Aspects of Holographic Induced Gravities

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Rohan Raghava Poojary

Dedicated To,

Mom.

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Contents

Synopsis

0.1 Overview

The AdS/CFT correspondence has emerged as one of the most powerful tools in theoretical physics in the past few years. In its simplest and most concrete form it is a statement of equivalence (a duality) between a d-dimensional conformal field theory (CFT$_d$) and a string theory in a background with a $(d+1)$-dimensional Anti-de Sitter (AdS_{d+1}) factor. The set of well studied examples of this correspondence include

- $\mathcal{N} = 4$, $d = 4$ $SU(N)$ Yang-Mills theory $\leftrightarrow$ type IIB string theory on $AdS_5 \times S^5$,
- $\mathcal{N} = (4,4)$, $d = 2$ D1-D5 CFT $\leftrightarrow$ type IIB string theory on $AdS_3 \times S^3 \times T^4$,

and various generalizations there of.

The symmetries of the vacuum of the CFT$_d$ get mapped to the isometries of the AdS_{d+1} space, while the symmetries of the CFT are isomorphic to the appropriately defined asymptotic symmetries of the string/gravity theory in such a background. A famous illustration of this aspect of the duality is the seminal work of Brown and Henneaux [1] that was a precursor to the AdS/CFT conjecture. This involved exhibiting that the asymptotic symmetry algebra of a $(d+1)$-dimensional gravity with negative cosmological constant (AdS_{d+1} gravity) is isomorphic to the algebra of conformal transformations of the d-dimensional CFT. Brown and Henneaux were concerned with finding asymptotic symmetries which included in them the set of global conformal transformations of the

CFT_d while demanding that the metric at the boundary of AdS_{d+1} is held fixed (Dirichlet condition). Their boundary conditions provided a definition of the so called asymptotically locally AdS ($AlAdS$) spaces. In more recent times the Brown-Henneaux analysis has been generalised to AdS_3 supergravities [2] and AdS_3 higher spin gauge theories as well.

In particular, when $d = 2$ the asymptotic symmetry algebra of AdS_3 gravity with Brown-Henneaux boundary conditions is the sum of two commuting copies of the Virasoro algebra with central charges of both the Virasoros given by $c = \frac{3l}{2G}$, where G is the Newton's constant and l is the radius of the AdS_3 space in AdS_3 gravity. The CFT_2 in this case is thus expected to be a standard $2d$ conformal field theory with the left and right moving sectors treated symmetrically. Similarly in the AdS_3 higher spin context (containing gauge fields with spins upto N) the asymptotic symmetry algebra is a sum of two commuting copies of W_N algebra with Brown-Henneaux central charges, again, with perfect parity between their left and right sectors.

However, there have been $2d$ CFTs which do not maintain such parity between their left and right sectors. Perhaps the simplest way to break this parity is to consider cases where the central charges appearing in the left and right Virasoro algebras are different. More dramatic breaking of the left-right parity would be when the symmetry algebra either on the left sector or the right sector is not a Virasoro/W_N algebra. In fact there have been such $2d$ CFTs in the literature and they are generically termed as chiral CFTs. Thus a natural question is how to generalise the Brown-Henneaux type computation to allow for asymptotic symmetry algebras of chiral CFTs.

This thesis concerns itself with studying different boundary conditions imposed on $3d$ AdS spaces and thus uncovering new asymptotic symmetry algebras. We do this in different settings, first involving pure AdS_3 gravity [3], then the $3d$ higher-spin gauge theories [4] and finally some AdS_3 supergravities. It turns out that we need to consider boundary conditions that are not the Dirichlet type, *i.e.* the fields and the metric at the AdS boundary can fluctuate. For the generalization to higher spin and supergravity it becomes necessary to work in the first order formalism of AdS_3 gravity/3d higher-spin theories. Our investigations fall into two different cases depending on whether or not the asymptotic symmetry algebra contains fully the global part of the 2d conformal/higher-spin algebra.

There have appeared some works in the literature that provide examples of generalizations of Brown-Henneaux that belong to both these categories. In [5] Troessaert provided boundary conditions that allow the conformal factor of the boundary metric to fluctuate with the boundary metric having zero scalar curvature. In this case the asymptotic algebra contains fully 2d conformal symmetry. In [6] Compère, Song and Strominger provided boundary conditions with a symmetry algebra being one copy of Virasoro and one copy of a $U(1)$ Kaĉ-Moody algebra. Our investigations presented in this thesis will provide results that are complementary to the results of [5], [6] and various generalizations.

The thesis contains the following results summarized in the next sections:

- **Chiral boundary condition:** Generalizations of Brown-Henneaux boundary conditions for AdS_3 gravity in the second order (metric) formulation, with asymptotic symmetry algebra equal to one copy of Virasoro algebra with Brown-Henneaux central charge and one copy of $sl(2,\mathbb{R})$ current algebra with level $k = c/6$ [3].
- **Liouville boundary condition:** Generalization of [5] where the conformal factor of the boundary metric satisfies the Liouville equation both in second order and first order formulations of AdS_3 gravity.
- **Duals to chiral induced W gravities:** Generalizations of the boundary conditions of [7] pertaining to 3d spin-3 gravity with symmetry algebra equal to one copy of W_3 algebra with Brown-Henneaux central charge and one copy of $sl(3,\mathbb{R})$ algebra with level $k = c/6$. Generalization of [6] to the 3d spin-3 gravity where the symmetry algebra is a copy of W_3 and two $u(1)$ Kaĉ-Moody algebras [4].
- **Chiral boundary condition for supergravities:** We generalize these boundary

conditions to minimal AdS_3 supergravities.

In what follows we give a brief summary of these results and juxtapose them with previous works studying $2d$ CFTs and their AdS_3 duals.

0.2 Boundary condition analysis of $AlAdS_3$ in second order formalism

The Brown-Henneaux(BH) boundary conditions on $AlAdS_3$ yield 2 copies of Virasoro as their asymptotic symmetry algebra. These boundary conditions can be summarized quite conveniently in the Fefferman-Graham gauge wherein $AlAdS$ space metrics admit a series expansion in a coordinate r:

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}(g^{(0)}_{ab} + O(\tfrac{l}{r})g^{(...)}_{ab})dx^a dx^b. \tag{1}$$

Here (ab) are the co-ordinates in the co-dimension one surface orthogonal to the radial coordinate r. The BH boundary conditions are the ones which hold $g^{(0)}_{ab}$ fixed to η_{ab}. The central term in the Virasoro algebra $c = 3l/2G$. These boundary conditions were crucial in understanding the relation between $AlAdS_3$ spaces and 2d CFT on the boundary, especially in the calculation of BTZ semiclassical black-hole entropy in the large charge limit from the asymptotic growth of states of the 2d CFT.

CSS boundary condition

Compère *et al* [6] had given an alternative set of boundary conditions which allow holomorphic fluctuations of the $g^{(0)}_{++}$ metric component. These set of boundary conditions reveal a semi-direct sum of Virasoro and an affine $u(1)$ Kaĉ-Moody algebra. The level of the $u(1)$ affine algebra is related to the central charge of the Virasoro $c = 6k$.

Chiral boundary condition

In [3], we generalized the CSS[6] boundary conditions to allow for *a priori* generic fluctuations of $g^{(0)}_{++}$, which when constrained by Einstein's eom restrict it to be of the form,

$$g^{(0)}_{++} = F = f(x^+) + g(x^+)e^{ix^-} + \tilde{g}(x^+)e^{-ix^-}. \tag{2}$$

For the theory in the bulk to be consistent with the variational principle, one is required to add a boundary term of the form:

$$\begin{aligned} S_{bndy} &= \tfrac{-1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}\tfrac{1}{2}\mathcal{T}_{ab}\gamma^{ab}, \\ \text{where } \mathcal{T}^{\mu\nu} &= \tfrac{-1}{2r^4}l^2\delta^\mu_+\delta^\nu_-. \end{aligned} \tag{3}$$

This term in turn fixes the T_{--} component of the Brown-York stress tensor for the bulk geometries. Such boundary term basically minimizes the on-shell bulk action only for those configurations which are allowed by the proposed boundary conditions. The Gibbons-Hawking term alone is enough to define the Brown-Henneaux (Dirichlet) type boundary condition. It is apparent that allowing any boundary metric component to fluctuate requires holding requisite component of the Brown-York stress tensor fixed.

$$\delta S_{total} = \int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}\tfrac{1}{2}T^{\mu\nu}\delta\gamma_{\mu\nu}. \tag{4}$$

A case where the boundary metric is completely free was considered in [8]. We would here be describing cases where only certain components of the boundary metric are allowed to fluctuate, thus corresponding to mixed (Robin) boundary conditions. For the CSS case, the boundary term is chosen to allow for fluctuations of the type $\delta g^{(0)}_{++} = F(x^+)$. The above boundary term similarly is chosen to allow for the fluctuations of the boundary metric component $g^{(0)}_{++}$ in accordance with (2).

The asymptotic symmetry algebra we obtained is a semi-direct product of a Virasoro with an affine Kaĉ-Moody $sl(2, \mathbb{R})$ algebra with level $k = \frac{c}{6} = \frac{l}{4G}$. Similar analysis is done in Poincaré AdS_3[9] where the boundary is spatially non-compact. The Ward identities thus derived from the bulk *eom* are seen to correspond to the chiral induced gravity theory first written down by Polyakov[10].

Troessaert's boundary condition

On the heels of [6] was a paper by Troessaert[5] which proposed boundary conditions which freed up the conformal factor of the boundary metric. This conformal factor was written as a product of holomorphic and anti-holomorphic components, thus restricting the boundary metric to be conformally flat. The asymptotic symmetry algebra that resulted from such considerations yielded left and right moving affine $u(1)$s along with two copies of Virasoro with the central charge $c = \frac{3l}{2G} = 6k$.

Liouville boundary condition

The Liouville theory *eom* is $\partial_+\partial_- \log F = 2\chi F$. One can ask what possibly could be a required set of boundary conditions on $AlAdS_3$ such that the conformal factor of the boundary metric is this Liouville field? To this end we find that one needs to add an appropriate boundary term of the form,

$$S_{bndy} = \frac{5\chi l}{4\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{-\gamma}. \tag{5}$$

apart from allowing the boundary metric to have an *a priori* arbitrary conformal factor $F(x^+, x^-)$. Upon imposing the variational principle on shell, F would obey the *eom* of the Liouville theory.

One finds that the asymptotic symmetry algebra in this case would be two copies of Virasoro corresponding to the Brown-Henneaux deformations, plus two copies of Virasoro corresponding to the holomorphic and anti-holomorphic components of the Liouville stress-tensor. One also finds that the Liouville central charge is $c_{Liouville} = -c_{BH} = -\frac{3l}{2G}$. Therefore the total central charge of such a theory is zero.

0.3 Analysis of $AlAdS_3$ as Chern-Simons theory

The second order formalism described previously is not amenable to analysis if one wants to generalize such mixed boundary conditions to higher-spins(hs) and supergravities in AdS_3. The second order action for hs in AdS_3 is not completely known and the computation of the asymptotic algebra for the boundary conditions of interest is difficult in the case of supergravity. 3d gravity with negative cosmological constant can be expressed as a difference of two Chern-Simons theories with gauge group $SL(2,\mathbb{R})$[11]. The left and right gauge fields are then given in terms of $A = \omega + \frac{e}{l}$ and $\tilde{A} = \omega - \frac{e}{l}$, with torsionless condition and Einstein's equations being imposed by flatness of gauge connections for the 2 gauge fields.

$$\begin{aligned} S_{CS}[A] &= \tfrac{k}{4\pi}\int dA\wedge A + \tfrac{2}{3}A\wedge A\wedge A, \\ S_{AdS_3} &= S_{CS}[A] - S_{CS}[\tilde{A}]. \end{aligned} \tag{6}$$

The analysis of the previous sub-section can then be repeated in this first order formalism by imposing different gauge field fall-off conditions.

AdS_3 gravity described thus was used to study the asymptotic dynamics by analysing WZNW in 2d [12, 13]. It was found that the suitable constraints on the WZNW conserved currents corresponded to the Brown-Henneaux boundary conditions. The CSS boundary conditions were also translated into the WZNW theory by the same authors. This formalism is particularly suitable to analyse supergravity with negative cosmological constant in 3d [2], where the asymptotic symmetry algebra now consists of two copies of super-Virasoro algebra *i.e.* the extended super conformal algebra with quadratic non-linearities in the current.

The first order formalism is the only way one can analyze the higher-spins in AdS_3. Here since $sl(3,\mathbb{R}) \subset hs[\lambda]$, one can analyze CS-CS theory with $SL(3,\mathbb{R})$ gauge group as a consistent truncation of the full $hs[\lambda]$ theory. The asymptotic symmetry algebra for such theory was studied in [7] and was found to be 2-copies of W_3 algebra, which are the hs analogues of Virasoro.

Chiral boundary conditions

The gauge fields corresponding to the metric configurations restricted by chiral boundary conditions are [4]:

$$\begin{aligned} A &= b^{-1}\partial_r b dr + b^{-1}[(L_1 + a_+^{(-)}L_{-1} + a_+^{(0)}L_0)dx^+(a_-^{(-)}L_{-1})dx^-]b, \\ \tilde{A} &= b\partial_r b^{-1}dr + b[(\tilde{a}_+^{(+)}L_+ + \tilde{a}_+^{(0)}L_0 + \tilde{a}_+^{(-)}L_-)dx^+ + (-L_- + \tilde{a}_-^{(+)}L_+)dx^-]b^{-1}, \\ \text{where } b &= e^{\log\frac{r}{l}L_0}. \end{aligned} \tag{7}$$

The A gauge field obeys the Brown-Henneaux type boundary condition but $\tilde{A}$ has mode $\tilde{a}_+^{(-)}$ which survives at boundary . The metric obtained from A and $\tilde{A}$ does relate $g_{++}^{(0)} = \tilde{a}_+^{(-)}$. The Ward identity is given by the remaining *eom* for $\tilde{A}$:

$$(\partial_+ + 2\partial_-\tilde{a}_+^{(-)} + \tilde{a}_+^{(-)}\partial_-)\tilde{a}_-^{(+)} = \tfrac{1}{2}\partial_-^3\tilde{a}_+^{(-)}. \tag{8}$$

Suitable boundary terms have to be added to make the theory variationally well defined:

$$\begin{aligned} S_{bdy} &= \tfrac{-k}{4\pi}\int dx^2 tr(L_0[a_+, a_-]) - \tfrac{k}{4\pi}\int dx^2 tr(L_0[\tilde{a}_+, \tilde{a}_-] - 2\tilde{\kappa}_0 L_+\tilde{a}_+), \\ \implies \delta S_{total} &= \tfrac{k}{2\pi}\int dx^2(\tilde{a}_-^{(+)} - \tilde{\kappa}_0)\delta\tilde{a}_+^{(-)}. \end{aligned} \tag{9}$$

For $\tilde{a}_-^{(+)} = \tilde{\kappa}_0 = -\frac{1}{4}$ one gets the desired asymptotic symmetries as in the second order formalism.

Liouville boundary condition

The Liouville theory obtained in the previous section, can also be cast in a first order form. The gauge fields corresponding to configurations in [5] are:

$$\begin{aligned} a &= (a_+^{(+)}(x^+)L_1 - \tfrac{\kappa(x^+)}{a_+^{(+)}}L_{-1})dx^+, & A &= b^{-1}ab + b^{-1}db, \\ \tilde{a} &= (\tilde{a}_-^{(-)}(x^-)L_{-1} - \tfrac{\tilde{\kappa}(x^-)}{\tilde{a}_-^{(-)}}L_1)dx^-, & \tilde{A} &= b\tilde{a}b^{-1} + bdb^{-1}, \\ & & b &= e^{\log\frac{r}{l}L_0}. \end{aligned} \tag{10}$$

If one needs the generic Liouville type of *eom* for the conformal factor $F = -a_+^{(+)}\tilde{a}_-^{(-)}$ of the boundary metric, then one would be forced to choose $(\tilde{a})a$ which is not (anti-)holomorphic.

$$\begin{aligned} a &= (a_+^{(+)}L_1 - \partial_+(\log a_-^{(-)})L_0 - \tfrac{\kappa(x^+)}{a_+^{(+)}}L_{-1})dx^+ + (\partial_-(\log a_+^{(+)})L_0 - a_-^{(-)}L_{-1})dx^-, \\ \tilde{a} &= (\tilde{a}_-^{(-)}L_{-1} + \partial_-(\log \tilde{a}_+^{(+)})L_0 - \tfrac{\tilde{\kappa}(x^-)}{\tilde{a}_-^{(-)}}L_1)dx^- + (-\partial_+(\log \tilde{a}_-^{(-)})L_0 - \tilde{a}_+^{(+)}L_1)dx^+. \end{aligned} \tag{11}$$

Duals to chiral induced *W* gravities

Massless higher-spin excitations can also be studied in a similar light in AdS_3 where the bulk action is written as CS-CS with the gauge group being $SL(3,\mathbb{R})$[1]. Here we propose a new set of boundary conditions which allow for either an $su(1,2)$ or $sl(3,\mathbb{R})$ or a $u(1)\times u(1)$ affine Kaĉ-Moody algebra along with a W_3 as an asymptotic symmetry algebra [4].

The gauge fields are[2]

$$\begin{aligned} a &= (L_1 - \kappa L_{-1} - \omega W_{-2})dx^+, \\ \tilde{a} &= (-L_{-1} + \tilde{\kappa}L_1 + \tilde{\omega}W_{-2})dx^- + (\sum_{a=-1}^{1} f^{(a)}L_a + \sum_{i=-2}^{2} g^{(i)}W_i)dx^+. \end{aligned} \tag{12}$$

The gauge algebra $sl(3,\mathbb{R})$ is in the principle embedding with the L_as denoting the $sl(2,\mathbb{R})$. The *eom* on the gauge fields yield a to be of the holomorphic form, while the components of $\tilde{a}$ can be solved in terms of each other yielding chiral W_3 induced gravity Ward identities. On the left gauge field we impose the usual constraints prevalent in literature [7, 14] which would yield a copy of W_3 algebra. The boundary term needed for the right gauge

[1]We restrict ourselves to the case of spins=2,3 although these can be generalized in a similar fashion to include arbitrary higher spins.

[2]The small case a and $\tilde{a}$ have their radial dependence gauged away as in (10).

field is:

$$\begin{aligned} S_{bndy} &= \frac{k}{4\pi}\int d^2x\, tr(-L_0[\tilde{a}_+,\tilde{a}_-] + 2\tilde{\kappa}_0 L_1\tilde{a}_+ + \frac{1}{2\alpha}W_0\{\tilde{a}_+,\tilde{a}_-\}) + \frac{1}{3}\tilde{a}_+\tilde{a}_- + 2\tilde{\omega}_0 W_2\tilde{a}_+), \\ \implies \delta S_{total} &= -\frac{k}{2\pi}\int d^2x\,[(\tilde{\kappa}-\tilde{\kappa}_0)\delta f^{(-1)} + 4\alpha^2(\tilde{\omega}-\tilde{\omega}_0)\delta g^{(-2)}] \end{aligned} \tag{13}$$

We impose $\tilde{\kappa} = \tilde{\kappa}_0$ and $\tilde{\omega} = \tilde{\omega}_0$ thus allowing for fluctuations of $f^{(-1)}$ and $g^{(-2)}$ which are the most leading order components in the radial co-ordinate r. Along with the left copy of W_3 algebra we get different affine Kaĉ-Moody algebras depending on the choice of parameters $\tilde{\kappa}_0$ and $\tilde{\omega}_0$:

- $\boldsymbol{\tilde{\kappa}_0 = 0}$, $\boldsymbol{\tilde{\omega}_0 = 0}$:
  This choice of the parameters yields an affine Kaĉ-Moody of $sl(3,\mathbb{R})$ at level $k = \frac{c}{6}$ and describes a field theory on a boundary with a non-compact spatial direction.

- $\boldsymbol{\tilde{\kappa}_0 = \frac{-1}{4}}$, $\boldsymbol{\tilde{\omega}_0 = 0}$:
  This choice of the parameters yields an affine Kaĉ-Moody of $su(1,2)$ at level $k = \frac{c}{6}$ with the boundary having a spatial circle. Both $sl(3,\mathbb{R})$ and $su(1,2)$ are non-compact real forms of $sl(3,\mathbb{C})$[3].

- $\boldsymbol{\tilde{\kappa}_0 \neq 0}$, $\boldsymbol{\tilde{\omega}_0 \neq 0}$:
  In this case one can make a choice of currents such that one gets a $u(1)\times u(1)$ affine Kaĉ-Moody as an asymptotic symmetry algebra. This generalizes results of [6] to the case of hs.

The asymptotic symmetries and the Ward identities make it apparent that a suitable candidate for the duals of such bulk configurations are chiral induced W_3 gravities. W-gravity is the higher-spin analogue of gravity in 2d based on the underlying W-algebra[15, 16]. Here, W-algebra plays a similar role as that of Virasoro algebra in pure 2d gravity[4]. We

[3]This ambiguity doesn't arise in pure gravity since the non-compact real forms of $sl(2,\mathbb{C})$ are $sl(2,\mathbb{R}) \equiv su(1,1)$

[4]It is worthwhile to point out that although 2d gravity admits a higher-spin extension with an underlying W-algebra, this is not a Lie algebra in the usual sense; the commutator of 2 generators generally consists

will concern ourselves to W_3 gravity in this text.

The induced (effective) action for pure 2d gravity in chiral gauge was shown to be derived from a constrained $sl(2,R)$ WZNW system[17, 18, 19, 20]. It was then derived to all orders in c. This approach made the hidden $sl(2,R)$ symmetry in the induced gravity manifest. An identical approach was used to find the induced W_3 gravity action to all orders in the chiral gauge [21]. It was obtained by constraining the $sl(3,R)$ WZNW field theory which can be further regarded as a reduced $sl(3,R)$ Chern-Simons theory in 3d.

0.4 Chiral boundary conditions for supergravity

It would be useful to see whether the boundary conditions in (0.2) can be extended to include minimal supergravity in $AlAdS_3$ spaces. This would ba a first step in realising such boundary conditions in the context of string theory. Analogues work was done for the case of Brown-Henneaux boundary conditions in [22, 23]. Extended $AlAdS_3$ supergravity were considered in [2] and their corresponding duals were proposed along the lines of [12]. Since the number of fields increases in supergravity, it would also be worthwhile to find the most general minimal supergravity extension which admits a bosonic truncation down to the chiral case.

$N = (1,0)$ and higher supergravity in AdS_3

The superalgebra for supergravity in locally AdS_3 with $N = (1,0)$ is $osp(1|2) \times sl(2,\mathbb{R})$. Here we show that there exists a unique extension of the chiral boundary condition to include fluctuations of the Rarita-Schwinger field on the boundary. The analysis in second order formalism is quite cumbersome as mentioned previously and we therefore analyse

of composites of generators. The restriction to W_3 gravity on the other hand has only linear and quadratic terms.

it in the first order formalism. A large part of the analysis done in section (0.3) in the first order formalism can be repeated, including the boundary terms to be added, since the only difference here would be the change in the gauge algebra from $sl(2,\mathbb{R})$ to $osp(1|2)$.

We show here that the asymptotic symmetry algebra consists of a Virasoro and an affine $osp(1|2)$ Kaĉ-Moody algebra with level $k = \frac{c}{6}$. The above generalizations of chiral boundary condition can be easily extended to the $N = (1,1)$ and more generally to the $N = (p,q)$ case in the first order formalism. When Dirichlet boundary conditions were imposed on these theories [22, 23], two copies of the super-Virasoro were found as the asymptotic symmetry algebra. It turns out that for the bosonic sector to be containing configurations corresponding to chiral boundary conditions of (0.2), it is enough to change one of the gauge fields ($\tilde{A}$ for ex.) to obey conditions similar to (0.3), while imposing Dirichlet (BH generalizations) boundary condition on the other (A). Then, the asymptotic symmetry algebra of A turns out to be the super extension of Virasoro while that obtained from $\tilde{A}$ gives rise to an affine Kaĉ-Moody algebra of the relevant super algebra under consideration. These boundary conditions on $N = (1,0)$ supergravity must correspond to the minimal supersymmetric extension of Polyakov's induced gravity in 2d analysed by [24]. The case for 2d chiral induced $N = (1,1)$ and generic $N = (p,q)$ supergravity was addressed in [25].

0.5 Conclusion

The above analysis shows that relaxing boundary conditions on $AlAdS_3$ in a systematic way does enrichen the AdS/CFT holography in relating it to different possible CFTs. It is known in AdS_d for $d > 3$, the Dirichlet(BH) boundary conditions only yield the global conformal transformations of the $d-1$ boundary. Infinite dimensional symmetries are know to exist in 3d and 4d asymptotically flat Minkowski spaces, known as the BMS group. The above analysis can be used as an indication that relaxing boundary conditions in these cases might yield new asymptotic symmetries and corresponding dual CFTs on the AdS boundary. This in turn may shed light on certain hidden infinite dimensional symmetries of these CFTs – previously known or otherwise, of the kind envisaged by Bershadsy and Ooguri [20, 26]. In a similar spirit it would be interesting to see how these considerations are realized in the context of string theory on AdS sapces, where infinite dimensional symmetries - like the Yangian, are know to occur in the CFTs.

The thesis would be organized as follows:

1. The first chapter would comprise of a brief review of $2d$ induced gravity theories.
2. The second chapter would summarize results in the second order formulation.
3. The third chapter would analyze the implications of different boundary conditions in the first order formalism and would include results in hs and supergravity.
4. The fourth chapter would discuss possible implications of the results stated and open problems.

Chapter 1

Introduction

The AdS/CFT correspondence has emerged as one of the most powerful tools from string theory in the past one and a half decades. In its strongest form, it proposes a duality between string theory (supergravity) on a product of $(d + 1)$ dimensional Anti-de Sitter space times a compact manifold ($AdS_{d+1} \times \Sigma_{compact}$) and the large N limit of certain CFT$_d$ living on its boundary. The well studied examples of this duality include

- $\mathcal{N} = 4$, $d = 4$ $SU(N)$ Yang-Mills theory $\leftrightarrow$ type IIB string theory on $AdS_5 \times S^5$,
- $\mathcal{N} = 6$, $d = 3$ $U(N) \times U(N)$ ABJM theory $\leftrightarrow$ type IIA string theory on $AdS_4 \times CP^3$,
- $\mathcal{N} = (4, 4)$, $d = 2$ D1-D5 CFT $\leftrightarrow$ type IIB string theory on $AdS_3 \times S^3 \times T^4$,

and other various generalizations.

This duality maps the symmetries of the two theories to each other. The symmetries vacuum of the CFT$_d$ gets mapped to the symmetries of the maximally symmetric global AdS_{d+1} space, whereas, the global symmetries of the CFT$_d$ are mapped to the appropriately defined asymptotic symmetries of the theory in $Asd_{d+1} \times \Sigma$ space. Much of the richness of this duality stems from some interesting properties of the AdS space itself. The first statement can be understood from the fact that the Killing symmetries of AdS_{d+1} - $SO(2, d)$, are exactly the conformal symmetries of the boundary - $\mathbb{R} \times S^{d-1}$ where the

CFT lives.

According to the AdS/CFT correspondence, there exists an operator O_Φ on boundary CFT for every field Φ in the bulk supergravity. The boundary value $\Phi_{(0)}$, of the bulk field, Φ can be identified as a source which couples to the operator O_Φ. The bulk partition function is then a functional of the boundary values $\{\Phi_{(0)}\}$ of the bulk fields $\{\Phi\}$. Under this duality, the bulk supergravity partition function is equal to the generating function of the correlation functions in the conformal field theory at spatial infinity ∂AdS [27].

$$Z_{sugra}[\{\Phi[0]\}] \simeq \langle exp\left(-\int_{\partial AdS}\sum_{\{\Phi\}}\Phi_{(0)}O_\Phi\right)\rangle_{\rm CFT} = Z_{\rm CFT}[\{\Phi_{(0)}\}] \tag{1.1}$$

When the bulk theory is weakly coupled, the *lhs* in the above equation can be approximated by the leading contribution from the on-shell bulk configurations $\{\Phi\}_{cl}$, which solve the classical equations of motion with the boundary values being $\{\Phi_{(0)}\}$, yielding

$$-S_{sugra}[\{\Phi\}_{cl}]\Big|_{\{\Phi_{(0)}\}} \simeq W_{\rm CFT}[\{\Phi_{(0)}\}] = -\log Z_{\rm CFT}[\{\Phi_{(0)}\}]. \tag{1.2}$$

The conformal dimension of the operators $\{O_\Phi\}$ in the CFT are given in terms of the masses and spins of the fields $\{\Phi\}$ in the supergravity in AdS[1]. Boundary conditions imposed on the bulk gravity theory play an important role as the duality requires one to specify boundary conditions at the spatial infinity of the AdS space, which in particular fixes the CFT to which its is dual to.

1.0.1 Brown-Henneaux analysis

A famous illustration of the relation between the boundary conditions on AdS_{d+1} space and global symmetries of the CFT_d was carried out in $d = 3$ by Brown and Henneaux [1]. Here although the Killing isometeries of AdS_3 form an SO(2,2), the conformal isome-

[1] We use the phrases (super)gravity with negative cosmological constant and (super)gravity in AdS space interchangeably.

tries of the boundary metric η_{ab} are given by two copies of the infinite dimensional Witt algebra.

$$ds^2 = \eta_{ab}dx^a dx^b = -dx^+ dx^- = -d\tau^2 + d\phi^2,$$

$$[L_m, L_n] = (m-n)L_{m+n} \qquad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n},$$

$$\text{where,} \quad L_n = e^{inx^+}\partial_+ \qquad \bar{L}_n = e^{inx^-}\partial_-, \tag{1.3}$$

where $x^\pm = \tau \pm \phi$. They were interested in finding the space of solutions to AdS_3 gravity which would include the maximally symmetric global AdS_3

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + r^2\left[-dx^+dx^- - \tfrac{\ell^2}{4r^2}(dx^{+2} + dx^{-2}) - \tfrac{\ell^4}{32r^4}dx^+dx^-\right], \tag{1.4}$$

of AdS radius ℓ and reproduce the above algebra from the bulk as an asymptotic symmetry algebra. To this end they demanded that all solutions to Einstein's equation with negative cosmological constant in the bulk have the same induced metric at spatial infinity. This amounted to imposing a Dirichlet boundary condition on the bulk geometries. The on-shell variation of the bulk Einstein-Hilbert action reads:

$$S = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\,\sqrt{-g}\left(R + \frac{2}{\ell^2}\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}\,\Theta + \frac{1}{8\pi G}S_{\text{ct}}(\gamma_{\mu\nu}),$$

$$\delta S = \int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}T^{ab}\delta\gamma_{ab}, \tag{1.5}$$

where T^{ab} is the Brown-York stress tensor defined on the co-dimension one time-like surface orthogonal to the radial direction r and γ_{ab} is the induced metric on the boundary. Here, S_{ct} contains counter terms that make the on-shell action finite and are determined through a procedure of holographic renormalization first outlined in [28]. Imposing Dirichlet boundary condition implied $\delta\gamma_{ab} = 0$ on the space of allowed solutions. The boundary fall-off conditions thus obtained are:

$$
\begin{aligned}
g_{rr} &= \frac{\ell^2}{r^2} + O(\frac{1}{r^4}) \\
g_{ra} &\approx O(\frac{1}{r^3}) \\
g_{ab} &= r^2 \eta_{ab} + O(r^0),
\end{aligned} \tag{1.6}
$$

since termed as the Brown-Henneaux boundary conditions[2]. The space of diffeomorphisms which respects these fall-off conditions are generated by:

$$
\begin{aligned}
\xi^r &= r\,R(\tau,\phi) + \frac{\ell^2}{r}\bar{R}(\tau,\phi) + O(\frac{1}{r^3}), \\
\xi^\tau &= T(\tau,\phi) - \frac{\ell^2}{2r^2}\partial_\tau R(\tau,\phi) + O(\frac{1}{r^4}), \\
\xi^\phi &= \Phi(\tau,\phi) + \frac{\ell^2}{2r^2}\partial_\phi R(\tau,\phi) + O(\frac{1}{r^4}),
\end{aligned} \tag{1.7}
$$

where $R = \partial_t T = \partial_\phi \Phi$, $\partial_t \Phi = \partial_\phi T$. These are termed as asymptotic Killing vectors since they leave the boundary metric unchanged. These of course are unique only upto those vector fields which are sub-leading in r *i.e.* begin at $O(1/r)$. The commutator defined via Lie derivative action of these vector fields obeys the Witt algebra. Brown and Henneaux computed the central extension to this algebra by computing the change in the asymptotic charge under such diffeomorphisms and uncovered two copies of Virasoro algebra:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0},
\end{aligned} \tag{1.8}
$$

with $c = \frac{3\ell}{2G}$. Here ℓ is the AdS length and G is the three dimensional Newton's constant. Conserved currents of a two dimensional CFT are generally expected to obey the above algebra where the central extension c is the central charge of the CFT. This was used to derive the Bekenstein-Hawking entropy for the BTZ black hole by using the Cardy

[2]Here we denote the indices corresponding to the boundary directions by lower case Roman alphabets a, b. The boundary metric γ_{ab} is taken to be the flat space metric η_{ab}

formula for the asymptotic growth of states in a CFT in the large central charge limit. Which in turn implies that the size of AdS_3 is much larger than the three dimensional Newton's constant; $\ell >> G$.

1.0.2 Generalizing the Brown-Henneaux analysis

The Brown-Henneaux boundary conditions have since been adapted to different settings in the context of AdS/CFT. Most notable works being the ones studying supergravity with negative cosmological constant in three dimensions [22, 23, 2], and more recently in the study of higher-spin gauge fields in AdS_3, which are conjectured to be duals to the coset models proposed by Gaberdiel and Gopakuamar [29]. In these cases, generalizations of the Virasoro have been obtained as the asymptotic symmetry algebra in d=3 dimensions.

The above works rely heavily on the fact that three dimensional AdS gravity can be cast as a difference of two Chern-Simons theories at level $k = \frac{\ell}{4G}$,

$$\begin{aligned} S_{AdS_3} &= S_{CS}[A]\big|_k - S_{CS}[\tilde{A}]\big|_k, \\ S_{CS}[A]\big|_k &= \frac{k}{4\pi}\int \mathrm{tr}(A \wedge A + \frac{2}{3} A \wedge A \wedge A), \end{aligned} \tag{1.9}$$

defined over a manifold $\Sigma_{disk} \times \mathbb{R}$, where $\mathbb{R}$ is the time co-ordinate. Here, the fields in the bulk are a derived concept as the vielbein and the spin connections are given in terms of the gauge fields as

$$\omega = \frac{1}{2}(A + \tilde{A}) \ \ \& \ \ e = \frac{\ell}{2}(A - \tilde{A}). \tag{1.10}$$

For the case of pure gravity, the gauge fields are valued in the adjoint of $sl(2,\mathbb{R})$. The equations of motion in this first order formalism translate to flatness of gauge connections for the two gauge fields. The Brown-Henneaux analysis can be done in this formalism too. Here, one imposes boundary fall-off conditions on the gauge fields such that the metric obeys the fall-off conditions of Brown-Henneaux,

$$a = [L_1 - \kappa(x^+)\, L_{-1}]\, dx^+, \qquad \tilde{a} = [-L_1 + \tilde{\kappa}(x^-)\, L_1]\, dx^-,$$

$$A = b^{-1}(d + a)b, \qquad \tilde{A} = b(d + \tilde{a})b^{-1},$$

$$\text{where,} \quad b = e^{L_0 \ln\left(\frac{r}{\ell}\right)} \qquad [L_m, L_n] = (m - n)L_{m+n},\ \ m, n \in \{-1, 0, 1\}. \quad (1.11)$$

The two copies of Virasoro algebra manifests themselves in the modes of $\kappa(x^+)$ and $\tilde{\kappa}(x^-)$ with the same central extension. In terms of the Poisson brackets fro the right-moving Virasoro look like

$$\{\kappa(x^+), \kappa(\tilde{x}^+)\} = -\kappa'(x^+)\, \delta(x^+ - \tilde{x}^+) - 2\, \kappa\, (x^+)\, \delta'(x^+ - \tilde{x}^+) - \tfrac{k}{4\pi}\, \delta'''(x^+ - \tilde{x}^+). \qquad (1.12)$$

The first order formalism is useful in that allowing the gauge algebra to contain $sl(2, \mathbb{R})$ as a subset of a bigger algebra leads to theories with other gauge fields coupled to gravity in $d = 3$ dimensions. As an example, supergravities in AdS_3 were studied to generalize Brown-Henneaux boundary conditions by Henneaux *et al* [2]. This was done by replacing the $sl(2, \mathbb{R})$ gauge group with a $\mathbb{Z}_2$ graded algebra for the $sl(2, \mathbb{R}) \oplus \tilde{G}$ bosonic algebra, where $\tilde{G}$ is the internal bosonic symmetry.

The guiding principle in the analysis of Henneaux *et al* [2] was to impose similar Dirichlet type boundary conditions on the supergravity fields such that the bulk metric is still asymptotically locally AdS_3 ($AlAdS_3$). This was done by demanding that the gauge fields obey the following fall of conditions[3]

$$\begin{aligned} \Gamma &= bdb^{-1} + bab^{-1}, \quad \tilde{\Gamma} \;=\; b^{-1}db + b^{-1}\tilde{a}b, \\ \text{where } b &= e^{\sigma^0 \ln(r/\ell)}, \\ a &= [\sigma^- + L(x^+)\sigma^+ + \psi_{+\alpha+}(x^+)R^{+\alpha} + B_{a+}(x^+)T^a]\, dx^+, \end{aligned}$$

[3]The conventions and notations are taken from chapter 3.

$$\tilde{a} \;=\; \Big[\sigma^{+} + \bar{L}(x^{-})\sigma^{-} + \bar{\psi}_{-\alpha-}(x^{-})R^{-\alpha} + \bar{B}_{a-}(x^{-})T^{a}\Big]dx^{-}. \tag{1.13}$$

These boundary conditions, which were extensions of the Brown-Henneaux boundary conditions to the supergravity case, uncovered two copies of super-Virasoro *i.e.* the extended super-conformal algebra as the asymptotic symmetry algebra[4]. This algebra, unlike the Virasoro, contains quadratic non-linearities in currents. There were other works which had reproduced special cases of the above result [22, 23].

In a similar light, one can obtain a bulk theory of higher-spin gauge fields coupled to gravity in AdS_3 by demanding that the two Chern-Simons gauge fields be valued in the adjoint of $sl(N,\mathbb{R})$. Here, one would end up with a bulk theory for gauge fields with spins ranging from $2, ...N$, with the spin-2 gauge field being the graviton.

The asymptotic symmetry of higher-spin gauge fields coupled to gravity in $d = 3$ dimensions was first done by Campoleoni *et al* [7], where they restricted to spin 3 fields coupled to gravity yielding 2-copies of classical W_3 algebra first written down by Zamalodchikov [15]. Here too, the authors were concerned only with finding the asymptotic symmetry algebra such that all the solutions admitted by the boundary conditions have a metric (spin-2 field) which is $AlAdS_3$. The boundary conditions are given as fall-off conditions on the gauge fields, as before

$$\begin{aligned} A = b^{-1}ab + b^{-1}db \qquad & \tilde{A} = b\tilde{a}b^{-1} + bdb^{-1}, \\ a = (L_1 - \kappa L_{-1} - \omega W_{-2})dx^{+} \qquad & \tilde{a} = (-L_{-1} + \tilde{\kappa}L_1 + \tilde{\omega}W_2)dx^{-}, \\ b = e^{L_0 ln\frac{r}{l}}, \qquad & \end{aligned} \tag{1.14}$$

where the gauge fields are valued in the adjoint of $sl(3,\mathbb{R})$ listed in the appendix 6.5. Here one finds that the gauge field a only depends on x^{+} and as a 1-form has a component only along the same coordinate. The same is true for $\tilde{a}$ and its x^{-} dependence. The analysis

[4]This analysis is contained in part in chapter 3.

is the same for either of the gauge fields. The spin fields are recovered from these gauge fields as

$$\begin{aligned} g_{\mu\nu} &= \frac{1}{2}Tr(e_\mu e_\nu) \qquad & \varphi_{\mu\nu\rho} &= \frac{1}{3!}Tr(e_{(\mu}e_\nu e_{\rho)}), \\ e &= \frac{\ell}{2}(A - \tilde{A}) \qquad & \omega &= \frac{1}{2}(A + \tilde{A}). \end{aligned} \tag{1.15}$$

Here, they uncovered two copies of the classical W_3 algebra as the asymptotic symmetry algebra, one each for the two gauge fields A and $\tilde{A}$. This algebra is represented by Poisson brackets between the functions parametrizing the phase-space of solutions *i.e.* for instance κ and ω for the gauge field A

$$\begin{aligned} -\tfrac{k}{2\pi}\{\kappa(x^+),\kappa(\tilde{x}^+)\} &= -\kappa'(x^+)\,\delta(x^+-\tilde{x}^+) - 2\,\kappa(x^+)\,\delta'(x^+-\tilde{x}^+) + \tfrac{1}{2}\,\delta'''(x^+-\tilde{x}^+), \\ -\tfrac{k}{2\pi}\{\kappa(x^+),\omega(\tilde{x}^+)\} &= -2\,\omega'(x^+)\,\delta(x^+-\tilde{x}^+) - 3\,\omega(x^+)\,\delta'(x^+-\tilde{x}^+), \\ -\tfrac{2k\alpha^2}{\pi}\{\omega(x^+),\omega(\tilde{x}^+)\} &= \tfrac{8}{3}[\kappa^2(x^+)\,\delta'(x^+-\tilde{x}^+) + \kappa(x^+)\,\kappa'(x^+)\delta(x^+-\tilde{x}^+)] \\ &\quad -\tfrac{1}{6}[5\kappa(x^+)\delta'''(x^+-\tilde{x}^+) + \kappa'''(x^+)\delta(x^+-\tilde{x}^+)] \\ &\quad -\tfrac{1}{4}[3\kappa''(x^+)\delta'(x^+-\tilde{x}^+) + 5\kappa'(x^+)\delta''(x^+-\tilde{x}^+)] + \tfrac{1}{24}\delta^{(5)}(x^+-\tilde{x}^+) \end{aligned} \tag{1.16}$$

Here, the Virasoro is a part of the above algebra. Also the W_3 algebra is not a Lie algebra in the conventional sense since the *rhs* does contain quadratic non-linearities. The procedure for obtaining the above algebra from the boundary conditions imposed is outlined in detail in chapter 4. As mentioned earlier, these boundary conditions are generalizations of the ones imposed by Brown and Henneaux in that for the above fall-off conditions on the gauge fields 1.14, the bulk solutions are $AlAdS_3$. In other words, the spin-3 field $\varphi_{\mu\nu\rho}$ doesn't survive till the boundary of AdS_3 while $g_{\mu\nu}$ contains all the configurations allowed by the Brown-Henneaux boundary condition.

The case of the higher-spin (hs) theory with spins ranging from 2....N with $N \to \infty$ in AdS_3 was dealt by Henneaux and Rey [14] where 2-copies of $\mathcal{W}_\infty$ were uncovered as the asymptotic symmetry algebra. The gauge group in the bulk is $hs[\lambda] \times hs[\lambda]$, which for suitable values of λ is equivalent to an $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$.[5] This as mentioned earlier, describes fields with spins ranging from 2 to N. The asymptotic symmetry group in such cases was shown to be two copies of W_N algebra [7, 14]. These are the generalizations of Virasoros to the higher-spin case with a similar peculiarity of having non-linearities in the right hand sides of the algebra for finite N. Although, these analyses were similar in spirit to the ones done by Brown and Henneaux [1], since they always demanded $AlAdS_3$ configurations; they differed in that they used the Chern-Simons formulation of gravity coupled to higher-spin fields in three dimensions as the second order formulation of the action is not yet completely known.

1.0.3 Deviations from the Brown-Henneaux type boundary conditions

We would now like to summarize some works which have required deviations from the Brown-Henneaux type boundary conditions in their analysis. Since- as already mentioned at the very beginning, boundary conditions play a major role in the AdS/CFT correspondence, it is worthwhile to see what kind of duality one unearths when one tries to impose different boundary conditions from that of Brown and Henneaux's.

