are given in a compact form as

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] = i\Omega_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \ldots, 2n, \tag{1.149}$$

where

$$\Omega = \begin{pmatrix} J & 0 & . & . & . & 0 \\ 0 & J & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.150}$$

Clearly, a canonical transformation S satisfies

$$S\Omega S^T = \Omega. \tag{1.151}$$

Suppose that the state has a nonzero mean. Then we can define the translated operators $\Delta\hat{\xi} = \hat{\xi} - \langle\hat{\xi}\rangle$, in which case, the variance matrix is defined as

$$\langle\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\rangle = \mathrm{Tr}(\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\hat{\rho}) = V_{\alpha\beta}. \tag{1.152}$$

The uncertainty principle is now stated as

$$V + \frac{i}{2}\Omega \geq 0. \tag{1.153}$$

We have the following theorem with regard to bipartite separability.

Theorem 1.10 *A necessary and sufficient condition for separability of a Gaussian state described by the variance matrix V is [280],*

$$\begin{aligned} V &\geq \frac{1}{2} S_a S_a^T \oplus S_b S_b^T, \\ S_a &\in Sp(2m, R), \quad S_b \in Sp(2n-2m, R). \end{aligned} \tag{1.154}$$

The inequality simply states that the variance matrix of a separable state is always greater than that of a pure product Gaussian state. In other words, a separable state can be made classical by local unitary canonical transformations. Such a thing is impossible if the Gaussian state was entangled across the a-b divide [26].

Partial transpose: As stated already in Section 1.8.2, partial transpose acts on the Wigner function as momentum reversal on Bob's side. Under partial transpose, the variance matrix of a multipartite state across a bipartite cut, undergoes the change $V \rightarrow \tilde{V} = \Lambda' V \Lambda'$, where

$$\Lambda = \mathrm{diag}(1_1, 1_1, \cdots, 1_m, 1_m; 1_{m+1}, -1_{m+1}, \cdots, 1_n, -1_n). \tag{1.155}$$

We can now implement partial transpose on the variance matrix and test for entanglement. In addition to the uncertainty principle, the variance matrix of a separable state has to obey

$$\tilde{V} + \frac{i}{2}\Omega \geq 0, \;\; \tilde{V} = \Lambda' V \Lambda', \;\; \Omega = \begin{pmatrix} \Omega_A & 0 \\ 0 & \Omega_B, \end{pmatrix}. \tag{1.156}$$

This can be alternatively stated as

$$V + \frac{i}{2}\tilde{\Omega} \geq 0, \;\; \tilde{\Omega} = \Lambda' \Omega \Lambda', \;\; \tilde{\Omega} = \begin{pmatrix} \Omega_A & 0 \\ 0 & -\Omega_B, \end{pmatrix}. \tag{1.157}$$

A variance matrix is said to be PPT, if it satisfies the above inequality.

Of particular interest are two-mode Gaussian states, whose variance matrix in block

form is given by

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{1.158}$$

where A, B, and C are 2×2 matrices. Such variance matrices can be brought to the following canonical form using local symplectic transformations, i.e., using the action of $S_{\rm local} \in Sp(2,R) \oplus Sp(2,R)$:

$$V \to V_0 = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}. \tag{1.159}$$

It turns out that in the two-mode case, PPT is a necessary and sufficient criterion for separability, and is stated in a local invariant form as [57]

$$\begin{aligned} &{\rm det}A{\rm det}B + \left(\frac{1}{4} - |{\rm det}C|\right)^2 - {\rm Tr}(AJCJBJC^TJ) \geq \frac{1}{4}({\rm det}A + {\rm det}B) \\ &\Leftrightarrow 4(ab - c_1^2)(ab - c_2^2) \geq (a^2 + b^2) + 2|c_1 c_2| - \frac{1}{4}. \end{aligned} \tag{1.160}$$

It is useful to note that Gaussian states with det(C) ≥ 0 are separable.

In the case of symmetric two-mode Gaussian states specified by parameters a, $b = a$, c_1 and c_2, it is possible to solve for the entanglement of formation. It is given by [70]

$$\begin{aligned} f(\Delta) &= c_+(\Delta){\rm log}[c_+(\Delta)] - c_-(\Delta){\rm log}[c_-(\Delta)], \quad \text{where} \\ C_\pm &= (\Delta^{-\frac{1}{2}} \pm \Delta^{\frac{1}{2}})^2, \\ \Delta &= 2\sqrt{(a - c_1)(a + c_2)}. \end{aligned} \tag{1.161}$$

The function f is a convex and decreasing function of Δ. We will study the EOF of general two-mode Gaussian states in Chapter 4.

1.9.3 Gaussian completely positive maps–Gaussian channels

A Gaussian completely positive map (or Gaussian channel) is any completely positive map which takes Gaussian states to Gaussian states. It was discussed initially in [149, 157], and more recently in [158, 159, 281–288]. In [158, 159, 281, 284, 286] single-mode Gaussian channels were discussed and classified, and issues regarding their channel capacity were studied. More recently multi-mode Gaussian channels have been classified in [287, 288].

Gaussian channels can be realised very much in the form of Eq. (1.26) and Eq. (1.27). But now one has to be careful to maintain the Gaussian character of the state. Given a Gaussian state $\hat{\rho}$, a simple way to generate the most general Gaussianity preserving map is to couple the given state to an auxiliary or ancilla Gaussian state, then evolve them together with a unitary evolution corresponding to the most general Gaussian preserving unitary, i.e., the unitary operator in this case consists of unitaries generated from canonical transformations corresponding to a symplectic group element and an arbitrary phase space translation, then trace away the auxiliary system. Clearly, all the operations done are canonical Gaussianity preserving operations. Thus the resultant state is Gaussian for every Gaussian input. The variance matrix of the resultant state is related to the variance matrix of the input state in the following way :

$$\begin{aligned} V &\rightarrow X^T V X + Y, \\ X, Y &\in R^{2n \times 2n}. \end{aligned} \tag{1.162}$$

Clearly, X acts on the variance matrix through congruence and Y is the additional noise matrix. Here X and Y are chosen so that the uncertainty principle is respected. We will have more to say on them in Chapter 6, where we derive the Kraus representation for a class of single-mode Gaussian channels.

2

Nonclassical photon statistics and entanglement

2.1 Introduction

In this Chapter we explore the relation between entanglement and nonclassicality. Nonclassicality and entanglement are two important features exhibited by continuous variable states. It is well known that nonclassicality is a prerequisite for entanglement. Two canonical manifestations of nonclassicality have been extensively studied in the quantum optics context: (1) squeezing [116–135], and (2) antibunching or sub-Poissonian fluctuation, which is a particular manifestation of nonclassical photon statistics [289–323]; these are respectively nonclassicalities of the phase-sensitive and phase-insensitive types. Whereas the former has been well explored as a source of entanglement in the context of Gaussian states, the same cannot be said in respect of the latter.

Recalling the definition of nonclassicality in Eq. (1.104), every classical state is a convex sum of coherent states, and hence separable. In general, nonclassicality does not imply entanglement, but there is a useful connection between nonclassicality and inseparability, and the beam splitter plays an important role as a bridge between these two attributes. The seminal work of Asboth *et al.* [171] shows that the output state $\hat{\rho}_{\rm out}^{(ab)}$ of a beamsplitter represented by a unitary operator U, whose input is a product state $\hat{\rho}_{\rm in}^{(ab)} = \hat{\rho}^{(a)} \otimes |0\rangle_{bb}\langle 0|$, is entangled if and only if the single-mode state $\hat{\rho}^{(a)}$ at the input is nonclassical. [Thus any measure of entanglement of the output state in this configuration is a computable measure of nonclassicality, the entanglement potential (EP) of $\hat{\rho}^{(a)}$].

For states of the form $\hat{\rho}_{\rm out}^{(ab)} = U\hat{\rho}^{(a)} \otimes |0\rangle_{bb}\langle 0|U^\dagger$, entanglement in $\hat{\rho}_{\rm out}^{(ab)}$ is clearly dictated by the nonclassicality of $\hat{\rho}^{(a)}$. Thus to test for entanglement of $\hat{\rho}_{\rm out}^{(ab)}$, one has to conclusively answer the question as to whether a given state $\hat{\rho}^{(a)}$ of a single mode of radiation field is classical or not. Furthermore, given the fact that negativity under partial transpose (NPT) is a prerequisite for distillability [45], an issue of interest is as to when is $\hat{\rho}_{\rm out}^{(ab)}$ NPT, and if so, is it distillable. If one is able to answer some of these

qualitative aspects of the entanglement of $\hat{\rho}_{\rm out}^{(ab)}$, then one may proceed further to address quantitative aspects like the measure or content of entanglement in $\hat{\rho}_{\rm out}^{(ab)}$. We attempt to answer these questions for a special class of states.

For the states $\hat{\rho}^{(a)}$ of the a-mode which are diagonal in the standard Fock basis and hence phase invariant, the issue of classicality/nonclassicality has already been settled [170], thanks to the Steiltjes moment problem [324]. We exploit this result to advantage. To begin with, we briefly review the earlier work in [170]. We introduce the notion of phase-insensitive nonclassicality and the classification of states that it leads to. This leads to demonstration of the the equivalence of the issue of classicality/nonclassicality of the phase invariant states to a Steiltjes moment problem.

We then briefly describe the two equivalent approaches in the description of the problem; one through the factorial moments $\{\gamma_m\}$, and the other through the photon number distribution or PND sequence $\{q_n\}$. We discuss two equivalent approaches to application of partial transpose in the context of continuous variables, firstly through its direct execution on the state through a chosen basis set, and then in a slightly indirect way through its manifestation on the moments using the technique introduced in Section 1.8.2.

This is followed by a brief discussion of the $SU(2)$ beamsplitter which serves our purpose as an entangling device. We show that with $\hat{\rho}_{\rm in}^{(ab)} = \hat{\rho}^{(a)} \otimes |0\rangle_{bb}\langle 0|$ as input, the two-mode state after the beam splitter is definitely NPT if $\hat{\rho}^{(a)}$ is nonclassical of the phase-insensitive type. That is, the output two-mode state is NPT if any one of a sequence of classicality conditions on the PND sequence $\{q_n\}$ is violated. We bring out the complete equivalence between PPT of $\hat{\rho}_{\rm out}^{(ab)}$ and classicality of $\hat{\rho}^{(a)}$, thus proving that for this restricted class, NPT is a necessary and sufficient criterion for entanglement. Having done so, we show that $\hat{\rho}_{\rm out}^{(ab)}$ is distillable if $\hat{\rho}^{(a)}$ violates any one of the three term classicality conditions we derive, or if $\hat{\rho}^{(a)}$ is antibunched. The establishment of the equivalence of inseparability and NPT of $\hat{\rho}_{\rm out}^{(ab)}$, ensures negativity as a good measure of entanglement in this context. We establish simple upper and lower bounds on the entanglement of formation (EOF) for two simple cases of nonclassical PND. For the simple case of $\hat{\rho}^{(a)}$ being a mixture of the ground state and the $n^{\rm th}$ Fock state, we show that we can distill more entanglement than indicated by a lower bound for EOF in [76, 325], which is based on the Terhal-Vollbrect formula [69], which returns the EOF exactly for isotropic states.

2.2 Phase-insensitive nonclassicality for a single mode of radiation field

Consider a single mode of quantised radiation field, with creation and annihilation operators $\hat{a}^\dagger$, $\hat{a}$ obeying the standard commutation relation $[\hat{a}, \hat{a}^\dagger] = \mathbb{1}$. A general state of the field is described by a density operator $\hat{\rho}$ which is hermitian nonnegative and has unit trace. According to the diagonal representation theorem [112], it can be expanded diagonally in the coherent state basis as

$$\hat{\rho}^{(a)} = \int \frac{d^2 z_a}{\pi} \phi(z_a) |z_a\rangle\langle z_a|, \tag{2.1}$$

the integration being done over the entire complex plane. Here $|z_a\rangle$ is the coherent state defined as the eigenvector of $\hat{a}$; expanded over the Fock basis it reads

$$|z_a\rangle = \exp\left(-\frac{1}{2}|z_a|^2\right) \sum_{n=0}^{\infty} \frac{{z_a}^n}{\sqrt{n!}} |n\rangle_a. \tag{2.2}$$

The weight function $\phi(z_a)$ is real on account of hermiticity of $\hat{\rho}^{(a)}$, and normalised to unit integral on account of unit trace of $\hat{\rho}^{(a)}$:

$$\int \frac{d^2 z_a}{\pi} \phi(z_a) = 1. \tag{2.3}$$

However it need not be pointwise positive on the complex plane (phase space). This fact leads to the classification of states as classical and nonclassical. States whose $\phi(z_a)$ is pointwise positive are said to be classical, and any state described by a $\phi(z_a)$ which is not pointwise positive is deemed nonclassical.

With $z_a = re^{i\theta}$, in general $\phi(z_a)$ depends on both r and θ. However for a special class of states, states which are invariant under evolution by the hamiltonian $\hat{a}^\dagger \hat{a}$, $\phi(z_a)$ is a function of $|z_a|$ alone. These states are diagonal in the Fock basis and are said to be phase-insensitive. The diagonal weight $\phi(z_a)$ describing such a state is radially symmetric in the complex plane and this radial dependence of $\phi(z_a)$ alone is sufficient to describe the state completely. We denote this family of states by D, and its elements by $\hat{\rho}_D^{(a)} \in D$:

$$\hat{\rho}_D^{(a)} = \sum_{n=0}^{\infty} p(n) |n\rangle\langle n|, \quad . \sum_{n=0}^{\infty} p(n) = 1, \quad p(n) \geq 0,$$
$$p(n) = \langle n|\hat{\rho}^{(a)}|n\rangle = \int_0^{\infty} dI_a \mathcal{P}(I_a) e^{-I_a} \frac{I_a^n}{n!}, \quad n = 0, 1, 2, \ldots$$

where

$$\phi(z_a) = \phi(|z_a|^2) = 2\pi\mathcal{P}(I_a), \quad \int_0^\infty dI_a \mathcal{P}(I_a) = 1. \tag{2.4}$$

Here $I_a = |z_a|^2$. The above relation is invertible, and $\mathcal{P}(I_a)$ can be written in terms of the $p(n)$'s:

$$\begin{aligned} \mathcal{P}(I_a) &= e^{I_a} \int_0^\infty dK \Lambda(K) J_0(2\sqrt{KI}), \\ \Lambda(K) &= \sum_{n=0}^{\infty} (-K)^n \frac{p(n)}{n!}, \end{aligned} \tag{2.5}$$

where J_0 is the Bessel function of order zero [170]. This invertible relationship implies every state $\hat{\rho}_D^{(a)}$ in the family D is completely specified by the radial weight $\mathcal{P}(I_a)$.

Alternatively, one can view these states from the perspective of observables. Any hermitian observable can always be written as a function of $\hat{a}$ and $\hat{a}^\dagger$ in normal ordered form, $F(\hat{a}^\dagger, \hat{a})$. Its expectation value in the state $\hat{\rho}^{(a)}$ is given by

$$\begin{aligned} \langle F(\hat{a}^\dagger \hat{a}) \rangle &= \mathrm{Tr}(\hat{\rho}^{(a)} F(\hat{a}^\dagger, \hat{a}) \\ &= \int \frac{d^2 z_a}{\pi} \phi(z_a) F(z_a^*, z_a). \end{aligned} \tag{2.6}$$

If in particular $F(\hat{a}^\dagger, \hat{a})$ is phase insensitive, i.e., $F(\hat{a}^\dagger, \hat{a}) = G(\hat{a}^\dagger \hat{a})$, then its expectation value does not require all the information in $\phi(z_a)$, a simpler phase averaged distribution $\mathcal{P}(I)$ suffices:

$$\begin{aligned} F(\hat{a}^\dagger e^{i\alpha}, \hat{a} e^{-i\alpha}) &= F(\hat{a}^\dagger, \hat{a}) \ \forall \ \alpha \ \in \ [0, 2\pi) \Rightarrow \\ \langle F(\hat{a}^\dagger \hat{a}) \rangle &= \int_0^\infty dI_a \mathcal{P}(I_a) F(I_a^{\frac{1}{2}}, I_a^{\frac{1}{2}}), \\ \mathcal{P}(I_a) &= \int_0^{2\pi} \frac{d\theta}{2\pi} \phi(I_a^{\frac{1}{2}} e^{i\theta}), \\ \int_0^\infty dI_a \mathcal{P}(I_a) &= 1. \end{aligned} \tag{2.7}$$

We can regard $\mathcal{P}(I_a)$ as the real marginal radial distribution obtained from the complete $\phi(z_a)$. Every phase-insensitive observable sees only the diagonal element of $\hat{\rho}^{(a)}$ in the Fock basis. This leads us to define the photon number distribution or PND of a state as these diagonal entries $p(n)$. The PND sequence $\{p(n)\}$ of a state is defined through phase averaging the state, which amounts to simply dropping the off-diagonal elements

of $\hat{\rho}^{(a)}$ in the Fock basis :

$$\begin{aligned} \hat{\rho}^{(a)} &\rightarrow \int e^{i\theta\hat{a}^\dagger\hat{a}}\hat{\rho}^{(a)}e^{-i\theta\hat{a}^\dagger\hat{a}}d\theta \\ &= \sum_{n=0}^{\infty} p(n)|n\rangle\langle n|, \\ \sum_{n=0}^{\infty} p(n) &= 1, \quad p(n) \geq 0. \end{aligned} \tag{2.8}$$

With the two quantities $\phi(z_a)$ and $\mathcal{P}(I_a)$, one can set up a three-fold classification of nonclassicality of states :

$$\begin{aligned} \hat{\rho}^{(a)} \text{ classical} &\Leftrightarrow \phi(z_a),\ \mathcal{P}(I_a) \geq 0; \\ \hat{\rho}^{(a)} \text{ weakly nonclassical} &\Leftrightarrow \phi(z_a) \ngeq 0,\ \mathcal{P}(I_a) \geq 0; \\ \hat{\rho}^{(a)} \text{ strongly nonclassical} &\Leftrightarrow \phi(z_a) \ngeq 0,\ \mathcal{P}(I_a) \ngeq 0. \end{aligned} \tag{2.9}$$

But when one is dealing with phase-invariant states $\hat{\rho}_D^{(a)}$ i.e., a PND $\{p(n)\}$ or alternatively only with phase-insensitive observables, the following classification of nonclassicality suffices :

$$\begin{aligned} \hat{\rho}^{(a)} \text{ classical} &\Leftrightarrow \mathcal{P}(I_a) \geq 0; \\ \hat{\rho}^{(a)} \text{ nonclassical} &\Leftrightarrow \mathcal{P}(I_a) \ngeq 0. \end{aligned} \tag{2.10}$$

Genuinely classical states cannot be distinguished from weakly nonclassical states at the phase insensitive level.

2.2.1 Phase-insensitive nonclassicality and moments

Given a phase-invariant state, it is useful to ask if it is classical or not, i.e., if its corresponding $\mathcal{P}(I_a)$ is pointwise positive or not. Clearly, $\mathcal{P}(I_a)$ is function on the real axis with the parameter I_a going from zero to infinity. A simple test of positivity of $\mathcal{P}(I_a)$ will be to take its overlap with a finite degree polynomial $f(I_a)$ which is itself pointwise positive in the real axis, and see if the overlap is positive, i.e.,

$$\begin{aligned} \mathcal{P}(I_a) \geq 0 \quad &\Rightarrow \quad \int_0^\infty dI_a \mathcal{P}(I_a) f(I_a) \geq 0, \\ f(I_a) &= \sum_{n=0}^{\infty} c_n {I_a}^n \geq 0, \quad 0 \leq I_a \leq \infty. \end{aligned} \tag{2.11}$$

The overlap integral returning a negative value would imply that $\mathcal{P}(I_a)$ could not have been pointwise positive and hence that the state in consideration was nonclassical. Thus every positive polynomial $f(I_a)$ results in a necessary condition for classicality. It is useful to note that the function $f(I_a)$ being pointwise positive does not require the coefficients in its expansion to be positive. A simple example is the case of square of a polynomial which has coefficients of both signs in its expansion.

There are several signatures of phase-insensitive nonclassicality. The most familiar is the Mandel's criterion, which distinguishes between super and sub-Poissonian PND's. A state with a sub-Poissonian PND is said to be antibunched. For an antibunched state

$$\begin{aligned}
\langle \Delta \hat{N}_a \rangle^2 - \langle \hat{N}_a \rangle &= \langle \hat{N}_a^2 \rangle - \langle \hat{N}_a \rangle^2 - \langle \hat{N}_a \rangle \\
&= \int_0^\infty dI_a \mathcal{P}(I_a)(I_a - \langle I_a \rangle)^2 \\
&= (\Delta I_a)^2 \leq 0, \quad \text{where} \\
\int_0^\infty dI_a \mathcal{P}(I_a)(I_a) &= \sum_{n=0}^\infty np(n) = \langle \hat{N}_a \rangle, \quad \text{and} \\
\int_0^\infty dI_a \mathcal{P}(I_a)(I_a+1)I_a &= \sum_{n=0}^\infty n^2 p(n) = \langle \hat{N}_a^2 \rangle, \qquad (2.12)
\end{aligned}$$

indicating that $\mathcal{P}(I_a) \ngeq 0$.

Clearly, Eqs. (2.11) and (2.12) are indicative of the fact that classicality/nonclassicality of a state $\hat{\rho}_D^{(a)}$ is a statement on the moments of its corresponding $\mathcal{P}(I_a)$. For a classical $\hat{\rho}_D^{(a)}$, its corresponding $\mathcal{P}(I_a)$ is a well defined probability distribution, and the set of all its moments completely specifies it. We note in the present context that a complete set of necessary and sufficient conditions expressing the content of the classicality condition $\mathcal{P}(I_a) \geq 0$ is the Stieltjes moment problem [170, 324]. The pointwise positivity of $\mathcal{P}(I_a)$ is completely captured in its moments (factorial moments) which are defined as

$$\begin{aligned}
\gamma_m &= \langle \hat{a}^{\dagger m} \hat{a}^m \rangle = \int_0^\infty dI_a \mathcal{P}(I_a) I_a^m \\
&= \sum_{n=0}^\infty \frac{(n+m)!}{n!} p(n+m)\,, \\
&\quad m = 0, 1, 2, \ldots . \qquad (2.13)
\end{aligned}$$

Given the factorial moments,

Theorem 2.1 *The necessary and sufficient conditions for a state $\hat{\rho}_D^{(a)}$ specified by facto-*

rial moments $\{\gamma_n\}$ *to be classical is*

$$M^{(N)} \geq 0, \quad \tilde{M}^{(N)} \geq 0, \quad N = 0, 1, 2, \ldots, \quad \text{where}$$
$$M^{(N)} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_N \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{N+1} \\ \vdots & & & & \\ \gamma_N & \gamma_{N+1} & \gamma_{N+2} & \cdots & \gamma_{2N} \end{pmatrix}, \quad \text{and}$$
$$\tilde{M}^{(N)} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{N+1} \\ \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_{N+2} \\ \vdots & & & & \\ \gamma_{N+1} & \gamma_{N+2} & \gamma_{N+3} & \cdots & \gamma_{2N+1} \end{pmatrix}. \tag{2.14}$$

Proof: The necessity part of the theorem follows from taking expectation value of a pointwise positive polynomial over $[0, \infty)$ with $\mathcal{P}(I_a)$. For example, consider the polynomial $f_1(I_a) = \sum_{n=0}^{N} c_n I_a^n$, where c_n are arbitrary real coefficients. If $\mathcal{P}(I)$ is pointwise nonnegative, we will expect $\langle (f_1(I_a))^2 \rangle_{\mathcal{P}} \geq 0$. That is

$$\begin{aligned} \langle (f_1(I_a))^2 \rangle_{\mathcal{P}} &= \sum_{m,n=0}^{N} c_m c_n \langle I_a^{m+n} \rangle_{\mathcal{P}} \\ &= \sum_{m,n=0}^{N} c_m c_n \gamma_{m+n} \\ &= \sum_{m,n=0}^{N} c_m c_n M_{mn}^{(N)} \geq 0 \\ &\Rightarrow \quad M^{(N)} \geq 0 \text{ for every } N. \end{aligned} \tag{2.15}$$

Similarly, defining the polynomial $f_2(I_a) = \sum_{n=0}^{N} d_n I_a^n$ and evaluating the expectation value of the nonnegative quantity $I_a (f_2(I_a))^2$ with respect to a pointwise positive function

$\mathcal{P}(I)$, we have

$$\begin{aligned}\langle I_a(f_2(I_a))^2\rangle_{\mathcal{P}} &= \sum_{m,n=0}^{N} d_m d_n \langle I_a^{m+n+1}\rangle_{\mathcal{P}} \\ &= \sum_{m,n=0}^{N} d_m d_n \gamma_{m+n+1} \\ &= \sum_{m,n=0}^{N} d_m d_n \tilde{M}^{(N)}_{mn} \geq 0 \\ &\Rightarrow \quad \tilde{M}^{(N)} \geq 0 \text{ for every } N. \end{aligned} \tag{2.16}$$

Thus nonnegativity of $\mathcal{P}(I_a)$ implies that $M^{(N)}$ and $\tilde{M}^{(N)}$ are positive semidefinite for every N. The sufficiency follows from the fact that every pointwise positive polynomial $f(I_a)$ over $[0,\infty)$ can always be written in terms of two perfect square polynomials as [324]

$$f(I_a) = (f_1(I_a))^2 + I_a(f_2(I_a))^2. \tag{2.17}$$

A simple subset of classicality conditions on the PND is given by the positivity of 2×2 sub-matrices (principal minors) of either $M^{(N)}$ or $\tilde{M}^{(N)}$. That is

$$\begin{pmatrix} \gamma_{2m} & \gamma_{m+m'} \\ \gamma_{m+m'} & \gamma_{2m'} \end{pmatrix} \geq 0, \quad \begin{pmatrix} \gamma_{2n+1} & \gamma_{n+n'+1} \\ \gamma_{n+n'+1} & \gamma_{2n'+1} \end{pmatrix} \geq 0\,. \tag{2.18}$$

The violation of the inequality for the choice $m = 0$ and $m' = 1$ corresponds to the Mandel's criterion encountered earlier in Eq. (2.12). Any classicality condition, involving the γ's can be deemed global in $\{p(n)\}$ in the sense that they involves all the $p(n)$'s in its description.

An equivalent approach to the problem is through the definition of the auxiliary distribution $\tilde{\mathcal{P}}(I_a) = \mathcal{P}(I_a)e^{-I_a}$. It is clear that if $\mathcal{P}(I_a)$ is pointwise positive, so is $\mathcal{P}(I_a)e^{-I_a}$. We have the following equivalence :

$$\text{Classical PND} \;\Leftrightarrow\; \mathcal{P}(I_a) \geq 0 \;\Leftrightarrow\; \tilde{\mathcal{P}}(I_a) \geq 0. \tag{2.19}$$

In contrast to the factorial moments, the moments of $\tilde{\mathcal{P}}(I_a)$ are well defined even if $\mathcal{P}(I_a)$

is not a well defined probability distribution, i.e.,

$$\begin{aligned} q_n = n!p(n) &= \int_0^\infty dI_a \tilde{\mathcal{P}}(I_a) I_a^n, \\ \tilde{\mathcal{P}}(I_a) &= \mathcal{P}(I_a) e^{-I_a}. \end{aligned} \tag{2.20}$$

So it is now possible to redefine the classicality requirement of $\hat{\rho}_D^{(a)}$ as the requirement of having $\tilde{\mathcal{P}}(I_a)$ to be well defined probability distribution. The necessary and sufficient conditions on $\tilde{\mathcal{P}}(I_a)$ to be true probability can now be cast in terms of moments q_n of $\tilde{\mathcal{P}}(I_a)$. Given the moment sequence $\{q_n\}$,

Theorem 2.2 *The necessary and sufficient conditions on the PND sequence* $\{q_n\}$ *in order that the associated distribution* $\tilde{\mathcal{P}}(I_a)$ *be a true probability distribution over* $[0, \infty)$ *are*

$$\begin{aligned} & L^{(N)} \geq 0, \quad \tilde{L}^{(N)} \geq 0, \quad N = 0, 1, 2, \ldots, \quad \text{where} \\ & L^{(N)} = \begin{pmatrix} q_0 & q_1 & q_2 & \cdots & q_N \\ q_1 & q_2 & q_3 & \cdots & q_{N+1} \\ \vdots & & & & \\ q_N & q_{N+1} & q_{N+2} & \cdots & q_{2N} \end{pmatrix}, \quad \text{and} \\ & \tilde{L}^{(N)} = \begin{pmatrix} q_1 & q_2 & q_3 & \cdots & q_{N+1} \\ q_2 & q_3 & q_4 & \cdots & q_{N+2} \\ \vdots & & & & \\ q_{N+1} & q_{N+2} & q_{N+3} & \cdots & q_{2N+1} \end{pmatrix}. \end{aligned} \tag{2.21}$$

This means that $L^{(N)}$ and $\tilde{L}^{(N)}$ are positive semidefinite for all N. The proof is analogous to the previous case.

A simple necessary condition on classicality of a PND $\{p(n)\}$ is a three term classicality condition, which is derived from the positivity requirement of any of the principal 2×2 minors of $L^{(N)}$ and $\tilde{L}^{(N)}$. That is,

$$\begin{pmatrix} q_{2m} & q_{m+m'} \\ q_{m+m'} & q_{2m'} \end{pmatrix} \geq 0, \quad \begin{pmatrix} q_{2n+1} & q_{n+n'+1} \\ q_{n+n'+1} & q_{2n'+1} \end{pmatrix} \geq 0\,. \tag{2.22}$$

At the next level in the hierarchy we get a five term local condition by requiring the

diagonal 3×3 blocks of $L^{(N)}$ and $\tilde{L}^{(N)}$ to be positive semidefinite, i.e.,

$$A_n = \begin{pmatrix} q_{n-2} & q_{n-1} & q_n \\ q_{n-1} & q_n & q_{n+1} \\ q_n & q_{n+1} & q_{n+2} \end{pmatrix} \geq 0 \tag{2.23}$$

An interesting situation that arises naturally in this context is that all the three term conditions contained in A_n could be satisfied but still A_n need not be positive semidefinite. Such a situation becomes interesting in the context of distillability which we discuss later. Any classicality condition involving the moments of $\tilde{\mathcal{P}}(I_a)$ can be deemed local in $\{p(n)\}$ to the conditions in terms of the factorial moments, as the former involve only a finite number of q_n's in their description.

To summarise, we have described two equivalent approaches in describing phase-insensitive nonclassicality, both of them leading to necessary and sufficient conditions on the PND $\{p(n)\}$. In the next Section we demonstrate how we could extract useful entanglement from a nonclassical PND. We convert the failure of the conditions given in Eq. (2.14) and Eq. (2.2) into witnessing of NPT entanglement.

2.3 Entanglement, Partial Transpose and Nonclassicality

We now turn to the partial transpose (PT) operation. The density matrix $\hat{\rho}_{AB}$ of a bipartite system $S = A + B$ is an operator on the tensor product Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$. Any density operator $\hat{\rho}_{AB}$ obeys three defining conditions: $\hat{\rho}_{AB}^{\dagger} = \hat{\rho}_{AB}$, $\mathrm{Tr}\,\hat{\rho}_{AB} = 1$, and $\hat{\rho}_{AB} \geq 0$. With respect to chosen orthonormal bases (ONB) $\{|j\rangle\}$, $\{|\alpha\rangle\}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, the product states $\{|j,\alpha\rangle \equiv |j\rangle\otimes\}|\alpha\rangle\}$ give an ONB for $\mathcal{H}_A \otimes \mathcal{H}_B$. The partial transpose of $\hat{\rho}_{AB}$ is the operator $\hat{\rho}_{AB}^{PT}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ defined in this ONB by the rule

$$\langle j,\alpha|\hat{\rho}_{AB}^{PT}|k,\beta\rangle = \langle j,\beta|\hat{\rho}_{AB}|k,\alpha\rangle. \tag{2.24}$$

The above definition of partial transpose is tied to the given choices of ONB's in $\mathcal{H}_A$ and $\mathcal{H}_B$, but the question of whether $\hat{\rho}_{AB}^{PT}$ is a valid density operator or not is independent of this choice.

In general $\hat{\rho}_{AB}^{PT}$ could fail to be positive, and the key result is: if $\hat{\rho}_{AB}^{PT}$ is not positive, then $\hat{\rho}_{AB}^{PT}$ is definitely an entangled state. In this case we may say that $\hat{\rho}_{AB}^{PT}$ is negative under partial transpose (NPT).

Any test which establishes the nonpositivity of $\hat{\rho}_{AB}^{PT}$ will suffice to reach the conclusion that $\hat{\rho}_{AB}$ is entangled. We may for instance be able to find an operator $\hat{A}$ of the total system S such that the 'expectation value' of the *positive* operator $\hat{A}^{\dagger}\hat{A}$ in $\hat{\rho}_{AB}^{PT}$ is negative.

In that case we can conclude,

$$\begin{aligned} \mathrm{Tr}\,(\hat{\rho}_{AB}^{PT}\hat{A}^{\dagger}\hat{A}) \quad &< \quad 0, \quad \text{for some} \quad \hat{A} \\ &\Longrightarrow \quad \hat{\rho}_{AB} \quad \text{entangled.} \end{aligned} \tag{2.25}$$

Alternatively, and in a sense more directly, we may be able to find some principal submatrix of the matrix $(\langle j,\alpha|\hat{\rho}_{AB}^{PT}|k,\beta\rangle)$ representing $\hat{\rho}_{AB}^{PT}$, i.e., a sub-matrix formed by intersections of any subset of *rows of this matrix and the corresponding columns*, such that this sub-matrix is not positive definite. Then again we can conclude,

$$\begin{aligned} &\text{Some principal submatrix of} \quad \hat{\rho}_{AB}^{PT} \ngeq 0 \\ &\Rightarrow \quad \hat{\rho}_{AB} \quad \text{entangled.} \end{aligned} \tag{2.26}$$

We will use both strategies in the sequel.

Our aim is to connect presence of entanglement *directly and transparently* to nonclassicality, atleast for a class of states of the radiation field. We therefore specialise the subsystem a and b to be two distinct mutually orthogonal single mode radiation fields, with respective creation and annihilation operator pairs $\hat{a}^{\dagger}$, $\hat{a}$ and $\hat{b}^{\dagger}$, $\hat{b}$ acting on Hilbert spaces $\mathcal{H}^a$, $\mathcal{H}^b$, the only non vanishing commutators being $[\hat{a},\hat{a}^{\dagger}] = [\hat{b},\hat{b}^{\dagger}] = \mathbb{1}$. The Fock or photon number states for the two modes provide ONB's for $\mathcal{H}^a$, $\mathcal{H}^b$ respectively,

$$|n\rangle = (n!)^{-1/2}(\hat{a}^{\dagger})^n|0\rangle_a\,, \quad |m\rangle = (m!)^{-1/2}(\hat{b}^{\dagger})^m|0\rangle_b\,, \quad n,m \;=\; 0,1,2,\ldots, \tag{2.27}$$

and the product states $|n,m\rangle \equiv |n\rangle_a \otimes |m\rangle_b$ form an ONB for $\mathcal{H}^a \otimes \mathcal{H}^b$. For definiteness and convenience we will implement the PT operation in the Fock or photon number basis, so that the transition $\hat{\rho}^{(ab)} \rightarrow \hat{\rho}^{(ab)PT}$ is defined by,

$$\langle n^{'}m^{'}|\hat{\rho}^{(ab)PT}|n,m\rangle \equiv \langle n^{'},m|\hat{\rho}^{(ab)}|n,m^{'}\rangle. \tag{2.28}$$

Then we easily find

$$\mathrm{Tr}\,(\hat{\rho}^{(ab)PT}\hat{a}^{\dagger j}\hat{a}^k\hat{b}^{\dagger l}\hat{b}^m) \;=\; \mathrm{Tr}\,(\hat{\rho}^{(ab)}\hat{a}^{\dagger j}\hat{a}^k\hat{b}^{\dagger m}\hat{b}^l). \tag{2.29}$$

The key point here is that in the Fock basis the operators $\hat{b}$, $\hat{b}^{\dagger}$ are represented by *real* matrices, so the transposition operation for the matrix of $\hat{b}^{\dagger l}\hat{b}^m$ coincides with hermitian conjugation. This result will be very useful in the sequel [227, 228].

2.4 Conversion of nonclassicality into entanglement – the Beamsplitter

We begin by recalling the action of a general linear lossless optical element on any input state of two mode a-b radiation field. Such an element is determined by some matrix $u(\theta, \phi, \eta)$ of the unitary unimodular groups $SU(2)$ [326],

$$u(\theta, \phi, \eta) = \begin{pmatrix} e^{i\phi}\cos\theta & e^{i\eta}\sin\theta \\ -e^{-i\eta}\sin\theta & e^{-i\phi}\cos\theta \end{pmatrix} \tag{2.30}$$

[More generally, including an overall phase, we have a matrix of $U(2)$]. The action of this optical element or gadget on the mode operators is by unitary operator $U(\theta, \phi, \eta)$ on the two-mode Hilbert space $\mathcal{H}^a \otimes \mathcal{H}^b$:

$$U(\theta, \phi, \eta) \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} U(\theta, \phi, \eta)^{-1} = u(\theta, \phi, \eta) \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}. \tag{2.31}$$

$U(\theta, \phi, \eta)$ can be written in the following form

$$\begin{aligned} U(\theta, \phi, \eta) &= e^{-i(\phi-\eta)\hat{L}_3} e^{-i\theta\hat{L}_2} e^{-i(\phi+\eta)\hat{L}_3}, \quad \text{where} \\ \hat{L}_2 &= \frac{1}{2i}(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger), \\ \hat{L}_3 &= \frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}). \end{aligned} \tag{2.32}$$

Such a transformation preserves the canonical commutation relations. The case of 50:50 beamsplitter (B-S) corresponds to $\theta = \pi/4$, $\phi = \eta = 0$, with $U_0 = e^{-i\frac{\pi}{4}\hat{L}_2}$ for the corresponding unitary operator, in which case the mode operators transform as

$$\begin{aligned} U_0\hat{a}{U_0}^{-1} &= \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}), \quad U_0\hat{b}{U_0}^{-1} = \frac{1}{\sqrt{2}}(\hat{b} - \hat{a}), \\ {U_0}^{-1}\hat{a}U_0 &= \frac{1}{\sqrt{2}}(\hat{a} - \hat{b}), \quad {U_0}^{-1}\hat{b}U_0 = \frac{1}{\sqrt{2}}(\hat{b} + \hat{a}). \end{aligned} \tag{2.33}$$

We will use only such a B-S for the purpose of demonstration, but our results hold for the general $U(2)$ beamsplitter. In Fig. (2.1) we give a schematic depiction of the action of a 50:50 beamsplitter.

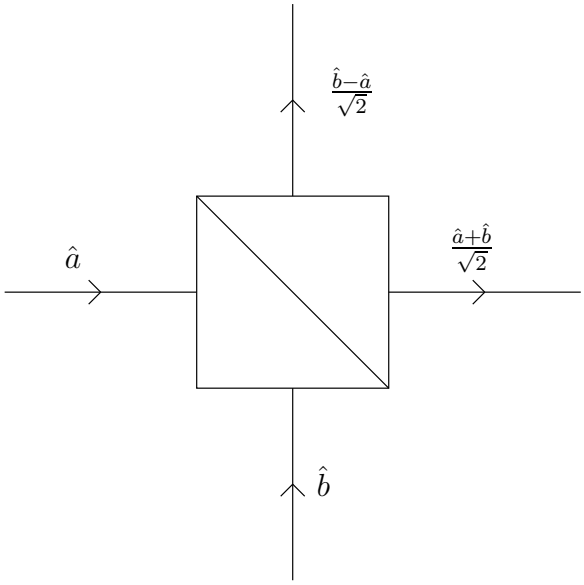

Figure 2.1: The action of a 50:50 beamsplitter on the mode operators.

2.5 The case of phase-insensitive nonclassicality

In this Section we demonstrate through selected tests, the equivalence of phase-insensitive nonclassicality at the input to the violation of positivity under partial transpose for the output state. We follow our first strategy, Eq. (2.25), to demonstrate the equivalence. Let us take the input state to be in the product form,

$$\hat{\rho}_{\rm in}^{(ab)} = \hat{\rho}^{(a)} \otimes |0\rangle_{bb}\langle 0|, \tag{2.34}$$

with the b-mode in the vacuum state. After passing through the beamsplitter, we have the output state

$$\hat{\rho}_{\rm out}^{(ab)} = U_0 \hat{\rho}_{\rm in}^{(ab)} U_0^{-1}. \tag{2.35}$$

We now perform the PT operation on the state $\hat{\rho}_{\rm out}^{(ab)}$ leading to operator $\hat{\rho}_{\rm out}^{(ab)PT}$, and test for its positive semidefiniteness. Making the choice

$$\hat{A} = \sum_{n=0}^{N} c_n \hat{a}^n \hat{b}^n, \tag{2.36}$$

and taking the expectation value of $\hat{A}^\dagger\hat{A}$ with respect to the partially transposed output state, we have

$$\mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)PT}\hat{A}^\dagger\hat{A}) = \sum_{n,m=0}^{N} c_m^* c_n \mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)PT}\hat{a}^{\dagger m}\hat{a}^n\hat{b}^{\dagger m}\hat{b}^n). \tag{2.37}$$

The expectation values of operators on the partially transposed state can now be related to expectation values of operators on $\hat{\rho}^{(a)}$, i.e.,

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)PT}\hat{a}^{\dagger m}\hat{a}^n\hat{b}^{\dagger m}\hat{b}^n) &= \mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)}\hat{a}^{\dagger m}\hat{a}^n\hat{b}^{\dagger n}\hat{b}^m) \\
&= \mathrm{Tr}(\hat{\rho}_{\mathrm{in}}^{(ab)}U_0^{-1}\hat{a}^{\dagger m}\hat{a}^n\hat{b}^{\dagger n}\hat{b}^m U_0) \\
&= \frac{1}{2^{2m+2n}}\mathrm{Tr}(\hat{\rho}_{\mathrm{in}}^{(ab)}(\hat{a}^\dagger-\hat{b}^\dagger)^m(\hat{a}-\hat{b})^n(\hat{a}^\dagger+\hat{b}^\dagger)^n(\hat{a}+\hat{b})^m) \\
&= \frac{1}{2^{2m+2n}}\mathrm{Tr}(\hat{\rho}^{(a)}\hat{a}^{\dagger(m+n)}\hat{a}^{(m+n)}).
\end{aligned} \tag{2.38}$$

Here we have used the fact that the b-mode is in the ground state initially. Thus we have

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)PT}\hat{A}^\dagger\hat{A}) &= \sum_{n,m=0}^{N} c_m^* c_n \mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)PT}\hat{a}^{\dagger m}\hat{a}^n\hat{b}^{\dagger m}\hat{b}^n) \\
&= \sum_{n,m=0}^{N} \frac{c_m^*}{2^{2m}}\frac{c_n}{2^{2n}}\mathrm{Tr}(\hat{\rho}^{(a)}\hat{a}^{\dagger(m+n)}\hat{a}^{(m+n)}) \\
&= \sum_{n,m=0}^{N} \frac{c_m^*}{2^{2m}}\frac{c_n}{2^{2n}}\gamma_{m+n}(\hat{\rho}^{(a)}) \\
&= \sum_{n,m=0}^{N} \frac{c_m^*}{2^{2m}}\frac{c_n}{2^{2n}}M_{mn}^{(N)}(\hat{\rho}^{(a)}),
\end{aligned} \tag{2.39}$$

which implies that if $M^{(N)}(\hat{\rho}^{(a)}) \not\geq 0$, then definitely $\hat{\rho}_{\mathrm{out}}^{(ab)PT} \not\geq 0$. Similarly for the choice

$$\hat{A} = \sum_{n=0}^{N} d_n \hat{a}^{(n+1)}\hat{b}^n, \tag{2.40}$$

we have

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)PT}\hat{A}^{\dagger}\hat{A}) &= \sum_{n,m=0}^{N} d_m^* d_n \mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)PT}\hat{a}^{\dagger(m+1)}\hat{a}^{(n+1)}\hat{b}^{\dagger(m+1)}\hat{b}^{n}) \\
&= \sum_{n,m=0}^{N} \frac{d_m^*}{2^{2m}}\frac{d_n}{2^{2n}}\mathrm{Tr}(\hat{\rho}^{(a)}\hat{a}^{\dagger(m+n+1)}\hat{a}^{(m+n+1)}) \\
&= \sum_{n,m=0}^{N} \frac{d_m^*}{2^{2m+1}}\frac{d_n}{2^{2n+1}}\gamma_{m+n}(\hat{\rho}^{(a)}) \\
&= \sum_{n,m=0}^{N} \frac{d_m^*}{2^{2m+1}}\frac{d_n}{2^{2n+1}}\tilde{M}_{mn}^{(N)}(\hat{\rho}^{(a)}),
\end{aligned} \tag{2.41}$$

which implies that if $\tilde{M}^{(N)}(\hat{\rho}^{(a)}) \not\geq 0$, then definitely $\hat{\rho}_{\mathrm{out}}^{(ab)PT} \not\geq 0$. Thus if the input in the a-mode $\hat{\rho}^{(a)}$ possesses any form of phase-insensitive nonclassicality, the output of the beamsplitter is NPT entangled. In particular if the input is antibunched, the output is NPT entangled.

