GAUGE THEORY OF GRAVITY WITH TOPOLOGICAL INVARIANTS

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STATEMENT BY THE AUTHOR

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Sandipan Sengupta

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institute/University.

Sandipan Sengupta

This thesis is dedicated to Soumenbabu, Ma, Baba, Dida and Chhotodadu

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As I take a downward perspective towards my days as a graduate student, it smells of anything but a wonderful springtime. The apparent feeling of revelry while being involved in a scientific pursuit also brings its own demons. It means a constant process of having to face and live with doubt and uncertainty. However, there are things which stand above such bizarre realizations, and there are people whose inspirational influence helps the unsettled mind to cling to its aspirations and shun the delusions.

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Contents

List of publications associated with the thesis

[**1**] ‘Topological interpretation of Barbero-Immirzi parameter’ ;
G. Date, R. K. Kaul, S. Sengupta; Phys. Rev. **D 79**, 044008 (2009)

[**2**] ‘Canonical supergravity with Barbero-Immirzi parameter’;
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[**3**] ‘Quantum realizations of Hilbert-Palatini second-class constraints’;
S. Sengupta; Class. Quantum Grav. **27**, 145008 (2010)

[**4**] ‘Topological parameters in gravity’;
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Chapter 1

Introduction

The discovery of Quantum Field Theory (QFT) alongwith its successful application towards the conception of the Standard Model of particle physics can be hailed as one of the most remarkable achievements in the history of physics. It was an endeavour which led to the emergence of a concrete framework, which consistently describes three of the four basic interactions as are observed to act between the elementary particles, namely, the strong and electro-weak forces. Although such a description is not strictly complete in the sense that some of its parameters are undetermined and also that it does not include the gravitational interactions, it has stood the test of experiments. The confirmation of its theoretical predictions through the legendary discovery of Lamb-shift and anomalous magnetic moment of the electron in QED exhibits its extraordinary power to calculate observables of microscopic scales to extremely high accuracy.

The perturbative framework of QFT underlying the Standard Model is based upon the principle of gauge invariance. This implies that the theory be characterised by a Lagrangian given in terms of local fields and remain invariant under local transformations acting on the fields, i.e., local gauge transformations. The standard model Lagrangian leads to a theory which is known as perturbatively renormalizable. This implies that the infinities which typically appear in the perturbative expansion can be absorbed away through the redefinitions of the parameters of the theory which can be measured through experiments. The robustness of the Standard Model against experiments has propelled the point of view that QFT should be regarded as a general framework which can incorporate all the known fundamental forces as observed in nature.

From this perspective, it seems natural to expect that such a potentially comprehensive construction should also apply to the case of gravity, the only known fundamental interaction which is not included in the standard model. In other words, one should be able to find a quantum field theoretic formulation of General Relativity (GR), which has been known to be the correct classical description of gravitational interactions. GR, when applied to various large scale phenomena as governed by gravitational forces, has enjoyed tremendous experimental success. To cite an example, the relativistic prediction for the centennial precession of the orbit of Mercury was $\Delta\phi \approx 43.03s$/century while the observed value was $\Delta\phi \approx 43.11s$/century. Looking at such remarkable successes of QFT and GR in their respective domains, one might wonder if they are compatible in some way, and can be embedded in one single framework to describe nature as it is.

According to a field-theoretic viewpoint, the theory of gravity should be characterised by a Lagrangian given in terms of local fields and exhibiting the appropriate local gauge symmetries. The corresponding quantum theory, at its naive best, would lead to particle-like excitations which would be the carriers of the gravitational force in a fixed background spacetime. For gravity, the corresponding local field theory is given by the Einstein-Hilbert action:

$$\mathcal{L} \;=\; -\frac{1}{\kappa}\int d^4x\ (-g)^{1/2}R$$

where κ is related to the Newton's constant as $\kappa = 16\pi G$. Here the metric $g_{\mu\nu}$ is treated as the basic field variable and is expanded around a fixed background:

$$g_{\mu\nu} \;=\; g^{(0)}_{\mu\nu} + \sqrt{\kappa}h_{\mu\nu}$$

Such a scheme is evidently perturbative in nature. Using this approach, it was shown by t'Hooft and Veltmann[1] that the one-loop s-matrix for four dimensional pure gravity is finite. However, catastrophe hits soon at the second order. Goroff and Sagnotti[2] provide an explicit demonstration of the divergence of pure Einstein gravity at two loops. Thus the same perturbative quantization scheme which consistently describes the strong and electroweak forces seems to fail when applied to GR, at least in the arena where renormalizabity holds the forte.

One can still take the point of view that such a failure might be an inherent feature of the perturbative approach alone. In a more liberal attempt, perhaps non-perturbative in nature, the usual problems plaguing the perturbative programmes might disappear altogether. In non-perturbative approaches, a convenient way to proceed is to write the theory of gravity in a Hamiltoninan form and then carry out the canonical quantization. The initial attempts to this end were developed using the spatial metric g_{ab} and its conjugate p^{ab}(related to the extrinsic curvature) as the canonical pair. Unfortunately, this seems to lead to a dead end as well because of the complicated nature of the non-polynomial Hamiltonian constraint. However, the hope of a non-perturbative quantization of gravity was revived soon, after it was demonstrated([3]) that gravity can be described as a complex SU(2) gauge theory. In this (Sen-Ashtekar) formulation, the theory of gravity is given entirely in terms of complex SU(2) connection gauge fields. The most important and attractive feature of this formulation was the polynomial form of the canonical constraints. Moreover, the rotation constraints take the form of Gauss' law as in non-abelian gauge theories, and generate the SU(2) gauge transformations on the basic fields. One can obtain a real section of the complex phase space by introducing suitable reality conditions. Subsequently, Barbero and Immirzi[4] showed that gravity can be directly formulated as a gauge theory in terms of real SU(2) connection fields, hence bypassing the need to deal with the non-trivial reality conditions. But in this real formulation, the Hamiltonian constraint takes a more complicated form and regains its original non-polynomial character. However, as noted by Thiemann[5], this feature does not pose a serious threat to quantization since the non-polynomial factors in the constraint can be represented consistently in terms of the basic operators in the quantum picture.

The relevance of the real SU(2) formulation does not end in being just another novel description of gravity as a gauge theory. Not only does this allow the application of standard tools of gauge theories, but this also leads to a framework which can naturally incorporate the notion of background independence. To emphasize, this opens up the possibility of studying the dynamics of spacetime as a whole without having to face any of the subtleties regarding the choice of a background spacetime. This is indeed desirable once we adopt the dictum that gravity should be quantized in a manifestly non-perturbative manner. The quantization proposal known as Loop quantum Gravity (LQG) was developed from such a perspective. In this formulation, one introduces Wilson-loop variables associated with the SU(2) gauge fields as in gauge theories. In the corresponding quantum theory, the kinematical states are associated with one dimensional, polymer-like excitations. However, while significant progress has been made along these lines towards developing a non-perturbative quantization programme, a full solution of the Hamiltonian constraint within such frameworks continues to be elusive.

Here in this thesis we investigate various aspects of the real SU(2) canonical formulation of gravity as mentioned above. The analyses involve both classical and quantum perspectives. The main results of these studies are summarised below.

As is well known, the real SU(2) formulation of gravity contains a free parameter, namely, (the inverse of) the Barbero-Immirzi parameter, η. We clarify and explain the exact origin of this parameter in chapter-1 (and in the subsequent analysis). The Lagrangian description containing η was originally given in terms of the Holst Lagrangian density [6]. In the first order form with the tetrads and the spin-connections as independent variables, this can be written as[1]:

$$\mathcal{L} \;=\; \frac{1}{2}e\Sigma^{\mu\nu}_{IJ}R_{\mu\nu}{}^{IJ}(\omega) \;+\; \frac{\eta}{2}e\Sigma^{\mu\nu}_{IJ}\tilde{R}_{\mu\nu}{}^{IJ}(\omega) \tag{1.1}$$

[1]here we assume that $\kappa = 1$

where,

$$\Sigma^{\mu\nu}_{IJ} := \frac{1}{2}(e^\mu_I e^\nu_J - e^\mu_J e^\nu_I)\ ,\ R_{\mu\nu}{}^{IJ}(\omega) := \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^{J}\ ,\ \tilde{R}_{\mu\nu}{}^{IJ}(\omega) := \frac{1}{2}\epsilon^{IJKL}R_{\mu\nu KL}(\omega).$$

The first term is the standard Hilbert-Palatini term. The second is the Holst term with η, the inverse of the Barbero-Immirzi parameter, appearing as its coeffiecient. This additional term and hence η, does not affect the classical equations of motion of the Hilbert-Palatini action. However, introduction of matter-coupling in the Lagrangian requires additional modifications other than the Holst term in order to preserve the classical equations of motion. These modifications are not universal and change with the matter content of the theory. In chapter-2 we develop a canonical formulation based on an action containing the Nieh-Yan topological density instead of the Holst term[7] (**publication 1**). This has the following advantages:

(a) While the new Lagrangian density leads to a real SU(2) theory as earlier, inclusion of matter now does not need any further modifications and the equations of motion continue to be independent of η for all couplings;

(b) The Nieh-Yan term provides a topological interpretation for η unlike the Holst term. For these reasons, addition of the Nieh-Yan density in the gravity action supercedes the Holst formulation. As an elucidation of these facts, the method has been applied to spin-$\frac{1}{2}$ fermions coupled to gravity in chapter-3[7] (**publication 1**). In chapter-4[8] (**publication 2**), we perform a similar analysis for spin-$\frac{3}{2}$ fermions and illustrate how the supergravity theories ($N = 1,\ 2,\ 4$ etc.) in general can be handled likewise.

In a separate investigation in chapter-5, we analyse the case of gravity coupled to antisymmetric tensor gauge fields of rank two. The particular form of the coupling is inspired from string theory and is non-linear in curvature. Hence it is essentially a higher derivative coupling in nature unlike the fermionic cases. Here we see that although the symplectic structure gets modified in a non-trivial manner, the SU(2) interpretation for the theory of gravity remains robust.

In four-dimensional gravity, there are two more topological densities other than the Nieh-Yan term, namely, the Euler and Pontryagin densities. In a complete theory of gravity, one needs to include all of them. A detailed study regarding how they affect the canonical structure of gravity is carried out in chapter-6[9] (**publication 4**). Here we demonstrate that one obtains a SU(2) theory of gravity where the canonical theory develops dependence on all three topological parameters associated with the three terms. The SU(2) gauge field contains the Barbaro-Immirzi parameter as the coupling constant.

We may recall that in QCD, the role of the θ parameter in the quantum theory can be studied through the method of wavefunction-rescaling. From this perspective, it is natural to ask whether there could be a similar procedure for gravity where the Nieh-Yan density may emerge as the source of a quantization ambiguity reflected by η. As it turns out, for gravity with or without matter, such a rescaling cannot be implemented using the Dirac quantization method. In other words, if the second-class constraints of Hilbert-Palatini theory are enforced before quantization, as is required by Dirac's procedure, the rescaling operator vanishes identically. To emphasize, such a situation does not arise in QCD where the corresponding classical theory has no second-class constraints. For the case of gravity, a non-vanishing operator, and hence a well-defined rescaling, can be invoked only if the second-class constraints are treated using alternative quantization schemes. We address this issue in chapter-7[10] (**publication 3**). Here we set up a general rescaling procedure for gravity with any arbitrary matter-coupling using the Gupta-Bleuler and coherent state quantization approaches. Such a construction provides a natural quantum framework for studying the possible non-perturbative vacua of gravity characterised by η, similar to the θ vacua in QCD.

Chapter 2

Topological interpretation of Barbero-Immirzi parameter

It is well-known that Einstein's General Theory of Relativity can be described as a gauge theory[11]. In other words, the theory can be characterised by a local Lagrangian which is invariant under certain local gauge transformations acting on the basic fields. These transformations involve both spacetime-diffeomorphisms and local Lorentz transformations. In the canonical formulation, the existence of such gauge symmetries implies that the canonical variables are not all independent. Thus, there exist relations between these variables which make some of them redundant. These relations, when expressed in a functional form in terms of the phase space variables, are known as constraints. It is convenient to analyse systems with constraints within a Hamiltonian framework. Within such a set up, the gauge symmetries of the theory get reflected through the constraints and the distinction between true and redundant variables becomes transparent. Since the theory of gravity exhibits gauge symmetries, it can be put into the Hamiltonian form with constraints. From the Hamiltonian description, one can pass over to the quantum theory.

Our subsequent analysis will be based on the standard approach for the constrained Hamiltonian systems as mentioned above. The general prescription has been discussed in detail in ref.[12, 13].

The theory of pure gravity is characterised by Hilbert-Palatini Lagrangian. This is written in terms of the connection field $\omega_\mu^{IJ}(x)$ and tetrad $e_\mu^I(x)$ as independent field variables (assuming $\kappa = 1$):

$$\mathcal{L}(e,\omega) \;=\; \frac{1}{2} e \Sigma_{IJ}^{\mu\nu} R_{\mu\nu}{}^{IJ}(\omega) \tag{2.1}$$

where, $\Sigma_{IJ}^{\mu\nu} := \frac{1}{2}(e_I^\mu e_J^\nu - e_J^\mu e_I^\nu)$ is the antisymmetric product of the inverse tetrads e_I^μ and $R_{\mu\nu}{}^{IJ}(\omega) := \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^{J}$ is the field-strength corresponding to ω_μ^{IJ}. Also,

$\tilde{R}_{\mu\nu}{}^{IJ}(\omega) := \frac{1}{2}\epsilon^{IJKL}R_{\mu\nu KL}(\omega)$. The variation of this Lagrangian density with respect to ω_μ^{IJ} and tetrad e_μ^I leads to the following expression (ignoring total derivatives):

$$\delta\mathcal{L}(e,\omega) \;=\; e\left(-e^\mu_K e^\alpha_I e^\nu_J \;+\; \frac{1}{2}e^\alpha_K e^\mu_I e^\nu_J\right)\; R^{IJ}_{\mu\nu}(\omega)\;\delta e^K_\alpha \;-\; D_\mu(\omega)\left(\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}e^I_\alpha e^J_\beta\right)\;\delta\omega^{KL}_\nu$$

Thus, we obtain two sets of equations of motion given by,

$$\left(e^\mu_K e^\alpha_I e^\nu_J \;-\; \frac{1}{2}e^\alpha_K e^\mu_I e^\nu_J\right)\; R^{IJ}_{\mu\nu} \;=\; 0 \tag{2.2}$$

$$D_{[\mu}(\omega)e^I_{\nu]} \;=\; 0 \tag{2.3}$$

Equation (2.3) implies the vanishing of torsion. These can be solved to write ω_μ^{IJ} in terms of the tetrads, i.e., $\omega_\mu^{IJ} = \omega_\mu^{IJ}(e)$. When this solution is inserted in (2.2), the equations of motion of standard Einstein gravity are recovered.

The Holst generalisation of Hilbert-Palatini formulation is given in terms of the Lagrangian density [6]:

$$\mathcal{L} \;=\; \frac{1}{2}e\Sigma^{\mu\nu}_{IJ}R_{\mu\nu}{}^{IJ}(\omega) \;+\; \frac{\eta}{2}e\Sigma^{\mu\nu}_{IJ}\tilde{R}_{\mu\nu}{}^{IJ}(\omega) \tag{2.4}$$

The second term is the Holst term with η^{-1} as the Barbero-Immirzi parameter [4]. For $\eta = -i$, this Lagrangian density leads to the canonical formulation in terms of the self-dual Ashtekar connection which is a *complex* $SU(2)$ connection [3]. For real η, we have a Hamiltonian formulation in terms of a *real* $SU(2)$ connection, which coincides with the Barbero formulation for $\eta = 1$ [4, 14].

Inclusion of Holst term does not change the classical equation of motion of the Hilbert-Palatini action; there is no dependence on η in the equations of motion. In fact, when the connection equation $\omega_\mu{}^{IJ} = \omega_\mu{}^{IJ}(e)$ is used, Holst term is identically zero.

Adding matter in the generalised Lagrangian density (2.4) needs special care. In particular when spin $\frac{1}{2}$ fermions are included through minimal coupling, the classical equations of motion acquire a dependence on η [15]. However it is possible to modify the Holst term in such a way that classical equations of motion remain unchanged. Such modification for spin $\frac{1}{2}$ fermionic matter and also those in the $N = 1, 2$ and 4 supergravities have been obtained [16, 17]. When the connection equation of motion is used, the modified Holst terms in each of these cases, become total divergences involving Nieh-Yan density and divergence of axial current densities involving the fermion fields. The modified Holst term used in these formulations *changes* with the matter content of the theory.

It has been suggested that the Barbero-Immirzi parameter should have a topological interpretation in the same manner as the θ parameter of QCD [18]. For this to be the case, η should be the coefficient of a term in the Lagrangian density which is a topological density.

Since such a term would be a total derivative for *all* field configurations, the classical equations of motion would remain unaltered. Such a term would be universal in the sense that it would not change when any matter coupling to gravity is introduced. The Holst term in (2.4) or any of its modifications mentioned above do not have such a property. However, there is a topological density in four dimensions which can be algebraically thought of as an extension of the Holst term. We introduce this term in the next section and discuss some of its important properties before going into the main part of our analysis.

2.1 The Nieh-Yan topological density

The Nieh-Yan density is given by [19]:

$$I_{NY} \ = \ \epsilon^{\mu\nu\alpha\beta}\left[D_\mu(\omega)e^I_\nu \ D_\alpha(\omega)e_{I\beta} - \frac{1}{2}\Sigma^{IJ}_{\mu\nu} \ R_{\alpha\beta IJ}(\omega)\right] \ , \ D_\mu(\omega)e^I_\nu \ := \ \partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu \ . \quad (2.5)$$

and the $SO(1,3)$ covariant derivative is: $D_\mu(\omega)e^I_\nu = \partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J$. This is a topological density, and can be written as a total divergence:

$$I_{NY} \ = \ \partial_\mu[\epsilon^{\mu\nu\alpha\beta} e^I_\nu D_\alpha(\omega) e_{I\beta}] \quad (2.6)$$

Note that in (2.5), the first term is quadratic in torsion and the second term is nothing but the Holst term itself. The Nieh-Yan density vanishes identically for a torsion free connection. In the Euclidean theory, this topological density, properly normalized, characterizes the *winding numbers* given by three integers associated with the homotopy groups $\Pi_3(SO(5)) = Z$ and $\Pi_3(SO(4)) = Z \ + \ Z$.

Since the Nieh-Yan density is a total derivative, its addition to the Hilbert-Palatini Lagrangian does not change the classical equations of motion. In other words, the Lagrangian density containing both the Hilbert-Palatini and Nieh-Yan terms, given by

$$\mathcal{L} \ = \ \frac{1}{2} e\Sigma^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2} I_{NY}, \quad (2.7)$$

leads to the same equations of motion as coming from the Hilbert-Palatini Lagrangian. This is to be contrasted with the case involving Holst term or its modifications used earlier.

We shall demonstrate that the canonical Hamiltonian formulation based on this new Lagrangian density (2.7) also leads to a theory of real SU(2) connections, exactly the same as that emerging from the theory with original Holst term. This in turn, for $\eta = 1$, is the Barbero formulation. Inclusion of matter now does not need any further modification and hence, in this sense, the Nieh-Yan term has a universal character. This also allows a direct interpretation η as a topological parameter, leading to a closer and more complete analogy with the θ-parameter in QCD. In this sense, the subsequent analysis here settles the long-standing issue of the exact origin of the Barbero-Immirzi parameter in the theory of gravity.

2.2 Hamiltonian Analysis

We propose the Lagrangian density for pure gravity to be that given in equation(2.7), rewritten as:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}e\Sigma^{\mu\nu}_{IJ}R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}\left[e\Sigma^{\mu\nu}_{IJ}\tilde{R}_{\mu\nu}{}^{IJ}(\omega) + \epsilon^{\mu\nu\alpha\beta}D_\mu(\omega)e^I_\nu\ D_\alpha(\omega)e_{I\beta}\right] \\ &= \frac{1}{2}e\Sigma^{\mu\nu}_{IJ}R^{(\eta)IJ}_{\mu\nu}(\omega) + \frac{\eta}{2}\ \epsilon^{\mu\nu\alpha\beta}D_\mu(\omega)e^I_\nu D_\alpha(\omega)e_{I\beta} \quad ,\end{aligned} \tag{2.8}$$

where $R^{(\eta)IJ}_{\mu\nu}(\omega)\ :=\ R_{\mu\nu}{}^{IJ}(\omega) + \eta\tilde{R}_{\mu\nu}{}^{IJ}(\omega)$ and we have used the identities,

$$\Sigma^{\mu\nu}_{IJ}\tilde{R}_{\mu\nu}{}^{IJ}(\omega)\ =\ \tilde{\Sigma}^{\mu\nu}_{IJ}R_{\mu\nu}{}^{IJ}(\omega) \quad , \quad e\tilde{\Sigma}^{\mu\nu}_{IJ}\ :=\ \frac{e}{2}\epsilon_{IJKL}\Sigma^{\mu\nu KL}\ =\ -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\Sigma_{\alpha\beta IJ}\ . \tag{2.9}$$

Introducing the notation, $t^a_I := \eta\epsilon^{abc}D_b(\omega)e_{Ic}$, $\epsilon^{abc} := \epsilon^{tabc}$, $\epsilon_{0ijk} := \epsilon_{ijk}$, $\epsilon^{0ijk} := -\epsilon^{ijk}$, the 3+1 decomposition is expressed as:

$$\mathcal{L}\ =\ e\Sigma^{ta}_{IJ}R^{(\eta)IJ}_{ta}(\omega) + \frac{e}{2}\Sigma^{ab}_{IJ}R^{(\eta)IJ}_{ab}(\omega) + t^a_I\left(D_t(\omega)e^I_a - D_a(\omega)e^I_t\right) \tag{2.10}$$

Defining $\omega^{(\eta)IJ}_a := \omega^{IJ}_a + \eta\tilde{\omega}^{IJ}_a$ and $\Sigma^{(\eta)ta}_{IJ} := \Sigma^{ta}_{IJ} + \eta\tilde{\Sigma}^{ta}_{IJ}$ we get,

$$\begin{aligned}\mathcal{L} &= e\Sigma^{ta}_{IJ}\partial_t\omega^{(\eta)IJ}_a + \omega^{IJ}_t D_a(\omega)\left(e\Sigma^{(\eta)ta}_{IJ}\right) + \frac{e}{2}\Sigma^{ab}_{IJ}R^{(\eta)IJ}_{ab}(\omega) \\ &\quad + t^a_I\partial_t e^I_a + \omega_t{}^{IJ}t^a_I e_{aJ} + e^I_t D_a(\omega)t^a_I - \partial_a\left(t^a_I e^I_t + e\Sigma^{(\eta)ta}_{IJ}\omega^{IJ}_t\right)\end{aligned} \tag{2.11}$$

We parametrize the tetrad fields as:

$$e^I_t\ =\ \sqrt{eN}M^I + N^aV^I_a\ ,\ e^I_a\ =\ V^I_a \quad ; \quad M_IV^I_a\ =\ 0\ ,\ M_IM^I\ =\ -1 \tag{2.12}$$

and then the inverse tetrad fields are:

$$\begin{aligned} e^t_I\ =\ -\frac{M_I}{\sqrt{eN}}\ ,\ e^a_I\ =\ V^a_I + \frac{N^aM_I}{\sqrt{eN}}\ ; \\ M^IV^a_I\ :=\ 0\ ,\ V^I_aV^b_I\ :=\ \delta^b_a\ ,\ V^I_aV^a_J\ :=\ \delta^I_J + M^IM_J\end{aligned} \tag{2.13}$$

Defining $q_{ab}\ :=\ V^I_aV_{bI}$ and $q := \det q_{ab}$ leads to $e := det(e^I_\mu) = Nq$. We may thus trade the 16 tetrad fields with the 9 fields V^a_I ($M^IV^a_I = 0$), the 3 fields M^I ($M_IM^I = -1$) and the 4 fields N and N^a.