In all the cases mentioned above, there is always a symmetry between the left and the right sectors. But there have been interesting examples of CFTs where this symmetry is broken, and further under the AdS/CFT correspondence these have been dual to interesting geometries in the bulk. There have been several works relating the statistical degeneracy of an extremal black hole to a thermal ensemble of a 1+1 dimensional chiral CFT [30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. It was observed that the near horizon ge-

[5] This requires moding the gauge algebra in the bulk by a suitable ideal.

ometry of many extremal black holes in many dimensions, in both flat and AdS spaces, contains an AdS_2 factor with a constant electric field [40]. A consistent application of the AdS/CFT conjecture leads to a dual theory in 1+1 dimensions which is a Discrete Light Cone Quantized (DLCQ) CFT [34, 35, 36, 37][41]. This is a chiral CFT with only one copy of Virasoro, where the states in the CFT are not charged with respect to the other Virasoro. Here, three dimensional extremal BTZ plays a crucial role since it appears in the near-horizon metric of many extremal black holes in asymptotically flat and AdS spaces,

$$\begin{aligned} \ell^{-2} ds^2_{BTZ} &= \frac{dr^2}{r^2} - r^2 dx^+ dx^- + \frac{(\ell M + J)}{2k}(dx^+)^2 + \frac{(\ell M - J)}{2k}(dx^-)^2 \\ & \quad - \frac{(\ell^2 M^2 - J^2)}{4k^2 r^2} dx^+ dx^-, \\ \text{where} \quad & k = \frac{\ell}{4G}, \qquad ds^2_{BTZ-extremal} = ds^2_{BTZ}\Big|_{\ell M = J}. \end{aligned} \tag{1.17}$$

The DLCQ CFT mentioned above is obtained by applying a dimensional reduction on the CFT dual to this BTZ [41]. The dual to this would be an AdS_2 with an electric flux [35] obtained from an AdS_3 as a $U(1)$ fibration over an AdS_2 base. The Virasoro of the chiral CFT was then obtained by showing that a consistent set of boundary conditions existed which enhanced this $U(1)$ isometery to a Virasoro [41]. Here, a chiral version of the Brown-Henneaux boundary conditions were imposed breaking the left-right symmetry. These analyses established that the DLCQ chiral CFTs are a feature of the extremal black holes since non-extremal black holes do not have an AdS_2 throat.

There have been some works analysing from the bulk perspective what happens when one tries to leave the domain of boundary conditions described by Brown and Henneaux. A work by Compère and Marolf [8] analysed the effect of allowing the boundary metric of the AdS space to fluctuate. The effect of completely freeing the boundary metric would require holding the Brown-York stress tensor fixed to $T^{ab} = 0$, as can be seen from the variation of the bulk action (1.5). This would correspond to imposing completely Neumann boundary conditions on the bulk gravitational dynamics in stead of the completely

Dirichlet ones, as done by Brown and Henneaux. The CFT on the boundary would be a theory of "induced gravity" where the boundary CFT partition function involves integrating over all possible fluctuations of the boundary metric.

$$\mathcal{Z}_{induced} := \int \mathcal{D}g^{(0)} \mathcal{Z}_{CFT}[g^{(0)}], \tag{1.18}$$

where $\ln \mathcal{Z}_{CFT}[g^{(0)}]$ defines the effective action for $g^{(0)}_{ab}$ after integrating out all the fields in the CFT. They further showed that such corresponding geometries in the bulk, described by $T^{ab} = 0$, are normalizable with respect to a simplectic structure defined by the full bulk action. Here, the contribution from the appropriate boundary counter term S_{ct} is included. The additional S_{ct} is added to make the variational principle well defined. This term plays a crucial role in that it minimises the on-shell bulk action for those bulk configurations which are allowed by the boundary conditions imposed. The authors of [8] reached to similar conclusions when a mix of Dirichlet and Neumann boundary conditions, also termed as Robin boundary conditions were imposed on the boundary metric. This allowed for certain components of the boundary metric $g^{(0)}$ to be held fixed while allowing the others to fluctuate. Again, this was achieved after suitable S_{ct} was added in accordance with the variational principle.

In [6], Compère *et al* introduced new boundary conditions for AdS_3 where they uncovered a copy of Virasoro with a central charge $c = 3\ell/2G$ along with an affine $u(1)$ Kaĉ-Moody algebra.[6] These boundary conditions are chiral in nature, in that they do not respect the left-right symmetry by allowing a component of the boundary metric $g^{(0)}_{++}$ to fluctuate along the x^+ boundary direction.

$$\begin{gathered} g_{rr} = \frac{l^2}{r^2} + O(r^{-4}), \quad g_{r\pm} = O(r^{-3}), \\ g_{+-} = -\frac{r^2}{2} + O(r^0), \quad g_{++} = r^2 f(x^+) + O(r^0), \quad g_{--} = -\frac{l^2}{4}N^2 + O(r^{-1}), \end{gathered} \tag{1.19}$$

[6]Since it is affine $u(1)$ current, the value of its level can always be absorbed by scaling the currents with real constants, but its sign remains unchanged.

A suitable boundary term was added to the bulk Einstein-Hilbert action so as to allow for the bulk solutions with such fall-of conditions be variationally consistent. They further construct the CFT dual to their boundary conditions by writing the AdS_3 gravity action in terms of difference of two Chern-Simons theories valued in $sl(2,\mathbb{R})$ (6.3) and interpreting their boundary conditions as constraints on WZNW theory dual to the Chern-Simon theory. This analysis is similar to the work done by Coussaert *et al* [12] in the context of Brown-Henneaux boundary conditions. Here they found a Liouville theory obtained from a constrained WZNW theory. On the heels of [6] was a paper by Troessaert [5] which allowed the boundary metric to fluctuate upto a conformal factor but restricted the boundary metric to be conformally flat.

$$\begin{aligned} ds^2 &= -e^{\Phi(x^+,x^-)}dx^+dx^-, \\ \text{where,} \qquad \Phi(x^+,x^-) &= \phi(x^+)+\phi(x^-). \end{aligned} \tag{1.20}$$

The asymptotic analysis of this theory revealed two copies of affine $u(1)$ Kaĉ-Moody $\times$ Virasoro as the asymptotic algebra. Here the central extension of the Virasoro sector from both halves is $c = 3\ell/2G$, while the two affine $u(1)$ currents have a level $k = -c/6$, implying that the total central charge vanishes. Although as pointed out earlier[6] that the affine $u(1)$ levels can be scaled by scaling the $u(1)$ currents.

In both the cases mentioned above there were boundary terms added to the bulk gravitational action so as to allow the required boundary metric component to fluctuate. The resultant dual CFT_2 theory would be an induced gravity theory as mentioned in [8] and the boundary terms are useful in defining a normalizable theory in the bulk. The induced gravity theory is an effective theory of these fluctuating boundary gravity modes which are obtained by first coupling them to a boundary CFT_2 and then integrating out the fields of the original CFT_2. The concept of induced gravity was first introduced in 1967 by Andrei Sakharov [42]. Here a psuedo-Riemannian manifold is considered as a background in which there are matter fields. The gravitational dynamics is not imposed but emerges at one loop order when the matter fields are integrated over. This procedure was shown to produce the Einstein-Hilbert term with a large cosmological constant along with higher derivative terms. The induced gravity action that arises in the context of AdS_3/CFT$_2$ would be the ones in which the original matter theory is a CFT. We would refer to this as induced gravity in this thesis.

Next, we give a brief summary of works that were carried out more than two decades ago when the subject of induced gravity was in vogue.

1.1 Induced gravity

A conformal field theory (CFT) in a background space-time with flat metric η_{ab} admits a spin-2 current namely the energy-momentum tensor T_{ab} which is symmetric, traceless and conserved ($\partial^a T_{ab} = 0$). In two dimensions it has two independent components $T_{++}(x^+)$ and $T_{--}(x^-)$ where $x\pm = t \pm x$ are the null coordinates. Classically one can couple the CFT to an arbitrary background metric making it a gravitational theory as well. Such a gravity coupled to matter in two dimensions plays an important role as the world-sheet theory of string theories. Classically this theory would be diffeomorphism and Weyl invariant though quantum mechanically these symmetries may become anomalous. Demanding that the Weyl symmetry is not anomalous constrains the matter content or the possible metric backgrounds. For instance, for a string propagating in the flat n-dimensional space-time the world-sheet theory being Weyl invariant at the quantum level means that the string should propagate in critical dimensions (n = 26) and one can gauge fix the world-sheet metric completely. Gauge fixing the world-sheet metric to the flat metric leaves one with a 2d CFT in flat background whose symmetries are two commuting copies of Virasoro algebra generated by the modes of the conserved stress-energy tensor of the

CFT. However, away from the critical dimension the 2d metric cannot be gauged away because of the Weyl anomaly - leaving one degree of freedom in the metric. Long ago, in a seminal paper [10], Polyakov addressed the problem of quantizing this theory.

When one integrates over the matter sector one obtains a non-local theory [43] of the metric referred to as the induced gravity theory. The covariant manifestation of such an effective action for the now dynamical metric (induced gravity action) is non-local as is made explicit in the expression for induced gravity in 2d arising in the quantization of string world sheet [10, 43],

$$\Gamma = \frac{c}{96\pi} \int d^2x \sqrt{g}(R\frac{1}{\nabla^2}R + \Lambda). \tag{1.21}$$

The induced gravity theory is diffeomorphism invariant but not Weyl invariant as expected. One can use the diffeomorphisms to gauge fix the metric down to one independent component. There are two standard gauge choices used in the literature:

- the conformal gauge: $ds^2 = -e^{\phi(x^+,x)}dx^+dx^-$
- the light-cone gauge: $ds^2 = -dx^+dx + F(x^+, x)(dx^+)^2$

The induced gravity theory becomes local in either of these gauge choices[7]. In the conformal gauge it is known to reduce to the Liouville theory.

$$S_{Liou} = \int d^2x \left(\partial_a\Phi\partial^a\Phi + e^{2\chi\Phi}\right). \tag{1.22}$$

In the light-cone gauge the induced gravity theory is called the chiral induced gravity (CIG) theory. Polyakov examined the CIG and uncovered an sl(2, R) current algebra worth of symmetries of it which in turn led to the determination of all correlation functions in that theory [10]. This was achieved by studying the Ward identity of the stress-tensor component in the chiral gauge. Subsequently people extended this analysis to the case

[7] In the light-cone gauge a further change of variables is required to make the action take a local form. These were termed as Polyakov's variables in the literature.

of $\mathcal{N} = (1,0)$,$\mathcal{N} = (1,1)$ and $\mathcal{N} = (p,q)$ super-gravities in 2d [25, 44]. Here for the chiral gauge[8] analysis of what one may call 2d chiral induced supergravity, the conserved currents obeyed the relevant super-Kaĉ-Moody current algebra.

On the other hand holography through the AdS/CFT correspondence has been a powerful tool to study CFTs. Since global isometries of a 2d CFT are identical to the asymptotic symmetry algebra of Brown-Henneaux, a natural question is whether one can generalize AdS/CFT to include gravity on the CFT side. To generalize the CFT to include gravity one requires to consider boundary conditions that are not Dirichlet type. We provide a solution to this question in chapter 2 where we make explicit the boundary conditions on gravity in AdS_3 which admits an $sl(2,\mathbb{R})$ Kaĉ-Moody current algebra as one of the asymptotic symmetries.

2d Chiral Induced W-gravity

A given conformal field theory in 2d flat space-time can admit more conserved currents than the energy-momentum tensor T_{ab}. The corresponding symmetry algebra should include the two commuting copies of Virasoro algebra. One such enhancement of symmetry algebra involves extending each copy of the Virasoro algebra to a W_N algebra first discovered by Zamolodchikov [15]. It is worthwhile to point out that although 2d gravity admits a higher-spin extension with an underlying W-algebra, this is not a Lie algebra in the usual sense; the commutator of 2 generators generally consists of composites of generators. A CFT with $W_N \oplus W_N$ symmetry (referred to as a WCFT) will have conserved currents given by completely symmetric and traceless rank-s tensors $\mathcal{W}_{a_1 \dots a_s}$ with the spin s ranging from 2 to N (here $\mathcal{W}_{ab} = T_{ab}$). The two independent components of such a traceless spin-s current are $\mathcal{W}_{+\dots+}(x^+)$ and $\mathcal{W}_{-\dots-}(x^-)$ in light-cone coordinates.

One can again couple such a CFT to background given by spin-s gauge fields. One then writes a $W-$covariant action thus promoting the global $W-$symmetries of the original

[8] Also termed as light-cone gauge.

theory to a local one; resulting in a higher spin extension of the 2d gravity referred to as the W-gravity theory (see [45] for a review). Therefore, along with the usual diffeomorphism, Weyl and Lorentz symmetries, W-gravity is supplemented with the W_N analogues of the above symmetries. But these symmetries are only obeyed classically, and at the quantum level some of these symmetries become anomalous; just as seen in the previous section. Again integrating out the WCFT field content will generically induce a dynamical theory for these higher spin fields which is the induced W-gravity[9].

Following Polyakov's work [10] and the discovery of W-symmetries (see [16, 45] for a review) people studied the induced W-gravity theories in a particular light-cone gauge. In this gauge the background spin-s gauge field is coupled only to one of the two independent components of the corresponding W-current, say $\mathcal{W}_{-\cdots-}(x^-)$. This is achieved by considering the action

$$S_{W-gravity} = S_{WCFT} + \int d^2x \sum_{s=2}^{N} \mu^{(s)}_{+\cdots+} \mathcal{W}^{(s)}_{-\cdots-} \tag{1.23}$$

before integrating out the WCFT fields, where S_{WCFT} denotes the action of the WCFT in 2d flat space-time. After integrating out the WCFT field content the resulting theory of $\mu^{(s)}_{+\cdots+}$ fields is dubbed the chiral induced W-gravity (CIWG). This can be done purturbatively in $1/c$ expansion where c was the central charge of the original matter system. It was shown in [20] that these CIWG theories are expected to have $sl(n,\mathbb{R})$-type current algebra symmetries generalizing the $sl(2,\mathbb{R})$ current algebra symmetry of the CIG of Polyakov. The Ward identities are then obtained as the functional differential equation obeyed by the variation of the induced gravity action. These generalize the Virasoro Ward identities of 2d CIG and are also known for quite some time. For details of how these symmetries and Ward identities emerge see, for instance, [20, 21]. It is an interesting question to ask if CIWG theories also admit holographic descriptions.

In this thesis we also generalize the results of chapter 2 and [9] towards describing chiral

[9]This induced gravity action can be written in a covariant manner for the case of pure gravity, as seen in the previous section. But no such W-covariant action is known for the case of induced W-gravity.

induced W-gravities (CIWG) holographically. It is natural to expect that the bulk theory should be a higher spin theory with one higher spin gauge field corresponding to each higher spin field in the induced W-gravity theory of interest. As mentioned before, such 3d theories have a description in terms of Chern-Simons theories with gauge algebra $sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})$ [46]. We therefore expect that the 3d higher spin gauge theory based on $sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})$ Chern-Simons action admits a set of boundary conditions that can describe a suitable chiral induced W-gravity with spins ranging from 2 to n.

We, in particular, provide and study a set of boundary conditions for the case of $n = 3$ and compute their asymptotic symmetry algebras. We verify that these boundary conditions give rise to the W_3 -Ward identities of the CIWG. We find that, in this case, the higher spin theory with our boundary conditions admits one copy of classical $\mathcal{W}_3$ algebra and an $sl(3,\mathbb{R})$ (or an $su(1,2)$) current algebra as its asymptotic symmetry algebra. As a by-product we also provide a generalization of the boundary conditions of [6] to this higher spin theory and compute the corresponding symmetry algebra.

Chapter 2

New boundary conditions in AdS_3 gravity

In his seminal 1987 paper [10], Polyakov provides a solution to the two-dimensional induced gravity theory (such as the one on a bosonic string worldsheet) [43],

$$S = \frac{c}{96\pi} \int d^2x \sqrt{-g}\, R \frac{1}{\nabla^2} R, \tag{2.1}$$

by working in a light-cone gauge. The gauge choice puts the metric into the form

$$ds^2 = -dx^+ dx^- + F(x^+, x^-)(dx^+)^2. \tag{2.2}$$

Polyakov shows that the quantum theory for the dynamical field $F(x^+, x^-)$ admits an $sl(2, \mathbb{R})$ current algebra symmetry with level $k = c/6$. We would like to find the AdS_3 duals which would exhibit essential features of such an induced gravity on the boundary. As espoused in the introduction, we seek this by allowing the $(dx^+)^2$ boundary term to fluctuate.

The action of three-dimensional gravity with negative cosmological constant [47] is given by

$$S = -\frac{1}{16\pi G}\int d^3x\,\sqrt{-g}\left(R+\frac{2}{l^2}\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}\,\Theta + \frac{1}{8\pi G}S_{\rm ct}(\gamma_{\mu\nu}), \tag{2.3}$$

where $\gamma_{\mu\nu}$ is the induced metric and Θ is trace of the extrinsic curvature of the boundary. Varying the action yields

$$\delta S = \int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}\,\frac{1}{2}T^{\mu\nu}\delta\gamma_{\mu\nu}\,, \tag{2.4}$$

where

$$T^{\mu\nu} = \frac{1}{8\pi G}\left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\right]. \tag{2.5}$$

The variational principle is made well-defined by imposing $\delta\gamma_{\mu\nu} = 0$ (Dirichlet) or $T^{\mu\nu} = 0$ (Neumann) at the boundary (see [8] for a recent discussion).

Recently Compère, Song and Strominger (CSS) [48, 6] and Troessaert [5] proposed new sets of boundary conditions for three-dimensional gravity, which differ from the well-known Dirichlet-type Brown–Henneaux boundary conditions [1].[1] Before delving into specifics, let us discuss the general strategy employed by [6]. One begins by adding a term of the type

$$S' = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}\,\frac{1}{2}\mathcal{T}^{\mu\nu}\gamma_{\mu\nu} \tag{2.6}$$

for a fixed ($\gamma_{\mu\nu}$-independent) symmetric boundary tensor $\mathcal{T}^{\mu\nu}$. The variation of this term is

$$\delta S' = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}\tilde{\mathcal{T}}^{\mu\nu}\delta\gamma_{\mu\nu}, \tag{2.7}$$

[1] In fact, the boundary conditions of [5] subsume those of [1].

where $\tilde{\mathcal{T}}^{\mu\nu} = \mathcal{T}^{\mu\nu} - \frac{1}{2}(\mathcal{T}^{\alpha\beta}\gamma_{\alpha\beta})\,\gamma^{\mu\nu}$. The variation of the total action then gives

$$\delta S + \delta S' = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}(T^{\mu\nu} - \tilde{\mathcal{T}}^{\mu\nu})\delta\gamma_{\mu\nu}. \tag{2.8}$$

Now the boundary conditions consistent with the variational principle depend on $\tilde{\mathcal{T}}^{\mu\nu}$. Generically, this leads to "mixed" type boundary conditions. If for a given class of boundary conditions some particular component of $T^{\alpha\beta} - \tilde{\mathcal{T}}^{\alpha\beta}$ vanishes sufficiently fast in the boundary limit such that its contribution to the integrand in (2.8) vanishes, then the corresponding component of $\gamma_{\alpha\beta}$ can be allowed to fluctuate. Since we want the boundary metric to match (2.2), we would like Neumann boundary conditions for γ_{++}. Therefore we choose $\mathcal{T}^{\mu\nu}$ such that the leading term of T^{++} equals $\tilde{\mathcal{T}}^{++}$ in the boundary limit.

This condition has been imposed in [6], with the addition of an extra boundary term with[2]

$$\mathcal{T}^{\mu\nu} = -\frac{1}{2r^4}N^2 l\delta^\mu_+\delta^\nu_+, \tag{2.9}$$

and the following boundary conditions are imposed on the metric:

$$\begin{gathered} g_{rr} = \frac{l^2}{r^2} + O(r^{-4}),\quad g_{r\pm} = O(r^{-3}), \\ g_{+-} = -\frac{r^2}{2} + O(r^0),\quad g_{++} = r^2 f(x^+) + O(r^0),\quad g_{--} = -\frac{l^2}{4}N^2 + O(r^{-1}), \end{gathered} \tag{2.10}$$

where $f(x^+)$ is a dynamical field and N^2 is fixed constant.[3] These boundary conditions give rise to an asymptotic symmetry algebra: a chiral $U(1)$ current algebra with level determined by N. They also ensure that T_{--} is held fixed in the variational problem, whereas g_{++} is allowed to fluctuate as long as its boundary value is independent of x^-.

In what follows, we show that (2.10) are not the most general boundary conditions consistent with the variational principle and the extra boundary term given by (2.9). For this, we introduce a weaker set of consistent boundary conditions that enhance the asymptotic

[2]The induced metric $\gamma_{\mu\nu}$ differs from $g^{(0)}_{\mu\nu}$ of [6] by a factor of r^2.
[3]To relate to the notation in [6], set $N^2 = -\frac{16G\Delta}{l}$ and $f(x^+) = l^2\partial_+\bar{P}(x^+)$.

symmetry algebra to an $sl(2,\mathbb{R})$ current algebra whose level is independent of N.

2.1 Chiral boundary conditions

In the new boundary conditions, the class of allowed boundary metrics coincides with that of (2.2). Since we want to allow γ_{++} to fluctuate, we keep T_{--} fixed in our asymptotically locally AdS_3 metrics. Therefore, we propose the following boundary conditions:

$$\begin{aligned} g_{rr} &= \frac{l^2}{r^2} + O(\tfrac{1}{r^4}), \;\; g_{r+} = O(\tfrac{1}{r}), \;\; g_{r-} = O(\tfrac{1}{r^3}), \\ g_{+-} &= -\frac{r^2}{2} + O(r^0), \;\; g_{--} = l^2\tilde{\kappa} + O(\tfrac{1}{r}), \\ g_{++} &= r^2 F(x^+, x^-) + O(r^0), \end{aligned} \tag{2.11}$$

where, as above, we take $F(x^+, x^-)$ to be a dynamical field and $\tilde{\kappa}(x^+, x^-)$ fixed. We were motivated to study this boundary condition after an analysis of the residual diffeomorphisms of linearised fluctuations of the metric in the covariant gauge (2.2, 6.1). The crucial difference between these boundary conditions and those in (2.10) is the different fall-off condition for g_{r+} which allows for the boundary component of g_{++} to depend on x^- as well. One must, of course, check the consistency of these conditions with the equations of motion. This involves constructing the non-linear solution in an expansion in inverse powers of r. Working to the first non-trivial order, one finds the following condition on $F(x^+, x^-)$:

$$2\,(\partial_+ + 2\,\partial_- F + F\,\partial_-)\,\tilde{\kappa} = \partial_-^3 F\,. \tag{2.12}$$

The above equation may be recognised as the Virasoro Ward identity of Polyakov[10] expected from the 2d CIG. This Ward identity is integrable. To find the solution, inspired by Polyakov, let us parametrize $F = -\frac{\partial_+ f}{\partial_- f}$. With this parametrization one can show that the above constraint (2.12) can be cast into the following form:

$$(\partial_- f\,\partial_+ - \partial_+ f\,\partial_-)\Big[4\,(\partial_- f)^{-2}\,\tilde{\kappa} - (\partial_- f)^{-4}\,[3\,(\partial_-^2 f)^2 - 2\,\partial_- f(\partial_-^3 f)]\Big] = 0 \tag{2.13}$$

For an arbitrary $f(x^+, x^-)$ the general solution to this equation is

$$\tilde{\kappa}(x^+, x^-) = \frac{1}{4}G[f](\partial_- f)^2 + \frac{1}{4}(\partial_- f)^{-2}\,[3\,(\partial_-^2 f)^2 - 2\,\partial_- f(\partial_-^3 f)] \tag{2.14}$$

where $G[f]$ is an arbitrary functional of $f(x^+, x^-)$. The second term in the solution may be recognized as the Schwarzian derivative of f with respect to x^-.

Along with this solution (2.14) for $\tilde{\kappa}$ the configurations in (2.34, 2.35, 2.36) provide the most general solutions consistent with the boundary conditions in (2.11).

The AdS_3 gravity with the boundary conditions (2.11) should provide a holographic description of the 2d CIG with F playing the role of its dynamical field. However, the classical solutions of the 2d CIG should correspond to bulk solutions with $\tilde{\kappa}$ either vanishing or an appropriate non-zero constant. In the latter case one needs to add additional boundary terms to the action (2.3), see [6, 3]. When $\tilde{\kappa} = 0$ one gets the solutions appropriate to asymptotically Poincare AdS_3, where as $\tilde{\kappa} = -1/4$ correspond to the solutions considered in [3].[4]

Now, the boundary term added to the action holds the value of the g_{--} component to be fixed at $-l^2/4$, *i.e.* $\tilde{\kappa} = -1/4$. This implies

$$\partial_- F(x^+, x^-) + \partial_-^3 F(x^+, x^-) = 0, \tag{2.15}$$

which forces $F(x^+, x^-)$ to take the form

$$F(x^+, x^-) = f(x^+) + g(x^+)e^{ix^-} + \bar{g}(x^+)e^{-ix^-} \tag{2.16}$$

[4]Let us note that these solutions also include those of [6] when one takes $\tilde{\kappa} = \Delta$ and $\partial_- F = 0$.

where $f(x^+)$ is a real function and $\bar{g}(x^+)$ is the complex conjugate of $g(x^+)$.

Let us note that this is directly analogous to the form of $F(x^+, x^-)$ derived in [10]. Throughout our discussion we think of $\phi = \frac{x^+ - x^-}{2}$ as 2π-periodic (and $\tau = \frac{x^+ + x^-}{2}$ as the time coordinate), and therefore we restrict our consideration to $\tilde{\kappa} < 0$. Similarly, we impose periodic boundary conditions on $f(x^+)$ and $g(x^+)$. If one takes the spatial part of the boundary to be instead of S^1, there are no such restrictions and one may even consider $\tilde{\kappa} > 0$ like in [6].

The form of eq.(2.12) is exactly the same as the Ward identity obtained in Polyakov's chiral induced gravity. It is interesting to note that this appears as an equation of motion in the bulk and therefore satisfing this Ward identity doesn't imply that one is on-shell with respect to the boundary theory. In stead, it implies that on-shell bulk configurations as described above, are dual to the space of solutions in the boundary theory which satisfy its quantum symmetries. Further, when the required boundary term is added so as to impose the variational principle on this space of solutions, the full bulk action is minimized for only a subset of these bulk on-shell geometries specified by (2.15). Thus reducing the space of solutions, which would now be dual to the boundary on-shell solutions with respect to the induced gravity theory.

2.1.1 The non-linear solution

One can write a general non-linear solution of AdS_3 gravity in Fefferman–Graham coordinates [49] as:

$$ds^2 = \frac{dr^2}{r^2} + r^2 \left[g^{(0)}_{ab} + \frac{l^2}{r^2} g^{(2)}_{ab} + \frac{l^4}{r^4} g^{(4)}_{ab} \right] dx^a dx^b. \tag{2.17}$$

Therefore, the full set of non-linear solutions consistent with our boundary conditions is obtained when

$$\begin{aligned}
g^{(0)}_{++} &= F(x^+, x^-),\;\; g^{(0)}_{+-} = -\frac{1}{2},\;\; g^{(0)}_{--} = 0,\\
g^{(2)}_{++} &= \kappa(x^+, x^-),\;\; g^{(2)}_{+-} = \sigma(x^+, x^-),\;\; g^{(2)}_{--} = \tilde{\kappa}(x^+, x^-),\\
g^{(4)}_{ab} &= \frac{1}{4} g^{(2)}_{ac}\, g^{cd}_{(0)} g^{(2)}_{db}\,,
\end{aligned} \tag{2.18}$$

where in the last line $g^{cd}_{(0)}$ is $g^{(0)}_{cd}$ inverse. Imposing the equations of motion $R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} = 0$ one finds that these equations are satisfied for $\mu, \nu = +, -$. Then the remaining three equations coming from $(\mu, \nu) = (r, r), (r, +), (r, -)$ impose the following relations:

$$\begin{aligned}
\sigma(x^+, x^-) &= \frac{1}{2}[\partial_-^2 F - 2\,\tilde{\kappa}\, F]\\
\kappa(x^+, x^-) &= \kappa_0(x^+) + \frac{1}{2}[\partial_+\partial_- F + 2\,\tilde{\kappa}\, F^2 - F\,\partial_-^2 F - \tfrac{1}{2}(\partial_- F)^2]
\end{aligned} \tag{2.19}$$

and

$$2\,(\partial_+ + 2\,\partial_- F + F\,\partial_-)\,\tilde{\kappa} = \partial_-^3 F\,, \tag{2.20}$$

with the general solution for $\tilde{\kappa}$ given in the last section 2.14. This solution reduces to the one given in [6] when $g(x^+) = \bar{g}(x^+) = 0$, $f(x^+) \to l^2 \partial_+ \bar{P}(x^+)$ and $\tilde{\kappa} \to -\frac{16 G\Delta}{\ell}$. To demonstrate the asymptotic symmetry of the simplest case we will confine ourselves to holding $\tilde{\kappa}(x^+, x^-)$ fixed at $-\frac{1}{4}$; this would imply that global AdS_3 would be part of the solution space. The metric then takes the form

$$\begin{aligned}
g^{(0)}_{++} &= f(x^+) + g(x^+)\, e^{ix^-} + \bar{g}(x^+)\, e^{-ix^-},\;\; g^{(0)}_{+-} = -\frac{1}{2},\;\; g^{(0)}_{--} = 0,\\
g^{(2)}_{++} &= \kappa(x^+) + \frac{1}{2}\left[g^2(x^+)\, e^{2ix^-} + \bar{g}^2(x^+)\, e^{-2ix^-}\right]\\
&\qquad + \frac{i}{2}\left[g'(x^+) e^{ix^-} - \bar{g}'(x^+) e^{-ix^-}\right],\\
g^{(2)}_{+-} &= \frac{1}{4}\left[f(x^+) - g(x^+)\, e^{ix^-} - \bar{g}(x^+)\, e^{-ix^-}\right],\;\; g^{(2)}_{--} = -\frac{1}{4},\\
g^{(4)}_{ab} &= \frac{1}{4} g^{(2)}_{ac}\, g^{cd}_{(0)} g^{(2)}_{db}\,,
\end{aligned} \tag{2.21}$$

where in the last line $g^{cd}_{(0)}$ is $g^{(0)}_{cd}$ inverse. As above, demanding that the solution respects the periodicity of ϕ-direction the functions $f(x^+)$, $g(x^+)$ and $\kappa(x^+)$ to be periodic.

Finally note that the global AdS_3 metric can be recovered by setting $f = g = \bar{g} = 0$ and $\kappa = -\frac{1}{4}$. Our solutions do not include the regular BTZ solutions. However by setting $f = g = \bar{g} = 0$ and $\kappa = -\frac{1}{4} + \Delta$ one can recover an extremal BTZ (with the horizon located at $r^2 = \Delta - 1/4$) solution.

As mentioned in the previous subsection one can take $\tilde{\kappa} > 0$ implying that the boundary spatial coordinate is not periodic. In this case too one can easily work out the the non-linear solution of the form (2.21) with $g(x^+)$ and $\bar{g}(x^+)$ treated as two real and independent functions. However, we will not consider this case further here.

2.1.2 Charges, algebra and central charges

It is easy to see that vectors of the form

$$\begin{aligned}
\xi^r &= -\frac{1}{2}\left[B'(x^+) + iA(x^+)e^{ix^-} - i\bar{A}(x^+)e^{ix^-}\right] r + O(r^0) \\
\xi^+ &= B(x^+) - \frac{l^2}{2r^2}\left[A(x^+)e^{ix^-} + \bar{A}(x^+)e^{-ix^-}\right] + O(\tfrac{1}{r^3}) \\
\xi^- &= A_0(x^+) + A(x^+)e^{ix^-} + \bar{A}(x^+)e^{-ix^-} + O(\tfrac{1}{r})
\end{aligned} \tag{2.22}$$

satisfy the criteria of [50], which allow us to construct corresponding asymptotic charges. If, on the other hand, one demands that the asymptotic symmetry generators ξ leave the space of boundary conditions invariant, one finds the same vectors but with the first sub-leading terms appearing at one higher order for each component. For either set of vectors, the Lie bracket algebra closes to the same order as one has defined the vectors.

Here, $B(x^+)$ and $A_0(x^+)$ are real and $A(x^+)$ is complex; therefore, there are four real, periodic functions of x^+ that specify this asymptotic vector. We take the following basis for the modes of the vector fields:

$$\begin{aligned} L_n &= ie^{i\,n\,x^+}[\partial_+ - \frac{i}{2} n\, r\partial_r] + \cdots \\ T_n^{(0)} &= ie^{i\,n\,x^+}\partial_- + \cdots \\ T_n^{(+)} &= ie^{i(n\,x^+ + x^-)}[\partial_- - \frac{i}{2}\, r\partial_r - \frac{1}{2r^2}\partial_+] + \cdots \\ T_n^{(-)} &= ie^{i(n\,x^+ - x^-)}[\partial_- + \frac{i}{2}\, r\partial_r - \frac{1}{2r^2}\partial_+] + \cdots\,, \end{aligned} \tag{2.23}$$

which satisfy the Lie bracket algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)\, L_{m+n}, \qquad & [L_m, T_n^{(a)}] &= -n\, T_{m+n}^{(a)}, \\ [T_m^{(0)}, T_n^{(\pm)}] &= \mp T_{m+n}^{(\pm)}, \qquad & [T_m^{(+)}, T_n^{(-)}] &= 2\, T_{m+n}^{(0)}\,. \end{aligned} \tag{2.24}$$

Thus, the classical asymptotic symmetry algebra is a Witt algebra and an $sl(2,\mathbb{R})$ current algebra.

The variation of the parameters labelling the space of solutions under the diffeomorphisms generated by the above vector fields can be summarized as

$$\begin{aligned} \delta_\xi J^a &= \partial_+\lambda^a - i\, f^a{}_{bc} J^b\, \lambda^c + \partial_+(J^a\, \lambda) \\ &= \partial_+(\lambda^a + J^a\, \lambda) - i\, f^a{}_{bc} J^b\, (\lambda^c + J^c\, \lambda), \end{aligned} \tag{2.25}$$

where $\{J^{(-1)}, J^{(0)}, J^{(1)}\} = \{\bar{g}, f, g\}$ and $\{\lambda^{(-1)}, \lambda^{(0)}, \lambda^{(1)}\} = \{\bar{A}, A_0, A\}$ and $\lambda = B$. Here f^a_{bc}s are the structure constants in $sl(2,\mathbb{R})$ written as $[L_m, L_n] = (m-n)L_{m+n}$ for $m, n \in \{-1, 0, 1\}$. One can note that the fluctuations of the currents can be cast in a form reminiscent of transformation of gauge field J^a under gauge transformation parameter

$$\Lambda^a = \lambda^a + \lambda. \tag{2.26}$$

This is should not be surprise since as mentioned earlier 3d pure gravity can be written as a Chern-Simons theory. We will return to this point after computing the central terms for the above algebra.

Since a boundary metric component varies for different solutions in the space of allowed geometries in the bulk, the determination of the conserved asymptotic charge is subtle. Were this not the case we could have read the change in the asymptotic charge from the holographically renormalized Brown-York stress tensor, first written down by Balasubramanian and Kraus in [47]. Fortunately there does exist a generalized prescription for computing asymptotic charges for gauge transformations given a set of boundary conditions developed by Barnich and Brandt in [50] and further studied in [51]. We have review it in the Appendix (6.2) were in some relevant examples have also been worked out.

Using this formalism for computing the corresponding charges of our geometry requires computing charge via (6.99) after computing the super-potential (6.98) for pure AdS_3 gravity. We find that the charges are integrable over the solution with

$$\begin{aligned}
\not{\delta} Q_\xi &= \frac{1}{8\pi G}\delta \int d\phi \Big\{B(x^+)\Big[\kappa(x^+) + (\frac{1}{2}f^2(x^+) - g(x^+)\bar{g}(x^+)) \\
&\qquad + \frac{1}{2}(e^{ix^-}g(x^+) + e^{-ix^-}\bar{g}(x^+)) \\
&\qquad + \frac{i}{2}\,\partial_+[B(x^+)\,(e^{ix^-}g(x^+) - e^{-ix^-}\bar{g}(x^+))]\Big]\Big\} \\
&\quad -\frac{1}{8\pi G}\delta \int d\phi \Big[\frac{1}{2}A_0(x^+)f(x^+) - (g(x^+)A(x^+) + \bar{g}(x^+)\bar{A}(x^+))\Big]\,.
\end{aligned} \tag{2.27}$$

These can be integrated between the configurations trivially in the solution space from $f(x^+) = g(x^+) = \kappa(x^+) = 0$ to general values of these fields to write down the charges

$$\begin{aligned}
Q_B &= \frac{1}{8\pi G}\int_0^{2\pi} d\phi \Big[B(x^+)\Big(\kappa(x^+) + \frac{1}{2}(f^2(x^+) - 2\,g(x^+)\bar{g}(x^+))\Big) \\
&\qquad + \frac{1}{2}(\partial_+ - \partial_-)\partial_-[e^{ix^-}g(x^+) + e^{-ix^-}\bar{g}(x^+)]\Big] \\
&= \frac{1}{8\pi G}\int_0^{2\pi} d\phi \Big[B(x^+)[\kappa(x^+) + \frac{1}{2}(f^2(x^+) - 2\,g(x^+)\bar{g}(x^+))] \\
&\qquad + \frac{1}{32\pi G}\partial_-[e^{ix^-}g(x^+) + e^{-ix^-}\bar{g}(x^+)]\Big|_{\phi=0}^{\phi=2\pi}\,,
\end{aligned} \tag{2.28}$$

$$Q_A = -\frac{1}{8\pi G}\int_0^{2\pi} d\phi \Big[\frac{1}{2}A_0(x^+)f(x^+) - (g(x^+)A(x^+) + \bar{g}(x^+)\bar{A}(x^+))\Big]. \tag{2.29}$$

The boundary term in (2.28) vanishes as we assumed $g(x^+)$ to be periodic. It is important to point out that these boundary conditions yield an expression asymptotic charge which on the space of solutions is finite. In other words, the change in the asymptotic charge while moving from one allowed solution to any other solution on the solution-space (via suitable residual diffeomorphism generated by (2.22)) is finite. Were this not true, then it would imply that the boundary conditions are weak in that they relate the space of bulk configurations to two or more phase spaces in the boundary separated by infinite conserved charges; which belong to different boundary theories.

Next we compute possible central extensions for the charges computed above. We find that the central term in the commutation relation between charges corresponding to two asymptotic symmetry vectors ξ and $\tilde{\xi}$ is given by

$$(-i)\frac{l}{32\pi G}\int_0^{2\pi} d\phi \Big[B'(x^+)\tilde{B}''(x^+) - B(x^+)\tilde{B}'''(x^+) \\ + 2\,A_0(x^+)\tilde{A}_0'(x^+) - 4\Big(A(x^+)\tilde{\bar{A}}'(x^+) + \bar{A}(x^+)\tilde{A}'(x^+)\Big)\Big]. \tag{2.30}$$

These give rise to the following algebra for the charges[5]

$$\begin{aligned} [L_m, L_n] &= (m-n)\,L_{m+n} + \frac{c}{12}m^3\,\delta_{m+n,0}\,, \\ [L_m, T^a_n] &= -n\,T^a_{m+n}\,, \\ [T^a_m, T^b_n] &= f^{ab}{}_c T^c_{m+n} + \frac{k}{2}\eta^{ab}\,m\,\delta_{m+n,0} \end{aligned} \tag{2.31}$$

with

$$c = \frac{3l}{2G},\;\; k = \frac{c}{6},\;\; f^{0+}{}_{+} = -1,\;\; f^{0-}{}_{-} = 1,\;\; f^{+-}{}_{0} = 2,\;\; \eta^{00} = -1,\;\; \eta^{+-} = 2. \tag{2.32}$$

[5] The bracket in (2.31) is i times the Dirac bracket.

This is precisely the $sl(2,\mathbb{R})$ current algebra found in [10].

We would like to mention here that the algebra (2.31) which consists of a semi direct product of the Virasoro modes; L_ms and the the $sl(2,\mathbb{R})$ current modes; $T^{(a)}_m$s can be decoupled by redefining the Virasoro upto a suitable Sugawara stress tensor constructed from the $sl(2,\mathbb{R})$ Kaĉ-Moody currents. This procedure does not change the central extensions of the the algebra but merely has the effect of putting the commutator between $[\hat{L}_m, T^a_n] = 0$, where $\hat{L} = L + L_{sug}(T^a)$. The fact that the cross commutators can be put to zero by suitable redefinitions without effecting the central extensions can be mimicked at by redefining the residual gauge transformation parameters as in (2.26). It would turn out that the analysis done in the Chern-Simons formalism naturally produces an asymptotic symmetry algebra for the boundary conditions of this section where the commutators between the Virasoro and the Kaĉ-Moody current algebra are zero.

It should be pointed out that although the boundary conditions imposed above seem to generalize the ones given in [6], the space of allowed solutions in the bulk are very different. The only solution common between the space of solutions in [6] and those allowed by the above boundary conditions is the extremal BTZ. In fact, the boundary conditions in [6] allow for non-extremal BTZs but do not allow for global AdS_3 to be in the space of solutions; therefore they do not realize the full set of isometries of the global AdS_3 as a subset of the resulting asymptotic symmetry algebra. The Brown-Henneaux boundary conditions are one such boundary conditions which do realize the full global AdS_3 isometeries as a subset of the asymptotic symmetry algebra; however they contain all the BTZ black-hole solutions too. The boundary conditions studied in this chapter on the other hand do allow for global AdS_3 but contain only the extremal BTZ as the allowed black hole solution.

Since the boundary metric in these boundary conditions is fixed to be in the light-cone gauge; the boundary CFT must be an induced gravity theory similar to the one studied by Polyakov in [10]. This is further lend credence to by the fact that the asymptotic symmetry algebra contains an $sl(2,\mathbb{R})$ Kaĉ-Moody current algebra spanned by the modes of chiral component of the boundary metric, just as in Polyakov's analysis of Chiral induced gravity in [10].

Next we analyse the result of holding the boundary metric fixed in the conformal gauge, and what happens when the conformal factor obeys the generic Liouville equation of motion.

2.2 Boundary conditions as a result of gauge fixing linearised fluctuations

There is an interesting way in which one can perceive how the chiral boundary conditions and the Dirichlet boundary conditions imposed by Brown and Henneaux, naturally arise as a result of gauge fixing linearised solutions to Einstein's equation. This is covered in the Appendix (6.1).

One begins with gauge fixing in de Donder gauge (6.15) (also called the covariant gauge) the gravitational fluctuations about global AdS_3. The residual gauge transformations would be those diffeomorphisms which respect this gauge choice. The vector fields generating these diffeomorphisms solve a either of the two first order linear differential equations (6.26) in the case of the background being globally AdS_3[6]. Of the solutions to these differential equations, interesting ones are those which survive till the boundary and these can be given exactly in terms of hyperbolic functions everywhere in the bulk of global AdS_3. These solutions can be categorized as to how they effect the boundary metric under the Lie action, and as a whole they yield a Diff×Weyl action on the boundary metric. In them one can find two copies of Witt algebra which commute with each other only asymptotically. Interestingly, the asymptotic values of the components of the solutions yield a semi-direct product of left moving Witt with left moving $sl(2,\mathbb{R})$ Kaĉ-Moody cur-

[6] These are identical to the left and right Maurer-Cartan equations obeyed by vector fields on S^3.

rent and a semi-direct product of right moving Witt with right moving $sl(2,\mathbb{R})$ Kaĉ-Moody current.

This gauge condition can be readily cast in the Chern-Simons formalism (6.1) which therefore would admit ready generalisation to the cases of super-gravity and higher-spins in AdS_3.

