Aspects of polymer quantization

By

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DEDICATIONS

To Ma, Baba and to the memory of Monima.

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Contents

Chapter 1: Introduction

1.1 Loop Quantum Gravity and low energy physics

Obtaining a quantum theory of gravity is perhaps the biggest challenge in theoretical physics today. The methods which work perfectly well for matter fields on fixed backgrounds fail in the case of gravity. In particular, attempts to perturbatively quantize gravity leads to a non-renormalizable theory. This failure is not so surprising - there are fundamental differences between other field theories and General Relativity. We think of the other field theories as living on a fixed background. Any algorithm to quantize these theories uses information about the particular background the field lives on. On the other hand the lesson of General Relativity has been that the background geometry is itself dynamical. There is thus no 'fixed background', the picture is one of dynamical fields living on other dynamical fields. There is therefore no preferred way of splitting the geometry into a 'non-dynamical background ' and an 'excitation propagating on this background' as one would do in the case of usual perturbative field theory. This suggests that applying background dependent perturbative approaches to quantization of gravity is fundamentally problematic. In more technical terms, incorporating the diffeomorphism invariance of General Relativity has been a challenge in obtaining a quantum theory of gravity.

But then how do we go about quantizing this theory? Could it be possible to develop a background independent, non-perturbative quantization for General Relativity? Loop Quantum Gravity is an approach to do just that to obtain a quantum theory of gravity (

See [1, 2, 3] for reviews). Indeed LQG has had considerable success in obtaining a background independent quantum theory of gravity, although many challenges remain. The starting point of LQG is a reformulation of General Relativity as a theory of connection (as opposed to a theory of metric). This allows one to cast General Relativity as a SU(2) gauge theory. Some of the important successes of Canonical Loop Quantum Gravity have been

- Obtaining a Kinematical Hilbert Space on which both finite spatial diffeomorphisms and SU(2) gauge transformations are represented as unitary operators [4, 5, 6, 7, 8]. Thus the theory overcomes the major challenge of incorporating spatial diffeomorphisms as a gauge symmetry of the theory.

- The prediction of discrete geometry. One can obtain operators corresponding to geometrical variables such as area [9, 10] and volume [9, 11] on the Kinematical Hilbert Space. These operators turn out to have *discrete* spectra - realizing the old expectation that the picture of smooth spatial geometry breaks down at the quantum level. Moreover this discreteness is an essential feature of the theory, the fundamental excitations in the quantum theory turn out to be a discrete set of one-dimensional spin networks [12].

- LQG provides an explanation for the black hole microstates and correctly predicts Bekenstein-Hawking entropy [13, 14, 15].

- Quantizing symmetry reduced models of General Relativity using the techniques of Loop Quantum Gravity leads one to Loop Quantum Cosmology [16, 17, 18]. LQC produces the striking result that the big bang singularity at the beginning of the universe is replaced by a 'big bounce'.

- Background independent quantizations for matter fields have also been developed, allowing us to quantize the combined matter-gravity system [19, 20, 21]. This is

called 'loop quantization' in the case of gauge fields and 'polymer quantization' for scalar fields.

- More recently, there have been hints that a non-trivial anomaly-free representation of the Dirac Algebra[22] may be realized in Loop Quantum Gravity [23, 24, 25, 26]. So far it has been realized in several toy models that closely mimic LQG. As Dirac Algebra encodes the general covariance of a theory, this means that we would be able to establish LQG as a generally covariant quantum theory

Despite these successes, several open questions remain. One of the important open issues is connecting LQG with the known low energy physics. In this thesis we take up certain aspects of this problem.

Why is there a problem connecting Loop Quantum Gravity with low energy physics? For one, the frameworks of LQG and that of perturbative field theory are widely different. LQG describes a picture when fields live on evolving graphs. The gravitational and gauge fields live on the edges of the 1-dimensional graphs while scalar fields reside on the vertices. This paints a polymer-like picture of the fundamental structure of spacetime. In perturbative field theory on the other hand the fields are 3-dimensional functions on smooth backgrounds. But we do know that semiclassical physics is correctly described by perturbative field theory on a fixed background. The question then is to how to connect these two pictures.

In this thesis we explore two different aspects of this question. The first is about the symmetries of low energy physics. How do we understand these symmetries from the perspective of LQG and matter theories coupled to it? Can this framework support symmetries? How may they arise in the low energy limit? We explore this question in two different cases. In one case we use a toy model (which we will call polymer Quantum Mechanics) which was introduced to explore how low energy physics can be extracted from a LQG-like theory. In another case we consider the case of a scalar field quantized in a background-independent manner. This is called the polymer quantized scalar field,

and we study the implementation of rotational symmetries here.

The second question is related to comparing the polymer quantized and Fock quantized scalar fields. The implementation of background independence in LQG demands that scalar fields must be polymer quantized. However the semiclassical picture is in terms of Fock quantized scalar fields. It is therefore essential to be able to compare the two theories. The Hilbert Space frameworks of the two theories are, however, widely different [20]. In this thesis we develop a path integral representation for polymer quantized scalar fields and show that in this representation the two theories look very similar, but with interesting differences. This provides a first step in further comparison of the two theories, as well as developing a perturbative framework for the polymer quantized scalar field.

We have mentioned two theories that we will use to explore the low energy limit of LQG in this thesis - polymer Quantum Mechanics and polymer quantized scalar field theory. The reason why the term 'polymer' appears in both the names is because the close resemblance of the quantization procedure used in both the cases. This procedure is called 'polymer quantization'. We now turn to introducing polymer quantization.

1.2 Introduction to Polymer Quantization

In this section we elaborate the ideas sketched in the previous section in some more detail. First we provide a brief semi-technical exposition to the ideas of polymer quantization. Then we discuss how this quantization resembles the one used for the gravitational field in Loop Quantum Gravity. Finally we return to the question of what the polymer quantized theories may teach us.

What is Polymer Quantization?

The name is used for the similar quantization procedures that have been applied to quantize a non-relativistic point particle and a scalar field - procedures which are analogous to (and inspired by) how the gravitational field is quantized in Loop Quantum Gravity[1]. That definition probably raises more questions than it answers. One would ask in what sense is a quantization of a scalar field or a point particle analogous to a quantization of gravity? Or what would be the motivation for introducing a new quantization for systems like scalar field theory or point particles, when the old one works perfectly well for all known physical systems? We'll address these questions in the next two sections. In this section we present a short, semi-technical introduction to polymer quantization (a more technical introduction will be presented in the next section). This will be helpful in understanding the analogy with Loop Quantum Gravity, which in turn will shed light on our motivations for introducing this new quantization.

Let us start by asking how this new quantization differs from the old one - the standard Schrodinger quantization that we are used to. In Schrodinger quantization one starts from the phase space formulation of the classical theory and elevates the canonically conjugate pair of variables to operators in the quantum theory[2]. In the point particle case, for example, the position and momentum variables are elevated to operators in the quantum theory. What distinguishes polymer quantization is that only one of those pair of variables can be elevated to an operator in the quantum theory. The other basic observable is the exponentiation of the remaining conjugate variable. In the case of the point particle, this means that we can't have both position and momentum as well defined observables. Suppose we choose to have the position operator $\hat{x}$ well defined. In this case the other basic operator will be the (family of) translation operator $\hat{U}(\alpha), \alpha \in \mathbb{R}$. Classically, this is the

[1] Polymer quantization for scalar field was first introduced in [20]. For non-relativistic point particles it was introduced in [29, 27, 28].

[2] It should be kept in mind that this only works when the phase space is $\mathbb{R}^{2n}$. For topologically nontrivial phase space this naive elevation does not work.

exponentiation of the momentum variable $U(\alpha) = e^{i\alpha p}$. However the momentum variable itself cannot be defined as an operator in the polymer quantized theory. Alternately one may choose the momentum operator and the family of momentum-translation operators as the basic variables in a polymer quantized theory. Similarly for the polymer quantized scalar field [20], either the field variable or the conjugate momentum variable may appear as an operator in the quantum theory.

It may seem odd that while the translation operators $\hat{U}(\alpha)$ can be defined but the momentum operator doesn't. Surely we could obtain the momentum operator by taking a limit like $\lim_{\alpha\to 0} \frac{\hat{U}(\alpha)-1}{\alpha}$. The reason why that doesn't work is $\hat{U}(\alpha)$ turns out to be discontinuous in α[3]. We'll see this explicitly in the next chapter where we describe polymer quantization in detail.

So it would seem that we are left one operator short, so to speak. For instance, how do we replicate the harmonic oscillator Hamiltonian in the polymer Hilbert space, if we have only have well-defined operators for either the position or the momentum? Similarly, what is the expression for the angular momenta? The only thing that can be done to replicate these observables is to *approximate* them. This would require us to introduce a scale μ. For example, we may define an approximate position operator as:

$$\hat{x}_\mu = \frac{\hat{U}(\mu) - \hat{U}(-\mu)}{2i\mu}$$

Then one may define a Hamiltonian or an angular momentum operator on the polymer Hilbert Space using these approximate operators. But then the natural question to ask is how does using approximate operators change the physical predictions of the theory? For instance, do the modified angular momentum operators have the same commutation relations? If not, how do we understand something like rotational symmetry in the polymer quantized theory? These are some of the questions this thesis is concerned with.

[3]This statement holds for the weak topology on the space of bounded operators on a Hilbert Space, as we'll see in the next chapter.

But before we get there, it is useful to take a brief tour through some of the ideas of LQG.

Loop Quantum Gravity and Polymer Quantization

Loop Quantum Gravity is an attempt to canonically quantize the gravitational field. The classical starting point for this approach is a Hamiltonian formulation of General Relativity where the basic variables are taken to be a gravitational connection $\mathcal{A}_a^i$ (called the Ashtekar Barbero connection) and the triads E_i^a. As is the case with gauge theories, this gives us a constrained system in the Hamiltonian formulation. There are seven independent constraints per phase space point, corresponding to the seven gauge freedoms that appear in our theory. Four of these correspond to the three spatial and one temporal diffeomorphisms that are the gauge symmetries of the General Relativity[4]. Our particular choice of variables in the Hamiltonian formulation - connections and triads - introduces an additional SU(2) gauge symmetry to General Relativity. The three remaining constraints are the generators of this gauge transformation.

One then follows the two step Dirac quantization process for quantizing constrained systems. The idea here is to first construct a Kinematical Hilbert Space on which the gauge transformations would be represented. Then one would look for states in this Hilbert Space (or rather in a dual of a subspace of this Hilbert Space) which are invariant under the transformations generated by the constraints. They would eventually form the physical Hilbert Space for the quantized theory of Gravity, after an appropriate new inner product has been chosen.

Let us focus on the first step - the construction of the Kinematical Hilbert Space. Usually in the construction of a quantum theory one elevates the basic variables of the classical theory to operators. In LQG things turn out not to be the slightly different. We have mentioned that the basic variables for us are a connection $\mathcal{A}_a^i$ and triad variables E_i^a. The basic

[4]The oft-repeated statement that these constraints *generate* the spacetime diffeomorphisms is only true on-shell. See Chapter 1 Section 4 of [2].

operators on the Kinematical Hilbert Space are however the holonomies of this connection along 1-dimensional closed curves and fluxes of the triads across 2-surfaces. The connection itself fails to be a well defined operator on this Hilbert Space, only the holonomy is a well-defined operator. Similarly, the constraints that generate spatial diffeomorphisms fail to be well defined operators, but finite spatial diffeomorphisms can be represented as Unitary operators. A convenient set of basis states of this space are spin networks. These can be thought of as basis states of quantum geometry and suggest a polymer-like picture of quantum geometry where the fundamental excitations are 1-dimensional [5] .

That last paragraph should remind us of our discussion of polymer quantization. There too, we saw that the position/momentum (in the point particle case) and the field variable/ conjugate momentum (in the scalar field case) fail to be well defined operators in the Quantum Theory. Thus we already see some ways in which polymer quantization mimics LQG. We'll see more ways in which the two quantizations resemble each other as we go on. Particularly, the feature of LQG that finite spatial diffeomorphisms can be represented as operators while infinitesimal generators of the same diffeomorphisms fail to be well defined operators has a counterpart in polymer quantization that will form one of the topics of investigation of this thesis.

To summarize, in this section we have presented a brief overview of Loop Quantum Gravity and drawn attention to its similarities with polymer quantized theories. This paves the way for us to get a better understanding of what polymer quantized theories can teach us.

[5] Actually these are a basis for the Hilbert space formed by gauge invariant subspace of the Kinematical Hilbert space. We are being loose in the distinction between these two spaces for the purpose of introduction here.

Why Polymer Quantize?

A. Motivations for polymer quantization of scalar field theory

As we have previously mentioned, usual quantum field theories require a background metric for their definition. When the background itself is quantized, as in LQG, no fixed metric is available. To co-exist with background independent quantization of gravity, matter fields should also be quantized in a background independent manner. For scalar fields, polymer quantization provides just that. Here the Hilbert Space is a space of functionals on scalar fields equipped with a *background independent* inner product. This quantization is thus background independent. From the perspective of background-independent quantization therefore, polymer quantization of scalar fields is *essential*.

In this thesis we will study a polymer quantized scalar field in a fixed background. As noted above, the construction of the Hilbert Space uses no information about the background metric. Rather, the metric information enters polymer quantized scalar field theory through the form of the Hamiltonian. Thus while all diffeomorphisms can be represented by unitary operators on the polymer Hilbert Space, not all of them will commute with the Hamiltonian. Only the isometries of the fixed metric may commute with the Hamiltonian. Let us now understand why it is important study polymer quantization in the presence of a fixed metric.

According to LQG the fundamental quantum picture of gravity coupled with scalar fields would be given in terms of states living on graphs. The gravitational excitations live on the edges of the graph while the scalar field excitations would live on the nodes. There are now two challenges to reproduce the familiar semiclassical picture of Fock quantized field theory on a fixed background which we know to be valid at low energies. One is to reproduce a fixed background. To do this one would need to produce semiclassical states of the gravitational field that are peaked on some fixed background metric. This is

still an open problem in LQG[6]. The second challenge is reproduce the predictions of the Fock quantized scalar field theory from the polymer quantized scalar field theory. Since the two field theories are very different in terms of Hilbert Space construction, this is a serious challenge. To tackle this problem, we may study polymer quantized field theory in a fixed background and see how to compare its predictions to those coming from a Fock quantized field theory.

B. Motivations for Polymer Quantum Mechanics

Unlike the case of the polymer quantized scalar fields, there is no motivation for a background independent quantization in a Quantum Mechanics context. The principle reason we are interested in polymer Quantum Mechanics is that it serves as a toy model that can help us develop intuition about the unfamiliar domain of loop quantized theories.

As we have seen, LQG presents a picture of quantized geometry that is *essentially discrete*. Spin network states provide a convenient basis. The basic operators are holonomies along 1-d curves and fluxes of triads across 2- surfaces. Thus the fields can be thought to be living on graphs - the gravity and gauge fields on the edges of the graphs and scalar fields on the nodes. This is a vastly different picture from the one we know from perturbative scalar field theory. There one pictures fields as continuous functions living on smooth 3-slices. A question then is to understand how these frameworks may be related to each other. This would be necessary to understand how semiclassical physics should arise from LQG in some low energy regime.

Furthermore in LQG, the diffeomorphism invariant states do not live in the Kinematical Hilbert Space but in the dual space Cyl* of a subspace Cyl of the Kinematical Hilbert Space. If the physical states live on Cyl*, should the semiclassical states not live on this also? However this space is not naturally equipped with an inner product. So a question is, what role would Cyl* play in connecting with semiclassical physics?

[6]See [30, 31] for some approaches to this issue.

Polymer Quantum Mechanics (also known as the polymer particle representation) was introduced in [29] as a toy model to explore exactly these issues[7]. Polymer Quantum Mechanics reproduces many of the essential features of LQG while being much simpler and mathematically more tractable. We've already seen that it replicates the feature that a basic variable fails to be well-defined as an operator while its exponentiation is well defined. As we'll see explicitly in the next chapter, the related fact that there is a discrete set of basis states is also reproduced in polymer Quantum Mechanics. Spaces analogous to Cyl and Cyl* appear here as well. All these features make polymer Quantum Mechanics an excellent toy model for exploring the issue of relating the LQG and perturbative field theory frameworks. The stand-in for perturbative field theory in this comparison is ordinary quantum mechanics i.e the Schrodinger representation. The connection between these two *unitarily inequivalent* representations is explored to obtain insights for the full theory.

In this section we discussed the motivations for the two polymer quantized theories we investigate in this thesis. We saw that polymer quantized scalar field theory has a more direct physical relevance in the context of gravity while polymer Quantum Mechanics is an important toy model. We are now ready to present the issues explored in this thesis. We do so in the next section.