Next using the identity,

$$\Sigma^{ab}_{IJ} \;=\; 2Ne\Sigma^{t[a}_{IK}\Sigma^{b]t}_{JL}\eta^{KL} + N^{[a}\Sigma^{b]t}_{IJ}\;, \tag{2.14}$$

and dropping the total space derivative terms,

$$\mathcal{L} = e\Sigma^{ta}_{IJ}\partial_t\omega^{(\eta)IJ}_a + t^a_I\partial_t e^I_a - NH - N^aH_a - \tfrac{1}{2}\omega^{IJ}_t G_{IJ} \tag{2.15}$$

where $2e\Sigma^{ta}_{IJ} = -\sqrt{q}M_{[I}V^a_{J]}$, $t^a_I := \eta\epsilon^{abc}D_b(\omega)V_{Ic}$ and

$$\begin{aligned} H &= 2e^2\Sigma^{ta}_{IK}\Sigma^{tb}_{JL}\eta^{KL}R^{(\eta)IJ}_{ab}(\omega) - \sqrt{q}M^I D_a(\omega)t^a_I, && (2.16)\\ H_a &= e\Sigma^{tb}_{IJ}R^{(\eta)IJ}_{ab}(\omega) - V^I_a D_b(\omega)t^b_I, && (2.17)\\ G_{IJ} &= -2D_a(\omega)\left(e\Sigma^{(\eta)ta}_{IJ}\right) - t^a_{[I}V_{J]}a\;. && (2.18)\end{aligned}$$

Introduce the fields,

$$E^a_i := 2e\Sigma^{ta}_{0i}\;,\;\; \chi_i := -M_i/M^0\;,\;\; A^i_a \;:=\; \omega^{(\eta)0i}_a - \chi_j\omega^{(\eta)ij}_a\;,\;\; \zeta^i \;:=\; -\,E^a_j\omega^{(\eta)ij}_a. \tag{2.19}$$

In terms of these, we have $2e\Sigma^{ta}_{ij} = -E^a_{[i}\chi_{j]}$ and $e\Sigma^{ta}_{IJ}\partial_t\omega^{(\eta)IJ}_a = E^a_i\partial_t A^i_a + \zeta^i\partial_t\chi^i$, and the Lagrangian density is:

$$\mathcal{L} \;=\; E^a_i\partial_t A^i_a + \zeta^i\partial_t\chi^i + t^a_I\partial_t V^I_a - NH - N^aH_a - \tfrac{1}{2}\omega^{IJ}_t G_{IJ}, \tag{2.20}$$

where, now we need to re-express H, H_a and G_{IJ} ($G^i_{\text{boost}} := G_{0i}\,,\; G^i_{\text{rot}} := \frac{1}{2}\epsilon^{ijk}G_{jk}$) in terms of these new fields:

$$\begin{aligned} G^i_{\text{boost}} &= -\partial_a\left(E^a_i - \eta\epsilon^{ijk}E^a_j\chi_k\right) + E^a_{[i}\chi_{k]}A^k_a + (\zeta^i - \chi\cdot\zeta\chi^i) - t^a_{[0}V_{i]a}\;, && (2.21)\\ G^i_{\text{rot}} &= \partial_a\left(\epsilon^{ijk}E^a_j\chi_k + \eta E^a_i\right) + \epsilon^{ijk}\left(A^j_aE^a_k - \zeta_j\chi_k - t^a_jV^k_a\right) && (2.22)\\ H_a &= E^b_i\left[R^{(\eta)0i}_{ab}(\omega) - \chi_j R^{(\eta)ij}_{ab}(\omega)\right] - V^I_a D_b(\omega)t^b_I && (2.23)\\ &= E^b_i\partial_{[a}A^i_{b]} + \zeta^i\partial_a\chi^i - V^I_a\partial_b t^b_I + t^b_I\partial_{[a}V^I_{b]}\\ &\quad -\frac{1}{1+\eta^2}\left[E^b_{[i}\chi_{l]}A^l_b + (\zeta_i - \chi\cdot\zeta\chi_i) - t^b_{[0}V_{i]b} - \eta\epsilon^{ijk}(A^j_bE^b_k - \zeta_j\chi_k - t^b_jV^k_b)\right]A^i_a\\ &\quad -\frac{1}{1+\eta^2}\left[\frac{1}{2}\epsilon^{ijk}(\eta G^k_{\text{boost}} + G^k_{\text{rot}}) - \chi^i(G^j_{\text{boost}} - \eta G^j_{\text{rot}})\right]\omega^{(\eta)ij}_a && (2.24)\\ H &= -E^a_k\chi_k H_a - \frac{1}{2}(1-\chi\cdot\chi)E^a_iE^b_jR^{(\eta)ij}_{ab}(\omega) - \left(E^a_k\chi_kV^I_a + \sqrt{q}M^I\right)D_b(\omega)t^b_I\\ &= -E^a_k\chi_k H_a + (1-\chi\cdot\chi)\left[E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_iE^a_iE^b_j\partial_aE^j_b\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\chi\cdot\chi}{2(1+\eta^2)}\zeta_i\left[-G^i_{\text{boost}}+\eta G^i_{\text{rot}}\right]-\left(E^a_k\chi_k V^I_a+\sqrt{q}M^I\right)\partial_b t^b_I \\
&-\frac{1-\chi\cdot\chi}{1+\eta^2}\left[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_aA^j_b+E^a_iA^i_a\chi\cdot\zeta+\eta\epsilon^{ijk}\zeta_iA^j_aE^a_k\right. \\
&\qquad\qquad\left.+\frac{3}{4}(\chi\cdot\zeta)^2-\frac{3}{4}(\zeta\cdot\zeta)+\frac{1}{2}\zeta_it^a_{[0}V_{i]a}-\frac{\eta}{2}\zeta_i\epsilon^{ijk}t^a_jV^k_a\right] \\
&+\frac{1-\chi\cdot\chi}{1+\eta^2}\left[\frac{1}{\sqrt{E}}A^i_at^a_i+\frac{1}{2}V^i_a\left(\zeta\cdot\chi t^a_i-\chi_i\zeta_jt^a_j+\eta\epsilon^{ijk}\zeta_jt^a_k\right)\right] \\
&+\frac{1-\chi\cdot\chi}{1+\eta^2}\left[-\frac{1}{\sqrt{E}}\chi_it^b_j+\frac{\eta}{2\sqrt{E}}\epsilon^{ijk}t^b_k+(1+\eta^2)E^a_i\partial_aE^b_j+E^a_i\chi_jE^b_mA^m_a\right. \\
&\qquad\qquad\left.-\eta\epsilon^{imn}E^a_mE^b_nA^j_a-\frac{\eta}{4}\left(\epsilon^{ijm}E^b_n+\epsilon^{ijn}E^b_m\right)\chi_m\zeta_n\right]u^{ij}_b \\
&+\frac{1-\chi\cdot\chi}{2(1+\eta^2)}\left(\chi_m\chi_n-\delta_{mn}\right)E^a_jE^b_iu^{im}_au^{jn}_b
\end{aligned}
\tag{2.25}
$$

In the above, $E^i_a := \sqrt{E}V^i_a$ is the inverse of E^a_i i.e. $E^i_aE^b_i = \delta^b_a$, $E^i_aE^a_j = \delta^i_j$ and $E^{-1} = q(M^0)^2$ equals $\det E^a_i$. Furthermore we have also set $u^{ij}_a := \omega^{(\eta)ij}_a - \frac{1}{2}E^{[i}_a\zeta^{j]}$. Notice that $E^b_iu^{ij}_b = 0$. The six independent fields in u^{ij}_a may be parametrized in terms a symmetric matrix M^{ij} as, $u^{ij}_a := \frac{1}{2}\epsilon^{ijk}E^l_aM^{kl}$ [14].

We have replaced the original 16 tetrad fields with 16 new fields: E^a_i, χ_i, N and N^a. In place of the original 24 connection fields ω^{IJ}_μ we use the new set of 24 fields $A^i_a, \zeta_i, M^{kl}, \omega^{ij}_t$ and ω^{0i}_t. The fields V^I_a and t^a_I are not independent; these are given in terms of the fundamental fields as: $V^I_a = \upsilon^I_a$ and $t^a_I = \tau^a_I$ where

$$
\upsilon^0_a := -\frac{1}{\sqrt{E}}E^i_a\chi_i \quad , \quad \upsilon^i_a := \frac{1}{\sqrt{E}}E^i_a \tag{2.26}
$$

$$
\tau^a_0 := \eta\epsilon^{abc}D_b(\omega)V_{0c} = \eta\sqrt{E}E^a_m\left[G^m_{\text{rot}}-\frac{\chi_l}{2}\left(\frac{2f_{ml}+N_{ml}}{1+\eta^2}+\epsilon_{mln}G^n_{\text{boost}}\right)\right] \tag{2.27}
$$

$$
\tau^a_k := \eta\epsilon^{abc}D_b(\omega)V_{ck} = -\frac{\eta}{2}\sqrt{E}E^a_m\left[\frac{2f_{mk}+N_{mk}}{1+\eta^2}+\epsilon_{kmn}G^n_{\text{boost}}\right] \tag{2.28}
$$

where,

$$
2f_{kl} := \epsilon_{ijk}E^a_i\left[(1+\eta^2)E^l_b\partial_aE^b_j+\chi_jA^l_a\right]+\eta\left(E^a_lA^k_a-\delta^{kl}E^a_mA^m_a-\chi_l\zeta_k\right)+(l\leftrightarrow k) \tag{2.29}
$$

$$
\begin{aligned}
N_{kl} &:= \epsilon^{ijk}(\chi_m\chi_j-\delta_{mj})E^a_iu^{lm}_a+(l\leftrightarrow k) \\
&= (\chi\cdot\chi-1)(M_{kl}-M_{mm}\delta_{kl})+\chi_m\chi_nM_{mn}\delta_{kl}+\chi_l\chi_kM_{mm}-\chi_m(\chi_kM_{ml}+\chi_lM_{mk})
\end{aligned}
\tag{2.30}
$$

We can upgrade V^I_a and t^a_I as independent fields through terms containing the Lagrange multiplier fields ξ^a_I and ϕ^I_a in the Lagrangian density:

$$\begin{aligned}\mathcal{L} &= E^a_i \partial_t A^i_a + \zeta^i \partial_t \chi^i + t^a_I \partial_t V^I_a - \mathcal{H} \\ \mathcal{H} &:= NH + N^a H_a + \frac{1}{2}\omega^{IJ}_t G_{IJ} + \xi^a_I (V^I_a - \upsilon^I_a) + \phi^I_a (t^a_I - \tau^a_I)\end{aligned} \quad (2.31)$$

where υ^I_a and τ^a_I are defined in equations (4.10 - 2.28). We have 24 pairs of canonically conjugate independent field variables $(E^a_i, A^j_b), (\zeta^i, \chi^j), (t^a_I, V^I_a)$. The remaining fields, namely, $N, N^a, \omega^{IJ}_t, \xi^a_I, \phi^I_a$ and M^{kl} have no conjugate momenta since in the Lagrangian their velocities do not appear. Preservation of these constraints (vanishing of the variation of the Hamiltonian with respect to the fields) leads to the secondary constraints. From variations with respect to the fields $\omega^{0i}_t, \omega^{ij}_t, N^a, N, \xi^a_I$ and ϕ^I_a, we get the constraints:

$$G^i_{\text{boost}} \approx 0 \;,\; G^i_{\text{rot}} \approx 0 \quad ; \quad H_a \approx 0 \;,\; H \approx 0\;; \quad (2.32)$$

$$V^I_a - \upsilon^I_a \approx 0 \quad , \quad t^a_I - \tau^a_I \approx 0. \quad (2.33)$$

From the variation with respect to M^{kl} or equivalently u^{ij}_a, we get:

$$\frac{\delta \mathcal{H}}{\delta M^{kl}}\delta M^{kl} \approx \frac{\delta H}{\delta M^{kl}}\delta M^{kl} = \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}[(\eta t^a_k - \epsilon^{ijk}\chi_i t^a_j)V^l_a + f_{kl} + \frac{1}{2}N_{kl}]\delta M^{kl} \approx 0.$$

This leads to

$$(\eta t^a_k - \epsilon^{ijk}\chi_i t^a_j)V^l_a + f_{kl} + \frac{1}{2}N_{kl} + (k \leftrightarrow l) \approx 0 \quad (2.34)$$

Using constraints (2.33), and the expressions (25-2.28) for τ^a_I, υ^I_a, equation (2.34) implies :

$$\left(\eta\epsilon^{ijk}\chi_i + \delta_{kj}\right)(2f_{jl} + N_{jl}) + \eta(1+\eta^2)\left(\delta^{kl}\chi_m G^m_{\text{boost}} - \chi_l G^k_{\text{boost}}\right) + (k \leftrightarrow l) \approx 0$$

Using (2.32), this in turn implies the constraint:

$$2f_{kl} + N_{kl} \approx 0 \quad (2.35)$$

where, f_{kl} and N_{kl} are given in (28,29). This constraint can be solved for M_{kl}. Furthermore it implies, from the definitions (2.27, 2.28), that $\tau^a_I \approx 0$ and hence,

$$t^a_I \approx 0\;. \quad (2.36)$$

Implementing this constraint then reduces the Hamiltonian density to

$$\mathcal{H} = NH + N^a H_a + \frac{1}{2}\omega^{IJ}_t G_{IJ} \quad (2.37)$$

where now,

$$
\begin{aligned}
G^i_{\rm boost} &= -\partial_a\left(E^a_i - \eta\epsilon^{ijk}E^a_j\chi_k\right) + E^a_{[i}\chi_{k]}A^k_a + (\zeta^i - \chi\cdot\zeta\zeta^i) \;\approx\; 0\;, \\
G^i_{\rm rot} &= \partial_a\left(\epsilon^{ijk}E^a_j\chi_k + \eta E^a_i\right) + \epsilon^{ijk}(A^j_a E^a_k - \zeta_j\chi_k) \;\approx\; 0\;, \\
H_a &= E^b_i\partial_{[a}A^i_{b]} + \zeta_i\partial_a\chi_i \\
&\quad -\frac{1}{1+\eta^2}\left[E^b_{[k}\chi_{l]}A^l_b + \zeta_i - \chi\cdot\zeta\chi^i - \eta\epsilon^{ijk}(A^j_a E^a_k - \zeta_j\chi_k)\right]A^i_a \\
&\quad -\frac{1}{1+\eta^2}\left[\frac{1}{2}\epsilon^{ijk}(\eta G^k_{\rm boost} + G^k_{\rm rot}) - \chi^i(G^j_{\rm boost} - \eta G^j_{\rm rot})\right]\omega^{(\eta)ij}_a \;\approx\; 0\;, \\
H &= -E^a_k\chi_k H_a + (1-\chi\cdot\chi)\left[E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_i E^a_i E^b_j\partial_a E^j_b\right] \\
&\quad +\frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}\zeta_i[-G^j_{\rm boost} + \eta G^j_{\rm rot}] \\
&\quad -\frac{(1-\chi\cdot\chi)}{1+\eta^2}\left[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_a A^j_b + E^a_i A^i_a\chi\cdot\zeta + \eta\epsilon_{ijk}\zeta_i A^j_a E^a_k + \frac{3}{4}(\chi\cdot\zeta)^2 - \frac{3}{4}(\zeta\cdot\zeta)\right] \\
&\quad +\frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}\left[f_{kl}M^{kl} + \frac{1}{4}(\chi\cdot\chi - 1)(M^{kl}M^{kl} - M^{kk}M^{ll})\right. \\
&\qquad\qquad \left. +\frac{1}{2}\chi_k\chi_l(M^{pp}M^{kl} - M^{kp}M^{lp})\right] \;\approx\; 0
\end{aligned}
$$

In the last equation we have M^{kl} given by the constraint $2f_{kl} + N_{kl} = 0$, which can be solved as:

$$
(1-\chi\cdot\chi)M_{kl} = 2f_{kl} + (\chi_m\chi_n f_{mn} - f_{mm})\delta_{lk} + (\chi_m\chi_n f_{mn} + f_{mm})\chi_k\chi_l - 2\chi_m(\chi_l f_{mk} + \chi_k f_{ml}) \tag{2.38}
$$

This is the same set of equations as those obtained by Sa [14] in his analysis of the action containing Holst term.

2.3 Time gauge

We may fix the boost gauge transformations (time gauge) by imposing $\chi^i \approx 0$ which together with the $G^i_{\rm boost} \approx 0$ forms a second class pair. Solving the boost constraint with $\chi^i = 0$ yields,

$$
\zeta_i \;=\; \partial_a E^a_i \tag{2.39}
$$

The constraints become-

$$
G^i_{\rm rot} \;=\; \eta\partial_a E^a_i + \epsilon^{ijk}A^j_a E^a_k \;\approx\; 0\;,
$$

$$
\begin{aligned}
H_a &= E^b_i \partial_{[a} A^i_{b]} - \frac{1}{1+\eta^2}\left[\zeta_i - \eta\epsilon^{ijk} A^j_b E^b_k\right] A^i_a - \frac{1}{2(1+\eta^2)}\epsilon^{ijk}\omega^{(\eta)ij}_a G^k_{\text{rot}} \approx 0 , \\
H &= E^a_i \partial_a \zeta_i + \frac{1}{2}\zeta_i E^a_i E^b_j \partial_a E^j_b - \frac{1}{1+\eta^2}\left[\frac{1}{2} E^a_{[i} E^b_{j]} A^i_a A^j_b + \eta\epsilon_{ijk}\zeta_i A^j_a E^a_k - \frac{3}{4}(\zeta\cdot\zeta)\right] \\
&\quad + \frac{1}{2(1+\eta^2)}\left[f_{kl} M^{kl} - \frac{1}{4}(M^{kl}M^{kl} - M^{kk}M^{ll})\right] + \frac{\eta}{2(1+\eta^2)}\zeta_i G^j_{\text{rot}} \approx 0
\end{aligned}
\tag{2.40}
$$

Note that the rotation constraint is exactly of the form of Gauss' law as in non-abelian gauge theories. These act as the generators of the SU(2) symmetry transformations. Thus, in time gauge we recover a canonical Hamiltonian formulation in terms of real SU(2) gauge fields A^i_a which reduces to the Barbero formulation for $\eta = 1$ [14].

2.4 Summary

We have demonstrated that the inclusion of Nieh-Yan topological density in the Lagrangian density of a theory of gravity allows us, in the time gauge, to describe gravity in terms of a real $SU(2)$ connection. The set of constraints so obtained in the Hamiltonian formulation, for $\eta = 1$, is the same as that in the Barbero formulation. For other real values of this parameter, we have the Immirzi formulation with Barbero-Immirzi parameter $\gamma = \eta^{-1}$.

Our analysis shows that the Barbero-Immirzi parameter η^{-1} has a topological origin, exactly like the θ-parameter in QCD. Such a topological interpretation for η does not exist in the Holst formulation. Like the famous theta vacua of QCD, whether the η-parameter also does lead to a rich vacuum structure in quantum gravity is an intriguing issue in itself. Questions regarding the possible non-trivial import(s) of the η-parameter in the (non-perturbative) quantum theory of gravity need a detailed study.

Chapter 3

Gravity with Spin-$\frac{1}{2}$ fermions

Here we consider the spin-$\frac{1}{2}$ Dirac fermion with its usual *minimal* coupling to gravity and demonstrate how the general prescription as discussed in chapter-2 applies to such a case and leads to a SU(2) interpretation for this theory.

3.1 Fermionic Lagrangian

The Lagrangian density corresponding to massless Dirac fermions coupled to gravity is[1],

$$\mathcal{L} \;=\; \frac{1}{2} e\ \Sigma^{\mu\nu}_{IJ}\ R_{\mu\nu}{}^{IJ}(\omega) \;+\; \frac{ie}{2}\ [\bar{\lambda}\gamma^\mu D_\mu(\omega)\lambda \;-\; \overline{D_\mu(\omega)\lambda}\gamma^\mu\lambda] \tag{3.1}$$

where,

$$D_\mu(\omega)\lambda \;:=\; \partial_\mu\lambda \;+\; \frac{1}{2}\ \omega_{\mu IJ}\ \sigma^{IJ}\ \lambda \;\;,\;\; \overline{D_\mu(\omega)\lambda} \;:=\; \partial_\mu\bar{\lambda} \;-\; \frac{1}{2}\ \bar{\lambda}\ \omega_{\mu IJ}\ \sigma^{IJ}$$

and the field λ represents a spin-$\frac{1}{2}$ majorana fermion. To this Lagrangian, we add the Nieh-Yan density-

$$\mathcal{L} \;=\; \frac{1}{2} e\ \Sigma^{\mu\nu}_{IJ}\ R_{\mu\nu}{}^{IJ}(\omega) \;+\; \frac{\eta}{2}\ I_{NY} \;+\; \frac{ie}{2}\ [\bar{\lambda}\gamma^\mu D_\mu(\omega)\lambda \;-\; \overline{D_\mu(\omega)\lambda}\gamma^\mu\lambda] \tag{3.2}$$

Notice that, unlike earlier attempts of setting up a theory of fermions and gravity with Barbero-Immirzi parameter [16, 17] where the Holst term was modified to include an additional non-minimal term for the fermions, the Lagrangian density here containing the Nieh-Yan density does not require any further modification, just the usual minimal fermion terms suffice. This is so because the Nieh-Yan term is universal and serves the purpose for any matter coupling without any need for modifications.

[1] Our Dirac matrices satisfy the Clifford algebra: $\gamma^I\gamma^J + \gamma^J\gamma^I = 2\eta^{IJ}$, $\eta^{IJ} := \mathrm{diag}(-1, 1, 1, 1)$. The chiral matrix $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{IJ} \;:=\; \frac{1}{4}\ [\gamma^I, \gamma^J]$.

3.2 Canonical analysis

We expand the fermion terms as

$$\begin{aligned}\mathcal{L}(F) &:= \frac{ie}{2}\,[\bar{\lambda}\gamma^{\mu}D_{\mu}(\omega)\lambda \;-\; \overline{D_{\mu}(\omega)\lambda}\gamma^{\mu}\lambda] \\ &= \left[\partial_t\bar{\lambda}\Pi \;-\; \bar{\Pi}\partial_t\lambda\right] \;-\; NH(F) \;-\; N^a H_a(F) \;-\; \frac{1}{2}\,\omega_t^{IJ}\,G_{IJ}(F) \qquad (3.3)\end{aligned}$$

where $\bar{\Pi}$ and Π are canonically conjugate momenta fields associated with λ and $\bar{\lambda}$ respectively. Explicitly [2],

$$\begin{aligned}\bar{\Pi} &= -\frac{ie}{2}\bar{\lambda}\gamma^t \;=\; \frac{i\sqrt{q}}{2}M_I\bar{\lambda}\gamma^I \;\;,\;\; \Pi \;=\; -\frac{ie}{2}\gamma^t\lambda \;=\; \frac{i\sqrt{q}}{2}M_I\gamma^I\lambda \qquad (3.4)\\ G^{IJ}(F) &= \bar{\Pi}\sigma^{IJ}\lambda \;+\; \bar{\lambda}\sigma^{IJ}\Pi \qquad (3.5)\\ H_a(F) &= \overline{D_a(\omega)\lambda}\,\Pi \;-\; \bar{\Pi}D_a(\omega)\lambda \qquad (3.6)\\ H(F) &= (-2\;e\;\Sigma^{ta}_{IJ})\;\left[\overline{D_a(\omega)\lambda}\;\sigma^{IJ}\;\Pi \;+\; \bar{\Pi}\;\sigma^{IJ}\;D_a(\omega)\lambda\right] \qquad (3.7)\end{aligned}$$

Incorporating these fermionic terms in the pure gravity Lagrangian density given in equation (2.31), we write the full Lagrangian density as,

$$\begin{aligned}\mathcal{L} \;=\;& E^a_i\;\partial_t A^i_a + \zeta^i\;\partial_t\chi_i + t^a_I\partial_t V^I_a + \partial_t\bar{\lambda}\;\Pi - \bar{\Pi}\;\partial_t\lambda - NH' - N^a H'_a - \frac{1}{2}\omega_t^{IJ}\;G'_{IJ} \\ &-\xi^a_I(V^I_a - \upsilon^I_a) - \phi^I_a(t^a_I - \tau^a_I) \qquad (3.8)\end{aligned}$$

where now

$$G'_{IJ} \;=\; G^{IJ} \;+\; G^{IJ}(F) \;\;,\;\; H'_a \;=\; H_a \;+\; H_a(F) \;\;,\;\; H' \;=\; H \;+\; H(F)\;, \qquad (3.9)$$

with G^{IJ}, H_a and H as the contributions from the pure gravity sector as given by the equations (2.16 – 2.18) or equivalently by the equations (2.22 – 2.25).

The various quantities above can then be rewritten in terms of the basic fields as:

$$\begin{aligned}G'^i_{\rm boost} &= -\partial_a(E^a_i \;-\; \eta\epsilon^{ijk}E^a_j\chi_k) \;+\; E^a_{[i}\chi_{k]}A^k_a \;+\; (\zeta^i \;-\; \chi\cdot\zeta\;\chi^i) - t'^a_{[0}V_{i]a} \\ &\quad + \left[\bar{\Pi}(1+i\eta\gamma_5)\sigma_{0i}\lambda \;+\; \bar{\lambda}(1+i\eta\gamma_5)\sigma_{0i}\Pi\right]; \qquad (3.10)\\ G'^i_{\rm rot} &= \partial_a(\epsilon^{ijk}E^a_j\chi_k \;+\; \eta E^a_i) \;+\; \epsilon^{ijk}(A^j_a E^a_k \;-\; \zeta_j\chi_k \;-\; t'^a_j V^k_a) \\ &\quad + \left[\bar{\Pi}(i\gamma_5-\eta)\sigma_{0i}\lambda \;+\; \bar{\lambda}(i\gamma_5-\eta)\sigma_{0i}\Pi\right]; \qquad (3.11)\end{aligned}$$

[2]The fermions are Grassmann valued and the functional differentiation is done on the left factor which accounts for the signs in the definitions of the conjugate momenta in (3.4).

$$
\begin{aligned}
H'_a \;=\;& E^b_i\partial_{[a}A^i_{b]} + \zeta_i\partial_a\chi_i \;-\; \partial_b(t'^b_I V^I_a) \;+\; t'^b_I\partial_a V^I_b \\
&+ \left[\partial_a\bar\lambda(1 \;+\; i\eta\gamma_5)\Pi \;-\; \bar\Pi(1 \;+\; i\eta\gamma_5)\partial_a\lambda\right] \;-\left[\bar\lambda\sigma_{0i}\Pi + \bar\Pi\sigma_{0i}\lambda\right] A^i_a \\
&- \frac{1}{1+\eta^2}\left[E^b_{[i}\chi_{l]}A^l_b + \zeta_i - \chi\cdot\zeta\chi^i - t'^b_{[0}V_{i]b} - \eta\epsilon^{ijk}\;(A^j_aE^b_k - \chi_j\zeta_k - t'^b_jV^k_b)\right] A^i_a \\
&- \frac{1}{1+\eta^2}\left[\frac{1}{2}\epsilon^{ijk}\left(\eta G'^k_{\text{boost}} + G'^k_{\text{rot}}\right) - \chi^i(G'^j_{\text{boost}} - \eta G'^j_{\text{rot}})\right]\omega^{(\eta)ij}_a \qquad (3.12)\\
H' \;=\;& -E^a_k\chi_k H'_a - (E^a_k\chi_kV^I_a + \sqrt{q}M^I)\;\partial_b t'^b_I + (1-\chi\cdot\chi)\left[E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_iE^a_iE^b_j\partial_aE^j_b\right] \\
&- \frac{(1-\chi\cdot\chi)}{1+\eta^2}\left[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_aA^j_b + E^a_iA^i_a\chi\cdot\zeta + \eta\epsilon_{ijk}\zeta_iA^j_aE^a_k + \frac{3}{4}(\chi\cdot\zeta)^2 - \frac{3}{4}(\zeta\cdot\zeta)\right] \\
&+ \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}\left[\zeta_i - 2V^i_b(t^b_0 - t'^b_0)\right]\left[-G'^i_{\text{boost}} + \eta G'^i_{\text{rot}} - t'^a_{[0}V_{i]a} + \eta\epsilon_{ijk}t'^a_jV^k_a\right] \\
&+ \frac{(1-\chi\cdot\chi)}{\sqrt{E}(1+\eta^2)}\left[t'^b_mA^m_b + \frac{1}{2}E^i_bt'^b_{[i}\chi_{j]}\zeta_j + \frac{\eta}{2}\epsilon_{ijk}t'^b_iE^j_b\zeta_k\right] \\
&- 2e\;\Sigma^{ta}_{IJ}\left[\partial_a\bar\lambda(1+i\eta\gamma_5)\sigma^{IJ}\Pi + \bar\Pi(1+i\eta\gamma_5)\sigma^{IJ}\partial_a\lambda\right] \\
&+ E^a_k\chi_k\left[\partial_a\bar\lambda(1+i\eta\gamma_5)\Pi - \bar\Pi(1+i\eta\gamma_5)\partial_a\lambda\right] \\
&- 2e\;\Sigma^{ta}_{IJ}\left[-\bar\lambda\sigma_{0l}\;\sigma^{IJ}\Pi + \bar\Pi\sigma^{IJ}\;\sigma_{0l}\lambda\right]A^l_a - E^a_k\chi_k\;[\bar\Pi\sigma_{0l}\lambda + \bar\lambda\sigma_{0l}\Pi]A^l_a \\
&+ \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}\left[\left(\eta t'^a_k - \epsilon^{ijk}\chi_it'^a_j\right)V^a_l + f_{kl} + (1+\eta^2)J_{kl} + \frac{1}{4}N_{kl}(M)\right]M^{kl} \qquad (3.13)
\end{aligned}
$$

where as earlier, $2e\;\Sigma^{ta}_{0i} \;=\; E^a_i$, $2e\;\Sigma^{ta}_{ij} \;=\; -\,E^a_{[i}\chi_{j]}$ and f_{kl} , $N_{kl}(M)$ are given by equations (4.11, 4.12) respectively. Also,

$$
\begin{aligned}
t'^a_I \;:=\;& t^a_I - \eta e\;\Sigma^{ta}_{IJ}\bar\lambda\gamma_5\gamma^J\lambda \qquad (3.14)\\
=\;& t^a_I + \frac{i\eta}{\sqrt{q}}\;e\;\Sigma^{ta}_{IJ}\left[M^J(\bar\Pi\gamma_5\lambda - \bar\lambda\gamma_5\Pi) + 2M_L(\bar\Pi\gamma_5\sigma^{LJ}\lambda + \bar\lambda\gamma_5\sigma^{LJ}\Pi)\right] \\
2J_{kl} \;:=\;& \frac{1}{2\sqrt{E}}\;\bar\lambda\gamma_5\left(\chi_k\gamma_l + \chi_l\gamma_k + 2\delta_{kl}\frac{M^I\gamma_I}{M^0}\right)\lambda \qquad (3.15)\\
=\;& \frac{i}{2}(\delta_{kl} + M_kM_l)(\bar\Pi\gamma_5\lambda - \bar\lambda\gamma_5\Pi) + iM_lM^J(\bar\Pi\gamma_5\sigma_{Jk}\lambda + \bar\lambda\gamma_5\sigma_{Jk}\Pi) + (k\leftrightarrow l)
\end{aligned}
$$

The Hamiltonian density now reads:

$$
\mathcal{H} \;=\; NH' \;+\; N^aH'_a \;+\; \frac{1}{2}\;\omega^{IJ}_t\;G'_{IJ} \;+\; \xi^a_I(V^I_a \;-\; \upsilon^I_a) + \phi^I_a(t^a_I \;-\; \tau^a_I) \qquad (3.16)
$$

The constraints associated with the fields $N^a, N, \omega^{0i}_t, \omega^{ij}_t, \xi^a_I$ and ϕ^I_a respectively are:

$$
H'_a \approx 0 \quad , \quad H' \approx 0 \quad , \quad G'^i_{\text{boost}} \approx 0 \quad , \quad G'^i_{\text{rot}} \approx 0 \qquad (3.17)
$$

$$V_a^I - \upsilon_a^I \approx 0 \quad , \quad t_I^a - \tau_I^a \approx 0. \tag{3.18}$$

The remaining fields M^{kl}, from $\frac{\delta H'}{\delta M^{kl}}\delta M^{kl} \approx 0$, lead to the constraint,

$$(\eta t_k'^a - \epsilon^{ijk}\chi_i t_j'^a)V_a^l + f_{kl} + \frac{1}{2}N_{kl} + (1+\eta^2)J_{kl} + (k \leftrightarrow l) \approx 0 \tag{3.19}$$

Using $t_I^a \approx \tau_I^a$, we write

$$\begin{aligned} t_k'^a &\approx -\frac{\eta}{2}\sqrt{E}\ E_l^a \left[\frac{2f_{kl} + N_{kl}}{1+\eta^2} + 2J_{kl} + \epsilon_{kln} G'^n_{\text{boost}} \right] \\ t_0'^a &\approx \eta\sqrt{E}\ E_l^a \left[G'^l_{\text{rot}} - \frac{\chi_k}{2} \left(\frac{2f_{kl} + N_{kl}}{1+\eta^2} + 2J_{kl} + \epsilon_{kln}\ G'^n_{\text{boost}} \right) \right] \end{aligned} \tag{3.20}$$

Using (3.20) in (3.19), leads to

$$2f_{kl} + N_{kl} + 2(1+\eta^2)J_{kl} \approx 0 \tag{3.21}$$

generalizing the constraint (2.35) of the pure gravity case. This in turn implies

$$t_I'^a \approx 0 \tag{3.22}$$

corresponding to the constraint (2.36) for pure gravity. Implementing this constraint along with those in (59) reduces the Hamiltonian density to

$$\mathcal{H} = NH' + N^a H_a' + \frac{1}{2}\omega_t^{IJ}\ G_{IJ}' \tag{3.23}$$

where the final set of constraints are obtained from equations (51-54) by substituting $t_I'^a = 0$ and dropping the terms containing $G'^i_{\text{boost}}, G'^i_{\text{rot}}$ in H_a', H'. The M_{kl} is given by the solution of the constraint (3.21).

