Some Problems in Quantum State Discrimination

By Tanmay Singal PHYS10200905003

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the Board of Studies in Physical Sciences In partial fulfillment of requirements For the Degree of DOCTOR OF PHILOSOPHY

of HOMI BHABHA NATIONAL INSTITUTE



March, 2016

Homi Bhabha National Institute Recommendations of the Viva Voce Board

As members of the Viva Voce Board, we certify that we have read the dissertation prepared by Tanmay Singal entitled Some Problems in Quantum State Discrimination and recommend that it maybe accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

	Date:
Chair	
	Date:
Guide/Convener	
	Date:
Member 1	
	Date:
Member 2	
	Date:
External Examiner	

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to HBNI.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it may be accepted as fulfilling the dissertation requirement.

Date:

Place:

Guide

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscription whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Tanmay Singal

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Tanmay Singal

List of Publications arising from the thesis

Journal

- Minimum error discrimination for an ensemble of linearly independent pure states Tanmay Singal and Sibasish Ghosh
 J. Phys. A: Math. Theor. 49 165304 (2016)
- 2. Framework for distinguishability of orthogonal bipartite states by one-way local operations and classical communication

Tanmay Singal

Phys. Rev. A 93, 030301(R) (2016)

Communicated

1. Algebraic structure of the minimum error discrimination problem for linearly independent density matrices

Tanmay Singal and Sibasish Ghosh

Uploaded on the arxiv at quant-ph/14127174 (To be sent for publication)

2. Complete analysis of perfect local distinguishability of ensemble of four Generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$

Tanmay Singal, Ramij Rahaman, Sibasish Ghosh, Guruprasad Kar

Uploaded on the arxiv at quant-ph/150603667

List of Papers Presented at Conference

a. Minimum Error Discrimination for an Ensemble of Linearly Independent Pure States

Tanmay Singal and Sibasish Ghosh

Poster presented at AQIS 2014, Kyoto Japan.

Listed as no. (20) on first poster session on 21/08/2014.

(No conference proceedings of the conference is published)

b. Minimum error discrimination of linearly independent pure states: analytic properties of optimal POVM

Tanmay Singal and Sibasish Ghosh

Oral Presentation at IPQI 2014, Bhubaneswar, Odisha, India

Listed for a talk on 19/02/2014 in schedule of conference.

(No conference proceedings of the conference is published)

List of Corrections Requested by Referees and Viva-Voce Examiners

- (Synopsis, p. 23) The notion of LI (linearly independent) has been defined on page 16 when it appears for the first time.
- (Synopsis, p. 29) $\log_2 n$ has been corrected to $\log_2 m$.
- (Synopsis, p. 44) The description of 2-LOCC has been changed to say that Alice's second measurement depends on outomes of all previous measurements performed.
- (Section 2.1, p. 51) Typo in citation [75] has been corrected.
- (Section 2.3, p. 53) Correction in equation (2.9) effected.
- (Section 2.4, p. 55) On the issue of whether equation (2.11) is sufficient to define the bijection $\mathscr{R}^{(\mathcal{G})^{-1}}$: this has now been clarified with an explanation placed just after where $\mathscr{R}^{(\mathcal{G})^{-1}}$ is introduced.
- (Section 2.4.1, p. 58) An explanation has been added as a footnote to verify the representation given in equation (2.14).
- (Section 2.4.3) The lack of any relationship between p_i and q_i (i.e., if one majorizes the other or if entropy of one is larger than the other) is true even for the case where n is small, for e.g. n = 2, 3. A statement that this is corroborated from examples for n = 2 has been added at the end of the paragraph.
- (Section 2.5, p. 94) A footnote has been added to the paragraph after equation
 (2.77) to clarify why condition rank(ρ_i) = r_i can't be relaxed to rank(ρ_i) ≤ r_i.
- (Section 2.5) i has been replaced with i throughout this section.
- (Section 3.1.3, p. 147) In a paragraph after equation (3.10), the fact that the apriori and posterior distribution over the states is the same owing to the fact that the states are MES, has been added.

- (Section 3.1.3, p. 151) The last two statements (which are italicized) have been appropriately changed.
- (Section 3.2, p. 169) Description in the introduction has been revised to make it simpler to read.
- (Section 3.2.1, p. 172) Definition of an extremal POVM has now been added as footnote. The proof of theorem 3.2.2 has still been retained for the sake of continuity and flow of argument.
- (Section 3.2.1, proof of theorem 3.2.3) In the beginning of 3.2.1 dim $\mathcal{H}_A = \dim \mathcal{H}_B$ is denoted by *n*, while the number of POVM elements is denoted by *d*.
- (Section 3.2.1, proof of theorem 3.2.3) Remedied the typo: $|s_{l'}\rangle_A \longrightarrow |l'\rangle_A$. $|\tilde{l}^*\rangle_A$ is the complex conjugate of $|\tilde{l}\rangle_A$ when represented in the ONB $\{|l'\rangle_A\}_{l=1}^n$ and $\{|\tilde{l}^*\rangle\langle\tilde{l}^*|\}_{l=1}^d \subset \mathcal{T}_{\perp}$ (for all ensembles).

I have added an explanation of the roles which \mathcal{T} and \mathcal{T}_{\perp} play, after the proof of theorem 3.2.3.

- (Section 3.2.1, p. 171) *mathcalHA* \longrightarrow \mathcal{H}_A typo corrected.
- (Section 3.2.1, p. 176) The correspondence $(a, b, c, d)^T$ is correct.
- (Section 4, p. 187) Corrected the line "subjected to the constraint that this error cannot be zero".

Tanmay Singal

Guide

ACKNOWLEDGEMENTS

I start by thanking my PhD. supervisor Sibasish Ghosh. Over the years we have developed a good working relationship, in which I've found him accessible and easy to discuss with. I owe him the debt and gratitude of giving me a great amount of academic freedom, even during the first half of my PhD. years when nothing which I pursued yielded any substantial results. I want to thank Prof. Simon for being a great source of inspiration and wisdom, for being a great teacher, and for reminding us (students in quantum information) of the standards of research we ought to aim for. I thank Professors Guruprasad Kar and Somshubro Banyopadhyay for being very accessible and easy to converse with. I want to thank Professor Chandrashekhar for his advice and help on many matters. Among students in the quantum information group, I thank, particularly, Rajarshi for stimulating a lot of intelligent discussions over a range of topics. I also enjoyed discussions with Sagnik and Manik.

I am grateful for the years I made Matscience my home for which I owe many folks from the institute - both individually and collectively. Among the first people I need to thank for this are those who entered the institute with me, in the physics stream and who were with me till the end - Abhra, Archana, Dibya, Nirmalya, and Rohan. I thank Abhra, Archana and Dibya particularly for all the fun times we had. Rohan and Champoo (Arghya) deserve a special mention for among the most memorable times I've had, Rohan, particularly for first year and Champoo for Thoraipakkam year and final years. I also thank Aarti with whom we all, as batch-mates, spent good times during the short period before she left. A significant part of my final two years in Matscience have been spent in a great deal of merry making for which I owe Aditi, Anup, Chandrashekhar, Divyarthi, Jyothnsaa, Madhu, Nitin, Prathamesh, Pratyush, Ramanathan, Sridhar and Varuni for funfilled memories. Along with Rohan and Champoo, I thank Madhu for among the best (and most intelligent) conversations I've had in my life. Rajarshi deserves a special mention in this list as well. I want to thank SK, Vandita and Aanchal who gave me reasons to want to visit Bangalore so many times during my PhD. years. I also thank Hershal and Pooja for the great times spent whenever I came home on a break from Chennai. I am grateful to have shared whatever time I did with all of the aforementioned people during my PhD. years, and I wish each of them a lifetime of happiness.

I thank Fahad, Chandan and Arya for being very co-operative office-mates - particularly Fahad for many discussions on issues related to academia and politics. There are many other people who I should thank: the mess workers in the kitchen, kiddo, many of the guards and cleaning ladies, the computer guys for always being there to help us students out... it's a long list and I can't recall everybody in this short space. That said I'll take some space to mention that I found Matscience to harbour very liberal values which I would be unlikely to find in most other institutes in this country, even in this day and age, and I'm very very grateful for that.

I am very grateful to authors like G. R. R. Martin, J. K. Rowling, the late Robert Jordan; manga artists like Kishimoto Masashi, Miura Kentaro, Isayama Hajime, Inoue Takehiko, Yukimura Makoto, Mashima Hiro; creators of the anime and animated series: Psycho Pass, Archer, Darker than Black, Black Lagoon, Rurouni Kenshin, Stein;s Gate, Chihayafuru, Avatar: Ang the last airbender, anything by Shinichiro Watanabe, Trigun, Ghost in the Shell, Neon Genesis Evangelion, Code Geass, Fullmetal Alchemist, Kabaneri of the Iron Fortress, anything by Studio Ghibli, Satoshi Kon, Death Note, Gungrave and Monster for creating whatever they did, taking me on great journeys across their imaginary landscapes and leaving me feeling empty every time I finished reading or watching whatever they created.

And lastly, I thank my family, whom I owe more than I can put in words.

Contents

Sy	nopsi	S	21			
Li	st of T	Fables	35			
1	Intro	troduction				
	1.1	Kinds of QSD Problems	39			
		1.1.1 Minimum Error Discrimination	40			
		1.1.2 Unambiguous State Discrimination	41			
		1.1.3 Discrimination by Local Operations and Classical Communication	43			
	1.2	Outline of Thesis	45			
2	Min	imum Error Discrimination	49			
	2.1	Introduction	49			
	2.2	Optimality Conditions	51			
	2.3	Structure of MED	52			
	2.4	MED for LI Pure States	53			
		2.4.1 Rotationally Invariant Necessary and Sufficient Conditions for MED	58			

		2.4.2	Solution for the MED of LI Pure State Ensembles	67	
		2.4.3	Remarks and Conclusion	93	
	2.5	.5 MED for Ensembles of LI Mixed States			
		2.5.1	Mathematical Structure for MED of Ensembles of LI States 9	97	
		2.5.2	Relation between MED for ensembles of LI pure states and LI		
			mixed states	27	
		2.5.3	Solution for the MED problem	32	
		2.5.4	Summary	35	
	2.6	Summa	ary	37	
		2.6.1	Future Directions	37	
3	Loca	l Dictin	michability of States	30	
5	LUCA	n Distin	guisnaulity of States 1.	39	
5	3.1	Necess	ary condition for LOCC of MES	39 41	
5	3.1	Necess 3.1.1	ary condition for LOCC of MES	39 41	
	3.1	Necess 3.1.1	ary condition for LOCC of MES	4141	
	3.1	Necess 3.1.1 3.1.2	ary condition for LOCC of MES	4141	
	3.1	Necess 3.1.1 3.1.2	ary condition for LOCC of MES	 41 41 41 41 	
	3.1	Necess 3.1.1 3.1.2 3.1.3	ary condition for LOCC of MES 14 Prerequisites: Classical and Quantum Entropic Quantities and Their Significance 14 Holevo-like Upper Bound for Locally Accessible Information of 14 The Necessary Condition 14	 41 41 41 41 46 47 	
	3.1	Necess 3.1.1 3.1.2 3.1.3 3.1.4	ary condition for LOCC of MES 14 Prerequisites: Classical and Quantum Entropic Quantities and Their Significance 14 Holevo-like Upper Bound for Locally Accessible Information of 14 The Necessary Condition 14 Example: Case When the Necessary Condition Becomes Sufficient 15	 41 41 41 46 47 51 	
	3.1	Necess 3.1.1 3.1.2 3.1.3 3.1.4 3.1.5	ary condition for LOCC of MES 14 Prerequisites: Classical and Quantum Entropic Quantities and Their Significance 14 Holevo-like Upper Bound for Locally Accessible Information of 14 Holevo-like Upper Bound for Locally Accessible Information of 14 The Necessary Condition 14 Example: Case When the Necessary Condition Becomes Sufficient Summary 16	 41 41 41 46 47 51 67 	
	3.1	Necess 3.1.1 3.1.2 3.1.3 3.1.4 3.1.5 3.1.6	guishability of States 14 ary condition for LOCC of MES 14 Prerequisites: Classical and Quantum Entropic Quantities and 14 Their Significance 14 Holevo-like Upper Bound for Locally Accessible Information of 14 Holevo-like Upper Bound for Locally Accessible Information of 14 The Necessary Condition 14 Example: Case When the Necessary Condition Becomes Sufficient 15 14 Summary 16 Future Directions 16	 41 41 41 46 47 51 67 68 	

		3.2.1	The Framework	171		
		3.2.2	Summary	183		
		3.2.3	Future Directions	184		
4	Disc	ussion		187		
	4.1	Summary and Significance				
	4.2	General Future Directions				
Aŗ	opend	ices		193		
A	LOC	CC of Fo	our Generalized Bell States in $\mathbb{C}^4\otimes\mathbb{C}^4$	195		
	A.1	Sets of states in theorem 3.1.2				
	A.2	LOCC	protocols for states in theorems 3.1.3 to 3.1.8	196		
		A.2.1	PMRS for each set in theorem 3.1.3	197		
		A.2.2	PMRS for each set in theorem 3.1.4	211		
		A.2.3	PMRS for each set in theorem 3.1.5	219		
		A.2.4	PMRS for each set in theorem 3.1.6	227		
		A.2.5	PMRS for each set in theorem 3.1.7	237		
		A.2.6	PMRS for each set in theorem 3.1.8	242		
В	Proc	of of Co	rollary 3.2.6.1	243		

References

Synopsis

Introduction

This thesis examines four distinct problems in Quantum State Discrimination (QSD). Two of these problems are in Minimum Error Discrimination (MED) and the remaining two in local distinguishability of quantum states.

Let \mathcal{H} be the Hilbert space of an *n* dimensional quantum system; then a quantum state of this system is represented by a density operator ρ , which is an observable on \mathcal{H} , with the properties $\rho \ge 0$ and $\operatorname{Tr}(\rho) = 1$. In a typical QSD problem one is given an unknown quantum state ρ_i with some probability p_i from an apriori agreed upon ensemble of quantum states $\widetilde{P} = \{p_i, \rho_i\}_{i=1}^m$, and one is then tasked with determining the value of *i* using measurement. Different kinds of QSD problems arise by imposing different kinds of constraints on the measurement-strategy. This thesis examines problems arising out of two different kinds of measurement constraints.

Minimum Error Discrimination: In this case the measurement strategy involves performing only one measurement, whose POVM elements E_i are in a one-to-one correspondence with the states ρ_i in the ensemble \tilde{P} in the sense that if the measurement yields the *i*-th measurement outcome, then it is inferred that one was given the *i*-th state ρ_i from \tilde{P} . Note that $\text{Tr}(\rho_i E_j)$ corresponds to the event where the *j*-th measurement outcome was obtained, conditioned on the fact that the *i*-th state was given. Such an event corresponds to an error. The average probability of error is given by $P_e = \sum_{\substack{i,j=1 \ i\neq j}}^m p_i \text{Tr}(\rho_i E_j)$. The scenario corresponding to success is when the *i*-th state is supplied and the *i*-th measurement outcome is obtained. The average probability of success is then given by

$$P_s = \sum_{i=1}^m p_i \operatorname{Tr} \left(\rho_i E_i \right).$$
(1)

It is easily seen that $P_s + P_e = 1$. This implies that every trial, in which a state is supplied and a measurement is performed, is either a success or results in an error. For the MED of a given ensemble of states \tilde{P} , the goal is to maximize the average probability of success P_s over the space of *m*-element POVMs (*m*-POVMs). The maximum success probability, denoted by P_s^{max} , is given by

$$P_s^{max} = \operatorname{Max} \left\{ P_s | \left\{ E_j \right\}_{j=1}^m \text{ is an } m \operatorname{POVM} \right\}.$$
(2)

Discrimination by Local Operations and Classical Communication: A situation often encountered in problems in QSD is when $\rho_1, \rho_2, \dots, \rho_m$ are bipartite or multipartite states of a composite system, whose subsystems are physically separated from each other. This physical separation imposes the constraint that parties possessing subsystems can only make local measurements on these subsystems and communicate classically their respective measurement outcomes to their peers. Protocols of this type are known as Local Operations and Classical Communication (LOCC) protocols. A significant question in QSD is to determine under what conditions can bipartite/multipartite states in an ensemble be discriminated amongst using an LOCC protocol, or in other words, can be discriminated locally?

Consider the instance where the composite system is bipartite, the Hilbert spaces of whose subsystems are given by \mathcal{H}_A and \mathcal{H}_B ; the Hilbert space of the composite system is then given by $\mathcal{H}_A \otimes \mathcal{H}_B$. Let Alice be in possession of the subsystem whose Hilbert space is \mathcal{H}_A and Bob the system whose Hilbert space is \mathcal{H}_B . LOCC protocols are classified as 1-LOCC, 2-LOCC, \cdots , *N*-LOCC - where *N* stands for the number of rounds of communication which the protocol demands Alice and Bob engage in.

In this thesis, I have taken up two problems each in minimum error discrimination and in local distinguishability of quantum states. I will summarize all the four problems over the coming sections of this synopsis.

After the introductory chapter, the second chapter covers problems which I worked on in the topic of minimum error discrimination. The third chapter covers problems which I worked on in the topic of local distinguishability of quantum states.

Minimum Error Discrimination

Minimum Error Discrimination is one of the oldest problems in QSD. There are very few ensembles for which MED has been solved analytically. The MED problem for ensembles of linearly independent (LI) pure states has an interesting structure which relates the ensemble to its optimal POVM. This was discovered by Belavkin and Maslov [1, 2] and Mochon [3] independently. Before our work, this structure wasn't used to obtain the optimal POVM for the MED of an ensemble (of LI pure states). Our¹ efforts were directed to use this structure to obtain the optimal POVM for the MED of an ensemble of LI pure states. Furthermore we were interested to see if a similar structure could also be found for the MED problem of other classes of ensembles, and when found, if said structure could then be exploited to obtain the optimal POVM for ensembles in those classes.

Minimum Error Discrimination for Linearly Independent Pure States

The work in this section is based on the following paper: "Minimum Error Discrimination for an Ensemble of Linearly Independent Pure States", Tanmay Singal, and Sibasish Ghosh. This paper has been published in Journal of Physics A: Mathematical and Theoretical².

Structure of the MED problem for ensembles of LI pure states: The MED problem for ensembles of LI pure states is known to have two interesting properties. Let $\tilde{P} = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^m$ be an ensemble of states, where $|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_m\rangle \in \mathcal{H}$ are LI and span \mathcal{H} . Then the number of states in \tilde{P} is equal to $dim\mathcal{H}$, i.e., m = n. Said properties of the MED problem for \tilde{P} are:

¹ Myself and Sibasish Ghosh

² http://dx.doi.org/10.1088/1751-8113/49/16/165304

- (i) The optimal POVM is a unique rank one projective measurement [4, 5, 3].
- (ii) The optimal POVM for MED of *P* is the pretty good measurement (PGM) of another ensemble, *Q* ≡ {*q_i* > 0, |*ψ_i*⟩⟨*ψ_i*]}ⁿ_{*i*=1} [1, 2, 3]. Note that the *i*-th states in *P* and *Q* are the same, whereas the probabilities are generally not. Additionally, in [3], it is explicitly shown that the ensemble pair (*P*, *Q*) are related through an invertible map.

To emphasize: it is the fact that \tilde{P} and \tilde{Q} are related by an invertible map that makes the mathematical structure of MED of LI pure state ensembles richer than that of MED of general ensembles. To formalize the invertible relation between \tilde{P} and \tilde{Q} we will now define certains sets and functions over these sets.

(1) \mathcal{E} is the set of all ensembles of the form $\widetilde{P} = \{p_i > 0, |\psi_i\rangle\langle\psi_i|\}_{i=1}^n$ of d LI pure states $|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_n\rangle$ which span \mathcal{H} . (2) \mathcal{P} is the set of rank one projective measurements; an element in \mathcal{P} is of the form $\{|v_i\rangle\langle v_i|\}_{i=1}^n$ where $\langle v_i|v_j\rangle = \delta_{ij}, \forall 1 \leq i, j \leq n$. (3) $\mathcal{P} : \mathcal{E} \longrightarrow \mathcal{P}$ is such that $\mathcal{P}(\widetilde{P})$ is the optimal POVM for the MED of $\widetilde{P} \in \mathcal{E}$. (4) Let *PGM* denote the PGM map, i.e., *PGM* : $\mathcal{E} \longrightarrow \mathcal{P}$ is such that $PGM(\widetilde{Q})$ is the PGM of any $\widetilde{Q} \in \mathcal{E}, PGM(\widetilde{Q}) = \left\{\rho_q^{-\frac{1}{2}}q_i|\psi_i\rangle\langle\psi_i|\rho_q^{-\frac{1}{2}}\right\}_{i=1}^n$, where $\rho_q = \sum_{i=1}^n q_i|\psi_i\rangle\langle\psi_i|$.

The point (ii) above says that there exists an invertible map, $\mathscr{R}: \mathcal{E} \longrightarrow \mathcal{E}$ such that

$$\mathscr{P}\left(\widetilde{P}\right) = PGM\left(\mathscr{R}\left(\widetilde{P}\right)\right), \ \forall \ \widetilde{P} \in \mathcal{E}.$$
(3)

Knowing the action of \mathscr{R} on a general point \widetilde{P} in \mathscr{E} solves the MED problem for LI pure state ensembles. While the action of \mathscr{R} on a general point in \mathscr{E} isn't known, the action of \mathscr{R}^{-1} on any general point in \mathscr{E} was derived in [2, 1, 3].

Motivation: To see if the knowledge of the action of \mathscr{R}^{-1} on \mathscr{E} could enable us to find the action of \mathscr{R} on \mathscr{E} .

Brief Summary of Work: Our motivation naturally guided us to use the inverse function theorem - a well-known result in functional analysis - to find the action of the inverse of \mathscr{R}^{-1} , i.e., to find the action of \mathscr{R} , on any point of \mathscr{E} . To do this we reformulated the MED problem in a rotationally invariant form. This was done in two steps:

(i.) The necessary and sufficient conditions, which a POVM must satisfy to maximize P_s for a fixed ensemble of LI pure state ensembles, was reformulated in a rotationally invariant form. While \tilde{P} itself doesn't feature in these rotationally invariant conditions, the gram matrix G, of the ensemble \tilde{P} , does. The matrix elements of G are given by

$$G_{ij} = \sqrt{p_i p_j} \langle \psi_i | \psi_j \rangle, \ \forall \ 1 \le i, j \le n.$$

(ii.) \mathscr{R} is a rotationally covariant function on \mathcal{E} , i.e.,

$$\mathscr{R}\left(\left\{p_{i}, U|\psi_{i}\rangle\langle\psi_{i}|U^{\dagger}\right\}_{i=1}^{n}\right) = \left\{q_{i}, U|\psi_{i}\rangle\langle\psi_{i}|U^{\dagger}\right\}_{i=1}^{n},$$

 \forall unitaries U acting on \mathcal{H} . Using the rotationally invariant necessary and sufficient condition, we define the rotationally invariant version of \mathscr{R} - denoted by $\mathscr{R}_{\mathcal{G}}$ - on the set \mathcal{G} of gram matrices for all ensembles in \mathcal{E} . In other words, if $\mathscr{R}(\widetilde{P}) = \widetilde{Q}$, then $\mathscr{R}^{(\mathcal{G})}(G) = G_q$, where G is the gram matrix for \widetilde{P} and G_q is the gram matrix for \widetilde{Q} .

Just as we don't know the action of \mathscr{R} on any $\widetilde{P} \in \mathscr{E}$, we don't know the action of $\mathscr{R}^{(\mathcal{G})}$ on any $G \in \mathcal{G}$. Similarly, just as we know the action of \mathscr{R}^{-1} on any $\widetilde{Q} \in \mathscr{E}$, we know the action of $(\mathscr{R}_{\mathcal{G}})^{-1}$ on any $G_q \in \mathcal{G}$. We then employ the implicit function theorem, which is the equivalent of the inverse function theorem, to obtain the inverse of the action of $(\mathscr{R}^{(\mathcal{G})})^{-1}$, i.e., of $\mathscr{R}^{(\mathcal{G})}$, on any $G \in \mathcal{G}$.

Further on, since the present technique is specific to the problem of MED for n-LI pure state ensembles, it is expected that the algorithm our technique offers is computationally

as expensive as or less expensive than existing techniques. We show that this is indeed the case.

Significance: It is seen that the computational complexity (time and space) of this algorithm is as efficient as those given by standard semidefinite programming methods, while being simpler to implement.

Analytic Structure of the Minimum Error Discrimination Problem for Linearly Independent Mixed States

The work in this section is based on the following paper: "Algebraic Structure of the Minimum Error Discrimination Problem for Linearly Independent Density Matrices", Tanmay Singal, and Sibasish Ghosh. This paper has been uploaded on the arxiv³, and will soon be sent for publication.

Motivation: It has been previously seen that the mathematical structure of the MED problem for LI pure state ensembles can be exploited to solve the corresponding MED problem. In this section we undertake the exercise to generalize the same structure to the MED problem of ensembles whose states are LI mixed states. We call a set of quantum states $\{\rho_i\}_{i=1}^m$ LI if any set of non-zero vectors $\{|\psi_i\rangle\}_{i=1}^m$, such that $|\psi_1\rangle \in \text{supp}(\rho_1), |\psi_2\rangle \in \text{supp}(\rho_2), \dots, |\psi_m\rangle \in \text{supp}(\rho_m)$, are LI [6]. Here the span of $\bigcup_{i=1}^m \text{supp}(\rho_i)$ is equal to \mathcal{H} . Let $r_i \equiv rank(\rho_i), \forall 1 \le i \le m$. Then $\sum_{i=1}^m r_i = n$.

Brief Summary of Work: The basic ingredients required for this mathematical structure are as follows:

(*i*) The set of all ensembles of LI mixed states in which the *i*-th state has rank r_i . We denote this set by \mathcal{E}_r , where $\underline{r} = (r_1, r_2, \cdots, r_m)$.

³ http://arxiv.org/abs/1412.7174

- (*ii*) The set of projective measurements, such that the rank of the *i*-th projector is r_i . We denote this set by \mathcal{P}_r .
- (*iii*) It has been shown that the optimal POVM for the MED of an ensemble of LI mixed states $\widetilde{P}_{r_{j}} \in \mathcal{E}_{r_{j}}$ is a unique projective measurement $\{\Pi_{i}\}_{i=1}^{m} \in \mathcal{P}_{r_{j}}$ [6, 7]. The uniqueness of the optimal POVM allows us to define $\mathscr{P}_{r_{j}} : \mathcal{E}_{r_{j}} \longrightarrow \mathscr{P}_{r_{j}}$ as the optimal POVM map on $\mathcal{E}_{r_{j}}$. Thus, $\mathscr{P}_{r_{j}}(\widetilde{P}_{r_{j}})$ is the unique optimal POVM in $\mathscr{P}_{r_{j}}$ for the MED of any ensemble $\widetilde{P}_{r_{j}} \in \mathcal{E}_{r_{j}}$.

(*iv*) $PGM_{r}: \mathcal{E}_{r} \longrightarrow \mathcal{P}_{r}$ is the PGM map on \mathcal{E}_{r} . So if $PGM_{r}(\widetilde{Q}_{r}) = {\{\Pi_{i}\}_{i=1}^{m}}$ then

$$\Pi_i = \left(\rho_q\right)^{-\frac{1}{2}} q_i \rho_i \left(\rho_q\right)^{-\frac{1}{2}},$$

where $\rho_q = \sum_{j=1}^n q_j \rho_j$.

Thus the objective of this work is to prove that for any positive integers r_1, r_2, \dots, r_m such that $\sum_{i=1}^m r_i = n$, there exists an invertible map $\mathscr{R}_r : \mathscr{E}_r \longrightarrow \mathscr{E}_r$ such that the following equation holds true for all $\widetilde{P}_r \in \mathscr{E}_r$.

$$\mathscr{P}_{\vec{r}}\left(\widetilde{P}_{\vec{r}}\right) = PGM_{\vec{r}}\left(\widetilde{Q}_{\vec{r}}\right) = PGM_{\vec{r}}\left(\mathscr{R}_{\vec{r}}\left(\widetilde{P}_{\vec{r}}\right)\right).$$
(4)

Equation (4) is the mixed state version of the equation (3). We say that this is a generalization of the pure state case because for the pure state case, i.e., for the case $r_i = 1$, $\forall 1 \le i \le n$, we know that such a function \mathscr{R} exists [1, 2, 3].

Next are listed the sequence of steps which we go through to prove the existence of the function \mathcal{R}_r .

(i) It is shown that the necessary and sufficient conditions for the MED of LI states (mixed or pure) are actually simpler than for the general case.

- (ii) Rotationally invariant form of these simplified necessary and sufficient conditions for MED of LI states are then obtained.
- (iii) Using the rotationally invariant conditions, it is shown that for each ensemble $\widetilde{P}_{r,\vec{r}}$ in $\mathcal{E}_{r,\vec{r}}$ one can associate another *unique* ensemble $\widetilde{Q}_{r,\vec{r}}$, also in $\mathcal{E}_{r,\vec{r}}$, and such that it satisfies

$$PGM_{\vec{r}}\left(\widetilde{Q}_{\vec{r}}\right) = \mathscr{P}_{\vec{r}}\left(\widetilde{P}_{\vec{r}}\right)$$

This allows us to define $\mathscr{R}_r : \mathscr{E}_r \longrightarrow \mathscr{E}_r$ such that

$$\mathscr{R}_{\vec{r}}\left(\widetilde{P}_{\vec{r}}\right) = \widetilde{Q}_{\vec{r}}$$

and

$$\mathscr{P}_{\underline{r}}\left(\widetilde{P}_{\underline{r}}\right) = PGM_{\underline{r}}\left(\widetilde{Q}_{\underline{r}}\right).$$

- (iv) Many results from above are used to prove that \mathscr{R}_r is one-to-one and onto, i.e., \mathscr{R}_r is invertible. A closed form expression for \mathscr{R}_r^{-1} is then easily obtained.
- (v) Finally it is shown that the rotationally invariant necessary and sufficient conditions suggest a numerical technique to obtain solve the MED problem for any ensemble $\widetilde{P}_{\vec{r}} \in \mathcal{E}_{\vec{r}}$. This technique is seen to be as computationally efficient as the barrier type interior point method, which is a standard SDP technique.

Significance: It is seen that the computational complexity (time and space) of this algorithm is as efficient as those given by standard SDP methods, while being simpler to implement.

Local Distinguishability of Quantum States

Finding the protocol which optimally discriminates among multipartite states using only LOCC is harder than doing so without the LOCC restriction. The reason for this is that the structure of LOCC isn't fully understood, and hence remains unexploited. Due to

this difficulty a significant amount of effort is limited to establishing when a set of pairwise orthogonal states, which are perfectly distinguishable under global operations, are distinguishable or indistinguishable under LOCC.

Motivation: Despite significant advances in the topic of local distinguishability of quantum states, very few results are independent of the dimension of the systems for which they are proven. Underlying our curiosity about the local (in)distinguishability of quantum states, is the belief that results independent of dimension do exist, and we sought to find a few such results.

Necessary Condition for Local Distinguishability of Maximally Entangled States

In [8], Badziag et al. introduced a Holevo-like-upper-bound, χ^{locc} , for the locally accessible information - information retrievable by performing only LOCC - of an ensemble of bipartite states. In the present work⁴ we⁵ show that χ^{locc} plays a significant role in the perfect local distinguishability of maximally entangled states (MES).

Brief Summary of Work: The fundamental principle underlying the work is that distinguishing among *m* orthogonal bipartite states $\{|\psi_i\rangle_{AB}\}_{i=1}^m$ by LOCC requires extracting log_2m bit of classical information from the set $\{|\psi_i\rangle_{AB}\}_{i=1}^m$ through some LOCC protocol. When the states are maximally entangled states (MES) we encounter three important features:

- (i) The Holevo-like upper bound on any set of *m* orthogonal bipartite MES $\{|\psi_i\rangle_{AB}\}_{i=1}^m$ in $\mathbb{C}^n \otimes \mathbb{C}^n$ is $\chi^{locc}(\{|\psi_i\rangle_{AB}\}_{i=1}^m) = log_2m$ bit.
- (ii) The first measurement cannot rule out any state from the set $\{|\psi_i\rangle_{AB}\}_{i=1}^m$. Thus after the first measurement of the LOCC protocol, both parties still need to extract

⁴ http://arxiv.org/abs/1506.03667, sent for review to Journal of Physics A: Mathematical and Theoretical.

⁵ Myself, Ramij Rahaman, Sibasish Ghosh, Guruprasad Kar

log₂m bit of classical information from the set of post-measurement states.

(iii) Let Alice be the party who initiates the LOCC protocol, and let α be the index corresponding to the outcome of her measurement. Let the post-measurement joint state be denoted by $|\psi_{i,\alpha}\rangle_{AB}^{6}$, and let $\rho_{\alpha}^{(B)}$ denote the average post-measurement reduced state (PMRS) on Bob's side, i.e., $\rho_{\alpha}^{(B)} = \frac{1}{m} \sum_{i=1}^{m} Tr_A (|\psi_{i,\alpha}\rangle \langle \psi_{i,\alpha}|)$. We show that after the first measurement, the Holevo like upper bound for the post-measurement states, $\{|\psi_{i,\alpha}\rangle_{AB}\}_{i=1}^{m}$ is given by $\chi^{locc} (\{|\psi_{i,\alpha}\rangle_{AB}\}_{i=1}^{m}) = S(\rho_{\alpha}^{(B)})$.

Thus, a necessary condition for the local distinguishability of the states $\{|\psi_i\rangle_{AB}\}_{i=1}^m$ is: $\chi^{locc}\left(\{|\psi_{i,\alpha}\rangle_{AB}\}_{i=1}^m\right) = S\left(\rho_{\alpha}^{(B)}\right) \ge \log_2 m$ bit, for all α outcomes.

When m = n, $S\left(\rho_{\alpha}^{(B)}\right)$ should be $log_{2}n$ bit exactly, which implies that the average PMRS on Bob's side should be maximally mixed, i.e., $\rho_{\alpha}^{(B)} = \frac{1}{d}\mathbb{1}_{B}$, where $\mathbb{1}_{B}$ is the identity operator on Bob's subsystem. The condition $\rho_{\alpha}^{(B)} = \frac{1}{d}\mathbb{1}_{B}$ imposes constraints on the α -th POVM effect of Alice's starting POVM. As a result of imposing the condition, if the α -th POVM effect is constrainted to be a multiple of $\mathbb{1}_{A}$, the only measurement Alice can perform is one whose POVM effects are multiples of the identity, i.e., she can only perform a trivial measurement. Hence a necessary condition for the local distinguishability of these n orthogonal bipartite MES is that the POVM - constrainted to satisfy the condition $\rho_{\alpha}^{(B)} = \frac{1}{d}\mathbb{1}_{B}$ - should not be trival.

The next question to ask is how strong this necessary condition is. We tested this condition for the local distinguishability of all sets of four Generalized Bell States in $\mathbb{C}^4 \otimes \mathbb{C}^4$. A significant number of sets (of four Generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$) failed the necessary condition, implying that they are locally indistinguishable. All remaining sets satisfied the necessary condition, but that does not imply that they are locally distinguishable. Therefore it comes as a big surprise that these sets, which satisfy the necessary condition, are *indeed* locally distinguishable, and that too by one-way LOCC and using only projective

⁶ For the α -th outcome, the measurement transforms the state as: $|\psi_i\rangle_{AB} \xrightarrow{\alpha \text{-th outcome}} |\psi_{i,\alpha}\rangle_{AB}$.

measurements; we proved this by finding LOCC protocols for *all* sets each of which satisfies the necessary condition. This implies that for four Generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, this condition is sufficient as well as necessary. This gives an indication that this necessary condition can be pretty powerful.

Significance: MES have acquired an elevated status in quantum information, particularly in quantum communication. Consequentially, when properties ascribable to an ensemble of quantum states are studied, there is a special focus on those cases wherein the ensemble comprises of MES. It is this that makes the perfect local distinguishability of MES a significant problem. The Holevo-liked upper bound χ^{locc} tells us that no more than *n bit* of classical information can be extracted from an ensemble of MES in $\mathbb{C}^n \otimes \mathbb{C}^n$ systems [9]. Hence, it is of particular interest to know when this upper limit can be attained, which is why the local distinguishability of *n* MES in $\mathbb{C}^n \otimes \mathbb{C}^n$ systems is among one of the more significant problems in the field of local distinguishability. Given this context, the significance of this work is as follows: hithero, the only known constraint on the measurements of the LOCC protocols were that they have to be orthogonality preserving. In this work, we have introduced a stronger constraint which subsumes orthogonality preservation.

Framework for Distinguishability of Orthogonal Bipartite States by Local Operations and One-Round of Classical Communication

In this work⁷ I propose a framework for the distinguishability of orthogonal bipartite states by local operations and one-round of classical communication. This is based on work previously done by Ye et al. [10] and arguments made by Michael Nathanson [11].

Brief Summary of Work: For a given set of orthogonal bipartite states in $n \otimes n$ systems, each party of the bipartite system is associated with a subspace of $n \times n$ hermitian matrices,

⁷ http://dx.doi.org/10.1103/PhysRevA.93.030301

denoted by \mathcal{T}_{\perp} ; \mathcal{T}_{\perp} contains information of all one-way LOCC protocols (1-LOCC) which this associated party can initiate to locally distinguish (probabilistic or perfectly) the given set of orthogonal bipartite states. I show that techniques and algorithms to extract such information from \mathcal{T}_{\perp} vary depending on $dim \mathcal{T}_{\perp}$. This gives a natural criterion to partition the set of all sets of orthogonal bipartite states in $n \otimes n$ into n^2 different classes, so that the dimension of \mathcal{T}_{\perp} , corresponding to a fixed party, is constant over all sets of orthogonal bipartite states in a class. It is seen that sweeping results for the local distinguishability of orthogonal quantum states in $n \otimes n$ systems have already been given for classes corresponding to the upper and lower extreme values of $dim \mathcal{T}_{\perp}$: Walgate et al's result [12] corresponds to the class for which $n^2 - 2 \le dim \mathcal{T}_{\perp} \le n^2$ and Ye et al's result [10] corresponds to the case $dim \mathcal{T}_{\perp} = 1$. I give similarly sweeping results for the local distinguishability by one-way LOCC for sets of orthogonal bipartite states which lie in classes corresponding to some intermediary values of $dim \mathcal{T}_{\perp}$, i.e., for $2 \leq dim \mathcal{T}_{\perp} \leq \sqrt{3n^2 - 3n + \frac{1}{4}} + \frac{3}{2}$; some of these results are necessary and sufficient conditions for local (in)distinguishability of all sets of orthogonal states in a class, while others are necessary but not sufficient. A significant corollary of one among the aforementioned results is the necessary and sufficient condition for the 1-LOCC distinguishability of almost all sets of n orthogonal bipartite pure states in $\mathbb{C}^n \otimes \mathbb{C}^n$.

Significance: While a lot of work has been done in the field of local distinguishability of orthogonal quantum states, most of these appear to be scattered because no underlying framework which these results emerge from, has been identified. Some very significant results in the local distinguishability of orthogonal quantum states [12, 13] are seen to emerge from the framework proposed here. This framework also gives *general* necessary and sufficient conditions for the 1-LOCC distinguishability for a large class of sets of orthogonal bipartite quantum states. Particularly, it gives the necessary and sufficient conditions for the 1-LOCC distinguishability of almost all sets of *n* orthogonal bipartite pure states in $\mathbb{C}^n \otimes \mathbb{C}^n$. To add a final comment on the usefulness of this framework:

List of Tables

2.1	Time and space complexity of various steps in the Taylor series algorithm.	86
2.2	Time and space complexity of various steps in Newton's method	89
2.3	Time and space complexity of steps in Barrier-Type Interior Point Method	92
3.1	Eigenvalues and Eigenvectors of $K_{\alpha}^{\dagger}K_{\alpha}$	157
3.2	Some sets of four Generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ which aren't distinguishable.	161
3.3	Remaining sets of four Generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ which aren't distinguishable.	161
3.4	Some sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.3.	164
3.5	Remaining sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.3.	164
3.6	Sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.4.	164
3.7	Sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.5.	165

3.8	Sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be	
	one-way locally distinguishable in theorem 3.1.6	56
3.9	Sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be	

one-way locally distinguishable in theorem 3.1.7.

A.1	Sets of states of the form	$\{ \psi_{a0}^{(4)}\rangle, \psi_{b1}^{(4)}\rangle,$	$ \psi_{c2}^{(4)}\rangle, \psi_{d3}^{(4)}\rangle\},$	where $a, b, c, d \in$	
	$\{0, 1, 2, 3\}$				5

A.2 Sets of states of the form $\{|\psi_{0a}^{(4)}\rangle, |\psi_{1b}^{(4)}\rangle, |\psi_{2c}^{(4)}\rangle, |\psi_{3d}^{(4)}\rangle, \}$, where $a, b, c, d \in \mathbb{Z}_4$ 196
Chapter 1

Introduction

In information theory, a message is encoded into a physical system, which is then transmitted by a sender across an information channel, and then decoded by a receiver to obtain the original message. Different messages are encoded as different states of the same system. For instance in radio communication, a carrier signal is *modulated* to encode different messages which need to be transmitted. Hence decoding the signal depends on the receiver's ability to distinguish among various states of the same physical system.

In quantum information theory, messages are encoded into states of a quantum system. These quantum states are then sent across through a quantum channel, and must finally be decoded by a receiver. If the quantum states don't lie on orthogonal supports, one has to distinguish among them by performing a measurement on them, whose measurement outcomes should yield the distinguishing criteria. To perfectly distinguish between different quantum states (i.e., successfully discriminate between quantum states in each trial) one requires that the quantum states should be orthogonal. Often, the states which come out of the channel aren't orthogonal to each other, owing to which they cannot be perfectly discriminated. In such a scenario one can choose one of many different imperfect strategies to try and distinguish among the states. Imperfect strategies of discrimination refer to strategies which can result in either errors or inconclusive outcomes. The field of

distinguishing non-orthogonal quantum states is known as **Q**uantum **S**tate **D**iscrimination (QSD).

Scenario in QSD: Technically, the scenario in QSD is as follows. Let \mathcal{H} be the Hilbert space of an *n* dimensional quantum system; then a quantum state of this system is represented by a density operator ρ , which is a linear operator on \mathcal{H} , with the properties $\rho \ge 0$ and $\operatorname{Tr}(\rho) = 1$. In a typical QSD problem one is given an unknown quantum state ρ_i from an apriori agreed upon ensemble of quantum states $\widetilde{P} = \{p_i, \rho_i\}_{i=1}^m$. p_i is the probability with which ρ_i is selected from \widetilde{P} . Note that $0 < p_i < 1$ and $\sum_{i=1}^m p_i = 1$. One has to perform a quantum measurement on the unknown quantum state ρ_i to obtain information about it. We now describe quantum measurements.

Quantum Measurements: Under a quantum measurement, the state of a quantum system ρ_i undergoes an instantaneous and, generally, a discontinuous transformation - a jump - to one among *d* number of other quantum states, i.e.,

$$\rho_i \longrightarrow \frac{1}{\operatorname{Tr}\left(M_j \rho_i M_j^{\dagger}\right)} M_j \rho_i M_j^{\dagger}, \qquad (1.1)$$

where $\{M_j\}_{j=1}^d$ is a set of operators acting on \mathcal{H} , with the property $\sum_{i=1}^d M_j^{\dagger} M_j = 1$, where 1 is the identity operator acting on \mathcal{H} . The operators $\{M_j\}_{j=1}^d$ are specific, but not unique to the measurement, and are called its Kraus operators. Note that there are as many measurement outcomes as there are Kraus operators for a quantum measurement. The jump in equation (1.1) is a random process. Given the state ρ_i , the probability that the measurement will transform ρ_i to $\frac{1}{\operatorname{Tr}(M_j\rho_i M_j^{\dagger})} M_j \rho_i M_j^{\dagger}$ is given by Born's rule.

$$p_{ji} = \operatorname{Tr}\left(M_{j}^{\dagger}M_{j}\rho_{i}\right) = \operatorname{Tr}\left(E_{j}\rho_{i}\right), \qquad (1.2)$$

where

 $p_{j|i}$ is the probability of obtaining the *j*-th outcome $\frac{1}{\text{Tr}(M_j\rho_i M_j^{\dagger})} M_j \rho_i M_j^{\dagger}$, conditioned

upon being given the *i*-th state from the ensemble \widetilde{P} , and

 E_j are positive semidefinite operators acting on \mathcal{H} , defined by $E_j \equiv M_j^{\dagger} M_j$. This implies that $\sum_{j=1}^d E_j = \mathbb{1}$.