2.6 The case of general nonclassical PND

Consider a state diagonal in the Fock basis which is completely determined by its PND probabilities $\{p(n)\}$,

$$\hat{\rho}_D^{(a)} = \sum_{n=0}^{\infty} p(n)|n\rangle\langle n|\,. \tag{2.42}$$

Now pass the two-mode state

$$\hat{\rho}_{\mathrm{in}}^{(ab)} = \hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|\,, \tag{2.43}$$

through a beamsplitter. The output two-mode state is

$$\begin{aligned}
\hat{\rho}_{\mathrm{out}}^{(ab)} = U_0\hat{\rho}_{\mathrm{in}}^{(a)}{U_0}^{-1} &= U_0 \sum_{n=0}^{\infty} \frac{p(n)}{n!}(\hat{a}^{\dagger})^n|0,0\rangle\langle 0,0|(\hat{a})^n{U_0}^{-1} \\
&= \sum_{n=0}^{\infty} \frac{p(n)}{2^n n!}(\hat{a}^{\dagger}+\hat{b}^{\dagger})^n|0,0\rangle\langle 0,0|(\hat{a}+\hat{b})^n \\
&= \sum_{n=0}^{\infty} \frac{p(n)n!}{2^n} \sum_{r,s=0}^{n} \frac{|r,n-r\rangle\langle s,n-s|}{\sqrt{r!(n-r)!s!(n-s)!}}\,.
\end{aligned} \tag{2.44}$$

The general matrix element of this density matrix is

$$\langle n^{'}, m^{'}|\hat{\rho}_{\rm out}^{(ab)}|n,m\rangle = \delta_{n^{'}+m^{'},n+m}\frac{(n+m)!p(n+m)}{2^{n+m}\sqrt{n^{'}!m^{'}!n!m!}}. \tag{2.45}$$

The partially transposed output state is given by

$$\hat{\rho}_{\rm out}^{(ab)PT} = \sum_{n=0}^{\infty}\frac{p(n)n!}{2^n}\sum_{r,s=0}^{n}\frac{|r,n-s\rangle\langle s,n-r|}{\sqrt{r!(n-r)!s!(n-s)!}}, \tag{2.46}$$

therefore the matrix elements of $\hat{\rho}_{\rm out}^{(ab)PT}$ are given by

$$\begin{aligned}\langle n^{'}, m^{'}|\hat{\rho}_{\rm out}^{(ab)PT}|n,m\rangle &= \delta_{n^{'}+m,n+m^{'}}\frac{q_{n+m^{'}}}{2^{n+m^{'}}\sqrt{n^{'}!m^{'}!n!m!}} \\ &= \delta_{n^{'}-m^{'},n-m}\frac{q_{n+m^{'}}}{2^{n+m^{'}}\sqrt{n^{'}!m^{'}!n!m!}}.\end{aligned} \tag{2.47}$$

We note that $\hat{\rho}_{\rm out}^{(ab)PT}$ commutes with the difference operator $\hat{N}_a - \hat{N}_b$, i.e.,

$$\begin{aligned}(\hat{N}_a - \hat{N}_b)\,\hat{\rho}_{\rm out}^{(ab)PT} &= \hat{\rho}_{\rm out}^{(ab)PT}\,(\hat{N}_a - \hat{N}_b) \\ &= \sum_{n=0}^{\infty}\frac{p(n)n!}{2^n}\sum_{r,s=0}^{n}\frac{(s+r-n)|r,n-s\rangle\langle s,n-r|}{\sqrt{r!(n-r)!s!(n-s)!}}.\end{aligned} \tag{2.48}$$

Thus the operator $\hat{\rho}_{\rm out}^{(ab)PT}$ is simply a direct sum of operators, each specified by the energy difference δ. Thus δ can be employed to label these invariant subspaces :

$$\hat{\rho}_{\rm out}^{(ab)PT} = \sum_{\delta}\oplus\hat{\rho}_{\rm out}^{(ab)PT(\delta)}. \tag{2.49}$$

A test for entanglement would be to look for violation of positivity in any of these subspaces.

It is clear that δ can take both negative and positive values. We define δ to be positive if the b-mode has more number of photons. The matrix elements of $\hat{\rho}_{\rm out}^{(ab)PT}$ in the subspace specified by δ is given by

$$\begin{aligned}\langle n^{'}, n^{'}+\delta|\hat{\rho}_{\rm out}^{(ab)PT}|n,n+\delta\rangle &= \langle n^{'}+\delta, n^{'}|\hat{\rho}_{\rm out}^{(ab)PT}|n+\delta,n\rangle \\ &= \frac{q_{n+n^{'}+\delta}}{2^{n+n^{'}+\delta}\sqrt{n^{'}!(n^{'}+\delta)!n!(n+\delta)!}}.\end{aligned} \tag{2.50}$$

With a slight modification of notation let us denote by $\hat{\rho}_{\rm out}^{(ab)PT(\delta)}$ the restriction of $\hat{\rho}_{\rm out}^{(ab)PT}$ to the subspace $\hat{N}_a - \hat{N}_b = \delta \geq 0$, and by $\hat{\rho}_{\rm out}^{(ab)PT(-\delta)}$ the restriction to $\hat{N}_b - \hat{N}_a = \delta \geq 0$.

With the definition of the matrix elements, it is easy to see that the operators $\hat{\rho}_{\rm out}^{(ab)PT(\delta)}$ and $\hat{\rho}_{\rm out}^{(ab)PT(-\delta)}$ for a given choice of δ are

$$\begin{aligned}\hat{\rho}_{\rm out}^{(ab)PT(\delta)} &= \sum_{n,n'} \frac{q_{n+n'+\delta}|n', n'+\delta\rangle\langle n, n+\delta|}{2^{n+n'+\delta}\sqrt{n'!(n'+\delta)!n!(n+\delta)!}}, \\ \hat{\rho}_{\rm out}^{(ab)PT(-\delta)} &= \sum_{n,n'} \frac{q_{n+n'+\delta}|n'+\delta, n'\rangle\langle n+\delta, n|}{2^{n+n'+\delta}\sqrt{n'!(n'+\delta)!n!(n+\delta)!}}. \end{aligned} \tag{2.51}$$

With the appropriate choices of basis vectors, both of them are given by

$$\begin{aligned}\hat{\rho}_{\rm out}^{(ab)PT(\pm\delta)} &= \mathrm{diag}\left(\frac{1}{2^{n'+\frac{\delta}{2}}\sqrt{n'!(n'+\delta)!}}\right) \times \\ &\quad \begin{pmatrix} q_\delta & q_{\delta+1} & q_{\delta+2} & \cdot\cdot & q_{\delta+N} & \cdot \\ q_{\delta+1} & q_{\delta+2} & q_{\delta+3} & \cdot\cdot & q_{\delta+N+1} & \cdot \\ \vdots & & & & & \\ q_{\delta+N} & q_{\delta+N+1} & q_{\delta+N+2} & \cdot\cdot & q_{\delta+2N} & \cdot \\ \vdots & & & & \vdots & \end{pmatrix} \times \\ &\quad \mathrm{diag}\left(\frac{1}{2^{n+\frac{\delta}{2}}\sqrt{n!(n+\delta)!}}\right). \end{aligned} \tag{2.52}$$

In particular, for the choice of $\delta = 0$ and 1, we have

$$\begin{aligned}
\hat{\rho}_{\rm out}^{(ab)PT(0)} &= {\rm diag}\left(\frac{1}{2^{n'}n'!}\right)\times \\
&\quad \begin{pmatrix} q_0 & q_1 & q_2 & \cdot\cdot & q_N & \cdot \\ q_1 & q_2 & q_3 & \cdot\cdot & q_{N+1} & \cdot \\ \vdots & & & & & \\ q_N & q_{N+1} & q_{N+2} & \cdot\cdot & q_{2N} & \cdot \\ \vdots & & & & \vdots & \end{pmatrix} {\rm diag}\left(\frac{1}{2^n n!}\right), \\
\hat{\rho}_{\rm out}^{(ab)PT(1)} &= {\rm diag}\left(\frac{1}{2^{n'+\frac{1}{2}}\sqrt{n'!(n'+1)!}}\right)\times \\
&\quad \begin{pmatrix} q_1 & q_2 & q_3 & \cdot\cdot & q_{N+1} & \cdot \\ q_2 & q_3 & q_4 & \cdot\cdot & q_{N+2} & \cdot \\ \vdots & & & & & \\ q_{N+1} & q_{N+2} & q_{N+3} & \cdot\cdot & q_{2N+1} & \cdot \\ \vdots & & & & \vdots & \end{pmatrix}\times \\
&\quad {\rm diag}\left(\frac{1}{2^{n+\frac{1}{2}}\sqrt{n!(n+1)!}}\right).
\end{aligned} \tag{2.53}$$

It is clear from Eqs. (2.52) and (2.53) that positivity of any $\hat{\rho}_{\rm out}^{(ab)PT(\delta)}$ is subsumed in the positivity of either $\hat{\rho}_{\rm out}^{(ab)PT(0)}$ or $\hat{\rho}_{\rm out}^{(ab)PT(1)}$, depending on whether δ is even or odd, which is in turn equivalent to the positivity of $L^{(N)}$ or $\tilde{L}^{(N)}$ for arbitrary N. Taking into account all N, we have the following equivalence

$$\begin{aligned}
&L^{(N)} \geq 0 \ \text{ and } \ \tilde{L}^{(N)} \geq 0 \ \Leftrightarrow \ \hat{\rho}_{\rm out}^{(ab)PT} \geq 0, \\
&L^{(N)} \ngeq 0 \ \text{ or } \ \tilde{L}^{(N)} \ngeq 0 \ \Leftrightarrow \ \hat{\rho}_{\rm out}^{(ab)PT} \ngeq 0.
\end{aligned} \tag{2.54}$$

Our conclusions may be summarised in the following theorem :

Theorem 2.3 *With $\hat{\rho}_{\rm in}^{(ab)} = \hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|$ as input to a beamsplitter, the output two-mode state is NPT entangled if the phase-insensitive $\hat{\rho}_D^{(a)}$ at the input is nonclassical. If the output is PPT, then it is separable, and the input $\hat{\rho}_D^{(a)}$ is classical.*

It follows that PPT bound entangled state can never arise at the output of the beamsplitter in this situation. Stated differently

Theorem 2.4 *For states of the form $\hat{\rho}_{\rm out}^{(ab)} = U_0\ (\hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|)\ U_0^{-1}$, the partial transpose map proves to be a necessary and sufficient test for entanglement.*

Proof: If $\hat{\rho}_{\rm out}^{(ab)PT}$ is not positive, then $\hat{\rho}_{\rm out}^{(ab)}$ is NPT and hence entangled. On the other hand by the equivalence that we have established above if $\hat{\rho}_{\rm out}^{(ab)PT}$ is positive semidefinite, then both $L^{(N)}$ and $\tilde{L}^{(N)}$ are positive definite for arbitrary N, and hence $\hat{\rho}_{\rm in}^{(ab)}$ is classical by Theorem (2.2). It follows that $\hat{\rho}_{\rm out}^{(ab)}$ is classical, and hence separable.

Corollary 2.1 *For states of the form* $\hat{\rho}_{\rm out}^{(ab)} = U_0\ (\hat{\rho}_D^{(a)} \otimes |\alpha\rangle_{bb}\langle\alpha|)\ U_0^{-1}$, *where* $|\alpha\rangle$ *is a coherent state, partial transpose gives a necessary and sufficient test for entanglement. That is, states of this form can never be PPT bound entangled.*

Proof: If $\hat{\rho}_D^{(a)}$ was classical, the output is classical, and hence separable. On the other hand if $\hat{\rho}_D^{(a)}$ is nonclassical, the output is definitely NPT entangled, as may be seen by carrying out partial transpose test in the b-mode with respect to the displaced Fock basis of the b-mode as compared to the standard Fock basis in the earlier case.

The proof outlined above can be easily translated to the case of general beamsplitter represented by an element of $U(2)$, even though we have demonstrated the proof only in the case of the 50:50 beamsplitter. The result in this Section is remarkable in the sense that one has been able to relate matrix conditions on nonclassicality to matrix conditions on the violation of positivity under partial transpose in a transparent manner. This is a manifest example in continuous variable entanglement theory where partial transpose has proved to capture entanglement completely, in a non-Gaussian context.

2.7 Nonclassical PND and distillable entanglement

Having produced NPT entangled output, it is pertinent to ask if the output is distillable, NPT being a necessary condition for distillability [45]. In this Section we demonstrate that $\hat{\rho}_{\rm out}^{(ab)}$ is distillable if the a-mode input state $\hat{\rho}_D^{(a)}$ violates one of the three term local classicality conditions. The strategy we use is the one outlined in [45], where distillability is defined in the following terms. *A state* $\hat{\rho}$ *is distillable if and only if, for some two dimensional projectors* P, Q *and for some number* n, *the* 2×2 *state* $P \otimes Q\hat{\rho}^{\otimes n}P \otimes Q$ *is entangled.* Such a definition is well motivated from the fact that for 2×2 systems, there are well defined protocols to distill entanglement from every entangled state [37]. In our case we show one copy distillability by projecting the output state $\hat{\rho}_{\rm out}^{(ab)}$ into an appropriate 2×2 subspace.

Consider the following 2×2 dimensional projectors

$$P \otimes Q = (|n\rangle\langle n| + |m\rangle\langle m|)_a \otimes (|n+\delta\rangle\langle n+\delta| + |m+\delta\rangle\langle m+\delta|)_b. \tag{2.55}$$

In this 2×2 subspace, the partially transposed output reads as

$$P \otimes Q\, \hat{\rho}_{\rm out}^{(ab)PT}\, P \otimes Q \;=\; D \begin{pmatrix} q_{2n+\delta} & q_{n+m+\delta} \\ q_{n+m+\delta} & q_{2m+\delta} \end{pmatrix} D, \text{ where}$$

$$D \;=\; \begin{pmatrix} \frac{1}{2^{n+\frac{\delta}{2}}\sqrt{n!(n+\frac{\delta}{2})!}} & 0 \\ 0 & \frac{1}{2^{m+\frac{\delta}{2}}\sqrt{m!(m+\frac{\delta}{2})!}} \end{pmatrix}. \qquad (2.56)$$

Clearly the positivity of $\hat{\rho}_{\rm out}^{(ab)PT}$ in this particular subspace is clearly determined by the three term classicality condition on the a-mode input state $\hat{\rho}_D^{(a)}$.

Theorem 2.5 *The output $\hat{\rho}_{\rm out}^{(ab)} = U_0\ (\hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|)\ U_0^{-1}$ is one-copy distillable if $\hat{\rho}_D^{(a)}$, fully specified by the PND $\{p(n)\}$, violates any one of the local three term classicality conditions.*

Though we have demonstrated distillability through the violation of a local three term classicality conditions on $\{p(n)\}$, the issue of distillability is much richer. There are states which respect all local three-term classicality conditions, but are nevertheless nonclassical by violation of a higher order nonclassicality condition, say a five term local classicality condition. Though such a violation would imply the violation of positivity of $\hat{\rho}_{\rm out}^{(ab)}$, this violation is not captured by a simple rank 2 state as earlier, indicating room for further exploration of the issue of distillability in the present context.

2.8 Distillable entanglement from antibunching

In the previous Section, we discussed the issue of distillability arising from the violation of a classicality condition in $\{p(n)\}$ which is local in n. In this Section,

we explore the possibility of distillable entanglement through the violation of three-term classicality condition in $\{\gamma_n\}$. Since antibunching is a viable resource within current technology, we explore the possibility of extracting distillable entanglement from antibunching, violation of a classicality condition involving γ_n for $n = 0, 1, 2$. The proof is then extended to more general nonclassicality conditions involving three γ_n's. Our strategy is as follows. For a given $\hat{\rho}_D^{(a)}$, we initially assume that none of the three term conditions on $\{p(n)\}$ is violated. We show that such a state can never be antibunched. Hence at least one three term conditions in $\{p(n)\}$ has to be violated for $\hat{\rho}_D^{(a)}$ to be antibunched, implying distillable entanglement at the output.

Our idea is to use the relation between the factorial moments $\{\gamma_n\}$ and the PND

probabilities $\{p(n)\}$ stated in Eq. (2.13) :

$$\gamma_n = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} p(n+m) = \sum_{n=0}^{\infty} \frac{q_{n+m}}{n!}. \tag{2.57}$$

It is easy to see that the most general $\{q_n\}$ which meets all the three-term classicality conditions is

$$\begin{aligned} & q_0 = a, \;\; q_1 = b, \;\; q_2 = \alpha_1 \frac{b^2}{a}, \;\; q_3 = \alpha_2 \alpha_1^2 \frac{b^3}{a^2}, \;\; q_4 = \alpha_3 \alpha_2^2 \alpha_1^3 \frac{b^4}{a^3} \\ & \cdots\cdots \quad q_n = \alpha_{n-1} \alpha_{n-1}^2 \cdots \alpha_2^{n-2} \alpha_1^{n-1} \frac{b^n}{a^{n-1}}, \cdots, \\ & a > 0, \;\; b > 0, \;\; \alpha_j \geq 1, \quad j = 1, 2, 3, \cdots . \end{aligned} \tag{2.58}$$

Clearly, $b = 0$ if and only if $\hat{\rho}_D^{(a)} = |0\rangle\langle 0|$, and all the α_j's are unity for a coherent state. It is readily verified that all the local three-term conditions are satisfied. It is useful to define the sequence of numbers $\{\lambda_n\}$ as functions of the α's :

$$\begin{aligned} & \lambda_0 = 1 \;\; \lambda_1 = \alpha_1, \;\; \lambda_2 = \alpha_2 \alpha_1^2, \;\; \cdots, \\ & \lambda_n = (\alpha_n \alpha_{n-1}^2 \cdots \alpha_2^{n-1} \alpha_1^n), \cdots \end{aligned} \tag{2.59}$$

With this definition, it is easy to see that the set of inequalities on $\{\lambda_n\}$

$$\lambda_{n-r+1} \lambda_{r-1} \geq \lambda_{n-r} \lambda_r \tag{2.60}$$

is equivalent to the set of all three-term classicality conditions on $\{q_n\}$. We can now write

$$\begin{aligned} \gamma_0 \gamma_2 - \gamma_1^2 = \sum_{n=0}^{\infty} \frac{b^{n+2}}{a^n} \Bigg[& \frac{(\lambda_{n+1} - \lambda_n)}{n!} + \\ & \sum_{r < \frac{n}{2}} (\lambda_{n-r+1} \lambda_{r-1} - \lambda_{n-r} \lambda_r) \left(\frac{1}{(n-r)! r!} - \frac{1}{(n-r+1)!(r-1)!} \right) \Bigg]. \end{aligned} \tag{2.61}$$

Clearly $\gamma_0 \gamma_2 - \gamma_1^2$ is positive if all the local three term conditions are satisfied. Thus a state cannot satisfy all the local three term classicality conditions on $\{q_n\}$ and still be antibunched. If a PND is antibunched, it definitely violates at least one of the three term classicality conditions. Hence the output generated from the beamsplitter from an antibunched $\hat{\rho}_D^{(a)}$ is necessarily distillable.

Theorem 2.6 *The output* $\hat{\rho}_{\rm out}^{(ab)} = U_0 \; (\hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|) \; U_0^{-1}$ *is distillable if* $\hat{\rho}_D^{(a)}$, *specified*

by the PND $\{p(n)\}$, is antibunched.

The proof can be extended along similar lines to demonstrate distillable entanglement from violation of any three-term classicality condition in the factorial moments $\{\gamma_n\}$.

2.9 Bounds on the EOF

It is well known that determining the EOF of a given state is an extremely difficult problem even in the finite dimensional case. Only in very special cases has one been able to solve for a closed form expression [67, 69–72] for EOF. In the continuous variable context progress has been achieved only in the case of two-mode Gaussian states [70, 327]. The issue of establishing bounds on the EOF for finite dimensional systems has been discussed in [76, 328–330]. In [328] the possibility of establishing bounds on EOF directly from experiments has been discussed. An analytic expression for a lower bound on the entanglement of formation for arbitrary $m \otimes n$ dimensional systems is given in [76, 325]. In [330] an improvement over the lower bound in [76, 325] is demonstrated. However this improvement holds only in the context of $4 \otimes n$ dimensional systems.

Our motivation here is to establish simple lower and upper bounds on the EOF on a class of entangled non-Gaussian states that could arise from a nonclassical PND. Our guiding principle tool is the fact that the average entanglement does not increase under LOCC [37]. Such an approach has already been used to estimate entanglement in continuous variable states [262, 331]. In [262] the estimation was done at the phase space level, and in [331] at the configuration space level. However the estimation in [331], being tied to the Wootters formula, has a drawback that it cannot estimate more than one ebit of entanglement, even if more was present. We work with the Fock basis since it seems to be the most appropriate choice in the present context.

Our scheme of things is as follows. By restricting ourselves to local Von Neumann measurements we evaluate the entanglement in each of the orthogonal subspaces. These subspaces are the probable outcomes of the local measurements, the average gives us a lower bound on the entanglement of formation. That we are able to calculate the entanglement in each of these subspaces is ensured by a careful choice of local measurements, so as to ensure that the outcome is a pure entangled state. We explore this procedure firstly on a finite dimensional entangled state generated from a mixture of the ground state and the n^{th} Fock state by coupling to an auxiliary mode in the ground state and passing through a 50:50 beamsplitter. In the process we explicitly demonstrate that we can distill more amount of entanglement for the cases $n > 6$ than given by the lower bound on EOF in [76], which uses the Terhal-Vollbrect formula [69], and is calculated

using the trace norm of the state subjected to either partial transpose or the realignment criterion [75, 332, 333]. We then go on to establish bounds on the EOF of a class of genuine continuous variable entangled states generated from a PND corresponding to a special superposition of coherent states, where we are able to provide an analytic expression. That we are able to do so, is again due to a careful choice of local measurement observables. Such an evaluation does not involve any optimisation whatsoever in terms of the choice of the local measurements involved or of additional improvements based on classical communication if possible, but it is nevertheless a reasonable estimate from a more practical point of view when one is dealing with such states. In the more general context, we outline a possible method which uses the Terhal-Vollbrect formula [69, 76], to establish lower bounds on the EOF. Before proceeding any further, it is appropriate to discuss negativity in our context, which will prove useful in our analysis later.

2.9.1 Negativity

As noted in Section 1.6.4, the logarithmic negativity $E(\hat{\rho}_{AB})$ of a state $\hat{\rho}_{AB}$ is [75]

$$E(\hat{\rho}_{AB}) = \log_2 ||\hat{\rho}_{AB}^{PT}||, \tag{2.62}$$

where $||\hat{\rho}_{AB}^{PT}||$ is the trace norm of $\hat{\rho}_{AB}^{PT}$:

$$||\hat{\rho}_{AB}^{PT}|| = 1 + 2\mathcal{N}(\hat{\rho}_{AB}^{PT}). \tag{2.63}$$

$\mathcal{N}(\hat{\rho}_{AB}^{PT})$ is the absolute value of the sum of the negative eigenvalues of $\hat{\rho}_{AB}^{PT}$, the negativity of $\hat{\rho}_{AB}$. For a state of the form

$$\hat{\rho}_{\rm out}^{(ab)} = U_0\, \hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|\, U_0^{-1}, \tag{2.64}$$

where U_0 corresponds to the unitary action of a 50:50 beamsplitter, we have already seen in Eq. (2.49) that the partially transposed matrix $\hat{\rho}_{\rm out}^{(ab)PT}$ splits into a direct sum of operators, labelled by δ, the number difference:

$$\hat{\rho}_{\rm out}^{(ab)PT} = \sum_{\delta} \oplus \hat{\rho}_{\rm out}^{(ab)PT(\delta)}. \tag{2.65}$$

Thus, for a mixture $\hat{\rho}_D^{(a)}$ truncated at the $k^{\rm th}$ Fock state,

$$\hat{\rho}_D^{(a)} = \sum_{n=1}^{k} p(n)|n\rangle\langle n|, \tag{2.66}$$

the operator $\hat{\rho}_{\rm out}^{(ab)PT}$ splits into a direct sum of finite number of operators,

$$\hat{\rho}_{\rm out}^{(ab)PT} = \sum_{\delta=-k}^{+k} \oplus \hat{\rho}_{\rm out}^{(ab)PT(\delta)}. \tag{2.67}$$

The negativity is therefore simply given by

$$||\hat{\rho}_{\rm out}^{(ab)PT}|| = \sum_{\delta=-k}^{+k} ||\hat{\rho}_{\rm out}^{(ab)PT(\delta)}|| = ||\hat{\rho}_{\rm out}^{(ab)PT(0)}|| + 2\sum_{\delta=1}^{+k} ||\hat{\rho}_{\rm out}^{(ab)PT(\delta)}||. \tag{2.68}$$

Each of the $\hat{\rho}_{\rm out}^{(ab)PT(\delta)}$ is seen to be a matrix whose entries below the anti-diagonal vanish, the dimension of the matrix itself decreasing with increasing δ. Thus one sees immediately that it is only those $\hat{\rho}_{\rm out}^{(ab)PT(\delta)}$'s for which the corresponding $L^{(k)}$ or $\tilde{L}^{(k)}$ (appropriate for the truncated form) fails to be positive, contribute to the negativity of $\hat{\rho}_{\rm out}^{(ab)PT}$. For the simple example of an entangled state generated from a nonclassical PND consisting of a mixture of the ground state and the n$^{\rm th}$ Fock state at the input,

$$\hat{\rho}^{(a)} = (1-p)|0\rangle\langle 0| + p|n\rangle\langle n|, \tag{2.69}$$

the output state $\hat{\rho}_{\rm out}^{(ab)}$ is

$$\hat{\rho}_{\rm out}^{(ab)} = (1-p)|00\rangle\langle 00| + \frac{p}{2^n}\sum_{r,s=0}^{n} \frac{n!|r,n-r\rangle\langle s,n-s|}{\sqrt{r!s!(n-r)!(n-s)!}}. \tag{2.70}$$

For this state the negativity is calculated to be

$$\begin{aligned} &||\hat{\rho}_{\rm out}^{(ab)PT}|| = f(p) \\ &= \frac{p}{2^n}\left(\sum_{r=0}^{n}\sqrt{\binom{n}{r}}\right)^2 - \frac{p}{2^{n-1}} + \sqrt{(1-p)^2 + 4\left(\frac{p}{2^n}\right)^2}. \end{aligned} \tag{2.71}$$

We note that for this kind of mixture, the realigned norm [332, 333] is same as negativity :

$$||\mathcal{R}(\hat{\rho}_{\rm out}^{(ab)})|| = ||\hat{\rho}_{\rm out}^{(ab)PT}|| = f(p). \tag{2.72}$$

Here $\mathcal{R}(\hat{\rho}_{\rm out}^{(ab)})$ is the realigned [332, 333] version of $\hat{\rho}_{\rm out}$. This can be seen for example from the matrix elements of $\mathcal{R}(\hat{\rho}_{\rm out}^{(ab)})$:

$$(\mathcal{R}(\hat{\rho}_{\rm out}^{(ab)}))_{ij,kl} = (1-p)\delta_{0,i+k}\delta_{i+k,j+l} + \frac{p}{2^n}\delta_{n,i+k}\delta_{i+k,j+l}\sqrt{\binom{n}{k}\binom{n}{l}}. \qquad (2.73)$$

We will use Eq. (2.72) in evaluating bounds on the entanglement of formation of states of this kind.

2.9.2 Example a

In this Section, we discuss the bounds on entanglement of formation of the states already defined in Eq. (2.70). It is also useful to note that the negativity $f(p)$ of this state is a monotonically increasing convex function of p. The monotonicity can be seen, for instance, from physical requirements such as avoiding the possibility of generating more entanglement by mixing lesser amount of a nonclassical state with the ground state. The convexity of the function $f(p)$ follows from the convexity of negativity [75]. Given the negativity of a two-mode state, a lower bound on the entanglement of formation can be evaluated [76]. In the example we have considered this lower bound is given by

$$\begin{aligned} E_{l_1}(\hat{\rho}_{\rm out}^{(ab)}) &\geq H_2(\gamma(f(p))) + (1-\gamma(f(p)))\log_2(n), \\ &\quad f(p) \in [1, \frac{4n}{n+1}], \\ &\geq \frac{\log_2(n)}{n-1}(f(p)-(n+1)) + \log_2(n+1), \\ &\quad f(p) \in [\frac{4n}{n+1}, (n+1)], \end{aligned}$$

where

$$\gamma(f(p)) = \frac{1}{(n+1)^2}\left[\sqrt{f(p)} + \sqrt{n(n+1-f(p))}\right]^2. \qquad (2.74)$$

Note that for $n = 1$ the Wootters formula already gives the exact entanglement of formation.

A simple upper bound on the entanglement of formation for states (2.70) is obtained by evaluating the expected entanglement in the spectral basis of $\hat{\rho}_{\rm out}$, and we have

$$E_{\rm up}(\hat{\rho}_{\rm out}^{(ab)}) = -\frac{p}{2^n}\left[\sum_{r=0}^{n}\binom{n}{r}\log_2\binom{n}{r} - 2^n n\right]. \qquad (2.75)$$

We now give a simple method which gives us an alternative lower bound on EOF for these states. Our method is based on the non-increase of EOF under LOCC. Our choice of the local measurement observable for both Alice's and Bob's side is of the form

$$\begin{aligned}\hat{O} &= \lambda_0|0\rangle\langle 0| + \lambda_n \sum_{j=1}^{n} |j\rangle\langle j| \\ &= \lambda_0 P_0 + \lambda_n P_n. \end{aligned} \tag{2.76}$$

For brevity, we denote the collapse of the state into the subspace P_0 as the event 0_c and the collapse into the subspace P_n by n_c, where c could be either Alice or Bob. In such a scenario, there are four possible outcomes for this local measurement done on both Alice's and Bob's side: 0_a0_b, 0_an_b, n_a0_b, and n_an_b. Denoting the probability of possible outcomes by $p_{ij} = \mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(ab)} P_i \otimes P_j)$, with i and j denoting the possible outcomes on Alice's and Bob's sides, we have

$$\begin{aligned} p_{0_a0_b} &= (1-p), \\ p_{0_an_b} &= p_{n_a0_b} = \frac{p}{2^n}, \\ p_{n_an_b} &= \frac{p}{2^n}\sum_{r=1}^{n-1} \left(\begin{array}{c} n \\ r \end{array} \right) = p\left(1 - \frac{1}{2^{n-1}}\right). \end{aligned} \tag{2.77}$$

We end up with a product state in the outcomes corresponding to 0_a0_b, 0_an_b, and n_a0_b, however we end up with a pure entangled state

$$\hat{\rho}_{\mathrm{out}}^{n_an_b} = \frac{1}{2^{n-1}} \sum_{r,s=1}^{n-1} \frac{n!|r, n-r\rangle\langle s, n-s|}{\sqrt{r!s!(n-r)!(n-s)!}}. \tag{2.78}$$

corresponding to the n_an_b outcome. Hence the average entanglement of $\hat{\rho}_{\mathrm{out}}$ subjected to such a local measurement scheme is simply given by

$$\begin{aligned} &E_{l_2}(\hat{\rho}_{\mathrm{out}}^{(ab)}) = p\left(1 - \frac{1}{2^{n-1}}\right) E(\hat{\rho}_{\mathrm{out}}^{n_an_b}) \\ &= -p\left(1 - \frac{1}{2^{n-1}}\right) \left[\sum_{r=1}^{n-1} \left(\frac{1}{2^n - 2}\left(\begin{array}{c} n \\ r \end{array} \right)\right) \log_2 \left(\frac{1}{2^n - 2}\left(\begin{array}{c} n \\ r \end{array} \right)\right)\right]. \end{aligned} \tag{2.79}$$

For the cases $n = 1$ and $n = 2$, such a distillation procedure does not yield any entanglement. However for the cases $n \geq 3$, it is easy to see that the lower bound given by $E_{l_2}(\hat{\rho}_{\mathrm{out}}^{(ab)})$ performs marginally better than $E_{l_1}(\hat{\rho}_{\mathrm{out}}^{(ab)})$ for small values of p. This can be seen, for instance, by the fact that $E_{l_2}(\hat{\rho}_{\mathrm{out}}^{(ab)})$ as a function of p is linear monotonically

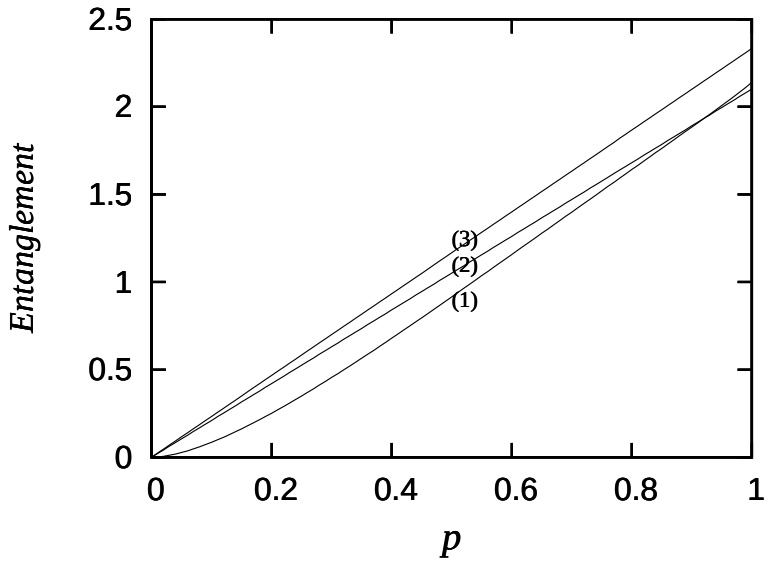

Figure 2.2: The case $n = 6$. Line 3 denotes the upper bound $E_{\rm up}(\hat{\rho}_{\rm out})$, line 2 the lower bound $E_{l_2}(\hat{\rho}_{\rm out})$, and the curve 1 the lower bound $E_{l_1}(\hat{\rho}_{\rm out})$

increasing in p, however $E_{l_1}(\hat{\rho}_{\rm out}^{(ab)})$ is a monotonically increasing convex function of $f(p)$ by construction, while $f(p)$ is a monotonically increasing convex function of p. Thus $E_{l_1}(\hat{\rho}_{\rm out}^{(ab)})$ is a monotonically increasing convex function of p. This simply means that for small values of p, $E_{l_2}(\hat{\rho}_{\rm out}^{(ab)}) > E_{l_1}(\hat{\rho}_{\rm out}^{(ab)})$, the range of values of p for which this is true being determined by n. What we actually observe is that for $n = 6$, $E_{l_2}(\hat{\rho}_{\rm out}^{(ab)}) > E_{l_1}(\hat{\rho}_{\rm out}^{(ab)})$ for most of values of p except in a small region close to 1. However for $n = 7$ we observe that $E_{l_2}(\hat{\rho}_{\rm out}^{(ab)}) > E_{l_1}(\hat{\rho}_{\rm out}^{(ab)})$ for all values of p. This is illustrated in Figure (2.2) and Figure (2.3). Another useful observation is that $E_{l_2}(\hat{\rho}_{\rm out}^{(ab)})$ tends to $E_{\rm up}(\hat{\rho}_{\rm out}^{(ab)})$ for larger n. This can be seen from the fact that the mixture in $\hat{\rho}_{\rm out}^{(ab)}$ is almost bi-orthogonal for large n, with the overlap in each of the local modes going as $\frac{1}{2^n}$, indicating that we can distill as much of the entanglement that is present [74].

2.9.3 Example b

We now establish bounds on another class of continuous variable states by extending the ideas of the previous example. These are states that arise from phase averaging of states of the form

$$|\Psi_n(\alpha)\rangle = \frac{1}{\sqrt{N}} \sum_{r=0}^{n-1} (\omega_n)^r |(\omega_n)^r \alpha\rangle, \tag{2.80}$$

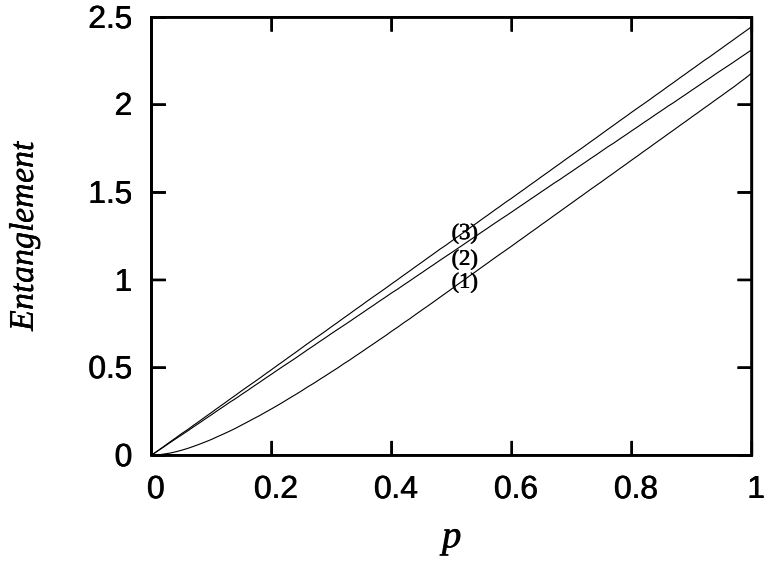

Figure 2.3: The case $n = 7$. Line 3 denotes the upper bound $E_{\rm up}(\hat{\rho}_{\rm out})$, line 2 the lower bound $E_{l_2}(\hat{\rho}_{\rm out})$, and the curve 1 the lower bound $E_{l_1}(\hat{\rho}_{\rm out})$

where ω_n is the primitive n$^{\rm th}$ root of unity and $|(\omega_n)^r\alpha\rangle$ is the coherent state with displacement $(\omega_n)^r\alpha$, and N is the appropriate normalisation. These states can be generated through nonlinear optical processes [180, 181]. With the use of the identity

$$\begin{aligned}\sum_{k=0}^{n-1}(\omega_n)^{jk} &= n \text{ if } j \equiv 0 \bmod n, \\ &= 0 \text{ otherwise,}\end{aligned} \tag{2.81}$$

we can rewrite the state as

$$|\Psi_n(\alpha)\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{\infty}\frac{\alpha^{nk+n-1}}{\sqrt{(nk+n-1)!}}|nk+n-1\rangle. \tag{2.82}$$

Denoting $|\alpha|^2$ by λ, the state after phase averaging is

$$\hat{\rho}_D^{(a)}(\lambda,n) = \frac{1}{N}\sum_{k=0}^{\infty}\frac{\lambda^{nk+n-1}}{(nk+n-1)!}|nk+n-1\rangle\langle nk+n-1|. \tag{2.83}$$

On combining this state with the ground state of an auxiliary b-mode and then passing through a 50:50 beamsplitter, we have

$$\hat{\rho}_{\rm out}^{(ab)}(\lambda,n) = \frac{1}{N}\sum_{k=0}^{\infty}\frac{\lambda^{nk+n-1}}{(nk+n-1)!}\frac{1}{2^{nk+n-1}}\times \sum_{r,s=0}^{nk+n-1}\frac{(nk+n-1)!|r,nk+n-1-r\rangle\langle s,nk+n-1-s|}{\sqrt{r!s!(nk+n-1-r)!(nk+n-1-s)!}}. \tag{2.84}$$

The local measurement observable we choose on both Alice's and Bob's side is of the form

$$\hat{O} = \sum_{\kappa=0}^{\infty}\gamma_\kappa\sum_{r=0}^{n-1}|n\kappa+r\rangle\langle n\kappa+r|. \tag{2.85}$$

For outcomes κ_1, κ_2 respectively on Alice's and Bob's side, the resultant state upto normalisation is given by

$$P_{\kappa_1}\otimes P_{\kappa_2}\hat{\rho}_{\rm out}^{(ab)}(\lambda,n)P_{\kappa_1}\otimes P_{\kappa_2},\quad \text{where}$$
$$P_{\kappa_1}\otimes P_{\kappa_2} = \sum_{r=0}^{n-1}|n\kappa_1+r\rangle\langle n\kappa_1+r|\otimes\sum_{r=0}^{n-1}|n\kappa_2+r\rangle\langle n\kappa_2+r|. \tag{2.86}$$

The total number of photons in this subspace can vary from $n(\kappa_1+\kappa_2)$ to $n(\kappa_1+\kappa_2)+2(n-1)$ The fact that $\hat{\rho}_D^{(a)}(\lambda,n)$ has nonvanishing projectors only for photon numbers of the form $nk+n-1$ implies

$$n(\kappa_1+\kappa_2)\le nk+n-1\le n(\kappa_1+\kappa_2)+2(n-1) \tag{2.87}$$

for the permitted values of k, for given κ_1, κ_2. It is easy to see that the only possible solution to this constraint is $k=\kappa_1+\kappa_2$. This means that for a given outcome κ_1, κ_2, the pure entangled state in $\hat{\rho}_{\rm out}(\lambda,n)$ corresponding to $nk+n-1$ with $k=\kappa_1+\kappa_2$ photons is singled out. Now for a fixed total number of photons $nk+n-1$, the possible values for (κ_1,κ_2) are $(k,0)$, $(k-1,1)$, $\cdots$, $(0,k)$ Given the outcomes corresponding to γ_{κ_1} on Alice's side and γ_{κ_2} on Bob's side, the photon number on Alice's side can run from $n\kappa_1$ to $n\kappa_1+(n-1)$ and similarly on Bob's side run from $n\kappa_2+(n-1)$ to $n\kappa_2$. Thus the resultant state corresponding to the outcome γ_{κ_1} on Alice's side and γ_{κ_2} on Bob's side is the pure state

$$|\psi(\kappa_1,\kappa_2)\rangle \;\propto\; \sum_{r=0}^{n-1}\frac{\sqrt{(n(\kappa_1+\kappa_2)+n-1)!}}{\sqrt{(n\kappa_1+r)!(n\kappa_2+n-1-r)!}}|n\kappa_1+r,n\kappa_2+n-1-r\rangle. \tag{2.88}$$

Thus, for a given k, the possible states after the local measurement are

$$|\psi(0,k)\rangle,\ |\psi(1,k-1)\rangle,\ \cdots,\ |\psi(k,0)\rangle. \tag{2.89}$$

Computation of average entanglement after the measurement, which is by definition the entanglement distilled by this protocol, is facilitated by the following facts. Firstly, each $|\psi(\kappa_1,\kappa_2)\rangle$ is already in the Schmidt form. Secondly, these $|\psi(\kappa_1,\kappa_2)\rangle$'s are biorthogonal [334]. Thus the entanglement distilled is

$$\begin{aligned}
&E_{\rm av}(nk+n-1) = \\
&-\sum_{r=0}^{nk+n-1}\left[\frac{1}{2^{nk+n-1}}\begin{pmatrix} nk+n-1 \\ r \end{pmatrix}\right]\times \\
&\log_2\left[\frac{1}{2^{nk+n-1}}\begin{pmatrix} nk+n-1 \\ r \end{pmatrix}\right]+ \\
&\sum_{r=0}^{k}\left[\frac{1}{2^{nk+n-1}}\sum_{d=0}^{n-1}\begin{pmatrix} nk+n-1 \\ nr+d \end{pmatrix}\right]\times \\
&\log_2\left[\frac{1}{2^{nk+n-1}}\sum_{d=0}^{n-1}\begin{pmatrix} nk+n-1 \\ nr+d \end{pmatrix}\right].
\end{aligned} \tag{2.90}$$

Thus the total average entanglement harvested over all possible photon number is

$$E_l(\hat{\rho}_{\rm out}^{(ab)}(\lambda,n)) = \frac{1}{N}\sum_{k=0}^{\infty}\frac{\lambda^{nk+n-1}}{(nk+n-1)!}E_{\rm av}(nk+n-1). \tag{2.91}$$

Clearly $E_l(\hat{\rho}_{\rm out}^{(ab)}(\lambda,n))$ is a lower bound on the EOF of $\hat{\rho}_{\rm out}^{(ab)}(\lambda,n)$.

A simple upper bound on the entanglement of formation of $\hat{\rho}_{\rm out}^{(ab)}(\lambda,n)$ is obtained by evaluating the average entanglement in its spectral basis, and is given by

$$E_{\rm up}(\hat{\rho}_{\rm out}^{(ab)}(\lambda,n)) = \frac{1}{N}\sum_{k=0}^{\infty}\frac{\lambda^{nk+n-1}}{(nk+n-1)!}E(nk+n-1),$$

where

$$\begin{aligned}
&E(nk+n-1) = \\
&-\sum_{r=0}^{nk+n-1}\left[\frac{1}{2^{nk+n-1}}\begin{pmatrix} nk+n-1 \\ r \end{pmatrix}\right]\log_2\left[\frac{1}{2^{nk+n-1}}\begin{pmatrix} nk+n-1 \\ r \end{pmatrix}\right].
\end{aligned} \tag{2.92}$$

In Figure (2.4), we plot the lower and upper bounds $E_l(\hat{\rho}_{\rm out}^{(ab)}(\lambda,n))$ and

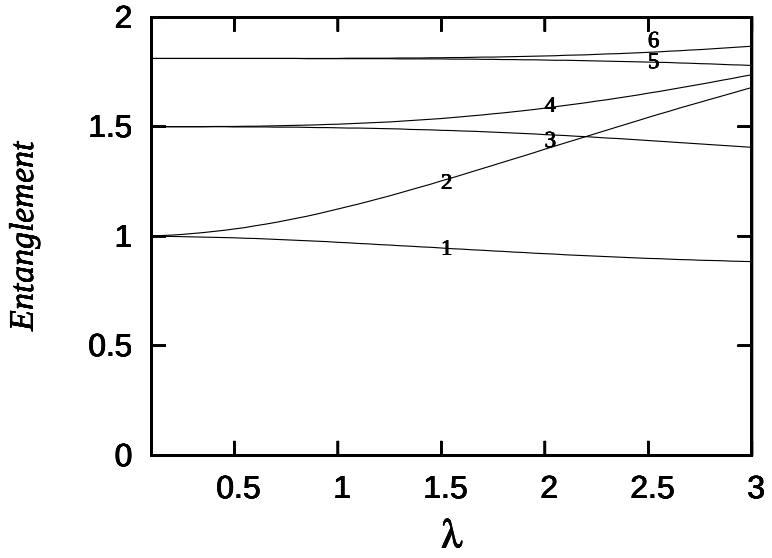

Figure 2.4: The curves 1 and 2 correspond to the lower and upper bounds for the case n = 2, the curves 3 and 4 correspond to the lower and upper bounds for the case n = 3, and the curves 5 and 6 correspond to the lower and upper bounds for the case n = 4

$E_{\rm up}(\hat{\rho}_{\rm out}^{(ab)}(\lambda, n))$ for the cases $n = 2$, $n = 3$, and $n = 4$.

We now discuss a more general method which establishes lower bounds using LOCC and the Terhal-Vollbrect formula. In most of the practical situations such as in [312, 318, 319, 322], the probabilities of the PND vanish for larger n. In such a case, one can do a truncation to evaluate a lower bound based on [76].

A truncation corresponds to a LOCC. To see this suppose the input on Alice's side is

$$\hat{\rho}_D^{(a)} = \sum_{n=1}^{\infty} p(n)|n\rangle\langle n|. \tag{2.93}$$

Let

$$\begin{aligned} {\rm Tr}(\hat{\rho}_{\rm out}^{(ab)} P_k \otimes P_k) &= 1 - \epsilon, \quad \text{where} \\ P_k &= \sum_{i=0}^{k} |i\rangle\langle i|. \end{aligned} \tag{2.94}$$

Since ϵ is a decreasing function of k, we can choose a value of k to achieve any small

$\epsilon > 0$. The truncated version of $\hat{\rho}_{\rm out}^{(ab)}$ is

$$\hat{\rho}_{\rm out}^{(ab)({\rm tr})} = \frac{1}{1-\epsilon}(P_k \otimes P_k \hat{\rho}_{\rm out}^{(ab)} P_k \otimes P_k). \tag{2.95}$$

The state $\hat{\rho}_{\rm out}^{(ab)({\rm tr})}$ is the $0_a 0_b$ outcome corresponding to the measurement of the local observable

$$\lambda_0 \sum_{i=0}^{k} |i\rangle\langle i| + \lambda_k \sum_{i=k+1}^{\infty} |i\rangle\langle i| \tag{2.96}$$

on both Alice's and Bob's side. Since negativity is an entanglement monotone under LOCC [75], we have

$$\mathcal{N}(\hat{\rho}_{\rm out}^{(ab)PT}) \geq (1-\epsilon)\mathcal{N}(\hat{\rho}_{\rm out}^{(ab)PT({\rm tr})}). \tag{2.97}$$

A lower bound on the EOF of $\hat{\rho}_{\rm out}^{(ab)}$ is now simply given by

$$E(\hat{\rho}_{\rm out}^{(ab)}) \geq F(\mathcal{N}(\hat{\rho}_{\rm out}^{(ab)PT})) \geq F[(1-\epsilon)\mathcal{N}(\hat{\rho}_{\rm out}^{(ab)({\rm tr})})], \tag{2.98}$$

where $F(.)$ is the Terhal-Vollbrect formula [76]. The last inequality follows from the fact that $F(.)$ is a monotonically increasing function of its argument.

