Studies in Loop Quantization of Cosmological Models

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

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Abstract

Loop Quantum Gravity(LQG) is an attempt to construct a mathematically rigorous, non-perturbative, background independent formulation of quantum General Relativity. In this thesis we present some studies on the Loop Quantization of cosmological models. We look at the the effect of loop quantization of FRW cosmology in the context of effective equations and WKB approximation in the first part of the thesis. In the second part of the thesis we undertake the initial steps in constructing a Loop quantized theory of a midisuperspace model known as the Gowdy Model on T^3.

One of the qualitatively distinct and robust implication of Loop Quantum Gravity is the underlying discrete structure. In the cosmological context elucidated by Loop Quantum Cosmology (LQC), this is manifested by the Hamiltonian constraint equation being a (partial) difference equation. One obtains an effective Hamiltonian framework by making the continuum approximation followed by a WKB approximation. In the large volume regime, these lead to the usual classical Einstein equation which is independent of both the Barbero-Immirzi parameter γ as well as $\hbar$. In this work we present an alternative derivation of the effective Hamiltonian by-passing the continuum approximation step. As a result, the effective Hamiltonian is obtained as a close form expression in γ. These corrections to the Einstein equation can be thought of as corrections due to the underlying discrete (spatial) geometry with γ controlling the size of these corrections. These corrections imply a bound on the rate of change of the volume of the isotropic universe. In most cases these are perturbative in nature but for cosmological constant dominated isotropic universe, there are significant deviations. Subsequent changes in the formalism of LQC are also discussed.

The vacuum Gowdy models provide much studied, non-trivial midi-superspace examples. Various technical issues within Loop Quantum Gravity can be studied in these models as well as one can hope to understand singularities and their resolution in the loop quantization. The first step in this program is to reformulate the model in real connection variables in a manner that is amenable to loop quantization. We begin with the unpolarized model and carry out a consistent reduction to the polarized case. Carrying out complete gauge fixing, the known solutions are recovered.

Then we introduce the kinematical Hilbert space on which the appropriate holonomies and fluxes are well represented. The quantization of the volume operator and the Gauss constraint is straightforward. Imposition of the Gauss constraint can be done on the kinematical Hilbert space to select subspace of gauge invariant states. We carry out the quantization of the Hamiltonian constraint making specific choices. Alternative choices are briefly discussed. It appears that to get spatial correlations reflected in the Hamiltonian constraint, one may have to adopt the so called 'effective operator viewpoint'.

LIST of PUBLICATIONS

- Kinjal Banerjee and Ghanashyam Date " Discreteness corrections to the effective Hamiltonian of isotropic loop quantum cosmology." Class. Quant. Grav. **22**, 2017 (2005) [arXiv:gr-qc/0501102].
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1 Introduction

General Relativity(GR) and Quantum Mechanics are two of the most successful theories of modern physics. While General Relativity has been spectacularly successful in explaining the universe at large scales, Quantum Mechanics has been equally successful for physics on small scales. However, one of the biggest unfulfilled challenges in physics remains to incorporate the two theories in the same framework. Ordinary Quantum Field Theories which have managed to describe the three other fundamental forces (Electromagnetic, Weak and Strong) have failed for General Relativity because it is not perturbatively renormalizable. In the absence of experimental evidence a number of candidates for a quantum theory of gravity have emerged (String Theory and Loop Quantum Gravity(LQG) being the two most developed ones), each with their relative strengths and weaknesses.

String Theory treats the gravitational field on the same footing as the other fundamental interactions and tries to incorporate all four into one fundamental unified theory. Gravitational interaction is sought to be understood as exchange of gravitons in analogy with other interactions. The spacetime metric is split into two parts, a background metric and a fluctuating one. Usually one chooses the background to be a flat Minkowskian metric $\eta_{\mu\nu}$. Then $g_{\mu\nu} = \eta_{\mu\nu} + Gh_{\mu\nu}$ where $h_{\mu\nu}$ is the dynamical variable with Newton's constant G acting as the coupling constant. The field $h_{\mu\nu}$ is then quantized on the $\eta_{\mu\nu}$ background and the perturbative machinery of QFT is applied to the Einstein-Hilbert action. Since the background enters manifestly, the background independence is to be obtained by summing over all possible backgrounds. Since the notion of gravitons is a perturbative one, this approach is perturbative in nature with non-perturbative effects sought subsequently.

String Theory [1, 2, 3, 4] is the only known consistent perturbative approach to quantum gravity. In this theory the point particles are replaced by one-dimensional extended objects which sweep out two-dimensional world sheets embedded in a D-dimensional manifold which represents the spacetime of the physical world. Matter and the mediators of the various interactions are all thought of as excitation modes of the strings.

The other approach is a relativist's approach viewing gravitational interactions as manifestations of curvature of a dynamical spacetime. In this view the metric plays the dual role of a mathematical object that defines spacetime geometry and encodes the physical gravitational field as well. Hence background independence lies at the heart of this approach. If we are going to quantize the gravitational field , i.e. the metric itself, it would make sense if the scheme does not explicitly depend on some background metric. Minkowski metric is not an externally prescribed eternal background structure but is only one possible example. In this formalism the ordinary Quantum Field Theory approach is no longer possible because it is defined when the background metric is fixed and hence it cannot handle variations of the background metric. New mathematical techniques have to be invented to go beyond the framework of perturbative quantum field theories.

LQG is an attempt to construct a mathematically rigorous, non-perturbative, background independent formulation of quantum General Relativity. General Relativity is reformulated in terms of the Sen-Ashtekar-Barbero-Immirizi connection variables and its dynamics is treated in a canonical framework. In a canonical framework, spacetime is viewed as an evolution of a 3 dimensional geometry. The 4 dimensional diffeomorphisms of general relativity are thus manifested as spatial diffeomorphisms of a 3 dimensional slice together with the Hamiltonian constraint generating time evolution.

In this thesis we will concentrate LQG and present some results of Loop Quantization in the cosmological context. In the rest of this chapter we will give a very brief introduction to the kinematic framework of Loop Quantum Gravity and the reasons for testing it in cosmological settings and then .

1.1 LQG: A Brief Introduction

The review presented in this section is based on [5, 6].

1.1.1 Classical Framework

The starting point in classical theory is the Hamiltonian formulation of GR written in terms of the Ashtekar variables, the densitized triad E^a_i and the Ashtekar connection A^i_a. (In this thesis we adopt the notation: $a = 1, 2, 3$ referring to the spatial indices while $i = 1, 2, 3$ refer to the internal SU(2) indices.) At each point on a spatial hypersurface Σ introduce a triad e^i_a such that it is related to the spatial metric as

$$q_{ab} = e^i_a e^j_b \delta_{ij} \tag{1.1}$$

The densitized triad is given by:

$$E^a_i := \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c \tag{1.2}$$

Let D_a be the derivative operator compatible with e^i_a. This implies

$$D_a e^j_b = 0 \Rightarrow \Gamma_{ajk} = -e^b_k \left[\partial_a e^j_b - \Gamma^c_{ab} e^j_c\right] \tag{1.3}$$

where Γ_{ajk} is called the spin connection and Γ^c_{ab} is the Christofel symbol associated with the metric q_{ab}. The spin connection can be written as $\Gamma_{ajk} := \Gamma^i_a \epsilon_{ijk}$ where ϵ_{ijk} is the totally antisymmetric tensor. From the definition of the densitized triad it also clear that $D_a E^b_j = 0$. Also define

$$K^i_a := K_{ab} E^b_j \delta^{ij} \tag{1.4}$$

where K_{ab} is the extrinsic curvature of Σ. Using this together with the spin connection Γ^i_a we can define a new connection A^i_a given by

$$A^i_a = \Gamma^i_a + \gamma K^i_a \tag{1.5}$$

where γ is a nonzero real number called Barbero-Immirizi parameter. It can be shown that E^a_i and A^i_a form a conjugate pair i.e.

$$\left\{E^a_i(x), E^b_j(y)\right\} = 0 = \left\{A^i_a(x), A^j_b(y)\right\} \tag{1.6}$$

$$\left\{E^a_i(x), A^j_b(y)\right\} = \kappa\gamma\delta^a_b\delta^i_j\delta(x,y) \tag{1.7}$$

where $\kappa = 8\pi G$.

The phase space of GR can be described in terms of these variables. The advantage of performing the canonical transformations to the Ashtekar variables is that gravity can now be formulated as a gauge theory.

In terms of these variables the constraints can be written as:

$$\text{Gauss}: \quad G_i := D_a E^a_i \approx 0 \tag{1.8}$$

$$\text{Diffeomorphism}: \quad V_a := F^i_{ab} E^b_i - (1+\gamma^2) K^i_a G_i \approx 0 \tag{1.9}$$

$$\text{Hamiltonian}: \quad H := \frac{E^a_i E^b_j}{\sqrt{\det E}} \left(\epsilon^{ij}_k F^k_{ab} - 2(1+\gamma^2) K^i_{[a} K^j_{b]}\right) \approx 0 \tag{1.10}$$

where F^i_{ab} is the curvature of the connection A^i_a. The Gauss constraint generating gauge transformations form a subalgebra in the the algebra of constraints and in fact forms an ideal of the constraint algebra. As a result the quotient algebra of connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$ can be completed to a C^* algebra.

1.1.2 Quantum Framework

In LQG the elementary classical configuration variables are taken to be SU(2) holonomies of the smooth connections along one dimensional edges while the momentum variables are obtained by smearing the densitized triad with a su(2) valued function on a two-surface. These are then promoted to operators in the quantum theory. The classical configuration space is the space of connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$. The quantum configuration space is a completion of it $\overline{\mathcal{A}/\mathcal{G}}$. The quantum configuration space as well as the kinematic Hilbert space $\mathcal{H} := L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ can be constructed in two ways.

In the first way, we use the fact that the configuration variables give rise to an Abelian C^* algebra $\mathcal{H}A$ called the holonomy algebra whose spectrum is $\overline{\mathcal{A}/\mathcal{G}}$. Then a diffeomorphism invariant positive linear functional is introduced and the GNS construction is used to construct the Hilbert space of states and a representation of $\mathcal{H}A$ on it. The electric flux operators are introduced using the standard quantum mechanical idea that the momentum operator E should be represented by $-i\hbar\delta/\delta A$.

The second method is more explicit and we shall explain it in a bit more detail. Given a smooth connection A and an oriented path $e : [0,1] \subset \mathbb{R} \to \Sigma$ we define a *holonomy* $h_e[A] \in \mathrm{SU}(2)$ as

$$h_e[A] = \mathcal{P}\exp - \int_e A \tag{1.11}$$

where $\mathcal{P}$ denotes a path ordered exponential.
A *generalized connection* $\bar{A}_e$ is an assignment of $\bar{A} \in \mathrm{SU}(2)$ to any analytic path $e \subset \Sigma$. A *graph* γ is a collection of analytic paths $e \subset \Sigma$ meeting at most at their endpoints. We will consider only closed graphs. The point at which two edges meet is called a *vertex*. Let n be the number of edges in γ. A function *cylindrical* with respect to γ is given by:

$$\psi_\gamma(\bar{A}) := f_\gamma\left(\bar{A}_{e_1}, \ldots \bar{A}_{e_n}\right) \tag{1.12}$$

where f_γ is a smooth function on $SU(2)^n$ such that $\psi_\gamma(\bar{A})$ is gauge invariant. The set of states cylindrical with respect to γ are denoted by Cyl_γ. For each $\gamma \in \Sigma$ construct Cyl_γ. The set of of all functions cylindrical with respect to some $\gamma \in \Sigma$ is denoted by Cyl and is given by

$$\mathrm{Cyl} = \bigcup_\gamma \mathrm{Cyl}_\gamma \tag{1.13}$$

Given a function $\psi_\gamma(\bar{A}) \in \mathrm{Cyl}$ the measure μ_0, on Cyl, known as the Ashtekar-Lewandowski measure is defined by

$$\int_{\overline{\mathcal{A}/\mathcal{G}}} \mathrm{d}\mu_0(\psi_\gamma(\bar{A})) := \int_{SU(2)^n} \prod_{e\subset\gamma} \mathrm{d}h^e f_\gamma\left(\bar{A}_{e_1}, \ldots \bar{A}_{e_n}\right) \qquad \forall \quad \psi_\gamma(\bar{A}) \tag{1.14}$$

where $\mathrm{d}h$ is the normalized Haar measure on SU(2).

Using this measure we can define an inner product on Cyl:

$$\begin{aligned} \langle \psi_\gamma, \psi'_\gamma \rangle &\equiv \langle f_\gamma (\bar{A}_{e_1}, \dots \bar{A}_{e_n}), g_\gamma' (\bar{A}_{e_1}, \dots \bar{A}_{e_m}) \rangle \\ &= \int_{SU(2)^n} \prod_{e \subset \Gamma_{\gamma\gamma'}} \mathrm{d}h^e \overline{f_\gamma (\bar{A}_{e_1}, \dots \bar{A}_{e_n})} g_\gamma' (\bar{A}_{e_1}, \dots \bar{A}_{e_m}) \end{aligned} \quad (1.15)$$

where $\Gamma_{\gamma\gamma'}$ is any graph such that $\gamma \subset \Gamma_{\gamma\gamma'}$ and $\gamma' \subset \Gamma_{\gamma\gamma'}$.

The Hilbert space is the Cauchy completion of Cyl along with the limit points of in the Ashtekar-Lewandowski norm.

To construct a basis on this Hilbert space we first consider a *spin network* which is constructed as follows: given a graph γ, each edge e is coloured by a non trivial irreducible representation π_{j_e} and to each vertex v associate a contraction matrix c_v which contracts the matrices π_{j_e} for all edges incident at the vertex in a gauge invariant way. This is denoted as $T_s := T_{\gamma,\vec{j},\vec{c}}$ where $\vec{j} = \{j_e\}$ and $\vec{c} = \{c_v\}$. States cylindrical with respect to spin networks are called *spin network states*. They provide an orthogonal basis on $\mathcal{H}_{kin}$.

This completes the construction of the kinematic Hilbert space $\mathcal{H}_{kin}$ in LQG. Note that the Gauss constraints generating gauge transformations have already been solved. The remaining constraints have to expressed in terms of the basic variables on $\mathcal{H}_{kin}$. To do that certain classical identities are used to rewrite them in a form suitable for Loop quantization. As an explicit example, we will discuss the Hamiltonian constraint. Let us first define:

- The total volume of Σ is given by

$$V := V(\Sigma) := \int_\Sigma \mathrm{d}^3x \sqrt{|\det(q)|} \quad (1.16)$$

- The integrated trace of the densitized extrinsic curvature of Σ is

$$K := \int_\Sigma \mathrm{d}^3x \sqrt{\det(q)} K_{ab} q^{ab} = \int_\Sigma \mathrm{d}^3x K_a^i E_i^a \quad (1.17)$$

Using these definitions we can invoke two *key* identities:

$$\frac{[E^a, E^b]_i}{\sqrt{\det(q)}} = \epsilon^{abc}(\mathrm{sgn}(\det(e))e_c^i = 2\epsilon^{abc}\frac{\delta V}{\delta E_i^a} = 2\epsilon^{abc}\left\{\frac{A_c^i}{\kappa}, V\right\} \quad (1.18)$$

$$K_a^i = \frac{\delta K}{\delta E_i^a} = \left\{\frac{A_c^i}{\kappa}, K\right\} \quad (1.19)$$

In the full theory it is convenient to break up the Hamiltonian constraint as

$$H = H^E - 2(1+\gamma^2)H^L \quad (1.20)$$

Using the above identities the *Euclidean Hamiltonian constraint* H^E can be written as:

$$H^E := \frac{E_i^a E_j^b}{\sqrt{\det E}} \left(\epsilon_k^{ij} F_{ab}^k\right) = \frac{2}{\kappa}\epsilon^{abc}\mathrm{tr}\left(F_{ab}\{A_c^i, V\}\right) \tag{1.21}$$

while the *Lorentzian* part H^L can be written as:

$$H^L := \frac{E_i^a E_j^b}{\sqrt{\det E}} \left(K^i_{[a}K^j_{b]}\right) = \frac{8}{\kappa^3}\epsilon^{abc}\mathrm{tr}\left(\{A_a^i, K\}\{A_b^j, K\}\{A_c^k, V\}\right) \tag{1.22}$$

Note that this allows us to remove the non polynomiality coming from the explicit factor of $1/\det E$ in the expression of H. These expressions will have to be written in terms of basic variables. These are then to be promoted to quantum operators and the Poisson brackets $\{.,.\}$ are replaced by $[.,.]/(i\hbar)$. general operators constructed in this fashion will obviously have operator ordering ambiguities.

The first step in constructing the quantum operator is *regularization*. We shall indicate some of the steps for H^E. The spatial manifold Σ is triangulated into elementary tetrahedra Δ each of whose edges are analytic. Let us denote this triangulation by T. For each tetrahedron choose one of its vertices and call it $v(\Delta)$. Let $s_i(\Delta), i = 1, 2, 3$ be the three edges of Δ meeting at $v(\Delta)$. Let $\alpha_{ij}(\Delta) := s_i(\Delta) \circ a_{ij} \circ s_j(\Delta)^{-1}$ be the loop based at $v(\Delta)$ where a_{ij} is the other edge of Δ connecting the other endpoints of s_i, s_j. Then

$$H_\Delta^E[N] := -\frac{2}{3\kappa}N_v\epsilon^{ijk}\mathrm{tr}\left(h_{\alpha_{ij}(\Delta)}[A]h_{s_k(\Delta)}[A]\left\{h^{-1}_{s_k(\Delta)}[A], V\right\}\right) \tag{1.23}$$

$$H_T^E[N] = \sum_{\Delta\in T} H_\Delta^E[N] \tag{1.24}$$

where $N_v := N(v(\Delta))$. This expression tends to the correct classical limit as all the Δ shrink to their base points $v(\Delta)$ and the number of tetrahedra tend to infinity, for any choice T. Similarly

$$H_\Delta^L[N] := -\frac{8}{3\kappa^3}N_v\epsilon^{ijk} \quad \mathrm{tr}\left(h_{s_i(\Delta)}[A]\left\{h^{-1}_{s_i(\Delta)}[A], K\right\}h_{s_j(\Delta)}[A]\left\{h^{-1}_{s_j(\Delta)}[A], K\right\}\right.$$
$$\left. h_{s_k(\Delta)}[A]\left\{h^{-1}_{s_k(\Delta)}[A], V\right\}\right) \tag{1.25}$$

$$H_T^L[N] = \sum_{\Delta\in T} H_\Delta^L[N] \tag{1.26}$$

These can be promoted to well defined quantum operators as there exists operators corresponding to V and K densely defined on $\mathcal{H}_{kin}$. Given a cylindrical function ψ_γ. Let $V(\gamma)$ be the set of vertices ofγ. Then the action of the volume operator is given by:

$$\hat{V}\ \psi_\gamma := \frac{l_P^3}{4}\sum_{v\in V(\gamma)}\sqrt{|\frac{i}{3!}\sum_{s_I\bigcap s_J\bigcap s_K}\varepsilon(s_I,s_J,s_K)\epsilon_{ijk}X_I^iX_J^jX_K^k|}\ \ \psi_\gamma \tag{1.27}$$

where $\varepsilon(s_I, s_J, s_K) = \text{sgn}(\det(\dot{s}_I(0), \dot{s}_J(0), \dot{s}_K(0))$. Given $g_I = A_{s_I}(\bar{A})$, $X_I = X(g_I)$ are the right invariant vector fields on SU(2). The operator for K can be constructed from noting that

$$K = \{H^E(1), V\} \tag{1.28}$$

where the lapse function is set $= 1$.

This concludes the main idea of the construction of the $\mathcal{H}_{kin}$ and the operators in LQG. We will not go into details of concrete implementations of these ideas in the full theory but will state some important results. Concrete implementation in the case of midisuperspace models will be seen later in this thesis.

- Eigen values of all operators constructed from fluxes of triads (eg. length, area, volume) are discrete.

- The gauge orbits of diffeomorphisms are not compact. Hence diffeomorphism invariant states are not contained in $\mathcal{H}_{kin}$ but are distributions on it. They belong to the the algebraic dual of Cyl, the space of linear functionals from Cyl to $\mathbb{C}$ denoted by Cyl*.

- Let $\hat{V}[\phi]$ be the operator representing the action of the diffeomorphism $\phi \in Diff(\Sigma)$. Its action on a spin network state is given by

  $$\hat{V}[\phi]\ T_s = T_{\phi\cdot s} \tag{1.29}$$

  where $\phi\cdot s := \left[\phi\cdot e := \phi^{-1}(e); (\phi\cdot\pi_{j_e})_{\phi^{-1}(e)} := \pi_{j_e}; (\phi\cdot c_v)_{\phi^{-1}(e)} := c_v\right]$ for all $e \in \gamma$.

  This is not weakly continuous and therefore there is no well defined operator on $\mathcal{H}_{kin}$ corresponding to infinitesimal diffeomorphisms. Since we are interested only in diffeo invariant states we can overcome the problem by imposing the requirement

  $$\hat{V}^\dagger[\phi]\ \psi = \psi \tag{1.30}$$

  for physical states $\psi \in$ Cyl*.

While the kinematical framework is well developed, there are many open issues which are difficult to handle in the full theory. For example, there are no known Dirac observables and no explicit realization of semiclassical states or recovering solutions of Einsteins equations in the limit where quantum corrections are small.

Also the question of anomaly in the quantum constraint algebra is not answered. One way of checking the validity of this quantization scheme would be to apply it to simpler models where explicit calculations can be performed.

1.2 Why Mini and Midi Superspace Models

In this thesis we have attempted to look at aspects of cosmological models which are loop quantized. This amounts to symmetry reduction before quantization rather than quantization before symmetry reduction. It is well known, these need not commute and the predictions of loop quantization of these symmetry reduced models may not be same as the results obtained from full LQG when these symmetry conditions are imposed.

However owing to the extremely complicated nature of general relativity, study of any theory of Quantum Gravity is extremely difficult. One approach usually taken is to apply the formal quantization prescriptions to symmetry-reduced space-time geometries. These serve as good toy models where questions difficult to address in the full theory can be explored. Since explicit calculations can be performed, one can hope for a comparison with known classical results. In this way one avoids the technical difficulties of the full theory while getting a flavour of the different features of loop quantization as opposed to Schroedinger Quantization of these models.

Let us emphasise once more that (symmetry) reduction after quantization and quantization after reduction may not lead to equivalent quantum theories. While one would like to view the reduced quantum theory as a 'sector' of the the full theory, how to do so is not yet clear. In the absence of such an identification, predictions of the reduced model may not necessarily be implications of the full theory. Nevertheless these toy models can be used as tests of how a particular quantization scheme works in some specific cases. Lessons learnt from these toy models can provide hints for tackling some of problems of the full theory.

Minisuperspace models refer to the models where all but a finite number of degrees of freedom in Einstein's equations are frozen. What we get is essentially a quantum mechanical model. In this case the classical observables and solutions are well known and quantum theory can be compared with classical results. Standard FRW cosmology is a good example of that. Its loop quantization (LQC) provides strikingly different results from standard Wheeler DeWitt quantization. One study of LQC in its semiclassical approximation is presented in Chapter 2.

The next level of complication is to study models where after symmetry reduction some degrees of freedom are gauge fixed but a number of field theory degrees of freedom are present. These correspond to field theories and are known as midisuperspace models. This is where complications from the field theoretic aspect of the theory are expected to show up and questions like the closure of the constraint algebra can be addressed. In Chapter 3 and 4 we present the initial steps in loop quantizing one of the simplest midisuperspace models, the polarized Gowdy model on T^3.

2

Loop Quantum Cosmology-Discreteness Corrections to the Effective Hamiltonian

In Loop Quantum Cosmology(LQC), the quantization techniques of LQG are applied to a simple minisuperspace model, the Friedmann-Robertson-Walker(FRW) cosmology. This is not the cosmological sector of LQG but the application of Loop quantization to a classically symmetry reduced model. As this model has only one quantum mechanical degree of freedom we can perform explicit calculations. Since the Gauss and the Diffeomorphism constraints are identically zero, we can avoid a lot of technical difficulties and explicitly determine the solutions of the Hamiltonian constraint. We can look for semiclassical states and how they behave close to singularity. The discreteness corrections coming from loop quantization significantly change the quantum behaviour from the traditional Wheeler-De-Witt(WdW) quantization. All these, however, come with significant caveats which will be mentioned later in the chapter.

In this chapter we will first give a brief description of the classical FRW metric in metric variables in section (2.1) and then look at the same in connection variables in section(2.2) in some detail to set the stage for quantization. Then in sections (2.3) and (2.4) we will give a brief introduction to the original formulation of LQC known as the Ashtekar-Bojowald-Lewandowski(ABL) quantization. In section (2.5) we shall derive an effective Hamiltonian for LQC and look at its equations of motion in (2.6). In section (2.7) we will look at some explicit examples of solutions of the modified equations. Subsequent to that work the physical Hilbert space has been constructed in LQC and new "improved" formulation has been developed by Ashtekar-Pawlowski-Singh(APS) which is different from the ABL formulation. We will briefly review the recent advances made in the section (2.8).

2.1 FRW Cosmology in metric variables

Based on observations on the cosmological scale, we can assume that the universe is *homogeneous* and *isotropic* on large scales. A spacetime is said to be spatially *homogeneous* if there exists a one-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime such that for each t and for any points $p, q \in \Sigma_t$ there exists an isometry of the spacetime metric g_{ab} which takes p to q. A spacetime is said to be spatially *isotropic* at each point if there exists a congruence of timelike curves with tangent vectors u^a filling the spacetime satisfying the following property: given any point $p \in \Sigma_t$ and any two unit spatial tangent vectors s_1^a, s_2^a perpendicular to u^a, there exists an isometry of the spacetime metric g_{ab} which leaves p and u^a at p fixed but rotates s_1^a into s_2^a. The metric for such a spacetime is the FRW metric and is given by [7]:

$$ds^2 = -N(t)^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-\eta r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] \tag{2.1}$$

where $\eta = 0, 1, -1$ corresponds to the flat, 3-sphere and hyperboloid universe respectively. The lapse function $N(t)$ does not play a dynamical role and a known as the *scale factor* is the only degree of freedom left after the imposition of homogeneity and isotropy. Note that the shift vector is zero and we do not have to worry about the Diffeo constraint.

This is a minisuperspace model with only one gravitational degree of freedom. The field equations are ordinary differential equations given by (setting lapse $N = 1$):

$$3\left(\frac{\dot{a}^2+\eta}{a^2}\right) = 8\pi\rho \qquad \text{(Friedmann equation)} \tag{2.2}$$

$$3\left(\frac{\ddot{a}}{a}\right) = -4\pi(\rho + 3P) \qquad \text{(Raychaudhuri equation)} \tag{2.3}$$

where the matter stress-energy tensor is taken to be of the perfect fluid form:

$$T_{ab} = \rho u_a u_b + P\left(g_{ab} + u_a u_b\right) \tag{2.4}$$

which is the most general form consistent with homogeneity and isotropy. It can be easily seen that universes of this type have an initial singularity at $a = 0$ known as the *big bang singularity*. In the next section we will look at FRW cosmology in connection variables.

2.2 FRW Cosmology in connection variables

LQC is the application of loop quantization techniques to the FRW cosmology. The first step is to express the theory in terms of the Ashtekar variables. In this section we will review ABL treatment of the classical variables which are subsequently loop quantized in LQC [8].

For simplicity, in this section we will consider the spatially flat case i.e. Σ is topologically $\mathbb{R}^3$, endowed with the action of the Euclidean group. This action can be used to introduce on Σ a fiducial flat metric ${}^0q_{ab}$. Let ${}^0e_i^a$ be the constant orthonormal triad and ${}^0\omega_a^i$ the co-triad compatible with ${}^0q_{ab}$. We shall fix the local diffeomorphism and gauge freedom. As a result we can choose basis such that the Ashtekar connection and the densitized triad can be written in the form

$$A = \tilde{c}\,{}^0\omega^i\,\tau_i \tag{2.5}$$
$$E = \tilde{p}\,\sqrt{{}^0q}\,{}^0e_i\,\tau^i \tag{2.6}$$

where $\tilde{c}$ and $\tilde{p}$ carry all the non-trivial information contained in A and E and τ_i are related to the Pauli matrices σ_i by the relation $2i\tau_i = \sigma_i$.