3.3 Time gauge:

We may now make the gauge choice $\chi_i = 0$ and solve the boost constraint $G'^i_{\text{boost}} = 0$ to obtain

$$\zeta_i = \partial_a E_i^a - i\eta \left[\bar{\Pi}\gamma_5\sigma_{0i}\lambda + \bar{\lambda}\gamma_5\sigma_{0i}\Pi\right] \tag{3.24}$$

Thus we have a canonical Hamiltonian formulation for a theory of gravity with fermions in terms of real SU(2) gauge fields A_a^i with the following constraints:

$$G'^i_{\text{rot}} = \eta\ \partial_a E_i^a + \epsilon^{ijk} A_a^j E_k^a + i[\ \bar{\Pi}\gamma_5\sigma_{0i}\lambda + \bar{\lambda}\gamma_5\sigma_{0i}\Pi] \approx 0;$$

$$
\begin{aligned}
H'_a &= E^b_i \partial_{[a} A^i_{b]} + [\ \partial_a \bar{\lambda}(1 + i\eta\gamma_5)\Pi - \bar{\Pi}(1 + i\eta\gamma_5)\partial_a\lambda\ ] \\
&\quad - \frac{1}{1+\eta^2}\left[\partial_a E^a_i - \eta\epsilon_{ijk} A^j_b E^b_k - i\eta\left(\ \bar{\Pi}\gamma_5\sigma_{0i}\lambda + \bar{\lambda}\gamma_5\sigma_{0i}\Pi\ \right)\right] A^i_a \quad \approx\ 0; \\
H' &= [E^a_i \partial_a \zeta_i + \frac{1}{2}\zeta_i E^a_i E^b_j \partial_a E^j_b] - \frac{1}{1+\eta^2}\left[\frac{1}{2} E^a_{[i} E^b_{j]} A^i_a A^j_b + \eta\epsilon_{ijk}\zeta_i A^j_a E^a_k - \frac{3}{4}\zeta\cdot\zeta\right] \\
&\quad + 2E^a_i\left[\partial_a\bar{\lambda}(1 + i\eta\gamma_5)\sigma_{0i}\Pi + \bar{\Pi}(1 + i\eta\gamma_5)\sigma_{0i}\partial_a\lambda\right] + E^a_i\ \left[\bar{\lambda}\sigma_{il}\Pi + \bar{\Pi}\sigma^{il}\lambda\right] A^l_a \\
&\quad + \frac{1}{2(1+\eta^2)}\left[\left\{f_{kl} + (1+\eta^2)J_{kl}\right\} M^{kl} - \frac{1}{4}\left(M^{kl}M^{kl} - M^{kk}M^{ll}\right)\right] \approx\ 0 \qquad (3.25)
\end{aligned}
$$

where ζ^i are given by (4.23) and

$$
M^{kl}\ =\ 2\left[f_{kl} + (1+\eta^2)J_{kl}\right] - \delta_{kl}\left[f_{mm} + (1+\eta^2)J_{mm}\right] \qquad (3.26)
$$

with

$$
\begin{aligned}
2f_{kl} &= (1+\eta^2)\epsilon^{ijk} E^a_i E^l_b \partial_a E^b_j + \eta\left(E^a_k A^l_a - \delta^{kl} E^a_m A^m_a\right) + (k \leftrightarrow l) \\
2J_{kl} &= i\delta_{kl}\left[\bar{\Pi}\gamma_5\lambda - \bar{\lambda}\gamma_5\Pi\right] \qquad (3.27)
\end{aligned}
$$

This completes our discussion of a Dirac fermion minimally coupled to gravity including the Nieh-Yan term. We have showed that the Nieh-Yan topological term indeed allows a $SU(2)$ gauge theoretic description.

Chapter 4

Supergravity

In this chapter we analyse the case of spin-$\frac{3}{2}$ fermions coupled to gravity. We illustrate how the general proposal as given in chapter-2 can be applied to this case, and hence to supergravity theories in general.

The theory of supergravity is based on the assumption that there exists a fundamental symmetry between bosons and fermions, called supersymmetry. Through this symmetry, every bosonic particle can be transmuted to a fermionic counterpart, and vice versa. Application of this principle to gravity naturally implies that there should be a fermionic partner of graviton, called gravitino. From the historical perspective, the original idea behind supergravity was to yield a perturbatively finite theory of quantum gravity, as it was noticed that some of the divergences as appearing in Einstein quantum gravity could be cancelled through the introduction of supersymmetry. However, keeping aside such details as these are not necessary for our original purpose here, we focus on the canonical formulation of the theory of gravity coupled to spin-$\frac{3}{2}$ fermions and develop the subsequent analysis.

4.1 N=1 supergravity Lagrangian

The Lagrangian density for gravity coupled to spin-$\frac{3}{2}$ Majorana fermions is given by [20]:

$$\mathcal{L} \ = \ \frac{1}{2} e \Sigma^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega) \ + \ \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\alpha(\omega) \psi_\beta \tag{4.1}$$

where[1],

$$D_\mu(\omega)\psi_a \ := \ \partial_\mu \psi_a \ + \ \frac{1}{2} \ \omega_{\mu IJ} \sigma^{IJ} \psi_a \ \ , \ \ D_\mu(\omega)\bar{\psi}_a \ := \ \partial_\mu \bar{\psi}_a \ - \ \frac{1}{2} \ \bar{\psi}_a \ \omega_{\mu IJ} \sigma^{IJ}$$

[1]The Dirac matrices here obey the Clifford algebra: $\gamma^I\gamma^J + \gamma^J\gamma^I = 2\eta^{IJ}$, $\eta^{IJ} := \text{diag}(-1,1,1,1)$. The chiral matrix $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{IJ} \ := \ \frac{1}{4}\ [\gamma^I, \gamma^J]$.

Here the gravitino field ψ_μ is the fermionic companion of graviton. The action is invariant under local supersymmetry transformations which have the following action on these fields:

$$\begin{aligned}\delta\psi_\mu &= D_\mu\epsilon \;=\; \partial_\mu\epsilon \;+\; \frac{1}{2}\omega_{\mu IJ}\sigma^{IJ}\epsilon \\ \delta e^I_\mu &= \bar{\epsilon}\gamma^I\psi_\mu\end{aligned}$$

The canonical treatment of this theory has been considered earlier in several contexts[22, 21]. In ref.[22], the Hamiltonian analysis of the corresponding Holst action has been carried out in time gauge. Here we consider a Lagrangian density describing the same theory, but containing the Nieh-Yan invariant instead of the Holst term in addition to the usual Hilbert-Palatini and spin-$\frac{3}{2}$ fermionic terms. In the next section, we demonstrate that the set of constraints corresponding to this Lagrangian leads to a real $SU(2)$ description of this theory in terms of the Barbero-Immirzi connection. We also add a few comments on how to recover the correct transformation properties of the fields under the action of the symmetry generators.

4.2 Hamiltonian analysis

Following the general proposal made in chapter-1 for any arbitrary matter-coupling, we add the Nieh-Yan density to the supergravity Lagrangian in (4.1) to write:

$$\mathcal{L} \;=\; \frac{1}{2}e\Sigma^{\mu\nu}_{IJ}R_{\mu\nu}{}^{IJ}(\omega) \;+\; \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\alpha(\omega)\psi_\beta \;+\; \frac{\eta}{2}I_{NY} \tag{4.2}$$

This can be recast as:

$$\mathcal{L} \;=\; \frac{1}{2}e\Sigma^{\mu\nu}_{IJ}R^{(\eta)IJ}_{\mu\nu}(\omega) \;+\; \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\alpha(\omega)\psi_\beta \;+\; \frac{\eta}{2}\;\epsilon^{\mu\nu\alpha\beta}D_\mu(\omega)e^I_\nu D_\alpha(\omega)e_{I\beta} \tag{4.3}$$

where $R^{(\eta)IJ}_{\mu\nu}(\omega) \;:=\; R_{\mu\nu}{}^{IJ}(\omega) + \eta\tilde{R}_{\mu\nu}{}^{IJ}(\omega)$

The Nieh-Yan density serves as the term through which η manifests itself as a topological parameter in the supergravity action, and does not show up in the classical equations of motion. This new Lagrangian density also preserves the supersymmetry properties as characterised by (4.1) since I_{NY} is a total derivative.

Next we develop the analysis in the same manner as done for gravity with spin-$\frac{1}{2}$ fermions in [7]. The 3+1 decomposition of (4.3) can be achieved through the following parametrisation for the tetrads and their inverses:

$$\begin{aligned}e^I_t &= \sqrt{e}NM^I + N^aV^I_a \;,\; e^I_a \;=\; V^I_a \quad ; \\ M_IV^I_a &= 0 \;,\; M_IM^I \;=\; -1\end{aligned}$$

$$e^t_I \;=\; -\frac{M_I}{\sqrt{eN}}\;,\; e^a_I \;=\; V^a_I + \frac{N^a M_I}{\sqrt{eN}}\;\;;$$
$$M^I V^a_I \;=\; 0\;,\; V^I_a V^b_I \;=\; \delta^b_a\;,\; V^I_a V^a_J \;=\; \delta^I_J + M^I M_J$$

Also, we define $q_{ab} \;:=\; V^I_a V_{bI}$ and $q := \det q_{ab}$ which leads to $e := det(e^I_\mu) = Nq$.

Ignoring the total spatial derivatives, the Lagrangian density can be written as:

$$\mathcal{L} = e\Sigma^{ta}_{IJ}\partial_t\omega^{(\eta)IJ}_a \;+\; t^a_I\partial_t e^I_a \;-\; \bar{\pi}^a\partial_t\psi_a \;-\; NH \;-\; N^a H_a \;-\; \frac{1}{2}\,\omega^{IJ}_t G_{IJ} \;-\; 2\bar{S}\psi_t$$

where $H,\ H_a,\ G_{IJ}$ and $\bar{S}$ are given below in equation (4.6) and

$$\begin{aligned} 2e\Sigma^{ta}_{IJ} &= -\sqrt{q}M_{[I}V^a_{J]} \\ t^a_I &= \eta\epsilon^{abc}D_b(\omega)V_{Ic} \\ \bar{\pi}^a &= -\frac{i}{2}\,\epsilon^{abc}\bar{\psi}_b\gamma_5\gamma_c \end{aligned} \tag{4.4}$$

Here $\bar{\pi}^a$ is the canonically conjugate momenta associated with ψ_a [2]. The last equation in (4.4) can be inverted as:

$$\bar{\psi}_a = \sqrt{q}\;\bar{\pi}^b\gamma_a\gamma_b \tag{4.5}$$

The action does not contain the velocities associated with the gravity fields N, N^a, ω_{tIJ} and the matter field ψ_t. Hence these are Lagrange multipliers, leading to the primary constraints $H,\ H_a, G_{IJ}$ and $\bar{S}$, respectively:

$$\begin{aligned} G_{IJ} &= -2D_a(\omega)\left(e\Sigma^{(\eta)ta}_{IJ}\right) - t^a_{[I}V_{J]}a \;+\; \bar{\pi}^a\sigma_{IJ}\psi_a \;\approx\; 0 \\ H_a &= e\Sigma^{tb}_{IJ}R^{(\eta)IJ}_{ab}(\omega) - V^I_a D_b(\omega)t^b_I \;+\; \frac{i}{2}\sqrt{q}\;\epsilon^{bcd}\bar{\pi}^e\gamma_b\gamma_e\gamma_5\gamma_a\;D_c(\omega)\psi_d \;\approx\; 0 \\ H &= 2e^2\Sigma^{ta}_{IK}\Sigma^{tb}_{JL}\eta^{KL}R^{(\eta)IJ}_{ab}(\omega) - \sqrt{q}M^I D_a(\omega)t^a_I \;+\; \frac{iq}{2}\epsilon^{abc}\bar{\pi}^d\gamma_a\gamma_d\gamma_5 M_I\gamma^I D_b(\omega)\psi_c \;\approx\; 0 \\ \bar{S} &= D_a(\omega)\bar{\pi}^a \;-\; \frac{i\sqrt{q}}{4\eta}\bar{\pi}^a\gamma_b\gamma_a\gamma_5\gamma^I t^b_I \;\approx\; 0 \end{aligned} \tag{4.6}$$

where γ_a is defined as :

$$\gamma_a \;=\; \gamma_I V^I_a \;=\; (\gamma_i - \gamma_0\chi_i)V_{ai} \tag{4.7}$$

While $H,\ H_a,\ G_{IJ}$ are the constraints for the pure gravity sector, $\bar{S}$ is the generator of the local supersymmetric transformations.

[2]The functional derivative involving the Grassmann variables (fermions) acts on the left factor resulting in a sign in the definition of the conjugate momenta in (4.4).

Following the general framework of ref.[7], we introduce the following set of convenient fields,

$$E^a_i := 2e\Sigma^{ta}_{0i} \ , \ \chi_i := -M_i/M^0 \ , \ A^i_a \ := \ \omega^{(\eta)0i}_a - \chi_j \omega^{(\eta)ij}_a \ , \ \zeta^i \ := \ - E^a_j \omega^{(\eta)ij}_a \tag{4.8}$$

alongwith the decomposition of the nine components of $\omega^{(\eta)ij}_a$ in terms of three ζ_i's and six M_{kl}'s $(M_{kl} = M_{lk})$:

$$\omega^{(\eta)ij}_a = \frac{1}{2}(E^{[i}_a \zeta^{j]} + \epsilon^{ijk} E_{al} M^{kl}) \tag{4.9}$$

In terms of the fields in (4.8), we have $2e\Sigma^{ta}_{ij} = -E^a_{[i}\chi_{j]}$ and $e\Sigma^{ta}_{IJ}\partial_t \omega^{(\eta)IJ}_a = E^a_i \partial_t A^i_a + \zeta^i \partial_t \chi^i$. Note that the eighteen coordinate variables ω^{IJ}_a have been reexpressed in terms of the twelve variables A_{ai} and χ_i. The remaining six variables are the M_{kl}'s, whose velocities do not appear in the Lagrangian density. Hence these are the additional Lagrange multiplier fields.

Thus the Lagrangian density takes a simple form as follows:

$$\mathcal{L} \ := \ E^a_i \partial_t A^i_a \ + \ \zeta^i \partial_t \chi^i \ + \ t^a_I \partial_t V^I_a \ - \ \bar{\pi}^a \partial_t \psi_a \ - \ NH \ - \ N^a H_a \ - \ \frac{1}{2} \ \omega^{IJ}_t G_{IJ} \ - \ 2\bar{S}\psi_t$$

The fields V^I_a and t^a_I are not really independent, these are given in terms of the basic fields as: $V^I_a = \upsilon^I_a$ and $t^a_I = \tau^a_I$ where

$$\begin{aligned}
\upsilon^0_a \ &:= \ -\frac{1}{\sqrt{E}} E^i_a \chi_i \ \ , \ \ \upsilon^i_a \ := \ \frac{1}{\sqrt{E}} E^i_a \\
\tau^a_0 \ &:= \ \eta \epsilon^{abc} D_b(\omega) \upsilon_{0c} \\
&= \ \eta \sqrt{E} E^a_m \left[G^m_{\text{rot}} - \frac{\chi_l}{2} \left(\frac{2f_{ml} + N_{ml}}{1+\eta^2} + \epsilon_{mln} G^n_{\text{boost}} \right) - \ i\bar{\pi}^b \gamma_5 (\sigma_{0m} + \frac{\chi_l}{2} \sigma_{ml}) \psi_b \right] \\
\tau^a_k \ &:= \ \eta \epsilon^{abc} D_b(\omega) \upsilon_{ck} \\
&= \ -\frac{\eta}{2} \sqrt{E} E^a_m \left[\frac{2f_{mk} + N_{mk}}{1+\eta^2} + \epsilon_{kmn} G^n_{\text{boost}} + \ i\bar{\pi}^b \gamma_5 \sigma_{km} \psi_b \right]
\end{aligned} \tag{4.10}$$

In the above, f_{kl} and N_{kl} are defined as:

$$2f_{kl} \ := \ \epsilon_{ijk} E^a_i \left[(1+\eta^2) E^l_b \partial_a E^b_j + \chi_j A^l_a \right] + \eta \left(E^a_l A^k_a - \delta^{kl} E^a_m A^m_a - \chi_l \zeta_k \right) + (l \leftrightarrow k) \tag{4.11}$$

$$N_{kl} \ := \ (\chi^2 - 1)(M_{kl} - M_{mm}\delta_{kl}) + \chi_m \chi_n M_{mn} \delta_{kl} + \chi_l \chi_k M_{mm} - \chi_m (\chi_k M_{ml} + \chi_l M_{mk}) \tag{4.12}$$

We shall treat V^I_a and t^a_I as independent variables and introduce associated Lagrange multipliers ξ^a_I and ϕ^I_a to express the equations in (4.10) as constraints.

Thus we write the full Lagrangian density as,

$$\begin{aligned}\mathcal{L} &= E^a_i \partial_t A^i_a + \zeta^i \partial_t \chi_i + t^a_I \partial_t V^I_a - \bar{\pi}^a \partial_t \psi_a \; - NH - N^a H_a - \frac{1}{2}\omega_t^{IJ} G_{IJ} \\ & \quad - \xi^a_I (V^I_a - v^I_a) - \phi^I_a (t^a_I - \tau^a_I) \; - \; 2\bar{S}\psi_t \end{aligned} \tag{4.13}$$

The constraints in (4.6) can now be rewritten in terms of the canonical fields. These can be worked out in an analogous manner as in ref.[7]. Thus the corresponding expressions for $G_i^{boost} := G_{0i}$, $G_i^{rot} := \frac{1}{2}\epsilon^{ijk} G_{jk}$, H_a, H and $\bar{S}$ are:

$$\begin{aligned}
G^i_{\text{boost}} &= -\partial_a (E^a_i \; - \; \eta \epsilon^{ijk} E^a_j \chi_k) \; + \; E^a_{[i}\chi_{k]} A^k_a \; + \; (\zeta^i \; - \; \chi \cdot \zeta \; \chi^i) - t'^a_{[0} V_{i]a} \\
&\quad + \bar{\pi}^a \sigma_{0i} \psi_a \; + \; \frac{\eta}{4M^0 E} \; \epsilon^{ijk} E_{al} E^b_k \bar{\pi}^a (\gamma_j - \gamma_0 \chi_j)(\gamma_l - \gamma_0 \chi_l)(\gamma_0 - \gamma_m \chi_m)\psi_b; \\
G^i_{\text{rot}} &= \partial_a (\epsilon^{ijk} E^a_j \chi_k \; + \; \eta E^a_i) \; + \; \epsilon^{ijk} (A^j_a E^a_k \; - \; \zeta_j \chi_k \; - \; t'^a_j V^k_a) \\
&\quad + i\bar{\pi}^a \gamma_5 \sigma_{0i} \psi_a \; - \; \frac{\eta}{4M^0 E} \; E_{al} \bar{\pi}^a (\gamma_{[i} - \gamma_0 \chi_{[i}) E^b_{j]} (\gamma_l - \gamma_0 \chi_l)\gamma_j \psi_b; \\
H_a &= E^b_i \partial_{[a} A^i_{b]} + \zeta_i \partial_a \chi_i \; - \; \partial_b (t^b_I V^I_a) \; + \; t^b_I \partial_a V^I_b \\
&\quad - \frac{1}{1+\eta^2} \left[E^b_{[i}\chi_{l]} A^l_b + \zeta_i - \chi \cdot \zeta \chi^i - t^b_{[0} V_{i]b} - \eta \epsilon^{ijk} \; (A^j_a E^b_k + \chi_j \zeta_k - t^b_j V^k_b) \right] A^i_a \\
&\quad - \frac{1}{1+\eta^2} \left[\frac{1}{2} \; \epsilon^{ijk} \left(\eta G^k_{\text{boost}} + G^k_{\text{rot}} \right) - \chi^i (G^j_{\text{boost}} - \eta G^j_{\text{rot}}) \right] \omega_a^{(\eta)ij} \\
&\quad - \frac{1}{4(1+\eta^2)} \frac{1}{M^0 \sqrt{E}} \epsilon^{bcd} \bar{\pi}^e \gamma_b \gamma_e (\eta - i\gamma_5) \gamma_{[a} \omega_{c]}^{(\eta)ij} (\sigma_{ij} + 2\sigma_{0i}\chi_j)\psi_d \\
&\quad - \frac{1}{2(1+\eta^2)} \; \frac{1}{M^0 \sqrt{E}} \; \epsilon^{bcd} \bar{\pi}^e \gamma_b \gamma_e \gamma_a (\eta + i\gamma_5) \sigma_{0i} A^i_c \psi_d \\
H &= -E^a_k \chi_k H_a + (1 - \chi \cdot \chi) \left[E^a_i \partial_a \zeta_i + \frac{1}{2} \zeta_i E^a_i E^b_j \partial_a E^j_b \right] \\
&\quad + \frac{1 - \chi \cdot \chi}{2(1+\eta^2)} \zeta_i \left[-G^i_{\text{boost}} + \eta G^i_{\text{rot}} \right] - \left(E^a_k \chi_k V^I_a + \sqrt{q} M^I \right) \partial_b t^b_I \\
&\quad - \frac{1 - \chi \cdot \chi}{1+\eta^2} \left[\frac{1}{2} E^a_{[i} E^b_{j]} A^i_a A^j_b + E^a_i A^i_a \chi \cdot \zeta + \eta \epsilon^{ijk} \zeta_i A^j_a E^a_k \right. \\
&\qquad\qquad \left. + \frac{3}{4} (\chi \cdot \zeta)^2 - \frac{3}{4} (\zeta \cdot \zeta) + \frac{1}{2} \zeta_i t^a_{[0} V_{i]a} - \frac{\eta}{2} \zeta_i \epsilon^{ijk} t^a_j V^k_a \right] \\
&\quad + \frac{1 - \chi \cdot \chi}{1+\eta^2} \left[\frac{1}{\sqrt{E}} A^i_a t^a_i + \frac{1}{2} V^i_a \left(\zeta \cdot \chi t^a_i - \chi_i \zeta_j t^a_j + \eta \epsilon^{ijk} \zeta_j t^a_k \right) \right] \\
&\quad + \frac{1 - \chi \cdot \chi}{2(1+\eta^2)} \left[(\eta t'^a_k \; - \; \epsilon^{ijk} \chi_i t'^a_j) \frac{E_{al}}{\sqrt{E}} \; + \; 2 f_{kl} \; + \; \frac{1}{2} N_{kl}(M) \; + \; 2(1+\eta^2) J_{kl} \right] M^{kl} \\
&\quad - \frac{1 - \chi . \chi}{2(1+\eta^2)} \; \frac{1}{M^0 E} \epsilon^{abc} \bar{\pi}^d \gamma_a \gamma_d \gamma_0 (\eta + i\gamma_5) \left(\sigma_{0i} A^i_b \psi_c \; + \; \frac{1}{4} (\sigma_{ij} + 2\sigma_{0i}\chi_j) E_{b[i} \zeta_{j]} \psi_c \right)
\end{aligned}$$

$$+ \frac{1-\chi\cdot\chi}{2(1+\eta^2)}\ \zeta_i\ \bar{\pi}^a(1-i\eta\gamma_5)\sigma_{0i}\psi_a$$

$$\bar{S} \ = \ \partial_a\bar{\pi}^a \ - \frac{1}{1+\eta^2}\bar{\pi}^a(1-i\eta\gamma_5)\left[\sigma_{0l}A_a^l \ + \ \frac{1}{2}(\sigma_{ij}+2\sigma_{0i}\chi_j)\omega_{aij}^{(\eta)}\right] - \frac{i}{4\eta M^0\sqrt{E}}\bar{\pi}^a\gamma_b\gamma_a\gamma_5\gamma^I t_I^b \tag{4.14}$$

where we have used the definitions:

$$\begin{aligned} t_I'^a \ &:= \ t_I^a \ - \ \frac{\eta}{4}\ \epsilon^{abc}\bar{\psi}_b\gamma_I\psi_c \\ &= \ t_I^a \ - \ \frac{\eta}{4M^0\sqrt{E}}\ \epsilon^{ijk}E_k^aE_j^cE_b^l\ \bar{\pi}^b(\gamma_i-\gamma_0\chi_i)(\gamma_l-\gamma_0\chi_l)\gamma_I\psi_c \end{aligned} \tag{4.15}$$

$$\begin{aligned} 2J_{kl} \ &:= \ \frac{1}{4}\epsilon^{abc}\bar{\psi}_b\gamma_k\psi_cE_{al} \ + \ (k\leftrightarrow l) \\ &= \ \frac{1}{4M^0\sqrt{E}}\ \epsilon^{iml}E_{aj}E_m^b\bar{\pi}^a(\gamma_i-\gamma_0\chi_i)(\gamma_j-\gamma_0\chi_j)\gamma_k\psi_b \ + \ (k\leftrightarrow l) \end{aligned} \tag{4.16}$$

The Hamiltonian density now reads:

$$\mathcal{H} \ = \ NH \ + \ N^aH_a \ + \ \frac{1}{2}\ \omega_t^{IJ}\ G_{IJ} \ + \ \xi_I^a(V_a^I \ - \ \upsilon_a^I) \ + \ \phi_a^I(t_I^a \ - \ \tau_I^a) \ + \ 2\bar{S}\psi_t$$

The constraints associated with the fields $N^a, N, \omega_t^{0i}, \omega_t^{ij}, \xi_I^a$, ϕ_a^I and ψ_t respectively are:

$$\begin{aligned} &H_a\approx 0 \ \ , \ \ H\approx 0 \ \ , \ \ G^i_{\rm boost}\approx 0 \ \ , \ \ G^i_{\rm rot}\approx 0 \\ &V_a^I-\upsilon_a^I\approx 0 \ \ , \ \ t_I^a-\tau_I^a\approx 0 \ \ , \ \ \bar{S}\approx 0. \end{aligned}$$

As mentioned earlier, the momenta conjugate to M_{kl} are zero. The preservation of this constraint requires:

$$\frac{\delta H}{\delta M_{kl}} \ \approx \ 0\ ,$$

which implies:

$$(\eta t_k'^a - \epsilon^{ijk}\chi_i t_j'^a)V_a^l + f_{kl} + \frac{1}{2}N_{kl} + (1+\eta^2)J_{kl} + (k\leftrightarrow l) \approx 0 \tag{4.17}$$

where, f_{kl} and N_{kl} are given in (4.11, 4.12). This constraint can be solved for M_{kl}. Next, using $t_I^a \ \approx \ \tau_I^a$, we write

$$\begin{aligned} t_k'^a \ &\approx \ -\frac{\eta}{2}\sqrt{E}\ E_l^a\left[\ \frac{2f_{kl}+N_{kl}}{1+\eta^2} + 2J_{kl} + \epsilon_{kln}G'^{\,n}_{\rm boost}\right] \\ t_0'^a \ &\approx \ \eta\sqrt{E}\ E_l^a\ \left[\ G'^{\,l}_{\rm rot} - \frac{\chi_k}{2}\ \left(\ \frac{2f_{kl}+N_{kl}}{1+\eta^2} + 2J_{kl} + \epsilon_{kln}\ G'^{\,n}_{\rm boost}\right)\right] \end{aligned} \tag{4.18}$$

Using (4.18) in (4.17), we obtain

$$2f_{kl} + N_{kl} + 2(1+\eta^2)J_{kl} \approx 0 \tag{4.19}$$

These constraints are second-class in themselves and therefore can be implemented strongly. Here the J_{kl} piece captures all the contribution coming from the spin-$\frac{3}{2}$ fermions. Note that this equation has the same form as the one for spin-$\frac{1}{2}$ fermions [7]. This constraint, from (4.18), further implies:

$$t'^a_I \approx 0 \quad => \quad t^a_I \;=\; \frac{\eta}{4}\,\epsilon^{abc}\bar{\psi}_b\gamma_I\psi_c \tag{4.20}$$

This is exactly same as the connection equation of motion which is obtained in the Lagrangian formulation by varying the standard supergravity action without the Nieh-Yan term (see [23], for example).