2.10 Conclusions

To conclude, we have demonstrated the effectiveness of partial transpose in the study of entanglement of a family of non-Gaussian continuous variable states. We have presented partial results on distillability of these states. We have demonstrated distillable entanglement from violation of three term conditions both on $\{p(n)\}$ and $\{\gamma_n\}$, particularly from an antibunched input. We have evaluated both upper and lower bounds on the entanglement of formation of a family of non-Gaussian continuous variable states. We have illustrated the possibility of going beyond the Terhal-Vollbrect formula in estimating entanglement through a careful choice of local observables to be measured. However, our analysis relies on partial transpose, and hence cannot answer issues in respect of PPT bound entangled states.

3

Nonclassicality, Mandel classification, and entanglement

3.1 Introduction

In the preceding Chapter quantum optical nonclassicality in single-mode fields was studied from the perspective of entanglement. In the case of phase invariant states, the nonclassicality was completely determined by the photon number distribution [170]. This was possible due to the connection with the classical Stieltjes moment problem [170, 324]. It was shown that if a phase-invariant single-mode state is coupled to a second mode in vacuum or some coherent state and then passed through a $U(2)$ beamsplitter, the resulting two-mode output state shows NPT entanglement [27] precisely when the input single-mode is quantum optically nonclassical. The signatures of the two, nonclassicality and entanglement, coincided exactly [194].

In this Chapter, we continue this exploration further. We begin with the study of nonclassicality of two-mode states and its potential to generate entanglement, our study being now restricted by the absence of results such as the result of the Stieltjes moment problem. We begin with the description of a single test, which if successful, is able to simultaneously establish both the nonclassicality and NPT entanglement of a given two-mode state. We then turn to Mandel matrix analysis of the two-mode fields and show that nonclassicality at this level naturally separates into distinct Types I and II, depending on whether the sub-Poissonian statistics is visible or not at a single-mode level. We then present three interesting examples of two-mode states to illustrate the idea. The first is a study of two-mode states obtained in the previous Chapter from single-mode states, the second and third are independently constructed. Along the way we develop a test at the Mandel matrix level to see whether the beamsplitter action on an initially non-entangled two-mode nonclassical state results in NPT entanglement, and apply it to the second

family of states. The third family is built upon the extensively studied squeezed vacuum states for one and two modes, where we bring out some of their nonintuitive features. We extend the method developed for single-mode states to two-mode states and construct NPT entanglement tests to test for entanglement when a two-mode nonclassical state is coupled to a third mode in vacuum and the result is passed through a '$U(3)$ beamsplitter'. The discussion and test is again at the Mandel matrix level and the test is illustrated with two families of states as examples. We show the possibility of generating genuine tripartite entanglement from two-mode Mandel type nonclassicality. We then extend the notion of Mandel's Q parameter to two modes. Based on the Mandel matrix construct, the Mandel parameters Q_1 and Q_2 are introduced to help differentiate the Type I and Type II states and gauge their Mandel level nonclassicality, and interesting possibilities are discussed.

3.2 Two-mode fields – Entanglement test, and Mandel classification

Our emphasis here is in new features encountered in the states of a two-mode system. The modes will be called the a-mode and the b-mode, with operator pairs $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$ obeying the commutation relations in Eq. (1.95). Some general points can be made right away.

Since we have a two-mode system, apart from examining whether a given state $\hat{\rho}^{(ab)}$ is quantum optically (QO-cl) or quantum optically nonclassical (QO-noncl), we can also ask whether it is entangled, and if so whether it is NPT type, distillable, etc. The latter questions become meaningful in the two-mode case. In fact we will develop an interesting criterion which can be witness *simultaneously* for QO-nonclassicality as well as entanglement of the NPT type.

With respect to the action of the beamsplitters representing general elements $u \in U(2)$ in the manner of Eq. (2.31), we note the following. Action by a beamsplitter is both nonlocal, in that the modes a and b get linearly mixed, and passive, as $\hat{N}_a + \hat{N}_b$ is conserved; since annihilation operators go to linear combinations of annihilation operators under this action, coherent states go into coherent states. Convex sums of coherent states go to convex sums of coherent states, and thus such action preserves the QO-cl or QO-noncl nature of a state $\hat{\rho}^{(ab)}$ [335]. On the other hand, while a QO-cl state has no entanglement, a QO-noncl state may possess entanglement or may be separable: entangled states are a proper subset of QO-noncl states. Thus beamsplitter action can cause a transition from a QO-noncl separable state to a QO-noncl entangled state, in

which case we can further enquire into the nature of the entanglement so obtained. This was in fact the case in the transition from $\hat{\rho}_{\rm in}^{(ab)}$ of Eq. (2.43) to $\hat{\rho}_{\rm out}^{(ab)}$ of Eq. (2.44).

As mentioned earlier, there is no generalisation of the results of the classical Stieltjes moment problem which can be used for two-mode systems, their PND's, etc. Therefore to study aspects of two-mode nonclassicality to an admittedly limited degree.

3.2.1 Nonclassicality and entanglement – a direct connection

A general two-mode state $\hat{\rho}^{(ab)}$ possesses the diagonal coherent state representation

$$\hat{\rho}^{(ab)} = \int\int \frac{d^2 z_a}{\pi}\,\frac{d^2 z_b}{\pi}\,\phi(z_a, z_b)|z_a, z_b\rangle\langle z_a, z_b| \tag{3.1}$$

in terms of the two-mode (product) coherent states $|z_a, z_b\rangle$. Of interest to us is a test which *simultaneously establishes* both QO-nonclassicality of $\hat{\rho}^{(ab)}$ and its NPT entanglement.

To this end, we set up an infinite matrix $\hat{N}$ with operator entries $\hat{N}_{jk,lm}$ where j, k, l, m run over the range $0, 1, 2, \cdots$ independently. The pair jk denotes a 'row index' and takes in sequence the values $00; 10, 01; 20, 11, 02; 30, 21,\ 12, 03; \cdots$. Similarly the 'column index' pair also takes these same values in the same sequence. We define the entries of $\hat{N}$ thus:

$$\hat{N}_{jk,lm} = \hat{N}^{\dagger}_{lm,jk} = \hat{a}^{\dagger j}\hat{b}^{\dagger k}\hat{a}^{l}\hat{b}^{m}. \tag{3.2}$$

Clearly $\hat{N} = \left(\left(\hat{N}_{jk,lm}\right)\right)$ is a 'hermitian' matrix of operator entries. Note that these entries are in normal-ordered form. Starting with the diagonal representation (3.1), for any set of complex coefficients $\{c_{jk}\}$ and the associated positive semidefinite operator $\sum_{jk,lm} c^{*}_{jk}\hat{N}_{jk,lm}c_{lm}$, we always have:

$$\begin{aligned}
&{\rm Tr}(\hat{\rho}^{(ab)}\sum_{jk,lm} c^{*}_{jk}\hat{N}_{jk,lm}c_{lm}) = {\rm Tr}(\hat{\rho}^{(ab)}(\sum_{jk} c_{jk}\hat{a}^{j}\hat{b}^{k})^{\dagger}(\sum_{lm} c_{lm}\hat{a}^{l}\hat{b}^{m})) \\
&= \int\int \frac{d^2 z_a}{\pi}\,\frac{d^2 z_b}{\pi}\,\phi(z_a, z_b)|\sum_{lm} c_{lm} z_a^{l} z_b^{m}|^2 \geq 0.
\end{aligned} \tag{3.3}$$

This is because we have here the expectation value of a positive semidefinite hermitian operator. On the other hand, if we pass to the partial transpose $\hat{\rho}^{(ab)PT}$ of $\hat{\rho}^{(ab)}$, by performing transposition only in the space of states of the b-mode, this will amount to everywhere replacing $\hat{b}^{\dagger j}\hat{b}^{m}$ by $\hat{b}^{\dagger m}\hat{b}^{j}$, since in the Fock basis $\hat{b}^{\dagger}$ and $\hat{b}$ are real [27, 57, 227].

Thus for the same positive semidefinite operator as in Eq. (3.3) we have

$$\mathrm{Tr}(\hat{\rho}^{(ab)PT}\sum_{jk,lm}c^*_{jk}\hat{N}_{jk,lm}c_{lm}) = \mathrm{Tr}(\hat{\rho}^{(ab)}\sum_{jk,lm}c^*_{jk}\hat{a}^{\dagger j}\hat{b}^{\dagger m}\hat{a}^l\hat{b}^k c_{lm})$$
$$= \int\int \frac{d^2z_a}{\pi}\frac{d^2z_b}{\pi}\,\phi(z_a,z_b)\,|\sum_{lm}c_{lm}z_a^l z_b^{*m}|^2. \tag{3.4}$$

Notice the difference in the integrands of the last integrals in Eqs. (3.3) and (3.4); the latter integral is sure to be positive if $\hat{\rho}^{(ab)PT}$ is a possible state, otherwise it could be negative.

Thus we arrive at a single test for QO-nonclassicality and NPT entanglement of $\hat{\rho}^{(ab)}$. The above expression being negative implies *two things simultaneously*:

$$\begin{aligned}&\text{(i) } \phi(z_a,z_b)\ngeq 0,\ \text{hence } \hat{\rho}^{(ab)} \text{ is QO}-\text{noncl};\\ &\text{(ii) } \hat{\rho}^{(ab)PT}\ngeq 0,\ \text{and hence } \hat{\rho}^{(ab)} \text{ is NPT entangled.}\end{aligned} \tag{3.5}$$

This interesting result is an indication of the possibility, in suitable circumstances, of bringing the characterisations of QO-nonclassicality and entanglement for the two-mode fields rather close to one another.

3.2.2 Mandel matrix

In Section 2.2.1, we introduced the concept of phase-insensitive nonclassicality for a single-mode radiation field. Of relevance were the matrix elements of operators conserving the number of photons. Generalising this concept to two-mode systems, if one is interested only in the total number conserving matrix elements, then given a $\hat{\rho}^{(ab)}$ it suffices to work with the state $\hat{\rho}^{(ab)}_D$ obtained from $\hat{\rho}^{(ab)}$ by phase averaging:

$$\begin{aligned}\hat{\rho}^{(ab)}_D &= \int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta(\hat{N}_a+\hat{N}_b)}\hat{\rho}^{(ab)}e^{-i\theta(\hat{N}_a+\hat{N}_b)}\\ &= \int\frac{d^2z_a}{\pi}\frac{d^2z_b}{\pi}P(I_a,I_b,\theta)|z_a,z_b\rangle\langle z_a,z_b|,\\ I_a &= |z_a|^2,\ \ I_b=|z_b|^2,\ \ \theta=\arg z_a^* z_b,\\ P(I_a,I_b,\theta) &= \int_0^{2\pi}\frac{d\theta'}{2\pi}\phi(z_a e^{-i\theta'},z_b e^{-i\theta'}).\end{aligned} \tag{3.6}$$

This state is clearly number conserving:

$$\begin{aligned}\hat{\rho}^{(ab)}_D(\hat{N}_a+\hat{N}_b) &= (\hat{N}_a+\hat{N}_b)\hat{\rho}^{(ab)}_D,\\ \langle n'm'|\hat{\rho}^{(ab)}_D|nm\rangle &= \delta_{n'+m',n+m}\langle n'm'|\hat{\rho}^{(ab)}|nm\rangle.\end{aligned} \tag{3.7}$$

Since $P(I_a, I_b, \theta)$ is the (real) diagonal representation weight of $\hat{\rho}_D^{(ab)}$, we have this QO-classification :

$$\begin{aligned} P(I_a, I_b, \theta) \geq 0 \;&\Leftrightarrow\; \hat{\rho}_D^{(ab)} \text{ is QO-cl,} \\ P(I_a, I_b, \theta) \ngeq 0 \;&\Leftrightarrow\; \hat{\rho}_D^{(ab)} \text{ is QO-noncl.} \end{aligned} \tag{3.8}$$

Now we present the two-mode generalisation of the Mandel matrix and the associated classification of states [187, 336]. We consider only states $\hat{\rho}_D^{(ab)}$ conserving, i.e., commuting with, $\hat{N}_a + \hat{N}_b$. Let us first develop the two-mode analogue of the Mandel criterion. Define a column and row vector with number conserving operator entries as follows :

$$\hat{C} = \begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix} \otimes \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} \hat{N}_a \\ \hat{a}^\dagger \hat{b} \\ \hat{b}^\dagger \hat{a} \\ \hat{N}_b \end{pmatrix}, \; \hat{C}^\dagger = \begin{pmatrix} \hat{N}_a & \hat{b}^\dagger \hat{a} & \hat{a}^\dagger \hat{b} & \hat{N}_b \end{pmatrix}. \tag{3.9}$$

With their help next define a 5×5 matrix with operator entries and which is 'hermitian' like $\hat{N}$ in Eq. (3.2), and also 'positive definite':

$$\begin{aligned} \hat{\Sigma} \;&=\; \begin{pmatrix} 1 \\ \hat{C} \end{pmatrix} \begin{pmatrix} 1 & \hat{C}^\dagger \end{pmatrix} = \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{C}\hat{C}^\dagger \end{pmatrix} \\ &= \begin{pmatrix} 1 & \hat{N}_a & \hat{b}^\dagger \hat{a} & \hat{a}^\dagger \hat{b} & \hat{N}_b \\ \hat{N}_a & \hat{N}_a^2 & \hat{N}_a \hat{b}^\dagger \hat{a} & \hat{N}_a \hat{a}^\dagger \hat{b} & \hat{N}_a \hat{N}_b \\ \hat{a}^\dagger \hat{b} & \hat{a}^\dagger \hat{b} \hat{N}_a & \hat{a}^\dagger \hat{b} \hat{b}^\dagger \hat{a} & (\hat{a}^\dagger \hat{b})^2 & \hat{a}^\dagger \hat{b} \hat{N}_b \\ \hat{b}^\dagger \hat{a} & \hat{b}^\dagger \hat{a} \hat{N}_a & (\hat{b}^\dagger \hat{a})^2 & \hat{b}^\dagger \hat{a} \hat{a}^\dagger \hat{b} & \hat{b}^\dagger \hat{a} \hat{N}_b \\ \hat{N}_b & \hat{N}_b \hat{N}_a & \hat{N}_b \hat{b}^\dagger \hat{a} & \hat{N}_b \hat{a}^\dagger \hat{b} & (\hat{N}_b)^2 \end{pmatrix}. \end{aligned} \tag{3.10}$$

Given a state $\hat{\rho}_D^{(ab)}$, we get the 5×5 numerical hermitian matrix Σ by taking entrywise expectation values of the opertors in $\hat{\Sigma}$, and the numerical matrix so obtained is clearly hermitian positive semidefinite :

$$\Sigma = \langle \hat{\Sigma} \rangle = \mathrm{Tr}(\hat{\rho}_D^{(ab)} \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{C}\hat{C}^\dagger \end{pmatrix}) \;=\; \begin{pmatrix} 1 & \langle \hat{C}^\dagger \rangle \\ \langle \hat{C} \rangle & \langle \hat{C}\hat{C}^\dagger \rangle \end{pmatrix} \geq 0 \,. \tag{3.11}$$

We define the *two-mode Mandel matrix* for the state $\hat{\rho}_D^{(ab)}$ by replacing $\hat{C}\hat{C}^\dagger$ in Eq. (3.11) by its normal ordered expression (entries of $\hat{C}$ and $\hat{C}^\dagger$ are already in the normal ordered

form) :

$$
\begin{aligned}
\hat{B} &= :\hat{C}\hat{C}^{\dagger}:, \\
M^{(2)}(\hat{\rho}_D^{(ab)}) &= \mathrm{Tr}(\hat{\rho}_D^{(ab)} \begin{pmatrix} 1 & \hat{C}^{\dagger} \\ \hat{C} & \hat{B} \end{pmatrix}) \\
&= \begin{pmatrix}
1 & \langle \hat{a}^{\dagger}\hat{a}\rangle & \langle \hat{b}^{\dagger}\hat{a}\rangle & \langle \hat{a}^{\dagger}\hat{b}\rangle & \langle \hat{b}^{\dagger}\hat{b}\rangle \\
\langle \hat{a}^{\dagger}\hat{a}\rangle & \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle & \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{a}^2\rangle & \langle \hat{a}^{\dagger 2}\hat{a}\hat{b}\rangle & \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{a}\hat{b}\rangle \\
\langle \hat{a}^{\dagger}\hat{b}\rangle & \langle \hat{a}^{\dagger 2}\hat{a}\hat{b}\rangle & \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{a}\hat{b}\rangle & \langle \hat{a}^{\dagger 2}\hat{b}^2\rangle & \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{b}^2\rangle \\
\langle \hat{b}^{\dagger}\hat{a}\rangle & \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{a}^2\rangle & \langle \hat{b}^{\dagger 2}\hat{a}^2\rangle & \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{a}\hat{b}\rangle & \langle \hat{b}^{\dagger 2}\hat{a}\hat{b}\rangle \\
\langle \hat{b}^{\dagger}\hat{b}\rangle & \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{a}\hat{b}\rangle & \langle \hat{b}^{\dagger 2}\hat{a}\hat{b}\rangle & \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{b}^2\rangle & \langle \hat{b}^{\dagger 2}\hat{b}^2\rangle
\end{pmatrix} \\
&= \int_0^{\infty} dI_a \int_0^{\infty} dI_b \int_0^{2\pi} \frac{d\theta}{2\pi} P(I_a, I_b, \theta) \\
&\quad \times \begin{pmatrix} 1 \\ I_a \\ \sqrt{I_a I_b}e^{i\theta} \\ \sqrt{I_a I_b}e^{-i\theta} \\ I_b \end{pmatrix} \begin{pmatrix} 1 & I_a & \sqrt{I_a I_b}e^{-i\theta} & \sqrt{I_a I_b}e^{i\theta} & I_b \end{pmatrix}.
\end{aligned}
\tag{3.12}
$$

The superscript 2 indicates that we are dealing with a two-mode state, and this Mandel matrix is 5×5 hermitian but not necessarily positive semidefinite.

Another useful construct is the 2×2 Mandel matrix associated with a general single-mode obtained as linear combination of the modes a and b, calculated again in the state $\hat{\rho}_D^{(ab)}$. The definition of the annihilation operator$\hat{A}$ of such a mode and then of its Mandel

matrix are :

$$\begin{aligned}
\hat{A} = \alpha\hat{a} + \beta\hat{b}, \qquad & |\alpha|^2 + |\beta|^2 = 1; \\
M^{(2,1)}(\hat{\rho}_D^{(ab)}; \alpha, \beta) &= \begin{pmatrix} 1 & \langle \hat{A}^\dagger \hat{A} \rangle \\ \langle \hat{A}^\dagger \hat{A} \rangle & \langle \hat{A}^{\dagger 2} \hat{A}^2 \rangle \end{pmatrix} \\
&= Y(\alpha, \beta)^\dagger M^{(2)}(\rho_D^{(ab)}) Y(\alpha, \beta), \\
Y(\alpha, \beta) &= \begin{pmatrix} 1 & 0 \\ 0 & \\ 0 & \psi_0(\alpha, \beta) \\ 0 & \\ 0 & \end{pmatrix}, \\
\psi_0(\alpha, \beta) &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} \alpha\alpha^* \\ \alpha\beta^* \\ \beta\alpha^* \\ \beta\beta^* \end{pmatrix}. \qquad (3.13)
\end{aligned}$$

The dependence of $\hat{A}$ on α, β is left implicit. The superscript $(2,1)$ at the start of the above equations indicates that we are dealing with a general single-mode Mandel matrix obtained from the two-mode Mandel matrix for the $a-b$ system in the state $\hat{\rho}_D^{(ab)}$, by focussing on the particular linear combination $\hat{A}$ of $\hat{a}$ and $\hat{b}$. It turns out that for two-mode states both $M^{(2)}(\hat{\rho}_D^{(ab)})$ and $M^{(2,1)}(\hat{\rho}_D^{(ab)}; \alpha, \beta)$ are important.

The two-mode definitions of Mandel-type nonclassicality, sub-Poissonian statistics (sub-PS), super-Poissonian statistics (super-PS), etc are now as follows :

$$\begin{aligned}
\{\hat{\rho}_D^{(ab)} \text{ is QO-cl} &\Leftrightarrow P(I_a, I_b, \theta) \geq 0\} \\
&\Rightarrow \{M^{(2)}(\hat{\rho}_D^{(ab)}) \geq 0 \Leftrightarrow \hat{\rho}_D^{(ab)} \text{has super-PS}\}; \\
M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0 &\Leftrightarrow \{\hat{\rho}_D^{(ab)} \text{is QO-noncl, has sub-PS}\}. \qquad (3.14)
\end{aligned}$$

In the definition of super-PS, we used Eq. (3.12). The sub-PS case can be usefully separated into two types, depending on whether or not the nonpositivity of the 5×5 matrix $M^{(2)}(\hat{\rho}_D^{(ab)})$ is visible already at the single-mode level for some choice of coefficients α, β. Thus we define :

$$\begin{aligned}
\hat{\rho}_D^{(ab)} \text{ has Type I sub-PS} &\Leftrightarrow M^{(2,1)}(\hat{\rho}_D^{(ab)}; \alpha, \beta) \ngeq 0 \text{ for some } \alpha,\ \beta; \\
\hat{\rho}_D^{(ab)} \text{ has Type II sub-PS} &\Leftrightarrow M^{(2,1)}(\hat{\rho}_D^{(ab)}; \alpha, \beta) \geq 0 \text{ for all } \alpha,\ \beta, \\
&\qquad M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0. \qquad (3.15)
\end{aligned}$$

The physical meaning is that in Type I sub-PS, the Mandel level of QO-nonclassicality is easy to detect already in terms of a suitable single-mode combination; while in Type II sub-PS, such nonclassicality is hidden or intrinsically two-mode in character.

For calculational purposes one can pass from the 5×5 Mandel matrix $M^{(2)}(\hat{\rho}_D^{(ab)})$ to a slightly simpler 4×4 matrix as follows. From Eq. (3.12),

$$\begin{aligned} M^{(2)}(\hat{\rho}_D^{(ab)}) &= \mathrm{Tr}(\hat{\rho}_D^{(ab)} \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{B} \end{pmatrix}) = \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix}, \\ C &= \mathrm{Tr}(\hat{\rho}_D^{(ab)} \hat{C}),\ B = \mathrm{Tr}(\hat{\rho}_D^{(ab)} \hat{B}). \end{aligned} \tag{3.16}$$

(When necessary the state will be indicated as argument of C, B). Then it is easy to see that

$$\begin{aligned} &M^{(2)}(\hat{\rho}_D^{(ab)}) \geq 0 \ \Leftrightarrow\ \Gamma = B - CC^\dagger \geq 0\,, \\ &M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0 \ \Leftrightarrow\ \Gamma \ngeq 0. \end{aligned} \tag{3.17}$$

Thus the 4×4 matrix Γ determines whether we have super-PS or sub-PS. For the separation of the latter into Type I and Type II, we have for any complex 2-vector $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\begin{aligned} &\phi^\dagger M^{(2,1)}(\hat{\rho}_D^{(ab)}; \alpha, \beta)\phi = \\ &|\phi_1 + \phi_2 C^\dagger \psi_0(\alpha,\beta)|^2 + |\phi_2|^2 \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta). \end{aligned} \tag{3.18}$$

So we are able to say, given $M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0$:

$$\begin{aligned} \text{Type I PS} &\Leftrightarrow \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta) < 0 \text{ for some } \alpha,\ \beta; \\ \text{Type II PS} &\Leftrightarrow \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta) \geq 0 \text{ for all } \alpha,\ \beta. \end{aligned} \tag{3.19}$$

Indeed we easily find from Eqs. (3.13, 3.17) that

$$\det M^{(2,1)}(\hat{\rho}^{(ab)}; \alpha, \beta) = \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta). \tag{3.20}$$

We will apply these definitions and classification of QO-nonclassicality to several families of states and examine the possible conversion of nonclassicality to entanglement in the next Section.

3.3 Some examples of two-mode states and their properties

We have seen in Section 2.6 that a single-mode QO-noncl state, when combined with a second mode in vacuum (or in a coherent state) and passed through a beamsplitter, always results at the output in a two-mode state exhibiting NPT entanglement. We study these two-mode states in the spirit of the preceding Section as a first example.

3.3.1 Example (a)

The two-mode state in question is given in Eq. (2.44). It is reasonable that its 5×5 Mandel matrix is obtainable from the 2×2 Mandel matrix associated with the single-mode input state $\hat{\rho}_D^{(a)}$. Straightforward calculation shows that:

$$\begin{aligned}
\hat{\rho}_D^{(ab)} &= \hat{U}(u)\{\hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|\}\hat{U}(u)^{-1}, \quad u \in U(2): \\
M^{(2)}(\hat{\rho}_D^{(ab)}) &= W(u)^\dagger M^{(1)}(\hat{\rho}_D^{(a)})W(u), \\
W(u) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & u_{11}^* u_{11} & u_{21}^* u_{11} & u_{11}^* u_{21} & u_{21}^* u_{21} \end{pmatrix}, \\
W(u)W(u)^\dagger &= I_{2\times2}.
\end{aligned} \tag{3.21}$$

Next using (3.13) we can immediately obtain the general single-mode projection of this two-mode Mandel matrix:

$$\begin{aligned}
M^{(2,1)}(\hat{\rho}_D^{(ab)}; \alpha, \beta) &= Y(\alpha,\beta)^\dagger W(u)^\dagger M^{(1)}(\hat{\rho}_D^{(a)})W(u)Y(\alpha,\beta) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & |\xi|^2 \end{pmatrix} M^{(1)}(\hat{\rho}_D^{(a)}) \begin{pmatrix} 1 & 0 \\ 0 & |\xi|^2 \end{pmatrix}, \\
\xi &= u_{11}\alpha + u_{22}\beta.
\end{aligned} \tag{3.22}$$

From these expressions and from the results of Section 2.6, we find that the two-mode states produced from single-mode states in the above manner have the following significant properties:

$$\begin{aligned}
&(i)\ \hat{\rho}_D^{(a)} \text{ has QO-noncl PND } \Rightarrow\ \hat{\rho}_D^{(ab)} \text{ has NPT entanglement;} \\
&(ii)\ \hat{\rho}_D^{(a)} \text{ has super-PS } \Rightarrow\ \hat{\rho}_D^{(ab)} \text{ has super-PS;} \\
&(iii)\ \hat{\rho}_D^{(a)} \text{ has sub-PS } \Rightarrow\ \hat{\rho}_D^{(ab)} \text{ has Type I sub-PS,} \\
&\qquad\qquad M^{(2,1)}(\rho_D^{(ab)}; \alpha, \beta) \ngeq 0 \text{ for every } \alpha, \beta.
\end{aligned} \tag{3.23}$$

Clearly only properties (*ii*) and (*iii*) involve the Mandel matrix analysis; and it is significant that in property (*iii*), *every* single-mode combination of the modes a and b displays sub-PS. Thus in the latter circumstance, we can say it is not just easy, but actually very easy, to detect the Mandel kind of QO-noncl of $\hat{\rho}_D^{(ab)}$. To these statements we can add the following: states $\hat{\rho}_D^{(ab)}$ obtained from states $\hat{\rho}_D^{(a)}$ in the manner of Eq. (3.21) can never display Type II sub-PS; and any sub-PS in $\hat{\rho}_D^{(a)}$ leads to both Type I sub-PS and NPT entanglement in $\hat{\rho}_D^{(ab)}$.

3.3.2 Example (b)

Hereafter we consider directly given two-mode states $\hat{\rho}_D^{(ab)}$, as opposed to the previous example. In case such a state is QO-noncl, even if it is of product or separable form, its passage through a $U(2)$ BS could result in an entangled state. We first set up the general framework to examine this, then illustrate it by an interesting example. For simplicity we use an elementary 50:50 BS rather than one corresponding to a general $u \in U(2)$.

We choose the $U(2)$ element and corresponding unitary operator action as follows:

$$u_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in U(2): \ \hat{U}_0^{-1} \begin{pmatrix} \hat{a} & \hat{a}^\dagger \\ \hat{b} & \hat{b}^\dagger \end{pmatrix} \hat{U}_0 \ = u_0 \begin{pmatrix} \hat{a} & \hat{a}^\dagger \\ \hat{b} & \hat{b}^\dagger \end{pmatrix}. \tag{3.24}$$

At the operator level, action by conjugation on $\hat{C}$, $\hat{C}^\dagger$, $\hat{B}$ of Eqs. (3.9, 3.12) is:

$$\begin{aligned} \hat{U}_0^{-1}\hat{C}\hat{U}_0 &= V_0\hat{C}, \ \hat{U}_0^{-1}\hat{C}^\dagger\hat{U}_0 = \hat{C}^\dagger V_0^T, \ \hat{U}_0^{-1}\hat{B}\hat{U}_0 = V_0\hat{B}V_0^T, \\ V_0 &= u_0 \otimes u_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \end{aligned} \tag{3.25}$$

Then if a state $\hat{\rho}_D^{(ab)}$ is transformed by this BS action to

$$\hat{\rho}'^{(ab)}_D = \hat{U}_0\hat{\rho}_D^{(ab)}\hat{U}_0^{-1}, \tag{3.26}$$

the change in the Mandel matrix is given by a transformation using V_0:

$$\begin{aligned} M^{(2)}(\hat{\rho}_D^{(ab)}) &= \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix} \rightarrow \\ M^{(2)}(\hat{\rho}'^{\,(ab)}_{\;D}) &= \begin{pmatrix} 1 & C'^\dagger \\ C' & B' \end{pmatrix} = \mathrm{Tr}(\hat{\rho}'^{\,(ab)}_{\;D} \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{B} \end{pmatrix}) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_0^T \end{pmatrix}, \\ C' &= V_0 C,\ B' = V_0 B V_0^T. \end{aligned} \tag{3.27}$$

Thus Γ' is related to Γ by *congruence*:

$$\Gamma' = B' - C'C'^\dagger = V_0 \Gamma V_0^T. \tag{3.28}$$

To test next whether $\hat{\rho}'^{\,(ab)}_{\;D}$ is NPT entangled, we pass to its partial transpose $\hat{\rho}'^{\,(ab)PT}_{\;D}$ and evaluate the 'expectation value' of a suitably chosen nonnegative hermitian operator with respect to it. If this turns out to be negative, then $\hat{\rho}'^{\,(ab)}_{\;D}$ is definitely NPT entangled. To construct such a test which involves as closely as possible the use of $M^{(2)}(\hat{\rho}'^{\,(ab)}_{\;D})$, hence of $M^{(2)}(\hat{\rho}_D^{(ab)})$, we should use a 'matrix of operators' similar in structure to $\begin{pmatrix} 1 \\ \hat{C} \end{pmatrix} \begin{pmatrix} 1 & \hat{C}^\dagger \end{pmatrix}$, i.e, making up a 'hermitian nonnegative' matrix of operator entries, such that when the partial transpose operation is switched from $\hat{\rho}'^{\,(ab)PT}_{\;D}$ to this 'matrix', we obtain essentially the expectation values of $\hat{C}$, $\hat{C}^\dagger$ and $\hat{B}$ in $\hat{\rho}'^{\,(ab)}_{\;D}$. We have seen in the passage from Eq. (3.3) to Eq. (3.4) that the PT operation converts $\hat{b}^{j\dagger}\hat{b}^k$ to $\hat{b}^{k\dagger}\hat{b}^j$, and $\hat{b}^j\hat{b}^{k\dagger}$ to $\hat{b}^k\hat{b}^{j\dagger}$. Keeping these motivations and facts in mind we construct a

5×5 matrix of operators as follows :

$$\hat{E} = \begin{pmatrix} \hat{a}^\dagger \hat{a} \\ \hat{a}^\dagger \hat{b}^\dagger \\ \hat{a}\hat{b} \\ \hat{b}^\dagger \hat{b} \end{pmatrix}, \; \hat{E}^\dagger = \begin{pmatrix} \hat{a}^\dagger \hat{a} & \hat{a}\hat{b} & \hat{a}^\dagger \hat{b}^\dagger & \hat{b}^\dagger \hat{b} \end{pmatrix} \rightarrow$$

$$\left\{ \begin{pmatrix} 1 \\ \hat{E} \end{pmatrix} \begin{pmatrix} 1 & \hat{E}^\dagger \end{pmatrix} \right\}^{PT} = \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{B} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \hat{Y} \end{pmatrix},$$

$$\hat{Y} = \begin{pmatrix} \hat{a}^\dagger \hat{a} & 0 & \hat{a}^\dagger \hat{b} & 0 \\ 0 & 0 & 0 & 0 \\ \hat{b}^\dagger \hat{a} & 0 & \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1 & \hat{b}^\dagger \hat{a} \\ 0 & 0 & \hat{a}^\dagger \hat{b} & \hat{b}^\dagger \hat{b} \end{pmatrix}. \quad (3.29)$$

We see that in the process of expressing the various operators involved in normal ordered form, an additional piece $\hat{Y}$ linear in the entries of $\hat{C}$ appears. Then a test for NPT entanglement of $\hat{\rho}'^{\,(ab)}_{D}$ is to evaluate

$$\mathrm{Tr}(\hat{\rho}'^{\,(ab)PT}_{D} \begin{pmatrix} 1 \\ \hat{E} \end{pmatrix} \begin{pmatrix} 1 & \hat{E}^\dagger \end{pmatrix}) = \mathrm{Tr}(\hat{\rho}'^{\,(ab)}_{D} \left\{ \begin{pmatrix} 1 \\ \hat{E} \end{pmatrix} \begin{pmatrix} 1 & \hat{E}^\dagger \end{pmatrix} \right\}^{PT})$$

$$= M^{(2)}(\hat{\rho}'^{\,(ab)}_{D}) + \begin{pmatrix} 0 & \vdots & 0 & 0 & 0 & 0 \\ \cdots & & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & & & & \\ 0 & \vdots & & & & \\ 0 & \vdots & & & Y' & \\ 0 & \vdots & & & & \end{pmatrix}$$

$$Y' = \begin{pmatrix} C'_1 & 0 & C'_2 & 0 \\ 0 & 0 & 0 & 0 \\ C'_3 & 0 & C'_1 + C'_4 + 1 & C'_3 \\ 0 & 0 & C'_2 & C'_4 \end{pmatrix}, \quad (3.30)$$

and see if this matrix is indefinite. The 5×5 matrix here is, by Eq. (3.27), a congruence transformation applied to the initial state Mandel matrix $M^{(2)}(\hat{\rho}^{(ab)}_{D})$ plus a 4×4 piece

coming from $\hat{Y}$, namely it is :

$$\begin{pmatrix} 1 & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_0^T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Y' \end{pmatrix} = \begin{pmatrix} 1 & C^\dagger V_0^T \\ V_0 C & V_0 B V_0^T + Y' \end{pmatrix}. \quad (3.31)$$

Therefore by Eq. (3.17) the positivity or otherwise of the matrix (3.30) is equivalent to the positivity or otherwise of either of the two following 4×4 matrices at the level of Γ :

$$\begin{aligned} \Omega &= \Gamma + V_0^T Y' V_0 \,, \\ V_0 \Omega V_0^T &= V_0 \Gamma V_0^T + Y'. \end{aligned} \quad (3.32)$$

Nonpositivity of either Ω or $V_0 \Omega V_0^T$ is proof of NPT entanglement of $\hat{\rho'}_D^{(ab)}$. There is some difference between the Mandel level QO-nonclassicality test for $\hat{\rho}_D^{(ab)}$ and the above developed NPT entanglement test for $\hat{\rho'}_D^{(ab)}$, the two being related but not identical. This is to be expected since, as mentioned, entangled states are a subset of QO-noncl states, and NPT states are a further subset.

We now illustrate the above scheme using an interesting family of states which is analytically quite simple. We begin with the family of two-mode pure states of infinite Schmidt rank,

$$|\mu\rangle = e^{-\frac{1}{2}|\mu|^2} \sum_{n=0}^{\infty} \frac{\mu^n}{\sqrt{n!}} |n, n\rangle, \ \mu \in \mathcal{C}, \quad (3.33)$$

form the density matrix $\hat{\rho}^{(ab)} = |\mu\rangle\langle\mu|$, and pass to $\hat{\rho}_D^{(ab)}$ via Eq. (3.6) :

$$\hat{\rho}_D^{(ab)} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |n, n\rangle\langle n, n|, \ \lambda = |\mu|^2 \geq 0. \quad (3.34)$$

This is clearly separable though not of product form. For the Mandel matrix analysis, $|\mu\rangle\langle\mu|$ and $\hat{\rho}_D^{(ab)}$ are equivalent.

The matrices C, $C^\dagger$, B, Γ involved in $M^{(2)}(\hat{\rho}_D^{(ab)})$ are easy to calculate since we have

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle \hat{b}^\dagger \hat{b} \rangle = \lambda, \ \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle = \langle \hat{b}^{\dagger 2} \hat{b}^2 \rangle = \lambda^2. \quad (3.35)$$

We thus have :

$$C = \begin{pmatrix} \lambda \\ 0 \\ 0 \\ \lambda \end{pmatrix}, \quad B = \begin{pmatrix} \lambda^2 & 0 & 0 & \lambda^2+\lambda \\ 0 & \lambda^2+\lambda & 0 & 0 \\ 0 & 0 & \lambda^2+\lambda & 0 \\ \lambda^2+\lambda & 0 & 0 & \lambda^2 \end{pmatrix};$$
$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & \lambda^2+\lambda & 0 & 0 \\ 0 & 0 & \lambda^2+\lambda & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix}. \tag{3.36}$$

The eigenvalues of Γ being $\lambda(\lambda+1)$, $\lambda(\lambda+1)$, λ, $-\lambda$, it follows that the state $\hat{\rho}_D^{(ab)}$ in (3.34) is QO-noncl. To find its type we compute

$$\psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta) = 2|\alpha|^2|\beta|^2\lambda(\lambda+2) \geq 0, \tag{3.37}$$

so these states display hidden or Type II- sub-PS.

In passing we note that the state $\hat{\rho}^{(a)}$ of mode a obtained from Eq. (3.34) by tracing over b alone is

$$\hat{\rho}_D^{(a)} = \mathrm{Tr}_b \hat{\rho}_D^{(ab)} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |n\rangle_{aa}\langle n|, \tag{3.38}$$

for which the diagonal weight $P(I_a)$ is

$$P(I_a) = \delta(I_a - \lambda). \tag{3.39}$$

Partial trace over a gives exactly similar results for mode b. Thus both $\hat{\rho}_D^{(a)}$ and $\hat{\rho}_D^{(b)}$ are QO-cl, with their PND coinciding exactly with that of a coherent state.

Now we pass the two-mode state $\hat{\rho}_D^{(ab)}$ of Eq. (3.34) through the BS $\hat{U}_0$ of Eq. (3.24); the resulting $\hat{\rho}'^{(ab)}_D$ is

$$\begin{aligned} \hat{\rho}'^{(ab)}_D &= \hat{U}_0 \hat{\rho}_D^{(ab)} \hat{U}_0^{-1} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \left(\frac{\lambda}{4}\right)^n \frac{1}{n!^3} (\hat{a}^{\dagger 2} - \hat{b}^{\dagger 2})^n |0,0\rangle\langle 0,0| (\hat{a}^2 - \hat{b}^2)^n. \end{aligned} \tag{3.40}$$

To apply the NPT entanglement test based on Eq. (3.32) it is convenient to examine $V_0 \Omega V_0^T$. Combining Eqs. (3.27, 3.36) we find the matrices Γ', Y' associated with $\hat{\rho}'^{(ab)}_D$

to be

$$\begin{aligned}\Gamma' = V_0 \Gamma V_0^T &= \begin{pmatrix} \frac{1}{2}\lambda^2+\lambda & 0 & 0 & -\frac{1}{2}\lambda^2 \\ 0 & \frac{1}{2}\lambda^2 & -\frac{1}{2}\lambda^2-\lambda & 0 \\ 0 & -\frac{1}{2}\lambda^2-\lambda & \frac{1}{2}\lambda^2 & 0 \\ -\frac{1}{2}\lambda^2 & 0 & 0 & \frac{1}{2}\lambda^2+\lambda \end{pmatrix} \\ Y' &= \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda+1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}. \end{aligned} \tag{3.41}$$

Therefore according to Eq. (3.32) we have to test the positivity or otherwise of

$$\begin{aligned} V_0 \Omega V_0^T &= V_0 \Gamma V_0^T + Y' \\ &= \begin{pmatrix} \frac{1}{2}\lambda^2+2\lambda & 0 & 0 & -\frac{1}{2}\lambda^2 \\ 0 & \frac{1}{2}\lambda^2 & -\frac{1}{2}\lambda^2-\lambda & 0 \\ 0 & -\frac{1}{2}\lambda^2-\lambda & \frac{1}{2}\lambda^2+2\lambda+1 & 0 \\ -\frac{1}{2}\lambda^2 & 0 & 0 & \frac{1}{2}\lambda^2+2\lambda \end{pmatrix} \end{aligned} \tag{3.42}$$

The $(2,3)$ submatrix here is indefinite as it has determinant $-\frac{1}{2}\lambda^2$. *This establishes that* $\hat{\rho'}_D^{(ab)}$ *of Eq. (3.40) is NPT entangled. The emphasis here was to show that the entanglement produced by BS action can indeed be witnessed by the Mandel matrix construct.*

Going back to the expression in Eq. (3.40), the terms for $n = 0$ and $n = 1$ are respectively :

$$\begin{aligned} & e^{-\lambda}|0,0\rangle\langle 0,0|; \\ & \frac{\lambda}{2}e^{-\lambda}(|2,0\rangle - |0,2\rangle)(\langle 2,0| - \langle 0,2|), \end{aligned} \tag{3.43}$$

giving the matrix elements

$$\begin{aligned} & (\hat{\rho'}_D^{(ab)})_{00,00} = e^{-\lambda}; \\ & (\hat{\rho'}_D^{(ab)})_{20,20} = (\hat{\rho'}_D^{(ab)})_{02,02} = -(\hat{\rho'}_D^{(ab)})_{20,02} = -(\hat{\rho'}_D^{(ab)})_{02,20} = \frac{\lambda}{2}e^{-\lambda}. \end{aligned} \tag{3.44}$$

One also obtains from the $n = 2$ term in Eq. (3.40) the matrix element

$$(\hat{\rho'}_D^{(ab)})_{22,22} = \frac{\lambda^2}{8}e^{-\lambda}. \tag{3.45}$$

If we now consider the partial transpose of $\hat{\rho'}_D^{(ab)}$ and look at the $00-22$ subspace of

dimension two, we have

$$\hat{\rho'}_{D}^{(ab)PT} \rightarrow e^{-\lambda} \begin{pmatrix} 1 & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & \frac{\lambda^2}{8} \end{pmatrix}. \tag{3.46}$$

Thus, as the determinant is negative in this subspace, $\hat{\rho'}_{D}^{(ab)PT}$ has an eigenvector of the form $\alpha|0,0\rangle + \beta|2,2\rangle$ with a negative eigenvalue. *This demonstrates that the NPT entanglement of* $\hat{\rho'}_{D}^{(ab)}$ *in Eq. (3.40) is distillable* [37].

3.3.3 Example (c)

The third example is built upon the two-mode squeezed (vacuum) states. For both single and multi-mode cases these states have been studied extensively in the literature [187, 237, 238]. Here we take them up in the context of the viewpoints of Sections 3.1, 3.2 and 3.3, in particular the Mandel level study of QO-nonclassicality. First we assemble the definition and important properties of a single (a) mode squeezed vacuum state. Such a state is obtained by applying a unitary (scaling) operator involving the exponential of a complex combination of $\hat{a}^{\dagger 2}$ and $\hat{a}^2$ to the Fock vacuum, and is parametrised by a complex variable $\xi = \xi_1 + i\xi_2$ or an equivalent complex variable ω :

$$\begin{aligned} |\psi^{(a)}(\omega)\rangle &= \exp\{\frac{1}{4}(\xi\hat{a}^{\dagger 2} - \xi^*\hat{a}^2)\}|0\rangle_a \\ &= (1-|\omega|^2)^{\frac{1}{4}} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+1/2)}{n!\sqrt{\pi}}}\, \omega^n\, |2n\rangle_a, \\ \omega &= \frac{\xi}{|\xi|}\tanh(|\xi|/2). \end{aligned} \tag{3.47}$$

Since only even photon number states are present, the probabilities $p(1)$, $p(3)$, $p(5)$, $\cdots$ in the PND vanish, which is immediate evidence that these states are QO-noncl. Some important expectation values are :

$$\begin{aligned} &\langle\psi^{(a)}(\omega)|\{\hat{a}^\dagger, \hat{a},\ \hat{N}_a,\ \hat{N}_a^2,\ \hat{a}^{\dagger 2}\hat{a}^2,\ \hat{a}^2\}|\psi^{(a)}(\omega)\rangle = \\ &\qquad \{0,\ 0,\ S^2,\ S^2(S^2+2C^2),\ S^2(2S^2+C^2),\ \frac{\xi}{|\xi|}\}, \\ &S = \sinh(|\xi|/2),\ C = \cosh(|\xi|/2). \end{aligned} \tag{3.48}$$

The 2×2 Mandel matrix for the state is thus :

$$\begin{aligned} M^{(1)}(|\psi^{(a)}(\omega)\rangle) &= \begin{pmatrix} 1 & S^2 \\ S^2 & S^2(2S^2+C^2) \end{pmatrix}, \\ \det M^{(1)}(|\psi^{(a)}(\omega)\rangle) &= S^2(S^2+C^2) \geq 0, \end{aligned} \tag{3.49}$$

where S and C are given in Eq. (3.48). Thus these states have super-PS, and the QO-nonclassicality does not show up at the Mandel level.