Since Σ is non-compact and the fields are spatially homogeneous, the integrals defined in the full theory diverge. Hence the spatial homogeneity requires us to bypass this problem by fixing a *cell* $\mathcal{V}$ and restrict all integrations to this cell. For simplicity we shall assume that $\mathcal{V}$ is cubical with respect to ${}^0q_{ab}$. Let V_0 be the volume of $\mathcal{V}$ with respect to ${}^0q_{ab}$. Then the symplectic structure is given by

$$\Omega = \frac{3V_0}{\kappa\gamma}\,\mathrm{d}\tilde{c}\wedge\mathrm{d}\tilde{p} \tag{2.7}$$

while the fundamental Poisson bracket is

$$\{\tilde{c},\tilde{p}\} = \frac{\kappa\gamma}{3V_0} \tag{2.8}$$

Note that there is a freedom of rescaling the fiducial metric by a constant ${}^0q_{ab}\mapsto k^2\,{}^0q_{ab}$. However $\tilde{c},\tilde{p}$ are not invariant under this rescaling. They transform as $\tilde{c}\mapsto k^{-1}\tilde{c}$ and $\tilde{p}\mapsto k^{-2}\tilde{p}$. Since rescaling of the fiducial metric should not change the physics it is convenient to use the elementary cell $\mathcal{V}$ to eliminate this additional freedom. We define new variables

$$c = V_0^{\frac{1}{3}}\tilde{c} \qquad \text{and} \qquad p = V_0^{\frac{2}{3}}\tilde{p} \tag{2.9}$$

Then the symplectic structure and the Poisson bracket in terms of these new variables become

$$\Omega = \frac{3V}{\kappa\gamma}\,\mathrm{d}c\wedge\mathrm{d}p \qquad \text{and} \qquad \{c,p\} = \frac{\kappa\gamma}{3} \tag{2.10}$$

These variables are independent of the choice of the fiducial metric ${}^0q_{ab}$ and the elementary cell $\mathcal{V}$. These variables describe the classical phase space of LQC. Since the Gauss and the Diffeomorphism constraint are already satisfied, the only remaining constraint is the the Hamiltonian constraint. Again setting the lapse as

1 and with some matter Hamiltonian H_m, it can be written as:

$$-\frac{6}{\gamma^2}c^2 \mathrm{sgn} p\sqrt{|p|} + \kappa H_m = 0 \tag{2.11}$$

2.3 LQC: Kinematics

In identifying the variables to be loop quantized we will follow the procedure used in the full theory [5, 6], i.e.

- Configuration variables are taken to be SU(2) holonomies of the connection along straight edges

$$h_k^{(\mu)}(A) := \mathcal{P}\exp\int_e A = \cos\frac{\mu c}{2}\mathbb{I} + 2\sin\frac{\mu c}{2}\tau_k \tag{2.12}$$

  where $\mu \in \mathbb{R}$ with $\mu V_0^{\frac{1}{3}}$ is the oriented length of the straight edge with respect to ${}^0q_{ab}$ and $\mathbb{I}$ is a 2×2 unit matrix.

- Momentum variables are triads smeared with constant test functions f^i over square cross sections S tangential to a fiducial triad ${}^0e_i^a$. Explicitly

$$E(S, f) := \int_S \epsilon_{abc}E_i^c f^i \mathrm{d}x^a \mathrm{d}x^b = pV_0^{-\frac{2}{3}} A_{S,f} \tag{2.13}$$

  where $A_{S,f}$ is the area of S as measured by ${}^0q_{ab}$ times an orientation factor which depends on f^i. Thus apart from a kinematic factor determined by the background metric the momenta are just given by p. In the classical geometry p is related to the volume of the elementary cell $\mathcal{V}$ as

$$V = |p|^{\frac{3}{2}} \tag{2.14}$$

Thus in LQC, $N_\mu := e^{i\mu c/2}$ and p are taken to be the elementary classical variables which are to be promoted to basic quantum operators [8].

To construct the Hilbert space we will stick as closely as possible to the path taken in full theory. The configuration variables generate the algebra of *almost periodic functions* of c. A typical element of this algebra can be written as

$$\psi(c) = \sum_j \xi_j e^{i\mu_j c/2} \tag{2.15}$$

where j runs over a finite number of integers labelling edges, $\mu_j \in \mathbb{R}$ and $\xi_j \in \mathbb{C}$. This is a C* algebra. The vector space of the almost cylindrical functions is the analog of Cyl in full theory which we shall denote by Cyl_{LQC}. The kinematic Hilbert space $\mathcal{H}$ constructed via GNS construction turns out to be $L^2\left(\overline{\mathbb{R}}_{\text{Bohr}}, \mathrm{d}\mu\right)$, where $\overline{\mathbb{R}}_{\text{Bohr}}$ is the *Bohr compactification* of the real line and $\mathrm{d}\mu$ is the Haar measure on it. This representation of the basic variables is inequivalent to the standard Schroedinger representation.

The inner product is given by

$$\langle N_{\mu_1}|N_{\mu_2}\rangle \;\equiv\; \langle e^{i\mu_1 c/2}|e^{i\mu_2 c/2}\rangle \;=\; \delta_{\mu_1,\mu_2} \tag{2.16}$$

where on the right hand side is a Kronecker delta. The almost periodic functions are the analogs of spin network functions of the full theory and they form an orthonormal basis on $\mathcal{H}$.

On this Hilbert space the configuration operator acts by multiplication and the momentum operator acts by differentiation:

$$\hat{N}_\mu\psi(c) \;=\; \exp\frac{i\mu c}{2}\psi(c) \tag{2.17}$$

$$\hat{p}\psi(c) \;=\; -i\frac{\gamma\ell_{\text{P}}^2}{3}\frac{\mathrm{d}\psi}{\mathrm{d}c} \tag{2.18}$$

where $\ell_{\text{P}}^2 = \kappa\hbar$.

It is convenient to use the Dirac bra-ket notation and set $e^{i\mu c/2} = \langle c|\mu\rangle$. In this notation the eigenstates of $\hat{p}$ are simply the basis vectors $|\mu\rangle$, i.e.

$$\hat{p}|\mu\rangle = \frac{\gamma\ell_{\text{P}}^2}{6}\mu|\mu\rangle \;\equiv\; p_\mu|\mu\rangle \tag{2.19}$$

Using eqn.(2.14) we can write the action of the operator $\hat{V}$ representing the volume of the elementary cell $\mathcal{V}$ as

$$\hat{V}|\mu\rangle = \left(\frac{\gamma\ell_{\text{P}}^2|\mu|}{6}\right)^{\frac{3}{2}}|\mu\rangle \;\equiv\; V_\mu|\mu\rangle \tag{2.20}$$

In this notation, general kinematical states $|s\rangle$, in the triad basis have the form

$$|s\rangle \;=\; \sum_{\mu\in\mathbb{R}} s_\mu\,|\mu\rangle \tag{2.21}$$

where μ run over some *countable* subset of $\mathbb{R}$.

We have thus defined the kinematic Hilbert space of LQC and the action of basic operators. The Hamiltonian constraint has to be classically expressed in terms of the basic variables and then quantized.

2.4 LQC: Hamiltonian Constraint

The gauge and the local diffeomorphism constraints have been fixed and the on[ly] constraint left in the theory is the Hamiltonian constraint. In the spatial flat cas[e] it takes the form [8]:

$$H = -\gamma^{-2} \int_{\mathcal{V}} \mathrm{d}x^3 N \epsilon_{ijk} F^i_{ab} E^{aj} E^{bk} e^{-1} \tag{2.2[2]}$$

where $e := \sqrt{|\det E|}$ and the integral is restricted to the elementary cell. In th[e] spatially homogeneous case, the lapse N is constant and will be set to one.

To obtain the quantum operator, we will need to express the above expressio[n] in terms of the elementary variables and their Poisson brackets. In that we wi[ll] follow the steps of the full theory. Consider square loops α_{ij} spanned by two [of] the triad vectors ${}^0e^a_i$, each of length $\mu_0 V_0^{1/3}$ with respect to the fiducial metric ${}^0q_{a[b]}$. Then the ab component of the curvature is given by

$$F^i_{ab} = -2 \lim_{Ar(\alpha_{ij})\to 0} \mathrm{Tr}\left(\frac{h^{(\mu_0)}_{\alpha_{ij}} - 1}{\mu_0^2 V_0^{2/3}}\right) \tau^{i0} \omega^{i0}_a \omega^j_b \tag{2.2[3]}$$

Then using equations (1.23) and (1.24), the constraint can be written as

$$H^{(\mu_0)}_{\text{grav}} = -\frac{4(\text{sgn}p)}{\kappa\gamma^3\mu_0^3} \sum_{ijk} \epsilon^{ijk} \mathrm{Tr}\left(h_i^{(\mu_0)} h_j^{(\mu_0)} h_i^{(\mu_0)-1} h_j^{(\mu_0)-1} h_k^{(\mu_0)} \left\{h_k^{(\mu_0)-1}, V\right\}\right) \tag{2.24}$$

$$H_{\text{grav}} = \lim_{\mu_0\to 0} H^{(\mu_0)}_{grav} \tag{2.25}$$

The Hamiltonian constraint now written in terms of basic variables can be pr[o]moted to the quantum operator

$$\begin{aligned} \hat{H}^{(\mu_0)}_{\text{grav}} &= \frac{4i(\text{sgn}p)}{\gamma^3 \ell_{\text{P}}^2 \mu_0^3} \sum_{ijk} \epsilon^{ijk} \mathrm{Tr}\left(\hat{h}_i^{(\mu_0)} \hat{h}_j^{(\mu_0)} \hat{h}_i^{(\mu_0)-1} \hat{h}_j^{(\mu_0)-1} \hat{h}_k^{(\mu_0)} \left[\hat{h}_k^{(\mu_0)-1}, V\right]\right) \\ &= \frac{24i(\text{sgn}p)}{\gamma^3 \ell_{\text{P}}^2 \mu_0^3} \sin^2(\mu_0 c)\left(\sin\left(\frac{\mu_0 c}{2}\right) \hat{V} \cos\left(\frac{\mu_0 c}{2}\right) - \cos\left(\frac{\mu_0 c}{2}\right) \hat{V} \sin\left(\frac{\mu_0 c}{2}\right)\right) \end{aligned} \tag{2.[26]}$$

Note that in the limit $\mu_0 \to 0$, (2.24) exists and goes over to the classical expressio[n] of the Hamiltonian constraint. However, since c is not a well defined operator o[n] $\mathcal{H}$, the quantum operator (2.26) does not exist in the limit $\mu_0 \to 0$. That is in keep[-] ing with the full theory where there is no well defined operator for the curvatur[e] F^i_{ab}. Therefore the dependence on μ_0 cannot be removed and has to be viewed a[s] a *regulator* in the reduced theory.The above treatment can be generalized for $\eta = 1$.

Remark on μ_0

The viewpoint taken in LQC is that the failure of the limit to exist is a reminder of the *underlying discreteness of the quantum geometry* coming from the full theory. Owing to the discreteness of the underlying space the square loop α_{ij} in equation (2.23) cannot be shrunk to zero but to a minimum nonzero value. Its value is motivated as follows. Consider the area operator $\hat{Ar} = |\hat{p}|$. Acting on a basic state of edge length μ_0 it gives

$$\hat{Ar}|\mu_0\rangle = \frac{\gamma\ell_{\rm P}^2}{6}\mu_0|\mu\rangle \tag{2.27}$$

We choose this eigenvalue to be the minimum nonzero eigenvalue of the Area operator coming from the full theory $\Delta = (\sqrt{3}\gamma\ell_{\rm P}^2)/4$. Equating these two we get

$$\frac{\gamma\ell_{\rm P}^2}{6}\mu_0 = \frac{\sqrt{3}\gamma\ell_{\rm P}^2}{4}, \quad \Rightarrow \quad \mu_0 = \frac{3\sqrt{3}}{2} \tag{2.28}$$

We shall discuss the new interpretation of μ_0 which has been subsequently proposed in the last section of this chapter which has major implications in the development of LQC. Till then in the rest of this chapter we shall treat the dimensionless real number μ_0 as an ambiguity parameter of order 1.

With the above comment in mind the action of the quantum Hamiltonian Constraint on a general kinematical state (2.21) is given by

$$\begin{aligned}\hat{H}^{(\mu_0)}_{\rm grav}|\mu\rangle &= \left(\frac{3}{8\kappa}\right)(\gamma^3\mu_0^3l_p^2)^{-1}(V_{\mu+\mu_0} - V_{\mu-\mu_0}) \\ &\quad \left(e^{-i\mu_0\eta}|\mu+4\mu_0\rangle - (2+4\mu_0^2\gamma^2\eta)|\mu\rangle + e^{i\mu_0\eta}|\mu-4\mu_0\rangle\right)\end{aligned} \tag{2.29}$$

Notice that the Hamiltonian connects states differing in their labels by $\pm 4\mu_0$. This is a direct consequence of the necessity of using holonomy operators in the quantization of the Hamiltonian operator and is responsible for leading to a difference equation below.

If we consider some matter Hamiltonian, the total constraint is given by

$$\hat{H}^{(\mu_0)}_{\rm grav} + \hat{H}^{(\mu_0)}_{\rm matter})|s\rangle = 0 \tag{2.30}$$

In terms of $\bar{s}_\mu := e^{\frac{i\mu}{4}\eta}s_\mu$ the Hamiltonian constraint (2.30) translates into a difference equation,

$$\begin{aligned}
0 &= A_{\mu+4\mu_0}\tilde{s}_{\mu+4\mu_0} - (2+4\mu_0^2\gamma^2\eta)A_\mu\tilde{s}_\mu + A_{\mu-4\mu_0}\tilde{s}_{\mu-4\mu_0} \\
&\quad +16\kappa\gamma^2{\mu_0}^3\left(\frac{1}{6}\gamma l_p^2\right)^{-\frac{1}{2}} H_m(\mu)\tilde{s}_\mu \quad , \qquad \forall\, \mu\, \in \mathbb{R} \qquad (2.31) \\
A_\mu &:= |\mu+\mu_0|^{\frac{3}{2}} - |\mu-\mu_0|^{\frac{3}{2}} \quad , \qquad \hat{H}^{(\mu_0)}_{\text{matter}}|\mu\rangle := H_m(\mu)|\mu\rangle \ .
\end{aligned}$$

$H_m(\mu)$ is a symbolic eigenvalue and we have assumed that the matter couples to the gravity via the metric component and *not* through the curvature component. In particular, $H_m(\mu = 0) = 0$. Some characteristics of the solutions of the difference equation are noteworthy.

- Although μ takes all possible real values, the equation connects the $\tilde{s}_\mu$ coefficients only in steps of $4\mu_0$ making it a difference equation for the coefficients. By putting $\mu := \nu + (4\mu_0)n,\ n \in \mathbb{Z},\ \nu \in (0, 4\mu_0)$, we have a continuous infinity of independent solutions of the difference equation, labelled by ν, $S_n(\nu) := \tilde{s}_{\nu+4\mu_0 n}$.

- For each ν an infinity of coefficients, $S_n(\nu), \forall n \in \mathbb{Z}$, are determined by 2 'initial conditions' since the order of the difference equation in terms of these coefficients is 2.

- Coefficients belonging to different ν are mutually decoupled. Denoting any two (fixed) independent solutions of the difference equation by $\rho_n(\nu)$ and $\sigma_n(\nu)$, a general solution of the second order difference equation, for *each* $\nu \in (0, 4\mu_0)$ can be expressed as $S_n(\nu) = S_0(\nu)\rho_n(\nu) + S_1(\nu)\sigma_n(\nu)$ where $S_0(\nu), S_1(\nu)$ are arbitrary complex numbers. Linearity of the equation implies that only their ratio is relevant and for $\nu = 0$, this ratio is fixed.

- A *general solution* of the fundamental equation (2.31) is given by a sum of the $S_n(\nu)$ solutions for a countable set of values of $\nu \in [0, 4\mu_0)$. Note that there are infinitely many ways of selecting the countable subsets of values of ν [9].

- Since the coefficients A_μ and the symbolic eigenvalues $H_m(\mu)$, both vanish for $\mu = 0$, the coefficient $\tilde{s}_0$ *decouples* from all other coefficients ($\tilde{s}_0$ of course occurs in the $\nu = 0$ sector). As the $\tilde{s}_\mu$ are defined for all $\mu \in \mathbb{R}$, there is no break down of the dynamics at zero volume.

- In ordinary Wheeler-DeWitt quantization, the evolution breaks down at $\mu = 0$ because the coefficients of the matter Hamiltonian in the differential equation blows up. In LQC, we can see from the difference equation that the although the coefficient $\tilde{s}_0$ remains undetermined, its knowledge is no necessary to obtain the evolution as the wave function vanishes at $\mu = 0$. As a result, the evolution can be continued beyond zero volume in a deterministic manner independent of $\bar{s}_0$. This is taken as one indication that the LQC dynamics is singularity free [10].

2.5 Derivation of Effective Hamiltonian

It is expected that the domain of validity of the effective Hamiltonian will be the large volume regime i.e. for large values of $\mu \gg 4\mu_0$ ($n \gg 1$).

The effective Hamiltonian can be obtained in various ways. One way is:

- For large volumes the coefficients A_μ in the equation (2.31) become almost constant (up to a common factor of $\sqrt{n}$) and that the matter contribution is also expected similarly to be almost constant. One then expects the coefficients to vary slowly as n is varied.

- This implies that $S_n(\nu)$ also vary slowly with n.

- Then we can interpolate these slowly varying *sequences* of coefficients by slowly varying, sufficiently differentiable, *functions* of the continuous variable $p(n) := \frac{1}{6}\gamma\ell_{\mathrm{P}}^2 n$ [11].

- Using Taylor expansion of the interpolating function, the difference equation for $S_n(\nu)$ then implies a *differential* equation for the interpolating function. This is referred to as a *continuum approximation.*

- The terms up to (and including) second order derivatives, turn out to be independent of γ and the differential equation, truncated to keep only these terms, matches with the usual Wheeler–DeWitt equation of quantum cosmology. This is referred to as a *pre-classical approximation* [12].

- Make a WKB ansatz for solutions of the differential equation. To the leading order in $\hbar$, we obtain a Hamilton-Jacobi equation for the phase from which the effective Hamiltonian is read-off.

- Improvement of the pre-classical approximation can be made by including higher derivative terms in the differential equation.

In [13], such an effective classical Hamiltonian was obtained from the pre-classical approximation and it was shown that the classical dynamics of implied by the effective Hamiltonian is also singularity free due to a generic occurrence of a bounce [14].

In this thesis, however, we will follow a different route [15]. We will make the WKB ansatz at the level of the difference equation itself. The slowly varying property of the interpolating function will be applied to the amplitude and the phase of the interpolating function. It is then very easy to obtain an effective Hamiltonian with nontrivial dependence on γ and also the domain of validity of WKB approximation. In the limit $\gamma \to 0$, one will recovers the results of [13].

In anticipation of making contact with a classical description, we introduce the dimensionful variable $p(\mu) := \frac{1}{6}\gamma\ell_{\mathrm{P}}^2\mu$ as the continuous variable and an interpolating function $\psi(p)$ via $\psi(p(\mu)) := \tilde{s}_\mu$. Correspondingly, define $p_0 := \frac{1}{6}\gamma\ell_{\mathrm{P}}^2\mu_0$ which provides a convenient *scale* to demarcate different regimes in p.

Defining $A(p) := (\frac{1}{6}\gamma\ell_{\mathrm{P}}^2)^{\frac{3}{2}}A_\mu$, $q := 4p_0$ and replacing $\tilde{s}_\mu$ by the interpolating function $\psi(p)$, the fundamental equation (2.31) becomes,

$$\begin{aligned} 0 &= A(p+q)\psi(p+q) - \left(2 + 9\frac{q^2}{\ell_{\mathrm{P}}^4}\eta\right)A(p)\psi(p) + A(p-q)\psi(p-q) \\ &\quad + 9\frac{q^3}{\ell_{\mathrm{P}}^4}\kappa H_m(p)\psi(p) \end{aligned} \tag{2.32}$$

This equation (2.32) can be thought of as a *functional equation* determining $\psi(p)$. It has infinitely many solutions, corresponding to $\{S_n(\nu)\}, \nu \in [0, 4\mu_0)$. Note that so far there has been no approximation, we have only used $p(\mu)$ instead of μ and $\psi(p)$ instead of $\tilde{s}_\mu$.

One can always write the complex function $\psi(p) := C(p)e^{i\frac{\Phi(p)}{\hbar}}$. In the spirit of a continuum description, we assume that the amplitude and the phase are Taylor expandable and write,

$$C(p \pm q) := C(p)(\delta C_+ \pm \delta C_-) \quad , \quad \Phi(p \pm q) := \Phi(p) + \delta\Phi_+ \pm \delta\Phi_- \tag{2.33}$$

$$\delta C_+ := \sum_{n=0}^{\infty}\frac{q^{2n}}{(2n)!}\frac{C_{2n}(p)}{C(p)} \quad , \quad \delta C_- := \sum_{n=0}^{\infty}\frac{q^{2n+1}}{(2n+1)!}\frac{C_{2n+1}(p)}{C(p)} \tag{2.34}$$

$$\delta\Phi_+ := \sum_{n=1}^{\infty}\frac{q^{2n}}{(2n)!}\Phi_{2n}(p) \quad , \quad \delta\Phi_- := \sum_{n=0}^{\infty}\frac{q^{2n+1}}{(2n+1)!}\Phi_{2n+1}(p) \tag{2.35}$$

We have just separated the even and odd number of derivative terms for later convenience. Substitution of $\psi(p \pm q) = C(p \pm q)e^{\frac{i}{\hbar}\Phi(p\pm q)}$ in (2.32) leads to,

$$
\begin{aligned}
0 &= H_m - \frac{1}{2\kappa}\frac{6}{\epsilon^3 \ell_{\mathrm{P}}^2}(2+\epsilon^2\eta)A(p) + \frac{1}{2\kappa}\frac{6}{\epsilon^3 \ell_{\mathrm{P}}^2} \times \\
& \left[(B_+(p,q)\delta C_+(p) + B_-(p,q)\delta C_-(p)) \left(\cos\frac{\delta\Phi_+}{\hbar}\cos\frac{\delta\Phi_-}{\hbar}\right) \right. \\
& -(B_-(p,q)\delta C_+(p) + B_+(p,q)\delta C_-(p)) \left(\sin\frac{\delta\Phi_+}{\hbar}\sin\frac{\delta\Phi_-}{\hbar}\right) \\
& +i(B_-(p,q)\delta C_+(p) + B_+(p,q)\delta C_-(p)) \left(\cos\frac{\delta\Phi_+}{\hbar}\sin\frac{\delta\Phi_-}{\hbar}\right) \\
& \left. +i(B_+(p,q)\delta C_+(p) + B_-(p,q)\delta C_-(p)) \left(\sin\frac{\delta\Phi_+}{\hbar}\cos\frac{\delta\Phi_-}{\hbar}\right) \right] \\
B_\pm(p,q) &:= A(p+q) \pm A(p-q) \qquad (2.36)
\end{aligned}
$$

Thus, one has *two equations* for the *four combinations*, $\delta C_\pm, \delta\Phi_\pm$.

Now we assume that the amplitude and the phase are slowly varying functions of p over a range q i.e. when compared over a range $\pm q$, succeeding terms of a Taylor expansion about p are smaller than the preceding terms. For example, $|\frac{qC'}{C}| \ll 1, |\frac{q^2C''}{2qC'}| \ll 1$ etc. and similarly for the phase.

For such slowly varying solutions of (2.32), we can approximate the equation by keeping only the first non-trivial derivatives i.e.

$$
\begin{aligned}
\delta C_+ = 1 \quad &, \quad \delta C_- = \frac{qC'}{C} \\
\text{and} \qquad \delta\Phi_- = q\Phi' \quad &, \quad \delta\Phi_+ = \frac{q^2\Phi''}{2}
\end{aligned}
$$

To arrive at a Hamilton-Jacobi equation, we identify $\Phi'(p) := \frac{3}{2\kappa}K$ which implies that $\frac{q\Phi'}{\hbar} = \frac{4\mu_0\gamma\ell_{\mathrm{P}}^2\Phi'}{3\hbar} = \epsilon K$, where we have used the definitions $\epsilon = 2\mu_0\gamma$ and $\ell_{\mathrm{P}}^2 = \kappa\hbar$. This combination of the first derivative of the phase thus has no explicit dependence on $\hbar$.

The terms in the real and imaginary equations can be organised according to powers of $\hbar$. For sine and cosine of $\delta\Phi_+$, we have to use the power series expansions and keep only the leading powers of $\hbar$. For sine and cosine of $\delta\Phi_-$, no expansion is needed since there is no $\hbar$ dependence. The real and imaginary equations can then be written as,

$$
\begin{aligned}
0 &= H_m - \frac{1}{2\kappa}\frac{6}{\epsilon^3\ell_{\mathrm{P}}^2}(2+\epsilon^2\eta)A(p) + \frac{1}{2\kappa}\frac{6}{\epsilon^3\ell_{\mathrm{P}}^2}\left[\left\{1+\frac{B_-(p,q)}{B_+(p,q)}\frac{qC'}{C}\right\}B_+(p,q)\ \cos(\epsilon\right. \\
&\qquad \left. -B_+(p,q)\left\{\frac{B_-(p,q)}{B_+(p,q)}+\frac{qC'}{C}\right\}\left(\frac{q^2\Phi''}{2\hbar}\right)\sin\epsilon K\right] \qquad (2 \\
0 &= \left(\frac{q^2\Phi''(p)}{\hbar\epsilon K}\right)\left\{1+\frac{B_-(p,q)}{B_+(p,q)}\frac{qC'}{C}\right\}+\left(\frac{B_-(p,q)}{B_+(p,q)}+\frac{qC'}{C}\right)\frac{\tan(\epsilon K)}{\epsilon K} \qquad (2
\end{aligned}
$$

On physical grounds, one expects a classical approximation ($\hbar^0$ terms), to b valid only for scales larger than the quantum geometry scale set by q and therefor we limit to the regime $p \geq q$. In this regime, the coefficients $A(p)$, $B_\pm(p,q)$ behav as

$$
\begin{aligned}
A(p) &\approx \frac{3}{4}q\sqrt{p} - o(p^{-\frac{3}{2}}) \\
B_+(p,q) &\approx \frac{3}{2}q\sqrt{p} - o(p^{-\frac{3}{2}}) \\
B_-(p,q) &\approx \frac{3}{4}q^2\ p^{-\frac{1}{2}} + o(p^{-\frac{5}{2}}) \qquad (2.39
\end{aligned}
$$

Noting that $q = \frac{1}{3}\epsilon\ell_{\mathrm{P}}^2$ and keeping only the leading powers of ℓ_{P}^2 (or $\hbar$), the rea and imaginary equations become (WKB approximation),

$$
\begin{aligned}
0 &= H_m - \frac{1}{2\kappa}\frac{6}{\epsilon^3\ell_{\mathrm{P}}^2}(2+\epsilon^2\eta)A(p) + \frac{1}{2\kappa}\frac{6}{\epsilon^3\ell_{\mathrm{P}}^2}\left[B_+(p,q)\cos(\epsilon K)\right] \qquad (2.40 \\
0 &= \left(\frac{q^2\Phi''(p)}{2\hbar\epsilon K}\right)+\left(\frac{B_-(p,q)}{B_+(p,q)}+\frac{qC'}{C}\right)\frac{\tan(\epsilon K)}{\epsilon K} \qquad (2.41
\end{aligned}
$$

The real equation, (2.40) is of $o(\hbar^0)$ and is a Hamilton-Jacobi equation for th phase. The right hand side, viewed as a function of p, K, it is the *effective Hami tonian constraint*. The imaginary equation, (2.41) is of $o(\hbar)$ and is a differenti equation for the amplitude, given the phase determined by the real equation. Th equations for the phase and the amplitude are decoupled. For self consistency the WKB approximation, the solutions have to be slowly varying. In general, th solutions will be slowly varying only over some intervals along the p-axis and suc interval(s) will be the domain of validity of the WKB approximation. One ca infer the domain of validity as follows.