The set of observables $\{E_i\}_{i=1}^d$ associated with the same measurement is known as a **P**ositive **O**perator **V**alued **M**easure (POVM). Often measurements are merely given in their POVM form, rather than their Kraus operator form. For the purpose of quantum state discrimination, we will need to use the POVM form more often than the Kraus operator form.

1.1 Kinds of QSD Problems

Being able to perfectly distinguish among the ρ_i 's requires that each measurement outcome corresponds to some unique state ρ_i from the ensemble. There don't always exist measurement strategies that always yield distinct outcomes for distinct states provided. In fact, as was mentioned in the beginning of this chapter, such a measurement strategy exists only when the ρ_i 's in \widetilde{P} are orthogonal, i.e., $\operatorname{Tr}(\rho_i \rho_j) = 0$, when $i \neq j$. When any two ρ_i 's in \widetilde{P} are non-orthogonal, there is inevitably some measurement outcome which corresponds to two or more of the ρ_i 's. In such a scenario, there are different strategies to decode the measurement outcome to know which of the ρ_i 's was sent. Usually, each strategy is associated with a real function f, which is a function of the ρ_i 's, and the POVM elements E_i 's. f measures how well the E_i 's can distinguish among the ρ_i 's. One can compare the distinguishing power of two distinct POVMs $\{E_i\}_{i=1}^d$ and $\{E'_i\}_{i=1}^d$, based on the value f takes over them, while keeping the ρ_i 's fixed. Finding the POVM which optimizes the value of f is then the goal of the QSD problem. There are different choices for distinguishability functions f. All such f's are required to satisfy the condition that in the limit that the ρ_i 's are orthogonal, they are perfectly distinguishable, and the perfect distinguishability is achieved by the measurement $\{E_i\}_{i=1}^m$, where E_i is a projector on

 $\operatorname{supp}(p_i\rho_i).$

Different f's correspond to different QSD problems. In the following we list three different kinds of QSD problems.

1.1.1 Minimum Error Discrimination

In the class of quantum state discrimination problems, minimum error discrimination (MED) is one of the oldest. The setting in MED is as follows. Alice has a fixed ensemble of states $\tilde{P} = \{p_i, \rho_i\}$. She selects one of these states (ρ_i , say) with probability p_i and gives it to Bob without telling him which state she gave him. Bob knows that Alice has selected the state from the set $\{\rho_i\}_{i=1}^m$ with apriori probabilities p_i and his job is to figure out which state he has been given using an *m*-element POVM. In MED, Bob's measurement strategy is constrained in the following way: there is a one-to-one correspondence between elements in Alice's ensemble $\{p_i, \rho_i\}_{i=1}^m$ and Bob's POVM elements $\{E_i\}_{i=1}^m$, so that when the *i*-th measurement outcome clicks, Bob infers Alice gave him the *i*-th state from her ensemble. When $\rho_1, \rho_2, \dots, \rho_m$ are not orthogonal, errors are likely to occur. Bob's job is to find the optimal POVM for minimizing the average probability of this error or equivalently maximizing the average probability of success.

History: The problem of MED was studied during a time which predates the time when quantum information came to be recognized as a field in its own right. The problem for two states was introduced by Bakut and Schurov in 1968. This was solved by Helstrom [5] in the 70's. Thereafter problems in MED of a more general nature were studied by Yuen et al. [14], Kennedy [4], Holevo [15] and Belavkin [1, 2] in the 70's, and have since been analytically solved for very few number of different kinds of ensembles. One notable example of this kind is an ensemble of geometrically uniform set of states [16]. A more complete list of the major developments in MED will be given in the introduction of chapter 2. Minimum error discrimination is alternatively referred to as Quantum Hypoth-

esis Testing. One of the reasons why this problem acquired significance in the 70's was that while radio communication systems could be modeled based on a classical description of electromagnetic radio waves, quantum mechanical description of light is necessary to accurately describe optical communication systems. The theories of classical hypothesis testing are incompatible with quantum optics, owing to which a completely new theory, based on the non-commutativity of measurement receptors had to be developed [5]. That said, MED also acquires significance in the classical wave pattern recognition problem. This was because of the incompatibility of optical filters to distinguish among non-orthogonal incoming signals reliably [2].

1.1.2 Unambiguous State Discrimination

For the unambiguous state discrimination of an ensemble of m states $\widetilde{P} = \{p_i, \rho_i\}_{i=1}^m$, the measurement performed is such that it has m + 1 outcomes, i.e., the POVM is of the form $\{E_i\}_{i=1}^{m+1}$, and is such that if the *i*-th state is given, the measurement yields either the *i*-th outcome or the (m + 1)-th outcome. This implies that $p_{ji} = 0$, when $j \neq m + 1$ and $j \neq i$. Thus if the measurement yields any of the first m outcomes one can conclude - without ambiguity - which state was provided, but if it yields the (m + 1)-th outcome then one states that the protocol is inconclusive. The average probability of yielding the inconclusive outcome will be denoted by P_{in} , and it is given by

$$P_{in} = \sum_{i=1}^{m} p_i \text{Tr}(\rho_i E_{m+1}), \qquad (1.3)$$

and similarly the average probability of yielding a conclusive outcome, which will be denoted by P_c , is given by

$$P_{c} = 1 - \sum_{i=1}^{m} p_{i} \operatorname{Tr} \left(\rho_{i} E_{m+1} \right) = \sum_{i=1}^{m} p_{i} \operatorname{Tr} \left(\rho_{i} E_{i} \right).$$
(1.4)

In the unambiguous state discrimination problem the distinguishability-measure for an ensemble of quantum states \widetilde{P} is P_c . Hence, the goal in unambiguous state discrimination is to find where P_c takes its maximum value over that subset of all (m + 1)-POVMs which is also constrained so that $p_{ji} = 0$, for all $j \neq m + 1$, $j \neq i$ and $1 \leq i \leq m$. This subset within the set of all (m + 1)-POVMs is called the feasible region.

History and Significant Results: In contrast to the MED problem, the unambiguous state discrimination problem acquired significance primarily out of theoretical interest, rather than practical concerns. The problem of unambiguous state discrimination was first proposed for an ensemble of two pure states by Ivanovic in [17] in 1987. In 1988, Dieks and Peres separately obtained the minimum value of P_{in} for an ensemble of two equiprobable pure states [18, 19]. In 1995, Jaegar and Shimoney obtained the minimum value of P_{in} for any ensemble of two pure states [20]. In 1998, Chefles showed that any set of *n* pure states can be unambiguously discriminated among only when they are linearly independent [21]. He and Barnett also gave the optimum unambiguous state discrimination strategy for pure linearly independent symmetric states [22]. Also, in 1998, Peres and Terno gave the optimal distinguishing strategy for any set of three states [23]. In 2001, Sun, Bergou and Hillery solved the problem for an ensemble of a mixed state and a pure states in two dimensions [24], and in 2003, Bergou, Herzog and Hillery solved it for general *n* pure states in [25]. In 2004, Eldar et al. reformulated unambiguous state discrimination into a semidefinite programming problem [26]. In 2005, Herzog and Bergou considered the problem for two mixed states [27]. In 2007, Rayden and Lutkenhäus considered the unambiguous state discrimination problem for two mixed states and obtained a second class of solutions [28]. Other significant developments can be found in [29, 30, 31, 32, 33, 34].

1.1.3 Discrimination by Local Operations and Classical Communication

A situation often encountered in quantum information is that the ρ_i 's are bipartite or multipartite states of a composite system, whose subsystems are geographically separated from each other thus disallowing joint operations to be performed on the composite system. If various parties in charge of their respective subsystems already share non-local resources - entanglement, for instance - among themselves they can use this non-locality to their advantage to discriminate the set of states. For example let the states be bipartite qubit states and let the two parties, Alice and Bob, share a single ebit of entanglement. Then Alice can teleport her state to Bob and he can perform all possible global operations on the joint system in his lab. In the absence of any non-local resources all parties can perform only local measurements on their respective subsystems and communicate classically their respective measurement outcomes to their peers. Protocols of this type are known as Local O perations and Classical Communication (LOCC) protocols. LOCC protocols play an integral role in quantum information, for example, LOCC gives the underlying argument which determines when a state is separable and when a state is entangled [35]. A significant question in QSD is to determine under what conditions can bipartite/multipartite states in an ensemble be discriminated amongst using an LOCC protocol. In QIT parlance this question is often phrased as "are the states *locally* distinguishable or *locally* indistinguishable?".

Consider the instance where the composite system is bipartite, the Hilbert spaces of whose subsystems are given by \mathcal{H}_A and \mathcal{H}_B ; the Hilbert space of the composite system is then given by $\mathcal{H}_A \otimes \mathcal{H}_B$. Let Alice be in possession of the subsystem whose Hilbert space is \mathcal{H}_A and Bob the system whose Hilbert space is \mathcal{H}_B . The LOCC protocols for this bipartite system can be classified as follows. In all the cases below we assume that Alice starts the protocol (someone has to start), but each of the situations referred to below also hold for protocols which Bob initiates. There is no loss of generality assumed in Alice starting the protocol (as is assumed throughout this thesis); it is merely a convention that is followed in quantum information theory and theoretical computer science.

- (i) 1-LOCC: Alice starts the protocol by performing a measurement on her subsystem, whose outcome she communicates to Bob (classical communication), who then performs a local measurement which is based on Alice's measurement outcome, and after which the protocol stops. This involves a single round of communication between Alice and Bob (in this case Alice to Bob).
- (ii) 2-LOCC: This proceeds the same way as 1-LOCC, with the difference that the protocol doesn't stop after Bob's measurement. Bob communicates his measurement outcome to Alice, who performs another local measurement which depends on Bob's measurement outcome and Alice's measurement outcome before that. The protocol stops after Alice's second measurement. The entire protocol involves two rounds of communication (Alice to Bob and Bob to Alice)
- (iii) General *N*-LOCC: Alice and Bob take turns to perform measurements on their respective subsystems and communicate their respective measurement outcomes to each other in the same way as in 1-LOCC and 2-LOCC. The protocol ends when there have been *N* instances of messages being communicated between Alice and Bob.
- (iv) Asymptotic LOCC: This refers to the closure of the set of *N*-LOCC where *N* is any natural number.

Sets of pairwise orthogonal states are always perfectly distinguishable under global operations (i.e., operations subsuming all possible joint measurements on the composite system) but are generally indistinguisable under LOCC operations. Since even pairwise orthogonal states cannot generally be perfectly distinguished using LOCC, a significant amount of focus in the literature has been on the deficit between perfect global distinguishability and local (in)distinguishability of the same sets of pairwise orthogonal states.

1.2. OUTLINE OF THESIS

History: The first paper in the topic of local distinguishability of quantum states was by Peres and Wootters [36]. Interestingly, a discussion over this paper lead to the discovery of quantum teleportation [37]. The question that lead the authors to the quantum teleporation paper was whether Alice and Bob could distinguish between two biparite qubit states better if they shared an ebit of entanglement¹. That said, it was only in the early 2000s when the problem started attracting more serious attention from researchers the world over due to the discovery of non-locality without entanglement [38] and the discovery that any two multipartite orthogonal pure states are locally distinguishable. A key feature of the progress made over the past one and a half decades is that most results have been limited to ensembles of states with certain symmetries, or systems whose dimensions are of specific values, rather than for general ensembles. A longer list of significant results in the topic is presented in chapter 3.

More comprehensive reviews on the aforementioned topics in QSD can be found in [39, 40, 41].

1.2 Outline of Thesis

This thesis examines four problems in all, two of which are in the topic of MED and the remaining two of which are in the topic of local distinguishability of quantum states. I will summarize all the four problems over the coming sections of this thesis.

The second chapter of the thesis covers problems which deal with the topic of minimum error discrimination. It is divided into six sections: in the first section I describe the MED problem with all relevant technical details, and make explicit the goal of the problem; in the second section I present the optimality conditions, in the third section I describe the structure of the problem as discovered by Belavkin and Maslov; in the fourth section I

¹ E. Chitambar, *Local Distinguishability of Quantum States*, Lecture 1, Institute of Mathematical Sciences, Chennai, 2014, January 27.

show how the mathematical structure of the MED of LI pure state ensembles, which is richer than the mathematical structure of general ensembles, is used to obtain the corresponding optimal POVM; in the fifth section I show how the richer mathematical structure, which relates a LI pure state ensemble to its optimal POVM, can be generalized to an ensemble of LI mixed states, and then be used to obtain the optimal POVM for MED of an ensemble of LI mixed states. In the final section of this chapter I briefly summarize the contents of the chapter and then describe future directions which emerge out of the problems considered in it.

The third chapter covers problems which I worked on in the topic of local distinguishability of quantum states. This chapter is divided into two sections: the first section details the derivation of a necessary condition for the local distinguishability of *m* pairwise orthogonal maximally entangled states in $\mathbb{C}^n \otimes \mathbb{C}^n$ systems, where $m \leq n$, and, also, provides examples of how strong the necessary condition is for certain classes of Lattice states, known as Generalized Bell states. The second section proposes a framework for the 1-LOCC distinguishability of sets of pairwise orthogonal bipartite states in an $n \otimes n$ bipartite system. It is shown that the results in Walgate et al's paper [12]², and the result in Cohen's paper [13]³ emerge out of this framework. Additionally, I also show that from the framework emerges, among other results, the necessary and sufficient conditions for the 1-LOCC distinguishability of almost all sets of *n* pairwise orthogonal bipartite states in $\mathbb{C}^n \otimes \mathbb{C}^n$ systems. At the end of each section I propose future directions which the content of the corresponding section could lead me to.

The fourth chapter summarizes the work in the thesis and discusses the relevance of this work in the field of quantum information community. After that I list more general prob-

² In this paper, Walgate et al. showed that any two orthogonal bipartite pure states are always distinguishable by 1-LOCC. In fact, the result by Walgate et al. is more general: they prove that any two orthogonal multipartite pure states are always locally distinguishable, but the result for the multipartite case follows trivially from the result for the bipartite case, which they establish earlier on in their paper.

³ In this paper Cohen showed that almost all sets of $m \ge n+1$ orthogonal multipartite states from $(\mathbb{C}^n)^{\otimes N}$ systems are locally indistinguishable.

1.2. OUTLINE OF THESIS

lems which I would like to pursue.

Some technical details of the third chapter are provided in the appendices A and B.

Chapter 2

Minimum Error Discrimination

2.1 Introduction

Minimum Error Discrimination (MED) is a kind of quantum state discrimination problem. The problem involves an *n*-dimensional quantum system, whose Hilbert space we will denote by \mathcal{H} . The setting is as follows: Alice selects a quantum state ρ_i (with $\rho_i \ge 0$, and $\operatorname{Tr}(\rho_i) = 1$) with probability p_i from an ensemble of *m* such states, $\widetilde{P} = \{p_i, \rho_i\}_{i=1}^m$ (where $0 < p_i < 1$ and $\sum_{i=1}^m p_i = 1$), and sends it to Bob, who has to find the value of *i* by performing measurement on the state ρ_i . This measurement is a generalized POVM of *m* elements, $E = \{\Pi_i\}_{i=1}^m$, where $\Pi_i \ge 0$ and $\sum_{i=1}^m \Pi_i = \mathbb{1}$ ($\mathbb{1}$ is the identity operator acting on \mathcal{H}). Furthermore his measurement strategy is based on a one-to-one correspondence between states in the ensemble \widetilde{P} and POVM elements $\{E_i\}_{i=1}^m$, i.e., $\rho_i \leftrightarrow \Pi_i$, such that if the measurement yields the *i*-th outcome, he will assume that he is given the *i*-th state from the ensemble \widetilde{P} . When the states $\rho_1, \rho_2, \dots, \rho_m$ are not all pairwise orthogonal, they aren't perfectly distinguishable, i.e., there exists no measurement such that $\operatorname{Tr}(\rho_i \Pi_j) = 0$, when $i \neq j$, $\forall 1 \le i, j \le m$. This implies that there may arise a situation where Alice sends the *i*-th state, ρ_i , but Bob concludes he was given the *j*-th state where $j \neq i$. This happens when even upon being given ρ_i , Bob's measurement yields the *j*-th outcome. Such a scenario corresponds to an error. The average probability of such errors is given by

$$P_e = \sum_{\substack{i,j=1\\i\neq j}}^{m} p_i \operatorname{Tr}(\rho_i \Pi_j).$$
(2.1)

Correspondingly, successful discrimination is when Bob is given the *i*-th state ρ_i and he also concludes that he was given the *i*-th state; this is happens when his measurement outcome is *i*. The average probability of success is given by

$$P_s = \sum_{i=1}^m p_i \operatorname{Tr}(\rho_i \Pi_i).$$
(2.2)

It is easy to see that $P_s + P_e = 1$.

The central objective in MED is to find the *m*-POVM(s) which maximize the value of P_s .

$$P_s^{max} = Max \left\{ P_s \mid \{\Pi_i\}_{i=1}^m \text{ where } \Pi_i \ge 0, \ \sum_i \Pi_i = \mathbb{1} \right\}$$
(2.3)

Developments in the MED problem: necessary and sufficient conditions for the optimal POVM for any ensemble were given by Holevo [15] and Yuen et al [14] independently. Yuen et al. cast MED into a convex optimization problem for which numerical solutions are given in polynomial time¹. While there are quite a number of numerical techniques to obtain the optimal POVM [42, 43, 44, 45], for very few ensembles has the MED problem been solved analytically. Some of these include an ensemble of two states [5], ensembles whose density matrix is maximally mixed state [14], equiprobable ensembles that lie on the orbit of a unitary, [16, 46, 47], and mixed qubit states [48, 49]. In [50], many interesting properties of the MED problem have been elucidated using geometry of

¹ That is, polynomial in $dim\mathcal{H}$, which is denoted by *n* here.

N qudit states. Elsewhere, an upper bound for the optimal success probability has been given [75].

2.2 **Optimality Conditions**

The set of *m*-POVMs is convex, i.e., if $\{E_i\}_{i=1}^m$ and $\{E_i'\}_{i=1}^m$ are *m*-POVMs, then so is $\{pE_i + (1-p)E_i'\}_{i=1}^m, \forall 0 \le p \le 1$. Hence MED is a constrained convex optimization problem. To every such a constrained convex optimization problem (called the primal problem) there is a corresponding dual problem which provides a lower bound (if the primal problem is a constrained minimization) or an upper bound (if the primal problem is a constrained maximization) to the quantity being optimized (called the objective function). Under certain conditions these bounds are tight implying that one can obtain the solution for the primal problem from its dual. We then say that there is no duality gap between both problems. For MED, there is no duality gap between the primal and dual problems; thus the dual problem can be solved to obtain the optimal POVM [14]. The dual problem is given as follows [14]:

Min Tr(Z), subject to:
$$Z \ge p_i \rho_i$$
, $\forall 1 \le i \le m$. (2.4)

Then P_s^{max} is given by $P_s^{max} = \text{Min Tr}(Z)$.

Also the optimal *m*-POVM, $\{E_i\}_{i=1}^m$ will satisfy the complementary slackness condition:

$$(Z - p_i \rho_i) E_i = 0, \forall 1 \le i \le m.$$

$$(2.5)$$

Now summing over *i* in equation (2.5) and using the fact that $\sum_{i=1}^{m} E_i = \mathbb{1}_n$ we get the following.

$$Z = \sum_{i=1}^{m} p_i \rho_i E_i = \sum_{i=1}^{m} E_i p_i \rho_i.$$
 (2.6)

From equation (2.5) we get

$$E_j \left(p_j \rho_j - p_i \rho_i \right) E_i = 0, \ \forall \ 1 \le i, j \le m.$$

Conditions (2.5) and (2.7) are equivalent to each other. Z, given by equation (2.6), has to satisfy another condition

$$Z \ge p_i \rho_i, \ \forall \ 1 \le i \le m.$$

Thus the *necessary and sufficient* conditions for the *m*-element POVM $\{E_i\}_{i=1}^m$ to maximize P_s are given by conditions (2.7) (or (2.5)) and (2.8) together.

2.3 Structure of MED

Belavkin and Maslov [1, 2] discovered a mathematical structure for the MED of general ensembles of states - a structure which relates said ensembles to their respective optimal POVM(s).

Let $\widetilde{P} = \{p_i, \rho_i\}_{i=1}^m$ be an ensemble of quantum states for whose MED we seek the corresponding optimal POVM(s); let the (an) optimal POVM(s) be denoted by $\{E_i\}_{i=1}^m$. The mathematical structure relates \widetilde{P} to another ensemble of quantum states $\widetilde{Q} \equiv \{q_i, \sigma_i\}_{i=1}^m$, which has the following properties.

(i.) $\{q_i\}_{i=1}^m$ is a probability (with $0 \le q_1, q_2, \cdots, q_n < 1$, and $\sum_{i=1}^m q_i = 1$), and

(ii.) σ_i are density operators on \mathcal{H} such that $\operatorname{supp}(q_i\sigma_i)$ is a subspace of $\operatorname{supp}(p_i\rho_i)$,

2.4. MED FOR LI PURE STATES

 $\forall \ 1 \leq i \leq m,$

(iii) The E_i can be written in terms of the states in the ensemble $\{q_i, \sigma_i\}_{i=1}^m$.

$$E_i = \left(\sum_{j=1}^m q_i \sigma_i\right)^{-\frac{1}{2}} q_i \sigma_i \left(\sum_{k=1}^m q_i \sigma_i\right)^{-\frac{1}{2}}, \ \forall \ 1 \le i \le m.$$
(2.9)

For a given ensemble of states $\{q_i, \rho_i\}_{i=1}^m$, the POVM $\{E_i\}_{i=1}^m$, whose POVM elements are given by equation (2.9), is called the **P**retty **G**ood **M**easurement of the ensemble $\{q_i, \sigma_i\}_{i=1}^m$.

2.4 MED for LI Pure States

(The work in this section can be found in a paper authored by myself and Sibasish Ghosh, and has been published in Journal of Physics A: Mathematical and Theoretical² [51].

The previous section ended with the salient features of the mathematical structure of the MED problem for general ensembles, i.e., ensembles whose states have no special properties. In case the states are pure and LI, said mathematical structure of the problem becomes richer. Note that \mathcal{H} , here, is the space which is spanned by the pure states, since any more dimensions of the quantum system won't play any active role in the problem. Thus for the MED of LI pure states, the number of states in the ensemble *m* is equal to the dimension of the Hilbert space, i.e., m = n.

Structure of the MED problem for ensembles of LI pure states: The MED problem for ensembles of LI pure states is known to have two interesting properties. Let $\tilde{P} =$ $\{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^n$ be an ensemble of states, where $|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_n\rangle \in \mathcal{H}$ are LI. Then said properties of the MED problem for \tilde{P} are:

(i) The optimal POVM is a unique rank one projective measurement [4, 5, 3].

² http://dx.doi.org/10.1088/1751-8113/49/16/165304

(ii) The optimal POVM for MED of *P* is the pretty good measurement (PGM) of another ensemble, *Q* ≡ {*q_i* > 0, |*ψ_i*⟩⟨*ψ_i*|}ⁿ_{*i*=1} [1, 2, 3]. Note that the *i*-th states in *P* and *Q* are the same, whereas the probabilities are generally not. Additionally, in [3], it is explicitly shown that the ensemble pair (*P*, *Q*) are related through an invertible map.

To formalize this invertible relation between \widetilde{P} and \widetilde{Q} we will now introduce a few definitions.

Definition 2.4.1. \mathcal{E} is the set of all ensembles comprising of *n* LI pure states. Hence, any ensemble in \mathcal{E} is of the form $\widetilde{P} = \{p_i > 0, |\psi_i\rangle\langle\psi_i|\}_{i=1}^n$ where $|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_n\rangle$ are LI.

 \mathcal{E} is a $(2n^2 - n - 1)$ real parameter set.

Definition 2.4.2. \mathcal{P} is the set of all rank one projective measurements on the states of \mathcal{H} ; an element in \mathcal{P} is of the form $\{|v_i\rangle\langle v_i|\}_{i=1}^n$ where $\langle v_i|v_j\rangle = \delta_{ij}, \forall 1 \le i, j \le n$.

 \mathcal{P} is an n(n-1) real parameter set. From point (i) above we see that the optimal POVM for $\widetilde{P} \in \mathcal{E}$ is a unique element in \mathcal{P} . Thus, one can define the *optimal POVM map*, \mathcal{P} , in the following way:

Definition 2.4.3. $\mathscr{P} : \mathscr{E} \longrightarrow \mathscr{P}$ is such that $\mathscr{P}(\widetilde{P})$ is *the* optimal POVM for MED of $\widetilde{P} \in \mathscr{E}$.

Let *PGM* denote the PGM map, i.e., *PGM* : $\mathcal{E} \longrightarrow \mathcal{P}$ is such that *PGM* (\widetilde{Q}) is the PGM of $\widetilde{Q} \in \mathcal{E}$, i.e. (refer to [3]), *PGM* $(\widetilde{Q}) = \left\{ \rho_q^{-\frac{1}{2}} q_i |\psi_i\rangle \langle \psi_i | \rho_q^{-\frac{1}{2}} \right\}_{i=1}^n$, where $\rho_q = \sum_{i=1}^n q_i |\psi_i\rangle \langle \psi_i |$ (see equation (2.9)).

Then (ii) above says that there exists an invertible map, which we label by \mathscr{R} , which can be defined in the following way:

Definition 2.4.4. $\mathscr{R}: \mathcal{E} \longrightarrow \mathcal{E}$ is a bijection such that

$$\mathscr{P}\left(\widetilde{P}\right) = PGM\left(\mathscr{R}\left(\widetilde{P}\right)\right), \ \forall \ \widetilde{P} \in \mathcal{E}.$$
(2.10)

Knowing \mathscr{R} would solve the problem of MED for LI pure state ensembles. While the existence of the invertible function \mathscr{R} has been proven [1, 3], unfortunately, it isn't known - neither analytically nor computationally for arbitrary ensemble \widetilde{P} . Fortunately \mathscr{R}^{-1} is known [2, 1, 3] i.e., having fixed the states $\{|\psi_i\rangle\}_{i=1}^n$ one can give p_i in terms of the q_i : let $G_q > 0$ represent the gram matrix of the ensemble \widetilde{Q} , i.e., $(G_q)_{ij} = \sqrt{q_i q_j} \langle \psi_i | \psi_j \rangle$, and let $G_q^{\frac{1}{2}}$ represent the positive square root of G_q ; let G denote the gram matrix of \widetilde{P} , i.e., $G_{ij} = \sqrt{p_i p_j} \langle \psi_i | \psi_j \rangle$, $\forall 1 \le i, j \le n$; then diagonal elements of G can be written as functions of q_i and matrix elements of $G_q^{\frac{1}{2}}$

$$G_{ii} = p_i = C \frac{q_i}{\left(G_q^{\frac{1}{2}}\right)_{ii}}, \ \forall \ 1 \le i \le n,$$
(2.11)

where C is the normalization constant³,

$$C = \left(\sum_{j=1}^{n} \frac{q_j}{\left(G_q^{\frac{1}{2}}\right)_{jj}}\right)^{-1}$$

This tells us what \mathscr{R}^{-1} is:

$$\mathscr{R}^{-1}\left(\left\{q_{i}, |\psi_{i}\rangle\langle\psi_{i}|\right\}_{i=1}^{n}\right) = \left\{p_{i}, |\psi_{i}\rangle\langle\psi_{i}|\right\}_{i=1}^{n},$$

where p_i and q_i are related by equation (2.11).

It is more convenient to define \mathscr{R}^{-1} and \mathscr{R} on the set of gram matrices, which we will denote by \mathcal{G} .

Definition 2.4.5. G is the set of all positive definite $n \times n$ matrices of trace one.

Note⁴ that \mathcal{G} is convex and is also open in \mathbb{R}^{n^2-1} .

³ $q_i > 0, \forall 1 \le i \le n$. This comes from the definition of \mathcal{E} and that $\{q_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^n \in \mathcal{E}$. Also, $\left(G^{\frac{1}{2}}\right)_{ii} > 0, \forall 1 \le i \le n$. This is because $G^{\frac{1}{2}}$, being the positive square root of G (gram matrix for the linearly independent vectors $\{\sqrt{q_i}|\psi_i\rangle\}_{i=1}^n$) is positive definite and the diagonal elements of a positive definite matrix have to be greater than zero.

⁴ Associating each $G \in \mathcal{G}$ with an $n \times n$ density matrix of rank *n*, we see that \mathcal{G} is the same as the interior of the generalized Bloch sphere for *n* dimensional systems. Hence $\mathcal{G} \subset \mathbb{R}^{n^2-1}$.

Define $\mathscr{R}^{(G)^{-1}}$: $\mathcal{G} \longrightarrow \mathcal{G}$ by $\mathscr{R}^{(G)^{-1}}(G_q) = G$, using relation (2.11). This is possible in the following manner: The matrix elements of G_q are given by $(G_q)_{ij} = \sqrt{q_i q_j} \langle \psi_i | \psi_j \rangle$ and that of G are given by $G_{ij} = \sqrt{p_i p_j} \langle \psi_i | \psi_j \rangle$, thus G_q and G are related by the following congruence transformation:

$$G = D''G_aD'',$$

where D'' is a diagonal matrix whose matrix elements are given by

$$(D'')_{ij} = \delta_{ij} \sqrt{\frac{p_i}{q_i}},$$

where the p_i 's are obtained from the q_i 's using equation (2.11). This defines $\mathscr{R}^{(G)^{-1}}$ unambiguously.

We know that \mathscr{R}^{-1} is invertible on \mathscr{E} (from [3]); this implies that $\mathscr{R}^{(\mathcal{G})^{-1}}$ is invertible on \mathscr{G} , i.e., $\mathscr{R}^{(\mathcal{G})}$ exists. Equation (2.11) tells us that $\mathscr{R}^{(\mathcal{G})^{-1}}$ is continuous in \mathscr{G} . Since $\mathscr{G} \subset \mathbb{R}^{n^{2}-1}$ is open⁵, the invariance of domain theorem [52] tells us that $\mathscr{R}^{(\mathcal{G})^{-1}}$ is a homeomorphism on \mathscr{G} . This means that $\mathscr{R}^{(\mathcal{G})}$ is also continuous on \mathscr{G} .

To be able to express what \mathscr{R} is one needs to be able to solve the *n* equations (2.11) for q_i in terms of p_j 's and $|\psi_j\rangle$'s. These equations are too complicated for one to hope to solve: to begin with one doesn't even have an explicit closed form expression for $G^{\frac{1}{2}}$ in terms of the matrix elements of *G* for arbitrary *n*. Even for the cases when n = 3, 4, where one *can* obtain such a closed form expression for $G^{\frac{1}{2}}$, the nature of the equations is too complicated to solve analytically. This tells us that it is hopeless to obtain q_i as a closed form expression in terms of $\{p_i, |\psi_i\rangle\}_{i=1}^n$. A similar sentiment was expressed earlier [53]. While a closed form expression of the solution seems too difficult to obtain (and even if obtained, too cumbersome to appreciate) giving an *efficient technique* to compute q_i from $\{p_i, |\psi_i\rangle\}_{i=1}^n$ establishes that the equation (2.10) along with technique (to compute q_i) provides a solution for MED of an ensemble of *n*-LIPs.

⁵ The topology of \mathcal{G} is that which is induced on it by the Hilbert-Schmidt norm. Note that this is equivalent to the Euclidean metric of \mathbb{R}^{n^2-1}

To achieve such a technique we recast the MED problem for any ensemble \tilde{P} in terms of a matrix equation and a matrix inequality using the gram matrix G of \tilde{P} . The matrix equation and the inequality are equivalent to the optimality conditions that the optimal POVM has to satisfy, i.e., the optimal conditions given by Yuen et al [14]. Recasting the problem in this form gives us three distinct benefits.

- (1) It helps us to *explicitly* establish that the optimal POVM for \tilde{P} is given by the PGM of another ensemble of the form \tilde{Q} (i.e., relation in equation (2.10) is made explicit in the matrix equality and matrix inequality conditions).
- (2) MED is actually a rotationally invariant problem, i.e., the optimal POVM, $\{E_i\}_{i=1}^n$, varies covariantly under a unitary transformation, U, of the states:

$$|\psi_i\rangle\langle\psi_i| \to U|\psi_i\rangle\langle\psi_i|U^{\dagger} \Longrightarrow E_i \to UE_iU^{\dagger}$$

This makes it desirable to subtract out the rotationally covariant aspect of the solution and, so, cast the problem in a rotationally invariant form. This is achieved through the aforesaid matrix equality and inequality.

(3) It gives us a technique to compute q_i .

For (3) we need to compute $\mathscr{R}^{(G)}(G)$ for a given $G \in \mathcal{G}$. This is done by using the analytic implicit function theorem which tells us that $\mathscr{R}^{(G)}$ is an analytic function on \mathcal{G} . Specifically, we will vary G from one point in \mathcal{G} , at which we know what the action of $\mathscr{R}^{(G)}$ is, to another point in \mathcal{G} , at which we want to know what the action of $\mathscr{R}^{(G)}$ is.

Further on, since our technique rests on the theory of the MED problem for *n*-LIP ensembles, it is expected that the algorithm our technique offers is computationally as efficient as or more efficient than existing techniques. We show that this is indeed the case, particularly by directly employing Newton's method to solve the matrix inequality. This adds to the utility of our technique.

The section for MED of LI pure states is divided into the following subsections: In subsection 2.4.1 we recast the MED problem for LI pure states in a rotationally invariant form. In subsection 2.4.2 IFT is employed to solve the rotationally invariant conditions, which were developed in the previous subsection; in subsubsection 2.4.2 of subsection 2.4.2 the computational complexity of our algorithm is compared with a standard SDP technique. We conclude this work in subsection 2.4.3.

2.4.1 Rotationally Invariant Necessary and Sufficient Conditions for MED

We wish to obtain the optimal POVM (which is a rank-one projective measurement) for MED of an ensemble $\widetilde{P} = \{p_i, |\psi_i\rangle \langle \psi_i|\}_{i=1}^n$, where $\{|\psi_i\rangle\}_{i=1}^n$ is a LI set. Let $|\widetilde{\psi_i}\rangle \equiv \sqrt{p_i} |\psi_i\rangle$, $\forall 1 \le i \le n$. Since $\{|\widetilde{\psi_i}\rangle\}_{i=1}^n$ is a LI set, corresponding to this set there exists a *unique* set of vectors $\{|\widetilde{u_i}\rangle\}_{i=1}^n \subset \mathcal{H}$ such that⁶:

$$\langle \widetilde{\psi}_i | \widetilde{u}_j \rangle = \delta_{ij}, \ \forall \ 1 \le i, j \le n.$$

Let G denote the gram matrix of $\{|\tilde{\psi}_i\rangle\}_{i=1}^n$. The matrix elements of G are hence given by

$$G_{ij} = \langle \widetilde{\psi}_i | \widetilde{\psi}_j \rangle, \ \forall \ 1 \le i, j \le n.$$
(2.13)

Tr(G) = 1. Since $\{|\tilde{\psi}_i\rangle\}_{i=1}^n$ is a LI set, G > 0. Any orthonormal basis $\{|v_i\rangle\}_{i=1}^n$ of \mathcal{H} can be represented as:

$$|v_i\rangle = \sum_{j=1}^n \left(G^{\frac{1}{2}}U\right)_{ji} |\widetilde{u}_j\rangle, \qquad (2.14)$$

⁶ Given a set of *n* LI vectors $\{|\widetilde{\psi}_i\rangle\}_{i=1}^n$ one can obtain the corresponding set of vectors $\{|\widetilde{u}_i\rangle\}_{i=1}^n$ in the following way: fix a basis to work in, arrange $\langle \widetilde{\psi}_i |$ as rows in a matrix which we call *V*. *V* is invertible since its rows are LI. The columns of V^{-1} correspond to the vectors $|\widetilde{u}_i\rangle$ in our chosen basis.

where $G^{\frac{1}{2}}$ is the positive square root of *G* and *U* is an $n \times n$ unitary matrix. This can be seen from the following: the gram matrix corresponding to the set $\{|\tilde{u}_i\rangle\}_{i=1}^n$ is G^{-17} . Using this, it can easily be verified that $\langle v_i | v_j \rangle = \delta_{ij}$, for all $1 \le i, j \le n$. Note that *U* captures the unitary degree of freedom of the orthonormal basis $\{|v_i\rangle\}_{i=1}^n$, in the sense that using different *U*'s will yield different ONBs $\{|v_i\rangle\}_{i=1}^n$. Any such orthonormal basis corresponds to a rank-one projective measurement:

$$\{|v_i\rangle\}_{i=1}^n \longrightarrow \{|v_i\rangle\langle v_i|\}_{i=1}^n.$$
(2.15)

Using this rank-one projective measurment for MED, the average probability of success is given by:

$$P_{s} = \sum_{i=1}^{n} |\langle \widetilde{\psi}_{i} | v_{i} \rangle|^{2} = \sum_{i=1}^{n} \left| \left(G^{\frac{1}{2}} U \right)_{ii} \right|^{2}.$$
 (2.16)

Let $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ be a rank-one projective measurement, which is also a solution for the *n*-POVM $\{E_i\}_{i=1}^n$ in condition (2.7). Here $\langle w_i|w_j\rangle = \delta_{ij}$ for $i, j = 1, 2, \dots, n$. Let an $n \times n$ unitary matrix W be related to the rank-one projective measurement $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ in the following way.

$$|w_i\rangle = \sum_{j=1}^n \left(G^{\frac{1}{2}}W\right)_{ji} |\widetilde{u}_j\rangle.$$
(2.17)

The unitary matrix W is fixed upto right-multiplication with a diagonal unitary matrix, which changes the phases of $|w_i\rangle$. We will soon fix the phases of $|w_i\rangle$ which will ensure that W will be *unique*.

Thus equation (2.7) can be rewritten as:

Note that $\sum_{j=1}^{n} \langle \widetilde{u}_i | \widetilde{u}_j \rangle \langle \widetilde{\psi}_j | \widetilde{\psi}_k \rangle = \delta_{ik}$, since $\sum_{j=1}^{n} | \widetilde{u}_j \rangle \langle \widetilde{\psi}_j | = \mathbb{1}$. This can be seen from the fact that if $|\eta\rangle = \sum_{j=1}^{n} \alpha_j | \widetilde{u}_j \rangle$ is any vector in \mathcal{H} , then $\left(\sum_{j=1}^{n} | \widetilde{u}_j \rangle \langle \widetilde{\psi}_j | \right) | \eta\rangle = |\eta\rangle$.

$$\langle w_j | \left(|\widetilde{\psi}_j\rangle \langle \widetilde{\psi}_j| - |\widetilde{\psi}_i\rangle \langle \widetilde{\psi}_i| \right) | w_i\rangle = 0, \ \forall \ 1 \le i, j \le n.$$

$$(2.18)$$

Using equation (2.17) in equation (2.18):

$$\left(G^{\frac{1}{2}}W\right)_{jj}^{*}\left(G^{\frac{1}{2}}W\right)_{ji} = \left(G^{\frac{1}{2}}W\right)_{ij}^{*}\left(G^{\frac{1}{2}}W\right)_{ii}, \ \forall \ 1 \le i, j \le n.$$
(2.19)

The diagonal elements of the matrix $G^{\frac{1}{2}}W$ can be made non-negative by appropriately fixing the phases of the $|w_i\rangle$ vectors in the following way: right-multiply W with a diagonal unitary W', whose diagonal elements will be phases. From equation (2.17) it is seen that right-multiplying W with W' merely changes the phases of the ONB vectors $|w_i\rangle$, and that they will still satisfy equation (2.18). We can vary the phases in W' so that the diagonals of $G^{\frac{1}{2}}WW'$ are non-negative. We absorb W' into W. After this absorption, the $n \times n$ unitary W which is associated with the rank-one projective measurement $\{|w_i\rangle\langle w_i|\}_{i=1}^n$, is unique. Continuing, we see that equations (2.19) now take the following form.

$$\left(G^{\frac{1}{2}}W\right)_{jj}\left(G^{\frac{1}{2}}W\right)_{ji} = \left(G^{\frac{1}{2}}W\right)_{ii}\left(G^{\frac{1}{2}}W\right)_{ij}^{*}, \ \forall \ 1 \le i, j \le n.$$
(2.20)

Let $D = Diag(d_{11}, d_{22}, \dots, d_{nn})$ be the real diagonal matrix of $G^{\frac{1}{2}}W$, i.e.,

$$d_{ii} = \left(G^{\frac{1}{2}}W\right)_{ii}, \ \forall \ 1 \le i \le n.$$
(2.21)

From equation (2.20) and the fact that the diagonals of $G^{\frac{1}{2}}W$ are all real, we infer that the matrix $DG^{\frac{1}{2}}W$ is hermitian.

$$DG^{\frac{1}{2}}W - W^{\dagger}G^{\frac{1}{2}}D = 0.$$
(2.22)

Left multiplying the LHS and RHS by $DG^{\frac{1}{2}}W$ gives

$$\left(DG^{\frac{1}{2}}W\right)^2 - DGD = 0$$

$$\implies X^2 - DGD = 0,$$
 (2.23)

where $X \equiv DG^{\frac{1}{2}}W$, $X^{\dagger} = X$ and (note that) D^2 is the diagonal of X.

In the MED problem, we are given the gram matrix G of the ensemble \widetilde{P} . To solve condition (2.7) for MED of \widetilde{P} we need to solve for X in equation (2.23). Knowing X tells us what $G^{\frac{1}{2}}W$ is, which can be used in equation (2.17) to obtain $\{|w_i\rangle\langle w_i|\}_{i=1}^n$. Equation (2.23) came from assuming that $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ represented some *n*-POVM which satisfied condition (2.7). For $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ to be the optimal POVM it needs to satisfy condition (2.8) too; this will impose another condition on the solution for X in equation (2.23).

Theorem 2.4.1. Let X be a solution for X in equation (2.23). Then X corresponds to the optimal POVM for MED of \tilde{P} if it is positive definite. Also, $\mathscr{R}^{(G)}(G) = \frac{1}{\operatorname{Tr}(D^2G)}DGD$, where D is the square root of the diagonal of X.