Using (4.20), the final set of constraints read:

$$\begin{aligned}
G^i_{\rm boost} &= -\partial_a(E^a_i \;-\; \eta\epsilon^{ijk}E^a_j\chi_k) \;+\; E^a_{[i}\chi_{k]}A^k_a \;+\; (\zeta^i \;-\; \chi\cdot\zeta\;\chi^i) \\
&\quad + \bar{\pi}^a\sigma_{0i}\psi_a \;+\; \frac{\eta}{4M^0E}\;\epsilon^{ijk}E_{al}E^b_k\bar{\pi}^a(\gamma_j-\gamma_0\chi_j)(\gamma_l-\gamma_0\chi_l)(\gamma_0-\gamma_m\chi_m)\psi_b \\
G^i_{\rm rot} &= \partial_a(\epsilon^{ijk}E^a_j\chi_k \;+\; \eta E^a_i) \;+\; \epsilon^{ijk}(A^j_aE^a_k \;-\; \zeta_j\chi_k) \;+\; i\bar{\pi}^a\gamma_5\sigma_{0i}\psi_a \\
&\quad - \frac{\eta}{4M^0E}\;E_{al}\bar{\pi}^a(\gamma_{[i}-\gamma_0\chi_{[i})E^b_{j]}(\gamma_l-\gamma_0\chi_l)\gamma_j\psi_b \\
H_a &= E^b_i\partial_{[a}A^i_{b]} + \zeta_i\partial_a\chi_i \;-\; \partial_b\left((\tau'^b_i-\chi_i\tau'^b_0)\frac{E^i_a}{\sqrt{E}}\right) \;-\; \tau'^b_0\partial_a\left(\chi_i\frac{E^i_b}{\sqrt{E}}\right) + \tau'^b_i\partial_a\left(\frac{E^i_b}{\sqrt{E}}\right) \\
&\quad -\frac{1}{1+\eta^2}\left[E^b_{[i}\chi_{l]}A^l_b + \zeta_i - \chi\cdot\zeta\chi^i - \frac{1}{\sqrt{E}}\tau'^b_{[0}E_{i]b} - \eta\epsilon^{ijk}\;(A^j_aE^b_k + \chi_j\zeta_k - \frac{1}{\sqrt{E}}\tau'^b_jE^k_b)\right]A^i_a \\
&\quad -\frac{1}{8(1+\eta^2)}\frac{1}{M^0\sqrt{E}}\epsilon^{bcd}\bar{\pi}^e\gamma_b\gamma_e(\eta-i\gamma_5)\gamma_{[a}(E^{[i}_{c]}\zeta^{j]} + \epsilon^{ijm}E^n_{c]}M_{mn})(\sigma_{ij}+2\sigma_{0i}\chi_j)\psi_d \\
&\quad -\frac{1}{2(1+\eta^2)M^0\sqrt{E}}\;\epsilon^{bcd}\bar{\pi}^e\gamma_b\gamma_e(\eta-i\gamma_5)\gamma_a\sigma_{0k}A^k_c\psi_d \\
H &= (1-\chi\cdot\chi)\left[E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_iE^a_iE^b_j\partial_aE^j_b\right] \;-\; \frac{1-\chi\cdot\chi}{\sqrt{E}}\partial_b\tau'^b_0 \\
&\quad -\frac{1-\chi\cdot\chi}{1+\eta^2}\left[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_aA^j_b + E^a_iA^i_a\chi\cdot\zeta + \eta\epsilon^{ijk}\zeta_iA^j_aE^a_k\right. \\
&\qquad\qquad \left. +\frac{3}{4}(\chi\cdot\zeta)^2 - \frac{3}{4}(\zeta\cdot\zeta) + \frac{1}{2\sqrt{E}}\zeta_i(\tau'^a_0-\chi_k\tau'^a_k)E^i_a - \frac{\eta}{2\sqrt{E}}\zeta_i\epsilon^{ijk}\tau'^a_jE^k_a\right] \\
&\quad +\frac{1-\chi\cdot\chi}{1+\eta^2}\left[\frac{1}{\sqrt{E}}A^i_a\tau'^a_i + \frac{1}{2\sqrt{E}}E^i_a\left(\zeta\cdot\chi\tau'^a_i - \chi_i\zeta_j\tau'^a_j + \eta\epsilon^{ijk}\zeta_j\tau'^a_k\right)\right]
\end{aligned}$$

$$
\begin{aligned}
& - \frac{1-\chi{\cdot}\chi}{2(1+\eta^2)} \; \frac{1}{M^0 E} \epsilon^{abc} \bar{\pi}^d \gamma_a \gamma_d \gamma_0 (\eta + i\gamma_5) \left(\sigma_{0i} A^i_b \psi_c \; + \; \frac{1}{4} (\sigma_{ij} + 2\sigma_{0i}\chi_j) E_{b[i}\zeta_{j]} \psi_c \right) \\
& + \frac{1-\chi\cdot\chi}{2(1+\eta^2)} \; \zeta_i \; \bar{\pi}^a (1 - i\eta\gamma_5) \sigma_{0i} \psi_a \; + \; \frac{1-\chi^2}{4(1+\eta^2)} [f_{kl} + (1+\eta^2) J_{kl}] M^{kl} \\
\bar{S} \; = \; & \partial_a \bar{\pi}^a \; - \frac{1}{1+\eta^2} \; \bar{\pi}^a (1 - i\eta\gamma_5) \left[\sigma_{0l} A^l_a \; + \; \frac{1}{4} (\sigma_{ij} + 2\sigma_{0i}\chi_j)(E_{a[i}\zeta_{j]} + \epsilon_{ijl} E_{am} M^{lm}) \right]
\end{aligned}
$$

where τ'^a_I is defined as

$$
\begin{aligned}
\tau'^a_I \; & := \; \frac{\eta}{4} \epsilon^{abc} \bar{\psi}_b \gamma_I \psi_c \\
& = \; \frac{\eta}{4M^0\sqrt{E}} \; \epsilon^{ijk} E^a_k E^c_j E^l_b \; \bar{\pi}^b (\gamma_i - \gamma_0\chi_i)(\gamma_l - \gamma_0\chi_l) \gamma_I \psi_c
\end{aligned} \tag{4.21}
$$

and f_{kl} , J_{kl} and M_{kl} are given by the (4.11), (4.12), (4.16) and (4.19). In writing $\bar{S}$, we have made use of the Fierz identity-

$$
\epsilon^{\mu\nu\alpha\beta} \bar{\psi}_\mu \gamma_I \psi_\nu \gamma^I \psi_\alpha \; = \; 0 \qquad ,
$$

which makes the piece proportional to t^a_I dissapear.

4.3 Time gauge:

One may adopt the time gauge through the choice $\chi_i = 0$. Since this condition forms a second-class pair with the boost constraint, both have to be implemented together. G^{boost}_i can be solved as:

$$
\zeta_i \; = \; \partial_a E^a_i \; - \; \bar{\pi}^a \sigma_{0i} \psi_a \; - \; \frac{\eta}{4M^0 E} \; \epsilon^{ijk} E_{al} E^b_k \bar{\pi}^a \gamma_j \gamma_l \gamma_0 \psi_b \tag{4.22}
$$

We can rewrite this as:

$$
\zeta_i \; = \; \partial_a E^a_i \; + \; \frac{1}{\sqrt{E}} \tau'^a_0 E_{ai} \; + \; \frac{1}{\eta\sqrt{E}} \epsilon^{ijk} \tau'^a_j E_{ak} \tag{4.23}
$$

with

$$
\tau'^a_I \; = \; \frac{\eta}{4\sqrt{E}} \epsilon^{ijk} E^a_k E^c_j E_{bl} \bar{\pi}^b \gamma_i \gamma_l \gamma_I \psi_c
$$

In this gauge the constraints, which are first-class, reduce to simpler expressions as follows:

$$
G^i_{\text{rot}} \; = \; \eta \; \partial_a E^a_i + \epsilon^{ijk} A^j_a E^a_k \; - \; \frac{1}{\eta\sqrt{E}} \tau'^a_0 E_{ai} \; - \; \frac{1}{\sqrt{E}} \epsilon^{ijk} \tau'^a_j E_{ak}
$$

$$
\begin{aligned}
H_a &= E^b_i F^i_{ab} + \frac{1}{2(1+\eta^2)} \left[A^k_a - E^i_a E^b_k A^i_b\right] \zeta_k - \frac{1}{\eta^2 \sqrt{E}} \tau'^b_0 E_{bi} A_{ai} - \frac{E_{ai}}{\sqrt{E}} \left[\partial_b \tau'^b_i + \frac{1}{\eta} \epsilon^{ijk} A^j_b \tau'^b_k\right] \\
H &= -\frac{\eta}{2} E^a_i E^b_j \epsilon^{ijk} \left[F^k_{ab} + (\eta + \frac{1}{\eta})\ R^k_{ab}\right] - \frac{1}{\sqrt{E}} \left[\partial_a \tau'^a_0 - \frac{1}{2\eta} (\epsilon^{ijk} E^j_a \zeta_k \tau'^a_i + E^j_a \tau'^a_i M^{ij})\right] \\
\bar{S} &= \partial_a \bar{\pi}^a - \frac{1}{1+\eta^2}\ \bar{\pi}^a (1 - i\eta\gamma_5) \sigma_{0k} \left[A^k_a + \frac{1}{2} i\gamma_5 (\epsilon_{jkl} \zeta_j + M^{kl}) E_{al}\right]
\end{aligned}
\tag{4.24}
$$

In these equations, we have used the following definitions:

$$
\begin{aligned}
\Gamma_{ai} &= \frac{1}{2} \epsilon^{ijk} \omega_{ajk} \\
F^k_{ab} &= \partial_{[a} A^k_{b]} + \frac{1}{\eta} \epsilon^{ijk} A_{ai} A_{bj} \quad , \quad R^k_{ab} = \partial_{[a} \Gamma^k_{b]} - \frac{1}{\eta} \epsilon^{ijk} \Gamma_{ai} \Gamma_{bj}
\end{aligned}
$$

and ζ^i is given by (4.23). Also, in the time gauge :

$$
\begin{aligned}
M_{kl} &= (1+\eta^2) \left(\epsilon^{ijk} E^l_b \partial_a E^b_j - \epsilon^{ijm} E^m_b \partial_a E^b_j \delta_{kl}\right) E^a_i + (1+\eta^2)\, (2J_{kl} - J_{mm}\delta_{kl}) \\
&\quad + \eta E^a_l A_{ak} + (k \leftrightarrow l) \\
2J_{kl} &= \frac{1}{4M^0 \sqrt{E}}\ \epsilon^{iml} E_{aj} E^b_m \bar{\pi}^a \gamma_i \gamma_j \gamma_k \psi_b + (k \leftrightarrow l)
\end{aligned}
$$

Here in (4.24) we have dropped terms proportional to rotation constraints from H_a and H.

As is evident, the dynamical variable which enters in the constraints apart from the fermionic degrees of freedom is the Barbero-Immirzi connection A^i_a. Thus in the time gauge we obtain a real $SU(2)$ formulation of the theory of gravity coupled to spin-$\frac{3}{2}$ fermions.

Notice that in the matter sector, $\bar{\pi}^a$ and ψ_a are not independent variables. These obey the second-class constraints:

$$
C^a := \bar{\pi}^a + \frac{i}{2}\ \epsilon^{abc} \bar{\psi}_b \gamma_5 \gamma_c \approx 0
$$

In order to implement these constraints, we need to go to corresponding Dirac brackets for the matter fields $\bar{\pi}^a, \psi_a$. This then leads to the correct transformations (modulo rotations) on the fields through their Dirac brackets with the corresponding generators. In particular, the Dirac brackets of the fields with the supersymmetry generator $\bar{S}$ make them transform properly under its action.

4.4 Summary

We have presented a framework to incorporate the Barbero-Immirzi parameter as a topological coupling constant in the classical theory of $N = 1$ supergravity. This is achieved through the inclusion of the Nieh-Yan density in the Lagrangian. This additional term, being a topological density, preserves the equations of motion and the supersymmetry of the original action. The canonical formulation has been first developed without going to any particular choice of gauge. This clarifies the structure of the theory exhibiting all of its gauge freedom. In the time gauge, the theory is shown to admit a real $SU(2)$ formulation in terms of the Barbero-Immirzi connection A_a^i.

The essential features for spin-$\frac{3}{2}$ fermions turn out to be very similar to those for spin-$\frac{1}{2}$ fermions as described in chapter-3, except that here we have the additional constraint $\tilde{S}$ which acts as the generator of local supersymmetry transformations. The cases for $N = 2, 4$ and higher supergravity theories can be treated in exactly similar fashion. There the constraint analysis leads to the same form of the connection equation of motion as given here (i.e., equation (4.19)), a fact which is evident from the structure of the fermionic terms in these theories. Only the expression for J_{kl} in terms of the matter fields gets modified.

Chapter 5

Antisymmetric tensor gauge fields

This chapter concerns the theory of gravity coupled to antisymmetric tensor gauge fields of rank two. This coupling is very different in nature as compared to the fermionic couplings studied in the previous chapters. Such couplings appear naturally in the perturbative formulations of quantum gravity, e.g. string theory and supergravity. Here the form of the coupling is taken to be exactly that as in string theory, i.e. as it appears in the context of anomaly cancellation mechanism. To elaborate, we deal with a gauge invariant field strength $H = dA + \omega_L$, where A is the antisymmetric tensor gauge field of rank two and ω_L is the Lorentz Chern-Simons three form.

From the viewpoint of the canonical SU(2) formulation of gravity, such a matter coupling poses some novel challenges. Firstly, since the Chern-Simons term is linear in curvature and the Lagrangian density is quadratic in the field strength H, one is essentially dealing with a higher curvature coupling. Also, Hamiltonian analysis of this theory shows that the canonical momenta E^a_i conjugate to the Barbero-Immirzi connection A^i_a gets modified unlike in the case of fermions. This implies a non-trivial modification in the corresponding symplectic structure. In what follows next, we investigate whether in presence of such a matter content one can use the general framework as developed in chapter-2 and obtain a SU(2) formulation for the theory of gravity with the Barbero-Immirzi coefficient as a topological parameter as in the earlier cases.

5.1 Antisymmetric tensor coupling

We introduce the coupling of gravity to antisymmetric tensor field of rank two through the following Lagrangian density in the first order formulation:

$$\mathcal{L} \;=\; \frac{1}{2} e\Sigma^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega) \;+\; \frac{1}{12}\, eH^{\mu\nu\alpha} H_{\mu\nu\alpha} \tag{5.1}$$

The field strength $H_{\mu\nu\alpha}$ is defined as:

$$H_{\mu\nu\alpha} = \partial_{[\mu} A_{\nu\alpha]} + C_{\mu\nu\alpha} \tag{5.2}$$

$$= \partial_{[\mu} A_{\nu\alpha]} + Tr\ (\omega_{[\mu}\partial_\nu \omega_{\alpha]} + \frac{2}{3}\omega_{[\mu}\omega_\nu \omega_{\alpha]}) \tag{5.3}$$

where $C_{\mu\nu\alpha}$ is the Lorentz Chern-Simons term. The field strength tensor is invariant under the tensor gauge transformations $\delta A_{\mu\nu} = \partial_{[\mu}\lambda_{\nu]}$. There is an additional local symmetry under the infinitesimal Lorentz transformations. Under this, the fields transform as follows, leaving $H_{\mu\nu\alpha}$ invariant:

$$\delta\omega_\mu^{IJ} = D_\mu \Theta^{IJ}, \tag{5.4}$$

$$\delta A_{\mu\nu} = \frac{1}{2}\Theta^{IJ}\partial_{[\mu}\omega_{\nu]}^{IJ} \tag{5.5}$$

As is evident, the coupling has a non-linear dependence on the spin-connection and also introduces a non-vanishing torsion in the theory.

Now as proposed in chapter-2, we add the Nieh-Yan topological density I_{NY} to (5.1):

$$L = \frac{1}{2}e\Sigma_{IJ}^{\mu\nu} R_{\mu\nu}{}^{IJ}(\omega) + \frac{1}{12}\ eH^{\mu\nu\alpha}H_{\mu\nu\alpha} + \frac{\eta}{2}I_{NY} \tag{5.6}$$

where the Nieh-Yan density is given by:

$$I_{NY} = \epsilon^{\mu\nu\alpha\beta}\left(D_\mu(\omega)e_\nu^I\ D_\alpha(\omega)e_{I\beta} - \frac{1}{2}\Sigma_{\mu\nu}^{IJ}\ R_{\alpha\beta IJ}(\omega)\right)\ ,\ D_\mu(\omega)e_\nu^I := \partial_\mu e_\nu^I + \omega_\mu{}^I{}_J e_\nu^J\ . \tag{5.7}$$

As discussed in [7], this is the general prescription which should provide a SU(2) formulation for a theory of gravity with (or without) any arbitrary matter coupling. The analyses in the cases for spin-$\frac{1}{2}$ fermion coupling and supergravity theories already substantiate this proposal. Here we discover that unlike these cases, this coupling changes the momenta conjugate to the real SU(2) connection A_a^i. This introduces additional degrees of freedom with additional constraints in the theory, leading to a non-trivial modification of the symplectic structure.

5.2 Canonical Analysis: Matter Sector

Next we perform the Hamiltonian decomposition of the matter part given by:

$$L_m = \frac{1}{12}eH^{\mu\nu\alpha}H_{\mu\nu\alpha} \tag{5.8}$$

This can be rewritten as:

$$
\begin{aligned}
L_m &= \frac{1}{12} e\Big(3H^{tab}H_{tab} + H^{abc}H_{abc}\Big) \\
&= \frac{1}{4}\Pi^{ab}H_{tab} + \frac{1}{12}\Big(-3N^a\Pi^{bc}H_{abc} + Nqq^{ad}q^{be}q^{cf}H_{abc}H_{def}\Big)
\end{aligned}
$$

where Π^{ab} is the canonical momenta conjugate to A_{ab}:

$$
\begin{aligned}
\Pi^{ab} &= \frac{\partial L}{\partial(\partial_t A_{ab})} = eH^{tab} \\
&= e\Big((g^{tt}g^{ac} - g^{tc}g^{ta})g^{bd} \; + \; g^{tc}g^{tb}g^{ad}\Big)H_{tcd} \; + \; eg^{td}g^{ae}g^{bc}H_{cde}
\end{aligned}
$$

The indices of Π^{ab} are lowered using the 3-metric q_{ab}; i.e.,$\Pi_{ab} = q_{ac}q_{bd}\Pi^{cd}$. Now, using the identities-

$$
\begin{aligned}
\Pi^{ab}H_{tab} &= \Pi^{ab}\Big(\partial_t A_{ab} + 2\partial_a A_{bt} + C_{tab}\Big) \\
&= N^a\Pi^{bc}H_{abc} \; - \; N\Pi^{ab}\Pi_{ab} \quad \text{and} \\
\Pi^{ab}C_{tab} &= 2\Pi^{ab}\omega_b^{0i}\partial_t\omega_a^{0i} - \Pi^{ab}\omega_b^{ij}\partial_t\omega_a^{ij} \; + \; 2\Pi^{ab}R_{ab}^{0i}\omega_t^{0i} - \Pi^{ab}R_{ab}^{ij}\omega_t^{ij},
\end{aligned}
$$

we arrive at the following expression:

$$
\begin{aligned}
L_m = \; & \mu_i^a\partial_t\omega_a^{(\eta)0i} \; + \; P_i^a\partial_t\omega_a^{0i} \; + \; \frac{1}{2}\Pi^{ab}\partial_t A_{ab} \; + \; \Pi^{ab}R_{ab}^{0i}\omega_t^{0i} \; - \; \frac{1}{2}\Pi^{ab}R_{ab}^{ij}\omega_t^{ij} \\
- \; & \frac{1}{2}N^a\Pi^{bc}H_{abc} \; + \; N\Big(\frac{1}{4}\Pi^{ab}\Pi_{ab} + \frac{1}{12}qq^{ad}q^{be}q^{cf}H_{abc}H_{def}\Big) \; - \; A_{bt}\partial_a\Pi^{ab} \qquad (5.9)
\end{aligned}
$$

where we have defined-

$$
\begin{aligned}
\mu_i^a &= \frac{1}{\eta^2}\Pi^{ab}\Big(\omega_b^{0i} - \omega_b^{(\eta)0i}\Big) \\
P_i^a &= \frac{1}{\eta^2}\Pi^{ab}\Big(\omega_b^{(\eta)0i} + (\eta^2 - 1)\omega_b^{0i}\Big)
\end{aligned}
$$

5.3 Canonical analysis: Full theory

Using equations (5.9), the full Lagrangian density in (5.6) can now be written as :

$$
\begin{aligned}
L = \; & \left(e\Sigma_{IJ}^{ta} \; - \; \frac{1}{2(1+\eta^2)}\,\Pi^{ab}(\omega_{bIJ} \; - \; \eta\,\tilde{\omega}_{bIJ})\right)\partial_t\omega_a^{(\eta)IJ} \; + \; t_I^a\partial_t V_a^I \; + \; \frac{1}{2}\Pi^{ab}\partial_t A_{ab} \\
- \; & \frac{1}{2}\,\omega_t^{IJ}\hat{G}^{IJ} \; - \; N^a H_a \; - \; NH \; - \; A_{bt}\partial_a\Pi^{ab} \\
= \; & \pi_{IJ}^a\partial_t\omega_a^{(\eta)IJ} \; + \; t_I^a\partial_t V_a^I \; + \; \frac{1}{2}\Pi^{ab}\partial_t A_{ab} \; - \; \frac{1}{2}\,\omega_t^{IJ}\hat{G}^{IJ} \; - \; N^a H_a \; - \; NH \\
- \; & A_{bt}\partial_a\Pi^{ab}
\end{aligned}
$$

where the constraints are given by the following expressions :

$$
\begin{aligned}
G_{IJ} &= 2D_a(\omega)(e\Sigma^{(\eta)ta}_{IJ}) \; - \; t^a_{[I}V_{J]a} \; + \; \Pi^{ab}R_{abIJ} \\
H_a &= e\Sigma^{(\eta)tb}_{IJ}R^{(\eta)IJ}_{ab} \; - \; V^I_a D_b(\omega)t^b_I \; + \; \frac{1}{2}\Pi^{bc}H_{abc} \\
H &= 2e^2\Sigma^{(\eta)ta}_{IK}\;\Sigma^{(\eta)tb}_{JL}\eta^{KL}R^{(\eta)IJ}_{ab} \; + \; \eta\sqrt{q}M^I D_a(\omega)t^a_I \; - \; \frac{1}{4}\Pi^{ab}\Pi_{ab} \\
&- \; \frac{1}{12}qq^{aa'}q^{bb'}q^{cc'}H_{abc}H_{a'b'c'}
\end{aligned} \tag{5.10}
$$

Here the momenta π^a_{IJ} conjugate to $\omega^{(\eta)IJ}_a$ are constrained, a fact reflected through the six primary constraints C^{ab}:

$$
C^{ab} \; := \; \epsilon^{IJKL}\Sigma^{ta}_{IJ}\;\Sigma^{tb}_{KL} \; \approx \; 0
$$

Preservation of these lead to the secondary constraints D^{ab}:

$$
\begin{aligned}
D^{ab} \; &:= \; 8e^2\tilde{\Sigma}^{tc}_{IJ}\Sigma^{ta}_{KI}D_c(\omega)\left(e\Sigma^{tb}_{JK}\right) \; + \; \frac{\eta}{1+\eta^2}\sqrt{q}M^I t^{bJ}\left(e\tilde{\Sigma}^{ta}_{IJ} \; + \; \eta\Sigma^{ta}_{IJ}\right) \\
&- \; \frac{1}{2}qq^{bb'}q^{cc'}q^{dd'}H_{b'c'd'}R^{IJ}_{cd}e\tilde{\Sigma}^{ta}_{IJ} \; + \; (\; a \leftrightarrow b \;)
\end{aligned}
$$