- Consider the eqn. (2.41). By definition of slowly varying phase, the absolu value of the first term must be much smaller than 1.
- Since smallest value of $\frac{\tan(\epsilon K)}{\epsilon K}$ is 1, (the absolute value of) the bracket in th second term must be smaller than 1.

- Since $\frac{qC'}{C}$ is also small by definition of slowly varying amplitude, we must have $|\frac{B_-}{B_+}| \leq 1$.

- This immediately requires $p \geq q = \frac{\epsilon \ell_P^2}{3}$ and is obviously true in the regime under consideration.

- For *smaller volumes*, $p \gtrsim q$, where $\frac{B_-}{B_+}$ is larger, we must have $\frac{\tan \epsilon K}{\epsilon K} \gtrsim 1$ i.e. $\epsilon K \approx 0$.

- For *larger volumes*, $p \gg q$, larger values of $\epsilon K (< \frac{\pi}{2})$ are permitted.

Thus, the equation (2.41) serves to identify a region of the classical *phase space* where the effective classical description is valid. The domain of validity of effective Hamiltonian and its relation to the usual general relativity (GR) Hamiltonian is discussed further in the remarks below. Also note that the WKB approximation is expected to break down near the turning points. The predictions from this approach are not expected to hold very near the regimes where the scale factor reaches an extremum.

It is convenient to write the real equation in the form,

$$
\begin{aligned}
0 \;=\; & -\frac{1}{2\kappa}\left[\left(\frac{3}{\epsilon \ell_P^2}\right)\left\{B_+(p,q)\left(\frac{4}{\epsilon^2}\sin^2\left(\frac{\epsilon}{2}K\right)\right) + 2A(p)\eta\right\}\right] \\
& + \frac{1}{2\kappa}\left[\left(\frac{6}{\epsilon^3 \ell_P^2}\right)\left\{B_+(p,q) - 2A(p)\right\}\right] \;+\; H_m \qquad (2.42)
\end{aligned}
$$

The second square bracket in eq.(2.42) is called the *quantum geometry potential*, $W_{qg}/2\kappa$. It is unaffected by the implicit inclusion of all terms of the Taylor expansion of $\psi(p)$. Since the leading terms in $B_+ - 2A$ vanish, the quantum geometry potential term is independent of ϵ. But it is also higher order in $\hbar$. Since we will be concentrating on $o(\hbar^0)$ corrections from now on, this term is suppressed. This term is also absent if we choose the symmetric ordering for the Hamiltonian instead of the ordering chosen in this work. This point is important and is discussed further later in this chapter. In the limit $\epsilon \to 0$, we get back the expression for the effective Hamiltonian constraint obtained from the pre-classical approximation.

2.6 Equations of Motion

Let us reiterate that as mentioned in the previous section

- We will look at the effective Hamiltonian at the large volume regime.

- We will look only at $o(\hbar^0)$ corrections to the classical Hamiltonian, i.e. look only at the discrete geometry corrections. The quantum geometry potential will also be suppressed because it is higher order in $\hbar$.

- The matter is assumed to couple only through metric components and thus the matter Hamiltonian has dependence on p but not on K. For the moment, we will take it to be a function p and matter degrees of freedom symbolically denoted by ϕ, p_ϕ.

The relevant Hamiltonian constraint then becomes:

$$0 \;=\; -\frac{3}{2\kappa}\sqrt{p}\left[\frac{4}{\epsilon^2}\sin^2\left(\frac{\epsilon}{2}K\right)+\eta\right] + \; H_m(p,\phi,p_\phi) \qquad (2.43)$$

To see what the modified Hamiltonian constraint implies for cosmological space-times, we have to obtain and solve the Hamilton's equations. We identify $|p| = \frac{1}{4}a^2$ and choose the synchronous time as the evolution parameter (Lapse = 1). It is straight forward to obtain the Hamilton's equations of motion as:

$$\frac{dp}{dt} \;=\; -\frac{\sqrt{p}}{\epsilon}\sin(\epsilon K) \;=\; \frac{a\dot{a}}{2} \qquad (2.44)$$

$$\frac{dK}{dt} \;=\; -\frac{\kappa}{3}\frac{\partial H_m(\phi,p_\phi,p)}{\partial p} + \frac{1}{4\sqrt{p}}\left[\frac{4}{\epsilon^2}\sin^2\left(\frac{\epsilon}{2}K\right)+\eta\right] \qquad (2.45)$$

$$\frac{d\phi}{dt} \;=\; \frac{\partial H_m(p,\phi,p_\phi)}{\partial p_\phi} \qquad (2.46)$$

$$\frac{dp_\phi}{dt} \;=\; -\frac{\partial H_m(p,\phi,p_\phi)}{\partial \phi} \qquad (2.47)$$

The equation (2.43) is the modification of the Friedmann equation while (2.44, 2.45) lead to modified Raychoudhuri equation *after eliminating K, p in favour of the scale factor and its time derivatives*. In particular, (2.44) gives $\dot{a} = -\frac{\sin(\epsilon K)}{\epsilon}$ and leads to $\cos(\epsilon K) = \sqrt{1-\epsilon^2\dot{a}^2}$ ($\epsilon \to 0$ limit fixes the sign of the square root). Thus the Hamiltonian constraint can be expressed in terms of $\dot{a}$. Furthermore, since the constraint is obviously preserved along the solutions of the Hamilton's equations, we can obtain $\dot{K}$ in terms of H_m.

Now it is straightforward to construct left hand sides of the Friedmann and the Raychoudhuri equation and comparing with usual Einstein equations, read-off the effective density and pressure. We get

$$\frac{\kappa}{2}\rho_{\text{eff}} := 3\left(\frac{\dot{a}^2+\eta}{a^2}\right) \;=\; \frac{\kappa}{2}\left[8a^{-3}H_m - \frac{3\epsilon^2}{2\kappa a^2}\left(\frac{4\kappa}{3a}H_m-\eta\right)^2\right] \qquad (2.48)$$

$$-\frac{\kappa}{4}\left(\rho_{\text{eff}}+3P_{\text{eff}}\right) := 3\frac{\ddot{a}}{a} \;=\; -\frac{\kappa}{4}\left[8a^{-3}\left(H_m - a\frac{\partial H_m}{\partial a}\right)\left\{1-\frac{\epsilon^2}{2}\left(\frac{4\kappa}{3a}H_m-\eta\right)\right\}\right] \qquad (2.49)$$

For future reference, we note that the effective density and pressure can be expressed as

$$\begin{aligned}
\rho_{\text{eff}} &= 8a^{-3}\bar{H}_m \\
(\rho_{\text{eff}} + 3P_{\text{eff}}) &= 8a^{-3}(\bar{H}_m - a\frac{\partial \bar{H}_m}{\partial a}) \qquad \text{where,} \\
\bar{H}_m &:= H_m - \frac{3\epsilon^2 a}{16\kappa}\left(\frac{4\kappa}{3a}H_m - \eta\right)^2
\end{aligned} \tag{2.50}$$

As $\epsilon \to 0$ ($\gamma \to 0$), the right hand sides of the above equations go over to the equations of [13] and so do the effective density and effective pressure, with the quantum geometry potential terms suppressed.

The advantage now is that the Raychoudhuri equation is automatically satisfied once the Hamilton's equations for the matter hold and the Friedmann equation holds. Thus we have the ϵ-corrected, three coupled (for a single matter degree of freedom), first order ordinary differential equations which go over to the classical equations when $\epsilon \to 0$. It is now a simple task to compare the solutions with/without ϵ corrections, for the same initial conditions consistent with small $\dot{a}$ and large a. Some solutions are discussed in the next section.

Several remarks are in order:

1. It is somewhat surprising that the parameters related to quantum geometry, μ_0 and γ (which have to be non-zero), appear in the effective Hamiltonian which is *independent* of $\hbar$ i.e., in a *classical* description.

   To trace how this happens, recall that the basic phase space variables in isotropic LQC (in the connection formulation) are c, p and the conjugate momentum variable K is related to the connection variable c by $c = \gamma K$ (for spatially flat model for definiteness). In loop quantizing the Hamiltonian constraint, the classical constraint is to be expressed in terms of the holonomies of the connection. In this process, the parameter μ_0 enters. The classical constraint so obtained, precisely contains the $\sin^2 \mu_0 c$ which is the same as $\sin^2 \frac{\epsilon K}{2}$. The parameter ϵ is thus present already at the classical level ($\hbar^0$). Within a strictly classical context, one can view it as a "regulator" and remove it at will, by taking $\epsilon \to 0$ to recover the classical Einstein Hamiltonian. Within a strictly WKB context also one can remove it since the corrections to WKB are positive powers of $\frac{q}{p} \sim o(\epsilon)$. However, from the perspective of the exact loop quantization, we cannot take $\epsilon \to 0$. While viewing the WKB solutions as an approximation to the exact LQC solutions, we retain the link to the exact solutions by keeping $\epsilon \neq 0$.

2. The classical GR Hamiltonian is the expression obtained from the effective Hamiltonian in the limit $\epsilon K \to 0$ in the large volume regime. This could

be achieved by either taking $\epsilon \to 0, K$ fixed, which is a natural classi GR perspective since the classical GR has no ϵ or by taking $K \to 0, \epsilon$ fix which is the appropriate quantum perspective in LQC. This opens up possibility that there could be modifications of classical GR especially wh the conjugate momentum (K) gets somewhat larger. As noted already, smaller volumes, the domain of WKB approximation restricted by equati (2.41), requires K to be small and the effective Hamiltonian reduces to GR Hamiltonian. For small K, $K \approx -\dot{a}$ and corresponds to the extrins curvature (a geometrical quantity). For larger values of K, the relati between K and the extrinsic curvature is given by equation (2.44).

3. Since for smaller volumes, the effective Hamiltonian goes over to the G Hamiltonian, the effective density and pressure defined in (2.48, 2.49) over to those defined by the preclassical approximation. Consequently, genericness of inflation is insensitive to the ϵ parameter. Furthermore, sin the quantum geometry potential is unaffected, the genericness of bounce al continues to hold (in this regime, the quantum geometry potential must retained in equation (2.42)) [14].

4. The occurrence of the trigonometric function of K immediately implies tw bounds. The effective Hamiltonian constraint, (2.43), implies that is th the matter Hamiltonian is necessarily bounded: $\eta \leq \frac{4\kappa}{3}a^{-1}H_m \leq \eta + \frac{4}{\epsilon^2}$ equivalently, $0 \leq \frac{4\kappa}{3}a^{-1}H_m - \eta \leq \frac{4}{\epsilon^2}$. Equation (2.44) on the other ha implies a bound on the expansion rate: $|\dot{a}| \leq \epsilon^{-1}$.

5. Finally, let us note that $\dot{a}$ is the extrinsic curvature of the symmetry adapte hypersurface (and is also the rate of change of the physical volume of th universe) and is gauge invariant. The modifications to GR due to the no zero parameter ϵ are manifested in the modified coupling (quadratic in matt Hamiltonian) between matter and geometry. The GR domain, containe within the domain of validity of WKB, can also be given as: $|p| = \frac{a^2}{4}$ $q = \frac{\epsilon \ell_P^2}{3}$ translates into $a \gg \sqrt{\frac{4\epsilon}{3}}\ell_P$ while $|\epsilon K| \ll \frac{\pi}{2}$ translates into 0 $\frac{4\kappa}{3a}H_m - \eta \ll \frac{\pi^2}{4\epsilon^2}$. The effective Hamiltonian and subsequent analysis bein $o(\hbar^0)$, is insensitive to factor ordering in the Hamiltonian constraint.

In the next sections we will look at some explicit examples of modified dynam ics.

2.7 Modified solutions

Let us consider a few solutions of the modified equations. These are not mean to be phenomenologically realistic solutions, but are to be viewed as indicatin modification to GR solutions. In particular we will consider the cases of a minimally coupled homogeneous scalar field, phenomenological matter with a constant equation of state and positive cosmological constant viewed as a special case of phenomenological matter.

2.7.1 Minimally coupled massive scalar field

For simplicity, let us take the matter sector consisting of a scalar field, ϕ, minimally coupled to gravity. Then its usual classical Hamiltonian is given by

$$H_m(\phi, p_\phi, p) = p^{-3/2}p_\phi^2 + 2p^{3/2}V(\phi) \tag{2.51}$$

Its quantization involves two parts:
- Quantization of $p^{-\frac{3}{2}}$ operator.
- Bohr quantization of the scalar field itself.

For large volume, the inverse volume operator in the triad representation just goes over to the classical expression. Its correction terms involve higher powers of $\hbar$ (the terms suppressed by inverse powers of p in the coefficients $B_+(p,q), A(p,q)$, likewise involve higher powers of $\hbar$). Thus, to $o(\hbar^0)$, these modifications are irrelevant in the large volume regime. The Bohr quantization appears not to introduce γ corrections to the usual classical matter Hamiltonian. In this paper we will assume that the matter is quantized in the usual Schrodinger quantization.

The solutions are obtained numerically with $V(\phi) := \frac{1}{2}m^2\phi^2$ and with initial conditions chosen to indicate different types of behaviours. We use geometrized units ($\kappa = 1$ and speed of light equal to 1). Introducing an arbitrary length scale $\bar{a}$, various quantities has following dimensions.

$$H_m \sim \bar{a}\ ,\ p_\phi \sim \bar{a}^2\ ,\ \phi \sim \bar{a}^0\ ,\ m \sim \bar{a}^{-1}\ ,\ t \sim \bar{a}\ . \tag{2.52}$$

Scaling the quantities by the appropriate power of $\bar{a}$ all equations are rendered dimensionless and integrated numerically. Since we use the usual classical form of the matter Hamiltonian without the corrections from the inverse volume, the length scale $\bar{a}$ is suitably large (eg $\bar{a} \gtrsim 100\ell_P$).

For the plots shown below in figure(2.1), figure(2.2) and figure(2.3), the dimensionless scale factor assumed to be 100 and the dimensionless mass is taken to be 0.001. Three different values of the discreteness parameter are chosen: $\epsilon = 0.0, 0.5, 0.7$. Three different initial conditions for the scalar field and its momentum are taken namely $(\phi, p_\phi) = (1, 10000), (10, 10000), (50, 50)$. As can be expected, the initial value of the scalar field has stronger effect on the evolution. For smaller value of the scalar field, the effect of non-zero ϵ is virtually absent while for larger values one sees the multiple re-collapse/bounce possibilities. For the plots, spatially flat model is considered ($\eta = 0$). Only the evolution of the scale factor is shown in the figures. The scalar field typically increases first and then decreases to smaller values. Both the increase and the decrease of the scalar field are steeper for larger values of ϵ. The scalar field momentum also shows similar behaviour. The decrease however is much sharper and occurs at later times for larger values of ϵ.

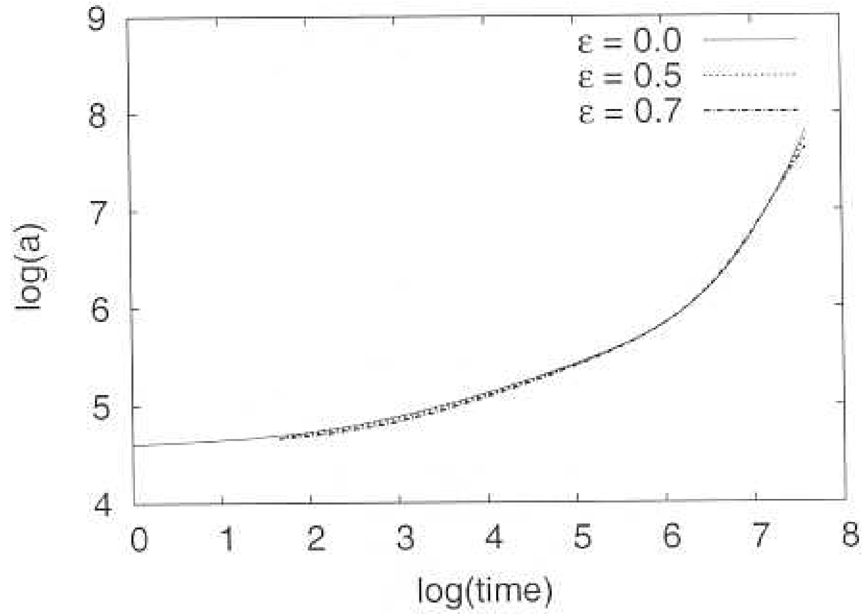

Figure 2.1: The plot is for the initial values $\phi = 1, p_\phi = 10000$. The discreteness corrections are very small.

2.7.2 Phenomenological Matter

In the current treatment, the matter is also supposed to be described by a Hamilto-nian. However, for the usual 'dust' and 'radiation' which are supposed to represe[nt] non-relativistic and relativistic matter respectively, one does not write a Ham[il]tonian. The underlying matter dynamics (with time scale shorter than that [of] expansion of the universe) is supposed to ensure a thermal equilibrium. Su[ch] matter is then thermodynamically described in terms of an energy density, ρ a[nd] pressure, P, with the equilibrium property implying an equation of state, $P = \omega$ The first law of thermodynamics in conjunction with adiabatic expansion impli[es] the 'continuity equation' $a\frac{d\rho}{da} = -3\rho(1+\omega)$ which can be integrated to give,

$$\rho(a) = \bar{\rho} e^{-3\int_{\bar{a}}^{a}\{1+\omega(a')\}\frac{da'}{a'}} = \bar{\rho}\left(\frac{a}{\bar{a}}\right)^{-3(1+\omega)}, \quad \text{when } \omega \text{ is constant.} \quad (2.5$$

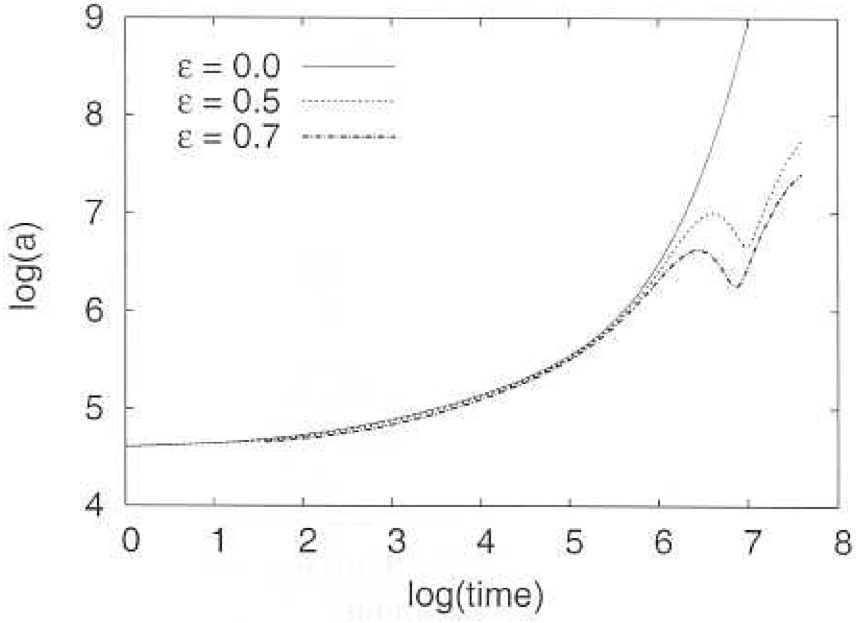

Figure 2.2: This plot is for the initial values $\phi = 10, p_\phi = 10000$. For late times, deviations due to non-zero ϵ are clearly visible.

The equilibrium property also implies that the density and pressure are homogeneous. Such matter in (approximate) equilibrium is coupled to gravity via a stress tensor of the form of the stress tensor of a prefect fluid (and the matter is correspondingly referred to as a perfect fluid) which is of course the most general form of stress tensor in the context of homogeneity and isotropy. This stress tensor is conserved by virtue of the continuity equation implied by adiabatic expansion and the first law of thermodynamics. This way of coupling matter to gravity is thus consistent with the Einstein equation.

It is conceivable that some of the mechanisms (eg microscopic matter dynamics) involved in establishing thermal equilibrium may be modified, especially if the expansion time scale becomes comparable to that of the matter processes responsible for establishing thermal equilibrium. However, for the large volume regime we are considering, one may justifiably assume that the microscopic matter dynamics are unchanged and lead to thermal equilibrium with the usual equations of states. If this assumption is granted then such sources can be incorporated by taking $H_m(a) := \rho(a)\frac{a^3}{8}$, with the scale factor dependence of matter Hamiltonian explicitly specified via that of the thermodynamical energy density, $\rho(a)$. In view

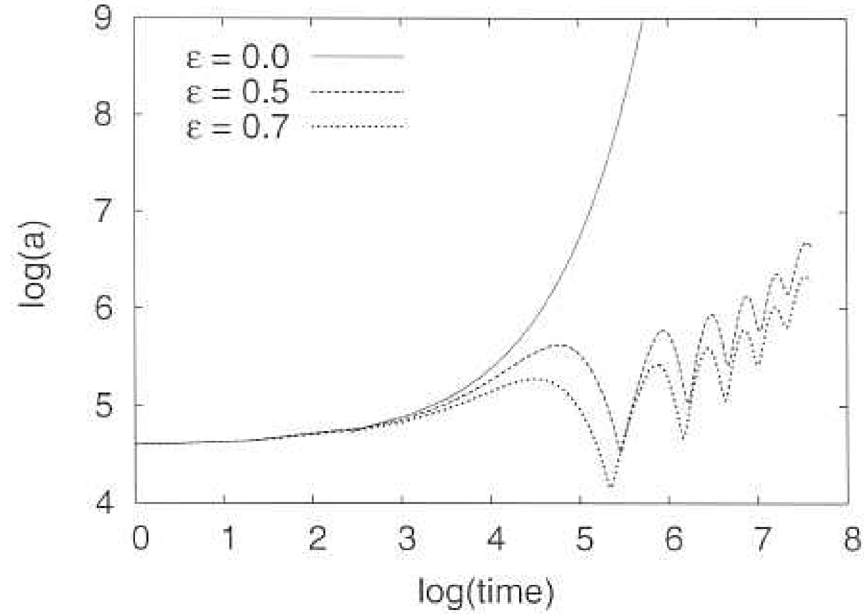

Figure 2.3: The initial values here are: $\phi = 50, p_\phi = 50$. For late times t
evolutions show multiple bounces and re-collapses for non-zero ϵ.

of the expressions in (2.50), this is equivalent to putting

$$\bar{H}_m = \rho_{\text{eff}} \frac{a^3}{16} \qquad (2.5$$

The assumption made above amounts to viewing the effective equation of sta $\omega_{\text{eff}} := P_{\text{eff}}/\rho_{\text{eff}}$ being determined from the usual equation of state, ω. The effect density and pressure are thus not ascribed any thermodynamic origin, but a viewed as prescribing a modified coupling of the matter to gravity.

The evolution of the scale factor is now obtained by solving just the (modifie Friedmann equation (2.48) for various choices of $\omega(a)$. For example, for consta equation of state, ω, and for expanding universe,

$$\dot{a} = + \left[\left\{ \frac{\kappa\bar{\rho}\bar{a}^2}{12} \left(\frac{a}{\bar{a}} \right)^{-1-3\omega} - \eta \right\} - \frac{\epsilon^2}{4} \left\{ \frac{\kappa\bar{\rho}\bar{a}^2}{12} \left(\frac{a}{\bar{a}} \right)^{-1-3\omega} - \eta \right\}^2 \right]^{\frac{1}{2}} \qquad (2.5$$

Introducing dimensionless quantities, 5

$$\sigma^2 := \frac{\kappa\bar{\rho}\bar{a}^2}{12} \ , \ \xi := \frac{a}{\bar{a}} \ , \ \tau := \frac{t}{\bar{a}} \ , \ ' := \frac{d}{d\tau} \ , \qquad (2.5$$

the Friedmann equation can be written as,

$$\xi' = \sigma\xi^{-(1+3\omega)}\sqrt{\xi^{(1+3\omega)} - \left(\frac{\sigma\epsilon}{2}\right)^2} \quad , \quad \text{(flat model)} \quad (2.57)$$

$$\xi' = \sqrt{\sigma^2\xi^{-(1+3\omega)} - 1}\sqrt{1 - \frac{\epsilon^2}{4}\left(\sigma^2\xi^{-(1+3\omega)} - 1\right)} \quad , \quad \text{(closed model)} \quad (2.58)$$

It is convenient to parameterize the equation of state variable as, $\omega(n) := \frac{3-n}{3(n-1)}$ so that

- $n = 0$ corresponds to the cosmological constant
- $n = 2$ corresponds to the radiation
- $n = 3$ corresponds to the dust

For the flat models, the substitution $\xi := (\frac{\sigma\epsilon}{2}\cosh\lambda)^{n-1}$, leads to,

$$\left[\frac{\sigma}{n-1}\left(\frac{2}{\sigma\epsilon}\right)^n\right](\tau - \tau_0) = \int(\cosh\lambda)^n d\lambda$$
$$= 2^{-n}\sum_{r=0}^{n} {}^nC_r\left(\frac{\sinh(2r-n)\lambda}{2r-n}\right) \quad (2.59)$$

For even integer n, the parenthesis in the $r = n/2$ term in the summation, is to be replaced by λ. We have thus obtained the solution of the Friedmann equation (2.57) in a parametric form, in particular for the cases of interest namely $n = 0, 2, 3$.

To recover the $\epsilon \to 0$ solutions, one has only to note that for non-zero n, the left hand side of (2.59) diverges for $\tau > \tau_0$ (say), and implies that one must have λ very large. The leading term on the right hand side is then $\frac{2^{-n}}{n}e^{n\lambda}$. Likewise, for large λ, $\xi \sim (\frac{\sigma\epsilon e^{\lambda}}{4})^{n-1}$. One can now eliminate ϵe^{λ} to get $a(t) \propto (t - t_0)^{\frac{n-1}{n}}$. Clearly, these match with the usual ($\epsilon = 0$) solutions. The case of the cosmological constant ($n = 0$) is discussed in the next subsection.

For the closed models, again one can obtain parametric form of the solutions which involves an integration. However the qualitative behavior can be seen easily. Now a different substitution is convenient. Putting $\xi := (\frac{\sigma}{\cosh\lambda})^{(n-1)}$ one obtains,

$$-\frac{1}{n-1}\sigma^{1-n}(\tau(\lambda) - \tau_0) = \int_{\lambda_0}^{\lambda} d\lambda' \frac{(\cosh\lambda')^{-n}}{\sqrt{1 - \left(\frac{\epsilon^2}{4}\right)(\sinh\lambda')^2}} \quad (2.60)$$

Clearly, λ must be bounded to ensure $0 \leq \sinh^2\lambda \leq \frac{4}{\epsilon^2}$. One can also check that the integrand is integrable at the maximum value of λ. Taking $\tau_0 = 0 = \lambda_0$ and

$\tau := \tau_{\max}$ for the upper limit of integration to be $\lambda_{\max}$ (defined by $\sinh^2 \lambda_{\max} =$
one obtains an oscillatory universe with a finite period given by $T := 2\tau_{\max}$ for the three cases of interest. The $\epsilon \to 0$ limit is simpler in this case since ξ does not depend explicitly on ϵ. Also $\lambda_{\max}$ diverges and therefore for any finite λ can simply take $\epsilon = 0$ in the integrand. For $n = 2, 3$, then one recovers the usual (parametric) form of the solutions.

For dust and radiation, the corrections due to non-zero ϵ are extremely small. This is to be expected since the densities decrease with the scale factor, making the $a^{-1}H_m^2$ term very small.

2.7.3 Cosmological Constant

This is obtained by taking

$$H_m := \Lambda p^{3/2} = \Lambda \frac{a^3}{8} \qquad (2.6$$

and as mentioned before, mathematically it corresponds to $n = 0$. In the equation (2.56), we replace $\bar{\rho}$ by Λ.