Proof. We relate d_{ii} , defined in equation (2.21), to the probability q_i mentioned in equation (2.11). In the introduction of this section it was mentioned that the optimal POVM for MED of \widetilde{P} is the PGM of an ensemble $\mathscr{R}(\widetilde{P}) = \widetilde{Q} = \{q_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^n$ (see definition (2.4.4)). This means that [1]

$$|w_i\rangle = \left(\sum_{j=1}^n |\widetilde{\psi'}_j\rangle \langle \widetilde{\psi'}_j|\right)^{-1/2} |\widetilde{\psi'}_i\rangle, \ \forall \ 1 \le i \le n,$$
(2.24)

where $|\widetilde{\psi'}_i\rangle \equiv \sqrt{q_i}|\psi_i\rangle$ and $\left(\sum_{j=1}^n |\widetilde{\psi'}_j\rangle\langle\widetilde{\psi'}_j|\right)^{-1/2} > 0$. Define $|\widetilde{u'}_i\rangle$ to be such that $\langle\widetilde{\psi'}_i|\widetilde{u'}_j\rangle = \delta_{ij}, \forall 1 \le i, j \le n$. G_q is the gram matrix corresponding to the ensemble \widetilde{Q} . It can be verified that G_q^{-1} is the gram matrix of the vectors $\{|\widetilde{u'}_i\rangle\}_{i=1}^n$, i.e., $\left(G_q^{-1}\right)_{ij} = \langle\widetilde{u'}_i|\widetilde{u'}_j\rangle, \forall 1 \le i, j \le n$. This implies that

$$\left(\sum_{j=1}^{n} |\widetilde{\psi'}_{j}\rangle\langle\widetilde{\psi'}_{j}|\right)^{-1/2} = \left(\sum_{j=1}^{n} |\widetilde{u'}_{j}\rangle\langle\widetilde{u'}_{j}|\right)^{1/2} = \sum_{i,j=1}^{n} \left(G_{q}^{\frac{1}{2}}\right)_{ij} |\widetilde{u'}_{i}\rangle\langle\widetilde{u'}_{j}|.$$
(2.25)

Note that since the LHS in equation (2.25) is positive definite, the RHS in equation (2.25) should also be positive definite and that can only be true if $G_q^{\frac{1}{2}} > 0$. One can verify the above equation by squaring on both sides ⁸. Substituting the above in equation (2.24) gives

$$|w_{i}\rangle = \sum_{j=1}^{n} \left(G_{q}^{\frac{1}{2}}\right)_{ji} |\widetilde{u'}_{j}\rangle$$
$$= \sum_{j=1}^{n} \frac{\sqrt{p_{j}}}{\sqrt{q_{j}}} \left(G_{q}^{\frac{1}{2}}\right)_{ji} |\widetilde{u}_{j}\rangle, \ \forall \ 1 \le i \le n,$$
(2.26)

where $|\widetilde{u'}_i\rangle = \frac{\sqrt{p_i}}{\sqrt{q_i}}|\widetilde{u}_i\rangle$, $\forall 1 \le i \le n$ (since $|\widetilde{\psi'}_i\rangle = \frac{\sqrt{q_i}}{\sqrt{p_i}}|\widetilde{\psi}_i\rangle$). Since $\{|\widetilde{u}_i\rangle\}_{i=1}^n$ is a basis for \mathcal{H} , on comparing equations (2.26) and (2.17) we obtain

$$\left(G^{\frac{1}{2}}W\right)_{ji} = \frac{\sqrt{P_j}}{\sqrt{q_j}} \left(G_q^{\frac{1}{2}}\right)_{ji}, \quad \forall \ 1 \le i, j, \le n,$$

$$(2.27a)$$

$$\implies d_{jj} = \frac{\sqrt{p_j}}{\sqrt{q_j}} \left(G_q^{\frac{1}{2}} \right)_{jj}, \ \forall \ 1 \le j \le n.$$
(2.27b)

Using equation (2.11) we get that

$$d_{jj}\frac{\sqrt{P_j}}{\sqrt{q_j}} = \frac{p_j}{q_j} \left(G_q^{\frac{1}{2}}\right)_{jj} = C, \ \forall \ 1 \le j \le n,$$

$$(2.28)$$

That $|\widetilde{\Psi'}_i\rangle$ and $|\widetilde{u'}_j\rangle$ are related by $\langle \widetilde{\Psi'}_i | \widetilde{u'}_j \rangle = \delta_{ij}$ implies that $\sum_{j=1}^n |\widetilde{u'}_j\rangle \langle \widetilde{\Psi'}_j| = \mathbb{1}_n$. That $\sum_{j=1}^n |\widetilde{u'}_j\rangle \langle \widetilde{\Psi'}_j| = \mathbb{1}_n$ is true implies that $\left(\sum_{j=1}^n |\widetilde{u'}_j\rangle \langle \widetilde{u'}_j|\right) \left(\sum_{k=1}^n |\widetilde{\Psi'}_k\rangle \langle \widetilde{\Psi'}_k|\right) = \mathbb{1}_n$. Hence $\sum_{j=1}^n |\widetilde{u'}_j\rangle \langle \widetilde{u'}_j|$ is the inverse of $\sum_{k=1}^n |\widetilde{\Psi'}_k\rangle \langle \widetilde{\Psi'}_k|$. Also, since G_q is the gram matrix of $\left\{|\widetilde{\Psi'}_j\rangle\right\}_{j=1}^n$ and G_q^{-1} is the gram matrix of $\left\{|\widetilde{u'}_j\rangle\right\}_{j=1}^n$ we get that $\left(\sum_{i,j=1}^n \left(G_q^{\frac{1}{2}}\right)_{ij} |\widetilde{u'}_i\rangle \langle \widetilde{u'}_j|\right)^2 = \sum_{j=1}^n |\widetilde{u'}_j\rangle \langle \widetilde{u'}_j|$.

where *C* is the positive constant that appears in equation (2.11). Since $d_{jj} \frac{\sqrt{p_j}}{\sqrt{q_j}} = C$, $\forall 1 \le j \le n$, using equations (2.27a) and (2.28) we get that

$$(X)_{ji} = \left(DG^{\frac{1}{2}}W \right)_{ji} = d_{jj} \left(G^{\frac{1}{2}}W \right)_{ji} = C \times \left(G_q^{\frac{1}{2}} \right)_{ji}, \ \forall \ 1 \le i, j, \le n,$$

that is, X is equal to the product of a positive constant C and $G_q^{\frac{1}{2}}$, which implies that X > 0. Also from equation (2.23) it follows that $DGD = C^2 \times G_q$, i.e., the gram matrix of $\mathscr{R}(\widetilde{P}) = \widetilde{Q}$ is given by $\frac{DGD}{\text{Tr}(D^2G)}$, i.e.,

$$\mathscr{R}^{(G)}(G) = \frac{\chi^2}{\operatorname{Tr}(\chi^2)} = \frac{DGD}{\operatorname{Tr}(D^2G)}.$$
(2.29)

The converse of theorem 2.4.1 is proved in the following.

Theorem 2.4.2. If X is a solution for X in equation (2.23) and X is positive definite, then X corresponds to the optimal POVM for MED of the ensemble \tilde{P} . Also, X is unique, i.e., there is no other X' such that it is a solution for X in equation (2.23) and X' > 0.

Proof. Let X be a solution for X in equation (2.23) and let X be positive definite. Equating $D^{-1}X$ with $G^{\frac{1}{2}}W$ (see below equation (2.23)) and employing it in equation (2.17), we obtain $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ to be the rank-one projective measurement corresponding to solution X. We want to prove that $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ is the optimal POVM. For this purpose we borrow a result from Mochon's paper. Equation (33) in Mochon's paper [3] tells us that the positive operator Z, defined in equation (2.6), is a scalar times the positive square root of the density matrix of the ensemble $\Re(\widetilde{P})$, i.e.,

$$Z = C \left(\sum_{i=1}^{n} q_i |\psi_i\rangle \langle \psi_i | \right)^{\frac{1}{2}}.$$
 (2.30)

We will compute $\sum_{i=1}^{n} p_i |w_i\rangle \langle w_i | \psi_i \rangle \langle \psi_i |$ and show that it is equal to $C(\sum_{i=1}^{n} q_i | \psi_i \rangle \langle \psi_i |)^{\frac{1}{2}}$,

thereby proving that $\sum_{i=1}^{n} p_i |w_i\rangle \langle w_i | \psi_i \rangle \langle \psi_i |$ is equal to Z. This will show that $\{|w_i\rangle \langle w_i|\}_{i=1}^{n}$ is the optimal POVM.

Since $|w_i\rangle = \sum_{k=1}^n (D^{-1}X)_{ki} |\widetilde{u}_k\rangle$ and $|\widetilde{u}_k\rangle = \sum_{j=1}^n (G^{-1})_{jk} |\widetilde{\psi}_j\rangle$, using equation (2.23) it's easily verified that $|w_i\rangle = \sum_{j=1}^n (DX^{-1})_{ji} |\widetilde{\psi}_j\rangle$. Then

$$\sum_{i=1}^{n} |w_i\rangle \langle w_i | \widetilde{\psi}_i \rangle \langle \widetilde{\psi}_i | = \sum_{j,i=1}^{n} \left(D \mathcal{X}^{-1} D \right)_{ji} | \widetilde{\psi}_j \rangle \langle \widetilde{\psi}_i | > 0.$$
(2.31)

Squaring the RHS in equation (2.31) and employing equation (2.23) we get that

$$\left(\sum_{i,j=1}^{n} \left(D \mathcal{X}^{-1} D \right)_{ij} |\widetilde{\psi}_i\rangle \langle \widetilde{\psi}_j| \right)^2 = \sum_{i,j=1}^{n} d_{ii}^2 |\widetilde{\psi}_i\rangle \langle \widetilde{\psi}_i|.$$
(2.32)

Consider the probability $k_i \equiv \frac{d_{ii}^2 p_i}{\sum_{j=1}^n d_{jj}^2 p_j}$, $\forall 1 \le i \le n$. Thus $\sum_{i=1}^n k_i |\psi_i\rangle \langle \psi_i|$ is the average state of the ensemble $\widetilde{K} = \{k_i, |\psi_i\rangle \langle \psi_i|\}_{i=1}^n$. The matrix elements of the gram matrix, G_k of \widetilde{K} are then given by

$$(G_k)_{ij} = \sqrt{k_i k_j} \langle \psi_i | \psi_j \rangle = \frac{1}{\sum_{l=1}^n d_{ll}^2 p_l} d_{ii} \langle \widetilde{\psi}_i | \widetilde{\psi}_j \rangle d_{jj}.$$

This tells us that $G_k = \frac{1}{\text{Tr}(D^2G)}DGD$; using equation (2.23) we get that the positive square root of G_k is $G_k^{\frac{1}{2}} = \frac{1}{\sqrt{\text{Tr}(D^2G)}}X$ and, hence, $d_{ii}^2 = X_{ii} = \sqrt{\text{Tr}(D^2G)}\left(G_k^{\frac{1}{2}}\right)_{ii}$ (see equation (2.21)). Thus k_i and p_i are related by the equation

$$p_i = C' \frac{k_i}{\left(G_k^{\frac{1}{2}}\right)_{ii}}, \ \forall \ 1 \le i \le n,$$
 (2.33)

where C' is the normalization constant given by $C' = \sqrt{\text{Tr}(D^2G)}$. We see that p_i are related to the k_i in the exact same way that p_i are related to q_i from equation (2.11). Below definition (2.10), it was mentioned that if \widetilde{P} and \widetilde{K} are two ensembles with the

same states with apriori probabilities p_i and k_i , which are related by equation (2.33), we get that $\mathscr{R}^{-1}(\widetilde{K}) = \widetilde{P}$. Since \mathscr{R}^{-1} is a bijection, this implies that $\widetilde{K} = \mathscr{R}(\widetilde{P}) = \widetilde{Q}$ and $k_i = q_i, \forall 1 \le i \le n$, where q_i is the apriori probability of states in \widetilde{Q} as given in equation (2.11). This also implies that C' = C.

From equation (2.30) we get that the RHS of equation (2.32) equates to

$$\left(\sum_{i=1}^{n} |w_i\rangle \langle w_i | \widetilde{\psi}_i \rangle \langle \widetilde{\psi}_i | \right)^2 = \sum_{i=1}^{n} d_{ii}^2 | \widetilde{\psi}_i \rangle \langle \widetilde{\psi}_i | = C^2 \left(\sum_{i=1}^{n} q_i | \psi_i \rangle \langle \psi_i | \right) = Z^2.$$

Then the fact that $\sum_{i=1}^{n} |w_i\rangle \langle w_i | \widetilde{\psi}_i \rangle \langle \widetilde{\psi}_i |$ is positive definite tells us that

$$\sum_{i=1}^{n} |w_i\rangle \langle w_i | \widetilde{\psi}_i \rangle \langle \widetilde{\psi}_i | = C \left(\sum_{i=1}^{n} q_i | \psi_i \rangle \langle \psi_i | \right)^{\frac{1}{2}} = Z.$$
(2.34)

Note that the ONB $\{|w_i\rangle\}_{i=1}^n$ was constructed from X, which solves for X in equation (2.23) and which is positive definite. That $\sum_{i=1}^n |w_i\rangle\langle w_i|\widetilde{\psi_i}\rangle\langle \widetilde{\psi_i}|$ is equal to $C\left(\sum_{i=1}^n q_i|\psi_i\rangle\langle \psi_i|\right)^{\frac{1}{2}}$, which we already know is equal to Z [3], implies that $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ is the optimal POVM. Thus $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ is the optimal POVM. Since $\{|w_i\rangle\langle w_i|\}_{i=1}^n$ is the unique optimal POVM for MED of \widetilde{P} so is the *n* tuple $(d_{11}, d_{22}, \dots, d_{nn})$ unique to the MED of \widetilde{P} , ⁹. This implies that $D = Diag(d_{11}, d_{22}, \dots, d_{nn})$ is unique, which implies that DGD is unique and since the positive square root of DGD is also unique, that tells us that X is unique too.

Theorem (3.2.7) tells us that for MED of any $\tilde{P} \in \mathcal{E}$, X is unique. But note that if $|\psi_i\rangle$ underwent a rotation by unitary U then it can be inferred from equation (2.23) that the solution for X won't change since G doesn't change. This implies that X is a function of G in \mathcal{G} .

Let the matrix elements of X be given by the following equation

⁹ Note that $d_{ii} = \langle w_i | \widetilde{\psi}_i \rangle$, thus if $\{ | w_i \rangle \langle w_i | \}_{i=1}^n$ is unique, so must the *n*-tuple $(d_{11}, d_{22}, \cdots, d_{nn})$.

$$\mathcal{X} = \begin{pmatrix} d_{11}^2 & d_{12} + id_{21} & d_{13} + id_{31} & \cdots & d_{1n} + id_{n1} \\ d_{12} - id_{21} & d_{22}^2 & d_{23} + id_{32} & \cdots & d_{2n} + id_{n2} \\ d_{13} - id_{31} & d_{23} - id_{32} & d_{33}^2 & \cdots & d_{3n} + id_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1n} - id_{n1} & d_{2n} - id_{n2} & d_{3n} - id_{n3} & \cdots & d_{nn}^2 \end{pmatrix},$$
 (2.35)

where d_{kl} are the real and imaginary parts of the matrix elements of X. Since X is a function on \mathcal{G} , d_{kl} are also functions on \mathcal{G} .

Definition 2.4.6. Let *Q* denote the set of all positive definite $n \times n$ matrices.

Thus $\mathcal{G} \subset \mathcal{Q}$. Using \mathcal{G} and \mathcal{Q} we formalize X as a function on \mathcal{G} .

Definition 2.4.7. $X : \mathcal{G} \longrightarrow Q$ is such that X(G) solves equation (2.23)

$$(X(G))^2 - D(G) G D(G) = 0.$$
(2.36a)

Let's denote $d_{kl} : \mathcal{G} \longrightarrow \mathbb{R}$ to be the real and imaginary parts of matrix elements of $\mathcal{X}(G)$, i.e.,

$$d_{kl}(G) \equiv Re((X(G))_{kl}), \ \forall \ 1 \le k < l \le n,$$
(2.36b)

$$d_{ii}(G) \equiv \sqrt{(\mathcal{X}(G))_{ii}}, \ \forall \ 1 \le i \le n,$$
(2.36c)

$$d_{kl}(G) \equiv Im((X(G))_{lk}), \ \forall \ 1 \le l < k \le n,$$
(2.36d)

and $D(G) \equiv Diag(d_{11}(G), d_{22}(G), \dots, d_{nn}(G)).$

Note that if one knows the real *n*-tuple $(d_{11}(G), d_{22}(G), \dots, d_{nn}(G))$, then using equation (2.23) one can compute X(G). Thus we have reformulated the MED problem for linearly independent pure states in a rotationally invariant way:

Rotationally Invariant Necessary and Sufficient Conditions: Let G be the gram matrix

corresponding to an n-LIP: $\{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^n$. To solve MED for this n-LIP, one needs to find real and positive n-tuple $(d_{11}(G), d_{22}(G), \dots, d_{nn}(G))$ such that, when arranged in the diagonal matrix $D(G) = Diag(d_{11}(G), d_{22}(G), \dots, d_{nn}(G))$, the diagonal of the positive square root of D(G)GD(G) is $(D(G))^2$.

2.4.2 Solution for the MED of LI Pure State Ensembles

X is a function on G such that X(G) is a solution for X in equation (2.23), and is positive definite. We need to compute X(G) for a given $G \in G$. We employ the Implicit Function Theorem (IFT) for this.

Functions and Variables for IFT

In this subsubsection, we will introduce the functions and variables which are part of the IFT technique.

Let the unknown hermitian matrix X in equation (2.23) be represented by

$$X = \begin{pmatrix} x_{11}^2 & x_{12} + ix_{21} & x_{13} + ix_{31} & \cdots & x_{1n} + ix_{n1} \\ x_{12} - ix_{21} & x_{22}^2 & x_{23} + ix_{32} & \cdots & x_{2n} + ix_{n2} \\ x_{13} - ix_{31} & x_{23} - ix_{32} & x_{33}^2 & \cdots & x_{3n} + ix_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1n} - ix_{n1} & x_{2n} - ix_{n2} & x_{3n} - ix_{n3} & \cdots & x_{nn}^2 \end{pmatrix},$$
(2.37)

where $x_{kl} \in \mathbb{R}$, $\forall 1 \le k, l \le n$.

Define *F* on $\mathcal{G} \times \mathcal{H}_n$, where \mathcal{H}_n is the real vector space of all $n \times n$ hermitian matrices.

Definition 2.4.8.

$$F(G, X) \equiv X^2 - D(X)GD(X), \qquad (2.38)$$

where *X* is of the form given in equation (2.37) and $D(X) \equiv Diag(x_{11}, x_{22}, \dots, x_{nn})$.

We define the matrix elements of *F* as functions of *G* and x_{ij} , $\forall 1 \le i, j \le n$.

$$F = \begin{pmatrix} f_{11} & f_{12} + if_{21} & f_{13} + if_{31} & \cdots & f_{1n} + if_{n1} \\ f_{12} - if_{21} & f_{22} & f_{23} + if_{32} & \cdots & f_{2n} + if_{n2} \\ f_{13} - if_{31} & f_{23} - if_{32} & f_{33} & \cdots & f_{3n} + if_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{1n} - if_{n1} & f_{2n} - if_{n2} & f_{3n} - if_{n3} & \cdots & f_{nn} \end{pmatrix},$$
(2.39)

where, for i < j, f_{ij} and f_{ji} represent the real and imaginary parts of F_{ij} respectively, and f_{ii} represents the diagonal element F_{ii} , and for j < i, f_{ji} and $-f_{ij}$ represent the real and imaginary parts of F_{ij} respectively. Then using definition (2.38) and equation (2.37) we get for i < j

$$f_{ij}(G, \underline{x}) = \sum_{k=1}^{i-1} \left(x_{ki} x_{kj} + x_{ik} y_{jk} \right) + \sum_{k=i+1}^{j-1} \left(x_{ik} x_{kj} - x_{ki} x_{jk} \right) + \sum_{k=j+1}^{n} \left(x_{ik} x_{jk} + x_{ki} x_{kj} \right) + x_{ij} \left(x_{ii}^2 + x_{jj}^2 \right) - x_{ii} x_{jj} Re \left(G_{ij} \right),$$
(2.40a)

$$f_{ji}(G, \underline{x}) = \sum_{k=1}^{i-1} \left(x_{ki} x_{jk} - x_{ik} x_{kj} \right) + \sum_{k=i+1}^{j-1} \left(x_{ik} x_{jk} + x_{ki} x_{kj} \right) + \sum_{k=j+1}^{n} \left(-x_{ik} x_{kj} + x_{ki} x_{jk} \right) + x_{ji} \left(x_{ii}^2 + x_{jj}^2 \right) - x_{ii} x_{jj} Im \left(G_{ij} \right),$$
(2.40b)

and for the diagonal elements we get

$$f_{ii}(G, \underline{x}) = \sum_{k=1}^{i-1} \left(x_{ki}^2 + x_{ik}^2 \right) + \sum_{k=i+1}^n \left(x_{ik}^2 + x_{ki}^2 \right) + 2x_{ii}^4 - x_{ii}^2 G_{ii}, \qquad (2.40c)$$

where $x \equiv (x_{11}, x_{12}, \dots, x_{nn})$ (i.e., x is the real n^2 -tuple of the x_{ij} -variables).

Finally, we define the Jacobian of the functions f_{ij} with respect to the variables x_{ij} ; this Jacobian matrix has the matrix elements

$$(J(G, \underline{x}))_{ij,kl} \equiv \frac{\partial f_{ij}(G, \underline{x})}{\partial x_{kl}}, \ \forall \ 1 \le i, j, k, l \le n.$$
(2.41)

Note that since the f_{ij} functions and the x_{ij} variables are both n^2 in number, this Jacobian matrix is an $n^2 \times n^2$ square matrix.

Implementing IFT

Let $G^{(0)} \in \mathcal{G}$ be a gram matrix whose MED for which we know the solution, that is, we know the values of $x_{ij} = d_{ij}(G^{(0)}), \forall 1 \le i, j \le n$ (see definition (2.4.7)). Substituting $x_{ij} = d_{ij}(G^{(0)}), \forall 1 \le i, j \le n$ in equation (2.37) gives us that $X = X(G^{(0)})$ (see equations (2.35) and definition (2.4.7)), and substituting $X = X(G^{(0)})$ into equation (2.38) gives (see theorem 3.2.7),

(i.) $F(G^{(0)}, X = X(G^{(0)})) = 0$. This equation tells us that $X = X(G^{(0)})$ is a solution for X in equation (2.23) when $G = G^{(0)}$.

(ii.)
$$X = X(G^{(0)}) > 0.$$

IFT, which is a well known result in functional analysis [54], then tells us the following.

Implicit Function Theorem: Consider the following inequality:

$$Det(J(G^{(0)}, d(G^{(0)}))) \neq 0,$$
(2.42)

where

$$d(G^{(0)}) = (d_{11}(G^{(0)}), d_{12}(G^{(0)}), \cdots, d_{1n}(G^{(0)}), d_{21}(G^{(0)}), \cdots, d_{nn}(G^{(0)}))$$

If the inequality (2.42) is true, then IFT tells us that there exists an open neighbourhood $I_{G^{(0)}}$ in \mathcal{G} containing $G^{(0)}$, such that for each i, j, where $1 \le i, j \le n$, there exists an open interval I_{ij} in \mathbb{R} containing the real number $d_{ij}(G^{(0)})$, such that one can define the function $\phi_{ij} : I_{G^{(0)}} \longrightarrow I_{ij}$, such that

- 1. ϕ_{ij} 's are continuously differentiable in $I_{G^{(0)}}$,
- 2. $\phi_{ij}(G^{(0)}) = d_{ij}(G^{(0)}), \forall 1 \le i, j \le n$, and
- 3. the following equation holds true for $\forall 1 \le i, j \le n$ and $\forall G \in I_{G^{(0)}}$:

$$f_{ij}(G, \phi(G)) = 0$$
, where $\phi(G) = (\phi_{11}(G), \phi_{12}(G), \cdots, \phi_{nn}(G)).$ (2.43)

Thus to use the IFT for our purpose we need to prove the following.

Theorem 2.4.3. $Det(J(G, d(G))) \neq 0, \forall G \in \mathcal{G}.$

Proof. This proof is divided into two parts:

(a.) To show that $J(G, \underline{d}(G))$ is a linear transformation on the real space \mathcal{H}_n of $n \times n$ complex hermitian matrices:

Proof of (a.): First note that J(G, d(G)) is the Jacobian of the function *F* with respect to the variable *X* (equation (2.38)).

Let x_{ij} be assigned the value $d_{ij}(G)$ for all $1 \le i, j \le n$. Now let $x_{ij} = d_{ij}(G) \longrightarrow x_{ij} = d_{ij}(G) + \epsilon \delta x_{ij}$ be an arbitrary perturbation, where ϵ is an infinitesimal positive real number and δx_{ij} are real, $\forall 1 \le i, j \le n$. As a result of this perturbation we have the following

(i)
$$(x_{ii}(G))^2 = (d_{ii}(G))^2 \longrightarrow (d_{ii}(G))^2 + 2\epsilon d_{ii}(G)\delta x_{ii} + O(\epsilon^2)$$
, and

2.4. MED FOR LI PURE STATES

(ii) $X = X(G) \longrightarrow X = X(G) + \epsilon \delta X + O(\epsilon^2)$ where

$$\delta X = \begin{pmatrix} 2d_{11}(G)\delta x_{11} & \delta x_{12} + i\delta x_{21} & \delta x_{13} + i\delta x_{31} & \cdots & \delta x_{1n} + i\delta x_{n1} \\ \delta x_{12} - i\delta x_{21} & 2d_{22}(G)\delta x_{22} & \delta x_{23} + i\delta x_{32} & \cdots & \delta x_{2n} + i\delta x_{n2} \\ \delta x_{13} - i\delta x_{31} & \delta x_{23} - i\delta x_{32} & 2d_{33}(G)\delta x_{33} & \cdots & \delta x_{3n} + i\delta x_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta x_{1n} - i\delta x_{n1} & \delta x_{2n} - i\delta x_{n2} & \delta x_{3n} - i\delta x_{n3} & \cdots & 2d_{nn}(G)\delta x_{nn} \end{pmatrix}.$$
(2.44a)

For the sake of brevity, for the rest of this proof, we will denote D(G) by D, X(G) by X, and J(G, d(G)) by J_G . Define:

$$D_{\delta} \equiv Diag(\delta x_{11}, \delta x_{22}, \cdots, \delta x_{nn}).$$
(2.44b)

Thus we get the following.

$$F(G, X + \epsilon \delta X)$$

$$= F(G, X) + \epsilon (\delta X X + X \delta X - D_{\delta} G D - D G D_{\delta}) + O(\epsilon^{2})$$

$$= \epsilon (\delta X X - D_{\delta} D^{-1} X^{2} + X \delta X - X^{2} D^{-1} D_{\delta}) + O(\epsilon^{2})$$

$$= \epsilon J_{G} (\delta X) + O(\epsilon^{2}), \qquad (2.44c)$$

where equation (2.23) was employed in the second step above, and

$$J_{G}(\delta X) = \delta X \mathcal{X} - D_{\delta} D^{-1} \mathcal{X}^{2} + \mathcal{X} \delta X - \mathcal{X}^{2} D^{-1} D_{\delta}$$

= $\left(\delta X \mathcal{X} - D_{\delta} D^{-1} \mathcal{X}^{2}\right) + \left(\delta X \mathcal{X} - D_{\delta} D^{-1} \mathcal{X}^{2}\right)^{\dagger}.$ (2.44d)

Thus it is seen that J_G is a linear transformation on the space of $n \times n$ complex hermitian matrices \mathcal{H}_n .

(b.) If the action of $J(G, \underline{d}(G))$ on some $n \times n$ complex hermitian matrix δX is 0 then δX

itself must be 0.

Proof of (b.): From equation (2.44d) it is clear that $J_G(\delta X) = 0$ if and only if $\delta X X - D_{\delta} D^{-1} X^2$ is anti-hermitian. Let's assume that $\delta X X - D_{\delta} D^{-1} X^2$ is anti-hermitian. That is,

$$\delta X \mathcal{X} - D_{\delta} D^{-1} \mathcal{X}^2 = -\mathcal{X} \delta X + \mathcal{X}^2 D^{-1} D_{\delta}$$
$$\implies \mathcal{X}^{-1} \delta X - \mathcal{X}^{-1} D_{\delta} D^{-1} \mathcal{X} = -\delta X \mathcal{X}^{-1} + \mathcal{X} D^{-1} D_{\delta} \mathcal{X}^{-1}.$$
(2.45a)

Let $X = \sum_{i=1}^{n} g_i |g_i\rangle \langle g_i|$ be the spectral decomposition of X. Then the *ij*-th matrix element of the matrix in equation (2.45a), in the $\{|g_i\rangle\}_{i=1}^{n}$ basis, is given by

$$\frac{1}{g_i} \langle g_i | \delta X | g_j \rangle - \frac{g_j}{g_i} \langle g_i | D_{\delta} D^{-1} | g_j \rangle$$

$$= -\frac{1}{g_j} \langle g_i | \delta X | g_j \rangle + \frac{g_i}{g_j} \langle g_i | D_{\delta} D^{-1} | g_j \rangle$$

$$\Longrightarrow \langle g_i | \delta X | g_j \rangle = \left(\frac{g_i^2 + g_j^2}{g_i + g_j}\right) \langle g_i | D_{\delta} D^{-1} | g_j \rangle.$$
(2.45b)

Multiplying the above number by $|g_i\rangle\langle g_j|$ and summing over *i*, *j* from 1 to *n* gives

$$\delta X = \sum_{i,j=1}^{n} \langle g_i | D_{\delta} D^{-1} | g_j \rangle \frac{g_i^2 + g_j^2}{g_i + g_j} | g_i \rangle \langle g_j |.$$
(2.45c)

Let $\{|k\rangle\}_{k=1}^{n}$ represent the standard basis, then $\langle k|g_{j}\rangle$ is complex number occuring in the *k*-th entry of $|g_{j}\rangle$. Using equations (2.44b) and (2.44a) we get $\langle g_{i} | D_{\delta}D^{-1} | g_{j}\rangle = \frac{1}{2} \sum_{l=1}^{n} \langle g_{i} | l \rangle \langle l | g_{j} \rangle \frac{(\delta X)_{ll}}{d_{ll}(G)^{2}}$. The diagonal elements of δX are then given by

$$(\delta X)_{kk} = \sum_{l=1}^{n} \left(\frac{1}{2} \sum_{i,j=1}^{n} \langle k | g_i \rangle \langle g_j | k \rangle \frac{g_i^2 + g_j^2}{g_i + g_j} \langle g_i | l \rangle \langle l | g_j \rangle \right) \frac{(\delta X)_{ll}}{(d_{ll}(G))^2}$$

$$= \sum_{l=1}^{n} \frac{1}{2} \left(O \Lambda O^{\dagger} \right)_{kl} \frac{(\delta X)_{ll}}{(d_{ll}(G))^2},$$
 (2.45d)
where *O* is an $n \times n^2$ matrix with matrix elements given by $O_{k,ij} = \langle k | g_i \rangle \langle g_j | k \rangle$, Λ is an $n^2 \times n^2$ diagonal matrix with matrix elements $\Lambda_{ij,kl} = \delta_{ik} \delta_{jl} \frac{g_i^2 + g_j^2}{g_i + g_j}$. It is easy to check that rows of *O* are orthogonal. Since $\Lambda > 0$ and *O* is of rank *n*, the matrix $\frac{1}{2}O\Lambda O^{\dagger}$ positive definite.

Consider

$$|D_{\delta X}\rangle \equiv \begin{pmatrix} (\delta X)_{11} \\ (\delta X)_{22} \\ \vdots \\ (\delta X)_{nn} \end{pmatrix}.$$
 (2.45e)

Then equation (2.45d) can be rewritten as

$$\left(\mathbb{1} - \frac{1}{2}O\Lambda O^{\dagger}D^{-2}\right)|D_{\delta X}\rangle = 0$$
$$\implies \left(D^{2} - \frac{1}{2}O\Lambda O^{\dagger}\right)D^{-2}|D_{\delta X}\rangle = 0 \qquad (2.45f)$$

Let Λ' be an $n^2 \times n^2$ diagonal matrix whose matrix elements are given by $\Lambda'_{ij,kl} = \delta_{ik}\delta_{jl}\frac{2g_ig_j}{g_i+g_j}$. Since $\Lambda' > 0$, $\frac{1}{2}O\Lambda'O^{\dagger}$ is positive definite. After some amount of tedious algebra we find that the following equation holds true.

$$D^{2} = \frac{1}{2}O\left(\Lambda + \Lambda'\right)O^{\dagger}.$$
 (2.45g)

Hence $D^2 - \frac{1}{2}O\Lambda O^{\dagger} = \frac{1}{2}O\Lambda' O^{\dagger} > 0$. This implies that for equation (2.45f) to be true $|D_{\delta X}\rangle = 0$. This implies (see equation (2.45e)) $(\delta X)_{ii} = 0$, which implies that $2d_{ii}(G)\delta x_{ii} = 0$, which implies that $x_{ii} = 0$, i.e., $D_{\delta} = 0$. Substituting $D_{\delta} = 0$ in equation (2.45c) gives $\delta X = 0$.

Hence, demanding $J_G(\delta X) = 0$ leads to the conclusion that $\delta X = 0$.

This means that J_G is non-singular, which then implies that $Det(J_G) \neq 0$. This proves the

theorem.

Theorem 2.4.3 implies that IFT holds true for all $G^{(0)} \in \mathcal{G}$, i.e., for all $G^{(0)} \in \mathcal{G}$ one can define these ϕ_{ij} functions so that the points 1., 2. and 3. mentioned in IFT are satisfied. The third point in IFT, i.e., equation (2.43), tells us that for any $G \in I_{G^{(0)}}$, F(G, X) = 0, when $x_{ij} = \phi_{ij}(G)$, $\forall 1 \leq i, j \leq n$. This is equivalent to stating that if one obtains the ϕ_{ij} functions, defined in some open neighbourhood $I_{G^{(0)}}$ of $G^{(0)}$, then $x_{ij} = \phi_{ij}(G)$, $\forall 1 \leq i, j \leq n$, gives us *some* solution for X in equation (2.23) for any $G \in I_{G^{(0)}}$. If it is true that assigning $x_{ij} = \phi_{ij}(G)$, $\forall 1 \leq i, j \leq n$, implies that X > 0, then obtaining the ϕ_{ij} functions in some open neighbourhood $I_{G^{(0)}}$ of $G^{(0)}$ gives us *the* solution for MED of all $G \in I_{G^{(0)}}$.

Theorem 2.4.4. When $G \in I_{G^{(0)}}$ and $x_{ij} = \phi_{ij}(G), \forall 1 \le i, j \le n$, then X > 0.

Proof. Suppose not.

Let there be some $G^{(1)} \in I_{G^{(0)}}$ such that when $x_{ij} = \phi_{ij}(G^{(1)}), \forall 1 \le i, j \le n$, then X has some non-positive eigenvalues.

Let $G(t) \equiv (1 - t)G^{(0)} + tG^{(1)}$ be a linear trajectory in \mathcal{G} . G(t) starts from G_0 when t = 0and ends at $G^{(1)}$ when t = 1. Note that eigenvalues of X are continuous functions of x_{ij} , and when restricting x_{ij} to be such that $x_{ij} = \phi_{ij}(G)$, $\forall 1 \le i, j \le n$, then x_{ij} are continuous functions over $I_{G^{(0)}}$. Thus the eigenvalues of X are continuous over $I_{G^{(0)}}$, whenever $x_{ij} = \phi_{ij}(G)$.

This implies the following.

- (i.) When $x_{ij} = \phi_{ij}(G(0)) \forall 1 \le i, j \le n$, all eigenvalues of X are positive.
- (ii.) When $x_{ij} = \phi_{ij}(G(1)) \forall 1 \le i, j \le n$, some eigenvalues of X are non-positive.

The intermediate value theorem tells us that since ϕ_{ij} 's are continuous over $I_{G^{(0)}}$, (i.) and (ii.) imply that there must be some $t' \in (0, 1]$, such that

- (i.) X > 0, when $x_{ij} = \phi_{ij}(G(t))$, for all $t \in [0, t')$,
- (ii.) for all $t \in (t', 1]$, X is not necessarily positive definite, when $x_{ij} = \phi_{ij}(G(t))$, and finally
- (iii.) *X* has some 0 eigenvalue(s) when $x_{ij} = \phi_{ij}(G(t'))$, i.e., when t = t'.

When t < t' then X > 0, which also implies that X = X(G(t)) holds true for the interval $t \in [0, t')$. Since $\frac{(X(G))^2}{\text{Tr}((X(G))^2)} = \mathscr{R}^{(\mathcal{G})}(G)^{-10}$ we get that $\frac{X^2}{\text{Tr}(X^2)} = \mathscr{R}^{(\mathcal{G})}(G(t))$, when t < t'. Since $\mathscr{R}^{(\mathcal{G})}$ is continuous on \mathcal{G}^{-11} and eigenvalues of X are continuous in $I_{G^{(0)}}$, it follows that when t = t', $\frac{X^2}{\text{Tr}(X^2)} = \mathscr{R}^{(\mathcal{G})}(G(t'))$. From (iii.) above it is seen that when t = t', X is singular; this implies that $\mathscr{R}^{(\mathcal{G})}(G(t'))$ is singular as well, which is a contradiction since we know that $\mathscr{R}^{(\mathcal{G})}$ is a function from \mathcal{G} to \mathcal{G} and all gram matrices in \mathcal{G} are positive definite.

This contradiction arose from the assumption that when $x_{ij} = \phi_{ij}(G^{(1)})$, *X* is not positive definite. This proves the theorem.

Theorem 2.4.4 tells us that for any starting point $G^{(0)} \in \mathcal{G}$, if we take any point $G \in I_{G^{(0)}}$, the ϕ_{ij} 's obey the equality: $\phi_{ij}(G) = d_{ij}(G)$, $\forall 1 \leq i, j \leq n$. Given this fact, from here onwards, we will represent the implicit functional dependence ϕ_{ij} by d_{ij} itself.

We can make a stronger statement about the behaviour of the functions d_{ij} on \mathcal{G} . It is easier to do so if we define trajectories, like the one defined in the proof of theorem 2.4.4 in \mathcal{G} , and prove results about the behaviour of the d_{ij} 's with respect to the independent variable *t*. For that purpose, let $G^{(0)}, G^{(1)} \in \mathcal{G}$ be distinct; define a linear trajectory in \mathcal{G} from $G^{(0)}$ to $G^{(1)}, G : [0, 1] \longrightarrow \mathcal{G}$ as

$$G(t) = (1-t)G^{(0)} + tG^{(1)}.$$
(2.46)

We now apply the implicit function theorem to F(G(t), X), where X represents the variables whose implicit dependence we seek and t is the independent variable.

¹⁰ See equation (2.29) in the proof of theorem 2.4.1 in subsection 2.4.1.

¹¹ See description below definition 2.4.5 in the beginning of this section.

The analytic implicit function theorem [54] tells us that if $f_{ij}(G(t), \underline{x})$ are analytic functions of the variables t and x_{kl} , then $\phi_{kl}(G(t))$ (which are equal to $d_{kl}(G(t))$) should also be analytic functions of the variable $t \in [0, 1]$. Equations (2.40a), (2.40b) and (2.40c) tell us that $f_{ij}(G(t), \underline{x})$ are multivariate polynomials in the variables t and x_{kl} , which implies that the f_{ij} 's are analytic functions of t and x_{kl} . Thus $d_{kl}(G(t))$ are analytic functions of the variable t. This implies that, more generally, d_{kl} are analytic functions over \mathcal{G} .

Taylor Series and Analytic Continuation

The fact that d_{kl} are analytic functions on \mathcal{G} allows us to Taylor expand d_{kl} from some point in \mathcal{G} to another point. Let us assume that we want to find the solution for MED of some gram matrix $G^{(1)} \in \mathcal{G}$, and that we know the solution for MED of $G^{(0)} \in \mathcal{G}$. Then we define a trajectory as was done in equation (2.46). We will now show that using equation (2.36a) we can obtain the derivatives of $d_{kl}(G(t))$, upto any order, with respect to t; this allows us to Taylor expand the $d_{kl}(G(t))$ function about the point t = 0. Analytically continuing from t = 0 to t = 1 allows us to obtain the values of $d_{kl}(G(t))$ at t = 1.

First we show how to obtain the first order derivatives of $d_{kl}(G(.))$ with respect to t. We will abbreviate $D(G(t)) = (d_{11}(G(t)), d_{22}(G(t)), \dots, d_{nn}(G(t)))$ as D(t) for convenience. Similarly X(G(t)) will be abbreviated as X(t). It will be useful to denote separately the matrix of off-diagonal elements of X(t). Thus define $N(t) \equiv X(t) - (D(t))^2$. Equation (2.36a) can be re-written as $(D(t)^2 + N(t))^2 = D(t)G(t)D(t)$. Let $\Delta \equiv \frac{dG(t)}{dt} = G^{(1)} - G^{(0)}$. Taking the total first order derivative on both sides of equation (2.36a) gives

$$\left(D(t)^{2} + N(t) \right) \left(2D(t) \frac{dD(t)}{dt} + \frac{dN(t)}{dt} \right) + \left(2D(t) \frac{dD(t)}{dt} + \frac{dN(t)}{dt} \right) \left(D(t)^{2} + N(t) \right) - \left(D(t)G(t) \right) \frac{dD(t)}{dt} - \frac{dD(t)}{dt}G(t)D(t) = D(t)\Delta D(t),$$

$$(2.47)$$

where

$$\frac{dD(t)}{dt} = \left(\frac{d \ (d_{11}(t))}{dt}, \frac{d \ (d_{22}(t))}{dt}, \cdots, \frac{d \ (d_{nn}(t))}{dt}\right),$$

$$\left(\frac{dN(t)}{dt}\right)_{kl} = \frac{d}{dt} \left(d_{kl}(t) + id_{lk}(t)\right) \text{ (when } k < l\text{)},$$
$$\left(\frac{dN(t)}{dt}\right)_{ii} = 0 \text{ and},$$
$$\left(\frac{dN(t)}{dt}\right)_{kl} = \frac{d}{dt} \left(d_{lk}(t) - id_{kl}(t)\right) \text{ (when } k > l\text{)}.$$

Thus we get n^2 coupled ordinary differential equations. By substituting the values of $d_{kl}(0)$ in equation (2.47) one can solve for $\frac{d d_{kl}(t)}{dt}|_{t=0}$.

The second order derivatives can be obtained in a similar fashion: taking the total derivative of LHS and RHS of the equation (2.47) with respect to *t* (i.e. the second order derivative of the LHS of equation (2.36a)) we get a set of n^2 coupled second order differential equations. Setting t = 0 and using the values of $d_{kl}(0)$ and $\frac{d d_{kl}(t)}{dt}|_{t=0}$, one can solve the resulting (linear) equations to obtain the values of the unknowns $\frac{n^2 d_{kl}(t)}{dt^2}|_{t=0}$.

Continuing in this manner one can obtain the values of the derivatives of $d_{kl}(t)$ upto any order, at the point t = 0. In the following equation we give the *k*-th order derivative of equation (2.36a) for this purpose.

$$\begin{split} & \left(D(t)^{2} + N(t)\right) \left(2D(t)\frac{d^{k}D(t)}{dt^{k}} + \frac{d^{k}N(t)}{dt^{k}}\right) + \left(2D(t)\frac{d^{k}D(t)}{dt^{k}} + \frac{d^{k}N(t)}{dt^{k}}\right) \left(D(t)^{2} + N(t)\right) \\ & - (D(t)G(t))\frac{d^{k}D(t)}{dt^{k}} - \frac{d^{k}D(t)}{dt^{k}}G(t)D(t) \\ & = -\left(\left(D(t)^{2} + N(t)\right) \left(\sum_{l_{1}=1}^{k-1} \binom{k}{l_{1}} \left((\frac{d}{dt})^{l_{1}}D(t)\right) \left((\frac{d}{dt})^{k-l_{1}}D(t)\right)\right) + \text{h.c.}\right) \\ & - \sum_{l_{2}=1}^{k-1} \binom{k}{l_{2}} \left(\left(\frac{d}{dt}\right)^{k_{1}} (D(t)^{2} + N(t))\right) \left(\left(\frac{d}{dt}\right)^{k-k_{1}} (D(t)^{2} + N(t))\right) \\ & + \sum_{m_{1}=1}^{k-1} \binom{k}{m_{1}} \left((\frac{d}{dt})^{m_{1}}D(t)\right) G(t) \left(\left(\frac{d}{dt}\right)^{k-m_{1}}D(t)\right) \\ & + k \sum_{m_{2}=0}^{k-1} \binom{k}{m_{2}} \left(\left(\frac{d}{dt}\right)^{m_{2}}D(t)\right) \Delta \left(\left(\frac{d}{dt}\right)^{k-m_{2}}D(t)\right). \end{split}$$

By substituting the values of all derivatives at t = 0, one can expand the d_{kl} functions about the point t = 0. Analytic continuation is straightforward: by using the aforementioned Taylor expansion about t = 0, one obtains the value of d_{kl} at some $t = \delta t > 0$; one can then use the aforementioned method to obtain the values of derivatives of d_{kl} at $t = \delta t$ and Taylor expand the d_{kl} functions from δt to $t > \delta t$. In this manner one can Taylor expand and analytically continue d_{kl} 's from t = 0 to the point t = 1.

The need for analytic continuation raises the following question: what is the radius of convergence for the Taylor series about some point *t* in the interval [0, 1]? The LHS of equations (2.47) and (2.48) gives us a hint: $\frac{d^k D(t)}{dt^k}$ and $\frac{d^k N(t)}{dt^k}$ scale proportionally to the *k*-th power of Δ i.e.,

$$\Delta \longrightarrow \nu \Delta \Longrightarrow \left(\frac{d^k D(t)}{dt^k}, \frac{d^k N(t)}{dt^k}\right) \to \left(\nu^k \frac{d^k D(t)}{dt^k}, \nu^k \frac{d^k N(t)}{dt^k}\right).$$
(2.49)

This tells us that we need to keep $\|\Delta\|_2$ small to ensure that either $G^{(1)}$ falls within the radius of convergence of the d_{kl} functions when expanded about the point $G^{(0)}$ or the number of times one is required to analytically continue from t = 0 to t = 1 is low. It is very difficult to obtain the exact radius of convergence for every point in \mathcal{G} since the value of the radius of convergence differs for different points in \mathcal{G}^{12} .

For a given $G^{(1)}$ for which we wish to find the solution, it is desirable to find a $G^{(0)}$ so that $G^{(1)}$ falls within the radius of convergence of the d_{kl} functions, when expanded about $G^{(0)}$. In the following we give a method to find such a $G^{(0)}$ for a given $G^{(1)}$.