2.3 Holographic Liouville theory

The analysis of the previous section can be carried out for a different set of boundary conditions which is such that the fluctuating field on the AdS_3 boundary obeys the Liouville equation $\partial_+\partial_- \log F = 2\chi F$. Here we would choose F to be the conformal factor of the metric on the AdS boundary. This is primarily motivated by the induced gravity obtained after fixing the boundary metric in the conformal gauge in (2.1). Therefore we propose the following boundary conditions on $AlAdS_3$:

$$\begin{aligned}
g_{rr} &= \frac{l^2}{r^2} + O(r^{-4}), \;\; g_{r+} = O(r^{-1}), \;\; g_{r-} = O(r^{-3}), \\
g_{+-} &= -\frac{r^2}{2}F(x^+, x^-) + O(r^0), \;\; g_{--} = O(r^0), \\
g_{++} &= O(r^0),
\end{aligned} \tag{2.33}$$

where x^+, x^- are treated to be the boundary coordinates and r is the radial coordinate with the asymptotic boundary at $r^{-1} = 0$. One can write a general non-linear solution of AdS_3 gravity in Fefferman–Graham coordinates [49] as:

$$ds^2 = l^2\frac{dr^2}{r^2} + r^2\left[g^{(0)}_{ab} + \frac{l^2}{r^2}g^{(2)}_{ab} + \frac{l^4}{r^4}g^{(4)}_{ab}\right]dx^a dx^b. \tag{2.34}$$

Therefore, the full set of non-linear solutions consistent with our boundary conditions is obtained when

$$\begin{aligned} & g^{(0)}_{++} = 0, \;\; g^{(0)}_{+-} = -\frac{1}{2}F(x^+, x^-), \;\; g^{(0)}_{--} = 0, \\ & g^{(2)}_{++} = \kappa(x^+, x^-), \;\; g^{(2)}_{+-} = \sigma(x^+, x^-), \;\; g^{(2)}_{--} = \tilde{\kappa}(x^+, x^-), \\ & g^{(4)}_{ab} = \frac{1}{4} g^{(2)}_{ac} g^{cd}_{(0)} g^{(2)}_{db}, \end{aligned} \tag{2.35}$$

where in the last line $g^{cd}_{(0)}$ is $g^{(0)}_{cd}$ inverse. Imposing the equations of motion $R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} = 0$ one finds that these equations are satisfied for $\mu, \nu = +, -$. Then the remaining three equations coming from $(\mu, \nu) = (r, r), (r, +), (r, -)$ impose the following relations:

$$\begin{aligned} & \sigma(x^+, x^-) - \frac{1}{2}\partial_+\partial_- \log F = 0, \\ & \partial_- \kappa = F\, \partial_+ \left(\frac{\sigma}{F}\right), \;\; \partial_+ \tilde{\kappa} = F\, \partial_- \left(\frac{\sigma}{F}\right) \end{aligned} \tag{2.36}$$

In general the equations can be solved for κ and $\tilde{\kappa}$ in terms of F as follows:

$$\begin{aligned} \kappa(x^+, x^-) &= \kappa_0(x^+) + \frac{1}{2}\partial_+^2 \log F - \frac{1}{4}(\partial_+ \log F)^2 \\ \tilde{\kappa}(x^+, x^-) &= \tilde{\kappa}_0(x^-) + \frac{1}{2}\partial_-^2 \log F - \frac{1}{4}(\partial_- \log F)^2 \end{aligned} \tag{2.37}$$

We now have to specialise to some subset of solutions such that we have Liouville equation satisfied by F. For this observe that when $\partial_- \kappa = \partial_+ \tilde{\kappa} = 0$ we have $\sigma = \chi F$ for some constant χ. Then the ward identity $\sigma = \frac{1}{2}\partial_+\partial_- \log F$ reads:

$$\frac{1}{2}\partial_+\partial_- \log F = \chi F \tag{2.38}$$

which is the famous Liouville's equation. So if we add boundary terms such that we keep $\sigma = \chi F$ then it follows that $\partial_- \kappa = \partial_+ \tilde{\kappa} = 0$. For this, it is useful to note that the boundary (holographic) stress tensor T_{ij} for the class of metrics we have is proportional to

$$g^{(2)}_{\mu\nu} - R^{(0)}\, g^{(0)}_{\mu\nu} = \begin{pmatrix} \kappa & 5\,\sigma \\ 5\,\sigma & \tilde{\kappa} \end{pmatrix} \tag{2.39}$$

Taking the trace with respect to the boundary metric gives

$$g^{\mu\nu}_{(0)}(g^{(2)}_{\mu\nu} - R^{(0)}\, g^{(0)}_{\mu\nu}) = -20\frac{\sigma}{F} \tag{2.40}$$

Therefore the constraint $\sigma = \chi\, F$ simply translates into demanding $g^{\mu\nu}_{(0)}(g^{(2)}_{\mu\nu} - R^{(0)}\, g^{(0)}_{\mu\nu}) = -20\chi$. The variation of the action along the solution space is

$$\delta S = \frac{1}{2}\int_{bdy.} d^2x\, \sqrt{|g^{(0)}|}\, T^{ij}\delta g^{(0)}_{ij} = -\frac{l}{8\pi\, G}\int_{bdy} d^2x\, \frac{5\sigma}{F}\delta F. \tag{2.41}$$

So we add the boundary term:

$$\frac{l}{8\pi G}\int_{bdy.} d^2x\, 10\chi\, \sqrt{|g^{(0)}|} = \frac{l}{8\pi\, G}\int_{bdy} d^2x\, 5\chi\, F \tag{2.42}$$

such that the total variation of the action is

$$\delta S_{\,total} = -\frac{l}{8\pi\, G}\int_{bdy} d^2x\, 5\,(\frac{\sigma}{F} - \chi)\,\delta F. \tag{2.43}$$

Now we could choose either $\delta F = 0$ (Dirichlet) or $\sigma = \chi\, F$ (Neumann). Choosing the latter gives rise to the Liouville equation as we desire.

2.3.1 Classical Solutions and asymptotic symmetries

It is well known that the general solution of the Liouville equation $\partial_+\partial_- \log F = 2\chi F$ is given by

$$F = \chi^{-1} \frac{\partial_+ f(x^+)\, \partial_- \tilde{f}(x^-)}{[1 + f(x^+)\, \tilde{f}(x^-)]^2} \;\text{ for } \chi \neq 0, \quad F = f(x^+)\, \tilde{f}(x^-) \;\text{ for } \chi = 0. \tag{2.44}$$

The latter case was considered by [5]. It will be interesting to work out the asymptotic symmetry algebra in the former case. We now proceed to get the asymptotic symmetries for the above boundary conditions. The residual diffeomorphisms that the leave the metric in the above form are:

$$\xi = r\xi^r \partial_r + (\xi^+ + (O)(\tfrac{1}{r}))\partial_+ + (\xi^- + O(\tfrac{1}{r}))\partial_-,$$

$$\text{where} \qquad \partial_-\xi^+ = \partial_+\xi^- = 0 \;,\; \partial_+\partial_-\xi^r = 2\xi^r F, \tag{2.45}$$

the subleading functions in r are all determined from the boundary values of components. For convenience of calculation, let us introduce a field $\Phi = \log(\chi F)$. The equation for Φ then is:

$$\partial_+\partial_-\Phi = 2e^{\Phi}. \tag{2.46}$$

The first order variation of the above differential equation satisfies:

$$\partial_+\partial_-\delta\Phi = 2e^{\Phi}\delta\Phi, \tag{2.47}$$

therefore $\delta\Phi$ satisfies the same equation as ξ^r. Reading off $\delta\Phi$ from the general solution of F and labelling $\delta f = gf'$ and $\delta\tilde{f} = \tilde{g}\tilde{f}'$, the expression for ξ^r reads:

$$\xi^r = g' + \tilde{g}' + g\partial_+\Phi + \tilde{g}\partial_-\Phi,$$

$$\text{where} \qquad \partial_- g = \partial_+\tilde{g} = 0. \tag{2.48}$$

The infinitesimal change in the asymptotic charge under such diffeomorphisms is given by:

$$\delta\!\!\!/ Q = -\frac{l}{8\pi G}\int_{\partial\mathcal{M}} d\phi \left\{2(\xi^{+}_{(0)}\delta\kappa + \xi^{-}_{(0)}\delta\tilde{\kappa}) + \frac{\delta F}{F^2}\xi^r(\partial_+ + \partial_-)F - \frac{\xi^r}{F}(\partial_+ + \partial_-)\delta F + \frac{\delta F}{F}(\partial_+ + \partial_-)\xi^r\right\} \tag{2.49}$$

Which can be simplified to be brought to an integrable form after throwing away terms which are total derivatives of ϕ:

$$\delta\!\!\!/ Q = -\frac{l}{8\pi G}\int_{\partial\mathcal{M}} d\phi \left\{2(\xi^{+}_{(0)}\delta\kappa + \xi^{-}_{(0)}\delta\tilde{\kappa}) + \delta\Phi(\partial_+ + \partial_-)\xi^r - \xi^r(\partial_+ + \partial_-)\delta\Phi\right\}. \tag{2.50}$$

One has to show that the above charge is integrable. The total integrated charge can be written again (after similarly throwing away total derivatives in ϕ):

$$Q = -\frac{l}{4\pi G}\int_{\partial\mathcal{M}} d\phi \left\{\xi^{+}_{(0)}\kappa + \xi^{-}_{(0)}\tilde{\kappa} + g(\partial_+^2\Phi - \tfrac{1}{2}(\partial_+\Phi)^2) + \tilde{g}(\partial_-^2\Phi - \tfrac{1}{2}(\partial_-\Phi)^2)\right\}. \tag{2.51}$$

The factors multiplying g and $\tilde{g}$ can be recognized as the stress-tensor modes of the Liouville theory. One can proceed to construct the classical Poisson brackets by demanding that the above charge gives rise to the fluctuations of the metric components F , κ and $\tilde{\kappa}$ produced by the residual diffeomorphisms (2.45). The change in the parameters under such boundary condition preserving gauge transformations are:

$$\begin{aligned}
\delta F &= 2F\xi^r + \partial_+(F\xi^{+}_{(0)}) + \partial_-(F\xi^{-}_{(0)}),\\
\delta f &= (g + \tfrac{1}{2}\xi^{+}_{(0)})f',\\
\delta \tilde{f} &= (\tilde{g} + \tfrac{1}{2}\xi^{-}_{(0)})\tilde{f}',\\
\delta\kappa &= \xi^{+}_{(0)}\kappa' + 2\kappa\xi^{+}_{(0)}{}' + g''' + g'\hat{f} + \tfrac{1}{2}g\partial_+\hat{f},\\
\delta\tilde{\kappa} &= \xi^{-}_{(0)}\tilde{\kappa}' + 2\tilde{\kappa}\xi^{-}_{(0)}{}' + \tilde{g}''' + \tilde{g}'\hat{\tilde{f}} + \tfrac{1}{2}\tilde{g}\partial_-\hat{\tilde{f}}.
\end{aligned} \tag{2.52}$$

Where $\hat{f} = 2\partial_+^2\Phi - (\partial_+\Phi)^2$ and $\hat{\bar{f}} = 2\partial_-^2\Phi - (\partial_-\Phi)^2$. The variation in F can be cast in terms of $\hat{f}$ and $\hat{\bar{f}}$ as:

$$\begin{aligned}
\tfrac{1}{2}\delta\hat{f} &= (2g + \xi^+_{(0)})''' + (2g + \xi^+_{(0)})'\hat{f} + \tfrac{1}{2}(2g + \xi^+_{(0)})\partial_+\hat{f}, \\
\tfrac{1}{2}\delta\hat{\bar{f}} &= (2\tilde{g} + \xi^-_{(0)})''' + (2\tilde{g} + \xi^-_{(0)})'\hat{\bar{f}} + \tfrac{1}{2}(2\tilde{g} + \xi^-_{(0)})\partial_-\hat{\bar{f}}.
\end{aligned} \tag{2.53}$$

Therefore the space of classical solutions allowed by the proposed boundary condition(2.33) are parametrized by the functions $(\hat{f}, \hat{\bar{f}}, \kappa, \tilde{\kappa})$, where as the diffeomorphisms that would keep the metric under Lie derivative in this form are parametrized by $(g, \tilde{g}, \xi^+_{(0)}, \xi^-_{(0)})$. Redefining functions as:

$$\hat{f} \to \tfrac{1}{2}\hat{f}, \qquad \hat{\bar{f}} \to \tfrac{1}{2}\hat{\bar{f}},$$

$$\kappa \to (\kappa - \tfrac{1}{2}\hat{f}), \qquad \tilde{\kappa} \to (\tilde{\kappa} - \tfrac{1}{2}\hat{\bar{f}}),$$

$$g \to (g + \tfrac{1}{2}\xi^+_{(0)}), \qquad \tilde{g} \to (\tilde{g} + \tfrac{1}{2}\xi^-_{(0)}), \tag{2.54}$$

the Poisson algebra reads:

$$\begin{aligned}
-\tfrac{k}{2\pi}\left\{\kappa(x^{+\prime}), \kappa(x^+)\right\} &= -[\kappa(x^+) + \kappa(x^{+\prime})]\delta'(x^{+\prime} - x^+) + \delta'''(x^{+\prime} - x^+), \\
-\tfrac{k}{2\pi}\left\{\hat{f}(x^{+\prime}), \hat{f}(x^+)\right\} &= -[\hat{f}(x^+) + \hat{f}(x^{+\prime})]\delta'(x^{+\prime} - x^+) - \delta'''(x^{+\prime} - x^+), \\
\left\{\hat{f}(x^{+\prime}, \kappa(x^+))\right\} &= 0.
\end{aligned} \tag{2.55}$$

Similarly for the left sector, which commutes with the right sector. This shows that central charge associated with the Virasoros of the Liouville theory to be negative of the central charge of the Virasoros obtained from the Brown-Henneaux boundary conditions. The space of bulk geometries allowed by the Brown-Henneaux boundary conditions is contained in the space of solutions allowed by the above boundary condition.

If one begins with a generic 3d asymptotically locally AdS_3 metric in the Fefferman and Graham gauge then the residual diffeomorphisms are the ones which would generate the Diff×Weyl[7] for the boundary metric. Restricting the boundary metric to have the form as the one in (2.11) restricts the residual diffeomorphisms further to have an algebra chiral-Diff×Witt to the leading order. Sub-leading order corrections to the residual diffeomorphisms further restrict it to an $sl(2,\mathbb{R})$×Virasoro. Similarly, if on the other hand one imposed the boundary conditions of section (2.3), then the Diff×Weyl reduces to two copies of left-right Virasoro with opposite central charges.

The analyses of the previous two sections were done in the second order formalism of gravity using the Einstein-Hilbert action. As mentioned in the Introduction of this thesis, generalizing these analyses to include super-gravity and higher-spin gauge fields in AdS_3 would require one to have the bulk action analysed in the first order formulation were the bulk theory can be written as a difference of two Chern-Simons gauge fields. Therefore, as a prelude to this we formulate the analyses of the previous two sections in the Chern-Simons formulation of pure gravity in AdS_3.

2.4 Holographic CIG in first order formalism

It is well-known [11, 52] that the AdS_3 gravity in the Hilbert-Palatini formulation can be recast as a Chern-Simons (CS) gauge theory with action[8]

$$S[A,\tilde{A}] = \frac{k}{4\pi}\int \mathrm{tr}(A\wedge A + \frac{2}{3}A\wedge A\wedge A) - \frac{k}{4\pi}\int \mathrm{tr}(\tilde{A}\wedge\tilde{A} + \frac{2}{3}\tilde{A}\wedge\tilde{A}\wedge\tilde{A}) \tag{2.56}$$

up to boundary terms, where the gauge group is $SL(2,\mathbb{R})$. These are related to veilbein and spin connection through $A = \omega^a + \frac{1}{l}e^a$ and $\tilde{A} = \omega^a - \frac{1}{l}e^a$. The equations of motion are $F = dA + A\wedge A = 0$ and $\tilde{F} := d\tilde{A} + \tilde{A}\wedge\tilde{A} = 0$. See appendix A for details on the most

[7] This is actually a semi-direct product where the commutator of Diff with Weyl is a Weyl.

[8] As is standard the symbol tr is understood to be $\frac{1}{2\mathrm{Tr}L_0^2}\mathrm{Tr}$ where Tr is the ordinary matrix trace.

general solutions to these flatness conditions.

Next we will write the solutions of AdS_3 gravity consistent with (2.11) in CS language. For this we simply specialize the flat connections given in appendix A to

$$\begin{aligned} A &= b^{-1}\partial_r b\, dr + b^{-1}[(L_1 + a_+^{(-)} L_{-1} + a_+^{(0)} L_0)\, dx^+ + (a_-^{(-)} L_{-1})\, dx^-]\, b \\ \tilde{A} &= b\,\partial_r b^{-1}\, dr + b\,[(\tilde{a}_+^{(0)} L_0 + \tilde{a}_+^{(+)} L_1 + \tilde{a}_+^{(-)} L_{-1})\, dx^+ + (\tilde{a}_-^{(+)} L_1 - L_{-1})\, dx^-]\, b^{-1} \end{aligned} \tag{2.57}$$

where $b = e^{L_0 \ln \frac{r}{l}}$ and all the functions are taken to be functions of both the boundary coordinates (x^+, x^-). The equations of motion impose the following conditions:

$$\begin{aligned} a_-^{(-)} &= \frac{1}{2}\partial_- a_+^{(0)}, \;\; a_+^{(-)} = -\kappa_0(x^+) + \frac{1}{4}(a_+^{(0)})^2 + \frac{1}{2}\partial_+ a_+^{(0)} \\ \tilde{a}_+^{(0)} &= -\partial_- \tilde{a}_+^{(-)}, \;\; \tilde{a}_+^{(+)} = -\tilde{a}_+^{(-)}\, \tilde{a}_-^{(+)} - \frac{1}{2}\partial_- \tilde{a}_+^{(0)}, \end{aligned} \tag{2.58}$$

and

$$(\partial_+ + 2\,\partial_- \tilde{a}_+^{(-)} + \tilde{a}_+^{(-)}\,\partial_-)\, \tilde{a}_-^{(+)} = \frac{1}{2}\partial_-^3 \tilde{a}_+^{(-)} \tag{2.59}$$

The last equation is a Virasoro Ward identity and it can be solved as before. To obtain metric in the FG gauge we need to impose $a_+^{(0)} = \tilde{a}_+^{(0)}$. With this choice it is easy to see that the metric obtained matches exactly with the solution given above in (2.34 - 2.36) with $F = \tilde{a}_+^{(-)}$ and $\tilde{\kappa} = \tilde{a}_-^{(+)}$.

To be able to define a variational principle that admits a fluctuating F, we add the following boundary action:

$$S_{bdy.} = -\frac{k}{4\pi}\int d^2x\, tr(L_0\,[A_+, A_-]) - \frac{k}{4\pi}\int d^2x\, tr(L_0\,[\tilde{A}_+, \tilde{A}_-] - 2\,\tilde{\kappa}_0\, L_1\, \tilde{A}_+) \tag{2.60}$$

where $\tilde{\kappa}_0$ is some constant. Then it is easy to see that the variation of the full action gives

$$\delta S_{total} = \frac{k}{2\pi}\int d^2x\,(\tilde{\kappa} - \tilde{\kappa}_0)\,\delta F. \tag{2.61}$$

In showing this we have to use all the constraints in (2.58) coming from the equations of motion except the Virasoro Ward identity. Therefore, we again have two ways to impose the variational principle $\delta S = 0$: (i) $\delta F = 0$ and (ii) $\tilde{\kappa} = \tilde{\kappa}_0$. The former is the usual Brown-Henneaux [1] type Dirichlet boundary condition. We therefore consider the latter.

2.4.1 Residual gauge transformations

To analyze the asymptotic symmetries in the CS language we seek the residual gauge transformations that leave $\tilde{\kappa}$ fixed, and the above flat connections form-invariant.

The gauge transformations act as $\delta_\Lambda A = d\Lambda + [A, \Lambda]$ which in turn act as $\delta_\lambda a = d\lambda + [a, \Lambda]$ where $A = b^{-1}\, a\, b + b^{-1}\, db$ with $\Lambda = b^{-1}\, \lambda\, b$ (and similarly on the right sector gauge fields $\tilde{a}$ with parameters labeled $\tilde{\lambda}$). The resulting gauge parameters are

$$\begin{aligned} \lambda &= \lambda^{(-)}(x^+, x^-)\, L_{-1} + [a_+^{(0)}\, \lambda^{(+)}(x^+) - \partial_+\lambda^{(+)}(x^+)]\, L_0 + \lambda^{(+)}(x^+)\, L_1 \\ \tilde{\lambda} &= \tilde{\lambda}^{(-)}\, L_{-1} - \partial_-\tilde{\lambda}^{(-)}\, L_0 - [\tilde{a}_-^{(+)}\, \tilde{\lambda}^{(-)} - \frac{1}{2}\partial_-^2\tilde{\lambda}^{(-)}]\, L_1, \end{aligned} \tag{2.62}$$

that induce the variations

$$\begin{aligned} \delta_\lambda a_+^{(0)} &= 2\,[\lambda^{(-)} - a_+^{(-)}\, \lambda^{(+)}] - \partial_+[\partial_+\lambda^{(+)} - a_+^{(0)}\, \lambda^{(+)}] \\ \delta_\lambda a_+^{(-)} &= \partial_+\lambda^{(-)} + a_+^{(0)}\, \lambda^{(-)} + a_+^{(-)}\, [\partial_+\lambda^{(+)} - a_+^{(0)}\lambda^{(+)}\,] \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} \delta_{\tilde{\lambda}}\tilde{a}_+^{(-)} &= (\partial_+ + \tilde{a}_+^{(-)}\, \partial_- - \partial_-\tilde{a}_+^{(-)})\, \tilde{\lambda}^{(-)} \\ \delta_{\tilde{\lambda}}\tilde{a}_-^{(+)} &= -\tilde{\lambda}^{(-)}\, \partial_-\tilde{a}_-^{(+)} - 2\,\tilde{a}_-^{(+)}\, \partial_-\tilde{\lambda}^{(-)} + \frac{1}{2}\, \partial_-^3\tilde{\lambda}^{(-)} \end{aligned} \tag{2.64}$$

respectively. In the global case when we hold $\tilde{a}_-^{(+)}$ fixed at $-1/4$ we find that $\tilde{\lambda}^{(-)} = \lambda_f + e^{i\,x^-}\lambda_g + e^{-i\,x^-}\, \bar{\lambda}_{\bar{g}}$. When we make a gauge transformation to ensure that we remain the FG coordinates for the metric we need to impose

$$(\delta_\lambda a_+^{(0)} - \delta_{\tilde{\lambda}} \tilde{a}_+^{(0)})\Big|_{a_+^{(0)}=\tilde{a}_+^{(0)}} = 0 \tag{2.65}$$

We find that this condition drastically reduces the number of independent residual gauge parameters down to four functions of x^+. In particular, the function $\lambda^{(-)}(x^+, x^-)$ is determined to be

$$\begin{aligned}\lambda^{(-)}(x^+, x^-) &= -\frac{1}{4}\lambda^{(+)}(\bar{g}\, e^{-i x^-} - g\, e^{i x^-})^2 - \kappa_0\, \lambda^{(+)} + \frac{1}{2}\partial_+^2 \lambda^{(+)} + \frac{i}{2}(g\, e^{i x^-} - \bar{g}\, e^{-i x^-})\partial_+ \lambda^{(+)} \\ &+\frac{1}{2}[(\lambda_g\, e^{i x^-} + \bar{\lambda}_{\bar{g}}\, e^{-i x^-})\, f - (g\, e^{i x^-} + \bar{g}\, e^{-i x^-})\, \lambda_f - i(\partial_+ \lambda_g\, e^{i x^-} - \partial_+ \bar{\lambda}_{\bar{g}}\, e^{-i x^-})]\end{aligned} \tag{2.66}$$

These induce the following transformations:

$$\begin{aligned}\delta_\lambda f &= \lambda_f' + 2i\,(\bar{g}\, \lambda_g - \bar{\lambda}_{\bar{g}}\, g), \;\; \delta_\lambda g = \lambda_g' + i\,(f\, \lambda_g - g\, \lambda_f), \;\; \delta_\lambda \bar{g} = \bar{\lambda}_{\bar{g}}' - i\,(f\, \bar{\lambda}_{\bar{g}} - \bar{g}\, \lambda_f), \\ \delta\kappa_0 &= \lambda^{(+)}\, \kappa_0' + 2\kappa_0\, \partial_+ \lambda^{(+)} - \frac{1}{2}\partial_+^3 \lambda^{(+)}\end{aligned} \tag{2.67}$$

We could have obtained this result starting with the left sector a to be $a = [L_1 - \kappa_0(x^+)\, L_{-1}]\, dx^+$.

Now, comparing this result with (2.25) one finds that $\{\lambda_f, \lambda_g, \bar{\lambda}_{\bar{g}}, \lambda^{(+)}\}$ are not quite the parameters in (2.25) that correspond to the asymptotic symmetry vector fields of [3]. For this it turns out that we have to redefine the gauge parameters

$$\{\lambda_f, \lambda_g, \bar{\lambda}_{\bar{g}}, \lambda\} \to \{\lambda_f + f\, \lambda, \lambda_g + g\, \lambda, \bar{\lambda}_{\bar{g}} + \bar{g}\, \lambda, \lambda\} \tag{2.68}$$

Then the transformations read

$$\begin{aligned}\delta_{\tilde{\lambda}} f &= \lambda_f' + 2i\,(\bar{g}\, \lambda_g - \bar{\lambda}_{\bar{g}}\, g) + (f\lambda^{(+)})', \;\; \delta_{\tilde{\lambda}} g = \lambda_g' + i\,(f\, \lambda_g - g\, \lambda_f) + (g\lambda^{(+)})', \\ \delta_{\tilde{\lambda}} \bar{g} &= \bar{\lambda}_{\bar{g}}' - i\,(f\, \bar{\lambda}_{\bar{g}} - \bar{g}\, \lambda_f) + (\bar{g}\lambda^{(+)})', \;\; \delta\kappa_0 = \lambda^{(+)}\, \kappa_0' + 2\kappa_0\, \partial_+ \lambda^{(+)} - \frac{1}{2}\partial_+^3 \lambda^{(+)}\end{aligned} \tag{2.69}$$

which match exactly with those in (2.25).

A method for computing the charges corresponding to residual gauge transformations is provided by the Barnich *et al* [50, 51]. Using their method one can show that the change in the charge $\not{\delta}Q$ along the space of solutions of one copy of the Chern-Simons theory to be:

$$\not{\delta}Q \;=\; -\tfrac{k}{2\pi}\int_0^{2\pi} d\phi\; tr[\Lambda\,\delta A_\phi]. \tag{2.70}$$

where Λ is the gauge transformation parameter. We will see that these charges are integrable for all the residual gauge transformations considered below.

Now, demanding that the charge (corresponding to a given residual gauge transformation) generates the right variations of the functions parametrizing the solutions via

$$\delta_\lambda f(x) = \{Q, f(x)\}, \tag{2.71}$$

allows one to read out the Poisson brackets between those functions.

If we compute the charges and the algebra of these symmetries it can be seen that they match with those of (2.28) and (2.31) in the second order formalism. We do not show this explicitly here since the above analyses in the first order formulation generalizing these boundary conditions to supergravity and higher-spins in AdS_3 would be thoroughly dealt with in the following chapters, and all these admit a consistent truncation to pure gravity in AdS_3.

2.5 Liouville boundary conditions in CS formulation

The starting point here - as in the previous section, is to first find the gauge fields which yield the metric proposed in (2.33). As before, one mods out the radial r dependence with a finite gauge transformation,

$$\begin{aligned} A &= b^{-1}ab + b^{-1}db, \\ \tilde{A} &= b\tilde{a}b^{-1} + bdb^{-1}, \\ b &= e^{\log\frac{r}{l}L_0}. \end{aligned} \tag{2.72}$$

Since the equations of motion for A and $\tilde{A}$ are flatness of their connections, one can equivalently work with a and $\tilde{a}$ for the rest of the analysis. Let us specialise the solution in the Appendix (6.105) to:

$$a = (a_+^{(+)}\, L_1 - a_+^{(-)}\, L_{-1} + a_+^{(0)}\, L_0)\, dx^+ + (-a_-^{(-)}\, L_{-1} \; + a_-^{(0)}\, L_0)\, dx^- \tag{2.73}$$

Assuming that $a_+^{(+)}$ does not vanish, the flatness conditions imply:

$$\begin{aligned} a_-^{(0)} &= \frac{1}{a_+^{(+)}}\partial_- a_+^{(+)}, \;\; a_+^{(+)} a_-^{(-)} = -\frac{1}{2}(\partial_- a_+^{(0)} - \partial_+ a_-^{(0)}) \\ a_+^{(+)}\, a_+^{(-)} &= \kappa_0(x^+) - \frac{1}{4}(a_+^{(0)})^2 - \frac{1}{2}\partial_+ a_+^{(0)} + \frac{1}{2} a_+^{(0)}\, \partial_+ \ln a_+^{(+)} + \frac{1}{2}\partial_+^2 \ln a_+^{(+)} - \frac{1}{4}(\partial_+ \ln a_+^{(+)})^2 \end{aligned} \tag{2.74}$$

Similarly if we consider the 1-form

$$\tilde{a} = (\tilde{a}_+^{(+)}\, L_1 + \tilde{a}_+^{(0)}\, L_0)\, dx^+ + (-\tilde{a}_-^{(+)}\, L_1 + \tilde{a}_-^{(-)}\, L_{-1} \; + \tilde{a}_-^{(0)}\, L_0)\, dx^- \tag{2.75}$$

Then, assuming now that $\tilde{a}_-^{(-)}$ does not vanish, the flatness conditions read

$$\begin{aligned} \tilde{a}_-^{(-)}\tilde{a}_+^{(+)} &= -\frac{1}{2}(\partial_- \tilde{a}_+^{(0)} - \partial_+ \tilde{a}_-^{(0)}), \;\; \tilde{a}_+^{(0)} = -\frac{1}{\tilde{a}_-^{(-)}}\, \partial_+ \tilde{a}_-^{(-)} \\ \tilde{a}_-^{(-)}\, \tilde{a}_-^{(+)} &= \tilde{\kappa}_0(x^-) - \frac{1}{4}(\tilde{a}_-^{(0)})^2 + \frac{1}{2}\partial_- \tilde{a}_-^{(0)} - \frac{1}{2}\tilde{a}_-^{(0)}\, \partial_- \ln \tilde{a}_-^{(-)} + \frac{1}{2}\partial_-^2 \ln \tilde{a}_-^{(-)} - \frac{1}{4}(\partial_- \ln \tilde{a}_-^{(-)})^2 \end{aligned} \tag{2.76}$$

The corresponding analysis in the second order formulation made us of the Fefferman and Graham (FG) gauge for the metric. One may impose this gauge on the above gauge fields by demanding that the metric corresponding to them be in the FG gauge. This is not strictly necessary but this has a benefit of reducing the number of solution space parameters by those ones which do not contribute to the asymptotic charge. Imposing the FG gauge on the metric translates to the following condition on the gauge field components:

$$a_+^{(0)} = \tilde{a}_+^{(0)}, \ \ \tilde{a}_-^{(0)} = a_-^{(0)} \tag{2.77}$$

This gives the same metric as in (2.35) with the following identifications:

$$F = a_+^{(+)}\tilde{a}_-^{(-)}, \ \ \kappa = a_+^{(+)}a_+^{(-)}, \ \ \tilde{\kappa} = \tilde{a}_-^{(-)}\tilde{a}_-^{(+)}, \ \ \sigma = a_+^{(+)}a_-^{(-)} = \tilde{a}_-^{(-)}\tilde{a}_+^{(+)} \tag{2.78}$$

2.5.1 Asypmtotic symmetry analysis in the first order formalism

Here we try and reproduce the results obtained in the second order formulation by starting out with the following gauge fields:

$$\begin{aligned} a &= (a_+^{(+)}L_1 - \partial_+(\log a_-^{(-)})L_0 - \tfrac{\kappa(x^+)}{a_+^{(+)}}L_{-1})dx^+ + (\partial_-(\log a_+^{(+)})L_0 - a_-^{(-)}L_{-1})dx^-, \\ \tilde{a} &= (\tilde{a}_+^{(+)}L_{-1} + \partial_-(\log \tilde{a}_+^{(+)})L_0 - \tfrac{\tilde{\kappa}(x^-)}{\tilde{a}_-^{(-)}}L_1)dx^- + (-\partial_+(\log \tilde{a}_-^{(-)})L_0 - \tilde{a}_+^{(+)}L_1)dx^+. \end{aligned} \tag{2.79}$$

Above we have relabelled the parameters for the sake of computational convenience. The above gauge fields reproduce the desired form of the metric with a FG constraint that $a^{(0)} = \tilde{a}^{(0)}$ along with the identifications

$$F = -a_+^{(+)}\tilde{a}_-^{(-)} \ , \ \partial_+\partial_- \log(a_+^{(+)}a_-^{(-)}) = 2a_+^{(+)}a_-^{(-)} \ , \ \partial_+\partial_- \log(\tilde{a}_+^{(+)}\tilde{a}_-^{(-)}) = 2\tilde{a}_+^{(+)}\tilde{a}_-^{(-)}. \tag{2.80}$$

The residual gauge transformations are:

$$\begin{aligned} \Lambda &= \Lambda^{(+)}a_+^{(+)}L_+ + \Lambda^{(0)}L_0 + (-\tfrac{\kappa\Lambda^{(+)}}{a_+^{(+)}} + (y - \tilde{\Lambda}^{(-)})a_-^{(-)})L_{-1}, \\ \tilde{\Lambda} &= \tilde{\Lambda}^{(-)}\tilde{a}_-^{(-)}L_- + \tilde{\Lambda}^{(0)}L_0 + (-\tfrac{\tilde{\kappa}\tilde{\Lambda}^{(-)}}{\tilde{a}_-^{(-)}} + (\tilde{y} - \Lambda^{(+)})\tilde{a}_+^{(+)})L_+, \\ \text{where} \quad & \partial_-\Lambda^{(+)} = 0 = \partial_+\tilde{\Lambda}^{(-)} \ , \ \partial_-(a_+^{(+)}a_-^{(-)}\partial_+ y) = 0 = \partial_+(\tilde{a}_+^{(+)}\tilde{a}_-^{(-)}\partial_-\tilde{y}). \end{aligned} \tag{2.81}$$

The solutions to y and $\tilde{y}$ can be given in terms of ξ^r (2.48):

$$y = -\frac{\partial_+ \xi^r}{a_+^{(+)} a_-^{(-)}}\ ,\ \tilde{y} = -\frac{\partial_- \xi^r}{\tilde{a}_+^{(+)} \tilde{a}_-^{(-)}}. \tag{2.82}$$

The FG constraint on the fluctuations yield the condition:

$$\tilde{\Lambda}^{(0)} - \Lambda^{(0)} = \partial_- y + y\partial_- \log(a_+^{(+)} a_-^{(-)}) = \partial_+ \tilde{y} + \tilde{y}\partial_+ \log(\tilde{a}_+^{(+)} \tilde{a}_-^{(-)}). \tag{2.83}$$

Therefore the residual gauge transformation parameters are labelled by $\{g, \tilde{g}, \Lambda^{(+)}, \Lambda^{(-)}\}$. After imposing the FG gauge the on can write

$$F = \tfrac{1}{\chi} a_+^{(+)} a_-^{(-)}\ ,\ \tilde{a}_+^{(+)} = -\tfrac{1}{\chi} a_+^{(+)}\ ,\ \tilde{a}_-^{(-)} = -\chi a_-^{(-)}. \tag{2.84}$$

It turns out that the fluctuations of the gauge field components yield the same result for the metric components$\{F, \kappa, \tilde{\kappa}\}$ as in the second order formalism with $\Lambda^{(+)} = \xi^+_{(0)}$ and $\Lambda^{(-)} = \xi^-_{(0)}$; explicitly given in (2.52). The asymptotic charge for such configurations can then be written as:

$$\begin{aligned}
\not{\delta} Q &= -\tfrac{k}{2\pi} \int d\phi\, (Tr[\Lambda.A_\phi] - Tr[\tilde{\Lambda}.\tilde{A}_\phi]), \\
&= \tfrac{\ell}{8\pi G} \int d\phi\, [\Lambda^{(+)}\delta\kappa + \tilde{\Lambda}^{(-)}\delta\tilde{\kappa}] \\
&\quad + \tfrac{(\tilde{\Lambda}^{(0)} - \Lambda^{(0)})}{2} \partial_- \delta \log a_+^{(+)} - \delta a_+^{(+)} a_-^{(-)} y + \tfrac{(\tilde{\Lambda}^{(0)} - \Lambda^{(0)})}{2} \partial_+ \delta \log a_-^{(-)} - \delta a_-^{(-)} a_+^{(+)} \tilde{y}, \\
&= \tfrac{\ell}{16\pi G} \int d\phi\, \left\{ 2(\xi^+_{(0)}\delta\kappa + \xi^-_{(0)}\delta\tilde{\kappa}) + \delta\Phi(\partial_+ + \partial_-)\xi^r - \xi^r(\partial_+ + \partial_-)\delta\Phi \right\}.
\end{aligned} \tag{2.85}$$

The above expression for charge turns out to be the same as (2.50), as that in the second order formalism. Since the expressions for the fluctuations and the charges are the same in both the formalisms, we get the same asymptotic symmetry algebra as expected.

We are now ready to study the effects of generalizing boundary conditions studied in this chapter to the case of supergravity and higher-spins in AdS_3. Since pure gravity in AdS_3 can be obtained as a consistent truncation of both supergravity and higher-spin gauge fields in AdS_3, demanding that the generalizations of the boundary conditions contain all the configurations allowed by the boundary conditions studied in pure gravity may not be enough. Tentatively one may consistently impose Dirichlet type boundary conditions on all additional fields in the supergravity and higher-spin settings in AdS_3, while imposing the boundary conditions studied in this chapter on the 3d metric components alone. We would not be interested in such generalizations in this thesis.

In the pure gravity case studied here, the dynamical fields on the boundary are the boundary metric(gauge field) components. In a similar spirit, we will demand that there be additional supergravity and higher-spin gauge field components on the boundary of AdS_3, apart from the metric(gauge field) components allowed by the boundary conditions studied in this chapter. In doing so we would be able study the properties of holographic duals to induced supergravities and induced W-gravities.

Chapter 3

Holographic induced supergravities

We would now like to generalize the chiral and conformal boundary conditions proposed for pure gravity in AdS_3 to a supersymmetric setting. This would primarily be a first step in understanding how these boundary conditions arise in supersymmetric string theories in $AdS_d \times \Sigma_{compact}$ spaces. Here, we would like to have configurations which allow the additional supergravity fields to fluctuate along the boundary of AdS. We first analyse this in a minimal set-up of $\mathcal{N} = (1, 1)$ supergarvity with negative cosmological constant in (2+1) dimensions, before commenting about generalizations to other more generic cases. Here we uncover a relevant graded version of the $sl(2, \mathbb{R})$ Kaĉ-Moody current algebra along with a super-Virasoro as the asymptotic symmetry algebra. This exercise was done to generalize the Brown-Henneaux boundary conditions in [22] for the case of $\mathcal{N} = (1, 1)$, while the boundary conditions for extended sugra in AdS_3 was done in [2]. Here, the authors were concerned with imposing boundary conditions consistent with the Brown-Henneaux result in the pure gravity sector while imposing similar Dirichlet conditions on the other superfields. In [2], the authors made full use of the Chern-Simons formulation of AdS_3 supergravity and showed that the asymptotic symmetry algebra is composed of two copies of super-Virasoros, *i.e.* supersymmetric generalization of two dimensional local conformal transformations (Witt algebras) with central extension $c = 3l/2G$. Although a supersymmetric extension of the Virasoro, these contained quadratic non-linearities on their right-hand sides, just like the W_3 algebra. We would also study in the last section of this chapter boundary conditions which generalize the the analysis of [5] to extended super-gravities in AdS_3. This boundary condition allowed for the boundary metric to have a dynamical conformal factor with flat curvature. The asymptotic algebra consists in this case two copies of a Virasoro with an algebra of what one may term as supersymmetrized harmonic Weyl transformations.

The contents of this chapter may be found in future in a paper yet to appear in [53].

3.1 $\mathcal{N} = (1,1)$ super-gravity in AdS_3

We first begin with a simple set up where we would like to see whether one can generalize the chiral boundary conditions of chapter 2. We will work in the Chern-Simons formulation as we would see that the calculations are easier in this formulation of super-gravity in AdS_3. The graded Lie algebra of interest for us is $osp(1|2)$ which contains in it the bosonic $sl(2,\mathbb{R})$. The commutation relations are as follows:

$$\begin{aligned} \sigma^0 &= \tfrac{1}{2}\sigma^3, & \left[\sigma^0, R^\pm\right] &= \pm\tfrac{1}{2}R^\pm, \\ \left[\sigma^0, \sigma^\pm\right] &= \pm\sigma^\pm, & [\sigma^\pm, R^\pm] &= 0, \\ [\sigma^+, \sigma^-] &= 2\sigma^0, & [\sigma^\pm, R^\mp] &= R^\pm, \\ \{R^\pm, R^\pm\} &= \pm\sigma^\pm, & \{R^\pm, R^\mp\} &= -\sigma^0. \end{aligned} \tag{3.1}$$

The gauge-invariant, bi-linear, non-degenerate metric on the algebra is:

$$Tr(\sigma^a\sigma^b) = h^{ab} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}, \qquad STr(R^-R^+) = -STr(R^+R^-) = 1. \tag{3.2}$$

3.1.1 Action

The $\mathcal{N} = (1,1)$ supergravity action can be written as a difference to two Chern-Simons actions,

$$\begin{aligned} S_{sugra-AdS_3} &= S_{CS}[\Gamma]\big|_k - S_{CS}[\tilde{\Gamma}]\big|_{k'}, \\ S_{CS}[\Gamma] &= \frac{k}{4\pi}\int STr(\Gamma\wedge d\Gamma + \tfrac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma), \\ \text{where } \Gamma &= \left[A_{a\mu}\sigma^a + \psi_{+\mu}R^+ + \psi_{-\mu}R^-\right]dx^\mu \end{aligned} \tag{3.3}$$

where the gauge algebra for the two CS terms is $osp(1|2)$[1]. One can rewrite the the above Chern-Simons action in an explicit form for As and ψs as [23]

$$\begin{aligned} S_{CS}[\Gamma] &= \frac{k}{4\pi}\int\left[Tr(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A) + i\bar{\psi}\wedge D\psi\right], \\ \bar{\psi} &= \psi^T\gamma_0, \quad \text{where } \gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ &\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad D\psi = d\psi + \tfrac{1}{2}A^a\wedge\gamma_a\psi. \end{aligned} \tag{3.4}$$

Note that the above definition for $\bar{\psi}$ would not be used from here on in this thesis. The analysis can equivalently be carried out in the above form of the action and. We would prefer to use the former form with the $osp(1|2)$ algebra listed in the previous section. The equation of motion- as mentioned earlier, is implied by the flatness condition imposed on the two gauge fields valued in the adjoint of $osp(1|2)$.

3.1.2 Boundary conditions

In order to obtain the required generalization, we first notice the form of the gauge fields corresponding to the chiral boundary conditions written down in chapter 2. Referring back

[1] The product of two fermions differs by a factor of i from the standard Grasmann product ($(ab)^* = b^*a^*$); this amounts to using $STr(R^-R^+) = -STr(R^+R^-) = -i$ and $\{R^\pm, R^\pm\} = \mp i\sigma^\pm$, $\{R^\pm, R^\mp\} = i\sigma^0$.

to the analysis in section 2.3 we notice that the Virasoro Ward identity can be obtained even when the components $a^{(0)}$ and $\tilde{a}_-^{(-)}$ are put to zero. Therefore for ease of analysis, we write the following gauge fields which also yields the same chiral boundary conditions on the metric but the metric will no longer be in Fefferman-Graham gauge.

$$\begin{aligned} a &= [L_1 - \kappa L_{-1}]\, dx^+, \\ \tilde{a} &= [-L_{-1} + \tilde{\kappa} L_1]\, dx^- + f^{(a)} L_a dx^+, \\ A &= b^{-1}(d + a)b, \\ \tilde{A} &= b(d + \tilde{a})b^{-1}, \end{aligned} \tag{3.5}$$

where we have used the notations and conventions of Appendix(6.3). The same conclusions of section 2.3 can be reached by working with the above ansatz for the gauge fields and they would indeed correspond to the chiral boundary conditions studied in sections (2.3) and (2.2).

Here we note that the fluctuating field at the boundary comes from the $f^{(-1)}$ component of $\tilde{a}_+$ in (3.5). The components of $\tilde{a}_-$ play the role of sources *i.e.* functions that need to specified like a chemical potential. We further notice that all the $\tilde{a}_+$ components are *a priori* turned on and are determined in terms of $f^{(-1)}$. On the other hand, the $\tilde{a}_-$ component with leading r dependence is fixed to be -1, while only the sub-leading component, L_1 of $\tilde{a}_-$ is allowed to have functional dependence.