For two modes we take the product of two such states, with independent complex ξ, ξ' or ω, ω' :

$$|\psi^{(ab)}(\omega,\omega')\rangle = |\psi^{(a)}(\omega)\rangle \otimes |\psi^{(b)}(\omega')\rangle. \tag{3.50}$$

The second factor obviously involves an exponential in $\hat{b}^{\dagger 2}$ and $\hat{b}^2$ applied to $|0\rangle_b$. This two-mode pure state is clearly also QO-noncl, but it is a manifest product state of Schmidt rank one. Unlike Eq. (3.49), however, now the QO-nonclassicality shows up at the Mandel level. The 5×5 Mandel matrix for the state (3.50) is easily found using Eqs. (3.48) and their analogues for the b-mode :

$$\begin{aligned}
&M^{(2)}(|\psi^{(ab)}(\omega,\omega')\rangle) = \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix}, \\
&C^\dagger = \begin{pmatrix} S^2 & 0 & 0 & S'^2 \end{pmatrix}, \\
&B = \begin{pmatrix} S^2(2S^2+C^2) & 0 & 0 & S^2 S'^{\,2} \\ 0 & S^2 S'^{\,2} & e^{i\eta} SCS'C' & 0 \\ 0 & e^{-i\eta} SCS'C' & S^2 S'^{\,2} & 0 \\ S^2 S'^{\,2} & 0 & 0 & S'^{\,2}(2S'^{\,2}+C'^{\,2}) \end{pmatrix}, \\
&\eta = \arg \xi' \xi^*
\end{aligned} \tag{3.51}$$

Here S' and C' are defined as in Eq. (3.48) but in terms of ξ'. The 4×4 matrix Γ of Eq. (3.17) is :

$$\Gamma = \begin{pmatrix} S^2(S^2+C^2) & 0 & 0 & 0 \\ 0 & S^2 S'^{\,2} & e^{i\eta} SCS'C' & 0 \\ 0 & e^{-i\eta} SCS'C' & S^2 S'^{\,2} & 0 \\ 0 & 0 & 0 & S'^{\,2}(S'^{\,2}+C'^{\,2}) \end{pmatrix}. \tag{3.52}$$

The eigenvalues of Γ are $S^2(S^2+C^2)$, $S'^2(S'^2+C'^2)$, $SS'(SS'+CC')$ and $SS'(SS'-CC')$. Assuming that ξ, ξ' are both non vanishing, the last eigenvalue is negative, leading by Eq. (3.17) to the conclusion that
$M^{(2)}(|\psi^{(ab)}(\omega,\omega')\rangle) \not\geq 0$ or that the state $|\psi^{(ab)}(\omega,\omega')\rangle$ has sub-PS. *This is an interesting and somewhat nonintuitive result* since we have seen in Eq. (3.49) that each factor in the product state $|\psi^{(ab)}(\omega,\omega')\rangle$ has super-PS. We must now see whether it is Type I or Type II. For this we must compute the 'expectation value' of Γ in Eq. (3.52) for the

four-component column vector $\psi_0(\alpha,\beta)$ as required by Eq. (3.19) :

$$\begin{aligned}\psi_0(\alpha,\beta)^\dagger\Gamma\psi_0(\alpha,\beta) = \\ (|\alpha|^2S^2+|\beta|^2S'^2)^2+|\alpha|^4S^2C^2+|\beta|^4S'^2C'^2+2SCS'C'\Re(e^{i\eta}(\alpha^*\beta)^2) \\ \geq (|\alpha|^2S^2+|\beta|^2S'^2)^2+(|\alpha|^2SC-|\beta|^2S'C')^2>0, \qquad (3.53)\end{aligned}$$

since $\Re(e^{i\eta}(\alpha^*\beta)^2)\geq -|\alpha|^2|\beta|^2$. It follows that the sub-PS of the product state $|\psi^{(ab)}(\omega,\omega')\rangle$ is of Type II, it is hidden or intrinsic as in the state in Eq. (3.34). That it is Type II is consistent with the fact that the individual states $|\psi^{(a)}(\omega)\rangle$ and $|\psi^{(b)}(\omega')\rangle$ are both super-PS.

It was noted above that the two-mode state $|\psi^{(ab)}(\omega,\omega')\rangle$ is both pure and of Schmidt rank one. The result of action by the BS $\hat{U}_0$ of Eq. (3.24) on it is seen upon inspection and without any calculations to be an entangled (pure) state :

$$\begin{aligned}\hat{U}_0|\psi^{(ab)}(\omega,\omega')\rangle &= \hat{U}_0\exp\{\frac{1}{4}(\xi\hat{a}^{\dagger 2}-\xi^*\hat{a}^2)+\frac{1}{4}(\xi'\hat{b}^{\dagger 2}-\xi'^*\hat{b}^2)\}\hat{U}_0^{-1}|0,0\rangle \\ &= \exp\{\frac{1}{8}(\xi(\hat{a}^\dagger-\hat{b}^\dagger)^2-\xi^*(\hat{a}-\hat{b})^2)+\frac{1}{8}(\xi'(\hat{a}^\dagger+\hat{b}^\dagger)^2-\xi'^*(\hat{a}+\hat{b})^2)\}|0,0\rangle. \qquad (3.54)\end{aligned}$$

This is because the final unitary operator acting on $|0,0\rangle$ is clearly not the tensor product of individual unitary operators acting separately on the two modes. On account of this simplicity in this sense of the initial state $|\psi^{(ab)}(\omega,\omega')\rangle$, there is no need to apply after the BS action the Mandel level NPT entanglement test developed in connection with Example (b). It is of course important that the states $|\psi^{(a)}(\omega)\rangle$, $|\psi^{(b)}(\omega')\rangle$ in the initial product are both QO-noncl. A two-mode pure product QO-cl state is necessarily a product of single-mode coherent states, and the product structure is maintained by BS action in this case because the factors are coherent states.

3.4 From two-mode nonclassicality to three-mode entanglement

We have studied the possibility of a $U(2)$ beamsplitter converting a two-mode QO-noncl separable state into an entangled one since for such systems both nonclassicality and entanglement are meaningful concepts. Now we present a treatment of two-mode states analogous to that given in Section 2.6 for single-mode systems. That is, we couple a given two-mode state $\hat{\rho}_D^{(ab)}$ to a third c-mode in vacuum, pass such an input state $\hat{\rho}_{\rm in}^{(abc)}$ through a '$U(3)$ beamsplitter', and obtain a three-mode output state $\hat{\rho}_{\rm out}^{(abc)}$. We then test whether this shows NPT entanglement as a consequence of Mandel type QO-

nonclassicality initially present in $\hat{\rho}_D^{(ab)}$, the partial transpose operation being applied to the c-mode.

We begin with $\hat{\rho}_D^{(ab)}$ for which $M^{(2)}(\hat{\rho}_D^{(ab)})$ possibly shows QO-nonclassicality. We take a third c-mode in vacuum and have an input three-mode state

$$\hat{\rho}_{\rm in}^{(abc)} = \hat{\rho}_D^{(ab)} \otimes |0\rangle_{cc}\langle 0|, \tag{3.55}$$

strictly analogous to Eq. (2.43). To a general matrix $u \in U(3)$ we associate a passive 'beamsplitter' which unitarily mixes the annihilation operators of the three modes in a manner analogous to Eq. (2.31), now conserving $\hat{N}_a + \hat{N}_b + \hat{N}_c$ [337]. In the three-mode Hilbert space this beamsplitter u acts through an unitary operator $\hat{U}$, and we have

$$\begin{aligned}
u &= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \in U(3) \;\rightarrow\; \hat{U}:\; \hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = 1, \\
\hat{U}\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix}\hat{U}^{-1} &= u^\dagger \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix}, \quad \hat{U}\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \\ \hat{c}^\dagger \end{pmatrix}\hat{U}^{-1} = u^T \begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \\ \hat{c}^\dagger \end{pmatrix}, \\
\hat{U}^{-1}\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix}\hat{U} &= u \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix}, \quad \hat{U}^{-1}\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \\ \hat{c}^\dagger \end{pmatrix}\hat{U} = u^* \begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \\ \hat{c}^\dagger \end{pmatrix}, \\
\hat{U}(\hat{N}_a + \hat{N}_b + \hat{N}_c) &= (\hat{N}_a + \hat{N}_b + \hat{N}_c)\hat{U}.
\end{aligned} \tag{3.56}$$

Therefore upon passage through this 'beamsplitter' the state in Eq. (3.55) changes to

$$\hat{\rho}_{\rm out}^{(abc)} = \hat{U}\hat{\rho}_{\rm in}^{(abc)}\hat{U}^{-1} = \hat{U}\{\hat{\rho}_D^{(ab)} \otimes |0\rangle_{cc}\langle 0|\}\hat{U}^{-1}. \tag{3.57}$$

To test this output state for NPT entanglement, we apply the partial transpose to the c-mode and then evaluate the 'expectation value' of a suitably chosen hermitian nonneg-

ative operator :

$$
\begin{aligned}
A &= \alpha_0 + \alpha_1 \hat{a}\hat{c} + \alpha_2 \hat{b}\hat{c} + \alpha_3 \hat{a}^\dagger \hat{c}^\dagger + \alpha_4 \hat{b}^\dagger \hat{c}^\dagger : \\
&\quad \mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(abc)PT} A^\dagger A) = \alpha^\dagger X \alpha, \\
X &= \mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(abc)PT} \begin{pmatrix} 1 \\ \hat{a}^\dagger \hat{c}^\dagger \\ \hat{b}^\dagger \hat{c}^\dagger \\ \hat{a}\hat{c} \\ \hat{b}\hat{c} \end{pmatrix} \begin{pmatrix} 1 & \hat{a}\hat{c} & \hat{b}\hat{c} & \hat{a}^\dagger \hat{c}^\dagger & \hat{b}^\dagger \hat{c}^\dagger \end{pmatrix}); \qquad \alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}. \quad (3.58)
\end{aligned}
$$

The 5×5 hermitian matrix X, constructed by taking entrywise expectation values as defined, is expected to be related to the input Mandel matrix $M^{(2)}(\hat{\rho}_D^{(ab)})$. Developing it we find

$$
\left\{ \begin{pmatrix} 1 \\ \hat{a}^\dagger \hat{c}^\dagger \\ \hat{b}^\dagger \hat{c}^\dagger \\ \hat{a}\hat{c} \\ \hat{b}\hat{c} \end{pmatrix} \begin{pmatrix} 1 & \hat{a}\hat{c} & \hat{b}\hat{c} & \hat{a}^\dagger \hat{c}^\dagger & \hat{b}^\dagger \hat{c}^\dagger \end{pmatrix} \right\}^{PT} =
$$

$$
: \begin{pmatrix} 1 \\ \hat{a}^\dagger \hat{c} \\ \hat{b}^\dagger \hat{c} \\ \hat{c}^\dagger \hat{a} \\ \hat{c}^\dagger \hat{b} \end{pmatrix} \begin{pmatrix} 1 & \hat{c}^\dagger \hat{a} & \hat{c}^\dagger \hat{b} & \hat{a}^\dagger \hat{c} & \hat{b}^\dagger \hat{c} \end{pmatrix} : + \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & \multicolumn{2}{c}{\multirow{2}{*}{$\hat{Z}$}} \\ 0 & 0 & 0 & \vdots & & \end{pmatrix},
$$

$$
\begin{aligned}
\hat{Z} &= \begin{pmatrix} \hat{a}^\dagger \hat{a} + \hat{c}^\dagger \hat{c} + 1 & \hat{b}^\dagger \hat{a} \\ \hat{a}^\dagger \hat{b} & \hat{b}^\dagger \hat{b} + \hat{c}^\dagger \hat{c} + 1 \end{pmatrix} \\
&= : \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \begin{pmatrix} \hat{a}^\dagger & \hat{b}^\dagger \end{pmatrix} : \; + \; (1 + \hat{c}^\dagger \hat{c}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (3.59)
\end{aligned}
$$

Using this in Eq. (3.58) we have after implementing the conjugation $\hat{U}^{-1}(\cdots)\hat{U}$:

$$
\begin{aligned}
X \quad &= \quad \mathrm{Tr}(\hat{\rho}_D^{(ab)} : \begin{pmatrix} 1 \\ \hat{a}'^\dagger \hat{c}' \\ \hat{b}'^\dagger \hat{c}' \\ \hat{c}'^\dagger \hat{a}' \\ \hat{c}'^\dagger \hat{b}' \end{pmatrix} \begin{pmatrix} 1 & \hat{c}'^\dagger \hat{a}' & \hat{c}'^\dagger \hat{b}' & \hat{a}'^\dagger \hat{c}' & \hat{b}'^\dagger \hat{c}' \end{pmatrix} :) \\
&\quad + \mathrm{Tr}(\hat{\rho}_D^{(ab)} \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & \multicolumn{2}{c}{\multirow{2}{*}{$\hat{Z}'$}} \\ 0 & 0 & 0 & \vdots & & \end{pmatrix}, \\
\hat{Z}' \quad &= \quad : \begin{pmatrix} \hat{a}' \\ \hat{b}' \end{pmatrix} \begin{pmatrix} \hat{a}'^\dagger & \hat{b}'^\dagger \end{pmatrix} : \; + \; (1 + \hat{c}'^\dagger \hat{c}') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\begin{pmatrix} \hat{a}' \\ \hat{b}' \\ \hat{c}' \end{pmatrix} \quad &= \quad \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}. \qquad (3.60)
\end{aligned}
$$

Here the fact that the c-mode is initially in vacuum has been used, and the appearance of the extra $\hat{Z}$, $\hat{Z}'$ terms is a result of normal ordering similar to the presence of $\hat{Y}$ in Eq. (3.29). One can now disentangle the u-dependences and express the result in terms

of $M^{(2)}(\hat{\rho}_D^{(ab)})$ and $C = \mathrm{Tr}(\hat{\rho}_D^{(ab)}\hat{C})$:

$$X = W(u)M^{(2)}(\hat{\rho}_D^{(ab)})W(u)^\dagger + \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & Z' & \\ 0 & 0 & 0 & \vdots & & \end{pmatrix},$$

$$W(u) = \begin{pmatrix} 1 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ \cdots & & \cdots & \cdots & & \cdots & \cdots \\ 0 & \vdots & \begin{pmatrix} u_{11}^* \\ u_{21}^* \end{pmatrix} & \begin{pmatrix} u_{31} & u_{32} \end{pmatrix} & \vdots & \begin{pmatrix} u_{12}^* \\ u_{22}^* \end{pmatrix} & \begin{pmatrix} u_{31} & u_{32} \end{pmatrix} \\ 0 & \vdots & & & \vdots & & \\ \cdots & & \cdots & \cdots & & \cdots & \cdots \\ 0 & \vdots & u_{31}^* \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} & & \vdots & u_{32}^* \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} & \\ 0 & \vdots & & & \vdots & & \end{pmatrix},$$

$$Z' = \mathrm{Tr}(\hat{\rho}_D^{(ab)}\hat{Z}'),$$

$$\hat{Z}' = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} : \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \begin{pmatrix} \hat{a}^\dagger & \hat{b}^\dagger \end{pmatrix} : \begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix}$$
$$+(1 + (u_{31}^*\hat{a}^\dagger + u_{32}^*\hat{b}^\dagger)(u_{31}\hat{a} + u_{32}\hat{b})) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (3.61)$$

If this matrix X, dependent on $\hat{\rho}_D^{(ab)}$ and $u \in U(3)$, is indefinite, the NPT entangled nature of $\hat{\rho}_{\mathrm{out}}^{(abc)}$ of Eq. (3.57) follows. Of course this can happen only if $\hat{\rho}_D^{(ab)}$ is QO-noncl, (at the level of its Mandel matrix), since the '$U(3)$ beamsplitter' $\hat{U}$ would map any QO-cl input into similar output.

As illustrations of this general procedure we consider two simple examples. The first is a two-mode state with only a finite number of photons, so that its QO-nonclassicality is a foregone conclusion :

$$\hat{\rho}_D^{(ab)} = p|2,0\rangle\langle 2,0| + q|1,1\rangle\langle 1,1| + r|0,2\rangle\langle 0,2|,$$
$$p,q,r \geq 0 \qquad p+q+r = 1. \qquad (3.62)$$

This is separable, though not a product state. The only non vanishing expectation values

needed to construct its Mandel matrix are

$$\begin{aligned} \langle \hat{a}^\dagger \hat{a} \rangle &= 2p+q,\ \langle \hat{b}^\dagger \hat{b} \rangle = 2r+q,\ \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle = 2p, \\ \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle &= q,\ \langle \hat{b}^{\dagger 2} \hat{b}^2 \rangle = 2r. \end{aligned} \tag{3.63}$$

Therefore the Mandel matrix is

$$M^{(2)}(\hat{\rho}_D^{(ab)}) = \begin{pmatrix} 1 & q+2p & 0 & 0 & q+2r \\ q+2p & 2p & 0 & 0 & q \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ q+2r & q & 0 & 0 & 2r \end{pmatrix}. \tag{3.64}$$

The determinants of various nontrivial 2×2 submatrices, the one nontrivial 3×3 submatrix, and finally of $M^{(2)}(\hat{\rho}_D^{(ab)})$ itself, are (indicating the submatrices by the relevant rows and columns) :

$$\begin{aligned} &(1,2):\ 2p-(q+2p)^2;\ \ (1,5):\ 2r-(q+2r)^2;\ \ (2,5):\ 4pr-q^2; \\ &(1,2,5):\ q^2-4pr; \\ &\det M^{(2)}(\hat{\rho}_D^{(ab)}) = q^2(q^2-4pr). \end{aligned} \tag{3.65}$$

One can easily imagine situations for which the $(1,2)$ and $(1,5)$ submatrices become indefinite, for instance q close to unity and p, r close to zero. In any case, since the $(2,5)$ subdeterminant is opposite in sign to the $(1,2,5)$ subdeterminant and to the full determinant, the state in Eq. (3.62) is always QO-noncl at the Mandel matrix level.

The type of sub-PS can be determined easily. From Eq. (3.64) we find the 4×4 matrix Γ to be

$$\begin{aligned} \Gamma &= \begin{pmatrix} \delta_a & 0 & 0 & q-(q+2p)(q+2r) \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ q-(q+2p)(q+2r) & 0 & 0 & \delta_b \end{pmatrix}, \\ \delta_a &= 2p-(q+2p)^2\,, \quad \delta_b = 2r-(2r+q)^2. \end{aligned} \tag{3.66}$$

Therefore also

$$\begin{aligned} &\psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta) = \\ &2p|\alpha|^4 + 4q|\alpha|^2|\beta|^2 + 2r|\beta|^4 - ((q+2p)|\alpha|^2 + (q+2r)|\beta|^2)^2. \end{aligned} \tag{3.67}$$

For $\alpha = 1$, $\beta = 0$ this becomes δ_a; for $\alpha = 0$, $\beta = 1$ it is δ_b. We now consider p running over its range $[0, 1]$ in successive portions and see what conclusions can be drawn:

$$\begin{aligned}
p = 0: \quad & q = 0 \;\Rightarrow\; \delta_b = -2;\; q > 0 \;\Rightarrow\; \delta_a < 0; \\
0 < p < \frac{1}{2}: \quad & \delta_a > 0 \;\Rightarrow\; 2p - (p - r + 1)^2 \geq 0 \;\Rightarrow \\
& (p - r)^2 + 1 - 2r < 0 \;\Rightarrow\; 2r > 1 \;\Rightarrow\; \delta_b < 0; \\
& \delta_a = 0 \;\Rightarrow\; (p - r)^2 + 1 - 2r = 0 \;\Rightarrow\; p \neq r, \\
& 2r > 1 \;\Rightarrow\; \delta_b < 0; \\
p = \frac{1}{2}: \quad & q = 0 \;\Rightarrow\; p = r = \frac{1}{2},\; \delta_a = \delta_b = 0; \\
& q > 0 \;\Rightarrow\; \delta_a < 0; \\
\frac{1}{2} < p \leq 1: \quad & 2p > 1 \;\Rightarrow\; \delta_a < 0.
\end{aligned} \tag{3.68}$$

Thus in every situation except $p = r = \frac{1}{2}$, $q = 0$, either δ_a or δ_b is negative. In this one exceptional case we find from Eq. (3.67):

$$p = r = \frac{1}{2},\; q = 0: \quad \psi_0(\alpha, \beta)^\dagger \Gamma \psi_0(\alpha, \beta) = -2|\alpha|^2|\beta|^2, \tag{3.69}$$

which is negative for $\alpha, \beta \neq 0$. This establishes that for the state (3.62) the sub-PS is of Type I.

Now we couple this state to the third c-mode in vacuum, and pass it through a particular $U(3)$ beamsplitter, namely a $50:50$ beamsplitter acting on the b and c modes alone. The final output state is calculated using Eq. (3.57), and to test its NPT entanglement we need to calculate the matrix X of Eq. (3.61) involving the $M^{(2)}(\hat{\rho}_D^{(ab)})$ term and the added Z' term. The choice of $u \in U(3)$, the resulting $W(u)$, and the two terms making

up X are as follows.

$$
\begin{aligned}
u &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \in U(3); \\
W(u) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & -1/2 \end{pmatrix}; \\
W(u)M^{(2)}(\hat{\rho}_D^{(ab)})W(u)^\dagger &= \left(\begin{array}{ccccc} 1 & 0 & -r-q/2 & 0 & -r-q/2 \\ 0 & q/2 & 0 & 0 & 0 \\ -r-q/2 & 0 & r/2 & 0 & r/2 \\ & & & . \;\; . & \;\; . \\ 0 & 0 & 0 & . \;\; q/2 & 0 \\ -r-q/2 & 0 & r/2 & . \;\; 0 & r/2 \end{array}\right); \\
Z' &= \begin{pmatrix} 2p+3q/2+r+1 & 0 \\ 0 & q+2r+1 \end{pmatrix}. \qquad (3.70)
\end{aligned}
$$

The dotted lines in (3.70) indicate where the 2×2 block Z' has to be added to this box. Leaving out the trivial second and fourth rows and columns as they do not couple to any others, the determinants of the various 2×2 and the one 3×3 submatrix in X are :

$$
\begin{aligned}
&(1,3): \; r/2-(q/2+r)^2; \;\; (1,5): \; 5r/2+q+1-(q/2+r)^2; \\
&(3,5): \; r(q+2r+1)/2; \;\; (1,3,5): \; (q+2r+1)(r/2-(q/2+r)^2). \qquad (3.71)
\end{aligned}
$$

Comparing Eqs. (3.65) and (3.71) we see: whenever the QO-nonclassicality of $\hat{\rho}_D^{(ab)}$ manifests itself in the $(1,5)$ submatrix of $M^{(2)}(\hat{\rho}_D^{(ab)})$ being indefinite, simultaneously the 3-mode state $\hat{\rho}_{\rm out}^{(abc)}$ displays NPT entanglement. If one had on the other hand an indefinite $(1,2)$ submatrix in $M^{(2)}(\hat{\rho}_D^{(ab)})$, Eq. (3.65), then by suitably altering the $U(3)$ element u in Eq. (3.70) we could achieve NPT entanglement of $\hat{\rho}_{\rm out}^{(abc)}$. In both situations, the signatures of QO-nonclassicality in $\hat{\rho}_D^{(ab)}$ and of NPT entanglement in $\hat{\rho}_{\rm out}^{(abc)}$ coincide.

The second example to illustrate the ideas of this Section is similar in structure to example (3.34) of the preceding Section, but its properties differ in certain details. For a

nonnegative real parameter η we define the separable state

$$\hat{\rho}_D^{(ab)} = \frac{1}{C}\sum_{n=0}^{\infty}\frac{\eta^{2n}}{(2n)!}|n,n\rangle\langle n,n|, \tag{3.72}$$

where $C = \mathrm{Cosh}\,\eta$, $S = \mathrm{Sinh}\,\eta$ and $t = \tanh\eta$. Clearly, the case $\eta = 0$ corresponds to the two-mode vacuum, and so we are interested in the parameter range $0 < \eta < \infty$. Using the elementary sums

$$\sum_{n=0}^{\infty}(n \text{ or } n^2)\frac{\eta^{2n}}{(2n)!} = \frac{\eta}{2}S \text{ or } \frac{\eta}{4}(S + \eta\, C), \tag{3.73}$$

the nonzero expectation values needed for the Mandel matrix are :

$$\begin{aligned}
&\langle\hat{a}^\dagger\hat{a}\rangle = \langle\hat{b}^\dagger\hat{b}\rangle = \frac{\eta}{2}t;\\
&\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle = \langle\hat{b}^{\dagger 2}\hat{b}^2\rangle = \frac{\eta}{4}(\eta - t);\\
&\langle\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}\rangle = \frac{\eta}{4}(\eta + t).
\end{aligned} \tag{3.74}$$

Therefore we have

$$M^2(\hat{\rho}_D^{ab}) = \begin{pmatrix}
1 & \frac{\eta t}{2} & 0 & 0 & \frac{\eta t}{2}\\
\frac{\eta t}{2} & \frac{\eta}{4}(\eta - t) & 0 & 0 & \frac{\eta}{4}(\eta + t)\\
0 & 0 & \frac{\eta}{4}(\eta + t) & 0 & 0\\
0 & 0 & 0 & \frac{\eta}{4}(\eta + t) & 0\\
\frac{\eta t}{2} & \frac{\eta}{4}(\eta + t) & 0 & 0 & \frac{\eta}{4}(\eta - t)
\end{pmatrix}. \tag{3.75}$$

Leaving out the third and fourth rows and columns, the remaining 2×2 submatrix determinants are :

$$(1,2) \text{ and } (1,5) : \ \frac{\eta}{4}(\frac{\eta}{C^2} - t); \ \ (2,5) : \ -\frac{\eta^3 t}{4}. \tag{3.76}$$

The combination $\frac{\eta}{C^2} - t$ decreases monotonically from 0 to -1 as η runs from zero to infinity. therefore the state (3.72) is QO-noncl for all $\eta > 0$. To determine its Type we

compute Γ and its 'expectation value' in $\psi_0(\alpha,\beta)$:

$$\begin{aligned}\Gamma &= \frac{\eta}{4}\begin{pmatrix} \frac{\eta}{C^2}-t & 0 & 0 & \frac{\eta}{C^2}+t \\ 0 & \eta+t & 0 & 0 \\ 0 & 0 & \eta+t & 0 \\ \frac{\eta}{C^2}+t & 0 & 0 & \frac{\eta}{C^2}-t, \end{pmatrix}, \\ \psi_0(\alpha,\beta)^\dagger\Gamma\psi_0(\alpha,\beta) &= \frac{\eta}{4}\{\frac{\eta}{C^2}-t+2|\alpha|^2|\beta|^2(\eta+3t)\}. \end{aligned} \tag{3.77}$$

At both $\alpha=1$, $\beta=0$ and $\alpha=0$, $\beta=1$ the last expression is negative, so the state (3.72) is QO-noncl Type I sub-PS. In this context we note that the single-mode state $\hat{\rho}^{(a)}$ obtained from (3.72) by tracing over b alone is

$$\hat{\rho}_D^{(a)} = \frac{1}{C}\sum_{n=0}^{\infty}\frac{\eta^{2n}}{(2n)!}|n\rangle_{aa}\langle n|, \tag{3.78}$$

and this has the Mandel matrix and determinant

$$\begin{aligned} M^{(1)}(\hat{\rho}_D^{(a)}) &= \begin{pmatrix} 1 & \frac{\eta t}{2} \\ \frac{\eta t}{2} & \frac{\eta}{4}(\eta-t) \end{pmatrix}, \\ \mathrm{det}M^{(1)}(\hat{\rho}_D^{(a)}) &= \frac{\eta}{4}(\frac{\eta}{C^2}-t) < 0. \end{aligned} \tag{3.79}$$

The properties of $\hat{\rho}_D^{(b)}$ are identical. Thus in contrast to the state (3.34), here both $\hat{\rho}_D^{(a)}$ and $\hat{\rho}_D^{(b)}$ are QO-noncl, accompanying the Type I nature of $\hat{\rho}_D^{(ab)}$.

We now apply the NPT entanglement test outlined in Eqs. (3.58, 3.60, 3.61). The necessary expressions are:

$$\begin{aligned} W(u)M^{(2)}(\hat{\rho}^{(ab)})W(u)^\dagger &= \begin{pmatrix} 1 & 0 & -\frac{\eta t}{4} & 0 & -\frac{\eta t}{4} \\ 0 & \frac{\eta}{8}(\eta+t) & 0 & 0 & 0 \\ -\frac{\eta t}{4} & 0 & \frac{\eta}{16}(\eta-t) & 0 & \frac{\eta}{16}(\eta-t) \\ 0 & 0 & 0 & \frac{\eta}{8}(\eta+t) & 0 \\ -\frac{\eta t}{4} & 0 & \frac{\eta}{16}(\eta-t) & 0 & \frac{\eta}{16}(\eta-t) \end{pmatrix}, \\ Z' &= \begin{pmatrix} 1+\frac{3\eta t}{4} & 0 \\ 0 & 1+\frac{\eta t}{2} \end{pmatrix}. \end{aligned} \tag{3.80}$$

According to Eq. (3.61), the 2×2 matrix Z' has to be 'added' at the lower right hand corner of the 5×5 matrix, thus leading to X of Eq. (3.61). Then the positivity or otherwise of X has to be examined. However, even without taking account of Z', the $(1,3)$ subdeterminant of X is $\frac{\eta}{16}(\frac{\eta}{C^2}-t)$, which is negative. This establishes the NPT

entanglement of $\hat{\rho}_{\rm out}^{(abc)}$ in this example, being 'caused' by the same expression $\frac{\eta}{C^2} - t$ whose negativity led to $\hat{\rho}_D^{(ab)}$ being QO-noncl in Eq. (3.76).

3.4.1 Genuine tripartite entanglement from Mandel nonclassicality

In this Section, we demonstrate the possibility of generating genuine residual tripartite entanglement from two-mode Mandel type nonclassicality. The entanglement so produced is residual in the sense of [275], whereby the end result is a tripartite state similar to the GHZ state, such that it has no bipartite entanglement when any one of the three modes is traced away. We demonstrate this using the state considered in example (b) of Section 3.3, where it was used to demonstrate Type-II sub-PS. Pass the state

$$\hat{\rho}_D^{(ab)} \otimes |0\rangle_{cc}\langle 0| = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |n,n\rangle_{ab\,ab}\langle n,n| \otimes |0\rangle_{cc}\langle 0|, \tag{3.81}$$

through a 50:50 $b-c$ beamsplitter, whose action on the mode operators $\hat{b}$ and $\hat{c}$ is

$$\hat{U} \begin{pmatrix} \hat{c} \\ \hat{b} \end{pmatrix} \hat{U}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{c} \\ \hat{b} \end{pmatrix}. \tag{3.82}$$

The resulting state is

$$\begin{aligned} \hat{\rho}_{\rm out}^{(abc)} &= \hat{U}(\hat{\rho}_D^{(ab)} \otimes |0\rangle_{cc}\langle 0|)\hat{U}^{-1} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{2^n n!} |n\rangle_{aa}\langle n| \otimes (\hat{b}^\dagger + \hat{c}^\dagger)^n |0,0\rangle_{bc\,bc}\langle 0,0| (\hat{b} + \hat{c})^n \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n n!}{2^n} |n\rangle_{aa}\langle n| \sum_{r,s=0}^{n} \frac{|r,n-r\rangle_{bc\,bc}\langle s,n-s|}{\sqrt{r!(n-r)!s!(n-s)!}}. \end{aligned} \tag{3.83}$$

Clearly the state $\hat{\rho}_{\rm out}^{(abc)}$ is separable in the a/bc cut. However it is entangled in both the c/ab cut and b/ac cuts as we show below. As a test for NPT entanglement in the c/ab cut, we evaluate the expectation value of a suitably chosen positive operator on the partially transposed output $\hat{\rho}_{\rm out}^{(abc)PT}$, the partial transpose being effected on the c mode. For the choice

$$\hat{A} = c_0 + c_1 \hat{b}\hat{c} + c_2 \hat{a}^\dagger \hat{a}, \tag{3.84}$$

the test for entanglement would be to check for violation of positivity of

$$\mathrm{Tr}(\hat{\rho}_{\rm out}^{(abc)PT} \hat{A}^\dagger \hat{A}) = \mathrm{Tr}(\hat{\rho}_{\rm out}^{(abc)} (\hat{A}^\dagger \hat{A})^{PT}). \tag{3.85}$$

The positivity or otherwise of $\mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(abc)}(\hat{A}^\dagger\hat{A})^{PT})$ is equivalent to the positivity or otherwise of

$$X = \mathrm{Tr}(\hat{\rho}_{\mathrm{out}}^{(abc)} \begin{pmatrix} 1 & \hat{b}\hat{c}^\dagger & \hat{a}^\dagger\hat{a} \\ \hat{b}^\dagger\hat{c} & \hat{b}^\dagger\hat{b}\hat{c}^\dagger\hat{c} & \hat{b}^\dagger\hat{c}\hat{a}^\dagger\hat{a} \\ \hat{a}^\dagger\hat{a} & \hat{a}^\dagger\hat{a}\hat{b}\hat{c}^\dagger & \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} \end{pmatrix}). \tag{3.86}$$

Using the beamsplitter relation and the fact that initially the c mode is in the ground state, we have

$$X = \begin{pmatrix} 1 & \frac{1}{2}\lambda & \lambda \\ \frac{1}{2}\lambda & \frac{1}{4}\lambda^2 & \frac{1}{2}(\lambda^2+\lambda) \\ \lambda & \frac{1}{2}(\lambda^2+\lambda) & \lambda^2+\lambda \end{pmatrix} \not\geq 0. \tag{3.87}$$

In particular, the (2,3) submatrix of X is not positive semidefinite, thus $\hat{\rho}_{\mathrm{out}}^{(abc)}$ is entangled across the c/ab cut. It is easy to see that a similar test with the same choice of $\hat{A}$, except that now the partial transpose is done on the b mode, yields the conclusion that the state $\hat{\rho}_{\mathrm{out}}^{(abc)}$ is entangled across the b/ac cut. Thus we have demonstrated bipartite entanglement in a tripartite setup.

Now to show that the entanglement is genuine tripartite, 'residual' in the sense of [275], we have the following.

$$\begin{aligned}
\hat{\rho}_{\mathrm{out}}^{(ab)} &= \mathrm{Tr}_c(\hat{\rho}_{\mathrm{out}}^{(abc)}) = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n n!}{2^n}|n\rangle_{a\,a}\langle n|\sum_{r=0}^{n}\frac{|r\rangle_{b\,b}\langle r|}{r!(n-r)!}, \\
\hat{\rho}_{\mathrm{out}}^{(ac)} &= \mathrm{Tr}_b(\hat{\rho}_{\mathrm{out}}^{(abc)}) = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n n!}{2^n}|n\rangle_{a\,a}\langle n|\sum_{r=0}^{n}\frac{|r\rangle_{c\,c}\langle r|}{r!(n-r)!}, \\
\hat{\rho}_{\mathrm{out}}^{(bc)} &= \mathrm{Tr}_a(\hat{\rho}_{\mathrm{out}}^{(abc)}) == e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n n!}{2^n}\sum_{r,s=0}^{n}\frac{|r,n-r\rangle_{bc\,bc}\langle s,n-s|}{\sqrt{r!(n-r)!s!(n-s)!}}.
\end{aligned} \tag{3.88}$$

Both $\hat{\rho}_{\mathrm{out}}^{(ab)}$ and $\hat{\rho}_{\mathrm{out}}^{(ac)}$ are manifestly separable. It may not be obvious at first glance that $\hat{\rho}_{\mathrm{out}}^{(bc)}$ is separable, but a closer look shows that $\hat{\rho}_{\mathrm{out}}^{(bc)}$ can be written in the following alternate form:

$$\hat{\rho}_{\mathrm{out}}^{(bc)} = e^{-\lambda}U(\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}|n\rangle_{b\,b}\langle n|\otimes|0\rangle_{cc}\langle 0|)U^{-1}, \tag{3.89}$$

where U corresponds to a 50:50 b-c beamsplitter. The output is a classical state passed through a 50:50 b-c beamsplitter. Thus the state $\hat{\rho}_{\mathrm{out}}^{(bc)}$ is classical and hence separable. An interesting question in the present context is the possibility of extension of monogamy

relations to this non-Gaussian case [254, 255, 275, 276].

3.5 Mandel Parameter

In Section 3.2.2, we had introduced the Mandel matrix construct and outlined the possible classification of states it leads to. It is desirable that we are able to gauge the Mandel type nonclassicality through the definition of a numerical measure. In the case of a single-mode radiation field, such a quantity was defined by Mandel in [295] through the Q parameter. We attempt to extend this definition to the case of two modes. A useful requirement of any such measure would be its invariance under beamsplitter action, as beamsplitter by itself does not produce nonclassicality, but rather transforms one form of nonclassicality to another, thus leaving invariant any reasonable quantitative measure of nonclassicality. Keeping this requirement in mind, we define the two-mode Mandel parameter as

$$Q_2 = \frac{\mathrm{Tr}(\Gamma) - ||\Gamma||}{2(\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle)}, \tag{3.90}$$

where Γ is the 4×4 matrix defined from the two-mode Mandel matrix $M^{(2)}(\hat{\rho}_D^{(ab)})$ in Eq. (3.17), and $||.||$ is the trace norm. In our case, since Γ is a hermitian matrix by definition, the trace norm is simply the sum of the absolute values of its eigenvalues. Thus our definition for the two-mode Mandel parameter is simply the sum of the negative eigenvalues of Γ divided by the total energy of the system. From Eq. (3.27) and Eq. (3.28), we know that under the action of a beamsplitter

$$\Gamma \to \Gamma' = V \Gamma V^T, \tag{3.91}$$

where $V = u \otimes u^*$, $u \in U(2)$. Clearly the trace norm $||\Gamma||$ and trace $\mathrm{Tr}(\Gamma)$ are both unitarily invariant, thus invariant under the action of a beamsplitter. The total energy is clearly also invariant under beamsplitter action. Thus our definition for the two-mode Mandel parameter given by Q_2 is manifestly beamsplitter action invariant. By definition it is zero for classical states, but can be non-zero for nonclassical states. As a simple example, for the case of two-mode product Fock states, $Q_2 = -1$. It is nonzero negative for states either with Type I or Type II nonclassicality. The two-mode Mandel parameter Q_2 can be useful in gauging the two mode Mandel nonclassicality only within the respective types, as the 'Type' is invariant under beamsplitter action.

To maintain the distinction between the two types, it is useful to introduce the single-mode Mandel parameter Q_1 defined at the two mode level. From Eq. (3.13), it is easy

to see that we can define Q_1 to be

$$Q_1 = \frac{\langle \hat{A}^{\dagger 2}\hat{A}^2\rangle - \langle \hat{A}^\dagger \hat{A}\rangle^2}{\langle \hat{A}^\dagger \hat{A}\rangle}. \tag{3.92}$$

It clear that the least possible value Q_1 can take is -1. Clearly for Type II states, Q_1 is positive but Q_2 is negative, .i.e., one can never choose a linear combination of the annihilation operators of the two modes to achieve Q_1 to be negative. On the other hand for Type I states, one can always choose a particular (atleast one) linear combination of the annihilation operators of the two modes to make Q_1 negative.

To be more precise, it is useful to define $Q_1^{\min}$, which is the minimum possible value Q_1 can take for a given two-mode state, the minimum being taken over all the possible linear combinations of the annihilation operators of the two modes under $U(2)$. With the definition of $Q_1^{\min}$, it is clear that for Type II states, Q_2 is negative but $Q_1^{\min}$ is positive. On the other hand for Type I states, both Q_2 and $Q_1^{\min}$ are negative.

Having thus introduced $Q_1^{\min}$, it is possible to classify states in the following manner.

(i) States with $Q_1^{\min} \geq 0$ and $Q_2 = 0$.

(ii) States with $Q_1^{\min} \geq 0$ but $Q_2 < 0$.

(iii) States with both $Q_1^{\min} < 0$ and $Q_2 < 0$.

For states denoted by (iii), a further sub-classification is possible :

(iiia) States with $Q_1^{\min} \leq Q_2$.

(iiib) States with $Q_2 \leq Q_1^{\min}$.

States classified as (i), show no two-mode Mandel type nonclassicality, Type II states come under (ii), and Type I states under (iii). Type I states can be further classified as (iiia) or (iiib).

All classical states are examples for states classified as (i). As simple examples for (ii), we consider states given as example (b) and example (c) in Section 3.3. These are Type II states, and thus are examples for (ii). From Eq. (3.35) and Eq. (3.36), it is easy to see that for the state (example (b)) given in Eq. (3.34), $Q_2 = -1/2$, and for the state (example (c)) in Eq. (3.50), $Q_2 = SS'(SS' - CC')/(S^2 + S'^2)$, which is greater than $-1/2$. At equal squeezings at both ends, Q_2 takes the value $-1/2$. Furthermore from Eq. (3.53), we see that the state in example (c) goes over to being a sample for (i) from (ii), when the squeeze parameter at one of the ends goes to zero.

All Type I states such as example (a) of Section 3.3 are examples for states classified as (iii). The subclassification into (iiia) and (iiib) is a bit subtle, nevertheless we provide examples for both. As an example for (iiia), we consider the product of a Fock state at the a-mode and a coherent state at the b-mode.

$$|\psi\rangle = |n\rangle \otimes |\beta\rangle. \tag{3.93}$$

For this state, the Γ matrix is diagonal with the diagonal entries given by $\{-n, n|\beta|^2, n|\beta|^2, 0\}$. Thus Q_2 for this state is

$$Q_2 = \frac{-n}{n + |\beta|^2} \geq -1. \tag{3.94}$$

On the other hand, the single-mode Mandel parameter Q_1 for the a-mode is -1. Thus $Q_1^{\min} \leq Q_2$ for this case.

As an example for (iiib), we consider the state in Eq. (3.72) of Section 3.4. For this state it is easy to see from Eqs. (3.77) and (3.74) that $Q_2 = -1/2$, however the expression for Q_1 for an arbitrary value of α and β is, by Eqs. (3.77) and (3.74),

$$\frac{\psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta)}{\langle \hat{A}^\dagger \hat{A}\rangle} = \frac{1}{2t}\{\frac{\eta}{C^2} - t + 2|\alpha|^2|\beta|^2(\eta + 3t)\}. \tag{3.95}$$

The minimum possible value for Q_1 is when α or β is zero, i.e.,

$$Q_1^{\min} = \frac{1}{2t}\{\frac{\eta}{C^2} - t\} = \frac{1}{2}\{\frac{\eta}{\mathrm{Sinh}\eta\mathrm{Cosh}\eta} - 1\} \geq -\frac{1}{2}. \tag{3.96}$$

Another interesting example for (iiib) is the class of states $|\psi_n\rangle$, obtained as an equal superposition of product Fock states with total number equal to n:

$$|\psi_n\rangle = \frac{1}{\sqrt{n+1}} \sum_{r=0}^{n} |r, n-r\rangle. \tag{3.97}$$

For the cases $n = 1, 2, 3, 4$ the numerically evaluated values of Q_2 are respectively -1, -1.085, -1.123, -1.143. However Q_1 is obviously bounded from below by -1. Thus $Q_2 < Q_1^{\min}$ for this example. Incidently, another interesting aspect should be noted, namely that Q_2 can take values less than -1 in comparison with Q_1 which cannot. An interesting aspect of these states is that they do not arise from beamsplitters, as entangled states, produced from product Fock states, with the exception of the case $n = 1$. Indeed, the value of the two-mode Mandel parameter Q_2 falling below -1 is precisely a signature of this aspect.

3.6 Concluding remarks

In this work, we have made an attempt to bring the notions of nonclassicality of two-mode states and entanglement as close as possible. We have given a transparent connection between normal ordering and NPT entanglement in the context of continuous variables. We have considered generalisation of the Mandel criterion to two-mode systems through the Mandel matrix construct, and exploited it to analyse entanglement. Such a construction leads to a natural classification of states as Type I and Type II. Type II states are special in the sense that their antibunching cannot be detected locally, i.e., through any single-mode process. We have considered several examples to illustrate this classification. We have shown the possibility of demonstrating NPT entanglement through the Mandel matrix construct. It is also shown that the demonstrated entanglement could as well be distilled. We have extended this idea to the tripartite case, where we have demonstrated through simple examples that the entanglement could be traced back to the Mandel matrix. We have introduced the two-mode Mandel parameter Q_2, through the Γ matrix construct, and discussed interesting situations that could arise in the case of two modes, by contrasting the value of Q_2 with that of the single-mode Mandel parameter Q_1. We have demonstrated the ability of the two-mode Mandel parameter Q_2 to detect entanglement that cannot arise from beamsplitters. We hope that the perspective developed here will help further interesting developments and generalisations.

4

Entanglement of Formation for Gaussian states

4.1 Introduction

Entanglement is an essential resource for many quantum information processing tasks, and hence it is important to be able to quantify this resource. In Section 1.6, we outlined a set of demands, that any good measure of entanglement should satisfy. In the case of bipartite pure states, the demands lead to a simple and unique measure for this resource: it is the von Neumann entropy of either subsystem [62, 338, 339]. For mixed states however, many different entanglement measures have been explored [340], and there is no measure which justifies itself to be unique. Of these measures, the entanglement of formation (EOF) [37] is the most natural extension of the pure state measure of entanglement, to the case of mixed states. To recall the definition of EOF in Eq. (1.87), the EOF for a bipartite state $\hat{\rho}^{(ab)}$ is defined as an infimum:

$$\mathrm{EOF}\,(\hat{\rho}^{(ab)}) \equiv \inf\{\sum_j p_j E(\psi_j) \mid \hat{\rho}^{(ab)} = \sum_j p_j |\psi_j\rangle\langle\psi_j|\}\,. \tag{4.1}$$

The infimum is to be taken over all possible ensemble realizations of the given mixed state $\hat{\rho}^{(ab)}$ as convex sum of pure states, and $E(\psi_j) \equiv S(\mathrm{tr}_B[|\psi_j\rangle\langle\psi_j|])$, where $S(\cdot)$ is the von Neumann entropy. The regularised version of EOF is the entanglement cost [37, 63]. EOF has been computed in closed form for *arbitrary* two-qubit states [67], and for highly symmetric states like the isotropic states [69] and the Werner states [71].

The role of Gaussian states in quantum information theory has already been outlined in Section 1.9. Their use in teleportation [83, 84] and quantum cryptography [97] has been demonstrated. Questions related to their separability [57, 58, 241, 242] and distillability [217] have been resolved. More recently, analytic expression for their EOF has been obtained in the *symmetric* case [70]. This notable achievement seems to be the first computation of EOF for states of infinite rank. These authors exploit a certain *extremal-*

ity that the two-mode-squeezed vacuum enjoys in respect of the Einstein-Podolsky-Rosen (EPR) correlation [341] on the one hand and entanglement on the other.

An interesting Gaussian-state-specific generalisation of EOF, the *Gaussian entanglement of formation*, has also been explored [342]. But the EOF of asymmetric Gaussian state has remained an open problem [343] in spite of considerable effort [344]. Naturally, the problem of EOF for general (asymmetric) Gaussian states should be solved before the important issue of additivity of entanglement for Gaussian states could be addressed [342].