Starting point which generally doesn't require analytic continuation: Let $G^{(0)} \in \mathcal{G}$ be some gram matrix with the property that the diagonal of the positive definite square root of $G^{(0)}$, i.e., $G^{(0)\frac{1}{2}}$ has the property $(G^{(0)\frac{1}{2}})_{11} = (G^{(0)\frac{1}{2}})_{22} = \cdots = (G^{(0)\frac{1}{2}})_{nn}$. Substituting

¹² Particularly as one gets closer to points near the boundary of \mathcal{G} (which lies outside \mathcal{G}), the radius of convergence becomes smaller.

 $G^{\frac{1}{2}} = G^{(0)\frac{1}{2}}$, along with $W = \mathbb{1}_n$ and $D = (G^{(0)\frac{1}{2}})_{11} \mathbb{1}_n$ in the LHS of equation (2.22) gives us the RHS of equation (2.22), i.e., they all satisfy equation (2.22). It is also seen that when $D = (G^{(0)\frac{1}{2}})_{11} \mathbb{1}_n$, then $X = DG^{(0)\frac{1}{2}}$ is a solution for the equation (2.23) for $G = G^{(0)}$, and since D is a multiple of $\mathbb{1}_n$, X > 0. Thus, when the diagonal of $G^{(0)\frac{1}{2}}$ is a multiple of $\mathbb{1}_n$, the solution for the MED of corresponding gram matrix $G^{(0)}$ is known¹³.

Thus for a given $G^{(1)}$, we want to find a starting point $G^{(0)}$ such that the diagonal elements of $G^{(0)\frac{1}{2}}$ are all equal. For this purpose expand the positive square root of $G^{(1)}$ i.e., $G^{(1)\frac{1}{2}}$ in an ONB of \mathcal{H}_n , which comprises of $\frac{1}{\sqrt{n}}$ and the generalized Gell-Man matrices $\frac{\sigma_{lk}}{\sqrt{2}}$ where $1 \le l, k \le n[55]$. Here the σ_{lk} matrices are defined as

$$\sigma_{lk} = \begin{cases} |l\rangle\langle k| + |k\rangle\langle l|, \text{ when } l < k, \\ i|l\rangle\langle k| - i|k\rangle\langle l|), \text{ when } l > k, \\ \sqrt{\frac{2}{l(l+1)}} \left(\sum_{j=1}^{l} |j\rangle\langle j| - l|l + 1\rangle\langle l + 1|\right) \delta_{lk}, \text{ when } 1 \le l \le n-1 \end{cases}$$

$$(2.50)$$

All generalized Gell-Mann matrices in equation (2.50) have Hilbert-Schmidt norm $\sqrt{2}$. Let $G^{(1)\frac{1}{2}}$ have the following expansion in these Gell-Mann matrices.

$$G^{(1)\frac{1}{2}} = \gamma \frac{\mathbb{1}}{\sqrt{n}} + \sum_{j=1}^{n-1} \beta_j \frac{\sigma_{jj}}{\sqrt{2}} + \sum_{l,k=1\atop l \neq k}^n \zeta_{lk} \frac{\sigma_{lk}}{\sqrt{2}},$$
(2.51)

where $\zeta_{lk}, \beta_j, \gamma$ are real numbers. Note that $\gamma > 0$ s $G^{(1)\frac{1}{2}}$. Based upon this define $G^{(0)\frac{1}{2}}$ as

$$G^{(0)\frac{1}{2}} = \kappa \frac{\mathbb{1}}{\sqrt{n}} + \sum_{l \neq k} \zeta_{lk} \frac{\sigma_{lk}}{\sqrt{2}},$$
(2.52)

where $\kappa = \sqrt{\gamma^2 + \sum_j \beta_j^2}$.

¹³ This result is well known. It corresponds to those cases when $mathscrR(\widetilde{P}) = \widetilde{P}$.

It is easily verified that $\operatorname{Tr}(G^{(0)}) = 1$. One needs to check if $G^{(0)\frac{1}{2}} > 0$ or not. Generally, it is true that $G^{(0)\frac{1}{2}} > 0$. But if some eigenvalues of $G^{(1)}$ are close to 0, this *may* not hold. Suppose it holds (as is generally the case), $d_{ii}(0) = \kappa$, $d_{kl}(0) = \operatorname{Re}\left(\left(G^{(0)\frac{1}{2}}\right)_{kl}\right)$, when k < l, $d_{kl}(0) = \operatorname{Im}\left(\left(G^{(0)\frac{1}{2}}\right)_{lk}\right)$, when k > l. $G^{(1)}$ generally falls within the radius of convergence of all d_{kl} functions about the starting point $G^{(0)}$. In such circumstances one doesn't need analytic continuation; one can straightforwardly calculate $d_{kl}(1)$ from the Taylor series about t = 0. If $G^{(0)\frac{1}{2}}$, obtained this way, isn't positive definite, then this method fails and one needs another starting point.

Starting points which generally require analytic continuation: Another possible starting point is an ensemble of equiprobable orthogonal states; this ensemble's gram matrix is $G^{(0)} = \frac{1}{n}$ where $d_{ij}(0) = \delta_{ij} \frac{1}{\sqrt{n}}$. To *drag* the solution from $G^{(0)}$ to $G^{(1)}$ one needs to divide the [0, 1] interval into subintervals and analytically continue the d_{kl} 's from the starting point of each subinterval to its corresponding ending point. Here it needs to be ensured that one doesn't overshoot beyond the radius of convergence of any of the d_{kl} functions at the starting point of each subinterval. For this purpose it was found that it generally suffices to divide [0, 1] into $\lceil n^2 ||\Delta||_2 \rceil$ subintervals. Generally the smaller the intervals, the lower the value of error.

Error-Estimation: There is a simple method to estimate the degree of error in the process; this is based on the fact that when the solution, i.e., $d_{kl}(1)$'s are substituted in the LHS of equation (2.23) one should obtain the zero matrix, which isn't what we get due to errors. Thus the value of the Hilbert-Schmidt norm of the quantity on the LHS, i.e., the value of $||(D(1)^2 + N(1))^2 - D(1)G(1)D(1))||_2$ gives us an estimate of the degree of error which has accumulated into the solution. The closer $||(D(1)^2 + N(1))^2 - D(1)G(1)D(1))||_2$ is to 0, the lower the error. Note that one cannot decrease the error significantly by increasing the order upto which the Taylor series is expanded beyond a order of expansion. On the other hand error rates can be substantially reduced by decreasing the size of the subintervals.

Thus having solved for $d_{kl}(1)$ with a high degree of accuracy, one can now obtain the optimal POVM. In the following we present an example for n = 5. Note that while the precision of the starting point is upto 20 significant digits, only the first 6 significant digits have been displayed. For lack of space sometimes quantities have been displayed with upto only 4 significant digits.

$$|\widetilde{\psi}_1\rangle = \begin{bmatrix} 0.320457 \\ 0.123687 + i0.0117558 \\ 0.117838 + i - 0.027942 \\ 0.109674 + i0.0167151 \\ 0.0860555 + i0.00780123 \end{bmatrix} |\widetilde{\psi}_2\rangle = \begin{bmatrix} 0.123687 - i0.0117558 \\ 0.397851 \\ 0.169692 - i0.0506685 \\ 0.125198 - i0.0244774 \\ 0.124106 - i0.0261114 \end{bmatrix}$$

$$|\widetilde{\psi}_{3}\rangle = \begin{bmatrix} 0.117838 + i0.027942 \\ 0.169692 + i0.0506685 \\ 0.404725 \\ 0.13847 + i0.0177653 \\ 0.122277 - i0.0249506 \end{bmatrix} |\widetilde{\psi}_{4}\rangle = \begin{bmatrix} 0.109674 - i0.0167151 \\ 0.125198 + i0.0244774 \\ 0.13847 - i0.0177653 \\ 0.373791 \\ 0.110387 - i0.013984 \end{bmatrix}$$

$$|\widetilde{\psi}_{5}\rangle = \begin{vmatrix} 0.0860555 - i0.00780123 \\ 0.124106 + i0.0261114 \\ 0.122277 + i0.0249506 \\ 0.110387 + i0.013984 \\ 0.33677 \end{vmatrix}$$

The corresponding $|\tilde{u}_i\rangle$ states are given by:

CHAPTER 2. MINIMUM ERROR DISCRIMINATION

$$|\widetilde{u}_{1}\rangle = \begin{bmatrix} 3.93887 \\ -0.668108 + i0.0313699 \\ -0.553671 + i0.331697 \\ -0.611991 - i0.234777 \\ -0.375925 - i0.264517 \end{bmatrix} |\widetilde{u}_{2}\rangle = \begin{bmatrix} -0.668108 - i0.0313699 \\ 3.52494 \\ -0.939093 + i0.353801 \\ -0.418308 + i0.204928 \\ -0.685643 + i0.0142212 \end{bmatrix}$$

$$|\widetilde{u}_{3}\rangle = \begin{bmatrix} -0.553671 - i0.331697 \\ -0.939093 - i0.353801 \\ 3.50731 \\ -0.634577 - i0.0903887 \\ -0.554402 + i0.418281 \end{bmatrix} |\widetilde{u}_{4}\rangle = \begin{bmatrix} -0.611991 + i0.234777 \\ -0.418308 - i0.204928 \\ -0.634577 + i0.0903887 \\ 3.42828 \\ -0.568152 + i0.0597921 \end{bmatrix}$$

$$|\widetilde{u}_{5}\rangle = \begin{bmatrix} -0.375925 + i0.264517 \\ -0.685643 - i0.0142212 \\ -0.554402 - i0.418281 \\ -0.568152 - i0.0597921 \\ 3.74634 \end{bmatrix}$$

The gram matrix for the ensemble $\{|\widetilde{\psi}_i\rangle\}_{i=1}^5$, i.e., $G^{(1)}$ is given by:

Then using equation (2.52), we have

 $\begin{array}{c} G^{(0)\frac{1}{2}} = \\ \\ \hline 0.36821 & 0.12368 \cdot i0.011755 & 0.11783 + i0.02794 & 0.10967 \cdot i0.016715 & 0.086055 \cdot i0.0078012 \\ \hline 0.12368 + i0.011755 & 0.36821 & 0.16969 + i0.050668 & 0.12519 + i0.024477 & 0.12410 + i0.026111 \\ \hline 0.11783 \cdot i0.02794 & 0.16969 \cdot i0.050668 & 0.36821 & 0.13847 \cdot i0.017765 & 0.12227 + i0.024950 \\ \hline 0.10967 + i0.016715 & 0.12519 \cdot i0.024477 & 0.13847 + i0.017765 & 0.36821 & 0.11038 + i0.013984 \\ \hline 0.086055 + i0.0078012 & 0.12410 \cdot i0.026111 & 0.12227 \cdot i0.024950 & 0.11038 \cdot i0.01398 & 0.36821 \\ \hline \end{array}$

We see that all the diagonal elements of $G^{(0)\frac{1}{2}}$ are all equal. Also $G^{(0)\frac{1}{2}} > 0$. Thus $d_{ii}(0)$ is equal to the diagonal elements of $G^{(0)\frac{1}{2}}$ and $d_{kl}(0)$ are assigned values of the off-diagonal elements of $G^{(0)\frac{1}{2}}$ (when $i \neq j$).

Here $||\Delta||_2 = ||G^{(1)} - G^{(0)}||_2 = 0.058777 \sim 1/5^2$ (= 0.04). This gives us the indication that t = 1 lies within the radius of convergence of the implicitly defined functions d_{kl} about the point t = 0 and that no analytic continuation is required at any intermittent point. Upon employing the Taylor series expansion and expanding the series upto 10-th term, we obtain the solution for $X(1) = D(1)^2 + N(1)$:

 $X(1) = D(1)^2 + N(1) =$

0.09627	0.04197 -i0.00407	0.04054+ i0.009487	0.03528-i0.005896	0.02484-i0.003121
0.04197 + i0.00407	0.1635	0.07237+ i0.02128	0.04981+ i0.009339	0.04439+ i0.008852
0.04054 -i0.009487	0.07237 -i0.02128	0.1710	0.05580 -i0.00729	0.04424+ i0.008926
0.03528 + i0.005896	0.04981 -i0.009339	0.05580+ i0.00729	0.1399	0.03732+ i0.004563
0.02484 + i0.003121	0.04439 + i-0.008852	0.04424-i0.008926	0.03732-i0.004563	0.1083

X(1) > 0 holds true.

 $d_{11}(1) = 0.310278, \ d_{22}(1) = 0.404377, \ d_{33}(1) = 0.413591, \ d_{44}(1) = 0.374064, \ d_{55}(1) = 0.329225.$

The maximum success probability, $P_s^{max} = \sum_{i=1}^n (d_{ii}(1))^2 = 0.679164$.

$$||(X(1))^2 - D(1)G(1)D(1)||_2 = 2.92337 \times 10^{-9}.$$

For lack of space the projectors $|w_i\rangle\langle w_i|$ aren't given here. Instead we give the ONB $\{|w_i\rangle\}_{i=1}^{n}$:

$$|w_{1}\rangle = \begin{vmatrix} 0.998902 - i0.000902941 \\ -0.0294294 - i0.00140465 \\ -0.0281464 + i0.0114238 \\ -0.0185558 - i0.00595048 \\ -0.00450716 - i0.00157192 \end{vmatrix} |w_{2}\rangle = \begin{vmatrix} 0.0295208 - i0.00161874 \\ 0.999231 - i0.000890303 \\ -0.00479151 + i0.00107801 \\ 0.00760396 - i0.00239694 \\ 0.0239121 - i0.00195944 \end{vmatrix}$$

$$|w_{3}\rangle = \begin{vmatrix} 0.0285947 + i0.0113073 \\ 0.00328547 + i0.000850588 \\ 0.999104 + i0.000230941 \\ 0.0125773 + i0.00210581 \\ 0.0237566 - i0.0103508 \end{vmatrix} 0.0179661 - i0.00612077 \\ -0.00850615 - i0.00226113 \\ -0.0132936 + i0.00235778 \\ 0.999616 + i0.00060647 \\ 0.0121594 - i0.000373086 \end{vmatrix}$$

$$|w_5\rangle = \begin{vmatrix} 0.00301285 - i0.00208885 \\ -0.0240121 - i0.00194482 \\ -0.0235693 - i0.0103215 \\ -0.0127196 - i0.000528417 \\ 0.99929 + i0.000965318 \end{vmatrix}$$

Despite having satisfied the rotationally invariant conditions (refer theorem 3.2.7), we would like to see if both the conditions (2.7) and (2.8) are satisfied. Instead of checking condition (2.7) we check if *Z*, from equation (2.6), is hermitian or not. We first use $\{|w_i\rangle\}_{i=1}^n$ to compute the operator *Z*. We measure the non-hermiticity of *Z* as $\frac{1}{2}||Z - Z^{\dagger}||_2$, which takes the value 2.22059×10^{-10} for our example. That *Z* is hermitian (within error) and satisfies equation (2.6) implies that equations (2.5) or equivalently equations (2.7) are satisfied. Additionally we find that $\forall 1 \le i \le n$, all except one eigenvalue of $Z - p_i |\psi_i\rangle \langle\psi_i|$ are positive. For each $i = 1, 2, \dots, n$ the non-positive eigenvalue of $Z - p_i |\psi_i\rangle \langle\psi_i|$ is either 0 or of the order 10^{-10} , showing that the condition (2.8) is also satisfied. Thus we have demonstrated an example of obtaining the optimal POVM for MED of an ensemble of 5 LI states.

Algorithms: Computational Complexity

In the following we outline the algorithm for the Taylor series expansion method, which gives us the solution for the MED of a given *n*-LIP ensemble. The method has already been elucidated in detail in subsubsection 2.4.2. After giving the algorithm, we give its time complexity¹⁴ and space complexity¹⁵. The acceptable tolerance error being assumed here is of the order 10^{-9} , and the time and space complexities are computed corresponding to this acceptable error margin.

Algorithm 1: Taylor Series The algorithm of the Taylor series method (subsubsection 2.4.2) is given in the following steps.

- (1) Construct the gram matrix $G^{(1)}$ from the given ensemble \widetilde{P} . Choose an appropriate starting point $G(0) = G^{(0)}$ (for which the solution $d_{ij}(0)$, for all $1 \le i, j \le n$, is known) and define the function $G(t) = (1 t)G^{(0)} + tG^{(1)}$. If $||\Delta||_2 n^2 \sim 1$ then there's no need to divide the interval [0, 1] into subintervals, but otherwise divide [0, 1] into $L \equiv \left[||\Delta||_2 n^2 \right]$ intervals.
- (2) For $l = 0, 1, 2, \dots, L 1$, set $t_l = \frac{l}{L}$ and iterate over each interval in the following manner:
 - (2.1) For k = 1 through k = K iterate the following: solve eqn (2.48) for $\frac{d^k d_{ij}}{dt^k}|_{t=t_l}$, for all $1 \le i, j \le n$, by using values of lower order derivatives as explained in subsubsection 2.4.2.

¹⁴ The time complexity of any algorithm is given by the order of the total number of elementary steps involved in completing said algorithm. In this work, each of the following are regarded as elementary steps: basic arithmetic operations (addition, subtraction, multiplication, division) of floating point variables, assigning a value to a floating-point variable, checking a condition and retrieving the value of a variable stored in memory.

¹⁵ The space complexity is the count of the total number of variables and constants used in algorithms. These variables and constants can be of floating point type, integer type, binary etc; in this work we treat them all alike while adding the number of variables to give us the final count. Similar to the case of the time complexity, space complexity too is given in terms of the *order* of the count, rather than the exact number.

(2.2) Having obtained the values of derivatives $\frac{d^k d_{ij}}{dt^k}|_{t=t_l}$ upto *K*-th order for all $1 \le i, j \le n$, substitute these derivatives in an expression of the Taylor series expansion of the d_{ij} functions about the point $t = t_l$, when expanded to *K*-th order. The resulting expressions will give *K*-th degree polynomials in the variable *t* for each $1 \le i, j \le n$, i.e., for each d_{ij} . Obtain the value of $d_{ij}(t_{l+1})$ by computing the value these polynomials take at $t = t_{l+1}$. Then increment *t* to t_{l+1} , go to (2) and iterate. Stop when l = L.

In the following table we give the time and space complexity of various steps in the aforementioned algorithm.

	Step in the algorithm	Time	Space
		Complexity	Complexity
1.	Computing $G^{(1)}$ from \widetilde{P}	$O(n^3)$	$O(n^2)$
2.	Computing $G^{(0)}$ from $G^{(1)}$, as in subsubsection 2.4.2	$O(n^3)$	$O(n^3)$
3.	Solving for $\frac{d^k d_{ij}}{dt^k} _{t=t_l}$ for $k = 1, 2, \cdots, K$	$O(Kn^6)$	$O(Kn^4)$
4.	Computing Taylor series expansion of $d_{ij}(t - t_l)$ upto	$O(Kn^2)$	$O(n^2)$
	K-th order at $t = t_{l+1}$		
5.	Repeating steps 3. and 4. over $L \simeq n^2 \Delta _2$ times	$O(K^2n^8)$	$O(n^6)$

Table 2.1: Time and space complexity of various steps in the Taylor series algorithm.

Note that the algorithm is polynomial in *n*. It is expected that to maintain the acceptable error margin (i.e., $\|(D(1)^2 + N(1))^2 - D(1)G(1)D(1)\|_2 \leq 10^{-9})$ as *n* increases, one would have to increase the value of *K* as well. While the numerical examples we checked support this hypothesis, the required increment of *K* to compensate the increase in the value of *n* is seen to be significant only over large variations of values of *n* (when *n* varies over a range of 20). Indeed, it remains almost constant for n = 3 to n = 10 for the error to remain within the margin of the order of 10^{-9} . As in the example given in the end of subsubsection 2.4.2, choosing K = 10 suffices to maintain the error within said margin.

If $\|\Delta\|_2 n^2 \simeq 1$, analytic continuation isn't required and then the total time complexity of the algorithm is $O(n^6)$ and the total space complexity of the algorithm is $O(n^4)$. In case $\|\Delta\|_2 n^2 > 1$, since the maximum value of $\|\Delta\|_2 \le 2$, analytic continuation is required, and

in that case, the worst case time and space complexities¹⁶ are given by $O(n^8)$ and $O(n^6)$ respectively.

While the Taylor series method is polynomial in time with a relatively low computational complexity, it is seen that directly employing Newton's method is simpler and more computationally efficient. We will now explain how to employ Newton's method.

Algorithm 2: Newton's Method This is a well known numerical technique for solving non-linear equations. We use it here to solve the equations $f_{ij}(G, \underline{x}) = 0, \forall 1 \le i, j \le n$, (see (2.40a), (2.40b) and (2.40c) for f_{ij}) where G is the gram matrix of the ensemble \widetilde{P} whose MED we want to solve for, and \underline{x} are the variables which - we demand - will converge to the solution $\underline{d}(G)$. This convergence is achieved over a few iterations which are part of the algorithm. The technique is based on a very simple principle which we will now elaborate.

The Taylor expansion of the $f_{ij}(G, .)$ functions, when expanded about the point d(G), can be approximated by the first order terms for small perturbations $d(G) \longrightarrow d(G) + \delta x$ as seen in the equation below.

$$f_{ij}(G, \underline{d}(G) + \delta \underline{x}) \approx f_{ij}(G, \underline{d}(G)) + \sum_{k,l=1}^{n} \left(\frac{\partial f_{ij}(G, \underline{x})}{\partial x_{kl}} \Big|_{\underline{x}=\underline{d}(G)} \right) \delta x_{kl}$$

$$= \sum_{k,l=1}^{n} (J_G)_{ij,kl} \, \delta x_{kl},$$
(2.53)

where we have used $f_{ij}(G, \underline{d}(G)) = 0$, $\forall 1 \le i, j \le n$, and where we denote $(J_G)_{ij,kl} \equiv \frac{\partial f_{ij}(G, \underline{x})}{\partial x_{kl}}|_{\underline{x}=\underline{d}(G)}$.

We want to obtain the value of the solution d(G). We assume that our starting point is $d(G) + \delta x$ which is close to d(G), so that $f_{ij}(G, d(G) + \delta x)$ can be approximated as the RHS of equation (2.53). Denote the inverse of the Jacobian J(G, d(G)) by $(J_G)^{-117}$. Then

¹⁶ That is, worst-case corresponding to the value of $\|\Delta\|_2$.

¹⁷ In theorem (2.4.3) we proved that the Jacobian is non-singular, so we know that the inverse *will* exist.

we get

$$\sum_{k,l=1}^{n} \left((J_G)^{-1} \right)_{ij,kl} f_{kl}(G, \underline{d}(G) + \delta \underline{x}) \approx \delta x_{ij}, \ \forall \ 1 \le i, j \le n.$$

$$(2.54)$$

Subtracting $\delta \underline{x}$ from $\underline{d}(G) + \delta \underline{x}$ gives us $\underline{d}(G)$, which is the required solution. The catch here is that since we do not know the solution $\underline{d}(G)$ to start with, we cannot compute the Jacobian $J(G, \underline{d}(G))$. But since $\underline{d}(G) + \delta \underline{x}$ is close to $\underline{d}(G)$, we can approximate $J(G, \underline{d}(G))$ by $J(G, \underline{d}(G) + \delta \underline{x})$, which we can compute. So we use $J(G, \underline{d}(G) + \delta \underline{x})$ in place of $J(G, \underline{d}(G))$ in the algorithm, particularly, instead of using $(J_G)^{-1}$, computed at $\underline{d}(G)$, in equation (2.54), we use it when computed at the point $\underline{d}(G) + \delta \underline{x}$.

The description of the principle behind Newton's method clarifies the algorithm, whose steps we list below.

Starting with
$$x_{ij}^{(0)} = \frac{1}{\sqrt{n}} \delta_{ij}$$
 (for all $1 \le i, j \le n$), $k = 1$ and assuming $\epsilon = 10^{-9}$, iterate

- (1) Substitute $x_{ij}^{(k-1)}$ into the functions $f_{ij}(G, .)$ defined in equations (2.40a), (2.40b) and (2.40c). Arrange all the f_{ij} 's in a single column, which will have n^2 rows; we will denote this n^2 -row long column by $\gamma^{(k-1)}$.
- (2) Stop when $\|\gamma^{(k-1)}\|_2 < \epsilon$.
 - (2.1) Compute the Jacobian, $J_G^{(k-1)}$, where $\left(J_G^{(k-1)}\right)_{ij,st} = \frac{\partial f_{ij}(G, \underline{x})}{\partial x_{st}}$ at the point $\underline{x} = \underline{x}^{(k-1)}$.

(2.2) Compute the inverse of $J_G^{(k-1)}$ i.e. $(J_G^{(k-1)})^{-1}$.

(2.3)
$$x_{ij}^{(k)} = x_{ij}^{(k-1)} - \left(\left(J_G^{(k-1)} \right)^{-1} \gamma^{(k-1)} \right)_{ij}$$

For each n = 3 to n = 20, we tested approximately twenty-thousand different examples, each of which for we obtained the required solution within the margin error. What's more, it was also seen that the maximum number of iterations required to maintain the error tolerance was constant over this range of n, more specifically, for each of these examples we required the number of iterations to be ten. Since the number of iterations required doesn't increase with n (or increases very slowly), the computational complexity (time and space) of this algorithm is determined by the cost of steps within an iteration. Keeping this in mind, we give the computational complexity (time and space) of this algorithm in the following table.

	Step in the algorithm	Time	Space
		Complexity	Complexity
1.	Computing $f_{ij}(G, \underline{x}^{(k-1)})$, by substituting $\underline{x}^{(k-1)}$ into	$O(n^2)$	$O(n^2)$
	equations (2.40a), (2.40b) and (2.40c), for all $1 \le i, j \le$		
	n		
2.	Computing the Jacobian, $J_G^{(k-1)}$ at the point $\vec{x}_{(k-1)}$	$O(n^4)$	$O(n^4)$
3.	Computing the inverse of the Jacobian $\left(J_G^{(k-1)}\right)^{-1}$ from	$O(n^6)$	$O(n^4)$
	the Jacobian, at the point $\vec{x}^{(k-1)}$		
4.	Computing $\vec{x}^{(k)}$ using $(J_G^{(k-1)})^{-1}$ and $\vec{x}^{(k-1)}$ (point 2.3 in	$O(n^4)$	$O(n^2)$
	the list of steps of this algorithm above)		

Table 2.2: Time and space complexity of various steps in Newton's method.

Thus we see that the time complexity of Newton's method is $O(n^6)$ and the space complexity is $O(n^4)$. The number of steps involved are lower than in the Taylor series method, making this algorithm simpler, and also the computational complexity (both time and space) of Newton's method is lower than that of the Taylor series' method when one cannot find a close enough starting point $G^{(0)}$ to the given point $G^{(1)}$ in the latter method.

Let's compare the efficiency of these methods to that of an SDP algorithm. We will employ an SDP algorithm known as the Barrier-type Interior Point Method (IPM) [56].

Algorithm 3: Barrier-type IPM (SDP) The SDP problem corresponding to MED is given by (2.4). The objective of this problem is to minimize the value of Tr(Z) subject to the constraints: $Z \ge p_i |\psi_i\rangle \langle \psi_i|, \forall i = 1, 2, \dots, n$.

In this method we obtain Z which solves (2.4) over a series of iterations, known as outer iterations. One starts the *k*-th such iteration with an input $Z^{(k-1)}$ - a candidate for Z - and ends with an output $Z^{(k)}$, which will serve as the input for the next iteration. The $Z^{(k)}$, which are successive approximations for Z, take values within the feasible region, i.e, the

region given by the set {*Z* is $n \times n$, positive definite $|Z \ge p_i|\psi_i\rangle\langle\psi_i|, \forall 1 \le i \le n$ }. If *Z* lies in the interior of this feasible region then it is such that $Z > p_i|\psi_i\rangle\langle\psi_i|, \forall 1 \le i \le n$, whereas if *Z* is a boundary point of the feasible region then there is some $i = 1, 2, \dots, n$ such that $Z - p_i|\psi_i\rangle\langle\psi_i|$ has at least one zero eigenvalue.

In the first iteration, one starts with some *strictly* feasible $Z = Z^{(0)}$, i.e., some $Z^{(0)}$ which lies in the interior of the feasible region. To ensure that $Z^{(k)}$ remains within the feasible region one perturbs the objective function which is being minimized: instead of performing an unconstrained minimization of Tr(Z), one performs an unconstrained minimization of $\text{Tr}(Z) - \frac{1}{w} \sum_{i=1}^{n} Log(Det(Z - p_i|\psi_i)\langle\psi_i|))$, where $\frac{1}{w}$ is a weight factor. The reason behind subtracting the expression $\frac{1}{w} \sum_{i=1}^{n} Log(Det(Z - p_i|\psi_i)\langle\psi_i|))$ from Tr(Z) for unconstrained minimization, is that the expression $Log(Det(Z - p_i|\psi_i)\langle\psi_i|))$ tends to infinity as Z approaches the boundary of the feasible region. Thus performing unconstrained minimization of $\text{Tr}(Z) - \frac{1}{w} \sum_{i=1}^{n} Log(Det(Z - p_i|\psi_i)\langle\psi_i|))$ will ensure that while the candidates for Z, viz, $Z^{(k)}$, may inch closer to the boundary of the feasible region, they will never cross it.

The unconstrained minimization of $\text{Tr}(Z) - \frac{1}{w} \sum_{i=1}^{n} Log(Det(Z - p_i | \psi_i \rangle \langle \psi_i |))$ is performed using Newton's method. The iterations within Newton's method are known as inner iterations. Newton's Method is performed as follows: using the generalized Gell-Mann basis, expand $Z = \sum_{i,j=1}^{n} y_{ij} \frac{\sigma_{ij}}{\sqrt{2}}$, where $\sigma_{nn} = \sqrt{\frac{2}{n}} \mathbb{1}_n$. Obtain the equations

$$h_{kl}(\underbrace{\mathbf{y}}_{\rightarrow}) \equiv \frac{\partial \left(\operatorname{Tr}(Z) - \frac{1}{w} \sum_{i=1}^{n} Log(Det(Z - p_i | \psi_i \rangle \langle \psi_i |)) \right)}{\partial y_{kl}}$$

$$= \sqrt{n} \delta_{k,n} \delta_{l,n} - \frac{1}{w} \sum_{i=1}^{n} \operatorname{Tr} \left((Z - p_i | \psi_i \rangle \langle \psi_i |)^{-1} \frac{\sigma_{kl}}{\sqrt{2}} \right).$$
(2.55)

We want to solve for the equations $h_{ij} = 0$, $\forall 1 \le i, j \le n$ using Newton's method. The algorithm is the same as the one described above. These equations give the stationary points of the functions h_{ij} , $\forall 1 \le i, j \le n$. The matrix elements of the Jacobian of the h_{ij}

functions with respect to the y_{kl} variables take the following form¹⁸.

$$H_{kl,st} \equiv \frac{\partial h_{kl}(y)}{\partial y_{st}} = \frac{\partial^2 \left(\operatorname{Tr}(Z) - \frac{1}{w} \sum_{i=1}^n Log(Det(Z - p_i | \psi_i \rangle \langle \psi_i |)) \right)}{\partial y_{kl} \partial y_{st}}$$

$$= \frac{1}{w} \sum_{i=1}^n \operatorname{Tr} \left((Z - p_i | \psi_i \rangle \langle \psi_i |)^{-1} \frac{\sigma_{kl}}{\sqrt{2}} (Z - p_i | \psi_i \rangle \langle \psi_i |)^{-1} \frac{\sigma_{st}}{\sqrt{2}} \right),$$
(2.56)

where $H_{kl,st}$ are the matrix elements of the Jacobian, as can be seen from equation (2.56). Let $\alpha \in \mathbb{C}^{n^2}$ be some non-zero complex n^2 -tuple, and let $A \equiv \sum_{i,j=1}^n \alpha_{ij} \frac{\sigma_{ij}}{\sqrt{2}}$. Then we have the equality

$$\sum_{k,l,s,t=1}^{n} \alpha_{kl}^{*} H_{kl,st} \alpha_{st}$$

$$= \frac{1}{w} \sum_{i=1}^{n} \operatorname{Tr} \left(\left((Z - p_i |\psi_i\rangle \langle \psi_i |)^{-\frac{1}{2}} A^{\dagger} (Z - p_i |\psi_i\rangle \langle \psi_i |)^{-\frac{1}{2}} \right) \left((Z - p_i |\psi_i\rangle \langle \psi_i |)^{-\frac{1}{2}} A(Z - p_i |\psi_i\rangle \langle \psi_i |)^{-\frac{1}{2}} \right) \right) > 0.$$
(2.57)

This inequality is true for all non-zero $\underline{\alpha} \in \mathbb{C}^{n^2}$. Thus the Jacobian *H*, whose matrix elements are given in equation (2.56), is positive definite throughout the feasible region. Thus the *only* stationary points in the feasible region can be local minima. But since H > 0 throughout the feasible region, there can only be *one* local minima in said region, i.e., the stationary point gives *the* minima which we are searching for¹⁹.

Thus the inner iterations give us the minima point $Z^{(k)} = \sum_{i,j=1}^{n} y_{ij}^{(k)} \frac{\sigma_{ij}}{\sqrt{2}}$ corresponding to some weight factor $\frac{1}{w^{(k-1)}}$. After having found the minima point $Z^{(k)}$ in the *k*-th iteraction, the k + 1-th iteration is commenced with changing the weight of the barrier function, i.e., $w^{(k-1)} \longrightarrow w^{(k)} > w^{(k-1)}$, and performing an unconstrained minimization of $\operatorname{Tr}(Z) - \frac{1}{w^{(k)}} \sum_{i=1}^{n} Log(Det(Z - p_i | \psi_i \rangle \langle \psi_i |))$, starting from the point $Z^{(k)}$. These iterations are continued until the weight of the barrier function decreases to an insignificantly small

¹⁸ This isn't difficult to derive; alternately the Barrier-type IPM algorithm for MED is also given in section 11.8.3 in [56] (p. 618), wherein expression for the matrix elements of the Jacobian has been explicitly given.

¹⁹ There is another way to appreciate this: since the function $\text{Tr}(Z) - \frac{1}{w} \sum_{i=1}^{n} Log (Det (Z - p_i |\psi_i\rangle\langle\psi_i|))$ is a convex function over the feasible region, there can only be one minima in said region, which corresponds to the point we want. The convexity of the Log-Determinant function -Log (Det(X))over the space {all $n \times n$ matrices $X | X \ge 0$ } is established in section 3.1 on p. 73 in [56].

number (i.e. given by the error tolerance). The final approximation $Z^{(k_f)}$ is then declared as the solution.

We briefly outline the steps in the algorithm below.

Let ϵ be the error tolerance for the algorithm. For starting the algorithm choose the following: the value of μ between ~ 3 to 100, the weight $w^{(0)} \sim 10$, the initial starting point for Z as $Z^{(0)} = \mathbb{1}_n$, then set k = 1 and iterate the following.

- (1) Perform unconstrained minimization of the function $\operatorname{Tr}(Z) \frac{1}{w^{(k-1)}} \sum_{i=1}^{n} Log(Det(Z p_i |\psi_i\rangle\langle\psi_i|))$ with starting point as $Z = Z^{(k-1)}$ (using Newton's Method).
- (2) Store the solution as $Z^{(k)}$. Update $w^{(k)} = \mu w^{(k-1)}$.
- (3) Stop when $w^{(k)} = \frac{n}{\epsilon}$.

The number of outer iterations for a given error tolerance is constant over *n*, but can vary with the factor μ by which the weights $w^{(k-1)}$ vary over the steps.²⁰. Thus the computational complexity of the algorithm is decided by the computational complexity of Newton's method within the inner iterations. In the following table we list the different steps as part of Newton's algorithm and give the computational complexity (time and space) for each step.

	Step in the algorithm	Time	Space
		Complexity	Complexity
1.	Computing values of $h_{ij}(y^{(k-1)})$, by substituting $y^{(k-1)}$	$O(n^2)$	$O(n^2)$
	into equations (2.55), for all $1 \le i, j \le n$.		
2.	Computing the Jacobian <i>H</i> , at the point $y^{(k-1)}$	$O(n^5)$	$O(n^4)$
3.	Computing the inverse of <i>H</i> at the point $\vec{y}^{(k-1)}$	$O(n^6)$	$O(n^4)$
4.	Computing $y_{ij}^{(k)} = y_{ij}^{(k-1)} - \sum_{s,t=1}^{n} (H^{-1})_{ij,st} \vec{h}_{st}^{(k-1)}, \forall i, j, \leq 1$	$O(n^4)$	$O(n^2)$
	ln statistical statist		

Table 2.3: Time and space complexity of steps in Barrier-Type Interior Point Method

²⁰ See section 11.5.3 of [56] for an upper bound on the number of outer iterations; particularly note figure 11.14. Also see the second example of section 11.6.3., figure 11.16 reveals the variation of the number of outer iterations with μ .

Comparing Different Methods: The table above shows that the computational complexity of the Barrier-type IPM is as costly as the direct application of Newton's method. In fact, a closer analysis shows that directly applying Newton's method is less costly than the SDP method, along with the advantage of being simpler to implement. Also, the Taylor series method is as costly as both Newton's method and the SDP method, when one can find a gram matrix $G^{(0)}$ in the close vicinity of the given gram matrix $G^{(1)}$. If one is interested in a one-time calculation for an ensemble of LI pure states Newton's method is the most desirable method to implement among all the three examined here.

2.4.3 Remarks and Conclusion

We showed how the mathematical structure of the MED problem for LI pure state ensembles could be used to obtain the solution for said problem. This was done by casting the necessary and sufficient conditions (2.7) and (2.8) into a rotationally invariant form which was employed to obtain the solution by using the implicit function theorem. We also showed that this technique is simpler to implement than standard SDP techniques.

As mentioned in the beginning of this section, for fixed states $|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_n\rangle$, \mathscr{R} induces an invertible map on the space of probabilities, $\{p_i\}_{i=1}^n \longrightarrow \{q_i\}_{i=1}^n$. This naturally begs a question on whether there is a relation between the two probabilities, for example does one majorize the other? Or, more generally, is the entropy of $\{q_i\}_{i=1}^n$ always larger than the entropy of $\{p_i\}_{i=1}^n$ or vice versa? The answer to this question is that there doesn't seem to be any simple property relating these two probabilities, vis-a-vis, one majorizing the other or that the $(\{p_i\}_{i=1}^n, \{q_i\}_{i=1}^n)$ -pair are either related by $H(p_i) \ge H(q_i)$ or $H(p_i) <$ $H(q_i)$; examples of both cases can be found. This is particularly easy to corroborate from for the cases n = 2.

In this work we studied only about the case for n-LIP ensembles. Naturally there is the question if a similar theory holds for more general ensembles. For the case of m-linearly

dependent pure state ensembles (where $m > dim\mathcal{H} = n$): it is explicitly shown that, while a map like \mathscr{R}^{-1} exists on the space of *m* linearly dependent pure (LDP) state ensembles, \mathscr{R}^{-1} isn't one-to-one [3]²¹. From the analysis in our work, it is clear that the one-toone nature of the map \mathscr{R}^{-1} (for *n*-LIPs) plays a crucial in formulating the rotationally invariant necessary and sufficient conditions for MED of said ensemble of states; thus it also plays a crucial role in the application of this necessary and sufficient condition to obtain the solution for MED of said ensemble. The non-invertibility of \mathscr{R}^{-1} also shows that the optimal POVM won't necessarily vary smoothly as one varies the ensemble from one *m*-LDP to another *m*-LDP. C. Mochon gave algebraic arguments for this [3] in his paper, and Ha et al. showed the same using the geometrical arguments for ensembles of three qubit states, as an example [49]. This has been shown for general qudits as well [50]. Besides this, there is also the fact that there are some LDPs for which the optimal POVM isn't even unique, i.e., two or more distinct POVMs give the maximum success probability for MED. This means that as the ensemble is varied in the neighbourhood of said ensemble, the optimal POVM can undergo discontinuous jumps. Hence, we conclude that such the technique which was used in subsection 2.4.2 for *n*-LIPs can't be generalized to *m*-LDPs. In the next section we see that such a technique can be generalized to mixed state ensembles of linearly independent states.

2.5 MED for Ensembles of LI Mixed States

The work done in this section has been detailed in a paper titled "Algebraic Structure of the Minimum Error Discrimination Problem for Linearly Independent Density Matrices", which has been uploaded on the arxiv at quant-ph/14127174 [57]. This paper is in preparation to be sent for publication soon.

In section 2.3 I gave the salient features of the mathematical structure which relates to

In that sense it defeats the purpose of denoting such a map by \mathscr{R}^{-1} , because \mathscr{R}^{-1} doesn't have an *inverse*.

any general ensemble of quantum states \tilde{P} , another ensemble of quantum states \tilde{Q} , whose PGM is the optimal POVM for the MED of \tilde{P} . In section 2.4 it was seen that if the states in \tilde{P} were restricted to being LI and pure, then \tilde{P} and \tilde{Q} can be related by a bijection \mathscr{R} (see equation (2.10)). In this section we generalize the result for ensembles of LI pure states to ensembles of LI mixed states.

Let $\{r_i\}_{i=1}^m \subset \mathbb{Z}^+$ be a subset of *m* real positive integers, with the property²²

$$\sum_{i=1}^{m} r_i = n.$$
 (2.58)

Let $\{\rho_i\}_{i=1}^m$ be a set of *m* quantum states such that $\operatorname{rank}(\rho_i) = r_i, \forall 1 \le i \le m$. We say that $\rho_1, \rho_2, \dots, \rho_m$ are LI if the following is true: $\forall 1 \le i \le m$, let $|\psi\rangle_i \in \operatorname{supp}(\rho_i)$ be non-zero vectors, then $|\psi\rangle_1, |\psi\rangle_2, \dots, |\psi\rangle_m$ are LI. Define $\underline{r} \equiv (r_1, r_2, \dots, r_m)$. From [6] we know that the optimal POVM for an ensemble of mixed states, like $\widetilde{P}_{\underline{r}} = \{p_i, \rho_i\}_{i=1}^m$, is a *unique* projective measurement $\{\Pi_i\}_{i=1}^m$, with $\Pi_i\Pi_j = \delta_{ij}\Pi_i, \forall 1 \le i, j \le m$, and $\operatorname{rank}(\Pi_i) = \operatorname{rank}(\rho_i) = r_i, \forall 1 \le i \le m$. Consider the following two definitions.

Definition 2.5.1. \mathcal{E}_r is the set of LI ensembles, such that if $\widetilde{P}_r \in \mathcal{E}_r$, then $\rho_i = r_i, \forall 1 \le i \le m$. \mathcal{E}_r is a $2n^2 - \sum_{i=1}^m r_i^2 - 1$ dimensional real manifold. If $r_k = r_{k+1} = \cdots = r_{k+s-1}$, then a single ensemble can be represented by *s*! points in the set \mathcal{E}_r , which differ only by the order in which states and corresponding probabilities are indexed²³.

Definition 2.5.2. Define $\mathcal{P}_{\underline{r}}$ to be the set of all *m*-element projective measurements, so that if $\{\Pi_i\}_{i=1}^m \in \mathcal{P}_{\underline{r}}$, then $\Pi_i \Pi_j = \delta_{ij} \Pi_i$, $\forall \ 1 \le i, j \le m$, and $\operatorname{rank}(\Pi_i) = r_i$, $\forall \ 1 \le i \le m$. $\mathcal{P}_{\underline{r}}$ is an $n^2 - \sum_{i=1}^m r_i^2$ dimensional real manifold.

Then, just like we defined the optimal POVM map $\mathscr{P}_{\underline{1}} : \mathscr{E}_{\underline{1}} \longrightarrow \mathscr{P}_{\underline{1}}$ for LI pure state ensembles, we can define the optimal POVM map $\mathscr{P}_{\underline{r}} : \mathscr{E}_{\underline{r}} \longrightarrow \mathscr{P}_{\underline{r}}$ for LI mixed state

²² The reason for keeping rank(ρ_i) = r_i fixed, and for instance not allowing, say, rank(ρ_i) $\leq r_i$ is will be made clear later (see equation (2.77)).

²³ Retaining this multiplicity is *merely* a matter of convenience, i.e., one could adopt more criteria to do away with such multiplicities but that complicates the description of \mathcal{E}_r ; this is avoided to keep things simple.

ensembles.