Therefore taking a cue from the above observation, we propose the following fall-off conditions:

$$\begin{aligned} \Gamma &= bdb^{-1} + bab^{-1}, \\ \tilde{\Gamma} &= b^{-1}db + b^{-1}\tilde{a}b, \\ \text{where } b &= e^{\sigma^0 \ln(r/\ell)}, \\ a &= [\sigma^- + L\sigma^+ + \psi_+ R^+]\, dx^+, \end{aligned}$$

$$\tilde{a} \;=\; \left[\sigma^+ + \bar{L}\sigma^- + \bar{\psi}_- R^-\right] dx^- + \left[\tilde{A}_{a+}\sigma^a + \tilde{\psi}_+ R^+ + \tilde{\psi}_- R^-\right] dx^+. \tag{3.6}$$

Here the dx^- component of the gauge field $\tilde{a}$ one form is that of a super-gauge field corresponding to Dirichlet boundary condition as given in [23, 22]. All functions above are *a priori* functions of both the boundary coordinates. The equations of motion imply

$$\begin{aligned}
\partial_+\Gamma_- - \partial_-\Gamma_+ + [\Gamma_+, \Gamma_-] &= 0, \\
\partial_+\tilde{\Gamma}_- - \partial_-\tilde{\Gamma}_+ + [\tilde{\Gamma}_+, \tilde{\Gamma}_-] &= 0.
\end{aligned} \tag{3.7}$$

For the left gauge field this implies that the functions are independent of the x^- co-ordinate. *i.e.* $\partial_- a = 0$.

$$\partial_- L = \partial_- \psi_+ = 0. \tag{3.8}$$

While for the right gauge field components, equation of motion allows one to express the sub-leading components of $\tilde{a}_+$ in r in terms of its leading ones *i.e.* $\tilde{A}_{++}, \tilde{\psi}_+$,

$$\begin{aligned}
\tilde{A}_{0+} &= \partial_- \tilde{A}_{++}, \\
\tilde{A}_{-+} &= \tilde{A}_{++}\bar{L} - \tfrac{1}{2}\partial_-^2 \tilde{A}_{++} + \tfrac{i}{2}\tilde{\psi}_+\bar{\psi}_-, \\
\tilde{\psi}_- &= \tilde{A}_{++}\bar{\psi}_- - \partial_-\tilde{\psi}_+.
\end{aligned} \tag{3.9}$$

The remaining relations imposed by equations of motion are differential equations relating components of $\tilde{a}_-$ and leading r components of $\tilde{a}_-$. These are interpreted as Ward identities for the boundary theory:

$$\begin{aligned}
\partial_+\bar{L} + \tfrac{1}{2}\partial_-^3\tilde{A}_{++} - 2\bar{L}\partial_-\tilde{A}_{++} - \tilde{A}_{++}\partial_-\bar{L} + i\bar{\psi}_-\left(\tilde{A}_{++}\bar{\psi}_- + \partial_-\tilde{\psi}_+\right) \;&+\; i\partial_-(\bar{\psi}_-\tilde{\psi}_+) = 0, \\
\partial_+\bar{\psi}_- - \partial_-[\tilde{A}_{++}\bar{\psi}_- - \partial_-\tilde{\psi}_+] - \tfrac{1}{2}\partial_-\tilde{A}_{++}\bar{\psi}_- - \bar{L}\tilde{\psi}_+ \;&=\; 0.
\end{aligned} \tag{3.10}$$

Here, conventionally (according to Brown-Henneaux analysis) the $\tilde{()}$ functions; $\tilde{A}_{++}, \tilde{\psi}_+$, are sources *i.e.* chemical potentials coupling to conserved currents labelled by $\bar{L}, \bar{\psi}_-$ respectively. But our boundary conditions would require that the currents $\bar{L}, \bar{\psi}_-$ play the role of sources. We will later choose these sources such that global AdS_3 is a part of the moduli space of bulk solutions, *i.e.* $\bar{L} = \frac{-1}{4}$ and $\bar{\psi} = 0$.

The boundary terms required to make the set of solutions considered above variationally well defined are:

$$S_{bndy} = \frac{k}{8\pi} \int_{\partial M} d^2x \; STr(-\sigma^0[\tilde{a}_+, \tilde{a}_-]) - 2\bar{L}_0 \sigma^- \tilde{a}_+ - \tfrac{1}{2}(\bar{\psi}_0)_- R^- \tilde{a}_+). \quad (3.11)$$

The variation of the total action therefore reads:

$$\delta S_{total} = \frac{k}{8\pi} \int_{\mathcal{M}} d^2x \; 2(\bar{L} - \bar{L}_0)\delta\tilde{A}_{++} + \tfrac{i}{2}(\bar{\psi}_- - (\bar{\psi}_0)_-)\delta\tilde{\psi}_+ \quad (3.12)$$

Here, choosing the fluctuations $\delta\tilde{A}_{++}$ and $\delta\tilde{\psi}_+$ to vanish would be similar to imposing Dirichlet type boundary condition, where as allowing for their fluctuations demands that the variational principal is satisfied when $\bar{L} = \bar{L}_0$ and $\bar{\psi}_- = \bar{\psi}_0$. We choose the later case as this would precisely imply fields fluctuating on the AdS boundary.

One may try and generalize the boundary conditions of Compère *et al* [6] by choosing only x^+ dependence for the fields fluctuating on the boundary -$\tilde{A}_{++}$ and $\tilde{\psi}_+$, for arbitrary values of the sources $\bar{L}_0$ and $\bar{\psi}_{0-}$; but when one solves the above Ward identities then one sees that $\tilde{\psi}_+$ cannot just be a function of x^+ unless $\bar{L} = 0$. Therefore extending the boundary conditions of [6] to the super-gravity case yields no super-symmetrization of the asymptotic symmetries of their boundary conditions in the pure gravity case.

3.1.3 Charges and asymptotic symmetry

We first solve the equation of motion (3.10) for a particular value of $\bar{L} = -\frac{1}{4}$ and $\bar{\psi}_- = 0$. This choice of $\bar{L}$ allows for global AdS_3 to be one of the allowed solutions. This implies that the boundary fields $\tilde{A}_{++}$ and $\tilde{\psi}_+$ take the following form:

$$\begin{aligned}\tilde{A}_{++} &= f(x^+) + g(x^+)e^{ix^-} + \bar{g}(x^+)e^{-ix^-},\\ \tilde{\psi}_+ &= \chi(x^+)e^{ix^-/2} + \bar{\chi}(x^+)e^{-ix^-/2}.\end{aligned} \tag{3.13}$$

The residual gauge transformations that leave $\tilde{a}$ form invariant are thus:

$$\begin{aligned}\tilde{\Lambda} &= \xi_a\sigma^a + \epsilon_\pm R^\pm,\\ \delta\tilde{a}_- &= d\tilde{\Lambda} + [\tilde{a}_-, \tilde{\Lambda}],\\ \Longrightarrow \xi_0 &= \partial_-\xi_+,\\ \xi_- &= -\tfrac{1}{4}(1+2\partial_-^2)\xi_+,\\ \epsilon_- &= -\partial_-\epsilon_+,\\ \partial_-(1+\partial_-^2)\xi_+ = \ 0 &= (\partial_-^2 + \tfrac{1}{4})\epsilon_+.\end{aligned} \tag{3.14}$$

One can solve for the residual gauge transformation parameters:

$$\begin{aligned}\xi_+ &= \lambda_f(x^+) + \lambda_g(x^+)e^{ix^-} + \bar{\lambda}_{\bar{g}}(x^+)e^{-ix^-},\\ \epsilon_+ &= \varepsilon_((x^+)e^{ix^-/2} + \bar{\varepsilon}_((x^+)e^{-ix^-/2}.\end{aligned} \tag{3.15}$$

The left gauge field components are independent of x^- and the corresponding residual gauge transformation are parametrized by:

$$\begin{aligned}\Lambda &= \zeta_a\sigma^a + \varepsilon_\pm R^\pm,\\ \delta a &= d\Lambda + [a, \Lambda],\\ \zeta_0 &= -\partial_+\zeta_-,\\ \zeta_+ &= -\tfrac{1}{2}\partial_+^2\zeta_- + \zeta_- L - i\psi_+\varepsilon_-,\\ \varepsilon_+ &= -\partial_+\varepsilon_- + \zeta_-\psi_+.,\\ 0 &= \partial_-\zeta_- = \partial_-\varepsilon_-,\end{aligned} \tag{3.16}$$

where all the parameters are determined in terms of $\zeta_-(x^+)$ and $\varepsilon_-(x^+)$. We observe that the arbitrary functions specifying the space of gauge fields a and $\tilde{a}$ and the space of residual gauge transformations are specified by functions of x^+ alone. Therefore the x^+ dependence of the functions will be suppressed further. The variation of the above parameters under the residual gauge transformations for the right sector are:

$$\begin{aligned}
\delta f &= \lambda_f' + 2i(g\bar{\lambda}_{\bar{g}} - \bar{g}\lambda_g) + i(\chi\bar{\varepsilon} + \bar{\chi}\varepsilon), \\
\delta g &= \lambda_g' + i(g\lambda_f - \lambda_g f) + i\chi\varepsilon, \\
\delta\bar{g} &= \bar{\lambda}_{\bar{g}}' - i(\bar{g}\lambda_f - \bar{\lambda}_{\bar{g}} f) + i\bar{\chi}\bar{\varepsilon}, \\
\delta\chi &= \varepsilon' + i[g\bar{\varepsilon} - \tfrac{f}{2}\varepsilon - \lambda_g\bar{\chi} + \tfrac{\lambda_f}{2}\chi], \\
\delta\bar{\chi} &= \bar{\varepsilon}' - i[\bar{g}\varepsilon - \tfrac{f}{2}\bar{\varepsilon} - \bar{\lambda}_{\bar{g}}\chi + \tfrac{\lambda_f}{2}\bar{\chi}]
\end{aligned} \tag{3.17}$$

Similar those for the left sector are:

$$\begin{aligned}
\delta L &= -\frac{1}{2}\zeta_-''' + [(\zeta_- L)' + \zeta_-' L] - i\left[\tfrac{1}{2}(\psi_+\varepsilon_-)' + \psi_+\varepsilon_-'\right] \\
\delta\psi_+ &= -\varepsilon_-'' + \left[(\zeta_-\psi_+)' + \tfrac{1}{2}\zeta_-'\psi_+\right] + L\varepsilon_- .
\end{aligned} \tag{3.18}$$

The charges corresponding to these transformation is given by:

$$\oint Q[\Lambda, \tilde{\Lambda}] = \frac{k}{2\pi}\int d\phi\, \left\{ Str[\Lambda, \delta a_\phi] - Str[\tilde{\Lambda}, \delta\tilde{a}_\phi]\right\}. \tag{3.19}$$

The above charge can be integrated and is finite. The charges for the two gauge fields decouple:

$$\begin{aligned}
Q[\tilde{\Lambda}] &= -\frac{k}{2\pi}\int d\phi[-\tfrac{f}{2}\lambda_f + g\bar{\lambda}_{\bar{g}} + \bar{g}\lambda_g + \chi\bar{\varepsilon} - \bar{\chi}\varepsilon], \\
Q[\Lambda] &= \frac{k}{2\pi}\int d\phi\, (\zeta_- L + i\varepsilon_-\delta\psi_+) .
\end{aligned} \tag{3.20}$$

This charge is the generator of canonical transformations on the space of solutions parametrized by set of functions F *via* the Poisson bracket.

$$\delta_\Lambda F = \{Q[\Lambda], F\} \tag{3.21}$$

Therefore the Poisson bracket algebra for the right sector is:

$$\begin{aligned}
&\{f(x^{+\prime}), f(x^+)\} = -2\alpha_Q\delta'(x^{+\prime} - x^+), && \{\chi(x^{+\prime}), f(x^+)\} = -i\alpha_Q\delta(x^{+\prime} - x^+)\chi, \\
&\{g(x^{+\prime}), f(x^+)\} = -2i\alpha_Q g(x^+)\delta(x^{+\prime} - x^+), && \{\bar\chi(x^{+\prime}), f(x^+)\} = i\alpha_Q\delta(x^{+\prime} - x^+)\bar\chi, \\
&\{\bar g(x^{+\prime}), f(x^+)\} = 2i\alpha_Q\bar g(x^+)\delta(x^{+\prime} - x^+), && \{\bar\chi(x^{+\prime}), g(x^+)\} = i\alpha_Q\delta(x^{+\prime} - x^+)\chi, \\
&\{\bar g(x^{+\prime}), g(x^+)\} = i\alpha_Q f(x^+)\delta(x^{+\prime} - x^+), && \{\chi(x^{+\prime}), \bar g(x^+)\} = -i\alpha_Q\delta(x^{+\prime} - x^+)\bar\chi. \\
&\qquad +\alpha_Q\delta'(x^{+\prime} - x^+) &&
\end{aligned} \tag{3.22}$$

While the fermionic Poisson brackets are:

$$\begin{aligned}
\{\bar\chi(x^{+\prime}), \chi(x^+)\} &= \tfrac{i\alpha_Q}{2} f(x^+)\delta(x^{+\prime} - x^+) + \alpha_Q\delta'(x^{+\prime} - x^+), \\
\{\chi(x^{+\prime}), \chi(x^+)\} &= i\alpha_Q g(x^+)\delta(x^{+\prime} - x^+), \\
\{\bar\chi(x^{+\prime}), \bar\chi(x^+)\} &= i\alpha_Q \bar g(x^+)\delta(x^{+\prime} - x^+).
\end{aligned} \tag{3.23}$$

where $\alpha_Q = \frac{2\pi}{k}$. Rescaling the above currents to:

$$f \to \tfrac{k}{4\pi} f, \quad g \to \tfrac{k}{2\pi} g, \quad \bar g \to \tfrac{k}{2\pi}\bar g, \quad \chi_\alpha \to \tfrac{k}{2\pi}\chi_\alpha, \quad \bar\chi_\alpha \to \tfrac{k}{2\pi}\bar\chi_\alpha, \tag{3.24}$$

and expanding them in the Fourier modes in x^+ yields the following commutators:

$$\begin{aligned}
&[f_m, f_n] = m\tfrac{k}{2}\delta_{m+n}, && [\chi_m, f_n] = \tfrac{1}{2}\chi_{m+n}, \\
&[g_m, f_n] = g_{m+n}, && [\bar\chi_m, f_n] = -\tfrac{1}{2}\bar\chi_{m+n}, \\
&[\bar g_m, f_n] = -\bar g_{m+n}, && [\bar\chi_m, g_n] = -\chi_{m+n}, \\
&[\bar g_m, g_n] = -2f_{m+n} - mk\delta_{m+n,0}, && [\chi_m, \bar g_n] = \bar\chi_{m+n},
\end{aligned} \tag{3.25}$$

and anti-commutators:

$$\{\chi_m, \chi_n\} = -g_{m+n}, \qquad \{\bar{\chi}_m, \bar{\chi}_n\} = -\bar{g}_{m+n},$$
$$\{\bar{\chi}_m, \chi_n\} = -f_{m+n} - km\delta_{m+n,0}. \tag{3.26}$$

This yields the familiar affine $sl(2,\mathbb{R})$ current algebra at level $k = c/6$ with two additional fermionic current parametrized by $\chi, \bar{\chi}$. These fermionic currents form a semi-direct sum with the $sl(2,\mathbb{R})$ current algebra elements with their anti-commutators yielding the later.

Similarly the left sector yields the familiar Brown-Henneaux result. We first redefine the currents by suitable scaling:

$$L \to \tfrac{k}{2\pi} L \ , \ \psi_+ \to \tfrac{k}{2\pi}\psi_+. \tag{3.27}$$

After these redefinitions one gets the following Poisson algebra:

$$\begin{aligned} \{L(x'^+), L(x^+)\} &= \frac{k}{4\pi}\delta'''(x'^+ - x^+) - \left(L(x'^+-) + L(x^+)\right)\delta'(x'^+ - x^+), \\ i\{\psi_+(x'^+), \psi_+(x^+)\} &= -\tfrac{k}{\pi}\delta''(x'^+ - x^+) + 2L(x'^+)\delta(x'^+ - x^+), \\ \{L(x'^+), \psi_+(x^+)\} &= -\left[\psi_+(x'^+) + \tfrac{1}{2}\psi_+(x^+)\right]\delta'(x'^+ - x^+). \end{aligned} \tag{3.28}$$

The Fourier modes for the above algebra satisfy the following Dirac brackets:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \tfrac{k}{2}m^3\delta_{m+n,0}, \\ \{\psi_{+m}, \psi_{+n}\} &= 2L_{m+n} + 2km^2\delta_{m+n,0}, \\ [L_m, \psi_{+n}] &= (\tfrac{m}{2} - n)\psi_{+(m+n)}. \end{aligned} \tag{3.29}$$

This is the Virasoro algebra with an affine super-current parametrized by ψ_+.

The above result may also be obtained by working in the Hilbert-Palatini action for super-gravity where the gauge transformations are recognized as diffeomorphisms and local Lorentz transformations. This exercise was done for the Brown-Henneaux case in

[54]. The bulk diffeomorphisms that respect the super-gravity generalization of Brown-Henneaux boundary condition would be generated by asymptotic Killing vectors and asymptotic Killing spinors. The spinor bi-linears of the later are again asymptotic Killing vectors. For the case of chiral boundary conditions dealt with in this chapter, it is difficult to define asymptotic charges via the formalism of Barnich, Brandt and Compère is one uses the approach of [54]. However one would essentially reach at the same set of asymptotic symmetries. Here too, one finds that bulk diffeomorphisms are generated by a set of generalized Killing spinors whose bi-linears are the vector fields generating residual diffeomorphisms found in chapter 2.

3.2 Generalization to extended AdS_3 super-gravity

Next, we will generalize the analysis of the last section to a extended super-gravity setting with negative cosmological constant. Since the number for gauge field components would now increase to include the once corresponding to the internal bosonic directions, it would be interesting to see whether the chiral boundary conditions proposed admit a unique non-trivial generalization. That is, we would seek boundary fall-off conditions for the gauge field components such that all of them admit a fluctuating mode at the AdS_3 boundary with the solutions elucidated in the pure gravity case being a subset. The $\mathcal{N}(4,4)$ case which would be of interest for realizing these boundary conditions in a string theoretic setting would therefore be a special case of this analysis.

One first begins by classifying the super algebras possible in AdS_3 [55, 2]. Let G denote the graded Lie algebra, such that $G = G_0 \oplus G_1$, where G_0 denotes the even part where as G_1 denotes the odd part. The even part, G_0 must contain a direct sum of $sl(2,\mathbb{R})$ and an internal symmetry algebra denoted by $\tilde{G}$. The fermions must transform in the 2-dimensional spinor representation of $sl(2,\mathbb{R})$. The dimension of the internal algebra $\tilde{G}$ is denoted by D while the fermions transform under a representation ρ ($dim\,\rho = d$) of $\tilde{G}$

which is real but not necessarily unitary. This is possible in the seven cases listed below:

G	$\tilde{G}$	D	ρ
$osp(N\|2,\mathbb{R})$	$so(N)$	$N(N-1)/2$	N
$su(1,1\|N)_{N\neq 2}$	$su(N)+u(1)$	N^2	$N+\bar{N}$
$su(1,1\|2)/u(1)$	$su(2)$	3	$2+\bar{2}$
$osp(4^*\|2M)$	$su(2)+usp(2M)$	M(2M+1)+3	$(2M,2)$
$D^1(2,1;\alpha)$	$su(2)+su(2)$	6	$(2,2)$
$G(3)$	G_2	14	7
$F(4)$	$spin(7)$	21	8_s

The super-traces defined above are consistent, invariant and non-degenerate with respect to the super-algebra defined above and would be used in defining the action and the charges. The detailed analysis is given in in Appendix (6.4).

3.2.1 Action

The gauge field is then written as a super gauge field valued in the adjoint of the above (graded Lie) super-algebra.:

$$\begin{aligned}\Gamma &= \left[A_{a\mu}\sigma^a + B_{a\mu}T^a + \psi_{+\alpha\mu}R^{+\alpha} + \psi_{-\alpha\mu}R^{-\alpha}\right]dx^\mu, \\ &= \Gamma_{a\mu}J^a dx^\mu. \end{aligned} \quad (3.30)$$

The gauge field as written above, separates as a sum of $sl(2,\mathbb{R})$, $\tilde{G}$ and fermionic one forms. The parameters A_a and B_a commute[2], while $\psi_{\pm\alpha}$s are anti-commuting Grasmann parameters. The gauge field Γ is a G valued one-form.

The super-gravity action is then given as the difference of two Chern-Simons action at

[2]It is understood in the above context that since A_a parametrizes the gauge one-form along the $sl(2,\mathbb{R})$, its index a runs from $\{0,+,-\}$. While the index a for the parameter B_a runs from $\{1..D\}$ along the internal $\tilde{G}$ basis.

level k written for two such gauge fields Γ and $\tilde{\Gamma}$.

$$S[\Gamma, \tilde{\Gamma}] = S_{CS}[\Gamma] - S_{CS}[\tilde{\Gamma}]. \tag{3.31}$$

We will concern ourselves with the case where both Γ and $\tilde{\Gamma}$ are valued in the same G. The cases where this is not so leads to chiral action and can be regarded as one of the ways to generate chiral asymptotic symmetries[3]. The detailed action is written out explicitly in terms of the specific super-gauge field components in Appendix (6.4.2).

3.2.2 Boundary conditions

Now, we would like to impose boundary conditions on the gauge fields- just as in the higher-spin case, which generalize the chiral boundary conditions on pure AdS_3 of chapter 2. Here we would allow one of the super-gauge fields- Γ, to obey the boundary conditions of [2] *i.e.* consistent with Brown-Henneaux, while proposing new boundary conditions on $\tilde{\Gamma}$. The review of Brown-Henneaux type boundary conditions would be done in the following analysis of Γ while the chiral boundary conditions generalized to extended super-gravity would be analysed in $\tilde{\Gamma}$.

The fall-off conditions in terms of the gauge fields are:

$$\begin{aligned}
\Gamma &= b d b^{-1} + b a b^{-1}, \\
\tilde{\Gamma} &= b^{-1} d b + b^{-1} \tilde{a} b, \\
\text{where } b &= e^{\sigma^0 \ln(r/\ell)}, \\
a &= [\sigma^- + L\sigma^+ + \psi_{+\alpha+} R^{+\alpha} + B_{a+} T^a]\, dx^+,
\end{aligned}$$

[3] At the level of the action there is no reason to suspect that the left and the right gauge fields represent left and right chiralities. A comment of such a nature can only be made in the light of suitable boundary conditions, which we know are necessary to be supplemented with a Chern-Simons theory on a non-compact gauge group. It so happens, that for the Brown-Henneaux boundary conditions the left and the right gauge fields do represent left and right chiralities, and it is with this understanding that we refer to them as two chiral sectors.

$$\tilde{a} = \left[\sigma^+ + \bar{L}\sigma^- + \bar{\psi}_{-\alpha-}R^{-\alpha} + \bar{B}_{a-}T^a\right]dx^- + \left[\tilde{A}_{a+}\sigma^a + \tilde{B}_{a+}T^a + \tilde{\psi}_{\pm\alpha+}R^{\pm\alpha}\right]dx^+. \tag{3.32}$$

Refer to Appendix (6.4.3) for more details. Provided they satisfy the following set of differential equations:

$$\begin{aligned}
&\partial_+\bar{L} + \tfrac{1}{2}\partial_-^3\tilde{A}_{++} - 2\bar{L}\partial_-\tilde{A}_{++} - \tilde{A}_{++}\partial_-\bar{L} \\
&+i\eta^{\alpha\beta}\bar{\psi}_{-\beta-}\left(\tilde{A}_{++}\bar{\psi}_{-\alpha-} + (\lambda^a)^\beta{}_\alpha\bar{B}_{a-}\tilde{\psi}_{+\beta+} + \partial_-\tilde{\psi}_{+\alpha+}\right) + i\eta^{\alpha\beta}\partial_-(\bar{\psi}_{-\beta-}\tilde{\psi}_{+\alpha+}) = 0, \\
&\partial_+\bar{B}_{a-} - \partial_-\tilde{B}_{a+} + f^{bc}{}_a\tilde{B}_{b+}\bar{B}_{c-} + i\tfrac{d-1}{2C_\rho}(\lambda^a)^{\alpha\beta}\tilde{\psi}_{+\alpha+}\bar{\psi}_{-\beta-} = 0, \\
&\partial_+\bar{\psi}_{-\alpha-} - \partial_-[\tilde{A}_{++}\bar{\psi}_{-\alpha-} - \partial_-\tilde{\psi}_{+\alpha+} + (\lambda^a)^\beta{}_\alpha\bar{B}_{a-}\tilde{\psi}_{+\beta+}] - \tfrac{1}{2}\partial_-\tilde{A}_{++}\bar{\psi}_{-\alpha-} \\
&+(\lambda^a)^\beta{}_\alpha\bar{B}_{a-}[\tilde{A}_{++}\bar{\psi}_{-\beta-} - \partial_-\tilde{\psi}_{+\beta+} + (\lambda^a)^\gamma{}_\beta\bar{B}_{b-}\tilde{\psi}_{+\gamma+}] - (\lambda^a)^\beta_\alpha\tilde{B}_{a+}\bar{\psi}_{-\beta-} - \bar{L}\tilde{\psi}_{+\alpha+} = 0.
\end{aligned} \tag{3.33}$$

These are the Ward identities expected to be satisfied by the induced gravity theory on the boundary. Here, conventionally, the $\tilde{()}$ functions $\tilde{A}_{++}, \tilde{B}_{a+}, \tilde{\psi}_{+\alpha+}$, are sources coupling to conserved currents $\bar{L}, \bar{B}_{a-}, \bar{\psi}_{-\alpha-}$, respectively. Since it is the sources which are the ones which survive on the boundary- as can be seen if one plugs back the r dependence, we would like to solve these differential equations for particular value of the currents. Therefore, as expected yielding an effective theory of of fluctuating sources on the boundary. We will later choose the $\bar{()}$ functions such that global AdS_3 is a part of the moduli space of bulk solutions.*i.e.* $\bar{L} = \frac{-1}{4}$ and $\bar{B} = 0 = \bar{\psi}$.

The boundary term to be added so as to make the right sector ansatz variationally well defined is given by:

$$\begin{aligned}
S_{bndy} = \frac{k}{8\pi}\int_{\partial M} & d^2x\, STr(-\sigma^0[\tilde{a}_+, \tilde{a}_-]) - 2\bar{L}_0\sigma^-\tilde{a}_+ + (\tfrac{d-1}{2C_\rho})^2 T^aT^b STr(\tilde{a}_+T_a)STr(\tilde{a}_-T_b) \\
& -2(\tfrac{d-1}{2C_\rho})\bar{B}_{0a}T^aT^b STr(\tilde{a}_+T^b) - \tfrac{1}{2}(\bar{\psi}_0)_{-\alpha}R^{-\alpha}\tilde{a}_+).
\end{aligned} \tag{3.34}$$

This implies the following desired variation of the total action:

$$\delta S_{total} = \frac{k}{8\pi}\int_{\mathcal{M}} d^2x\ 2(\bar{L}-\bar{L}_0)\delta\tilde{A}_{++} + 2(\tfrac{2C_\rho}{d-1})(\bar{B}_{a-}-\bar{B}_{0a})\delta\tilde{B}_{a+} + \tfrac{i}{2}(\bar{\psi}_{-\alpha-}-(\bar{\psi}_0)_{-\alpha})\delta\tilde{\psi}_{+\alpha+}\eta^{\alpha\beta} \tag{3.35}$$

Here, one has an option of choosing $\delta\tilde{()}$ functions to vanish at the AdS asymptote, implying a Brown-Henneaux type boundary condition where $\tilde{A}_{++}, \tilde{B}_{a+}, \tilde{\psi}_{+\alpha+}$ act as chemical potentials. Or, alternatively, treat $\bar{L}, \bar{B}_{0a}, (\bar{\psi}_0)_{-\alpha}$ as chemical potentials allowing $\tilde{A}_{++}, \tilde{B}_{a+}, \tilde{\psi}_{+\alpha+}$ to fluctuate. Thus describing a theory of induced gravity on the boundary. In our present case, we would be choosing the later by fixing $\bar{L} = -1/4, \bar{B}_{0a} = 0 = (\bar{\psi}_0)_{-\alpha}$. Thus the variational principle is satisfied for configurations with $\bar{L} = \frac{-1}{4}$ and $\bar{B}_{a-} = 0 = \bar{\psi}_{-\alpha-}$ which describes global AdS_3.

It turns out that one can choose the global AdS_3 configuration as the one with the zero asymptotic charge *i.e.* the vacuum for the above boundary condition. If on the other hand one chooses $\bar{L} = 0$ - which corresponds to an extremal BTZ blackhole, then one gets the generalization of Compère *et al*'s [48] to minimal supergravity. Here the vacuum can be chosen to be extremal BTZ with global AdS_3 configuration not being a part of the allowed space of solution. It is however interesting to note that allowing for global AdS_3 to be part of the solution space allows extremal BTZ to be in the space of solutions while disallowing non-extremal BTZ solutions. In other words, the only BTZ black-hole configuration allowed by the above boundary condition is an extremal one.

3.2.3 Charges and symmetries

Just as in the previous chapters, one needs to find the space of gauge transformations that maintains the above form of the gauge fields, thus inducing transformations on the functions $\tilde{A}_{a+}, \tilde{B}_{a+}, \tilde{\psi}_{+\alpha+}, L, B_a, \psi_{+\alpha+}$ which parametrize the space of solutions. Once this is achieved, one can define asymptotic conserved charge associated with the change induced by such residual gauge transformations on the space of solutions. For the boundary conditions to be well defined, this asymptotic charge must be finite and be integrable on the space of solutions.

Left sector

The analysis of the left sector is exactly as in [2] and is covered in the Appendix (6.4.4). One basically gets the super-Virasoro with quadratic non-linearities as the asymptotic algebra for the modes of parameters labelling the left sector gauge filed Γ.

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{k}{2}m^3\delta_{m+n,0}, \\
\left[B^a_m, B^b_n\right] &= -f^{abc}B^c_{m+n} + \tfrac{2kC_\rho}{d-1}m\delta^{ab}\delta_{m+n,0}, \\
[L_m, B^a_n] &= -nB^a_{m+n}, \\
\left\{(\psi_{+\alpha})_m, (\psi_{+\beta})_n\right\} &= 2\eta_{\alpha\beta}L_{m+n} - 2i\tfrac{d-1}{2C_\rho}(m-n)(\lambda^a)_{\alpha\beta}(B_a)_{m+n} \\
&+ 2k\eta_{\alpha\beta}m^2\delta_{m+n,0} \\
&- k(\tfrac{d-1}{2kC_\rho})^2\left[\left\{\lambda^a, \lambda^b\right\}_{\alpha\beta} + \tfrac{2C_\rho}{d-1}\eta_{\alpha\beta}\delta^{ab}\right](B_aB_b)_{m+n}, \\
[L_m, (\psi_{+\alpha})_n] &= (\tfrac{m}{2}-n)(\psi_{+\alpha})_{m+n}, \\
[B^a_m, (\psi_{+\alpha})_n] &= i(\lambda^a)^\beta_{\ \alpha}(\psi_{+\beta})_{m+n}.
\end{aligned} \tag{3.36}$$

This is the non-linear super-conformal algebra or the super-Virosoro algebra. The central extension is $k = c/6$, and is the same for all the seven cases listed in the table previously. This algebra, although a supersymmetric extension of the Virasoro algebra, is not a graded Lie algebra in the sense that the right-hand sides of the fermionic (Rarita-Schwinger) anti-commutators contains quadratic non-linearities in currents for the internal symmetry directions.

Right sector

The analysis of the right sector is similar to the one covered in the $\mathcal{N} = (1,1)$ case and is covered in the Appendix (6.4.4). Here we choose $\bar{L} = -\frac{1}{4}, \bar{B} = 0 = \bar{\psi}$ as the values for the chemical potential as it would allow for global AdS_3 as one of the solutions. The asymptotic symmetry algebra in Fourier modes of the parameters labelling the right gauge field $\tilde{\Gamma}$ is

$$
\begin{aligned}
&[f_m, f_n] = m\tfrac{k}{2}\delta_{m+n,0}, && [(\chi_\alpha)_m, f_n] = \tfrac{1}{2}(\chi_\alpha)_{(m+n)},\\
&[g_m, f_n] = g_{m+n}, && [(\bar{\chi}_\alpha)_m, f_n] = -\tfrac{1}{2}(\bar{\chi}_\alpha)_{(m+n)},\\
&[\bar{g}_m, f_n] = -\bar{g}_{m+n}, && [(\bar{\chi}_\alpha)_m, g_n] = -(\chi_\alpha)_{m+n},\\
&[\bar{g}_m, g_n] = -2f_{m+n} - mk\delta_{m+n,0}, && [(\chi_\alpha)_m, \bar{g}_n] = (\bar{\chi}_\alpha)_{m+n},
\end{aligned}
$$

$$
\begin{aligned}
&\{(\chi_\alpha)_m, (\chi_\beta)_n\} = -\eta_{\alpha\beta}g_{m+n}, && \{(\bar{\chi}_\alpha)_m, (\bar{\chi}_\beta)_n\} = -\eta_{\alpha\beta}\bar{g}_{m+n},\\
&[(\tilde{B}_{a+})_m, (\chi_\beta)_n] = -(\tfrac{d-1}{2C_\rho})(\lambda_a)^\alpha{}_\beta(\chi_\alpha)_{(m+n)}, && [(\tilde{B}_{a+})_m, (\bar{\chi}_\beta)_n] = -(\tfrac{d-1}{2C_\rho})(\lambda_a)^\alpha{}_\beta(\bar{\chi}_\alpha)_{(m+n)},
\end{aligned}
$$

$$
\begin{aligned}
&[(\tilde{B}_{a+})_m, (\tilde{B}_{b+})_n] = -i(\tfrac{d-1}{2C_\rho})f_{ab}{}^c(\tilde{B}_{c+})_{(m+n)} - (\tfrac{d-1}{2C_\rho})km\delta_{ab}\delta_{m+n,0},\\
&\{(\bar{\chi}_\alpha)_m, (\chi_\beta)_n\} = -\eta_{\alpha\beta}f_{(m+n)} + i(\lambda^a)_{\alpha\beta}(\tilde{B}_{c+})_{(m+n)} - km\eta_{\alpha\beta}\delta_{m+n,0}.
\end{aligned}
\tag{3.37}
$$

This is the affine Kaĉ-Moody super-algebra. Here, it is evident that the central extension to the $sl(2,\mathbb{R})$current sub-algebra spanned by $(f, g, \bar{g})$ is $k = c/6$. The quadratic non-linearities that occur in the super-Virasoro are not present here.

Thus demanding that one considers all types of fields $(\tilde{A}, B, \tilde{\psi})$ have fluctuating components on the boundary of asymptotic AdS_3, we have constructed a unique generalization of the boundary condition studied in chapter 2 to extended super-gravity in asymptotically AdS_3 spaces. In doing so we uncovered the expected super-Virasoro algebra with quadratic non-linearities for the left sector and a Kaĉ-Moody super-current algebra at level $k = c/6$ for the right sector. Here, we have demanded as before that the global AdS_3 remain in the space of allowed solutions.

3.3 Boundary conditions for holographic induced super-Liouville theory

We now turn to generalizing boundary conditions proposed by [5] to extended supergravity in AdS_3 The boundary conditions proposed by [5] allow the boundary metric to fluctuate upto a conformal factor; demanding that the boundary metric have vanishing Ricci curvature. This corresponds to the $\chi = 0$ case in chapter 2, in (2.44). The notations and conventions below are taken from Henneaux *et al* [2] and are summarized in Appendix (6.4.1).

We would like to carry out this analysis in the Chern-Simons formulation of gravity in $d = 3$, as the supergravity fields can be conveniently encapsulated as components of the gauge fields along the graded gauge algebra which comprise the super-gravity algebra. Like in the case for boundary conditions for induced Liouville theory in chapter 2, the analysis for the two gauge fields are identical; hence we will give the details of only one of the gauge fields - $\tilde{A}$. One begins with an ansatz,

$$\begin{aligned}
\tilde{A} &= b\tilde{a}b^{-1} + bdb^{-1}, \\
\tilde{a} &= \left[e^{-\tilde{\Phi}}\tilde{\kappa}\sigma^{-} + e^{\tilde{\Phi}}\sigma^{+} + \tilde{B}_{a-}T^{a} + \tilde{\psi}_{+\alpha-}R^{+\alpha} + \tilde{\psi}_{-\alpha-}R^{-\alpha}\right]dx^{-}, \\
\partial_{+}\tilde{a} &= 0, \quad b = e^{\sigma^{0}\ln(r/\ell)}. \qquad (3.38)
\end{aligned}$$

The equation of motion, $\partial_{+}\tilde{a}_{-} - \partial_{-}\tilde{a}_{+} + [\tilde{a}_{+}, \tilde{a}_{-}] = 0$ is readily satisfied. The above form of the gauge field ansatz doesn't need extra boundary terms added to the Chern-Simons action to make it variationally consistent. This is made apparent due to the fact that the gauge field $\tilde{a}$ as an 1-form only has a dx^{-} component along the boundary while the fluctuation of the action yields,

$$\delta S_{CS}[\tilde{A}] = \tfrac{k}{2\pi}\int_{\partial\mathcal{M}} Str[\tilde{a}\wedge\delta\tilde{a}]. \tag{3.39}$$

We now look for the space of gauge transformations which keep the above 1−form $\tilde{a}$ form invariant.

$$\delta_{\tilde{\Lambda}}\tilde{a} = d\tilde{\Lambda} + [\tilde{a},\tilde{\Lambda}] \implies \delta_{\tilde{\Lambda}}\tilde{a}\Big|_{\sigma^0} = 0,\ \partial_+\tilde{\Lambda} = 0 \tag{3.40}$$

where

$$\tilde{\Lambda} = \tilde{\xi}_a\sigma^a + \tilde{b}_aT^a + \tilde{\epsilon}_{\pm\alpha}R^{\pm\alpha}. \tag{3.41}$$

Solving these constraints on $\tilde{\Lambda}$, we get

$$\begin{aligned}
\delta_{\tilde{\Lambda}}\tilde{a}\Big|_{\sigma^0} &= 0,\\
\implies \tilde{\xi}_- &= -\tfrac{1}{2}e^{-\tilde{\Phi}}\partial_-\tilde{\xi}_0 + e^{-2\tilde{\Phi}}\tilde{\kappa}\tilde{\xi}_+ - \tfrac{1}{2}e^{-\tilde{\Phi}}\eta^{\alpha\beta}(\tilde{\psi}_{+\alpha-}\tilde{\epsilon}_{-\beta} + \tilde{\psi}_{-\alpha-}\tilde{\epsilon}_{+\beta}).
\end{aligned} \tag{3.42}$$

The variations these induce on the functions parametrizing the 1−form $\tilde{a}$ are

$$\begin{aligned}
\delta_{\tilde{\Lambda}}\tilde{\Phi} &= e^{-\tilde{\Phi}}\partial_-\tilde{\xi}_+ - \tilde{\xi}_0 + ie^{-\tilde{\Phi}}\eta^{\alpha\beta}\tilde{\psi}_{+\alpha-}\tilde{\epsilon}_{+\beta},\\
\delta_{\tilde{\Lambda}}\tilde{\kappa} &= -\tfrac{1}{2}\partial_-^2\tilde{\xi}_0 + \tfrac{1}{2}\partial_-\tilde{\Phi}\partial_-\tilde{\xi}_0 - 2\partial_-\tilde{\Phi}e^{-\tilde{\Phi}}\tilde{\kappa}\tilde{\xi}_+e^{-\tilde{\Phi}} + \partial_-(\tilde{\kappa}\tilde{\xi}_+),\\
&\quad -\tfrac{i}{2}\left[\partial_-\tilde{\Phi} - \partial_-\right]\eta^{\alpha\beta}(\tilde{\psi}_{+\alpha-}\tilde{\epsilon}_{-\beta} + \tilde{\psi}_{-\alpha-}\tilde{\epsilon}_{+\beta}) + e^{-\tilde{\Phi}}\tilde{\kappa}\partial_-\tilde{\xi}_+\\
&\quad -i\eta^{\alpha\beta}\left[e^{\tilde{\Phi}}\tilde{\psi}_{-\alpha-}\tilde{\epsilon}_{-\beta} - e^{-\tilde{\Phi}}\tilde{\kappa}\tilde{\psi}_{+\alpha-}\tilde{\epsilon_{+\beta}}\right],\\
\delta_{\tilde{\Lambda}}\tilde{B}_{a-} &= \partial_-\tilde{b}_a + f_a{}^{bc}\tilde{B}_{b-}\tilde{b}_c - i\tfrac{d-1}{2C_\rho}(\lambda^a)^{\alpha\beta}(\tilde{\psi}_{-\alpha-}\tilde{\epsilon}_{+\beta} - \tilde{\psi}_{+\alpha-}\tilde{\epsilon}_{-\beta}),\\
\delta_{\tilde{\Lambda}}\tilde{\psi}_{+\alpha-} &= \partial_-\tilde{\epsilon}_{+\alpha} - \tfrac{1}{2}\tilde{\psi}_{+\alpha-}\tilde{\xi}_0 + (e^{\tilde{\Phi}}\tilde{\epsilon}_{-\alpha} - \tilde{\psi}_{-\alpha-}\tilde{\xi}_+) - (\lambda^a)^\beta{}_\alpha(\tilde{B}_{a-}\tilde{\epsilon}_{+\beta} - \tilde{b}_a\tilde{\psi}_{+\beta-}),\\
\delta_{\tilde{\Lambda}}\tilde{\psi}_{-\alpha-} &= \partial_-\tilde{\epsilon}_{-\alpha} + \tfrac{1}{2}\tilde{\psi}_{-\alpha-}\tilde{\xi}_0 + e^{\tilde{\Phi}}\tilde{\kappa}\tilde{\epsilon}_{+\alpha} - (\lambda^a)^\beta{}_\alpha(\tilde{B}_{a-}\tilde{\epsilon}_{-\beta} - \tilde{b}_a\tilde{\psi}_{-\beta-})\\
&\quad -\tilde{\psi}_{+\alpha+}\left[-\tfrac{1}{2}e^{-\tilde{\Phi}}\partial_-\tilde{\xi}_0 + e^{-2\tilde{\Phi}}\tilde{\kappa}\tilde{\xi}_+ + \tfrac{i}{2}e^{-\tilde{\Phi}}\eta^{\alpha\beta}(\tilde{\psi}_{+\alpha-}\tilde{\epsilon}_{-\beta} - \tilde{\psi}_{-\alpha-}\tilde{\epsilon}_{+\beta})\right].
\end{aligned} \tag{3.43}$$

Associated to the above fluctuations that keeps the gauge field $\tilde{a}$ in the same form, are infinitesimal variations of a well defined asymptotic charge.

$$\begin{aligned} \not{\delta}\tilde{Q} &= -\tfrac{k}{2\pi}\int d\phi\, Str[\tilde{\Lambda}\delta\tilde{a}_\phi], \\ &= -\tfrac{k}{2\pi}\int d\phi \left\{\tilde{\xi}_-\delta e^{\tilde{\Phi}} + \tilde{\xi}_+\delta(e^{-\tilde{\Phi}}\tilde{\kappa}) - i\eta^{\alpha\beta}(\delta\tilde{\psi}_{+\alpha-}\tilde{\epsilon}_{-\beta} - \delta\tilde{\psi}_{-\alpha-}\tilde{\epsilon}_{+\beta}) + \tfrac{2C_\rho}{d-1}\delta\tilde{B}_{a-}b^a\right\} \end{aligned} \tag{3.44}$$

The next step is to be able to write the above change in the charge such that it is a total variation $\delta\tilde{Q}$ so that δ can be taken out of the integral and (the charge) can be integrated from a suitable point (vacuum) on the solution space to any arbitrary point. Therefore, it is important to recognize field independent parameters parametrizing the gauge transformations and accordingly functions of the fields parametrizing the space of solutions on whom a phase space can be defined via the Poisson brackets induced by the integrated charge.