For $\epsilon = 0$, one has the well known solutions: $a(t) = a(0)\exp(\sqrt{\Lambda/6}\,t)$ for $\eta =$ and $a(t) = \sqrt{6/\Lambda}\cosh(\sqrt{\Lambda/6}\,(t - t_0)\,)$ for $\eta = 1$.

For $\epsilon \neq 0$, the solutions are qualitatively different. For the spatially flat case $\eta = 0$, one has a re-collapsing solution given explicitly as,

$$a(t) = \frac{a_{\max}}{\cosh(\sqrt{\Lambda/6}\,(t_0 - t)\,)} \quad , \quad a_{\max}^2 := \frac{24}{\epsilon^2 \Lambda} \qquad (2.6$$

The scale factor vanishes only for $t \to \pm\infty$ and the solution is non-singular. As $\epsilon \to 0$, the maximum value for scale factor diverges and one is either in the expanding or contracting phase. One can obtain the $\epsilon = 0$ solution by taking the limit $\epsilon \to 0$, $t_0 \to \infty$ with $\epsilon e^{\sqrt{\frac{\Lambda}{6}} t_0}$ held constant.

For the closed model, $\eta = 1$, the scale factor is bounded from both above and below as $1 \leq \Lambda a^2/6 \leq (1 + 4/\epsilon^2)$ and the universe keeps oscillating between these values in finite time, with the period of oscillation given by,

$$T = 2\sqrt{6/\Lambda}\int_0^{\lambda_{\max}} \frac{d\lambda}{\sqrt{1 - \frac{\epsilon^2}{4}\sinh^2\lambda}} \quad , \quad \sinh(\lambda_{\max}) := 2/\epsilon \qquad (2.6$$

As $\epsilon \to 0$, the maximum value of ξ diverges, the period diverges. The usual de Sitter solution is recovered by just taking $\epsilon = 0$ in the integrand of equation (2.6 which gives $\lambda \sim \sigma(\tau - \tau_0)$ and eliminating λ. In both the cases above, one can check from the Raychoudhuri equation that smaller scale factor is indeed a local minimum while the larger one is a local maximum (as a function of t).

For the cosmological constant as the only source, for $\epsilon = 0$, the equation of state, $w := P/\rho = -1$ remains independent of the scale factor and the Raychoudhuri equation then implies an always accelerating universe, precluding the possibility of re-collapse. With the ϵ corrections incorporated, the effective density and pressure is modified and the modified equation of state acquires a scale factor dependence which is such that there is always a decelerating phase before re-collapse.

Had we taken $\epsilon \to 0$, we would have got the classical GR Hamiltonian leading to the usual exponentially expanding universe without a re-collapse ($\eta = 0$). The difference equation on the other hand shows that discrete wave function, in any *one sector* begins to oscillate over the quantum geometry scale $\sim \frac{4}{6}\mu_0\gamma\ell_{\mathrm{P}}^2$ for large values of the triad, implying a breakdown of pre-classicality (no small scale variations) of the exact solutions.

While deviations from pre-classicality are expected in the small volume regime due to quantum effects, deviations in the very large volume regime have no such physical reasons and are viewed as an 'infra-red problem' [16, 17]. A typical example of such large volume deviations is that of cosmological constant (and non-zero spatial curvature). No quantity which has a local interpretation such as energy density (Λ) is ill-behaved and the integrated quantity diverges due to integration over infinite volume. In such cases, the homogeneous idealization (integrand being a constant) is thought to become in-applicable beyond a finite volume. Noting that the wave function has an oscillation length of $(\Lambda a)^{-1}$, one can obtain a bound on the largest scale factor, $\tilde{a}$, by requiring the oscillation length to be larger than the scale of slow variation, $\sqrt{q} = \sqrt{\frac{\epsilon}{3}}\ell_{\mathrm{P}}$. This gives, $\tilde{a} \lesssim \sqrt{\frac{3}{\epsilon}}(\Lambda\ell_{\mathrm{P}})^{-1}$.

In the effective picture, we have a maximum value, a_{max} of the scale factor due to re-collapse and for $\eta = 0$ it is given by $a_{\mathrm{max}}^2 = \frac{24}{\epsilon^2\Lambda}$ in equation (2.62). Demanding that $a_{\mathrm{max}} \leq \tilde{a}$ gives a bound on the cosmological constant: $\Lambda\ell_{\mathrm{P}}^2 < \epsilon/8$. For such a cosmological constant, the effective picture can be consistent with pre-classical behaviour of exact solutions. Incidentally, for the currently favoured value of the cosmological constant, one has $\Lambda\ell_{\mathrm{P}}^2 \sim 10^{-120}$.

In the above discussion, we have used the large scale factor expressions for the equations, mainly because we were interested in seeing the re-collapse possibility. Secondly we also used the effective classical picture close to the WKB turning point which may lead to non physical predictions of the values of the variables at the turning points. The full effective classical equations together with the domain of validity of the WKB approximation have been given in the previous section. The main lesson is that the corrections due to discreteness can change some of the solutions qualitatively while for others these are perturbative in nature.

2.7.4 Criticisms

The work described above is based on the quantization available at the time it w done. Subsequently LQC has undergone significant changes based on a differe ordering of the Hamiltonian and with a different interpretation of the ambigui parameter μ_0. We will mention the recent progress in the next section. We e this section with some criticisms to the above approach.

- In the cosmological constant case, there is a recollapse in the classical regi i.e. in the regime where a is large and $\dot{a}$ is small. This is viewed in the abo framework as an infrared problem.

- The critical value of the scalar matter density ρ_{crit} at which bounce occu is very low and physically unrealistic.

  To see this consider the spatially flat case ($\eta = 0$) with a massless scal matter Hamiltonian $H_m(p_\phi, p) = p^{-3/2}p_\phi^2$ in the large volume regime whe the WKB approximation is supposed to be valid i.e. $p \gg p_0$. As we ca see from (2.44), the turning points occur at $p = p_*$ when $\sin(\epsilon K) = 0$. Th implies that the extremum occurs for either $\sin(\epsilon K/2) = 0$ or $\cos(\epsilon K/2) =$ If we choose the first option we can see from (2.43), $p_\phi \propto p_*$ and therefo from equation (2.50) we can see $\rho_{\text{eff}} \propto \frac{1}{p_*}$. Hence for large values of p_*, ρ is small. Note that, had we taken the symmetric ordering for the Hamilt nian (2.42) this is the only possibility available which leads to the physical unrealistic possibility of recollapse in the classical regime.

  However in the non-symmetric ordering chosen for this work, there is also t quantum geometry potential (i.e. the second term of (2.42)) which is sma for large volumes but which become important once the first term becom small. In that case it turns out that $p_\phi \propto \frac{1}{p_*}$ and therefore from equati (2.50) we can see $\rho_{\text{eff}} \propto p_*$ [18]. We get a bounce in the classical regime. Al strictly speaking, the WKB approximation is not valid near turning point

- All the results are obtained on the kinematic Hilbert space $\mathcal{H}_{kin}$. As we ha seen in the previous chapter the physical states i.e. solutions of the quantu Hamiltonian constraint is not normalizable on $\mathcal{H}_{kin}$. Therefore a new inn product has to be defined to construct the physical Hilbert space $\mathcal{H}_{phy}$ befo any the predictions coming from LQC can be put on a firm footing.

2.8 LQC: Recent Progress

Recently a lot of work has been done in the LQC framework to address some of t problems arising in the formulation described above. The first step is to constru a physical level understanding of the theory by constructing a physical Hilbe space. Initially this has been done for the $\eta = 0$ case with a massless scalar field [20] along the following lines:

- Choose a monotonically increasing classical variable as internal time and attempt to interpret the solutions of the Hamiltonian constraint as evolution with respect to this parameter. In this case the scalar field ϕ provides the time variable.
- The Hamiltonian is written in the symmetric ordering to construct a self adjoint quantum operator. An inner product is introduced on the space of solutions and the physical Hilbert space $\mathcal{H}_{phy}$ is constructed via "group averaging procedure".
- Dirac observables are chosen which are represented by self-adjoint operators on $\mathcal{H}_{phy}$. In this case they turn out to be the momentum conjugate to the scalar field i.e. p_ϕ and the value of p at any "instant" ϕ_0 i.e. $p|_{\phi_0}$.
- These observables are used to construct semi classical states at late times which are peaked around the classical trajectory.
- These states are evolved and the behaviour of these states near the classical singularity is observed

It turns out that the singularity is avoided even on the physical level in the sense that the wave functions 'tunnel' through the classical singularity. However the problems mentioned in the above section are accentuated. Because a symmetric ordered Hamiltonian is chosen there is a possibility of recollapse in the classical regime.

This has led to further investigation of the theory. In particular e nature of the regulator μ_0 has been reexamined and the theory has been reformulated where it is not a constant number but a dynamical quantity [19]. The crucial change is the change in the viewpoint that the size of the smallest loop used in defining the curvature term (2.23) should be measured by the area operator referring to physical geometry and not to the fiducial metric as before. That is if each edge of the basic cell is of length λ, then this is now chosen to be a function $\bar{\mu}(p)$. The equation (2.28) is therefore replaced by [19]

$$\bar{\mu}^2|p| = \Delta \equiv \frac{\sqrt{3}\gamma\ell_{\mathrm{P}}^2}{4} \tag{2.64}$$

Thus $\bar{\mu}$ is a non trivial function on phase space.

There is another heuristic justification of why $\bar{\mu}$ should be a function, coming from the full theory. As the scale factor grows, the number of vertices in a fixed fiducial cell should increase. In full LQG, the Hamiltonian constraint operator has to be evaluated at these vertices by considering elementary cubes around them. Since the number of vertices grow the area of the faces of the elementary cubes measured by the fiducial metric should decrease. Keeping the edge length of the elementary cubes to be constant amounts to ignoring the creation of the new vertices.

However making $\bar{\mu}$ to be a function of phase space means that the action of the operator $\exp \widehat{i(\bar{\mu}c/2)}$ on triad eigenstates is complicated. It turns out that volume eigenstates provide a more convenient basis for writing the action i.e. the basis is changed from $|\mu\rangle$ to $|v\rangle$ which also constitute an orthonormal basis in the LQC Hilbert space. It can be shown that the action of the volume operator on these states is given by

$$\hat{V}|v\rangle = \left(\frac{\gamma \ell_{\mathrm{P}}^2}{6}\right) \frac{|v|}{K}|v\rangle \qquad \text{where} \qquad K = \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}} \tag{2.65}$$

while $\exp \widehat{i(\bar{\mu}c/2)}$ acts by addition.

$$\widehat{e^{ik\frac{\bar{\mu}c}{2}}}|v\rangle = |v+k\rangle \tag{2.66}$$

The Hamiltonian operator written in symmetric ordering becomes

$$\begin{aligned} \hat{H}_{\mathrm{grav}} &= \sin(\bar{\mu}c)\hat{A}\sin(\bar{\mu}c) \qquad\qquad \text{where} \qquad (2.67)\\ \hat{A} &:= \frac{24i\mathrm{sgn}(\mu)}{\gamma^3 \ell_{\mathrm{P}}^2 \bar{\mu}^3}\left(\sin\left(\frac{\bar{\mu}c}{2}\right)\hat{V}\cos\left(\frac{\bar{\mu}c}{2}\right) - \cos\left(\frac{\bar{\mu}c}{2}\right)\hat{V}\sin\left(\frac{\bar{\mu}c}{2}\right)\right) \end{aligned}$$

In this new formalism, the difference equation for the $\eta = 0$ case with massless scalar becomes [19]

$$\partial_\phi^2 \psi(v,\phi) = [B(v)]^{-1}\left(C^+(v)\psi(v+4,\phi) + C^0(v)\psi(v,\phi) + C^-(v)\psi(v-4,\phi)\right) \tag{2.68}$$

where

$$\begin{aligned} B(v) &= \left(\frac{3}{2}\right)^3 K|v|\,\left||v+1|^{1/3} - |v-1|^{1/3}\right|^3 \\ C^+(v) &= \frac{3\kappa K}{64}|v+2|\,\left||v+1| - |v+3|\right| \\ C^-(v) &= C^+(v-4) \\ C^0(v) &= -C^+(v) - C^-(v) \end{aligned} \tag{2.69}$$

The effective Hamiltonian is now given by

$$0 = -\frac{3}{\kappa\gamma^2\bar{\mu}^2}|p|^{\frac{1}{2}}\sin^2(\bar{\mu}c) + \frac{1}{2}B(p)p_\phi^2 \tag{2.70}$$

where B(p) is the eigenvalue of the $\widehat{|p|^{-3/2}}$ operator.

This completes a brief overview of the new kinematic framework of LQC. One crucial change is that the difference equation, in the $|\mu\rangle$ basis, no longer has uniform step size of $4\mu_0$ but non uniform steps depending on the state. However once the $|v\rangle$ basis is chosen, the step size is again uniform. The Hamiltonian is self adjoint and the physical Hilbert space is constructed via group averaging techniques [20]. The fundamental features of singularity avoidance and bounce are recovered at the physical level [21]. Similar analysis has been carried out for the $\eta = 1$ case [22, 23]. In that model, with the new quantization bounces and re-collapses are neatly accommodated to get a cyclic evolution.

Work is on to extend the physical Hilbert space construction to cases other than the massless scalar field as well as in understanding the nature of the evolution of the universe across the big bang singularity.

3

Loop Quantization of Polarized Gowdy Model on T^3:Classical Theory

In the previous chapter we looked at studies on loop quantization of a Minisuperspace model, namely FRW cosmology. In this and the next chapter we will try to set up a loop quantized theory of a midisuperspace model. We will look at one of the simplest possible midisuperspace models studied in GR namely the Polarized Gowdy Model on T^3. This model has one field degree of freedom and two particle degrees of freedom. It is therefore more complicated than LQC but much less so than the full theory. In (3.1) we shall give a brief description of what are Gowdy cosmologies. In section (3.2) we shall briefly state the various properties and constructs used in the previous studies of the model. In section (3.3) we shall sketch the construction of the unpolarized T^3 model in Ashtekar variables. In sections (3.4,3.5)we will discuss the systematic reduction of the unpolarized model to the polarized model in variables which will be convenient for loop quantization. We shall reconstruct the classical spacetime solution from the Hamilton's equations of motion in sections (3.6). That will complete the study of the classical model. Loop quantization of this model will be discussed in the next chapter.

3.1 What are Gowdy Models

Gowdy Models [24] refer to the class of 4 dimensional spacetimes which are:

- globally hyperbolic i.e. we can foliate the spacetime $\mathcal{M}$ into spatial Cauchy slices Σ_t paramatrized by a global time function t
- spatially compact without boundary
- solutions of vacuum Einstein's equations
- is isometric under the action of an Abelian T^2 group which acts on the leaves, Σ_t, of the foliation

These conditions restrict the manifold Σ_t to be one of the following: T^3, S^3, $S^2 \otimes S^1$ (or any manifold covered by one of them). If we further assume that

- the Killing vectors generating the T^2 isometry can be chosen to be mutually orthogonal everywhere

we get what are known as Polarized Gowdy models. In this chapter we shall mainly deal with Polarized Gowdy models with the spatial slice taken to be T^3. In the next few sections we shall briefly review some of the properties of this model.

3.2 Polarised Gowdy T^3 Model: A Brief Review

We shall first discuss the spacetime picture of the model in terms of 4 dimensional metric variables. The metric is diagonal and can be written as [25]

$$\mathrm{d}s^2 = e^{2a}(-\mathrm{d}T^2 + \mathrm{d}\theta^2) + T(e^{2W}\mathrm{d}x^2 + e^{-2W}\mathrm{d}y^2) \tag{3.1}$$

where x, y, θ label points on the T^3 with $\partial/\partial x$ and $\partial/\partial y$ being the two Killing vectors. The two variables a and W are functions of T and periodic functions of θ.

The vacuum Einstein's equations can be organised so that W is obtained from the second order differential equation:

$$\frac{\partial^2 W}{\partial T^2} + \frac{1}{T}\frac{\partial W}{\partial T} - \frac{\partial^2 W}{\partial \theta^2} = 0 \tag{3.2}$$

and given a solution, $W(T, \theta)$, the function $a(T, \theta)$ is determined from:

$$\begin{aligned}
\frac{\partial a}{\partial \theta} &= 2T\frac{\partial W}{\partial T}\frac{\partial W}{\partial \theta} \\
\frac{\partial a}{\partial T} &= -\frac{1}{4T} + T\left[\left(\frac{\partial W}{\partial T}\right)^2 + \left(\frac{\partial W}{\partial \theta}\right)^2\right]
\end{aligned} \tag{3.3}$$

The equation (3.2) encodes the dynamics while the (3.3) encodes the constraints. Incidentally, this makes the initial value problem and the problem of preservation of constraints in numerical relativity trivial for this model. The initial values of the dynamical variable W can be freely specified and given a W the constraint a can be trivially determined [26]. The requirement of a being a periodic function of θ imposes a condition of the solutions of the W equation, namely, $\int d\theta \partial_T W \partial_\theta W = 0$. The general solution of (3.2) satisfying this condition is given by[25]

$$W = \alpha + \beta \ln T + \sum_{n=1}^{\infty} \left[a_n J_0(nT) \sin(n\theta + \gamma_n) + b_n N_0(nT) \sin(n\theta + \delta_n) \right] \quad (3$$

where α, β, a_n, b_n, γ_n and δ_n are real constants and J_0 and N_0 are regular a irregular Bessel functions of the zeroth order. The special case of *homogeneou* model is given by $\beta = \frac{1}{2}$ and $a_n = 0 = b_n$ and corresponds to the flat Kasn solution described as:

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}\theta^2 + T^2 \mathrm{d}x^2 + \mathrm{d}y^2 \quad (3.$$

It can be shown that the curvature invariant $C \equiv R_{abcd}R^{abcd}$ blows up alm everywhere as $T \to 0_+$. The solutions are therefore generically singular. Howev for the special choice,

$$b_n = 0 \ , \qquad \beta = \frac{1}{2} \quad (3.$$

the curvature invariant remains bounded and all components of R_{abcd} have fini limit as $T \to 0_+$. It can also be shown that these nonsingular solutions are analy ically extendible [25] but are causally ill-behaved (have closed time-like curves) i the extended portion. Thus there exists a infinite number of nonsingular solution which however form a set of measure zero in the space of solutions. Curvatur unboundedness is the generic behaviour.

The approach to classical singularity is well studied and is known to follo a special case of the BKL scenario known as asymptotically velocity term dom inated near singularity (AVTDS) [27]. At late times, the model is known to b asymptotically homogeneous [28].

These models have been analysed in the canonical framework in both metri variables as well as in terms of the complex Ashtekar variables. The first attempt of quantization, were carried out in ADM variables in [29, 30]. Another approac which has been more successful was based on an interesting property of the mod After a suitable (partial) gauge fixing, these models can be described by (modul a remaining global constraint) a "point particle" degree of freedom and by a scal field ϕ which is subject to the same equations of motion as a massless, rotationall symmetric, free scalar field propagating in a fictitious two dimensional expandin torus. This equivalence was used in the quantization carried out in [31]. Subse quent analysis has been carried out in a large number of works some of which ar listed in [32, 33, 34, 35, 36]. However in these quantizations, the evolution turne out to be non-unitary and in [37, 38, 39, 40, 41] a new parametrization was intro duced which implemented unitary evolution in quantum theory. More recently hybrid quantization wherein the homogeneous modes are loop quantized while th inhomogeneous ones are Fock quantized, has been proposed [42], claiming that loop quantization of the homogeneous modes suffices to resolve the Gowdy singularity.

In the next section we will briefly look at the construction of unpolarized Gowdy T^3 model in Ashtekar variables.

3.3 Gowdy T^3 Model in Ashtekar Variables

Canonical quantization of *unpolarised* Gowdy T^3 model in terms of the complex Ashtekar variables has been given in [43] and [44] which we will briefly sketch below in terms of the real Ashtekar variables.

Recall that owing to global hyperbolicity the spacetime can be decomposed as $\mathcal{M} = \Sigma_t \otimes \mathbb{R}$. Let the coordinates of Σ_t be (θ, x, y). Let the two commuting Killing vectors be $\xi_1^a = \frac{\partial}{\partial x}$ and $\xi_2^a = \frac{\partial}{\partial y}$. This implies that the Lie derivatives along these two Killing vectors vanish i.e.

$$\begin{aligned} \mathcal{L}_{\xi_1} A_a^i &= 0 = \mathcal{L}_{\xi_1} E_i^a \\ \mathcal{L}_{\xi_2} A_a^i &= 0 = \mathcal{L}_{\xi_2} E_i^a \end{aligned}$$

The phase space variables are therefore only functions of θ. The Gauss and the Diffeo constraint reduce to

$$G_i = \partial_\theta E_i^\theta + \epsilon_{ij}^k A_a^j E_k^a \tag{3.7}$$

$$V_a = (\partial_a A_b^i) E_i^b - (\partial_\theta A_a^i) E_i^\theta + \epsilon_{jk}^i A_a^j A_b^k E_i^b \tag{3.8}$$

The combination that generates spatial diffeomorphisms known as the *Vector constraint* is given by

$$\begin{aligned} C_a &= A_a^i G_i - V_a \\ &= A_a^i (\partial_\theta E_i^\theta) - (\partial_a A_b^i) E_i^b + (\partial_\theta A_a^i) E_i^\theta \end{aligned} \tag{3.9}$$

We now impose the following Gauge fixing conditions:

$$E_I^\theta = 0 = E_3^\rho \quad ; \quad \rho = x, y \; ; \; I = 1, 2 \tag{3.10}$$

The constraints G_I and C_ρ are then solved by

$$A_\theta^I = 0 = A_\rho^3 \tag{3.11}$$

Thus only one Gauss constraint ($G_3 := G$.) and one diffeomorphism constraint along the θ direction ($C_\theta := C$) remain along with the Hamiltonian constraint.

These are given by:

$$
\begin{aligned}
G &= \frac{1}{\kappa\gamma}\left[\partial_\theta E^\theta_3 + \epsilon_J{}^K A^J_\rho E^\rho_K\right] \; ; \qquad\qquad \epsilon_J{}^K := \epsilon_{3J}{}^K \\
C &= \frac{1}{\kappa\gamma}\left[(\partial_\theta A^I_\rho)E^\rho_I + \epsilon_J{}^K A^J_\rho E^\rho_K A^3_\theta - \kappa\gamma A^3_\theta G_3\right] \; ; \\
H &= \frac{1}{2\kappa}\frac{1}{\sqrt{|\det E|}}\left[\, 2A^3_\theta E^\theta_3 A^J_\rho E^\rho_J + A^J_\rho E^\rho_J A^K_\sigma E^\sigma_K - A^K_\rho E^\rho_J A^J_\sigma E^\sigma_K \right. \\
&- 2\epsilon_J{}^K(\partial_\theta A^J_\rho)E^\rho_K E^\theta_3 - (1+\gamma^2)\left(2K^3_\theta E^\theta_3 K^J_\rho E^\rho_J + K^J_\rho E^\rho_J K^K_\sigma E^\sigma_K - K^K_\rho E^\rho_J K^J_\sigma \ldots\right.
\end{aligned}
$$

Since none of the quantities depend on x or y we can integrate over the T^2 and write the symplectic structure and the total Hamiltonian as:

$$
\Omega = \frac{4\pi^2}{\kappa\gamma}\int \mathrm{d}\theta \left(\mathrm{d}A^3_\theta \wedge \mathrm{d}E^\theta_3 + \mathrm{d}A^I_\rho \wedge \mathrm{d}E^\rho_I\right) \tag{3.15}
$$

$$
H_{\text{tot}} = 4\pi^2 \int d\theta \left\{\lambda^3 G + N^\theta C + NH\right\} \tag{3.16}
$$

Note that this is basically a one dimensional theory. In one dimension, under orientation preserving coordinate transformations, a tensor density of contravariant rank p, covariant rank q and weight w, can be thought of as a scalar density of weight $= w + q - p$. Hence under the θ coordinate transformation E^θ_3 transforms as a scalar, E^ρ_I's transform as scalar densities of weight 1, A^3_θ transforms as a scalar density of weight 1 and A^I_ρ's transform as scalars.

3.4 Choice of New Variables

The results presented in this and the subsequent sections of this chapter are based on [45]. We will rewrite the model in terms of variables which turn out to be more convenient for loop quantization. Note that for each ρ, the A^I_ρ and E^ρ_I, rotate among themselves under the $U(1)$ gauge transformations generated by the Gauss constraint. These suggest that we can perform canonical transformations to define the following variables [46]:

$$
\begin{aligned}
E^x_1 &= E^x \cos\beta \quad ; \quad E^x_2 = E^x \sin\beta && (3.17) \\
E^y_1 &= -E^y \sin\bar{\beta} \quad ; \quad E^y_2 = E^y \cos\bar{\beta} && (3.18) \\
A^1_x &= A_x \cos(\alpha+\beta) \quad ; \quad A^2_x = A_x \sin(\alpha+\beta) && (3.19) \\
A^1_y &= -A_y \sin(\bar{\alpha}+\bar{\beta}) \quad ; \quad A^2_y = A_y \cos(\bar{\alpha}+\bar{\beta}) && (3.20)
\end{aligned}
$$

The angles for the connection components are introduced in a particular fashion for later convenience.

The radial coordinates, E^x, E^y, A_x, A_y, are gauge invariant and always strictly positive (vanishing radial coordinates correspond to trivial symmetry orbit which is ignored).

In terms of these variables, the symplectic structure (3.15) gets expressed as:

$$\Omega = \frac{4\pi^2}{\kappa\gamma}\int \mathrm{d}\theta \left[\mathrm{d}A_\theta^3 \wedge \mathrm{d}E_3^\theta + \mathrm{d}X \wedge \mathrm{d}E^x + \mathrm{d}Y \wedge \mathrm{d}E^y + \mathrm{d}\beta \wedge \mathrm{d}P^\beta + \mathrm{d}\bar{\beta} \wedge \mathrm{d}\bar{P}^\beta\right] \tag{3.21}$$

where:

$$X := A_x \cos(\alpha) \quad ; \quad Y := A_y \cos(\bar{\alpha}) \tag{3.22}$$

$$P^\beta := -E^x A_x \sin(\alpha) \quad ; \quad \bar{P}^\beta := -E^y A_y \sin(\bar{\alpha}) \tag{3.23}$$

The gauge transformations generated by the Gauss constraint shift $\beta, \bar{\beta}$ while α and $\bar{\alpha}$ are gauge invariant. *From now on we will absorb the $4\pi^2$ and use $\kappa' := \frac{\kappa}{4\pi^2} = \frac{2G_{\mathrm{Newton}}}{\pi}$.*

It is convenient to make a further canonical transformation:

$$\xi = \beta - \bar{\beta} \quad ; \quad \eta = \beta + \bar{\beta} \tag{3.24}$$

$$P^\xi = \frac{P^\beta - \bar{P}^\beta}{2} \quad ; \quad P^\eta = \frac{P^\beta + \bar{P}^\beta}{2} \tag{3.25}$$

In terms of these variables the Gauss and the diffeomorphism constraints can be written as:

$$G = \frac{1}{\kappa\gamma}\left[\partial_\theta E_3^\theta + 2P^\eta\right] \tag{3.26}$$

$$C = \frac{1}{\kappa\gamma}\left[(\partial_\theta X)E^x + (\partial_\theta Y)E^y - (\partial_\theta E_3^\theta)A_\theta^3 + (\partial_\theta \eta)P^\eta + (\partial_\theta \xi)P^\xi\right] \tag{3.27}$$

The Hamiltonian constraint is complicated but after putting $K_i^a = (A_i^a - \Gamma_i^a)/\gamma$, substituting the explicit expressions of Γ_i^a, and further simplification, turns out to be:

$$\begin{aligned}
H = & - \frac{\gamma^{-2}}{2\kappa}\frac{1}{\sqrt{E}}\left[E_3^\theta\left\{(XE^x + YE^y)\partial_\theta\eta + (XE^x - YE^y)\partial_\theta\xi - 2P^\xi\partial_\theta\left(\ln\frac{E^y}{E^x}\right)\right.\right. \\
& + \left. 2P^\eta(\partial_\theta \ln E_3^\theta + (\tan\xi)\ \partial_\theta\xi)\right\} + 2\left\{(\cos^2\xi)\left(XE^xYE^y + (P^\eta)^2 - (P^\xi)^2\right)\right. \\
& + \left.(XE^x + YE^y)E_3^\theta A_\theta^3\right\} + (\sin 2\xi)\left\{(XE^x + YE^y)P^\xi - (XE^x - YE^y)P^\eta\right\} \\
& + \left.\left(\frac{1+\gamma^2}{2}\right)\left\{(\partial_\theta E_3^\theta)^2 - \left(\frac{E_3^\theta\partial_\theta\xi}{\cos\xi}\right)^2 - \left(\frac{E_3^\theta\partial_\theta(\ln(E^y/E^x))}{(\cos\xi)}\right)^2\right\}\right] \\
& - \frac{1}{2\kappa}\partial_\theta\left(\frac{4E_3^\theta P^\eta}{\sqrt{E}}\right) \qquad (3.2
\end{aligned}$$

where $E = |E_3^\theta E^x E^y(\cos\xi)|$.