Now, we split up the 18 canonical pairs $(\omega^{(\eta)IJ}_a, \pi^a_{IJ})$ into two sets of 9 pairs, namely, $(A^i_a = \omega^{(\eta)oi}_a, E'^a_i)$ and (ω^{oi}_a, π^a_i). In terms of these, the Lagrangian density becomes:

$$
\begin{aligned}
L \; &= \; E'^a_i\partial_t A^i_a \; + \; \pi^a_i\partial_t\omega^{0i} \; + \; t^a_I\partial_t V^I_a \; + \; \frac{1}{2}\Pi^{ab}\partial_t A_{ab} \; - \; \frac{1}{2}\omega^{IJ}_t\hat{G}^{IJ} \; - \; N^a H_a \; - \; NH \\
&- \; A_{bt}\partial_a\Pi^{ab}
\end{aligned}
$$

$$
\begin{aligned}
\text{where,} \quad E'^a_i \; &= \; E^a_i \; + \; \frac{1}{\eta}\epsilon^{ijk}E^a_j\chi_k \; + \; \mu^a_i \; ; \;\; \mu^a_i \; = \; \frac{1}{\eta^2}\pi^{ab}(\omega^{0i}_b \; - \; A^i_b) \\
\pi^a_i \; &= \; -\left(\eta + \frac{1}{\eta}\right)\epsilon^{ijk}E^a_j\chi_k \; + \; P^a_i \; ; \;\; P^a_i \; = \; \frac{1}{\eta^2}\pi^{ab}\left(\omega^{0i}_b \; - \; (1-\eta^2)A^i_b\right)
\end{aligned}
$$

The constraints, when written in terms of the canonical variables, are given by the following expressions :

$$
\begin{aligned}
-G^{boost}_i \; := \; G^{0i} \; &= \; D_a(A)\left(E'^a_i + \pi^a_i - \mu^a_i - P^a_i\right) \; - \; \frac{1}{\eta}\epsilon^{ijk}\omega^{0j}_a\Big((1+\eta^2)E'^a_k + \pi^a_k \\
&- \; (1+\eta^2)\mu^a_k - P^a_k\Big) \; + \; t^a_{[0}V_{i]a} \; + \; \Pi^{ab}R^{0i}_{ab}
\end{aligned}
$$

$$
\begin{aligned}
G_i^{rot} \;&:=\; \frac{1}{2}\epsilon^{ijk}G_{jk} \;=\; \eta\; D_a(A)\,(E_i'^a \;-\; \mu_i^a) \;+\; \frac{1}{1+\eta^2}\epsilon^{ijk}\omega_a^{oj}(\pi_k^a - P_k^a) - \epsilon^{ijk}t_j^a V_{ka} \\
&+\; \frac{1}{2}\epsilon^{ijk}\Pi^{ab}R_{ab}^{jk} \\
H_a \;&=\; \left(E_i'^b + \frac{1}{1+\eta^2}\;\pi_i^b - \mu_i^b - \frac{1}{1+\eta^2}\;P_i^b\right)R_{ab}^{(\eta)0i} \;+\; \frac{1}{2(1+\eta^2)}\;\epsilon^{ijk}(\pi_k^b - P_k^b)R_{ab}^{(\eta)ij} \\
&-\; V_a^I D_b(\omega)t_I^b \;+\; \frac{1}{2}\Pi^{bc}H_{abc} \\
H \;&=\; \frac{\eta}{1+\eta^2}\;\epsilon^{ijk}\left(E_i'^a + \frac{1}{1+\eta^2}\;\pi_i^a - \mu_i^a - \frac{1}{1+\eta^2}\;P_i^a\right)(\pi_i^a - P_i^a)R_{ab}^{(\eta)0k} \\
&-\; \frac{1}{2}\Bigg((E_i'^a - \mu_i^a)(E_j'^b - \mu_j^b) + \frac{2}{1+\eta^2}\;(E_i'^a - \mu_i^a)(\pi_j^b - P_j^b) \\
&+\; \frac{1-\eta^2}{1+\eta^2}\;(\pi_i^a - P_i^a)(\pi_j^b - P_j^b)\Bigg)R_{ab}^{(\eta)ij} \;-\; \eta\sqrt{q}M^I D_a(\omega)t_I^a \;-\; \frac{1}{4}\Pi^{ab}\Pi_{ab} \\
&-\; \frac{1}{12}qq^{aa'}q^{bb'}q^{cc'}H_{abc}H_{a'b'c'}
\end{aligned}
$$

The constraints C^{ab}, which imply that the momenta π_i^a are not all independent, become:

$$
\left(\;E_i'^a - \mu_i^a + \frac{1}{1+\eta^2}\;(\pi_i^a - P_i^a)\;\right)(\pi_i^b - P_i^b) \;+\; (\;a \leftrightarrow b\;) \;\approx\; 0 \tag{5.11}
$$

Similarly, the expression for D^{ab} can also be obtained by replacing Σ_{IJ}^{ta}-s by π_{IJ}^a-s. To keep things simple, we do not write it explicitly here since that does not provide any conceptual insight as such.

5.4 SU(2) interpretation

We focus onto the expression of the rotation constraint to see how it can be interpreted as the generator of the SU(2) gauge transformations on the basic fields. After some algebra, this takes the following form:

$$
G_i^{rot} \;=\; \eta\; D_a(A)E_i'^a \;+\; \epsilon^{ijk}\omega_a^{0j}\pi_k^a \;-\; \epsilon^{ijk}t_j^a V_{ka} \;-\; \frac{1}{\eta}\;\Pi^{ab}(\partial_a A_b^i - \partial_a\omega_b^{0i})
$$

We note that the rotation constraint as above already provides an SU(2) gauge theoretic interpretation in terms of the field $\omega_a^{(\eta)0i}$. This can be seen explicitly from the transformation property of $\omega_a^{(\eta)0i}$. From equation (5.4), we obtain :

$$
\delta A_a^i \;=\; \delta\omega_a^{(\eta)0i} \;=\; \eta D_a\theta^i \tag{5.12}
$$

for the transformation parameters $\Theta_{jk} = -\epsilon_{ijk}\theta^i$ and $\Theta_{0i} = 0$. This implies that A^i_a transforms as a gauge field under a rotation. Now, under the rotation G^{rot}_i given by, $\omega^{(\eta)0i}_a$ transforms as :

$$\delta A^i_a \;=\; [A^i_a, \theta^k G^{rot}_k] \;=\; \eta D_a \theta^i \tag{5.13}$$

which agrees exactly with (5.12). Also, according to (5.4), the matter fields transform as

$$\delta A_{ab} \;=\; \frac{1}{2}\Theta^{ij}\partial_{[a}\omega^{ij}_{b]} \;=\; -\;\frac{1}{2\eta}\theta^i(\partial_{[a}A^i_{b]} - \partial_{[a}\omega^{oi}_{b]}) \tag{5.14}$$

which is exactly what is reproduced by G^{rot}_i:

$$\delta A_{ab} \;=\; [A_{ab}, \theta^k G^{rot}_k] \;=\; -\;\frac{1}{2\eta}\theta^i(\partial_{[a}A^i_{b]} - \partial_{[a}\omega^{oi}_{b]}) \tag{5.15}$$

Here we have exploited the fact that the pairs (A^i_a, E'^a_i) , (ω^{0i}_a, π^a_i) , (V^i_a, t^a_i) and (A_{ab}, π^{ab}) are canonical. Thus, we have an SU(2) theory already at this stage. To emphasize, this SU(2) description is independent of any gauge choice. One can nevertheless fix a gauge in order to exhaust some of the gauge freedom in the original theory. However, this is purely optional and the SU(2) interpretation would be preserved as it is in the gauge fixed theory as well. This completes the demonstration that the theory of gravity coupled to antisymmetric tensor gauge fields admits a real SU(2) description.

Chapter 6

Gravity with all possible topological parameters

Here we include all possible topological densities as in four-dimensional gravity and analyse the implications in the classical theory. As earlier, we first write this theory in the canonical form and investigate if it is possible to obtain a SU(2) gauge theoretic formulation.

6.1 Topological invariants in four dimensions

In (1+3) dimensions, apart from the Nieh-Yan term as discussed earlier, there are two other possible topological terms that can be added to the Hilbert-Palatini Lagrangian density. These are:

(i) Pontryagin class:

$$I_P \;=\; \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu IJ}(\omega) R_{\alpha\beta}{}^{IJ}(\omega) \tag{6.1}$$

This is the same topological density as in the case of QCD except that the gauge group here is $SO(1,3)$ instead of $SU(3)$. Again, it is a total divergence, given in terms of the $SO(1,3)$ Chern-Simons three-form:

$$I_P \;\equiv\; 4\partial_\mu \left[\epsilon^{\mu\nu\alpha\beta}\omega_\nu{}^{IJ}\left(\partial_\alpha\omega_{\beta IJ} + \frac{2}{3}\omega_{\alpha I}{}^{K}\omega_{\beta KJ}\right)\right] \tag{6.2}$$

For the Euclidean theory, this topological density, properly normalized, characterizes the *winding numbers* given by two integers corresponding to the homotopy group $\Pi_3(SO(4)) = Z\;+\;Z$.

(ii) Euler class:

$$I_E \;=\; \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu IJ}(\omega)\tilde{R}_{\alpha\beta}{}^{IJ}(\omega) \tag{6.3}$$

which again is a total divergence which can be explicitly written as:

$$I_E \;\equiv\; 4\partial_\mu \left[\epsilon^{\mu\nu\alpha\beta}\tilde{\omega}_\nu^{\;\;IJ}\left(\partial_\alpha \omega_{\beta IJ} + \frac{2}{3}\omega_{\alpha I}^{\;\;\;\;K}\omega_{\beta KJ}\right)\right] \tag{6.4}$$

For the Euclidean theory, integral of this topological density, properly normalized, over a compact four-manifold is an alternating sum of Betti numbers $b_0 - b_1 + b_2 - b_3$, characterizing the manifold.

Now we may construct the most general Lagrangian density by adding all three topological terms, namely the Nieh-Yan, Pontryagin and Euler, with the coefficients η, θ and ϕ respectively, to the Hilbert-Palatini Lagrangian density. Since all the topological terms are total divergences, the classical equations of motion are independent of the parameters η, θ and ϕ. However, the Hamiltonian formulation and the symplectic structure do see these parameters. Yet, classical dynamics are independent of them. But, quantum theory may depend on them.

All these topological terms in the action are functionals of local geometric quantities, yet they represent only the topological properties of the four-manifolds. These do not change under continous deformations of the four-manifold geometry.

Notice that, while the Nieh-Yan I_{NY} and Pontryagin I_P densities are P and T violating, the Euler density I_E is not. So in a quantum theory of gravity including these terms, besides the Newton's coupling constant, we can have three additional dimensionless coupling constants, two P and T violating (η, θ) and one P and T preserving (ϕ).

6.2 Hamiltonian formulation of gravity with Nieh-Yan, Pontryagin and Euler densities

Here we shall carry out the Hamiltonian analysis for the most general Lagrangian density containing all three topological terms besides the Hilbert-Palatini term:

$$\mathcal{L} \;=\; \frac{1}{2}\, e\, \Sigma^{\mu\nu}_{IJ}\, R_{\mu\nu}^{\;\;\;\;IJ}(\omega) \;+\; \frac{\eta}{2}\, I_{NY} \;+\; \frac{\theta}{4}\, I_P \;+\; \frac{\phi}{4}\, I_E \tag{6.5}$$

We shall use the following parametrization for tetrad fields[1]:

$$\begin{aligned} e_t^I \;&=\; NM^I + N^a V_a^I, \qquad e_a^I = V_a^I \;; \\ M_I V_a^I \;&=\; 0, \qquad M_I M^I = -1 \end{aligned} \tag{6.6}$$

[1]This parametrization differs from the one used earlier in [7]. To obtain the present parametrization replace eN by N^2 in the earlier parametrization.

with N and N^a as the lapse and shift fields. The inverse tetrads are:

$$\begin{aligned} e_I^t &= -\frac{M_I}{N} , \qquad e_I^a = V_I^a + \frac{N^a M_I}{N}; \\ M^I V_I^a &= 0 , \qquad V_a^I V_I^b = \delta_a^b , \qquad V_a^I V_J^a = \delta_J^I + M^I M_J \end{aligned} \tag{6.7}$$

The internal space metric is $\eta^{IJ} \equiv dia(-1,\ 1,\ 1,\ 1)$. The three-space metric is $q_{ab} \equiv V_a^I V_{bI}$ with $q = det(q_{ab})$ which leads to $e \equiv det(e_\mu^I) = N\sqrt{q}$. The inverse three-space metric is $q^{ab} = V_I^a V^{bI}, \quad q^{ab} q_{bc} = \delta_c^a$. Two useful identities are:

$$2e\Sigma_{IJ}^{ta} = -\sqrt{q}\ M_{[I} V_{J]}^a , \qquad e\Sigma_{IJ}^{ab} = \frac{2Ne^2}{\sqrt{q}} \Sigma_{IK}^{t[a} \Sigma_{JL}^{b]t} \eta^{KL} + e\ N^{[a} \Sigma_{IJ}^{b]t} \tag{6.8}$$

In this parametrization, we have, instead of the 16 tetrad components e_μ^I, the following 16 fields: 9 V_I^a ($M^I V_I^a = 0$), 3 M^I ($M^I M_I = -1$) and 4 lapse and shift vector fields N, N^a. From these, instead of the variables V_I^a and M^I, we define a convenient set of 12 variables, as:

$$E_i^a = 2e\Sigma_{0i}^{ta} \equiv e\left(e_0^t e_i^a - e_i^t e_0^a\right) = -\sqrt{q}\ M_{[0} V_{i]}^a , \qquad \chi_i = -M_i / M^0 \tag{6.9}$$

which further imply:

$$2e\Sigma_{ij}^{ta} = -\sqrt{q}\ M_{[i} V_{j]}^a = -E_{[i}^a \chi_{j]} \tag{6.10}$$

Now, using the parametrization (6.6, 6.7) for the tetrads, and the second identity in (6.8), we expand the various terms to write:

$$\frac{1}{2}\ e\ \Sigma_{IJ}^{\mu\nu}\ R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}\ I_{NY} = e\Sigma_{IJ}^{ta} \partial_t \omega_a^{(\eta)IJ} + t_I^a \partial_t V_a^I - NH - N^a H_a - \frac{1}{2}\omega_t^{IJ} G_{IJ} \tag{6.11}$$

where we have dropped the total space derivative terms. Here $t_I^a \equiv \eta \epsilon^{abc} D_b(\omega) V_{cI}$ with $\epsilon^{abc} \equiv \epsilon^{tabc}$ and, for any internal space antisymmetric tensor, $X_{IJ}^{(\eta)} \equiv X_{IJ} + \eta \tilde{X}_{IJ} = X_{IJ} + \frac{\eta}{2} \epsilon_{IJKL} X^{KL}$. Further,

$$\begin{aligned} H &= \frac{2e^2}{\sqrt{q}} \Sigma_{IK}^{ta} \Sigma_{JL}^{tb} \eta^{KL} R_{ab}^{IJ}(\omega) = \frac{2e^2}{\sqrt{q}} \Sigma_{IK}^{ta} \Sigma_{JL}^{tb} \eta^{KL} R_{ab}^{(\eta)IJ}(\omega) - M^I D_a(\omega) t_I^a \\ H_a &= e\Sigma_{IJ}^{tb} R_{ab}^{IJ}(\omega) = e\Sigma_{IJ}^{tb} R_{ab}^{(\eta)IJ}(\omega) - V_a^I D_b(\omega) t_I^b \\ G_{IJ} &= -2D_a(\omega)\left\{e\Sigma_{IJ}^{ta}\right\} = -2D_a(\omega)\left\{e\Sigma_{IJ}^{(\eta)ta}\right\} - t_{[I}^a V_{J]a} \end{aligned} \tag{6.12}$$

where we have used the following identities:

$$\begin{aligned} M^I D_a(\omega) t_I^a &\equiv \frac{2\eta e^2}{\sqrt{q}} \Sigma_{IK}^{ta} \Sigma_{JL}^{tb} \eta^{KL} \tilde{R}_{ab}{}^{IJ}(\omega) \\ V_a^I D_b(\omega) t_I^b &\equiv \eta e \Sigma_{IJ}^{tb} \tilde{R}_{ab}{}^{IJ}(\omega) , \qquad t_{[I}^a V_{J]a} \equiv -2\eta D_a(\omega)\left\{e\tilde{\Sigma}_{IJ}^{ta}\right\} \end{aligned}$$

Next notice that, dropping the total space derivative terms and using the Bianchi identity, $\epsilon^{abc}D_a(\omega)R_{bcIJ} \equiv 0$, we can write

$$\frac{\theta}{4}\, I_P \;+\; \frac{\phi}{4}\, I_E \;=\; e^a_{IJ}\;\partial_t \omega_a^{(\eta)IJ} \tag{6.13}$$

where e^a_{IJ} are given by

$$\left(1+\eta^2\right) e^a_{IJ} \;=\; \epsilon^{abc}\left\{\left(\theta+\eta\phi\right) R_{bcIJ}(\omega) + \left(\phi-\eta\theta\right) \tilde{R}_{bcIJ}(\omega)\right\} \tag{6.14}$$

Thus, collecting terms from (6.11) and (6.13), full Lagrangian density (6.5) assumes the following form:

$$\mathcal{L} \;=\; \pi^a_{IJ}\partial_t \omega_a^{(\eta)IJ} \;+\; t^a_I \partial_t V^I_a \;-\; NH \;-\; N^a H_a \;-\; \frac{1}{2}\omega_t^{IJ} G_{IJ} \tag{6.15}$$

with

$$\pi^a_{IJ} \;=\; e\;\Sigma^{ta}_{IJ} \;+\; e^a_{IJ} \tag{6.16}$$

In this Lagrangian density, the fields $\omega_a^{(\eta)IJ}$ and π^a_{IJ} form canonical pairs. Then, H, H_a and G_{IJ} of (6.12) can be expressed in terms of these fields as:

$$G_{IJ} \;=\; -\,2D_a(\omega)\pi^{a(\eta)}_{IJ} - t^a_{[I}V_{J]a} \tag{6.17}$$

$$H_a \;=\; \pi^b_{IJ}R^{(\eta)IJ}_{ab}(\omega) \;-\; V^I_a D_b(\omega)t^b_I \tag{6.18}$$

$$H \;=\; \frac{2}{\sqrt{q}}\left(\pi^{a(\eta)}_{IK} - e^{a(\eta)}_{IK}\right)\left(\pi^{b(\eta)}_{JL} - e^{b(\eta)}_{JL}\right)\eta^{KL}R_{ab}{}^{IJ}(\omega) - M^I D_a(\omega)t^a_I \tag{6.19}$$

where we have used the relations: $D_a(\omega)e^a_{IJ} = 0$ and $D_a(\omega)\tilde{e}^a_{IJ} = 0$ which result from the Bianchi identity $\epsilon^{abc}D_a(\omega)R_{bcIJ}(\omega) = 0$, and also used $e^b_{IJ}R_{ab}{}^{IJ}(\omega) = 0$ and $\tilde{e}^b_{IJ}R_{ab}{}^{IJ}(\omega) = 0$ which follow from the fact that $2q\,(\theta^2+\phi^2)\,R_{abIJ} = \epsilon_{abc}\left\{(\theta+\eta\phi)\,e^c_{IJ} - (\phi-\eta\theta)\,\tilde{e}^c_{IJ}\right\}$.

Now, in order to unravel the $SU(2)$ gauge theoretic framework for the Hamiltonian formulation, from the 24 $SO(1,3)$ gauge fields $\omega_\mu{}^{IJ}$, we define, in addition to 6 field variables ω^{IJ}_t, the following suitable set of 18 field variables:

$$A^i_a \;\equiv\; \omega_a^{(\eta)0i} = \omega_a^{0i} + \eta\tilde{\omega}_a^{0i}, \qquad K^i_a \;\equiv\; \omega_a^{0i} \tag{6.20}$$

The fields A^i_a transform as the connection and the extrinsic curvature K^i_a as adjoint representations under the $SU(2)$ gauge transformations. In terms of these, it is straight forward to check that:

$$\pi^a_{IJ}\partial_t\omega_a^{(\eta)IJ} \;=\; 2\pi^a_{0i}\partial_t\omega_a^{(\eta)0i} \;+\; \pi^a_{ij}\partial_t\omega_a^{(\eta)ij} \;=\; \hat{E}^a_i\partial_t A^i_a \;+\; \hat{F}^a_i\partial_t K^i_a \tag{6.21}$$

with

$$\hat{E}_a^i \equiv -\frac{2}{\eta}\,\tilde{\pi}_{0i}^{a(\eta)} \equiv -\frac{2}{\eta}\,(\tilde{\pi}_{0i}^a - \eta\pi_{0i}^a) = E_i^a - \frac{2}{\eta}\,\tilde{e}_{0i}^{a(\eta)}(A,K) + \frac{1}{\eta}\,\epsilon^{ijk}E_j^a\chi_k \quad (6.22)$$

$$\hat{F}_i^a \equiv 2\left(\eta + \frac{1}{\eta}\right)\tilde{\pi}_{0i}^a = \left(\eta + \frac{1}{\eta}\right)\left\{-\epsilon^{ijk}E_j^a\chi_k + 2\tilde{e}_{0i}^a(A,K)\right\} \quad (6.23)$$

where e_{0i}^a and $\tilde{e}_{oi}^a \equiv \frac{1}{2}\epsilon^{ijk}e_{jk}^a$ as defined in (6.14) and $\tilde{e}_{0i}^{a(\eta)} \equiv \tilde{e}_{0i}^a - \eta e_{0i}^a$ are written as functions of the gauge field A_a^i and the extrinsic curvature K_a^i using

$$R_{ab}{}^{0i}(\omega) = D_{[a}(A)K_{b]}^i - \frac{2}{\eta}\,\epsilon^{ijk}K_a^jK_b^j$$

$$R_{ab}{}^{ij}(\omega) = -\frac{1}{\eta}\,\epsilon^{ijk}F_{ab}^k(A) + \frac{1}{\eta}\,\epsilon^{ijk}D_{[a}(A)K_{b]}^k - \left(\frac{\eta^2-1}{\eta^2}\right)K_{[a}^iK_{b]}^j \quad (6.24)$$

with the $SU(2)$ field strength and covariant derivative respectively as:

$$F_{ab}^i(A) \equiv \partial_{[a}A_{b]}^i + \frac{1}{\eta}\,\epsilon^{ijk}A_a^jA_b^k\,, \qquad D_a(A)K_b^i \equiv \partial_aK_b^i + \frac{1}{\eta}\,\epsilon^{ijk}A_a^jK_b^k \quad (6.25)$$

Now, using (6.21), the Lagrangian density (6.15) can be written as:

$$\mathcal{L} = \hat{E}_i^a\partial_tA_a^i + \hat{F}_i^a\partial_tK_a^i + t_I^a\partial_tV_a^I - NH - N^aH_a - \frac{1}{2}\,\omega_t^{IJ}G_{IJ} \quad (6.26)$$

Thus, we have the canonically conjugate pairs $(A_a^i,\ \hat{E}_i^a)$, $(K_a^i,\ \hat{F}_i^a)$ and $(V_a^I,\ t_I^a)$. We may write G_{IJ}, H_a and H of (6.17)-(6.19) in terms of these fields. For example, from (6.17):

$$G_i^{rot} \equiv \frac{1}{2}\,\epsilon_{ijk}G_{jk} = \eta D_a(A)\hat{E}_i^a + \epsilon_{ijk}\left(K_a^j\hat{F}_k^a - t_j^aV_a^k\right) \quad (6.27)$$

$$G_i^{boost} \equiv G_{0i} = -D_a(A)\left(\hat{E}_i^a + \hat{F}_i^a\right) + \epsilon^{ijk}K_a^j\left\{\left(\eta + \frac{1}{\eta}\right)\hat{E}_k^a + \frac{1}{\eta}\hat{F}_k^a\right\} - t_{[0}^aV_{i]a} \quad (6.28)$$

$$= -D_a(A)\hat{F}_i^a + \epsilon^{ijk}K_a^j\left\{\left(\eta + \frac{1}{\eta}\right)\hat{E}_k^a + \frac{2}{\eta}\hat{F}_k^a\right\} - \frac{1}{\eta}\epsilon^{ijk}t_j^aV_{ak} - t_{[0}^aV_{i]a} - \frac{1}{\eta}\,G_i^{rot}$$

where the covariant derivatives are: $D_a(A)\hat{E}_i^b = \partial_a\hat{E}_i^b + \eta^{-1}\,\epsilon^{ijk}A_a^j\hat{E}_k^b$ and $D_a(A)\hat{F}_i^b = \partial_a\hat{F}_i^b + \eta^{-1}\,\epsilon^{ijk}A_a^j\hat{F}_k^b$. Next, for the generators of spatial diffeomorphisms H_a from (6.18):

$$H_a = \hat{E}_i^bF_{ab}^i(A) + \hat{F}_i^bD_{[a}(A)K_{b]}^i - K_a^iD_b(A)\hat{F}_i^b + t_i^bD_{[a}(A)V_{b]}^i - V_a^iD_b(A)t_i^b$$
$$+ t_0^b\partial_{[a}V_{b]}^0 - V_a^0\partial_bt_0^b - \frac{1}{\eta}\left(G_i^{rot} + \eta G_i^{boost}\right)K_a^i$$
$$= \hat{E}_i^b\partial_{[a}A_{b]}^i - A_a^i\partial_b\hat{E}_i^b + \hat{F}_i^b\partial_{[a}K_{b]}^i - K_a^i\partial_b\hat{F}_i^b + t_i^b\partial_{[a}V_{b]}^i - V_a^i\partial_bt_i^b$$
$$+ t_0^b\partial_{[a}V_{b]}^0 - V_a^0\partial_bt_0^b + \frac{1}{\eta}G_i^{rot}A_a^i - \frac{1}{\eta}\left(G_i^{rot} + \eta G_i^{boost}\right)K_a^i \quad (6.29)$$

where we have used $-V_a^I D_b(\omega) t_I^b \equiv -V^I \partial_b t_I^b + t_I^b \partial_{[a} V_{b]}^I + t_I^b V_{Jb} \omega_a^{IJ}$. Similarly we can express H of (6.19) in terms of these fields.

Now, notice that all the fields $(A_a^i,\ \hat{E}_i^a)$, $(K_a^i,\ \hat{F}_i^a)$ and $(V_a^I,\ t_I^a)$ in the Lagrangian density (6.26) are not independent. Of these, the fields V_a^I and t_I^a are given in terms of others as: $V_a^I = v_a^I$ and $t_I^a = \tau_I^a$ with

$$v_a^i \ \equiv \ \frac{1}{\sqrt{E}} E_a^i \ , \qquad\qquad v_a^0 \equiv -\frac{1}{\sqrt{E}} E_a^i \chi_i \tag{6.30}$$

where E_a^i is inverse of E_i^a, *i.e.*, $E_a^i E_i^b = \delta_a^b$, $E_a^i E_j^a = \delta_j^i$ and $E \ \equiv \ det(E_a^i) = q^{-1}(M^0)^{-2}$ and

$$\begin{aligned} \tau_i^a \ &\equiv \ \eta \epsilon^{abc} D_b(\omega) v_c^i \ = \ \epsilon^{abc} \left(\eta D_b(A) v_c^i - \epsilon^{ijk} K_b^j v_c^k + K_b^i v_c^0 \right) \ , \\ \tau_0^a \ &\equiv \ -\eta \epsilon^{abc} D_b(\omega) v_c^0 = -\eta \epsilon^{abc} \left(\partial_b v_c^0 + K_b^j v_c^j \right) \end{aligned} \tag{6.31}$$

In addition, the fields $\hat{F}_i^a$, which are conjugate to the extrinsic curvature K_a^i, are also not independent; these are given in terms of other fields by (6.23) .