The above expression for $\not{\delta}\tilde{Q}$ can be simplified if one redefines

$$\tilde{\psi}_{\pm\alpha-} = \tilde{\Psi}_{\pm\alpha-}e^{\pm\tilde{\Phi}}, \quad \tilde{\epsilon}_{\pm\alpha} = \varepsilon_{\pm\alpha}e^{\pm\tilde{\Phi}}, \quad \tilde{\xi}_+ = \tilde{\Xi}_+ e^{\tilde{\Phi}}. \tag{3.45}$$

$$\delta\tilde{Q} = -\tfrac{k}{2\pi}\int d\phi\, \left\{\tfrac{1}{2}\delta\tilde{\Phi}'\tilde{\xi}_0 + \tilde{\Xi}_+\delta\tilde{\kappa} + \tfrac{2C_\rho}{d-1}\delta\tilde{B}_{a-}\tilde{b}^a - i\eta^{\alpha\beta}\left[\delta\tilde{\Psi}_{+\alpha-}\tilde{\varepsilon}_{-\beta} - \delta\tilde{\Psi}_{-\alpha-}\tilde{\varepsilon}_{+\beta}\right]\right\} \tag{3.46}$$

Therefore the total integrated charge is

$$\tilde{Q} = -\tfrac{k}{2\pi}\int d\phi\, \left\{\tfrac{1}{2}\tilde{\Phi}'\tilde{\xi}_0 + \tilde{\Xi}_+\tilde{\kappa} + \tfrac{2C_\rho}{d-1}\tilde{B}_{a-}\tilde{b}^a - i\eta^{\alpha\beta}\left[\tilde{\Psi}_{+\alpha-}\tilde{\varepsilon}_{-\beta} - \tilde{\Psi}_{-\alpha-}\tilde{\varepsilon}_{+\beta}\right]\right\} \tag{3.47}$$

The redefined gauge transformation parameters in (3.45) are therefore to be considered field independent. Also, one sees that it is $\tilde{\Phi}'$ and not $\tilde{\Phi}$ which is appropriate for defining the phase space structure on the space of solutions. The variations of the redefined fields in terms of the field independent gauge parameters are,

$$\begin{aligned} \delta_{\tilde{\Lambda}}\tilde{\Phi}' &= \partial_-(\tilde{\Phi}'\tilde{\Xi}_+) + \partial_-^2\tilde{\Xi}_+ - \partial_-\tilde{\xi}_0 + i\eta^{\alpha\beta}(\partial_-\tilde{\Psi}_{+\alpha-}\tilde{\varepsilon}_{+\beta} + \tilde{\Psi}_{+\alpha-}\partial_-\tilde{\varepsilon}_{+\beta}), \\ \delta_{\tilde{\Lambda}}\tilde{\kappa} &= 2\tilde{\kappa}\partial_-\tilde{\Xi}_+ + \partial_-\tilde{\kappa}\tilde{\Xi}_+ - \tfrac{1}{2}\partial_-^2\tilde{\xi}_0 + \tfrac{1}{2}\tilde{\Phi}'\partial_-\tilde{\xi}_0 \\ &\quad -\tfrac{i}{2}\eta^{\alpha\beta}\Big[(\tilde{\Phi}'\tilde{\Psi}_{+\alpha-} - \partial_-\tilde{\Psi}_{+\alpha-} - \tilde{\Psi}_{+\alpha-}\partial_- + 2\tilde{\Psi}_{-\alpha-})\tilde{\varepsilon}_{-\beta} \end{aligned}$$

$$
\begin{aligned}
& +(\tilde{\Phi}'\tilde{\Psi}_{-\alpha-} - \partial_-\tilde{\Psi}_{-\alpha-} - 2\tilde{\kappa}\tilde{\Psi}_{+\alpha-} - \tilde{\Psi}_{-\alpha-}\partial_-)\tilde{\varepsilon}_{+\beta}\Big], \\
\delta_{\tilde{\Lambda}}\tilde{\Psi}_{+\alpha-} &= -\tfrac{1}{2}\tilde{\Psi}_{+\alpha-}\Big[\tilde{\Phi}'\tilde{\Xi}_+ + i\eta^{\rho\sigma}\tilde{\Psi}_{+\rho-}\tilde{\varepsilon}_{+\sigma}\Big] + \partial_-\tilde{\varepsilon}_{+\alpha} + \tfrac{1}{2}\tilde{\Phi}'\tilde{\varepsilon}_{+\alpha} + \tilde{\varepsilon}_{-\alpha} - \tilde{\Psi}_{-\alpha-}\tilde{\Xi}_+ \\
& -(\lambda^a)^\beta{}_\alpha(\tilde{B}_{a-}\tilde{\varepsilon}_{+\beta} - \tilde{b}_a\tilde{\Psi}_{+\alpha-}), \\
\delta_{\tilde{\Lambda}}\tilde{\Psi}_{-\alpha-} &= \partial_-\tilde{\varepsilon}_{-\alpha} - \tfrac{1}{2}\tilde{\Phi}'\tilde{\varepsilon}_{-\alpha} + \tilde{\kappa}\tilde{\varepsilon}_{+\alpha} - (\lambda^a)^\beta{}_\alpha(\tilde{B}_{a-}\tilde{\varepsilon}_{-\beta} - \tilde{b}_a\tilde{\Psi}_{-\beta-}) \\
& -\tilde{\Psi}_{+\alpha-}\Big[-\tfrac{1}{2}\partial_-\tilde{\xi}_0 + \tilde{\kappa}\tilde{\Xi}_+ + i\eta^{\alpha\beta}(\tilde{\Psi}_{+\alpha-}\tilde{\varepsilon}_{-\beta} - \tilde{\Psi}_{-\alpha-}\tilde{\varepsilon}_{+\beta})\Big] \\
& +\tfrac{1}{2}\tilde{\Psi}_{-\alpha-}\Big[\tilde{\Psi}'\tilde{\Xi}_+ + \partial_-\tilde{\Xi}_+ + i\eta^{\alpha\beta}\tilde{\Psi}_{+\alpha-}\tilde{\varepsilon}_{+\beta}\Big], \\
\delta_{\tilde{\Lambda}}\tilde{B}_{a-} &= \partial_-\tilde{b}_a + f_a{}^{bc}\tilde{B}_{a-}\tilde{b}^a - i\tfrac{d-1}{2C\rho}(\lambda^a)^{\alpha\beta}(\tilde{\Psi}_{-\alpha-}\tilde{\varepsilon}_{+\beta} - \tilde{\Psi}_{+\alpha-}\tilde{\varepsilon}_{-\beta}). \qquad (3.48)
\end{aligned}
$$

This leads to the following Poisson brackets amongst the solution space variables

$$
\begin{aligned}
\tfrac{-k}{2\pi}\left\{\tilde{\kappa}(x^{-\prime}), \tilde{\kappa}(x^-)\right\} &= -2\tilde{\kappa}(x^-)\delta'(x^{-\prime} - x^-) + \tilde{\kappa}'(x^-)\delta(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{\Phi}'(x^{-\prime}), \tilde{\kappa}(x^-)\right\} &= -\tilde{\Phi}'(x^-)\delta'(x^{-\prime} - x^-) - \delta''(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{\Psi}_{+\alpha-}(x^{-\prime}), \tilde{\kappa}(x^-)\right\} &= \Big[\tfrac{1}{2}\tilde{\Phi}'\tilde{\Psi}_{+\alpha-} - \tfrac{1}{2}\partial_-\tilde{\Psi}_{+\alpha-} + \tilde{\Psi}_{-\alpha-}\Big](x^-)\delta(x^{-\prime} - x^-) \\
& +\tfrac{1}{2}\tilde{\Psi}_{+\alpha-}(x^-)\delta'(x^{-\prime} - x^-),
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{-k}{2\pi}\left\{\tilde{\Psi}_{-\alpha-}(x^{-\prime}), \tilde{\kappa}(x^-)\right\} &= \Big[-\tfrac{1}{2}\tilde{\Phi}'\tilde{\Psi}_{-\alpha-} + \tfrac{1}{2}\partial_-\tilde{\Psi}_{-\alpha-} - \tilde{\kappa}\tilde{\Psi}_{+\alpha-}\Big](x^-)\delta(x^{-\prime} - x^-) \\
& -\tfrac{1}{2}\tilde{\Psi}_{-\alpha-}\delta'(x^{-\prime} - x^-),
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{-k}{2\pi}\left\{\tilde{\Phi}'(x^{-\prime}), \tilde{\Phi}'(x^-)\right\} &= 2\delta'(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{\Psi}_{-\alpha-}(x^{-\prime}), \tilde{\Phi}'(x^-)\right\} &= -\tilde{\Psi}_{+\alpha-}(x^{-\prime})\delta'(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{\Psi}_{+\alpha-}(x^{-\prime}), \tilde{\Psi}_{+\beta-}(x^-)\right\} &= -i\eta_{\alpha\beta}\delta(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{\Psi}_{-\alpha-}(x^{-\prime}), \tilde{\Psi}_{+\beta-}(x^-)\right\} &= -i\eta_{\alpha\beta}\delta'(x^{-\prime} - x^-) \\
& +\Big[\tfrac{i}{2}\eta_{\alpha\beta}\tilde{\Phi}' - i(\lambda^a)_{\alpha\beta}\tilde{B}_{a-} + \tfrac{1}{2}\tilde{\Psi}_{+\beta-}\tilde{\Psi}_{+\alpha-}\Big](x^-)\delta(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{\Psi}_{-\alpha-}(x^{-\prime}), \tilde{\Psi}_{-\beta-}(x^-)\right\} &= \Big[i\tilde{\kappa}\eta_{\alpha\beta} - \tilde{\Psi}_{+\beta-}\tilde{\Psi}_{-\alpha-} - \tfrac{1}{2}\tilde{\Psi}_{-\beta-}\tilde{\Psi}_{+\alpha-}\Big](x^-)\delta(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{B}_{a-}(x^{-\prime}), \tilde{\Psi}_{+\alpha-}(x^-)\right\} &= \tfrac{d-1}{2C\rho}(\lambda_a)^\beta{}_\alpha\tilde{\Psi}_{+\beta-}(x^-)\delta(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{B}_{a-}(x^{-\prime}), \tilde{\Psi}_{-\alpha-}(x^-)\right\} &= \tfrac{d-1}{2C_\rho}(\lambda_a)^\beta{}_\alpha\tilde{\Psi}_{-\beta-}(x^-)\delta(x^{-\prime} - x^-), \\
\tfrac{-k}{2\pi}\left\{\tilde{B}_{a-}(x^{-\prime}), \tilde{B}_{b-}(x^-)\right\} &= -\tfrac{d-1}{2C_\rho}\Big[\delta_{ab}\delta'(x^{-\prime} - x^-) - f_{ab}{}^c\tilde{B}_{c-}\delta(x^{-\prime} - x^-)\Big]. \qquad (3.49)
\end{aligned}
$$

The first Possion bracket among $\tilde{\kappa}$s can be easily recognized as that of the Witt algebra *i.e*. Virasoro without the central extension. The phase-space variable $\tilde{\Phi}'$ can be used to generate the central term in the Witt algebra of $\tilde{\kappa}$ by redefining

$$\hat{\tilde{\kappa}} = \tilde{\kappa} + \alpha_c \tilde{\Phi}''. \qquad (3.50)$$

After further rescaling $\hat{\tilde{\kappa}} \rightarrow \frac{k}{2\pi}$ one can show the central term in the Virasoro of $\hat{\tilde{\kappa}}$ to be $2\alpha_c(\alpha_c + 1)k/(2\pi)$; which for $\alpha_c = -\frac{1}{2} \pm \frac{1}{\sqrt{2}}$ yields the central extension of the Virasoro found in Brown-Henneaux analysis.

We conclude this chapter mentioning some salient points. The Brown-Henneaux type boundary conditions adapted to super-gravity in AdS_3 [2] yield the super-Virasoro algebra which has quadratic non-linearities, while the current algebra obtained as a result of imposing chiral boundary conditions yielding the super-Kaĉ-Moody current algebra do not have such non-linearities on the *rhs*. We will see that this seems to be a recurrent feature of the chiral boundary conditions imposed on gauge theories on AdS_3. We further shown that the super-symmetrizing the boundary conditions of [5] yields a super extension of the harmonic Weyl transformations as the asymptotic algebra. Thus we see that the Virosoro can be a sub-algebra of a new infinite dimensional super-algebra apart from the super-Virasoro uncovered in [2].

Chapter 4

Holographic chiral induced W-gravities

In this chapter we turn to generalization of chiral boundary conditions studied in chapter 2 to the case of higher-spin gauge fields in AdS_3. We expect these boundary conditions on such gauge fields in AdS_3 to be dual to a dynamical theory of higher-spin currents *i.e.* induced theory of W-gravity, since according to the AdS/CFT prescription it is the gauge fields in the bulk which couple to the boundary conserved currents. We study the case of higher-spin fields with spin-2 and spin-3 fields are turned on; spin-2 field being the usual metric or the graviton. As mentioned in the introduction, the only known complete action for such theories when spins greater than 2 are included is in the Chern-Simons formalism of gauge fields in AdS_3. For the higher spin theory based on $sl(3,\mathbb{R})\oplus sl(3,\mathbb{R})$ algebra, our boundary conditions give rise to one copy of classical W_3 and a copy of $sl(3,\mathbb{R})$ or $su(1,2)$ Kaĉ-Moody as the asymptotic symmetry algebra. We propose that the higher spin theories with these boundary conditions describe appropriate chiral induced W-gravity theories on the boundary. We also consider boundary conditions of spin-3 higher spin gravity that admit a $u(1)\oplus u(1)$ current algebra.

4.1 Chiral boundary conditions for $SL(3,\mathbb{R})$ higher spin theory

We are interested in proposing boundary conditions for higher spin theories such that they holographically describe appropriate chiral induced W-gravity theories. In the first order formalism the theory is formulated on the same lines as AdS_3 gravity but with the gauge group replaced by $SL(3,\mathbb{R})$ [46, 56]. The Dirichlet boundary conditions of this theory were considered by Campoleoni *et al* [56] and they showed that the asymptotic symmetry algebra is two commuting copies of classical W_3 algebra with central charges.

We now turn to generalising the boundary conditions of the section (2.1) to the 3-dimensional higher spin theory based on two copies of $sl(3,\mathbb{R})$ or $su(1,2)$ algebra [46, 56]. Motivated by the CIG boundary conditions of chapter 2 we write the connections again as deformations of AdS_3 solution. We work in the principal embedding basis for the gauge algebra. Our conventions very closely follow those of [56] and may be found in Appendix (6.5). The action is given as difference of two chern-Simons theories valued with gauge fields - A and $\tilde{A}$, valued in the adjoint of $sl(3,\mathbb{R})$. The generalized dribein and spin connections are still given by the same relation,

$$e = \frac{\ell}{2}(A - \tilde{A}) \qquad \omega = \frac{1}{2}(A + \tilde{A}), \tag{4.1}$$

with the spin fields given by,

$$g_{\mu\nu} = \frac{1}{2}Tr(e_\mu e_\nu) \qquad \varphi_{\mu\nu\rho} = \frac{1}{3!}Tr(e_{(\mu}e_\nu e_{\rho)}). \tag{4.2}$$

We make use of the same observation; as in the super-gravity analysis, to propose boundary fall-off conditions on the higher-spin gauge fields. The gauge fields given below

$$a = [L_1 - \kappa L_{-1}]\, dx^+,$$

$$\begin{aligned}
\tilde{a} &= [-L_{-1} + \tilde{\kappa} L_1]\, dx^- + f^{(a)} L_a dx^+, \\
A &= b^{-1}(d + a)b, \\
\tilde{A} &= b(d + \tilde{a})b^{-1}, \;\; b = e^{L_0 \ln(\frac{r}{\ell})},
\end{aligned} \tag{4.3}$$

reproduce the same Ward identity as in (2.59) and asymptotic symmetry algebra as in the case of chiral induced gravity boundary conditions of chapter 2. They can be constructed as follows:

Begin with the generic solution corresponding to the Brown-Henneaux boundary conditions; which after getting rid of their radial dependence have 1-form components in either x^+ or x^- directions. Then add to one of the gauge fields; $\tilde{a}$ in the above case, a generic adjoint valued component in the other direction; in this case $f^{(a)} L_a dx^+$. Solving *eom* with *a priori* generic functional dependence would subsequently yield the required Ward identity.

Therefore we start with the following ansatz for the gauge connections:

$$\begin{aligned}
A &= b^{-1}\partial_r b\, dr + b^{-1}\left[(L_1 - \kappa\, L_{-1} - \omega\, W_{-2})\; dx^+\right] b \qquad (4.4)\\
\tilde{A} &= b\, \partial_r b^{-1}\, dr + b\left[(-L_{-1} + \tilde{\kappa}\, L_1 + \tilde{\omega}\, W_2)\; dx^- + \left(\sum_{a=-1}^{1} f^{(a)} L_a + \sum_{i=-2}^{2} g^{(i)} W_i\right) dx^+\right] b^{-1}
\end{aligned}$$

where b is again $e^{L_0 \ln\frac{r}{l}}$. Note that, as in $sl(2,\mathbb{R})$ case, our ansatz is the Dirichlet one of [56] for the left sector. Similarly, the right sector includes the right sector ansatz of previous section as a special case.[1] All the coefficients of the algebra generators above are understood to be functions of x^+ and x^-.

Imposing flatness conditions on A and $\tilde{A}$ demand

$$\partial_- \kappa \;=\; 0, \;\; \partial_- \omega = 0, \tag{4.5}$$

[1] A similar ansatz has been considered in [57].

$$
\begin{aligned}
\partial_- f^{(-1)} + f^{(0)} &= 0, \\
\partial_- f^{(0)} + 2\, f^{(1)} + 2\,\tilde{\kappa}\, f^{(-1)} - 16\,\alpha^2\,\tilde{\omega}\, g^{(-2)} &= 0, \\
-\partial_+ \tilde{\kappa} + \partial_- f^{(1)} + \tilde{\kappa}\, f^{(0)} - 4\,\alpha^2\,\tilde{\omega}\, g^{(-1)} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\partial_- g^{(-2)} + g^{(-1)} &= 0, \\
\partial_- g^{(-1)} + 2\, g^{(0)} + 4\,\tilde{\kappa}\, g^{(-2)} &= 0, \\
\partial_- g^{(0)} + 3\, g^{(1)} + 3\,\tilde{\kappa}\, g^{(-1)} &= 0, \\
\partial_- g^{(1)} + 4\, g^{(2)} + 2\,\tilde{\kappa}\, g^{(0)} + 4\,\tilde{\omega}\, f^{(-1)} &= 0, \\
-\partial_+ \tilde{\omega} + \partial_- g^{(2)} + \tilde{\kappa}\, g^{(1)} + 2\,\tilde{\omega}\, f^{(0)} &= 0.
\end{aligned}
\tag{4.6}
$$

These equations enable one to solve for $\{f^{(0)}, f^{(1)}, g^{(-1)}, g^{(0)}, g^{(1)}, g^{(2)}\}$ in terms of $\{\tilde{\kappa}, \tilde{\omega}, f^{(-1)}, g^{(-2)}\}$ and their derivatives, provided the functions $\{\tilde{\kappa}, \tilde{\omega}, f^{(-1)}, g^{(-2)}\}$ satisfy the constraints coming from the 3rd and the 8th equations:

$$
(\partial_+ + 2\,\partial_- f^{(-1)} + f^{(-1)}\,\partial_-)\,\tilde{\kappa} - \alpha^2\,(12\,\partial_- g^{(-2)} + 8\, g^{(-2)}\partial_-)\,\tilde{\omega} = \frac{1}{2}\partial_-^3 f^{(-1)}, \tag{4.7}
$$

$$
\begin{aligned}
12\,(\partial_+ + 3\,\partial_- f^{(-1)} + f^{(-1)}\partial_-)\,\tilde{\omega} + (10\,\partial_-^3 g^{(-2)} + 15\,\partial_-^2 g^{(-2)}\,\partial_- + 9\,\partial_- g^{(-2)}\partial_-^2 + 2\, g^{(-2)}\partial_-^3)\,\tilde{\kappa} \\
- 16\,(2\,\partial_- g^{(-2)} + g^{(-2)}\,\partial_-)\,\tilde{\kappa}^2 = \frac{1}{2}\partial_-^5 g^{(-2)}.
\end{aligned}
\tag{4.8}
$$

We point out that these are the Ward identities that the induced W_3 gravity action is expected to satisfy. See Ooguri *et al* [21] for a comparison. These have also appeared recently in [57] in a related context.

Next, we seek the residual gauge transformations of our solutions. Defining the gauge parameter to be $\lambda = \lambda^{(a)} L_a + \eta^{(i)}\, W_i$ and imposing the conditions that the gauge field configuration $a = (L_1 - \kappa\, L_{-1} - \omega\, W_{-2})\, dx^+$ is left form-invariant leads to the following conditions [56]:

$$
\partial_- \lambda^{(a)} = \partial_- \eta^{(i)} = 0,
$$

$$\begin{aligned}
\partial_+\lambda^{(0)} + 2\,\lambda^{(-1)} + 2\,\kappa\,\lambda^{(1)} - 16\,\alpha^2\,\omega\,\eta^{(2)} &= 0,\\
\partial_+\lambda^{(1)} + \lambda^{(0)} &= 0,\\
\partial_+\eta^{(-1)} + 4\,\eta^{(-2)} + 2\,\kappa\,\eta^{(0)} + 4\,\omega\,\lambda^{(1)} &= 0,\\
\partial_+\eta^{(0)} + 3\,\eta^{(-1)} + 3\,\kappa\,\eta^{(1)} &= 0,\\
\partial_+\eta^{(1)} + 2\,\eta^{(0)} + 4\,\kappa\,\eta^{(2)} &= 0,\\
\partial_+\eta^{(2)} + \eta^{(1)} &= 0.
\end{aligned} \tag{4.9}$$

Under these we have

$$\delta\kappa = \lambda\,\kappa' + 2\,\lambda'\,\kappa - \frac{1}{2}\lambda''' - 8\,\alpha^2\,\eta\,\omega' - 12\,\alpha^2\,\omega\,\eta' \tag{4.10}$$

$$\delta\omega = \lambda\,\omega' + 3\,\omega\,\lambda' - \frac{8}{3}\,\kappa\,(\kappa\,\eta' + \eta\,\kappa') + \frac{1}{4}(5\,\kappa'\,\eta'' + 3\,\eta'\,\kappa'') + \frac{1}{6}\,(5\,\kappa\,\eta''' + \eta\,\kappa''') - \frac{1}{24}\eta''''' \tag{4.11}$$

We parametrize the residual gauge transformations of $\tilde{a}$ by the gauge parameters $\tilde{\lambda} = \tilde{\lambda}^{(a)}L_a + \tilde{\eta}^{(i)}\,W_i$. The constraints on this parameter are

$$\begin{aligned}
\tilde{\lambda}_0 + \partial_-\tilde{\lambda}^{(-1)} &= 0,\\
\partial_-\tilde{\lambda}_0 + 2\,\tilde{\lambda}^{(1)} + 2\,\tilde{\kappa}\,\tilde{\lambda}^{(-1)} - 16\,\alpha^2\,\tilde{\omega}\,\tilde{\eta}^{(-2)} &= 0,\\
\partial_-\tilde{\eta}^{(-2)} + \tilde{\eta}^{(-1)} &= 0,\\
\partial_-\tilde{\eta}^{(-1)} + 2\,\tilde{\eta}^{(0)} + 4\,\tilde{\eta}^{(-2)}\,\tilde{\kappa} &= 0,\\
\partial_-\tilde{\eta}^{(0)} + 3\,\tilde{\eta}^{(1)} + 3\,\tilde{\eta}^{(-1)}\,\tilde{\kappa} &= 0,\\
\partial_-\tilde{\eta}^{(1)} + 4\,\tilde{\eta}^{(2)} + 2\,\tilde{\eta}^{(0)}\,\tilde{\kappa} + 4\,\tilde{\lambda}^{(-1)}\,\tilde{\omega} &= 0.
\end{aligned} \tag{4.12}$$

These induce the following variations:

$$\delta\tilde{\kappa} = -2\,\tilde{\kappa}\,\partial_-\tilde{\lambda}^{(-1)} - \tilde{\lambda}^{(-1)}\partial_-\tilde{\kappa} + 8\,\alpha^2\,\tilde{\eta}^{(-2)}\,\partial_-\tilde{\omega} + 12\,\alpha^2\,\tilde{\omega}\,\partial_-\tilde{\eta}^{(-2)} + \frac{1}{2}\partial_-^3\tilde{\lambda}^{(-1)} \tag{4.13}$$

$$\begin{aligned}\delta\tilde{\omega} &= -\tilde{\lambda}^{(-1)}\,\partial_-\tilde{\omega} - 3\,\tilde{\omega}\,\partial_-\lambda^{(-1)} + \frac{8}{3}\tilde{\kappa}\,(\tilde{\kappa}\,\partial_-\tilde{\eta}^{(-2)} + \tilde{\eta}^{(-2)}\,\partial_-\tilde{\kappa}) \\ &\quad - \frac{1}{4}(5\,\partial_-\tilde{\kappa}\,\partial_-^2\tilde{\eta}^{(-2)} + 3\,\partial_-\tilde{\eta}^{(-2)}\,\partial_-^2\tilde{\kappa}) - \frac{1}{6}(5\,\tilde{\kappa}\,\partial_-^3\tilde{\eta}^{(-2)} + \tilde{\eta}^{(-2)}\,\partial_-^3\tilde{\kappa}) + \frac{1}{24}\partial_-^5\tilde{\eta}^{(-2)}\end{aligned} \tag{4.14}$$

$$\begin{aligned}\delta f^{(-1)} &= \partial_+\tilde{\lambda}^{(-1)} + f^{(-1)}\,\partial_-\tilde{\lambda}^{(-1)} - \tilde{\lambda}^{(-1)}\,\partial_- f^{(-1)} + \frac{32}{3}\,\alpha^2\,\tilde{\kappa}\,(g^{(-2)}\,\partial_-\tilde{\eta}^{(-2)} - \tilde{\eta}^{(-2)}\,\partial_- g^{(-2)}) \\ &\quad + \alpha^2(\partial_- g^{(-2)}\,\partial_-^2\tilde{\eta}^{(-2)} - \partial_-\tilde{\eta}^{(-2)}\,\partial_-^2 g^{(-2)}) - \frac{2}{3}\,\alpha^2\,(g^{(-2)}\,\partial_-^3\tilde{\eta}^{(-2)} - \tilde{\eta}^{(-2)}\,\partial_-^3 g^{(-2)})\end{aligned} \tag{4.15}$$

$$\delta g^{(-2)} = \partial_+\tilde{\eta}^{(-2)} + f^{(-1)}\,\partial_-\tilde{\eta}^{(-2)} - \tilde{\lambda}^{(-1)}\,\partial_- g^{(-2)} + 2\,(g^{(-2)}\,\partial_-\tilde{\lambda}^{(-1)} - \tilde{\eta}^{(-2)}\,\partial_- f^{(-1)}) \tag{4.16}$$

For the residual gauge transformations to be global symmetries of the boundary theory one needs to impose the variational principle $\delta S = 0$ as well. We add the following boundary action:

$$\begin{aligned}&S_{bdy.} \\ &= \frac{k}{4\pi}\int d^2x\,tr\,(-L_0[\tilde{A}_+,\tilde{A}_-] + 2\,\tilde{\kappa}_0\,L_1\,\tilde{A}_+ + \frac{1}{2\,\alpha}\,W_0\{\tilde{A}_+,\tilde{A}_-\} + \frac{1}{3}\tilde{A}_+\,\tilde{A}_- + 2\,\tilde{\omega}_0\,W_2\tilde{A}_+)\end{aligned} \tag{4.17}$$

With this the variation of the total action can be seen to be:

$$\delta S_{total} = -\frac{k}{2\pi}\int d^2x\,\left[(\tilde{\kappa} - \tilde{\kappa}_0)\,\delta f^{(-1)} + 4\,\alpha^2\,(\tilde{\omega} - \tilde{\omega}_0)\,\delta g^{(-2)}\right] \tag{4.18}$$

where $\tilde{\kappa}_0$ and $\tilde{\omega}_0$ are some real numbers. Again we have several ways to satisfy $\delta S = 0$:

1. $\delta f^{(-1)} = 0$ and $\delta g^{(-2)} = 0$.

   This is the Dirichlet condition again (has been considered by [57] recently for constant $f^{(-1)}$ and $g^{(-2)}$) and leads to W_3 as the asymptotic symmetry algebra.

2. $\tilde{\kappa} = \tilde{\kappa}_0$ ($\tilde{\omega} = \tilde{\omega}_0$) and $\delta g^{(-2)} = 0$ ($\delta f^{(-1)} = 0$).

These are the other mixed boundary conditions – we will not consider them further here.

3. $\tilde{\kappa} = \tilde{\kappa}_0$ and $\tilde{\omega} = \tilde{\omega}_0$ are the free boundary conditions we will consider below.

4.2 Solutions of W_3 Ward identities

The W_3 Ward identities (4.7, 4.8) are also expected to be integrable (just as the Virasoro one in section 2 was) and general solutions can be written down by appropriate reparametrization of $f^{(-1)}$ and $g^{(-2)}$.

However, we restrict to the case of $\tilde{\kappa} = \tilde{\kappa}_0$ and $\tilde{\omega} = \tilde{\omega}_0$ for constant $\tilde{\kappa}_0$ and $\tilde{\omega}_0$, and holding them fixed. This allows for classical solutions with fluctuating $f^{(-1)}$ and $g^{(-2)}$. Let us solve the W_3 Ward identities in this case. These read

$$\begin{aligned} \partial_-^3 f^{(-1)} + 24\,\alpha^2\,\tilde{\omega}_0\,\partial_- g^{(-2)} - 4\,\tilde{\kappa}_0\,\partial_- f^{(-1)} &= 0 \\ \partial_-^5 g^{(-2)} - 20\,\tilde{\kappa}_0\,\partial_-^3 g^{(-2)} + 64\,\tilde{\kappa}_0^2\,\partial_- g^{(-2)} - 72\,\tilde{\omega}_0\,\partial_- f^{(-1)} &= 0 \end{aligned} \tag{4.19}$$

There are two distinct cases: $\tilde{\omega}_0 = 0$ and $\tilde{\omega}_0 \neq 0$. When $\tilde{\omega}_0 = 0$ there are further two distinct cases:

1. $\tilde{\omega}_0 = 0$ and $\tilde{\kappa}_0 = 0$ gives:

$$\begin{aligned} f^{(-1)} &= f_{-1}(x^+) + x^-\,f_0(x^+) + (x^-)^2 f_1(x^+), \qquad (4.20) \\ g^{(-2)} &= g_{-2}(x^+) + x^-\,g_{-1}(x^+) + (x^-)^2\,g_0(x^+) + (x^-)^3\,g_1(x^+) + (x^-)^4\,g_2(x^+) \end{aligned}$$

This solution is suitable for non-compact x^+ and x^- (such as the boundary of Poincare or Euclidean AdS_3)

2. $\tilde{\omega}_0 = 0$ and $\tilde{\kappa}_0 \neq 0$:

$$\begin{aligned} f^{(-1)} &= f_\kappa(x^+) + g_\kappa(x^+)\, e^{2\sqrt{\tilde{\kappa}_0}\, x^-} + \bar{g}_\kappa(x^+)\, e^{-2\sqrt{\tilde{\kappa}_0} x^-}, \qquad (4.21)\\ g^{(-2)} &= f_\omega(x^+) + g_\omega(x^+)\, e^{2\sqrt{\tilde{\kappa}_0}\, x^-} + \bar{g}_\omega(x^+)\, e^{-2\sqrt{\tilde{\kappa}_0}\, x^-} + h_\omega(x^+)\, e^{4\sqrt{\tilde{\kappa}_0}\, x^-} + \bar{h}_\omega(x^+)\, e^{-4\sqrt{\tilde{\kappa}_0}\, x^-} \end{aligned}$$

Again any positive value for $\tilde{\kappa}_0$ is suitable for non-compact boundary coordinates. Among the negative values $\tilde{\kappa}_0 = -\frac{1}{4}$ (times square of any integer) is suitable for compact boundary coordinates (such as the boundary of global AdS_3).

3. When $\tilde{\kappa}_0 \neq 0$ and $\tilde{\omega}_0 \neq 0$, again the Ward identities can be solved. The general solutions involve eight arbitrary functions of x^+ (just as in the cases with $\tilde{\omega}_0 = 0$). Here we will only consider the special case where the solutions do not depend on x^-:

$$f^{(-1)} = f(x^+), \quad g^{(-2)} = g(x^+). \qquad (4.22)$$

This case is the analogue of [6] in the higher spin context.

Next we analyse these cases one by one and find the asymptotic symmetries.

4.3 Asymptotic symmetries, charges and Poisson brackets

To find the asymptotic symmetries to which we can associate charges one needs to look for the residual gauge transformations of the solutions of interest. Just as in the $sl(2,\mathbb{R})$ case we can look at the residual gauge transformations of a and $\tilde{a}$ and translate the corresponding gauge parameters λ and $\tilde{\lambda}$ using $\Lambda = b^{-1}\lambda\, b$ and $\tilde{\Lambda} = b\,\tilde{\lambda}\, b^{-1}$. After finding these one can compute the corresponding charges.

We are now ready to carry out this exercise for left sector and each of the cases (4.20, 4.21, 4.22) in the right sector one by one.

4.3.1 The left sector symmetry algebra

The left sector is common for all of the cases we consider in this paper. The corresponding $\not\delta Q$ is

$$\not\delta Q_\lambda \;=\; -\tfrac{k}{2\pi}\int_0^{2\pi} d\phi\,(\lambda\,\delta\kappa - 4\,\alpha^2\,\eta\,\delta\omega) \tag{4.23}$$

This when integrated between ($\kappa = 0$, $\omega = 0$) and generic (κ, ω) gives

$$Q_{(\lambda,\eta)} = -\tfrac{k}{2\pi}\int_0^{2\pi} d\phi\,[\lambda\,\kappa - 4\,\alpha^2\,\eta\,\omega] \tag{4.24}$$

This charge generates the variations (4.10, 4.11) provided we take the Poisson brackets amongst the κ and ω to be:

$$\begin{aligned}
-\tfrac{k}{2\pi}\,\{\kappa(x^+),\kappa(\tilde{x}^+)\} &= -\kappa'(x^+)\,\delta(x^+-\tilde{x}^+) - 2\,\kappa(x^+)\,\delta'(x^+-\tilde{x}^+) + \tfrac{1}{2}\,\delta'''(x^+-\tilde{x}^+),\\
-\tfrac{k}{2\pi}\,\{\kappa(x^+),\omega(\tilde{x}^+)\} &= -2\,\omega'(x^+)\,\delta(x^+-\tilde{x}^+) - 3\,\omega(x^+)\,\delta'(x^+-\tilde{x}^+),\\
-\tfrac{2k\alpha^2}{\pi}\,\{\omega(x^+),\omega(\tilde{x}^+)\} &= \tfrac{8}{3}[\kappa^2(x^+)\,\delta'(x^+-\tilde{x}^+) + \kappa(x^+)\,\kappa'(x^+)\delta(x^+-\tilde{x}^+)]\\
&\quad -\tfrac{1}{6}[5\kappa(x^+)\delta'''(x^+-\tilde{x}^+) + \kappa'''(x^+)\delta(x^+-\tilde{x}^+)]\\
&\quad -\tfrac{1}{4}[3\kappa''(x^+)\delta'(x^+-\tilde{x}^+) + 5\kappa'(x^+)\delta''(x^+-\tilde{x}^+)] + \tfrac{1}{24}\delta^{(5)}(x^+-\tilde{x}^+)
\end{aligned} \tag{4.25}$$

These brackets were computed by Campoleoni *et al.* [7]. To compare with their answers one has to take $\kappa \to -\frac{2\pi}{k}\kappa$, $\omega \to \frac{\pi}{2k\alpha^2}\omega$, $\alpha^2 \to -\sigma$ in the expressions here.

Next we turn to computing the charges and Poisson brackets on the right sector for all the cases of interest.

4.3.2 $\tilde{\kappa}_0 = 0$ and $\tilde{\omega}_0 = 0$

In this case the residual gauge transformation parameters are

$$\begin{aligned}
\tilde{\lambda}^{(-1)} &= \lambda_{-1}(x^+) + x^- \lambda_0(x^+) + (x^-)^2 \lambda_1(x^+) \\
\tilde{\eta}^{(-2)} &= \eta_{-2}(x^+) + x^- \eta_{-1}(x^+) + (x^-)^2 \eta_0(x^+) + (x^-)^3 \eta_1(x^+) + (x^-)^4 \eta_2(x^+)
\end{aligned} \tag{4.26}$$

The corresponding action on the fields gives

$$\begin{aligned}
\delta f_0 &= \lambda_0' + 2\,(f_{-1}\,\lambda_1 - \lambda_{-1}\,f_1) - 2\,\alpha^2\,(\eta_{-1}\,g_1 - \eta_1\,g_{-1}) - 16\,\alpha^2\,(\eta_2\,g_{-2} - \eta_{-2}\,g_2) \\
\delta f_1 &= \lambda_1' + (\lambda_1\,f_0 - \lambda_0\,f_1) - 2\,\alpha^2\,(\eta_0\,g_1 - \eta_1\,g_0) - 4\,\alpha^2\,(\eta_2\,g_{-1} - \eta_{-1}\,g_2) \\
\delta f_{-1} &= \lambda_{-1}' + (\lambda_0\,f_{-1} - \lambda_{-1}\,f_0) - 2\,\alpha^2\,(\eta_{-1}\,g_0 - \eta_0\,g_{-1}) - 4\,\alpha^2\,(\eta_1\,g_{-2} - \eta_{-2}\,g_1) \\
\delta g_0 &= \eta_0' + 3\,(\eta_1\,f_{-1} - \eta_{-1}\,f_1) + 3\,(\lambda_1\,g_{-1} - \lambda_{-1}g_1) \\
\delta g_1 &= \eta_1' + (\eta_1\,f_0 - \lambda_0\,g_1) + 2\,(\lambda_1\,g_0 - \eta_0\,f_1) + 4\,(\eta_2\,f_{-1} - \lambda_{-1}\,g_2) \\
\delta g_{-1} &= \eta_{-1}' + (\lambda_0\,g_{-1} - \eta_{-1}\,f_0) + 2\,(\eta_0\,f_{-1} - \lambda_{-1}\,g_0) + 4\,(\lambda_1\,g_{-2} - \eta_{-2}\,f_1) \\
\delta g_2 &= \eta_2' + (\lambda_1\,g_1 - \eta_1\,f_1) + 2\,(\eta_2\,f_0 - \lambda_0\,g_2) \\
\delta g_{-2} &= \eta_{-2}' + (\eta_{-1}\,f_{-1} - \lambda_{-1}\,g_{-1}) + 2\,(\lambda_0\,g_{-2} - \eta_{-2}\,f_0)
\end{aligned} \tag{4.27}$$

Defining

$$\begin{aligned}
\{J^a, a = 1, \cdots, 8\} &= \{f_{-1}, f_0, f_1, g_{-2}, g_{-1}, g_0, g_1, g_2\} \\
\{\lambda^a, a = 1, \cdots, 8\} &= \{\lambda_{-1}, \lambda_0, \lambda_1, \eta_{-2}, \eta_{-1}, \eta_0, \eta_1, \eta_2\}
\end{aligned} \tag{4.28}$$

these expressions can also be written in a compact form:

$$\delta J^a = \partial_+ \lambda^a - f^a{}_{bc} J^b \lambda^c \tag{4.29}$$

where $f^a{}_{bc}$ are structure constants of our gauge algebra. The charge in this case is integrable and has the expression:

$$Q[\tilde{\lambda}] = \tfrac{k}{4\pi} \int dx^+ \, \eta_{ab} J^a \lambda^b \tag{4.30}$$

The Poisson brackets can be read out and we find:

$$\{J^a(x^+), J^b(\tilde{x}^+)\} = f^{ab}{}_c \, J^c(x^+) \, \delta(x^+ - \tilde{x}^+) - \tfrac{k}{4\pi} \, \eta^{ab} \, \delta'(x^+ - \tilde{x}^+) \tag{4.31}$$

where we have redefined: $J^a \to -\frac{4\pi}{k} J^a$. This may be recognized as level $-k$ Kaĉ-Moody extension of the algebra used in defining the higher spin theory.

4.3.3 $\tilde{\kappa}_0 = -\frac{1}{4}$ and $\tilde{\omega}_0 = 0$

In this case the residual gauge transformation parameters are

$$\begin{aligned}
\tilde{\lambda}^{(-1)} &= \lambda_f(x^+) + \lambda_g(x^+) \, e^{i\,x^-} + \bar{\lambda}_{\bar{g}}(x^+) \, e^{-i\,x^-} \\
\tilde{\eta}^{(-2)} &= \eta_f(x^+) + \eta_g(x^+) \, e^{i\,x^-} + \bar{\eta}_{\bar{g}}(x^+) \, e^{-i\,x^-} + \eta_h(x^+) \, e^{2i\,x^-} + \bar{\eta}_{\bar{h}}(x^+) \, e^{-2i\,x^-}
\end{aligned} \tag{4.32}$$

The symmetry transformations are:

$$\begin{aligned}
\delta f_\kappa &= \lambda_f' + 2\,i\,(\bar{g}_\kappa \, \lambda_g - \bar{\lambda}_{\bar{g}} \, g_\kappa) + 2\,i\,\alpha^2\,(\bar{\eta}_{\bar{g}} \, g_\omega - \eta_g \, \bar{g}_\omega) + 16\,i\,\alpha^2\,(\eta_h \, \bar{h}_\omega - \bar{\eta}_{\bar{h}} \, h_\omega) \\
\delta g_\kappa &= \lambda_g' + i\,(\lambda_g \, f_\kappa - \lambda_f \, g_\kappa) + 2\,i\,\alpha^2\,(\eta_f \, g_\omega - \eta_g \, f_\omega) + 4\,i\,\alpha^2\,(\eta_h \, \bar{g}_\omega - \bar{\eta}_{\bar{g}} \, h_\omega) \\
\delta \bar{g}_\kappa &= \bar{\lambda}_{\bar{g}}' + i\,(\lambda_f \, \bar{g}_\kappa - \bar{\lambda}_{\bar{g}} \, f_\kappa) + 2\,i\,\alpha^2\,(\bar{\eta}_{\bar{g}} \, f_\omega - \eta_f \, \bar{g}_\omega) + 4\,i\,\alpha^2\,(\eta_g \, \bar{h}_\omega - \bar{\eta}_{\bar{h}} \, g_\omega) \\
\delta f_\omega &= \eta_f' + 3i\,(\eta_g \, \bar{g}_\kappa - \bar{\eta}_{\bar{g}} \, g_\kappa) + 3i\,(\lambda_g \, \bar{g}_\omega - \bar{\lambda}_{\bar{g}} g_\omega) \\
\delta g_\omega &= \eta_g' + i\,(\eta_g \, f_\kappa - \lambda_f \, g_\omega) + 2i\,(\lambda_g \, f_\omega - \eta_f \, g_\kappa) + 4i\,(\eta_h \, \bar{g}_\kappa - \bar{\lambda}_{\bar{g}} \, h_\omega) \\
\delta \bar{g}_\omega &= \bar{\eta}_{\bar{g}}' + i\,(\lambda_f \, \bar{g}_\omega - \bar{\eta}_{\bar{g}} \, f_\kappa) + 2i\,(\eta_f \, \bar{g}_\kappa - \bar{\lambda}_{\bar{g}} \, f_\omega) + 4i\,(\lambda_g \, \bar{h}_\omega - \bar{\eta}_{\bar{h}} \, g_\kappa) \\
\delta h_\omega &= \eta_h' + i(\lambda_g \, g_\omega - \eta_g \, g_\kappa) + 2i\,(\eta_h \, f_\kappa - \lambda_f \, h_\omega) \\
\delta \bar{h}_\omega &= \bar{\eta}_{\bar{h}}' + i\,(\bar{\eta}_{\bar{g}} \, \bar{g}_\kappa - \bar{\lambda}_{\bar{g}} \, \bar{g}_\omega) + 2i\,(\lambda_f \, \bar{h}_\omega - \bar{\eta}_{\bar{h}} \, f_\kappa)
\end{aligned} \tag{4.33}$$

Defining the currents J^a and parameters λ^a as

$$\begin{aligned}\{J^a, a = 1, \cdots, 8\} &= \{\bar{g}_\kappa, f_\kappa, g_\kappa, \bar{h}_\omega, \bar{g}_\omega, f_\omega, g_\omega, h_\omega\} \\ \{\lambda^a, a = 1, \cdots, 8\} &= \{\bar{\lambda}_{\bar{g}}, \lambda_f, \lambda_g, \bar{\eta}_{\bar{h}}, \bar{\eta}_{\bar{g}}, \eta_f, \eta_g, \eta_h\} \end{aligned} \tag{4.34}$$

these expressions can also be written in a compact form:

$$\delta J^a = \partial_+\lambda^a - i\,\hat{f}^a{}_{bc} J^b \lambda^c \tag{4.35}$$

where (some what surprisingly) $\hat{f}^a{}_{bc}$ are obtained from the structure constants $f^a{}_{bc}$ by replacing $\alpha^2 \to -\alpha^2$. In this case the charge is:

$$Q[\lambda^a] = -\tfrac{k}{4\pi}\int_0^{2\pi} d\phi\, \hat{\eta}_{ab}\, \lambda^a J^b \tag{4.36}$$

where $\hat{\eta}_{ab}$ is the one obtained from η_{ab} by replacing α^2 by $-\alpha^2$. The corresponding Poisson brackets are

$$\{J^a(x^+), J^b(\tilde{x}^+)\} = i\hat{f}^{ab}_{\ \ c}\, J^c(x^+)\,\delta(x^+ - \tilde{x}^+) + \frac{k}{4\pi}\hat{h}^{ab}\delta'(x^+ - \tilde{x}^+). \tag{4.37}$$

where again we have redefined: $J^a \to \frac{4\pi}{k} J^a$. This again is a level-$k$ Kaĉ-Moody algebra, but for the difference that it is obtained from the gauge algebra by $\alpha^2 \to -\alpha^2$ replacement.