1.3 Aspects of Polymer Quantization

The central concern of this thesis is comparing polymer and Schrodinger quantizations. We've investigated two different aspects of polymer quantization which shed some light on this issue. We have -

(i) Studied how symmetries may be incorporated in polymer quantized theories. Important differences from Schrodinger quantized theories show up here. One also discovers

[7]The same representation had been earlier introduced in a very different context in [27].

that if one starts from polymer quantization and demands the incorporation of symmetries, there is a sense in which one is led to the Schrodinger quantization.
(ii) Obtained a path integral representation of polymer quantized theories. We have obtained path integral formulations for polymer Quantum Mechanics and polymer quantized scalar field theory. The path integral language is another avenue for comparing the two quantizations.
Let us understand these statements above in some more detail.

A. Polymer Quantization and Symmetries

In trying to relate LQG with the known low -energy description in terms of fields on fixed backgrounds, an important question is - how do we obtain the symmetries of these field theories (for instance Poincare symmetry for a field theory on Minkowski space) from the description in terms of discrete quantum geometry? The toy model of polymer Quantum Mechanics can be used to shed light on this issue. Here one compares the implementation of symmetries in Schrodinger and polymer Quantum Mechanics. We have already seen a hint that polymer Quantum Mechanics may run into trouble on this issue. We saw that the absence of either position or momentum operator meant that angular momentum would not be well-defined as an operator. As angular momenta are generators of rotation, a natural question to ask would be how do we then implement rotational symmetries in polymer Quantum Mechanics? It turns out that one *can* define unitary operators that implement finite rotations even though infinitesimal generators fail to be well defined.

This had been first explored by Dah-Wei Chiou in the more general context of Galilean symmetries in [38], which was the first paper to explore the question of symmetries in polymer Quantum Mechanics. Chiou then explored the approximated forms of the usual generators (which do not form a closed algebra) and concluded that the deviations are small within the domain of validity of the non-relativistic model.

In this thesis we present further investigations of the subject of polymer quantization and symmetries. We focus on the particular case of rotational symmetries for both polymer Quantum Mechanics and polymer quantized scalar field theory. First let us state our findings for polymer Quantum mechanics. As we noted above Rotation groups are unitarily represented on the polymer Hilbert space, but the corresponding generators cannot be defined. Such representations of Lie groups are the *discontinuous* representations - representations of a Lie group which do not induce a representation of the corresponding Lie algebra. Discontinuous representations are rare in physics[8] and some of their properties can be surprising, as we shall see. What is the physical relevance of such a representation? We show in this thesis that this leads to infinite degeneracies. That is, a rotationally invariant Hamiltonian will have infinite dimensional eignesubspaces!

This is obviously an undesirable feature for a theory. We show that there are two escape routes. One is to break the symmetry. This can be done explicitly by introducing a rotationally non-invariant Hamiltonian. Or we can break the symmetry spontaneously, by choosing to work in one of the (symmetry breaking) superselection sectors. There is a different, less obvious escape route as well which we explore in our thesis. This is to work in the dual space Cyl* of a certain subspace Cyl of the Hilbert Space. We show that the symmetry generators can be defined as operators on Cyl*. The position operator can be represented as well. By demanding that these operators are self-adjoint one in fact uniquely recovers the Schrodinger representation. This leads to a new understanding of polymer Quantum Mechanics. We can now understand it as an intermediate step in a multi-step quantization procedure that finally leads to a physical theory. Thus polymer quantization resembles the construction of Kinematical Hilbert Space in LQG. There the other steps in quantization are related to the implementation of the constraints, here they appear to be related to the implementation of the symmetries.

For polymer quantized scalar field theory we see that once again rotational symmetry has a discontinuous representation on the Hilbert space of the theory. We show that here too

[8] In fact we do not know any other physical situation where they occurred

one may recover the symmetry generators by working in a dual space Cyl*.

B. Path Integral Formulation for Polymer Quantization

We have already emphasized the importance of studying polymer Quantum field theories in fixed backgrounds and comparing the dynamics with that predicted by Schrodinger/Fock quantization[9]. Now we know that the field variable is unavailable as an operator for the polymer theory and one has to work with an approximate field operator. What is the effect of using approximate operators on the dynamics? In this thesis we take a step towards addressing this issue by developing a path integral representation for a polymer quantized scalar field theory in a Minkowski background. Here we build on previous work by Ashtekar et al [32]who obtained a path integral representation for Loop Quantum Cosmology, which is a polymer quantized theory with a single degree of freedom. We see that while the polymer Quantized and Schrodinger quantized theories differ vastly in the Hilbert space language, the path integral representations of the two theories can be made to look very similar. The path integral representation is very convenient to compare the two quantizations. Despite the similarities, there are some important differences in the two quantizations. One major difference arises from the use of approximate variables in the polymer quantized theory. We find that the the use of approximate field variables indeed modifies the dynamics. In particular, we show that the polymer quantized theory in the path integral representation is Lorentz non-invariant. We also find that an entirely new global symmetry appears in the action in the case of polymer quantized theories. We explore the origins of this symmetry in our thesis.

This concludes our introduction to the investigations presented in this thesis. The chapters of the thesis are organized as follows. After an introductory chapter 1, in chapter 2 we review polymer quantum mechanics and polymer quantized scalar field theory. Our

[9]In the context of scalar field theory, we will use the terms Schrdinger and Fock quantizations interchangeably, as they refer to equivalent representations [33].

investigation of polymer quantization and symmetry are presented in Chapter 3. Chapter 4 describes our work on the path integral representation of polymer quantized scalar field theory. In chapter 5 we outline open issues and possible future directions.

Chapter 2: Polymer Quantization

In the last chapter we introduced the reader to the ideas behind polymer Quantum Mechanics and polymer quantized Scalar Field Theory. In this chapter we introduce these two theories in details. As we have indicated earlier, these two theories closely mimic the quantization procedure of LQG. For the reader interested in the details of this resemblance, we have provided an introduction to construction of the Kinematical Hilbert Space in LQG in the appendix.

2.1 Polymer Quantum Mechanics

Polymer Quantum Mechanics was first introduced in [27, 28, 29]. The object of study here is the non-relativistic point particle. We study the general case of a point particle moving in N dimensions. Classically it is described by the configuration space, $\mathbb{R}^N$ coordinatized by $\vec{q} \leftrightarrow q^i, i = 1, 2, 3, ..., N$.

Both polymer and Schrodinger quantizations are based on Unitary representations of the Weyl Algebra. We now introduce this algebra.

The Weyl Algebra

The familiar starting point for presentations of quantum mechanics is the Heisenberg Commutation relations: $[\hat{q}, \hat{p}] = i\hbar$, where these operators are understood to act on the

space of square integrable functions L^2. From a technical point of view, however, this is not quite the best formulation. For one thing, operators like $\hat{q}, \hat{p}$ generally take an element of L^2 out of L^2. This is easily seen in the one dimensional case from the action of $\hat{x}$ on the function $g(x) = \frac{1}{x+i}$. The function $g(x)$ belongs to L^2 but $xg(x)$ does not. This means that a commutation relation like the one above can only really make sense in the common domain of operators $\hat{q}\hat{p}$ and $\hat{p}\hat{q}$. Furthermore, these operators are unbounded - $\|\hat{x}g(x)\| = \infty$ from the example above. The powerful techniques of C*-algebras would be unavailable to us then.

To see how a better formulation can be given we first look at the unitary operators associated with the 'exponentiations' of the position and momentum operators (this is possible-for self adjoint operators):

$$\hat{U}(\vec{\lambda}) = e^{i\vec{\lambda}\cdot\hat{\vec{q}}} \qquad \text{and} \qquad \hat{V}(\vec{\mu}) = e^{i\frac{\vec{\mu}}{\hbar}\cdot\hat{\vec{p}}}$$

These are called Weyl operators. These operators can be used to generate a $C^\star$-algebra. The product is defined via the Heisenberg commutation relations:

$$\begin{aligned} \hat{U}(\vec{\lambda_1})\hat{U}(\vec{\lambda_2}) &= \hat{U}(\vec{\lambda_1} + \vec{\lambda_2}) \\ \hat{V}(\vec{\mu_1})\hat{V}(\vec{\mu_2}) &= \hat{V}(\vec{\mu_1} + \vec{\mu_2}) \\ \hat{U}(\vec{\lambda})\hat{V}(\vec{\mu}) &= e^{-i\vec{\lambda}\cdot\vec{\mu}}\hat{V}(\vec{\mu})\hat{U}(\vec{\lambda}) \end{aligned}$$

The involution relations follow from the self-adjointness relations of the $\hat{q}, \hat{p}$ operators:

$$[\hat{U}(\vec{\lambda})]^\star = \hat{U}(-\vec{\lambda}),\ [\hat{V}(\vec{\mu})]^\star = \hat{V}(-\vec{\mu})$$

The abstract $C^\star$-algebra generated by the Weyl operators is called the *Weyl Algebra*. Now we are ready to state the more technically satisfactory starting point for quantum mechan-

ics. *A quantum mechanical system is described by a unitary, irreducible representation of the Weyl Algebra.* This formulation has the advantage of avoiding domain problems and making the tools of $C^\star$-algebra available. This is also a more general formulation in the sense that a representation of Weyl operators can be well-defined even when the position and/or momentum operator does not exist. We'll see an example of this in a moment.

The Schrodinger Representation

On any representation of the Weyl Algebra, the position and momentum operators can be recovered from the Weyl Operators when the criterion of *weak continuity* is satisfied, that is, when all matrix elements of $\hat{U}(\vec{\lambda}), \hat{V}(\vec{\mu})$ are continuous in $\vec{\lambda}$ and $\vec{\mu}$ respectively. The Heisenberg commutation relations given above then follow from the Weyl algebra.[1] This is the case in the standard Schrodinger representation. Here in our case of a particle in $\mathbb{R}^N$ this representation is given by the the Hilbert space $L^2(\mathbb{R}^N, dx)$ of functions on $\mathbb{R}^N$ which are square integrable with respect to the standard Lebesgue measure. The action of the Weyl Operators on this space is given by operators is given by

$$\hat{U}(\vec{\lambda})\,\psi(\vec{q}) = e^{i\vec{\lambda}\cdot\vec{q}}\,\psi(\vec{q}) \quad \text{and} \quad \hat{V}(\vec{\mu})\,\psi(\vec{q}) = \psi(\vec{q}+\vec{\mu})$$

for all $\psi \in L^2(\mathbb{R}^N, dx)$. As this action is weakly continuous, there exist self-adjoint operators $\hat{x}$ and $\hat{p}$ on the Hilbert Space such that the Weyl operators are the exponentiations of these operators.

$$\hat{U}(\vec{\lambda}) = e^{i\vec{\lambda}\cdot\hat{\vec{q}}} \qquad \text{and} \qquad \hat{V}(\vec{\mu}) = e^{i\frac{\vec{\mu}}{\hbar}\cdot\hat{\vec{p}}}$$

where $\vec{\lambda}, \vec{\mu}$ take values in $\mathbb{R}^N$.

A very powerful statement can be made about weakly continuous representations of the

[1] For more details on the relation between Weyl and Heisenberg algebras see [34]

Weyl algebra. This is the Stone- Von Neumann theorem according to which every irreducible unitary representation of the Weyl algebra which satisfies weak continuity is unitarily equivalent to the standard Schrodinger representation.

The Polymer Representation

Now we come to the polymer representation, a representation which is not equivalent to the Schrodinger representation defined above. By the Stone-Von Neumann theorem this means that the weak continuity of Weyl operators must be violated in this representation. In other words, both position and momentum cannot exist as well-defined operators. Here we present the polymer representation where momentum is well-defined. We will work in this representation all through this thesis. To construct Polymer quantization, choose a countable set, γ, of N-dimensional vectors $\vec{k}_j$ and define a set Cyl_γ of linear combinations of functions of $\vec{q}$ of the form: $\mathrm{Cyl}_\gamma := \{\sum_j f_j e^{i\vec{k}_j\cdot\vec{q}}, f_j \in \mathbb{C}\}$ [2].

Next, define the set of functions of $\vec{q}$, $\mathrm{Cyl} := \cup_\gamma \mathrm{Cyl}_\gamma$. Clearly, $\{e^{i\vec{k}\cdot\vec{q}} \ /\vec{k} \in \mathbb{R}^N\}$ form an uncountable set and we denote them as the kets $|\vec{k}\rangle$. An inner product on Cyl is defined by demanding that the kets $|\vec{k}\rangle$ form an orthonormal basis:

$$(e^{i\vec{k}\cdot\vec{q}}, e^{i\vec{k'}\cdot\vec{q}}) = \delta_{\vec{k},\vec{k'}}. \tag{2.1}$$

We denote the completion of Cyl w.r.t. this inner product as H_{poly}. As the basis set is uncountable the Hilbert space is non-separable. We also have the natural triple, $\mathrm{Cyl} \subset \mathrm{H}_{\mathrm{poly}} \subset \mathrm{Cyl}^*$, where Cyl^* denote the algebraic dual of Cyl.

It is clear that $\vec{q}$ *cannot* be represented on the polymer Hilbert space as a multiplicative operator since q^i acting on a basis element does not produce a countable linear combination of the basis elements (exponentials). The exponentials of the form, $e^{il_j q^j}$ however

[2]Here the coefficients f_j must satisfy the regularity conditions (i) The f_j do not contain sequences with accumulation points in $\mathbb{R}^{\mathbb{N}}$ (ii) There exist constants v_γ and ρ_γ such that the number $n(I)$ of points in any subset I of volume $v(I)\gamma v_\gamma$ is bounded by $n(I) \leq \rho_\gamma v(I)$ [29].

do form multiplicative (and *unitary*) operators. The derivatives too act invariantly on Cyl and $p_i := -i\hbar\frac{\partial}{\partial q^i}$ are self-adjoint operators representing the momenta. The exponentials are the eigenfunctions of the momenta: $\hat{p}_i|\vec{k}\rangle = \hbar k_i|\vec{k}\rangle$.

That the self-adjoint position operators $\hat{q}^i$ do not exist can be seen more formally as well. Consider a 1-parameter family of unitary operators, defined by $\hat{U}(\alpha, \vec{m})|\hat{k}\rangle := |\hat{k} + \alpha\vec{m}\rangle \; \forall \; \vec{k} \in \mathbb{R}^N$. For any vector $\vec{\ell}$, $\langle\vec{\ell}|\hat{U}(\alpha, \vec{m})|\vec{\ell}\rangle = \langle\vec{\ell}|\vec{\ell} + \alpha\vec{m}\rangle = \delta_{\alpha\vec{m},\vec{0}}$, as implied by the orthonormality. Hence, the family of unitary operators is *not* weakly continuous at $\alpha = 0$. *If* a self-adjoint operator of the form $\vec{m} \cdot \vec{q}$ existed, then we could define a one parameter family of unitary operators $\hat{V}(\alpha, \vec{m}) := e^{i\alpha\vec{m}\cdot\vec{q}}$ which *is continuous at* $\alpha = 0$ and precisely matches the $\hat{U}(\alpha, \vec{m})$ family, thus reaching a contradiction. Hence, on the polymer Hilbert space, the momenta and exponentials of positions are well defined operators but there are *no* self-adjoint operators representing positions.

We can however define an *approximate* position operator by choosing some scale (with the dimension of inverse length). μ_0:

$$\hat{q}_j = (2i\mu_0)^{-1}(\hat{V}(\mu_0\hat{e}_j) - \hat{V}(-\mu_0\hat{e}_j))$$

where $\hat{e}_j$ is the unit vector in the j direction and$\hat{V}$ is the generator of translation. Once the approximate position operator is defined with a particular choice μ_0, starting from a given $|\vec{p}_0\rangle$ and acting on it with $\hat{V}(\mu_0)$ we'll generate a set of basis vectors $\{|\vec{p}\rangle = |\vec{p_0} + n^j\mu_0\hat{e}_j\hbar\rangle\}$ (Here n^j denotes the number of times an operator $\hat{V}(\mu_0)$ has acted on the state). This gives a proper subspace of the Hilbert Space and the action of observables built from $\hat{q}_j$ and $\hat{p}_j$ will leave the subspace invariant.

Thus we have exhibited H_{poly}, defined candidate basic operators and identified an invariant subspace which is separable. This will be useful later.