In this work we compute, under a conjecture, the EOF for arbitrary two-mode Gaussian states. Our analysis rests on two principal ingredients. The first one is a four-parameter canonical form we develop for the covariance matrix; one of these parameters, the squeeze parameter, proves to be a measure of EOF. The second one is a family of generalised EPR correlations for *noncommuting* pairs of nonlocal variables; this family is indexed by a continuous parameter θ. And the conjecture is in respect of an extremal property of this generalised EPR correlation.

4.2 Canonical Form for Covariance Matrix

Given a two-mode Gaussian state, with the mode on Alice's side described by canonical quadrature variables $\hat{x}_a$, $\hat{p}_a$ and that on Bob's side by $\hat{x}_b$, $\hat{p}_b$, we can assume without loss of generality that the first moments of all four variables vanish [57, 70]. Such a zero-mean Gaussian state is fully described by the covariance matrix [57, 70]

$$V_G = \frac{1}{2}\begin{bmatrix} \alpha\beta n & 0 & \beta k_x & 0 \\ 0 & \alpha^{-1}\beta^{-1} n & 0 & -\beta^{-1} k_p \\ \beta k_x & 0 & \alpha^{-1}\beta m & 0 \\ 0 & -\beta^{-1} k_p & 0 & \alpha\beta^{-1} m \end{bmatrix}, \tag{4.2}$$

where the phase space variables are assumed to be arranged in the order (x_a, p_a, x_b, p_b) $\equiv \xi$, and we have retained through the parameters α, $\beta > 0$ the freedom of independent local unitary (i.e., symplectic) scalings on the Alice's and Bob's sides. This freedom will be used shortly.

Note that V_G is left with no correlation between the 'spatial' variables $\hat{x}_a$, $\hat{x}_b$ and the 'momentum' variables $\hat{p}_a$, $\hat{p}_b$. Thus it is sometimes convenient to view V_G as the direct

sum of 2×2 matrices:

$$V_G = X_G \oplus P_G\,,$$
$$X_G = \tfrac{\beta}{2}\begin{bmatrix} \alpha n & k_x \\ k_x & \alpha^{-1} m \end{bmatrix},\ P_G = \tfrac{\beta^{-1}}{2}\begin{bmatrix} \alpha^{-1} n & -k_p \\ -k_p & \alpha m \end{bmatrix}. \tag{4.3}$$

Let $|\Psi_r\rangle$ denote the standard two-mode-squeezed vacuum state with squeeze parameter r. It takes the Schmidt form in the standard Fock basis:

$$\begin{aligned} |\Psi_r\rangle &= \sum_{n=0}^{\infty} c_n |n\rangle_A \otimes |n\rangle_B \equiv \sum_{n=0}^{\infty} c_n |n,n\rangle\,, \\ c_n &= \tanh^n r/\cosh r\,. \end{aligned} \tag{4.4}$$

Denoting by E_r the entanglement of $|\Psi_r\rangle$, we have

$$E_r = \cosh^2 r \log_2(\cosh^2 r) - \sinh^2 r \log_2(\sinh^2 r)\,. \tag{4.5}$$

The covariance matrix of $|\Psi_r\rangle$ has the form

$$\begin{aligned} V_{\Psi_r} &= X_{\Psi_r} \oplus P_{\Psi_r}\,, \\ X_{\Psi_r} &= \frac{1}{2}\begin{bmatrix} C & S \\ S & C \end{bmatrix},\ P_{\Psi_r} = \frac{1}{2}\begin{bmatrix} C & -S \\ -S & C \end{bmatrix}, \\ C &\equiv \cosh 2r,\quad S \equiv \sinh 2r\,. \end{aligned} \tag{4.6}$$

Proposition 4.1 *Given a two-mode covariance matrix V_G, the local scale parameters α, β can be so chosen that V_G gets recast in the form*

$$V_0 = \frac{1}{2}\begin{bmatrix} C + u\,c^2 & 0 & S + u\,cs & 0 \\ 0 & C + v\,c^2 & 0 & -S - v\,cs \\ S + u\,cs & 0 & C + u\,s^2 & 0 \\ 0 & -S - v\,cs & 0 & C + v\,s^2 \end{bmatrix},$$
$$C \equiv \cosh 2r_0,\ S \equiv \sinh 2r_0;\quad c \equiv \cos\theta_0,\ s \equiv \sin\theta_0\,. \tag{4.7}$$

Note: We will call V_0 *the canonical form* of a two-mode covariance matrix; our results below will justify this elevated status. We assume without loss of generality $n \geq m$ or, equivalently, $0 < \theta_0 \leq \pi/4$. For a given V_G there will be two solutions for the above form. Canonical form will always refer to the one with the smaller squeeze parameter r, which

is ensured by the restriction

$$\tan\theta_0 \geq \tanh r_0\,. \tag{4.8}$$

This condition proves central to our analysis. Its origin may be appreciated by inverse two-mode-squeezing the Gaussian state V_0 until it becomes just separable, and noting that there exists a range of further squeezing in which the *mixed* Gaussian state remains separable before becoming inseparable again. The parameters $u,\, v \geq 0$.

Theorem 4.1 *The essence of the canonical form is that V_0 differs from the covariance matrix of a two-mode-squeezed vacuum $|\Psi_{r_0}\rangle$ by a positive matrix which is a direct sum of two singular 2×2 matrices which are, modulo signature of the off-diagonal elements, multiples of one another.*

Proof: The canonical form demands, as a necessary condition, that α, β, and r be chosen to meet

$$\det(X_G - X_{\Psi_r}) = 0\,, \qquad \det(P_G - P_{\Psi_r}) = 0\,. \tag{4.9}$$

These being two constraints on three parameters, one will expect to get a one-parameter family of solutions to these constraints. For each such solution we may denote the vector annihilated by the singular matrix $X_G - X_{\Psi_r}$ by $(\sin\theta,\, -\cos\theta)$, and that annihilated by $P_G - P_{\Psi_r}$ by $(\sin\theta^{'},\, \cos\theta^{'})$. The canonical form corresponds to that solution for which $\theta^{'} = \theta$; it is this degenerate value that equals θ_0 of the canonical form.

That there exists such a degenerate value can be seen as follows. We may fix the scale parameter α through $\alpha = \sqrt{m/n}$, and then solve Eqs. (4.9) for β and r, the smaller r being the relevant one. We will find $\theta = \pi/4$ and $\theta^{'} < \pi/4$ in this case. On the other hand if we take $\alpha = \sqrt{n/m}$ and then solve Eqs. (4.9), we will find $\theta^{'} = \pi/4$ and $\theta < \pi/4$. It follows from continuity that there exists an intermediate value α_0 for the parameter α, in the range $\sqrt{m/n} < \alpha < \sqrt{n/m}$, for which $\theta^{'} = \theta$ ($< \pi/4$ since $n > m$). And this yields the canonical form.

Viewed alternatively, the canonical form places the following two requirements on the scale factors α, β :

$$\begin{aligned} &\frac{\det X_G - 1/4}{\det P_G - 1/4} = \frac{\mathrm{tr}(\sigma_3 X_G)}{\mathrm{tr}(\sigma_3 P_G)}\,, \\ &\det(X_G - \sigma_3 P_G \sigma_3) = 0\,, \end{aligned} \tag{4.10}$$

where σ_3 is the diagonal Pauli matrix. These are simultaneous equations in α, β, and

solving these equations yields, in terms of n, m, k_x, k_p, the values of α, β corresponding to the canonical form.

Two special cases may be noted. If $m = n$ we have $\alpha = 1$ (since $\sqrt{n/m} = \sqrt{m/n}$), and hence $\beta = \sqrt{(n-k_p)/(n-k_x)}$, so that the canonical squeeze parameter r_0 is given by $e^{-2r_0} = \sqrt{(n-k_x)(n-k_p)}$, reproducing the results of Ref. [70]. The parameter θ_0 always equals $\pi/4$ in this (symmetric) case. On the other hand, if $k_x = k_p = k$, the canonical form corresponds to $\alpha = \beta = 1$, and one obtains r_0 by simply solving

$$\det \begin{bmatrix} n - \cosh 2r_0 & k - \sinh 2r_0 \\ k - \sinh 2r_0 & m - \cosh 2r_0 \end{bmatrix} = 0\,, \tag{4.11}$$

which yields this closed-form expression for r_0 :

$$\begin{aligned} \cosh(2\eta - 2r_0) &= \frac{nm - k^2 + 1}{\sqrt{(n+m)^2 - 4k^2)}}, \\ \mathrm{e}^{\pm 2\eta} &\equiv \frac{(n+m) \pm 2k}{\sqrt{(m+n)^2 - 4k^2}}. \end{aligned} \tag{4.12}$$

4.3 Generalised EPR Correlation

To proceed further, we need to generalise the familiar EPR correlation in Eq. (1.120) [70]. Given any bipartite state $|\psi\rangle$, define

$$\begin{aligned} \hat{x}_\theta &= \sin\theta\, \hat{x}_a - \cos\theta\, \hat{x}_b\,, \\ \hat{p}_\theta &= \sin\theta\, \hat{p}_a + \cos\theta\, \hat{p}_b\,, \\ \Lambda_\theta(\psi) &= \langle\psi|(\hat{x}_\theta)^2|\psi\rangle + \langle\psi|(\hat{p}_\theta)^2|\psi\rangle\,. \end{aligned} \tag{4.13}$$

In defining $\Lambda_\theta(\psi)$ we have assumed $\langle\psi|\hat{x}_\theta|\psi\rangle = 0 = \langle\psi|\hat{p}_\theta|\psi\rangle$; if this is not the case then $\hat{x}_\theta$ and $\hat{p}_\theta$ in $\hat{\Lambda}_\theta(\psi)$ should be replaced by $\hat{x}_\theta - \langle\psi|\hat{x}_\theta|\psi\rangle$ and $\hat{p}_\theta - \langle\psi|\hat{p}_\theta|\psi\rangle$ respectively. Clearly, the usual EPR correlation in Eq. (1.120) [70] corresponds to $\theta = \pi/4$. While $\hat{x}_{\pi/4}$, $\hat{p}_{\pi/4}$ commute, the generalised EPR (nonlocal) variables $\hat{x}_\theta$, $\hat{p}_\theta$ *do not commute*, and hence the name generalised EPR correlation for $\hat{\Lambda}_\theta(\Psi_r)$; indeed, we have

$$[\hat{x}_\theta,\, \hat{p}_\theta] = -i\cos 2\theta. \tag{4.14}$$

For the two-mode-squeezed vacuum $|\Psi_r\rangle$ the generalised EPR correlation reads

$$\Lambda_\theta(\Psi_r) = \cosh 2r - \sin 2\theta \sinh 2r\,. \tag{4.15}$$

Let us combine the quadrature variables of the oscillators of Alice and Bob into boson operators $\hat{a} = (\hat{x}_a + i\hat{p}_a)/\sqrt{2}$ and $\hat{b} = (\hat{x}_b + i\hat{p}_b)/\sqrt{2}$. Then, $\Lambda_\theta(\psi)$ has this expression quadratic in the boson variables :

$$\begin{aligned} \Lambda_\theta(\psi) &= \langle\psi|\hat{\Lambda}_\theta|\psi\rangle, \\ \hat{\Lambda}_\theta &= 1 + 2\sin^2\theta\, \hat{a}^\dagger\hat{a} + 2\cos^2\theta\, \hat{b}^\dagger\hat{b} \\ &\quad - 2\cos\theta\sin\theta(\hat{a}\hat{b} + \hat{a}^\dagger\hat{b}^\dagger)\,. \end{aligned} \tag{4.16}$$

We may call $\hat{\Lambda}_\theta$ the *generalised EPR operator*.

The entanglement of Ψ_r monotonically increases with increasing value of the squeezing parameter r. In order that $\Lambda_\theta(\Psi_r)$ be useful as an entanglement measure of Ψ_r it should, for fixed value of θ, decrease with increasing r. The restriction $\tan\theta \geq \tanh r$, encountered earlier in Eq. (4.8) from a different perspective, simply ensures this. Through the monotonic relationship (3) between r and E_r, we will view this constraint as a restriction on the allowed range of values of θ, for a fixed value of entanglement.

Given a squeezed state $|\Psi_r\rangle$, let us denote by $|\Psi'_r\rangle$ the state obtained from $|\Psi_r\rangle$ by independent local canonical transformations [57] S_a, $S_b \in Sp(2, R)$ acting respectively on the oscillators of Alice and Bob.

Proposition 4.2 *We have* $\Lambda_\theta(\Psi'_r) \geq \Lambda_\theta(\Psi_r)$, $\forall\,\theta$ *in the range* $1 \geq \tan\theta \geq \tanh r$ *and for all* S_a, $S_b \in Sp(2, R)$.

Proof: Clearly,

$$\begin{aligned} \Lambda_\theta(\Psi'_r) &= \frac{1}{2}\,\{\cosh 2r[\,\sin^2\theta\,\mathrm{tr}(S_a S_a^T) + \cos^2\theta\,\mathrm{tr}(S_b S_b^T)\,] \\ &\quad - \sin 2\theta \sinh 2r\,\mathrm{tr}\,(\sigma_3 S_a \sigma_3 S_b^T)\,\}. \end{aligned} \tag{4.17}$$

If $e^{\pm\gamma_a}$ are the singular values of S_a, and $e^{\pm\gamma_b}$ those of S_b, then

$$\begin{aligned} \mathrm{tr}(S_a S_a^T) &= 2\cosh 2\gamma_a, \\ \mathrm{tr}(S_a S_b^T) &= 2\cosh 2\gamma_b, \quad \text{and} \\ \mathrm{tr}(\sigma_3 S_a \sigma_3 S_b^T) &\leq 2\cosh(\gamma_a + \gamma_b). \end{aligned} \tag{4.18}$$

Thus the difference $\Delta(\gamma_a,\, \gamma_b) \equiv \Lambda_\theta(\Psi'_r) - \Lambda_\theta(\Psi_r)$ obeys

$$\begin{aligned} \Delta(\gamma_a,\, \gamma_b) &\geq \cosh 2r[\,\sin^2\theta(\cosh 2\gamma_a - 1) + \cos^2\theta(\cosh 2\gamma_b - 1)\,] \\ &\quad - \sin 2\theta \sinh 2r[\,\cosh(\gamma_a + \gamma_b) - 1\,]. \end{aligned} \tag{4.19}$$

It is easily seen that $\Delta(\gamma_a, \gamma_b)$ is expremal at $\gamma_a = \gamma_b = 0$ corresponding to the standard squeezed state $|\Psi_r\rangle$. To show that this extremum is indeed minimum, we note that the determinant of the Hessian matrix of the right hand side, evaluated at $\gamma_a = 0 = \gamma_b$, is proportional to $\sin 2\theta \cosh 2r - \sinh 2r$, and hence is positive if and only if $\tan\theta \geq \tanh r$.

Once again we see a role for the requirement $\tan\theta \geq \tanh r$. Let the equivalence $V_G \sim V_0$ denote the fact that the corresponding Gaussian states are connected by a local canonical transformation. The fact that $M \equiv V_0 - V_{\Psi_{r_0}} \geq 0$ implies $\Lambda_{\theta_0}(\rho_{V_0}) \geq \Lambda_{\theta_0}(\Psi_{r_0})$. In view of Proposition 6.2, this implies

$$\Lambda_{\theta_0}(\rho_{V_G}) \geq \Lambda_{\theta_0}(\rho_{V_0}) \geq \Lambda_{\theta_0}(\Psi_{r_0}) = \cosh 2r_0 - \sin 2\theta \, \sinh 2r_0, \tag{4.20}$$

for any Gaussian state V_G connected to V_0 by local canonical transformation. This assigns an alternative meaning to the canonical parameter r_0 :

Proposition 4.3 *Given a Gaussian state described by $V_G \sim V_0$, the canonical squeeze parameter r_0 is the smallest r for which the matrix inequality $V_G - V_{\Psi'_r} \geq 0$ is true.*

It is well known that the two-mode-squeezed vacuum has several extremal properties of interest to entanglement [70, 253]. It seems that this state enjoys one more such distinction, this time in respect of our generalised EPR correlation.

Conjecture 4.1 *Among all bipartite states of fixed entanglement numerically equalling E_r, and for every θ in the range $\tanh r \leq \tan\theta$, the two-mode-squeezed vacuum $|\Psi_r\rangle$ yields the least value for the generalised EPR correlation $\Lambda_\theta(\cdot)$. In other words, no state $|\psi\rangle$ with entanglement $E(|\psi\rangle) \leq E_r$ can yield a generalised EPR correlation $\Lambda_\theta(\psi) < \Lambda_\theta(\Psi_r)$, for any θ in the range $\tan\theta \geq \tanh r$*

The special case $\theta = \pi/4$ is the basis of the important work of Ref. [70]. Hence the present assertion can be viewed as a generalisation of their Proposition 1. The original EPR correlation $\Lambda_{\pi/4}(\cdot)$ continuously decreases to zero with increasing entanglement. But this is not true of the generalised EPR correlation $\Lambda_\theta(\cdot)$.

Let us denote by r_θ the value of r determined by a given value of θ through the equation $\tan\theta = \tanh r$, and let θ_r denote the value of θ so determined by r. Then, for a given numerical E_r, the relevant range for θ in Conjecture 1 is $\theta_r \leq \theta \leq \pi/4$.

Proposition 4.4 *The generalised EPR correlation $\Lambda_\theta(\cdot)$ obeys the basic inequality $\Lambda_\theta(\cdot) \geq \cos 2\theta$. The two-mode-squeezed vacuum saturates this inequality if and only if the squeeze parameter r respects $\tanh r = \tan\theta$.*

Proof: It is clear that the relations $\tan\theta = \tanh r$, $\sin 2\theta = \tanh 2r$, and $\cos 2\theta = (\cosh 2r)^{-1}$ are equivalent to one another, and so also are the inequalities $\tan\theta \geq \tanh r$, $\sin 2\theta \geq \tanh 2r$, and $\cos 2\theta \leq (\cosh 2r)^{-1}$. Now consider the transformation $(\hat{a}, \hat{b}) \rightarrow U(r)(\hat{a}, \hat{b})U(r)^\dagger$ where $U(r) = \exp\{ r(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}) \}$ is the unitary two-mode-squeeze operation:

$$\hat{a} \rightarrow \hat{a}\cosh r - \hat{b}^\dagger \sinh r, \quad \hat{b} \rightarrow \hat{b}\cosh r - \hat{a}^\dagger \sinh r. \tag{4.21}$$

This implies the following transformation for the anticommutator $\{\hat{b}, \hat{b}^\dagger\} \equiv \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b}$:

$$\begin{aligned} \{\hat{b}, \hat{b}^\dagger\} \quad \rightarrow \quad & (\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a}) + \frac{1}{2}(\{\hat{a}, \hat{a}^\dagger\} + \{\hat{b}, \hat{b}^\dagger\})\cosh 2r \\ & - (\hat{a}\hat{b} + \hat{a}^\dagger\hat{b}^\dagger)\sinh 2r \\ & = \cosh 2r\, \hat{\Lambda}_{\theta_r}, \quad \theta_r \equiv \arctan(\tanh r). \end{aligned} \tag{4.22}$$

Since $\{\hat{b}, \hat{b}^\dagger\} \geq 1$, so is also its unitary transform $\cosh 2r\, \hat{\Lambda}_{\theta_r}$. That is, $\hat{\Lambda}_{\theta_r} \geq (\cosh 2r)^{-1} = \cos 2\theta_r$.

Thus, saturation of the inequality $\Lambda_{\theta_r}(\psi') \geq \cos 2\theta_r$ is equivalent to the condition $\langle\psi|\{\hat{b}, \hat{b}^\dagger\}|\psi\rangle = 1$, where $|\psi'\rangle = U(r)|\psi\rangle$. A pure state which satisfies $\langle\psi|\{\hat{b}, \hat{b}^\dagger\}|\psi\rangle = 1$, is of the form $|\psi\rangle = |\phi\rangle_a \otimes |0\rangle_b$, where $|\phi\rangle_a$ is *any* vector in Alice's Hilbert space $\mathcal{H}^b$. It follows that states saturating the inequality $\Lambda_{\theta_r}(\hat{\rho}) \geq \cos 2\theta_r$ constitute the set $\{\hat{\rho}^{(ab)} = U(r)\hat{\rho}^{(a)} \otimes |0\rangle_{bb}\langle 0|U(r)^\dagger\}$, where $\hat{\rho}^{(a)}$ is any (pure or mixed) state of Alice's oscillator. Finally, Conjecture 6.1 claims that among all these states saturating this inequality the two-mode-squeezed vacuum $|\Psi_{r_\theta}\rangle$, corresponding to the choice $\hat{\rho}^{(a)} = |0\rangle_{aa}\langle 0|$, has the least entanglement.

4.4 Entanglement of Formation

With the canonical form and the generalised EPR correlations in hand, we are now fully equipped to compute the EOF of an arbitrary two-mode Gaussian state.

Proposition 4.5 *Given an inseparable zero-mean two-mode Gaussian state ρ_{V_0} with covariance matrix V_0 specified in the canonical form by u, v, θ_0 and r_0 with $u, v \geq 0$ and $0 < \tanh r_0 \leq \tan\theta_0 \leq 1$, its EOF equals E_{r_0}, the entanglement of the squeezed vacuum $|\Psi_{r_0}\rangle$.*

Proof: The fact that $M \equiv V_0 - V_{\Psi_{r_0}} \geq 0$ guaranties that ρ_{V_0} can be realized as a convex sum of displaced versions $D(\xi)|\Psi_{r_0}\rangle$ of the squeezed vacuum states $|\Psi_{r_0}\rangle$, all of which

have the same entanglement E_{r_0} as $|\Psi_{r_0}\rangle$:

$$\rho_{V_0} \sim \int \mathrm{d}^2\xi D(\xi)|\Psi_{r_0}\rangle\langle\Psi_{r_0}|D^\dagger(\xi)\exp(-\frac{1}{2}\xi^T M^{-1}\xi). \tag{4.23}$$

Here $D(\xi)$ is the unitary phase space displacement operator. The rank of M equals 2, and both M^{-1} and the two-dimensional integral refer to the restriction of the phase space variable ξ to the range of M.

Since a specific ensemble realization with average entanglement E_{r_0} is exhibited, $\mathrm{EOF}(\rho_{V_0}) \leq E_{r_0}$. On the other hand, evaluation of the generalised EPR correlation $\Lambda_\theta(\rho_{V_0}) = \mathrm{tr}\,(\hat{\Lambda}_\theta\rho_{V_0})$, for the particular value of θ occurring in V_0 shows that $\Lambda_{\theta_0}(\rho_{V_0}) = \cosh 2r_0 - \sin 2\theta_0 \sinh 2r_0$. And by Conjecture 6.1, this implies $\mathrm{EOF}(\rho_{V_0}) \geq E_{r_0}$. We have thus proved $\mathrm{EOF}(\rho_{V_0}) = E_{r_0}$.

An attractive feature of the canonical form of the covariance matrix is that the two-mode-squeezing $U(r)$ acts on it in a covariant or form-preserving manner.

Proposition 4.6 *Under the two-mode-squeezing transformation $U(r)$ we have*

$$\begin{aligned} V_0(r_0,\theta_0,u,v) &\rightarrow V_0(r_0',\theta_0',u',v')\,; \\ r_0' = r_0 + r, &\qquad \sin 2\theta_0' = \frac{\sinh 2r + \cosh 2r \sin 2\theta_0}{\cosh 2r + \sin 2\theta_0 \sinh 2r}, \\ (u',\, v') &= (u,\, v) \times (\cosh 2r + \sin 2\theta_0 \sinh 2r). \end{aligned} \tag{4.24}$$

This is easily verified by direct computation. While the canonical squeeze parameter r_0 simply gets translated by r, the parameters u and v get scaled by a *common factor*. If we define r_{θ_0}, $r_{\theta_0'}$ through $\tan\theta_0 \equiv \tanh r_{\theta_0}$ and $\tan\theta_0' \equiv \tanh r_{\theta_0'}$, the transformation law for θ_0 takes the form of translation: $r_{\theta_0'} = r_{\theta_0} + r$.

As a consequence of this covariance, the convex decomposition which minimizes the average entanglement goes covariantly to such a decomposition under two-mode-squeezing: the minimal decomposition commutes with squeezing. This implies, in particular, the following simple behaviour of EOF under squeezing: $E_{r_0} \rightarrow E_{r_0+r}$.

Finally, the just separable Gaussian states on the separable-inseparable boundary, correspond to the canonical form with $r_0 = 0$ [57]. As was to be expected, the condition (4.8) places no restriction on θ_0 in this case.

5

Compatibility conditions on local and global spectra for n-mode Gaussian states

5.1 Introduction

The quantum marginal problem has attracted considerable interest in quantum information theory [17, 34, 345–351]. Given a multipartite system, it asks: what kind of spectra for the subsystem density operators are consistent with a given spectrum for the density operator of the full system? The Gaussian quantum marginal problem (detailed below) has been solved recently [285, 352] (As noted in Ref. [352], the three-mode case was known earlier [254]). Our approach to this problem makes effective use of beam splitter and two-mode squeezing transformations. In the case of two modes it is shown that *every Gaussian state is uniquely determined, modulo local canonical transformations, by its global spectrum and local spectra*; *in particular, the entanglement is fully determined by these spectra.*

Consider a Gaussian state of a system of n-modes, represented by density operator $\hat{\rho}$. The mean values of the position and momentum variables q_j, p_j have no role to play in our considerations, and so we assume that these mean values vanish. Such a zero-mean Gaussian state is fully described by its $2n \times 2n$ covariance matrix V.

The reduced state $\hat{\rho}_j$ of the j^{th} mode, obtained by tracing out from $\hat{\rho}$ all other modes, is also a zero-mean Gaussian state. With the phase space variables assumed arranged in the order $q_1, p_1\,;\, q_2, p_2\,;\, \cdots\,;\, q_n, p_n$ the j^{th} 2×2 block along the leading diagonal of V represents precisely the covariance matrix of the reduced state $\hat{\rho}_j$. Through (independent) local canonical transformations $\in Sp(2, R)$ on each mode we make all the 2×2 blocks along the diagonal of V multiples of identity. The covariance matrix of the j^{th} mode will then be of the form $\text{diag}(m_j, m_j)$. It corresponds to a thermal state, with temperature $T(m_j)$ which is a monotone increasing function of m_j. Being thermal, $\hat{\rho}_j$ has the spectral

resolution $\hat{\rho}_j = [1-\xi(m_j)]\sum_{k=0}^{\infty}\xi(m_j)^{n_{jk}}|n_{jk}\rangle\langle n_{jk}|$. The parameter $\xi(m_j)$ is another monotone increasing function of m_j, and $|n_{jk}\rangle$'s are the energy eigenstates of the j^{th} oscillator. Clearly, the eigenvalue spectra of the $\hat{\rho}_j$'s are determined by, and determine, the *local spectral parameters* m_j.

Using an appropriate (nonlocal) canonical transformation $S \in Sp(2n,R)$ the covariance matrix V can be decoupled and brought into the canonical form $V^{(0)}$ of independent oscillators in thermal states [57]: $V^{(0)} = SVS^T$ =diag$(\kappa_1,\kappa_1\,;\,\kappa_2,\kappa_2\,;\,\cdots\,;\,\kappa_n,\kappa_n)$. The associated density operator $\hat{\rho}^{(0)}$ thus has the spectral decomposition

$$\hat{\rho}^{(0)} = \prod_{j=1}^{n}[1-\xi(\kappa_j)]\sum_{k=0}^{\infty}\xi(\kappa_j)^{n_{jk}}|n_{jk}\rangle\langle n_{jk}|. \tag{5.1}$$

Since $\hat{\rho}^{(0)}$ and the original $\hat{\rho}$ are unitarily related, the spectrum of $\hat{\rho}$ is the same as that of $\hat{\rho}^{(0)}$. It is clear that this global spectrum and the n-tuple of *global spectral parameters* $(\kappa_1,\kappa_2,\cdots,\kappa_n)$ determine each other.

We may now ask what are the constraints connecting the global spectrum of a Gaussian state to its local spectra. In view of the invertible relationships just noted this *Gaussian quantum marginal problem* is equivalent to seeking the compatibility constraints between the global spectral parameters $\{\,\kappa_j\,\}$ and the local spectral parameters $\{\,m_j\,\}$. Interestingly, the answer can be given in the form of necessary and sufficient conditions.

Theorem 5.1 *Let $m=(m_1,m_2,m_3,\cdots,m_n)$ and $\kappa=(\kappa_1,\kappa_2,\cdots,\kappa_n)$ be the local and global spectral parameters of an n-mode Gaussian state, written in nondecreasing order. These are compatible iff*

$$\sum_{j=1}^{k} m_j \geq \sum_{j=1}^{k}\kappa_j, \quad k=1,2,\cdots,n\,, \tag{5.2}$$

$$m_n - \sum_{j=1}^{n-1} m_j \leq \kappa_n - \sum_{j=1}^{n-1}\kappa_j. \tag{5.3}$$

Remarks: What this claim means can be clarified by stating it in two parts. Suppose a Gaussian state is given. Its local spectral parameters $m_1,m_2,\cdots,m_n$, and global spectral parameters $\kappa_1,\kappa_2,\cdots,\kappa_n$ are certain to meet these inequalities (with $\kappa_1 \geq 1$). Conversely, given a set of local and global spectral parameters meeting these inequalities (with $\kappa_1 \geq 1$), we can certainly construct a physical Gaussian state with these parameters.

The first part of the theorem was essentially proved by Hiroshima [285]. But the full theorem in this form was formulated by Eisert et al. [352] who presented an inductive proof for the second part. Our proof of both parts will be seen to be constructive, consistent

with the elementary nature of the theorem, and *it rests in an essential manner on a fuller appreciation of the two-mode situation.*

Given two vectors $m, \kappa \in R^n$, we will say κ *dominates* m if m and κ, *after their components are rearranged in the nondecreasing order*, obey the set of $n+1$ inequalities (5.2), (5.3). This definition is such that *permutation of the components of* m *or* κ *does not affect dominance.* Thus $(9,7,8,6,12,11,10)$ is dominated by $(5,2,18,4,1,12,3)$, since $(1,2,3,4,5,12,18)$ manifestly dominates $(6,7,8,9,10,11,12)$. Further, dominance so defined is transitive: κ dominates m, and m dominates m', together imply κ dominates m'.

In the Schur-Horn case [353] wherein m corresponds to the diagonal entries of a hermitian matrix and κ to its eigenvalues, the last inequality in (5.2) becomes an equality. It is clear that (5.3) is subsumed by (5.2) in that case.

5.2 The two-mode case

This case is of interest in its own right. Further, it possesses an aspect which seems to be unique, not shared by any other system. Finally, our analysis of the n-mode case relies critically on repeated applications of the two-mode result. Hence we begin with a direct proof of the theorem in the two-mode case.

Lemma 5.1 *The parameters* $m_1 \leq m_2$ *and* $1 \leq \kappa_1 \leq \kappa_2$ *are compatible for two-mode Gaussian states iff*

$$\begin{aligned} m_1 + m_2 &\geq \kappa_1 + \kappa_2, \\ m_2 - m_1 &\leq \kappa_2 - \kappa_1. \end{aligned} \tag{5.4}$$

Note that the condition $m_1 \geq \kappa_1$ is subsumed by (5.4).

Proof of Lemma: The covariance matrix can, through local unitary (canonical) transformation $\in Sp(2,R) \times Sp(2,R)$, be brought to the form

$$V = \begin{pmatrix} m_1 & 0 & k_x & 0 \\ 0 & m_1 & 0 & k_p \\ k_x & 0 & m_2 & 0 \\ 0 & k_p & 0 & m_2 \end{pmatrix}. \tag{5.5}$$

The global spectral parameters κ_1, κ_2 are related to the local m_1, m_2 through the symplectic invariants [57]

$$\begin{aligned} \frac{1}{2}\mathrm{tr}\,(\Omega V \Omega^T V) &= \kappa_1^2+\kappa_2^2 = m_1^2+m_2^2+2k_xk_p, \\ \det V = (\kappa_1\kappa_2)^2 &= (m_1m_2-k_x^2)(m_1m_2-k_p^2). \end{aligned} \tag{5.6}$$

These immediately imply

$$\begin{aligned} \kappa_1\kappa_2 &\leq m_1m_2, \\ \kappa_1^2+\kappa_2^2 &\geq m_1^2+m_2^2, \ \text{ if } \ k_xk_p \geq 0, \\ \kappa_1^2+\kappa_2^2 &\leq m_1^2+m_2^2, \ \text{ if } \ k_xk_p \leq 0, \end{aligned} \tag{5.7}$$

equality in the first inequality holding if $k_x = 0 = k_p$. These inequalities imply

$$\begin{aligned} \kappa_2-\kappa_1 &\geq m_2-m_1, \ \text{ when } \ k_xk_p \geq 0, \\ \kappa_2+\kappa_1 &\leq m_2+m_1, \ \text{ when } \ k_xk_p \leq 0. \end{aligned} \tag{5.8}$$

This much is immediate from the symplectic invariants. What remain to be proved are : $\kappa_2-\kappa_1 \geq m_2-m_1$ when $k_xk_p \leq 0$ and $\kappa_2+\kappa_1 \geq m_2+m_1$ when $k_xk_p \geq 0$.

To prove these we reinterpret (5.6) as simultaneous expressions for k_x, k_p in terms of κ_1, κ_2; m_1, m_2 :

$$k_xk_p = [\,(\kappa_1^2+\kappa_2^2)-(m_1^2+m_2^2)\,]/2, \tag{5.9}$$

$$k_x^2+k_p^2 = \frac{1}{m_1m_2}[\,m_1^2m_2^2-\kappa_1^2\kappa_2^2+k_x^2k_p^2\,]. \tag{5.10}$$

It is clear that real solutions for k_x and k_p will exist iff ' $k_x^2+k_p^2$ ' $\geq$ ' $2|\,k_xk_p\,|$ '. That is, iff

$$m_1m_2-|\,k_xk_p\,| \geq \kappa_1\kappa_2. \tag{5.11}$$

With use of (5.9) for k_xk_p, this last condition reads

$$\begin{aligned} \kappa_2-\kappa_1 &\geq m_2-m_1, \ \text{ when } \ k_xk_p \leq 0, \\ \kappa_2+\kappa_1 &\leq m_2+m_1, \ \text{ when } \ k_xk_p \geq 0. \end{aligned} \tag{5.12}$$

Proof of the Lemma is thus complete.

Two types of simple transformations on any pair of modes characterised by annihilation operators a_j, a_k deserve particular mention; they play a key role in our proof of the theorem. The first, S_θ, corresponds to the *compact* transformations $a_j \to \cos\theta\, a_j + \sin\theta\, a_k$, $a_k \to -\sin\theta\, a_j + \cos\theta\, a_k$, and therefore is represented by $S_\theta = \cos\theta\, \sigma_0 \otimes \sigma_0 + \sin\theta\, i\sigma_2 \otimes \sigma_0 \in Sp(4,R)$, $0 \le \theta < 2\pi$, where σ_0 is the 2×2 unit matrix and σ_2 is the antisymmetric Pauli matrix. Physically, S_θ is a *beam splitter with transmitivity* $\cos^2\theta$. The second one, S_μ, is *noncompact* and corresponds to squeezing transformations $a_j \to \cosh\mu\, a_j + \sinh\mu\, a_k^\dagger$, $a_k \to \cosh\mu a_k + \sinh\mu\, a_j^\dagger$, and is represented by $S_\mu = \cosh\mu\, \sigma_0 \otimes \sigma_0 + \sinh\mu\, \sigma_1 \otimes \sigma_3 \in Sp(4,R)$, $0 \le \mu < \infty$.

It is easily verified that when the covariance matrix V, Eq. (5.5), has $k_p = k_x \equiv k$, it can be diagonalised by the beam splitter transformation $V \to S_\theta V S_\theta^T$, with θ fixed through $\tan 2\theta = 2k/(m_2 - m_1)$. And $\kappa_2 + \kappa_1$ will precisely equal $m_2 + m_1$ in this case. Similarly, if $k_p = -k_x = k > 0$, then V is diagonalised by the squeezing transformation $V \to S_\mu V S_\mu^T$, with $\tanh 2\mu = 2k/(m_2 + m_1)$, and one will find $\kappa_2 - \kappa_1 = m_2 - m_1$ in this case.

Conversely, suppose we start with the canonical form $V^{(0)} = \text{diag}\ (\kappa_1,\kappa_1;\ \kappa_2,\kappa_2)$, and we wish to achieve through symplectic congruence $V^{(0)} \to S V^{(0)} S^T$, $S \in Sp(4,R)$, a covariance matrix with diagonals m_1, m_2. If $m_1 < m_2$ are such that $m_2 < \kappa_2$ and $\kappa_2 + \kappa_1 = m_2 + m_1$, such a *redistribution* of κ_1, κ_2 among m_1, m_2 can always be achieved through a beam splitter transformation S_θ. Under S_θ we have $m_2 + m_1 = \kappa_2 + \kappa_1$ and $m_2 - m_1 = \cos 2\theta\, (\kappa_2 - \kappa_1)$. On the other hand, if $m_2 > \kappa_2$ and $\kappa_2 - \kappa_1 = m_2 - m_1$, so that κ_1 and κ_2 are enhanced by equal amounts to m_1, m_2, this can be achieved through a squeezing transformation S_μ. Under S_μ we have $m_2 - m_1 = \kappa_2 - \kappa_1$ and $m_2 + m_1 = \cosh 2\mu\, (\kappa_2 + \kappa_1)$.

Our Lemma is similar to Lemma 5 of Ref. [352], but our proof is direct and constructive. There is an important distinction in content as well: while theirs claims that $m_2 - m_1 = \kappa_2 - \kappa_1$ iff $m_2 = \kappa_2$ and $m_1 = \kappa_1$, we have just demonstrated that if $m_2 - m_1 = \kappa_2 - \kappa_1$ then $m_2 + m_1$ could equal $\cosh 2\mu\, (\kappa_2 + \kappa_1)$ for any $0 \le \mu < \infty$, not just $\mu = 0$. Indeed, this distinction is central to Stage 2 of our proof of the second part of the main theorem, the part which distinguishes the present symplectic situation from the Schur-Horn case.

Returning to Eq. (5.10), if we are given values for the expressions '$a^2 + b^2$' and 'ab' with $a^2 + b^2 \ge 2|ab|$, the solution for (a,b) is *unique* [(a,b) and (b,a) are not distinct solutions for our purpose]. This innocent looking observation leads to a surprising conclusion.

Proposition 5.1 *Specification of the local and global spectra of a two-mode Gaussian*

state determines uniquely *the state itself, modulo local unitary transformations.*

States of a pair of qubits share a similarity with two-mode Gaussian states in important respects. For instance, positivity under partial transpose is a necessary and sufficient condition for separability and nondistillability in both cases. But a statement analogous to the above proposition is not true for a pair of qubits!

5.3 Proof of main theorem

Assume we are given a (zero-mean) Gaussian state, or equivalently, an acceptable covariance matrix V, the 2×2 blocks along the leading diagonal of V being of the form $\mathrm{diag}(m_j, m_j)$. The global spectral parameters $\{\kappa_j\}$ are immediately defined by V [57, 192]. It is assumed that $m = (m_1, m_2, \cdots m_n)$ and $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ are arranged in nondecreasing order. Let P_κ denote the product $\kappa_1\kappa_2\cdots\kappa_n$ and let $P_m = m_1 m_2 \cdots m_n$. Clearly, $P_\kappa = \det V \leq P_m$, equality holding iff V is diagonal, i.e., iff $m_j = \kappa_j$, $j = 1, 2, \cdots, n$. Our task is to prove that κ dominates m.

Choose a pair $1 \leq j < k \leq n$ such that the 2×2 block (in the off-diagonal location) connecting the j^{th} and k^{th} modes is nonzero. We can arrange (through local rotations) this block to be diagonal. Let us 'diagonalise' this 4×4 part of the covariance matrix using an appropriate two-mode canonical transformation $\in Sp(4, R)$, so that m_j and m_k are transformed to $\tilde{m}_j$ and $\tilde{m}_k$ respectively, the other diagonal parameters remaining unaffected.

It is be noted that the new m dominates the original m. That this is so follows, in the case $k < n$, from the facts $\tilde{m}_j < m_j$ and $\tilde{m}_j + \tilde{m}_k \leq m_j + m_k$. In the case $k = n$ it follows from the additional fact that if $\tilde{m}_k$ is less that m_k it is so by a magnitude which does not exceed the magnitude by which $\tilde{m}_j$ is less than m_j ($\tilde{m}_k - \tilde{m}_j \geq m_k - m_j$). Further, $\tilde{m}_j\tilde{m}_k < m_j m_k$.

Denote by m' the new diagonal m-parameters arranged in nondecreasing order by correspondingly permuting the oscillators. Since $\tilde{m}_j\tilde{m}_k < m_j m_k$ we have $P_{m'} < P_m$.

For purpose of clarity, let us carry out this process one more time. The parameters m' will then go to m'' dominating m', with $P_{m''} < P_{m'}$. It follows from the transmitivity of dominance that m'' dominates m.

It is now clear that when this process is iterated, m goes through a sequence of intermediate values, the value at every stage dominating the previous value, and correspondingly P_m steadily decreasing, until P_m reaches P_κ or, equivalently, until V becomes diagonal. This completes proof of the first part of the theorem.

The elementary nature of our proof may be compared with that of Ref. [285]. P_m played the role of 'profit function' monitoring progress of this diagonalisation process.

To prove the second part assume, conversely, that we are given the global and local spectral parameters κ, $m \in R^n$. Assume that these are compatible: i.e., κ dominates m, with $\kappa_1 \geq 1$. Our task is to construct a Gaussian state with these properties. In other words we have to present a canonical transformation $S \in Sp(2n, R)$ which acting on a covariance matrix $V = \mathrm{diag}(\kappa_1, \kappa_1\,;\, \kappa_2, \kappa_2\,;\, \cdots\,;\, \kappa_n, \kappa_n)$ will produce a covariance matrix SVS^T with the target diagonal values m. We build such an S as a product of $n-1$ *specific* two-mode transformations, evolving $m^{(0)} \equiv \kappa$ successively through a sequence of intermediates $m^{(1)}$, $m^{(2)}$, $\cdots$ to finally $m^{(n-1)} = m$. It will be manifestly clear that $m^{(k)}$ dominates $m^{(k+1)}$ at each stage. For clarity, this process is implemented through four elementary stages.

5.3.1 Stage 1

Since $m^{(0)} \equiv \kappa$ dominates m, we have $m_1 \geq m_1^{(0)} = \kappa_1$. Suppose $m_1 = m_1^{(0)} + \epsilon_1$, $\epsilon_1 > 0$ (one will move to the next step if $m_1 = m_1^{(0)}$). Let j_1 be the least integer $< n$ such that $m_{j_1}^{(0)} \geq m_1$. Carry out a beam splitter transformation S_θ between the first and $j_1{}^{\text{th}}$ mode so that the corresponding diagonal elements $(m_1^{(0)}, m_{j_1}^{(0)})$ get redistributed to $(m_1^{(0)} + \epsilon_1, m_{j_i}^{(0)} - \epsilon_1) = (m_1, m_{j_1}^{(0)} - \epsilon_1)$, with no change in the other diagonal entries: $m^{(0)} = (m_1^{(0)}, m_2^{(0)}, \cdots, m_n^{(0)}) \rightarrow m^{(1)} = (m_1, m_2^{(0)}, \cdots, m_{j_1}^{(0)} - \epsilon_1, \cdots, m_n^{(0)}) \equiv (m_1, m_2^{(1)}, m_3^{(1)}, \cdots, m_n^{(1)})$.

We can repeat this process. Let $m_2 = m_2^{(1)} + \epsilon_2$. By hypothesis $\epsilon_2 \geq 0$ (this is so even if j_1 had equalled 2). Assume $\epsilon_2 > 0$ (if $\epsilon_2 = 0$, one moves to the next step). Let j_2 be the smallest integer $< n$ such that $m_{j_2}^{(1)} \geq m_2$ [Clearly, j_2 can be as small as j_1, but not any smaller]. Carry out a beam splitter transformation on the 2^{nd} and j_2^{th} modes so that the corresponding diagonal elements $(m_2^{(1)}, m_{j_2}^{(1)})$ get redistributed to $(m_2, m_{j_2}^{(1)} - \epsilon_2)$ to produce $m^{(2)}$, leaving the other diagonals unaffected.

If we are able to repeat this process only ℓ times we have, at the end of it,

$$m^{(\ell)} = (\, m_1, m_2, \cdots, m_\ell; m_{\ell+1}^{(\ell)}, m_{\ell+2}^{(\ell)}, \cdots, m_n^{(\ell)}\,), \tag{5.13}$$

with $m_j^{(\ell)} < m_j$, $\forall\ \ell + 1 \leq j \leq n-1$, and $m_n^{(l)} = m_n^{(0)} = \kappa_n$. What we have done so far is identical to what one would have done in the Schur-Horn situation. Clearly, the beam splitter transformations carried out so far affected neither the sum of the diagonal entries of $m^{(\cdot)}$ nor its n^{th} entry. Consequently, the difference $m_n^{(k)} - \sum_{j=1}^{n-1} m_j^{(k)}$ has remained the same for all $0 \leq k \leq \ell$.

5.3.2 Stage 2

Define $\delta^{(k)} = \sum_{j=1}^{n} m_j - \sum_{j=1}^{n} m_j^{(k)}$. It is clear that $\delta^{(k)} = \delta^{(0)}$, for $k = 1, 2, \cdots, l$. In the Schur-Horn situation $\delta^{(0)}$ vanishes by hypothesis. We will now employ two-mode squeezing transformations S_μ to rectify this 'departure' from the Schur-Horn situation.

We know that $\delta^{(\ell)} = \delta^{(0)}$ is nonnegative. Assume $\delta^{(0)} > 0$ (if $\delta^{(0)} = 0$, one will move directly to Stage 4, as will become evident below). Define $\epsilon_{\ell+1} = m_{\ell+1} - m_{\ell+1}^{(\ell)}$. Assume $\delta^{(\ell)} \geq 2\epsilon_{\ell+1}$ (if this is not the case one will move to Stage 3). Carry out a two-mode squeezing transformation S_μ between the $(\ell+1)^{\text{th}}$ and n^{th} modes, raising the corresponding diagonal entries $m_{\ell+1}^{(\ell)}$, $m_n^{(\ell)} = m_n^{(0)} = \kappa_n$ by equal magnitude to $m_{\ell+1}$, $m_n^{(\ell)} + \epsilon_{\ell+1}$ with no change in the other diagonal entries, so that

$$\begin{aligned} m^{(\ell+1)} &= (m_1, \cdots, m_{\ell+1}, m_{\ell+2}^{(\ell+1)}, \cdots, m_n^{(\ell+1)}), \\ m_j^{(\ell+1)} &= m_j^{(\ell)}, \quad \forall\, \ell+2 \leq j \leq n-1, \\ m_n^{(\ell+1)} &= m_n^{(\ell)} + \epsilon_{\ell+1} = \kappa_n + \epsilon_{\ell+1}. \end{aligned} \tag{5.14}$$

We can now repeat this kind of two-mode squeezing transformation between the $(\ell+2)^{\text{th}}$ mode and the n^{th} mode, and so on. Assume we are able to carry out this process only r times. We will have, at the end of it,

$$\begin{aligned} m^{(\ell+r)} &= (m_1, \cdots, m_{\ell+r}, m_{\ell+r+1}^{(\ell+r)}, \cdots, m_n^{(\ell+r)}), \\ m_j^{(\ell+r)} &= m_j^{(\ell)}, \quad \forall\, \ell+r+1 \leq j \leq n-1, \\ m_n^{(\ell+r)} &= \kappa_n + \epsilon_{\ell+1} + \epsilon_{\ell+2} + \cdots + \epsilon_{\ell+r}, \end{aligned} \tag{5.15}$$

so that $\delta^{(\ell+r)} = \delta^{(0)} - 2(\epsilon_{\ell+1} + \epsilon_{\ell+2} + \cdots + \epsilon_{\ell+r})$. Clearly, $0 \leq \delta^{(\ell+r)} < 2\epsilon_{\ell+r+1} = 2(m_{\ell+r+1} - m_{\ell+r+1}^{(\ell+r)})$ (the last inequality encodes the fact that we could not carry out the Stage 2 operation one more time).