Under the action of the diffeomorphism constraint X, Y, E_3^θ, η and ξ transform as scalars while E^x, E^y, A_θ^3, P^η and P^ξ transform as scalar densities of weight 1.

This completes the description of the *unpolarised* Gowdy T^3 Model in the variables we have defined. The number of canonical field variables is 10 while there is a 3-fold infinity of first class constraints. There are therefore 2 field degrees of freedom. We now need to impose two second class constraints such that the number of field degrees of freedom are reduced from two to one (as it should be in the polarized case).

3.5 Reduction to Polarized model

In terms of the variables defined above, the spatial 3-metric is given by,

$$ds^2 = \cos\xi\frac{E^xE^y}{E_3^\theta}d\theta^2 + \frac{E_3^\theta}{\cos\xi}\frac{E^y}{E^x}dx^2 + \frac{E_3^\theta}{\cos\xi}\frac{E^x}{E^y}dy^2 - 2\frac{E_3^\theta}{\cos\xi}\sin\xi\ dxdy \quad (3.2$$

For the Killing vectors $\partial/\partial x$ and $\partial/\partial y$ to be orthogonal to each other, the $dxdy$ term in the metric should be zero. This implies that the polarization condition is implemented by restricting to $\xi = 0$ sub-manifold of the phase space of the unpolarized model. For getting a non-degenerate symplectic structure, one needs to have one more condition. This condition should be chosen *consistently* in the following sense.

We expect the two conditions to reduce a field degree of freedom. This can be viewed in two equivalent ways. The condition $\xi = 0$ makes the metric diagonal and this property should be preserved under evolution (i.e. the extrinsic curvature should also be diagonal). Alternatively, the unpolarized model is a constrained system and we want to impose two conditions such that one *physical* (field) degree of freedom is reduced. The extra conditions to be imposed should therefore be *first class* with respect to the constraints of the unpolarized model i.e. should weakly Poisson-commute with them.

The choice $\xi = 0$ also requires $E^\theta_3 > 0$ for the spatial metric to have signature $(+,+,+)$. The choice $\xi = \pi$ would require $E^\theta_3 < 0$. From now on $E^\theta_3 > 0$ will be assumed. We shall take $\xi = 0$ as a constraint and demand its preservation under the evolution generated by the total Hamiltonian. Since $\xi = 0$ weakly Poisson commutes with the Gauss and the diffeomorphism constraints, only the Poisson bracket with the Hamiltonian constraint is needed.

$$\xi(\theta) \approx 0 \quad , \quad \{\xi(\theta), \int d\theta' N(\theta') H(\theta')\} \approx 0 \tag{3.30}$$

It follows that,

$$\dot{\xi}(\theta) \approx 0 \quad \Rightarrow \quad \chi(\theta) := 2P^\xi + E^\theta_3 \partial_\theta (\ln E^y/E^x) \ \approx 0 \tag{3.31}$$

The Poisson Bracket of χ with the Hamiltonian turns out to be zero on the constraint surface i.e. $\dot{\chi} \approx \chi \ \approx 0$. Thus, the reduction to the Polarized model is obtained by imposing the two *polarization constraints*

$$\xi \approx 0 \quad ; \quad \chi \approx 0 \quad ; \quad \{\xi(\theta), \chi(\theta')\} = 2\kappa\gamma\delta(\theta - \theta') \tag{3.32}$$

The fact that the $\chi \approx 0$ condition follows from preservation of $g_{xy} = 0$, can be seen by noting that for the present case, it implies that $\dot{g}_{xy} \sim K_{xy} = K^i_x e^i_y (= e^i_x K^i_y) = 0$. Using the definition $K^i_a = \gamma^{-1}(A^i_a - \Gamma^i_a)$ and putting in the explicit expressions one can check directly that $K_{xy} = 0 \Leftrightarrow \chi = 0$. Note that this is *not* equivalent to requiring orthogonality of components of the connection, $A^i_x A^i_y = 0$ which would imply $\alpha = \bar{\alpha}$ (see eqns. (3.19 and 3.20). This condition is not preserved under evolution.

It follows from (3.26) that $\{\chi, G\} = 0$ And using (3.27), one can see that :

$$\left\{\xi, \int N^\theta C_\theta\right\} = N^\theta \partial_\theta \xi \approx 0 \ , \ \left\{\chi, \int N^\theta C_\theta\right\} = \partial_\theta (N^\theta \chi) \approx 0 \ . \tag{3.33}$$

We can solve the polarization constraints strongly and use Dirac brackets. Symbolically,

$$\{f, g\}^* = \{f, g\} - \{f, \xi\} \odot \{\xi, \chi\}^{-1} \odot \{\chi, g\} - \{f, \chi\} \odot \{\chi, \xi\}^{-1} \odot \{\xi, g\}$$

Here $\odot$ denotes appropriate integrations since we have field degrees of freedom.

Since the polarization constraints weakly commute with all the other con-

straints, the constraint algebra in terms of Dirac brackets is same as that in ter[ms]
of the Poisson brackets and thus remains unaffected. Furthermore, equations [of]
motions for all the variables other than ξ, P_ξ also remain unaffected. We can th[us]
set the polarization constraints strongly equal to zero in all the expressions [and]
continue to use the original Poisson brackets.

The expressions of the constraints simplify greatly. In particular the Hamil[to]nian constraint simplifies to,

$$
\begin{aligned}
H &= -\frac{\gamma^{-2}}{2\kappa}\frac{1}{\sqrt{E}}\left[\frac{(\kappa\gamma G)^2}{2} + (XE^x + YE^y)E^\theta_3\partial_\theta\eta \right.\\
&+ 2\Big\{XE^xYE^y + (XE^x + YE^y)E^\theta_3 A^3_\theta\Big\} + \frac{\gamma^2}{2}\Big\{(\partial_\theta E^\theta_3)^2 - (E^\theta_3\partial_\theta \ln(E^y/E^x) \ldots \\
&+\frac{1}{2\kappa}\partial_\theta\left\{\frac{2E^\theta_3\left(\partial_\theta E^\theta_3 - \kappa\gamma G\right)}{\sqrt{E}}\right\}
\end{aligned}
$$

where $E = |E^\theta_3|E^xE^y$.
P_η has also been eliminated in terms of the Gauss constraint using
$2P^\eta = (\kappa\gamma G - \partial_\theta E^\theta_3)$.

Note that η is translated under a gauge transformation. we can therefore se[t it]
to any constant and fix the gauge transformation freedom. Explicitly, imposing η [≈]
0 as a constraint, we can fix the λ^3 from preservation of this gauge fixing conditi[on].
Once again we can use Dirac brackets with respect to the Gauss constraint a[nd]
the $\eta \approx 0$ constraint and impose these constraints strongly. Then the he first t[wo]
terms and the G dependent piece in the last term in the Hamiltonian, drop o[ut]
and so do the degrees of freedom η, P_η. We are left with six canonical degrees [of]
freedom and the two first class constraints, leaving one field degree of freedo[m].
Thus our final variables and constraints for the *polarized* Gowdy model are (wi[th]
$\kappa' := \kappa$) :

$$
\begin{aligned}
\mathcal{E} := E^\theta_3 \quad &, \quad \mathcal{A} := \gamma^{-1}A^3_\theta \\
K_x := \gamma^{-1}X \quad &, \quad K_y := \gamma^{-1}Y
\end{aligned}
\qquad (3.3\ldots)
$$

(3.3…)

The Poisson brackets are given by

$$
\{K_x(\theta), E^x(\theta')\} = \kappa\delta(\theta - \theta')
$$

and similarly for $(K_y, E^y), (\mathcal{A}, \mathcal{E})$ pairs.

The explicit expressions of the remaining constraints are

$$C = \frac{1}{\kappa}\left[(\partial_\theta K_x)E^x + (\partial_\theta K_y)E^y - (\partial_\theta \mathcal{E})\mathcal{A}\right] \tag{3.37}$$

$$\begin{aligned} H = \frac{1}{\kappa}\Bigg[& -\frac{1}{\sqrt{E}}\Big\{(K_x E^x K_y E^y) + (K_x E^x + K_y E^y)\mathcal{E}\mathcal{A}\Big\} \\ & -\frac{1}{4\sqrt{E}}\Big\{(\partial_\theta \mathcal{E})^2 - (\mathcal{E}\partial_\theta(\ln(E^y/E^x)))^2\Big\} + \partial_\theta\left(\frac{\mathcal{E}\partial_\theta\mathcal{E}}{\sqrt{E}}\right)\Bigg] \end{aligned} \tag{3.38}$$

The constraint algebra among the $C[N^\theta], H[N]$ is:

$$\{\, C[N^\theta]\,, C[M^\theta]\,\} = C\,[\, N^\theta\partial_\theta M^\theta - M^\theta\partial_\theta N^\theta\,] \tag{3.39}$$

$$\{\, C[N^\theta]\,, H[N]\,\} = H[\, N^\theta\partial_\theta N\,] \tag{3.40}$$

$$\{\, H[M]\,, H[N]\,\} = C\left[\, (M\partial_\theta N - N\partial_\theta M)\mathcal{E}^2 E^{-1}\,\right] \tag{3.41}$$

Since each term in the Hamiltonian constraint is a scalar density of weight +1 and each term in the diffeomorphism constraint is of density weight +2, the first two brackets are easily verified. The last one also follows with a bit longer computation. We have thus verified the constraint algebra of polarized model showing the consistency of the reduction procedure.

This completes the construction of the Polarized Gowdy T^3 model in the variables which we had defined. Before proceeding with quantization we will first reconstruct the classical spacetime solutions in the next section.

3.6 Space-time construction

The next task is to find the set of gauge *inequivalent* solutions of the Hamilton's equations of motion, satisfying the two sets of constraints and obtain the space-time interpretation. The total Hamiltonian being a constraint, the Lagrange multipliers – the lapse function and the shift vector – also enter in the Hamilton's equations of motion. These need to be either *prescribed* or deduced via a gauge-fixing procedure. Once this is done, one can obtain the solution curves in the phase space with "initial points" lying on the constrained surface.

3.6.1 Solutions with a Chosen Lapse and Shift

The space-time metric, solving Einstein equation is paramatrized by,

$$ds^2 = -N^2(t,x^i)dt^2 + g_{ij}(t,x^i)\left(dx^i - N^i(t,x^i)dt\right)\left(dx^j - N^j(t,x^i)dt\right) \tag{3.42}$$

For our case, the metric is diagonal, $(x^1,x^2,x^3) \leftrightarrow (\theta,x,y)$, $N^i \leftrightarrow (N^\theta,0,0)$ and the metric is independent of the coordinates (x,y). The t = constant, hypersurfaces are diffeomorphic to the 3-torus. The metric components are given by $g_{\theta\theta} = E^x E^y \mathcal{E}^{-1} = E\mathcal{E}^{-2}, g_{xx} = \mathcal{E}E^y/E^x, g_{yy} = \mathcal{E}E^x/E^y$ (eq.(3.29)). The Gowdy form of the metric (3.1) is realized if one *prescribes* $N^\theta = 0$ and $N^2 = g_{\theta\theta}$.

Such a prescription is eminently consistent since any metric on a 2 dimensional manifold (coordinatized by (t, θ)), can always be (locally) chosen to be conformally flat. This however does *not* fix the coordinates t, θ completely – one can still make the conformal diffeomorphisms: $t \to t' = t + \xi^t(t,\theta), \theta \to \theta' = \theta + \xi^\theta(t,\theta)$ with ξ satisfying the conformal Killing equations: $\partial_t\xi^t - \partial_\theta\xi^\theta = 0 = \partial_\theta\xi^t - \partial_t\xi^\theta$.

We will first take the above prescription for the lapse and the shift, obtain the Hamilton's equations of motion, use the freedom of conformal diffeomorphisms and reduce the equations to those given in section (3.2). Subsequently, we will also determine gauge fixing functions to arrive at the same result. This will complete the identification of inequivalent solutions of the Einstein equation.

With the choices $N^\theta = 0, N = \sqrt{E}\mathcal{E}^{-1}$, the space-time metric (3.42) is

$$ds^2 = E\mathcal{E}^{-2}\left(-dt^2 + d\theta^2\right) + \mathcal{E}\left(\frac{E^y}{E^x}dx^2 + \frac{E^x}{E^y}dy^2\right) \tag{3.43}$$

and the time evolution is governed by the Hamiltonian given by,

$$\begin{aligned} H[\mathcal{E}^{-1}\sqrt{E}] &= \frac{1}{\kappa}\int d\theta\left[-\frac{1}{\mathcal{E}}\left\{(K_xE^xK_yE^y) + (K_xE^x + K_yE^y)\mathcal{E}\mathcal{A}\right\}\right. \\ &\left. -\frac{1}{4\mathcal{E}}\left\{(\partial_\theta\mathcal{E})^2 - (\mathcal{E}\partial_\theta(\ln(E^y/E^x)))^2\right\} + \frac{\sqrt{E}}{\mathcal{E}}\partial_\theta\left(\frac{\mathcal{E}\partial_\theta\mathcal{E}}{\sqrt{E}}\right)\right] \end{aligned} \tag{3.44}$$

In anticipation let us define $2W := \ln(E^y/E^x)$ and $2a := \ln(E^xE^y/\mathcal{E})$. Then

$$\begin{aligned} \frac{\dot{E}^x}{E^x} &= \mathcal{E}^{-1}(K_yE^y + \mathcal{A}\mathcal{E}) \\ \frac{\dot{E}^y}{E^y} &= \mathcal{E}^{-1}(K_xE^x + \mathcal{A}\mathcal{E}) \\ \dot{\mathcal{E}} &= (K_xE^x + K_yE^y) \end{aligned}$$

Therefore

$$2\partial_t W := \partial_t \ln\frac{E^y}{E^x} = \frac{(K_xE^x - K_yE^y)}{\mathcal{E}} \tag{3.45}$$

$$2\partial_t a := \partial_t \ln\frac{E^xE^y}{\mathcal{E}} = 2\mathcal{A} \tag{3.46}$$

The Poisson brackets of K_xE^x, K_yE^y with the Hamiltonian are given by,

$$\{K_xE^x, H[\mathcal{E}^{-1}\sqrt{E}]\} = \frac{1}{2}\partial_\theta\left(\mathcal{E}\partial_\theta \ln\frac{E^y}{E^x}\right) + \frac{1}{2}\partial_\theta^2\mathcal{E} \tag{3.47}$$

$$\{K_yE^y, H[\mathcal{E}^{-1}\sqrt{E}]\} = -\frac{1}{2}\partial_\theta\left(\mathcal{E}\partial_\theta \ln\frac{E^y}{E^x}\right) + \frac{1}{2}\partial_\theta^2\mathcal{E} \tag{3.48}$$

$$\tag{3.49}$$

This implies

$$\{K_xE^x - K_yE^y, H[\mathcal{E}^{-1}\sqrt{E}]\} = \partial_\theta\left(\mathcal{E}\partial_\theta \ln\frac{E^y}{E^x}\right) \tag{3.50}$$

$$\{K_xE^x + K_yE^y, H[\mathcal{E}^{-1}\sqrt{E}]\} = \partial_\theta^2\mathcal{E} \tag{3.51}$$

From these, we get second order equations for $\mathcal{E}$, W as,

$$\partial_t^2\mathcal{E} = \partial_\theta^2\mathcal{E} \tag{3.52}$$

$$\partial_t^2 W = \frac{1}{\mathcal{E}}\partial_\theta\left(\mathcal{E}\partial_\theta W\right) - \left(\frac{1}{\mathcal{E}}\partial_t\mathcal{E}\right)\partial_t W \tag{3.53}$$

The equation for $\mathcal{E}$ is a simple wave equation and given a solution of this, the equation for W can be solved determining W or the ratio E^y/E^x. From the first order equations, one determines the $K_xE^x \pm K_yE^y$ as well. The Hamiltonian constraint then determines $\mathcal{A}$ in terms of known quantities, $\mathcal{E}, W$ *and* the θ-derivatives of a. Using the equation $\partial_t a = \mathcal{A}$, one obtains,

$$\partial_t a = \mathcal{A} = -\frac{1}{4}\frac{\partial_t\mathcal{E}}{\mathcal{E}} + \frac{\mathcal{E}}{\partial_t\mathcal{E}}\left((\partial_t W)^2 + (\partial_\theta W)^2\right) - \frac{\partial_\theta\mathcal{E}}{\partial_t\mathcal{E}}\partial_\theta a - \frac{1}{4}\frac{(\partial_\theta\mathcal{E})^2}{\mathcal{E}\partial_t\mathcal{E}} + \frac{\partial_\theta^2\mathcal{E}}{\partial_t\mathcal{E}} \tag{3.54}$$

One can also obtain, by direct computation and using the diffeomorphism constraint,

$$\partial_\theta a = \frac{\mathcal{E}}{\partial_t\mathcal{E}}\left[2\partial_t W\partial_\theta W - \frac{\partial_\theta\mathcal{E}}{\mathcal{E}}\partial_t a + \frac{\partial_{t\theta}^2\mathcal{E}}{\mathcal{E}} - \frac{\partial_t\mathcal{E}\partial_\theta\mathcal{E}}{2\mathcal{E}^2}\right] \tag{3.55}$$

From these two equations, one can obtain $\partial_\theta a, \partial_t a$ in terms of $\mathcal{E}$ and W which can be integrated. Thus the metric can be completely determined starting from a solution for $\mathcal{E}$.

However, all these solutions are not gauge inequivalent corresponding to the fact that the coordinates can still be subjected to conformal diffeomorphisms. Under these coordinate transformations, $\mathcal{E}$ which is the determinant of the metric on the symmetry torus, is a scalar. Under conformal diffeomorphisms, the wave operator gets scaled by a prefactor. Hence, under the transformations generated by conformal Killing vectors, solutions of the wave equation transform among themselves. In fact the conformal Killing vectors also satisfy the wave equati and on the cylinder (t, θ), both $\mathcal{E}$ and ξ satisfy the same boundary conditio Thus, their general solutions are linear combinations of $exp\{in(t \pm \theta)\}, n \neq 0$ an a solution of the form $A + Bt$. The Killing vectors however satisfy first ord coupled equations. This removes the θ-independent, linear in t piece from th general solution. Consequently, one can use conformal diffeomorphisms to remo the θ-dependence from the solutions for $\mathcal{E}$ as well as the constant piece. In oth words, all solutions for $\mathcal{E}$, except $\mathcal{E} = \#t$ are related to each other by conform diffeomorphisms. The gauge inequivalent solutions are thus obtained from th choice $\mathcal{E} = t$. Equivalently, one has finally fixed the (t, θ) coordinates complete The time coordinate so fixed will be denoted by T.

With this choice, $\mathcal{E} = T$, the constraints also simplify to,

$$
\begin{aligned}
0 &= \mathcal{E}(\partial_\theta W)^2 - \mathcal{E}^{-1}\{K_x E^x K_y E^y + (K_x E^x + K_y E^y)\mathcal{A}\mathcal{E}\} && (3.56) \\
0 &= E^x \partial_\theta K_x + E^y \partial_\theta K_y \, , && (3.57)
\end{aligned}
$$

one gets $K_x E^x + K_y E^y = 1$ and the equations (3.53, 3.54, 3.55) go over to th equations (3.2, 3.3) in section 3.2.

Remark:
Up to the derivation of the equations for $\mathcal{E}$ and W, the constraints are not used The T^3 topology has also not been used! Thus these expressions are also valid fo polarized versions of Gowdy models with other topologies. In Gowdy's origina analysis, the three allowed topologies are distinguished by different choices of solu tions of the equation for $\mathcal{E}$ (R, the determinant of the two metric on the T^2 orbits in Gowdy's notation). The different topologies get distinguished by the bound ary conditions on $\mathcal{E}$ and on the conformal Killing vectors. For non-T^3 topologies $\theta \in [0, \pi]$ and $\mathcal{E}, \xi^\theta$ have to vanish at the end-points. With these taken into ac count, the gauge inequivalent solutions are obtained by choosing $\mathcal{E} = \sin(t)\sin(\theta)$ [24].

3.6.2 Solutions via Gauge Fixing on Phase Space

We reproduced the known results by obtaining the solutions of the Hamilton's equations with chosen lapse and shift, motivated by comparison with the space time form of Gowdy model, *and* invoking the 'residual' freedom in the space-tim coordinates to obtain the gauge inequivalent solutions. Thus we used the canonica structure as well as the anticipated form of space-time geometry to arrive at th distinct solutions. We would like to see if the same result can also be derived by using only the phase space view.

Within a phase space view, the lapse and the shift are to be determined by doing an explicit gauge fixing. To do this, we will keep the lapse and the shift as unspecified and look at the evolution generated by the total Hamiltonian,

$$
\begin{aligned}
H_{\text{tot}}[N^\theta, N] &= \frac{1}{\kappa}\int d\theta N^\theta \left\{E^x\partial_\theta K_x + E^y\partial_\theta K_y - \mathcal{A}\partial_\theta\mathcal{E}\right\} \\
&\quad + \frac{1}{\kappa}\int d\theta N \left[-\frac{1}{\sqrt{E}}\left\{(K_x E^x K_y E^y) + (K_x E^x + K_y E^y)\mathcal{E}\mathcal{A}\right\}\right. \\
&\quad \left. - \frac{1}{4\sqrt{E}}\left\{(\partial_\theta\mathcal{E})^2 - (\mathcal{E}\partial_\theta(\ln(E^y/E^x)))^2\right\} + \partial_\theta\left(\frac{\mathcal{E}\partial_\theta\mathcal{E}}{\sqrt{E}}\right)\right] \qquad (3.58)
\end{aligned}
$$

Denoting by over-dots, the Poisson brackets with the total Hamiltonian, it is straight forward to see,

$$\frac{\dot{E}^x}{E^x} = \frac{N}{\sqrt{E}}(K_y E^y + \mathcal{A}\mathcal{E}) + \frac{\partial_\theta(N^\theta E^x)}{E^x} \qquad (3.59)$$

$$\frac{\dot{E}^y}{E^y} = \frac{N}{\sqrt{E}}(K_x E^x + \mathcal{A}\mathcal{E}) + \frac{\partial_\theta(N^\theta E^y)}{E^y} \qquad (3.60)$$

$$\frac{\dot{\mathcal{E}}}{\mathcal{E}} = \frac{N}{\sqrt{E}}(K_x E^x + K_y E^y) + \frac{N^\theta\partial_\theta\mathcal{E}}{\mathcal{E}} \qquad (3.61)$$

$$\dot{(K_x E^x)} = \frac{1}{2}\partial_\theta\left\{\frac{N\mathcal{E}}{\sqrt{E}}\left(\mathcal{E}\partial_\theta\ln\frac{E^y}{E^x} + \partial_\theta\mathcal{E}\right)\right\} + \partial_\theta(N^\theta K_x E^x) \qquad (3.62)$$

$$\dot{(K_y E^y)} = \frac{1}{2}\partial_\theta\left\{\frac{N\mathcal{E}}{\sqrt{E}}\left(-\mathcal{E}\partial_\theta\ln\frac{E^y}{E^x} + \partial_\theta\mathcal{E}\right)\right\} + \partial_\theta(N^\theta K_y E^y) \qquad (3.63)$$

The following combinations are convenient for looking at gauge fixing.

$$\dot{(K_x E^x + K_y E^y)} = \partial_\theta\left\{\frac{N\mathcal{E}}{\sqrt{E}}\partial_\theta\mathcal{E}\right\} + \partial_\theta\left\{N^\theta\left(K_x E^x + K_y E^y\right)\right\} \qquad (3.64)$$

$$\dot{\mathcal{E}} = \frac{N\mathcal{E}}{\sqrt{E}}\left(K_x E^x + K_y E^y\right) + N^\theta\partial_\theta\mathcal{E} \qquad (3.65)$$

$$\dot{(K_x E^x - K_y E^y)} = \partial_\theta\left\{\frac{N\mathcal{E}}{\sqrt{E}}\mathcal{E}\partial_\theta\ln\frac{E^y}{E^x}\right\} + \partial_\theta\left\{N^\theta\left(K_x E^x - K_y E^y\right)\right\} \qquad (3.66)$$

$$\dot{\left(\ln\frac{E^y}{E^x}\right)} = \frac{N}{\sqrt{E}}\left(K_x E^x - K_y E^y\right) + N^\theta\partial_\theta\ln\frac{E^y}{E^x} \qquad (3.67)$$

The first two equations above show that we can consistently impose $K_x E^x + K_y E^y = C_1$, a constant, and $\partial_\theta\mathcal{E} = 0$ as two gauge fixing conditions. Preservation of the first leads to $N^\theta = f(t)$ while that of the second leads to $N\mathcal{E}/\sqrt{E} = g(t)$. Since $\partial_\theta\mathcal{E} = 0$ already requires $\mathcal{E}$ to be a function of t alone, we can strengthen the gauge fixing condition by specifying $\mathcal{E} = t$. This determines $N = C_1\sqrt{E}\mathcal{E}^{-1}$. Evidently, we must have a non-zero lapse and therefore $C_1 \neq 0$ must be chosen.

The sign of C_1 will determine if $\mathcal{E}$ increases or decreases with t and by convent
we can take the sign to be positive and without any loss of generality, we cho
$C_1 = +1$ and denote the t by T as before.

The shift is however determined to be a function of T alone. With such a sh
$C[N^\theta] = f(T)\int C$ generates T-dependent *translations* of the θ-coordinate.
tensor densities on the spatial slice, transform as scalars under these translati
and there is no way to fix the left over constraint $\int C$, by any gauge fixing conditi
However, we can always redefine the θ-coordinate such that $d\theta - f(t)dt =: d\theta'$. T
means that solutions inequivalent with respect to translations, can be determin
by effectively choosing shift $= 0$. Incidentally, for other admissible topologies,
shift has to vanish at $\theta = 0, \pi$ and hence $f(t) = 0$ is the only admissible soluti
We have thus achieved our goal of determining the *same* lapse and shift, by expli
gauge fixing. The inequivalent solutions are then obtained as in section 3.2.

One can make the physical degrees of freedom explicit by noting that $2W$
$\ln(E^y/E^x)$ and $\pi_W := K_x E^x - K_y E^y$ are canonically conjugate. Similarly, $2\bar{a}$
$-\ln(E^x E^y)$ and $\pi_{\bar{a}} := K_x E^x + K_y E^y$ are also conjugate variables. The gauge fixi
conditions are: $\mathcal{E} = T$, $\pi_{\bar{a}} = 1$ while the gauge-fixed form of constraints become

$$C = \frac{1}{\kappa}\left[\pi_W \partial_\theta W + \partial_\theta \bar{a}\right]$$ (3.6

$$\left(T^{-1}\sqrt{E}\right) H = \frac{1}{\kappa}\left[-\frac{1-\pi_W^2}{4T} - \mathcal{A} + T(\partial_\theta W)^2\right]$$ (3.6

The Hamiltonian constraint determines $\mathcal{A}$ completely in terms of W, π_W wh
the diffeomorphism constraints determines the $\bar{a}$ except for the homogeneous (
independent) part. The periodicity of $\bar{a}$ also requires the $\int \pi_W \partial_\theta W = 0$ whi
is a constraint on the W, π_W. The physical degrees of freedom are thus describ
by W, π_W together with one constraint and the homogeneous pieces of $\bar{a}, \pi_{\bar{a}}$. O
gauge fixing has fixed the homogeneous part of $\pi_{\bar{a}}$ to be 1. These are of course th
well known results [44].