Definition 2.5.3. $\mathscr{P}_{\underline{r}} : \mathscr{E}_{\underline{r}} \longrightarrow \mathscr{P}_{\underline{r}}$ is the optimal POVM map on $\mathscr{E}_{\underline{r}}$. Thus $\mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}})$ is the unique optimal POVM for the MED of any ensemble $\widetilde{P}_{\underline{r}} \in \mathscr{E}_{\underline{r}}$.

One can trivially generalize the pretty good measurement map as well.

Definition 2.5.4. $PGM_{\underline{r}} : \mathcal{E}_{\underline{r}} \longrightarrow \mathcal{P}_{\underline{r}}$ is the pretty good measurement map on $\mathcal{E}_{\underline{r}}$. So $PGM_{\underline{r}}(\widetilde{Q}_{\underline{r}}) = {\Pi_i}_{i=1}^m$ is the PGM associated with the ensemble $\widetilde{Q}_{\underline{r}} = {q_i, \sigma_i}_{i=1}^m$, where Π_i is given by

$$\Pi_{i} = \left(\sum_{j=1}^{m} q_{j} \sigma_{j}\right)^{-1/2} q_{i} \sigma_{i} \left(\sum_{k=1}^{m} q_{k} \sigma_{k}\right)^{-1/2}, \ \forall \ 1 \le i \le m.$$
(2.59)

In this work we show that one can define a map $\mathscr{R}_{\vec{r}} : \mathscr{E}_{\vec{r}} \longrightarrow \mathscr{E}_{\vec{r}}$, which allows us to generalize the relation (4) for arbitrary \underline{r}^{24} .

In the process we also show that $\mathscr{R}_{\underline{r}}$ is invertible over $\mathscr{E}_{\underline{r}}$, and we give a closed-form expression for $\mathscr{R}_{\underline{r}}^{-1}$. The functions $\mathscr{R}_{\underline{r}}$ exhibit the mathematical structure of the problem of MED for ensembles in $\mathscr{E}_{\underline{r}}$. We relate $\mathscr{R}_{\underline{r}}$ to $\mathscr{R}_{\underline{1}}$, which will show how the problem of the MED of ensembles in $\mathscr{E}_{\underline{r}}$ is related to the problem of MED of ensembles in $\mathscr{E}_{\underline{1}}$.

The work is divided into various subsections as follows: subsection 2.5.1 establishes the main result of the work in the following steps: we first simplify the known optimality conditions for the MED of an ensemble of LI mixed states and then, use these conditions to arrive at the rotationally invariant necessary and sufficient conditions for MED of said ensemble; using the rotationally invariant optimality conditions we define the function $\mathscr{R}_{\underline{r}}$, which is related to $\mathscr{P}_{\underline{r}}$ by equation (4). We also prove that $\mathscr{R}_{\underline{r}}$ is an invertible function and give the action of $\mathscr{R}_{\underline{r}}^{-1}$ on any ensemble of LI mixed states in $\mathcal{E}_{\underline{r}}$. In subsection 2.5.2 we compare the problem of MED for general LI mixed ensembles with the problem of MED for LI pure state ensembles. In subsection 2.5.3, we use the results from subsection

r should satisfy equation (2.58).

2.5.1 to give an efficient and simple numerical technique to obtain the optimal POVM for the MED of any LI ensemble. Section 2.5.4 concludes the work.

2.5.1 Mathematical Structure for MED of Ensembles of LI States

Let U be some unitary acting on \mathcal{H} . It is easy to see that

$$\mathscr{P}_{\vec{J}}\left(\{p_i, U^{\dagger}\rho_i U\}_{i=1}^m\right) = \{U^{\dagger}\Pi_i U\}_{i=1}^m.$$

Thus MED is a rotationally covariant problem. To strip the problem of its rotational covariance and to retain only the rotationally invariant aspect of the problem, we will now recast the optimality conditions given in section 2.2 in a rotationally invariant form.

Rotationally Invariant Conditions for MED of an Ensemble of LI Mixed States

First we define the index sets $I_i \equiv \{(i, j) \mid 1 \le j \le r_i\}, \forall 1 \le i \le m$. Thus when $i \in I_i$, then it takes the form $\underline{i} = (i, j)$, where $1 \le j \le r_i$. Also define the index set $I = \bigcup_{i=1}^m I_i$. We will use this indicing convention throughout this work. This convention will also be used to denote matrix elements of $r_i \times r_{i'}$ matrices, where $1 \le i, i' \le m$ or $n \times n$ matrices. For instance if $X_{ii'}$ is an $r_i \times r_{i'}$ matrix, and if $\underline{i} = (i, j)$ and $\underline{i}' = (i', j')$, then $(X_{ii'})_{\underline{i}\underline{j}'}$ will be the matrix element occuring at the intersubsection of the *j*-th row and *j'*-th column of $X_{ii'}$. Similarly, if *G* is an $n \times n$ matrix, then $G_{\underline{i}\underline{j}'}$ is the matrix element occuring at the intersubsection of the $(\sum_{k=1}^{i-1} r_k + j)$ -th row and $(\sum_{k=1}^{i'-1} r_k + j')$ -th column.

Consider a spectral decomposition for Π_i , for all $1 \le i \le m$.

$$\Pi_{i} = \sum_{\substack{i \in \mathcal{I}_{i} \\ i \neq i}} |w_{i}\rangle \langle w_{i}|.$$
(2.60)

Then $\{|w_i\rangle\}_{i \in I}$ is an ONB for \mathcal{H} , i.e., $\langle w_i | w_{i'}\rangle = \delta_{ii'}$, for all $i, i' \in I$. If $r_i \ge 2$, Π_i is degenerate and hence there is a $U(r_i)$ degree of freedom in choosing the spectral decomposition

of Π_i , i.e., when

$$|w'_{\vec{j}}\rangle = \sum_{\substack{j \in \mathcal{I}_i \\ \vec{j}}} (U_i)_{\vec{j}} |w_{\vec{j}}\rangle, \ \forall \ \vec{i} \in \mathcal{I}_i,$$
(2.61a)

where U_i is an $r_i \times r_i$ unitary matrix, then

$$\Pi_{i} = \sum_{\substack{i \in I_{i} \\ j \neq i}} |w_{i}'\rangle \langle w_{i}'|, \ \forall \ 1 \le i \le m.$$
(2.61b)

For now assume that the choice of spectral decomposition of Π_i is arbitrary for all $1 \le i \le m$. Later in this subsection, a special choice of spectral decomposition of Π_i for each $1 \le i \le m$ will be made.

Consider a pure state decomposition of the unnormalized states $p_i \rho_i$:

$$p_i \rho_i = \sum_{\substack{i \in I_i \\ i \notin I_i}} |\widetilde{\psi}_i\rangle \langle \widetilde{\psi}_i|.$$
(2.62a)

The vectors $|\widetilde{\psi}_i\rangle$ are unnormalized. Since $\operatorname{supp}(p_i\rho_i)$ is spanned by the r_i vectors $\{|\widetilde{\psi}_i\rangle\}_{i\in I_i}$ and since $\operatorname{rank}(p_i\rho_i) = r_i$, $\{|\widetilde{\psi}_i\rangle\}_{i\in I_i}$ is a LI set. There is a $U(r_i)$ degree of freedom in the choice of decomposition of the unnormalized state $p_i\rho_i$ into the pure states $|\widetilde{\psi}_i\rangle\langle\widetilde{\psi}_i|$, i.e., let U_i be an $r_i \times r_i$ unitary matrix; define the vectors

$$|\widetilde{\psi'}_{\underline{j}}\rangle \equiv \sum_{\substack{j \in \mathcal{I}_i \\ \underline{j} \in \mathcal{I}_i}} (U_i)_{\underline{j}\underline{i}} |\widetilde{\psi}_{\underline{j}}\rangle, \ \forall \ \underline{i} \in \mathcal{I}_i,$$
(2.62b)

then

$$p_i \rho_i = \sum_{\substack{i \in I_i \\ i \neq j}} |\widetilde{\psi'}_i\rangle \langle \widetilde{\psi'}_i|$$
(2.62c)

is another pure state decomposition of $p_i\rho_i$, similar to the one in equation (2.62a). For now we make an arbitrary choice of pure state decomposition in equation (2.62a); in subsection 2.5.2 we will make a specific choice of pure state decomposition for the states $p_i\rho_i$. Let the gram matrix corresponding to $\{|\tilde{\psi}_i\rangle\}_{i\in I}$ be denoted by *G*, whose matrix elements are given by

$$G_{\underline{i}\underline{i}'} = \langle \widetilde{\psi}_{\underline{i}} | \widetilde{\psi}_{\underline{i}'} \rangle, \ \forall \ \underline{i}, \underline{i}' \in \mathcal{I}.$$

$$(2.63)$$

G is an $n \times n$ positive semidefinite matrix. Since the states $\rho_1, \rho_2, \cdots, \rho_m$ are LI, the set $\{|\widetilde{\psi_i}\rangle\}_{i \in I}$ is also LI. This implies that *G* is a positive definite matrix. Define the set of *n* vectors $\{|\widetilde{u_i}\rangle\}_{i \in I}$ in the following way

$$|\widetilde{u}_{\vec{j}}\rangle \equiv \sum_{\substack{j \in I \\ \vec{j}}} \left(G^{-1} \right)_{\substack{ji \\ \vec{j}}} |\widetilde{\psi}_{\vec{j}}\rangle, \ \forall \ \vec{i} \in I.$$
(2.64a)

Equation (2.64a) implies that

$$\langle \widetilde{\psi}_{\underline{i}} | \widetilde{u}_{\underline{j}} \rangle = \delta_{\underline{i}\underline{j}}, \ \forall \ \underline{i}, \underline{j} \in I.$$
(2.64b)

Using equation (2.64a), one can corroborate that

$$|\widetilde{\psi}_{\vec{j}}\rangle = \sum_{\substack{j \in I \\ \vec{j}}} G_{\underline{j}\underline{i}} |\widetilde{u}_{\vec{j}}\rangle, \ \forall \ \underline{i} \in I.$$
(2.64c)

Since G^{-1} is non-singular, equation (2.64a) implies that $\{|\widetilde{u}_{i}\rangle\}_{i \in I}$ is a set of *n* LI vectors, hence it is a basis for \mathcal{H} . Thus the orthonormal basis vectors $\{|w_{i}\rangle\}_{i \in I}$, given by equation (2.60), can be expanded in terms of the $|\widetilde{u}_{i}\rangle$ vectors.

$$|w_{i}\rangle = \sum_{\substack{j \in I \\ \downarrow}} \left(G^{\frac{1}{2}}W\right)_{ji} |\widetilde{u}_{j}\rangle, \ \forall \ \underline{i} \in I,$$
(2.65)

where *W* is an $n \times n$ unitary matrix.

Substituting the expression for $|w_i\rangle$ from equation (2.65) into equation (2.60) we get

$$\Pi_{i} = \sum_{\substack{l,k \in \mathcal{I} \\ \downarrow \neq \mathcal{I}}} \left(\sum_{\substack{i \in \mathcal{I}_{i}}} \left(G^{\frac{1}{2}} W \right)_{li} \left(W^{\dagger} G^{\frac{1}{2}} \right)_{ik} \right) [\widetilde{u}_{i}\rangle \langle \widetilde{u}_{k}|.$$
(2.66)

Rotationally Invariant Form of Condition (2.7)

Since $\mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}}) = {\Pi_i}_{i=1}^m$, the POVM elements Π_i should satisfy equation (2.7). Substituting the expression for Π_i and Π_j from equation (2.66) into equation (2.7) we get

$$\Pi_{i}\left(p_{i}\rho_{i}-p_{j}\rho_{j}\right)\Pi_{j}=\sum_{\substack{l,k\in I\\ \vec{x}\neq\vec{x}}}\xi_{lk}\widetilde{[u_{l}]}\langle\widetilde{u_{k}}|=0, \qquad (2.67)$$

where $\xi_{\underline{lk}}$ is given by

$$\xi_{\overset{j}{\leftrightarrow}} = \sum_{\substack{i \in \mathcal{I}_i \\ \downarrow \neq \mathcal{I}_j}} \sum_{j \in \mathcal{I}_j} \left(G^{\frac{1}{2}} W \right)_{\overset{j}{\leftrightarrow}} \left(\sum_{\substack{i' \in \mathcal{I}_i \\ \downarrow \neq \mathcal{I}_i}} \left(W^{\dagger} G^{\frac{1}{2}} \right)_{\overset{j}{\rightarrow}} \left(G^{\frac{1}{2}} W \right)_{\overset{j}{\rightarrow}} - \sum_{\substack{j' \in \mathcal{I}_j \\ \downarrow \neq \mathcal{I}_j}} \left(W^{\dagger} G^{\frac{1}{2}} \right)_{\overset{j}{\rightarrow}} \left(G^{\frac{1}{2}} W \right)_{\overset{j}{\rightarrow}} \right) \left(W^{\dagger} G^{\frac{1}{2}} \right)_{\overset{j}{\rightarrow}} \left(G^{\frac{1}{2}} W \right)_{\overset{j}{\rightarrow}} \right)$$
(2.68)

The expression for $\xi_{\underline{k}}$ in equation (2.68) is complicated. To simplify we do the following.

i. First partition the matrix $G^{\frac{1}{2}}W$ into matrix blocks in the following way.

$$G^{\frac{1}{2}}W = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1m} \\ X_{21} & X_{22} & \cdots & X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mm} \end{pmatrix},$$
(2.69a)

where $X_{ii'}$ is the $r_i \times r_{i'}$ matrix block in $G^{\frac{1}{2}}W$, and it is located at the intersubsection of the rows with indices $\underline{i} \in I_i$ and the columns with indices $\underline{i'} \in I_{i'}$. ii. Define:

$$C_{i} \equiv \begin{pmatrix} X_{1i} \\ X_{2i} \\ \vdots \\ X_{mi} \end{pmatrix}, \ 1 \le i \le m.$$

$$(2.69b)$$

Thus C_i is the *i*-th column block of $G^{\frac{1}{2}}W$.

iii. Similarly, partition $W^{\dagger}G^{-\frac{1}{2}}$ into matrix blocks in the following way.

$$W^{\dagger}G^{-\frac{1}{2}} = \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1m} \\ Y_{21} & Y_{22} & \cdots & Y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{m1} & Y_{m2} & \cdots & Y_{mm} \end{pmatrix}, \qquad (2.69c)$$

where $Y_{ii'}$ is the $r_i \times r_{i'}$ matrix block in $W^{\dagger}G^{-\frac{1}{2}}$, located at the intersubsection of the rows with indices $\underline{i} \in I_i$ and the columns with indices $\underline{i'} \in I_{i'}$.

iv. Define

$$R_i \equiv \begin{pmatrix} Y_{i1} & Y_{i2} & \cdots & Y_{im} \end{pmatrix}, \ 1 \le i \le m.$$
(2.69d)

Thus R_i is the *i*-th row block of $W^{\dagger}G^{-\frac{1}{2}}$.

Substituting equations (2.69a) and (2.69b) in equation (2.67) we obtain a simplified expression for equation (2.68)

$$\xi_{\underline{l}\underline{k}} = \left(C_i \left(X_{ii}^{\dagger} X_{ij} - X_{ji}^{\dagger} X_{jj} \right) C_j^{\dagger} \right)_{\underline{l}\underline{k}} = 0, \ \forall \ \underline{l}, \underline{k} \in \mathcal{I},$$
(2.70)

where X_{ji}^{\dagger} is the *ij*-th matrix block of $W^{\dagger}G^{\frac{1}{2}}$. Equations (2.69b) and (2.69d) imply that $R_iC_i = \mathbb{1}_{r_i}, \forall 1 \le i \le m$, where $\mathbb{1}_{r_i}$ is the $r_i \times r_i$ identity matrix. Left multiplying by R_i and

right multiplying by R_i^{\dagger} , the LHS and RHS of equation (2.70) become

$$R_{i}C_{i}\left(X_{ii}^{\dagger}X_{ij} - X_{ji}^{\dagger}X_{jj}\right)C_{j}^{\dagger}R_{j}^{\dagger} = 0$$

$$\implies X_{ii}^{\dagger}X_{ij} - X_{ji}^{\dagger}X_{jj} = 0, \ \forall \ 1 \le i, j \le m.$$
(2.71)

Let U_{D_r} be an $n \times n$ block diagonal unitary matrix of the form

$$U_{D_{\underline{r}}} = \begin{pmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_m \end{pmatrix},$$
(2.72)

where U_i is an $r_i \times r_i$ unitary matrix for $i = 1, 2, \dots, m$. Equations (2.61) imply that there is a $U(r_i)$ degree of freedom in the choice of spectral decomposition of Π_i in equation (2.60). Expanding the vectors $|w'_i\rangle$ in the basis $\{|\tilde{u}_i\rangle\}_{i \in I}$ gives (compare with equation (2.65))

$$|w_{i}'\rangle = \sum_{\substack{j \in I \\ j \neq i}} \left(G^{\frac{1}{2}} W U_{D_{i}} \right)_{ji} |\widetilde{u}_{j}\rangle.$$
(2.73)

Due to the block-diagonal form of U_{D_r} , $|w_i\rangle \longrightarrow |w'_i\rangle$ will leave Π_i invariant in equation (2.66). We exploit this degree of freedom to make a specific choice of U_{D_r} : choose U_i such that $X_{ii}U_i \ge 0$, $\forall 1 \le i \le m^{25}$. To simplify the notation, assume that U_{D_r} is absorbed within W, i.e., $WU_{D_r} \rightarrow W$, $X_{ij}U_j \rightarrow X_{ij}$ and $|w'_i\rangle \longrightarrow |w_i\rangle$, such that $X_{ii} \ge 0$, $\forall 1 \le i \le m$.

Thus we have established that for any given pure state decomposition of the unnormalized states $p_i\rho_i$ (see equations (2.62)), there is a $n \times n$ unitary W such that

1. the ONB $\{|w_i\rangle\}_{i\in I}$, defined by equation (2.65), can be partitioned into subsets of m

²⁵ Using the singular value decomposition of X_{ii} , it can be seen that there is always some U_i such that $X_{ii}U_i \ge 0$.

vectors, $\{|w_{ij}\rangle\}_{i \in I_i}$, for $i = 1, 2, \dots, m$, so that $\{|w_{ij}\rangle\}_{i \in I_i}$ are the eigenvectors of Π_i , i.e., $\{|w_{ij}\rangle\}_{i \in I_i}$ satisfy equation (2.60), and

the matrix G^{1/2}W, which occurs in the equation (2.65), has positive semi-definite block diagonal matrices, i.e., X_{ii} ≥ 0, ∀ 1 ≤ i ≤ m.

Thus equation (2.71) becomes

$$X_{ii}X_{ij} - X_{ji}^{\dagger}X_{jj} = 0, \ \forall \ 1 \le i, j \le m.$$
(2.74)

Define $D_{\vec{r}}$ as the block diagonal matrix, containing diagonal matrix blocks of $G^{\frac{1}{2}}W$.

$$D_{\underline{r}} \equiv \begin{pmatrix} X_{11} & 0 & \cdots & 0 \\ 0 & X_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{mm} \end{pmatrix}.$$
 (2.75)

`

Left multiplying $G^{\frac{1}{2}}W$ by D_{r} gives (see equation (2.69a))

1

$$X_{r} \equiv D_{r}G^{\frac{1}{2}}W = \begin{pmatrix} (X_{11})^{2} & X_{11}X_{12} & \cdots & X_{11}X_{1m} \\ X_{22}X_{21} & (X_{22})^{2} & \cdots & X_{22}X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{mm}X_{m1} & X_{mm}X_{m2} & \cdots & (X_{mm})^{2} \end{pmatrix}.$$
(2.76)

Thus, equation (2.74) tells us that $X_r \equiv D_r G^{\frac{1}{2}} W$ is a hermitian matrix. This implies that²⁶

$$\left(X_{\vec{r}}\right)^{2} = \left(X_{\vec{r}}\right) \left(X_{\vec{r}}^{\dagger}\right) = D_{\vec{r}}GD_{\vec{r}}.$$
(2.77)

²⁶ An explanation for why the r_i 's are kept constant: consider n = 3, let *G* be a 3×3 gram matrix. Then the solution (of equation (2.77)) for this gram matrix will be different for the case $r_1 = 1$, $r_2 = 2$ compared to the case $r_1 = 2$, $r_2 = 1$, and also the case $r_1 = r_2 = r_3 = 1$. Each of these cases corresponds to MED problems for different LI ensembles, hence their solutions will generally differ.

Next we prove that $D_{\vec{x}}$ is non-singular. This is equivalent to proving that X_{ii} are non-singular for all $1 \le i \le m$, i.e., rank $(X_{ii}) = r_i$ for all $1 \le i \le m$.

Theorem 2.5.1. rank(X_{ii}) = r_i , $\forall 1 \le i \le m$.

Proof. Substituting the expression for $p_i\rho_i$ from equation (2.62a) and the expression for Π_i from equation (2.66) in the operator $p_i^2\rho_i\Pi_i\rho_i$, and using the partition of $G^{\frac{1}{2}}W$ into the X_{ij} matrix blocks (equation (2.69a)), one obtains

$$p_i^2 \rho_i \Pi_i \rho_i = \sum_{\substack{i,j \in I_i \\ i,j \in I_i}} \left((X_{ii})^2 \right)_{ij} |\widetilde{\psi}_i\rangle \langle \widetilde{\psi}_j|.$$
(2.78)

The fact that rank(Π_i) = r_i , $\forall 1 \le i \le m$ implies that rank($p_i^2 \rho_i \Pi_i \rho_i$) $\le r_i$, and rank($p_i \rho_i \Pi_i$) = rank($p_i \Pi_i \rho_i$) $\le r_i$, $\forall 1 \le i \le m$. We first establish that rank($p_i \rho_i \Pi_i$) = rank($p_i \Pi_i \rho_i$) = r_i . Suppose not, i.e., let rank ($p_i \rho_i \Pi_i$) $< r_i$. This implies that $\exists | u \rangle \in$ supp (Π_i) – {0} such that $p_i \rho_i \Pi_i | u \rangle = 0$. But since $\Pi_j | u \rangle = 0$ when $j \ne i^{27}$, we get that $Z | u \rangle = \sum_{j=1}^m p_j \rho_j \Pi_j | u \rangle = 0$, which implies that Z cannot be non-singular. Hence the assumption that rank ($p_i \rho_i \Pi_i$) $< r_i$ cannot be true for any $1 \le i \le m$. This implies that rank ($p_i \rho_i \Pi_i$) = r_i , $\forall 1 \le i \le m$.

Since $p_i \Pi_i \rho_i$ is the adjoint of $p_i \rho_i \Pi_i$, rank $(p_i \Pi_i \rho_i) = \operatorname{rank}(p_i \rho_i \Pi_i)$. That rank $(p_i \rho_i \Pi_i) = \operatorname{rank}(\Pi_i)$ implies that any non-zero vector belonging to $\sup (\Pi_i)$ has a non-zero component in $\sup (p_i \rho_i)$. Similarly, that $\operatorname{rank}(p_i \Pi_i \rho_i) = \operatorname{rank}(p_i \rho_i)$ implies that any non-zero vector in $\operatorname{supp}(p_i \rho_i)$ has a non-zero component in $\operatorname{supp}(\Pi_i)$. Let $|u\rangle \in \operatorname{supp}(p_i \rho_i)$, thus $p_i \rho_i |u\rangle \in \operatorname{supp}(p_i \rho_i)$. The arguments above tell us that $p_i \Pi_i \rho_i |u\rangle \neq 0$, and since $p_i \Pi_i \rho_i |u\rangle \in \operatorname{supp}(\Pi_i)$, they also tell us that $p_i^2 \rho_i \Pi_i \rho_i |u\rangle \neq 0$, $\forall |u\rangle \in \operatorname{supp}(p_i \rho_i) - \{0\}$. If $|u_1\rangle, |u_2\rangle, \cdots, |u_k\rangle \in \operatorname{supp}(p_i \rho_i)$ are *k* LI vectors such that $\{p_i^2 \rho_i \Pi_i rho_i |u_j\}_{j=1}^k$ are linearly dependent, then there must exist some non-zero $|u\rangle \in \operatorname{supp}(p_i \rho_i)$ such that $p_i^2 \rho_i \Pi_i \rho_i |u\rangle = 0$. But we know that this isn't true. Thus $p_i^2 \rho_i \Pi_i \rho_i$ preserves the linear independence of any set of vectors in $\operatorname{supp}(p_i \rho_i)$, which implies that $\operatorname{rank}(p_i^2 \rho_i \Pi_i \rho_i) = \operatorname{rank}(p_i \rho_i) = r_i$, $\forall 1 \leq i \leq m$. Using equation (2.78), this implies that $\operatorname{rank}(X_{ii})^2 = r_i, \forall 1 \leq i \leq m$, which

²⁷ $|u\rangle \in \operatorname{supp}(\Pi_i)$ and $\Pi_i \Pi_j = \delta_{ij} \Pi_i$, $\forall 1 \le i, j \le m$ implies that $|u\rangle \notin \operatorname{supp}(\Pi_j) \forall j \ne i$.

implies that $\operatorname{rank}(X_{ii}) = r_i, \forall 1 \le i \le m$.

Theorem (2.5.1) implies that D_r is non-singular, which implies that $X_r = D_r G^{\frac{1}{2}} W$ is non-singular.

Thus the rotationally invariant form of condition (2.7) can be given in two equivalent forms.

(I) For **any** pure state decomposition of the $p_i\rho_i$'s, for e.g., the pure states $\{|\tilde{\psi}_i\rangle\}_{i\in I}$ (see equation (2.62a)), one needs to find an $n \times n$ hermitian matrix X_r with the following properties

(a) When $X_{r_{r_{i}}}$ is partitioned into matrix blocks, as $G^{\frac{1}{2}}W$ is partitioned in equation (2.69a), then the diagonal matrix blocks of $X_{r_{i}}$ are positive definite. Represent these diagonal blocks by $(X_{11})^2, (X_{22})^2, \dots, (X_{mm})^2$. Define $D_{r_{i}}$ as given by equation (2.75), where the diagonal blocks X_{ii} are positive square roots of the diagonal blocks in $X_{r_{i}}$.

(b) $X_{\underline{r}}$ and $D_{\underline{r}}$ solve equation (2.77), where G is the gram matrix of $\{|\widetilde{\psi}_{\underline{i}}\rangle\}_{\underline{i}\in I}$.

(II) For **any** given pure state decomposition of the $p_i \rho_i$'s (for example as given by equation (2.62a)), one needs to find a block diagonal positive definite matrix D_r , of the form in equation (2.75), so that the diagonal blocks of **a** hermitian square root of the matrix $D_r GD_r$ are $(X_{11})^2$, $(X_{22})^2$, \cdots , $(X_{mm})^2$ respectively.

It is easily seen that (I) and (II) are both equivalent. To say that we solved the rotationally invariant form of condition (2.7) would mean that corresponding to some pure state decomposition of the $p_i\rho_i$'s, we either found X_r that solves (I) or we found D_r that solves (II).

Next we corroborate the claim that the above conditions are equivalent to the condition (2.7). Let $D_r > 0$ be of the form given in equation (2.75), and be the solution for (II),

corresponding to the pure state decomposition of the $p_i \rho_i$'s as given by equation (2.62a). Thus there is some hermitian square root of $D_r G D_r$, whose diagonal blocks are given by $(X_{11})^2, (X_{22})^2, \dots, (X_{mm})^2$. This hermitian square root must then have the form of X_r , as given by equation (2.76), where the X_{ij} 's satisfy equation (2.74). Left multiplying X_r by $(D_r)^{-1}$, gives us $G^{\frac{1}{2}}W$. We obtain the $|\tilde{u}_i\rangle$ vectors using equation (2.64a). We then obtain $|w_i\rangle$ from equation (2.65), and we obtain the Π_i 's using equation (2.60). Substituting the expression for Π_i 's in the LHS of equation (2.7) gives us the RHS of equation (2.7).

We claim that the aforementioned conditions are rotationally invariant because the solution for these conditions, i.e., X_r and/or D_r , are invariant under any unitary transformation of the states $p_i \rho_i \longrightarrow U p_i \rho_i U^{\dagger}$, for all $1 \le i \le m$, where U is any unitary operator on \mathcal{H} .

Theorem 2.5.2. For the pure state decomposition of $p_i\rho_i$'s given by equation (2.62a), let D_r be the solution for the aforementioned rotationally invariant form of condition (2.7). Let equation (2.62c) give another pure state decomposition for the $p_i\rho_i$'s into the pure states $|\tilde{\psi}'_i\rangle$, where $|\tilde{\psi}'_i\rangle$ and $|\tilde{\psi}_i\rangle$ are related by equation (2.62b). Then for the pure state decomposition given by equation (2.62c), the solution transforms to $(U_{D_r})^{\dagger}D_r U_{D_r}$, where U_{D_r} is given by equation (2.72). Similarly X_r transforms to $(U_{D_r})^{\dagger}X_r U_{D_r}$.

Proof. Note that the transformations $|\widetilde{\psi}_{i}\rangle \longrightarrow |\widetilde{\psi}_{i}'\rangle$, $\forall i \in I$, is accompanied by the following transformations.

$$G \longrightarrow G' \equiv (U_{D_r})^{\dagger} G U_{D_r}, \qquad (2.79a)$$

and

$$G^{\frac{1}{2}} \longrightarrow (G')^{\frac{1}{2}} \equiv (U_{D_r})^{\dagger} G^{\frac{1}{2}} U_{D_r}.$$
 (2.79b)

 D_{r_i} is a solution for the rotationally invariant form of the condition (2.7), corresponding to the pure state decomposition of the $p_i \rho_i$'s given by equation (2.62a). This implies that there is a hermitian square root of the matrix $D_{r_i}GD_{r_i}$, whose diagonal matrix blocks are given by $X_{11}^2, X_{22}^2, \dots, X_{mm}^2$. Define

$$D'_r \equiv (U_{D_r})^{\dagger} D_r U_{D_r}.$$
(2.79c)

Thus $D'_{r_{i}}G'D'_{r_{j}} = (U_{D_{i}})^{\dagger} (D_{r_{i}}GD_{r_{j}}) U_{D_{r_{i}}}$. The diagonal blocks of the hermitian square root transform as $X_{ii}^{2} \longrightarrow (U_{i}^{\dagger}X_{ii}U_{i})^{2}$, $\forall 1 \le i \le m$. Thus $D'_{r_{i}} = (U_{D_{i}})^{\dagger}D_{r_{i}}U_{D_{r_{i}}}$ gives the solution for the rotationally invariant form of condition (2.7) for the pure state decomposition of the $p_{i}\rho_{i}$'s given by equation (2.62b). It is also easily seen that $X_{r_{i}}$ transforms to $(U_{D_{r_{i}}})^{\dagger}X_{r_{i}}U_{D_{r_{i}}}$.

Simplification of Condition (2.8) for MED of LI Mixed State Ensembles

We want to obtain the rotationally invariant form of the necessary and sufficient conditions (2.7) and (2.8). Earlier, we gave the rotationally invariant form of condition (2.7). To solve this condition, corresponding to some pure state decomposition of the $p_i\rho_i$'s, one needs to find some X_r which is hermitian, and satisfies (a) and (b). We will later show that condition (2.8) is satisfied when X_r is positive definite. In other words, if we find X_r which satisfies the rotationally invariant conditions, and is positive definite, then upon constructing²⁸ Π_i 's from X_r , these Π_i 's would satisfy condition (2.8), along with satisfying condition (2.7). But to show this we need to simplify condition (2.8) for the MED of LI mixed states. For this purpose consider the following: let D_r be a solution for the rotationally invariant form of equation (2.7), corresponding to the pure state decomposition of the $p_i\rho_i$'s given by equation (2.62a). Define $\{|\tilde{\chi}_i\rangle\}_{i\in I}$ by

$$|\widetilde{\chi}_{\vec{i}}\rangle \equiv \frac{1}{\sqrt{\mathrm{Tr}(D_{\vec{j}}GD_{\vec{j}})}} \sum_{\substack{j \in I \\ \vec{j}}} \left(D_{\vec{j}} \right)_{\vec{j}} |\widetilde{\psi}_{\vec{j}}\rangle, \ \forall \ \vec{j} \in I.$$
(2.80a)

The inner product of $|\tilde{\chi}_i\rangle$ and $|\tilde{\chi}_j\rangle$ is

²⁸ Refer to the second paragraph before theorem 2.5.2 for instructions on how to construct the Π_i 's from either X_r or D_r .

$$\langle \widetilde{\chi}_{\underline{i}} | \widetilde{\chi}_{\underline{j}} \rangle = \frac{1}{\operatorname{Tr}(D_{\underline{r}} G D_{\underline{r}})} \left(D_{\underline{r}} G D_{\underline{r}} \right)_{\underline{i}\underline{j}}.$$
 (2.80b)

Thus the gram matrix of the vectors $\{|\tilde{\chi}_{\vec{i}}\rangle\}_{\vec{i}\in I}$ is $\frac{D_r G D_r}{\text{Tr}(D_r G D_r)}$.

Since $|\widetilde{\psi}_i\rangle_{i\in I_i}$ is a basis for $\operatorname{supp}(p_i\rho_i)$, and since $\operatorname{rank}(X_{ii}) = r_i$, equation (2.80a) implies that $\{|\widetilde{\chi}_i\rangle_{i\in I_i}\}$ is a basis for $\operatorname{supp}(p_i\rho_i)$. Since $\operatorname{supp}(p_1\rho_1)$, $\operatorname{supp}(p_2\rho_2)$, \cdots , $\operatorname{supp}(p_m\rho_m)$ are LI and since they together span \mathcal{H} , $\{|\widetilde{\chi}_i\rangle_{i\in I}\}$ is a basis for \mathcal{H} .

We will now show that for MED of LI mixed state ensembles, the condition (2.8) is subsumed in the condition $Z = \sum_{i=1}^{m} p_i \rho_i \Pi_i > 0.$

Theorem 2.5.3. The necessary and sufficient conditions for $\mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}}) = {\Pi_i}_{i=1}^m$ is that the Π_i 's should satisfy condition (2.7) and the condition $Z = \sum_{i=1}^m p_i \rho_i \Pi_i > 0$. That is, given that the Π_i 's satisfy condition (2.7), Z > 0 implies $Z \ge p_i \rho_i$, $\forall 1 \le i \le m$ (condition (2.8)).

Proof. Given that Π_i 's satisfy equation (2.7) and $Z = \sum_{i=1}^m p_i \rho_i \Pi_i > 0$. Theorem 2.5.1 tells us that the non-singularity of Z implies that X_{ii} are non-singular. The non-singularity of X_{ii} allowes us to define a set of *n* LI vectors $\{|\tilde{\chi}_i\rangle\}_{i\in I}$, as shown in equation (2.80a). We want to expand Z and $p_i\rho_i$ in the operator basis $\{|\tilde{\chi}_i\rangle\langle\tilde{\chi}_j|\}_{i,j\in I}$. For that note the following.

Equation (2.80a) implies that

$$|\widetilde{\psi}_{\underline{k}}\rangle = \sqrt{\mathrm{Tr}(D_{\underline{r}}GD_{\underline{r}})} \sum_{\underline{j}\in I} \left(\left(D_{\underline{r}}\right)^{-1} \right)_{\underline{j}\underline{k}} |\widetilde{\chi}_{\underline{j}}\rangle, \ \forall \ \underline{k} \in I,$$
(2.81a)

and since $|\widetilde{u}_{i}\rangle = \sum_{k \in I} (G^{-1})_{ki} |\widetilde{\psi}_{k}\rangle$ (see equation (2.64a)), we get that

$$|\widetilde{u}_{i}\rangle = \sqrt{\mathrm{Tr}(D_{\vec{j}}GD_{\vec{j}})} \sum_{\substack{j \in I \\ \vec{j}}} \left((D_{\vec{j}})^{-1}G^{-1} \right)_{ji} |\widetilde{\chi}_{\vec{j}}\rangle, \ \forall \ \underline{i} \in I.$$
(2.81b)
Substituting the expression for $|\tilde{\psi}_{\underline{k}}\rangle$ from equation (2.81a) into equation (2.62a) we get

$$p_i \rho_i = \sqrt{\mathrm{Tr}(D_{\vec{j}} G D_{\vec{j}})} \sum_{\substack{j,k \in \mathcal{I}_i \\ j \neq i}} \left((X_{ii})^{-1} \right)_{jk} |\widetilde{\chi}_{\vec{j}}\rangle \langle \widetilde{\psi}_{\vec{k}}|$$
(2.82a)

$$= \operatorname{Tr}(D_{\vec{j}}GD_{\vec{j}}) \sum_{\substack{j,k \in I_i \\ \downarrow \downarrow}} \left((X_{ii})^{-2} \right)_{jk} |\widetilde{\chi}_{\vec{j}}\rangle \langle \widetilde{\chi}_{\vec{k}}|.$$
(2.82b)

Substituting the expression for $p_i\rho_i$ from equation (2.82a), and for Π_i from equation (2.66), into $p_i\rho_i\Pi_i$, we get

$$p_i \rho_i \Pi_i = \sqrt{\mathrm{Tr}(D_{\vec{j}} G D_{\vec{j}})} \sum_{\substack{j \in I_i \\ \vec{j} \in I}} \sum_{\substack{k \in I}} \left(W^{\dagger} G^{\frac{1}{2}} \right)_{\substack{jk \\ \vec{j} \neq i}} |\widetilde{\chi}_{\vec{j}}\rangle \langle \widetilde{u}_{\vec{k}}|.$$
(2.82c)

Now substituting the expression for $|\tilde{u}_i\rangle$ in terms of $|\tilde{\chi}_i\rangle$ (see equation (2.81b)) in the expression for $p_i\rho_i\Pi_i$ in equation (2.82c) and summing over *i* gives

$$Z = \operatorname{Tr}(D_{\vec{r}}GD_{\vec{r}}) \sum_{\substack{j,k \in I\\ \vec{r} \neq \vec{r}}} \left(\left(X_{\vec{r}}\right)^{-1} \right)_{jk} |\widetilde{\chi}_{\vec{j}}\rangle \langle \widetilde{\chi}_{\vec{k}}|.$$
(2.82d)

Equation (2.82d) implies that the following statements are equivalent.

$$Z > 0 \Leftrightarrow \left(X_{\underline{r}}\right)^{-1} > 0. \tag{2.83}$$

The aim of this theorem is to prove the following statement

$$Z > 0 \Longrightarrow Z \ge p_i \rho_i, \ \forall \ 1 \le i \le m.$$
(2.84a)

Substituting the expression for $p_i\rho_i$ from equation (2.82b) and the expression for Z from equation (2.82d), into the statement (2.84a), gives us a statement, which is equivalent to the statement (2.84a) (note that we have used the RHS of equation (2.76) to obtain the

statement (2.84b) below):

$$\left(X_{\underline{r}}\right)^{-1} > 0 \Longrightarrow \begin{pmatrix} (X_{11})^2 & \cdots & X_{11}X_{1i} & \cdots & X_{11}X_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{ii}X_{i1} & \cdots & (X_{ii})^2 & \cdots & X_{ii}X_{im} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{mm}X_{m1} & \cdots & X_{mm}X_{mi} & \cdots & (X_{mm})^2 \end{pmatrix}^{-1} \ge \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & (X_{ii})^{-2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

$$(2.84b)$$

$$\forall 1 \le i \le m.$$

$$\left\{ \begin{array}{ccc} \Pr \text{ermute:} \begin{cases} k \to m + k - (i - 1), & \forall 1 \le k \le i - 1 \\ k \to k - (i - 1), & \forall i \le k \le m \end{cases} \right\} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{cccc} (X_{ii})^2 & X_{ii}X_{i\,i+1} & \cdots & X_{ii}X_{i\,i-1} \\ X_{i+1\,i+1}X_{i+1\,i} & (X_{i+1\,i+1})^2 & \cdots & X_{i+1\,i+1}X_{i+1\,i-1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{i-1\,i-1}X_{i-1\,i} & X_{i-1\,i-1}X_{i-1\,i+1} & \cdots & (X_{i-1\,i-1})^2 \end{array} \right)^{-1} \ge \left(\begin{array}{cccc} (X_{ii})^{-2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} \right), \quad (2.84c)$$

$$\forall 1 \le i \le m.$$

To simplify the notation define the following.

$$\begin{pmatrix} A & B \\ \\ B^{\dagger} & C \\ \\ \end{pmatrix} = \begin{pmatrix} (X_{ii})^2 & X_{ii}X_{i\,i+1} & \cdots & X_{ii}X_{i\,i-1} \\ \hline X_{i+1\,i+1}X_{i+1\,i} & (X_{i+1\,i+1})^2 & \cdots & X_{i+1\,i+1}X_{i+1\,i-1} \\ \vdots & \vdots & \ddots & \vdots \\ \hline X_{i-1\,i-1}X_{i-1\,i} & X_{i-1\,i-1}X_{i-1\,i+1} & \cdots & (X_{i-1\,i-1})^2 \end{pmatrix}, \quad (2.85)$$

where A is of dimension $r_i \times r_i$, B is of dimension $r_i \times (n - r_i)$ and C is of dimension $(n - r_i) \times (n - r_i)$. Note that the off diagonal blocks of X_r are adjoints of each other, since X_r is hermitian.

Thus the statement (2.84b) is equivalent to the following statement.

$$\begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}^{-1} > 0 \Longrightarrow \begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}^{-1} \ge \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$
 (2.86)

Since Z > 0, equations (2.83), (2.76) and (2.85) tell us that $\binom{A}{B^{\dagger}} \binom{B}{C} > 0$. This proves the inequality on the LHS of (2.86). We now have to establish that the inequality on the RHS of the statement (2.86) is true as well. Thus we have to prove that

$$\begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}^{-1} - \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} \ge 0.$$
 (2.87)

Now, note that $[56]^{29} \begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix} > 0 \Longrightarrow A > 0$ and $C - B^{\dagger}A^{-1}B > 0$, where $C - B^{\dagger}A^{-1}B$ is the Schur complement of A in $\begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}$. The inverse of $\begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}$ is given by [56] ³⁰

$$\begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + QS_AQ^{\dagger} & -QS_A \\ -S_AQ^{\dagger} & S_A \end{pmatrix}$$
$$= \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} QS_AQ^{\dagger} & -QS_A \\ -S_AQ^{\dagger} & S_A \end{pmatrix}$$
(2.88)

where $S_A \equiv (C - B^{\dagger} A^{-1} B)^{-1}$ is the inverse of the Schur complement of A in $\begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}$ and $Q \equiv A^{-1}B$. Substituting the expression for $\begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}^{-1}$ from equation (2.88) into the inequality (2.87) gives us

$$\begin{pmatrix} QS_A Q^{\dagger} & -QS_A \\ -S_A Q^{\dagger} & S_A \end{pmatrix} \ge 0.$$
(2.89)

²⁹ See subsection **A.5.5 Schur Complement** of the appendix (page 651) in [56].

³⁰ See subsection **A.5.5 Schur Complement** of the appendix (page 650) in [56].

Note that

$$\begin{pmatrix} QS_AQ^{\dagger} & -QS_A \\ -S_AQ^{\dagger} & S_A \end{pmatrix} = \begin{pmatrix} -Q & 0 \\ 0 & \mathbb{1}_{n-r_i} \end{pmatrix} \begin{pmatrix} S_A & S_A \\ S_A & S_A \end{pmatrix} \begin{pmatrix} -Q^{\dagger} & 0 \\ 0 & \mathbb{1}_{n-r_i} \end{pmatrix},$$

where $\mathbb{1}_{n-r_i}$ is the $(n-r_i) \times (n-r_i)$ identity matrix. Since $S_A > 0$, we get that $\begin{pmatrix} QS_AQ^{\dagger} & -QS_A \\ -S_AQ^{\dagger} & S_A \end{pmatrix} \ge 0$. Hence the inequality (2.89) is true. This proves the theorem.

Hence the necessary and sufficient conditions (2.7) (or equivalently equation (2.5)) and (2.8) are subsumed in the statement: $X_{\underline{r}} > 0$, where $X_{\underline{r}}$ is a solution for the rotationally invariant form of condition (2.7). In other words:

A: Let the ensemble $\widetilde{P}_{r_{i}} = \{p_{i}, \rho_{i}\}_{i=1}^{m}$ be in $\mathcal{E}_{r_{i}}$. Let a pure state decomposition of the $p_{i}\rho_{i}$'s into the pure states $\{|\widetilde{\psi}_{i}\rangle\}_{i\in I}$ be given by equation (2.62a), and let G denote the gram matrix of the set of vectors $\{|\widetilde{\psi}_{i}\rangle\}_{i\in I}$. Then for the MED of $\widetilde{P}_{r_{i}}$, one needs to find a block diagonal matrix, $D_{r_{i}}$, of the form as given in equation (2.75), where X_{ii} is an $r_{i} \times r_{i}$ positive definite matrix, so that the diagonal blocks of the **positive square root** of $D_{r_{i}}GD_{r_{i}}$ are given by $(X_{11})^{2}, (X_{22})^{2}, \cdots, (X_{mm})^{2}$ respectively.