2.2 Polymer Quantized Scalar Field Theory

In this section we describe the polymerised scalar field theory. This theory was first presented in [20], following previous ideas of [35, 36] In this section we'll set $\hbar = 1$. We'll follow the notation of [37]. First define a vertex set $V = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)$ of finitely many, distinct points $\in \mathbb{R}^3$. The corresponding vector space Cyl_V is generated by basis vectors:

$$\mathcal{N}_{\mathcal{V},\vec{\lambda}}(\phi) := e^{i\sum_j \lambda_j \phi(\vec{x_j})}$$

where λ_j are *non-zero* real numbers and $\vec{\lambda}$ denotes the set $\{\lambda_j\}$. Then define $\mathrm{Cyl} := \cup_V \mathrm{Cyl}_V$ and on Cyl define the inner product

$$\int d\mu(\phi)\mathcal{N}^*_{\mathcal{V},\lambda}(\phi)\mathcal{N}_{\mathcal{V},\lambda}(\phi) = \delta_{\mathcal{V},\mathcal{V}'}\delta_{\lambda,\lambda'} \tag{2.2}$$

The Cauchy completion of Cyl w.r.t this inner product gives the Hilbert Space H_{poly}: $\overline{\mathrm{Cyl}} =: H_{\mathrm{poly}}$. The basic operators here are $\hat{U}(\lambda, \vec{x})$ and $\hat{\pi}(\vec{x})$. The former acts as:

If $\vec{x}$ is not in $\{\vec{x_j}\}$

$$\hat{U}(\lambda, \vec{x})e^{i\sum_j \lambda_j \phi(\vec{x_j})} = e^{i\sum_j \lambda_j \phi(\vec{x_j}) + \lambda\phi(\vec{x})} \tag{2.3}$$

If $x = \vec{x_i} \in \{\vec{x_j}\}$ and $\lambda_j + \lambda \neq 0$

$$\hat{U}(\lambda, \vec{x})e^{i\sum_j \lambda_j \phi(\vec{x_j})} = e^{i\sum_j (\lambda_j + \lambda\delta_{\vec{x_j},\vec{x_i}})(\phi(\vec{x_j}))} \tag{2.4}$$

If $x = \vec{x_i} \in \{\vec{x_j}\}$ and $\lambda_j + \lambda = 0$

$$\hat{U}(\lambda, \vec{x})e^{i\sum_j \lambda_j \phi(\vec{x_j})} = e^{i\sum'_j \lambda_j (\phi(\vec{x_j})} \tag{2.5}$$

where the sum $\sum'_j$ excludes the vertex $\vec{x_i}$.

The action of the latter is given by

$$\hat{\pi}(\vec{x}) = \frac{\delta}{\delta\phi(\vec{x})} \tag{2.6}$$

Here we have described the representation where the field operator is not well defined. We may define an approximate field operator using a scale μ:

$$\phi_\mu(\vec{x}) = \frac{\hat{U}(\mu, \vec{x}) - \hat{U}(-\mu, \vec{x})}{2\mu i} \tag{2.7}$$

Notice that the choice of a μ and a basis vector gives a proper subspace of the Hilbert space on which the basic operators act invariantly.

Now we'll introduce a new notation which we believe helps underline the similarities between polymer and Schrodinger frameworks. This notation was introduced in [43].Henceforth we will consider space to be discretized into a (not necessarily regular) lattice. Let us consider a state $e^{i\sum_j \lambda_j \phi(\vec{x_j})}$. We can specify this state by specifying the vertex set V and the values of λ_i. But we could alternately specify the state by we defining a field $\pi((\vec{x})$ for all $\vec{x}$ in our discretized space such that

$$\pi(\vec{x_j}) = \lambda_j \qquad \text{if } \vec{x_j} \in V$$

$$\pi(\vec{x_j}) = 0 \qquad \text{otherwise}$$

Equivalently, $\pi(\vec{x_i})$ is non zero at finitely many points, with values λ_i. Then the same state may be written as

$$e^{i\sum_{\vec{y}} \pi(\vec{y})\phi(\vec{y})} =: |\{\pi(\vec{y})\}\rangle \leftrightarrow |\pi\rangle$$

where the sum is over all values of $\vec{y}$.

We note that the field π has no relation with the conjugate momentum at this stage. However the states $|\pi\rangle$ are eigenstates of the momentum operator $\hat{\pi}$, as we will now show. Now

the action of the basic operators maybe represented as

$$\hat{U}(\lambda, \vec{x})|\{\pi(\vec{y})\}\rangle = |\{\pi(\vec{y}) + \lambda\delta_{\vec{x},\vec{y}}\}\rangle \qquad \text{where } \vec{y} \in \text{support}(\pi) \tag{2.8}$$

and

$$\hat{\pi}(\vec{x})|\{\pi(\vec{y})\}\rangle = \pi(\vec{x})|\{\pi(\vec{y})\}\rangle \tag{2.9}$$

Note that 2.8 now incorporates all the cases 2.3, 2.4, 2.5. Also note that discretizing space has turned the functional derivative of 2.6 into ordinary partial derivative in 2.9.

In our notation, we can write the inner product as:

$$\langle\{\pi(\vec{y})\}|\{\pi'(\vec{y})\}\rangle = \prod_{\vec{y}} \delta_{\pi(\vec{y}),\pi'(\vec{y})} \tag{2.10}$$

To understand that this is the same as $\delta_{V,V'}\delta_{\lambda,\lambda'}$ note that this expression equals 1 only if all the values of $\pi(\vec{x_i})$ and $\pi'(\vec{x_i})$ agree, that is they should (i) both be non zero on the same points i.e vertex sets V and V' must coincide and (ii) The values of λ s must agree on this set. Else it vanishes.

We may write $|\{\pi(\vec{x})\}\rangle = \prod_{\vec{x}}|\pi(\vec{x})\rangle$ where $|\pi(\vec{x})\rangle$ is shorthand for $e^{i\pi(\vec{x})\phi(\vec{x})}$.

We notice that we may then write

$$\mathbb{1} = \prod\nolimits_{\vec{x}} \sum\nolimits_{\pi'(\vec{x})} |\pi'(\vec{x})\rangle\langle\pi'(\vec{x})| \tag{2.11}$$

We verify this using 2.10:

$$\left(\prod\nolimits_{\vec{x}} \sum\nolimits_{\pi'(\vec{x})} |\pi'(\vec{x})\rangle\langle\pi'(\vec{x})\right)|\pi\rangle = \prod\nolimits_{\vec{x}} \sum\nolimits_{\pi'(\vec{x})} \delta_{\pi(\vec{x}),\pi'(\vec{x})} e^{i\pi'(\vec{x})\phi(\vec{x})} = \prod_{\vec{x}} e^{i\pi(\vec{x})\phi(\vec{x})} = |\pi\rangle \tag{2.12}$$

This notation will prove convenient in Chapter 4. In the next two chapters we discuss symmetry realization in H_{poly} and path integral representation for transition amplitudes in

polymer quantized theories.

Chapter 3: Polymer Quantization and Rotational Symmetries

In this chapter we present our work on Polymer quantization and Rotational Symmetries, based on work done with G. Date [37]. This chapter is divided into three sections. In the first section, we present our investigations on Polymer Quantum Mechanics and rotational symmetries. We turn to the case of polymer quantized Scalar field theory in the second section. In the final section we provide a discussion of the results.

3.1 Polymer Quantum Mechanics and Rotational Symmetries

In this section we study the interplay between polymer Quantum Mechanics and the rotational symmetry. We'll see that the representation of the rotation group on the Polymer Hilbert Space is non-continuous. Finite rotations are perfectly well represented but infinitesimal generators are absent. Approximate generators may however be defined. These were studied in [38] by Dah Wei Chiou. Working in the more general context of Galilean symmetries, he established that deviations from usual Quantum predictions are small within the domain of validity of the non-relativistic model. In our thesis we start by exploring how far we can go *without* introducing approximations, that is without breaking the symmetries. We are asking if there is some sense in which the polymer representation

can support rotational symmetries, even if the generators are absent. And if so, what are the physical implications of such a realization of rotational symmetry? To answer this, we first explore the representation of rotational group on the polymer Hilbert Space.

Representation of Rotational Symmetries

We know that any group of symmetries is represented in a quantum theory by *unitary* operators[1], with the states transforming as $|\psi\rangle \to |\psi_g\rangle := U(g)|\psi\rangle$ and the operators transforming as, $A \to A_g := U(g)A\,U^\dagger(g)$ for each group element $g \in \mathcal{G}$. The specific unitary operators representing specific symmetry operation can be determined by *stipulating how the basic observables transform*. For example, with q^i, p_i being the basic observables in the usual quantization, the unitary operators corresponding to rotations are determined by:

$$q^i_\Lambda := U(\Lambda)q^i\,U(\Lambda)^\dagger = \Lambda^i{}_j q^j \quad , \quad p_i^\Lambda := U(\Lambda)p_i\,U(\Lambda)^\dagger = \Lambda^j{}_i p_j \quad , \quad \Lambda^i{}_m\Lambda^j{}_n\delta^{mn} = \delta^{ij} \tag{3.1}$$

For infinitesimal rotations, $\Lambda^i{}_j := \delta^i{}_j + \epsilon^i{}_j$, $U(\mathbb{1}+\epsilon) := \mathbb{1} - \frac{i}{\hbar}\epsilon\cdot\hat{J}$ we get,

$$-\frac{i}{\hbar}[\epsilon\cdot\hat{J}, q^i] \;=\; \epsilon^i{}_j q^j \quad , \quad -\frac{i}{\hbar}[\epsilon\cdot\hat{J}, p_i] \;=\; \epsilon^j{}_i p_j. \tag{3.2}$$

With the identifications $\epsilon^i{}_j := \epsilon_k\mathcal{E}^{ki}{}_j$, $\epsilon\cdot\hat{J} := \epsilon_k\hat{J}^k$, we deduce $\hat{J}^k := \mathcal{E}_m^{nk}q^m p_n$ as the operators representing the infinitesimal generators.

Alternatively, the operators $U(\Lambda)$ could also be determined by specifying their action on *wavefunctions* - explicit functions on the *configuration space* (say), eg. $\Psi_\Lambda(\vec{q}) := \Psi(\overrightarrow{\Lambda q})$.

[1] We will not be considering time reversal or charge conjugation symmetries, so we will not consider anti-unitary operators.

For the polymer quantization, the defining stipulations for the action of rotations are:

$$\left(e^{i\vec{k}\cdot\vec{q}}\right)_\Lambda := U(\Lambda)\left(e^{i\vec{k}\cdot\vec{q}}\right)U(\Lambda)^\dagger = \left(e^{ik_i\Lambda^i{}_j q^j}\right) \quad , \quad p_i^\Lambda := U(\Lambda)p_i\,U(\Lambda)^\dagger = \Lambda^j{}_i p_j \tag{3.3}$$

Noting that $|\hat{k}\rangle$ are eigenstates of $\hat{p}_i$, it follows,

$$\begin{aligned} U^\dagger(\Lambda)\hat{p}_i U(\Lambda)|\vec{k}\rangle &= (\Lambda^{-1})^j{}_i\hat{p}_j|\vec{k}\rangle = (\Lambda^{-1})^j{}_i k_j|\vec{k}\rangle \\ \therefore \hat{p}_i\left[U(\Lambda)|\vec{k}\rangle\right] &= \left[(\Lambda^{-1})^j{}_i k_j\right]\left[U(\Lambda)|\vec{k}\rangle\right] \\ \therefore U(\Lambda)|\vec{k}\rangle &= |(\Lambda^{-1})^j{}_i k_j\rangle \end{aligned} \tag{3.4}$$

Evidently, this action of rotation group on the polymer Hilbert space is *reducible*, with the orbit through any $\vec{k}$ being spanned by the orthonormal kets $\{|\vec{k}'\rangle\}$ with $\vec{k}'$ lying on the 2-sphere through $\vec{k}$. The *subspace* spanned by $\{|\vec{k}\rangle, \vec{k}\cdot\vec{k} = \text{constant}\}$, forms an *irreducible representation* and is clearly infinite dimensional.

This may come as a surprise as one recalls the theorem that all unitary, irreducible representations of the rotation group (indeed any compact group) are finite dimensional. However it is to be noted that the theorem is proved only for *continuous* representations of the group (which arise from and also induce, representations of the corresponding Lie algebra). It is also a theorem that *if G is a locally compact topological group whose every irreducible representation on a Hilbert space is continuous, then the group itself is discrete* [39]. Since the rotation group is locally compact and is not a discrete group, it must have discontinuous representations as well and these do not have to be finite dimensional. What we have is an explicit example of just such a representation whose discontinuous nature is shown below.

This action of the rotation group coupled, with the fact that the kets $|\vec{k}\rangle$ are orthonormalised, implies that $U(\Lambda)$ also *cannot* be weakly continuous. Unlike the case of the momentum-translation operator discussed in the previous chapter where the group action

necessarily transformed a basis vector to another basis vector, here we have the possibility that $\vec{k}$ could be along the axis of rotation represented by $U(\Lambda)$ and hence invariant under $U(\Lambda)$. To show discontinuity, consider any one parameter subgroup of rotations. All of these will leave some particular axis invariant. Choose any $\vec{k}$, orthogonal to this axis. Now the subgroup action transforms a basis ket to another distinct basis ket. The lack of weak continuity for every 1-parameter subgroup follows as before and we cannot write $U(\lambda) = \mathbb{1} - \frac{i}{\hbar}\epsilon \cdot \hat{J}$.

Thus we see here that in the polymer Hilbert Space the representation of rotational group is non-continuous. Infinitesimal generators of rotation do *not* exist as operators on this Hilbert Space, although finite rotations are perfectly well defined. What consequences, if any, does the discontinuity of the representation of rotational group have on dynamics? We explore this question in the next section.

Symmetries and Dynamics

For rotations to be a *symmetry*, their action must also preserve the dynamics. Classically, we have three 'elementary' rotational invariants: $\vec{p}\cdot\vec{p}$, $\vec{q}\cdot\vec{q}$ and $\vec{p}\cdot\vec{q}$. Only first of these can be promoted to operator on the polymer Hilbert space. A Hamiltonian which is a function of p^2 alone will describe only a 'free' dynamics. Is this the only possible rotationally invariant dynamics supported by the polymer Hilbert space? Not quite. As noticed in the context of the 'improved quantization' of LQC, exponentials of arbitrary functions of momenta, times q^i (i.e. functions *linear* in q^i) can also be promoted to well defined operators. This is because the Hamiltonian vector field X_{q^i} generates translations along p_i and any function of $\vec{p}$ multiplying X_{q^i} generates more general infinitesimal transformations, also along p_i. While X_{q^i} cannot be promoted to an operator, its exponential which generates *finite diffeomorphisms* can be! Incorporating rotational invariance, we can thus have unitary operators of the form $e^{\pm i f(p^2)\, p_i q^i}$. From these, the corresponding *sin* and *cos* self-adjoint operators can be defined. A candidate rotationally invariant Hamiltonian will

be a function of p^2 and the sin, cos operators. There is no corresponding trick to use the $q \cdot q$ invariant.

To compute the action of finite diffeomorphism, say by unit parameter, consider the integral curves defined by,

$$\begin{aligned} \frac{dp_i}{d\lambda} &= f(p\cdot p)\, p_i \quad \Rightarrow \quad \frac{dp\cdot p}{d\lambda} = 2(p\cdot p) f(p\cdot p) \\ \int_0^1 d\lambda &= \frac{1}{2}\int_{p^2_{\text{initial}}}^{p^2_{\text{final}}} \frac{dp^2}{p^2 f(p^2)} \end{aligned} \tag{3.5}$$

This defines the change in the $p \cdot p$ for unit change in the parameter. Notice that the vector field is *radial*, and therefore the integral curves are in the radial direction (in 'p'-space) and for unit change in the parameter, connect two spheres of radii p^2_{initial} and $p^2_{\text{final}} := \xi^2 p^2_{\text{initial}}$. The corresponding unitary operator is then defined by,

$$\widehat{e^{-if(p^2)p\cdot q}}|\vec{k}\rangle := |\vec{k}' = \xi\vec{k}\rangle \,, \tag{3.6}$$

the scale ξ being determined by eq.(3.5).

Thus, we *can* have non-trivial rotationally invariant dynamics. However, there is now a different problem. As noted before, the unitary representation of SO(3) on the polymer Hilbert space is reducible with irreducible representations carried by $\mathcal{H}_\sigma := \text{span}\{|\vec{k}\rangle\,, k\cdot k = \sigma^2 > 0\}$. Each of these is infinite dimensional. Each eigenspace of any invariant Hamiltonian will carry a representation of SO(3) which has to be infinite dimensional, being made up of some of the irreducible representations together possibly with the trivial representation ($\sigma = 0$). Thus we face the problem of *infinite degeneracy* which is physically untenable: the partition function of such a system will be undefined. We are led to the conclusion that in any physically acceptable model, rotational symmetry must be broken.

Breaking Symmetries

We have now two possibilities: (a) rotations cease to be a symmetry (explicit breaking of symmetry) or (b) *spontaneous breaking* of rotational symmetry.