4.3.4 $\tilde{\kappa}_0 \neq 0$, $\tilde{\omega}_0 \neq 0$, $\partial_- f^{(-1)} = \partial_- g^{(-2)} = 0$

In this case the residual gauge transformation parameters are

$$\tilde{\lambda}^{(-1)} = \tilde{\lambda}(x^+), \quad \tilde{\eta}^{(-2)} = \tilde{\eta}(x^+). \tag{4.38}$$

Under these gauge transformations the fields transform as

$$\delta f^{(-1)} = \partial_+ \tilde{\lambda}, \quad \delta g^{(-2)} = \partial_+ \tilde{\eta}. \tag{4.39}$$

Thus the residual gauge symmetries generate two commuting copies of $U(1)$ classically. Restricted to the $sl(2,\mathbb{R})$ sub-sector this case corresponds to [6]. The charge is

$$Q_{\tilde{a}} = \frac{k}{2\pi}\int_0^{2\pi} d\phi\, 2\,[\tilde{\lambda}\,(\tilde{\kappa}_0\, f - 6\,\alpha^2\,\tilde{\omega}_0\, g) + \tilde{\eta}\, 2\,\alpha^2\,(\tfrac{8}{3}\tilde{\kappa}_0^2\, g - 3\,\tilde{\omega}_0\, f)] \tag{4.40}$$

This leads to Poisson brackets:

$$\begin{aligned}
\{\tilde{\kappa}_0\, f(x^+) - 6\,\alpha^2\,\tilde{\omega}_0\, g(x^+),\, f(\tilde{x}^+)\} &= -\tfrac{\pi}{k}\delta'(x^+ - x^{+\prime}),\ \{\tfrac{8}{3}\tilde{\kappa}_0^2\, g(x^+) - 3\,\tilde{\omega}_0\, f(x^+),\, f(\tilde{x}^+)\} = 0 \\
\{\tilde{\kappa}_0\, f(x^+) - 6\,\alpha^2\,\tilde{\omega}_0\, g(x^+),\, g(\tilde{x}^+)\} &= 0,\ \{\tfrac{8}{3}\tilde{\kappa}_0^2\, g(x^+) - 3\,\tilde{\omega}_0\, f(x^+),\, g(\tilde{x}^+)\} = -\tfrac{\pi}{2k\alpha^2}\delta'(x^+ - \tilde{x}^+)
\end{aligned} \tag{4.41}$$

These four relations are solved by the following three equations:

$$\{f(x^+),\, f(\tilde{x}^+)\} = -\tfrac{\pi}{k}\tfrac{\tilde{\kappa}_0^2}{\Delta}\delta'(x^+ - \tilde{x}^+), \qquad \{g(x^+),\, g(\tilde{x}^+)\} = -\tfrac{\pi}{k}\tfrac{3\,\tilde{\kappa}_0}{16\,\Delta\alpha^2}\delta'(x^+ - \tilde{x}^+),$$

$$\{f(x^+),\, g(\tilde{x}^+)\} = -\tfrac{\pi}{k}\tfrac{9\,\tilde{\omega}_0}{8\,\Delta}\delta'(x^+ - \tilde{x}^+) \tag{4.42}$$

where $\Delta = \tilde{\kappa}_0^3 - \frac{27}{4}\,\alpha^2\,\tilde{\omega}_0^2$ which we have to assume not to vanish.[2]

The diagonal embedding of $sl(2,\mathbb{R})$ in $sl(3,\mathbb{R})$ gives rise to an asymptotic symmetry algebra consisting of two copies of $\mathcal{W}_3^{(2)}$ also known as the Polyakov-Bershadsky algebra when one generalizes Brown-Henneaux boundary condition to this case [58]. The central

[2]Taking linear combinations $f + \chi\, g$ and $f - \chi\, g$ (for some constant χ) as the currents one can decouple these two $u(1)$ Kaĉ-Moody algebras.

charge for the algebra turns out to be $c/4$. The gauge field choosen in [58] is of the type

$$A = e^{\hat{L}_0 \ln(r/\ell)}(\hat{W}_2 - \mathcal{T}\hat{W}_{-2} + jW_0 + g_1 L_{-1} + g_2 W_{-1})e^{-\hat{L}_0 \ln(r/\ell)}dx^+ + \hat{L}_0 \frac{dr}{\ell},$$
$$\text{where,} \quad \hat{W}_{\pm 2} = \pm\frac{1}{4}W_{\pm 2}, \;\; \hat{L}_0 = \frac{1}{2}L_0. \tag{4.43}$$

Here the equations of motion forces the gauge field parameters to be a function of of x^+ alone. One can propose chiral boundary conditions in a manner similar to the one illustrated in this chapter; one considers a gauge field of the type

$$A = e^{\hat{L}_0 \ln(r/\ell)}(a)e^{-\hat{L}_0 \ln(r/\ell)} + \hat{L}_0 \frac{dr}{\ell},$$
$$\text{where} \quad a = (\hat{W}_2 - \mathcal{T}\hat{W}_{-2} + jW_0 + g_1 L_{-1} + g_2 W_{-1})dx^+ + f^{(a)}J_a dx^-,$$
$$\& \quad J_a \in \{\hat{W}_{-2}, W_{-1}, L_{-1}, \hat{L}_0, W_0, L_1, W_1, \hat{W}_2\}. \tag{4.44}$$

Here as before, the parameters are *a priori* functions of both the boundary co-ordinates. The equation of motion can be used to to solve for $f^{(a)}$s yielding differential equations which are interpreted as Ward identities. The parameters $\{\mathcal{T}, g_1, g_2, j\}$ can be fixed to a specific values by choosing boundary terms to be added to the Chern-Simons action fro the gauge field A. One again expects to get an $sl(3,\mathbb{R})$ and $su(1,2)$ current algebra for certain values of these parameters $\{\mathcal{T}, g_1, g_2, j\}$.

Thus the chiral boundary conditions studied in this chapter can be adapted to various known W-algebras [45] yielding Ward identities for the corresponding chiral induced W-gravities.

In this chapter, generalizing the results of chiral boundary conditions studied in chapter 2 and [9], we proposed boundary conditions for higher spin gauge theories in 3d in their first order formalism that are different from the usual Dirichlet boundary conditions.[3] The left sector is treated with the usual Dirichlet boundary conditions where as in the right

[3]It should be noted that the ansatz for the right sector gauge field (4.4) studied here also appeared recently in [59, 57] where the authors were still interested in generalizations of Dirichlet type boundary conditions.

sector we chose free boundary conditions. We restricted our attention to the spin-3 case for calculational convenience. The Dirichlet boundary conditions for general higher spin theory based on $sl(n, \mathbb{R})$ Chern-Simons was discussed in [56] and for $hs[\lambda]$ case in [14]. One should be able to generalize our considerations to these other higher spin theories as well.

The boundary conditions considered here give one copy of W_3 and a copy of $sl(3, \mathbb{R})$ (or $su(1, 2)$ or $u(1) \oplus u(1)$) Kaĉ-Moody algebra. This matches with the symmetry algebra expected of the 2d chiral induced W-gravity with an appropriate field content.

Let us emphasize that there appears to be a surprising difference between the asymptotic symmetry algebras of section (4.3.2) and section (4.3.3) namely the maximal finite subalgebra of (4.31) is isomorphic to the gauge algebra of the higher spin theory where as for that in (4.37) differs from the gauge algbra by $\alpha^2 \to -\alpha^2$ (this interchanges $sl(3, \mathbb{R})$ and $su(1, 2)$). It will be interesting to understand the source of this possibility of getting a different real-form of the complexified gauge algebra out of our boundary conditions.

The Poisson brackets between κ or ω of the left sector and any of the right sector currents vanish. Recall that in the $sl(2, \mathbb{R})$ case, motivated by how the asymptotic vector fields in the second order formalism [3] acted on the fields, we made (current dependent) redefinitions of the residual gauge parameters. Here too one can do such a redefinition. For instance, if we change variables

$$\lambda^a \to \lambda^a + \alpha_1 J^a \lambda + \alpha_2 \, d^a{}_{bc} J^b J^c \eta + \cdots \tag{4.45}$$

where λ^a are the parameters defined in (4.28) and λ and η are the gauge parameters of the left sector, $d_{abc} \sim \mathrm{Tr}(T_a\{T_b, T_c\})$, then one finds that

$$\kappa \to \kappa + \#\eta_{ab} J^a J^b + \cdots , \quad \omega \to \omega + \#d_{abc} J^a J^b J^c + \cdots , \tag{4.46}$$

where the dots in these redefinitions represent possible terms of higher orders in J^as or terms involving derivatives of currents. The additional terms here may be recognized as the (classical analogues) of Sugawara constructions of spin-2 and spin-3 currents out of the Kaĉ-Moody currents. After such redefinitions the generators of the asymptotic symmetry algebras of the left and the right sectors will not commute any longer.

Chapter 5

Conclusion and discussions

5.1 Conclusion

In this thesis we uncovered the effect of imposing a novel set of boundary conditions on gravity in AdS_3 which allow the boundary metric - and other fields if present, to fluctuate along the boundary of AdS_3. This we further divide into two cases; one when the fluctuating component of the boundary metric is chiral; another when it is a conformal factor of the boundary metric.

- For the chiral boundary conditions imposed on pure gravity in AdS_3, we uncovered a Virasoro with $c = \frac{3\ell}{2G}$ times an $sl(2,\mathbb{R})$ Kaĉ-Moody current algebra with level $k = c/6$ as the asymptotic symmetry algebra. We proposed that such chiral boundary conditions make the bulk theory dual to a chiral induced gravity in 2d first studied by Polyakov [10]. This is also born out by the Ward identity one obtains as the bulk equations of motion for generic configurations obeying the chiral boundary conditions. We generalized the chiral boundary conditions to extended super-gravities and higher-spin gauge fields in AdS_3.
- In the super-gravity case we found a copy of super-Virasoro along with a Kaĉ-Moody super-current algebra with level $k = c/6$ as the asymptotic symmetry alge-

bra corresponding to super-symmetric extension of chiral induced gravity. Here we also obtained the Ward identities obeyed by the Kaĉ-Moody super-current algebra.

- In the higher-spin case we restricted the analysis to spins= $2, 3$ and uncovered a classical W_3 algebra along with an $su(1, 2)$ Kaĉ-Moody current algebra with level $k = c/6$ when the solution space contains global AdS_3; and an $sl(3, \mathbb{R})$ Kaĉ-Moody current algebra with $k = -c/6$ when Poincaré AdS_3 is in the solution-space.

- We also generalized the boundary conditions of Compère *et al* [6] to the higher-spin case and uncover a $u(1) \times u(1)$ along with a W_3 as the asymptotic symmetry algebra.

- For the case of conformal boundary conditions, we first provide a set of boundary conditions for pure gravity in AdS_3 such that the boundary metric has non vanishing curvature and the boundary conformal factor satisfies the Liouville equations of motion. Here we uncovered 2-copies of Virasoro corresponding to the Brown-Henneaux analysis with $c = \frac{3\ell}{2G}$ and 2-copies of Virasoro corresponding to the Liouville field stress tensor with $c = -\frac{3\ell}{2G}$. The conformal boundary conditions with vanishing boundary curvature was studied by Troessaert [5] were 2-copies of Virasoro$\ltimes u(1)$ were uncovered. We generalized these to extended super-gravity and uncovered 2-copies of super-Virasoro$\ltimes$super-current which is a super-symmetric extension of the harmonic Weyl current obtained in [5].

5.2 Discussions

The chiral boundary conditions for pure gravity studied in this thesis have global AdS_3 in the space of allowed solutions, implying that the asymptotic symmetry algebra contains the $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$ algebra; the symmetries of the maximally symmetric geometry in 3d with negative cosmological constant. This is not true for the boundary conditions of Compère *et al* [6]. In this sense it is a solution to the question as to what boundary conditions allow for asymptotic symmetries which admit an $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$; to which Brown-Henneaux [1], Troessaert's boundary condition [5] and the conformal boundary condition studied in chapter 2 are other answers.

Since the Ward identity uncovered for the chiral boundary conditions are exactly the same as those of the chiral induced gravity in 2d [10], it would be interesting to know how one can recover the correlation functions of the boundary currents from the bulk analysis. The chiral boundary conditions studied here are peculiar in having a non-vanishing boundary curvature and therefore a Weyl anomaly. This is also true for chiral induced gravity whose effective action is usually interpreted as 'integrated anomaly'[1]. The action for the 2d induced gravity can be seen as an effective action for the boundary currents [45, 21, 60] and can be obtained from the bulk on-shell action after the addition of suitable boundary terms as written down in chapter 2.

The Chern-Simons formalism of 3d gravity gives a better understanding of this as one can write the boundary conditions imposed as constraints on the conserved currents of a WZNW action derived from the Chern-Simons theory on the group $SL(2,\mathbb{R})$ [61]. Therefore one may view the different boundary conditions on gravity in AdS_3 being dual to different constrained WZNW theories on the boundary. Similarly, induced gravities can be obtained from constraining WZNW field theories with an appropriate level. For ordinary gravity this was first studied in [62] and was generalized to $\mathcal{W}-$gravity in [21]. These comments also apply for the generalization of chiral boundary conditions to the case of super-gravities and higher-spin gauge fields in AdS_3 with appropriate gauge groups.

An interesting exercise would be generalizing the conformal boundary conditions to include the higher-spin gauge fields in AdS_3 which would correspond to the conformal factors of the higher-spin fields obeying Toda equations of motion. Since the case studied in [5] corresponds to vanishing of the boundary Ricci scalar, generalizing to higher-spin gauge fields in AdS_3 would require the knowledge of higher-spin curvature tensors, a

[1] See chapter 8 of the review on '$\mathcal{W}$-symmetry in conformal field theory' - Bouwknegt and Schoutens [16].

form of which was proposed in [63]. It is worthwhile to point out that the Liouville action and similarly the Toda action can be derived from a constrained WZNW action on appropriate gauge group[2], where in suitable constraints are placed on the conserved currents of the latter action; this was first demonstrated by Balog *et al* [64] and Forgacs *et al* [65].

It would also be of interest to work out the case of super-higher-spin gauge fields in AdS_3 in a similar light; generalizing both, the chiral and conformal boundary conditions. The work generalizing Brown-Henneaux boundary conditions to super-higher-spins in AdS_3 was done by Henneaux *et al* [66] where they discovered two copies of a non-linear (N, M) extended super-W_∞ algebra when the Chern-Simons theory defining the super-higher-spin theory in the bulk is based on an infinite dimensional super-higher-spin algebra $shs^E(N|2, \mathbb{R}) \times shs^E(M|2, \mathbb{R})$. Further investigations of higher-spin theories in AdS_3 super-gravities and their CFT duals were considered in [67],[68].

It would be interesting to see how the considerations of the boundary conditions studied in this thesis can be adapted when the bulk dimensions is d > 3 with negative cosmological constant. Here if one tries to use the Dirichlet type boundary conditions and repeat the Brown-Henneaux analysis, one simply uncovers the conformal Killing algebra of the boundary which is finite dimensional $so(2, d-1)$ algebra when the boundary has dimensions greater than two. As was seen in the chiral boundary conditions studied in this thesis, bulk equations of motions on generic bulk configurations obeying a certain chiral boundary condition may yield Ward identities for induced gravities defined on the dim>2 boundary. As mentioned in the introduction, these would be the Ward identities obeyed by the generating function of the stress-tensor correlators of the boundary CFT.

The Ward identities and asymptotic symmetry analysis done in the bulk corresponds to the large central charge limit. This central charge must correspond to the original CFT on the boundary which was integrated out after coupling to gauge fields yielding an effective theory of the these fields which is the induced gravity action on the boundary. People

[2] $SL(2, \mathbb{R})$ for Liouville and $SL(N, \mathbb{R})$ higher-spins ranging from $2, ..., N$ corresponding to $N-1$ fields of the Toda theory.

have found a $1/c$ expansion for this effective action, and in the chiral induced gravities an exact action in all orders of c on the boundary was found by Ooguri *et al* [21]. It would be interesting to get a perturbative or if possible exact in c Ward identities and asymptotic symmetries from the bulk analysis.

The notion of small and large gauge transformations or equivalently diffeomorphisms depends on the boundary conditions imposed on the theory in question. When considering Brown-Henneaux type boundary conditions; in vogue for over two decades, large diffeomorphisms where those which changed the boundary metric. When one uses boundary conditions studied in this thesis and in [6, 5], large diffeomorphisms would be those which would change the fall-off behaviour of the bulk configurations such that the change in the asymptotic charge computed by the Barnich and Brandt's approach [50] is infinite[3].

It has been known for a long time that 2d unitary, Poincare and rigid scale invariant QFT with a discrete non-negative spectrum of scaling dimensions has an extended global symmetry of left and right semi-local conformal symmetry [69]. It was shown by Hofman and Strominger [70] that if one began with 2D translations and just scale invariance in the left sector *i.e.* $x^- \to \lambda x^-$, then one finds that the left sector scale symmetry gets enhanced to an infinite dimensional left conformal symmetry, whereas the right translations either can be enhanced to a right conformal symmetry or a $U(1)$ Kaĉ-Mood symmetry. This seems to suggest the existence of 2d CFTs with infinite dimensional symmetries which are not only of the left and right conformal symmetries *i.e.* the Virasoros. Given the fact that 2d induced pure gravity has an $SL(2,\mathbb{R})$ Kaĉ-Moody symmetry it should be possible to generalize the considerations in [70] to see the rigid right translations get enhanced to either of an $U(1)$ or an $SL(2,\mathbb{R})$ or a right conformal symmetry. Moreover one could also find minimum symmetries to be demanded by a unitary, Poincare and scale invariant 2D super-symmetric QFT and higher-spin QFT so that one can generalize the results of [69] and [70] and further seek minimal conditions on these 2d QFTs to have relevant super-symmetric and higher-spin Kaĉ-Moody currents obtained in chapters 3 and 4.

[3]See Appendix (6.2) for a brief review.

The asymptotic symmetries of flat space pure gravity and super gravity have been computed using techniques similar to the ones employed in this thesis[4]. For pure gravity in 3d asymptotically flat space-times the BMS_3 algebra emerges as the asymptotic symmetry algebra, its supergravity extension was studied in [71] where in a super-symmetric extension of BMS_3 was uncovered. The BMS algebra in 4d asymptotically flat space times is closely related to Weinberg's soft theorems and was studied by Strominger [72, 73] and by Laddha and Campiglia [74]. The implications of BMS group in critical bosonic string theory was studied by Avery and Burkhard [75] were they showed that the generalized Ward identity for the action of this group yields the Weinberg's soft theorem. The BMS_3 algebra and its super-symmetric extension was shown to be related to the flat-space limit of gravity and similarly super-gravity in AdS_3 with Dirichlet boundary conditions. It would be interesting to work out the possible flat-space limits of for the chiral and conformal boundary conditions studied here for gravity and super-gravity in AdS_3.

It would be interesting to investigate as to where in the world-sheet formulation of string theory in space times with AdS factors does one have the information about the boundary conditions imposed. For the case of string theories on AdS_3 there have been works were operators corresponding to the boundary Virasoro and super-Virasoro algebra have been constructed in the world-sheet theory [76, 77, 78, 79]. It would be exciting to investigate how such analysis can be carried out in the string world-sheet when the boundary conditions studied in this thesis are considered.

The boundary conditions of [6] pick a certain class of BTZ configurations in the bulk with the vacuum being the extremal BTZ solution. While the chiral boundary conditions of chapter 2 have the global AdS_3 as the vacuum and the only black-hole solution allowed is the extremal BTZ. The bulk theory of gravity in AdS_3 with boundary conditions of [6] were interpreted as an effective theory on AdS_2 after dimensional reduction describing the near horizon geometry of a (near)extremal black-hole. It would be interesting to see whether the chiral boundary conditions studied here can have any application in such

[4] Various other approaches also exist.

a regard, especially since the boundary conditions of [8] do not yield a super-current when generalized to super-gravities in AdS_3 but the chiral boundary conditions studied in chapter 3 do.

It would in general be interesting to see how this duality between induced gravities on the boundary with boundary condition similar to the ones proposed here on locally asymptotic AdS spaces furthers our understanding of AdS/CFT and how it may give new insights into quantities such as entanglement entropy, black-hole entropy etc.

Chapter 6

Appendix

6.1 Gauge fixing linearised AdS_3 gravity

In this section we linearise n-dimensional gravity[1] with negative cosmological constant around its global AdS_n vacuum. Our conventions are

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \;\; R^{\lambda}{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\delta\sigma}\Gamma^{\delta}_{\mu\nu} - \Gamma^{\lambda}_{\delta\nu}\Gamma^{\delta}_{\mu\sigma}$$
$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \quad R = R_{\mu\nu}g^{\mu\nu}\,. \tag{6.1}$$

Then the variations of various terms are

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\; g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\,\delta g^{\mu\nu}, \tag{6.2}$$

One useful variation is to find the physical fluctuations around any given background one needs to consider the linearisation of GR:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{6.3}$$

[1] We later fix $n = 3$.

and $h^{\mu\nu} = -\bar{g}^{\mu\lambda}\bar{g}^{\nu\sigma}h_{\lambda\sigma}$. The background metric $\bar{g}_{\mu\nu}$ is taken to satisfy the Einstein equation $\bar{R}_{\mu\nu} = x\,\bar{g}_{\mu\nu}$. This leads to the following variation of the Christoffel connections

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}(\bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}_\nu h_{\mu\sigma} - \bar{\nabla}_\sigma h_{\mu\nu}), \tag{6.4}$$

for Riemann tensor

$$\delta R^{\lambda}{}_{\mu\sigma\nu} = 2\bar{\nabla}_{[\sigma}\delta\Gamma^{\lambda}_{\nu]\mu}, \tag{6.5}$$

and for Ricci tensor

$$\begin{aligned} \delta R_{\mu\nu} &= \bar{\nabla}_\sigma\delta\Gamma^{\sigma}_{\mu\nu} - \bar{\nabla}_\nu\delta\Gamma^{\sigma}_{\mu\sigma} \\ &= \bar{\nabla}^\lambda\bar{\nabla}_{(\mu}h_{\nu)\lambda} - \frac{1}{2}(\bar{\nabla}^2 h_{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}_\nu h) := R^L_{\mu\nu} \\ \delta R^{\mu}_{\nu} &= h^{\mu\sigma}\bar{R}_{\sigma\nu} + \bar{g}^{\mu\sigma}R^L_{\sigma\nu}, \end{aligned} \tag{6.6}$$

where $h = \bar{g}^{\mu\nu}h_{\mu\nu}$. The variation of the Ricci scalar is

$$\begin{aligned} \delta R &= h^{\mu\nu}\bar{R}_{\mu\nu} + \bar{g}^{\mu\nu}\delta R_{\mu\nu} \\ &= h^{\mu\nu}\bar{R}_{\mu\nu} + \bar{\nabla}^\lambda\bar{\nabla}^\sigma h_{\lambda\sigma} - \bar{\nabla}^2 h := R^L \end{aligned} \tag{6.7}$$

To simplify this further let us note the identity

$$[\nabla_\mu, \nabla_\nu]T^{\sigma\kappa\cdots}{}_{\alpha\beta\cdots} = R^{\sigma}{}_{\lambda\mu\nu}T^{\lambda\kappa\cdots}{}_{\alpha\beta\cdots} + R^{\kappa}{}_{\lambda\mu\nu}T^{\sigma\lambda\cdots}{}_{\alpha\beta\cdots} + \cdots - R^{\lambda}{}_{\alpha\mu\nu}T^{\sigma\kappa\cdots}{}_{\lambda\beta\cdots} - R^{\lambda}{}_{\beta\mu\nu}T^{\sigma\kappa\cdots}{}_{\alpha\lambda\cdots} - \cdots \tag{6.8}$$

The equation that we would like to linearlize is

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \frac{(n-1)(n-2)}{2l^2}g_{\mu\nu} \tag{6.9}$$

which is equivalent to

$$R_{\mu\nu} = -\frac{(n-1)}{l^2}g_{\mu\nu}. \tag{6.10}$$

We assume that the background solution is a locally AdS_n space with $R = -n(n-1)/l^2$. Then we can write the Riemann tensor as

$$\bar{R}^{\alpha}_{\beta\mu\nu} = -\frac{1}{l^2}(\delta^{\alpha}_{\mu}\,\bar{g}_{\beta\nu} - \delta^{\alpha}_{\nu}\,\bar{g}_{\beta\mu}) \tag{6.11}$$

and we have

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu]\,T_{\alpha\beta} = -\bar{g}_{\mu\alpha}T_{\nu\beta} - \bar{g}_{\mu\beta}T_{\alpha\nu} + \bar{g}_{\nu\beta}T_{\alpha\mu} + \bar{g}_{\nu\alpha}T_{\mu\beta}\,. \tag{6.12}$$

Then the linearized equation is

$$-\frac{1}{2}(\bar{\nabla}^2 h_{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}_\nu h) + \bar{\nabla}^\lambda\bar{\nabla}_{(\mu}h_{\nu)\lambda} = -\frac{n-1}{l^2}h_{\mu\nu}\,. \tag{6.13}$$

Contracting this equation with $\bar{g}^{\mu\nu}$ gives

$$\bar{\nabla}^\lambda\bar{\nabla}^\sigma h_{\lambda\sigma} - \bar{\nabla}^2 h = -\frac{n-1}{l^2}h\,. \tag{6.14}$$

We try the gauge $\bar{\nabla}^\lambda h_{\lambda\sigma} = \kappa\,\bar{\nabla}_\sigma h$. The most popular choices are $\kappa = 0$ and $\kappa = 1$. We choose the latter

$$\bar{\nabla}^\lambda h_{\lambda\sigma} = \bar{\nabla}_\sigma h \tag{6.15}$$

which when contracted by ∇^σ gives

$$\bar{\nabla}^\lambda\bar{\nabla}^\sigma h_{\lambda\sigma} = \bar{\nabla}^2 h$$

Substituting this into the linearized equation gives $h = 0$ (for $n \geq 2$). Then we use (6.12) to write

$$\bar{g}^{\lambda\sigma}\bar{\nabla}_\lambda\bar{\nabla}_\mu h_{\nu\sigma} = -n\,h_{\mu\nu} + \bar{g}_{\mu\nu}\,h\,.$$

Finally we have

$$(\frac{2}{l^2} + \bar{\nabla}^2)h_{\mu\nu} = 0, \;\; \bar{\nabla}^\mu h_{\mu\nu} = 0, \;\; \bar{g}^{\mu\nu} h_{\mu\nu} = 0. \tag{6.16}$$

Let us note that the gauge condition: $\bar{\nabla}^\mu h_{\mu\nu} = 0$ does not fix the diffeomorphism gauge completely. If we perform a diffeomorphism by a vector ξ: $h_{ij} \to h_{ij} + \bar{g}_{ik}\bar{\nabla}_j\xi^k + \bar{g}_{jk}\bar{\nabla}_i\xi^k$ then if ξ^k satisfies $\nabla_k\xi^k = 0, \;\; \bar{\nabla}^2\,\xi^i + \bar{g}^{ik}\bar{R}_{kj}\xi^j = 0$. On the AdS_n background we have

$$\bar{\nabla}_k\xi^k = \bar{\nabla}^2\,\xi^i - \frac{n-1}{l^2}\xi^i = 0. \tag{6.17}$$

The former comes from demanding that $h = 0$ is preserved and the second follows from requiring that the gauge condition is respected.[2]

From here on we restrict to $n = 3$ case and will solve these equations completely. The equations $(\bar{\nabla}^2 + \frac{2}{l^2})h_{ij} = \bar{g}^{jk}\bar{\nabla}_j h_{ki} = h = 0$ are implied by the first order equations:

$$\frac{1}{l}h_{ij} + \epsilon_i{}^{mn}\bar{\nabla}_m h_{nj} = 0 \;\text{ or }\; \frac{1}{l}h_{ij} - \epsilon_i{}^{mn}\bar{\nabla}_m h_{nj} = 0 \tag{6.18}$$

In fact as noted by Li *et al* [80] one can write $(\bar{\nabla}^2 + \frac{2}{l^2})h_{ij} = 0$ as $\mathcal{D}^L\mathcal{D}^R h_{ij} = 0$ where $\mathcal{D}^{L/R}h_{ij} = 0$ are the first order equations above. Therefore the most general solution for h_{ij} can be written as a linear combination of solutions to these first order equations.

Now, every solution to the fluctuation equations of AdS_3 gravity should be locally a pure gauge - diffeomorphic to the background. This is true also of any solution to the above first order equations where the diffeomorphism is generated by one of the solutions to the residual gauge transformation equations. To see this one can show easily that $g_{ij} = \bar{g}_{ij} + h_{ij}$ where h_{ij} is taken to be a solution of either of the first order equations has

$$\delta R^\lambda{}_{\mu\sigma\nu} = -\frac{1}{l^2}(\delta^\lambda_\sigma h_{\mu\nu} - \delta^\lambda_\nu h_{\mu\sigma})\,. \tag{6.19}$$

[2] These equations are precisely that satisfied by any Killing vector, but not necessary that ξ^i is Killing.

This in turn implies that the Riemann tensor of g_{ij} takes the form expected of a locally AdS_3 metric up to higher order terms. But the background metric $\bar{g}_{ij}$ is already that of AdS_3. Therefore h_{ij} should be writable locally as $h_{ij} = \mathcal{L}_\xi \bar{g}_{ij}$ for some vector field ξ^i. However, there are some diffeos that act non-trivially in the asymptotes and others don't. To discern which diffeomorphisms take one solution to a physically different solution, one would have to take into account the boundary condition one imposes on the theory. It is with regards to the boundary condition imposed that one would be able to determine whether finite, infinite or zero charges are associated with these gauge transformations rendering them as allowed, not allowed or trivial respectively. This would be dealt in the next chapter of this Appendix.

Since we are working in a gauge we first look for all solutions to the residual gauge transformation equations.

6.1.1 Solutions to residual gauge vector equation in AdS_3:

We work with the global coordinates for AdS_3:

$$ds^2_{AdS_3} = l^2(d\rho^2 - \cosh^2\rho \, d\tau^2 + \sinh^2\rho \, d\phi^2). \tag{6.20}$$

The Killing vectors of this metric are:

$$\begin{aligned} L_{30} &= -\partial_\tau, \;\; L_{12} = \partial_\phi \\ (L_{31} + i\eta_1 L_{32}) + i\eta_2(L_{01} + i\eta_1 L_{02}) &= -e^{i(\eta_1\phi+\eta_2\tau)}[\partial_\rho + i(\eta_1 \coth\rho \, \partial_\phi + \eta_2 \tanh\rho \, \partial_\tau)] \end{aligned} \tag{6.21}$$

where η_1, η_2 are ± 1. Here the algebra involved is:

$$[L_{mn}, L_{pq}] = \eta_{mq}L_{np} + \eta_{np}L_{mq} - \eta_{mp}L_{nq} - \eta_{nq}L_{mp} \tag{6.22}$$

where $m, n \in \{3, 0, 1, 2\}$ and $\eta_{mn} = \text{diag}\{-1, -1, 1, 1\}$. We also have for any Killing vector:

$$\bar{\nabla}_i \bar{\nabla}_j \xi^k = \bar{R}^k{}_{ijm} \xi^m, \quad \bar{\nabla}^2 \xi^i = -\bar{g}^{ik} \bar{R}_{km} \xi^m \,. \tag{6.23}$$

The isometry algebra $SO(2,2)$ can be separated into $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$:

$$\begin{aligned} L_0 &= \frac{i}{2}(\partial_\tau + \partial_\phi), \quad L_{\pm 1} = \pm\frac{1}{2} e^{\pm i(\tau+\phi)}(\partial_\rho \pm i(\tanh\rho\, \partial_\tau + \coth\rho\, \partial_\phi)\,, \\ \bar{L}_0 &= \frac{i}{2}(\partial_\tau - \partial_\phi), \quad \bar{L}_{\pm 1} = \pm\frac{1}{2} e^{\pm i(\tau-\phi)}(\partial_\rho \pm i(\tanh\rho\, \partial_\tau - \coth\rho\, \partial_\phi)\,. \end{aligned} \tag{6.24}$$

The equations satisfied by the vector fields that generate residual gauge transformations are:

$$\bar{\nabla}_k \xi^k = 0, \quad \bar{\nabla}^2 \xi^k = \frac{2}{l^2} \xi^k \,. \tag{6.25}$$

One can in principle solve the above second order equation for ξ^k by solving for the Green's function[3] for a massive vector field and using the vanishing divergence condition in AdS_3, but we choose a different approach below. These can be decomposed into two first order equations on ξ^i:

$$\epsilon_i^{mn} \nabla_m \xi_n = -\frac{1}{l} \xi_i, \quad \epsilon_i^{mn} \nabla_m \xi_n = \frac{1}{l} \xi_i \tag{6.26}$$

Where $\epsilon_{\tau\rho\phi} = l^3 \cosh\rho \sinh\rho$; we call these left and right movers respectively. Since these are each a set of three coupled linear differential equations in ρ, the independent data must comprise of three arbitrary functions of the boundary co-ordinate. This is exactly the number of parameters which are required to parametrize a Diff×Weyl action on the boundary metric. We consider the following ansatz for vector fields and decompose each of the above equations to a second order ODE in one of the components.

$$\xi^i = e^{i(E_0 \tau + S\phi)} V^i(\rho) \tag{6.27}$$

[3]This would correspond to a bulk to bulk propagator for the massive vector field in AdS_3.

The left equation reduces to:

$$
\begin{aligned}
-iS\,V^{\rho} - 2\sinh\rho\cosh\rho\left(V^{\tau} - V^{\phi}\right) + \sinh^2\rho\,\partial_{\rho}V^{\phi} &= 0,\\
2V^{\tau} + i\left(E_0\tanh\rho\,V^{\phi} + S\,\coth\rho\,V^{\tau}\right) &= 0,\\
iE_0\,V^{\tau} + 2\sinh\rho\cosh\rho\left(V^{\phi} + V^{\tau}\right) + \cosh^2\rho\;\partial_{\rho}V^{\tau} &= 0,
\end{aligned}
\tag{6.28}
$$

Eliminating V^{ρ} and V^{τ} we get :

$$
\begin{aligned}
&2(3E_0^2 - E_0S + 4S^2 - E_0^2S^2 - S^4 + (E_0^2 - S^2)(-4 + S^2)\cosh 2\rho\\
&+E_0(E_0 + S)\cosh 4\rho)V^{\phi} + (\sinh 2\rho\cosh 2\rho(-4 + 6S^2)\\
&+4\sinh 2\rho(S^2 + \cosh 4\rho))\partial_{\rho}V^{\phi}(\rho) + \sinh^2 2\rho(-2 + S^2 + 2\cosh 2\rho)\;\partial_{\rho}^2V^{\phi} \;=\; 0
\end{aligned}
\tag{6.29}
$$

The above differential equation has the boundary $\rho \to \infty$ as a regular singular point, hence the solutions can be expanded in a series about the boundary. There are two solutions, one which dies out at the boundary and one which doesn't. Further we find that asymptotically finite solutions exist when certain conditions between S and E_0 hold true (6.31). We try an ansatz for V^{ϕ} of the type:

$$V^{\phi}(\rho) = \mathrm{sech}^2\rho\;\tanh^m\rho(a + \cosh^2\rho). \tag{6.30}$$

It turns out that the solution of this form implies the same conditions on E_0 and S for which it is asymptotically finite. Values of a and m for these solutions are:

$$
\begin{aligned}
E_0 = S &\implies \left\{m = -2 + S, a = 1/2(-2 + S + S^2)\right\} \\
&\quad \& \left\{m = -2 - S, a = 1/2(-2 - S + S^2)\right\}, \\
E_0 = -S &\implies \{m = -2 + S, a = 1/2(-2 + S)\} \\
&\quad \& \{m = -2 - S, a = 1/2(-2 - S)\}, \\
E_0 = 2 - S &\implies \{m = -2 + S, a = 0\} \\
&\quad \& \{m = -4 + S, a = -1\}, \\
E_0 = -2 - S &\implies \{m = -2 - S, a = 0\} \\
&\quad \& \{m = -4 - S, a = -1\}.
\end{aligned} \tag{6.31}
$$

Similar conditions are implied by the existence of asymptotic solutions of the right moving equation:

$$
\begin{aligned}
E_0 = -S &\implies \left\{m = -2 + S, a = 1/2(-2 + S + S^2)\right\} \\
&\quad \& \left\{m = -2 - S, a = 1/2(-2 - S + S^2)\right\}, \\
E_0 = S &\implies \{m = -2 + S, a = 1/2(-2 + S)\} \\
&\quad \& \{m = -2 - S, a = 1/2(-2 - S)\}, \\
E_0 = -2 + S &\implies \{m = -2 + S, a = 0\} \\
&\quad \& \{m = -4 + S, a = -1\}, \\
E_0 = 2 + S &\implies \{m = -4 - S, a = -1\} \\
&\quad \& \{m = -2 - S, a = 0\}.
\end{aligned} \tag{6.32}
$$

The first of the conditions in (6.31) and (6.32) give the left and right Virasoros in the bulk. The 3rd and 4th conditions give only one solution instead of two, therefore there exist more solutions linearly independent from those above for $E_0 = 2 - S$, $E_0 = -2 - S$ for left movers and $E_0 = -2+S$, $E_0 = 2+S$ for right movers. Also (6.29) has a symmetry of $E_0 \to -E_0$, $S \to -S$ which enables us to write down solutions for condition 4 (in (6.31) & (6.32)) interms of solution for condition 3. The 4 copies of Virasoros are:

$$
\begin{aligned}
L_n &= \frac{1}{4} i e^{ni(\tau+\phi)}(1+n+n^2+\cosh 2\rho)\mathrm{sech}^2\rho \tanh^n\rho\, \partial_\tau \\
&+\frac{1}{2} e^{ni(\tau+\phi)} n(n+\cosh 2\rho)\mathrm{csch}2\rho \tanh^n\rho\, \partial_\rho \\
&+\frac{1}{4} i e^{ni(\tau+\phi)}(-1+n+n^2+\cosh 2\rho)\mathrm{csch}^2\rho \tanh^n\rho\, \partial_\phi, \;\; \& \; L_n^\dagger \\
\bar{L}_n &= -\frac{1}{4} i e^{in(\tau-\phi)}(-1+n+n^2-\cosh 2\rho)\mathrm{sech}^2\rho \coth^n\rho\, \partial_\tau + \\
&\frac{1}{2} e^{in(\tau-\phi)} n(-n+\cosh 2\rho)\mathrm{csch}2\rho \; \coth^n\rho\, \partial_\rho \\
&-\frac{1}{4} i e^{ni(\tau-\phi)}(-1-n+n^2+\cosh 2\rho)\mathrm{csch}^2\rho \coth^n\rho\, \partial_\phi, \;\; \& \; \bar{L}_n^\dagger
\end{aligned} \tag{6.33}
$$

L_n and $L_n^\dagger$ correspond to the left moving Virasoro with $E_0 = S$ and $\bar{L}_n$ and $\bar{L}_n^\dagger$ correspond to the right moving Virasoro with $E_0 = -S$. The remaining solutions to the left moving equations are:

$$
\begin{aligned}
T_n &= e^{ni(\tau-\phi)}[\frac{1}{4} i(1-n+\cosh 2\rho)\mathrm{sech}^2\rho \coth^n\rho \partial_\tau - \frac{1}{2} n\,\mathrm{csch}2\rho \coth^n\rho\, \partial_\rho \\
&+\frac{1}{4} i(-1-n+\cosh 2\rho)\,\mathrm{csch}^2\rho \coth^n\rho\, \partial_\phi] \; \& \\
T_n^\dagger &\;\; \text{for} \;\; E_0 = -S. \\
B_n &= \frac{1}{2} e^{i(\tau(2-n)+n\phi)}[i\tanh^n\rho\, \partial_\tau + \tanh^{(n-1)}\rho\, \partial_\rho + i\tanh^{(n-2)}\rho\, \partial_\phi] \\
&\;\; \text{for } E_0 = 2-S \;\; \& \\
B_n^\dagger &\;\; \text{for } E_0 = -2-S; \\
[L_{-1}, T_{n-1}] &= D_n \\
&= \tfrac{1}{16} e^{i(\tau(n-2)-\phi n)} \\
&\;\; [-i(11-10n+2n^2-2(n-3)\cosh 2\rho+\cosh 4\rho)\mathrm{sech}^4\rho \; \coth^n\rho\, \partial_\tau \\
&\;\; +(-3+4n-2n^2-4(n-1)\cosh 2\rho+\cosh 4\rho)\mathrm{csch}\rho\, \mathrm{sech}^3\rho \coth^n\rho\, \partial_\rho \\
&\;\; -i(-5+2n+2n^2-2(n-1)\cosh 2\rho+\cosh 4\rho) \\
&\;\; \mathrm{csch}^2 2\rho\, \mathrm{sech}^2 2\rho \coth^n\rho\, \partial_\phi] \\
&\;\; \text{for } E_0 = 2-S \;\; \&
\end{aligned}
$$

$$[L_{-1}, T_{n-1}]^\dagger = [T^\dagger_{n-1}, L_1] = D^\dagger_n \text{ for } E_0 = -2 - S.$$

Similarly the ones for the right moving equation are:

$$\begin{aligned}
\bar{T}_n &= e^{in(\tau+\phi)}[\frac{i}{4}(1+n+\cosh 2\rho)\text{sech}^2\rho\ \tanh^n\rho\,\partial_\tau - \frac{1}{2}n\,\text{csch}2\rho\ \tanh^n\rho\,\partial_\rho \\
&\quad -\frac{i}{4}(-1+n+\cosh 2\rho)\,\text{csch}^2\rho\ \tanh^n\rho\,\partial_\phi]\ \& \\
\bar{T}^\dagger_n &\text{ for } E_0 = S \\
\bar{B}_n &= \frac{1}{2}e^{i(\tau(2+n)+\phi n)}[i\coth^n\rho\,\partial_\tau + \coth^{1+n}\rho\,\partial_\rho - i\coth^{2+n}\rho\,\partial_\phi], \\
&\quad \text{for } E_0 = 2+S\,\& \\
\bar{B}^\dagger_n &\quad \text{for } E_0 = -2+S \\
[\bar{L}_1, \bar{T}_n] &= \bar{D}_n \\
&= \frac{1}{8}(1+n)e^{i((n+2)\tau+n\phi)} \\
&\quad [i(5+2n+\cosh 2\rho+\cosh 4\rho)\,\text{csch}\rho\,\text{sech}^3\rho\ \tanh^n\rho\,\partial_\tau \\
&\quad (-3-2n+\cosh 2\rho\cosh 4\rho)\,\text{csch}^2 2\rho\ \tanh^n\rho\,\partial_\rho \\
&\quad +i(1+2n+\cosh\rho)\,\text{csch}^3\rho\,\text{sech}\rho\ \tanh^n\rho\,\partial_\phi] \\
&\quad \text{for } E_0 = 2+S \\
[\bar{L}_1, \bar{T}_n]^\dagger &= [\bar{T}^\dagger_n, \bar{L}_{-1}] = \bar{D}^\dagger_n \text{ for } E_0 = -2+S.
\end{aligned}$$

Their commutation relations with the killing vectors are(to be read along with their hermitian conjugates):

$$\begin{aligned}
&[T_n, T_m] = 0 &\qquad& [\bar{T}_n, \bar{T}_m] = 0 \\
&[L_0, T_n] = 0 && [\bar{L}_0, \bar{T}_n] = 0 \\
&[\bar{L}_0, T_n] = -nT_n && [L_0, \bar{T}_n] = -n\bar{T}_n \\
&[L_1, T_n] = B_{1-n} && [\bar{L}_{-1}, \bar{T}_n] = -\bar{B}^\dagger_{-n-1} \\
&[\bar{L}_1, T_n] = -nT_{n+1} && [L_1, \bar{T}_n] = -n\bar{T}_{n+1} \\
&[\bar{L}_{-1}, T_n] = -nT_{n-1} && [L_{-1}, \bar{T}_n] = -n\bar{T}_{n-1}
\end{aligned} \tag{6.34}$$

$$
\begin{aligned}
&[B_n, B_m] = 0 && [\bar{B}_n, \bar{B}_m] = 0 \\
&[L_0, B_n] = -B_n && [\bar{L}_0, \bar{B}_n] = -\bar{B}_n \\
&[\bar{L}_0, B_n] = (n-1)B_n && [L_0, \bar{B}_n] = -(n+1)\bar{B}_n \\
&[L_1, B_n] = 0 && [\bar{L}_1, \bar{B}_n] = 0 \\
&[L_{-1}, B_n] = -2T_{1-n} && [\bar{L}_{-1}, \bar{B}_n] = -2\bar{T}^{\dagger}_{-n-1} \\
&[\bar{L}_1, B_n] = (n-1)B_{n-1} && [L_1, \bar{B}_n] = -(n+1)\bar{B}_{n+1} \\
&[\bar{L}_{-1}, B_n] = (n-1)B_{n+1} && [L_{-1}, \bar{B}_n] = -(n+1)\bar{B}_{n-1}
\end{aligned}
\tag{6.35}
$$

Also note that $B_1 = L_1$ & $\bar{B}_{-1} = \bar{L}_1$.