Theorem 2.84 implies that the necessary and sufficient conditions (2.7) and (2.8) can be alternatively given by corollary 2.5.3.1.

Corollary 2.5.3.1. If $\{\Pi'_i\}_{i=1}^m \in \mathcal{P}_r$, and $\sum_{i=1}^m p_i \rho_i \Pi'_i > 0$, then $\{\Pi'_i\}_{i=1}^m$ is the optimal POVM for MED of \widetilde{P}_r , i.e., $\Pi'_i = \Pi_i$, $\forall 1 \le i \le m$.

Proof. We need to prove that if $\{\Pi'_i\}_{i=1}^m \in \mathcal{P}_{\underline{r}}$ and if $\sum_{i=1}^m p_i \rho_i \Pi'_i > 0$, then the Π'_i 's satisfy the conditions (2.7) and (2.8). First note that since $\{\Pi'_i\}_{i=1}^m$ is a projective measurement, $\sum_{j=1}^m p_j \rho_j \Pi'_j \Pi'_i = p_i \rho_i \Pi'_i$. Also, note that $\sum_{i=1}^m p_i \rho_i \Pi'_i = \sum_{i=1}^m p_i \Pi'_i \rho_i$, since $\sum_{i=1}^m p_i \rho_i \Pi'_i$ is hermitian. Thus we get

$$\sum_{l=1}^{m} p_l \rho_l \Pi'_l - \sum_{k=1}^{m} p_k \Pi'_k \rho_k = 0$$

Left multiplying the above expression by Π'_i and right-multiplying it by Π'_i gives

$$\Pi'_i \left(p_j \rho_j - p_i \rho_i \right) \Pi'_j = 0, \ \forall \ 1 \le i, j \le m,$$

which is condition (2.7). Theorem (2.5.3) tells us that if we have a POVM $\{\Pi_i'\}_{i=1}^m$ such that it satisfied equation (2.7) and so that $\sum_{i=1}^m p_i \rho_i \Pi_i' > 0$, then the following matrix inequalities are also satisfied: $\sum_{j=1}^m p_j \rho_j \Pi_j' \ge p_i \rho_i \forall 1 \le i \le m$, which is condition (2.8). Hence proved.

Note that the optimal success probability (see equation (2.3)) P_s^{max} is given by

$$P_{s}^{\max} = \sum_{i=1}^{m} \operatorname{Tr}\left(p_{i}\rho_{i}\Pi_{i}\right),$$

$$= \sum_{i=1}^{m} \operatorname{Tr}\left(\left(\sum_{\substack{j \in I_{i} \\ \downarrow \notin \downarrow \end{pmatrix}} |\widetilde{\psi}_{j}\rangle\langle\widetilde{\psi}_{j}|\right) \left(\sum_{\substack{l,k \in I \\ \downarrow \notin \downarrow \end{pmatrix}} \left(G^{\frac{1}{2}}W\right)_{li} \left(W^{\dagger}G^{\frac{1}{2}}\right)_{ik} |\widetilde{u}_{l}\rangle\langle\widetilde{u}_{k}|\right)\right) \qquad (2.90)$$

$$= \sum_{i=1}^{m} \operatorname{Tr}\left((X_{ii})^{2}\right),$$

where we used equations (2.66) and (2.62a) to obtain the second line from the first and equation (2.69a) to obtain the third line from the second.

Also, from theorem 2.5.2, it is easily seen that as $|\tilde{\psi}_{i_{j}}\rangle \longrightarrow |\tilde{\psi}_{i_{j}}\rangle$ (equation (2.62b)), the solution for **A** transforms as $D_{r_{j}} \longrightarrow D'_{r_{j}} = (U_{D_{r_{j}}})^{\dagger} D_{r_{j}} U_{D_{r_{j}}}$ and $X_{r_{j}} \longrightarrow X'_{r_{j}} = (U_{D_{r_{j}}})^{\dagger} X_{r_{j}} U_{D_{r_{j}}}$, where $U_{D_{r_{j}}}$ is given by equation (2.72). From equation (2.90) it is seen that under this transformation, the optimal success probability remains the same, as it should. In theorem 2.5.7 we show that corresponding to each choice of the pure state decomposition of the $p_i \rho_i$'s, there is a unique D_r which satisfies the conditions in **A**.

Constructing $\mathscr{R}_{\underline{r}}$

We will now construct the ensemble $\widetilde{Q}_{r} = \{q_i, \sigma_i\}_{i=1}^m \in \mathcal{E}_{r}$, such that

- 1. supp $(q_i \sigma_i) = \text{supp}(p_i \rho_i), \forall 1 \le i \le m$, and
- 2. $PGM_{\vec{r}}(\widetilde{Q}_{\vec{r}}) = \mathscr{P}_{\vec{r}}(\widetilde{P}_{\vec{r}}) = {\{\Pi_i\}_{i=1}^m}.$

Using equation (2.80a), define the following:

$$\sigma_{i} \equiv \frac{\sum_{i \in I_{i}} |\widetilde{\chi_{i}}\rangle \langle \widetilde{\chi_{i}}|}{\sum_{j \in I_{i}} \langle \widetilde{\chi_{j}} | \widetilde{\chi_{j}} \rangle}, \quad \forall \ 1 \le i \le m,$$
(2.91a)

and

$$q_{i} \equiv \sum_{\substack{i \in I_{i} \\ i \notin \widetilde{\mathcal{X}}_{i} \mid \widetilde{\mathcal{X}}_{i}}} \langle \widetilde{\mathcal{X}}_{i} \mid \widetilde{\mathcal{X}}_{i} \rangle, \ \forall \ 1 \le i \le m.$$
(2.91b)

Equation (2.91a) tells us that σ_i are density matrices on \mathcal{H} , and equation (2.80b) tells us that $\{q_i\}_{i=1}^m$ is an *m*-outcome probability with $q_i > 0$ for all $1 \le i \le m$.

Since the set $\{|\tilde{\chi}_i\rangle\}_{i \in I_i}$ spans $\sup(p_i\rho_i)$, we have that $\sup(q_i\sigma_i) = \sup(p_i\rho_i), \forall 1 \le i \le m$. It remains to be shown that $\mathscr{P}_r(\widetilde{P}_r)$ is the PGM of \widetilde{Q}_r .

Theorem 2.5.4. $\mathscr{P}_{r}(\widetilde{P}_{r})$ is the PGM of \widetilde{Q}_{r} , i.e.,

$$\Pi_i = \left(\sum_{j=1}^m q_j \sigma_j\right)^{-\frac{1}{2}} q_i \sigma_i \left(\sum_{k=1}^m q_k \sigma_k\right)^{-\frac{1}{2}}, \ \forall \ 1 \le i \le m.$$

Proof. Corresponding to the set of LI vectors $\{|\tilde{\chi}_i\rangle\}_{i\in I}$, define the vectors $\{|\tilde{y}_i\rangle\}_{i\in I}$ by

$$|\widetilde{y}_{\vec{j}}\rangle \equiv \operatorname{Tr}(D_{\vec{j}}GD_{\vec{j}})\sum_{\substack{j\in I\\ j\neq i}} \left(\left(D_{\vec{j}}GD_{\vec{j}}\right)^{-1} \right)_{\substack{ji\\ j\neq i}} |\widetilde{\chi}_{\vec{j}}\rangle, \ \forall \ \underline{i} \in I,$$
(2.92a)

where, using (2.80b), it is seen that

$$\langle \widetilde{\mathcal{Y}}_{\underline{i}} | \widetilde{\mathcal{X}}_{\underline{j}} \rangle = \delta_{\underline{i}\underline{j}}, \ \forall \ \underline{i}, \ \underline{j} \in I.$$
(2.92b)

Note that since $\{|\tilde{\chi}_{\underline{j}}\rangle\}_{\underline{j}\in\mathcal{I}}$ is a basis for \mathcal{H} and $D_{\underline{r}}$ is non-singular, $\{|\tilde{y}_{\underline{j}}\rangle\}_{\underline{j}\in\mathcal{H}}$ is a basis for \mathcal{H} . Equations (2.92a) and (2.92b) imply that

$$\langle \widetilde{y}_{\vec{j}}[\widetilde{y}_{\vec{j}}\rangle = \operatorname{Tr}(D_{\vec{j}}GD_{\vec{j}})\left(\left(D_{\vec{j}}GD_{\vec{j}}\right)^{-1}\right)_{\substack{ij\\ \vec{j}}}, \ \forall \ \underline{i}, \underline{j} \in I.$$
(2.92c)

Thus, the gram matrix of the set $\{|\widetilde{y_i}\rangle\}_{i \in I}$ is given by $\text{Tr}(D_r GD_r) (D_r GD_r)^{-1}$. Equation (2.92a) can also be used to corroborate that

$$|\widetilde{\chi}_{\vec{i}}\rangle = \frac{1}{\mathrm{Tr}(D_{\vec{i}}GD_{\vec{i}})} \sum_{\substack{j \in I \\ \vec{j} \neq \vec{i}}} \left(D_{\vec{i}}GD_{\vec{j}} \right)_{\substack{j \in I \\ \vec{j} \neq \vec{i}}} |\widetilde{y}_{\vec{j}}\rangle, \ \forall \ \vec{i} \in I.$$
(2.92d)

We can relate the set of vectors $\{|\widetilde{u}_{ij}\rangle\}_{j\in I}$ and $\{|\widetilde{y}_{jj}\rangle\}_{j\in I}$ in the following way: substitute the expansion of $|\widetilde{\psi}_{jj}\rangle$ in terms of $|\widetilde{\chi}_{kj}\rangle$ from equation (2.81a) in equation (2.64a), and in the resulting expression, substitute the expansion of $|\widetilde{\chi}_{kj}\rangle$ in terms of $|\widetilde{y}_{jj}\rangle$ from equation (2.92d). Thus we get that

$$|\widetilde{u}_{i}\rangle \equiv \frac{1}{\sqrt{\mathrm{Tr}(D_{r}GD_{r})}} \sum_{j \in I} \left(D_{r}\right)_{ji} |\widetilde{y}_{j}\rangle, \ \forall \ i \in I.$$
(2.93)

Expand Π_i in terms of $[\tilde{y}_i \rangle \langle \tilde{y}_j |$, by substituting equation (2.93) in equation (2.66):

$$\Pi_{i} = \frac{1}{\mathrm{Tr}(D_{r}GD_{r})} \sum_{\substack{j,k \in I \\ \downarrow \not \downarrow}} \left(\sum_{\substack{i \in I_{i} \\ \downarrow \not \downarrow}} \left(X_{r} \right)_{ji} \left(X_{r} \right)_{jk} \right) |\widetilde{y}_{j}\rangle \langle \widetilde{y}_{k}|.$$
(2.94)

We will prove that

$$\left(\sum_{j=1}^m q_j \sigma_j\right)^{-\frac{1}{2}} q_i \sigma_i \left(\sum_{k=1}^m q_k \sigma_k\right)^{-\frac{1}{2}}$$

is equal to the RHS of equation (2.94), $\forall 1 \le i \le m$. That proves the theorem.

From the definition of σ_i and q_i in equations (2.91), we get that

$$\sum_{i=1}^{m} q_i \sigma_i = \sum_{i \in \mathcal{I}} |\widetilde{\chi}_i\rangle \langle \widetilde{\chi}_i|.$$
(2.95)

Using equation (2.95), it can easily be verified that³¹

$$\left(\sum_{i=1}^{m} q_i \sigma_i\right)^{-1} = \sum_{\substack{i \in I \\ i \neq i}} \left| \widetilde{y_i} \right\rangle \langle \widetilde{y_i} |.$$
(2.96)

Bearing in mind that $\frac{1}{\sqrt{\operatorname{Tr}(D_r,GD_r)}}X_r$ is the positive square root of the matrix $\frac{1}{\operatorname{Tr}(D_r,GD_r)}D_r,GD_r$, and that $\frac{1}{\operatorname{Tr}(D_r,GD_r)}D_r,GD_r$ is the gram matrix of the set of vectors $\{|\tilde{\chi}_i\rangle\}_{i\in I}$, it can easily be verified that³²

$$\left(\sum_{i=1}^{m} q_i \sigma_i\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{\operatorname{Tr}(D_r G D_r)}} \sum_{\substack{j,k \in \mathcal{I} \\ r \neq i}} \left(X_r\right)_{\substack{j,k \in \mathcal{I} \\ r \neq i}} [\widetilde{y}_j\rangle \langle \widetilde{y}_k|.$$
(2.97)

Substituting the expression for $(\sum_{i=1}^{m} q_i \sigma_i)^{-\frac{1}{2}}$ from equation (2.97), the expression for $q_i \sigma_i$ from equations (2.91a) and (2.91b) in the expression $(\sum_{j=1}^{m} q_j \sigma_j)^{-\frac{1}{2}} q_i \sigma_i (\sum_{k=1}^{m} q_k \sigma_k)^{-\frac{1}{2}}$, and after some tedious algebra, it is seen that the result is equal to the RHS of equation (2.94), $\forall 1 \le i \le m$. This establishes that $\mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}}) = PGM_{\underline{r}}(\widetilde{Q}_{\underline{r}})$. Hence proved. \Box

Thus we have shown that for $\widetilde{P}_{\vec{r}} = \{p_i, \rho_i\}_{i=1}^m$ there exists an ensemble $\widetilde{Q}_{\vec{r}} = \{q_i, \sigma_i\}_{i=1}^m \in \mathcal{E}_{\vec{r}}$ such that

1. supp $(q_i \sigma_i) = \text{supp}(p_i \rho_i), \forall 1 \le i \le m \text{ and}$

³¹ Using equation (2.92b), we get $\left(\sum_{i \in I} |\widetilde{y}_{i}\rangle\langle \widetilde{y}_{i}|\right) \left(\sum_{j \in I} |\widetilde{\chi}_{j}\rangle\langle \widetilde{\chi}_{j}|\right) = \sum_{i \in I} \left(|\widetilde{y}_{i}\rangle\langle \widetilde{\chi}_{i}|\right) = 1$. This can be seen by acting $\sum_{i \in I} \left(|\widetilde{y}_{i}\rangle\langle \widetilde{\chi}_{i}|\right)$ on any $|v\rangle \in \mathcal{H}$ using the expansion of $|v\rangle$ in the $\{|\widetilde{y}_{i}\rangle\}_{i \in I}$ basis, i.e., $|v\rangle = \sum_{i \in I} \alpha_{i}|\widetilde{y}_{i}\rangle$.

³² This verification can be done by squaring the RHS of equation (2.97), which gives the LHS of equation (2.96) (using equation (2.92c)). This tells us that the RHS of equation (2.97) is *some* self-adjoint square root of $(\sum_{i=1}^{m} q_i \sigma_i)^{-1}$. Also note that since $X_r > 0$, the RHS of equation (2.97) is positive definite. Hence the RHS of equation (2.97) is the positive square root of $(\sum_{i=1}^{m} q_i \sigma_i)^{-1}$.

2.
$$PGM_{\underline{r}}(\widetilde{Q}_{\underline{r}}) = \mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}}).$$

This establishes the $\widetilde{P}_{r_{j}} \longrightarrow \widetilde{Q}_{r_{j}}$ correspondence mentioned in subsection (1).

Is there another ensemble $\widetilde{Q}'_{\underline{r}} \equiv \{q'_i, \sigma'_i\}_{i=1}^m$, which satisfies the properties 1. and 2.? Theorem 2.5.5 tells us that this is not so.

Theorem 2.5.5. There is a unique $\widetilde{Q}_{\vec{r}}$ of the form $\{q_i, \sigma_i\}_{i=1}^m$, also in $\mathcal{E}_{\vec{r}}$ such that the following points hold true.

1. $\operatorname{supp}(p_i\rho_i) = \operatorname{supp}(q_i\sigma_i), \forall 1 \le i \le m.$

2.
$$\mathscr{P}_{\vec{r}}(\widetilde{P}_{\vec{r}}) = PGM_{\vec{r}}(\widetilde{Q}_{\vec{r}}).$$

Proof. To prove this, note that equation (2.78) tells us that³³

$$p_i^2 \rho_i \Pi_i \rho_i = \operatorname{Tr}(D_{\underline{r}} G D_{\underline{r}}) \sum_{\underline{i} \in I_i} |\widetilde{\chi}_{\underline{i}}\rangle \langle \widetilde{\chi}_{\underline{i}}|$$

which is equal to³⁴ Tr $(D_r GD_r)q_i\sigma_i$. Thus the states and the probabilities of the ensemble $\widetilde{Q}_r = \{q_i, \sigma_i\}_{i=1}^m$ are equal to

$$q_i = \frac{\operatorname{Tr}\left(p_i^2 \rho_i \Pi_i \rho_i\right)}{\operatorname{Tr}\left(\sum_{j=1}^m p_j^2 \rho_j \Pi_j \rho_j\right)},$$
(2.98a)

$$\sigma_i = \frac{1}{\operatorname{Tr}\left(p_i^2 \rho_i \Pi_i \rho_i\right)} p_i^2 \rho_i \Pi_i \rho_i, \qquad (2.98b)$$

for all $1 \le i \le m$. Assume that there exists some ensemble $\widetilde{Q}'_{r} = \{q'_{i}, \sigma'_{i}\}_{i=1}^{m} \in \mathcal{E}_{r}$ which satisfies 1. and 2., then q'_{i} must satisfy

$$q'_{i} = \frac{\operatorname{Tr}\left(p_{i}^{2}\rho_{i}\Pi'_{i}\rho_{i}\right)}{\operatorname{Tr}\left(\sum_{j=1}^{m}p_{j}^{2}\rho_{j}\Pi'_{j}\rho_{j}\right)},$$
(2.99a)

³³ This can be seen by substituting the expression for $|\tilde{\psi}_i\rangle$ from equation (2.81a) into the RHS of equation (2.78).

³⁴ See equation (2.91).

and σ'_i must satisfy

$$\sigma_i' = \frac{1}{\operatorname{Tr}\left(p_i^2 \rho_i \Pi_i' \rho_i\right)} p_i^2 \rho_i \Pi_i' \rho_i, \qquad (2.99b)$$

where $\{\Pi'_i\}_{i=1}^m$ is another optimal POVM for the MED of $\widetilde{P}_{\underline{r}}$. But since the optimal POVM for MED of $\widetilde{P}_{\underline{r}}$ is unique, $\Pi'_i = \Pi_i$, $\forall \ 1 \le i \le m$. Thus, $q'_i = q_i$ and $\sigma'_i = \sigma_i$, $\forall \ 1 \le i \le m$. Thus $\widetilde{Q'}_{\underline{r}} = \widetilde{Q}_{\underline{r}}$, which proves the theorem.

Theorem 2.5.5 tells us that there is a *unique* $\widetilde{Q}_{\underline{r}} \in \mathcal{E}_{\underline{r}}$, which satisfies the conditions 1. and 2., this implies that we can now define the map $\mathscr{R}_{\underline{r}}$.

Definition 2.5.5. The map $\mathscr{R}_{\underline{r}} : \mathscr{E}_{\underline{r}} \longrightarrow \mathscr{E}_{\underline{r}}$ is such that, for any $\widetilde{P}_{\underline{r}} = \{p_i, \rho_i\}_{i=1}^m \in \mathscr{E}_{\underline{r}}, \mathscr{R}_{\underline{r}}(\widetilde{P}_{\underline{r}}) = \widetilde{Q}_{\underline{r}}$, where $\widetilde{Q}_{\underline{r}} = \{q_i, \sigma_i\}_{i=1}^m \in \mathscr{E}_{\underline{r}}$ and satisfies 1. and 2...

In the following we give an application of proving the existence of \mathscr{R}_{r} .

The optimal POVM for the MED of an ensemble of LI pure states $\{k_{i}, |\psi_{i}\rangle\langle\psi_{j}|\}_{i\in I} \in \widetilde{P}_{1}$, where $0 < k_{i} < 1$ and $\sum_{i\in I} k_{i} = 1$, is given by its own PGM if and only if all diagonal elements of $G_{k}^{\frac{1}{2}}$ are equal [3], where $G_{k}^{\frac{1}{2}}$ is the positive square root of G_{k} , the gram matrix of the vectors $\{\sqrt{k_{i}}|\psi_{i}\rangle\}_{i\in I}$. We now generalize this result to the mixed state ensemble case.

Theorem 2.5.6. Choose **any** pure state decomposition for the $p_i \rho_i$'s (equation (2.62a)). *G* is then the gram matrix of the $|\tilde{\psi}_i\rangle$'s, whose matrix elements are given by equation (2.63). $G^{\frac{1}{2}}$ is the positive definite square root of *G*. Partition $G^{\frac{1}{2}}$ into blocks of matrices of the form

$$G^{\frac{1}{2}} = \begin{pmatrix} \dot{X}_{11} & \dot{X}_{12} & \cdots & \dot{X}_{1m} \\ \dot{X}_{21} & \dot{X}_{22} & \cdots & \dot{X}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{X}_{m1} & \dot{X}_{m2} & \cdots & \dot{X}_{mm} \end{pmatrix},$$
(2.100)

where \dot{X}_{ij} is an $r_i \times r_j$ dimensional matrix block of $G^{\frac{1}{2}}$. Note that since $G^{\frac{1}{2}}$ is hermitian, we get $\dot{X}_{ji} = (\dot{X}_{ij})^{\dagger}$, $\forall \ 1 \le i, j \le m$. Then, $\mathscr{R}(\widetilde{P}_{r}) = \widetilde{P}_{r}$ holds true if and only if equations (2.101) hold true.

$$\left(\dot{X}_{ii}\right)_{\substack{ij\\j \neq j}} = 0, \ \forall \ \underline{i} \neq \underline{j}, \ \forall \ 1 \le i \le m.$$
(2.101a)

$$\left(\dot{X}_{ii}\right)_{\underset{\rightarrow}{ii}} = \left(\dot{X}_{jj}\right)_{\underset{\rightarrow}{jj}}, \ \forall \ 1 \le i, j \le m, \ \forall \ \underset{\rightarrow}{i} \in \mathcal{I}_i, \ \forall \ \underset{\rightarrow}{j} \in \mathcal{I}_j.$$
(2.101b)

Proof. **ONLY IF part:** Let $\mathscr{R}(\widetilde{P}_{r_{i}}) = \widetilde{P}_{r_{i}}$. Let equation (2.62a) give any pure state decomposition for the $p_{i}\rho_{i}$'s. Let the solution for the MED of $\widetilde{P}_{r_{i}}$ corresponding to this pure state decomposition be $D_{r_{i}}$. Since $\widetilde{Q}_{r_{i}} = \widetilde{P}_{r_{i}}$, we require that $p_{i}\rho_{i} = \sum_{\substack{i \in I_{i} \\ i \neq i \\ i$

IF part: The equations (2.101) say that $\check{X}_{ii} = a \mathbb{1}_{r_i}$, $\forall 1 \le i \le m$, where *a* is some positive real number. Choosing $W = \mathbb{1}_n$, $G^{\frac{1}{2}}W \longrightarrow G^{\frac{1}{2}}$. Note that the matrix blocks of $G^{\frac{1}{2}}$, viz., the \check{X}_{ij} 's, satisfy the equations (2.71). Then $D_r = a \mathbb{1}_n$. Substituting this in equation (2.80a) we get $|\tilde{\chi}_i\rangle = |\tilde{\psi}_i\rangle$, $\forall i \in I$. Thus $q_i\sigma_i = \sum_{i\in I_i} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = p_i\rho_i$, $\forall 1 \le i \le m$. Thus $\widetilde{P}_r = \widetilde{Q}_r = \mathscr{R}(\widetilde{P}_r)$. Hence proved.

\mathscr{R}_r is Invertible

First we show that corresponding to each choice of the pure state decomposition of the $p_i \rho_i$'s, there is a unique D_r which satisfies the condition **A**.

Theorem 2.5.7. There is a unique positive definite block diagonal matrix D_r of the form in equation (2.75), which solves the condition **A** for a given choice of the pure state

decomposition of the $p_i \rho_i$'s.

Proof. Using equations (2.78) and (2.98) we see that

$$q_i \sigma_i = \frac{1}{\mathrm{Tr}(D_{\vec{j}} G D_{\vec{j}})} \sum_{\substack{i,j \in I_i \\ \vec{j}, \vec{j} \in I_i}} \left((X_{ii})^2 \right)_{\substack{ji \\ \vec{j}, \vec{j} \in I_i}} |\widetilde{\psi}_{\vec{j}}\rangle \langle \widetilde{\psi}_{\vec{j}}|, \ \forall \ 1 \le i \le m.$$
(2.102)

Assume that for the same pure state decomposition of the $p_i \rho_i$'s, as given by equation (2.62a), there is a positive definite block diagonal matrix, D'_r , of the form

$$D'_{r} \equiv \begin{pmatrix} X'_{11} & 0 & \cdots & 0 \\ 0 & X'_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X'_{mm} \end{pmatrix},$$
(2.103)

so that $D'_{\underline{r}}$ is a solution other than $D_{\underline{r}}$ for condition **A**. Using the X'_{ii} 's define

$$q'_{i}\sigma'_{i} = \frac{1}{\operatorname{Tr}(D'_{\vec{r}}GD'_{\vec{r}})} \sum_{\substack{i,j \in I_{i} \\ \vec{r}, \vec{r} \in I_{i}}} \left((X'_{ii})^{2} \right)_{ji} |\widetilde{\psi}_{\vec{r}}\rangle \langle \widetilde{\psi}_{\vec{r}}|, \ \forall \ 1 \le i \le m.$$
(2.104)

Note that $\operatorname{supp}(q'_i\sigma'_i) = \operatorname{supp}(p_i\rho_i), \forall 1 \le i \le m.$

Define the following

$$\begin{split} \widetilde{Q}'_{\vec{r}} &\equiv \{q'_i, \sigma'_i\}_{i=1}^m. \text{ Note that } \widetilde{Q}'_{\vec{r}} \in \mathcal{E}_{\vec{r}}. \\ |\widetilde{\chi}'_{\vec{i}}\rangle &\equiv \frac{1}{\sqrt{\operatorname{Tr}(D'_{\vec{r}}GD'_{\vec{r}})}} \sum_{\vec{j} \in I_i} (X'_{ii})_{\vec{j},\vec{j}} |\widetilde{\psi}_{\vec{j}}\rangle, \ \forall \ \vec{i} \in I \text{ (compare with equation (2.80a))}. \\ |\widetilde{y}'_{\vec{i}}\rangle &\equiv \operatorname{Tr}(D'_{\vec{r}}GD'_{\vec{r}}) \sum_{\vec{j} \in I} \left((D'_{\vec{r}}GD'_{\vec{r}})^{-1} \right)_{\vec{j},\vec{j}} |\widetilde{\chi}'_{\vec{j}}\rangle, \ \forall \ \vec{i} \in I \text{ (with equation (2.92a))}. \end{split}$$

In theorem 2.5.4 we showed that $\mathscr{P}_{r}(\widetilde{P}_{r}) = PGM_{r}(\widetilde{Q}_{r})$. Using the above defined quantities, one can also corroborate that $\mathscr{P}_{r}(\widetilde{P}_{r}) = PGM_{r}(\widetilde{Q}_{r}')$. This can be done by expanding the Π_{i} 's, given in equation (2.60) in the operator basis $\{[\widetilde{y}_{i}'], \langle \widetilde{y}_{j}']\}_{i,j\in I}$, and then verify-

ing that this equals the operator $\left(\sum_{j=1}^{m} q'_{j}\sigma'_{j}\right)^{-\frac{1}{2}} q'_{i}\sigma'_{i} \left(\sum_{k=1}^{m} q'_{k}\sigma'_{k}\right)^{-\frac{1}{2}}$, when expanded in the $\{[\widetilde{y}'_{j}\rangle\langle\widetilde{y}'_{j}]\}_{i,j\in I}$ operator basis.

This implies that the ensemble \widetilde{Q}'_{r} satisfies conditions 1. and 2... Then theorem 2.5.5 implies that $\widetilde{Q}'_{r} = \widetilde{Q}_{r}$. Thus

$$q'_i \sigma'_i = q_i \sigma_i, \ \forall \ 1 \le i \le m.$$

$$(2.105)$$

Note that $\{|\tilde{\psi}_{i}\rangle\}_{i \in I_{i}}$ is a basis for $\operatorname{supp}(p_{i}\rho_{i})$. This implies that $\{|\tilde{\psi}_{i}\rangle\langle\tilde{\psi}_{j}|\}_{i,j\in I_{i}}$ is a basis for all operators acting on $\operatorname{supp}(p_{i}\rho_{i})$. Then substituting the expression for $q_{i}\sigma_{i}$ from equation (2.102) and the expression for $q'_{i}\sigma'_{i}$ from equation (2.104) in equation (2.105) gives us the following.

$$\frac{1}{\text{Tr}(D_{\vec{r}}GD_{\vec{r}})}(X_{ii})^{2} = \frac{1}{\text{Tr}(D_{\vec{r}}GD_{\vec{r}})}(X_{ii}')^{2}, \ \forall \ 1 \le i \le m.$$
(2.106)

Equation (2.90) implies that $P_s^{\max} = \sum_{i=1}^m \operatorname{Tr}\left((X_{ii})^2\right)$. Since D'_r is also a solution for condition **A**, we get that (see equation (2.103)) $P_s^{\max} = \sum_{i=1}^m \operatorname{Tr}\left(\left(X'_{ii}\right)^2\right)$. Using equation (2.106) to substitute $\frac{\operatorname{Tr}(D'_r G D'_r)}{\operatorname{Tr}(D'_r G D'_r)} (X_{ii})^2$ in place of $\left(X'_{ii}\right)^2$, we get that

$$P_s^{\max} = \frac{\operatorname{Tr}(D_r'GD_r')}{\operatorname{Tr}(D_r'GD_r)} \sum_{i=1}^m \operatorname{Tr}\left((X_{ii})^2\right).$$

Thus $\operatorname{Tr}(D_r G D_r) = \operatorname{Tr}(D'_r G D'_r)$, and hence $(X_{ii})^2 = (X'_{ii})^2$. Since X_{ii} and X'_{ii} are positive definite, $X_{ii} = X'_{ii}$, $\forall 1 \le i \le m$. Hence we have proved that for a given choice of the pure state decomposition of the $p_i \rho_i$'s, as given by equation eqrefrhodecomposition, there is a unique D_r , of the form in equation (2.75), which solves the condition **A**.

Next, we show that \mathscr{R}_r is a bijection.

Theorem 2.5.8. \mathscr{R}_r is a bijection.

Proof. We first prove that \mathscr{R}_r is onto.

Given some $\widetilde{Q}_{r} = \{q_{i}, \sigma_{i}\}_{i=1}^{m} \in \mathcal{E}_{r}$, we need to prove that there exists some $\widetilde{P}_{r} \in \mathcal{E}_{r}$ such that $\mathscr{R}_{r}(\widetilde{P}_{r}) = \widetilde{Q}_{r}$. For this, we will first construct an ensemble which we will denote by $\widetilde{P}_{r} = \{p_{i}, \rho_{i}\}_{i=1}^{m}$ and which lies in \mathcal{E}_{r} . Later we show that this ensemble will be such that $\mathscr{R}_{r}(\widetilde{P}_{r}) = \widetilde{Q}_{r}$.

We construct $\widetilde{P}_{\underline{r}}$ in the following three steps.

- (i) Choose a pure state decomposition for the q_iσ_i's: Since rank(q_iσ_i) = r_i, we can decompose q_iσ_i as a convex sum of r_i pure states, i.e., q_iσ_i = Σ_{i∈I_i} | 𝔅_i ⟩ ⟨𝔅_i| such that the set of vectors {|𝔅_i⟩}_{i∈I_i} is LI. Note that there is a unitary degree of freedom for choosing this pure state decomposition consider U_i to be an r_i × r_i unitary matrix, define | 𝔅_i ≥ Σ_{j∈I_i} (U_i)_{ji} | 𝔅_j ⟩, then the set of vectors {|𝔅_i⟩ ⟨𝔅_i| is LI and is such that q_iσ_i = Σ_{j∈I_i} | 𝔅_i ⟩ ⟨𝔅_i'|. Assume that q_iσ_i = Σ_{j∈I_i} | 𝔅_j ⟩ ⟨𝔅_i'| is a random choice for the pure state decomposition of the q_iσ_i's.
- (ii) Corresponding to the |X_i⟩'s, define a new set of vectors { | ψ_i⟩ }_{i∈I}: Let's denote the gram matrix for the vectors { |X_i⟩ }_{i∈I} by G_q. The matrix elements of G_q are given by

$$\left(G_q\right)_{\substack{ij\\j}} = \langle \widetilde{\chi}_i | \widetilde{\chi}_j \rangle, \ \forall \ \underline{i}, \underline{j} \in I.$$

$$(2.107)$$

Note that $G_q > 0$ because the set $\{|\tilde{\chi}_i\rangle\}_{i \in I}$ is LI.

Denote by $G_q^{\frac{1}{2}}$ the positive square root of G_q . Partition $G_q^{\frac{1}{2}}$ into matrix blocks, the same way $G^{\frac{1}{2}}W$ was partitioned into matrix blocks in equation (2.69a).

$$G_{q}^{\frac{1}{2}} = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1m} \\ H_{21} & H_{22} & \cdots & H_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ H_{m1} & H_{m2} & \cdots & H_{mm} \end{pmatrix},$$
(2.108)

where H_{ij} is of dimension $r_i \times r_j$, $\forall 1 \le i, j \le m$. Note that since $G_q^{\frac{1}{2}} > 0$, $H_{ii} > 0$, $\forall 1 \le i \le m$.

Define

$$D_{q} \equiv \begin{pmatrix} (H_{11})^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & (H_{22})^{\frac{1}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (H_{mm})^{\frac{1}{2}} \end{pmatrix}.$$
 (2.109)

Note that $D_q > 0$.

Define the states

$$|\widetilde{\psi}_{\vec{i}}\rangle \equiv \frac{1}{\sqrt{\mathrm{Tr}\left(D_q^{-1}G_q D_q^{-1}\right)}} \sum_{j \in \mathcal{I}} \left(D_q^{-1}\right)_{ji} |\widetilde{\chi}_{\vec{j}}\rangle, \ \forall \ \vec{i} \in \mathcal{I}.$$
(2.110)

Note that the states $\{|\widetilde{\psi}_{i}\rangle\}_{i\in I}$, as defined in equation (2.110), are LI. This is because $(D_q)^{-1}$ is non-singular and the set of vectors $\{|\widetilde{\chi}_i\rangle\}_{i\in I}$ is LI.

(iii) Define the states $p_i\rho_i$, so that $\widetilde{P}_r = \{p_i, \rho_i\}_{i=1}^m \in \mathcal{E}_r$: Using the states $|\widetilde{\psi}_i\rangle$, as defined in equation (2.110), define $p_i\rho_i \equiv \sum_{i \in I_i} |\widetilde{\psi}_i\rangle \langle \widetilde{\psi}_i|$, where $p_i \equiv \sum_{i \in I_i} \langle \widetilde{\psi}_i| \widetilde{\psi}_i\rangle$, and $\rho_i \equiv \frac{1}{p_i} \sum_{i \in I_i} |\widetilde{\psi}_i\rangle \langle \widetilde{\psi}_i|$. Since the set of vectors $\{|\widetilde{\psi}_i\rangle\}_{i \in I}$ is LI, rank $(p_i\rho_i) = r_i$ and supp $(p_1\rho_1)$, supp $(p_2\rho_2)$, \cdots , supp $(p_m\rho_m)$ are LI. Define $\widetilde{P}_r \equiv \{p_i, \rho_i\}_{i=1}^m$. Thus, $\widetilde{P}_r \in \mathcal{E}_r$.

Note that supp $(p_i \rho_i) = \text{supp}(q_i \sigma_i), \forall 1 \le i \le m.$

Corresponding to the set of LI vectors $\{|\widetilde{\chi_{i}}\rangle\}_{i \in I}$, there exists another set of LI vectors $\{|\widetilde{\gamma_{i}}\rangle\}_{i \in I}$, defined by $|\widetilde{\gamma_{i}}\rangle \equiv \sum_{i \in I} ((G_q)^{-1})_{ji} |\widetilde{\chi_{j}}\rangle$, and which satisfy the equation $\langle \widetilde{\chi_{i}}|\widetilde{\gamma_{j}}\rangle = \delta_{ij}$, $\forall i, j \in I$. The gram matrix of the set $\{|\widetilde{\gamma_{i}}\rangle\}_{i \in I}$ is G_q^{-1} .

Let $PGM_{\vec{r}}(\widetilde{Q}_{\vec{r}}) = {\Pi_i}_{i=1}^m$, thus ${\Pi_i}_{i=1}^m \in \mathcal{P}_{\vec{r}}$. In the body of the proof of theorem (2.5.4) we constructed the PGM for an ensemble of mixed states using the pure state decomposition of the corresponding mixed states. Following the same sequence of steps gives us the

expansion of the Π_i projectors in the $\{[\widetilde{y}_i] \land (\widetilde{y}_j) \}_{i,j \in I}$ operator basis.

$$\Pi_{i} = \left(\sum_{j=1}^{m} q_{j}\sigma_{j}\right)^{-\frac{1}{2}} q_{i}\sigma_{i} \left(\sum_{k=1}^{m} q_{k}\sigma_{k}\right)^{-\frac{1}{2}}$$

$$= \sum_{\substack{j,k\in I\\ \rightarrow}} \left(\sum_{i\in I_{i}} \left(G_{q}^{-\frac{1}{2}}\right)_{ji} \left(G_{q}^{-\frac{1}{2}}\right)_{ik} \right) \widetilde{[y_{j}]} \langle \widetilde{y_{k}}|, \ \forall \ 1 \leq i \leq m.$$

$$(2.111)$$

Substituting the expression for $p_i \rho_i$ in terms of the $|\tilde{\chi}_i\rangle$'s from equation (2.110) and the expression for Π_i from (2.111) into the term $\sum_{i=1}^m p_i \rho_i \Pi_i$, and after a tedious bit of algebra we get³⁵

$$\sum_{i=1}^{m} p_i \rho_i \Pi_i$$

$$= \frac{1}{\operatorname{Tr}\left(D_q^{-1} G_q D_q^{-1}\right)} \sum_{\substack{j,k \in I \\ j \neq i}} \left(G_q^{-\frac{1}{2}}\right)_{\substack{j,k \in I \\ j \neq i}} |\widetilde{\chi}_j\rangle \langle \widetilde{\chi}_k| > 0.$$

So, note that $\{\Pi_i\}_{i=1}^m$ is a projective measurment and $\sum_{i=1}^m p_i \rho_i \Pi_i > 0$. Then corollary (2.5.3.1) tells us that $\mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}}) = {\{\Pi\}_{i=1}^m}$.

Hence we have proved two things: 1. $\operatorname{supp}(p_i\rho_i) = \operatorname{supp}(q_i\sigma_i), \forall 1 \le i \le m$, and 2. $PGM_r(\widetilde{Q}_r) = \{\Pi_i\}_{i=1}^m = \mathscr{P}_r(\widetilde{P}_r)$. Thus by theorem (2.5.5) \widetilde{Q}_r is *the unique* ensemble in \mathscr{E}_r which satisfies the conditions 1. and 2. for the ensemble \widetilde{P}_r . The definition of \mathscr{R}_r then tells us that $\mathscr{R}_r(\widetilde{P}_r) = \widetilde{Q}_r$.

Thus for any $\widetilde{Q}_{r_{i}}$ in $\mathcal{E}_{r_{i}}$ we can find a corresponding $\widetilde{P}_{r_{i}}$ in $\mathcal{E}_{r_{i}}$ so that $\mathscr{R}_{r_{i}}(\widetilde{P}_{r_{i}}) = \widetilde{Q}_{r_{i}}$. Hence \overline{S}^{35} For completeness, we compare the quantities appearing in the (onto part of) proof of theorem (2.5.8) with quantities derived earlier. From equations (2.81a) and (2.110), we see that $H_{ii}^{\frac{1}{2}} = \frac{1}{(\operatorname{Tr}(D_{r_{i}}GD_{r_{i}}))^{\frac{1}{4}}}X_{ii}$, so $D_{q}^{-1} = (\operatorname{Tr}(D_{r_{i}}GD_{r_{i}}))^{\frac{1}{4}}D_{r_{i}}^{-1}$. This, together with the fact that $G_{q} = \frac{D_{r}GD_{r_{i}}}{\operatorname{Tr}(D_{r_{i}}GD_{r_{i}})}$, implies that $D_{q}^{-1}G_{q}D_{q}^{-1} = \frac{1}{\sqrt{\operatorname{Tr}(D_{r_{i}}GD_{r_{i}})}}G$. Also note that $X_{r_{i}} = \sqrt{\operatorname{Tr}(D_{r_{i}}GD_{r_{i}})}G_{q}^{\frac{1}{2}}$. Using these relations we can see that equations (2.82d) and (2.112) are consistent with each other. \mathscr{R}_r is onto.

Next we prove that \mathscr{R}_{r} is one-to-one.

Suppose $\widetilde{P}'_{\vec{r}} = \{p'_i, \rho'_i\}_{i=1}^m$, $\widetilde{P}_{\vec{r}} = \{p_i, \rho_i\}_{i=1}^m \in \mathcal{E}_{\vec{r}}$ such that $\mathscr{R}_{\vec{r}}(\widetilde{P}'_{\vec{r}}) = \mathscr{R}_{\vec{r}}(\widetilde{P}_{\vec{r}}) = \mathcal{Q}_{\vec{r}} = \{q_i, \sigma_i\}_{i=1}^m$.

Let equation (2.62a) give a pure state decomposition for the $p_i \rho_i$'s. *G* is the gram matrix of the set of vectors $\{|\widetilde{\psi}_{ij}\rangle\}_{i \in I}$. Let the solution for MED of \widetilde{P}_r , corresponding to the pure state decomposition in equation (2.62a), be D_r , which is of the form as the RHS in equation (2.75). Then define the LI vectors $\{|\widetilde{\chi}_{ij}\rangle\}_{i \in I}$ as shown in equation (2.80a). Since $\mathscr{R}_r(\widetilde{P}_r) = \widetilde{Q}_r = \{q_i, \sigma_i\}_{i=1}^m$, we get that $q_i \sigma_i = \sum_{i \in I_i} |\widetilde{\chi}_i\rangle\langle\widetilde{\chi}_i|$.

Similarly, let

$$p_i'\rho_i' = \sum_{\substack{i \in I_i \\ i}} |\widetilde{\psi}_i'\rangle \langle \widetilde{\psi}_i'|$$

give a pure state decomposition for the $p'_i \rho'_i$'s into the LI pure states $\{|\widetilde{\psi}'_i\rangle\}_{i \in I_i}$. Let G' be the gram matrix of the vectors $\{|\widetilde{\psi}'_i\rangle\}_{i \in I}$. Since the set of vectors $\{|\widetilde{\psi}'_i\rangle\}_{i \in I}$ is LI, G' > 0. Corresponding to this pure state decomposition, let the solution for the MED of \widetilde{P}'_i be D'_i which is for the form of the RHS in equation (2.103). Define the set of LI vectors $\{|\widetilde{\chi}'_i\rangle\}_{i \in I}$ as

$$|\widetilde{\chi}'_{i}\rangle = \frac{1}{\sqrt{\mathrm{Tr}(D'_{r}G'D'_{r})}} \sum_{\substack{j \in I \\ j \neq i}} \left(D'_{j}\right)_{ji} |\widetilde{\psi}'_{j}\rangle, \ \forall \ \underline{i} \in I.$$

Since $\mathscr{R}_{r}(\widetilde{P}'_{r}) = \widetilde{Q}_{r} = \{q_{i}, \sigma_{i}\}_{i=1}^{m}$, we get that $q_{i}\sigma_{i} = \sum_{i \in I_{i}} |\widetilde{\chi}'_{i}\rangle \langle \widetilde{\chi}'_{i}|$.