To see both possibilities, we first seek an approximate substitute for the position operators. The operators, $e^{i\vec{k}\cdot\vec{q}}$, allow us to define families of self-adjoint operators. For instance, choosing $\vec{k}_j := \delta\hat{e}_j$, $\hat{e}_j$ a unit vector, we can define $\sin_{\delta\hat{e}_j} := (2i)^{-1}(e^{i\delta\hat{e}_j\cdot\vec{q}} - e^{-i\delta\hat{e}_j\cdot\vec{q}})$ and a cos operator analogously[2]. We could choose several triplets of linearly independent unit vectors $\hat{e}_j$ and also choose many different parameters δ's (equivalently, finitely many $\vec{k}_j$). If we collect finitely many of such sets and restrict ourselves to observables which are functions of these (and the momentum) operators, then from any given $|\vec{k}_0\rangle$, we will generate a collection of basis vectors, $\{|\vec{k}_0 + \sum_j n_j\vec{k}_j\rangle, n_j \in \mathbb{Z}\}$. The closed subspace generated by this set will be a *proper subspace* of the polymer Hilbert space and is clearly *separable*. If we also include operators which are exponentials in $p \cdot q$, discussed above, then the lattice generated will also involve scaling determined by the choices for $f(p^2)$. As long as the number of such operators is finite, we will continue to have separable sectors. The chosen set of observables, will act invariantly on each such subspace and will provide *superselection sectors*. Observe that among the chosen class of observables, we can also have an invariant Hamiltonian. Action of rotations however mixes different sectors and we have *spontaneous breaking of rotational invariance*. If we chose a Hamiltonian involving the approximated position operators, we have *explicit breaking* of rotations controlled by the δ-parameter(s). The example of spherically symmetric harmonic oscillator in three dimensions illustrates this. For a economical parametrization of violation, we can choose a single common δ. For sufficiently small values of this, at a certain level of observational precision, it is of course possible to have the illusion of rotational invariance.

To summarise, having made a choice of the polymer Hilbert space $H_{\rm poly}$, we *can* have

[2]These operators however do not suffice to represent the Lie algebra of rotations [38].

exact rotational symmetry with a some what restricted form of dynamics (no $q \cdot q$ dependence) but with uncountably infinite degeneracy. To avoid the problem of infinite degeneracy, the symmetry must be broken - either explicitly *or* spontaneously. By introducing separable sectors we can see both possibilities.

It is however possible to define symmetry operators if we choose to work in Cyl*. Using the triple, we can try to select a suitable *subspace* of Cyl* on which infinitesimal generators can be defined. With a suitable choice of a new inner product, we can obtain a 'physical' Hilbert space with a rotationally invariant dynamics. This would imply a view of polymer quantization as an intermediate step in a multi-step quantization procedure. We explore this avenue in the next section.

Regaining Symmetries

The possibility of looking to Cyl^* for a home to a suitable quantum theory is inspired by analogous steps taken in the context of LQG. In LQG, the step is motivated for a very different reason. The kinematical quantization is essentially forced upon us by the demand of SU(2) invariance and diffeomorphism covariance. Since there are constraints whose kernels are in general distributional, an appropriate diffeo-invariant subspace of corresponding Cyl^* is a natural arena. In our case, the polymer quantization itself is not a compulsion, but is a useful illustration of a *multi-step construction of a quantum theory.*

Recall that construction of $\mathrm{H}_{\mathrm{poly}}$ naturally gave us the triple $\mathrm{Cyl} \subset \mathrm{H}_{\mathrm{poly}} \subset \mathrm{Cyl}^*$. This structure provides us with a convenient representation of the elements $(\Psi|$ of Cyl^* by complex valued linear functions $\psi^*(\vec{\mathrm{k}}) := (\Psi|\vec{\mathrm{k}}\rangle$. No smoothness properties are assumed at this stage for these functions. Furthermore, for every operator $\mathrm{A} : \mathrm{Cyl} \to \mathrm{Cyl}$, we can define an operator $\tilde{\mathrm{A}} : \mathrm{Cyl}^* \to \mathrm{Cyl}^*$ by the 'dual action', eg.

$$(\tilde{\mathrm{A}}\Psi|\mathrm{f}\rangle := (\Psi| \dagger \mathrm{Af}\rangle, \quad \forall\, |\mathrm{f}\rangle \in \mathrm{Cyl}, \quad \forall\, (\Psi| \in \mathrm{Cyl}^*$$

Conversely, given an operator $\tilde{\mathrm{A}}$ defined on *all* of Cyl*, we can define an operator A on Cyl by the same equation as above (read backwards). In particular this means that we have the operators $\tilde{\mathrm{U}}(\Lambda)$ defined on Cyl*. We will use these to define infinitesimal generators on Cyl*. We will also define the position operators.

We begin with infinitesimal rotation generators. Let us consider a rotation along the x^l axis.

$$\begin{aligned}
(\Psi|\mathrm{U}(\mathbb{1}+\epsilon)-\mathrm{U}(\mathbb{1}-\epsilon)|\vec{\mathrm{k}}\rangle &= (\Psi|\vec{\mathrm{k}}-\overrightarrow{\epsilon\mathrm{k}}\rangle-(\Psi|\vec{k}+\overrightarrow{\epsilon\mathrm{k}}\rangle \\
&\approx 2\epsilon_{\mathrm{l}}\mathcal{E}^{\mathrm{li}}{}_{\mathrm{j}}\mathrm{k}^{\mathrm{j}}\frac{\partial\psi^*}{\partial\mathrm{k}^{\mathrm{i}}} \\
\therefore \lim_{\epsilon_{\mathrm{l}}\to 0}(\Psi|\frac{\mathrm{U}(\mathbb{1}-\epsilon)-\mathrm{U}(\mathbb{1}+\epsilon)}{2\epsilon_{\mathrm{l}}}|\vec{\mathrm{k}}\rangle &= -\mathcal{E}^{\mathrm{li}}{}_{\mathrm{j}}\mathrm{k}^{\mathrm{j}}\frac{\partial\psi^*}{\partial\mathrm{k}^{\mathrm{i}}} \\
\therefore (\mathrm{J}^{\mathrm{l}}\Psi|\vec{\mathrm{k}}\rangle &:= -\mathrm{i}\hbar\mathcal{E}^{\mathrm{li}}{}_{\mathrm{j}}\,\mathrm{k}^{\mathrm{j}}\,\frac{\partial\psi^*}{\partial\mathrm{k}^{\mathrm{i}}}
\end{aligned} \tag{3.7}$$

Notice that these operators are defined only on a *subspace* of Cyl*, consisting of those $(\Psi|$ whose corresponding $\psi^*(\vec{\mathrm{k}})$ are *differentiable functions*. Hence, by dual action we *cannot* define the corresponding operators on Cyl.

Next, recall the $\mathrm{sin}_{\delta\hat{\mathrm{e}}_{\mathrm{j}}}$ operators defined in the previous section. For each orthonormal triad, $\hat{e}_j,\ j=1,2,3,\ \hat{e}_i\cdot\hat{e}_j=\delta_{ij}$ and a small parameter δ, we have, $U_{\delta\hat{e}_j}(\vec{q}) := e^{i\delta\hat{e}_j\cdot\vec{q}}$ and $\mathrm{sin}_{\delta\hat{e}_j} := (2i)^{-1}(U_{\delta\hat{e}_j}(\vec{q})-U_{-\delta\hat{e}_j}(\vec{q}))$. Now,

$$\begin{aligned}
(\Psi|\mathrm{sin}_{\delta\hat{\mathrm{e}}_{\mathrm{j}}}|\vec{\mathrm{k}}\rangle &= -\frac{(\Psi|\vec{\mathrm{k}}-\delta\hat{\mathrm{e}}_{\mathrm{j}}\rangle-(\Psi|\vec{\mathrm{k}}+\delta\hat{\mathrm{e}}_{\mathrm{j}}\rangle}{2i} \\
&= \frac{\psi^*(\vec{\mathrm{k}}+\delta\hat{\mathrm{e}}_{\mathrm{j}})-\psi^*(\vec{\mathrm{k}}-\delta\hat{\mathrm{e}}_{\mathrm{j}})}{2i} \\
&\approx \frac{2\delta\hat{e}_j}{2i}\frac{\partial\psi^*}{\partial\vec{k}} \\
\therefore \lim_{\delta\to 0}(\Psi|\frac{\mathrm{sin}_{\delta\hat{\mathrm{e}}_{\mathrm{j}}}}{\delta}|\vec{\mathrm{k}}\rangle &= -\mathrm{i}\hat{\mathrm{e}}_{\mathrm{j}}\cdot\vec{\nabla}_{\vec{\mathrm{k}}}\psi^*
\end{aligned} \tag{3.8}$$

Thus, by restricting to functions ψ^* which are at least differentiable, we *can* define a

position operator on a *subspace* of Cyl* via the dual action,

$$(\hat{e}_j \cdot \vec{q}\, \Psi|\vec{k}\rangle \;:=\; -i\hat{e}_j \cdot \vec{\nabla}_{\vec{k}}\psi^* \;\;,\;\; \forall\, (\Psi| \in \mathrm{Cyl}^* \;\text{ such that } \psi^*(\vec{k}) \text{ is differentiable .} \qquad (3.9)$$

It is easy to see that the position operators defined above and the momentum operators defined by dual action, also satisfy,

$$(\,[\hat{e}_m \cdot \vec{q}\,,\; \hat{e}_n \cdot \vec{p}]\; \Psi|\vec{k}\rangle \;=\; (\,\{i\hbar\hat{e}_m \cdot \hat{e}_n\}\; \Psi|\vec{k}\rangle\;.$$

So far we have not specified any subspace of Cyl* except to say that it should consists of, at least, differentiable functions. The space of all differentiable functions is too large a subspace to choose. We are guided in our choice of a subspace by the requirement that the 'position' and the 'momentum' operators be self-adjoint with respect to a suitable inner-product and satisfy the canonical commutation relation on an invariant, common dense domain. Representations of the canonical commutation relations are usually analyzed by going to the bounded, unitary operators (exponentials of the positions and momenta) , satisfying the Weyl-Heisenberg relations. The Stone-von Neumann theorem then guarantees a unique continuous representation of the Weyl-Heisenberg relations and the corresponding canonical commutation relations. This representation corresponds to the choice of Schwartz space as the subspace of Cyl* and the usual inner product with the Lebesgue measure. The Hilbert space is then obtained by completing Schwartz space in the L_2 norm. Making this choice, we just get back the usual Schrodinger quantization using functions of "momenta", $\vec{k}$ instead of functions of "positions", $\vec{q}$. The intermediate polymer quantization has only led us to the Heisenberg (or momentum) representation instead of the Schrodinger (or position) representation. The measure being invariant under rotations also admits (unitary) representation of infinitesimal rotations. It is interesting to note that one can also choose a subspace which is *larger*, eg space of $\psi^*(\vec{k})$ which are normalizable with respect to a Sobolev norm, and choose the Lorentz invariant measure

$\frac{d^3k}{2\sqrt{\vec{k}\cdot\vec{k}+m^2}}$, to construct Hilbert space of a *free, relativistic particle of mass m*[40]. This is not our primary concern though.

This is obviously a roundabout way of arriving at the usual quantization. But it shows that (a) not every choice of quantization may be flexible enough for physical modeling and (b) we can reach a satisfactory quantum theory by modifying the quantization algorithm. In principle, if the quantum theory constructed from a subspace of Cyl* were *not* satisfactory, we could repeat the process forming a new triple. This is further discussed in the last section. In the next section we discuss the case of a scalar field theory.

3.2 Polymer Scalar Field Theory and Symmetries

Can rotational invariance be supported in a 'polymerised scalar field theory'? Consider the example of a scalar field $\phi(\vec{x})$ defined on $\mathbb{R}^3$. The rotations act on the space which in turn induces a transformation on the field: $\phi'(\vec{x}) = \phi(\overrightarrow{\Lambda^{-1}x})$.

Let us recall the polymer quantization of scalar fields as defined in the previous chapter. Define a vertex set $V = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)$, of finitely many, *distinct* points. For *non-zero* real numbers $\lambda_j, j = 1, \ldots, n$, define the functions $\mathcal{N}_{\mathcal{V},\vec{\lambda}}(\phi) := \rceil^{\rangle \sum_| \lambda_| \phi(\vec{\S}_|)}$. For each fixed set V, let Cyl_V denote the set of finite, complex linear combinations of these functions. Let $\mathrm{Cyl} := \cup_V \mathrm{Cyl}_V$. Thus every element of Cyl is a function of ϕ which is a *finite* linear combination of functions $\mathcal{N}_{\mathcal{V},\vec{\lambda}}$ for *some* vertex set V and *some* choice of $\vec{\lambda}$. That is, any element of Cyl may be written as

$$\psi(\phi) = \sum_{V,\vec{\lambda}} C_{V,\vec{\lambda}} \mathcal{N}_{\mathcal{V},\vec{\lambda}}(\phi)$$

where $C_{V,\vec{\lambda}} \in \mathbb{C}$ The inner product had been defined in 2.2. This is given by

$$\langle\psi|\psi'\rangle \;:=\; \int d\mu(\phi)\psi^*(\phi)\psi'(\phi) = \int d\mu(\phi) \sum_{V,\vec{\lambda},V',\vec{\lambda}'} C^*_{V,\vec{\lambda}} C'_{V',\vec{\lambda}'} e^{i\sum_k \lambda'_k \phi(\vec{x}'_k) - i\sum_j \lambda_j \phi(\vec{x}_j)} \tag{3.10}$$

Observe that each term in the summand is again of the form $\mathcal{N}_{V\cup V',\vec{\lambda},\vec{\lambda}'}$, except that all vertices in the union $V \cup V'$ are not necessarily distinct. If $\vec{x}'_k = \vec{x}_j$, then the exponent would be $(\lambda'_k - \lambda_j)\phi(\vec{x}_j)$. If the $\lambda' s$ are equal, then the exponent is identically zero and the integral contributes to the sum. Otherwise, the integral gives zero. It follows that $\mathcal{N}_{V,\vec{\lambda}}$ and $\mathcal{N}_{V',\vec{\lambda}'}$ are orthogonal unless the two sets of vertices coincide *and* their corresponding λ's are equal.

The Hilbert space $\mathrm{H}_{\mathrm{poly}}$, is obtained as the Cauchy completion of Cyl with respect to this inner product. The functions $\mathcal{N}_{V,\vec{\lambda}}(\phi)$, with *every* $\lambda \neq 0$, form an orthonormal basis for the polymer Hilbert space. The constant function corresponding to empty vertex set, $\mathcal{N}(\phi) = 1$, is also included in the basis.

Action of rotations on Cyl is defined by $[U_\Lambda\psi](\phi) := \psi(\Lambda^{-1}\phi)$. Evaluating it on the elementary functions lead to,

$$\mathcal{N}_{V,\vec{\lambda}}(\phi) \;\to\; \mathcal{N}'_{V',\vec{\lambda}'}(\phi) \;:=\; \mathcal{N}_{V,\vec{\lambda}}(\phi') \;=\; \mathcal{N}_{V',\vec{\lambda}}(\phi) \tag{3.11}$$

The middle equality is the definition of the action, $\phi' = \Lambda \circ \phi$ and we have used the scalar nature of ϕ, $\phi'(\vec{x}) = \phi(\overrightarrow{\Lambda x})$, in the last equality.

Observe that under the action of rotation Λ, a vertex set $V = (\vec{x}_1, \ldots, \vec{x}_n)$ changes to a new vertex set $V' := (\overrightarrow{\Lambda x_1}, \ldots, \overrightarrow{\Lambda x_n})$. The $\lambda' s$ are unchanged and the field is evaluated at the transformed points. Now the inner product among two elementary functions depends only on their respective λ s. Therefore the inner product among elementary functions is invariant under the action of the rotations and rotations are represented unitarily on the Hilbert space.

That this unitary action is also non-weakly-continuous can be seen easily. For non-trivial rotation, a diagonal matrix element between basis states is zero while for the identity rotation, the matrix element is 1. Thus, infinitesimal generators have no representation on the polymer Hilbert space.

The momenta variables are defined as,

$$P_g := \int d^3x g(\vec{x})\pi_\phi(x) \;=\; -i\hbar \int d^3x g(\vec{x}) \frac{\delta}{\delta\phi(\vec{x})} \tag{3.12}$$

Here $g(\vec{x})$ is a 'suitably smooth' function (π_ϕ has density weight 1, though not relevant here). It is easy to see that,

$$P_g \mathcal{N}_{V,\vec{\lambda}} \;=\; \left[\hbar \sum_j \lambda_j g(\vec{x}_j)\right] \mathcal{N}_{V,\vec{\lambda}} \;,\quad \left[P_f, P_g\right] = 0 \;,\quad P_f^\dagger \;=\; P_f \;. \tag{3.13}$$

Thus the momentum representation exists and the elementary functions $\mathcal{N}_{V,\vec{\lambda}}$ are simultaneous eigenstates of the momenta variables P_g. Under the action of rotation, $U(\Lambda)$, the momentum variables transform as,

$$\begin{aligned} U_\Lambda P_g(\pi) U_\Lambda^\dagger \;&:=\; P_g(\Lambda \circ \pi) \\ &=\; P_{\Lambda^{-1}\circ g}(\pi) \quad \text{(from the definition)} \quad \Rightarrow \\ U_\Lambda P_g(\pi) \;&=\; P_{\Lambda^{-1}\circ g}(\pi) U_\Lambda \end{aligned} \tag{3.14}$$

This is consistent with (3.11). Let us use the notation $|V, \vec{\lambda}\rangle \leftrightarrow \mathcal{N}_{V,\vec{\lambda}}(\phi)$.