Observe that in the homogeneous limit (all variables independent of θ), one ge
the Bianchi I model. The Hamiltonian constraint, for each θ looks like a Bianc
model with a potential and is highly suggestive of the BKL scenario and has bee
explored numerically as well [47].

3.7 Discussion

In this chapter we did two main reformulations of the polarized Gowdy model
real connection variables.

- First is the choice of the gauge invariant variables: $A_x, A_y, E^x, E^y, \alpha, \bar{\alpha}$ an
  the subsequent canonical transformation to the variables X, Y, P^ξ, P^η. Th

has already been done in the case of spherical symmetry and also mentioned for cylindrical waves in [46]. The main advantages of these variables are that the volume becomes a functional of the momenta variables alone and the components of the connection along the homogeneous directions are separated neatly and gauge invariantly, into extrinsic curvature components (X, Y) and the spin-connection components (Γ_x, Γ_y). In the next chapter we will see that in the quantum theory, both the features allow a simpler choice of edge and point holonomies, simpler form for the volume operator and also a more tractable form of the Hamiltonian constraint.

- The second aspect, obtains the polarized model from the unpolarized by a simple systematic reduction (Dirac procedure) ensuring a consistent reduction at the level of *physical degrees of freedom*. Getting this reduction consistently is important since the form of the reduced constraints, depend on the reducing conditions. In contrast to the second polarization condition mentioned in the literature, namely orthogonality of the connection components in analogy with that of the triad components, our $\chi \approx 0$ condition, (3.31) is consistent with dynamics. The consistency is seen in three ways:
  - from a systematic derivation
  - verifying the constraint algebra of the reduced constraints
  - reproducing the known space-times, obtained by directly solving the Einstein equations for polarized ansatz.

We are thus confident to use these constraint expressions in the passage to quantization which will be described in the next chapter.

4

Loop Quantization of Polarized Gowdy Model on T^3: Kinematical States and Constraint Operators

In this chapter, we specifically focus on the loop quantization of the Gowdy model. The methods and steps used here follow closely those used in LQG and are to be viewed as first steps towards constructing a background independent quantum theory of the Gowdy model. Analogous steps have been carried out in the case of spherical symmetry [46, 48]. The work presented in this chapter is based upon [49].

As seen in the previous chapter, the classically reduced Gowdy model has all the ingredients of the full general relativity: it is a generally covariant field theory on $\mathbb{R} \times S^1$, its basic fields are 0-forms and connection 1-forms, it has the three sets of first class constraints – Gauss, diffeo and Hamiltonian. It is simpler than the full $1+3$ dimensional theory in that its graphs will be 1-dimensional, its gauge group is Abelian ($U(1)$) and flux or triad representation exist (so the volume operator is simpler). It differs from the full theory in that certain limits available in the full theory are not available here. For example, in the full theory one gets back the classical expression of the constraints in the limit of shrinking the tetrahedra to their base points (continuum limit). This also shrinks the loops appearing in the (edge) holonomies, thereby ensuring that the exponents in the holonomies can be taken to be small. In the reduced theory, however, we have point holonomies and the exponents are not necessarily small in the continuum limit. (When the exponents *are* components of extrinsic curvature, they are indeed small in the classical regime as is the case in the present context.) Nevertheless, the strategies of background independent quantization continue to be available and are discussed below in detail.

In section (4.1), we will introduce the basic notation to be used in this chapter. In section (4.2), we define the basic states, the flux operators in section (4.3). After defining the basic quantum variables we look at the construction of the more general operators in section (4.4). In section (4.5) we will carry out the regularization of the Hamiltonian constraint. We make specific choices for the partitions as well as for transcribing the expressions in terms of the basic variables. Section (4.6) is devoted to the action of the Hamiltonian constraint on basis states. Section (4.7) contains a discussion of ambiguities in the transcriptions as well as in the choices of partitions. These have a bearing on incorporating the spatial correlations in the classical constraint (spatial derivatives) also in the quantum operator.

4.1 Notation and Preliminary remarks

In this section we will state the results from the previous chapter rewritten in the notation to be used for Loop quantization. The basic canonical variables are a real, $SU(2)$ connection $A := A^i_a \tau_i dx^i$ and a densitized triad $E := \tau^i E^a_i \partial_a$ with the Poisson bracket given by $\{A^i_a(x), E^b_j(y)\} = (8\pi G_{\text{Newton}})\gamma\delta^b_a\delta^i_j\delta^3(x,y)$. There are three sets of constraints which can be conveniently presented in matrix notation as follows. Introduce:

$$
\begin{aligned}
\kappa &:= 8\pi G_{\text{Newton}} \ , \quad \tau_i := -i\sigma_i/2 \ , \quad \tau_i\tau_j = -(1/4)\delta_{ij}\mathbb{I} + (1/2)\epsilon_{ijk}\tau_k, \\
A_a &:= A^i_a\tau_i \quad , \quad E^a := E^a_i\tau^i \quad , \quad F_{ab} := \partial_a A_b - \partial_b A_a + [A_a, A_b] \ . \quad (4.1)
\end{aligned}
$$

Then,

$$
G(x) := G_i\tau^i = \frac{1}{\kappa\gamma}\left[\partial_a E^a + [A_a, E^a]\right] \quad (4.2)
$$

$$
C_a(x) = \frac{1}{\kappa\gamma}\left[(-2)\mathrm{Tr}\left(F_{ab}E^b - A_a G\right)\right] \quad (4.3)
$$

$$
\begin{aligned}
H(x) = \ & \frac{1}{\kappa}(|\mathrm{det}E^a_i|)^{-1/2}\left[(-\mathrm{Tr})(F_{ab}[E^a, E^b]) \right. \\
& \left. -2(1+\gamma^2)\left(\mathrm{Tr}(E^a K_a)\mathrm{Tr}(E^b K_b) - \mathrm{Tr}(E^a K_b)\mathrm{Tr}(E^b K_a)\right)\right] \quad (4.4)
\end{aligned}
$$

For the polarized Gowdy model, the connection and triad variables get restricted to satisfy $E^x_3 = E^y_3 = E^\theta_1 = E^\theta_2 = 0$, $A^3_x = A^3_y = A^1_\theta = A^2_\theta = 0$. These can then be expressed in the form:

$$
\tau_x(\theta) := \cos\beta(\theta)\ \tau_1 + \sin\beta(\theta)\ \tau_2 \ , \ \tau_y(\theta) := -\sin\beta(\theta)\ \tau_1 + \cos\beta(\theta)\ \tau_2 \quad (4.5)
$$

$$
A(\theta) := \tau_3\mathcal{A}(\theta)d\theta + \left\{\tau_x(\theta)X(\theta) + \tau_y(\theta)\tilde{X}(\theta)\right\}dx + \left\{\tau_y(\theta)Y(\theta) + \tau_x(\theta)\tilde{Y}(\theta)\right\}dy \quad (4.6)
$$

$$
E(\theta) := \tau_3\mathcal{E}(\theta)\ \partial_\theta + \tau_x(\theta)E^x(\theta)\ \partial_x + \tau_y(\theta)E^y(\theta)\ \partial_y \quad (4.7)
$$

In the above, we have essentially defined $\sum_{i=1,2} E^x_i\tau^i := E^x\tau_x, \sum_{i=1,2} E^x_i\tau^i := E^y\tau_y$ and demanded that $\tau_x^2 = -(1/4) = \tau_y^2$. It follows that $[\tau_x, \tau_y] = \tau_3$ *iff* polarization condition, $\sum_{i=1,2} E^x_i E^y_i = 0$, holds. This allows us to identify E^x, E^y as the magnitudes of the two dimensional vectors $\vec{E^x}, \vec{E^y}$ and introduce an angular coordinate β so that $E^x_1 := E^x \cos\beta, E^x_2 := E^x \sin\beta, E^y_1 := -\sin\beta, E^y_2 := E^y \cos\beta$. From these, the definitions of the $\beta-$dependent τ matrices follows. The matrices $E^a_i(\theta)$ are now "diagonal" for each θ. This fact together with the properties of $\beta-$dependent τ matrices, simplifies the computations. In particular, the co-triad e, the spin connection Γ and the extrinsic curvature $K := \gamma^{-1}(A - \Gamma)$ are obtained as,

$$
\begin{aligned}
e &= \tau_3 \frac{\sqrt{E}}{\mathcal{E}} d\theta + \frac{\sqrt{E}}{E^x}\tau_x dx + \frac{\sqrt{E}}{E^y}\tau_y dy, \; E := E^x E^y |\mathcal{E}| && (4.8)\\
\Gamma &= \tau_3 \Gamma^3_\theta d\theta + \tau_y \Gamma_x dx + \tau_x \Gamma_y dy \qquad \text{where,} &&\\
\Gamma^3_\theta &= -\partial_\theta, \Gamma_x := \frac{1}{2}\frac{E^\theta_3}{E^x}\partial_\theta \ln\left(E^\theta_3 \frac{E^y}{E^x}\right), \Gamma_y := \frac{1}{2}\frac{E^\theta_3}{E^y}\partial_\theta \ln\left(\frac{1}{E^\theta_3}\frac{E^y}{E^x}\right) \; ; && (4.9)\\
\gamma K &= \tau_3\left(\mathcal{A} + \partial_\theta \beta\right)\, d\theta + \left(\tau_x X + \tau_y(\tilde{X} - \Gamma_x)\right) dx + \left(\tau_y Y + \tau_x(\tilde{Y} - \Gamma_y)\right) dy && (4.10)
\end{aligned}
$$

The preservation of the polarization condition or equivalently diagonal form of the extrinsic curvature K^i_a requires, $\tilde{X} = \Gamma_x$ $\tilde{Y} = \Gamma_y$.

Thus, the basic variables are $X, Y, \mathcal{A}, \eta := \beta$ and $E^x, E^y, \mathcal{E}, P^\eta$ with the Poisson brackets of the form $\{X, E^x\} = (2G_{\mathrm{Newton}}/\pi)\gamma\delta(\theta - \theta')$. We have relabelled β by η for conformity with the notation of the previous chapter (modulo a factor of 2).

As before, putting $\kappa' := \kappa/(4\pi^2)$, the constraints take the form,

$$
G := G_3 = \frac{1}{\kappa'\gamma}\left[\partial_\theta \mathcal{E} + P^\eta\right] \qquad (4.11)
$$

$$
C_\theta = \frac{1}{\kappa'\gamma}\left[E^x \partial_\theta X + E^y \partial_\theta Y - \mathcal{A}\partial_\theta \mathcal{E} + P^\eta \partial_\theta \eta\right] \qquad (4.12)
$$

$$
\begin{aligned}
H = &-\frac{1}{\kappa'}\frac{1}{\sqrt{E}}\left[\frac{1}{\gamma^2}\left(XE^x YE^y + \mathcal{A}\mathcal{E}(XE^x + YE^y) + \mathcal{E}\partial_\theta \eta (XE^x + YE^y)\right)\right.\\
&\left.- E^x \Gamma_x E^y \Gamma_y\right] + \frac{1}{2\kappa'}\partial_\theta \left\{\frac{2\mathcal{E}\left(\partial_\theta \mathcal{E}\right)}{\sqrt{E}}\right\} - \frac{\kappa'}{4}\frac{G^2}{\sqrt{E}} - \frac{\gamma}{2}\partial_\theta\left(\frac{G}{\sqrt{E}}\right)
\end{aligned} \qquad (4.13)
$$

It is obvious from these definitions that $X, Y, \mathcal{E}, \eta$ are scalars while $E^x, E^y, \mathcal{A}, P^\eta$ are scalar densities of weight 1. The Gauss constraint shows that $\mathcal{A}$ transforms as a $U(1)$ connection while η is *translated* by the gauge parameter. All other variables are gauge invariant.

4.2 Basic States

The configuration variable $\mathcal{A}$ is a $U(1)$ connection 1-form, so we integrate it along an edge (an arc along the S^1) and by taking its exponential we define the (edge) holonomy variable valued in $U(1)$:

$$h_e^{(k)}(\mathcal{A}) := \exp(i\frac{k}{2}\int_e \mathcal{A}), k \in \mathbb{Z} \tag{4.14}$$

The integer label k denotes the representation and the factor of 1/2 is introduced for later convenience. The Hilbert space can be constructed via projective families labelled by closed, oriented graphs in S^1. The graphs are just n arcs with n vertices. Associated with each arc is an edge holonomy in the representation k. For a given graph γ, consider functions ψ of n group elements $h_{e_i}^{(k_i)}(\mathcal{A})$ and define an inner product using the Haar measure on $U(1)$. The projective methods then allow one to obtain the Hilbert space as a completion of the projective limits of the graph Hilbert spaces.

The configuration variables $X, Y \in \mathbb{R}$ and $\eta \in \mathbb{R}/\mathbb{Z}$ are scalars and hence no smearing is needed. For these we define the point holonomies (at points v):

$$h_v^{(\mu)}(X) := \exp(i\frac{\mu}{2}X(v)) \tag{4.15}$$
$$h_v^{(\nu)}(Y) := \exp(i\frac{\nu}{2}Y(v)) \tag{4.16}$$
$$h_v^{\lambda}(\eta) := \exp(i\lambda\eta(v)) \tag{4.17}$$

where $\mu, \nu \in \mathbb{R}$ and $\lambda \in \mathbb{Z}$.

Again, the factor of 1/2 is introduced for later convenience. A similar factor is *not* introduced for the η holonomy since η is already an angle. The X, Y point holonomies are interpreted as unitary representation of the compact, Abelian group $\mathbb{R}_{\text{Bohr}}$ which is the Bohr compactification of the additive group of real numbers, $\mathbb{R}$.

Note

The functions $\{\exp(i\mu X), \mu \in \mathbb{R}\}$ form a separating set of functions to separate points in $\mathbb{R}$. These are also characters of the topological group $\mathbb{R}$. Their finite linear combinations give almost periodic functions of X. From these one constructs a commutative C* algebra with unity. The spectrum of this algebra happens to be the Bohr compactification, $\mathbb{R}_{\text{Bohr}}$, of the topological group $\mathbb{R}$. Its (unitary) irreducible representations are one dimensional and are labelled by real numbers. The point holonomies are the representatives. The Haar measure on this compact group can be presented as: $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}$. With this measure, the Hilbert space of functions on the group is defined via the inner product: $\langle f, g\rangle := \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dX f^*(X)g(X)$.

The representation labels μ, ν take values in $\mathbb{R}$. By contrast, η is an angle variable, so the corresponding point holonomy is valued in $U(1)$. The representation label then takes only integer values, $\lambda \in \mathbb{Z}$. The corresponding Hilbert spaces are constructed again via projective families – now labelled by finite sets of points which can be taken to be the vertices of the graphs used in the previous paragraph.

The kinematical Hilbert space for the model is thus a tensor product of the Hilbert spaces constructed for $\mathcal{A}, X, Y, \eta$ variables. A convenient orthonormal basis for this is provided by the "charge network functions" which are labelled by a closed oriented graph G with n edges e and n vertices v, a $U(1)$ representation k_e for each edge, a $U(1)$ representation $\lambda_v \in \mathbb{Z}$ for each vertex and $\mathbb{R}_{\mathrm{Bohr}}$ representations μ_v, ν_v for each vertex:

$$
\begin{aligned}
T_{G,\vec{k},\vec{\mu},\vec{\nu},\vec{\lambda}}(\mathcal{A}, X, Y, \eta) &:= \prod_{e\in G} k_e(h^{(e)}) \prod_{v\in V(G)} \mu_v(h_v(X))\nu_v(h_v(Y))\lambda_v(h_v(\eta)) \\
&= \prod_{e\in G} \exp\left(i\frac{k_e}{2}\int_e \mathcal{A}\right) \prod_{v\in V(G)} \\
&\qquad \left(\exp\left(i\frac{\mu_v}{2}X\right)\exp\left(i\frac{\nu_v}{2}Y\right)\exp\left(i\lambda_v\eta\right)\right) \qquad (4.18)
\end{aligned}
$$

where $V(G)$ represents the set of vertices belonging to the graph G. The functions with any of the labels different, are orthogonal – in particular two graphs must coincide for non-zero inner product. These basis states provide an orthogonal decomposition for the kinematical Hilbert space when all the representation labels are non-zero.

4.3 Flux Operators

The conjugate variables are represented as

$$
E^x(\theta) \sim -i\gamma\ell_{\mathrm{P}}^2 \frac{\delta h_\theta(X)}{\delta X(\theta)} \frac{\partial}{\partial h_\theta(X)} \qquad (4.19)
$$

where $\ell_{\mathrm{P}}^2 := \kappa' \hbar$.

The flux variables corresponding to E^x, E^y, P^η are defined by integrating these densities on an interval $\mathcal{I}$ of the circle, eg $\mathcal{F}_{x,\mathcal{I}} := \int_{\mathcal{I}} E^x, \mathcal{F}_{y,\mathcal{I}} := \int_{\mathcal{I}} E^y$. $\mathcal{E}$ being a scalar, is already a suitable variable. Their actions on the basis functions (4.18) are:

$$\begin{aligned}
\hat{\mathcal{E}}(\theta)T_{G,k,\mu,\nu,\lambda} &= \frac{\gamma\ell_{\rm P}^2}{2}\,\frac{k_{e^+(\theta)}+k_{e^-(\theta)}}{2}T_{G,k,\mu,\nu,\lambda} && (4.20)\\
\int_I \hat{E}^x T_{G,k,\mu,\nu,\lambda} &= \frac{\gamma\ell_{\rm P}^2}{2}\sum_{v\in V(G)\cap\mathcal{I}} \mu_v T_{G,k,\mu,\nu,\lambda} && (4.21)\\
\int_I \hat{E}^y T_{G,k,\mu,\nu,\lambda} &= \frac{\gamma\ell_{\rm P}^2}{2}\sum_{v\in V(G)\cap\mathcal{I}} \nu_v T_{G,k,\mu,\nu,\lambda} && (4.22)\\
\int_I \hat{P}^y T_{G,k,\mu,\nu,\lambda} &= \gamma\ell_{\rm P}^2\sum_{v\in V(G)\cap\mathcal{I}} \lambda_v T_{G,k,\mu,\nu,\lambda} && (4.23)
\end{aligned}$$

where $\mathcal{I}$ is an interval on S^1. The $e^{\pm}(\theta)$ refer to the two oriented edges of the graph G, meeting at θ if there is a vertex at θ or denote two parts of the same edge if there is no vertex at θ. The k labels in such a case are the same. The $\cap$ allows the case where a vertex may be an end-point of the interval $\mathcal{I}$. In such a case, there is an additional factor of $\frac{1}{2}$ for its contribution to the sum. This follows because

$$\int_a^b dx\delta(x-x_0) = \begin{cases} 1 & \text{if} \quad x_0 \in (a,b)\,; \\ \frac{1}{2} & \text{if} \quad x_0 = a \text{ or } x_0 = b; \\ 0 & \text{if} \quad x_0 \notin [a,b]. \end{cases} \qquad (4.24)$$

Note that classically the triad components, E^x, E^y are positive. The fluxes however can take both signs since they involve integrals which depend on the orientation.

This completes the specification of the kinematical Hilbert space together with the representation of the basic background independent variables. Next we turn to the construction of composite operators.

4.4 Construction of more general Operators

The diffeomorphism covariance requires all operators of interest (constraints and observables) are integrals of expressions in terms of the basic operators. Secondly, operators of interests also involve products of elementary operators at the same point (same θ) and thus need a "regularization". As in LQG, the general strategy to define such operators is

1. replace the integral by a Riemann sum using a "cell-decomposition" (or partition) of S^1
2. for each cell, define a regulated expression choosing suitable ordering of the basic operators and evaluate the action on basis states;

3. check "cylindrical consistency" of this action in (ii) so that the (regulated) operator can be densely defined on the kinematical Hilbert space via projective limit

4. finally one would like to remove the regulator.

One would like to do this in such a manner that the constructed limiting operator has the same properties under the diffeomorphism. To achieve this, usually one has to restrict the cell-decomposition in relation to a graph.

In the present case of one dimensional spatial manifold, both the cell-decomposition and the graphs underlying the basis states are characterised by finitely many points and the arcs connecting the consecutive points. As in LQG [5, 6], the products of elementary variables are regulated by using a "point splitting" and then expressing the fields in terms of the appropriate holonomies and fluxes both of which need at most edges and at each point there are precisely two edges (in the full theory one needs edges as well as close loops and there can be an arbitrary number of these). A regulator, for each given graph G, then consists of a family of partitions Π^G_ϵ, such that for each ϵ, the partition is such that each vertex of G is contained in exactly one cell [1]. There is also a choice of representation labels $k_0, \mu_0, \nu_0, \lambda_0$ made which can be taken to be the same for all ϵ. The regulated expressions constructed depend on ϵ and are such that one recovers the classical expressions in the limit of removing the regulator ($\epsilon \to 0$). There are of course infinitely many such regulators. A diffeomorphism covariant regulator is one such that if under a diffeomorphism the graph $G \to G'$, then the corresponding partitions also transform similarly. Since each Π_G can also be thought of as being defined by a set of points such that each vertex is flanked by two points (between two consecutive points, there need not be any vertex), any orientation preserving diffeomorphism will automatically preserve the order of the vertices and cell boundaries. Every sufficiently refined partition then automatically becomes a diffeomorphism covariant regulator. This is assumed in the following.

As in LQG, the issue of cylindrical consistency is automatically sorted out by referring to the orthogonal decomposition of $\mathcal{H}_{\rm kin}$ i.e. by specifying the action of the operators on basis states with all representation labels being non-zero.

With these preliminaries, we proceed to define some specific operators of interest.

4.4.1 Volume Operator

In the classical expression for the Hamiltonian constraint, powers of $E := |\mathcal{E}|E^xE^y$, occur in the same manner as in the full theory. It is therefore natural to consider

[1]Note that this is one possible natural choice of a class of partitions adapted to a graph. The vertices of G always lie in the interior of the cells and some cells have no vertices. We discuss this further in the last section.

the expression for the volume of a region $\mathcal{I} \times T^2$ and construct the corresponding operator. With the canonical variables chosen, the volume involves only the conjugate momenta whose quantization is already done. The volume operator written in terms of the basic operators :

$$
\begin{aligned}
\mathcal{V}(\mathcal{I} \times T^2) &= \int_{\mathcal{I}\times T^2} d^3x\sqrt{g} \\
&= 4\pi^2 \int_{\mathcal{I}} d\theta \sqrt{|\mathcal{E}|E^x E^y} \\
\mathcal{V}_\epsilon(\mathcal{I}) &\approx \sum_{i=1}^{n} \int_{\theta_i}^{\theta_i+\epsilon} \sqrt{|\mathcal{E}|E^x E^y} \\
&\approx \sum_{i=1}^{n} \epsilon \sqrt{|\mathcal{E}(\bar{\theta}_i)|}\sqrt{E^x(\bar{\theta}_i)}\sqrt{E^y(\bar{\theta}_i)} \\
&\approx \sum_{i=1}^{n} \sqrt{|\mathcal{E}|}\sqrt{\epsilon E^x}\sqrt{\epsilon E^y} \\
&\approx \sum_{i=1}^{n} \sqrt{|\mathcal{E}|(\bar{\theta}_i)}\sqrt{\left|\int_{\theta_i}^{\theta_i+\epsilon} E^x\right|}\sqrt{\left|\int_{\theta_i}^{\theta_i+\epsilon} E^y\right|} \qquad (4.25)
\end{aligned}
$$

The right hand side is expressed in terms of flux variables, so the regulated volume operator can be defined as:

$$
\hat{\mathcal{V}}_\epsilon(\mathcal{I}) := \sum_{i=1}^{n} \sqrt{|\hat{\mathcal{E}}|(\bar{\theta}_i)}\sqrt{\left|\widehat{\int_{\mathcal{I}_i} E^x}\right|}\sqrt{\left|\widehat{\int_{\mathcal{I}_i} E^y}\right|} \qquad (4.26)
$$

Clearly this is diagonal in the basis states. and its action on a basis state $T_{G,\vec{k},\vec{\mu},\vec{\nu},\vec{\lambda}}$ gives the eigenvalue,

$$
V_{\vec{k},\vec{\mu},\vec{\nu},\vec{\lambda}} = \frac{1}{\sqrt{2}}\left(\frac{\gamma l_P^2}{2}\right)^{3/2} \sum_{v\in\mathcal{I}\cap V(G)} \left(|\mu_v|\;|\nu_v|\;|k_{e^+(v)} + k_{e^-(v)}|\right)^{\frac{1}{2}} \qquad (4.27)
$$

Remarks:

- In the above, $\mathcal{I}_i$ denotes the i^{th} cell of the partition and $\bar{\theta}_i$ denotes a point in that cell – it need not be an end-point of the interval. We have also assumed the "length of the intervals" to be same and equal to ϵ. This corresponds to a "cubic" partition and is chosen for convenience only. We will always use such partitions in all the operators below.

- Although we could restrict to $\mu_v, \nu_v > 0$, it will be more convenient (eg in the Hamiltonian constraint below) to allow both signs (corresponding to the orientation of the interval). The eigenvalues of the volume operator then must have explicit absolute values. We have thus used the absolute value operators defined from the flux operators.

- For a given graph, the partition (of $\mathcal{I}$) is so chosen that each vertex is included in one and only one interval $\mathcal{I}_i$. For those intervals which contain no vertex of the graph, there is no contribution to the summation since flux operators have this property. Hence, the sum collapses to contributions only from the vertices, independent of the partition. The action is manifestly independent of ϵ and even though the number of intervals go to infinity as $\epsilon \to 0$, the action remains *finite* and well defined.

  Because of this property of the fluxes, we can choose the $\bar{\theta}_i$ point in a cell to coincide with a vertex of a graph if $\mathcal{I}_i$ contains a vertex or an arbitrary point if $\mathcal{I}_i$ does not contain a vertex. Such a choice will be understood in the following.

- For intervals $\mathcal{I} \neq S^1$, a choice of diffeo-covariant regulator retains the $v \in \mathcal{I} \cap V(G)$ and hence the action is diffeo-invariant. The eigenvalues are also manifestly independent of "location labels" of the states. For the total volume (which is diffeo-invariant), the operator is manifestly diffeo-invariant.

4.4.2 Gauss Constraint

Consider the Gauss constraint (4.11):

$$
\begin{aligned}
G_3 &= \int_{S^1} d\theta(\partial_\theta \mathcal{E} + P^\eta) \\
&\approx \sum_{i=1}^{n} \int_{\theta_i}^{\theta_i+\epsilon} (\partial_\theta \mathcal{E} + P^\eta) d\theta \\
&\approx \sum_{i=1}^{n} \left[\int_{\mathcal{I}_i} P^\eta + \mathcal{E}(\theta_i + \epsilon) - \mathcal{E}(\theta_i) \right] && (4.28) \\
\hat{G}_3^\epsilon &:= \sum_{i=1}^{n} \left[\widehat{\int_{\mathcal{I}_i} P^\eta} + \hat{\mathcal{E}}(\theta_i + \epsilon) - \hat{\mathcal{E}}(\theta_i) \right] && (4.29)
\end{aligned}
$$

Again, this is easily quantized with its action on a basis state $T_{G,\vec{k},\vec{\mu},\vec{\theta},\vec{\lambda}}$ giving the eigenvalue,

$$
\gamma l_P^2 \sum_{v \in V(G)} \left[\lambda_v + \frac{k_{e^+}(v) - k_{e^-}(v)}{2} \right] \qquad (4.30)
$$

Notice that in the limit of infinitely fine partitions, for a given graph, if there is a vertex $v \in \mathcal{I}_i$, then there is *no vertex* in the adjacent cells. As a result, $\mathcal{E}(\theta_{i+1})$ gives the $k_{e^+}(v)/2$ and $-\mathcal{E}(\theta_i)$ gives $-k_{e^-}(v)/2$, since θ_i divides the same edge and so does θ_{i+1}.