Note that $\{|\tilde{\chi}_{i}\rangle\}_{i \in I_{i}}$ and $\{|\tilde{\chi}_{i}'\rangle\}_{i \in I_{i}}$ are LI vectors such that

$$q_i \sigma_i = \sum_{\underline{i} \in \mathcal{I}_i} |\widetilde{\chi}_{\underline{i}}\rangle \langle \widetilde{\chi}_{\underline{i}}| = \sum_{\underline{i} \in \mathcal{I}_i} |\widetilde{\chi}_{\underline{i}}'\rangle \langle \widetilde{\chi}_{\underline{i}}'|.$$

Thus, there exists some $r_i \times r_i$ unitary U_i such that $|\widetilde{\chi}_i\rangle$ and $|\widetilde{\chi}'_i\rangle$ are related by $|\widetilde{\chi}'_i\rangle \equiv \sum_{j \in I_i} (U_i)_{ji} |\widetilde{\chi}_j\rangle$. Let U_{D_r} be the $n \times n$ block diagonal unitary given by the RHS of equa-

tion (2.72). Then note that $\langle \widetilde{\chi'_{i}} | \widetilde{\chi'_{j}} \rangle = \sum_{\substack{k,l \in I}} \left(U^{\dagger}_{D_{r_{j}}} \right)_{\substack{i,k \\ j\neq j}} \langle \widetilde{\chi_{k}} | \widetilde{\chi_{l}} \rangle \left(U_{D_{r_{j}}} \right)_{\substack{l,l \\ j\neq j}}$. The gram matrix of the set of vectors $\{ | \widetilde{\chi'_{i}} \rangle \}_{\substack{i \in I}}$ is $\frac{1}{\operatorname{Tr}(D_{r_{i}}^{G} D_{r_{i}})} D_{r_{i}}^{r_{i}} G D_{r_{i}}$, and that of the set of vectors $\{ | \widetilde{\chi'_{i}} \rangle \}_{\substack{i \in I}}$ is $\frac{1}{\operatorname{Tr}(D_{r_{i}}^{r_{i}} O_{r_{i}})} D_{r_{i}}^{r_{i}} G D_{r_{i}}$, and that of the set of vectors $\{ | \widetilde{\chi'_{i}} \rangle \}_{\substack{i \in I}}$ is $\frac{1}{\operatorname{Tr}(D_{r_{i}}^{r_{i}} O_{r_{i}})} D_{r_{i}}^{r_{i}} G' D_{r_{i}}^{r_{i}}$. Thus both matrices are related by

$$\frac{1}{\operatorname{Tr}(D_{\underline{r}}'G'D_{\underline{r}}')}D_{\underline{r}}'G'D_{\underline{r}}' = \frac{1}{\operatorname{Tr}(D_{\underline{r}}GD_{\underline{r}})}U_{D_{\underline{r}}}^{\dagger}D_{\underline{r}}GD_{\underline{r}}U_{D_{\underline{r}}}.$$
(2.112)

Taking the positive square root on both sides gives

$$\frac{1}{\sqrt{\mathrm{Tr}(D_{\underline{r}}'G'D_{\underline{r}}')}} \left(D_{\underline{r}}'G'D_{\underline{r}}'\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\mathrm{Tr}(D_{\underline{r}}'GD_{\underline{r}})}} U_{D_{\underline{r}}}^{\dagger} \left(D_{\underline{r}}'GD_{\underline{r}}\right)^{\dagger} U_{D_{\underline{r}}}.$$
 (2.113)

Comparing the diagonal blocks of the LHS and RHS in equation (2.113) gives

$$\frac{1}{\sqrt{\text{Tr}(D_{r}'G'D_{r}')}} (X_{ii}')^2 = \frac{1}{\sqrt{\text{Tr}(D_{r}GD_{r})}} U_i^{\dagger} (X_{ii})^2 U_i.$$
(2.114)

Since X'_{ii} and X_{ii} are both positive definite, we get

$$\frac{1}{\left(\operatorname{Tr}(D_{r}'G'D_{r}')\right)^{\frac{1}{4}}}X_{ii}' = \frac{1}{\left(\operatorname{Tr}(D_{r}GD_{r})\right)^{\frac{1}{4}}}U_{i}^{\dagger}X_{ii}U_{i}.$$

$$\Longrightarrow \frac{1}{\left(\operatorname{Tr}(D_{r}'G'D_{r}')\right)^{\frac{1}{4}}}D_{r}' = \frac{1}{\left(\operatorname{Tr}(D_{r}GD_{r})\right)^{\frac{1}{4}}}U_{D_{r}}^{\dagger}D_{r}U_{D_{r}}.$$
(2.115)

Substituting the expression for $\frac{1}{\left(\operatorname{Tr}(D'_{r}G'D'_{r})\right)^{\frac{1}{4}}}D'_{r}$ from equation (2.115) into equation (2.112) gives

$$\frac{1}{\operatorname{Tr}(D'_{r}G'D'_{r})}D'_{r}G'D'_{r} = \frac{1}{\sqrt{\operatorname{Tr}(D'_{r}G'D'_{r})\operatorname{Tr}(D_{r}GD_{r})}}U^{\dagger}_{D_{r}}D_{r}U_{D_{r}}G'U^{\dagger}_{D_{r}}D_{r}U_{D_{r}}$$

$$= \frac{1}{\operatorname{Tr}(D_{r}GD_{r})}U^{\dagger}_{D_{r}}D_{r}GD_{r}U_{D_{r}}$$
(2.116)

$$\implies \frac{1}{\sqrt{\operatorname{Tr}(D'_{r}G'D'_{r})}} U_{D_{r}}G'U^{\dagger}_{D_{r}} = \frac{1}{\sqrt{\operatorname{Tr}(D_{r}GD_{r})}}G.$$
(2.117)

Taking trace on both sides of equation (2.117) tells us that $\text{Tr}(D_r GD_r) = \text{Tr}(D'_r G'D'_r)$.

This implies that

$$G' = U_{D_{r}}^{\dagger} G U_{D_{r}} \Longrightarrow |\widetilde{\psi}_{i}'\rangle = \sum_{\substack{j \in I \\ j \notin I}} \left(U_{D_{r}} \right)_{ji} |\widetilde{\psi}_{j}\rangle, \; \forall \; \underline{i} \in I,$$
(2.118)

which implies that

$$p_{i}\rho_{i} = \sum_{\substack{i \in I_{i} \\ i \neq j}} |\widetilde{\psi}_{i}\rangle \langle \widetilde{\psi}_{i}| |$$

$$= \sum_{\substack{i \in I_{i} \\ i \neq j}} |\widetilde{\psi}_{i}'\rangle \langle \widetilde{\psi}_{i}'| = p_{i}'\rho_{i}', \quad \forall \ 1 \le i \le m,$$

$$(2.119)$$

which proves that $\widetilde{P}_{r} = \widetilde{P}'_{r}$. Hence $\mathscr{R}_{r}(\widetilde{P}_{r}) = \mathscr{R}_{r}(\widetilde{P}'_{r}) \Longrightarrow \widetilde{P}_{r} = \widetilde{P}'_{r}$. Hence \mathscr{R}_{r} is one-to-one.

 \mathscr{R}_{r} being one-to-one and onto means that it is a bijection.

The steps (i), (ii) amd (iii) in theorem 2.5.8 give us the action of $\mathscr{R}_{r_{j}}^{-1}$ on any ensemble $\widetilde{Q}_{r_{j}}$ in $\mathscr{E}_{r_{j}}$.

2.5.2 Relation between MED for ensembles of LI pure states and LI mixed states

In the beginning of subsection (2.5.1) it was mentioned that MED is a rotationally covariant problem. In the previous subsection we stripped the problem of MED for ensembles of LI states from its rotational covariance, which left us with the rotationally invariant necessary and sufficient condition **A**. **A** demands that we find a block diagonal positive definite matrix D_r , of the form given by the RHS of equation (2.75), for any $n \times n$ gram matrix *G*, where *G* is associated with some ensemble $\widetilde{P}_r \in \mathcal{E}_r$.

Definition 2.5.6. \mathcal{G} is the set of all $n \times n$ positive definite matrices with trace one.

Hence, \mathcal{G} is the set of all gram matrices which one can correspond any ensemble in $\mathcal{E}_{\vec{r}}$ with, for arbitrary \vec{r} , such that $\sum_{i=1}^{m} r_i = n$.

Definition 2.5.7. Let $\mathscr{R}_{\underline{r}}^{(\mathcal{G})} : \mathcal{G} \longrightarrow \mathcal{G}$ be a bijection so that

$$\mathscr{R}_{r}^{(G)}(G) = \frac{1}{\mathrm{Tr}(D_r G D_r)} D_r G D_{r}, \qquad (2.120)$$

where *G* is the gram matrix of the vectors $\{|\widetilde{\psi}_{ij}\rangle\}_{i\in I}$, which gives a pure state decomposition of the states $p_i\rho_i$ (equation (2.62a)), and D_r is that solution for the MED of \widetilde{P}_r which corresponds to the aforementioned pure state decomposition of the states $p_i\rho_i$.

 $\mathscr{R}_{\underline{r}}^{(g)}$ reproduces the action of $\mathscr{R}_{\underline{r}}$ on $\mathscr{E}_{\underline{r}}$, at the level of gram matrices. Since $\mathscr{R}_{\underline{r}}$ is a bijection on $\mathscr{E}_{\underline{r}}$, $\mathscr{R}_{\underline{r}}^{(g)}$ is a bijection on \mathscr{G} . Note that \mathscr{G} can be partitioned into an equivalence class of gram matrices so that any ensemble $\widetilde{P}_{\underline{r}}$ in $\mathscr{E}_{\underline{r}}$ is represented by a single class of gram matrices, each gram matrix of which, corresponds to the LI pure states which the $p_i\rho_i$'s can be decomposed into (e.g. equation (2.62a)). Hence gram matrices in the same class are related to each other by an $n \times n$ unitary matrix $U_{D_{\underline{r}}}$ of the form given in the RHS of equation (2.72). Theorem 2.5.2 tells us that

$$\mathscr{R}_{\underline{r}}^{(\mathcal{G})}(U_{D_{\underline{r}}}^{\dagger}GU_{D_{\underline{r}}}) = U_{D_{\underline{r}}}^{\dagger}\mathscr{R}_{\underline{r}}^{(\mathcal{G})}(G)U_{D_{\underline{r}}}.$$
(2.121)

Thus $\mathscr{R}_{r}^{(G)}$ maps all gram matrices from one class to gram matrices in another class.

We will see that the comparison of actions of $\mathscr{R}_{\underline{r}}^{(\mathcal{G})}$ and $\mathscr{R}_{\underline{l}}^{(\mathcal{G})}$ on \mathcal{G} will establish the relation between the MED of ensembles in $\mathcal{E}_{\underline{r}}$ and $\mathcal{E}_{\underline{l}}$. Define $\mathcal{G}^{\star} \subset \mathcal{G}$ to be such that if $\mathcal{G}^{\star} \in \mathcal{G}^{\star}$, then the solution $D_{\underline{r}}$ and $D_{\underline{l}}$ for \mathcal{G}^{\star} are equal. This also implies that $\mathscr{R}_{\underline{r}}^{(\mathcal{G})}(\mathcal{G}^{\star}) = \mathscr{R}_{\underline{l}}^{(\mathcal{G})}(\mathcal{G}^{\star})$.

Theorem 2.5.9. $G^* \in \mathcal{G}^*$ if and only if the solution D_r is diagonal.

Proof. **ONLY IF part:** We have to prove that $G^* \in \mathcal{G}^*$ implies D_r is diagonal. Note that $G^* \in \mathcal{G}^*$ implies that $D_r = D_1$ for G^* , where D_1 is diagonal.

IF part: We have to prove that if D_r is diagonal then $G^* \in \mathcal{G}$. D_r is such that the diagonal blocks of $D_r G^* D_r$ are $(X_{11})^2, (X_{22})^2, \dots, (X_{mm})^2$, where the X_{ii} 's are also diagonal since

 $D_{\vec{r}}$ is diagonal. Hence $D_{\vec{r}}$ also satisfies the conditions for **A** when $\vec{r} = 1$. Hence $D_{\vec{r}} = D_{1}$. Hence $G^* \in \mathcal{G}^*$.

Since, for any $G^* \in \mathcal{G}^*$, $D_{\underline{r}} = D_{\underline{1}}$, we will denote such $D_{\underline{r}} (= D_{\underline{1}})$ by $D_{\underline{r}}^*$.

Theorem 2.5.10. For \widetilde{P}_r , one can find a corresponding gram matrix $G^* \in \mathcal{G}^*$ such that $\mathscr{R}_r^{(\mathcal{G})}(G^*) = \mathscr{R}_{\downarrow}^{(\mathcal{G})}(G^*).$

Proof. Let $G \in \mathcal{G}$ correspond to some pure state decomposition of states of \widetilde{P}_{r} . Let D_{r} satisfy the condition **A** corresponding to said pure state decomposition. Choose $U_{D_{r}}$ to be such that $U_{D_{r}}^{\dagger} D_{r} U_{D}$ is diagonal. Denote $U_{D_{r}}^{\dagger} G U_{D_{r}}$ by G^{\star} . Then theorems 2.5.9 and 2.5.2 imply that $\mathscr{R}_{r}^{(G)}(G^{\star}) = \mathscr{R}_{1}^{(G)}(G^{\star})$.

Let $\{|\widetilde{\psi}_{i}^{\star}\rangle\}_{i\in I}$ be a set of *n* LI vectors such that $p_{i}\rho_{i} = \sum_{i\in I_{i}} |\widetilde{\psi}_{i}^{\star}\rangle\langle\widetilde{\psi}_{i}^{\star}|$, for all $1 \leq i \leq m$, and such that the gram matrix of $\{|\widetilde{\psi}_{i}^{\star}\rangle\}_{i\in I}$, denoted by G^{\star} , lies in \mathcal{G}^{\star} . Since $\operatorname{Tr}(G^{\star}) = 1$, the set of real numbers $\{\langle\widetilde{\psi}_{i}^{\star}|\widetilde{\psi}_{i}^{\star}\rangle\}_{i\in I}$ is an *n* outcome probability, and since $G^{\star} > 0$, $\langle\widetilde{\psi}_{i}^{\star}|\widetilde{\psi}_{i}^{\star}\rangle > 0$, $\forall i \in I$. Hence $\widetilde{P}_{1}^{\star} \equiv \{\langle\widetilde{\psi}_{i}^{\star}|\widetilde{\psi}_{i}^{\star}\rangle, |\psi_{i}^{\star}\rangle\langle\psi_{i}^{\star}|\}_{i\in I}$ is a LI pure state ensemble in \mathcal{E}_{1} , where $|psi_{i}^{\star}\rangle = \frac{1}{\sqrt{\langle\widetilde{\psi}_{i}^{\star}|\widetilde{\psi}_{i}^{\star}\rangle}, \forall i \in I$. Note that G^{\star} is a gram matrix associated with $\widetilde{P}_{1}^{\star}$.

We want to associate $\mathscr{P}_{r}(\widetilde{P}_{r})$ with $\mathscr{P}_{1}(\widetilde{P}_{1}^{\star})$. For this, first note that $(D_{r}^{\star})^{-1}(D_{r}^{\star}G^{\star}D_{r}^{\star})^{\frac{1}{2}}$ is a (generally, non-hermitian) square root of G^{\star} , in the sense that

$$\left((D_{r,}^{\star})^{-1}(D_{r,}^{\star}G^{\star}D_{r,}^{\star})^{\frac{1}{2}}\right)\left((D_{r,}^{\star})^{-1}(D_{r,}^{\star}G^{\star}D_{r,}^{\star})^{\frac{1}{2}}\right)^{\dagger}=G^{\star}.$$

Define

$$|w_{\underline{j}}^{\star}\rangle = \sum_{\underline{j}\in\overline{I}} \left((D_{\underline{r}}^{\star})^{-1} (D_{\underline{r}}^{\star}G^{\star}D_{\underline{r}}^{\star})^{\frac{1}{2}} \right)_{\underline{j}\underline{i}} |\widetilde{u}_{\underline{j}}^{\star}\rangle, \qquad (2.122)$$

where $\langle \widetilde{\psi}_{i}^{\star} | \widetilde{u}_{j}^{\star} \rangle = \delta_{ij}, \forall i, j \in I$. Thus $\{|w_{i}^{\star}\rangle\}_{i \in I}$ is an ONB for \mathcal{H} . Also define $\Pi_{i}^{\star} \equiv \sum_{i \in I_{i}} |w_{i}^{\star}\rangle \langle w_{i}^{\star}|$. Note that $\{\Pi_{i}^{\star}\}_{i=1}^{m} \in \mathcal{P}_{r}$.

Theorem 2.5.11. $\mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}}) = \{\Pi_i^{\star}\}_{i=1}^m$, and $\mathscr{P}_{\underline{l}}(\widetilde{P}_{\underline{l}}^{\star}) = \{|w_{\underline{l}}^{\star}\rangle\langle w_{\underline{l}}^{\star}|\}_{\underline{l}\in I}$.

Proof. Since $\{\Pi_i^{\star}\}_{i=1}^m \in \mathcal{P}_{\underline{r}}$, and $\{|w_{\underline{i}}^{\star}\rangle\langle w_{\underline{i}}^{\star}|\}_{\underline{i}\in I} \in \mathcal{P}_{\underline{i}}$, if we proved that $\sum_{i=1}^m p_i \rho_i \Pi_i^{\star} > 0$ and $\sum_{\underline{i}\in I} \langle \widetilde{\psi}_{\underline{i}}^{\star} | \widetilde{\psi}_{\underline{i}}^{\star} \rangle | \psi_{\underline{i}}^{\star}\rangle \langle \psi_{\underline{i}}^{\star} | w_{\underline{i}}^{\star}\rangle \langle w_{\underline{i}}^{\star} | > 0$, then by corollary 2.5.3.1 we would have proved this theorem.

We start with the proof of $\sum_{i=1}^{m} p_i \rho_i \prod_i^* > 0$.

$$\sum_{i=1}^{m} p_{i}\rho_{i}\Pi_{i}^{\star}$$

$$= \sum_{i=1}^{m} \left(\sum_{\substack{i \in \mathcal{I}_{i} \\ i \notin \mathcal{I}_{i} \notin \mathcal{I}_{i} \notin \mathcal{I}_{i} \notin \mathcal{I}_{i} \notin \mathcal{I}_{i} \notin \mathcal{I}_{i} = 1} \left| \left((D_{j}^{\star})^{-1} (D_{j}^{\star}G^{\star}D_{j}^{\star})^{\frac{1}{2}} \right)_{jl} \left((D_{j}^{\star}G^{\star}D_{j}^{\star})^{\frac{1}{2}} (D_{j}^{\star})^{-1} \right)_{jk} |\widetilde{u}_{j}^{\star}\rangle \langle \widetilde{u}_{k}^{\star}| \right)$$

$$(2.123a)$$

Summing over all $j \in I$ we get

$$\sum_{\substack{j\in\mathcal{I}\\ \downarrow}} \left((D_{\underline{r}}^{\star})^{-1} (D_{\underline{r}}^{\star} G^{\star} D_{\underline{r}}^{\star})^{\frac{1}{2}} \right)_{\underbrace{jl}} \langle \widetilde{\psi}_{\underline{j}}^{\star} | \widetilde{u}_{\underline{j}}^{\star} \rangle = \left((D_{\underline{r}}^{\star})^{-1} (D_{\underline{r}}^{\star} G^{\star} D_{\underline{r}}^{\star})^{\frac{1}{2}} \right)_{\underbrace{il}}.$$

Note that $\underline{i}, \underline{l} \in I_i$. Hence $\left((D_{\underline{r}}^{\star})^{-1}(D_{\underline{r}}^{\star}G^{\star}D_{\underline{r}}^{\star})^{\frac{1}{2}}\right)_{\underline{i}\underline{l}}$ is a matrix element of a diagonal block of $(D_{\underline{r}}^{\star})^{-1}(D_{\underline{r}}^{\star}G^{\star}D_{\underline{r}}^{\star})^{\frac{1}{2}}$, which implies that it is equal to $\left(D_{\underline{r}}^{\star}\right)_{\underline{i}\underline{l}}$. But note that $D_{\underline{r}}^{\star}$ is diagonal. Hence upon summing over $\underline{l} \in I_i$ gives $\sum_{\underline{l} \in I_i} \left(D_{\underline{r}}^{\star}\right)_{\underline{i}\underline{l}} = \left(D_{\underline{r}}^{\star}\right)_{\underline{i}\underline{l}}$. Thus we get

$$\sum_{i=1}^{m} p_{i}\rho_{i}\Pi_{i}^{\star}$$

$$= \sum_{i=1}^{m} \sum_{\substack{k \in \mathcal{I} \\ j \neq i}} \sum_{\substack{i \in \mathcal{I}_{i} \\ j \neq i}} \left(D_{r}^{\star} G^{\star} D_{r}^{\star} \right)^{\frac{1}{2}} (D_{r}^{\star})^{-1} \Big)_{\substack{ik \\ j \neq i}} |\widetilde{\Psi}_{i}^{\star} \rangle \langle \widetilde{u}_{k}^{\star} |$$

$$(2.123b)$$

Note that $|\widetilde{u}_{k}^{\star}\rangle = \sum_{j \in I} \left((G^{\star})^{-1} \right)_{jk} |\widetilde{\psi}_{j}^{\star}\rangle.$

Thus we get

$$\sum_{\substack{k \in I \\ \downarrow \notin I}} \left((D_{r}^{\star}G^{\star}D_{r}^{\star})^{\frac{1}{2}} (D_{r}^{\star})^{-1} \right)_{ik} \langle \widetilde{u}_{k}^{\star} |$$

$$= \sum_{\substack{k \in I \\ \downarrow \notin I}} \left((D_{r}^{\star}G^{\star}D_{r}^{\star})^{\frac{1}{2}} (D_{r}^{\star}G^{\star}D_{r}^{\star})^{-1} \right)_{ik} (D_{r}^{\star})_{kk} \langle \widetilde{\psi}_{k}^{\star} |$$

$$= \sum_{\substack{k \in I \\ \downarrow \notin I}} \left((D_{r}^{\star}G^{\star}D_{r}^{\star})^{-\frac{1}{2}} \right)_{ik} (D_{r}^{\star})_{kk} \langle \widetilde{\psi}_{k}^{\star} |.$$

Thus we get

$$\sum_{i=1}^{m} p_{i}\rho_{i}\Pi_{i}^{\star}$$

$$= \sum_{\substack{i,k\in \mathcal{I}\\ i\neq j}} \left(D_{r_{j}}^{\star} \right)_{ii} \left((D_{r_{j}}^{\star}G^{\star}D_{r_{j}}^{\star})^{-\frac{1}{2}} \right)_{ik} \left(D_{r_{j}}^{\star} \right)_{kk} |\widetilde{\psi}_{i}^{\star}\rangle \langle \widetilde{\psi}_{k}^{\star}| > 0.$$

$$(2.123c)$$

This proves that $\mathscr{P}_{r}(\widetilde{P}_{r}) = \{\Pi_{i}^{\star}\}_{i=1}^{m}$. Since $\mathscr{P}_{r}(\widetilde{P}_{r}) = \{\Pi_{i}^{\star}\}_{i=1}^{m}$, we remove the ' \star ' symbol from the Π_{i}^{\star} 's.

Using similar arguments one can prove that

$$\sum_{\substack{i \in I \\ j \neq i}} \langle \widetilde{\psi}_{i}^{\star} | \widetilde{\psi}_{i}^{\star} \rangle | \psi_{i}^{\star} \rangle \langle \psi_{i}^{\star} | w_{i}^{\star} \rangle \langle w_{i}^{\star} |$$

$$= \sum_{\substack{i,k \in I \\ j \neq i}} \left(D_{r}^{\star} \int_{i}^{j} \left((D_{r}^{\star} G^{\star} D_{r}^{\star})^{-\frac{1}{2}} \right)_{ik} \left(D_{r}^{\star} \int_{i}^{j} \langle \widetilde{\psi}_{i}^{\star} \rangle \langle \widetilde{\psi}_{k}^{\star} | > 0. \right)$$

$$(2.124)$$

Hence $\mathscr{P}_{\underline{l}}(\widetilde{P}_{\underline{l}}^{\star}) = \{ |w_{\underline{l}}^{\star} \rangle \langle w_{\underline{l}}^{\star} | \}_{\underline{l} \in I}$. This proves the theorem.

Thus, $\mathscr{P}_{\underline{l}}(\widetilde{P}_{\underline{l}}^{\star})$ and $\mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}})$ are related by the fact that the projectors $\{|w_{\underline{l}}^{\star}\rangle\langle w_{\underline{l}}^{\star}|\}_{\underline{l}\in I_{i}}$ give a spectral decomposition for $\Pi_{i}, \forall 1 \leq i \leq m$. The significance of this lies in the fact that the optimal discrimination among the pure states (by measuring with the projective measurement $\mathscr{P}_{\underline{l}}(\widetilde{P}_{\underline{l}}^{\star})$) subsumes the optimal discrimination of the mixed states (by measuring with the projective measurement $\mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}})$). This establishes the final theorem of

this subsection.

Theorem 2.5.12. There is a pure state decomposition of the $p_i \rho_i$'s: $p_i \rho_i = \sum_{i \in I_i} |\widetilde{\psi}_i^{\star}\rangle \langle \widetilde{\psi}_i^{\star}|$, $\forall 1 \le i \le m$, and a corresponding spectral decomposition of the Π_i 's: $\Pi_i = \sum_{i \in I_i} |w_{i}^{\star}\rangle \langle w_{i}^{\star}|$, $\forall 1 \le i \le m$, such that the rank-one projective measurement $\{|w_{i}^{\star}\rangle \langle w_{i}^{\star}|\}_{i \in I}$ is the optimal POVM for the MED of the LI pure state ensemble $\{\langle \widetilde{\psi}_i^{\star} | \widetilde{\psi}_i^{\star}\rangle, |\psi_{i}^{\star}\rangle \langle \psi_i^{\star}|\}_{i \in I}$, where $|\psi_{i}^{\star}\rangle \equiv \frac{1}{\sqrt{\widetilde{\psi}_i^{\star} |\widetilde{\psi}_i^{\star}\rangle}} |\widetilde{\psi}_{i}^{\star}\rangle, \forall i \in I$.

2.5.3 Solution for the MED problem

In this subsection we show how equation (2.77) can be used to obtain the solution for the MED of any ensemble in $\mathcal{E}_{\underline{r}}$. Consider an ensemble $\widetilde{P}_{\underline{r}}^{(1)} = \{p_i^{(1)}, \rho_i^{(1)}\}_{i=1}^m \in \mathcal{E}_{\underline{r}}$, where $p_i^{(1)}\rho_i^{(1)} = \sum_{\underline{i} \in I_i} \lambda_{\underline{i}}^{(1)} |\psi_{\underline{i}}^{(1)}\rangle \langle \psi_{\underline{i}}^{(1)}|$ is a pure state decomposition of $p_i^{(1)}\rho_i^{(1)}$ into r_i LI pure states. Let $G^{(1)}$ be the gram matrix of the set of vectors $\{\sqrt{\lambda_{\underline{i}}^{(1)}} |\psi_{\underline{i}}^{(1)}\rangle\}_{\underline{i} \in I}$. Corresponding to this pure state decomposition of the $p_i^{(1)}\rho_i^{(1)}$'s, we want to find $D_{\underline{r}}(1)$ and $X_{\underline{r}}(1)$ which satisfies the condition **A** for the MED of $\widetilde{P}_{\underline{r}}^{(1)}$.

Consider an ensemble $\widetilde{P}_{\underline{r}}^{(0)} = \{p_i^{(0)}, \rho_i^{(0)}\}_{i=1}^m \in \mathcal{E}_{\underline{r}}, \text{ where } p_i^{(0)}\rho_i^{(0)} = \sum_{\underline{i}\in I_i} \lambda_{\underline{i}}^{(0)} |\psi_{\underline{i}}^{(0)}\rangle \langle \psi_{\underline{i}}^{(0)}|.$ Let $G^{(0)}$ be the gram matrix of the set of vectors $\{\sqrt{\lambda_{\underline{i}}^{(0)}} |\psi_{\underline{i}}^{(0)}\rangle \}_{\underline{i}\in I}$. Corresponding to this pure state decomposition, suppose that we know $D_{\underline{r}}(0)$ and $X_{\underline{r}}(0)$, which satisfies the condition **A** for the MED of $\widetilde{P}_{r}^{(0)}$.

Define

$$G(t) \equiv (1-t)G^{(0)} + tG^{(1)}.$$
(2.125)

Firstly, note that $G(0) = G^{(0)}$ and $G(1) = G^{(1)}$. Next, note that for any $t \in [0, 1]$, G(t) > 0and $\operatorname{Tr}(G(t)) = 1$. This implies that $G(t) \in \mathcal{G}$, $\forall t \in [0, 1]$. This implies that there exists some ensemble $\widetilde{P}_{\underline{r}_{\downarrow}}^{(t)} = \{p_{i}^{(t)}, \rho_{i}^{(t)}\}_{i=1}^{m} \in \mathcal{E}_{\underline{r}_{\downarrow}}$, where $p_{i}^{(t)}\rho_{i}^{(t)} = \sum_{\underline{i} \in I_{i}} \lambda_{\underline{i}}^{(t)} |\psi_{\underline{i}}^{(t)}\rangle \langle \psi_{\underline{i}}^{(t)}|$, such that G(t) is the gram matrix of the set of vectors $\{\sqrt{\lambda_{\underline{i}}^{(t)}} |\psi_{\underline{i}}^{(t)}\rangle \}_{\underline{i} \in I}$. Corresponding to this pure state decomposition, let $D_{\underline{r}}(t)$ and $X_{\underline{r}}(t)$ denote the solution for the MED of $\widetilde{P}_{\underline{r}}^{(t)}$. For convenience denote $X_{\underline{r}}(t) = (D_{\underline{r}}(t))^{2} + N_{\underline{r}}(t)$, where $N_{\underline{r}}(t)$ is a hermitian matrix, whose diagonal blocks are 0. We rewrite equation (2.77) in the following form.

$$\left(\left(D_{r}(t)\right)^{2} + N_{r}(t)\right)^{2} - D_{r}(t)G(t)D_{r}(t) = 0.$$
(2.126)

Taylor Series Method

Differentiating with respect to t we get

$$\begin{pmatrix} D_{\underline{r}}(t)^2 + N_{\underline{r}}(t) \end{pmatrix} \begin{pmatrix} 2D_{\underline{r}}(t) \frac{dD_{\underline{r}}(t)}{dt} + \frac{dN_{\underline{r}}(t)}{dt} \end{pmatrix} + \begin{pmatrix} 2D_{\underline{r}}(t) \frac{dD_{\underline{r}}(t)}{dt} + \frac{dN_{\underline{r}}(t)}{dt} \end{pmatrix} \begin{pmatrix} D_{\underline{r}}(t)^2 + N(t) \end{pmatrix} = 0,$$

$$- (D_{\underline{r}}(t)G(t)) \frac{dD_{\underline{r}}(t)}{dt} - D_{\underline{r}}(t)\Delta D_{\underline{r}}(t) - \frac{dD_{\underline{r}}(t)}{dt}G(t)D_{\underline{r}}(t)$$

$$(2.127)$$

where $\Delta \equiv \frac{dG(t)}{dt} = G^{(1)} - G^{(0)}$. Assume that we know $D_r(t)$ and $N_r(t)$, for some $t \in [0, 1]$. Substituting these values in LHS of equation (2.127) turns it into a set of n^2 simultaneous linear equations in n^2 unknowns, which are the block diagonal matrix elements of $\frac{dD_r(t)}{dt}$ and the off block diagonal matrix elements of $37 \frac{dN_r(t)}{dt}$. This set of n^2 simultaneous linear equations in n^2 unknown has a unique solution, which can be easily computed. The justification for this statement will be given later.

Next, upon taking the derivative of the LHS and RHS of equation (2.127) with respect to t again gives us an equation whose LHS contains the terms $D_r(t)$, $N_r(t)$, G(t), $\frac{dD_r(t)}{dt}$, $\frac{dN_r(t)}{dt}$, Δ , $\frac{d^2 D_r(t)}{d^2 t}$, $\frac{d^2 N_r(t)}{dt^2}$. Substituting the values of $D_r(t)$, $N_r(t)$, G(t), $\frac{dD_r(t)}{dt}$, $\frac{dN_r(t)}{dt}$ and Δ into the LHS of this equation, gives us another set of n^2 linear equations in n^2 unknowns, which are the matrix elements of $\frac{d^2 D_r(t)}{d^2 t}$, $\frac{d^2 N_r(t)}{dt^2}$. Again, this set of linear equations has a unique solution.

Similarly, one can obtain the general *N*-th order derivatives of $D_{\underline{r}}(t)$ and $N_{\underline{r}}(t)$ with respect to *t*.

³⁶ This is because the off-block diagonal elements of $D_r(t)$ are 0, for all $t \in [0, 1]$.

³⁷ This is because the block diagonal elements of $N_r(t)$ are 0, for all $t \in [0, 1]$.

One can use these derivatives to Taylor expand $D_r(t+\delta)$ and $N_r(t+\delta)$ about the point *t*. Now note that we know the solution $D_r(0)$ and $N_r(0)$ at t = 0. Thus, our algorithm to obtain $D_r(1)$ and $N_r(1)$ begins with obtaining the Taylor expansion for $D_r(.)$ and $N_r(.)$ about the point t = 0 and analytically continuing these functions until t = 1. It is generally sufficient to divide the [0, 1] interval into³⁸ $\lceil n^2 ||\Delta||_2 \rceil$ subintervals, and analytically continue the $D_r(.)$ and $N_r(.)$ functions at the end point of each subinterval. This allows us to finally obtain $D_r(1)$ and $N_r(1)$.

In subsection 2.4.2 (and subsubsection 4.3.2 in [51]) we developed the same algorithm to obtain the Taylor series expansion of the $D_{\underline{1}}(.)$ and $N_{\underline{1}}(.)$ functions, for the special case where $\underline{r} = \underline{1}$, i.e., when m = n and $r_i = 1$, $\forall 1 \leq i \leq n$. The theory underpinning this algorithm is based using the analytic implicit function theorem to show that $D_{\underline{1}}(.)$ and $N_{\underline{1}}(.)$ are analytic functions on \mathcal{G} , and that the aforementioned algorithm is justifiably usable to obtain the Taylor series expansion for $D_{\underline{1}}(.)$ and $N_{\underline{1}}(.)$ about *any* point in \mathcal{G} . The generalization of this theory to the general \underline{r} case is straightforward and is hence not detailed here.

Newton-Raphson Method

Newton Raphson's method is a well-known technique to solve a set of simultaneous nonlinear equations. In this case the set of equations we want to solve are given by the matrix equation (2.126), where t = 1 and the unknowns are $D_r(1)$ and $N_r(1)$. We initiate Newton's method with the initial point $D_r(t) = D_r(0)$ and $N_r(t) = N_r(0)$, which upon substituting in the LHS of equation (2.126) gives $\left(\left(D_r(0)\right)^2 + N_r(0)\right)^2 - D_r(0)G(1)D_r(0) \neq$ 0. The algorithm of the technique is elaborated in subsection (2.4.2) (see the paragraph titled "Algorithm 2: Newton's Method", right above equation (2.53)), hence it won't be reproduced here.

³⁸ See the paragraph: "Starting points which generally require analytic continuation", after equation (2.52), in subsection 2.4.2, or see subsubsection 4.3.2 in [51].

Comparing Different Techniques

Another technique to solve the MED problem any ensemble in $\mathcal{E}_{r_{i}}$ is the standard barrier type interior point method employed in semidefinite programming problems [56] (see subsection 2.4.2, see the paragraph titled "Algorithm 3: Barrier-type IPM (SDP)"). The computational complexity of this technique is $O(n^{6})$ (see table 2.4.2). It is seen that the worst case computational complexity of the Taylor series algorithm is $O(n^{8})$, but for most ensembles in $\mathcal{E}_{r_{i}}$ the computational complexity is $O(n^{6})$ (see table (2.4.2)). The computational complexity for Newton's Raphson's method is $O(n^{6})$ (see table (2.4.2)). Newton-Raphson's method is also the simplest to implement. Hence for a one-time solution of the MED of some ensemble in $\mathcal{E}_{r_{i}}$, employing the Newton-Raphson's method is more desirable than the other methods mentioned here.

2.5.4 Summary

The necessary and sufficient conditions which the POVM elements of the optimal POVM have to satisfy, where simplified. These simplified conditions were then used to obtain rotationally invariant necessary and sufficient conditions for the MED of an ensemble. This allowed us to establish that there exists a bijective function \mathscr{R}_r which maps an ensembles $\widetilde{P}_r \in \mathscr{E}_r$ to another ensemble $\widetilde{Q}_r \in \mathscr{E}_r$ such that the PGM of \widetilde{Q}_r is the optimal POVM of for the MED of \widetilde{P}_r . We also obtained a closed form expression for \mathscr{R}_r^{-1} . This is a generalization of a similar result that was hitherto only proved for LI pure state ensembles in [1, 2, 3]. The rotationally invariant conditions were then exploited to show two things.

i) The MED of a mixed state ensemble $\widetilde{P}_{\underline{r}} = \{p_i, \rho_i\}_{i=1}^m \in \mathcal{E}_{\underline{r}}$ is related to the MED of a pure state ensemble $\widetilde{P}_{\underline{1}}^{\star} = \{\langle \widetilde{\psi}_{\underline{i}}^{\star} | \widetilde{\psi}_{\underline{i}}^{\star} \rangle, |\psi_{\underline{i}}^{\star} \rangle \langle \psi_{\underline{i}}^{\star} | \}_{\underline{i} \in I} \in \mathcal{E}_{\underline{1}}, \text{ where } p_i \rho_i = \sum_{\underline{i} \in I_i} |\widetilde{\psi}_{\underline{i}}^{\star} \rangle \langle \widetilde{\psi}_{\underline{i}}^{\star} |, \forall 1 \leq i \leq m, \text{ in the following way: if } \mathscr{P}_{\underline{r}}(\widetilde{P}_{\underline{r}}) = \{\Pi_i\}_{i=1}^m, \text{ and } \mathscr{P}_{\underline{1}}(\widetilde{P}_{\underline{1}}^{\star}) = \{|w_{\underline{i}}^{\star} \rangle \langle w_{\underline{i}}^{\star} |\}_{\underline{i} \in I}, \text{ then } \Pi_i = \sum_{\underline{i} \in I_i} |w_{\underline{i}}^{\star} \rangle \langle w_{\underline{i}}^{\star} |, \forall 1 \leq i \leq m. \text{ Thus the optimal discrimination of states in } \widetilde{P}_{\underline{1}}^{\star} \text{ subsumes optimally discriminating states in } \widetilde{P}_{\underline{i}}^{\star}.$ ii) We employ this rotationally invariant form of the necessary and sufficient conditions in a technique which gives us the optimal POVM for an ensemble. Our technique is found to be as computationally efficient as a standard SDP technique.

2.6 Summary

Inspired by work done earlier by Belavkin, Maslov and Mochon [1, 2, 3], we simplified the necessary and sufficient conditions for a POVM to be the optimal POVM for the MED of LI states. Also, MED being a rotationally covariant problem, we obtained the rotationally invariant versions of the aforementioned necessary and sufficient conditions, and then used these rotationally invariant conditions along with the well known implicit functional theorem from functional analysis to give a simple and efficient algorithm to compute the optimal POVM for the MED of an ensemble of LI states. We show that the efficiency of this algorithm is at par with the efficiency of a standard barrier-type SDP algorithm.

2.6.1 Future Directions

I would like to explore if it is possible to extend the results obtained above to other problems which are variants of the MED problem. For instance, a significant open problem in QIT is to establish if the maximum success probability of states $|\psi_1\rangle^{\otimes N}$, $|\psi_2\rangle^{\otimes N}$, \cdots , $|\psi_n\rangle^{\otimes N}$ can be obtained asymptotically as $N \longrightarrow \infty$, by performing measurements on the individual copies separately, rather than performing collective measurements on all Ncopies simultaneously. I anticipate that the rotationally invariant versions of the necessary and sufficient conditions can give us results for the case when the states $\{|\psi_i\rangle\}_{i=1}^n$ are LI.

Chapter 3

Local Distinguishability of States

A common scenario encountered in quantum information theory and processing is one where two or more physically separated parties share some set of bipartite or multipartite (with more than two subsystems) states, on which they are supposed to perform certain tasks. The nature of these tasks is as follows: each party of the whole system can perform quantum operations on his/her subsystem of the joint quantum system and, classically, communicate to his/her peers about the quantum operation, for e.g. tell them what operation was performed and what the outcome of the operation was. An LOCC protocol is a protocol of local quantum operations which the various parties agree to perform to achieve a certain task. Here LOCC stands for local operations and classical communication. Some significant tasks which two parties can accomplish using LOCC are entanglement distillation [58], quantum key distribution [59], quantum teleportation [37]. In fact, the classification of any quantum state as being separable or entangled is based on whether there exists an LOCC protocol by which said state can be prepared or not. This exemplifies the significance of LOCC in quantum information processing. Needless to say, there are many tasks which can be performed only jointly on the joint quantum system, i.e., these tasks cannot be achieved by any LOCC protocol. Thus, understanding what can and what can't be accomplished by LOCC protocols is a significant area of study in quantum

information theory and processing. The distinguishability of multipartite quantum states is one such task.

Developments in local distinguishability of quantum states: Among the first papers on this topic was one by Bennet et al. [38], which showed that nine orthogonal product states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ cannot be perfectly distinguished by LOCC. This exemplifies the intriguing phenomenon of non-locality without entanglement. In another celebrated paper Walgate et al. [12] showed that any two pure orthogonal multipartite states can be perfectly distinguished. In particular they showed that any pure orthogonal bipartite states can be distinguished by one-way LOCC. Fan [60] showed that, for $\mathbb{C}^n \otimes \mathbb{C}^n$ systems, when n is a prime number and m is a positive integer such that $m(m-1) \leq 2n$, then any *m* number of mutually orthogonal Generalized Bell states are perfectly distinguishable by LOCC only. The question of perfect local discrimination of pairwise orthogonal Generalized Bell states in $\mathbb{C}^n \otimes \mathbb{C}^n$ was later raised by Ghosh et al. [61] for general n. They showed that *no* set of *m* number of Generalized Bell states in $\mathbb{C}^n \otimes \mathbb{C}^n$ can be perfectly distinguished by LOCC if m > n. In the context of *general* maximally entangled states (MES) in $\mathbb{C}^n \otimes \mathbb{C}^n$, it is known that no set of m > n pairwise orthogonal MES in $\mathbb{C}^n \otimes \mathbb{C}^n$ are perfectly distinguishable by LOCC [9]. Moreoever, as a general result, it has been shown in [9] that any three pairwise orthogonal MES in $\mathbb{C}^3 \otimes \mathbb{C}^3$ are perfectly distinguishable by LOCC, whose generalization in higher dimensions was open till Yu et al [62] provided the first example of a set of four pairwise orthogonal ququad-ququad states (namely, MES of the form $\frac{1}{2} \sum_{i=0}^{1} |ij\rangle \otimes (\sigma_{\alpha}|i\rangle \otimes \sigma_{\beta}|j\rangle)$, where $\sigma_{\alpha}, \sigma_{\beta} \in \{\mathbb{1}_{2}, \sigma_{x}, \sigma_{y}, \sigma_{z}\})$ which are not perfectly distinguisable by LOCC. In fact, they have shown that these four states are not distinguisable even by PPT POVM - an operation more general than LOCC.

Despite significant advances in the topic of local distinguishability of quantum states, very few results are independent of the dimension of the systems for which they are proven. Underlying our curiosity about the local (in)distinguishability of quantum states, is the belief that results independent of dimension do exist, and we sought to find a few such

results.

Maximally Entangled States

In the following we give a definition of Maximally Entangled States (MES).

Definition 3.0.1. Maximally Entangled States: In a bipartite quantum system *AB*, where both subsystems *A* and *B* are *d* dimensional, a quantum state $|\psi\rangle$ is said to be a MES if it has the form

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B, \qquad (3.1)$$

where $\{|i\rangle_{A}\}_{i=0}^{d-1}$ and $\{|i\rangle_{B}\}_{i=0}^{d-1}$ are ONB for systems A and B respectively.

MES have acquired an elevated status in quantum information. This is because they play a distinguished role in some very significant applications of quantum information theory, for example: quantum teleportation [37], quantum superdense coding [63], giving a standard for quantifying entanglement [58], etc. Consequentially, when properties ascribable to an ensemble of quantum states are studied, there is a special focus on those cases wherein the ensemble comprises of MES. It is for this reason that perfectly local distinguishability of MES is a significant problem.

3.1 Necessary condition for LOCC of MES

I start by giving a brief introduction to entropic quantities which are relevant to the work.