In the Lagrangian density (6.26), there are no velocity terms associated with $SO(1,3)$ gauge fields ω_t^{IJ}, shift vector field N_a and lapse field N. Hence these fields are Lagrange multipliers. Associated with these are as many constraints: $G_{IJ} \approx 0$, $H_a \approx 0$, and $H \approx 0$ where the weak equality $\approx$ is in the sense of Dirac theory of constrained Hamiltonian systems. Here from the form of $G_i^{rot} = \frac{1}{2}\ \epsilon_{ijk} G_{jk}$ in (6.27), it is clear that these generate $SU(2)$ rotations on various fields. The boost transformations are generated by $G_i^{boost} = G_{0i}$, spatial diffeomorphisms by H_a and $H \approx 0$ is the Hamiltonian constraint. This, thus can already be viewed, *without fixing the boost degrees of freedom and without solving the second class constraints (6.30) and (6.31)*, as a $SU(2)$ gauge theoretic framework. Here, besides the three $SU(2)$ generators G_i^{rot}, we have seven constraints, G_i^{boost}, H_a and H. We may, however, fix the boost gauge invariance by choosing a time gauge. Then we are left with only the $SU(2)$ gauge invariance besides the diffeomorphism H_a and Hamiltonian H constraints. This we do in the next section.

6.3 Time gauge

We work in the time (boost) gauge by choosing the gauge condition $\chi_i = 0$ which then implies for the tetrad components $e_a^0 \equiv V_a^0 = 0$. Correspondingly the boost generators (6.28) are also set equal to zero strongly, $G_i^{boost} = 0$. In this gauge, the Lagrangian density (6.26) takes the simple form:

$$\mathcal{L} = \ \hat{E}_i^a \partial_t A_a^i \ + \ \hat{F}_i^a \partial_t K_a^i \ + \ t_i^a \partial_t V_a^i - \mathcal{H} \tag{6.32}$$

with the Hamiltonian density as:

$$\mathcal{H} \; = \; NH \; + \; N^a H_a \;\; + \; \frac{1}{2} \, \epsilon^{ijk} \omega_t^{ij} G_k^{rot} \; + \; \xi_i^a \left(V_a^i - v_a^i \right)$$
$$+ \; \phi_a^i \left(t_i^a - \tau_i^a \right) \; + \; \lambda_a^i \left\{ \hat{F}_i^a - 2 \left(\eta + \frac{1}{\eta} \right) \tilde{e}_{0i}^a (A, K) \right\} \tag{6.33}$$

where all the fields involved are not independent. In particular, the fields V_a^i, t_i^a and $\hat{F}_i^a$ depend on other fields. This fact is reflected in $\mathcal{H}$ above through terms with Lagrange multiplier fields ξ_i^a, ϕ_a^i and λ_a^i. Now, in this time gauge, expressions for G_i^{rot}, H_a and H are:

$$G_i^{rot} \; \equiv \; \eta D_a(A) \hat{E}_i^a + \epsilon_{ijk} \left(K_a^j \hat{F}_k^a - t_j^a V_a^k \right)$$
$$H_a \equiv \; \hat{E}_i^b F_{ab}^i(A) + \hat{F}_i^b D_{[a}(A) K_{b]}^i - K_a^i D_b(A) \hat{F}_i^b + t_i^b D_{[a}(A) V_{b]}^i - V_a^i D_b(A) t_i^b - \eta^{-1} G_i^{rot} K_a^i$$
$$= \; \hat{E}_i^b \partial_{[a} A_{b]}^i - A_a^i \partial_b \hat{E}_i^b + \hat{F}_i^b \partial_{[a} K_{b]}^i - K_a^i \partial_b \hat{F}_i^b + t_i^b \partial_{[a} V_{b]}^i - V_a^i \partial_b t_i^b + \eta^{-1} G_i^{rot} \left(A_a^i - K_a^i \right)$$
$$H \; \equiv \; \frac{\sqrt{E}}{2\eta} \epsilon^{ijk} E_i^a E_j^b \left\{ F_{ab}^k(A) - \left(1 + \eta^2 \right) \left(D_{[a}(A) K_{b]}^k - \eta^{-1} \, \epsilon^{kmn} K_a^m K_b^n \right) \right\}$$
$$+ \; K_a^i t_i^a \; - \eta \; \partial_a \left(\sqrt{E} G_k^{rot} E_k^a \right) \tag{6.34}$$

where $D_a(A)$ is the $SU(2)$ gauge covariant derivative. In the last line, we have used the time-gauge identity: $t_0^a = \tau_0^a = \eta \sqrt{E} G_k^{rot} E_k^a$. Also E_i^a are functions of $\hat{E}_i^a$, A_a^i and K_a^i:

$$E_i^a \; = \; E_i^a(\hat{E}, A, K) \; \equiv \; \hat{E}_i^a \; + \; \frac{2}{\eta} \; \tilde{e}_{0i}^{a(\eta)}(A, K) \tag{6.35}$$

.

Associated with the Lagrange multiplier fields ω_t^{ij}, N^a and N in (6.33), we have the constraints:

$$G_i^{rot} \; \approx \; 0, \qquad H_a \; \approx \; 0 \; , \qquad H \; \approx \; 0 \tag{6.36}$$

In addition, corresponding to Lagrange multiplier fields ξ_i^a and ϕ_a^i, we have more constraints:

$$V_a^i \; - \; v_a^i(E) \; \approx \; 0 \; , \qquad t_i^a \; - \; \tau_i^a(A, K, E) \; \approx \; 0 \tag{6.37}$$

where, from (6.30) and (6.31), in the time gauge:

$$v_a^i \; \equiv \; \frac{1}{\sqrt{E}} \; E_a^i \; , \qquad \tau_i^a \equiv \eta \epsilon^{abc} D_b(\omega) v_c^i \; = \; \epsilon^{abc} \left(\eta D_b(A) v_c^i - \epsilon^{ijk} K_b^j v_c^k \right) \tag{6.38}$$

Similarly, from the last term in (6.33), there are the additional constraints:

$$\chi_i^a \;\equiv\; \hat{F}_i^a \;-\; 2\left(\eta+\frac{1}{\eta}\right)\tilde{e}_{0i}^a(A,K) \;\approx\; 0 \tag{6.39}$$

Here e_{0i}^a and $\tilde{e}_{0i}^a$ of (6.14), with the help of Eqn.(6.24), are written as functions of the gauge fields A_a^i, extrinsic curvature K_a^i and the topological parameters θ, ϕ besides η as follows:

$$\begin{aligned}
\eta^2\left(1+\eta^2\right)e_{0i}^a(A,K) \equiv -\epsilon^{abc}\Big\{&\eta\left(\phi-\eta\theta\right)F_{bc}^i(A) - 2\eta\left(\left(1-\eta^2\right)\phi - 2\eta\theta\right)D_b(A)K_c^i \\
&- \left(\eta\left(3-\eta^2\right)\theta + \left(3\eta^2-1\right)\phi\right)\epsilon^{ijk}K_b^jK_c^k\Big\} \\
\eta^2\left(1+\eta^2\right)\tilde{e}_{0i}^a(A,K) \equiv -\epsilon^{abc}\Big\{&\eta\left(\theta+\eta\phi\right)F_{bc}^i(A) - 2\eta\left(\left(1-\eta^2\right)\theta + 2\eta\phi\right)D_b(A)K_c^i \\
&- \left(\left(3\eta^2-1\right)\theta - \eta\left(3-\eta^2\right)\phi\right)\epsilon^{ijk}K_b^jK_c^k\Big\}
\end{aligned} \tag{6.40}$$

From these we can construct for $e_{0i}^{a(\eta)} \equiv e_{0i}^a + \eta\tilde{e}_{0i}^a$ and $\tilde{e}_{0i}^{a(\eta)} \equiv \tilde{e}_{0i}^a - \eta e_{0i}^a$:

$$\begin{aligned}
e_{0i}^{a(\eta)} &= -\frac{1}{\eta}\,\epsilon^{abc}\left\{\phi F_{bc}^i(A) - \left(\phi-\eta\theta\right)D_{[b}(A)K_{c]}^i - \left(\frac{(\eta^2-1)\phi+2\eta\theta}{\eta}\right)\epsilon^{ijk}K_b^jK_c^k\right\} \\
\tilde{e}_{0i}^{a(\eta)} &= -\frac{1}{\eta}\,\epsilon^{abc}\left\{\theta F_{bc}^i(A) - \left(\theta+\eta\phi\right)D_{[b}(A)K_{c]}^i - \left(\frac{(\eta^2-1)\theta-2\eta\phi}{\eta}\right)\epsilon^{ijk}K_b^jK_c^k\right\}
\end{aligned} \tag{6.41}$$

The χ_i^a constraints (6.39) are of particular interest. To study their effect, we note that $(A_a^i,\ \hat{E}_j^b)$ and $(K_a^i, \hat{F}_j^b)$ are canonically conjugate pairs. They have accordingly the standard Poisson brackets. From these, using the relation (6.35) expressing E_i^a in terms of $\hat{E}_i^a$, A_a^i and K_a^i, as indicated in the Appendix, the following Poisson brackets can be calculated with respect to phase variables $(A_a^i, \hat{E}_i^a)$ and $(K_a^i, \hat{F}_i^a)$:

$$\begin{aligned}
&\left[A_a^i(x),\ E_j^b(y)\right] = \left[A_a^i(x),\ \hat{E}_j^b(y)\right] = \delta_j^i\delta_a^b\ \delta^{(3)}(x,y), \\
&\left[K_a^i(x),\ E_j^b(y)\right] = 0\ , \qquad \left[E_i^a(x),\ E_j^b(y)\right] \;=\; 0
\end{aligned}$$

These then imply the Poisson bracket relations:

$$\begin{aligned}
\left[\chi_i^a(x),\ A_b^j(y)\right] &\;=\; 0\ , \qquad \left[\chi_i^a(x),\ K_b^j(y)\right] \;=\; -\,\delta_j^i\delta_b^a\delta^{(3)}(x,y)\ , \\
\left[\chi_i^a(x),\ E_j^b(y)\right] &\;=\; 0\ , \qquad \left[\chi_i^a(x),\ \chi_j^b(y)\right] \;=\; 0
\end{aligned} \tag{6.42}$$

Using these, we notice that the Poisson brackets of Hamiltonian constraint H and χ_i^a are non-zero. Requiring $[\chi_i^a(x), H(y)] \approx 0$ leads us to the secondary constraints as:

$$t_i^a - \left(\frac{1+\eta^2}{\eta^2}\right)\left\{\eta\epsilon^{ijk}D_b(A)\left(\sqrt{E}E_j^aE_k^b\right) + \sqrt{E}E_j^{[a}E_i^{b]}K_b^j\right\} \;\approx\; 0$$

which can be rewritten as:

$$t_i^a - \left(\frac{1+\eta^2}{\eta^2}\right)\epsilon^{abc}\left\{\eta D_b(A)v_c^i - \epsilon^{ijk}K_b^j v_c^k\right\} \;\approx\; 0$$

Next, since from (6.37) and (6.38), $t_i^a \approx \tau_i^a \;\equiv\; \epsilon^{abc}\left\{\eta D_b(A)v_c^i - \epsilon^{ijk}K_b^j v_c^k\right\}$, this implies $t_i^a \;\approx\; 0$. Thus we have the constraints:

$$\epsilon^{abc}\left\{\eta D_b(A)v_c^i - \epsilon^{ijk}K_b^j v_c^k\right\} \;\approx\; 0$$

These can be solved for the extrinsic curvature K_a^i and recast as the following secondary constraints:

$$\begin{aligned}
\psi_a^i \;&\equiv\; K_a^i - \kappa_a^i(A,E) \;\approx\; 0\ , \\
\kappa_a^i(A,E) \;&\equiv\; \frac{\eta}{2}\,\epsilon^{ijk}E_a^j D_b(A)E_k^b \\
&\quad - \frac{\eta}{2E}E_a^k\epsilon^{bcd}\left\{E_b^k D_c(A)E_d^i + E_b^i D_c(A)E_d^k - \delta^{ik}E_b^m D_c(A)E_d^m\right\}
\end{aligned} \qquad (6.43)$$

These are additional constraints and have the important property that these form second class pairs with the constraints χ_i^a of (6.39):

$$\left[\chi_i^a(x),\ \psi_b^j(y)\right] \;=\; -\,\delta_b^a\delta_i^j\delta^{(3)}(x,y) \qquad (6.44)$$

To implement these second class constraints, χ_i^a and ψ_a^i, we need to go over from Poisson brackets to the corresponding Dirac brackets and then impose the constraints strongly, $\psi_a^i = 0$ (which also implies $t_i^a = 0$) and $\chi_i^a = 0$, in accordance with Dirac theory of constrained Hamiltonian systems. As outlined in the Appendix, *the Dirac brackets of fields* A_a^i *and* E_i^a *turn out to be the same as their Poisson brackets; these are displayed in (6.71). On the other hand, those for* $(A_a^i,\ \hat{E}_i^a;\ K_a^i,\ \hat{F}_i^a)$ *are different; these have been listed in (6.73) and (6.74).*

Finally, after implementing these second class constraints, we have the Lagrangian density in the time-gauge as:

$$\mathcal{L} = \hat{E}_i^a\partial_t A_a^i \;+\; \hat{F}_i^a\partial_t K_a^i \;-\mathcal{H} \qquad (6.45)$$

with the Hamiltonian density

$$\mathcal{H} \;=\; NH + N^a H_a + \frac{1}{2}\,\epsilon^{ijk}\omega_t^{jk}G_k^{rot} \qquad (6.46)$$

and a set of seven first class constraints:

$$
\begin{aligned}
G_i^{rot} &\equiv \eta\, D_a(A)\hat{E}_i^a \;+\; \epsilon^{ijk}K_a^j\hat{F}_k^a \;\approx\; 0 \\
H_a &\equiv \hat{E}_i^b F_{ab}^i(A) \;+\; \hat{F}_i^b D_{[a}(A)K_{b]}^i \;-\; K_a^i D_b(A)\hat{F}_i^b - \eta^{-1}\, G_i^{rot}K_a^i \;\approx\; 0 \\
H &\equiv \frac{\sqrt{E}}{2\eta}\epsilon^{ijk}E_i^aE_j^bF_{ab}^k(A) - \left(\frac{1+\eta^2}{2\eta^2}\right)\sqrt{E}E_i^aE_j^bK_{[a}^iK_{b]}^j + \frac{1}{\eta}\,\partial_a\left(\sqrt{E}G_k^{rot}E_k^a\right) \approx\; 0
\end{aligned} \tag{6.47}
$$

with E_i^a in the last equation given by: $E_i^a \;=\; E_i^a(\hat{E},A,K) \equiv \hat{E}_i^a + \frac{2}{\eta}\; \tilde{e}_{0i}^{a(\eta)}(A,K)$. The fields $(A_a^i,\ \hat{E}_i^a,\ K_a^i,\ \hat{F}_i^a)$ have non-trivial Dirac brackets as listed in (6.73) and (6.74). The second class constraints χ_i^a and ψ_a^i are now set strongly equal to zero:

$$
\begin{aligned}
K_a^i &= \kappa_a^i(A,E) \;\equiv\; \frac{\eta}{2}\,\epsilon^{ijk}\; E_a^jD_b(A)E_k^b - \frac{\eta}{2E}\; E_a^k\,\epsilon^{bcd}\Big\{E_b^kD_c(A)E_d^i + E_b^iD_c(A)E_d^k \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \delta^{ik}E_b^mD_c(A)E_d^m\Big\} \\
\hat{F}_i^a &= 2\left(\eta+\frac{1}{\eta}\right)\tilde{e}_{0i}^a(A,K)
\end{aligned} \tag{6.48}
$$

In writing the Hamiltonian constraint H in (6.47) from (6.34), we have used the identity:

$$
\sqrt{E}\;\epsilon^{ijk}E_i^bE_j^c\left(D_b(A)K_c^k - \frac{1}{\eta}\;\epsilon^{kmn}K_b^mK_c^n\right) = -\;\partial_a\left(\sqrt{E}E_i^aG_i^{rot}\right) \tag{6.49}
$$

which holds due to the time gauge relation $EE_i^aG_i^{rot} \;=\; \epsilon^{abc}E_b^iK_c^i$ with the constraints $K_a^i = \kappa_a^i(A,E)$ imposed strongly.

6.4 SU(2) interpretation

To evaluate the effect of generators (6.47) on various fields, we need to use the Dirac brackets instead of the Poisson brackets. For example, for the $SU(2)$ gauge generators, using the results listed in the Appendix, we obtain:

$$
\begin{aligned}
\left[G_i^{rot}(x),\ \hat{E}_j^a(y)\right]_D &= \epsilon^{ijk}\hat{E}_k^a\;\delta^{(3)}(x,y)\;, \\
\left[G_i^{rot}(x),\ A_a^j(y)\right]_D &= -\eta\left(\delta^{ij}\partial_a\; + \eta^{-1}\;\epsilon^{ikj}A_a^k\right)\;\delta^{(3)}(x,y)
\end{aligned} \tag{6.50}
$$

reflecting the fact G_i^{rot} are generators of $SU(2)$ transformations: A_a^i transform as the $SU(2)$ connection and fields $\hat{E}_i^a$ as adjoint representations. Besides, the fields $\hat{F}_i^a,\ K_a^i$ and E_i^a also behave as adjoint representations under $SU(2)$ rotations:

$$\begin{aligned}
\left[G_i^{rot}(x),\ \hat{F}_j^a(y)\right]_D &= \epsilon^{ijk}\hat{F}_k^a\ \delta^{(3)}(x,y) \\
\left[G_i^{rot}(x),\ K_a^j(y)\right]_D &= \epsilon^{ijk}K_a^k\ \delta^{(3)}(x,y) \\
\left[G_i^{rot}(x),\ E_j^a(y)\right]_D &= \epsilon^{ijk}E_k^a\ \delta^{(3)}(x,y)
\end{aligned} \tag{6.51}$$

Similar discussion is valid for the spatial diffeomorphism generators H_a. The Dirac brackets of H_a with various fields yield the Lie derivatives of these fields respectively, modulo $SU(2)$ gauge transformations.

As stated earlier and demonstrated in the Appendix, Dirac brackets for the fields $(A_a^i,\ \hat{E}_i^a;\ K_a^i,\ \hat{F}_i^a)$ are different from their Poisson brackets (see (6.72), (6.73) and (6.74)). This is so because the transition from Poisson brackets to Dirac brackets, except for some special cases, in general, does not preserve canonical structure of the algebra. When the second class constraints are imposed strongly, the algebraic structure of the Dirac brackets of phase variables $(A_a^i,\ \hat{E}_i^a)$ of the final theory is different from those of the phase variables $(A_a^i,\ E_i^a)$ of the standard canonical theory. Thus the variables $(A_a^i,\ \hat{E}_i^a)$ are not related to $(A_a^i,\ E_i^a)$ through a canonical transformation. However, it is possible to construct a set of new phase space field variables whose Dirac bracket algebra has the same structure as that of the standard canonical variables $(A_a^i,\ E_i^a)$.

In fact, in general, for theories with second class constraints as is the case here, instead of the ordinary canonical transformations, what is relevant are the Gitman D-transformations, which preserve the form invariance of Dirac brackets and equations of motion [24]. Thus, in the present context also, new phase variables can be constructed through these D-transformations. These transformations change both the gauge fields as well as their conjugate momentum fields.

6.5 Poisson and Dirac brackets

In the time-gauge Lagrangian density (6.45), the fields $(A_a^i,\ \hat{E}_i^a)$ and $(K_a^i,\ \hat{F}_i^a)$ are canonical pairs which have the standard Poisson bracket relations:

$$[A_a^i(t,\vec{x}),\ \hat{E}_j^b(t,\vec{y})] = \delta_j^i\delta_a^b\ \delta^{(3)}(\vec{x},\vec{y})\ , \qquad [K_a^i(t,\vec{x}),\ \hat{F}_j^b(t,\vec{y})] = \delta_j^i\delta_a^b\ \delta^{(3)}(\vec{x},\vec{y}) \tag{6.52}$$

and all other brackets amongst these fields are zero. Thus the Poisson bracket for any two arbitrary fields P and Q is given by:

$$[P(x),\ Q(y)] = \int d^3z \left(\frac{\delta P(x)}{\delta A_a^i(z)}\,\frac{\delta Q(y)}{\delta \hat{E}_i^a(z)} - \frac{\delta P(x)}{\delta \hat{E}_i^a(z)}\,\frac{\delta Q(y)}{\delta A_a^i(z)}\right)$$

$$+ \int d^3z \left(\frac{\delta P(x)}{\delta K^i_a(z)} \frac{\delta Q(y)}{\delta \hat{F}^a_i(z)} - \frac{\delta P(x)}{\delta \hat{F}^a_i(z)} \frac{\delta Q(y)}{\delta K^i_a(z)} \right) \tag{6.53}$$

From these, using $E^a_i = E^a_i(\hat{E}, A, K) \equiv \hat{E}^a_i + 2\eta^{-1}\ \tilde{e}^{a(\eta)}_{0i}(A, K)$, we have the Poisson bracket relations

$$\left[A^i_a(x),\ E^b_j(y)\right] = \left[A^i_a(x),\ \hat{E}^b_j(y)\right] = \delta^i_j \delta^b_a\ \delta^{(3)}(x, y), \qquad \left[K^i_a(x),\ E^b_j(y)\right] = 0 \tag{6.54}$$

Using the expressions for $e^a_{0i}(A, K)$ and $\tilde{e}^a_{0i}(A, K)$ as functions of A^i_a and K^i_a as in (6.40), the following relations obtain:

$$\begin{aligned}
\left[\hat{E}^a_i(x), E^b_j(y)\right] &= \frac{2}{\eta}\left[\hat{E}^a_i(x), \tilde{e}^{b(\eta)}_{0j}(y)\right] = -\frac{4}{\eta^2}\ \epsilon^{abc}\left\{\theta D^{ij}_c - \left(\frac{\theta + \eta\phi}{\eta}\right)\epsilon^{ikj} K^k_c\right\}\delta^{(3)}(x, y) \\
\left[\hat{F}^a_i(x), E^b_j(y)\right] &= \frac{2}{\eta}\left[\hat{F}^a_i(x), \tilde{e}^{b(\eta)}_{0j}(y)\right] \\
&= \frac{4}{\eta^2}\ \epsilon^{abc}\left\{(\theta + \eta\phi) D^{ij}_c - \left(\frac{(1-\eta^2)\theta + 2\eta\phi}{\eta}\right)\epsilon^{ikj} K^k_c\right\}\delta^{(3)}(x, y) \\
\left[\tilde{e}^{a(\eta)}_{0i}(x), E^b_j(y)\right] &= \left[\tilde{e}^{a(\eta)}_i(x), \hat{E}^b_j(y)\right] = \frac{2}{\eta}\ \epsilon^{abc}\left\{\theta D^{ij}_c - \left(\frac{\theta + \eta\phi}{\eta}\right)\epsilon^{ikj} K^k_c\right\}\delta^{(3)}(x, y) \\
(1+\eta^2)\left[\tilde{e}^a_{0i}(x), E^b_j(y)\right] &= (1+\eta^2)\left[\tilde{e}^a_{0i}(x), \hat{E}^b_j(y)\right] \\
&= \frac{2}{\eta}\ \epsilon^{abc}\left\{(\theta + \eta\phi) D^{ij}_c - \left(\frac{(1-\eta^2)\theta + 2\eta\phi}{\eta}\right)\epsilon^{ikj} K^k_c\right\}\delta^{(3)}(x, y)
\end{aligned} \tag{6.55}$$

where the $SU(2)$ gauge covariant derivative is: $D^{ij}_c \equiv \delta^{ij}\partial_c + \eta^{-1}\epsilon^{ikj} A^k_c$. These Poisson bracket relations imply for $E^a_i = E^a_i(\hat{E}, A, K) \equiv \hat{E}^a_i + 2\eta^{-1}\ \tilde{e}^{a(\eta)}_{0i}(A, K)$:

$$\left[E^a_i(x),\ E^b_j(y)\right] = 0 \tag{6.56}$$

Now, using these Poisson bracket relations along with (6.54), yields:

$$\left[\kappa^i_a(x),\ E^b_j(y)\right] = \left[A^i_a(x),\ E^b_j(y)\right] = \delta^b_a \delta^i_j\ \delta^3(x, y) \tag{6.57}$$

where $\kappa^i_a(E, A)$ is given by (6.43) and can be rewritten explicitly as:

$$\begin{aligned}
\kappa^i_a(A, E) &= A^i_a + f^i_a(E) \\
f^i_a(E) &= \frac{\eta}{2}\ \epsilon^{ijk} E^j_a \partial_b E^b_k - \frac{\eta}{2E}\ E^k_a\ \epsilon^{bcd}\left(E^i_b \partial_c E^k_d + E^k_b \partial_c E^i_d - \delta^{ik} E^l_b \partial_c E^l_d\right) \\
&= -\eta\ E^j_a\ \epsilon^{bcd}\left(v^i_b \partial_c v^j_d - \frac{1}{2}\ \delta^{ij}\ v^r_b \partial_c v^r_d\right)
\end{aligned} \tag{6.58}$$

with $v_a^i \equiv E_a^i/\sqrt{E}$. It is straight forward to check that f_a^i satisfy the identity:

$$\epsilon^{abc}\left\{\partial_b E_c^i - \partial_b(\ln\sqrt{E})\ E_c^i - \eta^{-1}\epsilon^{ijk} f_b^j E_c^k\right\} = 0$$

Equivalently, this relation can also be written as:

$$\partial_a E_i^a \ - \ \eta^{-1}\epsilon^{ijk} f_a^j E_k^a \ \equiv \ D_a(A)E_i^a \ - \ \eta^{-1}\epsilon^{ijk}\kappa_a^j E_k^a \ = \ 0 \tag{6.59}$$

These relations can be used to calculate the variation δf_a^i to be:

$$\delta f_a^i \ = \ \mathcal{S}_{ab}^{il}\delta E_l^b \ - \ \eta\ \partial_c\left(\mathcal{A}_{ab}{}^{cil}\delta E_l^b\right) + \frac{\eta}{2}\left(\partial_c \mathcal{A}_{ab}{}^{cil}\right)\delta E_l^b \tag{6.60}$$

with

$$\begin{aligned}
\mathcal{S}_{ab}^{il} \ &= \ - \ \left(E_a^l f_b^i + E_b^i f_a^l\right) \ + \ \frac{3}{4}\left(E_a^i f_b^l + E_b^l f_a^i\right) \ - \ \frac{1}{2}\ E_a^m E_b^m\left(E_i^c f_c^l + E_l^c f_c^i\right) \\
&\quad + \ \frac{1}{4}\left(E_a^m E_b^l E_i^c + E_b^m E_a^i E_l^c\right) f_c^m \ + \ \left(E_b^i E_a^l - E_a^i E_b^l + \delta^{li} E_a^n E_b^n\right) E_m^c f_c^m \\
&\quad - \ \frac{\eta}{4}\ \epsilon^{imk}\left(\partial_c E_a^m E_b^l - \partial_c E_b^l E_a^m\right) E_k^c \ - \ \frac{\eta}{4}\ \epsilon^{lmk}\left(\partial_c E_b^m E_a^i - \partial_c E_a^i E_b^m\right) E_k^c \\
&\quad + \ \frac{\eta}{2}\ \epsilon^{ilk}\left(\partial_c E_a^m E_b^m - \partial_c E_b^m E_a^m\right) E_k^c \\
\mathcal{A}_{ab}{}^{cil} \ &= \ \left(\epsilon^{ilk} E_a^m E_b^m - \frac{1}{2}\ \epsilon^{imk} E_a^m E_b^l + \frac{1}{2}\ \epsilon^{lmk} E_b^m E_a^i\right) E_k^c
\end{aligned} \tag{6.61}$$