Observe that $e^{i\lambda\phi(\vec{x})}$, a '*point holonomy operator*', clearly acts as a multiplication operator:

$$e^{i\lambda\phi(\vec{x})}|V,\vec{\lambda}\rangle \;:=\; \begin{cases} |\vec{x}_1,\ldots,\vec{x}_n,\vec{x}\,;\lambda_1,\ldots,\lambda_n,\lambda\rangle & \text{if } \vec{x} \neq \vec{x}_i \text{ for any i} \\ |\vec{x}_1,\ldots,\vec{x}_k\ldots\vec{x}_n\,;\lambda_1,\ldots,\lambda_k+\lambda,\ldots\lambda_n\rangle & \text{if } \vec{x}=\vec{x}_k\,,\ \lambda+\lambda_k\neq 0 \\ |\vec{x}_1,\ldots,,\ldots\vec{x}_n\,;\lambda_1,\ldots,,\ldots\lambda_n\rangle & \text{if } \vec{x}=\vec{x}_k\,,\ \lambda+\lambda_k=0 \end{cases}$$

In the last equation, the $\vec{x}_k, \lambda_k$ labels are missing on the right hand side.

What about the scalar field operator itself? It does *not* exist since the point holonomy operators are not weakly continuous, exactly as in the point particle case. In the usual Schrodinger type representation too, a scalar field operator exists *only* as an operator valued distribution. This has to do with the presence of Dirac delta in the canonical commutation relations. In the polymer representation it does not exist even as an operator valued distribution.

Now consider an element $(\Psi| \in \mathrm{Cyl}^*$. Its action on an elementary function $\mathcal{N}_{V,\vec{\lambda}}(\phi)$ is given by,

$$(\Psi|V,\vec{\lambda}\rangle =: \psi^*(\vec{x}_1,\cdots,\vec{x}_n,\lambda_1,\cdots,\lambda_n) \ , \ \text{distinct } \vec{x}\,'s \text{ and non-zero } \lambda' s.$$

Under the action of rotations, the arguments of the elementary function change: $|V,\vec{\lambda}\rangle \to |V',\vec{\lambda}\rangle$. Thus, if we choose the functions ψ^*'s to be differentiable, we can define infinitesimal rotations as operators on a subspace of Cyl^*, exactly as before. Explicitly,

$$\begin{aligned}
(\Psi|\mathrm{U}(\mathbb{1}+\epsilon)-\mathrm{U}(\mathbb{1}-\epsilon)|\mathrm{V},\vec{\lambda}\rangle &= (\Psi|\mathrm{V}'_+,\vec{\lambda}\rangle - (\Psi|\mathrm{V}'_-,\vec{\lambda}\rangle \\
&= \psi^*(\vec{\mathrm{x}}_1+\overrightarrow{\epsilon x_1},\cdots,\vec{\mathrm{x}}_\mathrm{n}+\overrightarrow{\epsilon x_n},\vec{\lambda}) - \\
&\qquad \psi^*(\vec{\mathrm{x}}_1-\overrightarrow{\epsilon x_1},\cdots,\vec{\mathrm{x}}_\mathrm{n}-\overrightarrow{\epsilon x_n},\vec{\lambda}) \\
&\approx 2\epsilon_\mathrm{k}\mathcal{E}^{\mathrm{ki}}{}_\mathrm{j}\sum_{\mathrm{m}=1}^{\mathrm{n}}\mathrm{x}^\mathrm{j}_\mathrm{m}\frac{\partial\psi^*}{\partial\mathrm{x}^\mathrm{i}_\mathrm{m}} \\
\therefore \lim_{\epsilon_\mathrm{k}\to 0}(\Psi|\frac{\mathrm{U}(\mathbb{1}+\epsilon)-\mathrm{U}(\mathbb{1}-\epsilon)}{2\epsilon_\mathrm{k}}|\mathrm{V},\vec{\lambda}\rangle &= \mathcal{E}^{\mathrm{ki}}{}_\mathrm{j}\sum_{\mathrm{m}=1}^{\mathrm{n}}\mathrm{x}^\mathrm{j}_\mathrm{m}\frac{\partial\psi^*}{\partial\mathrm{x}^\mathrm{i}_\mathrm{m}} \\
\therefore (J^k\Psi|V,\vec{\lambda}\rangle &:= -\mathrm{i}\hbar\,\mathcal{E}^{\mathrm{ki}}{}_\mathrm{j}\sum_{\mathrm{m}=1}^{\mathrm{n}}\mathrm{x}^\mathrm{j}_\mathrm{m}\frac{\partial\psi^*}{\partial\mathrm{x}^\mathrm{i}_\mathrm{m}}
\end{aligned} \tag{3.15}$$

Thus, by restricting to a subspace of Cyl^*, corresponding functions $\psi^*(V,\vec{\lambda})$ suitably differentiable w.r.t x, we can define the generator of the infinitesimal rotations.

Likewise, to define a smeared operator scalar field on Cyl^*, consider,

$$(\Psi|\phi^\delta_f|V,\vec{\lambda}\rangle := \int d^3x f(\vec{x})(\Psi|\frac{e^{i\delta\phi(\vec{x})}-e^{-i\delta\phi(\vec{x})}}{2i\delta}|V,\vec{\lambda}\rangle \tag{3.16}$$

$$= \int d^3x \frac{f(\vec{x})}{2i\delta} \Big((\Psi|V, \vec{x}, \vec{\lambda}, \delta\rangle - (\Psi|V, \vec{x}, \vec{\lambda}, -\delta\rangle \Big)$$

For a generic $\vec{x}$, assuming differentiability of ψ^*, we will get a function of the vertices of V and the corresponding $\lambda's$ together with the additional point $\vec{x}$ and the corresponding 'δ' = 0. This function cannot come from any element of Cyl* acting on $|V, \vec{\lambda}\rangle$. Hence we should avoid getting a contribution from a generic $\vec{x}$. If however, $\vec{x}$ coincides with one of the vertices in V, then the resultant function (derivative) *is* a function of $(V, \vec{\lambda})$ and we can interpret the right hand side as a new element of Cyl* evaluated on the basis element $|V, \vec{\lambda}\rangle$. This can be made more precise by employing the commonly used procedure of defining the integral by introducing a cell decomposition adapted to the 'graph' (vertices of V) and *demanding that*

$$\frac{\partial \psi^*}{\partial \lambda_j}(\vec{x}_1, \ldots, \vec{x}_n, \lambda_1, \ldots, \lambda_j, \ldots, \lambda_n)\Big|_{\lambda_j=0} = 0 \ \ , \forall \ j = 1, 2, \ldots, n \ . \tag{3.17}$$

This condition ensures that there is no contribution from cells that do not contain a vertex of V and we are led to the definition:

$$(\widetilde{\phi_f}\Psi|V, \vec{\lambda}\rangle \ := \ \lim_{\delta\to 0}(\Psi|\phi_f^\delta|V, \vec{\lambda}\rangle \ := \ -i \sum_j f(\vec{x}_j) \frac{\partial \psi^*(\vec{x}_1, \ldots, \vec{x}_n, \lambda_1, \ldots, \lambda_n)}{\partial \lambda_j} \ . \tag{3.18}$$

It is now easy to verify that

$$\begin{aligned} (\widetilde{[\phi_f, P_g]}\Psi|V, \vec{\lambda}\rangle \ &:= \ (\widetilde{P_g}\widetilde{\phi_f}\Psi|V, \vec{\lambda}\rangle - (\widetilde{\phi_f}\widetilde{P_g}\Psi|V, \vec{\lambda}\rangle \\ &= \ -i\hbar \left(\sum_{j=1}^n f(\vec{x}_j) g(\vec{x}_j) \right) (\Psi|V, \vec{\lambda}\rangle \\ &= \ (\left\{ +i\hbar \left(\sum_{j=1}^n f(\vec{x}_j) g(\vec{x}_j) \right) \right\} \Psi|V, \vec{\lambda}\rangle \ . \end{aligned} \tag{3.19}$$

We have thus succeeded in defining the smeared versions of the field operators ϕ_f, P_g in a subspace of Cyl*.

We can also verify that the infinitesimal generators J^k induce expected actions on the smeared fields operators.

$$\begin{aligned}
(\widetilde{[J^k, \phi_f]}\Psi|V,\vec{\lambda}\rangle &= (\widetilde{\phi_f}\widetilde{J_k}\Psi|V,\vec{\lambda}\rangle - (\widetilde{J_k}\widetilde{\phi_f}\Psi|V,\vec{\lambda}\rangle \\
&= -i\sum_{m=1}^{N} f(\vec{x}_m)\frac{\partial\psi^*_{J_k}(V,\vec{\lambda})}{\partial\lambda_m} - (-i\hbar)\mathcal{E}^{ki}{}_j\sum_{n=1}^{N} x_n^j\frac{\partial\psi^*_{\phi_f}(V,\vec{\lambda})}{\partial x_n^i} \\
&= -\hbar\sum_{m,n=1}^{N} f(\vec{x}_m)\mathcal{E}^{ki}{}_j x_n^j\frac{\partial\psi^*(V,\vec{\lambda})}{\partial x_n^i} \\
&\qquad + \hbar\mathcal{E}^{ki}{}_j\sum_{m,n=1}^{N} x_n^j\frac{\partial}{\partial x_n^i}\left\{f(\vec{x}_m)\frac{\partial\psi^*(V,\vec{\lambda})}{\partial\lambda_m}\right\} \\
&= i\hbar\left[-i\sum_{n=1}^{N}\left(\mathcal{E}^{ki}{}_j x_n^j\frac{\partial f(\vec{x}_n)}{\partial x_n^i}\right)\cdot\frac{\partial\psi^*(V,\vec{\lambda})}{\partial\lambda_n}\right] \\
&= i\hbar\,(\widetilde{\phi_{\mathcal{L}_k f}}\Psi|V,\vec{\lambda}\rangle \qquad , \qquad \mathcal{L}_k f(\vec{x}) := \mathcal{E}^{ki}{}_j x^j\frac{\partial f}{\partial x^i}
\end{aligned} \tag{3.20}$$

Similar computation can be done for commutator of $[J^k, P_g]$.

We have now identified the *minimal conditions*, namely differentiability in all arguments and the condition of equation (3.17), on functions $\psi^*(\vec{x}_1,\ldots,\vec{x}_n,\lambda_1,\ldots,\lambda_n)$ in order that the smeared field operators and the infinitesimal rotation actions are well defined. Since such an element of Cyl^* can be viewed as a sequence of *differentiable*, complex functions defined on $(\mathbb{R}^{3n} - \text{diagonal}) \times ((\mathcal{R}-0)^n)$ where diagonal is the subset of $\mathbb{R}^{3n}$ with two or more points coinciding, we are restricted to a subspace of Cyl^*.

The next step is to choose a suitable inner product on this subspace, possibly restricted further with additional conditions. Let us denote such a subspace by Cyl_1. Here we initiate first steps. For notational simplicity, let us denote elements of Cyl^* generically by underlined letters such as as $\underline{\Psi}, \underline{\Phi}, \underline{[V,\vec{\lambda}]}, \ldots$ etc.

Heuristically, we can represent each element of Cyl_1 and a yet to be defined inner product

as,

$$\underline{\Psi} := \sum_{V,\vec{\lambda}} \psi^*(V,\vec{\lambda})\underline{[V,\vec{\lambda}]}\,, \tag{3.21}$$

$$\begin{aligned}\langle\underline{\Psi},\underline{\Phi}\rangle &:= \sum_{V,\vec{\lambda}}\sum_{V',\vec{\lambda}'} \psi(V,\vec{\lambda})\phi^*(V',\vec{\lambda}')\langle\underline{[V,\vec{\lambda}]},\underline{[V',\vec{\lambda}']}\rangle \\ &:= \sum_{V,\vec{\lambda}}\sum_{V',\vec{\lambda}'} \psi(V,\vec{\lambda})\phi^*(V',\vec{\lambda}')\mathcal{G}(V,\vec{\lambda};V',\vec{\lambda}')\end{aligned} \tag{3.22}$$

The coefficients $\psi^*(V,\vec{\lambda})$ in the first line, contain the information about the subspace, Cyl_1. The $\mathcal{G}$ denotes the inner product between 'basis' elements.

For example, Cyl is a subspace of Cyl* through the natural embedding $|V,\vec{\lambda}\rangle \in \mathrm{Cyl} \to \underline{[V,\vec{\lambda}]} \in \mathrm{Cyl}^*$. If Cyl_1 were to be this subspace, then the $\psi^*(V,\vec{\lambda})$ in eqn.(3.21) would be non-zero only for finitely many $(V,\vec{\lambda})$ sets and $\mathcal{G}(V,\vec{\lambda};V',\vec{\lambda}')$ would equal $\delta_{V,V'}\delta_{\vec{\lambda},\vec{\lambda}'}$. The double summation would then collapse to a *finite* sum over $(V,\vec{\lambda})$ (compare eqn.(3.10)). Likewise, if Cyl_1 were to echo the Hilbert space of the r-Fock construction [**?**, 36, 20], the $\mathcal{G}(V,\vec{\lambda};V',\vec{\lambda}')$ would be $\sim \exp[-\frac{1}{4}\sum_{ij} G_{ij}(\vec{x}_i,\vec{x}_j)\lambda^i\lambda^j]$, where the sum over (i,j) is over the vertices of $V \cup V'$ and we use the notation of [20]. The double sum will be a finite sum since $\psi^*(V,\vec{\lambda})$ is non-zero for finitely many $(V,\vec{\lambda})$ sets.

More generally, we could have uncountably many non-zero $\psi^*(V,\vec{\lambda})$ and then each $\underline{\Psi}$ can be thought of as a potentially infinite sequence of functions, ψ_n, on $\sim \mathbb{R}^{4n}$. If we choose an inner product so that the 'basis states' are orthonormal ($\mathcal{G} \propto \delta_{V,V'}\delta_{\vec{\lambda},\vec{\lambda}'}$), then we may write the inner product as,

$$\begin{aligned}\langle\underline{\Psi},\underline{\Phi}\rangle &:= \sum_{V,\vec{\lambda}} \psi(V,\vec{\lambda})\phi^*(V',\vec{\lambda}') \\ &:\simeq \sum_{n=0}^{\infty}\int_{\mathbb{R}^{3n}} d^{3n}x \int_{\mathbb{R}^n} d^n\lambda\, \psi(\vec{x}_1,\ldots,\vec{x}_n,\lambda_1,\ldots,\lambda_n)\phi^*(\vec{x}_1,\ldots,\vec{x}_n,\lambda_1,\ldots,\lambda_n)\end{aligned} \tag{3.23}$$

The $:\sim$ indicates that the integration measures need to be defined and we need to put conditions to ensure convergence of the sum.

Assuming that we can choose suitable weights in the sum and measures in the integrations, what further conditions we need to put on the ψ_n's so that our basic operators and generators are self-adjoint? It is easy to see that we need only the usual fall-off conditions on these so that surface terms resulting from the partial integrations drop out. Roughly, we make each member ψ_n as an element of $L_2(\mathbb{R}^{4n})$. This indicates that it is, at least heuristically, *conceivable* to choose suitable definitions to construct a new Hilbert space.

3.3 Summary and Discussion

We began by exploring symmetries and their violations in polymer quantized systems. Specifically, we focused on three dimensional rotations and explored the polymer quantized particle in three dimensions and a scalar field defined on $\mathbb{R}^3$. It is certainly possible to have a unitary representation of SO(3) on the polymer Hilbert space but the representation is discontinuous and consequently does not admit representation of its Lie algebra. The non-availability of configuration space operators - position operators - severely restricts the possible invariant Hamiltonians and *every one of these* has infinitely degenerate eigenvalues. In effect, physically acceptable dynamics on polymer Hilbert space *must necessarily violate rotational symmetry, either explicitly or spontaneously*. In case of explicit breaking, one can then look for economical parametrization of symmetry violations and put bounds on the parameters. As noted in the introduction, this route has already been followed in [29, 38, 41, 42]. We explored another route to see if acceptable quantization, *with infinitesimal symmetries*, can be arrived at viewing polymer quantization as an intermediate step. This was done by looking for suitable subspace(s) of the dual member of the Gelfand triple with a hope of defining a new inner product and a new Hilbert space. For the point particle case we verified that it is possible to construct a new Hilbert space which carries continuous representations of the rotation group as well as continuous representation of the Heisenberg group. By the Stone-von Neumann theorem, this is of course the usual Schrodinger representation which supports the usual rotationally in-

variant non-trivial Hamiltonians. The case of scalar field revealed greater richness. There can be infinitely many choices of inner products, all of which can support infinitesimal rotations as well as elementary smeared field.