Once again, the eigenvalues are manifestly independent of ϵ and the action is diffeo-invariant. Imposition of Gauss constraint can be done simply by restriction to basis states with labels satisfying $\lambda_v = -(k_{e^+(v)} - k_{e^-(v)})/2, \forall v \in V(G)$. Since $\lambda_v \in \mathbb{Z}$, the difference in the k labels at each vertex must be an *even* integer. We will assume these restrictions on the representation labels and from now on deal with gauge invariant basis states. The label $\vec{\lambda}$ will be suppressed and terms proportional to the Gauss law constraint in the Hamiltonian constraint will also be dropped.

Substituting for λ_v for each of the vertices and rearranging the holonomy factors, one can write the gauge invariant basis states are explicitly given by,

$$T_{G,\vec{k},\vec{\mu},\vec{\nu}} = \prod_{e\in G} \exp\left\{i\frac{k_e}{2}\int_e (\mathcal{A}(\theta) - \partial_\theta\eta)\right\} \prod_{v\in V(G)} \left(\exp\left\{i\frac{\mu_v}{2}X(v)\right\} \exp\left\{i\frac{\nu_v}{2}Y(v)\right\}\right) \tag{4.31}$$

We have also used, $\eta(v^+(e)) - \eta(v^-(e)) = \int_e \partial_\theta\eta$, where $v^\pm(e)$ denote the tip and tail of the edge e.

4.5 Hamiltonian Constraint

The Hamiltonian constraint is a more complicated object. Let us write (4.13) as a sum of a kinetic term and a potential term,

$$H := -\frac{1}{\kappa'}[H_K + H_P] \tag{4.32}$$

$$H_K := \frac{1}{\gamma^2}\int_{S^1} \mathrm{d}\theta N(\theta)\frac{1}{\sqrt{E}}\left[XE^xYE^y + (\mathcal{A} + \partial_\theta\eta)\,\mathcal{E}(XE^x + YE^y)\right] \tag{4.33}$$

$$\begin{aligned} H_P := & -\int_{S^1} \mathrm{d}\theta N(\theta)\frac{1}{\sqrt{E}}\left[-\frac{1}{4}(\partial_\theta\mathcal{E})^2 + \frac{(\mathcal{E})^2}{4}\left(\frac{\partial_\theta E^x}{E^x} - \frac{\partial_\theta E^y}{E^y}\right)^2\right] \\ & -\int_{S^1} \mathrm{d}\theta N(\theta)\frac{1}{2}\partial_\theta\left[\frac{2\mathcal{E}\,(\partial_\theta\mathcal{E})}{\sqrt{E}}\right] \end{aligned} \tag{4.34}$$

In the above we have used the expressions of Γ_x and Γ_y and suppressed the terms dependent on the Gauss constraint which will drop out on gauge invariant basis states. Only H_K depends on the configuration variables and all terms have two powers of momenta in the numerator and the $\sqrt{E}$ in the denominator whose vanishing is a potential problem.

The kinetic term has a structure similar to the Euclidean term in the full theo[ry] $\sim EEF/\sqrt{q}$ (but it is *not* the simplification of the Euclidean term of LQG). Th[is] will be treated in a manner similar to the full theory, using appropriate holonomi[es] in the form $h_i h_j h_i^{-1} h_j^{-1} h_k \{h_k^{-1}, V\}$. The remaining terms are functions of momen[ta] alone and the $E^{-1/2}$ is treated using Poisson bracket of the volume with suitab[le] holonomy.

Although the holonomies defined before, eg in the basis states (4.18), are a[ll] phases (Abelian gauge theory), it is convenient to introduce their $SU(2)$ value[d] analogues using the η dependent τ matrices defined in eq. (4.1) and in eq. (4.5[)]. Thus,

$$
\begin{aligned}
h_\theta(\mathcal{I}) &:= \exp\left(\tau_3 \int_{\mathcal{I}} \mathcal{A}(\theta')\, \mathrm{d}\theta'\right) = \cos\left(\frac{1}{2}\int_{\mathcal{I}} \mathcal{A}\right) + 2\tau_3 \sin\left(\frac{1}{2}\int_{\mathcal{I}} \mathcal{A}\right) \\
h_x(\theta) &:= \exp\left(\mu_0 X(\theta)\, \tau_x(\theta)\right) = \cos\left(\frac{\mu_0}{2} X(\theta)\right) + 2\tau_x(\theta) \sin\left(\frac{\mu_0}{2} X(\theta)\right) \\
h_y(\theta) &:= \exp\left(\nu_0 Y(\theta)\, \tau_y(\theta)\right) = \cos\left(\frac{\nu_0}{2} Y(\theta)\right) + 2\tau_y(\theta) \sin\left(\frac{\nu_0}{2} Y(\theta)\right)
\end{aligned}
\qquad (4.35)
$$

Each of the sin, cos are well defined on the kinematical Hilbert space (this wa[s] the reason for the factors of 1/2 in the definitions of the holonomies in the bas[is] states) and therefore so are the above $SU(2)$-valued holonomies. The interval $\mathcal{I}$ will typically be a cell of a partition, $(\theta_i, \theta_i + \epsilon)$. The parameters μ_0, ν_0 are th[e] chosen and fixed representations of $\mathbb{R}_{\mathrm{Bohr}}$, $k_0 = 1$ is the fixed representation of th[e] $U(1)$, while ϵ is a small parameter which will also play the role of the regulato[r] parameter. Let us also define, the volume function labelled by an interval $\mathcal{I}$ and a point θ inside the interval:

$$
V(\mathcal{I}, \theta) := \sqrt{|\mathcal{E}(\theta)| \left|\int_{\mathcal{I}} E^x\right| \left|\int_{\mathcal{I}} E^y\right|} \qquad (4.36)
$$

For brevity of notation we will suppress the label θ and denote the above volum[e] function as $V(\mathcal{I})$.

Consider expression of the form $\mathrm{Tr}(h_i h_j h_i^{-1} h_j^{-1} h_k \{h_k^{-1}, \sqrt{E}\})$ for distinct i, j, k taking values θ, x, y. For small values of $X, Y, \int_{\mathcal{I}} \mathcal{A}$, the holonomies can be ex[-]panded in a power series. Because of the trace, it is enough to expand each holonomy up to 1st order. The surviving terms are quadratic terms arising from products of the linear ones and a linear term coming from h_k. If one interchanges the $i \leftrightarrow j$ holonomies, the linear term retains the sign while the quadratic one changes the sign. Thus taking the difference of the two traces, leaves us with only the quadratic terms which are exactly of the form needed in H_K. Explicitly,

$$\mathrm{Tr}\left[\left\{\ h_x(\theta)h_y(\theta)h_x^{-1}(\theta)h_y^{-1}(\theta) - h_y(\theta)h_x(\theta)h_y^{-1}(\theta)h_x^{-1}(\theta)\ \right\} h_\theta(\mathcal{I})\{h_\theta^{-1}(\mathcal{I}), V(\mathcal{I})\}\right]$$

$$\approx \left(-\frac{\kappa'\gamma}{2}\,\mu_0\nu_0\right)\frac{\epsilon\, X(\theta)Y(\theta)E^x(\theta)E^y(\theta)}{\sqrt{E(\theta)}} \tag{4.37}$$

$$\mathrm{Tr}\left[\left\{\ h_y(\theta)h_\theta(\mathcal{I})h_y^{-1}(\theta+\epsilon)h_\theta^{-1}(\mathcal{I}) - h_\theta(\mathcal{I})h_y(\theta+\epsilon)h_\theta^{-1}(\mathcal{I})h_y^{-1}(\theta)\ \right\} h_x(\theta)\{h_x^{-1}(\theta), V(\mathcal{I})\}\right]$$

$$\approx \left(-\frac{\kappa'\gamma}{2}\mu_0\nu_0\right)\frac{\epsilon\, Y(\theta)(\mathcal{A}(\theta)+\partial_\theta\eta(\theta))E^y(\theta)\mathcal{E}(\theta)}{\sqrt{E(\theta)}} \tag{4.38}$$

$$\mathrm{Tr}\left[\left\{\ h_\theta(\mathcal{I})h_x(\theta+\epsilon)h_\theta^{-1}(\mathcal{I})h_x^{-1}(\theta) - h_x(\theta)h_\theta(\mathcal{I})h_x^{-1}(\theta+\epsilon)h_\theta^{-1}(\mathcal{I})\ \right\} h_y(\theta)\{h_y^{-1}(\theta), V(\mathcal{I})\}\right]$$

$$\approx \left(-\frac{\kappa'\gamma}{2}\mu_0\nu_0\right)\frac{\epsilon\, (\mathcal{A}(\theta)+\partial_\theta\eta(\theta))X(\theta)\mathcal{E}(\theta)E^x(\theta)}{\sqrt{E(\theta)}} \tag{4.39}$$

In equations (4.38) and (4.39) $\mathcal{I}$ is the interval between θ and $\theta+\epsilon$. The derivatives of η arise from the position dependence of the τ_x, τ_y matrices which satisfy:

$$\begin{aligned}\tau_x(\theta+\epsilon) - \tau_x(\theta) &\approx \epsilon\partial_\theta\tau_x = \epsilon\partial_\theta\eta\ \tau_y(\theta)\\ \tau_y(\theta+\epsilon) - \tau_y(\theta) &\approx \epsilon\partial_\theta\tau_y = -\,\epsilon\partial_\theta\eta\ \tau_x(\theta)\end{aligned} \tag{4.40}$$

In the above, we have also used:

$$\begin{aligned} h_x(\theta)\{h_x(\theta)^{-1}, V(\mathcal{I})\} &= -\frac{\kappa'\gamma}{2}\mu_0\tau_x\frac{\mathcal{E}(\theta)\int_{\mathcal{I}}E^y}{V(\mathcal{I})} \approx -\frac{\kappa'\gamma}{2}\mu_0\tau_x\frac{E^y(\theta)\mathcal{E}(\theta)}{\sqrt{E(\theta)}}\\ h_y(\theta)\{h_y(\theta)^{-1}, V(\mathcal{I})\} &= -\frac{\kappa'\gamma}{2}\nu_0\tau_y\frac{\mathcal{E}(\theta)\int_{\mathcal{I}}E^x}{V(\mathcal{I})} \approx -\frac{\kappa'\gamma}{2}\nu_0\tau_y\frac{E^x(\theta)\mathcal{E}(\theta)}{\sqrt{E(\theta)}}\\ h_\theta\{h_\theta^{-1}, V(\mathcal{I})\} &= -\frac{\kappa'\gamma}{2}\tau_3\frac{\int_{\mathcal{I}}E^x\int_{\mathcal{I}}E^y}{V(\mathcal{I})} \approx -\frac{\kappa'\gamma}{2}\epsilon\tau_3\frac{E^x(\theta)E^y(\theta)}{\sqrt{E(\theta)}}\end{aligned} \tag{4.41}$$

$$\int_{\mathcal{I}}\mathcal{A} \approx \epsilon\mathcal{A}(\theta)\quad,\quad \int_{\mathcal{I}}E^x \approx \epsilon E^x(\theta)\,,\ \int_{\mathcal{I}}E^y \approx \epsilon E^y(\theta) \tag{4.42}$$

In the quantization of the H_P, H_T, we also use the following identities repeatedly (in the form LHS/RHS = 1):

$$
\begin{aligned}
\mathcal{Z}(\mathcal{I}) &:= \epsilon^{abc}\mathrm{Tr}\left[\, h_a\{h_a^{-1}, V(\mathcal{I})\}\; h_b\{h_b^{-1}, V(\mathcal{I})\}\; h_c\{h_c^{-1}, V(\mathcal{I})\}\,\right] \\
&= \frac{3}{2}\left(\frac{\kappa'\gamma}{2}\right)^3 \mu_0\nu_0 V(\mathcal{I}) \\
\mathcal{Z}_\alpha(\mathcal{I}) &:= \epsilon^{abc}\mathrm{Tr}\left[\, h_a\{h_a^{-1}, (V(\mathcal{I}))^\alpha\}\; h_b\{h_b^{-1}, (V(\mathcal{I}))^\alpha\}\; h_c\{h_c^{-1}, (V(\mathcal{I}))^\alpha\}\,\right] \\
&= \frac{3}{2}\left(\frac{\kappa'\gamma}{2}\right)^3 \mu_0\nu_0\; \alpha^3 (V(\mathcal{I}))^{3\alpha-2} \\
&= \alpha^3 (V(\mathcal{I}))^{3(\alpha-1)}\; \mathcal{Z}(\mathcal{I})
\end{aligned}
$$

These are essentially versions of the identity $1 = \left(\frac{|\det(e^i_a)|}{\sqrt{E}}\right)^n$ [50].

Having noted the ingredients common to the quantization of the different pieces of the Hamiltonian constraint, we turn to each one in explicit details.

4.5.1 Quantization of H_K

Choosing a partition of S^1 with a sufficiently large number of n points at $\theta_i, i = 1, \cdots, n, \theta_n = 2\pi$, $\epsilon = \theta_{i+1} - \theta_i$, we write the integral as a sum,

$$
\begin{aligned}
H_K &\approx \frac{1}{\gamma^2}\sum_{i=1}^{n} \epsilon N(\bar\theta_i)\frac{1}{\sqrt{E(\bar\theta_i)}}\left[XE^xYE^y + (\mathcal{A}+\partial_\theta\eta)\,\mathcal{E}(XE^x + YE^y)\right](\bar\theta_i) \\
&= \frac{1}{\gamma^2}\sum_{i=1}^{n} N(\bar\theta_i)\frac{1}{\sqrt{\epsilon^2 E(\bar\theta_i)}}\left[X(\epsilon E^x)Y(\epsilon E^y) + \epsilon\,(\mathcal{A}+\partial_\theta\eta)\,\mathcal{E}(X(\epsilon E^x) + Y(\epsilon E^y\right. \\
&= \frac{1}{\gamma^2}\sum_{i=1}^{n} N(\bar\theta_i)\frac{1}{V(\mathcal{I}_i)}\left[X(\bar\theta_i)\left(\int_{\mathcal{I}_i} E^x\right)Y(\bar\theta_i)\left(\int_{\mathcal{I}_i} E^y\right) + \right. \\
&\qquad \left.\left(\int_{\mathcal{I}_i}\mathcal{A}+\partial_\theta\eta\right)\mathcal{E}(\bar\theta_i)\left\{X(\bar\theta_i)\int_{\mathcal{I}_i} E^x + Y(\bar\theta_i)\int_{\mathcal{I}_i} E^y\right\}\right]
\end{aligned}
$$

From the equations (4.37, 4.38, 4.39), one sees immediately that for small values of the extrinsic curvature components ($\sim X, Y$, classical regime) and sufficiently refined partition ($\epsilon \ll 1$, continuum limit), the i^{th} term in the sum can be written in terms of the traces of the $SU(2)$ valued holonomies. In other words, the expression in terms of holonomies and fluxes, does go over to the classical expression in the classical regime and can be promoted to an operator by putting hats on the holonomies and fluxes and replacing Poisson brackets by $(i\hbar)^{-1}$ times the commutators. Here, there are possibilities for choosing the ordering of various factors but we will make the "standard choice" of putting the holonomies on the left. Thus the regulated quantum operator corresponding to the kinetic piece is (suppressing the hats on the holonomies and using $\ell_{\rm P}^2 := \kappa'\hbar$),

$$\begin{aligned}
\widehat{H}_K^{\rm reg} \quad &:= i\frac{2}{\ell_{\rm P}^2\gamma^3}\frac{1}{\mu_0\nu_0}\sum_{i=1}^{n} N(\bar{\theta}_i){\rm Tr}\Big(\\
&\quad \{h_x h_y h_x^{-1}h_y^{-1} - h_y h_x h_y^{-1}h_x^{-1}\}\, h_\theta(\mathcal{I}_i)\left[h_\theta^{-1}(\mathcal{I}_i), \hat{V}(\mathcal{I}_i)\right] \qquad (4.47)\\
&+ \{h_y h_\theta(\mathcal{I}_i)h_y^{-1}(\bar{\theta}_i+\epsilon)h_\theta^{-1}(\mathcal{I}_i) - h_{\theta_i}(\mathcal{I}_i)h_y(\bar{\theta}_i+\epsilon)h_\theta^{-1}(\mathcal{I}_i)h_y^{-1}\}\, h_x\left[h_x^{-1}, \hat{V}(\mathcal{I}_i)\right] \\
&+ \{h_\theta(\mathcal{I}_i)h_x(\bar{\theta}_i+\epsilon)h_\theta^{-1}(\mathcal{I}_i)h_x^{-1} - h_x h_\theta(\mathcal{I}_i)h_x^{-1}(\bar{\theta}_i+\epsilon)h_\theta^{-1}(\mathcal{I}_i)\}\, h_y\left[h_y^{-1}, \hat{V}(\mathcal{I}_i)\right]\Big)
\end{aligned}$$

In the above equation, the point holonomies without an explicit argument, are at $\bar{\theta}_i$.

At this point it is convenient to define the following families of operators:

$$\begin{aligned}
\hat{\mathcal{O}}^x_\alpha(\mathcal{I},\theta) &:= \Big[\cos\left(\frac{1}{2}\mu_0 X(\theta)\right)\hat{V}^\alpha(\mathcal{I})\sin\left(\frac{1}{2}\mu_0 X(\theta)\right) - \\
&\qquad \sin\left(\frac{1}{2}\mu_0 X(\theta)\right)\hat{V}^\alpha(\mathcal{I})\cos\left(\frac{1}{2}\mu_0 X(\theta)\right)\Big] \\
\hat{\mathcal{O}}^y_\alpha(\mathcal{I},\theta) &:= \Big[\cos\left(\frac{1}{2}\mu_0 Y(\theta)\right)\hat{V}^\alpha(\mathcal{I})\sin\left(\frac{1}{2}\mu_0 Y(\theta)\right) - \\
&\qquad \sin\left(\frac{1}{2}\mu_0 Y(\theta)\right)\hat{V}^\alpha(\mathcal{I})\cos\left(\frac{1}{2}\mu_0 Y(\theta)\right)\Big] \\
\hat{\mathcal{O}}^\theta_\alpha(\mathcal{I},\theta) &:= \Big[\cos\left(\frac{1}{2}\int_{\mathcal{I}}\mathcal{A}\right)\hat{V}^\alpha(\mathcal{I})\sin\left(\frac{1}{2}\int_{\mathcal{I}}\mathcal{A}\right) - \\
&\qquad \sin\left(\frac{1}{2}\int_{\mathcal{I}}\mathcal{A}\right)\hat{V}^\alpha(\mathcal{I})\cos\left(\frac{1}{2}\int_{\mathcal{I}}\mathcal{A}\right)\Big] \qquad (4.48)
\end{aligned}$$

In the above, θ is a point in the interval $\mathcal{I}$ and $\alpha > 0$ is the power of the volume operator. Again for simplicity of notation we will suppress the θ labels in the above operators.

The operator form of $\mathcal{Z}_\alpha(\mathcal{I})$ can be obtained as:

$$\begin{aligned}
\hat{\mathcal{Z}}_\alpha(\mathcal{I}) &:= \epsilon^{abc}{\rm Tr}\Big(\,\hat{h}_a\,[\,\hat{h}_a^{-1}\,,\,\hat{V}(\mathcal{I})^\alpha\,]\,\hat{h}_b\,[\,\hat{h}_b^{-1}\,,\,\hat{V}(\mathcal{I})^\alpha\,]\,\hat{h}_c\,[\,\hat{h}_c^{-1}\,,\,\hat{V}(\mathcal{I})^\alpha\,]\,\Big) \\
&= -12\,\hat{\mathcal{O}}^x_\alpha(\mathcal{I})\,\hat{\mathcal{O}}^y_\alpha(\mathcal{I})\,\hat{\mathcal{O}}^\theta_\alpha(\mathcal{I}) \qquad (4.49)
\end{aligned}$$

Using the expressions for the holonomies in terms of the "trigonometric" operators given in the eq. (4.35), and evaluating the traces etc, one can see that,

$$
\begin{aligned}
\widehat{H}_K^{\rm reg} \;=\; & -i\frac{4}{\ell_{\rm P}^2\gamma^3}\frac{1}{\mu_0\nu_0}\sum_{i=1}^{n} N(\bar\theta_i)\Bigg[\Big\{ \sin\left(\mu_0 X(\bar\theta_i)\right)\sin\left(\nu_0 Y(\bar\theta_i)\right)\Big\} \times \mathcal{O}_1^\theta(\mathcal{I}_i) + \\
& \left\{2\sin\left(\frac{1}{2}\nu_0 Y(\bar\theta_i+\epsilon)\right)\cos\left(\frac{1}{2}\nu_0 Y(\bar\theta_i)\right)\sin\left(\int_{\mathcal{I}_i}\mathcal{A}-\Delta_i\right)\right\} \times \mathcal{O}_1^x(\mathcal{I} \\
& \left\{2\sin\left(\frac{1}{2}\mu_0 X(\bar\theta_i+\epsilon)\right)\cos\left(\frac{1}{2}\mu_0 X(\bar\theta_i)\right)\sin\left(\int_{\mathcal{I}_i}\mathcal{A}-\Delta_i\right)\right\} \times \mathcal{O}_1^y(\mathcal{I}
\end{aligned}
$$

In the above $\Delta_i := \eta(\bar\theta_i) - \eta(\bar\theta_i+\epsilon)$ and is outside the integral.

4.5.2 Quantization of H_P

All the three terms of H_P are functions of the momenta (triad) only. These have to be expressed in terms of fluxes and holonomies alone. Furthermore, the power(s) of momenta in the denominators will make the action on some states to be singular. The first part is easy to take care of thanks to the density weight 1. For the second part we use the by now familiar procedure of using the identities (4.43, 4.44). Due to the spatial dimension being 1, it is easier to convert triads in terms of fluxes directly, without explicitly doing any point-splitting (one could of course do that if so desired [50]).

The terms in the H_P will be manipulated in the following steps:

1. introduce sufficient number, $k > 0$, of positive powers of $1 = 16(3(\kappa'\gamma)^3\mu_0\nu_0)^{-1}\mathcal{Z}(\mathcal{I})/V(\mathcal{I})$ and express $\mathcal{Z}$ in terms of Z_α. This introduces further powers of the volume

2. choose $\alpha(k)$ such that explicit multiplicative factors of the volume become and further choose k

Now the expression can be promoted to an operator. Here are the details.

<u>*The first term of H_P* :</u>

$$
\begin{aligned}
-\int_{S^1} N(\theta)\frac{1}{\sqrt{E(\theta)}}\left[-\frac{1}{4}(\partial_\theta\mathcal{E})^2\right] &\approx +\frac{1}{4}\sum_{i=1}^{n} N(\bar{\theta}_i)\epsilon\frac{\left(\partial_\theta\mathcal{E}(\bar{\theta}_i)\right)^2}{\sqrt{E(\bar{\theta}_i)}} \\
&= \frac{1}{4}\sum_{i=1}^{n} N(\bar{\theta}_i)\frac{\left(\epsilon\partial_\theta\mathcal{E}(\bar{\theta}_i)\right)^2}{\epsilon\sqrt{E(\bar{\theta}_i)}} \\
&= \frac{1}{4}\sum_{i=1}^{n} N(\bar{\theta}_i)\frac{\left(\mathcal{E}(\bar{\theta}_i+\epsilon)-\mathcal{E}(\bar{\theta}_i)\right)^2}{\sqrt{\mathcal{E}(\bar{\theta}_i)\ \int_{\mathcal{I}_i}E^x\ \int_{\mathcal{I}_i}E^y}}\,(1)^k
\end{aligned}
\tag{4.50}
$$

$$
\begin{aligned}
\text{RHS} &= \frac{1}{4}\sum_{i=1}^{n} N(\bar{\theta}_i)\frac{\left(\mathcal{E}(\bar{\theta}_i+\epsilon)-\mathcal{E}(\bar{\theta}_i)\right)^2}{V(\mathcal{I}_i)}\left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0}\right)^k\left(\frac{\mathcal{Z}(\mathcal{I}_i)}{V(\mathcal{I}_i)}\right)^k \qquad (4.51)\\
&= \frac{1}{4}\left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0}\right)^k\sum_{i=1}^{n} N(\bar{\theta}_i)\frac{\left(\mathcal{E}(\bar{\theta}_i+\epsilon)-\mathcal{E}(\bar{\theta}_i)\right)^2}{V(\mathcal{I}_i)}\left(\frac{\mathcal{Z}_\alpha(\mathcal{I}_i)}{\alpha^3(V(\mathcal{I}_i))^{(3\alpha-2)}}\right)^k \\
&= \frac{1}{4}\left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0\alpha^3}\right)^k\sum_{i=1}^{n} N(\bar{\theta}_i)\left(\mathcal{E}(\bar{\theta}_i+\epsilon)-\mathcal{E}(\bar{\theta}_i)\right)^2\left(\mathcal{Z}_\alpha(\mathcal{I}_i)\right)^k\Big|_{\alpha:=\frac{2}{3}-\frac{1}{3k}}
\end{aligned}
$$

In the last line we have chosen $\alpha := \frac{2}{3} - \frac{1}{3k}$ which removes the explicit factors of the volume. The choice of $k > 0$ is limited by $\alpha > 0$ (being a power of the volume appearing in $\mathcal{Z}_\alpha$). Some convenient choices would be $k = 1$ ($\alpha = 1/3$), $k = 2$ ($\alpha = 1/2$) etc. For all such choices, the above expression can be promoted to a well defined operator.