Using these vector fields one can generate solutions to the metric fluctuation equation: $h_{ij} = \bar{\nabla}_i \xi_j + \bar{\nabla}_j \xi_i$. From L_n and $\bar{L}_n$ we find that the behaviour of h_{ij} near the asymptotes is

$$h_{\rho\rho} \approx O(e^{-4\rho}), \;\; h_{\rho\tau} \approx O(e^{-2\rho}) \approx h_{\rho\phi}, \;\; h_{\tau\tau} \approx h_{\tau\phi} \approx h_{\phi\phi} \approx O(1)\,. \tag{6.36}$$

Note these are consistent with Brown-Henneaux boundary conditions (in fact $h_{rr} \sim 1/r^6$ seems to fall faster then BH requirement). Whereas for T_n and $\bar{T}_n$ they are:

$$h_{\rho\rho} \approx O(e^{-2\rho}), \;\; h_{\rho\tau} \approx O(e^{-2\rho}) \approx h_{\rho\phi}, \;\; h_{\tau\tau} \approx h_{\tau\phi} \approx h_{\phi\phi} \approx O(e^{2\rho})\,. \tag{6.37}$$

and for B_n, D_n and their $\bar{B}_n, \bar{D}_n$ they are:

$$h_{\rho\rho} \approx O(e^{-2\rho}), \;\; h_{\rho\tau} \approx O(1) \approx h_{\rho\phi}, \;\; h_{\tau\tau} \approx h_{\tau\phi} \approx h_{\phi\phi} \approx O(e^{2\rho})\,. \tag{6.38}$$

If one tries to categorize these vector fields as to how they act on the boundary metric then one can see that except for the Virasoros $L_n, \bar{L}_n$, all other vectors effect the boundary metric. $\{\bar{T}_n, \bar{B}_n, \bar{B}^{\dagger}_n\}$ and $\{T_n, B_n, B^{\dagger}_n\}$ change the dx^{+2} and dx^{-2} components while $D_n, \bar{D}_n$ change the boundary metric by a mix of chiral Diff and Weyl transformation. Therefore, the residual gauge transformations in the covariant gauge have an action of Diff×Weyl on the boundary metric, as was also expected from the analysis done in the Fefferman-Graham gauge.

The asymptotic values of the components of $\{\bar{T}_n, \bar{B}_n, \bar{B}_n^\dagger\}$ match exactly in form to the asymptotic values of $\{T_n^{(0)}, T_n^{(+)}, T_n^{(-)}\}$[4] in (2.23). The essential data defining these vector fields is their most leading asymptotic value, the subleading terms into the bulk away from the boundary are basically an artefact of which gauge one tries to work in.

Therefore in the covariant gauge we do obtain close form expressions for the vector fields which comprise the two copies of Witt algebra and the two copies of of the $sl(2,\mathbb{R})$ Kaĉ-Moody current algebra. Which of these get choosen as the asymptotic algebra depends on the boundary conditions and the boundary terms one adds to make the variational problem well defined as discussed in all the chapters.

As was mentioned in chapter (2), one may have proceeded to work in the Fefferman-Graham gauge where the residual gauge transformations generate a Diff×Weyl transformations of the boundary metric.

6.1.2 Gauge fixing in the Chern-Simons formalism

Since the the first order formalism for AdS_3 written as a difference of two Chern-Simons theories can be a very useful, and in some cases the only construction, it would be interesting to present the covariant gauge fixing approach outlined above in this formalism. To this end it simply becomes necessary to write down the equivalent equations to (6.26) on the gauge parameters Λ and $\tilde{\Lambda}$. The first order formalism is reviewed in the Appendix (6.3) for details.

[4] The exact values of n might be different for either sides.

The equations (6.26) on the vector fields is equivalent to

$$e^a_{[\mu}\mathcal{D}_{\nu]}\Lambda_a = 0 \quad \& \quad e^a_{[\mu}\tilde{\mathcal{D}}_{\nu]}\tilde{\Lambda}_a = 0, \tag{6.39}$$

where

$$e^a_\mu = (A - \tilde{A})^a_\mu,$$

$\mathcal{D}_\mu$ and $\tilde{\mathcal{D}}_\nu$ are the respective gauge covariant derivatives. Since the Chern-Simons formalism admits natural generalisations to super-gravity and higher-spins in AdS_3, the above gauge fixing condition generalises as it stands to these cases. One may read out what the above condition translates to the non-$sl(2,\mathbb{R})$ components of the gauge transformation parameters in these cases. For example, in the case of $N = (1,1)$ super-gravity in AdS_3, the above condition translates to

$$\begin{aligned}
\gamma^\mu\mathcal{D}_\mu\epsilon &= 0. \\
\epsilon &= e^{-\rho\gamma^2/2}(\epsilon_0(x^\pm) + e^{-\rho}\epsilon_1(x^\pm) + e^{-2\rho}\epsilon_2(x^\pm)\ldots).
\end{aligned}$$

where :

$$\begin{aligned}
\epsilon_0^- &= \epsilon_{0+}^-(x^+)e^{ix^-/2} + \epsilon_0^-(x^+)e^{-ix^-/2} + \epsilon_{00}^-(x^-), \\
\epsilon_0^+ &= -2\partial_-\epsilon_0^-, \\
\epsilon_{1.3.5\ldots} &= 0, \\
\epsilon_2^- &= -4\partial_-\partial_+\epsilon_0^-, \\
\epsilon_4^- &= \tfrac{1}{2}\epsilon_0^- + 4\partial_-\epsilon_0^- + 2\partial_+\epsilon_2^+(x^\pm), \\
\epsilon_4^+ &= \partial_-\epsilon_{00}^- + 4i(e^{ix^-/2}\partial_+^2\epsilon_{0+}^- - e^{-ix^-/2}\partial_+^2\epsilon_{0-}^-) + 4\partial_-^3\epsilon_{00}^- + 2\partial_+\partial_-\epsilon_2^+.
\end{aligned} \tag{6.40}$$

Where

$$\mathcal{D}_\mu\epsilon = (\partial_\mu + \frac{1}{4}\omega_\mu{}^{\ddot{ab}}\Gamma_{\underset{..}{ab}})\epsilon + \frac{\eta}{2l}\gamma_\mu\epsilon, \tag{6.41}$$

and the frames are

$$e^0 = l\,\cosh\rho\,d\tau,\;\; e^1 = l\,\sinh\rho\,d\phi,\;\; e^2 = l\,d\rho. \tag{6.42}$$

Here we work with the following gamma matrices:

$$\Gamma^{0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \Gamma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \Gamma^{2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.43}$$

Above, the ± index denotes 2 dimensional chirality w.r.t. γ^2. The spinor bi-linears constructed out of these residual gauge spinors - $\bar{\chi}\gamma^\mu\epsilon$ are:

$$\begin{aligned} \xi^\mu \partial_\mu &= -\tfrac{1}{2}\partial_- f(x^\pm)\partial_\rho + f(x^\pm)\partial_- + O(e^{-\rho}). \\ &\text{where :} \\ -f(x^\pm)\tfrac{l}{4} &= \chi^-_{00}\epsilon^-_{00} + \chi^-_{0-}\epsilon^-_{0+} + \chi^-_{0+}\epsilon^-_{0-} + e^{-ix^-}\chi^-_{0}\epsilon^-_{0-} + e^{ix^-}\chi^-_{0+}\epsilon^-_{0+} + e^{-ix^-/2}(\chi^-_{0-}\epsilon^-_{00} + \chi^-_{00}\epsilon^-_{0-}) \\ &\quad + e^{-ix^-/2}(\chi^-_{0+}\epsilon^-_{00} + \chi^-_{00}\epsilon^-_{0+}). \end{aligned} \tag{6.44}$$

The above expression for the vector field coincides with the vector fields (2.23) if the spinor parameters χ^-_{00} and ϵ^-_{00} are put to zero. Therefore:

$$-f(x^\pm)\tfrac{l}{4} = \chi^-_{00}\epsilon^-_{00} + \chi^-_{0-}\epsilon^-_{0+} + \chi^-_{0+}\epsilon^-_{0-} + e^{-ix^-}\chi^-_{0-}\epsilon^-_{0-} + e^{ix^-}\chi^-_{0+}\epsilon^-_{0+}. \tag{6.45}$$

6.1.3 Gauge fixing the the Fefferman-Graham gauge

The considerations of the previous sub-sections are easily reflected if the analysis were carried out in the Fefferman-Graham gauge. Since we already know that the fluctuation $h_{\mu\nu}$ will be given by a Lie derivative of the background metric $\bar{g}_{\mu\nu}$ we can start with the ansatz

$$h_{\mu\nu} = -\bar{\nabla}_\mu \xi_\nu - \bar{\nabla}_\nu \xi_\mu \tag{6.46}$$

and impose the gauge fixing conditions to find ξ. We write the background metric of global AdS_3 as

$$ds^2_{AdS_3} = l^2\left[-\cosh^2\!\rho\, d\tau^2 + d\rho^2 + \sinh^2\!\rho\, d\phi^2\right] \tag{6.47}$$

Then the gauge conditions $h_{\rho\rho} = h_{\rho\tau} = h_{\rho\phi} = 0$ give the following equations:

$$\partial_\rho \xi^\rho = 0, \quad -\cosh^2\rho\,\partial_\rho\xi^\tau + \partial_\tau\xi^\rho = 0, \quad \sinh^2\rho\,\partial_\rho\xi^\phi + \partial_\phi\xi^\rho = 0. \tag{6.48}$$

The general solution to these equations is given by the vector

$$\xi = \left[a^\tau(\tau,\phi) + (\tanh\rho - 1)\,\partial_\tau\sigma(\tau,\phi)\right]\partial_\tau + \left[a^\phi(\tau,\phi) + (\coth\rho - 1)\,\partial_\phi\sigma(\tau,\phi)\right]\partial_\phi + \sigma(\tau,\phi)\,\partial_\rho\,. \tag{6.49}$$

The corresponding $h_{\mu\nu}$ is

$$\begin{aligned} h_{\tau\tau} &= 2\,\cosh\rho\left[\,\sinh\rho\,(\sigma + \partial_\tau^2\sigma) + \cosh\rho\,\partial_\tau(a^\tau - \partial_\tau\sigma)\right] \\ h_{\tau\phi} &= \cosh^2\rho\,\partial_\phi(a^\tau - \partial_\tau\sigma) - \sinh^2\rho\,\partial_\tau(a^\phi - \partial_\phi\sigma) \\ h_{\phi\phi} &= -2\,\sinh\rho\left[\,\cosh\rho\,(\sigma + \partial_\phi^2\sigma) + \sinh\rho\,\partial_\phi(a^\phi - \partial_\phi\sigma)\right] \end{aligned} \tag{6.50}$$

The physical fluctuations therefore are characterized by a boundary scalar σ and a boundary vector a^μ. We have done the fluctuation analysis around the global AdS_3. The new boundary metric changes and is given by

$$g^{(0)}_{\tau\tau} = -\frac{1}{4} + \frac{1}{2}(\partial_\tau a^\tau + \sigma), \;\; g^{(0)}_{\tau\phi} = \frac{1}{4}(\partial_\phi a^\tau - \partial_\tau a^\phi), \;\; g^{(0)}_{\phi\phi} = \frac{1}{4} - \frac{1}{2}(\partial_\phi a^\phi + \sigma) \tag{6.51}$$

It turns out that the physical fluctuations around asymptotically locally AdS_3 space are also characterized by a boundary scalar and a boundary vector. To see this we start with a asymptotically locally AdS_3 space in Fefferman-Graham expansion:

$$ds^2 = l^2\frac{dr^2}{r^2} + r^2\left[g^{(0)}_{ab} + \frac{l^2}{r^2}g^{(2)}_{ab} + \cdots\right]dx^a dx^b \tag{6.52}$$

where x^a are the boundary coordinates with boundary metric $g^{(0)}$. Then the gauge conditions $h_{rr} = h_{ra} = 0$ give rise to the residual gauge vector

$$\xi = \sigma(x)\,r\partial_r + a^a(x)\,\partial_a + \frac{1}{2r^2}g^{ab}_{(0)}(x)\partial_b\sigma(x)\partial_a + O(\tfrac{1}{r^4}) \tag{6.53}$$

with

$$h_{ab} = -r^2 \left[\mathcal{L}_a g^{(0)}_{ab} + 2\sigma\, g^{(0)}_{ab} + O(\tfrac{1}{r^2}) \right] \tag{6.54}$$

Therefore the residual diffeomorphisms in the Fefferman-Graham gauge have an action of Diff×Weyl on the boundary metric $g^{(0)}$. The Diff×Weyl algebra in terms of the parameters $a^a(x)$ and $\sigma(x)$ is

$$\left[(a^a_{(1)}\partial_a, \sigma_1), (a^a_{(2)}\partial_a, \sigma_2)\right] = \left((a^c_{(2)}\partial_c a^a_{(1)} - a^c_{(1)}\partial_c a^a_{(2)})\, \partial_a, a^c_{(1)}\partial_c\sigma_2 - a^c_{(2)}\partial_c\sigma_1\right), \tag{6.55}$$

where $a^a(x)$ parametrises the vector fields generating infinitesimal diffeomorphisms and $\sigma(x)$ denotes the Weyl transformation parameter.

The 2d Lorentzian boundary metric is conformally flat and can always be locally written as

$$ds^2 = e^{\varphi'(x^+,x^-)} dx^+ dx^-. \tag{6.56}$$

The Lorentzian version of the Beltrami transform on (x^+, x^-) to $(z(x, x^+), \bar{z}(x^+, x^-))$ such that

$$\partial_{\bar{z}} x^+ = \mu \partial_z x^+, \;\; \partial_z x^- = \bar{\mu} \partial_{\bar{z}} x^-, \tag{6.57}$$

brings the boundary metric in the form

$$ds^2 = e^{\varphi}(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz). \tag{6.58}$$

Here the parameters μ and $\bar{\mu}$ are such that $|\mu| < 1$ and $|\bar{\mu}| < 1$. When the boundary has just one spatial periodic direction, this parametrization captures arbitrary metric on the boundary with fixed co-ordinates $(z, \bar{z})$. For the case when $\mu = 0 = \bar{\mu}$ one recovers the conformal metric. While for the case of chiral or light-cone gauge corresponds to $\bar{\mu} = 0 = \phi$. For either of these gauge choices one can recover the induced gravity action on the boundary by considering appropriate boundary terms added to the bulk action. A similar exercise was done by Bañados *et al* in [13].

6.2 Review of the asymptotic covariant charge formalism

6.2.1 Introduction

Noether's first theorem gives a systematic way of associating conserved charges to continuous global symmetries of a theory. By global we mean that the parameters of these symmetries are constants and have no space-time dependence. Gauge theories on the other hand posses gauge symmetries which are parametrized by local functions of space-time, and are a commonplace in physics. If one tries to associate conserved currents to gauge symmetries following the usual procedure then one runs into a problem; the associated current vanishes on-shell. This was first noticed by Noether herself and addressed in Noether's second theorem. We briefly outline the consequence of this theorem elucidating the problem.

Consider a Lagrangian $\mathcal{L}(\phi^i)$ which is function of fields ϕ^i and it's space time derivatives at a point in n dimensions. We denote by X^i the character of a symmetry of the Lagrangian $\mathcal{L}$ *i.e.* $X^i = \delta_X \phi^i$. By definition a symmetry of the Lagrangian $\mathcal{L}$ satisfies

$$X^i \frac{\delta \mathcal{L}}{\delta \phi^i} = \partial_\mu j^\mu_X. \tag{6.59}$$

The current j^μ_X is conserved when Euler-Lagrange equations (eom) hold *i.e.* $\frac{\delta \mathcal{L}}{\delta \phi^i} = 0$. Consider now a gauge symmetry of the same Lagrangian $\mathcal{L}$ whose character we denote by $\delta_f \phi = R^i_\alpha(f^\alpha)$, where $R^i_\alpha = \sum_{k=0} R^{i(\mu_1\mu_2 \ldots \mu_k)}_\alpha \partial_{\mu_1} \ldots \partial_{\mu_k}$ and f^α are some arbitrary (possibly field dependent) local functions. Since this is a symmetry therefore

$$R^i_\alpha(f^\alpha) \frac{\delta \mathcal{L}}{\delta \phi^i} = \partial_\mu j^\mu_f. \tag{6.60}$$

Now, since the gauge transformation character $R^i_\alpha(.)$ is linear in its argument, one can shift derivatives from the later onto $\frac{\delta\mathcal{L}}{\delta\phi^i}$

$$\begin{aligned} R^i_\alpha(f^\alpha)\frac{\delta\mathcal{L}}{\delta\phi^i} &= f^\alpha R^{+i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}) + \partial_\mu S^{\mu i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}, f^\alpha) = \partial_\mu j^\mu_f, \\ \implies f^\alpha R^{+i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}) &= \partial_\mu\left[j^\mu_f - S^{\mu i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}, f^\alpha)\right]. \end{aligned} \tag{6.61}$$

The last equation implies that the *r.h.s.* is a total divergence for arbitrary local function f^α multiplying the *l.h.s*, implying

$$R^{+i}_\alpha\left(\frac{\delta\mathcal{L}}{\delta\phi^i}\right) = 0. \tag{6.62}$$

Therefore, in the presence of gauge symmetries there exists associated identities among the Euler-Lagrange equations of motion. It is easily seen that the Euler-Lagrange equations satisfy a system of equations which are same in number to the independent gauge parameters, and of same order in derivatives as is the character of gauge transformations. This is Noether's second theorem. Therefore for any conserved current j^μ_f associated with gauge transformations,

$$\partial_\mu(j^\mu_f - S^{\mu i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}, f^\alpha)) = 0 \tag{6.63}$$

holds true off-shell. The current $S^{\mu i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}, f^\alpha)$ depends linearly on the Euler-Lagrange equations of motion and the gauge transformation parameters. Using the algebraic Poincaré lemma one concludes that the conserved current can be improved on-shell by a divergence of a super-potential $k^{[\mu\nu]}_f$

$$j^\mu_f = S^{\mu i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}, f^\alpha) - \partial_\nu k^{[\mu\nu]}_f \tag{6.64}$$

for space-time dimensions greater than one. The form of the super potential is chosen such that it drops out when the divergence of j^μ_f is computed. The associated Noether charge is then given by

$$Q_f[\phi] = \int_\Sigma j_f|_\phi = \int_{\partial\Sigma} k_f|_\phi \tag{6.65}$$

evaluated on-shell. Here, Σ is a $n-1$ dimensional space-like surface with a boundary $\partial\Sigma$. This implies that the Noether charge associated with gauge transformation may be given by some seemingly arbitrary super-potential thus giving rise to the fore mentioned problem.

The work by Barnich and Brandt allows one to associate a certain equivalence class of conserved $n-2$ forms $k^{[\mu\nu]}$ to an equivalence class of gauge transformations respecting a given set of boundary conditions. They further give conditions enabling one to associate finite charges with such conserved $n-2$ forms. Their analysis applies to quite a general set of Lagrangian field theories as it is carried out independent of the specific details Lagrangian in question. Prior to this work, prescriptions for computing finite conserved charges associated to gauge theories prescribed with a particular set of boundary conditions have been worked out for some specific Lagrangians, such as the Chern-Simons theory defined on a non-compact gauge group and Einstein-Hilbert action for gravity in $2+1$ dimensions. These results are readily reproduced from the general prescription outlined in their work.

6.2.2 Definitions and result

We begin with a summary of definitions and notations required to state the result.

Let $\bar{\phi}^i$ denote a solution to the Euler-Lagrange equations near the boundary. We denote a field configuration by $\phi^i = \bar{\phi}^i + \varphi^i$, where $\varphi^i \to O(\chi^i)$ as one approaches the boundary. Here, χ^i denote a set of functions which prescribes a desired fall-off condition, for instance $\chi^i = 1/r^{m^i}$, where the boundary is reached by taking the limit $r \to \infty$ and m^is being some integers. Therefore $\{\chi^i\}$ describe a set of boundary conditions.

We further assume that the theory is "asymptotically linear". More precisely the leading order contribution to the Euler-Lagrange equations as one approaches the boundary comes from the Euler-Lagrange equations linearised around a background solution. Let the behaviour of the linearised Euler-Lagrange equations evaluated on arbitrary $\varphi^i \to O(\chi^i)$ be denoted by $O(\chi_i)$ *i.e.*

$$\forall \varphi^i \to O(\chi^i): \quad \left.\frac{\delta \mathcal{L}^{free}}{\delta \varphi^i}\right|_{\varphi(x)} dx^n \to O(\chi_i), \tag{6.66}$$

where $\mathcal{L}^{free}$ is the Lagrangian $\mathcal{L}$ linearised about a background solution $\bar{\phi}^i$. Further, any φ^i would be termed as an asymptotic solution if

$$\left.\frac{\delta \mathcal{L}^{free}}{\delta \varphi^i}\right|_{\varphi(x)} dx^n \to o(\chi_i), \tag{6.67}$$

where $o(\chi_i)$ denotes asymptotic behaviour of a lower degree than that of χ_i. The theory being asymptotically linear would imply

$$\forall \varphi^i \longrightarrow O(\chi^i): \quad \left.\left[\frac{\delta \mathcal{L}}{\delta \phi^i} - \frac{\delta \mathcal{L}^{free}}{\delta \varphi^i}\right]\right|_{\varphi(x)} dx^n \to o(\chi_i). \tag{6.68}$$

We call those gauge transformation parameters as reducibility parameters which vanish when the Euler-Lagrange equations hold and denote them by f^α *i.e.*

$$R^i_\alpha(f^\alpha) \approx 0 \tag{6.69}$$

Just like the Lagrangian, both $R^i_\alpha(.)$ and f^α can be expanded in a power series in φ^i about a particular background solution $\bar{\phi}^i$. We denote with a superscript as to how many powers of φ^i they contain *i.e.*

$$\begin{aligned} R^i_\alpha &= R^{i0}_\alpha + R^{i1}_\alpha + \dots\,, \\ f^\alpha &= f^{\alpha 0} + f^{\alpha 1} + \dots\,. \end{aligned} \tag{6.70}$$

Since $\delta_f \phi^i = R^i_\alpha(f^\alpha)$ is a gauge symmetry of the full theory therefore we have

$$R^i_\alpha(f^\alpha)\frac{\delta\mathcal{L}}{\delta\phi^i} = \partial_\mu S^{\mu i}_{\ \alpha}(\frac{\delta\mathcal{L}}{\delta\phi^i}, f^\alpha). \tag{6.71}$$

Expanding the above equation in powers of φ^i about $\bar{\phi}^i$ one can show that $R^{+i0}_\alpha(f^\alpha)$ generates gauge symmetries of $\mathcal{L}^{free}$. Further if $f^\alpha = \tilde{f}^\alpha(x)$ is a field independent reducibility parameter of the free theory, then the defining equation (6.69) for reducibility parameter yields

$$R^{i0}_\alpha(\tilde{f}^\alpha) = 0^5 \tag{6.72}$$

and $R^{i1}_\alpha(f^\alpha)$ generates symmetries of the free theory.

As shown in the previous section, the Noether operators R^{+i}_α can be obtained from the generating set of gauge transformations R^i_α by integration by parts and ignoring the total derivative term that vanishes on-shell. These operators too furnish an asymptotic expansion. Let R^{+i}_α denote the generating set of all Noether identities for the full theory *i.e*.

$$R^{+i}_\alpha(\frac{\delta\mathcal{L}}{\delta\phi^i}) = 0. \tag{6.73}$$

Then, the generating set of Noether identities for the linearised theory is given by R^{+i0}_α such that

$$R^{+i0}_\alpha(\frac{\delta\mathcal{L}^{free}}{\delta\phi^i})dx^n \longrightarrow 0. \tag{6.74}$$

Here as before, R^{+i0}_α is the φ^i independent term in the linearisation of R^{+i}_α. Further we define χ_α as

$$\forall\psi_i \longrightarrow O(\chi_i): \ \ R^{+i0}_\alpha(\psi_i) \longrightarrow O(\chi_\alpha). \tag{6.75}$$

[5]Here the equality is evaluated while being on-shell w.r.t. the background field $\bar{\phi}^i$ but being off shell w.r.t. to the linearised Lagrangian $\mathcal{L}^{free}$ for φ^i

Since the theory linearises as we approach the boundary therefore we assume

$$\forall \psi_i \longrightarrow O(\chi_i) : \;\; [R^{+i}_{\alpha}(\psi_i) - R^{+i0}_{\alpha}(\psi_i)] \longrightarrow o(\chi_\alpha). \tag{6.76}$$

The operators R^{+i}_{α} and R^{+i0}_{α} are termed as the Noether operators of the full and the linearised (free) theory respectively and act upon the left-hand sides of the respective Euler-Lagrange equations of motion. The above statements on the linearisation of the full theory implies that the linearised Euler-Lagrange equations and the corresponding linearised Noether operators capture the leading order behaviour of the corresponding quantities of the full theory as one approaches the boundary.

As explained in the previous section, a conserved current j^{μ}_{f} defined in the usual manner for any gauge transformation vanishes on-shell and can only be improved by a superpotential $k^{[\mu\nu]}_{f}$, the origin of which seems arbitrary. But one can systematically associate equivalence class of conserved $n-2$ forms $\tilde{k}^{[\mu\nu]}_{f}$ for equivalence class of asymptotic reducibility parameters $\tilde{f}^{\alpha}$.

Let us now define what one means by an asymptotic reducibility parameter in the context of a linearisable theory. We begin by expanding the definition for a reducibility parameter (6.69) about a background solution $\bar{\phi}^i$ in powers of the fluctuation φ^i.

$$\begin{aligned} R^{i}_{\alpha}(f^{\alpha}) &= M^{+ij}(\frac{\delta\mathcal{L}}{\delta\phi^j}), \\ \phi^i &= \bar{\phi}^i + \varphi^i, \\ \left.\frac{\delta\mathcal{L}}{\delta\phi^i}\right|_{\bar{\phi}^i} &= 0 \end{aligned} \tag{6.77}$$

and the operator $M^{+ij}(.)$ is linear and homogeneous in its arguments. Here we assume in all generality that the operators R^{i}_{α}, M^{+ij} and parameters f^{α} depend on ϕ^i. At the zeroth order in φ^i we find that the *r.h.s.* vanishes giving

$$R^{i0}_{\alpha}(f^{\alpha 0}) = 0 \tag{6.78}$$

Further, if f^{α} were field independent *i.e.* independent of φ^{i}[6] then this would imply $R^{i0}_{\alpha}(f^{\alpha}) = 0$. This should be seen as a condition defining a field independent reducibility parameter of the full theory linearised about a background solution. Therefore, in a sense this would be similar to finding Killing vectors of a background solution in general relativity about which a theory is linearised.

Since we are interested in studying the symmetries of the linearised theory as it approaches a boundary, we therefore relax this condition and define asymptotic reducibility parameters as a field independent parameters $\tilde{f}^{\alpha}$ *s.t.*

$$\forall \psi_i \longrightarrow O(\chi_i) : \;\; \psi_i R^{i0}_{\alpha}(\tilde{f}^{\alpha}) \longrightarrow 0. \tag{6.79}$$

Since $R^{+i0}_{\alpha}(\psi_i) \longrightarrow O(\chi_{\alpha})$ and assuming integration by parts does not change the asymptotic degree, the above condition holds trivially for

$$\tilde{f}^{\alpha} \longrightarrow o(\chi^{\alpha}) \iff \tilde{f}^{\alpha} \sim 0, \tag{6.80}$$

where $|\chi_{\alpha}| = -|\chi^{\alpha}|$. Therefore, non-trivial asymptotic reducibility parameters: $\tilde{f}^{\alpha} \longrightarrow O(\chi^{\alpha})$ - are defined upto asymptotic reducibility parameters with the above fall-off behaviour.

An asymptotically conserved $n-2$ form is defined as

$$\forall \varphi^i(x) : \;\; d_H \tilde{k}|_{\varphi(x)} \longrightarrow \tilde{s}^i(\frac{\delta \mathcal{L}}{\delta \varphi^i})|_{\varphi(x)}, \tag{6.81}$$

[6]Since we have linearised the theory about a background solution, field independence would refer to not depending upon the fluctuation φ^i.

where $\tilde{s}^i(.)$ is an $n-1$ form which depends linearly and homogeneously on its argument and its derivatives. A conserved $n-2$ form is trivial if

$$\forall\varphi^i(x):\ \ \tilde{k}|_{\varphi(x)} \longrightarrow \tilde{t}^i(\frac{\delta\mathcal{L}^{free}}{\delta\varphi^i})\Big|_{\varphi(x)} + d_H\tilde{l}|\varphi(x) \tag{6.82}$$

where $\tilde{t}^i(.)$ is an $n-2$ form which depends linearly and homogeneously on its argument and its derivatives.

We use the above definitions to state the following bijective correspondence. An $n-1$ form s^i_α can be defined for a set of functions Q_i as

$$\forall Q_i:\ \ d^n x\, Q_i R^{i0}_\alpha(\tilde{f}^\alpha) = d^n x\, R^{+i0}_\alpha(Q_i)\tilde{f}^\alpha + d_H s^i_\alpha(Q_i,\tilde{f}^\alpha) \tag{6.83}$$

For the case of $Q_i = \frac{\delta\mathcal{L}^{free}}{\delta\varphi^i}$ the above equation reduces to

$$d_H\tilde{s}^i(\frac{\delta\mathcal{L}^{free}}{\delta\varphi^i},\tilde{f}^\alpha) = d^n x\,\frac{\delta\mathcal{L}^{free}}{\delta\varphi^i}R^{i0}_\alpha(\tilde{f}^\alpha). \tag{6.84}$$

Further if $\tilde{f}^\alpha$ is an asymptotic reducibility parameter as defined above then

$$\begin{aligned} \forall\varphi^i \quad &: \quad d_H\tilde{s}_{\tilde{f}}|_{\varphi(x)} \longrightarrow 0, \\ \Longrightarrow \forall\varphi^i \quad &: \quad \tilde{s}_{\tilde{f}}|_{\varphi(x)} \longrightarrow -d_H\tilde{k}_{\tilde{f}}|_{\varphi(x)}. \end{aligned} \tag{6.85}$$

As $\tilde{s}_{\tilde{f}}$ is linear and homogeneous in the linearised field equations, $\tilde{k}_{\tilde{f}}$ is therefore an asymptotically conserved $n-2$ form. The explicit expression for $\tilde{k}_{\tilde{f}}$ can be obtained from applying the contracting homotopy of the algebraic Poincaré lemma to $\tilde{s}_{\tilde{f}}$.

$$\tilde{k}^{[\mu\nu]}_{\tilde{f}} = \frac{1}{2}\varphi^i\frac{\partial^S\tilde{s}^\nu_{\tilde{f}}}{\partial\varphi^i_\mu} + \left(\frac{2}{3}\varphi^i_\lambda - \frac{1}{3}\varphi^i\partial_\lambda\right)\frac{\partial^S\tilde{s}^\nu_{\tilde{f}}}{\partial\varphi^i_{\lambda\mu}} - (\mu\leftrightarrow\nu), \tag{6.86}$$

where $\varphi^i_\lambda = \partial_\mu\varphi^i$ and $\partial^S\varphi^i_{\mu_1\mu_2\ldots\mu_k}/\partial\varphi^j_{\nu_1\nu_2\ldots\nu_k} = \delta^i_j\delta^{\nu_1\nu_2\ldots\nu_k}_{(\mu_1\mu_2\ldots\mu_k)}$, where indices are symmetrised with weight one.

6.2.3 Some examples

As mentioned in the last section, the above result - which applies for a generic Lagrangian systems that admits linearisation, yields the same expression for asymptotic charges as was known in the case of certain gauge theories. We illustrate this in the case of Chern-Simons theory defined on non-compact gauge group $Sl(2,\mathbb{R})$. Next, the prescription can be used to define an asymptotically conserved $D-2$ current for the case of Einstein-Hilbert gravity with arbitrary cosmological constant and boundary fall-off conditions.

Chern-Simons theory for a non-compact gauge group

We illustrate this in the case of Chern-Simons theory with a non-compact gauge group $Sl(2,\mathbb{R})$ and level k

$$S_{cs} = \frac{k}{4\pi}\int A\wedge dA + \frac{2}{3}A\wedge A\wedge A. \tag{6.87}$$

The theory is defined on a disk times a time coordinate $\Sigma\times\mathbb{R}$ parametrized by (r,ϕ,t). The asymptotic charge in this case was derived from Hamiltonian constraint analysis to be

$$Q_\lambda = -\frac{k}{2\pi}\int d\phi\ tr(\lambda\, a_\phi). \tag{6.88}$$

Where $A = \bar{A} + a$ with $\bar{A}$ being flat $\bar{F} = d\bar{A} + \bar{A}\wedge\bar{A} = 0$ satisfies the Euler-Lagrange equations. The gauge transformations and the linearised equation of motion for a are described by

$$\begin{aligned} \delta_\lambda a = d\lambda + [\bar{A},\lambda] &= \bar{\nabla}\lambda, \\ \epsilon^{\sigma\mu\nu}\bar{\nabla}_\mu a_\nu &= 0, \end{aligned} \tag{6.89}$$

respectively. Therefore, the analogue of $R^i_\alpha(f^\alpha)\frac{\delta\mathcal{L}}{\delta\phi^i}$ is

$$R^i_\alpha(f^\alpha)\frac{\delta\mathcal{L}}{\delta\phi^i} \sim (\bar{\nabla}_\sigma\lambda_a)\epsilon^{\sigma\mu\nu}\bar{\nabla}_\mu a^a_\nu. \tag{6.90}$$

One can show the existence of Noether's identities by shifting the gauge covariant derivative from the gauge transformation parameter λ to the rest of the expression:

$$
\begin{aligned}
(\bar{\nabla}_\sigma \lambda_a)\epsilon^{\sigma\mu\nu}\bar{\nabla}_\mu a^a_\nu &= \lambda_a \epsilon^{\sigma\mu\nu}\bar{\nabla}_\sigma \bar{\nabla}_\mu a^a_\nu + \bar{\nabla}_\sigma(\lambda_a \epsilon^{\sigma\mu\nu}\bar{\nabla}_\mu a^a_\nu), \\
&= 2\lambda_a \epsilon^{\sigma\mu\nu}[\bar{F}_{\sigma\mu}, a^a_\nu] + \bar{\nabla}_\sigma(\lambda_a \epsilon^{\sigma\mu\nu}\bar{\nabla}_\mu a^a_\nu). \qquad (6.91)
\end{aligned}
$$

The first term on the *r.h.s.* vanishes off-shell for arbitrary a_ν, therefore $\epsilon^{\sigma\mu\nu}[F_{\sigma\mu}, a^a_\nu] = 0$ is the Noether's identity. The remaining term is interpreted as the divergence of

$$
\tilde{s}^\mu_\lambda = -\frac{k}{2\pi}\epsilon^{\mu\rho\sigma}\lambda_a \bar{\nabla}_\mu a^a_\nu \qquad (6.92)
$$

which is the required $n-1$ form. Using (6.86) one gets

$$
\begin{aligned}
\tilde{k}^{[\mu\nu]}_\lambda &= -\frac{k}{2\pi}\epsilon^{\mu\nu\sigma}\lambda_a a^a_\sigma, \\
&, \\
Q_\lambda &= -\frac{k}{2\pi}\int d\phi \; tr(\lambda\, a_\phi) \qquad (6.93)
\end{aligned}
$$

as the super-potential and the conserved charge respectively.

Einstein-Hilbert gravity

We would now like to apply this technique to the case of classical gravity as defined by the Einstein-Hilbert action in D space-time dimensions.

$$
\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}(R - 2\Lambda) + \mathcal{L}_{matter}. \qquad (6.94)
$$

Here the field is the metric $g_{\mu\nu}$ and the gauge transformations on it are the diffeomorphisms generated infinitesimally by vector fields via the Lie derivative *i.e.* $\delta g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$. We will work with configurations where the matter fields do not survive till the boundary. In this case only the asymptotic dynamics of the theory is not governed by $\mathcal{L}_{matter}$.

The equation of motion linearised about a solution $\bar{g}_{\mu\nu}$ for $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ reads as:

$$\mathcal{H}^{\mu\nu}[h, \bar{g}] = \frac{\sqrt{-g}}{32\pi}\Bigg(\frac{2\Lambda}{D-2}(2h^{\mu\nu`} - \bar{g}^{\mu\nu}h) + \bar{D}^{\mu}\bar{D}^{\nu}h + \bar{D}^{\lambda}\bar{D}_{\lambda}h^{\mu\nu} \\ -2\bar{D}_{\lambda}\bar{D}^{(\mu}h^{\nu)\lambda} - \bar{g}^{\mu\nu}(\bar{D}^{\lambda}\bar{D}_{\lambda}h - \bar{D}_{\lambda}\bar{D}\rho h^{\rho\lambda})\Bigg) \quad (6.95)$$

where $h = h_{\mu\nu}\bar{g}^{\mu\nu}$ and $h^{\mu\nu} = \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}h_{\alpha\beta}$[7]. The gauge transformations are given by $\delta_{\xi}h_{\mu\nu} = \bar{g}_{\rho(\nu}\bar{\nabla}_{\mu)}\xi^{\rho}$. Therefore

$$R^{i}_{\alpha}(f^{\alpha})\frac{\delta\mathcal{L}}{\delta\phi^{i}} \longleftrightarrow \bar{\nabla}_{(\mu}\xi_{\nu)}\mathcal{H}^{\mu\nu} \quad (6.96)$$

Thus the Noether identities which are obeyed for arbitrary fluctuations over the background $h_{\mu\nu}$ are given by $\bar{\nabla}_{\mu}\mathcal{H}^{\mu\nu} = 0$, which can be easily verified by using equations of motion $\bar{R}_{\mu\nu} = 2\Lambda(D-2)^{-1}\bar{g}_{\mu\nu}$ satisfied by $\bar{g}_{\mu\nu}$ and general properties of the Riemann curvature tensor.

The asymptotic reducibility parameters are vector fields such that

$$\forall h_{\mu\nu} \to O(\chi_{\mu\nu}): \quad d^{D}x\bar{\nabla}_{\mu}\xi_{\nu}\mathcal{H}^{\mu\nu} \longrightarrow 0. \quad (6.97)$$

Applying the general formula (6.86) to the $D-1$ form conserved current $\xi_{\nu}\mathcal{H}^{\mu\nu}$ we get

$$\tilde{k}^{[\nu\mu]}_{\xi}[h, \bar{g}] = -\frac{\sqrt{-\bar{g}}}{16\pi}[\bar{\nabla}^{\nu}(h\xi^{\mu}) + \bar{\nabla}_{\sigma}(h^{\mu\sigma}\xi^{\nu}) + \bar{\nabla}^{\mu}(h^{\nu\sigma}\xi_{\sigma}) \\ +\tfrac{3}{2}h\bar{\nabla}^{\mu}\xi^{\nu} + \tfrac{3}{2}h^{\sigma\mu}\bar{\nabla}^{\nu}\xi_{\sigma} + \tfrac{3}{2}h^{\nu\sigma}\bar{\nabla}_{\sigma}\xi^{\mu} - (\mu \leftrightarrow \nu)] \quad (6.98)$$

as the asymptotically conserved $D-2$ form. The asymptotic conserved charge is accordingly given by the integral of the above current over the $D-2$ space-like surface at the asymptote.

The above charges yield the expected charges (mass, angular momenta *etc.*) for the global

[7]Here $\bar{()}$ indicates quantities computed *w.r.t.* the background metric $\bar{g}_{\mu\nu}$.

Killing vectors of the space-time *w.r.t.* a background metric (typically chosen to be the maximally symmetric solution.).

$$\not\delta Q_\xi = \int_{\partial\Sigma} d^{D-2}x \; \tilde{k}_\xi[h = \delta g, \bar{g}] \tag{6.99}$$

$h_{\mu\nu}$ consists of infinitesimal parameters $h_{\mu\nu}[\delta f_1, \delta f_2 \ldots] = \delta g_{\mu\nu}$ that takes one away from the background metric $\bar{g}_{\mu\nu}$. For the boundary conditions implied on $h_{\mu\nu}$ to be consistent the expression for the infinitesimal variation of the asymptotic charge, $\not\delta Q_\xi$ must be integrable, *i.e.* $\not\delta Q_\xi$ must have an expression as $\delta(Q_\xi)$.