Notice that $\mathcal{S}_{ab}^{il}$ and $\mathcal{A}_{ab}{}^{cil}$ are respectively symmetric and antisymmetric under the interchange of the pair of indices (a,i) and (b,l):

$$\mathcal{S}_{ab}^{il} \ = \ \mathcal{S}_{ba}^{li}\ , \qquad \mathcal{A}_{ab}{}^{cil} \ = \ - \ \mathcal{A}_{ba}{}^{cli} \tag{6.62}$$

These properties, immediately, lead to the relation:

$$\frac{\delta f_a^i(x)}{\delta E_l^b(y)} \ = \ \frac{\delta f_b^l(y)}{\delta E_i^a(x)} \tag{6.63}$$

Next, using $\chi_i^a(x) \equiv \hat{F}_i^a(x) - \frac{2(1+\eta^2)}{\eta}\ \tilde{e}_{0i}^a(x)$ from (6.39), equations (6.52) also imply the following:

$$\begin{aligned}
\left[\chi_i^a(x), \hat{E}_j^b(y)\right] &= - \ \frac{2(1+\eta^2)}{\eta}\ \left[\tilde{e}_{0i}^a(x), \hat{E}_j^b(y)\right] \\
&= - \ \frac{4}{\eta^2}\epsilon^{abc}\left\{(\theta+\eta\phi)D_c^{ij} - \left(\frac{(1-\eta^2)\theta + 2\eta\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x,y)
\end{aligned}$$

$$\left[\chi_i^a(x), \hat{F}_j^b(y)\right] = -\frac{2(1+\eta^2)}{\eta} \ \left[\tilde{e}_{0i}^a(x), \hat{F}_j^b(y)\right]$$
$$= \frac{4}{\eta^2}\epsilon^{abc}\left\{\left((1-\eta^2)\theta + 2\eta\phi\right) D_c^{ij} + \left(\frac{(3\eta^2-1)\theta - \eta(3-\eta^2)\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x,y)$$
$$\left[\chi_i^a(x),\ A_b^j(y)\right] \ = \ 0\ , \qquad \left[\chi_i^a(x),\ K_b^j(y)\right] \ = \ -\,\delta_i^j\delta_b^a\ \delta^{(3)}(x,y)$$
$$(1+\eta^2)\left[\chi_i^a(x),\ e_{0j}^b(y)\right] = (1+\eta^2)\left[\hat{F}_i^a(x),\ e_{0j}^b(y)\right]$$
$$= \ \frac{2}{\eta}\ \epsilon^{abc}\left\{\left((1-\eta^2)\phi - 2\eta\theta\right) D_c^{ij} \ + \left(\frac{\eta(3-\eta^2)\theta + (3\eta^2-1)\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x,y)$$
$$(1+\eta^2)\left[\chi_i^a(x),\ \tilde{e}_{0j}^b(y)\right] = (1+\eta^2)\left[\hat{F}_i^a(x),\ \tilde{e}_{0j}^a(y)\right]$$
$$= \frac{2}{\eta}\ \epsilon^{abc}\left\{\left((1-\eta^2)\theta + 2\eta\phi\right) D_c^{ij} \ + \left(\frac{(3\eta^2-1)\theta - \eta(3-\eta^2)\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x,y)$$
$$[\chi_i^a(x),\ \tilde{e}_j^{(\eta)b}(y)] = [\hat{F}_i^a(x),\ \tilde{e}_j^{(\eta)b}(y)]$$
$$= \frac{2}{\eta}\ \epsilon^{abc}\left\{(\theta+\eta\phi)D_c^{ij} - \left(\frac{((1-\eta^2)\theta + 2\eta\phi}{\eta}\right)\epsilon^{ikj}K_c^k\right\}\delta^{(3)}(x,y) \qquad (6.64)$$

which further imply:

$$\left[\chi_i^a(x),\ E_j^b(y)\right] \ = \ 0\ , \quad \left[\chi_i^a(x),\ \kappa_b^i(y)\right] \ = \ 0, \quad \left[\chi_i^a(x),\ \chi_j^b(y)\right] \ = \ 0 \qquad (6.65)$$

For $\psi_a^i \equiv K_a^i - \kappa_a^i(A,E)$ as given by (6.43), using (6.54) and (6.58), we have the following useful relations:

$$\begin{aligned}
\left[\psi_a^i(x),\ E_j^b(y)\right] &= -\left[\kappa_a^i(x)\ , E_j^b(y)\right] = -\delta_a^b\delta_j^i\ \delta^3(x,y), \\
\left[\psi_a^i(x),\ A_b^j(y)\right] &= -\left[\kappa_a^i(x),\ A_b^j(y)\right] \ = \ \frac{\delta\kappa_a^i(x)}{\delta E_i^b(y)} \\
\left[\psi_a^i(x),\ E_b^j(y)\right] &= -\ \left[\kappa_a^i(x),\ E_b^j(y)\right] = \ E_a^j E_b^i\ \delta^3(x,y), \\
\left[\psi_a^i(x),\ E(y)\right] &= -\ \left[\kappa_a^i(x),\ E(y)\right] \ = \ EE_a^i\ \delta^3(x,y)
\end{aligned} \qquad (6.66)$$

The Poisson bracket relations among χ_i^a and ψ_a^i, obtained by using the properties listed above, can be summarized as:

$$\left[\chi_i^a(x), \chi_j^b(y)\right] = 0\ , \quad \left[\chi_i^a(x), \psi_b^j(y)\right] = -\delta_b^a\delta_i^j\ \delta^{(3)}(x,y)\ , \quad \left[\psi_a^i(x), \psi_j^b(y)\right] = 0 \qquad (6.67)$$

where the last equation follows from the relation:

$$\left[\kappa_a^i(x),\ \kappa_b^j(y)\right] \ = \ \left[A_a^i(x),\ f_b^j(y)\right] + \left[f_a^i(x),\ A_b^j(y)\right] \ = \ \frac{\delta f_b^j(y)}{\delta E_i^a(x)} \ - \ \frac{\delta f_a^i(x)}{\delta E_j^b(y)} \ = \ 0 \qquad (6.68)$$

Here the Poisson brackets involving $f_a^i(E)$ are calculated by using their expressions as functions of E_a^i as given by (6.58). The identity (6.68) further implies the following Poisson bracket relations:

$$\begin{aligned}
&\left[F_{ab}^i(x),\ \kappa_d^j(y)\right] + \left[D_{[a}(A)\kappa_{b]}^i(x),\ A_d^j(y)\right] = 0, \\
&\left[F_{ab}^i(x),\ D_{[c}(A)\kappa_{d]}^j(y)\right] + \left[D_{[a}(A)\kappa_{b]}^i(x),\ F_{cd}^j(y)\right] = 0, \\
&\eta\left[D_{[a}(A)\kappa_{b]}^i(x), D_{[c}(A)\kappa_{d]}^j(y)\right] + \left[F_{ab}^i(x), \epsilon^{jmn}\kappa_c^m(y)\kappa_d^n(y)\right] + \left[\epsilon^{ikl}\kappa_a^k(x)\kappa_b^l(x), F_{cd}^j(y)\right] = 0 \\
&\left[D_{[a}(A)\kappa_{b]}^i(x),\ \epsilon^{jmn}\kappa_c^m(y)\kappa_d^n(y)\right] + \left[\epsilon^{ikl}\kappa_a^k(x)\kappa_b^l(x),\ D_{[c}(A)\kappa_{d]}^j(y)\right] = 0,
\end{aligned} \tag{6.69}$$

To implement the second-class constraints $\chi_i^a \approx 0$ and $\psi_i^a \approx 0$, we need to go over to the corresponding Dirac brackets and then put $\chi_i^a = 0$ and $\psi_i^a = 0$ strongly. From the Poisson bracket relations of these constraints (6.67), the Dirac bracket of any two fields C and D can be constructed to be:

$$[C,\ D]_D \ = \ [C,\ D] - [C,\ \chi] \ \ [\psi,\ D] + [C,\ \psi] \ \ [\chi,\ D] \tag{6.70}$$

Using the Poisson bracket relations listed above, it is straight forward to check that the Dirac brackets amongst A_a^i and E_i^a are the same as their Poisson brackets:

$$\begin{aligned}
&\left[E_i^a(x), E_j^b(y)\right]_D = \left[E_i^a(x), E_j^b(y)\right] = 0\ , \quad \left[A_a^i(x), A_b^j(y)\right]_D = \left[A_a^i(x), A_b^j(y)\right] = 0 \\
&\left[A_a^i(x), E_j^b(y)\right]_D = \left[A_a^i(x), E_j^b(y)\right] = \delta_a^b\delta_j^i\ \delta^3(x,y)
\end{aligned} \tag{6.71}$$

Also we note that,

$$\begin{aligned}
&\left[K_a^i(x), E_j^b(y)\right]_D = \left[\kappa_a^i(x), E_j^b(y)\right]_D = \left[\kappa_a^i(x), E_j^b(y)\right] = [A_a^i(x), E_j^b(y)] = \delta_a^b\delta_j^i\ \delta^3(x,y), \\
&\left[K_a^i(x), A_b^j(y)\right]_D = \left[\kappa_a^i(x), A_b^j(y)\right]_D = \left[\kappa_a^i(x), A_b^j(y)\right] = \left[f_a^i(x), A_b^j(y)\right] = -\ \frac{\delta f_a^i(x)}{\delta E_j^b(y)}, \\
&\left[K_a^i(x), K_b^j(y)\right]_D = \left[\kappa_a^i(x), \kappa_b^j(y)\right]_D = \left[\kappa_a^i(x), \kappa_b^j(y)\right] \\
&\qquad = \left[A_a^i(x), f_b^j(y)\right] \ + \ \left[f_a^i(x), A_b^j(y)\right] = 0
\end{aligned} \tag{6.72}$$

where in the last terms of second and third equations, the Poisson brackets are to be evaluated using (6.58) which express $f_a^i(E)$ as functions of E_a^i.

The Dirac brackets of $(A_a^i, \hat{E}_i^a)$ and $(\hat{E}_i^a, \hat{E}_j^b)$ are not same as their Poisson brackets:

$$\left[A_a^i(x),\ \hat{E}_j^b(y)\right]_D \ = \ \left[A_a^i(x),\ E_j^b(y) - \frac{2}{\eta}\ \tilde{e}_{0j}^{b(\eta)}(y)\right]_D = \delta_a^b\delta_j^i\ \delta^3(x,y) - \frac{2}{\eta}\ \left[A_a^i(x),\ \tilde{e}_{0j}^{b(\eta)}(\kappa;y)\right],$$

$$
\begin{aligned}
\left[\hat{E}^a_i(x),\ \hat{E}^b_j(y)\right]_D &= \frac{4}{\eta^2}\ \left[\tilde{e}^{a(\eta)}_{0i}(\kappa;x),\ \tilde{e}^{b(\eta)}_{0j}(\kappa;y)\right] \\
&= -\ \frac{4(\theta^2+\phi^2)}{\eta^3}\ \epsilon^{acd}\epsilon^{bef}\left(\left[F^i_{cd}(x),\ \epsilon^{jmn}\kappa^m_e(y)\kappa^n_f(y)\right]\right. \\
&\qquad\qquad \left.+\left[\epsilon^{imn}\kappa^m_c(x)\kappa^n_d(x),\ F^j_{ef}(y)\right]\right) \\
&= \frac{4(\theta^2+\phi^2)}{\eta^2}\ \epsilon^{acd}\epsilon^{bef}\left[D_{[c}(A)\kappa^i_{d]}(x),\ D_{[e}(A)\kappa^j_{f]}(y)\right] \qquad (6.73)
\end{aligned}
$$

Here the argument κ in $e^a_{0i}(\kappa)$ and $\tilde{e}^a_{0i}(\kappa)$ is to indicate that these are as in (6.40) with K^i_a replaced by κ^i_a which in turn are given by (6.58) as functions of A^i_a and E^a_i. Further, here in the second equation, we have used:

$$
\left[E^a_i(x),\ \tilde{e}^{b(\eta)}_{0j}(\kappa;y)\right]\ +\ \left[\tilde{e}^{a(\eta)}_{0i}(\kappa;x),\ E^b_j(y)\right]\ =\ 0
$$

Also,

$$
\begin{aligned}
\left[A^i_a(x),\ \hat{F}^b_j(y)\right]_D &= \frac{2(1+\eta^2)}{\eta}\left[A^i_a(x),\ \tilde{e}^b_{0j}(y)\right]_D = \frac{2(1+\eta^2)}{\eta}\ \left[A^i_a(x),\ \tilde{e}^b_{0j}(\kappa;y)\right] \\
\left[E^a_i(x),\ \hat{F}^b_j(y)\right]_D &= \frac{2(1+\eta^2)}{\eta}[E^a_i(x),\ \tilde{e}^b_{0j}(y)]_D = \frac{2(1+\eta^2)}{\eta}\left[E^a_i(x),\ \tilde{e}^b_{0j}(\kappa;y)\right] \\
\left[\hat{F}^a_i(x),\ \hat{F}^b_j(y)\right]_D &= \frac{4(1+\eta^2)^2}{\eta^2}\left[\tilde{e}^a_{0i}(x),\ \tilde{e}^b_{0j}(y)\right]_D = \frac{4(1+\eta^2)^2}{\eta^2}\left[\tilde{e}^a_{0i}(\kappa;x),\ \tilde{e}^b_{0j}(\kappa;y)\right] \\
&= \frac{4(1+\eta^2)}{\eta^2}\left(\theta^2+\phi^2\right)\epsilon^{acd}\epsilon^{bef}\left[D_{[c}(A)\kappa^i_{d]}(x),\ D_{[e}(A)\kappa^j_{f]}(y)\right] \\
\left[\hat{E}^a_i(x),\hat{F}^b_j(y)\right]_D + \left[\hat{F}^a_i(x),\hat{E}^b_j(y)\right]_D &= -\ \frac{8\,(1+\eta^2)}{\eta^2}\left[\tilde{e}^a_{0i}(\kappa;x),\ \tilde{e}^b_{0j}(\kappa;y)\right] \\
&= -\ \frac{8\,(\theta^2+\phi^2)}{\eta^2}\epsilon^{acd}\epsilon^{bef}\left[D_{[c}(A)\kappa^i_{d]}(x),\ D_{[e}(A)\kappa^j_{f]}(y)\right] \\
\left[K^i_a(x),\ \hat{F}^b_j(y)\right]_D &= \frac{2(1+\eta^2)}{\eta}\left[K^i_a(x),\ \tilde{e}^b_{0j}(y)\right]_D = \frac{2(1+\eta^2)}{\eta}\left[\kappa^i_a(x),\ \tilde{e}^b_{0j}(\kappa;y)\right] \qquad (6.74)
\end{aligned}
$$

6.6 Summary

We have analysed the most general gravity Lagrangian with three topological densities, namely, the Nieh-Yan, Euler and Pontryagin, from a classical canonical perspective. The canonical theory develops a dependence on all three parameters which are coefficients of these terms. In the time gauge, we obtain a real SU(2) gauge theoretic description with a set of seven first class constraints corresponding to three SU (2) rotations, three spatial diffeomorphism and one to evolution in a timelike direction. Inverse of the coefficient of the Nieh-Yan term, identified as the Barbero-Immirzi parameter, acts as the coupling constant of the gauge theory.

It would be of interest to see how any or all of these three topological parameters affect the quantum theory of gravity, if at all. Also, since two of the topological densities, namely, the Nieh-Yan and Pontryagin, are parity odd, it is possible that they might leave their signatures through some parity-violating effect(s) in gravity.

Chapter 7

Second-class constraints and quantization

Here we explore the issue of constructing an appropriate quantum framework where the imports of the topological Nieh-Yan term can be seen in a way as is typical in gauge theories.

It is well-known that the presence of total divergences in the Lagrangian does not affect the classical dynamics. However, when the divergences are of topological origin, the corresponding quantum theory might be affected by such additions. For example, exactly such a situation is realized in QCD. The classical vacua there are given by the condition of a vanishing field strength and are characterised by pure gauge configurations for the gauge fields:

$$A_\mu = g^{-1}\partial_\mu g$$

The group elements g fall into homotopy classes labelled by integers n. Thus, the classical ground states are infinitely degenerate. The naive quantum vacua, corresponding to each of these states, would also be infinitely degenerate. However, due to tunneling effects, the actual quantum vacuum is actually a linear superposition of the perturbative vacua associated with these ground states and is essentially non-perturbative. The existence of such a non-trivial vacuum structure can also be understood through an effective Lagrangian, which contains a topological density in addition to the standard kinetic term. This is known as the Pontryagin density of SU(3) gauge theory with a constant coefficient θ. This parameter, although classically irrelevant, shows up in the quantum theory through the true non-perturbative vacuum state which can be written as a superposition of all the perturbative ground states:

$$|\psi>_{vac} \;=\; \sum_{n=-\infty}^{+\infty} e^{in\theta}|\psi_n>$$

The integers n correspond to the charges given by the integral of the Pontryagin density and are associated with the homotopy maps $s^3 \to s^3$. These are characterised by the homotopy group $\pi_3(SU(3)) = Z$. Moreover, when this theory is coupled to massive fermions, physical quantities like the electron dipole moment of the neutron depend on θ.

In a quantum framework, the implications of a topological term can also be understood through a rescaling of the wavefunctional by a topologically non-trivial phase factor. This procedure is used in non-abelian gauge theories[25] where the wavefunctional is rescaled by the exponential of the Chern-Simons three-form with iθ as its coefficient:

$$\psi' \;=\; e^{i\theta\int d^3xj^t}\;\psi_{phy} \;\;\text{with}\;\; j^t = \epsilon^{abc}(A^i_a\partial_b A^i_c + \frac{2}{3}\epsilon_{ijk}A^i_a A^j_b A^k_c)$$

Here j^t is the time component of the 4-vector j^μ, the Chern-Simons 4-current and i, j, k denote the SU(N) group indices. For Yang-Mills theory and QCD, N=2 and 3, respectively. The physical states ψ_{phy} are invariant under 'small' gauge transformations built out of infinitesimal ones, but not under the 'large' gauge transformations which cannot be constructed out of infinitesimal ones. Under these, the gauge fields transform as follows:

$$A'^i_\mu T_i \;=\; g^{-1}A^i_\mu T_i\; g \;-\; g^{-1}\partial_\mu g$$

Thus, when g, the SU(N) group element, belongs to the trivial homotopy class, it is deformable to identity and corresponds to the 'small' gauge transformations. On the other hand, the 'large' gauge transformations arise when the g-s correspond to the non-trivial homotopy classes. These cannot be deformed to identity and under these the physical state ψ_{phy} is only phase-invariant. To be precise, under a representative gauge transformation belonging to the n-th homotopy class, ψ_{phy} transforms with a winding number n:

$$g_n\psi_{phy} \;=\; e^{in\theta}\psi_{phy}$$

Note that the rescaled wavefunctional ψ' is invariant under both 'small' and 'large' transformations by construction. This can be seen from the fact that under 'large' transformations, the functional as in the exponent in equation (7.1) transforms as:

$$\int d^3xj^{t'} \;=\; \int d^3xj^t \;-\; n$$

Through ψ', the non-trivial import of the topological parameter θ in the quantum theory gets manifested in a transparent manner. The study of the underlying non-perturbative vacuum structure in non-abelian gauge theories becomes particularly simpler within such a framework.

Motivated by the gauge theory example, we might invoke a similar construction for gravity in order to study the potentially non-trivial import of the topological Nieh-Yan density which can be added to the gravity Lagrangian. However, it turns out that it is not possible to do so when one uses the standard quantization procedure of Dirac. To see why this is so, let us study the rescaled wavefunctional in the case of gravity in analogy to gauge theories:

$$\psi' \;=\; e^{i\eta\int d^3x j^t_{NY}}\;\psi \;\;\text{with}\;\; j^t_{NY} = \epsilon^{abc} V^I_a D_b V^I_c \tag{7.1}$$

Here ψ is the wavefunctional representing the formally quantized Hilbert-Palatini theory and j^t_{NY} is the time component of the Nieh-Yan 4-current j^μ_{NY}. Its divergence gives the Nieh-Yan topological density: $\partial_\mu j^\mu_{NY} = I_{NY}$. Here j^t_{NY} and I_{NY} are to be seen as the analogues of the Chern-Simons functional j^t and Pontryagin topological density in the gauge theory case, respectively. However, the canonical theory of gravity contains second-class constraints (unlike the gauge theories as above) in addition to the first-class constraints. The standard quantization method of Dirac requires these second-class pairs to be implemented before quantization. When this is done for pure gravity, this implies vanishing of torsion, and results in a vanishing rescaling functional j^t_{NY}. Thus the rescaling in equation (7.1) becomes trivial. This precludes any possibility of studying the import of the Nieh-Yan topological term in the quantum theory through the rescaling framework.

Hence, for this purpose, one must adopt alternative quantization methods instead of the Dirac procedure. Here we use the Gupta-Bleuler and coherent state quantization approaches which involve a different treatment of the second-class constraints. These methods are quite general and can be used for gravity theory with or without matter.

We note that the procedure of rescaling has been applied earlier to gravity coupled to spin-$\frac{1}{2}$ fermions [16]. However, the approach in ref.[16] uses Dirac's method to solve the second-class constraints before quantization, or in other words, uses the connection equation of motion. As explained already, this method cannot be applied to pure gravity, since the rescaling functional j^t_{NY} vanishes when the connection equation is used.

The general idea of the Gupta-Bleuler quantization [26] is to split the original set of first and second-class constraints into a holomorphic and an anti-holomorphic set of first-class constraints related through the hermitian conjugation. The physical subspace contains only those ket states which are annihilated by the holomorphic set. Here we apply this method to Hilbert-Palatini theory. The resulting space of physical wavefunctions is then used to employ the rescaling. This leads to the canonical formulation based on the Lagrangian containing both Hilbert-Palatini and Nieh-Yan terms, as desired. Next we repeat this exercise using the coherent state quantization for constrained systems[27].

There we consider a squared sum of the original second-class constraints to define the physical Hilbert space. Note that such squared combinations also appear in the context of the Master constraint programme[28], where the constraints are enforced in a different way than above.

In contrast to Dirac's approach, our analysis in either cases does not require the use of the connection equation of motion for the rescaling. This particular feature is essential in order to recover a complete topological interpretation of the Barbero-Immirzi parameter, independent of any matter coupling.

In the following section we demonstrate the rescaling procedure in time gauge, first in the Gupta-Bleuler and then in the coherent state approach. We work in a representation diagonal in the densitized triad operators. Subsequently, we generalise our construction for any choice of gauge.

7.1 Hilbert-Palatini canonical theory

First-class constraints

The Hilbert-Palatini Lagrangian density is given by :

$$L \;=\; \frac{1}{2} e \Sigma^{\mu\nu}_{IJ} R^{IJ}_{\mu\nu} \tag{7.2}$$

We parametrize the tetrad fields as [23] :

$$e^I_t \;=\; \sqrt{eN} M^I + N^a V^I_a \;,\; e^I_a \;=\; V^I_a \quad ; \quad M_I V^I_a \;=\; 0 \;,\; M_I M^I \;=\; -1 \tag{7.3}$$

and then the inverse tetrad fields are :

$$\begin{aligned} e^t_I \;&=\; -\frac{M_I}{\sqrt{eN}} \;,\; e^a_I \;=\; V^a_I + \frac{N^a M_I}{\sqrt{eN}} \;; \\ M^I V^a_I \;&=\; 0 \;,\; V^I_a V^b_I \;=\; \delta^b_a \;,\; V^I_a V^a_J \;=\; \delta^I_J + M^I M_J \end{aligned} \tag{7.4}$$

Introducing the fields

$$E^a_i \;= 2e\Sigma^{ta}_{0i} \;\;,\;\; \chi_i \;=\; \frac{M_i}{M_0} \;\;,\;\; \tilde{\omega}_b{}^{0i} \;=\; \omega_b{}^{0i} \;-\; \chi_m \omega_b{}^{im} \;\;,\;\; \zeta_j \;=\; \omega_{aij}\chi^i \tag{7.5}$$

the Lagrangian density in (7.2) can be written as :

$$L \;=\; E^a_i \partial_t \tilde{\omega}_a{}^{0i} \;+\; \zeta^i \partial_t \chi_i \;-\; \omega^{0i}_t G_{0i} \;-\; \frac{1}{2}\,\omega^{ij}_t G_{ij} \;-\; NH \;-\; N^a H_a \tag{7.6}$$

H, H_a and G^{boost}_i and G^{rot}_i are the scalar, vector, boost and rotation constraints, respectively ($G^{\rm boost}_i := G_{0i}\,,\; G^{\rm rot}_i := \frac{1}{2}\epsilon_{ijk} G^{jk}$) .

In terms of the canonical variables, the Nieh-Yan functional j^t_{NY} becomes :

$$\begin{aligned} j^t_{NY} \;\;&=\;\; \frac{1}{2}\,\sqrt{E}\epsilon^{ijk} E^a_j E^b_k \;\; \Big[-\,\chi_i \partial_a \left(\sqrt{E}\epsilon^{lmn}\epsilon_{bcd}\;\chi_l E^c_m E^d_n\right) \;+\; \partial_a\left(\sqrt{E}\epsilon^{imn}\epsilon_{bcd} E^c_m E^d_n\right)\;\Big] \\ &\quad -\, M_{kk} - \;\frac{1}{2(1-\chi^2)}\chi_k\chi_l M_{kl} \;-\; \epsilon^{ijk}\chi_j \tilde{\omega}_b{}^{0i} E^b_k \end{aligned} \tag{7.7}$$

where we have used the identities

$$\epsilon^{abc} V_{kc} \;=\; \sqrt{E}\epsilon_{ijk} E^{ai} E^{bj}, \;\; \sqrt{E} V_{ck} \;=\; E_{ck} \;\text{ and }\; \sqrt{E} V_{c0} \;=\; \chi^k E_{ck}$$

and also the decomposition[1]:

$$\omega_{aij} \;=\; \frac{1}{2} E_{a[i}\zeta_{j]} \;+\; \frac{1}{2(1-\chi^2)}\epsilon_{ijk}E_{al}M^{kl}, \quad M^{kl} \;=\; M^{lk}$$

which is just a way to represent the nine components of ω_{aij} in the basis of three ζ_i's and six M_{kl}'s.

One can choose the time gauge by putting $\chi_i \approx 0$. As this condition forms a second class pair with the boost constraint, they have to be imposed together.

The boost constraint is given by -

$$\begin{aligned} G_i^{boost} &= -\;\partial_a E_i^a \;-\; \omega_{aij}E_j^a \\ &= -\;\partial_a E_i^a \;+\; \zeta_i \end{aligned}$$

which is solved by the condition

$$\zeta_i = \partial_a E_i^a \quad . \tag{7.8}$$

The first-class set of constraints are given by the following expressions :

$$\begin{aligned} G_i^{rot} &= \epsilon_{ijk}\omega_a^{0j}E_k^a \\ H_a &= E_i^b R_{ab}^{\;0i} \\ &= E_i^b\partial_{[a}\omega_{b]}^{0i} \;-\; \omega_a^{0i}\zeta_i \;+\; [\;\epsilon_{ijl}E_{aj}\zeta_i \;-\; E_{ak}M^{kl}\;]\;G_l^{rot} \\ H &= -\;\frac{1}{2}\;E_i^aE_j^bR_{ab}^{\;\;ij} \\ &= E_i^a\partial_a\zeta_i \;+\; \frac{1}{2}\;\zeta_i\zeta_i \;-\; \frac{1}{2}\;E_i^aE_{bj}\zeta_i\partial_aE^{bj} \;+\; \frac{1}{2}\epsilon_{ijm}\;E_i^aE_{bn}\partial_aE^{bj}M^{mn} \\ &\quad +\; \frac{1}{8}\;[\;2\zeta_i\zeta_i \;+\; M^{kk}M^{ll} \;-\; M^{kl}M^{kl}] \;-\; \frac{1}{2}E_i^aE_j^b\;\omega_{[a}^{0i}\omega_{b]}^{0j} \end{aligned} \tag{7.9}$$

where ζ_i is given by (7.8) .