5.3.3 Stage 3

Assume $\delta^{(\ell+r)} > 0$ (if $\delta^{(\ell+r)} = 0$, we move directly to Stage 4). Carry out a two-mode canonical transformation between the $(\ell+r+1)^{\text{th}}$ mode and the n^{th} mode, taking the corresponding diagonal entries $m_{\ell+r+1}^{(\ell+r)}$, $m_n^{(\ell+r)}$ to $m_{\ell+r+1} = m_{\ell+r+1}^{(\ell+r)} + \epsilon_{r+\ell+1}$ and $m_n^{(\ell+r+1)} = m_n^{(\ell+r)} + \delta^{(\ell+r)} - \epsilon_{r+\ell+1}$ respectively, leaving the other diagonals invariant,

so that we have

$$\begin{aligned} m^{(\ell+r+1)} &= (m_1, \cdots, m_{\ell+r+1}, m_{\ell+r+2}^{(\ell+r+1)}, \cdots, m_n^{(\ell+r+1)}), \\ m_j^{(\ell+r+1)} &= m_j^{\ell} < m_j, \quad \forall\, \ell + r + 2 \leq j \leq n-1, \\ & \qquad \sum_{j=\ell+r+2}^{n} m_j^{(\ell+r+1)} = \sum_{j=\ell+r+2}^{n} m_j. \end{aligned} \tag{5.16}$$

i.e., the situation in respect of the remaining $n-(\ell+r+1)$ (or $n-\ell-r$ if $\delta^{(\ell+r)}=0$) modes is precisely of the Schur-Horn type, suggesting that we deploy the beam splitter transformation $n-l-r-2$ (or $n-l-r-1$) times.

5.3.4 Stage 4

Note that at the end of Stage 3 we have $m_n^{(\ell+r+1)}$ larger than m_n precisely by the sum of the amounts by which $m_{\ell+r+1+j}^{(\ell+r+1)}$, for $1 \leq j \leq n-\ell-r-2$, are less than $m_{\ell+r+1+j}$. Therefore, for each value of j in this range, we effect a beam splitter transformation connecting the $(\ell+r+1+j)^{\text{th}}$ mode to the n^{th} mode, raising $m_{\ell+r+1+j}^{(\ell+r+1)}$ to m_j and correspondingly pulling $m_n^{(\ell+r+1+j)}$ down by an equal amount. It is clear that at the end of these $n-\ell-r-2$ (or $n-\ell-r-1$) redistributions, the diagonals will be precisely m. That is, $m^{(n-1)}=m$. This completes proof of the theorem.

We have taken maximal advantage of the simpler two-mode transformations S_θ, S_μ. The former was deployed r times in Stage 1 and $n-\ell-r-2$ (or $n-\ell-r-1$) times in Stage 4, and the latter ℓ times in Stage 2. The more general two-mode transformation was deployed (at the most) once in Stage 3.

As illustration, and for comparison with Ref. [352], we apply our procedure to the example noted after the statement of the theorem. The difference between $\sum_{j=1}^{7} m_j = 63$ and $\sum_{j=1}^{7} \kappa_j = 45$ indicates the amount of squeezing that will have to be deployed at Stages 3 and 4. We have $m^{(0)} \equiv \kappa = (1,2,3,4,5,12,18)$; $m^{(1)} = (6,2,3,4,5,7,18)$; $m^{(2)} = (6,7,3,4,5,2,18)$; $m^{(3)} = (6,7,8,4,5,2,23)$; $m^{(4)} = (6,7,8,9,5,2,26)$; $m^{(5)} = (6,7,8,9,10,2,21)$; and $m^{(6)} = (6,7,8,9,10,11,12) = m$. The number of two-mode transformations required at the four stages are 2, 1, 1, and 2 respectively. Note that $m^{(k)}$ dominates $m^{(k+1)}$, for $k = 0, 1, \cdots, 5$.

6

Operator-sum representation for Bosonic Gaussian channels

6.1 Introduction

Gaussian states are fully specified by their first and second moments. Since the first moments play no significant role in our study, we may assume that they vanish (this can indeed be ensured using the unitary Weyl-Heisenberg displacement operators), so that a Gaussian state for our purpose is fully described by its covariance matrix [192, 202, 226, 354]. The symplectic group of real linear canonical transformations (acting through its unitary metaplectic representation) and the Weyl-Heisenberg group of phase space translations are the only unitary evolutions which preserve Gaussianity, and these groups are generated by hermitian Hamiltonians which are respectively quadratic and linear in the creation and annihilation operators [192, 202, 226].

Any physical evolution that maps an input Gaussian state to a Gaussian state at the output is a Gaussian channel. In other words, Gaussian channels are those trace preserving completely positive (CP) maps which image every input Gaussian state into a Gaussian state at the output. The feasibility of processing information using Gaussian channels was originally explored in [355, 356]. More recently, the problem of evaluating the classical capacity of Gaussian channels was addressed in [141, 149, 282], and the quantum capacities in [157, 159, 281, 283, 284]. In particular, the classical capacity of the attenuator channel was evaluated in [282], and the quantum capacity of a class of channels was studied in [159]. A systematic study of the structure of the family of all Gaussian channels has been carried out in [158, 286–288, 357]; single-mode Gaussian channels have been classified in [158, 286], and the case of multimodes in [287, 288, 357].

Gaussian channels may be realized as Gaussianity preserving unitaries on a suitably

enlarged system :

$$\rho_A \rightarrow \rho_A^{'} = \mathrm{Tr}_B \left(U_{AB} \left(\rho_A \otimes \rho_B \right) U_{AB}^{\dagger} \right). \tag{6.1}$$

Here ρ_B is a Gaussian state of the ancilla B, and U_{AB} is a linear canonical transformation on the enlarged composite system consisting of the system of interest A and the ancilla B. That all Gaussian channels can indeed be realized in this manner has been shown by the work of Holevo and coauthors [158, 286, 288, 357].

It is clear that the most general trace-preserving linear map Ω which takes Gaussian characteristic functions to Gaussian, taking states with vanishing first moments to ones with vanishing first moments, are necessarily of the form $\Omega : \chi(\xi) \rightarrow \chi^{'}(\xi) = \chi(X\xi) \exp[-\frac{1}{2}\xi^T Y \xi]$, where X, Y are real matrices with $Y = Y^T \geq 0$. And X, Y need to obey an appropriate matrix inequality to ensure that the trace-preserving map Ω is completely positive [157, 159, 358, 359]. For a given X, the minimal Y, say Y_0, meeting this inequality represents the threshold Gaussian noise that needs to be added to $\chi(X\xi)$ to make atonement for the failure of X to be a symplectic matrix, and thus rendering the map completely positive; if X happens to be a symplectic matrix, then the corresponding minimal $Y_0 = 0$.

Now, given a Gaussian channel Ω we can construct, 'quite cheaply', an entire family of Gaussian channels by simply preceding and following Ω with unitary (symplectic) Gaussian channels $U(S_1), U(S_2)$ corresponding respectively to symplectic matrices S_1, S_2. Therefore in classifying Gaussian channels it is sufficient to classify these orbits or double cosets and, further, we may identify each orbit with the 'simplest' looking representative element of that orbit (the canonical form). Since

$$U(S_1)\, \Omega\, U(S_2) \; : \; \chi(\xi) \rightarrow \chi(S_2 X S_1\, \xi) \exp[-\frac{1}{2}\xi^T S_1^T Y S_1 \xi], \tag{6.2}$$

the task actually reduces to enumeration of the orbits of (X, Y) under the transformation $(X, Y) \rightarrow (X^{'}, Y^{'}) = (S_2 X S_1,\, S_1^T Y S_1)$.

The injection of an arbitrary amount of classical (Gaussian) noise into the state is obviously a Gaussian channel : $\chi(\xi) \rightarrow \chi(\xi) \exp[-\frac{a}{2}\xi^T \xi]$, $a > 0$. It is called the classical noise channel. Now, given a Gaussian channel we may follow it up with a classical noise channel to obtain another Gaussian channel. A Gaussian channel will be said to be *quantum-limited* if it cannot be realized as another Gaussian channel followed by a classical noise channel. Conversely, the most general Gaussian channel is a quantum-limited Gaussian channel followed by a classical noise channel, and it follows that quantum-limited channels are the primary objects which need to be classified into orbits.

In the single-mode case where X, Y are 2×2 matrices, $S_1, S_2 \in Sp(2, R)$ can be so

chosen that $X^{'}$ equals a multiple of identity, a multiple of σ_3, or $(\mathbb{1}+\sigma_3)/2$ while $Y^{'}$ equals a multiple of identity or $(\mathbb{1}+\sigma_3)/2$. Thus the canonical form of a Gaussian channel X, Y is fully determined by the rank and determinant of X, Y and we have the following classification of *quantum-limited bosonic Gaussian channels* [158, 286]

$$\begin{aligned}
\mathcal{D}(\kappa;0)\ &:\quad X=-\kappa\sigma_3, && Y_0=(1+\kappa^2)\mathbb{1},\ \kappa>0;\\
\mathcal{C}_1(\kappa;0)\ &:\quad X=\kappa\mathbb{1}, && Y_0=(1-\kappa^2)\mathbb{1},\ 0\le\kappa\le 1;\\
\mathcal{C}_2(\kappa;0)\ &:\quad X=\kappa\mathbb{1}, && Y_0=(\kappa^2-1)\mathbb{1},\ \kappa\ge 1;\\
\mathcal{A}_1(0)\ &:\quad X=0, && Y_0=\mathbb{1};\\
\mathcal{A}_2(0)\ &:\quad X=(\mathbb{1}+\sigma_3)/2, && Y_0=\mathbb{1};\\
\mathcal{B}_2(0)\ &:\quad X=\mathbb{1}, && Y_0=0;\\
\mathcal{B}_1(0)\ &:\quad X=\mathbb{1}, && Y_0=0.
\end{aligned} \tag{6.3}$$

It may be noted that the quantum-limited end of both the $\mathcal{B}_1$ and $\mathcal{B}_2$ families is the trivial identity channel.

By following the above listed quantum-limited channels by injection of classical noise of magnitude a we get respectively $\mathcal{D}(\kappa;a)$, $\mathcal{C}_1(\kappa;a)$, $\mathcal{C}_2(\kappa;a)$, $\mathcal{A}_1(a)$, $\mathcal{A}_2(a)$, and $\mathcal{B}_2(a)$; the last case $\mathcal{B}_1(a)$ is special in that it is obtained from $\mathcal{B}_1(0)$ by injection of noise into *just one quadrature*: $\chi(\xi)\to\chi(\xi)\,\exp[-\frac{a}{4}\xi^T(\mathbb{1}+\sigma_3)\xi]$.

It is clear in the case of $\mathcal{D}(\kappa;0)$ that $X=-\kappa\sigma_3$ corresponds to (scaled) phase conjugation or matrix transposition of the density operator. And the phase conjugation is the most famous among positive maps which are not CP [27, 28, 57]; it is the injection of additional classical noise of magnitude (not less than) $1+\kappa^2$, represented by Y_0, that mends it into a CP map.

It is well known that every trace-preserving completely positive map has an operator-sum representation of the form

$$\rho\to\rho^{'}=\sum_\alpha W_\alpha\,\rho\,W_\alpha^\dagger,\qquad \sum_\alpha W_\alpha^\dagger W_\alpha=\mathbb{1}, \tag{6.4}$$

often called Kraus representation [7]. It may be noted, however, that this representation appears as Theorem 4 of a much earlier work of Sudarshan et al [31]. It has been presented also by Choi [6], apparently independently. Mathematicians seem to view it as a direct and immediate consequence of the dilation theorem of Stinespring [9].

In this Chapter we obtain the operator-sum representation of all the quantum limited single-mode Bosonic Gaussian channels. Our analysis lends insight into how unphysical processes such as the transposition map, or the scaling of Weyl-ordered characteristic function, or a combination of both can be rendered physical through a threshold Gaussian

noise. The motive here is to bring out this aspect in a transparent manner through the operator-sum representation. We have that scaling of the diagonal weight function and scaling of the Husimi Q function correspond to physical processes. As will be seen in the following Chapter, the fact that scaling of the Q function is physical is of critical relevance when one defines a measure of non-Gaussianity for quantum states. This Chapter further explores the notion of nonclassicality breaking and the notion of entanglement breaking in light of the operator-sum representation.

We begin with the illustration a general scheme for computation of Kraus operators, and this scheme applies uniformly to all quantum-limited Gaussian channels. This scheme takes particular advantage of the fact that the symplectic two-mode transformation which realizes the channel in the sense of (6.1) does not couple, in the Holevo canonical form, the position variables with the momentum variables. With the ancilla mode assumed to be in its vacuum state initially, it turns out that the Kraus operators for each channel can be simply read off from the matrix elements of the appropriate two-mode metaplectic operator. Even though the single-quadrature classical noise channels $\mathcal{B}_1(a)$, $a \neq 0$ [$\mathcal{B}_1(0)$ is the identity channel] are not quantum-limited, we deal with them briefly just to bring out the fact that this case too is obedient to our general computational scheme.

6.2 Kraus representation: Some general considerations

Given density operator $\rho^{(a)}$ describing the state of a single-mode radiation field, the action of a quantum-limited Gaussian channel takes it to [158, 286]

$$\rho^{'(a)} = \mathrm{Tr}_b(U^{(ab)}\,(\rho^{(a)} \otimes |0\rangle_{bb}\langle 0|)\,U^{(ab)\,\dagger}). \tag{6.5}$$

Here $|0\rangle_b$ is the vacuum state of the ancilla mode b, and $U^{(ab)}$ is the unitary operator corresponding to a suitable two-mode linear canonical transformation. It is convenient to perform the partial trace in the Fock basis of mode b. We have

$$\begin{aligned} \rho^{'(a)} &= \sum_\ell {}_b\langle \ell|U^{(ab)}\,(\rho^{(a)} \otimes |0\rangle_{bb}\langle 0|)\,U^{(ab)\,\dagger}|\ell\rangle_b \\ &= \sum_\ell {}_b\langle \ell|U^{(ab)}|0\rangle_b\,\rho^{(a)}\,{}_b\langle 0|U^{(ab)\,\dagger}|\ell\rangle_b\,. \end{aligned} \tag{6.6}$$

Clearly, ${}_b\langle \ell|U^{(ab)}|0\rangle_b$ is an operator acting on the Hilbert space of mode a. The last expression thus leads us to the Kraus representation of the channel [7] :

$$\rho \to \rho^{'(a)} = \sum_\ell W_\ell\,\rho^{(a)} W_\ell^\dagger, \qquad W_\ell = {}_b\langle \ell|U^{(ab)}|0\rangle_b. \tag{6.7}$$

It follows that once the Fock basis matrix elements of $U^{(ab)}$ are known, the Kraus operators W_ℓ can be easily read off. Let $\langle m_1 m_2|U^{(ab)}|n_1 n_2\rangle \equiv C_{n_1 n_2}^{m_1 m_2}$ be the matrix elements of $U^{(ab)}$ in the two-mode Fock basis. Since the ancilla mode b is assumed to be in the vacuum state, the W_ℓ's are obtained by setting $n_2 = 0$ and $m_2 = \ell$:

$$W_\ell = \sum_{n_1, m_1 = 0}^{\infty} C_{n_1 0}^{m_1 \ell} |m_1\rangle\langle n_1|. \tag{6.8}$$

Now, in evaluating $C_{n_1 n_2}^{m_1 m_2}$ it proves useful to employ a resolution of identity in the position basis [205] :

$$\begin{aligned} C_{n_1 n_2}^{m_1 m_2} &= \langle m_1 m_2|U^{(ab)}|n_1 n_2\rangle \\ &= \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1 m_2|x_1 x_2\rangle\langle x_1 x_2|U^{(ab)}|n_1 n_2\rangle. \end{aligned} \tag{6.9}$$

Under conjugation by $U^{(ab)}$ the quadrature variables $q_j, p_j \;\; (j = 1, 2)$ undergo a linear canonical transformation $S \in Sp(4, R)$, of which $U^{(ab)}$ is the (metaplectic) unitary representation [192]. Let us assume that this canonical transformation does not mix the position variables with the momentum variables. That is,

$$\begin{aligned} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \;\; &\to U^{(ab)\dagger} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} U^{(ab)} = \;\; \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \\ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \;\; &\to U^{(ab)\dagger} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} U^{(ab)} = \;\; \begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = (M^{-1})^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \end{aligned} \tag{6.10}$$

where M is a real non-singular 2×2 matrix. This assumption that our $S \in Sp(4, R)$ has the direct sum structure $S = M \oplus (M^{-1})^T$ will prove to be of much value in our analysis. We have

$$\begin{aligned} C_{n_1 n_2}^{m_1 m_2} &= \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1 m_2|x_1 x_2\rangle\langle x_1 x_2|U^{(ab)}|n_1 n_2\rangle \\ &= \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1 m_2|x_1 x_2\rangle \psi_{n_1}(x_1')\psi_{n_2}(x_2') \\ &= \int_{-\infty}^{\infty} dx_1 dx_2 \psi_{m_1}^*(x_1)\psi_{m_2}^*(x_2)\psi_{n_1}(x_1')\psi_{n_2}(x_2'), \end{aligned} \tag{6.11}$$

where (x_1', x_2') is linearly related to (x_1, x_2) through M. These wavefunctions are the familiar Hermite functions, the Fock states in the position representation. The above

integral may be evaluated using the generating function for Hermite polynomials [205] :

$$\begin{aligned}\psi_n(x) &= \frac{\pi^{-1/4}}{\sqrt{2^n n!}} e^{-x^2} H_n(x) \\ &= \frac{\pi^{-1/4}}{\sqrt{n!}} \frac{\partial^n}{\partial z^n} \exp\left(-\frac{1}{2}[(x - z\sqrt{2})^2 - z^2]\right)\Big|_{z=0}. \end{aligned} \tag{6.12}$$

Inserting in Eq. (6.11) the generating function for each of the four wavefunctions we have

$$C_{n_1 n_2}^{m_1 m_2} = \frac{1}{\sqrt{n_1! n_2! m_1! m_2!}} \frac{\partial^{m_1}}{\partial \eta_1^{m_1}} \frac{\partial^{m_2}}{\partial \eta_2^{m_2}} \frac{\partial^{n_1}}{\partial z_1^{n_1}} \frac{\partial^{n_2}}{\partial z_2^{n_2}} F(z_1, z_2, \eta_1, \eta_2)\Big|_{z_1, z_2, \eta_1, \eta_2 = 0}, \tag{6.13}$$

where

$$\begin{aligned} F(z_1, z_2, \eta_1, \eta_2) = \pi^{-1} \int_{-\infty}^{\infty} dx_1\, dx_2 \exp\Big\{ -\frac{1}{2}[(x_1 - \eta_1\sqrt{2})^2 + (x_2 - \eta_2\sqrt{2})^2 \\ + (x_1' - z_1\sqrt{2})^2 + (x_2' - z_2\sqrt{2})^2 - \eta_1^2 - \eta_2^2 - z_1^2 - z_2^2] \Big\}. \end{aligned} \tag{6.14}$$

The Gaussian integration over the variables x_1 and x_2 can be easily carried out to obtain $F(z_1, z_2, \eta_1, \eta_2)$, and from $F(z_1, z_2, \eta_1, \eta_2)$ we may readily obtain $C_{n_1 n_2}^{m_1 m_2}$, and hence the Kraus operators. This is the general scheme we will employ in what follows to obtain Kraus representation for quantum-limited Gaussian channels of the various families.

6.3 Phase conjugation or transposition channel $\mathcal{D}(\kappa),\ \kappa \geq 0$

We now use the above scheme to evaluate a set of Kraus operators representing the phase conjugation channel. The metaplectic unitary operator $U^{(ab)}$ appropriate for this case induces on the quadrature operators of the bipartite phase space a linear canonical transformation corresponding to the following $S \in Sp(4, R)$[158] :

$$S = \begin{pmatrix} \sinh\mu & 0 & \cosh\mu & 0 \\ 0 & -\sinh\mu & 0 & \cosh\mu \\ \cosh\mu & 0 & \sinh\mu & 0 \\ 0 & \cosh\mu & 0 & -\sinh\mu \end{pmatrix}. \tag{6.15}$$

Written in detail, the phase space variables undergo, under the action of this channel, the transformation

$$\begin{aligned}
\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &\rightarrow \begin{pmatrix} q_1^{'} \\ q_2^{'} \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \\
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} &\rightarrow \begin{pmatrix} p_1^{'} \\ p_2^{'} \end{pmatrix} = (M^{-1})^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \\
M &= \begin{pmatrix} -\sinh\mu & \cosh\mu \\ \cosh\mu & -\sinh\mu \end{pmatrix}. \qquad (6.16)
\end{aligned}$$

It is seen that the above S is indeed of the form $S = M \oplus (M^{-1})^T \in Sp(4,R)$, and does not mix the position variables with the momentum variables, and so our general scheme above readily applies.

It is clear from the structure of S that the parameter μ is related to κ in $\mathcal{D}(\kappa)$ through $\kappa = -\sinh\mu > 0$, so that $\cosh\mu = \sqrt{\kappa^2+1}$. Thus (6.14) translates, for the present case, to the following expression :

$$\begin{aligned}
F(z_1,z_2,\eta_1,\eta_2) &= \pi^{-1}\int_{-\infty}^{\infty} dx_1\,dx_2 \exp\Big\{ -\frac{1}{2}[(x_1-\eta_1\sqrt{2})^2+(x_2-\eta_2\sqrt{2})^2 \\
&\quad +(-\kappa x_1+\sqrt{1+\kappa^2}\,x_2 - z_1\sqrt{2})^2 + (\sqrt{1+\kappa^2}\,x_1 - \kappa x_2 - z_2\sqrt{2})^2 \\
&\quad -\eta_1^2-\eta_2^2-z_1^2-z_2^2]\Big\}. \qquad (6.17)
\end{aligned}$$

Performing the Gaussian integrals in x_1 and x_2 we obtain

$$\begin{aligned}
F(z_1,z_2,\eta_1,\eta_2) = (\sqrt{1+\kappa^2})^{-1} \exp\Big\{ &(\sqrt{1+\kappa^{-2}})^{-1}(\eta_1\eta_2 - z_1 z_2) \\
&+(\sqrt{1+\kappa^2})^{-1}(\eta_1 z_2 + \eta_2 z_1)\Big\}. \qquad (6.18)
\end{aligned}$$

To obtain the matrix elements $C_{n_1 n_2}^{m_1 m_2}$ we need to carry out the procedure indicated in Eq. (6.13). This may be done in two steps. We begin by rewriting the function $F(z_1,z_2,\eta_1,\eta_2)$ as

$$\begin{aligned}
F(z_1,z_2,\eta_1,\eta_2) = (\sqrt{\kappa^2+1})^{-1} \exp\Big\{ &z_2[(\sqrt{1+\kappa^2})^{-1}\eta_1 - (\sqrt{1+\kappa^{-2}})^{-1} z_1] \\
&+\eta_2[(\sqrt{1+\kappa^{-2}})^{-1}\eta_1 + (\sqrt{1+\kappa^2})^{-1} z_1]\Big\}. \qquad (6.19)
\end{aligned}$$

Performing the z_2 and η_2 differentiations respectively n_2 and m_2 times on $F(z_1,z_2,\eta_1,\eta_2)$,

we obtain

$$[(\sqrt{1+\kappa^2})^{-1}\eta_1 - (\sqrt{1+\kappa^{-2}})^{-1}z_1]^{n_2} \times$$
$$[(\sqrt{1+\kappa^{-2}})^{-1}\eta_1 + (\sqrt{1+\kappa^2})^{-1}z_1]^{m_2} F \equiv GF. \tag{6.20}$$

The remaining differentiations can be carried out using the Leibniz rule. Since we finally set $z_1, z_2, \eta_1, \eta_2 = 0$, and since $F(0) = 1$, the only terms that could possibly survive are necessarily of the form

$$\frac{\partial^{m_1}}{\partial \eta_1^{m_1}} \frac{\partial^{n_1}}{\partial z_1^{n_1}} [(\sqrt{1+\kappa^2})^{-1}\eta_1 - (\sqrt{1+\kappa^{-2}})^{-1}z_1]^{n_2} \times$$
$$[(\sqrt{1+\kappa^{-2}})^{-1}\eta_1 + (\sqrt{1+\kappa^2})^{-1}z_1]^{m_2}. \tag{6.21}$$

To evaluate the above expression we set $x = (\sqrt{\kappa^2+1})^{-1}\eta_1 - (\sqrt{1+\kappa^{-2}})^{-1}z_1$ and $y = (\sqrt{1+\kappa^{-2}})^{-1}\eta_1 + (\sqrt{1+\kappa^2})^{-1}z_1$, and compute

$$[(\sqrt{1+\kappa^2})^{-1}\partial_x + (\sqrt{1+\kappa^{-2}})^{-1}\partial_y]^{m_1} \times$$
$$[-(\sqrt{1+\kappa^{-2}})^{-1}\partial_x + (\sqrt{1+\kappa^2})^{-1}\partial_y]^{n_1}\ x^{n_2}\, y^{m_2}|_{x,y=0}. \tag{6.22}$$

Straight forward algebra leads, in view of Eq. (6.13), to

$$C_{n_1 n_2}^{m_1 m_2} = \frac{(\sqrt{1+\kappa^2})^{-1}}{\sqrt{n_1! n_2! m_1! m_2!}} \sum_{j=0}^{n_1} \sum_{r=0}^{m_1} {}^{n_1}C_j\, {}^{m_1}C_r\, (-\sqrt{1+\kappa^{-2}})^{-(m_1+j-r)}\, (\sqrt{1+\kappa^2})^{-(n_1-j+r)}$$
$$\times (-1)^{m_1-r}\ n_2! m_2! \delta_{n_2, r+j}\, \delta_{m_2, n_1-j+m_1-r}\,. \tag{6.23}$$

The Kraus operators W_ℓ, denoted $T_\ell(\kappa)$ in this case, are obtained from these matrix elements by setting $n_2 = 0$ and $m_2 = \ell$. Since $n_2 = 0 \Rightarrow r, j = 0$, we have,

$$\begin{aligned} T_\ell(\kappa) &= (\sqrt{1+\kappa^2})^{-1} \sum_{n_1, m_1 = 0}^{\infty} \frac{(\sqrt{1+\kappa^2})^{-n_1} (-\sqrt{1+\kappa^{-2}})^{-m_1} \sqrt{\ell!}}{\sqrt{n_1! m_1!}} \times \\ &\quad \delta_{\ell, n_1+m_1} (-1)^{m_1}\ |m_1\rangle\langle n_1|. \end{aligned} \tag{6.24}$$

We set $n_1 + m_1 = \ell$ and denote $n_1 = n$, leading to

$$\begin{aligned} T_\ell(\kappa) &= (\sqrt{1+\kappa^2})^{-1} \sum_{n=0}^{\ell} (\sqrt{1+\kappa^2})^{-n} (\sqrt{1+\kappa^{-2}})^{-(\ell-n)} \times \\ &\quad \sqrt{{}^{\ell}C_n}|\ell - n\rangle\langle n|, \quad \ell = 0, 1, 2, \cdots \end{aligned} \tag{6.25}$$

as our final form for the Kraus operators of the phase conjugation channel. We note that the $T_\ell(\kappa)$'s are real and manifestly trace-orthogonal: $\mathrm{tr}(T_\ell(\kappa)^\dagger T_{\ell'}(\kappa)) = 0$ if $\ell \neq \ell'$.

6.3.1 The dual of $\mathcal{D}(\kappa)$

As is well known (and also obvious), if a set of Kraus operators $\{W_\ell\}$ describes the completely positive map $\Phi : \rho \to \rho' = \sum_\ell W_\ell \rho W_\ell^\dagger$, then the dual map $\tilde{\Phi} : \rho \to \rho' = \sum_\ell W_\ell^\dagger \rho W_\ell$, described by the dual or adjoint set of operators $\{W_\ell^\dagger\}$, is also completely positive. It is clear that the dual map $\tilde{\Phi}$ is unital or trace-preserving according as Φ is trace-preserving or unital.

For the present case of $\mathcal{D}(\kappa)$, it is readily verified that the Kraus operators $\{T_\ell(\kappa)\}$ presented in (6.25) meet $\sum_\ell T_\ell{}^\dagger(\kappa) T_\ell(\kappa) = \mathbb{1}$, consistent with the expected trace-preserving nature of $\rho \to \rho' = \sum_\ell T_\ell(\kappa)\, \rho\, T_\ell{}^\dagger(\kappa)$. But the phase conjugation channel is not unital in general, for we have

$$\sum_\ell T_\ell(\kappa)\, T_\ell{}^\dagger(\kappa) = \kappa^{-2}\mathbb{1}. \tag{6.26}$$

We may say that it is 'almost unital' to emphasise the minimal nature of the failure: the unit element is taken by the channel into a scalar multiple of itself. However, the scale factor κ^{-2} can not be transformed away by absorbing κ^{-1} into the Kraus operators, for the Kraus operators so modified would not then respect the trace-preserving property of the map.

It is thus of interest to understand the nature of the *unital channel* described by the set of Kraus operators $\{T_\ell(\kappa)^\dagger\}$. We have

$$\begin{aligned}
T_\ell(\kappa)^\dagger &= (\sqrt{1+\kappa^2})^{-1} \sum_{n=0}^{\ell} (\sqrt{1+\kappa^2})^{-n} \, (\sqrt{1+\kappa^{-2}})^{-(\ell-n)} \, \sqrt{{}^\ell C_n} |n\rangle\langle \ell - n| \\
&= (\sqrt{1+\kappa^2})^{-1} \sum_{n'=\ell}^{0} (\sqrt{1+\kappa^2})^{-(\ell-n')} (\sqrt{1+\kappa^{-2}})^{-n'} \, \sqrt{{}^\ell C_{\ell-n'}} \, |\ell - n'\rangle\langle n'| \\
&= (\sqrt{1+\kappa^2})^{-1} \sum_{n=0}^{\ell} (\sqrt{\kappa^2+1})^{-(\ell-n)} \, (\sqrt{1+\kappa^{-2}})^{-n} \sqrt{{}^\ell C_n} |\ell - n\rangle\langle n| \\
&= \kappa^{-1} T_\ell(\kappa^{-1}).
\end{aligned} \tag{6.27}$$

Thus the dual $\{T_\ell(\kappa)^\dagger\}$ differs from the original $\{T_\ell(\kappa)\}$ in two elementary aspects. The multiplicative factor κ^{-1} *is the same* for all Kraus operators, independent of ℓ. Thus the only significant difference is change in the argument of T_ℓ, from κ to κ^{-1}. We conclude that the 'dual' channel whose Kraus operators are $\kappa\, T_\ell(\kappa)^\dagger$ is the (trace-preserving) phase conjugation channel $\mathcal{D}(\kappa^{-1})$. We have thus proved

Theorem 6.1 *While the Kraus operators* $\{T_\ell(\kappa)\}$ *describe* $\mathcal{D}(\kappa)$*, the 'dual' channel described by Kraus operators* $\{\kappa T_\ell(\kappa)^\dagger\}$ *is the trace-preserving phase conjugation channel* $\mathcal{D}(\kappa^{-1})$ *with reciprocal scale parameter.*

6.3.2 Action of the Kraus operators

The expected or defining action of the phase conjugation channel on the characteristic function is [158] :

$$\chi_W(\xi) \to \chi_W^{'}(\xi) = \chi_W(-\kappa\,\xi^*)\ \exp[-(1+\kappa^2)|\xi|^2/2]. \tag{6.28}$$

It is of interest to understand how the 'antilinear' phase conjugation ($\xi \to \xi^*$) action of this channel on the characteristic function emerges from the linear action of the Kraus operators. To this end, it is sufficient to establish such an action on the 'characteristic function' corresponding to the operators $|n\rangle\langle m|$, for arbitrary pairs of integers $n,\, m \geq 0$. The 'characteristic function' of $|n\rangle\langle m|$ is given by [169]

$$\begin{aligned}\chi_{W|n\rangle\langle m|}(\xi) &\equiv \langle m|D(\xi)|n\rangle \\ &= \sqrt{\frac{m!}{n!}}(-\xi^*)^{n-m}L_m^{n-m}(|\xi|^2)\ \exp[-|\xi|^2/2] \quad \text{for} \quad n \geq m, \\ &= \sqrt{\frac{n!}{m!}}(\xi)^{m-n}L_n^{m-n}(|\xi|^2)\ \exp[-|\xi|^2/2 \quad \text{for} \quad n \leq m. \end{aligned} \tag{6.29}$$

Assuming $n \geq m$, the action of the phase conjugation channel on the operator $|n\rangle\langle m|$ is

$$\begin{aligned}\sum_{\ell=0}^{\infty} T_\ell(\kappa)|n\rangle\langle m|T_\ell^\dagger(\kappa) \quad &= \quad (1+\kappa^2)^{-1}\sum_{\ell=n}^{\infty}(\sqrt{1+\kappa^2})^{-(n+m)}\,(\sqrt{1+\kappa^{-2}})^{-(2\ell-n-m)}\ \times \\ &\qquad \sqrt{{}^{\ell}C_n\,{}^{\ell}C_m}\,|\ell-n\rangle\langle\ell-m|. \end{aligned} \tag{6.30}$$

Denoting $n = m+\delta$ and $\ell - n = \lambda$, we have

$$\begin{aligned}\sum_{\ell=0}^{\infty} T_\ell(\kappa)|m+\delta\rangle\langle m|T_\ell^\dagger(\kappa) &= (1+\kappa^2)^{-1}(\sqrt{1+\kappa^2})^{-(2m+\delta)}(\sqrt{1+\kappa^{-2}})^{-\delta} \\ &\qquad \times \sum_{\lambda=0}^{\infty}\frac{(\lambda+m+\delta)!(1+\kappa^{-2})^{-\lambda}}{\sqrt{(m+\delta)!m!\lambda!(\lambda+\delta)!}}|\lambda\rangle\langle\lambda+\delta|. \end{aligned} \tag{6.31}$$

The manner in which $\mathcal{D}(\kappa)$, matrix transposition accompanied by threshold Gaussian noise $\exp[-(1+\kappa^2)|\xi|^2/2]$, acts as a channel may now be appreciated. Every operator M can be written in the Kronecker delta basis $\{|j\rangle\langle\ell|\}$ as $M = \sum_{j,\ell} c_{j\ell}|j\rangle\langle\ell|$. The coefficient

matrix C associated with $|5\rangle\langle 3|$, for instance, is $c_{j,k} = \delta_{5j}\delta_{\ell 3}$, with non-zero entry only at the lower-diagonal location $(5,3)$ marked $\otimes$ in the matrix below.

$$\begin{pmatrix} 0 & 0 & \times & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \times & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \oplus & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \times & 0 & 0 & \cdots \\ 0 & 0 & \otimes & 0 & 0 & 0 & \times & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

On transposition this entry moves to the upper-diagonal location $(3,5)$ marked $\oplus$, and the threshold noise then spreads it along the parallel upper diagonal $(3+r, 5+r)$, $-3 \leq r < \infty$ marked $\times$.

Let the Weyl-ordered characteristic function $\mathrm{tr}(D(\xi)|m+\delta\rangle\langle m|)$ where $D(\xi) = \exp[\xi a^\dagger - \xi^* a]$ is the displacement operator, be denoted $\chi_{W|m+\delta\rangle\langle m|}(\xi)$, and that of the output $\sum_{\ell=0}^{\infty} T_\ell(\kappa)|m+\delta\rangle\langle m|T_\ell(\kappa)^\dagger$ be denoted $\chi_{W|m+\delta\rangle\langle m|}^{'}(\xi)$. Then we have from Eq. (6.31)

$$\begin{aligned} \chi_{W|m+\delta\rangle\langle m|}^{'}(\xi) &= (1+\kappa^2)^{-1}(\sqrt{1+\kappa^2})^{-(2m+\delta)}(\sqrt{1+\kappa^{-2}})^{-\delta} \\ &\times \sum_{\lambda=0}^{\infty} \frac{(\lambda+m+\delta)!(1+\kappa^{-2})^{-\lambda}}{\sqrt{(m+\delta)!m!\lambda!(\lambda+\delta)!}}\langle\lambda+\delta|D(\xi))|\lambda\rangle \\ &= \frac{(1+\kappa^2)^{-1}e^{-|\xi|^2/2}}{\sqrt{(m+\delta)!m!}}(\sqrt{1+\kappa^2})^{-(2m+\delta)}(\sqrt{1+\kappa^{-2}})^{-\delta} \\ &\times \sum_{\lambda=0}^{\infty}(1+\kappa^{-2})^{-\lambda}\frac{(\lambda+m+\delta)!}{(\lambda+\delta)!}\xi^\delta L_\lambda^\delta(|\xi|^2), \end{aligned} \tag{6.32}$$

where we used (6.29), the Fock basis representation of the displacement operator. While no 'phase conjugation' is manifest as yet, we expect from Eq. (6.28) that the channel should take the characteristic function of $|m+\delta\rangle\langle m|$ to

$$\begin{aligned} \chi_{W|m+\delta\rangle\langle m|}^{''}(\xi) &= \langle m|D(-\kappa\xi^*)|m+\delta\rangle\ \exp\left[-\frac{1}{2}(1+\kappa^2)|\xi|^2\right] \\ &= \langle m+\delta|D(\kappa\xi^*)|m\rangle^*\ \exp\left[-\frac{1}{2}(1+\kappa^2)|\xi|^2\right] \\ &= \sqrt{\frac{m!}{m+\delta!}}(\kappa\xi)^\delta L_m^\delta(\kappa^2|\xi|^2)\ \exp\left[-\left(\frac{1}{2}+\kappa^2\right)|\xi|^2\right]. \end{aligned} \tag{6.33}$$

Thus the problem reduces to one of establishing equality of $\chi^{'}_{W|m+\delta\rangle\langle m|}(\xi)$ in (6.32) and $\chi^{''}_{W|m+\delta\rangle\langle m|}(\xi)$ in (6.33). That is, it remains to prove

$$\begin{aligned}
& \sqrt{\frac{m!}{m+\delta!}}(\kappa\xi)^{\delta} L_m^{\delta}(\kappa^2|\xi|^2)\, \exp\left[-(1/2+\kappa^2)|\xi|^2\right] \\
= \;& \frac{(1+\kappa^2)^{-1} e^{-|\xi|^2/2}}{\sqrt{(m+\delta)!m!}} \sum_{\lambda=0}^{\infty} (1+\kappa^{-2})^{-\lambda} (\sqrt{1+\kappa^2})^{-(2m+\delta)} (\sqrt{1+\kappa^{-2}})^{-\delta} \\
\times \;& \frac{(\lambda+m+\delta)!}{(\lambda+\delta)!} \xi^{\delta} L_{\lambda}^{\delta}(|\xi|^2), \qquad (6.34)
\end{aligned}$$

for all m, $\delta \geq 0$ [the case of $|m\rangle\langle m+\delta|$ can be handled similarly].

Since the associated Laguerre functions form a complete orthonormal set, we may expand the LHS of Eq. (6.34) in the Laguerre basis. That is, we multiply both sides of Eq. (6.34) by $(\xi^*)^{\delta} L_{\ell}^{\delta}(|\xi|^2)\, e^{-\frac{|\xi|^2}{2}}$ and evaluate the overlap integrals. We use the following two standard results : (i) orthogonality relation among Laguerres, and (ii) the overlap between a Laguerre and a scaled Laguerre function [360] :

$$\begin{aligned}
& \int_0^{\infty} e^{-|\xi|^2} |\xi|^{2\delta} L_n^{\delta}(|\xi|^2)\, L_m^{\delta}(|\xi|^2) d|\xi|^2 = \frac{(n+\delta)!}{n!}\delta_{n,m}. \\
& \int_0^{\infty} e^{-t|\xi|^2} |\xi|^{2\delta} L_m^{\delta}(\eta^2|\xi|^2) L_{\ell}^{\delta}(|\xi|^2) d|\xi|^2 = \frac{(m+\ell+\delta)!}{m!\ell!} \frac{(t-\eta^2)^m\,(t-1)^{\ell}}{t^{m+\ell+\delta+1}} \times \\
& F\left[-m, -\ell; -m-\ell-\delta, \frac{t(t-\eta^2-1)}{(t-1)(t-\eta^2)}\right]. \qquad (6.35)
\end{aligned}$$

Here $F[\cdot]$ is the hypergeometric function. In our case $t = \eta^2 + 1$, which implies that the last argument of $F[\cdot]$ in Eq. (6.35) is zero, and thereby $F[\cdot] = 1$. Performing the overlap integrals, we obtain for the left and right hand sides of (6.34)

$$\begin{aligned}
\text{LHS} &= \frac{(m+\ell+\delta)!}{\ell!\sqrt{(m+\delta)!m!}} \frac{\kappa^{2\ell+\delta}}{(1+\kappa^2)^{m+\ell+\delta+1}} \quad , \\
\text{RHS} &= \frac{(m+\ell+\delta)!}{\ell!\sqrt{(m+\delta)!m!}} (\sqrt{1+\kappa^2})^{-(2+2m+\delta)} (\sqrt{1+\kappa^{-2}})^{-(2\ell+\delta)}. \qquad (6.36)
\end{aligned}$$

These two expressions obviously equal one another for all ℓ. We have thus established Eq. (6.34), and the fact that the Kraus operators indeed effect the 'completely positive phase conjugation' operation, transforming the characteristic function as expected in (6.28).

Theorem 6.2 *The scaled phase conjugation transformation* $\chi_W(\xi) \rightarrow \chi^{'}{}_W(\xi) = \chi_W(-\kappa\,\xi^*)\, \exp[-(1+\kappa^2)\frac{|\xi|^2}{2}]$ *is, in view of the threshold noise* $\exp[-(1+\kappa^2)|\xi|^2/2]$ *a completely positive map, and is implemented linearly by the Kraus operators* $\{T_{\ell}(\kappa)\}$ *in*

Eq. (6.25).

The phase conjugation channel has an interesting property in respect of classicality/nonclassicality of the output states. We may say a channel is *nonclassicality breaking* if the output of the channel is classical for every input state. That is, if the normal-ordered characteristic function $\chi_N^{'}(\xi)$ of the output, related to the Weyl-ordered characteristic function $\chi_W^{'}(\xi)$ of (6.28) through $\chi_N^{'}(\xi) = \chi_W^{'}(\xi)\exp[|\xi|^2/2]$, is such that its Fourier transform, called the diagonal 'weight' function $\phi(\alpha)$ [112], is a genuine probability density.

Now, Eq. (6.28) written in terms of the normal-ordered characteristic function reads

$$\begin{aligned}\chi_N(\xi) \to \chi_N^{'}(\xi) &= \chi_W(-\kappa\xi^*)\exp[-\kappa^2|\xi^*|^2/2] \\ &= \chi_A(-\kappa\xi^*),\end{aligned} \tag{6.37}$$

where $\chi_A(\xi) = \chi_N(\xi)\exp[-|\xi|^2]$ is the antinormal-ordered characteristic function corresponding to the Q or Husimi distribution.

Under Fourier transformation this important relation (6.37), namely $\chi_N^{'}(\xi) = \chi_A(-\kappa\xi^*)$, reads that the *output* diagonal weight function $\phi^{'}(\alpha)$ evaluated at α equals the *input* $Q(\alpha)$ evaluated at $\kappa^{-1}\alpha^*$. Thus $\phi^{'}(\alpha)$ is a genuine probability density for every input state, and we have

$$\mathcal{D}(\kappa) \,:\, \phi_{\rm in}(\alpha) \to \phi_{\rm out}(\alpha) = \kappa^{-2}Q_{\rm in}(\kappa^{-1}\alpha^*). \tag{6.38}$$

Since the Q-distribution of a density operator is given by $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle$, it is a genuine probability distribution for all states including nonclassical states. We have thus proved

Theorem 6.3 *The phase conjugation channel is a nonclassicality breaking channel.*

6.3.3 Entanglement breaking property

It is known that the phase conjugating channel is entanglement breaking [361, 362]. It is also known that every entanglement breaking channel has a description in terms of rank one Kraus operators [363]. We demonstrate these aspects using our Kraus operators $\{T_\ell(\kappa)\}$.

The Kraus operators $T_\ell(\kappa)$ presented in (6.25) are not of unit rank; indeed, rank $T_\ell(\kappa) = \ell + 1$, $\ell = 0, 1, 2, \cdots$. We noted immediately following (6.25) that $T_\ell(\kappa)$ are trace-orthogonal. In the generic case, trace-orthogonality requirement would render the Kraus operators unique, but this is not true with the present situation. The reason is that all these trace-orthogonal $T_\ell(\kappa)$'s have the same Frobenius norm: tr $\left(T_\ell(\kappa)T_\ell(\kappa)^\dagger\right) =$

$(1+\kappa^2)^{-1}$, independent of ℓ. Thus the set $\{T_r^{'}\}$ defined through $T_r^{'}(\kappa) = \sum_\ell U_{\ell r} T_\ell(\kappa)$, for any unitary matrix $(U_{\ell r})$ will be a set of trace-orthogonal Kraus operators describing the same channel as the original trace-orthogonal set $\{T_\ell(\kappa)\}$.