6.3 AdS_3 gravity in first order formulation

The AdS_3 gravity in the Hilbert-Palatini formulation can be recast as a gauge theory with action

$$S[A,\tilde{A}] = \frac{k}{4\pi}\int \mathrm{tr}(A\wedge A + \frac{2}{3}A\wedge A\wedge A) - \frac{k}{4\pi}\int \mathrm{tr}(\tilde{A}\wedge\tilde{A} + \frac{2}{3}\tilde{A}\wedge\tilde{A}\wedge\tilde{A}) \quad (6.100)$$

up to boundary terms, where the gauge group is $SL(2,\mathbb{R})$. These are related to vielbein and spin connection through $A = \omega^a + \frac{1}{l}e^a$ and $\tilde{A} = \omega^a - \frac{1}{l}e^a$. The equations of motion are $F = dA + A\wedge A = 0$ and $\tilde{F} := d\tilde{A} + \tilde{A}\wedge\tilde{A} = 0$. We work with the following defining representation of the $sl(2,\mathbb{R})$ algebra.

$$L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ , \quad (6.101)$$

Satisfying $[L_m, L_n] = (m-n)L_{m+n}$. The metric defined by $\mathrm{Tr}(T_a, T_b) = \frac{1}{2}h_{ab}$ is

$$h_{ab} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix} \quad (6.102)$$

It is known that the connections

$$\begin{aligned} A &= b^{-1}\partial_r b\, dr + b^{-1}(L_1 - \kappa(x^+)\, L_{-1})\, b\, dx^+ \\ \tilde{A} &= b\,\partial_r b^{-1}\, dr + b\,(\tilde{\kappa}(x^-)\, L_1 - L_{-1})\, b^{-1}\, dx^- \end{aligned} \quad (6.103)$$

represent all the solutions of AdS_3 gravity satisfying Brown-Henneaux (Dirichlet) boundary conditions (in FG coordinates) where $b = e^{L_0 \ln\frac{r}{l}}$. In fact, any solution of the Chern-Simons theory (locally) can be written as

$$A = b^{-1}\partial_r b\, dr + b^{-1}\, a\, b, \quad \tilde{A} = b\,\partial_r b^{-1}\, dr + b\,\tilde{a}\, b^{-1} \quad (6.104)$$

where a and $\tilde{a}$ are flat connections in two dimensions with coordinates (x^+, x^-). The general solution can be written as $a = g^{-1}\, dg$ and $\tilde{a} = \tilde{g}\, d\tilde{g}^{-1}$ where g and $\tilde{g}$ are $SL(2,\mathbb{R})$ group elements that depend on (x^+, x^-). We now present general solution to this flatness condition in a different parametrization that will be useful to us. Consider the most general $sl(2,\mathbb{R})$ 1-form on the boundary

$$a = (a_+^{(+)}\, L_1 + a_+^{(-)}\, L_{-1} + a_+^{(0)}\, L_0)\, dx^+ + (a_-^{(+)}\, L_1 + a_-^{(-)}\, L_{-1} + a_-^{(0)}\, L_0)\, dx^- \tag{6.105}$$

Assuming that $a_+^{(+)}$ does not vanish, the flatness conditions imply:

$$\begin{aligned} a_-^{(0)} &= \frac{1}{a_+^{(+)}}\left(a_-^{(+)}\, a_+^{(0)} + \partial_- a_+^{(+)} - \partial_+ a_-^{(+)}\right), \quad a_-^{(-)} = \frac{1}{2a_+^{(+)}}\left(2\, a_-^{(+)}\, a_+^{(-)} + \partial_- a_+^{(0)} - \partial_+ a_-^{(0)}\right) \\ &\frac{1}{2}\partial_+^3 f = \partial_- \kappa - 2\,\kappa\, \partial_+ f - f\, \partial_+ \kappa \end{aligned} \tag{6.106}$$

where $\kappa = a_+^{(+)}\, a_+^{(-)} - \frac{1}{4}(a_+^{(0)})^2 - \frac{1}{2}\partial_+ a_+^{(0)} + \frac{1}{2}a_+^{(0)}\, \partial_+ \ln a_+^{(+)} + \frac{1}{2}\partial_+^2 \ln a_+^{(+)} - \frac{1}{4}(\partial_+ \ln a_+^{(+)})^2$ and $f = \frac{a_-^{(+)}}{a_+^{(+)}}$. Similarly if we consider the 1-form

$$\tilde{a} = (\tilde{a}_+^{(+)}\, L_1 + \tilde{a}_+^{(-)}\, L_{-1} + \tilde{a}_+^{(0)}\, L_0)\, dx^+ + (\tilde{a}_-^{(+)}\, L_1 + \tilde{a}_-^{(-)}\, L_{-1} + \tilde{a}_-^{(0)}\, L_0)\, dx^- \tag{6.107}$$

Then, assuming now that $\tilde{a}_-^{(-)}$ does not vanish, the flatness conditions read

$$\begin{aligned} \tilde{a}_+^{(+)} &= \frac{1}{2\,\tilde{a}_-^{(-)}}(2\,\tilde{a}_-^{(+)}\,\tilde{a}_+^{(-)} + \partial_- \tilde{a}_+^{(0)} - \partial_+ \tilde{a}_-^{(0)}), \quad \tilde{a}_+^{(0)} = \frac{1}{\tilde{a}_-^{(-)}}\,(\tilde{a}_-^{(0)}\,\tilde{a}_+^{(-)} + \partial_- \tilde{a}_+^{(-)} - \partial_+ \tilde{a}_-^{(-)}) \\ &\frac{1}{2}\partial_-^3 \tilde{f} = \partial_+ \tilde{\kappa} - 2\,\tilde{\kappa}\,\partial_- \tilde{f} - \tilde{f}\,\partial_- \tilde{\kappa} \end{aligned} \tag{6.108}$$

where $\tilde{f} = \frac{\tilde{a}_+^{(-)}}{\tilde{a}_-^{(-)}}$ and $\tilde{\kappa} = \tilde{a}_-^{(-)}\,\tilde{a}_-^{(+)} - \frac{1}{4}(\tilde{a}_-^{(0)})^2 + \frac{1}{2}\partial_- \tilde{a}_-^{(0)} - \frac{1}{2}\tilde{a}_-^{(0)}\,\partial_- \ln \tilde{a}_-^{(-)} + \frac{1}{2}\partial_-^2 \ln \tilde{a}_-^{(-)} - \frac{1}{4}(\partial_- \ln \tilde{a}_-^{(-)})^2$. The last equation is again the famous Virasoro Ward identity that can be solved explicitly as in section 2. Some special cases of the above formulae have appeared before, for instance, in [61].

6.4 Generalization to extended AdS_3 super-gravity

We give the detail analysis of generalizing the chiral induced boundary conditions introduced in chapter 2 to extended super-gravity in AdS_3. Here, the left moving gauge field Γ obeys the boundary conditions of the Dirichlet type studied in [2] and we repeat their analysis as it is for the left sector while imposing generalization of chiral boundary condition on the right moving gauge field $\tilde{\Gamma}$. The conventions are taken as it is from [2].

6.4.1 Conventions

We follow the conventions of [2]. The structure constants for the $\tilde{G}$ are f_{abc} which are completely anti-symmetric. The representation ρ has the basis $(\lambda^a)^\alpha{}_\beta$ where a counts the dimension of $\tilde{G}$ *i.e.* D. Therefore, $[\lambda^a, \lambda^b] = f^{ab}{}_c\lambda^c$. the Killing metric on $\tilde{G}$ is denoted by $g^{ab} = -f^{acd}f^{bcd} = -C_v\delta^{ab}$, where C_v is the eigenvalue of the second Casimir in the adjoint representation of $\tilde{G}$. Similarly $tr(\lambda^a\lambda^b) = -\frac{d}{D}C_\rho\delta^{ab}$, where C_ρ is the eigenvalue of the second Casimir in the representation ρ. We denote by $\eta^{\alpha\beta}$ the $\tilde{G}-$invariant symmetric metric on the representation ρ which is orthogonal. Its inverse is $\eta_{\alpha\beta}$, this is used to raise and lower the supersymmetric (Greek) indices.

The list of all possible super-gravities in AdS_3 is given in section (3.2); we consider any such generic extended sugra in AdS_3. Below we list all the super-algebra generators:

- The $sl(2,\mathbb{R})$ generators are denoted as before by $(\sigma^0, \sigma^\pm)$

$$\begin{aligned}\sigma^0 &= \tfrac{1}{2}\sigma^3,\\ \left[\sigma^0, \sigma^\pm\right] &= \pm\sigma^\pm,\\ [\sigma^+, \sigma^-] &= 2\sigma^0.\end{aligned} \tag{6.109}$$

The Killing form on $sl(2,\mathbb{R})$ is:

$$Tr(\sigma^a\sigma^b) = h^{ab} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \tag{6.110}$$

- The generators of $\tilde{G}$ which commute with σ^as are[8]:

$$\begin{aligned}
\left[T^a, T^b\right] &= f^{ab}_{\ \ c}\, T^c, \quad \text{where } a\epsilon\{1, D\}, \\
\left[T^a, \sigma^b\right] &= 0, \quad \text{where } b\epsilon\{+,-,0\}, \\
STr\left(T^a T^b\right) &= \tfrac{2C_\rho}{d-1}\delta^{ab}.
\end{aligned} \tag{6.111}$$

- The fermionic generators are denoted by $R^{\pm\alpha}$, where $\pm$ denotes the spinor indices with respect to the $sl(2,\mathbb{R})$ and α (Greek indices) denotes the vector index in the representation ρ of $\tilde{G}$.

$$\begin{aligned}
\left[\sigma^0, R^{\pm\alpha}\right] &= \pm\tfrac{1}{2}R^{\pm\alpha}, \quad \text{where } \alpha\epsilon\{1, d\}, \\
[\sigma^\pm, R^{\pm\alpha}] &= 0, \\
[\sigma^\pm, R^{\mp\alpha}] &= R^{\pm\alpha}, \\
[T^a, R^{\pm\alpha}] &= -(\lambda^a)^\alpha_{\ \beta} R^{\pm\beta}, \\
\left\{R^{\pm\alpha}, R^{\pm\beta}\right\} &= \pm\eta^{\alpha\beta}\sigma^\pm, \\
\left\{R^{\pm\alpha}, R^{\mp\beta}\right\} &= -\eta^{\alpha\beta}\sigma^0 \pm \tfrac{d-1}{2C_\rho}(\lambda^a)^{\alpha\beta}T^a,
\end{aligned}$$

$$STr\left(R^{-\alpha}R^{+\beta}\right) = -STr\left(R^{+\alpha}R^{-\beta}\right) = \eta^{\alpha\beta}. \tag{6.112}$$

Since the underlining algebra is now promoted to a graded Lie algebra, its generators satisfy the generalized Jacobi identity. The three fermion Jacobi identity thus

[8]The indices on σ always run over $(0,+,-)$ while those on T run from $(1...D)$, this is to be understood from the context.

yields an identity for the matrices in the representation ρ of the internal algebra $\tilde{G}$:

$$(\lambda^a)^{\alpha\beta}(\lambda^a)^{\gamma\delta} + (\lambda^a)^{\gamma\beta}(\lambda^a)^{\alpha\delta} = \frac{C_\rho}{d-1}(2\eta^{\alpha\gamma}\eta^{\beta\delta} - \eta^{\alpha\beta}\eta^{\gamma\delta} - \eta^{\gamma\beta}\eta^{\alpha\delta}) \tag{6.113}$$

The super-traces defined above are consistent, invariant and non-degenerate with respect to the super-algebra defined above and would be used in defining the action and the charges.

6.4.2 Action

The super Chern-Simons action is defined as:

$$S_{CS}[\Gamma] = \frac{k}{2\pi}\int_{\mathcal{M}} Str[\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma]. \tag{6.114}$$

The above integration is over a three manifold $\mathcal{M} = D \times \mathbb{R}$, where D has a topology of a disk. The level k of the Chern-Simons action is related to the Newton's constant G in three dimension and the AdS length ℓ through $k = \ell/(4G)$. The product of two fermions differs by a factor of i from the standard Grasmann product ($(ab)^* = b^*a^*$).This basically requires one the multiply a factor of $-i$ where ever $\eta^{\alpha\beta}$ occurs, and where ever $\frac{d-1}{2C\rho}(\lambda^a)^{\alpha\beta}$ occurs while evaluating anti-commutator between fermionic generators in the calculations below[9].

In the Chern-Simons formulation of (super-)gravity, the metric (and other fields) which occur in Einstein-Hilbert (Hilbert-Palatini) action are a derived concept. The equations of motion for the Chern-Simons action can for example be satisfied by gauge field configu-

[9]This basically so because the product of two real Grasmann fields is imaginary. This is equivalent to using

$$\begin{aligned}\left\{R^{\pm\alpha}, R^{\pm\beta}\right\} &= \mp i\eta^{\alpha\beta}\sigma^{\pm},\\ \left\{R^{\pm\alpha}, R^{\mp\beta}\right\} &= i\eta^{\alpha\beta}\sigma^0 \mp i\tfrac{d-1}{2C_\rho}(\lambda^a)^{\alpha\beta}T^a,\\ STr\left(R^{-\alpha}R^{+\beta}\right) &= -STr\left(R^{+\alpha}R^{-\beta}\right) = -i\eta^{\alpha\beta},\end{aligned} \tag{6.115}$$

instead of the one stated in the commutation relations of the extended super-algebra.

rations which may yield a non-singular metric. There fore one has to make sure that such configurations are not considered in the analysis.

The super-gravity action for the above super-algebra can be written in full detail yielding the action in the Hilbert-Palatini form:

$$\begin{aligned} S[\Gamma,\tilde{\Gamma}] &= \tfrac{1}{8\pi G}\int_{\mathcal{M}} d^3x\{\tfrac{1}{2}eR + \frac{e}{\ell^2} + \\ &\quad -\frac{i\ell}{2}\varepsilon^{ijk}(\psi_i)\mathcal{D}^{\mu\nu}_j(\psi_k)_\nu + \frac{i\ell}{2}\varepsilon^{ijk}(\tilde{\psi}_i)\tilde{\mathcal{D}}^{\mu\nu}_j(\tilde{\psi}_k)_\nu\} \\ &\quad +\frac{C_\rho}{d-1}\ell\varepsilon^{ijk}(B^a_i\partial_j B^a_k + \tfrac{1}{3}f_{abc}B^a_iB^b_jB^c_k) \\ &\quad -\frac{C_\rho}{d-1}\ell\varepsilon^{ijk}(\tilde{B}^a_i\partial_j\tilde{B}^a_k + \tfrac{1}{3}f_{abc}\tilde{B}^a_i\tilde{B}^b_j\tilde{B}^c_k) \\ &\quad -\frac{i}{2}\varepsilon^{ijk}\eta^{\alpha\beta}e^a_i([\bar{\psi}_j]_\alpha t^a[\psi_k]_\beta - [\tilde{\bar{\psi}}_j]_\alpha t^a[\tilde{\psi}_k]_\beta)\} \end{aligned} \tag{6.116}$$

The square brackets denote the two-component $sl(2,\mathbb{R})$ spinor representations. The spin covariant operators $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are:

$$\mathcal{D}^{\mu\nu}_j = \begin{pmatrix} 2\,(\,\eta^{\alpha\beta}\partial_j + (\lambda^a)^{\alpha\beta}B^a_j\,)\;\delta^\mu_{+\alpha}\delta^\nu_{-\beta} + \\ -\eta^{\alpha\beta}(\,\tfrac{1}{2}\omega^3_j[\delta^\mu_{+\alpha}\delta^\nu_{-\beta} + \delta^\mu_{-\alpha}\delta^\nu_{+\beta}] + \omega^+_j\delta^\mu_{-\alpha}\delta^\nu_{-\beta} - \omega^-_j\delta^\mu_{+\alpha}\delta^\nu_{+\beta}\,) \end{pmatrix},$$

$$\tilde{\mathcal{D}}^{\mu\nu}_j = \begin{pmatrix} 2\,(\,\eta^{\alpha\beta}\partial_j + (\lambda^a)^{\alpha\beta}\tilde{B}^a_j\,)\;\delta^\mu_{+\alpha}\delta^\nu_{-\beta} + \\ -\eta^{\alpha\beta}(\,\tfrac{1}{2}\omega^3_j[\delta^\mu_{+\alpha}\delta^\nu_{-\beta} + \delta^\mu_{-\alpha}\delta^\nu_{+\beta}] + \omega^+_j\delta^\mu_{-\alpha}\delta^\nu_{-\beta} - \omega^-_j\delta^\mu_{+\alpha}\delta^\nu_{+\beta}\,) \end{pmatrix} \tag{6.117}$$

From the form of the above action it is quite evident that the analysis done in the Hilbert-Palatini formulation of super-gravity would be quite cumbersome if not difficult. Further, it was found that computation of the asymptotic charge associated with gauge transformations which vary the super-gauge field at the AdS asymptote[10] via the prescription of Barnich *et al* [50] for the above form of the action is too difficult. The same prescription of computing asymptotic charges in the Chern-Simons formalism yields a know expression for asymptotic charge in Chern-Simons theory. Therefore we proceed as before with the analysis in the Chern-Simons prescription.

[10] By this we mean the boundary of the disk D

6.4.3 Boundary conditions

The fall-off conditions in terms of the gauge fields are:

$$\begin{aligned}
\Gamma &= bdb^{-1} + bab^{-1}, \\
\tilde{\Gamma} &= b^{-1}db + b^{-1}\tilde{a}b, \\
\text{where } b &= e^{\sigma^0 \ln(r/\ell)}, \\
a &= [\sigma^- + L\sigma^+ + \psi_{+\alpha+}R^{+\alpha} + B_{a+}T^a]\, dx^+, \\
\tilde{a} &= \left[\sigma^+ + \bar{L}\sigma^- + \bar{\psi}_{-\alpha-}R^{-\alpha} + \bar{B}_{a-}T^a\right] dx^- \\
&\quad + \left[\tilde{A}_{a+}\sigma^a + \tilde{B}_{a+}T^a + \tilde{\psi}_{+\alpha+}R^{+\alpha} + \tilde{\psi}_{-\alpha+}R^{-\alpha}\right] dx^+. \qquad (6.118)
\end{aligned}$$

Here the dx^- component of the gauge field $\tilde{a}$ one form is that of a super-gauge field corresponding to Dirichlet boundary condition as given in [2]. All functions above are *a priori* functions of both the boundary coordinates. The equation of motion- as mentioned earlier, is implied by the flatness condition imposed on the two gauge fields. For the right gauge field this implies that the functions are independent of the x^- co-ordinate. *i.e.* $\partial_- a = 0$.

$$\partial_- L = \partial_- \psi_{+\alpha+} = \partial_- B_{a+} = 0 \qquad (6.119)$$

For the left gauge field we would like to use the equations of motion to solve for the $\tilde{a}_+$ components. This gives the $\tilde{a}_+$ components in terms of $\tilde{A}_{++}, \tilde{B}_{a+}$, $\tilde{\psi}_{+\alpha+}$ and the $\tilde{a}_-$ components:

$$\begin{aligned}
\tilde{A}_{0+} &= \partial_- \tilde{A}_{++}, \\
\tilde{A}_{-+} &= \tilde{A}_{++}\bar{L} - \tfrac{1}{2}\partial_-^2 \tilde{A}_{++} + i\tfrac{\eta^{\alpha\beta}}{2}\tilde{\psi}_{+\alpha+}\bar{\psi}_{-\beta-}, \\
\tilde{\psi}_{-\alpha+} &= \tilde{A}_{++}\bar{\psi}_{-\alpha-} - \partial_- \tilde{\psi}_{+\alpha+} + (\lambda^a)^\beta_{\ \alpha}\bar{B}_a\tilde{\psi}_{+\beta+}. \qquad (6.120)
\end{aligned}$$

Provided they satisfy the following set of differential equations:

$$\partial_+ \bar{L} + \tfrac{1}{2}\partial_-^3 \tilde{A}_{++} - 2\bar{L}\partial_- \tilde{A}_{++} - \tilde{A}_{++}\partial_- \bar{L}$$

$$
\begin{aligned}
+i\eta^{\alpha\beta}\bar{\psi}_{-\beta-}\left(\tilde{A}_{++}\bar{\psi}_{-\alpha-}+(\lambda^a)^\beta_{\ \alpha}\bar{B}_{a-}\tilde{\psi}_{+\beta+}+\partial_-\tilde{\psi}_{+\alpha+}\right)+i\eta^{\alpha\beta}\partial_-(\bar{\psi}_{-\beta-}\tilde{\psi}_{+\alpha+}) &= 0,\\
\partial_+\bar{B}_{a-}-\partial_-\tilde{B}_{a+}+f^{bc}_{\ \ a}\tilde{B}_{b+}\bar{B}_{c-}+i\tfrac{d-1}{2C_\rho}(\lambda^a)^{\alpha\beta}\tilde{\psi}_{+\alpha+}\bar{\psi}_{-\beta-} &= 0,\\
\partial_+\bar{\psi}_{-\alpha-}-\partial_-[\tilde{A}_{++}\bar{\psi}_{-\alpha-}-\partial_-\tilde{\psi}_{+\alpha+}+(\lambda^a)^\beta_{\ \alpha}\bar{B}_{a-}\tilde{\psi}_{+\beta+}]-\tfrac{1}{2}\partial_-\tilde{A}_{++}\bar{\psi}_{-\alpha-} &\\
+(\lambda^a)^\beta_{\ \alpha}\bar{B}_{a-}[\tilde{A}_{++}\bar{\psi}_{-\beta-}-\partial_-\tilde{\psi}_{+\beta+}+(\lambda^a)^\gamma_{\ \beta}\bar{B}_{b-}\tilde{\psi}_{+\gamma+}]-(\lambda^a)^\beta_\alpha\tilde{B}_{a+}\bar{\psi}_{-\beta-}-\bar{L}\tilde{\psi}_{+\alpha+} &= 0.
\end{aligned}
\tag{6.121}
$$

These are the Ward identities expected to be satisfied by the induced gravity theory on the boundary. We will later choose the $\bar{()}$ functions such that global AdS_3 is a part of the moduli space of bulk solutions.*i.e.* $\bar{L}=\frac{-1}{4}$ and $\bar{B}=0=\bar{\psi}$.

In the following analysis we will consider the sources *i.e.* the bared functions as constants along the boundary directions. There is no need to assume this, and we have done so only for simplicity in the expressions for change in the moduli space parameters. Either ways, demanding that the bared functions- $\bar{L},\bar{B},\bar{\psi}$, be treated as sources which determine aspects of the theory requires adding of specific boundary term to the bulk action. As explained previously, this is done so that the required set of bulk solutions obey the variational principle.

The boundary term to be added is given by:

$$
\begin{aligned}
S_{bndy} &= \frac{k}{8\pi}\int_{\partial M} d^2x\ STr(-\sigma^0[\tilde{a}_+,\tilde{a}_-])-2\bar{L}_0\sigma^-\tilde{a}_++(\tfrac{d-1}{2C_\rho})^2T^aT^bSTr(\tilde{a}_+T_a)STr(\tilde{a}_-T_b)\\
&\qquad -2(\tfrac{d-1}{2C_\rho})\bar{B}_{0a}T^aT^bSTr(\tilde{a}_+T^b)-\tfrac{1}{2}(\bar{\psi}_0)_{-\alpha}R^{-\alpha}\tilde{a}_+).
\end{aligned}
\tag{6.122}
$$

This implies the following desired variation of the total action:

$$
\delta S_{total} = \frac{k}{8\pi}\int_{\mathcal{M}} d^2x\ 2(\bar{L}-\bar{L}_0)\delta\tilde{A}_{++}+2(\tfrac{2C_\rho}{d-1})(\bar{B}_{a-}-\bar{B}_{0a})\delta\tilde{B}_{a+}+\tfrac{i}{2}(\bar{\psi}_{-\alpha-}-(\bar{\psi}_0)_{-\alpha})\delta\tilde{\psi}_{+\alpha+}\eta^{\alpha\beta}
\tag{6.123}
$$

In our present case, we would be choosing the later by fixing $\bar{L}=-1/4$, $\bar{B}_{0a}=0=(\bar{\psi}_0)_{-\alpha}$. Thus the variational principle is satisfied for configurations with $\bar{L}=\frac{-1}{4}$ and $\bar{B}_{a-}=0=$

$\bar{\psi}_{-\alpha-}$ which describes global AdS_3.

6.4.4 Charges and symmetries

Just as in the previous chapters, one needs to find the space of gauge transformations that maintains the above form of the gauge fields, thus inducing transformations on the functions $\tilde{A}_{a+}, \tilde{B}_{a+}, \tilde{\psi}_{+\alpha+}, L, B_a, \psi_{+\alpha+}$ which parametrize the space of solutions. Once this is achieved, one can define asymptotic conserved charge associated with the change induced by such residual gauge transformations on the space of solutions. For the boundary conditions to be well defined, this asymptotic charge must be finite and be integrable on the space of solutions.

Left sector

The analysis of the left sector *i.e.* on the gauge field Γ is exactly the one done in [2]. We first find the space of residual gauge transformations that maintain the form of Γ. The residual gauge transformations acting on a are parametrized as $\Lambda = \zeta_a\sigma^a + \omega_a T^a + \varepsilon_{\pm\alpha}R^{\pm\alpha}$ where there is no explicit radial dependence. The radial dependence can be introduced just as for the gauge fields $b^{-1}\Lambda b$. The variation of a under such gauge transformation is:

$$\begin{aligned}
\delta a_- &= 0, \\
\Longrightarrow \partial_-\Lambda &= 0, \\
\delta a_+ &= \partial_+\Lambda + [a_+, \Lambda], \\
&= \left(\partial_+\zeta_0 + 2L\zeta_- - 2\zeta_+ - i\eta^{\alpha\beta}\psi_{+\alpha}\varepsilon_{-\beta}\right)\sigma^0 \\
&+ \left(\partial_+\zeta_+ - L\zeta_0 + i\eta^{\alpha\beta}\psi_{+\alpha}\varepsilon_{+\beta}\right)\sigma^+ \\
&+ \left(\partial_+\zeta_- + \zeta_0\right)\sigma^- \\
&+ \left(\partial_+\omega_c + f^{abc}B_a\omega_b + i\tfrac{d-1}{2C_\rho}(\lambda^c)^{\alpha\beta}\psi_{+\alpha}\varepsilon_{-\beta}\right)T^c \\
&+ \left(\partial_+\varepsilon_{+\beta} + L\varepsilon_{-\beta} - (\lambda^a)^\alpha_{\ \beta}\varepsilon_{+\alpha}B_a - \zeta_0\psi_{+\beta} + (\lambda^a)^\alpha_{\ \beta}\omega_a\psi_{+\alpha}\right)R^{+\beta} \\
&+ \left(\partial_+\varepsilon_{-\beta} + \varepsilon_{+\beta} - (\lambda^a)^\alpha_{\ \beta}\varepsilon_{-\alpha}B_a - \zeta_-\psi_{+\beta}\right)R^{-\beta} \qquad (6.124)
\end{aligned}$$

The above change in a under gauge transformation must preserve the form of a. This allows for three independent gauge transformation parameters in terms of which the rest of the gauge transformation parameters are determined.

$$\begin{aligned}
\zeta_0 &= -\partial_+\zeta_-, \\
\zeta_+ &= -\tfrac{1}{2}\partial_+^2\zeta_- + \zeta_- L - i\eta^{\alpha\beta}\psi_{+\alpha}\varepsilon_{-\beta}, \\
\varepsilon_{+\alpha} &= -\partial_+\varepsilon_{-\alpha} + \zeta_-\psi_{+\alpha} + (\lambda^a)^\beta_{\ \alpha}\varepsilon_{-\beta}B_a.
\end{aligned} \tag{6.125}$$

Here, all the residual gauge parameters are independent of x^-, just like the solution space parameters for the gauge field Γ. The gauge transformation parameter Λ with the above substitution parametrizes the space of residual gauge transformations for Γ. The corresponding changes in the solution space parameters are:

$$\begin{aligned}
\delta L &= -\frac{1}{2}\zeta_-''' + [(\zeta_- L)' + \zeta_-' L] - i\eta^{\alpha\beta}\left[\tfrac{1}{2}(\psi_{+\alpha}\varepsilon_{-\beta})' + \psi_{+\alpha}\varepsilon_{-\beta}'\right] \\
&\quad -(\lambda^a)^{\alpha\beta}\psi_{+\alpha}B_a\varepsilon_{-\beta}, \\
\delta B_a &= \omega_a' + f_a^{\ bc}B_b\omega_c + i\tfrac{d-1}{C_\rho}(\lambda^a)^{\alpha\beta}\psi_{+\alpha}\varepsilon_{-\beta}, \\
\delta\psi_{+\alpha} &= -\varepsilon_{-\alpha}'' + \left[(\zeta_-\psi_{+\alpha})' + \tfrac{1}{2}\zeta_-'\psi_{+\alpha}\right] + L\varepsilon_{-\alpha} + (\lambda^a)^\beta_{\ \alpha}\left[(\varepsilon_{-\beta}B_a)' + \varepsilon_{-\beta}'B_a\right] + \\
&+ (\lambda^a)^\beta_{\ \alpha}\omega_a\psi_{+\beta} - \zeta_-(\lambda^a)^\beta_{\ \alpha}\psi_{+\beta}B_a - \tfrac{1}{2}\left\{\lambda^a, \lambda^b\right\}^\gamma_{\ \alpha}\varepsilon_{-\gamma}B_aB_b.
\end{aligned} \tag{6.126}$$

The asymptotic charge associated with the full bulk geometry splits as difference for the ones corresponding to the left and the right sector, just like the action. The asymptotic charge associated with the left sector is given by:

$$\begin{aligned}
\not\delta Q &= \frac{k}{2\pi}\int d\phi STr[\Lambda\delta a_\phi], \\
\delta Q &= \frac{k}{2\pi}\int d\phi\left(\zeta_-\delta L + \tfrac{d-1}{2C_\rho}\omega_a\delta B^a + i\eta^{\alpha\beta}\varepsilon_{-\alpha}\delta\psi_{+\beta}\right).
\end{aligned} \tag{6.127}$$

Note that in the conventions adapted in this section there is no minus sign in front of the asymptotic charge. We redefine the currents as follows:

$$L \to \tfrac{k}{2\pi} L \ , \ \ B_a \to \tfrac{kC_\rho}{\pi(d-1)} B_a \ , \ \ \psi_{+\alpha} \to \tfrac{k}{2\pi}\psi_{+\alpha}. \tag{6.128}$$

Further, we add a Sugawara energy-momentum operator related to B_a to L.

$$L \to L + \tfrac{2\pi(d-1)}{4C_\rho} B_a B^a, \quad \psi_{+\alpha} \to \sqrt{2}\psi_{+\alpha} \tag{6.129}$$

After suitable redefinitions one gets the following Poisson algebra:

$$\begin{aligned}
\{L(x'^+), L(x^+)\} &= \frac{k}{4\pi}\delta'''(x'^+ - x^+) - \left(L(x'^+-) + L(x^+)\right)\delta'(x'^+ - x^+), \\
\{B_a(x'^+), B_b(x^+)\} &= -\frac{k}{2\pi}\tfrac{2C_\rho}{d-1}\delta^{ab}\delta'(x'^+ - x^+) + f_{ab}{}^{c}\delta(x'^+ - x^+)B_c(x'^+), \\
\{L(x'^+), B_a(x^+)\} &= -B_a(x'^+)\delta'(x'^+ - x^+), \\
i\left\{\psi_{+\alpha}(x'^+), \psi_{+\beta}(x^+)\right\} &= -\tfrac{k}{\pi}\eta_{\alpha\beta}\delta'''(x'^+ - x^+) - 2(\lambda^a)_{\alpha\beta}\tfrac{d-1}{2C_\rho}\delta'(x'^+ - x^+)\left[B_a(x'^+) + B_a(x^+)\right] + \\
&- 2\pi k(\tfrac{d-1}{2C_\rho})^2\left[\left\{\lambda^a, \lambda^b\right\}_{\alpha\beta} + \tfrac{2C_\rho}{d-1}\eta_{\alpha\beta}\delta^{ab}\right]B_a(x'^+)B_b(x^+)\delta(x'^+ - x^+) \\
&+ 2\eta_{\alpha\beta}L(x'^+)\delta(x'^+ - x^+), \\
\{L(x'^+), \psi_{+\alpha}(x^+)\} &= -\left[\psi_{+\alpha}(x'^+) + \tfrac{1}{2}\psi_{+\alpha}(x^+)\right]\delta'(x'^+ - x^+), \\
\{B_a(x'^+), \psi_{+\alpha}(x^+)\} &= (\lambda^a)^\beta{}_\alpha\psi_{+\beta}(x'^+)\delta(x'^+ - x^+).
\end{aligned} \tag{6.130}$$

The Fourier modes for the above algebra satisfy the following Dirac brackets:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{k}{2}m^3\delta_{m+n,0}, \\
\left[B^a_m, B^b_n\right] &= if^{abc}B^c_{m+n} + \tfrac{2kC_\rho}{d-1}m\delta^{ab}\delta_{m+n,0}, \\
[L_m, B^a_n] &= -nB^a_{m+n}, \\
\left\{(\psi_{+\alpha})_m, (\psi_{+\beta})_n\right\} &= 2\eta_{\alpha\beta}L_{m+n} - 2i\tfrac{d-1}{2C_\rho}(m-n)(\lambda^a)_{\alpha\beta}(B_a)_{m+n} \\
&+ 2k\eta_{\alpha\beta}m^2\delta_{m+n,0} \\
&- k(\tfrac{d-1}{2kC_\rho})^2\left[\left\{\lambda^a, \lambda^b\right\}_{\alpha\beta} + \tfrac{2C_\rho}{d-1}\eta_{\alpha\beta}\delta^{ab}\right](B_aB_b)_{m+n}, \\
[L_m, (\psi_{+\alpha})_n] &= (\tfrac{m}{2} - n)(\psi_{+\alpha})_{m+n},
\end{aligned}$$

$$[B^a_m, (\psi_{+\alpha})_n] \;=\; i(\lambda^a)^\beta{}_\alpha(\psi_{+\beta})_{m+n}. \tag{6.131}$$

This is the non-linear super-conformal algebra or the super-Virosoro algebra. The central extension is $k = c/6$, and is the same for all the seven cases listed in the table chapter 3. This algebra, although a supersymmetric extension of the Virasoro algebra, is not a graded Lie algebra in the sense that the right-hand sides of the fermionic (Rarita-Schwinger) anti-commutators contains quadratic non-linearities in currents for the internal symmetry directions.

Right sector

For the choice of $\bar{L} = -\frac{1}{4}, \bar{B} = 0 = \bar{\psi}$ the *eom* can be solved and the solutions can be parametrized as below:

$$\begin{aligned} \tilde{A}_{++} &= f(x^+) + g(x^+)e^{ix^-} + \bar{g}(x^+)e^{-ix^-}, \\ \tilde{B}_{a+} &\cong \tilde{B}_{a+}(x^+), \\ \tilde{\psi}_{+\alpha+} &= \chi_\alpha(x^+)e^{ix^-/2} + \bar{\chi}_\alpha(x^+)e^{-ix^-/2}. \end{aligned} \tag{6.132}$$

We would now seek the residual gauge tranformation parameters that would keep the above form of the gauge field $\tilde{\Gamma}$ form invariant. The residual gauge transformations are generated by $\tilde{\Lambda} = \xi_a\sigma^a + b_aT^a + \epsilon_{+\alpha}R^{+\alpha} + \epsilon_{-\alpha}R^{-\alpha}$ with the constraint that $\delta\tilde{a}_- = 0$:

$$\begin{aligned} \delta\tilde{a}_- &= d\tilde{\Lambda} + [\tilde{a}_-, \tilde{\Lambda}], \\ \Longrightarrow \xi_0 &= \partial_-\xi_+, \\ \xi_- &= -\tfrac{1}{4}(1 + 2\partial_-^2)\xi_+, \\ \epsilon_{-\alpha} &= -\partial_-\epsilon_{+\alpha}, \\ \partial_-(1 + \partial_-^2)\xi_+ &= 0, \\ \partial_-b_a = \quad 0 &= (\partial_-^2 + \tfrac{1}{4})\epsilon_{+\alpha}. \end{aligned} \tag{6.133}$$

One can solve for the residual gauge transformations:

$$\begin{aligned}
\xi_+ &= \lambda_f(x^+) + \lambda_g(x^+)e^{ix^-} + \bar{\lambda}_{\bar{g}}(x^+)e^{-ix^-}, \\
b_a &\cong b_a(x^+), \\
\epsilon_{+\alpha} &= \varepsilon_\alpha(x^+)e^{ix^-/2} + \bar{\varepsilon}_\alpha(x^+)e^{-ix^-/2}.
\end{aligned} \tag{6.134}$$

Here too, one finds that the functions parametrizing the space of solutions and residual gauge transformations are functions of x^+ alone. The x^+ dependence of the functions will be suppressed from here on for neatness. The variation of the above parameters under the residual gauge transformations are:

$$\begin{aligned}
\delta f &= \lambda'_f + 2i(g\bar{\lambda}_{\bar{g}} - \bar{g}\lambda_g) + i\eta^{\alpha\beta}(\chi_\alpha\bar{\varepsilon}_\beta + \bar{\chi}_\alpha\varepsilon_\beta), \\
\delta g &= \lambda'_g + i(g\lambda_f - \lambda_g f) + i\eta^{\alpha\beta}\chi_\alpha\varepsilon_\beta, \\
\delta\bar{g} &= \bar{\lambda}'_{\bar{g}} - i(\bar{g}\lambda_f - \bar{\lambda}_{\bar{g}} f) + i\eta^{\alpha\beta}\bar{\chi}_\alpha\bar{\varepsilon}_\beta, \\
\delta\tilde{B}_{a+} &= b'_a + f_a{}^{bc}\tilde{B}_{b+}b_c + \tfrac{d-1}{2C_\rho}(\lambda_a)^{\alpha\beta}(\bar{\chi}_\alpha\varepsilon_\beta - \chi_\alpha\bar{\varepsilon}_\beta), \\
\delta\chi_\alpha &= \varepsilon'_\alpha - (\lambda^a)^\beta{}_\alpha[\tilde{B}_{a+}\varepsilon_\beta - b_a\chi_\beta] + i[g\bar{\varepsilon}_\alpha - \tfrac{f}{2}\varepsilon_\alpha - \lambda_g\bar{\chi}_\alpha + \tfrac{\lambda_f}{2}\chi_\alpha], \\
\delta\bar{\chi}_\alpha &= \bar{\varepsilon}'_\alpha - (\lambda^a)^\beta{}_\alpha[\tilde{B}_{a+}\bar{\varepsilon}_\beta - b_a\bar{\chi}_\beta] - i[\bar{g}\varepsilon_\alpha - \tfrac{f}{2}\bar{\varepsilon}_\alpha - \bar{\lambda}_{\bar{g}}\chi_\alpha + \tfrac{\lambda_f}{2}\bar{\chi}_\alpha]
\end{aligned} \tag{6.135}$$

The charges corresponding to these transformation is given by:

$$\not\delta Q[\tilde{\Lambda}] = -\frac{k}{2\pi}\int d\phi\, S\,tr[\tilde{\Lambda}, \delta\tilde{a}_\phi]. \tag{6.136}$$

The above charge can be integrated to

$$Q[\Lambda] = -\tfrac{k}{2\pi}\int d\phi[-\tfrac{f}{2}\lambda_f + g\bar{\lambda}_{\bar{g}} + \bar{g}\lambda_g + \tfrac{2C_\rho}{d-1}\tilde{B}_{a+}b^a + \eta^{\alpha\beta}(\chi_\alpha\bar{\varepsilon}_\beta - \bar{\chi}_\alpha\varepsilon_\beta)]. \tag{6.137}$$

This charge is the generator of canonical transformations on the space of solutions parametrized by set of functions F *via* the Poisson bracket.

$$\delta_{\tilde{\Lambda}}F = \{Q[\tilde{\Lambda}], F\} \tag{6.138}$$

Therefore the Poisson bracket algebra is:

$$\{f(x^{+\prime}), f(x^+)\} = -2\alpha_Q\delta'(x^{+\prime} - x^+), \qquad \{\chi_\alpha(x^{+\prime}), f(x^+)\} = -i\alpha_Q\delta(x^{+\prime} - x^+)\chi_\alpha,$$
$$\{g(x^{+\prime}), f(x^+)\} = -2i\alpha_Q g(x^+)\delta(x^{+\prime} - x^+), \qquad \{\bar\chi_\alpha(x^{+\prime}), f(x^+)\} = i\alpha_Q\delta(x^{+\prime} - x^+)\bar\chi_\alpha,$$
$$\{\bar g(x^{+\prime}), f(x^+)\} = 2i\alpha_Q\bar g(x^+)\delta(x^{+\prime} - x^+), \qquad \{\bar\chi_\alpha(x^{+\prime}), g(x^+)\} = i\alpha_Q\delta(x^{+\prime} - x^+)\chi_\alpha,$$
$$\{\bar g(x^{+\prime}), g(x^+)\} = i\alpha_Q f(x^+)\delta(x^{+\prime} - x^+) + \alpha_Q\delta'(x^{+\prime} - x^+), \quad \{\chi_\alpha(x^{+\prime}), \bar g(x^+)\} = -i\alpha_Q\delta(x^{+\prime} - x^+)\bar\chi_\alpha,$$

$$\{\tilde B_{a+}(x^{+\prime}), \tilde B_{b+}(x^+)\} = -\alpha_Q(\tfrac{d-1}{2C_\rho})\delta(x^{+\prime} - x^+)f_{ab}{}^c\tilde B_{c+}(x^+) + \alpha_Q(\tfrac{d-1}{2C_\rho})\delta'(x^{+\prime} - x^+)\delta_{ab}, \tag{6.139}$$

while those among the fermions is:

$$\{\bar\chi_\alpha(x^{+\prime}), \chi_\beta(x^+)\} = \tfrac{i\alpha_Q}{2}\eta_{\alpha\beta}f(x^+)\delta(x^{+\prime} - x^+) + \alpha_Q(\lambda^a)_{\alpha\beta}\tilde B_{a+}\delta(x^{+\prime} - x^+)$$
$$+\alpha_Q\eta_{\alpha\beta}\delta'(x^{+\prime} - x^+),$$

$$\{\chi_\alpha(x^{+\prime}), \chi_\beta(x^+)\} = i\alpha_Q\eta_{\alpha\beta}g(x^+)\delta(x^{+\prime} - x^+),$$
$$\{\bar\chi_\alpha(x^{+\prime}), \bar\chi_\beta(x^+)\} = i\alpha_Q\eta_{\alpha\beta}\bar g(x^+)\delta(x^{+\prime} - x^+),$$
$$\{\tilde B_{a+}(x^{+\prime}), \chi_\beta(x^+)\} = -\alpha_Q(\tfrac{d-1}{2C_\rho})(\lambda_a)^\alpha{}_\beta\chi_\alpha(x^+)\delta(x^{+\prime} - x^+),$$
$$\{\tilde B_{a+}(x^{+\prime}), \bar\chi_\beta(x^+)\} = -\alpha_Q(\tfrac{d-1}{2C_\rho})(\lambda_a)^\alpha{}_\beta\bar\chi_\alpha(x^+)\delta(x^{+\prime} - x^+). \tag{6.140}$$

where $\alpha_Q = \frac{2\pi}{k}$. Rescaling the above currents to:

$$f \to \tfrac{k}{4\pi}f, \qquad g \to \tfrac{k}{2\pi}g, \qquad \bar g \to \tfrac{k}{2\pi}\bar g,$$
$$\tilde B_{a+} \to \tfrac{k}{2\pi}\tilde B_{a+}, \quad \chi_\alpha \to \tfrac{k}{2\pi}\chi_\alpha, \quad \bar\chi_\alpha \to \tfrac{k}{2\pi}\bar\chi_\alpha, \tag{6.141}$$

and expanding it in the modes yields the following commutators:

$$[f_m, f_n] = m\tfrac{k}{2}\delta_{m+n,0}, \qquad [(\chi_\alpha)_m, f_n] = \tfrac12(\chi_\alpha)_{(m+n)},$$
$$[g_m, f_n] = g_{m+n}, \qquad [(\bar\chi_\alpha)_m, f_n] = -\tfrac12(\bar\chi_\alpha)_{(m+n)},$$
$$[\bar g_m, f_n] = -\bar g_{m+n}, \qquad [(\bar\chi_\alpha)_m, g_n] = -(\chi_\alpha)_{m+n},$$

$$[\bar{g}_m, g_n] = -2f_{m+n} - mk\delta_{m+n,0}, \qquad [(\chi_\alpha)_m, \bar{g}_n] = (\bar{\chi}_\alpha)_{m+n},$$

$$\{(\chi_\alpha)_m, (\chi_\beta)_n\} = -\eta_{\alpha\beta} g_{m+n}, \qquad \{(\bar{\chi}_\alpha)_m, (\bar{\chi}_\beta)_n\} = -\eta_{\alpha\beta}\bar{g}_{m+n},$$
$$[(\tilde{B}_{a+})_m, (\chi_\beta)_n] = i(\tfrac{d-1}{2C_\rho})(\lambda_a)^\alpha{}_\beta(\chi_\alpha)_{(m+n)}, \quad [(\tilde{B}_{a+})_m, (\bar{\chi}_\beta)_n] = i(\tfrac{d-1}{2C_\rho})(\lambda_a)^\alpha{}_\beta(\bar{\chi}_\alpha)_{(m+n)},$$

$$[(\tilde{B}_{a+})_m, (\tilde{B}_{b+})_n] = -i(\tfrac{d-1}{2C_\rho})f_{ab}{}^c(\tilde{B}_{c+})_{(m+n)} - (\tfrac{d-1}{2C_\rho})km\delta_{ab}\delta_{m+n,0},$$
$$\{(\bar{\chi}_\alpha)_m, (\chi_\beta)_n\} = -\eta_{\alpha\beta}f_{(m+n)} + i(\lambda^a)_{\alpha\beta}(\tilde{B}_{c+})_{(m+n)} - km\eta_{\alpha\beta}\delta_{m+n,0}. \tag{6.142}$$

This is the affine Kaĉ-Moody super-algebra. Here, it is evident that the central extension to the $sl(2,\mathbb{R})$current sub-algebra spanned by $(f, g, \bar{g})$ is $k = c/6$. The quadratic non-linearities that occur in the super-Virasoro are not present here.

6.5 $sl(3,\mathbb{R})$ conventions

We work with the following basis of 3×3 matrices (see [7]) for the fundamental representation of the gauge group used in the definition of the higher spin theory:

$$L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \; L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; W_{-2} = \alpha \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$W_{-1} = \alpha \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \; W_0 = \alpha\,\tfrac{2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; W_1 = \alpha \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \; W_2 = \alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}. \tag{6.143}$$

The algebra satisfied by these matrices is

$$[L_m, L_n] = (m-n)\,L_{m+n}, \quad [L_m, W_n] = (2m-n)\,w_{m+n},$$
$$[W_m, W_n] = -\frac{\alpha^2}{3}(m-n)(2m^2 + 2n^2 - mn - 8)\,L_{m+n} \tag{6.144}$$

For $\alpha^2 = -1$ this is the $su(1,2)$ algebra and for $\alpha^2 = 1$ this is the $sl(3,\mathbb{R})$ algebra. We take the Killing metric as $\eta_{ab} = \frac{1}{2}\,\mathrm{Tr}(T_a T_b)$ where T_a are the above matrices. The structure constants are $f_{abc} = \frac{1}{2}\mathrm{Tr}(T_a\,[T_b, T_c])$.

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