In principle neither of the two routes is unnatural. It is not certain, that continuous symmetries need be realised *exactly* in nature even if observations support their existence to excellent approximation, eg Lorentz symmetry. Symmetries help to exercise tighter control over theoretical frameworks but physical system may not exactly respect the implicit idealization. The in-built, non-invariant dynamics of a polymer quantized system, suggests a particular parametrization of symmetry violation eg the use of the 'trigonometric' operators to build the Hamiltonian. At least in the cases explored, such violations are viable.

The second alternative is anyway needed in the context of theories with first class constraints. It could well be thought of as a multi-step quantization procedure. Just as in a classical theory, specified by an action, the variables we begin with need not represent the physical states (eg when there are constraints). However following a systematic procedure - the Dirac algorithm of constraint analysis - we can arrive at a formulation which is either a theory with a first class constraint algebra or a theory without any constraints. Likewise, one could begin with a set of basic functions on the configuration space forming a Cyl_0, choose an inner product $\langle|\rangle_0$, obtain a Cyl_0^* as well as a Hilbert space H_0 forming a triple: $\mathrm{Cyl}_0 \subset \mathrm{H}_0 \subset \mathrm{Cyl}_0^*$. If the model is satisfactory, we are done. If not, look for a *subspace* $\mathrm{Cyl}_1 \subset \mathrm{Cyl}_0^*$, define a new inner product $\langle|\rangle_1$ and obtain a new triple $\mathrm{Cyl}_1 \subset \mathrm{H}_1 \subset \mathrm{Cyl}_1^*$. Hopefully the process would terminate after a finite number of iterations. This procedure offers a flexibility to refine the class of observables we wish to be supported on the quantum state space. It is constructive and could help keep the focus on physical observables. This possibility needs to be examined further to see its viability/utility.

Chapter 4: Path Integral Representation for Polymer Quantization

One of the most important open problems in LQG is to understand how known semi-classical physics arises from it. One aspect of the problem that pertains to scalar field theories is understanding how to go from a description of these theories in the polymer language to a description in the Schrodinger-Fock language. In this thesis, we take a step towards this by obtaining a path integral representation for a polymer quantized scalar field theory. The path integral language is in general well-suited for obtaining semiclassical approximations. The path integral representation for polymer quantized field theory should likewise be useful for that purpose. The work presented in this chapter is based on [43].

Although the Hilbert Space frameworks of the two theories are very different, we find that in the continuum limit, the path integral representation of polymer quantized scalar field theory can be put in a form closely resembling the standard path integral form for a Schrodinger quantized theory. Thus the path integral framework provides a convenient language to compare the two quantizations and indicate the points of difference. In this chapter we will particularly consider the free Klein-Gordon Lagrangian, although the derivation goes through for any arbitrary polynomial term (in the field or its derivatives) added to this Lagrangian.

A path integral formulation for a polymer quantized theory was first provided by Ashtekar,

Campigilia and Henderson in [32] for loop quantum cosmology, a theory with a single degree of freedom. We follow their technique and extend the path integral description to scalar field theories. For simplicity we'll derive the path integral formulation for a 1+1 dimensional field theory. The extension to higher dimensions is straightforward. Our strategy for this derivation will be to start from a scalar field theory defined on a one dimensional lattice, obtain a path integral formulation for this and finally take the 'continuum limit' by making the lattice spacing go to zero. This follows the standard quantum field theory approach [44].

One distinguishing feature of the polymer quantized theory is that both momentum and the field cannot be well defined operators. We will choose the representation where the field is not a well-defined operator, only momentum and the exponentiated field ('holonomy') operators are well-defined. This choice will allow us to get a closed form expression for the configuration space path integral. With the alternate choice one can proceed up to deriving a phase space path integral. Now to construct a Hamiltonian, an approximate field operator has to be constructed from the holonomy operator by introducing a scale μ. We will find that in the path integral expression the term corresponding to the approximate field operator is $\frac{\sin\mu\phi}{\mu}$. We will see that in the end one can simply go from the Schrodinger to the polymer path integral form by replacing ϕ by $\frac{\sin\mu\phi}{\mu}$ in the potential term in the action, while leaving the kinetic term unchanged. This means that the $\dot{\phi}^2$ term in the action remains unchanged but the $(\nabla\phi)^2$ term becomes $\cos^2(\mu\phi)(\nabla\phi)^2$. Thus Lorentz invariance is lost in the polymer formulation (for a different approach to Lorentz violation in polymer quantized scalar field theory see[45]). Another consequence of the replacement is that as only powers of sine terms appear in the potential, the action becomes periodic in ϕ with a periodicity of $2\pi/\mu$. A further point of difference is that in the polymer case we would be looking to obtain the correlation functions for the approximate field operator and these would enter the path integral as $\frac{\sin\mu\phi}{\mu}$. That is, the expressions for

say a 2 point correlation function would look like:

$$\int [\mathcal{D}\phi] \frac{\sin \mu\phi(x_1)}{\mu} \frac{\sin \mu\phi(x_2)}{\mu} e^{iS}$$

We'll see that the features mentioned above prevent a straightforward derivation of a perturbative framework from this path integral representation. We'll discuss some possible strategies to overcome this issues in the next chapter.

There are three sections in this chapter. The first section discusses path integral representation for polymer Quantum Mechanics. We derive a path integral representation for the simple harmonic oscillator, following the steps of [43]. In the second section we extend the path integral to the case of polymer quantized scalar field theory. In the last section we summarise our findings and discuss some general features of the path integral representations of polymer quantized theories.

4.1 Path Integral representation for polymer Quantum Mechanics

In this section we'll follow the steps of [32] to derive a path integral expression for an one dimensional polymer quantized simple harmonic oscillator. We'll consider momentum to be well-defined while an approximation will be used for the position. This choice helps get a closed form expression for the configuration space path integral. Also we will make some choice of μ_0. As we discussed in the previous chapter this choice will restrict us to a separable subspace of the Hilbert Space on which our basic operators act invariantly. In this derivation we will always work in one such subspace. Thus the completeness relation will be

$$\sum_n |p_0 + \mu_0 n\hbar\rangle\langle p_0 + \mu_0 n\hbar|$$

We'll choose $p_0 = 0$ for convenience.

As is usual for a derivation of the path integral representation, our starting point will be the transition amplitude $\langle p_f|e^{-\frac{i}{\hbar}t\hat{H}}|p_i\rangle$. We divide t into N pieces ϵ = t/N. So

$$e^{-\frac{i}{\hbar}\hat{H}t} = \Pi_{n=1}^{N} e^{-\frac{i}{\hbar}\hat{H}\epsilon} \tag{4.1}$$

Inserting complete basis of the form $\mathbb{1} = \sum |p\rangle\langle p|$ in between each factor we have

$$\langle p_f|e^{-\frac{i}{\hbar}\hat{H}t}|p_i\rangle = \sum_{p_{N-1},\ldots,p_1} \langle p_f|e^{-\frac{i}{\hbar}\hat{H}\epsilon}|p_{N-1}\rangle.....\langle p_1|e^{-\frac{i}{\hbar}\hat{H}\epsilon}|p_i\rangle \tag{4.2}$$

Taking N very large ($\epsilon << 1$) the nth term of the series is approximated as

$$\langle p_{n+1}|e^{-\frac{i}{\hbar}\hat{H}\epsilon}|p_n\rangle = \delta_{p_{n+1},p_n} - \frac{i}{\hbar}\epsilon\langle p_{n+1}|\hat{H}|p_n\rangle + O(\epsilon^2) \tag{4.3}$$

The matrix elements of H are:

$$\langle p_{n+1}|\hat{H}|p_n\rangle = \frac{{p_n}^2}{2m}\langle p_{n+1}|p_n\rangle - \langle p_{n+1}|\left(\frac{\hat{V}(\mu_0) - \hat{V}(-\mu_0)}{2\mu_0 i}\right)^2|p_n\rangle \tag{4.4}$$

$$= \frac{{p_n}^2}{2m}\delta_{p_{n+1},p_n} - \frac{1}{4{\mu_0}^2}(2\delta_{p_{n+1},p_n} - \delta_{p_{n+1},p_n-2\mu_0\hbar} - \delta_{p_{n+1},p_n+2\mu_0\hbar}) \tag{4.5}$$

We would have obtained the same expression using the Hamiltonian given in [9]. We'll make use of the identity:

$$\delta_{p',p} = \frac{\mu_0}{2\pi}\int_0^{\frac{2\pi}{\mu_0}} \mathrm{d}x e^{-\frac{i}{\hbar}x(p'-p)} \tag{4.6}$$

From 4.6, 4.5 and 4.3 we obtain up to $O(\epsilon^2)$ terms:

$$\langle p_{n+1}|e^{-\frac{i}{\hbar}\hat{H}\epsilon}|p_n\rangle = \frac{\mu_0}{2\pi}\int_0^{\frac{2\pi}{\mu_0}} \mathrm{d}x_{n+1} e^{-\frac{i}{\hbar}x_{n+1}(p_{n+1}-p_n)}\left[1-\frac{i\epsilon}{\hbar}\left(\frac{p_n^2}{2m}+\frac{\sin^2(\mu_0 x_{n+1})}{{\mu_0}^2}\right)\right] \tag{4.7}$$

$$= \frac{\mu_0}{2\pi}\int_0^{\frac{2\pi}{\mu_0}} \mathrm{d}x_{n+1} e^{-\frac{i}{\hbar}x_{n+1}(p_{n+1}-p_n)-\frac{i\epsilon}{\hbar}(\frac{p_n^2}{2m}+\frac{\sin^2(\mu_0 x_{n+1})}{{\mu_0}^2})} \tag{4.8}$$

Substituting this in 4.2 we have

$$\langle p_f|e^{-\frac{i}{\hbar}\hat{H}t}|p_i\rangle = \sum_{p_{N-1},\ldots,p_1}\left(\frac{\mu_0}{2\pi}\right)^N \int dx_N ... dx_1 e^{-\frac{i}{\hbar}S_N} \tag{4.9}$$

where

$$S_N = \epsilon\sum_{n=0}^{N-1} x_{n+1}\frac{(p_{n+1}-p_n)}{\epsilon}+\frac{p_n^2}{2m}+\frac{\sin^2(\mu_0 x_{n+1})}{{\mu_0}^2} \tag{4.10}$$

This is the discrete sum version of the path integral. The differences from the usual case are : (i)Here we have sums over momenta instead of integrals (ii) The integrals over positions are bounded whereas in the usual case these are unbounded and finally (iii) The x_n^2 term that appears in the action in the usual case has been replaced by $\frac{\sin^2(\mu_0 x_n)}{{\mu_0}^2}$.

The next step is to take the$N \to \infty$ limit. However we cannot interpret $\frac{(p_{n+1}-p_n)}{\epsilon}$ as a derivative as p takes discrete values. So we use:

$$\epsilon\sum_{n=0}^{N-1} -x_{n+1}\frac{(p_{n+1}-p_n)}{\epsilon} = \epsilon\sum_{n=0}^{N-1} p_n\frac{(x_{n+1}-x_n)}{\epsilon}+(x_1p_0-x_Np_N) \tag{4.11}$$

Now the limit $N \to \infty$ may be taken. This gives us the path integral expression: $K(p_f, p_i) = \int[\mathcal{D}p_q][\mathcal{D}x_q]e^{\frac{i}{\hbar}S'}$ where S′ given by

$$S' = \int_0^t \mathrm{d}\tau\ p\dot{x} - \left(\frac{p^2}{2m}+\frac{\sin^2(\mu_0 x)}{{\mu_0}^2}\right) \tag{4.12}$$

and

$$\int[\mathcal{D}p_q][\mathcal{D}x_q] = \lim_{N\to\infty}\prod_{n=1}^{N}\frac{\mu_0}{2\pi}\sum_{p_n}\int_0^{\frac{2\pi}{\mu_0}} dx_n \tag{4.13}$$

The subscript q in the path integral expression is to indicate that this expression is different from the usual path integral expression. Here p takes only discrete values and x is bounded. However, this can be converted into the familiar form by using the following identity [46]:

$$\sum_{m\epsilon\mathbb{Z}} \int_0^{2\pi} \mathrm{d}\theta\ f(\theta, m) e^{im\theta} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}\theta\ f(\theta, x) e^{ix\theta} \tag{4.14}$$

This is true for any continuous function $f(\theta, x)$ with a period of 2π in θ. One can see that integrand in 4.9 has the requisite property. Using this we can now replace

$$\frac{\mu_0}{2\pi} \sum_{p_n} \int_0^{\frac{2\pi}{\mu_0}} \rightarrow \int_{-\infty}^{\infty} \mathrm{d}p_n \int_{-\infty}^{\infty} \mathrm{d}x_n \tag{4.15}$$

Now we have our result for the phase space path integral:

$$K(p_f, p_i) = \int [\mathcal{D}p][\mathcal{D}x] e^{\frac{i}{\hbar}S} \tag{4.16}$$

where

$$S = \int_0^t \mathrm{d}\tau p\dot{x} - \left(\frac{p^2}{2m} + \frac{\sin^2(\mu_0 x)}{{\mu_0}^2} \right) \tag{4.17}$$

As the momentum integral is Gaussian one may integrate it out from the phase space transition amplitude to obtain the configuration space transition amplitude

$$Z = \int [\mathcal{D}x] e^{\frac{i}{\hbar}S} \tag{4.18}$$

with

$$S = \int_0^T \mathrm{d}\tau \frac{m\dot{x}^2}{2} - \frac{\sin^2(\mu_0 x)}{{\mu_0}^2} \tag{4.19}$$

We note that in the path integral formulation, going from Schrodinger to polymer representation for any polynomial potential the only change will be a replacement of x by $sin(\mu_0 x)/\mu_0$ in the action. This will be true in the case of polymer field theory as well.

The appearance of $sin(\mu_0 x)/\mu_0$ in the action has the novel consequence that $x \rightarrow x + \frac{2\pi}{\mu_0}$ is a symmetry of the action. Where did this periodicity in the action come from? Recall that originally the position variable took values on a circle. Using 4.14 we extended the range of x to the entire real line. But the memory of the original configuration space is contained in the periodicity. Therefore one would expect winding modes to be present as well in the path integral.

This can be seen more transparently in the path integral formulation of transition amplitude between *position* eigenstates (eigenstates of $\hat{x}_{\mu_0}$) in polymer quantum mechanics. The eigenstates of $\hat{x}_{\mu_0}$ are given by

$$|x\rangle = \sum_{n\in\mathbb{Z}} e^{-i\mu_0\hbar x n}|p_0 + n\mu\hbar\rangle \qquad x \in [0, 2\pi/\mu_0]$$

The inner product between them can be seen to be :

$$\langle x|x'\rangle = \delta(x - x')$$

Thus these are non-normalizable states of the Hilbert Space. Now this is the same situation as a particle on a circle. The transition amplitude in this case can be given a path integral expression in terms of unbounded paths as well (See 6. 1 of [46] for the detailed derivation). This expression reads-

$$\langle x_f, t_f|x_i, t_i\rangle = \sum_{l=-\infty}^{\infty} \langle x_f + 2\pi l, t_f|x_i, t_i\rangle_{\text{noncyclic}}$$

Where

$$\langle x_f, t_f|x_i, t_i\rangle_{\text{noncyclic}} = \int [\mathcal{D}x] e^{\frac{i}{\hbar}\int_0^T \mathrm{d}\tau \frac{m\dot{x}^2}{2} - \frac{\sin^2(\mu_0 x)}{{\mu_0}^2}}$$

is the usual path integral between two points where the sum is over non-cyclic paths, except with the modified polymer potential. We thus see that the winding modes are present

in the polymer path integral, given by the different values of l. This is a consequence of the fact that the effective configuration space for polymer quantum mechanics is a circle.