The second term of H_P :

To begin with one observes that E^y/E^x is a scalar, $\partial_\theta \ln(E^y/E^x)$ is a scalar density. This term is then manipulated as:

$$
-\frac{1}{4}\int_{S^1} N(\theta)\frac{(\mathcal{E}(\theta))^2}{\sqrt{E(\theta)}}\left(\frac{\partial_\theta E^x}{E^x}-\frac{\partial_\theta E^y}{E^y}\right)^2 = -\frac{1}{4}\int_{S^1} N(\theta)\frac{(\mathcal{E}(\theta))^2}{\sqrt{E(\theta)}}\left(\partial_\theta \ln\left(\frac{E^y}{E^x}\right)\right)^2
\tag{4.52}
$$

$$\begin{aligned}
\text{RHS} &\approx -\frac{1}{4}\sum_{i=1}^{n} N(\bar\theta_i)\epsilon\frac{(\mathcal{E}(\bar\theta_i))^2}{\sqrt{E(\bar\theta_i)}}\left(\partial_\theta \ln\left(\frac{E^y}{E^x}(\bar\theta_i)\right)\right)^2 \\
&= -\frac{1}{4}\sum_{i=1}^{n} N(\bar\theta_i)\frac{(\mathcal{E}(\bar\theta_i))^2}{\sqrt{\epsilon^2 E(\bar\theta_i)}}\left(\frac{E^x(\bar\theta_i)}{E^y(\bar\theta_i)}\,\epsilon\partial_\theta\left(\frac{E^y}{E^x}(\bar\theta_i)\right)\right)^2 \\
&= -\frac{1}{4}\sum_{i=1}^{n} N(\bar\theta_i)\frac{(\mathcal{E}(\bar\theta_i))^2}{V(\mathcal{I}_i)}\left[\frac{E^x(\bar\theta_i)}{E^y(\bar\theta_i)}\left\{\left.\frac{E^y}{E^x}\right|_{\bar\theta_i+\epsilon}-\left.\frac{E^y}{E^x}\right|_{\bar\theta_i}\right\}\right]^2 \\
&= -\frac{1}{4}\sum_{i=1}^{n} N(\bar\theta_i)\frac{(\mathcal{E}(\bar\theta_i))^2}{V(\mathcal{I}_i)}\left[\frac{\int_{\mathcal{I}_i}E^x}{\int_{\mathcal{I}_i}E^y}\left\{\frac{\int_{\mathcal{I}_{i+1}}E^y}{\int_{\mathcal{I}_{i+1}}E^x}-\frac{\int_{\mathcal{I}_i}E^y}{\int_{\mathcal{I}_i}E^x}\right\}\right]^2 \qquad (4.53)
\end{aligned}$$

Now we have the fluxes in the denominator which can be defined exactly as the inverse triad operators of LQC [51]. To be explicit, denoting the fluxes as $\mathcal{F}_{x,\mathcal{I}} := \int_{\mathcal{I}} E^x, \mathcal{F}_{y,\mathcal{I}} := \int_{\mathcal{I}} E^y$.

$$\begin{aligned}
\mathcal{F}^{-1}_{x,\mathcal{I}} &= \left(\frac{1}{\kappa'\gamma l}\right)^{\frac{1}{1-l}}\{X(v),\mathcal{F}^l_{x,\mathcal{I}}\}^{\frac{1}{1-l}}\quad, \qquad\qquad l\in(0,1) \\
&= \left(\frac{2i}{\kappa'\gamma l\mu_0}\right)^{\frac{1}{1-l}}\left(h_v^{(\mu_0/2)}(X)\left\{h_v^{(-\mu_0/2)}(X),\mathcal{F}^l_{x,\mathcal{I}}\right\}\right)^{\frac{1}{1-l}} \qquad (4.54)
\end{aligned}$$

and similarly for $\mathcal{F}^{-1}_{y,\mathcal{I}}$. These can be promoted to a well defined operator. Continuing with the equation above,

$$\begin{aligned}
\text{RHS} &= -\frac{1}{4}\sum_{i=1}^{n} N(\bar\theta_i)\frac{(\mathcal{E}(\bar\theta_i))^2}{V(\mathcal{I}_i)}\left[\mathcal{F}^{-1}_{y,\mathcal{I}_i}\mathcal{F}_{x,\mathcal{I}_i}\left(\mathcal{F}^{-1}_{x,\mathcal{I}_{i+1}}\mathcal{F}_{y,\mathcal{I}_{i+1}}-\mathcal{F}^{-1}_{x,\mathcal{I}_i}\mathcal{F}_{y,\mathcal{I}_i}\right)\right]^2 \\
&= -\frac{1}{4}\left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0\alpha^3}\right)^k\sum_{i=1}^{n} N(\bar\theta_i)(\mathcal{E}(\bar\theta_i))^2\times \qquad (4.55) \\
&\qquad \left[\mathcal{F}^{-1}_{y,\mathcal{I}_i}\mathcal{F}_{x,\mathcal{I}_i}\left(\mathcal{F}^{-1}_{x,\mathcal{I}_{i+1}}\mathcal{F}_{y,\mathcal{I}_{i+1}}-\mathcal{F}^{-1}_{x,\mathcal{I}_i}\mathcal{F}_{y,\mathcal{I}_i}\right)\right]^2\left.(\mathcal{Z}_\alpha(\mathcal{I}_i))^k\right|_{\alpha=\frac{2}{3}-\frac{1}{3k}}
\end{aligned}$$

where, in the last step, we have manipulated,

$$\begin{aligned}
\frac{1}{V(\mathcal{I}_i)} &= \frac{1}{V(\mathcal{I}_i)}(1)^k = \frac{1}{V(\mathcal{I}_i)}\left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0}\right)^k\left(\frac{\mathcal{Z}(\mathcal{I}_i)}{V(\mathcal{I}_i)}\right)^k \\
&= \left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0\alpha^3}\right)^k\left.(\mathcal{Z}_\alpha(\mathcal{I}_i))^k\right|_{\alpha=\frac{2}{3}-\frac{1}{3k}} \qquad (4.56)
\end{aligned}$$

The choice of α would be same as that in the first term.

The third term of H_P :

$$H_T = -\int_{S^1} N(\theta)\partial_\theta \left[\frac{\mathcal{E}\partial_\theta \mathcal{E}}{\sqrt{E(\theta)}}\right] \approx -\sum_{i=1}^{n} N(\bar{\theta}_i)\epsilon\partial_\theta \left[\frac{\mathcal{E}(\bar{\theta}_i)\partial_\theta \mathcal{E}}{\sqrt{E(\bar{\theta}_i)}}\right]$$

$$\begin{aligned}
\text{RHS} &= \sum_{i=1}^{n} N(\bar{\theta}_i)\left[\left\{\frac{\mathcal{E}\epsilon\partial_\theta\mathcal{E}}{\epsilon\sqrt{E}}\right\}\Big|_{\bar{\theta}_i+\epsilon} - \left\{\frac{\mathcal{E}\epsilon\partial_\theta\mathcal{E}}{\epsilon\sqrt{E}}\right\}\Big|_{\bar{\theta}_i}\right] \\
&= -\sum_{i=1}^{n} N(\bar{\theta}_i)\left[\frac{\mathcal{E}(\bar{\theta}_i+\epsilon)\left\{\mathcal{E}(\bar{\theta}_i+2\epsilon) - \mathcal{E}(\bar{\theta}_i+\epsilon)\ \right\}}{V(\mathcal{I}_{i+1})}\right. \\
&\qquad\qquad \left. - \frac{\mathcal{E}(\bar{\theta}_i)\left\{\mathcal{E}(\bar{\theta}_i+\epsilon) - \mathcal{E}(\bar{\theta}_i)\ \right\}}{V(\mathcal{I}_i)}\right] \\
&= -\left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0\alpha^3}\right)^k \\
&\qquad \sum_{i=1}^{n} N(\bar{\theta}_i)\left[\mathcal{E}(\bar{\theta}_i+\epsilon)\left\{\mathcal{E}(\bar{\theta}_i+2\epsilon) - \mathcal{E}(\bar{\theta}_i+\epsilon)\right\}(\ \mathcal{Z}_\alpha(\mathcal{I}_{i+1})\ )^k\right. \\
&\qquad\qquad \left. - \mathcal{E}(\bar{\theta}_i)\left\{\mathcal{E}(\bar{\theta}_i+\epsilon) - \mathcal{E}(\bar{\theta}_i)\ \right\}(\ \mathcal{Z}_\alpha(\mathcal{I}_i)\ )^k\right]\Big|_{\alpha=\frac{2}{3}-\frac{1}{3k}}
\end{aligned} \tag{4.57}$$

At this point we have expressed the H_P in terms of the holonomy-flux variables and quantization can be carried out simply by replacing the $(\mathcal{Z}_\alpha)^k \to (-i/\hbar)^{3k}(\hat{\mathcal{Z}}_\alpha)^k$. This correctly combines with the powers of κ' to give $(\ell_{\mathrm{P}}^2)^{3k}$ in the denominator. The $(\mathcal{Z}_\alpha)^k$, will give $(\ell_{\mathrm{P}}^{3\alpha})^{3k}$ since each factor of volume gives 3α, there are 3 factors of volume in each $\mathcal{Z}$ and there is the overall power of k. Substituting for α one sees that each of the terms in H_P, H_T has ℓ_{P}^{-3} apart from the ℓ_{P}^4 supplied by the factors of momenta/fluxes, thus giving the correct dimensions.

The operators $\mathcal{O}^a_\alpha := [\cos(\cdots)\hat{V}^\alpha\sin(\cdots) - \sin(\cdots)\hat{V}^\alpha\cos(\cdots)]$, $a = x, y, \theta$, appear in all the terms and is a function of both holonomies and fluxes. To see that this is actually diagonal in the charge network basis, write the cos and sin operators as sums and differences of the exponentials (i.e. holonomies). It then follows that,

$$\cos(\cdots)\hat{V}^\alpha\sin(\cdots) - \sin(\cdots)\hat{V}^\alpha\cos(\cdots)] = \frac{1}{2i}\left[e^{-i(\cdots)}\hat{V}e^{+i(\cdots)} - e^{+i(\cdots)}\hat{V}e^{-i(\cdots)}\right] \tag{4.58}$$

It is now obvious that the operators are diagonal and thus commute with all the flux operators. Thus there are no ordering issues in quantization of H_P operato In the H_K, however, operators of the above type are ordered on the right as in t full theory.

4.6 Action on States

To make explicit the action of the Hamiltonian constraint on the basis states, it useful to make a couple of observations.

- Every gauge-invariant *basis* state can be thought of as a collection of $m-$vert with a quadruple of labels $(k_v^{\pm}, \mu_v, \nu_v)$, all non-zero. The $k_v^{\pm}$ denoting the U representations on the two edges meeting at v with $+$ referring to the exiti edge and $-$ to the entering edge. A partition may also be viewed as a gra except that at its "vertices" all representation labels are zero.

- The action of the flux operators labelled by $\mathcal{I}$, on a basis state is necessari zero if none of the vertices of the state have an intersection with the lab interval. Note that the operator $\mathcal{E}(\theta_i)$, however always has a non-zero actio on a basis state. This is because, all graphs are closed and hence there always an edge (and non-zero label for a basis state) which overlaps with θ

- The volume operator associated with an interval $\mathcal{I}$ gives a non-zero contri bution on a basis state *only if* $\mathcal{I}$ contains a vertex of the graph. Recall th our partition is sufficiently refined so that each cell contains *at most on vertex* (two vertices at the cell boundaries are counted as a single vertex i the interior).

- The full Hamiltonian has been written as a sum using a partition of S Consider the i^{th} term in each of the H_K, H_P. Each of these contains $\mathcal{O}$ operators either separately (as in H_K) or as a product through the $\mathcal{Z}_\alpha$ (as i H_P). Since these contain the volume operator, it ensures that the action o each of these terms is necessarily zero unless the $\mathcal{I}_i$ contains a vertex of th basis state. Evidently the action of the full constraint is *finite* regardless o the chosen partition.

- The factors of trigonometric operators multiplying the $\mathcal{O}^a_\alpha$ on the left in H_K can be thought of as "creating new vertices" at the points $\bar{\theta}_i$ of the partition Notice however that at these new vertices one has either an edge holonom or *one* of the point holonomies only i.e. the volume operator acting at thes vertices will give zero.

Summarizing, thanks to the $\hat{\mathcal{O}}, \hat{\mathcal{Z}}$ operators acting first, only those intervals o a partition will contribute which contain at least one vertex of the graph of a basi state. This immediately implies that in the second term of H_P (eq. 4.55), only one of the terms in square bracket will contribute. We will return to this later. Let us denote the factor associated with a vertex v of a basis state by $|k_v^\pm, \mu_v, \nu_v\rangle$. Here are the actions of all the 6 terms of the Hamiltonian constraint restricted to the interval containing v:

$$\hat{H}_K^\theta |k_v^\pm, \mu_v, \nu_v\rangle = \frac{\sqrt{\gamma \ell_{\rm P}^2}}{4\gamma^2 \mu_0 \nu_0} \Bigg[\sqrt{|\mu_v||\nu_v|} \left(\sqrt{|k_v^+ + k_v^- + 1|} - \sqrt{|k_v^+ + k_v^- - 1|} \right) \times \sin(\mu_0 X(\bar{\theta}_i)) \sin(\nu_0 Y(\bar{\theta}_i)) \Bigg] |k_v^\pm, \mu_v, \nu_v\rangle \tag{4.59}$$

$$\hat{H}_K^x |k_v^\pm, \mu_v, \nu_v\rangle = \frac{\sqrt{\gamma \ell_{\rm P}^2}}{4\gamma^2 \mu_0 \nu_0} \Bigg[\sqrt{|k_v^+ + k_v^-||\nu_v|} \left(\sqrt{|\mu_v + \mu_0|} - \sqrt{|\mu_v - \mu_0|} \right) \times \tag{4.60}$$
$$2 \sin\left(\frac{1}{2}\nu_0 Y(\bar{\theta}_i + \epsilon) \right) \cos\left(\frac{1}{2}\nu_0 Y(\bar{\theta}_i) \right) \sin\left(\int_{\mathcal{I}_i} \mathcal{A} - \Delta_i \right) \Bigg] |k_v^\pm, \mu_v, \nu_v\rangle$$

$$\hat{H}_K^y |k_v^\pm, \mu_v, \nu_v\rangle = \frac{\sqrt{\gamma \ell_{\rm P}^2}}{4\gamma^2 \mu_0 \nu_0} \Bigg[\sqrt{|k_v^+ + k_v^-||\mu_v|} \left(\sqrt{|\nu_v + \nu_0|} - \sqrt{|\nu_v - \nu_0|} \right) \times \tag{4.61}$$
$$2 \sin\left(\frac{1}{2}\nu_0 X(\bar{\theta}_i + \epsilon) \right) \cos\left(\frac{1}{2}\nu_0 X(\bar{\theta}_i) \right) \sin\left(\int_{\mathcal{I}_i} \mathcal{A} - \Delta_i \right) \Bigg] |k_v^\pm, \mu_v, \nu_v\rangle$$

$$\hat{H}_P^{(1)} |k_v^\pm, \mu_v, \nu_v\rangle = \left[\frac{\sqrt{\gamma \ell_{\rm P}^2}}{2} \left(\frac{1}{8\mu_0 \nu_0 \alpha^3} \right)^k \right] \left\{ (k_v^+ - k_v^-) \right\}^2 \times$$
$$\Big[\left\{ \left(|\mu_v + \mu_0|^\alpha - |\mu_v - \mu_0|^\alpha \right) |\nu_v|^\alpha |k_v^+ + k_v^-|^\alpha \right\} \times \tag{4.62}$$
$$\left\{ |\mu_v|^\alpha \left(|\nu_v + \nu_0|^\alpha - |\nu_v + \nu_0|^\alpha \right) |k_v^+ + k_v^-|^\alpha \right\} \times$$
$$\left\{ |\mu_v|^\alpha |\nu_v|^\alpha \left(|k_v^+ + k_v^- + 1|^\alpha - |k_v^+ + k_v^- + 1|^\alpha \right) \right\} \Big]^k |k_v^\pm, \mu_v, \nu_v\rangle$$

$$
\begin{aligned}
\hat{H}_P^{(2)}|k_v^\pm,\mu_v,\nu_v\rangle \;=\; & \left[-\frac{\sqrt{\gamma\ell_{\mathrm{P}}^2}}{2}\left(\frac{1}{8\mu_0\nu_0\alpha^3}\right)^k\right]\Bigg\{(k_v^+ + k_v^-)^2 \times \\
& \qquad \left(\widehat{\mathcal{F}^{-1}_{x,\mathcal{I}_i}}\hat{\mathcal{F}}_{x,\mathcal{I}_i}(\mu_v)\right)^2\left(\widehat{\mathcal{F}^{-1}_{y,\mathcal{I}_i}}\hat{\mathcal{F}}_{y,\mathcal{I}_i}(\nu_v)\right)^2\Bigg\}\times \\
& \Big[\left\{\left(|\mu_v+\mu_0|^\alpha - |\mu_v-\mu_0|^\alpha\right)|\nu_v|^\alpha|k_v^+ + k_v^-|^\alpha\right\}\times \qquad (4.63)\\
& \left\{|\mu_v|^\alpha\left(|\nu_v+\nu_0|^\alpha - |\nu_v+\nu_0|^\alpha\right)|k_v^+ + k_v^-|^\alpha\right\}\times \\
& \left\{|\mu_v|^\alpha|\nu_v|^\alpha\left(|k_v^+ + k_v^- + 1|^\alpha - |k_v^+ + k_v^- + 1|^\alpha\right)\right\}\Big]^k\;|k_v^\pm,\mu_v,\nu_v\rangle \\
\hat{H}_P^{(3)}|k_v^\pm,\mu_v,\nu_v\rangle \;=\; & \left[-2\sqrt{\gamma\ell_{\mathrm{P}}^2}\left(\frac{1}{8\mu_0\nu_0\alpha^3}\right)^k\right]\left\{-k_v^-(k_v^+ - k_v^-)\right\}\times \\
& \Big[\left\{\left(|\mu_v+\mu_0|^\alpha - |\mu_v-\mu_0|^\alpha\right)|\nu_v|^\alpha|k_v^+ + k_v^-|^\alpha\right\}\times \qquad (4.64)\\
& \left\{|\mu_v|^\alpha\left(|\nu_v+\nu_0|^\alpha - |\nu_v+\nu_0|^\alpha\right)|k_v^+ + k_v^-|^\alpha\right\}\times \\
& \left\{|\mu_v|^\alpha|\nu_v|^\alpha\left(|k_v^+ + k_v^- + 1|^\alpha - |k_v^+ + k_v^- + 1|^\alpha\right)\right\}\Big]^k\;|k_v^\pm,\mu_v,\nu_v\rangle
\end{aligned}
$$

In the above, factors of $N(\bar{\theta})$ are suppressed.

In the first three equations, we have explicitly evaluated only the action of the diagonal operators and kept the holonomies which "create new vertices" as operators acting on $|k_v^\pm,\mu_v,\nu_v\rangle$. In the terms involving α, we have to use $\alpha = \frac{2}{3} - \frac{1}{3k}$. The last square brackets in the last three terms is the action of the $\mathcal{Z}_\alpha(\mathcal{I}_i)$ after the dimensional and numerical factors are collected together in the first square bracket.

In $H_P^{(2)}$, the products of inverse flux and flux operators approach 1 only for large values of μ_v, ν_v while for smaller values, these products vanish.

The above actions have to be summed over all the vertices of the graph. These being finite, the action is finite as noted before. There is no explicit appearance of ϵ. Reference to cells enclosing the vertices (eg $\bar{\theta}_i, \mathcal{I}_i$), will again transfer only to the vertices in the limit of infinite refinement. The technical issue of limiting operator on Cyl* can be done in the same manner as in the full theory eg as in [5, 6].

The above definitions of the quantization of the Hamiltonian constraint constitute *a choice* and there are many choices possible. There is also the issue related to "local degrees of freedom". In the next section, a preliminary discussion of these features is presented.

4.7 Discussion

Let us quickly recapitulate where we made various choices.

- We made a cell decomposition with the understanding of taking the limit of infinitely many cells. At this stage, no reference to any state or graph is made.
- In the regularization of the kinetic term we used the 'inverse volume' and 'plaquette holonomies'. We could have introduced inverse flux operators and $\mathcal{E}$ operators to replace $1/\sqrt{E}$ and also replaced the $X, Y, \int_{\mathcal{I}_i} \mathcal{A}$ by $\sin(\mu_0 X)/\mu_0$ and similarly for the others. Such a replacement would still give the classical expression back, in the limit of small X, Y, ϵ. The quantum operator however would be different. This procedure will also deviate from the full theory. From the point of view of the reduced theory, this is an ambiguity.
- In the transcription of H_P in terms of holonomies and fluxes certain choices have been made. For example, the second term in the H_P, could have been manipulated in terms of inverse powers of $\sqrt{E}$ instead of introducing inverse flux operators (eg by replacing $1/E^x = \mathcal{E}E^y/(\sqrt{E})^2)$). This would lead to $\mathcal{E}^2\left[\mathcal{F}_{x,\mathcal{I}_i}\mathcal{F}_{y,\mathcal{I}_{i+1}} - \mathcal{F}_{y,\mathcal{I}_i}\mathcal{F}_{x,\mathcal{I}_{i+1}}\right]^2$ and lead to $\alpha(k) = 2/3 - 5/(3k)$. In the limit of infinite refinement, each cell will contain *at most* one vertex and the cells adjacent to such a cell will always be empty. Consequently, the second term of H_P, regulated in the above manner will always give a zero action.
- Over and above these different transcriptions, we also have the ambiguity introduced by the arbitrary positive power k (and $\alpha(k)$) as well as that introduced by the arbitrary power $l \in (0, 1)$ in the definition of inverse flux operators.

All these ambiguities refer to the transcription stage.

There are also issues related to the choice of partitions, subsequent $\epsilon \to 0$ limit and the presence/absence of local degrees of freedom. This is most dramatically brought out by the second term of H_P. Classically, this is the term which reveals spatial correlations in a solution space-time through $\partial_\theta \ln(E^y/E^x)$ [45] and reflect the infinitely many, physical solutions. In the (vacuum) spherically symmetric case, such a term is absent and so are local physical degrees of freedom. We would like to see if there is a quantization of this term which reflects these correlations. The quantization chosen above does not correlate μ, ν labels at different vertices.

In general, given a graph, a partition may be chosen to have

1. every cell containing *at least* one vertex
2. every cell containing *exactly* one vertex

3. every cell containing *at most* one vertex.

In this classification, we assume that a vertex is never a boundary-point of a cell, which is always possible to choose. Infinite refinement is possible only for (iii) which we have been assuming so far. This is the reason that in the contribution from the $\mathcal{I}_i$ cell, the terms referring to $\mathcal{I}_{i+1}$ drop out. We could introduce a fourth case by requiring

4. every vertex to be a boundary point of a cell.

Then we would receive contributions from two adjacent cells. However, in $H_P^{(2)}$ (eq. 4.55), the two terms with labels $\mathcal{I}_{i+1}, \mathcal{I}_i$, both give equal contribution such that the total is zero! The same would happen in the alternative expression given above. It seems that in either of (3) or (4) type partitions, we will either get a zero or a contribution depending only on a single vertex. Note that these are the only partitions which allow infinite refinement ($\epsilon \to 0$) in a diffeo-covariant manner.

We can give up on the infinite partitions (and $\epsilon \to 0$ limit) and consider instead case (2) partitions – each cell contains exactly one vertex (say in the interior). Now the contributions will explicitly depend upon μ, ν labels of adjacent vertices and in this sense, spatial correlations will survive in the constraint operator. An even more restrictive choice would be to choose the partition defined by the graph itself - cells defined by the edges and the boundary points of cells as vertices. In this case, the new vertices created by H_K would be the already present vertices and the constraint equation would lead to a (partial) difference equation among the labels. This case has been considered in the spherical symmetry [48] and corresponds to 'effective operator viewpoint' discussed by Thiemann in [5]. The $\epsilon \to 0$ limit may then be thought to be relevant when states have support on graphs with very large (but finite) number vertices, heuristically for semiclassical states. Whether requiring the constraint algebra to be satisfied on diffeomorphism invariant states chooses/restricts the alternatives and ambiguities remains to be seen.

5 Conclusion

In this chapter we will give a brief summary of the work done in this thesis. We will also discuss a few of the open issues and further directions of research. In section (5.1) we will look at the work presented in Chapter 2 of this thesis and in section (5.2) we will consider the work presented in Chapters 3 and 4.

5.1 Effective Dynamics in LQC

In chapter 2, we obtained the effective dynamics of LQC in a route different from that available in literature. The perturbative corrections were expressed in terms of a small parameter ϵ which is proportional to the Barbero-Immirizi parameter γ and an ambiguity parameter μ_0. These were summed up by making the WKB approximation at the level of the difference equation itself and the effective Hamiltonian incorporating these corrections was determined. The Hamilton's equations of motion of the effective Hamiltonian give the effective evolution equations, which were of the form of Raychoudhuri equations plus corrections. The modified dynamics were numerically calculated for the minimally coupled massive scalar field and exactly for phenomenological matter (dust and radiation) and cosmological constant. It was shown that in most cases the corrections are perturbative in nature but significant deviations from the classical picture occur in a cosmological constant dominated universe. All this was done using the quantization proposed in [8] which was the one available at that time.

However two criticisms of the previous work, namely, the low value of critical density required for bounce at large volumes and the absence of a physical Hilbert space led to further developments in LQC subsequently. In [19] a different quantization was proposed for the Hamiltonian constraint, where the parameter μ_0 is replaced by a function $\bar{\mu}$ which instead of being a constants is considered to be a function of the triad. This has led to improved dynamics in LQC which retains the good features of LQC like singularity resolution while avoiding the problems mentioned above. Also the physical Hilbert space has been constructed for LQC with a massless scalar field [20, 21] and it has been analysed at the level of physical pectation values. These new developments have been commented upon, in the al section of chapter 2.

In the light of these new developments, there are lot of issues still under dis- sion. The physical Hilbert space has been constructed explicitly only for FRW mologies with a massless scalar field. The construction has to be extended to ssive scalar fields. Also the issue of what happens to the universe after evolution ough the classical singularity is not fully settled. In particular, it is not clear the ct amount of correlation on either side of the big bang (see [52, 53]). There is an attempt to introduce perturbative inhomogeneities into FRW cosmology to ax the strict restrictions of homogeneity and generalise the Loop quantization to omogeneous universes [54]. There is also an effort to derive effective equations notion based on the framework developed in [55, 56]. In short, the field of LQC lot more open issues both from the context of the reduced model as well as nection to the full theory.

2 Loop Quantization of Gowdy Models

chapters 3 and 4 we have described the preliminary steps in Loop quantizing Gowdy Model on T^3.

In Chapter 3 we carried out a classical analysis to prepare the ground for setting the quantum theory. We defined the Unpolarized model in terms of variables ch will be convenient for Loop quantization. In particular we make a canoni- transformation such that the variables have proper transformation properties ler gauge transformations, the volume is a simple function of the conjugate mo- nta and the Hamiltonian constraint is simplified. Unlike in the full theory, the venient configuration variables conjugate to the triad components turn out to connection components in the inhomogeneous direction and extrinsic curvature nponents in the homogeneous directions. A systematic Hamiltonian reduction hen carried out to obtain the Polarized Gowdy model from the Unpolarized e. Interestingly, diagonalization of the metric to obtain the polarized model s not imply diagonalization of the conjugate Ashtekar connection but that of extrinsic curvature. Validity of the reduction is further verified by reproduc- the known classical spacetime solutions and by verification of the constraint bra for the reduced constraints.

In Chapter 4 the Loop quantization of the model is attempted. In the full theory elementary variables are taken to be traces of Ashtekar connections along closed fluxes of the conjugate variables. In the context of this model however, the figuration variables in the classical theory are scalar fields and scalar densities of ght 1. These are therefore quantized as point holonomies and edge holonomies ectively. A basis for the Hilbert space is obtained by defining the "Charge work functions". These are also eigenstates of the momenta operators which

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are just the fluxes of the triad components. The Gauss law constraint is easily solved by restricting to gauge invariant states which involve only a restriction in the labels of the charge networks. The action of the Volume operator is trivially obtained since it is just a function of the momentum operators.

Finally the Hamiltonian constraint is addressed. In LQC, the configuration variables are point holonomies while in (vacuum) LQG, the configuration variables are edge holonomies. In this model, however, configuration variables consist of both point and edge holonomies and therefore the transcription of the Hamiltonian in terms of the basic variables is somewhat different from the full theory. The Hamiltonian contains three types of terms: a 'kinetic' term, a 'potential' and a total derivative term. The last two involve the triad components, the inverse triad components and their derivatives. Tools developed in the regularization of the matter Hamiltonians in the full theory are used to obtain a well defined operator modulo operator ordering ambiguities.

Since these are just the preliminary steps in loop quantization of a midisuperspace model, which has not been attempted before, a lot of issues are, as expected, still under investigation. The operator ordering issues as well as the correct choice of partitions to be chosen so as to reflect the local degrees of freedom are not clear. One way to clarify these would be by calculating the quantum commutator algebra. Determination of observables and identification of semi classical states (even at the kinematic level) are other interesting issues. The diffeomorphism constraint has to be imposed. Construction of the physical Hilbert space by means of group averaging procedure requires the Hamiltonian constraint to be self adjoint. However, the commutator algebra may determine the ordering of the Hamiltonian constraint and it will be interesting to determine whether that picks out the symmetric ordered operator as the correct ordering. Finally, it is hoped that since the classical solutions are well known, the Hamiltonian constraint can be solved and the question of whether the classical singularity is resolved can be addressed. Moreover, in the light of recent developments in LQC, the labels μ_o and ν_0 in the homogeneous directions may have to made into phase space functions to reflect the features of the full theory as discussed in section (2.8). Some of these issues are currently under investigation.

5.3 Finally ...

One important issue we have not discussed in this thesis is the issue of embeddebility of the reduced theories into the larger theory. It is well known that symmetry reduction after quantization and quantization after reduction may not lead to equivalent quantum theories. In this thesis, we have followed the second path with the attitude that lessons learnt from these toy models can provide hints for tackling some of problems of the full theory. While one would like to view the reduced quantum theory as a 'sector' of the the full theory, how to do so is not yet clear [57, 58, 59] In the absence of such an identification, predictions of the reduced model may not necessarily be implications of the full theory. In the context of this thesis, the identification of the 'sector' may be done at two levels:

- viewing LQC and Gowdy model as a sector of the full theory
- retrieving the homogeneous Bianchi I model from the Gowdy model

It might be easier to attempt the second identification which may provide hints to do the first. Further work is required to clarify these issues and progress along them will lead to a better understanding of Loop Quantum Gravity.

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