More generally, and independent of trace-orthogonality, the map $\rho \to \rho^{'} = \sum_\alpha T_\alpha^{'}(\kappa)\rho T_\alpha^{'\dagger}(\kappa)$ describes the same channel as $\rho \to \rho^{'} = \sum_\ell T_\ell(\kappa)\rho T_\ell^\dagger(\kappa)$ if the matrix U connecting the sets $\{T_\ell(\kappa)\}$ and $\{T_\alpha^{'}(\kappa)\}$ is an isometry [6, 364] :

$$\begin{aligned} T_\alpha^{'}(\kappa) = \sum_\alpha U_{\ell\alpha} T_\ell(\kappa), \qquad \sum_\alpha U_{\ell\alpha} U_{r\alpha}^* = \delta_{\ell r} \\ \Rightarrow \sum_\ell T_\ell(\kappa)\, \rho\, T_\ell^\dagger(\kappa) = \sum_\alpha T_\alpha^{'}(\kappa)\, \rho\, T^{'\dagger}_{\ \alpha}(\kappa). \end{aligned} \tag{6.39}$$

If the index set α is continuous, as in the case below, then $\sum_\alpha$ is to be understood, of course, as an integral. Now, the matrix elements between coherent states $|\alpha\rangle$ and Fock states $|k\rangle$ define such an isometry

$$U_{\ell\alpha} \equiv \langle \ell|\alpha\rangle = \exp[-|\alpha|^2/2]\frac{\alpha^\ell}{\sqrt{\ell!}}. \tag{6.40}$$

The resulting new Kraus operators $T_\alpha^{'}(\kappa)$ are

$$\begin{aligned} T_\alpha^{'}(\kappa) &= e^{-\frac{|\alpha|^2}{2}} \sum_{\ell=0}^{\infty} \frac{\alpha^\ell}{\sqrt{\ell!}} T_\ell(\kappa) \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{\ell=0}^{\infty} \frac{\alpha^\ell}{\sqrt{\ell!}} (\sqrt{1+\kappa^2})^{-1} \sum_{n=0}^{\ell} \sqrt{{}^\ell C_n}\, (\sqrt{1+\kappa^2})^{-n}\, (\sqrt{1+\kappa^{-2}})^{-(\ell-n)} |\ell-n\rangle\langle n| \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{\ell=0}^{\infty} (\sqrt{1+\kappa^2})^{-1} \sum_{n=0}^{\ell} \frac{[(\sqrt{1+\kappa^2})^{-1}\alpha]^n\, [(\sqrt{1+\kappa^{-2}})^{-1}\alpha]^{\ell-n}}{\sqrt{(\ell-n)!n!}}\, |\ell-n\rangle\langle n| \\ &= \frac{1}{\sqrt{1+\kappa^2}} |\alpha/\sqrt{1+\kappa^{-2}}\rangle\langle \alpha^*/\sqrt{1+\kappa^2}|, \ \forall \alpha \in \mathcal{C}. \end{aligned} \tag{6.41}$$

It is manifest that rank $T_\alpha^{'}(\kappa) = 1$ for all $\alpha \in \mathcal{C}$, the complex plane, showing that the phase conjugation channel is indeed entanglement breaking. However $\{T_\alpha^{'}(\kappa)\}$ are not trace-orthogonal even though $\{T_\ell(\kappa)\}$ from which the former are constructed were trace-orthogonal. This is due to the fact that the isometry U defined in (6.40) is not an unitary, which in turn is a consequence of the overcompleteness of the coherent states.

This brings us to another aspect of $\mathcal{D}(\kappa)$. In terms of these new Kraus operators the

phase conjugation channel $\mathcal{D}(\kappa)$ reads

$$\begin{aligned}\rho \to \rho^{'} &= \pi^{-1} \int d^2\alpha\, T_{\alpha}^{'}(\kappa)\, \rho\, T_{\alpha}^{'\dagger}(\kappa) \\ &= \pi^{-1}(1+\kappa^2)^{-1} \int d^2\alpha\, Q((\sqrt{1+\kappa^2})^{-1}\alpha^*)|\alpha/\sqrt{1+\kappa^{-2}}\rangle\langle\alpha/\sqrt{1+\kappa^{-2}}|. \quad (6.42)\end{aligned}$$

Thus the diagonal weight function of the output state of the channel is the Q-distribution of the input state ρ: $\phi_{\rm out} = \kappa^{-2} Q_{\rm in}(\kappa^{-1}\alpha^*)$. We may combine this result with the earlier one on rank one Kraus operators to state

Theorem 6.4 *The diagonal weight of the output of the quantum-limited phase conjugation channel is essentially the Q-distribution of the input state. The channel $\mathcal{D}(\kappa)$ is not only classicality breaking, but also entanglement breaking.*

The diagonal weight of the *output state* at α is the Q-distribution of the *input state* evaluated at $\kappa^{-1}\alpha^*$. Since $Q(\alpha) \geq 0$ for all α and for any ρ, the channel is nonclassicality breaking. The intimate relationship between this result and the earlier one on nonclassicality breaking may be noted. While the former followed directly from the behaviour of the characteristic function, the present one required consideration of the Kraus operators.

6.4 Beamsplitter/attenuator channel $\mathcal{C}_1(\kappa),\ 0 < \kappa < 1$

The two-mode unitary operator corresponding to the beamsplitter channel induces the following symplectic transformation on the quadrature operators of the bipartite phase space [158]:

$$S = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}. \quad (6.43)$$

Note that S is a direct sum of identical two-dimensional rotations: as in the case of $\mathcal{D}(\kappa)$, the position and momentum operators are not mixed by this transformation. The position variables transform as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} q_1^{'} \\ q_2^{'} \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad (6.44)$$

and, consequently, the momentum variables as

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rightarrow \begin{pmatrix} p_1^{'} \\ p_2^{'} \end{pmatrix} = (M^{-1})^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = M \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{6.45}$$

It is evident from S that the parameter κ in $\mathcal{C}_1(\kappa)$ is related to θ through $\cos\theta = \kappa$, $\sin\theta = \sqrt{1-\kappa^2}$. The function $F(z_1, z_2, \eta_1, \eta_2)$ of (6.14) for the present case is given by

$$F(z_1, z_2, \eta_1, \eta_2) = \exp\left[\eta_2(\sqrt{1-\kappa^2}\, z_1 + \kappa z_2) + \eta_1(\kappa z_1 - \sqrt{1-\kappa^2}\, z_2)\right]. \tag{6.46}$$

As in the previous case of $\mathcal{D}(\kappa)$, the differentiation on $F(z_1, z_2, \eta_1, \eta_2)$ can be performed in a straight forward manner to obtain the matrix elements of the unitary operator [326], leading to

$$\begin{aligned} C_{n_1 n_2}^{m_1 m_2} &= \frac{1}{\sqrt{n_1! n_2! m_1! m_2!}} \sum_{r=0}^{n_1} \sum_{j=0}^{n_2} {}^{n_1}C_r \, {}^{n_2}C_j \, (-1)^{n_2-j} \, \kappa^{n_1-r+j} \, (\sqrt{1-\kappa^2})^{r+n_2-j} \\ &\quad \times m_1! m_2! \, \delta_{m_2, r+j} \, \delta_{m_1, n_1+n_2-r-j}. \end{aligned} \tag{6.47}$$

Now, to obtain the Kraus operators from these matrix elements we set, as in the case of $\mathcal{D}(\kappa)$, $n_2 = 0$ and $m_2 = \ell$. Setting $n_2 = 0 \Rightarrow j = 0$, and we have

$$B_\ell(\kappa) = \sum_{m=0}^{\infty} \sqrt{{}^{m+\ell}C_\ell} \, (\sqrt{1-\kappa^2})^\ell \, \kappa^m |m\rangle\langle m+\ell|, \quad \ell = 0, 1, 2, \cdots \tag{6.48}$$

as the Kraus operators of the beamsplitter or quantum-limited attenuator channel. It is easy to see that the Kraus operators are real and pairwise trace-orthogonal, as in the case of $\mathcal{D}(\kappa)$.

6.4.1 Action of the Kraus operators

Recall that the beamsplitter channel induces the following transformation on the characteristic function [158]:

$$\begin{aligned} \chi_W(\xi) \rightarrow \chi_W{}'(\xi) &= \chi_W(\kappa\,\xi) \;\; \exp[-(1-\kappa^2)|\xi|^2/2] \\ &= \chi_W(\kappa\,\xi) \, \exp[\kappa^2|\xi|^2/2] \exp[-|\xi|^2/2]. \end{aligned} \tag{6.49}$$

Thus the normal ordered characteristic function $\chi_N(\xi)$ transforms as

$$\chi_N(\xi) \equiv \chi_W(\xi) \exp(|\xi|^2/2) \rightarrow \chi_N^{'}(\xi) = \chi_N(\kappa\,\xi). \tag{6.50}$$

Since $\chi_N(\xi)$ and the diagonal weight $\phi(\alpha)$ form a Fourier transform pair, it is immediately seen that $\phi(\alpha)$ gets simply scaled under the action of the $\mathcal{C}_1(\kappa)$ channel: $\phi(\alpha) \rightarrow \phi^{'}(\alpha) = \kappa^{-2}\phi(\kappa^{-1}\alpha)$ [365].

It is instructive to bring out this fact from the perspective of the Kraus operators. Since every state ρ can be expressed through a diagonal 'weight' $\phi(\alpha)$ as [112]

$$\rho = \pi^{-1} \int d^2\alpha \, \phi(\alpha) |\alpha\rangle\langle\alpha|, \tag{6.51}$$

to exhibit the action of the channel on an arbitrary state it is sufficient to consider its action on a generic coherent state. We have

$$\begin{aligned} |\alpha\rangle\langle\alpha| \rightarrow & \sum_{\ell=0}^{\infty} B_\ell(\kappa)|\alpha\rangle\langle\alpha|B_\ell^\dagger(\kappa) \\ = & \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((1-\kappa^2)|\alpha|^2)^\ell}{\ell!}(\kappa\alpha^*)^m(\kappa\alpha)^n \frac{e^{-|\alpha|^2}}{\sqrt{m!n!}}|m\rangle\langle n|, \end{aligned} \tag{6.52}$$

where we used the fact that the operator

$$\begin{aligned} |m\rangle\langle n| \rightarrow & \sum_{\ell=0}^{\infty} B_\ell(\kappa)|m\rangle\langle n|B_\ell^\dagger(\kappa) \\ = & \sum_{\ell=0}^{\min\{m,n\}} \sqrt{{}^mC_\ell \, {}^nC_\ell}\,(1-\kappa^2)^\ell \kappa^{m+n-2\ell}|m-\ell\rangle\langle n-\ell|. \end{aligned} \tag{6.53}$$

Carrying out the summations in Eq. (6.52), one finds [366]

$$\sum_{\ell=0}^{\infty} B_\ell(\kappa)|\alpha\rangle\langle\alpha|B_\ell^\dagger(\kappa) = |\kappa\alpha\rangle\langle\kappa\alpha|. \tag{6.54}$$

With this the action of the channel $\mathcal{C}_1(\kappa)$ reads

$$\begin{aligned} \rho \;\; \rightarrow \;\; \rho^{'} &= \pi^{-1} \int d^2\alpha \, \phi(\alpha)|\kappa\alpha\rangle\langle\kappa\alpha| \\ &= \pi^{-1}\kappa^{-2} \int d^2\alpha \, \phi(\kappa^{-1}\alpha)|\alpha\rangle\langle\alpha|, \end{aligned} \tag{6.55}$$

which means

$$\mathcal{C}_1(\kappa) \, : \, \phi(\alpha) \rightarrow \kappa^{-2}\phi\left(\kappa^{-1}\alpha\right). \tag{6.56}$$

We have thus proved in the Kraus representation

Theorem 6.5 *The scaling* $\phi_\rho(\alpha) \rightarrow \phi_\rho^{'}(\alpha) = \kappa^{-2}\phi_\rho(\kappa^{-1}\alpha)$, $0 < \kappa < 1$, *is a completely positive map whose Kraus decomposition is given by* $\{B_\ell(\kappa)\}$ *of* (6.48).

As an immediate consequence we have

Corollary 6.1 *The beamsplitter channel cannot generate or destroy nonclassicality.*

Proof: By definition a state is classical if and only if its diagonal weight function $\phi(\alpha)$ is pointwise nonnegative everywhere in the complex plane [112]. Since a pointwise positive function goes to a pointwise positive function under the above scaling transformation, it follows that a classical state (and a classical state alone) is taken to a classical state under the action of the (quantum-limited) attenuator channel.

6.4.2 The issue of Entanglement breaking

It is known that the beamsplitter channel is not entanglement breaking [361]. It should thus be possible, as it is obligatory, to demonstrate that this channel cannot be represented using a set of rank one Kraus operators. We begin by noting that in the limiting case $\kappa = 0$, all our Kraus operators $B_\ell(0)$ are of rank one. Indeed, $(B_\ell(0))_{mn} = \delta_{m0}\delta_{n\ell}$. This singular limit corresponds to the quantum-limited $\mathcal{A}_1$ channel which is known to be entanglement breaking. We consider therefore the nontrivial case $\kappa \neq 0$. It is manifestly clear that rank $B_\ell(\kappa) = \infty$ for all ℓ (for $\kappa \neq 0$). If we represent this channel using another set of Kraus operators $\{B_r^{'}(\kappa)\}$, then these new operators should necessarily be in the support of the set of operators $\{B_\ell(\kappa)\}$. Thus a necessary condition that one is able to represent the channel $\{B_\ell(\kappa)\}$ using rank one Kraus operators is that there be (sufficient number of) rank one operators in the support of $\{B_\ell(\kappa)\}$. It turns out that there is not even one rank one operator in this support. Indeed, a much stronger result is true.

Theorem 6.6 *: There exists no finite rank operator in the support of the set* $\{B_\ell(\kappa)\}$, $\kappa \neq 0$.

Proof follows immediately from the structure of the $B_\ell(\kappa)$'s : $B_0(\kappa)$ is diagonal, and the mn^{th} entry of $B_\ell(\kappa)$ is nonzero iff $n = m + \ell$. Any matrix in the linear span of $\{B_\ell(\kappa)\}$ is of the form $M = \sum_\ell c_\ell B_\ell(\kappa)$, and is upper diagonal. Let N be the smallest ℓ for which the c-number coefficient $c_\ell \neq 0$. Let $\tilde{M}$ be the matrix obtained from the upper-diagonal M by deleting the first N columns. Clearly, rank $\tilde{M}$ = rank M. Further, the diagonal entries of the upper triangular $\tilde{M}$ are all nonzero, being the nonzero entries of $B_N(\kappa)$. Now, the rank of an upper triangular matrix is not less than that of its diagonal part. Thus, rank $\tilde{M}$ is not less than rank $B_N(\kappa) = \infty$, thus completing the proof.

6.5 Amplifier channel $\mathcal{C}_2(\kappa)$, $\kappa \geq 1$

The two-mode metaplectic unitary operator describing a single-mode quantum-limited amplifier channel corresponds to the following symplectic transformation on the mode operators [158] :

$$S = \begin{pmatrix} \cosh\nu & 0 & \sinh\nu & 0 \\ 0 & \cosh\nu & 0 & -\sinh\nu \\ \sinh\nu & 0 & \cosh\nu & 0 \\ 0 & -\sinh\nu & 0 & \cosh\nu \end{pmatrix}. \tag{6.57}$$

As in the earlier two cases of $\mathcal{D}(\kappa)$ and $\mathcal{C}_1(\kappa)$, the position and momentum variables do not mix under the action of $\mathcal{C}_2(\kappa)$. The position variables transform as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow \begin{pmatrix} q_1^{'} \\ q_2^{'} \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cosh\nu & -\sinh\nu \\ -\sinh\nu & \cosh\nu \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{6.58}$$

and the momentum variables transform according to M^{-1}. Thus the parameter κ in $\mathcal{C}_2(\kappa)$ is related to the two-mode squeeze parameter ν through $\kappa = \cosh\nu$. The function $F(z_1, z_2, \eta_1, \eta_2)$ in (6.14) is readily computed to be

$$F(z_1, z_2, \eta_1, \eta_2) = \kappa^{-1} \exp\left\{\kappa^{-1}(\eta_1 z_1 + \eta_2 z_2) + (\sqrt{1-\kappa^{-2}})(\eta_1\eta_2 - z_1 z_2)\right\}. \tag{6.59}$$

As in the earlier cases of $\mathcal{D}(\kappa)$ and $\mathcal{C}_1(\kappa)$, the differentiation on $F(z_1, z_2, \eta_1, \eta_2)$ can be performed to obtain the matrix elements of the unitary operator corresponding to the symplectic S in (6.57). We obtain, after some algebra patterned after the earlier two cases,

$$\begin{aligned} C_{n_1 n_2}^{m_1 m_2} &= \frac{\kappa^{-1}}{\sqrt{n_1! n_2! m_1! m_2!}}\, n_1! m_2! \sum_{r=0}^{n_2} \sum_{j=0}^{m_1} {n_2}C_r\, {m_1}C_j\, (-1)^r\, (\sqrt{1-\kappa^{-2}})^{r+m_1-j} \times \\ &\quad (\kappa^{-1})^{n_2+j-r} \delta_{n_1, r+j}\, \delta_{m_2, n_2+m_1-r-j}\,. \end{aligned} \tag{6.60}$$

The Kraus operators are obtained from $C_{n_1 n_2}^{m_1 m_2}$ by setting $n_2 = 0$, and $m_2 = \ell$. Setting $n_2 = 0 \Rightarrow r = 0$, and we have

$$A_\ell(\kappa) = \kappa^{-1} \sum_{m=0}^{\infty} \sqrt{m+\ell C_\ell} \left(\sqrt{1-\kappa^{-2}}\right)^\ell (\kappa^{-1})^m |m+\ell\rangle\langle m|, \quad \ell = 0, 1, 2, \cdots \tag{6.61}$$

as the Kraus operators of the quantum-limited amplifier channel $\mathcal{C}_2(\kappa)$, $\kappa > 1$ [367].

6.5.1 Duality between the attenuator family $\mathcal{C}_1(\cdot)$ and the amplifier family $\mathcal{C}_2(\cdot)$

The Kraus operators $A_\ell(\kappa)$, $\kappa > 1$ of the amplifier channel $\mathcal{C}_2(\kappa)$ have an interesting dual relationship to the Kraus operators $B_\ell(\kappa^{-1})$, $\kappa > 1$ of the attenuator channel $\mathcal{C}_1(\kappa^{-1})$. While $\sum_{\ell=0}^{\infty} A_\ell^\dagger(\kappa)A_\ell(\kappa) = \mathbb{1},\ \kappa > 1$ and $\sum_{\ell=0}^{\infty} B_\ell^\dagger(\kappa')B_\ell(\kappa') = \mathbb{1},\ \kappa' < 1$, consistent with the trace-preserving property of $\mathcal{C}_2(\kappa)$ and $\mathcal{C}_1(\kappa')$, we have

$$\begin{aligned}\sum_{\ell=0}^{\infty} A_\ell(\kappa)A_\ell^\dagger(\kappa) &= \kappa^{-2}\mathbb{1},\\ \sum_{\ell=0}^{\infty} B_\ell(\kappa')B_\ell^\dagger(\kappa') &= (\kappa')^{-2}\mathbb{1}.\end{aligned} \tag{6.62}$$

Thus the (trace-preserving) families $\mathcal{C}_1$ and $\mathcal{C}_2$ are not unital. But they are 'almost unital', for the failure to be unital is by just a scalar factor. This shows that the family $\{\kappa A_\ell(\kappa)^\dagger,\ \kappa > 1\}$ and the family $\{\kappa'^{-1}B_\ell(\kappa')^\dagger,\ \kappa' < 1\}$ too describe trace-preserving CP maps, and we may ask what these 'new' channels stand for.

The meaning of these channels may be easily seen by considering the adjoints $A_\ell(\kappa)^\dagger$, $\kappa > 1$ of the Kraus operators of the amplifier channel:

$$\begin{aligned}A_\ell(\kappa)^\dagger &= \kappa^{-1}\sum_{m=0}^{\infty}\sqrt{m+\ell C_\ell}\left(\sqrt{1-\kappa^{-2}}\right)^\ell \kappa^{-m}|m\rangle\langle m+\ell|\\ &= \kappa^{-1}B_\ell(\kappa^{-1})\end{aligned} \tag{6.63}$$

Thus $\{\kappa A_\ell(\kappa)^\dagger\}$, $\kappa > 1$ are the Kraus operators of the beamsplitter channel $\mathcal{C}_1(\kappa')$ with $\kappa' = \kappa^{-1} < 1$. Similarly it can be seen that $\{\kappa' B_\ell(\kappa')^\dagger\}$, $\kappa' < 1$ represents the amplifier channel $\mathcal{C}_2(\kappa)$ with $\kappa = (\kappa')^{-1} > 1$. Thus we have

Theorem 6.7 *The amplifier family $\mathcal{C}_2(\kappa)$ and the attenuator family $\mathcal{C}_1(\kappa^{-1})$, $\kappa > 1$ are mutually dual: their Kraus operators are connected through the adjoint operation.*

6.5.2 Action of the Kraus operators

Under the action of the amplifier channel $\mathcal{C}_2(\kappa)$ the Weyl-ordered characteristic function transforms as follows, and this may be identified with the very definition of the channel:

$$\chi_W(\xi) \to \chi_W'(\xi) = \chi_W(\kappa\,\xi)\ \exp\left[-(\kappa^2-1)|\xi|^2/2\right]. \tag{6.64}$$

Given a Weyl-ordered characteristic function $\chi_W(\xi)$, the corresponding antinormal ordered characteristic function corresponding to the Q-distribution is [169]

$$\chi_A(\xi) = \chi_W(\xi)\ \exp\left[-|\xi|^2/2\right]. \tag{6.65}$$

Therefore the channel action Eq. (6.64), written in terms of $\chi_A(\xi)$, reads

$$\chi_A(\xi) \to \chi_A^{'}(\xi) = \chi_A(\kappa\,\xi). \tag{6.66}$$

That is, $\chi_A(\xi)$ simply scales under the action of the amplifier channel, a fact that should be profitably compared with the scaling behaviour (6.50) for the attenuator channel. Since $\chi_A(\xi)$ and the Q- function form a Fourier transform pair, the action of the amplifier channel is fully described as a scaling transformation of the Q-function : $Q(\alpha) \to Q^{'}(\alpha) = \kappa^{-2}Q(\kappa^{-1}\alpha)$, $\kappa > 1$ [368].

It is instructive to see in some detail how our Kraus operators $A_\ell(\kappa)$ bring out this behaviour. Given a state

$$\rho = \sum_{n,m=0}^{\infty} |n\rangle\langle n|\rho|m\rangle\langle m| = \sum_{n,m=0}^{\infty} \rho_{nm}|n\rangle\langle m|, \tag{6.67}$$

its corresponding Q function is [169]

$$Q_\rho(\alpha) = \langle\alpha|\rho|\alpha\rangle = \exp[-|\alpha|^2] \sum_{n,m=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} \frac{(\alpha)^m}{\sqrt{m!}} \rho_{nm}. \tag{6.68}$$

To see the action of the linear map $\mathcal{C}_2(\kappa)$ on an arbitrary ρ, it is sufficient to exhibit its action on the operators $|n\rangle\langle m|$, for all $n, m \geq 0$. We have

$$\begin{aligned} |n\rangle\langle m| &\to \sum_{\ell=0}^{\infty} A_\ell(\kappa)|n\rangle\langle m|A_\ell^\dagger(\kappa) \\ &= \kappa^{-2}\frac{(\kappa)^{-(n+m)}}{\sqrt{n!m!}} \sum_{\ell=0}^{\infty} \frac{(1-\kappa^{-2})^\ell}{\ell!} \sqrt{(n+\ell)!}\sqrt{(m+\ell)!}|n+\ell\rangle\langle m+\ell|. \end{aligned} \tag{6.69}$$

Thus, under the action of the channel $\mathcal{C}_2(\kappa)$, ρ goes to

$$\rho^{'} = \kappa^{-2} \sum_{n,m=0}^{\infty} \rho_{nm} \frac{\kappa^{-(n+m)}}{\sqrt{n!m!}} \sum_{\ell=0}^{\infty} \frac{(1-\kappa^{-2})^\ell}{\ell!} \sqrt{(n+\ell)!}\sqrt{(m+\ell)!}\,|n+\ell\rangle\langle m+\ell|. \tag{6.70}$$

The Q function of the resultant or output state $\rho^{'}$ is

$$\begin{aligned}\langle\alpha|\rho^{'}|\alpha\rangle &= \kappa^{-2}\exp[-|\alpha|^2]\sum_{n,m=0}^{\infty}\rho_{nm}\frac{\kappa^{-(n+m)}}{\sqrt{n!m!}}(\alpha^*)^n(\alpha)^m\left(\sum_{\ell=0}^{\infty}\frac{(1-\kappa^{-2})\ell}{\ell!}|\alpha|^{2\ell}\right)\\ &= \kappa^{-2}\exp[-|\kappa^{-1}\alpha|^2]\sum_{n,m=0}^{\infty}\frac{(\kappa^{-1}\alpha^*)^n}{\sqrt{n!}}\frac{(\kappa^{-1}\alpha)^m}{\sqrt{m!}}\rho_{nm}\\ &= \kappa^{-2}Q(\kappa^{-1}\alpha). \qquad (6.71)\end{aligned}$$

We thus conclude

Theorem 6.8 *The scaling $Q_\rho(\alpha)\rightarrow Q_{\rho'}(\alpha)=\kappa^{-2}Q_\rho(\kappa^{-1}\alpha)$, $0<\kappa^{-1}<1$, is a completely positive map whose Kraus decomposition is given by $\{A_\ell(\kappa)\}$.*

This result may be compared with Theorem 6 for the $\mathcal{C}_1(\cdot)$ family of channels.

The amplifier channel has the following property in respect of nonclassicality of the output states :

Corollary 6.2 *The amplifier channel cannot generate nonclassicality.*

Proof: By Eq. (6.64), the normal ordered characteristic function transforms as follows

$$\mathcal{C}_2(\kappa)\,:\chi_N(\xi)\rightarrow\chi_N^{'}(\xi)=\chi_W(\kappa\xi)\exp\left[-(\kappa^2-2)|\xi|^2/2\right]. \qquad (6.72)$$

This may be rewritten in the suggestive form

$$\chi_N(\xi)\rightarrow\chi_N^{'}(\xi)=\chi_N(\kappa\xi)\exp[-(\kappa^2-1)|\xi|^2]. \qquad (6.73)$$

Fourier transforming, we see that the diagonal weight $\phi\,(\alpha)$ of the output state is the convolution of the (scaled) input diagonal weight with a Gaussian (corresponding to the last factor), and hence it is pointwise nonnegative whenever the input diagonal weight $\phi(\alpha)$ is pointwise nonnegative.

Remark : We are not claiming that the amplifier channel cannot destroy nonclassicality [compare the structure of Corollary 2 with that of Corollary 1 following Theorem 6]. Indeed, it is easy to show that nonclassicality of every Gaussian state will be destroyed by any $\mathcal{C}_2(\kappa)$ with $\kappa\geq\sqrt{2}$ [184, 367–369]. It is also easy to show that there are states whose nonclassicality will survive $\mathcal{C}_2(\kappa)$ even for arbitrarily large κ [184, 367, 368]. To see this, note first of all, that any state ρ whose Q-function $Q(\alpha)=\langle\alpha|\rho|\alpha\rangle$ vanishes for some α is necessarily nonclassical. The assertion simply follows from the fact that under the scaling $Q(\alpha)\rightarrow\kappa^{-2}Q(\kappa^{-1}\alpha)$ a zero α_0 of $Q(\alpha)$ goes to a zero at $\kappa\alpha_0$.

Remark on entanglement breaking: It is well known that the quantum-limited amplifier channel is not entanglement breaking [361]. It may be pointed out in passing that this fact follows also from the structure of our Kraus operators $\{A_\ell(\kappa)\}$. Since these operators coincide with the transpose of the beamsplitter channel Kraus operators $\{B_\ell(\kappa^{-1})\}$, apart from a ℓ-independent multiplicative factor, there exists *no finite rank operator* in the support of the set of operators $\{A_\ell(\kappa)\}$. In particular, there are no rank one operators in the support of $\{A_\ell(\kappa)\}$. Hence, $\mathcal{C}_2(\kappa)$ is not an entanglement breaking channel.

6.6 The Singular case $\mathcal{A}_2$

We now consider briefly $\mathcal{A}_2$, the last of the quantum limited Bosonic Gaussian channels. The two-mode metaplectic unitary operator representing $\mathcal{A}_2$ produces a symplectic transformation on the quadrature variables which does not mix the position variables with the momentum variables [158]:

$$\begin{aligned}
\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &\rightarrow \begin{pmatrix} q_1^{'} \\ q_2^{'} \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \\
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} &\rightarrow \begin{pmatrix} p_1^{'} \\ p_2^{'} \end{pmatrix} = (M^{-1})^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \\
M &= \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.
\end{aligned} \tag{6.74}$$

Therefore, our general scheme applies to this case as well. Unlike in the earlier cases of $\mathcal{D}(\kappa)$, $\mathcal{C}_1(\kappa)$, and $\mathcal{C}_2(\kappa)$, in the present case it turns out to be more convenient to evaluate the matrix elements of $U^{(ab)}$ in a mixed basis:

$$C_{n_1 n_2}^{m_1 q} = \langle m_1 | \langle q | U^{(ab)} | n_1 \rangle | n_2 \rangle. \tag{6.75}$$

Here $|q\rangle$ labels the position basis of the ancilla mode. With this mixed choice, the Kraus operators are labelled by a continuous index 'q', and are given by

$$V_q = \langle q | U^{(ab)} | 0 \rangle = \sum_{m_1, n_1} C_{n_1 0}^{m_1 q} |m_1\rangle\langle n_1|, \tag{6.76}$$

where

$$C_{n_1 0}^{m_1 q} = \int dq_1 \langle m_1|q_1\rangle\langle q_1|\langle q|U^{(ab)}|n_1\rangle|0\rangle$$
$$= \int dq_1 \langle m_1|q_1\rangle\langle q, q_1 - q|n_1, 0\rangle. \tag{6.77}$$

Here we have used, as in the earlier cases, the action of the unitary operator in the position eigenstates of the two-mode system. Employing the position space wavefunctions of the Fock states, we have

$$C_{n_1 0}^{m_1 q} = \frac{\pi^{-3/4}}{\sqrt{2^{n_1+m_1} n_1! m_1!}} H_{n_1}(q) e^{-\frac{q^2}{2}} \int dq_1 H_{m_1}(q_1) e^{-\frac{(q_1-q)^2}{2} - \frac{q_1^2}{2}}. \tag{6.78}$$

The above integral is easily evaluated [370], and we have

$$C_{n_1 0}^{m_1 q} = \frac{\pi^{-1/4}}{\sqrt{2^{n_1+m_1} n_1! m_1!}} q^{m_1} H_{n_1}(q) \exp[-3q^2/4]$$
$$= \langle m_1|q/\sqrt{2})\langle q|n_1\rangle, \tag{6.79}$$

where $|q/\sqrt{2})$ is the coherent state $|\alpha\rangle$ for $\alpha = q/\sqrt{2}$, and the purpose of the round bracket being to distinguish the same from the position eigenket $|q/\sqrt{2}\rangle$. With this notation the Kraus operators are

$$V_q = |q/\sqrt{2})\,\langle q|. \tag{6.80}$$

That the trace-preserving condition on the Kraus operators is satisfied emerges from the fact that the position kets are complete : $\int dq\, V_q^\dagger V_q = \int dq|q\rangle\langle q| = \mathbb{1}$.

To connect these Kraus operators V_q to the action of the channel in the phase space picture, we examine the behaviour of an arbitrary pure state $|\psi\rangle$ under passage through the channel. We have

$$\mathcal{A}_2 : \rho = |\psi\rangle\langle\psi| \to \rho' = \int dq\ |q/\sqrt{2})\,\langle q|\psi\rangle\langle\psi|q\rangle\,(q/\sqrt{2}|$$
$$= \int dq\ |\psi(q)|^2\,|q/\sqrt{2})\,(q/\sqrt{2}|$$
$$= \int dq\,dp\ |\psi(q)|^2\delta(p)\,|[q+ip]/\sqrt{2})\ ([q+ip]/\sqrt{2}|. \tag{6.81}$$

The last expression is already in the 'diagonal' form in the coherent states basis, with $|\psi(q)|^2\delta(p)$, $\alpha = (q+ip)/\sqrt{2}$ forming the diagonal weight function $\phi(\alpha)$. It follows by

convexity that for an arbitrary input state ρ the output of the channel is given by

$$\rho^{'} = \pi^{-1} \int d^2\alpha\, \phi(\alpha)\, |\alpha\rangle\langle\alpha|, \quad \phi(\alpha) = \langle q|\rho|q\rangle\, \delta(p). \tag{6.82}$$

It is seen that this transformation is the same as $\chi_W(\xi) \rightarrow \chi_W\left(\frac{(\mathbb{1}+\sigma_3)}{2}\,\xi\right) \exp[-|\xi|^2/2]$, the expected behaviour of the characteristic function under passage through $\mathcal{A}_2$ [371].

The above results can be alternatively understood through the action of the channel in the Fock basis. Under passage through the channel,

$$\begin{aligned} |n\rangle\langle m| &\rightarrow \int dq V_q |n\rangle\langle m| V_q^\dagger \\ &= \int dq\, |q/\sqrt{2})\, \langle q|n\rangle\langle m|q\rangle\, (q/\sqrt{2}| \\ &= \int dq\, \frac{\pi^{-1/2}}{\sqrt{2^{n+m} n! m!}}\, H_n(q) H_m(q)\, e^{-q^2} |q/\sqrt{2})\, (q/\sqrt{2}|, \end{aligned} \tag{6.83}$$

for all n, m. The outcome for an arbitrary input state ρ follows by linearity, and we have

Theorem 6.9 *The channel $\mathcal{A}_2$ is both nonclassicality breaking and entanglement breaking.*

Proof: We note from Eq. (6.80) that the Kraus operators are already in rank one form, thereby showing that the channel is entanglement breaking. And from Eq. (6.82) we see that the output of the channel, for every input state ρ, supports a diagonal representation with nonnegative weight $\langle q|\rho|q\rangle\, \delta(p) \geq 0$, for all $\alpha = (q+ip)/\sqrt{2}$, showing that the output is classical for all input states.

6.7 Single Quadrature classical noise channel $\mathcal{B}_1(a)$, $a \geq 0$

The channel $\mathcal{B}_1(a)$, whose action is to simply inject Gaussian noise of magnitude a into one quadrature of the oscillator, and is not quantum limited. It can be realized in the form

$$\mathcal{B}_1(a) \,:\, \rho \rightarrow \rho^{'} = \frac{1}{\sqrt{\pi a}} \int dq\, \exp[-q^2/a]\, D(q/\sqrt{2})\, \rho\, D(q/\sqrt{2})^\dagger, \tag{6.84}$$

where $D(\alpha)$'s are the unitary displacement operators. $\mathcal{B}_1(a)$ is thus a case of the so-called random unitary channels [364], a convex sum of unitary channels. The continuum

$$Z_q \equiv (\pi a)^{-1/4} \exp[-^2/2a]\, D(q/\sqrt{2}) \tag{6.85}$$

are the Kraus operators of this realization. The quantum-limited end of $\mathcal{B}_1(a)$ is obviously the identity channel, corresponding to $a \to 0$ [$\lim_{a\to 0} \sqrt{\pi a}^{-1} \exp[-q^2/a] = \delta(q)$, and $Z_{q=0}$ = identity]. One may assume $a = 1$ without loss of generality. The reason we present a brief treatment of this channel here is just to demonstrate that this case too subjects itself to our general scheme.

The two-mode metaplectic unitary operator representing $\mathcal{B}_1$ produces a symplectic transformation on the quadrature variables which, as in the earlier cases of $\mathcal{D}(\kappa)$, $\mathcal{C}_1(\kappa)$, $\mathcal{C}_2(\kappa)$, and $\mathcal{A}_2$, does not mix the position variables with the momentum variables [158] :

$$\begin{aligned} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &\to \begin{pmatrix} q_1^{'} \\ q_2^{'} \end{pmatrix} = M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \\ M &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{6.86}$$

And p_1, p_2 transform according to $(M^{-1})^T$.

As in the immediate previous case $\mathcal{A}_2$, the matrix elements of $U^{(ab)}$ are

$$C_{n_1 n_2}^{m_1 q} = \langle m_1 | \langle q | U^{(ab)} | n_1 \rangle | n_2 \rangle, \tag{6.87}$$

where $|q\rangle$'s are the position eigenvectors. In view of this the Kraus operators are labelled by a continuous index 'q' and are given by

$$\langle q | U^{(ab)} | 0 \rangle = \sum_{m_1, n_1} C_{n_1 0}^{m_1 q} |m_1\rangle\langle n_1|, \tag{6.88}$$

where

$$\begin{aligned} C_{n_1 0}^{m_1 q} &= \int dq_1 \langle m_1 | q_1 \rangle \langle q_1 | \langle q | U^{(ab)} | n_1 \rangle | 0 \rangle \\ &= \int dq_1 \langle m_1 | q_1 \rangle \langle q_1 - q, q | n_1, 0 \rangle. \end{aligned} \tag{6.89}$$

Here we made the two-mode metaplectic unitary operator act on the position basis. To evaluate the Kraus operator, it is sufficient to evaluate the matrix elements

$$C_{m_1 0}^{n_1 q} = \frac{\pi^{-3/4}}{\sqrt{2^{n_1+m_1} n_1! m_1!}} e^{-\frac{q^2}{2}} \int dq_1 H_{n_1}(q_1 - q) H_{m_1}(q_1) e^{-\frac{q_1^2}{2}} e^{-\frac{(q_1-q)^2}{2}}. \tag{6.90}$$

The above integral can be readily performed [372], and we obtain

$$C_{m_1 0}^{n_1 q} = \pi^{-1/4} e^{-\frac{q^2}{2}} \left[e^{-\frac{q^2}{4}} \sqrt{\frac{m_1!}{n_1!}} \left(\frac{-q}{\sqrt{2}} \right)^{n_1 - m_1} L_{m_1}^{n_1 - m_1}(q^2/2) \right]$$
$$\equiv \pi^{-1/4} e^{-\frac{q^2}{2}} \langle m_1 | D(q/\sqrt{2}) | n_1 \rangle = Z_q. \tag{6.91}$$

We have thus recovered (6.85), but staying entirely within our general scheme.

6.8 Summary

We have obtained operator-sum representations for all single-mode Bosonic Gaussian channels presented in their respective canonical forms. Evidently, the operator-sum representation of a channel not in the canonical form follows by adjoining of appropriate unitary Gaussian evolutions before and after the channel. The Kraus operators were obtained from the matrix elements of the two-mode metaplectic unitary operator which effects the channel action on a single mode. The two-mode symplectic transformation in each case did not mix the position and momentum variables and this fact proved valuable for our study. The Kraus operators for the quantum-limited channels except the singular case were found to have a simple and sparse structure in the Fock basis.

It was shown that the phase conjugation channels $\mathcal{D}(\kappa)$ and $\mathcal{D}(\kappa^{-1})$ are dual to one another, and the attenuator and the amplifier families $\mathcal{C}_1(\kappa)$ and $\mathcal{C}_2(\kappa^{-1})$, $\kappa < 1$ are mutually dual. The channels $\mathcal{D}(\kappa)$, $\mathcal{C}_1(\kappa)$, and $\mathcal{C}_2(\kappa)$ were found to be almost unital; in the sense that the unit operator was taken to a scalar times the unit operator.

In the case of the phase conjugation channel, the action in phase space was brought out explicitly through the action of the Kraus operators on the Fock basis. The attenuator channel resulted in the scaling of the diagonal weight function $\phi(\alpha)$ and the amplifier channel resulted in the scaling of the Husimi Q-function as expected. Further, the output of the channel with respect to classicality/nonclassicality was studied. It was found that the phase conjugation channel $\mathcal{D}(\kappa)$ and the singular channel $\mathcal{A}_2$ are classicality breaking while the attenuator channel $\mathcal{C}_1(\kappa)$ and the amplifier channel $\mathcal{C}_2(\kappa)$ do not generate nonclassicality.

The Kraus operators of the phase conjugation channel was brought to a rank one form, thereby explicitly bringing out the entanglement breaking nature of this channel. It was further shown that there is no finite rank operator in the support of the Kraus operators of either the amplifier or the attenuator channel, and this explicitly demonstrates that the quantum-limited attenuator and the amplifier families of channels are not entanglement breaking. The Kraus operators of the singular channel $\mathcal{A}_2$ was also obtained in

the rank one form thereby manifestly showing that this channel is entanglement breaking.

Note: A more detailed analysis on the operator-sum representation of single-mode Bosonic Gaussian channels can be found at [373]. This includes an analysis on fixed points, an analysis on interrupted evolution, a proof of the extremality of all quantum limited single-mode Gaussian channels, and the operator-sum representation of composite channels.

7

A measure of non-Gaussianity for quantum states

7.1 Introduction

Quantum information theory of continuous variable systems has been actively pursued in recent years, especially in the context of Gaussian states [245, 256, 258]. Such states are the ones which occur naturally in most experimental situations, particularly in quantum optics. While these states live in an infinite-dimensional Hilbert space, they are remarkably easy to handle since they are fully described by their covariance matrix (and first moments). Further, their evolution under quadratic hamiltonians is easily cast in the language of symplectic groups and (classical) phase space [57, 226, 280]. The fundamental protocol of quantum teleportation has been achieved using these states [36, 83]. However, there are situations wherein one deals with (nonclassical) non-Gaussian resources to generate entanglement [130, 170, 171, 186, 194, 318, 319, 374]. They arise naturally in nonlinear evolutions like passage through a Kerr medium [375, 376].

It has been shown recently that teleportation fidelities can be improved with the use of non-Gaussian resources [377]. It is thus important that one is able to quantify the non-Gaussianity of such resources. Effort in this direction has been initiated in some recent publications [378–380].

From the perspective of classical probability theory, Gaussian distributions are those probability distributions which are completely specified by their first and second moments; all their higher-order moments are determined by these lower-order moments. Non-Gaussian probabilities do not enjoy this special property. An easier, and possibly more effective, way to distinguish the two is through cumulants: every non-vanishing cumulant of order greater than two serves as an indicator of non-Gaussianity of the probability distribution under consideration [381, 382].

The purpose of any good measure of non-Gaussianity in the context of classical probability theory is thus to capture the essence of the non-vanishing higher-order cumulants.

A non-Gaussianity measure should thus manifestly depend on the higher-order cumulants. Yet another desirable feature one would like to have is *invariance of the measure under scaling. Ultimately, non-Gaussianity measure is a quantitative statement of the departure of the shape of a probability distribution from Gaussian.* But *uniform scaling of all the variables of a probability distribution does not alter the 'shape' of the distribution, and hence it should not affect its non-Gaussianity.*

The notion of non-Gaussianity can be extended to a quantum mechanical state through its definition on the associated Q function, a member of the one-parameter family of s-ordered quasi-probabilities [168]. That this is an appropriate route is endorsed by the fact that the Marcinkiewicz theorem [see below] holds for the s-ordered quasi-probabilities as well. It turns out that the cumulants of order greater than 2 for the various s-ordered quasi-probabilities corresponding to a fixed state $\hat{\rho}$ are independent of s, indicating that the higher order cumulants are intrinsic to the state. Moreover, all higher-order cumulants of order greater than 2 vanish identically for Gaussian states. Thus any non-vanishing higher-order cumulant of the quasi-probability indicates non-Gaussianity of the state, and this conclusion is independent of the ordering parameter s.

The above considerations will suggest that any good measure of non-Gaussianity relevant in the context of classical probability theory can, with suitable modification, lead to a good measure of non-Gaussianity of quantum mechanical states, provided a state is identified through its Q function (For a brief review on such measures in classical probability theory, see [382]). The purpose of such a quantum measure would be to capture the essence of the non-vanishing higher-order cumulants of the Q function associated with the state. And invariance of the measure under an overall scaling of the Q function is a desirable feature worth insisting on. The desirability for scale invariance is endorsed by the fact that scaling of the Q function is physical as shown in the preceding Chapter.

In this chapter we motivate and present such a measure of non-Gaussianity of quantum states. Our measure is based on the Wehrl entropy [383], the quantum analogue of differential entropy [384] well-known from the context of classical information theory of continuous variables [Differential entropy itself is a generalisation of Shannon entropy from discrete to continuous variables].

The photon-added thermal states [182] play a key role in our considerations. These nonclassical states have been generated experimentally [320–323]. Their special importance to the present work arises from the fact that the Q functions of these states are scaled versions of those of the Fock states, and therefore one will expect any good measure of non-Gaussianity to return the same values for both classes of states.

The plan of the Chapter is as follows. We begin with a brief introduction to moments

and cumulants, and recall two well-known theorems in the context of these notions. The one-parameter family of s-ordered quasi-probabilities corresponding to quantum density operators is then briefly discussed, with particular emphasis on the Q function, we then review the relationship between differential entropy and the Kullback-Leibler distance of classical probability theory. We then review briefly the Wehrl entropy [383] and some of its properties. In the preceding Chapter, we already showed that scaling of the Q function is physical. With these preparations, we introduce our non-Gaussianity measure and explore some of its important properties, including its invariance under uniform scaling of the underlying phase space. We then evaluate this measure for three families of quantum states, and we compare our measure with two other measures of non-Gaussianity available in the literature. Finally we end with some concluding remarks.

7.2 Moments and cumulants

For a multivariate probability distribution $\mathcal{P}(x)$, where $x = (x_1, x_2, \cdots, x_n) \in \mathcal{R}^n$, the characteristic function $\chi(\xi)$, $\xi \in \mathcal{R}^n$, is given by the Fourier transform of $\mathcal{P}(x)$ [381]:

$$\begin{aligned}\chi(\xi) &= \int d^n x\, \mathcal{P}(x) \exp[\, i\xi \cdot x\,] \\ &= \sum_{m_1 m_2 \cdots m_n} \left(\prod_{k=1}^{n} \frac{(i\xi_k)^{m_k}}{m_k!} \right) \langle x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \rangle\,,\end{aligned}$$

$$\langle x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \rangle = \int d^n x\, x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \mathcal{P}(x)\,. \tag{7.1}$$

It follows from the invertibility of Fourier transformation that the characteristic function retains all the information contained in the probability distribution. The characteristic function is often called the moment generating function, since one obtains from it all the moments of the underlying probability distribution through this compact expression:

$$\langle x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \rangle = \left(\prod_{k=1}^{n} \frac{d^{m_k}}{d(i\xi_k)^{m_k}} \right) \chi(\xi)\,|_{\xi=0}\,. \tag{7.2}$$

Another equivalent description of a probability distribution is through the cumulant

generating function. This is defined through the logarithm of the characteristic function

$$\begin{aligned} \Gamma(\xi) &= \log \chi(\xi) \\ &= \sum_{m_1 m_2 \cdots m_n} \left(\prod_{k=1}^{n} \frac{(i\xi_k)^{m_k}}{m_k!} \right) \gamma_{m_1, m_2, \cdots, m_n} , \end{aligned} \tag{7.3}$$

or, equivalently, through

$$\chi(\xi) = \exp(\Gamma(\xi)) . \tag{7.4}$$

From Eq. (7.3), it is easy to see that the cumulants $\gamma_{m_1,m_2,\cdots,m_n}$ can be expressed as

$$\gamma_{m_1,m_2,\cdots,m_k} = \left(\prod_{k=1}^{n} \frac{d^{m_k}}{d(i\xi_k)^{m_k}} \right) \Gamma(\xi) \, |_{\xi=0} . \tag{7.5}$$

Thus, the cumulants are related to $\Gamma(\cdot)$ in precisely the same way as the moments are related to $\chi(\cdot)$. The set of all moments $\langle x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \rangle$ gives a complete characterisation of a probability distribution $\mathcal{P}(x)$, and the same is true of the set of all cumulants $\gamma_{m_1,m_2,\cdots,m_n}$ as well. Indeed, one can describe one set in terms of the other [324, 381, 385].