4.2 Path Integral representation for polymer Quantized scalar field theory

Having seen how a path integral representation can be obtained for the polymer quantized simple harmonic oscillator, we are now ready to tackle the more complicated case of the scalar field. Our derivation will be facilitated by the use of the new notation we introduced in Chapter 2, which we'll use throughout this section. We also set $\hbar = 1$. Now we'll calculate $\langle \pi_f | e^{-i\hat{H}t} | \pi_i \rangle$ by discretizing t into N pieces $e^{-i\hat{H}t} = \Pi_{n=1}^{N} e^{-i\hat{H}\epsilon_t}$ The Hamiltonian we consider will be the approximate Klein Gordon Hamiltonian, where the approximate field variable (defined using some number μ) introduced in the last chapter will be used. We recall that the choice of a μ and a basis vector gives a proper subspace of the Hilbert space on which the basic operators act invariantly. We'll restrict ourselves to one such subspace i.e a lattice in the $\vec{\lambda}$ labels. Our strategy will be to do this first for a scalar field theory which lives on a lattice and finally take the continuum limit. To this end we discretize space with lattice spacing ϵ_x and call the entire lattice L from now on. That is all vertex sets are now subsets of L. The completeness relation 2.11 will now have x ∈ L

So we have

$$\langle \pi_f | e^{-i\hat{H}t} | \pi_i \rangle = \langle \pi_f | \prod_{n=1}^{N} e^{-i\hat{H}\epsilon_t} | \pi_i \rangle \tag{4.20}$$

Using 2.11 this is rewritten as:

$$\langle \pi_f | e^{-i\hat{H}t} | \pi_i \rangle = \prod_{x_1 \in L} \sum_{\pi_1(x_1)} ..... \prod_{x_{N-1} \in L} \sum_{\pi_{N-1}(x_{N-1})} \langle \pi_f(x_{N-1}) | e^{-i\hat{H}\epsilon_t} | \pi_{N-1}(x_{N-1}) \rangle ..... \langle \pi_1(x_1) | e^{-i\hat{H}\epsilon_t} | \pi_i(x_1) \rangle \tag{4.21}$$

Taking N very large ($\epsilon_t << 1$) and expanding the nth factor in ϵ_t we have

$$\langle \pi_{n+1}(x)|e^{-i\hat{H}\epsilon}|\pi_n(x)\rangle = \langle \pi_{n+1}(x)|\pi_n(x)\rangle + i\epsilon_t\langle \pi_{n+1}(x)|\hat{H}|\pi_n(x)\rangle + O(\epsilon_t^2) \tag{4.22}$$

Our Hamiltonian will be a discretization of $\int dx \frac{1}{2}(\pi(x)^2 + (\nabla\phi(x))^2 + m^2\phi^2(x))$. We'll proceed term by term.

(i) The π^2 term:

First we consider a Hamiltonian with only the π^2 term. This helps demonstrate most simply the steps in the derivation of the path integral. In this case the discrete Hamiltonian reads $\sum_{x\in L} \epsilon_x \frac{\pi(x)^2}{2\epsilon_x^2}$, π here being the canonical conjugate to the discrete field ϕ. We'll suppress $O(\epsilon_t^2)$ terms in the following. We have, therefore

$$\prod_L \langle \pi_{n+1}(x)|e^{-i\hat{H}\epsilon_t}|\pi_n(x)\rangle = \prod_L \delta_{\pi_n+1(x),\pi_n(x)} + \epsilon_t \sum_L \frac{\pi_n(x)^2}{2\epsilon_x} \prod_L \delta_{\pi_n+1(x),\pi_n(x)} \tag{4.23}$$

As we have restricted ourselves to a subspace with a choice of μ it follows that the values of the π fields at any point will differ by an integer multiple of μ, just as the momenta in quantum mechanics differed by factors of μ_0. Using this fact, we may write:

$$\delta_{\pi(x),\pi'(x)} = \frac{\mu_0}{2\pi}\int_0^{\frac{2\pi}{\mu_0}} \mathrm{d}\phi(x) e^{i\phi(x)(\pi'(x)-\pi(x))} \tag{4.24}$$

This can be used to rewrite 4.22 as:

$$\langle \pi_{n+1}|e^{-i\hat{H}\epsilon_t}|\pi_n\rangle = \left(\prod_L \frac{\mu_0}{2\pi}\int_0^{\frac{2\pi}{\mu_0}} d\phi_{n+1}(x)\right) e^{i\sum_L(\phi(x)(\pi_{n+1}(x)-\pi_n(x))+(\pi_n(x))^2/2\epsilon_x)} \tag{4.25}$$

Then we have

$$\langle \pi_f|e^{-i\hat{H}t}|\pi_i\rangle = \left(\prod_L \sum_{\pi_1,\pi_2,....\pi_N} \int d\phi_1(x)...d\phi_n(x)\right) e^{-iS_{N,L}} \tag{4.26}$$

where

$$S_{N,L} = \sum_L \epsilon_x \left(\epsilon_t \sum_{n=0}^{N-1} \frac{\phi(x)(\pi_{n+1}(x) - \pi_n(x))}{\epsilon_t \epsilon_x} + \frac{(\pi_n(x))^2}{2\epsilon_x^2} \right) \tag{4.27}$$

To take the limit ϵ_t going to zero we use, as before:

$$-\epsilon_t \sum_{n=0}^{N-1} \phi_{n+1}(x) \frac{(\pi_{n+1}(x) - \pi_n(x))}{\epsilon_t} = \epsilon_t \sum_{n=0}^{N-1} \pi_n(x) \frac{(\phi_{n+1}(x) - \phi_n(x))}{\epsilon_t} + (\phi_1(x)\pi_0(x) - \phi_N(x)\pi_N(x)) \tag{4.28}$$

And using 4.14 as before we'll turn the sums over paths into path integrals. We finally have the following expression for the transition amplitude

$$\langle \pi_f | e^{-i\hat{H}t} | \pi_i \rangle = \left(\prod_{x \epsilon L} \prod_k^N \int d\phi_k(x) \int d\pi_k(x) \right) e^{iS_{N,L}} \tag{4.29}$$

where

$$S_{N,L} = \sum_L \epsilon_x \left(-\epsilon_t \sum_{n=0}^{N-1} \pi(x)_n \frac{(\phi_{n+1}(x) - \phi_n(x))}{\epsilon_t} + (\phi_1(x)\pi_0(x) - \phi_N(x)\pi_N(x)) - \frac{(\pi_n(x))^2}{2\epsilon_x^2} \right) \tag{4.30}$$

Now we can take the $\epsilon_t \to 0$ limit. This gives us:

$$\langle \pi_f | e^{-i\hat{H}t} | \pi_i \rangle = \left(\prod_{x \epsilon L} \int d\phi(x) \int d\pi_k(x) \right) e^{iS_L} \tag{4.31}$$

where

$$S_L = \sum_L \epsilon_x \left(- \int dt \sum_{n=0}^{N-1} \int dt \pi \dot{\phi} - \frac{\pi^2}{2\epsilon_x^2} \right) \tag{4.32}$$

In the transition amplitude we can integrate out the π field and obtain the expression

$$Z = \prod_{x \epsilon L} \int d\phi(x) e^{-i \int dt \sum_L \epsilon_x \frac{\phi^2}{2}} \tag{4.33}$$

Finally taking the ϵ_x going to zero limit we obtain the configuration space path integral

expression for the transition amplitude:

$$Z = \int [\mathcal{D}\phi] e^{iS} \tag{4.34}$$

where

$$S = \int dxdt\ \dot{\phi}^2/2 \tag{4.35}$$

and

$$\int [\mathcal{D}\phi] = \prod_x d\phi(x) \tag{4.36}$$

This demonstrates how the path integral for a polymer quantized scalar field theory is obtained. Now we'll look at the contributions from the $(\nabla\phi)^2$ and $m^2\phi^2$ terms respectively. We start with the latter.

(ii) The $m^2\phi^2$ term:

The discretized expression for this is:

$$\sum_{x,z} \epsilon_x \delta_{x,z} \hat{\phi}_\mu(x) \hat{\phi}_\mu(z) \tag{4.37}$$

where

$$\hat{\phi}_\mu(x) = \frac{\hat{U}(\mu, x) - \hat{U}(-\mu, x)}{2\mu i} \tag{4.38}$$

So we want to calculate

$$\sum_{x,z} \epsilon_x \delta_{x,z} \langle \{\pi_{n+1}\} | \hat{\phi}_\mu(x) \hat{\phi}_\mu(z) | \pi_n \rangle \tag{4.39}$$

Using 4.38, 2.8, 2.10 and carrying out the sum over the z index we obtain:

$$-\frac{\epsilon_x m^2}{4\mu^2} \prod_y \delta_{\pi_n+1(y), \pi_n(y)+2\mu\delta_{x,y}} + \delta_{\pi_n+1(y), \pi_n(y)-2\mu\delta_{x,y}} - 2\delta_{\pi_n+1(y), \pi_n(y)} \tag{4.40}$$

Then using 4.24 we obtain

$$\epsilon_x m^2 \int_0^{2\pi/\mu} e^{i\sum_y \phi_{n+1}(y)(\pi_{n+1}(y)-\pi_n(y))} \sum_{x\in L} \frac{e^{2i\mu\phi_{n+1}(x)} + e^{-2i\mu\phi_{n+1}(x)} - 2}{(2i\mu)^2} \tag{4.41}$$

The continuum limit of the sum

$$\sum_{x\in L} \epsilon_x m^2 \frac{e^{2i\mu\phi_{n+1}(x)} + e^{-2i\mu\phi_{n+1}(x)} - 2}{(2i\mu)^2}$$

is $\int dx(m^2 \frac{\sin^2(\mu\phi)}{\mu^2})$. Therefore including 4.37 in the Hamiltonian along with the π^2 term and repeating the same steps as before(4.28-4.34) we obtain the following path integral expression:

$$Z = \int [\mathcal{D}\phi] e^{i\int dxdt \frac{1}{2}(\dot{\phi}^2 - m^2(\sin^2(\mu\phi)/\mu^2))} \tag{4.42}$$

(iii) The $(\nabla\phi)^2$ term:

Finally we come to the $(\nabla\phi)^2$ term. The discretized expression for this is:

$$\sum_{x,y} \epsilon_x \delta_{x,y} \widehat{\nabla_{\epsilon_x}\phi_\mu}(x) \widehat{\nabla_{\epsilon_x}\phi_\mu}(y) \tag{4.43}$$

and we'll be calculating:

$$\langle \pi_{n+1}| \sum_{x,y} \epsilon_x \delta_{x,y} \widehat{\nabla_{\epsilon_x}\phi_\mu}(x) \widehat{\nabla_{\epsilon_x}\phi_\mu}(y) |\pi_n\rangle \tag{4.44}$$

where

$$\nabla_{\epsilon_x} f(x) = \frac{f(x+\epsilon_x) - f(x)}{\epsilon_x} \tag{4.45}$$

Let us evaluate

$$\sum_{x,y} \epsilon_x \delta_{x,y} \widehat{\nabla_{\epsilon_x}\phi_\mu}(x)\widehat{\nabla_{\epsilon_x}\phi_\mu}(y)|\pi_n\rangle = \sum_{x,y\epsilon L} \epsilon_x \delta_{x,y} \widehat{\nabla_{\epsilon_x}\phi_\mu}(x)\widehat{\nabla_{\epsilon_x}\phi_\mu}(y) e^{i\sum\limits_{x_j}\pi_n(x_j)\phi(x_j)} \tag{4.46}$$

Using 2.7, 2.8, 2.10 and carrying out the sum over the y label, this gives:

$$\sum_L \frac{\epsilon_x}{-4\epsilon_x\mu^2} e^{i\sum\limits_{x_j}\pi_n(x_j)\phi(x_j)} (e^{2i\mu\phi(x+\epsilon_x)} - 2e^{i\mu(\phi(x+\epsilon_x)+\phi(x))} + e^{2i\mu\phi(x)} + 2e^{i\mu(\phi(x+\epsilon_x)-\phi(x))}$$

$$+2e^{i\mu(-\phi(x+\epsilon_x)+\phi(x))} + e^{-2i\mu\phi(x+\epsilon_x)} - 2e^{-i\mu(\phi(x+\epsilon_x)+\phi(x))} + e^{-2i\mu\phi(x)} - 4) \tag{4.47}$$

Now let us consider a term like $e^{i\sum\limits_{x_j}\pi_n(x_j)\phi(x_j)} e^{2i\mu\phi(x+\epsilon_x)}$ that appears in the above expression. We may rewrite this as

$$e^{i\sum\limits_{y}(\pi_n(y)+2\mu\delta_{x+\epsilon_x,y})\phi_n(y)} = |\{\pi_n(y) + 2\mu\delta_{x+\epsilon_x,y}\}\rangle \tag{4.48}$$

Using this and acting on 4.47 with $\langle\pi_{n+1}|$ from left we have:

$$\begin{aligned}
\sum_L \frac{\epsilon_x}{-4\epsilon_x^2\mu^2} \quad & \Big(\langle\{\pi_{n+1}(y)\}|\{\pi_n(y) + 2\mu\delta_{x+\epsilon_x,y}\}\rangle + \langle\{\pi_{n+1}(y)\}|\{\ \pi_n(y) + 2\mu\delta_{x,y}\}\rangle \\
+ \quad & \langle\{\pi_{n+1}(y)\}|\{\pi_n(y) - 2\mu\delta_{x+\epsilon_x,y}\}\rangle + \langle\{\pi_{n+1}(y)\}|\{\pi_n(y) - 2\mu\delta_{x,y}\}\rangle \\
- \quad & 2\langle\{\pi_{n+1}(y)\})|\{\pi_n(y) + \mu\delta_{x+\epsilon_x,y} + \mu\delta_{x,y}\}\rangle - 2\langle\{\pi_{n+1}(y)\}|\{\pi_n(y) - \mu\delta_{x+\epsilon_x,y} - \mu\delta_{x,y}\}\rangle \\
- \quad & 4\langle\{\pi_{n+1}(y)\}|\{\pi_n(y)\})\rangle + 2\langle\{\pi_{n+1}(y)\}|\{\pi_n(y) + \mu\delta_{x+\epsilon_x,y} - \mu\delta_{x,y}\}\rangle \\
+ \quad & 2\langle\{\pi_{n+1}(y)\}|\{\pi_n(y) - \mu\delta_{x+\epsilon_x,y} + \mu\delta_{x,y}\}\rangle + 2\langle\{\pi_{n+1}(y)\}|\{\pi_n(y) \\
+ \quad & 2\langle\{\pi_{n+1}(y)\}|\{\pi_n(y) - \mu\delta_{x+\epsilon_x,y} + \mu\delta_{x,y}\}\rangle\Big)
\end{aligned} \tag{4.49}$$

Using 4.24 this becomes

$$\prod_y \frac{2\mu_0}{\pi} \int_0^{\frac{\pi}{2\mu_0}} d\phi_{n+1}(y) e^{i\sum_y \phi_{n+1}(y)(\pi_{n+1}(y)-\pi(y))} \sum_{x\in L} \frac{\epsilon_x}{-4\epsilon_x^2\mu^2}$$
$$\Big(e^{2i\mu\phi(x+\epsilon_x)} - 2e^{i\mu(\phi(x+\epsilon_x)+\phi(x))} + e^{2i\mu\phi(x)} + 2e^{i\mu(\phi(x+\epsilon_x)-\phi(x))}$$
$$+2e^{i\mu(-\phi(x+\epsilon_x)+\phi(x))} + e^{-2i\mu\phi(x+\epsilon_x)} - 2e^{-i\mu(\phi(x+\epsilon_x)+\phi(x))} + e^{-2i\mu\phi(x)} - 4\Big) \tag{4.50}$$

In the continuum limit($\epsilon_x, \epsilon_t \to 0$) the sum goes to $\int dx(\nabla\phi)^2(\cos\mu\phi)^2$. Therefore including this term in the Hamiltonian with the previous two terms and following the same steps once again (4.28-4.34) one has the path integral expression for the full free Klein Gordon Hamiltonian:

$$Z = \int [\mathcal{D}\phi] e^{iS} \tag{4.51}$$

where

$$S = \int dxdt \frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2(cos\mu\phi)^2 - m^2(\sin^2(\mu\phi)/\mu^2)) \tag{4.52}$$

We see again that in the path integral formulation one can go from the usual form to the polymer form by replacing ϕ by $\sin(\mu\phi)/\mu$. In the polymer quantized theory we would be interested in calculating time ordered n-point correlation functions in the approximate field operator. From the derivation it is clear that these would be obtained by inserting $\sin(\mu\phi)/\mu$ at appropriate times.

There are two interesting features that appear in the polymer path integral. One is that the action is no longer Lorentz invariant. We'll discuss the origin of this difference in the last section. Another new feature is that the action has become periodic in ϕ with a periodicity of $2\pi/\mu$.

4.3 Summary and Discussion

In this work we obtained a path integral representation for a polymer quantized Klein Gordon field. We found that the path integral representation can be put in a form that closely resembles the Schrodinger form with the only difference being a replacement of ϕ by $\sin(\mu\phi)/\mu$ in the action. This replacement has two immediate consequences - loss of Lorentz invariance and periodic symmetry in action.