From (7.7) , we get the following expression for j_{NY}^t in time gauge :

$$j_{NY}^t \;=\; \frac{1}{2}\sqrt{E}\epsilon^{ijk}E_j^aE_k^b\partial_a\left(\sqrt{E}\epsilon^{imn}\epsilon_{bcd}E_m^cE_n^d\right) \;-\; M_{kk} \tag{7.10}$$

Second-class constraints

[1]The parametrisation for ω_{aij} here is different from that in ref.[23]

As the Lagrangian density (7.6) is independent of the velocities associated with M_{kl}, we have the primary constraints involving the corresponding momenta :

$$\pi_{kl} \approx 0 \tag{7.11}$$

These in turn imply secondary constraints, which essentially lead to the vanishing of torsion (see [23] for details) :

$$\begin{aligned}
& [H, \pi_{kl}] \approx 0 \\
=> \quad & \epsilon_{ijk} E^a_i E_{bl} \partial_a E^b_j + \frac{1}{2} (M^{ii} \delta_{kl} - M_{kl}) + (k \leftrightarrow l) \approx 0 \\
=> \quad & M_{kl} - F_{kl}(E^a_i) \approx 0
\end{aligned} \tag{7.12}$$

where we have defined F_{kl} as [2] :

$$F_{kl}(E^a_i) = \frac{1}{2} \epsilon_{ijm} E^a_i E^b_m \partial_a E_{bj} \, \delta_{kl} - \epsilon_{ijk} E^a_i E^b_l \partial_a E_{bj} + (k \leftrightarrow l) \tag{7.13}$$

Dirac's prescription leads to the next step where the second-class constraints are solved before quantization or are eliminated through Dirac brackets. This is equivalent to imposing them 'strongly' as operator conditions[12]. Thus the physical subspace of the original Hilbert space would be obtained through the states which are annihilated by the operators corresponding to the remaining set of first-class constraints. However, here the second-class pair in (7.11) and (7.12), when enforced strongly, leads to the vanishing of the rescaling functional j^t_{NY}. Thus, the Dirac quantization procedure as it is cannot provide any passage to the new set of constraints corresponding to the Lagrangian density containing the Nieh-Yan term.

Hence, one must adopt alternative quantization procedures to impose the second-class constraints in the quantum state space. Here we first employ a method which is a slight generalisation of the Gupta-Bleuler approach in electrodynamics, and then repeat the exercise using the coherent state quantisation. Both the cases result in a non-vanishing rescaling functional through which the canonical transformation can be carried out.

7.2 Gupta-Bleuler quantization

Following the general idea of Gupta-Bleuler quantization, we have to find suitable holomorphic and anti-holomorphic sets containing all the constraints. Here the sets can be

[2] In the presence of matter coupling this would be $[M_{kl} - F_{kl}(E^a_i, \phi_m)] \approx 0$, where ϕ_m denotes the matter fields, eg. fermions when gravity is coupled to fermions

defined as:

$$
\begin{aligned}
C &:= \Big(G^i_{rot},\ H_a,\ \tilde{H},\ (Q_{kl} + i\alpha\pi_{kl})\Big) \\
C^\dagger &:= \Big(G^i_{rot},\ H_a,\ \tilde{H}^\dagger,\ (Q_{kl} - i\alpha\pi_{kl})\Big)
\end{aligned}
\tag{7.14}
$$

where α is a constant and $\tilde{H},\ \tilde{H}^\dagger,\ Q_{kl}$ are defined as:

$$
\begin{aligned}
\tilde{H} &= E^a_i\partial_a\zeta_i \ + \ \frac{1}{2}\ \zeta_i\zeta_i \ - \ \frac{1}{2}\ E^a_i E_{bj}\zeta_i\partial_a E^{bj} \ + \ \frac{1}{2}\epsilon_{ijm}\ E^a_i E_{bn}\partial_a E^{bj}(M_{mn} + i\alpha\pi_{mn}) \\
&+ \frac{1}{8}\,[\ 2\zeta_i\zeta_i \ + \ (M_{kk} + i\alpha\pi_{kk})(M_{ll} + i\alpha\pi_{ll}) \ - \ (M_{kl} + i\alpha\pi_{kl})(M_{kl} + i\alpha\pi_{kl})] \\
&- \frac{1}{2}E^a_i E^b_j\ \omega^{0i}_{[a}\omega^{0j}_{b]} \\
\tilde{H}^\dagger &= E^a_i\partial_a\zeta_i \ + \ \frac{1}{2}\ \zeta_i\zeta_i \ - \ \frac{1}{2}\ E^a_i E_{bj}\zeta_i\partial_a E^{bj} \ + \ \frac{1}{2}\epsilon_{ijm}\ E^a_i E_{bn}\partial_a E^{bj}(M_{mn} - i\alpha\pi_{mn}) \\
&+ \frac{1}{8}\,[\ 2\zeta_i\zeta_i \ + \ (M_{kk} - i\alpha\pi_{kk})(M_{ll} - i\alpha\pi_{ll}) \ - \ (M_{kl} - i\alpha\pi_{kl})(M_{kl} - i\alpha\pi_{kl})] \\
&- \frac{1}{2}E^a_i E^b_j\ \omega^{0i}_{[a}\omega^{0j}_{b]} \\
Q_{kl} &= M_{kl} - F_{kl}(E^a_i)
\end{aligned}
$$

Thus we have two sets of first-class constraints which satisfy the required algebra given by[26]:

$$
\begin{aligned}
[C_A, C_B] &\approx 0 \approx [C^\dagger_A, C^\dagger_B], \\
[C_A, C^\dagger_B] &\approx Z_{AB}
\end{aligned}
\tag{7.15}
$$

where Z_{AB}, the central charge is a function of α in our case.

In the definitions (7.14), instead of H, one needs to take the classically equivalent constraints $\tilde{H}$ and $\tilde{H}^\dagger$ in order to ensure the abelian property of the individual sets, and hence to reproduce the correct algebra as in (7.15). $\tilde{H}$ and $\tilde{H}^\dagger$ are obtained by replacing M_{kl} in H by $(M_{kl} + i\alpha\pi_{kl})$ and $(M_{kl} - i\alpha\pi_{kl})$, respectively.

Next we define a representation based on the fundamental commutation relations as:

$$
\begin{aligned}
\hat{E}^a_i\ |\Psi> \ &= \ E^a_i\ |\Psi> \quad , \quad \hat{\omega}_a{}^{0i}\ |\Psi> \ = \ -\ i\frac{\delta}{\delta\hat{E}^a_i}\ |\Psi> \\
\hat{M}_{kl}\ |\Psi> \ &= \ M_{kl}\ |\Psi> \quad , \quad \hat{\pi}^{kl}\ |\Psi> \ = \ -\ i\frac{\delta}{\delta\hat{M}_{kl}}\ |\Psi>
\end{aligned}
$$

Here $|\Psi>$ represents the formally quantized Hilbert-Palatini theory. The physical subspace is obtained through the realisation of the set C on the ket states:

$$
\hat{C}\ |\Psi> \ = \ 0 \tag{7.16}
$$

Thus, the constraints involving the canonical pair $(\hat{M}_{kl},\ \hat{\pi}_{kl})$ act as:

$$(\hat{Q}_{kl} + i\alpha\hat{\pi}_{kl})\ |\Psi> \ = \ 0 \tag{7.17}$$

The hermitian conjugation of the above implies that the physical bra states are annihilated by $C^\dagger$.

¿From (7.17), it follows that the original second-class constraints are satisfied individually through the expectation values with respect to the physical states:

$$< \Psi|\hat{Q}_{kl}|\Psi > \ = \ < \Psi|\hat{\pi}^{kl}|\Psi > \ = \ 0$$

This is how the correspondence with the classical formulation emerges in this framework.

Equation (7.17) completely specifies the dependence of the wavefunctional on the variables M_{kl}, whereas the constraints G_i^{rot}, H_a and $\tilde{H}$ determine the E_i^a dependence. Note that in $\tilde{H}$, M_{kl} appears only through the corresponding constraint. Thus the full wavefunctional can be written as:

$$\Psi(M, E) \ = \ \tilde{\phi}(M - F)\ \phi(E) \tag{7.18}$$

where $\tilde{\phi}(M - F)$ is a Gaussian functional of $(M_{kl} - F_{kl}) = Q_{kl}$. Thus the Gupta-Bleuler wavefunctional differs from the one obtained through Dirac's procedure by the vacuum of the oscillator in the Q space. The integral representation for the inner product becomes:

$$\int dM\ (\tilde{\phi}(M - F))^2 \int dE\ \phi'^*(E)\phi(E) \ = \ \int dQ\ (\tilde{\phi}(Q))^2 \int dE\ \phi'^*(E)\phi(E) \tag{7.19}$$

where we have used the fact that the Jacobian corresponding to the change of variables from M to Q is identity. The Q integration can be performed trivially, leaving only the E integral. This then becomes equivalent to the reduced space integral as would be obtained by Dirac's procedure, upto a normalisation.

Importantly, the above expression contains no delta-function corresponding to the constraint in (7.17), which usually appears as a projector in the inner product for first-class constrained systems (see chapter-13 in [13], for example). Also, note that the presence of the Gaussian functional $\tilde{\phi}(M - F)$ in (7.19) leads to normalisable states in the M_{kl} sector.

7.3 Rescaling

Next we proceed to perform the rescaling in the quantized phase space. Note that j^t_{NY} depends only on the operators corresponding to the configuration variables (E_i^a, M_{kl}). Thus, the new momenta conjugate to $\hat{E}_i^a$ are thus given by :

$$
\begin{aligned}
\hat{\omega}_d^{\prime 0l}\ \Psi' &= e^{i\eta\int d^3x j^t_{NY}}\ \hat{\omega}_d^{\ 0l}\ e^{-i\eta\int d^3x j^t_{NY}}\ \Psi' \\
&= \left(\frac{\delta}{\delta \hat{E}_l^d} \ - \ \eta\ \frac{\delta j^t_{NY}(\hat{E}_i^a, \hat{M}_{jk})}{\delta \hat{E}_l^d} \right)\ \Psi' \\
&= \left(\frac{\delta}{\delta \hat{E}_l^d} \ - \ \frac{\eta}{2}\ \epsilon_{ijl}\hat{\omega}_{dij}\ + \ \eta\ \left[\frac{\hat{E}_{di}\hat{E}_{al}\ - \ \frac{1}{2}\hat{E}_{dl}\hat{E}_{ai}}{\sqrt{\hat{E}}} \right]\ \hat{t}_i^a \right)\ \Psi'
\end{aligned}
$$

where we have used the expression

$$
\begin{aligned}
\frac{\delta j^t_{NY}}{\delta E_l^d} &= \epsilon_{ijk}\partial_b \left(\frac{E_{ci}}{\sqrt{E}} \right)\ \frac{\delta(\sqrt{E}E^{bj}E^{ck}\ )}{\delta E_l^d} \\
&= \left(\frac{1}{2}\epsilon_{ijl}\omega_{dij}\ - \ \left[\frac{E_{di}E_{al}\ - \ \frac{1}{2}E_{dl}E_{ai}}{\sqrt{E}} \right]\ t_i^a \right)
\end{aligned}
$$

with t_i^a defined as :

$$
\begin{aligned}
t_i^a &= \epsilon^{abc}D_b V_{ci} \\
&= \epsilon^{abc}\partial_b \left(\frac{E_{ci}}{\sqrt{E}} \right)\ + \ \frac{\sqrt{E}}{2}\ [\ \epsilon^{ijk}E_k^a\partial_b E_j^b\ + \ E_k^a M_{ik}\ - \ E_i^a M_{kk}\ ]
\end{aligned}
\tag{7.20}
$$

The new $\hat{\pi}_{kl}$'s are obtained as -

$$
\begin{aligned}
\hat{\pi}'_{kl}\ \Psi' &= e^{i\eta\int d^3x j^t_{NY}}\ \hat{\pi}_{kl}\ e^{-i\eta\int d^3x j^t_{NY}}\ \Psi' \\
&= \left(\frac{\delta}{\delta \hat{M}^{kl}} \ - \ \eta\ \frac{\delta j^t_{NY}(\hat{E}_i^a, \hat{M}_{jk})}{\delta \hat{M}^{kl}} \right)\ \Psi' \\
&= \left(\frac{\delta}{\delta \hat{M}_{kl}} \ + \ \eta\ \delta_{kl} \right)\ \Psi'
\end{aligned}
$$

Note that this procedure goes through in presence of matter couplings which lead to non-vanishing torsion.

As already mentioned, the expectation value of the constraint (7.17) among the physical states in the M_{kl} sector (i.e., the states $\tilde{\phi}(M-F)$ in (7.18)) leads to the relation-

$$
<\hat{M}_{kl}>_M \ = \ <F_{kl}(\hat{E}_i^a)>_M \quad ,
$$

which is the analogue of the classical constraint in (7.12). To emphasize, here torsion as an operator in (7.20) does not annihilate Ψ, rather its expectation value vanishes. This is to be contrasted with the Dirac procedure where the torsion operator vanishes 'strongly'.

New constraints

The new constraints, which annihilate the rescaled wavefunctional Ψ', can be found by introducing the new momenta in the expressions as given in (7.9) . We illustrate this for $\hat{G}_i^{rot}$ below.

The rotation constraint for the action in (**??**) containing the Hilbert-Palatini and Nieh-Yan terms is given by :

$$\begin{aligned} \hat{G}_i^{\prime rot} &= e^{i\eta \int d^3x j^{\hat{t}}_{NY}} \; \hat{G}_i^{rot} \; e^{-i\eta \int d^3x j^{\hat{t}}_{NY}} \\ &= \eta \; \partial_a \hat{E}_i^a \; + \; \epsilon^{ijk} \hat{\omega}_a^{\prime 0j} \hat{E}_k^a \; - \; \frac{\eta}{\sqrt{\hat{E}}} \; \epsilon_{ijk} \hat{E}_{bj} \hat{t}_k^b \end{aligned}$$

Taking the expectation value with respect to the states $\tilde{\phi}(M-F)$, we arrive at the familiar $SU(2)$ Gauss' law :

$$< \hat{G}_i^{\prime rot} >_M \; = \; \eta \partial_a \hat{E}_i^a \; + \; \epsilon^{ijk} \hat{A}_a^j \hat{E}_k^a$$

where,

$$\hat{A}_d^l \; = \; < \hat{\omega}_d^{\prime \; 0l} >_M \; = \; \hat{\omega}_d^{\; 0l} \; - \; \frac{\eta}{2} \epsilon^{ijl} \hat{\omega}_{dij}$$

Without going into the detailed algebraic expressions of the remaining constraints corresponding to (**??**) as they are not relevant for our purpose here, we observe that they can be obtained in a similar manner as shown for $\hat{G}_i^{rot}$.

7.4 Coherent state quantization

We now demonstrate another approach, namely, the coherent state quantization for constrained systems[27]. Although this was originally designed to develop an alternative path-integral formulation using coherent states, here we use the essential idea to enforce the appropriate 'quantum' constraints. This would allow a consistent rescaling formulation for gravity with or without matter.

Following the general construction developed by Klauder [27] , we seek the states for which

$$< \Psi | \left((\hat{M}_{kl} - \hat{F}_{kl}(\hat{E}_i^a))^2 + \hat{\pi}_{kl}^2 \right) | \Psi > \; = \; < \Psi | (\hat{Q}_{kl}^2 + \hat{\pi}_{kl}^2) | \Psi > \; = \; 0 \tag{7.21}$$

However, since we have $< \hat{A}^2 >= (\Delta \hat{A})^2 + < \hat{A} >^2$ for any operator $\hat{A}$, equation (7.21) cannot be satisfied by any $\hat{A}$ with non-zero uncertainty $\Delta \hat{A}$. Hence, as suggested in ref.[27], one has to modify the above criterion as

$$< \Psi | (\hat{Q}_{kl}^2 + \hat{\pi}_{kl}^2) | \Psi > \; \leq \; \lambda_0 \; (\mathrm{O}(\hbar)) \tag{7.22}$$

where, λ_0 denotes the minimum eigenvalue in the spectrum of the constraint operator. The modification, being of the order of $\hbar$, is a purely quantum feature. The rest of the constraints G_i^{rot}, H_a and H, as given by equation(7.9), are imposed as they are on the physical states:

$$\hat{G}_i^{rot}|\Psi> \;=\; \hat{H}_a|\Psi> \;=\; \hat{H}|\Psi> \;=\; 0 \;\; . \tag{7.23}$$

Now, there is a family of minimum uncertainty states, namely, the canonical coherent states, defined as:

$$|M,\pi> \;=\; e^{-iM\hat{\pi}}e^{i\pi\hat{M}}|\beta> \tag{7.24}$$

where $M =< \hat{M}_{kl} >$, $\pi =< \hat{\pi}_{kl} >$ and $|\beta >$ is some fiducial state for which $M = 0,\ \pi = 0$ (we supress the indices just to simplify the notation). Among these, the one satisfying (7.22) is the coherent state for which $Q = (M - F(E)) = 0,\ \pi = 0$, with $F(E) =< \hat{F}_{kl}(\hat{E}_i^a) >$. Using (7.24) , the explicit form of this state reads:

$$\Psi(M) \;=\; e^{-iF(E)\hat{\pi}}\ \beta(M) \;=\; \beta(M - F(E))$$

where $\beta(M)$ is the fiducial state functional in a representation diagonal in $\hat{M}_{kl}$.

In this formulation, one can define a projection operator P onto the physical Hilbert space, requiring the following properties[27]:

$$P^\dagger = P, \quad P^2 = P$$

In our case P (in the M_{kl} sector) becomes simply $|\Psi(M) >< \Psi(M)|$.

The full wavefunctional representing the physical subspace can thus be written as:

$$\Psi(M,E) \;=\; \beta(M - F)\ \phi(E)$$

The inner product in this space reads:

$$\int dMdE\ \Psi'^*(M,E)\Psi(M,E) \;=\; \int dQ\ \beta^*(Q)\beta(Q)\int dE\ \phi'^*(E)\phi(E)$$

As in the Gupta-Bleuler case, here also the Q_{kl} (or, M_{kl}) sector factors out leaving only the E integral in the product. In particular, we can choose the fiducial state to be the oscillator ground state in this case. Then this expression reproduces the Gupta-Bleuler product as in (7.19). Thus the two Hilbert spaces are equivalent. The rescaling can now be implemented along the lines of our previous discussion.

7.5 Rescaling for any gauge choice

Now we provide a brief outline of the rescaling procedure without choosing any gauge. For non-zero χ_i, the canonical coordinates in the Lagrangian density in (7.6) are $\tilde{\omega}_a{}^{0i}$, M_{kl} and χ_i where $\tilde{\omega}_a{}^{0i}$ is given by equation (7.5) .

Y in (7.7) can be rewritten as :

$$
\begin{aligned}
j^t_{NY} \;&=\; \frac{1}{2}\,\sqrt{E}\epsilon^{ijk}E^a_jE^b_k\,\Big[-\,\chi_i\,\partial_a\left(\sqrt{E}\epsilon^{lmn}\epsilon_{bcd}\,\chi_lE^c_mE^d_n\right)\;+\;\partial_a\left(\sqrt{E}\epsilon^{imn}\epsilon_{bcd}E^c_mE^d_n\right)\Big] \\
&-\; M_{kk}\;-\;\frac{1}{(1-\chi^2)}\chi_k\chi_lM_{kl}\;-\;2\epsilon^{ijk}E^a_j\chi_k\partial_a\chi_i\;+\;2\chi_kG^{rot}_k \qquad (7.25)
\end{aligned}
$$

The structure of Y suggests that we can choose the representation to be diagonal in the operators $\hat{E}^a_i$, $\hat{M}_{kl}$ and $\hat{\chi}_i$. In the above equation the last term involving $\hat{G}^{rot}_i$ commutes with all other remaining terms and acts trivially on $|\Psi>$ to give zero. Hence this term can be ignored at this stage itself.

One can follow exactly the same procedure as earlier to define a suitable physical subspace using either the Gupta-Bleuler or the coherent state method, and then find the new set of canonical operators through the rescaling. Thus, the new momenta $\tilde{\omega}'^{0i}_a$ conjugate to E^a_i read:

$$
\begin{aligned}
\tilde{\omega}'^{0l}_d \;&=\; \omega^{(\eta)\;0l}_d\;-\;\chi_j\omega^{(\eta)\;lj}_d\;-\;\eta\epsilon^{ikl}\chi_k\partial_d\chi_i \\
&+\;\eta\,\left(\frac{E_{ai}E_{dl}}{2\sqrt{E}}\;+\;\sqrt{E}\epsilon^{ilk}\epsilon_{abd}E^b_k\right)(t^a_i\;-\;\chi_it^a_0) \qquad (7.26)
\end{aligned}
$$

where we have defined :

$$
\begin{aligned}
\omega^{(\eta)\;0l}_d \;&=\; \omega^{\;0l}_d\;-\;\frac{\eta}{2}\,\epsilon^{jkl}\omega_{djk}\;\;,\;\;\omega^{(\eta)\;lj}_d\;=\;\omega^{\;lj}_d\;-\;\eta\,\epsilon^{jkl}\omega_{d0k}\;\;, \\
t^a_i \;&=\; \epsilon^{abc}\,D_bV_{ci}\;=\;\epsilon^{abc}\,[\;\partial_bV_{ci}\;+\;\omega_{bij}V^j_c\;-\;\omega^{0i}_bV_{c0}\;]\;\;, \\
t^a_0 \;&=\; \epsilon^{abc}\,D_bV_{c0}\;=\;\epsilon^{abc}\,[\;\partial_bV_{c0}\;-\;\omega^{0i}_bV^i_c\;]
\end{aligned}
$$

As is evident, we recover the new momenta in time gauge when $\chi_i = 0$ and $\zeta_i = \partial_aE^a_i$.

The momenta corresponding to M_{kl} and χ_i transform as follows:

$$
\begin{aligned}
\pi'_{kl} \;&=\; \pi_{kl}\;+\;\eta\left(\delta_{kl}+\frac{\chi_k\chi_l}{1-\chi^2}\right) \\
\zeta'^i \;&=\; \zeta^i\;-\;\eta\epsilon^{ijk}E^a_j\left[\partial_a\chi_k\;-\;\sqrt{E}E^b_k\chi_m\partial_a\left(\frac{E^m_b}{\sqrt{E}}\right)\right]\;+\;\frac{\eta}{1-\chi^2}\left(\delta_{ij}+\frac{\chi_i\chi_j}{1-\chi^2}\right)\chi_kM^{jk}
\end{aligned}
$$

The new set of constraints, which annihilate the rescaled wavefunctional Ψ' can now be obtained in a manner as demonstrated for the time-gauge fixed theory.

7.6 Summary

We have illustrated how to arrive at the canonical formulation corresponding to the action containing the Hilbert-Palatini and Nieh-Yan terms starting from the Hilbert-Palatini canonical theory through a generic rescaling procedure. The constraint operators, through their action on the physical states, reproduce the real Ashtekar-Barbero formulation.

As it turns out, one cannot invoke such a rescaling to obtain the Ashtekar-Barbero constraints if the Hilbert-Palatini second-class constraints are eliminated before quantization, as is done in Dirac's method. Here we have provided a remedy to this problem by using alternative approaches, namely, the Gupta-Bleuler and the coherent state quantizations. These two cases result in the same physical Hilbert space. Also, here the torsional degrees of freedom as associated with the second-class constraints emerge as relevant canonical operators through $\hat{M}_{kl}$ and $\hat{\pi}^{kl}$. To emphasize, these do not appear in the Dirac-quantized phase space where the second-class constraints are eliminated beforehand. It would be interesting to see the implications of this particular feature. The framework as developed here provides a natural arena where the potential role of the torsional fluctuations in quantum gravity can be studied further. Any progress along such lines might also shed some light on the possible import of the Nieh-Yan density in the quantum theory. Note that a concrete understanding of this particular issue might be difficult, if not impossible, within the Dirac quantization scheme where the Nieh-Yan term is trivially zero. Thus, the alternative frameworks as proposed here might provide some insight in this regard.

As both the quantization methods lead to a non-vanishing rescaling functional j^t_{NY}, they apply to any arbitrary matter coupling. When such couplings lead to nonzero torsion (e.g. fermion coupled to gravity), one can obtain the new canonical constraints by writing j^t_{NY} in terms of the geometric variables (i.e., tetrads and spin connections). Using the connection equation of motion to write j^t_{NY} in terms of matter fields there becomes purely optional. Thus our analysis provides a complete topological interpretation of the Barbero-Immirzi parameter in a quantum framework, whether or not matter is coupled to gravity.

We have also shown that the rescaling can be carried out without having to choose any particular gauge (e.g., time gauge). Thus the appearance of η as a topological parameter in this quantum description is not an artifact of some special gauge choice, as also shown in chapter-2 from a classical perspective.

Chapter 8

Concluding remarks

The investigations and results as presented in this thesis provide some perspective into what could be potentially worthwhile issues to explore.

In the case of gauge theories, the import of the topological Pontryagin term in the quantum theory is non-trivial, as already explained in the previous chapter[25, 29]. It leads to a ground state which is very different from the one as would be obtained through a naive analysis. The (degenerate) minimum energy states all conspire to form a new vacua. Each of them contribute with a phase which depends on the topological θ parameter. This effect is non-perturbative in nature as semiclassical perturbation theory for small coupling constant cannot sense such state of affairs. Also, some of the observables in the quantum theory do depend on the topological parameter. For example, the electric dipole moment of the neutron depend on the θ parameter of QCD.

Whether some or all of the three topological terms in gravity theory imply such rich vacuum structure demands a careful investigation. Such questions are also tied up with issues like a suitable classification of the vacuum configuration in gravity and the precise analogue of the large gauge transformations as in non-abelian gauge theories. To emphasize, it would be important to study what might be the winding numbers associated with the topological invariants and whether there are instanton-like configurations in gravity which lead to tunneling effects. Such analyses might unravel some important features of the quantum theory of gravity, a complete understanding of which still eludes us.

Also, since the Nieh-Yan term vanishes on-shell for pure gravity, it is not a priori clear how its coefficient η might contribute to the tunneling amplitude, if at all. A rigorous investigation along these lines would require a consistent path-integral formulation in place for gravity in the first-order framework. This seems to be one possible way to understand if the torsional fluctuations play any role in the quantum theory of gravity.

The analyses in this thesis are sufficiently general to suggest that the the topological parameters should manifest themselves in any quantum theory of gravity. In particular, one can study how the Nieh-Yan invariant, and hence its coefficient η, does emerge in the low energy effective action of string theory, which is a candidate theory for quantum gravity. In such a framework, η might show up as a dynamical field, leaving open the issue of possible mechanisms through which it can be fixed at some expectation value.

The results here indicate that a complete theory of gravity would contain two other topological parameters, i.e., the coefficients of the Euler and Pontryagin densities, along with the Barbero-Immirzi parameter. As is well-known, the spectrum of the area operator in Loop Quantum Gravity depends on η[30]. It is intriguing to see whether the other two also do leave their imprints on any geometrical operators or show up through other non-perturbative effects as such.

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