With these notations and definitions on hand, we now recall two important results from classical probability theory.

Theorem 7.1 *The cumulant generating function of a Gaussian probability distribution in n variables is a multinomial of degree equal to 2 [381].*

Theorem 7.2 *(Marcinkiewicz Theorem). If the cumulant generating function of a (normalised) function in n variables is a multinomial of finite degree greater than 2, then the function will not be point wise non-negative, and hence will fail to be a probability distribution [386, 387].*

Theorem 1 is a statement of the fact that a Gaussian probability is fully determined by its moments of order ≤ 2; all the higher-order cumulants are identically zero for a Gaussian probability. Theorem 2 is a much stronger statement. It implies that any true probability distribution other than the Gaussian distribution has a cumulant generating function which cannot truncate at any (finite) order. That is, a non-Gaussian probability distribution has non-vanishing cumulants of arbitrarily high order. We note in passing that *non-vanishing cumulants of order greater than* 2 *serve as indicators of the non-Gaussianity of the underlying probability.*

7.3 Quasi-probabilities and the Q function

A state of a quantum mechanical system specified by density operator $\hat{\rho}$ can be faithfully described by any member of the one-parameter family of s-ordered quasi-probability distributions $-1 \leq s < 1$ [168]. In other words, an s-ordered quasi-probability captures all the information present in the density operator $\hat{\rho}$. However, it is not a genuine probability distribution in general; in particular, it is not point wise non-negative. The prefix *quasi* underscores precisely this aspect. Nevertheless, the s-ordered family of quasi-probability distributions gives us a framework wherein one could give a phase space description of quantum mechanical systems in the language of classical probability theory.

To recollect Section 1.7, for a quantum state describing the radiation field of n modes (n oscillators) the characteristic function of the s-ordered quasi-probability, for any $-1 \leq s \leq 1$, is defined through [168]

$$\chi_\rho(\xi, s) = \exp[\,\frac{s}{2}|\xi|^2\,]\,\mathrm{Tr}(\hat{\rho}D(\xi))\,, \tag{7.6}$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathcal{C}^n$, and $D(\xi)$ is the n-mode (phase space) displacement operator :

$$D(\xi) \;\; = \;\; \exp[\sum_j (\xi_j \hat{a}_j^\dagger - \xi^* \hat{a}_j)\,]\,. \tag{7.7}$$

The s-ordered quasi-probability itself is just the Fourier transform of this characteristic function $\chi_\rho(\xi, s)$:

$$W_\rho(\alpha, s) = \frac{1}{\pi^n} \int \exp[\sum_j (\alpha_j^* \xi_j - \alpha_j \xi_j^*)\,]\chi_\rho(\xi, s) \prod_j d^2\xi_j. \tag{7.8}$$

Here $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the annihilation and creation operators of the jth mode, α_j represents the (c-number) phase space variables q_j, p_j corresponding to the jth mode through $\alpha_j = (q_j + ip_j)/\sqrt{2}$, and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathcal{C}^n$. The particular cases $s = -1, 0, 1$ correspond, respectively, to the better known Q function, the Wigner function, and the P function.

The Q function corresponding to a density operator $\hat{\rho}$ has a particularly simple expression in terms of coherent state projections :

$$Q_\rho(\alpha) = \langle\alpha|\rho|\alpha\rangle, \quad \alpha \in \mathcal{C}^n\,. \tag{7.9}$$

It may be noted that the Q function is manifestly nonnegative for all $\alpha \in \mathcal{C}^n$.

Reality of $W_\rho(\alpha, s)$ is equivalent to hermiticity of the density operator $\hat{\rho}$, and the fact that $\hat{\rho}$ is of unit trace faithfully transcribes to

$$\frac{1}{\pi^n} \int W_\rho(\alpha, s) d^2\alpha = 1 \,. \tag{7.10}$$

While these two properties hold for every s-ordered quasi-probability, point wise non negativity for all states is a distinction which applies to the Q function alone. In other words, *the Q function is a genuine probability distribution; every other $W_\rho(\alpha; s)$ is only a quasi-probability.* Gaussian pure states are the only pure states for which the Wigner function is a classical probability [199]; in the case of P function, the coherent states are the only pure state with this property.

However, not every probability distribution is a Q function. This is evident, for instance, from the obvious fact that $Q(\alpha) \leq 1, \ \forall\, \alpha \in \mathcal{C}^n$.

The next result captures, in a concise form, the manner in which members of the one-parameter family of s-ordered quasi-probabilities $W_\rho(\alpha, s)$ differ from one another for a given state $\hat{\rho}$.

Theorem 7.3 *Only the second order cumulants of the quasi-probability of a given state depend on the order parameter s; all the other cumulants are independent of the quasi-probability under consideration.*

This result is already familiar in the case of a single-mode radiation field [375]. But the proof is, as outlined below, immediate in the multi-mode case as well. The characteristic functions of a state $\hat{\rho}$ for two different values of the order parameter s_1 and s_2 are obviously related in the following manner [168] :

$$\chi_\rho(\xi, s_1) = \exp\left((s_1 - s_2)|\xi|^2\right) \chi_\rho(\xi, s_2) \,. \tag{7.11}$$

On taking logarithm of both sides to obtain the corresponding cumulant generating functions we have

$$\log\chi_\rho(\xi, s_1) = (s_1 - s_2)|\xi|^2 + \log\chi_\rho(\xi, s_2) \,.$$

That is,

$$\Gamma_\rho(\xi, s_1) = (s_1 - s_2)|\xi|^2 + \Gamma_\rho(\xi, s_2) \,. \tag{7.12}$$

Thus the cumulant generating function for different s-ordered quasi-probabilities differ only in second order, completing proof of the theorem.

In these equations $|\xi|^2$ stands, as usual, for $\sum_{j=1}^{n} |\xi_j|^2$. As an immediate consequence of this theorem we have

Theorem 7.4 *For no quasi-probability can the cumulant generating function be a multinomial of finite order* > 2.

Proof: Since the Q function, for every state ρ, is a genuine probability distribution, it follows from the Marcinkiewicz theorem that the cumulant generating function of Q cannot be a multinomial of finite order > 2. Since the different s-ordered quasi-probabilities differ only in second-order cumulants, this conclusion holds for all s-ordered quasi-probabilities, thus proving the theorem.

We conclude this Section with the following remarks. The above considerations show that quasi-probabilities fail to be true probabilities only in this limited sense: they differ from genuine probabilities only in cumulants of order two. The distributions, however, can be quite different from classical probabilities, particularly for $s > 0$, and they can become as subtle as Fourier transform of $\exp[\sigma y^2]$, $\sigma > 0$, a Gaussian with the wrong signature for the variance.

Since the higher-order cumulants, which should play an essential role in any reasonable definition of non-Gaussianity measure, do not depend on the value of the parameter s, they may be viewed as attributes intrinsic to the state under consideration; we may therefore use any convenient quasi-probability to capture their essence.

7.4 Differential entropy and the Kullback-Leibler distance

The role of Shannon entropy of probability distributions over discrete random variables is taken over by differential entropy in the case of continuous variables. Given a multivariate probability distribution $\mathcal{P}(x)$ in n variables $(x_1, x_2, \cdots, x_n) \in \mathcal{R}^n$, the associated differential entropy $H(\mathcal{P}(x))$ is defined by [384]

$$H(\mathcal{P}(x)) = -\int d^n x \mathcal{P}(x) \log \mathcal{P}(x)\,. \tag{7.13}$$

But unlike the Shannon entropy, the differential entropy can be negative. This is manifest, for instance, for uniform distribution over a region of less than unit volume in $\mathcal{R}^n$.

Among all the probability distributions with a fixed set of first and second moments, the Gaussian probability distribution has the maximum differential entropy [384]. This fact may be used to modify differential entropy to result in a non-negative quantity

$$J(\mathcal{P}(x)) = H(\mathcal{P}_G(x)) - H(\mathcal{P}(x))\,. \tag{7.14}$$

Here $\mathcal{P}_G(x)$ is the Gaussian probability distribution *with the same first and second moments* as the given probability distribution $\mathcal{P}(x)$.

It may be recalled that Kullback-Leibler distance between two probabilities $P_1(x)$ and $P_2(x)$ is defined as the difference of their differential entropies [384] :

$$\begin{aligned} S(\mathcal{P}_1(x)||\mathcal{P}_2(x)) &= H(\mathcal{P}_2(x)) - H(\mathcal{P}_1(x)) \\ &= -\int \mathcal{P}_1(x)\log(\mathcal{P}_1(x))d^n x \\ &\quad + \int \mathcal{P}_2(x)\log(\mathcal{P}_2(x))d^n x\,. \end{aligned} \tag{7.15}$$

Thus $J(\mathcal{P}(x))$ can be regarded as the Kullback-Leibler distance between the given probability $P(x)$ and the associated Gaussian distribution $P_G(x)$:

$$J(\mathcal{P}(x)) = S(\mathcal{P}_G(x)||\mathcal{P}(x))\,. \tag{7.16}$$

$J(\mathcal{P}(x))$ is sometimes known by the name negentropy.

7.5 Wehrl entropy

Wehrl entropy [383, 388] may be viewed as the extension of differential entropy to the quantum mechanical context, but the Wehrl entropy has interesting properties which distinguish it from differential entropy. The distinction arises from the fact that while every Q function certainly qualifies to be a classical probability distribution, every classical probability is not a Q function. The uncertainty principle has a fundamental role to play in this aspect [383]. The potential use of Wehrl entropy as a measure of the 'coherent' component of a state has been discussed in Ref [389]. And its possible role in defining an entanglement measure has also been explored [390, 391].

For a state $\hat{\rho}$ describing n modes of radiation field, the Wehrl entropy is defined as

$$H_W(\hat{\rho}) = -\int \prod d^2\alpha_j Q_\rho(\alpha)\log Q_\rho(\alpha)\,, \tag{7.17}$$

where $Q_\rho(\alpha)$ is the Q function corresponding to $\hat{\rho}$. This definition may be compared with that of differential entropy; the role of $\mathcal{P}(x)$ in differential entropy is played by $Q_\rho(\alpha)$ in Wehrl entropy.

However, in contradistinction to differential entropy, the Wehrl entropy is always positive. This is an immediate consequence of the fact that the Q function is bounded from above by unity. It turns out that the Wehrl entropy is always greater than or equal

to unity [392]; indeed, it attains its least value of unity for the coherent states and only for these states. This property can be thought of as a manifestation of the uncertainty principle, which the coherent states saturate. Further, the Wehrl entropy is always greater than the von Neumann entropy [383] :

$$H_W(\hat{\rho}) \geq S(\hat{\rho}) = -\mathrm{Tr}(\hat{\rho}\log\hat{\rho})\,. \tag{7.18}$$

While the von Neumann entropy is zero for pure states, we have just noted that the Wehrl entropy $H_W(\hat{\rho})$ is greater than or equal to unity for all states. Several aspects of the Wehrl entropy have been explored in Ref. [389].

7.6 A non-Gaussianity measure for quantum states

As is well-known, a quantum state $\hat{\rho}$ is said to be Gaussian iff the associated Wigner distribution is Gaussian. This will suggest that the non-Gaussianity of a state is coded into the non-vanishing cumulants of order > 2 of the Wigner function. Since the Wigner and Q functions are related by convolution by a Gaussian, the Q function of a state is Gaussian iff the Wigner function is, and the non-Gaussianity should thus be found coded in the higher-order cumulants of the Q function as well. The consistency of these statements is ensured by the fact that the higher-order cumulants are the same for the Wigner and the Q functions [Indeed, as we have shown earlier, the higher-order cumulants are intrinsic to the state, and hence are the same for all s-ordered quasi-probabilities].

Non-Gaussianity can thus be described using either the Wigner function or the Q function. The fact that the Q function is everywhere non-negative, rendering it a genuine probability in the classical sense, makes it our preferred choice. We employ therefore the Wehrl entropy to capture the essence of the higher-order cumulants.

Given a state $\hat{\rho}$, our measure of non-Gaussianity $\mathcal{N}(\hat{\rho})$ is defined as the difference of two Wehrl entropies :

$$\mathcal{N}(\hat{\rho}) \;\; = \;\; H_W(\hat{\rho}_G) - H_W(\hat{\rho})\,. \tag{7.19}$$

Here $H_W(\hat{\rho})$ is the Wehrl entropy of the given state $\hat{\rho}$ and $H_W(\hat{\rho}_G)$ is the Wehrl entropy of the Gaussian state $\hat{\rho}_G$ that has the same first and second moments as $\hat{\rho}$. Since $\mathcal{N}(\hat{\rho})$ measures the departure of the Wehrl entropy of $\hat{\rho}$ from that of its Gaussian partner $\hat{\rho}_G$, it can be viewed as a quantum Kullback-Leibler distance. $\mathcal{N}(\hat{\rho})$ could also be viewed as a relative Wehrl entropy. But we prefer to call it simply a non-Gaussianity measure.

This measure of non-Gaussianity enjoys several interesting properties. We will now

list some of them :

(i) $\mathcal{N}(\hat{\rho}) \geq 0$, equality holding iff $\hat{\rho}$ is Gaussian.

Proof: This is a restatement of the fact that the Wehrl entropy of a Gaussian state is greater than that of all states with the same first and second moments as the Gaussian [384].

(ii) $\mathcal{N}(\hat{\rho})$ is invariant under phase space displacements :

$$\mathcal{N}(\hat{\rho}) = \mathcal{N}(\, D(\xi)\, \hat{\rho}\, D(\xi)^{\dagger}\,)\,. \tag{7.20}$$

Proof: Let $D(\xi)\, \hat{\rho}\, D(\xi)^{\dagger}$ be denoted, for brevity, by $\hat{\rho}'$. The Q function of $\hat{\rho}'$ is related to that of $\hat{\rho}$ in this simple manner :

$$Q_{\rho'}(\alpha) = Q_{\rho}(\alpha - \xi)\,. \tag{7.21}$$

That is, displacement $D(\xi)$ acts as a rigid translation in phase space [192, 383, 392]. Thus it has no effect on the Wehrl entropy of any state, and hence leaves $\mathcal{N}(\hat{\rho})$ invariant for every state.

(iii) $\mathcal{N}(\hat{\rho})$ is invariant under passage through any passive linear system.

Proof: A passive linear system is represented by a $n \times n$ unitary matrix U. It maps a coherent state $|\alpha\rangle$ into a new coherent state $|\alpha'\rangle = |U\,\alpha\rangle$ [192, 383, 392], where $\alpha \in \mathcal{C}^n$ is to be viewed as a column vector. Let $\hat{\mathcal{U}}_U$ be the unitary operator in the n-mode Hilbert space which represents the passive linear system labelled by the matrix U. Let us denote by $\hat{\rho}'$ the transformed state $\hat{\mathcal{U}}_U\, \hat{\rho} \hat{\mathcal{U}}_U^{\dagger}$ at the output of this passive system. Then the output Q function is related to the input Q function in this manner :

$$Q_{\rho'}(\alpha) = Q_{\rho}(U^{-1}\alpha) = Q_{\rho}(U^{\dagger}\alpha)\,. \tag{7.22}$$

That is, the action of a passive linear system is a rigid $SO(2n)$ rotation in the $2n$-dimensional phase space. It follows immediately that this transformation does not change the Wehrl entropy of any state, and hence does not affect $\mathcal{N}(\hat{\rho})$.

Remark: While in the single-mode case of two-dimensional phase space all proper rotations are canonical transformations, this is not true in the multi-mode case. That is, $\mathrm{Sp(2n,\mathcal{R})} \cap \mathrm{SO(2n)}$ is a proper subgroup of SO(2n) isomorphic to U(n), the n^2-parameter group of $n \times n$ unitary matrices, whereas SO(2n) is a much larger $(2n^2 - n)$-parameter group [192]. Only those phase space rotations which are elements of this intersection act as unitary transformations in the Hilbert space of n oscillators.

(iv) $\mathcal{N}(\hat{\rho})$ is invariant under a uniform phase space scaling λ defined at the level of

the Q function in the following manner :

$$\lambda : \quad Q_\rho(\alpha) \to Q_{\rho'}(\alpha) = \lambda^{2n} Q(\lambda\alpha)\,. \tag{7.23}$$

Proof: Under this uniform phase space scaling of the Q function, the Wehrl entropy changes by a simple additive part *that is independent of the state* :

$$\begin{aligned} H_W(\hat{\rho}) &= -\frac{1}{\pi^n}\int Q_\rho(\alpha)\log Q_\rho(\alpha)\prod_{j=1}^{n} d^2\alpha_j \\ &\to -\frac{1}{\pi^n}\int \lambda^{2n} Q_\rho(\lambda\alpha)\log\left(\lambda^{2n} Q_\rho(\lambda\alpha)\right)\prod_{j=1}^{n} d^2\alpha_j \\ &= H_W(\hat{\rho}) - 2n\log\lambda\,. \end{aligned} \tag{7.24}$$

Note that in arriving at the last equation we have made a change of variables in the integral and made use of the normalisation of the Q function. Now it trivially follows from this result that $\mathcal{N}(\hat{\rho})$, being a difference of two Wehrl entropies, remains invariant. *Remark*: While the above conclusion holds mathematically for all $\lambda > 0$, the scaled Q function fails to be a physical Q function if $\lambda > 1$. Therefore we restrict this scale parameter to the physically relevant range $0 < \lambda \leq 1$. This may be seen from the analysis in Section 6.5.

(v) $\mathcal{N}(\hat{\rho})$ is additive on tensor product states :

$$\mathcal{N}(\hat{\rho}_1 \otimes \hat{\rho}_2) = \mathcal{N}(\hat{\rho}_1) + \mathcal{N}(\hat{\rho}_2)\,. \tag{7.25}$$

Proof: Under tensor product the Q functions go as product probabilities by definition. This is true of their associated Gaussian probabilities as well.

(vi) For a bipartite state of the form $\hat{\rho} = \hat{\rho}_a \otimes \hat{\rho}_G$, where $\hat{\rho}_G$ is a Gaussian state

$$\mathcal{N}(\hat{\rho}) = \mathcal{N}(\hat{\rho}_a \otimes \hat{\rho}_G) = \mathcal{N}(\hat{\rho}_a)\,. \tag{7.26}$$

Proof: From (v) we have

$$\mathcal{N}(\hat{\rho}) = \mathcal{N}(\hat{\rho}_a \otimes \hat{\rho}_G) = \mathcal{N}(\hat{\rho}_a) + \mathcal{N}(\hat{\rho}_G)\,.$$

and from (i)

$$\mathcal{N}(\hat{\rho}_a) + \mathcal{N}(\hat{\rho}_G) = \mathcal{N}(\hat{\rho}_a)\,. \tag{7.27}$$

(vii) For a bipartite state of the form $\hat{\rho}_{\text{out}} = \hat{\mathcal{U}}_U\,(\hat{\rho}_a \otimes |\alpha\rangle\langle\alpha|)\,\hat{\mathcal{U}}_U^\dagger$, where U represents

a passive linear system and $|\alpha\rangle$ is a coherent state, we have

$$\mathcal{N}(\hat{\rho}_{\rm out}) = \mathcal{N}(\hat{\rho}_a)\,. \tag{7.28}$$

Proof: From (iii) we have

$$\mathcal{N}(\hat{\rho}_{\rm out}) = \mathcal{N}(\mathcal{U}_U\,(\hat{\rho}_a \otimes |\alpha\rangle\langle\alpha|)\,\mathcal{U}_U^\dagger) = \mathcal{N}(\hat{\rho}_a \otimes |\alpha\rangle\langle\alpha|)\,.$$

We have from (v)

$$\mathcal{N}(\hat{\rho}_a \otimes |\alpha\rangle\langle\alpha|) = \mathcal{N}(\hat{\rho}_a) + \mathcal{N}(|\alpha\rangle\langle\alpha|)\,.$$

Since the coherent state $|\alpha\rangle$ is Gaussian, we have from (i)

$$\mathcal{N}(\hat{\rho}_a) + \mathcal{N}(|\alpha\rangle\langle\alpha|) = \mathcal{N}(\hat{\rho}_a)\,. \tag{7.29}$$

This result is useful in evaluating the non-Gaussianity of bipartite states produced by the action of beamsplitters, as we shall illustrate in the next Section.

7.6.1 Shape criterion for good measure of non-Gaussianity

Properties (ii), (iii), and (iv) deal with transformations which do not change the shape of the Q functions. Since non-Gaussianity is a quantitative statement regarding the departure of the shape of the Q function from Gaussian, it will appear that any good measure of non-Gaussianity should return the same value for all states connected by these transformations. In particular, two quantum states whose Q functions are related by a uniform scaling of all the phase space coordinates should be assigned the same amount of non-Gaussianity. We will call this the *shape criterion*, and we have seen that our measure $\mathcal{N}(\hat{\rho})$ meets this requirement.

7.7 Examples

In this Section we evaluate our non-Gaussianity measure $\mathcal{N}(\hat{\rho})$ for three families of states, namely the Fock states, the photon-added thermal states, and the phase-averaged coherent states of a single-mode of radiation. While the first two families consist of nonclassical states, the third one is a family of classical states.

7.7.1 Photon number states

The Q function of the Fock state (energy eigenstate) $\hat{\rho} = |m\rangle\langle m|$ of the oscillator is given by the phase space distribution

$$Q_{|m\rangle}(\alpha) = \frac{|\alpha|^{2m}}{m!} \exp(-|\alpha|^2)\,, \tag{7.30}$$

whose only non-vanishing moment of order ≤ 2 is $\langle |\alpha|^2 \rangle = \mathrm{Tr}(\hat{\rho}\hat{a}\hat{a}^\dagger) = m + 1$. The phase space average $\langle |\alpha|^2 \rangle$ is with respect to the probability distribution $Q_{|m\rangle}(\alpha)$ and, by definition, it equals the (quantum) expectation value of the associated anti-normally ordered operator $\hat{a}\hat{a}^\dagger$. The Gaussian state which has the same moments of order ≤ 2 as $\hat{\rho}_{|m\rangle} = |m\rangle\langle m|$ is clearly the thermal state with mean photon number $\langle \hat{n} \rangle \equiv \langle \hat{a}^\dagger \hat{a} \rangle = m$. The Q function of such a thermal state $\hat{\rho}_G$ is given by

$$Q_G(\alpha) = \frac{1}{\langle \hat{n} \rangle + 1} \exp\left(-\frac{|\alpha|^2}{\langle \hat{n} \rangle + 1} \right), \;\; \langle \hat{n} \rangle = m\,. \tag{7.31}$$

The Wehrl entropy corresponding to ρ_G is easily computed :

$$\begin{aligned} H_W(\hat{\rho}_G) &= 1 + \log(1 + \langle \hat{n} \rangle) \\ &= 1 + \log(1 + m)\,. \end{aligned} \tag{7.32}$$

The Wehrl entropy of the photon number state $\hat{\rho} = |m\rangle\langle m|$ is

$$H_W(\hat{\rho}_{|m\rangle}) = -\frac{1}{\pi} \int d^2\alpha Q_{|m\rangle}(\alpha) \log Q_{|m\rangle}(\alpha)\,. \tag{7.33}$$

This can be computed explicitly by going to the polar coordinates, and one obtains [389]

$$\begin{aligned} H_W(\hat{\rho}_{|m\rangle}) &= 1 + m + \log m! - m\psi(m+1), \\ & \qquad \psi(m+1) = \sum_{k=1}^{m} \frac{1}{k} - \gamma\,, \end{aligned} \tag{7.34}$$

where $\psi(m)$ is the digamma function, and $\gamma = 0.5772\cdots$ is the Euler constant. Hence the non-Gaussianity of the photon number state $\hat{\rho} = |m\rangle\langle m|$ is

$$\begin{aligned} \mathcal{N}(\hat{\rho}_{|m\rangle}) &= H_W(\hat{\rho}_G) - H_W(\hat{\rho}_{|m\rangle}), \\ &= \ln(m+1) - m - \log m! + m\psi(m+1). \end{aligned} \tag{7.35}$$

In Figure (7.1) we have plotted this non-Gaussianity as a function of the photon

number m. It is clear that the non-Gaussianity of $|m\rangle$ increases monotonically with the photon number m, and goes to ∞ as m tends to ∞. That this was to be expected can be seen as follows. For large m values $\psi(m+1) \sim \ln(m+1)$, and $\log m! \sim m\log m - m$, and hence $\mathcal{N}(\hat{\rho}_{|m\rangle}) \sim \log(m+1)$. We shall be returning to this result in the next Section.

Now consider a bipartite state of two modes with one mode in the Fock state and the other in the vacuum. Non-Gaussianity of this product state is the same as that of the Fock state, and this follows from Eq. (7.26) . Let this bipartite state be passed through a beamsplitter. The state at the output will be entangled due to the nonclassicality of the Fock state [171, 194], but in view of Eq. (7.28), this two-mode state will have the same non-Gaussianity as the original single-mode Fock state.

7.7.2 Photon-added thermal states

In this subsection we evaluate the non-Gaussianity of the photon-added thermal state (PATS) [182]. The PATS is defined through

$$\hat{\rho} = C\,\hat{a}^{\dagger m}\hat{\rho}_{\rm th}\hat{a}^{m}\,, \tag{7.36}$$

where C is the normalisation constant which ensures $\mathrm{Tr}\,(\hat{\rho}) = 1$, and $\hat{\rho}_{\rm th}$ is the thermal state given by

$$\hat{\rho}_{\rm th} = (1-x)\sum_{n=0}^{\infty} x^k |k\rangle\langle k|\,;\;\; x = \exp\left[-\frac{\hbar\omega}{kT}\right]\,. \tag{7.37}$$

One can alternatively define the PATS through parametric differentiation :

$$\hat{\rho} = \frac{(1-x)^{m+1}}{m!}\frac{d^m}{dx^m}\sum_{k=0}^{\infty} x^k |k\rangle\langle k|\,. \tag{7.38}$$

PATS are thus parametrised by two parameters: $0 \le x < 1$, and $m = 0, 1, 2, \cdots$. The limit $x \to 0$ corresponds to Fock states, and the limit $m \to 0$ corresponds to thermal states.

We may note that PATS (with $m \ge 1$) is nonclassical for all values of x [194]. Indeed, it violates a three-term classicality condition [170].

The Q function of PATS can be easily calculated and is given by

$$Q^{(m,x)}_{\rm PATS}(\alpha) = \frac{(1-x)^{m+1}}{m!}|\alpha|^{2m}\exp[-(1-x)|\alpha|^2]\,. \tag{7.39}$$

It is evident that *the Q function of the PATS is a scaled version of the Q function of the*

Fock state:

$$Q^{(m,x)}_{\rm PATS}(\alpha) = \lambda^2 Q_{|m\rangle}(\lambda\,\alpha), \quad \lambda = \sqrt{1-x}\,. \tag{7.40}$$

Since our measure of non-Gaussianity respects the shape criterion put forward in the previous Section, it is immediate that the non-Gaussianity of the PATS is the same as that of the photon number state:

$$\begin{aligned} N(\,\hat{\rho}^{(m,x)}_{\rm PATS}\,) &= \ln(m+1) - m - \log m! + m\psi(m+1) \\ &= N(\hat{\rho}_{|m\rangle})\,. \end{aligned} \tag{7.41}$$

It is worth emphasising here that the PATS is a special state with regard to the question of verifying whether a given measure of non-Gaussianity is a good measure, *i.e.*, whether it satisfies the shape criterion. The test is as simple as checking whether the measure in question evaluated for the PATS is independent of the temperature parameter x or not.

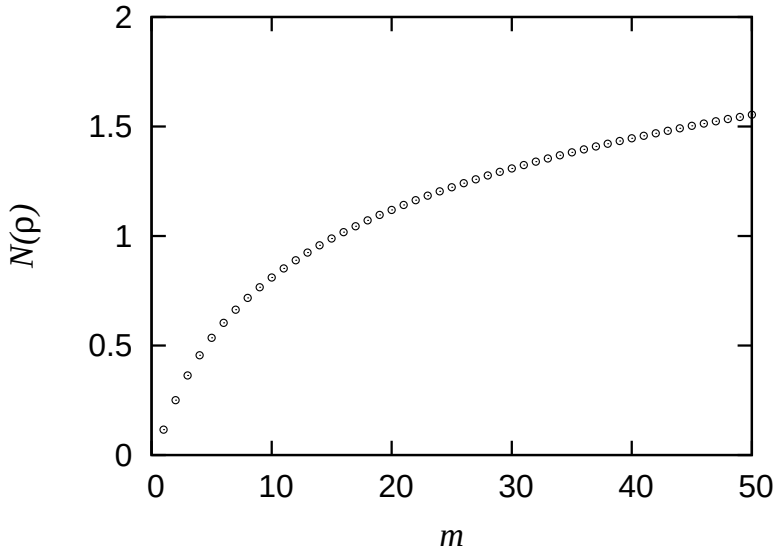

Figure 7.1: Variation of $\mathcal{N}(\rho)$ with number of photons m for the Fock state $\rho = |m\rangle\langle m|$.

Finally we consider, as in the previous Subsection, a bipartite state of two modes, with one mode in the PATS $\hat{\rho}^{(m,x)}_{\rm PATS}$ and the other in the vacuum state. Let us pass this two-mode state through a beamsplitter. That the state at the output of the beamsplitter is entangled follows from the nonclassicality of the PATS [171, 194]. It follows from Eq. (7.28) that non-Gaussianity of this entangled state is the same as that of the PATS, and

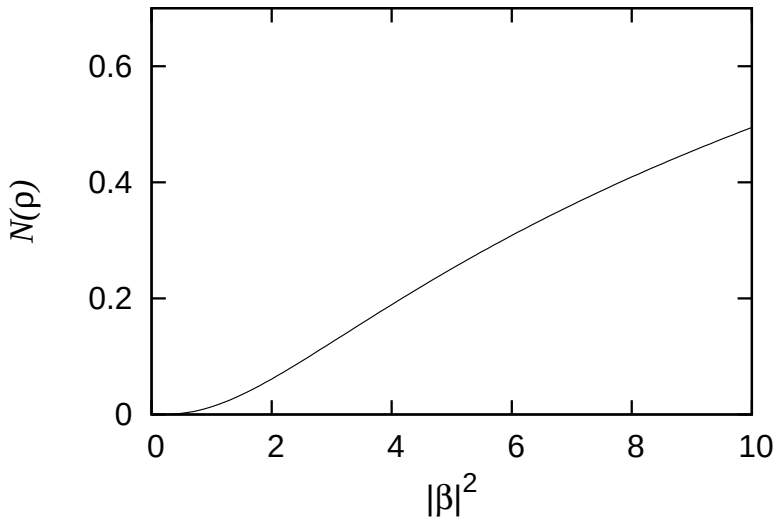

Figure 7.2: Variation of $\mathcal{N}(\rho)$ with energy $|\beta^2|$ for the phase-averaged coherent state.

hence is fully determined by m.

We have already noted that PATS violates a three-term classicality condition. This implies that the output state is entangled [194].

7.7.3 Phase-averaged coherent states

As our final example, we evaluate the non-Gaussianity for the phase-averaged coherent states. Given a coherent state $|\beta\rangle$ its phase-averaged version is

$$\begin{aligned} \hat{\rho}_{|\beta|} &\equiv \int \frac{d\theta}{2\pi} \exp[-i\,\theta\,\hat{a}^\dagger\hat{a}\,]\,|\beta\rangle\langle\beta|\,\exp[\,i\,\theta\,\hat{a}^\dagger\hat{a}\,] \\ &= \exp(-|\beta|^2)\sum_{n=0}^{\infty}\frac{|\beta|^{2n}}{n!}|n\rangle\langle n|\,. \end{aligned} \tag{7.42}$$

Since $\hat{\rho}_{|\beta|}$ is a convex sum of Fock states, its Q function is a corresponding convex sum :

$$\begin{aligned} Q^{|\beta|}(\alpha) &= \exp[-(|\alpha|^2+|\beta|^2)]\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}|\beta|^{2n}}{n!n!} \\ &= \exp[-(|\alpha|^2+|\beta|^2)]I_0(2|\alpha||\beta|)\,, \end{aligned} \tag{7.43}$$

where $I_0(.)$ is the modified Bessel function of integral order zero. The only non-zero moment of order ≤ 2 is $\langle|z|^2\rangle = \mathrm{Tr}(\hat{\rho}_{|\beta|}\hat{a}\hat{a}^\dagger) = 1+|\beta|^2$. The associated Gaussian prob-

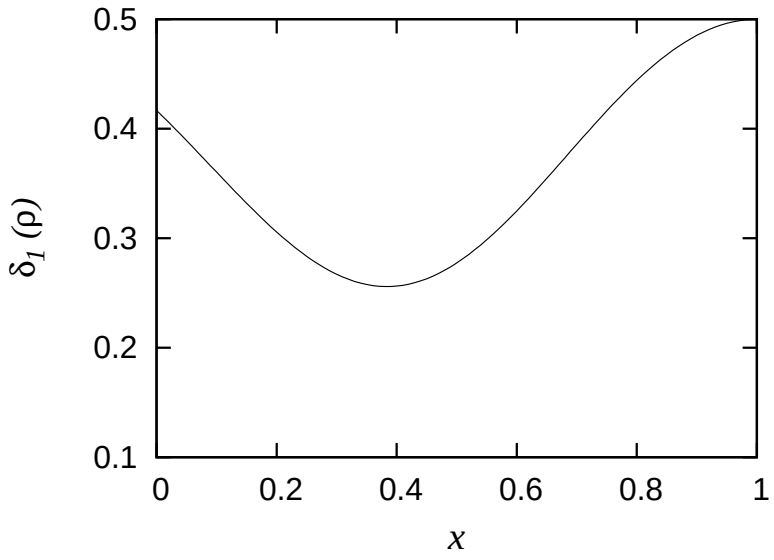

Figure 7.3: Variation of $\delta_1(\hat{\rho})$ as a function of the Boltzmann parameter x for the photon-added thermal state.

ability $Q_G^{|\beta|}(\alpha)$ that has the same first and second moments is thus the thermal state with average photon number $\langle \hat{n} \rangle = |\beta|^2$. As we have shown earlier in Eq. (7.32), the Wehrl entropy of this Gaussian state is $H_W(\hat{\rho}_G^{|\beta|}) = 1 + \log(1 + |\beta|^2)$. To compute the Wehrl entropy corresponding to the original phase-averaged coherent state, however, we resort to numerical evaluation. In Figure (7.2) we present the non-Gaussianity of $\hat{\rho}_{|\beta|}$ as a function of $|\beta|^2$, the energy of the state. It is seen to be a monotone increasing function of $|\beta|^2$.

Note that the phase-averaged coherent states are classical since they are, by definition, convex sums of coherent states. Thus if a bipartite state consisting of a phase-averaged coherent state in one mode and vacuum in the other is passed through a beamsplitter, the two-mode mixed state at the output will remain separable (since the phase-averaged coherent state is classical[194]), with the same non-Gaussianity as the original phase-averaged coherent state.

7.8 Comparison with other measures

In this Section we compare our non-Gaussianity measure $\mathcal{N}(\hat{\rho})$ with two non-Gaussianity measures which have been proposed recently.

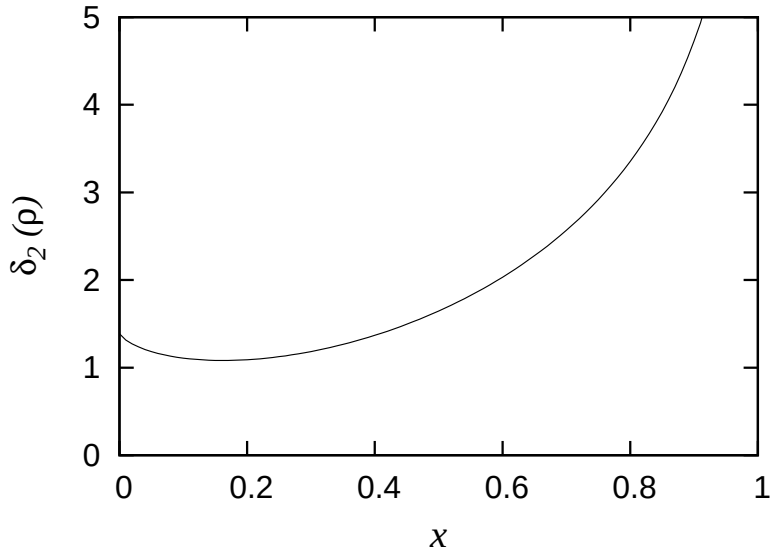

Figure 7.4: Variation of $\delta_2(\hat{\rho})$ as a function of the Boltzmann parameter x for the photon-added thermal state.

7.8.1 Measure based on Hilbert-Schmidt distance

Genoni *et al* [378], have proposed a non-Gaussianity measure based on the Hilbert-Schmidt distance. They define non-Gaussianity of a state $\hat{\rho}$ as

$$\delta_1(\hat{\rho}) = \frac{\mathrm{Tr}[(\hat{\rho} - \hat{\tau})^2]}{2\mathrm{Tr}(\hat{\rho}^2)}\,, \tag{7.44}$$

where $\hat{\tau}$ is the Gaussian state with the same first and second moments as $\hat{\rho}$. Let us compare this measure with ours in the specific case of the PATS $\hat{\rho}^{(m,x)}_{\mathrm{PATS}}$. In Figure (7.3) we plot $\delta_1(\hat{\rho}^{(m,x)}_{\mathrm{PATS}})$ as a function of the Boltzmann parameter x, for fixed value of $m = 1$. It is seen that $\delta_1(\hat{\rho}^{(m,x)}_{\mathrm{PATS}})$, for $m = 1$, is not a constant but varies with the temperature parameter x. This shows that this measure of Genoni *et al.* does not satisfy our shape criterion.

Another interesting difference appears when one compares our measure $\mathcal{N}(\hat{\rho})$ with $\delta_1(\hat{\rho})$ in the case of the photon number states $\hat{\rho} = |m\rangle\langle m|$. As we have shown earlier [see Figure (7.1)], our measure monotonically increases with the photon number m and tends to infinity as m tends to infinity. In contrast, as Genoni *et al.* have shown and emphasised [378], their measure $\delta_1(\hat{\rho})$ saturates at the value $\frac{1}{2}$.

7.8.2 Measure based on quantum relative entropy

Genoni *et al.* [379] have proposed, in a subsequent paper, a second measure of non-Gaussianity, this one based on quantum relative entropy. They define non-Gaussianity of a state $\hat{\rho}$ as

$$\delta_2(\hat{\rho}) = S(\hat{\tau}) - S(\hat{\rho}), \tag{7.45}$$

where $S(\cdot)$ is the von Neumann entropy of the state in question and $\hat{\tau}$ is the Gaussian state with the same first and second moments as the given state $\hat{\rho}$.

At first sight it would seem that $\delta_2(\hat{\rho})$ and our measure $\mathcal{N}(\hat{\rho})$ are very similar, the only difference being that $\hat{\rho}$ is replaced by $Q_\rho(z)$ and that the trace operation in the formula for the von Neumann entropy is replaced in our measure by a phase space integral. A closer look reveals that this is not the case; $\delta_2(\hat{\rho})$ does not reduce to $\mathcal{N}(\hat{\rho})$ under this kind of 'quantum-classical correspondence'. And $\delta_2(\hat{\rho})$ and $\mathcal{N}(\hat{\rho})$ turn out to be quite different entities.

A qualitative difference between $\delta_2(\hat{\rho})$ and $N(\hat{\rho})$ becomes manifest when one compares these two measures in the context of a pure state. As the von Neumann entropy of a pure state is zero, $\delta_2(\hat{\rho})$ reduces to $S(\hat{\tau})$, the von Neumann entropy of the Gaussian state with the same first and second moments as $\hat{\rho}$. In other words $\delta_2(\hat{\rho})$ does not consult, in the case of pure states, moments or cumulants of $\hat{\rho}$ of order higher than 2. Consequently, all pure states which have the same set of first and second moments but differ in higher moments will get assigned the same non-Gaussianity $\delta_2(\rho)$. This is not the case with our measure $\mathcal{N}(\hat{\rho})$.

To bring out a second qualitative difference we check if $\delta_2(\hat{\rho})$ satisfies the shape criterion. To this end we ask if $\delta_2(\hat{\rho})$ will ascribe the same amount of non-Gaussianity to the PATS and the photon number state, *i.e.,* , whether $\delta_2(\rho)$ evaluated for the PATS $\hat{\rho}^{(m,x)}_{\rm PATS}$ is independent of the temperature parameter x. We find that this is not the case. This is shown in Figure (7.4) wherein we present $\delta_2(\hat{\rho}^{(m,x)}_{\rm PATS})$, for fixed value $m = 1$, as a function of x.

We conclude this Section with a further remark. With reference to Figures (7.3) and (7.4), while the non-Gaussianity measures $\delta_1(\hat{\rho}^{(m,x)}_{\rm PATS})$ and $\delta_2(\hat{\rho}^{(m,x)}_{\rm PATS})$, for fixed m, vary with the temperature (or scale) parameter x, thus failing the shape criterion, the variation is not monotone. The significance of the temperatures at which these measures assume their respective minimum values is not clear.

7.9 Concluding remarks

We have presented a measure of non-Gaussianity of quantum states based on the Q function. In doing so we have been guided by the fundamental principle that any measure of non-Gaussianity is an attempt to make a quantitative statement on the departure of the shape of the Q function from Gaussian, and the measure must therefore remain invariant under all transformations which do not change the shape of the Q function.

Uniform scaling of all the phase space coordinates at the level of the Q function has proved to be an important shape preserving transformation, and our shape criterion demands that non-Gaussianity of the photon-added thermal states should be independent of temperature.

We have explored various properties our measure which meets the shape criterion. We have presented analytical and numerical results on the non-Gaussianity of a few families of quantum states. We have also compared our measure with other measures of non-Gaussianity available in the literature.

Our measure $\mathcal{N}(\hat{\rho})$ meets the shape criterion which, in our opinion, should be respected by every good measure of non-Gaussianity. We hasten to add, however, that this is not the only measure that meets this criterion. For instance, if $\gamma^{(2n)}$ is an appropriate linear combination of the cumulants of order $2n$, and $\gamma^{(2)}$ an appropriate linear combination of the cumulants of order 2, it is clear that the ratio between $\gamma^{(2n)}$ and the n^{th} power of $\gamma^{(2)}$ will meet this criterion, for every $n \geq 2$. Our choice $\mathcal{N}(\hat{\rho})$ has the attraction of being immediately related to well-known entities like the Wehrl entropy and Kullback-Leibler distance.

In the case of classical probability defined on a $2n$ dimensional space $\mathcal{C}^n$, one would have required the non-Gaussianity measure to be invariant under the full Euclidean group consisting of translations and all SO(2n) rotations. In the case of phase space, SO(2n) rotations which fall outside the subgroup $\mathrm{Sp}(2n,\mathcal{R}) \cap \mathrm{SO}(2n)$ are unphysical, and hence the restriction to this subgroup of passive linear systems.

Our shape criterion rests on the invariance semi-group of Q functions which is different from the invariance semi-group of the Gaussian family of states – operations which map Gaussian states into Gaussians. The latter semigroup includes the full $\mathrm{Sp}(2n,\mathcal{R})$, and not just the intersection subgroup $\mathrm{Sp}(2n,\mathcal{R}) \cap \mathrm{SO}(2n)$. It further includes a whole family of completely positive maps known as Gaussian channels.

8

Discussion

Quantum information theory in the context of continuous variables, has been primarily explored in Gaussian states. Two immediate reasons for this has been the fact that they have been produced in many laboratories across the world and are experimentally viable, and secondly that their characterisation has been relatively easy in the sense that their variance matrix tells everything about them [226]. The notion of nonclassicality as well as entanglement in their context is by and large well understood [57, 192]. This is not so for non-Gaussian states. The characterisation of non-Gaussian states through their moments is an on going study in quantum optics and any progress in quantum optics is based on our progress with regard to the theory of moments. A classic example of this has been the case of states of a single-mode of radiation field which are diagonal in the Fock basis. The characterisation of these states in terms of moments has been found to be equivalent to the Stieltjes moment problem [170]. We, in our present work, have made an attempt to make use of this knowledge. A possible future program would be to find possible continuous variable systems which can be mapped to well studied problems in the context of classical probability theory, and if such a thing has been carried out, then see if it helps us understand entanglement. We might attempt this bearing in mind that the theory of moments tells us of some practical limitations, and an example of such is as implied by the Marcinkiewicz theorem [375].

Another area of possible exploration is the extension of the use known positive maps to the continuous variable case. Only the partial transpose map and the reduction map have been extended [57, 217]. Recent methods, illustrated in [233] give us a window to this aspect. The direct extension of positive maps is still a problem though. The current knowledge seems to be nascent regarding this issue.

The problem of separability/entanglement has been settled for the case of multi-mode Gaussians [242]. We, in our work, have settled this issue for a restricted class of non-Gaussian states. But, the issue of separability/entanglement is still open in the more

general context. One major tool has been the use of uncertainty relations [57, 58, 230]. Such relations have been seen to be strong enough in certain contexts to detect bound entanglement [60, 61, 241]. Such a study has been by and large restricted to the level of the variance matrix. Recently in [232], a more general approach was devised. Further exploration along these lines would be of great interest.

One of the foremost problems in quantum information theory has been the estimation of entanglement. In the continuous variable case, the EOF has been evaluated for the case of two-mode Gaussians. But for the general multi-mode Gaussian case, there is not yet a coherent understanding of entanglement. Extremal properties of Gaussian states have already played their role in the two-mode case, but a deeper exploration of such properties in the multi-mode context is much awaited. In the non-Gaussian case, the problem is even more intruiging by the very lack of our understanding with regard to these states. We have in our work, outlined a possible context dependant procedure, which estimates entanglement in very special non-Gaussian states, but a more general approach is much awaited Recent methods outlined in [328], tell us of the possibility of estimating entanglement with incomplete knowledge of the state. It would be of great interest to extend these ideas to the non-Gaussian continuous variable state.

A primary concern in this work, has been the study on nonclassicality, and its relationship to entanglement. The squeezing nonclassicality, a nonclassicality associated with Gaussian states, has been well explored as a resource in the context of quantum information processing. The same cannot be said of other available nonclassicalities. The potential role of the other non-Gaussian nonclassicalities is yet to be realised. Recently, exploration in these lines have started emerging[377, 393, 394], the advantages are also being spelt out. More exploration needs to be done along these directions.

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