The origin of the Lorentz violation is easy to understand. The Hamiltonian has terms $(\nabla\phi)^2$ and π^2, the latter equaling $(\dot{\phi})^2$ in the classical theory. In the transition to the polymer theory, the latter term remains as it is while a scale μ enters the polymerisation of the former. This spoils the symmetry between the space and time derivatives that existed in the classical theory.

The other consequence is the appearance of a discrete global symmetry in the action: $\phi \rightarrow \phi + \frac{2n\pi}{\mu}, n \in \mathbb{Z}$. A similar discrete symmetry $x \rightarrow x + \frac{2n\pi}{\mu_0}, n \in \mathbb{Z}$ appeared in quantum mechanics as well. The origin of this periodicity can be traced to the fact that in both cases, we chose a single scale while defining the position/field observable, and thereafter restricted ourselves to a subspace of the Hilbert Space generated by the action of the observables. We may recall that originally the position/field variables took values on a circle and it was through the use of identity 4.14 that the configurations spaces were extended. The memory of the original configuration space is retained through this periodicity. In fact was shown in [47] that the classical phase space corresponding to a polymer quantum mechanical theory with one degree of freedom is a cylinder when one restricts to observables defined with a single scale, as we do here. This explains the periodicity in the action that we have here.

This periodicity in the action gives our theory a resemblance with sine Gordon theories (except of course for the Lorentz invariance). The difference is that in sine Gordon theory one would look to calculate correlators in the variable ϕ whereas here we do not have a

field operator and it is instead correlators in $\sin(\mu\phi)/\mu$ that we are looking to calculate.

An important open issue here is the construction of a perturbative framework which will allow us to make predictions. The feature of global symmetry, as well as the fact that we need to calculate correlators for $\sin(\mu\phi)/\mu$ instead of ϕ presents some difficulties in this regard. We discuss this issue in more detail in the last chapter.

Chapter 5: Outlook

In this chapter we discuss some of the issues in our investigation that remain open. There are several open issues here which are both interesting and important. We discuss them in the next two sections.

5.1 Outlook on Polymer Quantization and Symmetries

Our analysis of polymer quantization and symmetries was considerably more extensive for polymer Quantum mechanics than polymer quantized scalar field theory. In the former case we showed that non-continuous representations of the rotation group are physically untenable because they give rise to infinite degenracy. In the case of the polymer quantized scalar field, we did not investigate if the representation of the rotation group is infinite dimensional. We know infinite dimensional representations of non-continuous groups exist [39], but we have not checked if they are realised for the plymer quantized scalar field theory

Secondly, for polymer Quantum Mechanics we showed how we may regain infinitesimal generators in Cyl*. We showed that by imposing the demand that the position operator is self-adjoint on Cyl*, we were led back to the Schrodinger representation on Cyl*. This was a consequence of the Stone-Von Neumann theorem, which states that all weakly continuous unitary irreducible representations of the Weyl Algebra are unitarily equivalent to the standard Schrodinger representation. In the case of the scalar field, however, it is

known that *even after invoking the Weyl-Heisenberg relations*, there are infinitely many inequivalent representations of the canonical commutation relations. This means that there can be infinitely many choices of inner products, all of which can support infinitesimal rotations as well as permit non-trivial dynamics.In the usual case, Poincare invariance is additionally invoked to uniquely single out the Fock representation [48]. The analysis of the various possibilities remains an open issue.

We have not considered Fermions or gauge fields in this analysis. We have only considered scalar field theory where 'point holonomies' are the basic functions generating the commutative C^* algebra. Fermions are similar to point holonomies as far as the label sets are concerned. For gauge fields, we will have the $H_{poly} := H_{kin}$ with the basis labeled by discrete labels. Hence the analogues of $(V, \vec{\lambda})$ will now have embedded graphs and representation labels of the gauge group. One will have to impart a 'manifold structure' for these spaces of labels to attempt a definition of infinitesimal generators in the manner discussed above. This is another avenue to be explored.

We would also like to draw a parallel with recent work on polymer quantization of parametrized field theory (PFT) [49, 50, 51]. Parametrized field theories are field theories with a background geometry which however are presented in diffeomorphism covariant form by promoting the background coordinates to fields. The diffeomorphism covariance introduces constraints and the physical sector of the theory is the old theory with a background. Let's consider for definiteness a free field theory on the flat Minkowski space-time and its parametrized form. In the non-parametrized form, isometries of the Minkowski metric are the symmetries of the theory. One may choose the usual Fock quantization and see the representations of the infinitesimal symmetries. In the parametrized form however, the diffeomorphism covariance would suggest polymer quantization, not just for the embedding variables but also for the scalar field. One can now ask, how the isometries are represented in such a quantization.

If we insist that Dirac quantization of the PFT *should* produce a physical sector which

is same as the quantized non-parametrized theory and it is possible to realise this, then the quantization of the matter sector chosen in the non-parametrized form will already determine the symmetry realization, regardless of its parametrized version. However, it is conceivable that that there is a (different) Dirac quantization of the polymer quantized PFT such that the physical states carry the usual Fock representation. There is no definitive statement available on this as yet[1]. Such a possibility could be quite relevant for LQG, at least in a 'semiclassical approximation'. This is one context in which the discussion of this work, especially the Cyl^* alternative, could be directly relevant.

Finally, let us note that the constructions can be carried out in any dimension, for any group - in particular the Lorentz or Poincare groups - with the same qualitative conclusions: unitary, discontinuous representation of the group[2], non-existence of infinitesimal generators, non-existence of an invariant dynamics which is both non-trivial and physically acceptable and subspaces of Cyl^* as possible arenas for further searches of acceptable models.

5.2 Outlook on Path Integral Representation of Polymer Quantized Scalar Field Theory

The most important open issue for polymer quantized scalar field theory is the construction of a perturbative framework which will enable us to obtain predictions from polymer quantized Scalar field theory. We highlight the challenges to this and discuss our efforts to overcome them in this section.

We saw that the path integral representation for the polymer quantized scalar field theory

[1] In two dimensional space-time, $\mathbb{R} \times S^1$, the work of Laddha and Madhavan [51], obtains the isometry group being spontaneously broken to its discrete subgroup.

[2] When we speak of discontinuity of representations of Lorentz or Poincare groups what we mean is that the rotational subgroup of these groups will be discontinuously represented.

is given by:

$$Z = \int [\mathcal{D}\phi] e^{iS} \tag{5.1}$$

where

$$S = \int dxdt \frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2 (cos\mu\phi)^2 - m^2(\sin^2(\mu\phi)/\mu^2)) \tag{5.2}$$

We would like to obtain a perturbation expansion for this integral, proceeding order by order in μ. One might think of expanding the sine and cosine terms in a power series expansion and then assembling the series in orders of μ. One can easily see that this would give the free Klein Gordon Lagrangian in the zeroth order in μ and an infinite series of 'interaction terms' in higher powers in μ. One might want to adopt the strategy of truncating the series at some finite order in μ - this would now look like a standard Schrodinger path integral for a scalar field with some interaction terms - and follow the standard derivation to obtain a perturbative series in Feynman diagrams. There are however two problems with this strategy. First, truncating the series expansion of the Lagrangian at any finite order would mean that the periodic symmetry of the action will disappear. Thus we would miss out on any physical predictions that would have come as consequences of this symmetry encoding the underlying polymer representation. Second, the standard strategy is convenient for perturbatively calculating, say, a term like $\int [\mathcal{D}\phi]\phi(x)\phi(y)e^{iS}$ which comes $\langle\phi(x)\phi(y)\rangle$. In the polymer theory field operators are replaced by approximate field operators, and for instance a 2 point correlator of approximate field operators would have a path integral expression $\int [\mathcal{D}\phi] \frac{\sin(\mu\phi(x))}{\mu} \frac{\sin(\mu\phi(y))}{\mu} e^{iS}$.

To get around this issues we could try to to recast the path integral in terms of a new variable χ:

$$\chi = \sin(\mu\phi)/\mu = \phi_\mu$$

In terms of χ the path integral reads:

$$Z = \int [\mathcal{D}\chi]_\mu e^{iS} \tag{5.3}$$

where

$$\int [\mathcal{D}\chi]_\mu = \prod_x \int_{-1/\mu}^{1/\mu} d\chi(x) \tag{5.4}$$

and

$$S = \int dxdt(1-\mu^2\chi^2)^{-1}(\dot{\chi})^2 - (\nabla\chi)^2 - m^2\chi^2 - \frac{1}{2}ln(1+\mu^2\chi^2) \tag{5.5}$$

The log term that appears in the action originates from the change of the integration measure due to the change in variables. We note that in terms of χ variables, the action is no longer periodic but the field is now bounded and so the integration now proceeds from $-1/\mu$ to $1/\mu$ instead of $-\infty$ to ∞ as in the usual case. Since the periodicity has been 'absorbed', we can now proceed to expand the Lagrangian in a power series and truncate to some finite order. We expand the Lagrangian as a power series in μ -

$$\mathcal{L} = \mathcal{L}_0 + \mu^2\mathcal{L}_I^{(1)} + \mu^4\mathcal{L}_I^{(2)} + ... \tag{5.6}$$

where

$$\mathcal{L}_0 = (\dot{\chi})^2 - (\nabla\chi)^2 - m^2\chi^2 \tag{5.7}$$

$$\mathcal{L}_I^{(1)} = \chi^2\dot{\chi}^2 - \chi^2/2 \tag{5.8}$$

$$\mathcal{L}_I^{(2)} = \chi^4\dot{\chi}^2 + \chi^4/4 \tag{5.9}$$

Here one notes that the $\mathcal{L}_0$ term is Lorentz invariant and Lorentz violation comes from the *time-derivative* terms that appear in the coefficients of higher powers of μ. Next, one takes the usual step of expanding the exponential in a power series in the 'interaction terms'. We can then re-order the obtained series in powers of μ:

$$Z = \int [\mathcal{D}\chi_\mu] \quad e^{i\int dxdt\mathcal{L}_0} \quad \left(1+\mu^2\int dxdt\mathcal{L}_I^{(1)} + \mu^4\left(\int dxdt\mathcal{L}_I^{(1)}\right)^2 + ...\right)$$
$$\left(1+\mu^4\int dxdt\mathcal{L}_I^{(2)} + \mu^8\left(\int dxdt\mathcal{L}_I^{(2)}\right)^2 + ...\right)....$$

$$= \int [\mathcal{D}\chi_\mu] e^{i\int dxdt\mathcal{L}_0} \left(1 + \mu^2 \int dxdt\mathcal{L}_I^{(1)} + +\mu^4 \left(\int dxdt\mathcal{L}_I^{(2)} + \left(\int dxdt\mathcal{L}_I^{(1)}\right)^2\right) + ...\right) \tag{5.10}$$

To proceed with perturbative calculations the standard step is to introduce:

$$Z[J] = \int [\mathcal{D}\chi_\mu] e^{i\int \mathcal{L}+J\chi}$$

Now an n point correlator in the *approximate field operators* can be written as:

$$\langle\chi(x_1)...\chi(x_n)\rangle = \frac{1}{Z[0]}\left(\frac{\delta}{\delta J(x_1)}\right)...\left(\frac{\delta}{\delta J(x_n)}\right) Z[J]|_{J=0} \tag{5.11}$$

One similarly introduces

$$Z_0[J] = \int [\mathcal{D}\chi_\mu] e^{i\int \mathcal{L}_0+J\chi} = \int [\mathcal{D}\chi_\mu] e^{i\int (\dot{\chi})^2-(\nabla\chi)^2-m^2\chi^2+J\chi} \tag{5.12}$$

To carry out a perturbative calculation for any n-point correlation function up to any order in μ all one needs to be able to do is obtain functional derivatives of $Z_0[J]$ with J, because any term appearing in the perturbative series can always be written in terms of such a derivative.

In the Schrodinger case, it is easy to evaluate the functional dependence of $Z_0[J]$ on J. One has the term $\int [\mathcal{D}\phi] e^{i\int (\dot{\phi})^2-(\nabla\phi)^2-m^2\phi^2+J\phi}$ where one can complete the squares and carry out a Gaussian integral over the field variable to obtain a term whose functional dependence on J is of the form $e^{-\frac{i}{2}\int\int J(x)D_F(x-y)J(y)}$ where $D_F(x-y)$ is the usual Feynman propagator. A consequence of this particular form of functional dependence is that the terms in the perturbation series - which are obtained by taking derivatives of this term with respect to J and evaluating at J=0 - admit a diagrammatic representation in terms of vertices and propagators. Here in the polymer case, operations of completing the square and carrying out the Gaussian integral are not feasible because of the bounded nature of the field χ.

However it should be possible to calculate functional derivatives on 5.12 and obtain a perturbative series. One notes that such a series would not have the usual diagrammatic interpretation in terms of vertices and propagators. Evaluation of this series remains an important open issue.

To summarise, we saw that while the path integral representation for the polymer quantized field theory has a close formal resemblance with the usual path integral representation of the Schrodinger quantized theory, there are important differences as well. In particular, the action that appears in the path integral representation of the polymer scalar field theory is periodic in the field variable. As a result of this property, the usual derivation of a perturbative series from the path integral representation does not go through. We have outlined here an approach to deriving a perturbation series that appears appropriate for the polymer path integral, but more work remains to be done. We expect that further investigation of this question will help illuminate the possibly deep differences between polymer and Schrodinger quantizations.

Appendix A: The Kinematical Hilbert Space in Loop Quantum Gravity

In this section we sketch the construction of the Kinematical Hilbert Space in LQG. We eschew all proofs and details and focus instead on a schematic presentation. The interested reader is referred to [1, 2, 3] for the details. We use the notation of [29], whose treatment we closely follow.

The starting point for LQG is a re-formulation of General Relativity as a theory of connections. The phase space consists of pairs of fields $(\mathcal{A}, E)$ on a 3-manifold Σ, where $\mathcal{A}_a^i$ are connection 1-forms which take values and E_i^a are the triad variables. The $\mathcal{A}_a^i$ take values in the Lie Algebra $\mathfrak{su}(2)$ while the triads take value in its dual. The basic observables in the quantum theory are however not the connection or the triads. Instead they are taken to be the holonomies $\mathcal{A}_e$ along paths e defined by $\mathcal{A}$ (these will take values in the group SU(2)and fluxes E_S of the triads across 2-surfaces. That is to say, one looks for a representation of the Poisson bracket algebra of the holonomy and flux variables to obtain a quantum theory.

To obtain the Hilbert Space one first defines a space Cyl of functions of connection. To define Cyl one first fixes a closed, analytic graph γ on the 3-manifold Σ . Let the graph have N edges 1,...,N. A given connection $\mathcal{A}_a^i$ will associate to each edge e a holonomy

$\mathcal{A}_e$. Let us consider the N-tuples $(\mathcal{A}_1, \ldots, \mathcal{A}_N)$. These will take value in $SU(2)^N$. We call the space of all such N-tuples $\mathcal{A}_\gamma$. This space is isomorphic to $SU(2)^N$. Now we construct a function Ψ on a given connection $\mathcal{A}$ such that

$$\Psi(\mathcal{A}) = \psi(\mathcal{A}_1, \ldots, \mathcal{A}_N).$$

where ψ is a smooth function on $SU(2)^N$. Such functions are called cylindrical functions (on the graph γ) and the space of all such functions is defined to be Cyl_γ.

Now one considers all possible graphs γ and defines the space Cyl of all cylindrical functions as

$$\mathrm{Cyl} = \bigcup_\gamma \mathrm{Cyl}_\gamma.$$

To construct a Hilbert Space we must define an inner product on Cyl. There is a natural definition coming from the fact that Cyl_γ is isomorphic to the space of smooth functions on $SU(2)^N$. One may then use the Haar measure on $SU(2)^N$ to induce a measure $\mu_H^{(N)}$ on $\mathcal{A}_\gamma$ and define an inner product:

$$(\Psi_1, \Psi_2) = \int_{\mathcal{A}_\gamma} \bar{\psi}_1 \, \psi_2 \; d\mu_H^{(N)}$$

where Ψ_1, Ψ_2 are functions on Cyl. This works for the entire space Cyl because we can always find a graph big enough to support any two functions.

Cauchy completing Cyl with respect to this inner product leads us to the Kinematical Hilbert Space $\mathcal{H}_{\mathrm{Kin}}$. This is a non-separable Hilbert Space since the graphs give an uncountable basis. We also have the natural Gelfand Triple $\mathrm{Cyl} \subset \mathcal{H}_{\mathrm{Kin}} \subset \mathrm{Cyl}^*$ where Cyl* is the algebraic dual Cyl*. It is easy to show that the inner product is invariant under spatial diffeomorphisms, thus leading to a unitary representation of finite spatial diffeo-

morphisms. Diffeomorphism invariant states may also be constructed, but they live not in Cyl, but in the dual Cyl*. This is where the space Cyl* plays an important role in LQG.

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