3.1.1 Prerequisites: Classical and Quantum Entropic Quantities and Their Significance

In 1948, Claude E. Shannon published an article titled "A Mathematical Theory of Communication", in two parts [64, 65]. The content of this article was to be the founding basis for information theory. The transmission of information from a sender (say, Alice) to a receiver (say, Bob) is based on the model that Alice and Bob share some channel, which Alice can use to send Bob messages with. This is achieved via the following: Alice can pick one of *m* different letters in an alphabet, $\{x_j\}_{j=1}^m$, and send the letter to Bob via the channel. Using said channel *N* times successively allows Alice to send Bob a string of *N* letters of the form $x_{i_1}x_{i_2}\cdots x_{i_N}$ where $i_1, i_2, \cdots, i_N \in \{1, 2, \cdots, m\}$. Usually there is a probability p_i associated with selecting a letter from *X* which Alice would send to Bob, i.e., Alice will select x_i with the probability p_i . The alphabet with the associated probability $\tilde{P}_c \equiv \{p_i, x_i\}_{i=1}^m$ is a random variable¹. Shannon defined a mathematical quantity called Shannon Entropy, $H(\tilde{P}_c)$, for any random variable.

Definition 3.1.1. The Shannon Entropy associated with the random variable \tilde{P}_c , denoted by $H(\tilde{P}_c)$, is quantified by

$$H(\tilde{P}_c) = -\sum_{i=1}^m p_i Log_2(p_i) \quad bit.$$
(3.2)

When $p_i = 0$ for some $1 \le i \le m$, one assumes that $p_i Log_2 p_i = 0$.

The significance of Shannon Entropy lies in the fact that it quantifies the *rate* as which Alice can reliably send messages to Bob using the random variable \tilde{P}_c , as $N \rightarrow \infty$. This can be heuristically explained in the following way [66]: As $N \rightarrow \infty$ one would expect to find the alphabet x_1 occurring approximately Np_1 times, x_2 occurring approximately Np_2 times, \cdots , and x_m occurring approximately Np_m times in the N letter string. Strings of this type are known as *typical strings*. Then Shannon's noiseless channel coding theorem tells us the following:

- (i) Shannon's noiseless channel coding theorem tells us that as $N \longrightarrow \infty$, the probability that an *N*-letter string is a typical string tends to $2^{-NH(\tilde{P}_c)}$.
- (ii) As $N \rightarrow \infty$ the probability of an N letter string being atypical tends to 0, which

1

The subscript c in \tilde{P}_c stands for 'classical' random variable.

implies that as $N \longrightarrow \infty$, the probability of an N letter string being a typical string tends to 1.

(iii) Note that (i) and (ii) together imply that as $N \to \infty$, the number of distinct strings generated by the random variable is $2^{NH(\tilde{P}_c)}$.

Thus, Shannon's noiseless channel coding theorem says that as $N \longrightarrow \infty$, Alice can communicate to Bob one of $2^{NH(\tilde{P}_c)}$ different messages over N uses of the channel.

To distinguish letters received by Bob from letters sent by Alice, we denote the former by primed quantities: x'_i . Assume that Alice sends the letter x_i from her end of the channel, and Bob receives the letter x'_j at his end. If $p_{j|i}$ is the probability of such an event (where $0 \le p_{j|i} \le 1$ and $\sum_{j=1}^{m} p_{j|i} = 1$), then the probability of the event: Alice sends x_i and Bob receives x'_j is $p_{ij} = p_i p_{j|i}$. To quantify the amount of information which Alice can send Bob over this noisy channel, Shannon defined the following quantity.

Definition 3.1.2. The **Mutual Information** of two random variables x_i and x'_j with the joint probability distribution p_{ij} is given by

$$H(\tilde{P}_c:\tilde{P}_c') = H(\tilde{P}_c) + H(\tilde{P}_c') - H(\tilde{P}_c,\tilde{P}_c'), \qquad (3.3)$$

where

 $H(\tilde{P}_c)$ is the Shannon entropy of the random variable \tilde{P}_c , whose probability over the input alphabet $\{x_i\}_{i=1}^m$ is given by $p_i = \sum_{j=1}^m p_{ij}$.

 $H(\tilde{P}'_{c})$ is the Shannon entropy of the random variable \tilde{P}'_{c} , whose probability over the output alphabet $\{x_i\}_{i=1}^m$ is given by $p'_i = \sum_{i=1}^m p_{ij}$.

And $H(\tilde{P}_c, \tilde{P}'_c)$ is the Shannon entropy of \tilde{P}_c and \tilde{P}'_c jointly.

Note that $H(\tilde{P}_c : \tilde{P}'_c) \leq H(\tilde{P}_c)$. Shannon's noisy channel coding theorem says that as $N \longrightarrow \infty$, of the $2^{NH(\tilde{P}_c)} N$ letter strings which Alice can send Bob, Bob can distinguish

a maximum of $2^{NH(\tilde{P}_c:\tilde{P}'_c)}$ among them due to unfaithful transmission resulting from the channel being noisy.

Shannon's noiseless channel coding theorem and noisy channel coding theorem, give an operational definition to Shannon entropy of an random variable $H(\tilde{P}_c)$ and to mutual information $H(\tilde{P}_c : \tilde{P}'_c)$. We now give the quantum version of these quantities.

Consider a classical random variable $\tilde{P}_c = \{p_i, x_i\}_{i=1}^m$ in Alice's possession. Assume that Alice and Bob now share a quantum channel, so designed that Alice can input quantum states into the channel from one end and Bob receives quantum states at the other end. Let $\rho_1, \rho_2, \dots, \rho_m$ be *m* quantum states in a one-to-one correspondence with letters in \tilde{P}_c , such that if Alice intends to send x_i to Bob, she sends the *i*-th quantum state ρ_i . We assume that the channel Alice and Bob share is noiseless, so that Bob receives the state ρ_i at his end. To know which letter Alice meant to send him, Bob has to establish that he was sent ρ_i from the ensemble $\widetilde{P} = \{p_i, \rho_i\}_{i=1}^m$. To establish this he has to perform a quantum measurement on his state. Let the Kraus operators of Bob's measurement be $\{K_i\}_{i=1}^d$, and let the POVM operators of this measurement be $\{E_i\}_{i=1}^d$, where $E_i = K_i^{\dagger}K_i$. The input states correspond to the x_i 's and the measurement outcomes correspond to the x'_i 's of the classical channel setting. It is convenient to represent an entire event quantum mechanically, therefore we represent the x_i 's by ket vectors $|x_i\rangle_A$, which lie in some m dimensional auxiliary space in Alice's possession, with $\langle x_i | x_j \rangle = \delta_{ij}$, $\forall 1 \le i, j \le m$; and x'_i 's by $|x'_i\rangle_B$, which lie in some d dimensional auxiliary space in Bob's possession, with $\langle x'_i | x'_j \rangle = \delta_{ij}, \forall 1 \le i, j \le d$. Then the operator $p_i | x_i \rangle \langle x_i | \otimes K_j \rho_i K_j^{\dagger} \otimes | x'_j \rangle \langle x'_j |$ represents the event that Alice sent Bob the *i*-th state and Bob's measuring device yielded the *j*-th outcome. Tracing over this operator gives us the probability of the occurrence of this event: $p_{ij} = p_i \text{Tr}(\rho_i E_j)$. Note that Bob can use one of many different POVMs for the purpose of identifying the state he was sent. Shannon's noisy channel coding theorem tells us that as $N \longrightarrow \infty$, Alice can reliably send Bob one of $2^{NH(\tilde{P}_c:\tilde{P}'_c)}$ messages. Bob can increase this number by appropriately choosing his measurement so that $H(\tilde{P}_c : \tilde{P}'_c)$
is greater. The maximum value that $H(\tilde{P}_c : \tilde{P}'_c)$ can attain over the space of POVMs is known as the accessible information I_{acc} of the ensemble \tilde{P} . This is defined as follows.

$$I_{acc}(\tilde{P}) \equiv Max \left\{ H(\tilde{P}_c : \tilde{P}'_c) | \text{ over all POVMs} \{ E_i \}_{i=1}^d \right\}.$$
(3.4)

Note that owing to the maximization in equation (3.4), I_{acc} is a function of the ensemble \tilde{P} . For a random ensemble \tilde{P} , it is generally very difficult to optimize the quantity given in equation (3.4).

Definition 3.1.3. Let ρ be a density matrix with spectral decomposition $\rho = \sum_{i=1}^{m} \lambda_i |i\rangle \langle i|$, where $\langle i|j\rangle = \delta_{ij}$. Then the **von Neuman entropy** $S(\rho)$ of ρ is given by the following equation.

$$S(\rho) = -\operatorname{Tr}\left(\rho Log_2(\rho)\right),\tag{3.5}$$

where $Log_2(\rho) = \sum_{i=1}^m Log_2(\lambda_i) |i\rangle \langle i|$.

Thus, $S(\rho) = H(\{\lambda_i, x_i\}_{i=1}^m)$. Just as Shannon entropy is a functional over the space of probability distributions, the von Neumann entropy is a functional over the space of density matrices. Define the following functional on an ensemble of quantum states.

Definition 3.1.4. The Holevo bound for the ensemble $\tilde{P} = \{p_i, \rho_i\}_{i=1}^m$ is given by the expression

$$\chi(\widetilde{P}) = S(\rho) - \sum_{i=1}^{m} p_i S(\rho_i), \qquad (3.6)$$

where $\rho \equiv \sum_{i=1}^{m} p_i \rho_i$.

The Holevo bound functional is one of the pillars on which quantum information theory stands. That said, for the purpose of this thesis, it's significance lies in the fact that $\chi(\tilde{P})$ is an upper bound for $I_{acc}(\tilde{P})$, i.e.,

$$\chi\left(\widetilde{P}\right) \ge I_{acc}\left(\widetilde{P}\right). \tag{3.7}$$

3.1.2 Holevo-like Upper Bound for Locally Accessible Information of an Ensemble of Bipartite States

Consider an ensemble of bipartite quantum states $\tilde{P} = \{p_i, \rho_i^{(AB)}\}_{i=1}^m$. Assume that Alice and Bob are provided an unknown state from this ensemble, and their task is to figure out which state using the output generated from an LOCC protocol.

Let Alice start the measurement with the measurement Kraus operators $\{K_i\}_{i=1}^{d_1}$, where d_1 is the number of Kraus operators in the measurement. Let her measuring device yield the outcome $\alpha_1 \in \{1, 2, \dots, d_1\}$. Alice lets Bob know that her measurement outcome is α_1 . Conditioned upon the outcome α_1 , Bob performs a measurement on this quantum system with the Kraus operators $\{K_i(\alpha_1)\}_{i=1}^{d_2}$, where d_2 is the number of Kraus operators in his measurement. Let the measurement outcome of his measurement device be $\alpha_2 \in \{1, 2, \dots, d_2\}$. He informs Alice of his measurement outcome. Conditioned upon $q^{(2)} \equiv (\alpha_1, \alpha_2)$, let Alice perform her next measurement with the Kraus operators $\{K_i(q^{(2)})\}_{i=1}^{d_3}$, which yields the measurement outcome α_3 . After being told the measurement outcome α_3 , Bob performs his next measurement conditioned upon $q^{(3)} = (alpha_1, \alpha_2, \alpha_3)$ and so on. Assume that the LOCC protocol has a maximum of M steps. If a certain branch of the LOCC protocol has fewer than M steps in said branch. Let \mathcal{A} denote the set of all $\alpha_i = (\alpha_1, \alpha_2, \dots, \alpha_M)$, and define

$$L_{\alpha} \equiv \left(\mathbb{1}_{A} \otimes K_{\alpha_{M}}(\alpha)\right) \cdots \left(K_{\alpha_{3}}(\alpha^{(2)}) \otimes \mathbb{1}_{B}\right) \left(\mathbb{1}_{A} \otimes K_{\alpha_{2}}(\alpha_{1})\right) \left(K_{\alpha_{1}} \otimes \mathbb{1}_{B}\right),$$

where $\mathbb{1}_A$ and $\mathbb{1}_B$ are the identity operators on \mathcal{H}_A and \mathcal{H}_B . It's seen that $\{L_{\underline{\alpha}}\}_{\underline{\alpha}\in\mathcal{A}}$ are Kraus operators for a measurement on the joint system of Alice and Bob, i.e., $\sum_{\underline{\alpha}\in\mathcal{A}} L_{\underline{\alpha}}^{\dagger}L_{\underline{\alpha}} = \mathbb{1}_{AB}$, where $\mathbb{1}_{AB}$ is the identity operator on the joint *AB* system.

Thus the output generated from the LOCC protocol is the 'letter' α from the alphabet \mathcal{A} . Assuming that Alice and Bob were given $\rho_i^{(AB)}$, let the conditional probability of the

outcome $\underline{\alpha}$ be denoted by $p_{\underline{\alpha}|i}$. Let $p_{i,\underline{\alpha}}$ be the probability associated with the event that Alice and Bob are given the *i*-th state $\rho_i^{(AB)}$ and their LOCC protocol generates the output $\underline{\alpha}$. The output random variable \tilde{P}'_c is then $\{p_{i,\underline{\alpha}},\underline{\alpha}\}_{\underline{\alpha}\in\mathcal{A}}$. In a paper by Badziag et al. [8], it was shown that the mutual information between \tilde{P}_c and \tilde{P}'_c is bounded above by a Holevo-like upper bound, which we denote by χ^{LOCC} , and which is defined in the following.

Definition 3.1.5. The Holevo like upper bound for a given ensemble $\widetilde{P} = \{p_i, \rho_i^{(AB)}\}_{i=1}^m$ of *m* bipartite quantum states is quantified by

$$\chi^{LOCC}(\widetilde{P}) \equiv S\left(\rho^{(A)}\right) + S\left(\rho^{(B)}\right) - \max_{X=A,B} \left\{ \sum_{i=1}^{m} p_i S\left(\rho_i^{(X)}\right) \right\},$$
(3.8)

where

 $\rho_i^{(A)}$ is obtained by partially tracing the state $\rho_i^{(AB)}$ over system *B*, $\rho_i^{(B)}$ is obtained by partially tracing the state $\rho_i^{(AB)}$ over system *A*, $\rho^{(A)} \equiv \sum_{i=1}^m p_i \rho_i^{(A)}$, and $\rho^{(B)} \equiv \sum_{i=1}^m p_i \rho_i^{(B)}$.

Thus, the result in [8] tells us that for any ensemble of bipartite quantum states $\widetilde{P} = \{p_i, \rho_i^{(AB)}\}_{i=1}^m$, we get the following inequality:

$$\chi^{LOCC}(\widetilde{P}) \ge H\left(\widetilde{P}_c : \widetilde{P}'_c\right). \tag{3.9}$$

3.1.3 The Necessary Condition

The work done in this section has been detailed in a paper by myself, Ramij Rahaman, Sibasish Ghosh and Guruprasad Kar. This paper has been uploaded on the arxiv in [67], and has been sent for review to Journal of Physics A: Mathematical and Theoretical. Consider a set of *m* pairwise orthogonal MES $|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_m\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. Let Alice control one subsystem and Bob the other. Let $\rho_i^{(A)} = Tr_B(|\psi_i\rangle\langle\psi_i|)$ and $\rho_i^{(B)} = Tr_A(|\psi_i\rangle\langle\psi_i|)$ be the *i*-th reduced state on Alice's subsystem and Bob's subsystem respectively. Let Alice start the LOCC protocol with some measurement, whose Kraus operators are $\{K_i\}_{i=1}^d$, where $d \ge 2$ and $\sum_{i=1}^d K_i^{\dagger} K_i = \mathbb{1}_A$, where $\mathbb{1}_A$ is the identity operator acting on Alice's subsystem. Let the measurement yield the α -th outcome. Thus the post-measurement state is given by

$$|\psi_i\rangle \longrightarrow |\psi_{i,\alpha}\rangle = \frac{K_\alpha \otimes \mathbb{1}_B}{\sqrt{\langle \psi_i | K_\alpha^{\dagger} K_\alpha \otimes \mathbb{1}_B | \psi_i \rangle}} |\psi_i\rangle, \qquad (3.10)$$

for all $1 \le i \le m$, and where $\mathbb{1}_B$ is the identity operator acting on Bob's subsystem. Let the post measurement reduced states (PMRS) on Alice's and Bob's sides be denoted by $\rho_{i,\alpha}^{(A)}$ and $\rho_{i,\alpha}^{(B)}$, respectively. Then the average PMRS on Alice's and Bob's sides are $\rho_{\alpha}^{(A)} = \sum_{i=1}^{m} \frac{1}{m} \rho_{i,\alpha}^{(A)}$ and $\rho_{\alpha}^{(B)} = \sum_{i=1}^{m} \frac{1}{m} \rho_{i,\alpha}^{(B)}$, respectively, where the $\frac{1}{m}$ factor denotes the probability which with each state appears in ensemble. The apriori probability of the *i*-th state $|\psi_i\rangle$ and the post-measurement probability of the *i*-th state, conditioned upon the α -th outcome (where α can be any outcome) are equal. Note that this is not because we chose the apriori probability of the states to be $\frac{1}{m}$, but due to the fact that the $|\psi_i\rangle$'s are MES. In fact, any other choice of apriori probabilities can be made without affecting the results; our choice of $\frac{1}{m}$ is merely for convenience.

Lemma 3.1.0.1. If Alice starts the protocol to distinguish m MES by LOCC, the post measurement reduced states (PMRS) on her side are completely indistinguishable.

Proof. Since $\{|\psi_i\rangle\}_{i=1}^m$ are MES, the corresponding reduced states on Alice's subsystem are maximally mixed, i.e., $\rho_i^{(A)} = \frac{1}{n} \mathbb{1}_A$. The states on Alice's subsystem transforms as $\rho_i^{(A)} = \frac{1}{n} \mathbb{1}_A \longrightarrow \rho_{i,\alpha}^{(A)} \propto K_\alpha K_\alpha^{\dagger}, \forall 1 \le i \le m$. This implies that (even) after the first measurement, the PMRS on Alice's side are completely indistinguishable.

For the post-measurement joint states $\{|\psi_{i,\alpha}\rangle\}_{i=1}^m$ to still be distinguishable, the indistinguishability of PMRS on Alice's side imposes constraints on the average PMRS on Bob's

side. This is made clear in theorem (3.1.1).

Theorem 3.1.1. If the PMRS on Alice's side are completely indistinguishable, the von Neuman entropy of the average PMRS on Bob's side has to be at least log_2m bit for the states to be perfectly distinguishable by LOCC.

Proof. The Holevo-like upper bound for the locally accessible information of the set of states $\{|\psi_{i,\alpha}\rangle\}_{i=1}^{m}$ is given by (see equation (3.8) and the inequality (3.9))

$$I_{acc}^{LOCC} \leq S\left(\rho_{\alpha}^{(A)}\right) + S\left(\rho_{\alpha}^{(B)}\right) - Max \left\{\frac{1}{m}\sum_{i=1}^{m}S\left(\rho_{i,\alpha}^{(X)}\right) : X = A, B\right\}.$$

Globally, each of the post-measurement states are pure (equation (3.10)), which implies that the spectrum of $\rho_{i,\alpha}^{(A)}$ is equal to the spectrum of $\rho_{i,\alpha}^{(B)}$, $\forall \ 1 \le i \le m$. Also, that Alice's states are completely indistinguishable implies that $\rho_{\alpha}^{(A)} = \rho_{i,\alpha}^{(A)}$, $\forall \ 1 \le i \le m$. This implies that $I_{acc}^{LOCC} \le S(\rho_{\alpha}^{(B)})$. Since we need to distinguish between *m* different states, the aforementioned inequality tells us that we require $S(\rho_{\alpha}^{(B)})$ to be at least log_2m bit. \Box

With respect to the standard ONB $\{|j\rangle_A\}_{j=1}^n$ of Alice's system, every MES $|\psi_i\rangle$ from the shared ensemble $\{|\psi_i\rangle\}_{i=1}^m$ can be expressed as

$$|\psi_i\rangle = \frac{1}{\sqrt{n}} \sum_{j=1}^n |j\rangle_A |b_j^{(i)}\rangle_B,$$
 (3.11)

where $\{|b_j^{(i)}\rangle_B\}_{j=1}^n$ is an ONB for Bob's system for each $i = 1, 2, \dots, m$.

The *i*-th PMRS on Bob's side is then given by

$$\rho_{i,\alpha}^{(B)} = \frac{1}{\operatorname{Tr}\left(K_{\alpha}^{\dagger}K_{\alpha}\right)} \sum_{j,k=1}^{n} \langle j | K_{\alpha}^{\dagger}K_{\alpha} | k \rangle | b_{k}^{(i)} \rangle \langle b_{j}^{(i)} | = U_{i}^{(B)} \frac{K_{\alpha}^{T}K_{\alpha}^{*}}{\operatorname{Tr}\left(K_{\alpha}^{T}K_{\alpha}^{*}\right)} U_{i}^{(B)^{\dagger}}, \qquad (3.12)$$

where $U_i^{(B)}$ are $n \times n$ unitaries such that $U_i^{(B)} | j \rangle_B = | b_j^{(i)} \rangle_B$, for $j = 1, 2, \dots, n$, where $i = 1, 2, \dots, m$, and where $K_{\alpha}^T K_{\alpha}^*$ are operators on Bob's system, whose matrix elements with

respect to the ONB $\{|j\rangle_B\}_{j=1}^n$ are the same as the complex conjugate of matrix elements of Alice's POVM effect $K_{\alpha}^{\dagger}K_{\alpha}$ when represented with respect to the ONB $\{|j\rangle_A\}_{j=1}^n$. The average PMRS corresponding to the set on Bob's side is thus given by

$$\rho_{\alpha}^{(B)} = \sum_{i=1}^{m} \frac{1}{m} U_{i}^{(B)} \frac{K_{\alpha}^{T} K_{\alpha}^{*}}{\operatorname{Tr} (K_{\alpha}^{T} K_{\alpha}^{*})} U_{i}^{(B)^{\dagger}}.$$
(3.13)

We require that $\rho_{\alpha}^{(B)}$ satisfies theorem (3.1.1). This requirement puts a constraint on Alice's starting measurement.

We already know one constraint on Alice's starting measurement, i.e., it should be OP. Hence whenever $i \neq j$,

$$\langle \psi_i | K_{\alpha}^{\dagger} K_{\alpha} \otimes \mathbb{1} | \psi_j \rangle = 0.$$
(3.14)

It is easy to see that condition (3.14) should be subsumed in the requirement that $\rho_{\alpha}^{(B)}$ should satisfy theorem (3.1.1).

Consider the special case when m = n.

Corollary 3.1.1.1. If m = n in Theorem (3.1.1), then the average PMRS on Bob's side has to be maximally mixed.

Proof. When m = n, we require log_2n bit of information to distinguish between n states. The maximal value that $S(\rho_{\alpha}^{(B)})$ can take is log_2n and it can take this value *only* when $\rho_{\alpha}^{(B)}$ is maximally mixed.

Thus, requiring that $S(\rho_{\alpha}^{(B)})$ be at least $log_2(n)$ bit implies that $\rho_{\alpha}^{(B)}$ has to be a maximally *mixed state*, i.e., we require that

$$\sum_{i=1}^{n} \frac{1}{n} U_{i}^{(B)} \frac{K_{\alpha}^{T} K_{\alpha}^{*}}{\operatorname{Tr} (K_{\alpha}^{T} K_{\alpha}^{*})} U_{i}^{(B)^{\dagger}} = \frac{1}{n} \mathbb{1}_{n}.$$
(3.15)

After having imposed the condition (3.15) on the matrix elements of the effects of the POVM $\{K_i^{\dagger}K_i\}_{i=1}^d$, if the resulting POVM is trivial, i.e., if all its effects are multiples of $\mathbb{1}_A$, the set $\{|\psi_i\rangle\}_{i=1}^m$ fails the necessary condition for local distinguishability. If not, then the set of states may still be distinguishable by LOCC.

3.1.4 Example: Case When the Necessary Condition Becomes Sufficient

The necessary condition (3.15) has to be tested for protocols initiated by both Alice and Bob, separately. Consider the case of Generalized Bell states.

Definition 3.1.6. Generalized Bell states are bipartite MES in $\mathbb{C}^n \otimes \mathbb{C}^n$ of the form

$$|\psi_{lk}^{(n)}\rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{\frac{2\pi i j l}{n}} |j\rangle_A |j \oplus_4 k\rangle_B, \qquad (3.16)$$

where $l, k \in \{0, 1, \dots, n-1\}$ and where $\{|j\rangle_A\}_{j=0}^{n-1}$ is an ONB for Alice's subsystem and $\{|j\rangle_B\}_{j=0}^{n-1}$ is an ONB for Bob's subsystem.

Note that $\langle \psi_{l'k'} | \psi_{nm} \rangle = \delta_{l'l} \delta_{k'k}, \forall l, l', k, k' \in \{0, 1, \cdots, l-1\}.$

When l = 4, there are 122 local-unitarily inequivalent equivalence classes of sets of four Generalized Bell states. Testing the necessary condition (3.15) on representative sets from these 122 distinct equivalence classes we find that 39 such sets are locally indistinguishable. An explicit proof of one such a set is given in example (3.1.1).

Example 3.1.1. The states $|\psi_{00}^{(4)}\rangle$, $|\psi_{11}^{(4)}\rangle$, $|\psi_{31}^{(4)}\rangle$, $|\psi_{32}^{(4)}\rangle$ are locally indistinguishable.

Proof. In [68] (example 1, p 6) it has already been shown that the given set of states are indistinguishable by one-way LOCC using only projective measurements. Here we will generalize the result for all possible LOCC protocols. Also, we show that the condition (3.15) is stronger than the OP condition (3.14).

Let Alice commense the protocol by applying a measurement, whose Kraus operators are $\{K_i\}_{i=1}^d$ on her subsystem, and obtain the α -th outcome. We impose the conditions (3.14) on $K_{\alpha}^{\dagger}K_{\alpha}$.

The orthogonality preserving condition (3.14) is given by:

$$\langle \psi_{00}^{(4)} | \left(K_{\alpha}^{\dagger} K_{\alpha} \otimes \mathbb{1} \right) | \psi_{11}^{(4)} \rangle = 0.$$
(3.17a)

$$\langle \psi_{00}^{(4)} | \left(K_{\alpha}^{\dagger} K_{\alpha} \otimes \mathbb{1} \right) | \psi_{31}^{(4)} \rangle = 0.$$
(3.17b)

$$\langle \psi_{00}^{(4)} | \left(K_{\alpha}^{\dagger} K_{\alpha} \otimes \mathbb{1} \right) | \psi_{32}^{(4)} \rangle = 0.$$
 (3.17c)

$$\langle \psi_{11}^{(4)} | \left(K_{\alpha}^{\dagger} K_{\alpha} \otimes \mathbb{1} \right) | \psi_{31}^{(4)} \rangle = 0.$$
(3.17d)

$$\langle \psi_{11}^{(4)} | \left(K_{\alpha}^{\dagger} K_{\alpha} \otimes \mathbb{1} \right) | \psi_{32}^{(4)} \rangle = 0.$$
 (3.17e)

$$\langle \psi_{31}^{(4)} | \left(K_{\alpha}^{\dagger} K_{\alpha} \otimes \mathbb{1} \right) | \psi_{32}^{(4)} \rangle = 0.$$
 (3.17f)

Let the spectral decomposition of $K_{\alpha}^{\dagger}K_{\alpha}$ be given by

$$K_{\alpha}^{\dagger}K_{\alpha} = |u\rangle\langle u| + |v\rangle\langle v| + |w\rangle\langle w| + |x\rangle\langle x|, \qquad (3.18)$$

where $\langle u|v\rangle = \langle u|w\rangle = \langle u|x\rangle = \langle v|w\rangle = \langle v|x\rangle = \langle w|x\rangle = 0$, but $|u\rangle$, $|v\rangle$, $|w\rangle$ and $|x\rangle$ aren't normalized.

Additionally, let's $|u\rangle$, $|v\rangle$, $|w\rangle$ and $|w\rangle$ have the following expansions in the standard ONB.

$$|u\rangle = \sum_{i=0}^{3} u_i |i\rangle, \qquad (3.19a)$$

$$|v\rangle = \sum_{i=0}^{3} v_i |i\rangle, \qquad (3.19b)$$

$$|w\rangle = \sum_{i=0}^{3} w_i |i\rangle, \qquad (3.19c)$$

$$|x\rangle = \sum_{i=0}^{3} x_i |i\rangle.$$
(3.19d)

Then equations (3.17) become (using the definition (3.1.6))

$$(u_{1}u_{0}^{*} + v_{1}v_{0}^{*} + w_{1}w_{0}^{*} + x_{1}x_{0}^{*})$$

+ $i(u_{2}u_{1}^{*} + v_{2}v_{1}^{*} + w_{2}w_{1}^{*} + x_{2}x_{1}^{*})$
- $(u_{3}u_{2}^{*} + v_{3}v_{2}^{*} + w_{3}w_{2}^{*} + x_{3}x_{2}^{*})$
- $i(u_{0}u_{3}^{*} + v_{0}v_{3}^{*} + w_{0}w_{3}^{*} + x_{0}x_{3}^{*}) = 0.$ (3.20a)

$$(u_{1}u_{0}^{*} + v_{1}v_{0}^{*} + w_{1}w_{0}^{*} + x_{1}x_{0}^{*})$$

- $i(u_{2}u_{1}^{*} + v_{2}v_{1}^{*} + w_{2}w_{1}^{*} + x_{2}x_{1}^{*})$
- $(u_{3}u_{2}^{*} + v_{3}v_{2}^{*} + w_{3}w_{2}^{*} + x_{3}x_{2}^{*})$
+ $i(u_{0}u_{3}^{*} + v_{0}v_{3}^{*} + w_{0}w_{3}^{*} + x_{0}x_{3}^{*}) = 0.$ (3.20b)

$$(u_{2}u_{0}^{*} + v_{2}v_{0}^{*} + w_{2}w_{0}^{*} + x_{2}x_{0}^{*})$$

- $i(u_{3}u_{1}^{*} + v_{3}v_{1}^{*} + w_{3}w_{1}^{*} + x_{3}x_{1}^{*})$
- $(u_{0}u_{2}^{*} + v_{0}v_{2}^{*} + w_{0}w_{2}^{*} + x_{0}x_{2}^{*})$
+ $i(u_{1}u_{3}^{*} + v_{1}v_{3}^{*} + w_{1}w_{3}^{*} + x_{1}x_{3}^{*}) = 0.$ (3.20c)

$$(u_0u_0^* + v_0v_0^* + w_0w_0^* + x_0x_0^*)$$

- $(u_1u_1^* + v_1v_1^* + w_1w_1^* + x_1x_1^*)$
+ $(u_2u_2^* + v_2v_2^* + w_2w_2^* + x_2x_2^*)$
- $(u_3u_3^* + v_3v_3^* + w_3w_3^* + x_3x_3^*) = 0.$ (3.20d)

$$(u_{1}u_{0}^{*} + v_{1}v_{0}^{*} + w_{1}w_{0}^{*} + x_{1}x_{0}^{*})$$

$$-(u_{2}u_{1}^{*} + v_{2}v_{1}^{*} + w_{2}w_{1}^{*} + x_{2}x_{1}^{*})$$

$$+(u_{3}u_{2}^{*} + v_{3}v_{2}^{*} + w_{3}w_{2}^{*} + x_{3}x_{2}^{*})$$

$$-(u_{0}u_{3}^{*} + v_{0}v_{3}^{*} + w_{0}w_{3}^{*} + x_{0}x_{3}^{*}) = 0.$$
(3.20e)

$$(u_{1}u_{0}^{*} + v_{1}v_{0}^{*} + w_{1}w_{0}^{*} + x_{1}x_{0}^{*})$$

$$+(u_{2}u_{1}^{*} + v_{2}v_{1}^{*} + w_{2}w_{1}^{*} + x_{2}x_{1}^{*})$$

$$+(u_{3}u_{2}^{*} + v_{3}v_{2}^{*} + w_{3}w_{2}^{*} + x_{3}x_{2}^{*})$$

$$+(u_{0}u_{3}^{*} + v_{0}v_{3}^{*} + w_{0}w_{3}^{*} + x_{0}x_{3}^{*}) = 0.$$
(3.20f)

If we expand $K_{\alpha}^{\dagger}K_{\alpha}$ in the $|i\rangle\langle j|$ basis, obtained by using equations (3.19) we get the following

$$\xi_{ij} \equiv \left(K_{\alpha}^{\dagger}K_{\alpha}\right)_{ij} = u_i u_j^* + v_i v_j^* + w_i w_j^* + x_i x_j^*, \ \forall \ 0 \le i, j \le 3.$$
(3.21)

Using equation (3.21) we can rewrite equations (3.20) in an even more condensed form:

$$\begin{pmatrix} \xi_{01}^{*} & \xi_{12}^{*} & \xi_{23}^{*} & \xi_{30}^{*} \end{pmatrix} \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} = 0.$$
(3.22a)

$$\begin{pmatrix} \xi_{01}^{*} & \xi_{12}^{*} & \xi_{23}^{*} & \xi_{30}^{*} \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = 0.$$
(3.22b)

$$\begin{pmatrix} \xi_{02}^* & \xi_{13}^* & \xi_{02} & \xi_{13} \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = 0.$$
(3.22c)

$$\begin{pmatrix} \xi_{00} & \xi_{11} & \xi_{22} & \xi_{33} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 0.$$
 (3.22d)

$$\begin{pmatrix} \xi_{01}^{*} & \xi_{12}^{*} & \xi_{23}^{*} & \xi_{30}^{*} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 0.$$
(3.22e)
$$\begin{pmatrix} \xi_{01}^{*} & \xi_{12}^{*} & \xi_{23}^{*} & \xi_{30}^{*} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0.$$
(3.22f)

Equations (3.22a), (3.22b), (3.22e) and (3.22f) collectively imply that

$$(\xi_{01},\xi_{12},\xi_{23},\xi_{30}) = 0. \tag{3.23}$$

(3.22f)

Equation (3.22d) implies that

$$(\xi_{00}, \xi_{11}, \xi_{22}, \xi_{33}) = a_0(1, 1, 1, 1) + a_1(1, 1, -1, -1) + a_2(1, -1, -1, 1),$$
(3.24)

where a_0 , a_1 and a_2 are real. This is because ξ_{ii} are diagonal matrix elements of $K_{\alpha}^{\dagger}K_{\alpha}$. Equation (3.22b) implies that $(\xi_{02}, \xi_{13}, \xi_{20}, \xi_{31})$ has to be of the form

$$(\xi_{02}, \xi_{13}, \xi_{20}, \xi_{31}) = b_0(1, 1, 1, 1) + b_1(1, i, -1, -i) + b_2(1, -1, 1, -1).$$
(3.25)

Since $K_{\alpha}^{\dagger}K_{\alpha}$ is hermitian, $\xi_{ij} = \xi_{ji}^{*}$ must hold true. This implies that $(b_0 - b_1 + b_2)^{*} =$

 $b_0 + b_1 + b_2$ (from i = 0, j = 2) and $(b_0 + ib_1 - b_2)^* = b_0 - ib_1 - b_2$ (from i = 1, j = 3) and these imply that $b_1 = 0$ and b_0 and b_2 are real.

Thus putting the constraints imposed by equations (3.23), (3.24) and (3.25), tells us that in the $|i\rangle\langle j|$ basis $K_{\alpha}^{\dagger}K_{\alpha}$ is given by equation:

$$K_{\alpha}^{\dagger}K_{\alpha} = \begin{pmatrix} a_0 + a_1 + a_2 & 0 & b_0 + b_2 & 0 \\ 0 & a_0 + a_1 - a_2 & 0 & b_0 - b_2 \\ b_0 + b_2 & 0 & a_0 - a_1 - a_2 & 0 \\ 0 & b_0 - b_2 & 0 & a_0 - a_1 + a_2 \end{pmatrix}.$$
 (3.26)

The eigensystem of $K_{\alpha}^{\dagger}K_{\alpha}$ in equation (3.26) is given in the following table

Eigenvalue	Eigenvector
$\lambda_u = a_0 + \mu_0$	$ u\rangle = N_u \begin{pmatrix} a_1 + a_2 + \mu_0 \\ 0 \\ b_0 + b_1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \cos \frac{\zeta}{2} \\ 0 \\ \sin \frac{\zeta}{2} \\ 0 \end{pmatrix}$
$\lambda_{\nu}=a_0-\mu_0$	$ v\rangle = N_{v} \begin{pmatrix} a_{1} + a_{2} - \mu_{0} \\ 0 \\ b_{0} + b_{1} \\ 0 \end{pmatrix} \equiv \begin{pmatrix} -\sin\frac{\zeta}{2} \\ 0 \\ \cos\frac{\zeta}{2} \\ 0 \end{pmatrix}$
$\lambda_w = a_0 + \mu_1$	$ w\rangle = N_{w} \begin{pmatrix} (a_{1} - a_{2}) - \mu_{1} \\ 0 \\ b_{0} - b_{1} \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \cos \frac{\eta}{2} \\ 0 \\ \sin \frac{\eta}{2} \\ 0 \end{pmatrix}$
$\lambda_x = a_0 + \mu_1$	$ x\rangle = N_{x} \begin{pmatrix} (a_{1} - a_{2}) - \mu_{1} \\ 0 \\ b_{0} - b_{1} \\ 0 \end{pmatrix} \equiv \begin{pmatrix} -\sin\frac{\eta}{2} \\ 0 \\ \cos\frac{\eta}{2} \\ 0 \end{pmatrix}$

Table 3.1: Eigenvalues and Eigenvectors of $K_{\alpha}^{\dagger}K_{\alpha}$.

where μ_0 and μ_1 are given by $\sqrt{(a_1 + a_2)^2 + (b_0 + b_2)^2}$ and $\sqrt{(a_1 - a_2)^2 + (b_0 - b_2)^2}$ and N_u , N_v , N_w and N_x are normalization factors. For $K_{\alpha}^{\dagger}K_{\alpha}$ to be a positive semidefinite operator it is necessary that $a_0 \ge |\mu_0|, |\mu_1|$.

Using the table above,

$$K_{\alpha}^{\dagger}K_{\alpha} = a_{0}\mathbb{1} + \mu_{0} \begin{pmatrix} \cos\zeta & 0 & \sin\zeta & 0 \\ 0 & 0 & 0 & 0 \\ \sin\zeta & 0 & -\cos\zeta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mu_{1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\eta & 0 & \sin\eta \\ 0 & 0 & 0 & 0 \\ 0 & \sin\eta & 0 & -\cos\eta \end{pmatrix}.$$
(3.27)

Imposing condition (3.14) doesn't conclude anything about the local (in)distinguishability of $|\psi_{00}^{(4)}\rangle$, $|\psi_{11}^{(4)}\rangle$, $|\psi_{31}^{(4)}\rangle$, $|\psi_{32}^{(4)}\rangle$, since the solution for $K_{\alpha}^{\dagger}K_{\alpha}$ (equation (3.27)) is not a multiple of the identity, i.e., the measurement isn't constrained to be trivial. Hence, at this point we do not know if the states are distinguishable or not.

We now obtain the post-measurement joint states $|\psi_{\alpha,nm}^{(4)}\rangle$ using necessary condition (3.14) for OP.

 $K_{\alpha}^{\dagger}K_{\alpha}$ enables us to determine K_{α} upto a left-unitary, i.e., $K_{\alpha} = U\sqrt{K_{\alpha}^{\dagger}K_{\alpha}}$, where U is a 4×4 unitary matrix. This unitary U is irrelevant because physically it implies Alice performing a unitary after her measurement and we know that such a unitary transformation on Alice's side (or Bob's side) doesn't alter the local distinguishability of the set of states. Hence we can assume the $U = \mathbb{1}_4$. Using the above table of eigenvalues and eigenvectors, we get that $K_{\alpha} \left(\equiv \sqrt{K_{\alpha}^{\dagger}K_{\alpha}}\right)$ is given by

$$\begin{aligned} K_{\alpha} &= \\ \sqrt{a_{0} + \mu_{0}} \left(\cos \frac{\zeta}{2} | 0 \rangle + \sin \frac{\zeta}{2} | 2 \rangle \right) \left(\cos \frac{\zeta}{2} \langle 0 | + \sin \frac{\zeta}{2} \langle 2 | \right) \\ &+ \sqrt{a_{0} - \mu_{0}} \left(-\sin \frac{\zeta}{2} | 0 \rangle + \cos \frac{\zeta}{2} | 2 \rangle \right) \left(-\sin \frac{\zeta}{2} \langle 0 | + \cos \frac{\zeta}{2} \langle 2 | \right) \\ &+ \sqrt{a_{0} + \mu_{1}} \left(\cos \frac{\eta}{2} | 1 \rangle + \sin \frac{\eta}{2} | 3 \rangle \right) \left(\cos \frac{\eta}{2} \langle 1 | + \sin \frac{\eta}{2} \langle 3 | \right) \\ &+ \sqrt{a_{0} - \mu_{1}} \left(-\sin \frac{\eta}{2} | 1 \rangle + \cos \frac{\eta}{2} | 3 \rangle \right) \left(-\sin \frac{\eta}{2} \langle 1 | + \cos \frac{\eta}{2} \langle 3 | \right). \end{aligned}$$
(3.28)

Using equation (3.28) we now give the Schmidt decomposition of the states $|\psi_{a,00}^{(4)}\rangle$, $|\psi_{a,11}^{(4)}\rangle$, $|\psi_{a,31}^{(4)}\rangle$,

$$\begin{split} |\psi_{\alpha,00}^{(4)}\rangle &= \frac{1}{2}\sqrt{1+\frac{\mu_{0}}{a_{0}}} |\chi\rangle_{A} \left(\cos\frac{\zeta}{2}|0\rangle_{B} + \sin\frac{\zeta}{2}|2\rangle_{B}\right) + \frac{1}{2}\sqrt{1-\frac{\mu_{0}}{a_{0}}} |\kappa\rangle_{A} \left(-\sin\frac{\zeta}{2}|0\rangle_{B} + \cos\frac{\zeta}{2}|2\rangle_{B}\right) \\ &+ \frac{1}{2}\sqrt{1+\frac{\mu_{1}}{a_{0}}} |\omega\rangle_{A} \left(\cos\frac{\eta}{2}|1\rangle_{B} + \sin\frac{\eta}{2}|3\rangle_{B}\right) + \frac{1}{2}\sqrt{1-\frac{\mu_{1}}{a_{0}}} |\tau\rangle_{A} \left(-\sin\frac{\eta}{2}|1\rangle_{B} + \cos\frac{\eta}{2}|3\rangle_{B}\right), \end{split}$$
(3.29a)

$$\begin{split} |\psi_{a,11}^{(4)}\rangle &= \frac{1}{2}\sqrt{1+\frac{\mu_{0}}{a_{0}}} |\chi\rangle_{A} \left(\cos\frac{\zeta}{2}|1\rangle_{B} - \sin\frac{\zeta}{2}|3\rangle_{B}\right) - \frac{1}{2}\sqrt{1-\frac{\mu_{0}}{a_{0}}} |\kappa\rangle_{A} \left(\sin\frac{\zeta}{2}|1\rangle_{B} + \cos\frac{\zeta}{2}|3\rangle_{B}\right) \\ &+ i\frac{1}{2}\sqrt{1+\frac{\mu_{1}}{a_{0}}} |\omega\rangle_{A} \left(-\sin\frac{\eta}{2}|0\rangle_{B} + \cos\frac{\eta}{2}|2\rangle_{B}\right) - i\frac{1}{2}\sqrt{1-\frac{\mu_{1}}{a_{0}}} |\tau\rangle_{A} \left(\cos\frac{\eta}{2}|0\rangle_{B} + \sin\frac{\eta}{2}|2\rangle_{B}\right), \end{split}$$
(3.29b)

$$\begin{split} |\psi_{\alpha,31}^{(4)}\rangle &= \frac{1}{2}\sqrt{1+\frac{\mu_0}{a_0}} |\chi\rangle_A \left(\cos\frac{\zeta}{2}|1\rangle_B - \sin\frac{\zeta}{2}|3\rangle_B\right) - \frac{1}{2}\sqrt{1-\frac{\mu_0}{a_0}} |\kappa\rangle_A \left(\sin\frac{\zeta}{2}|1\rangle_B + \cos\frac{\zeta}{2}|3\rangle_B\right) \\ &-i\frac{1}{2}\sqrt{1+\frac{\mu_1}{a_0}} |\omega\rangle_A \left(-\sin\frac{\eta}{2}|0\rangle_B + \cos\frac{\eta}{2}|2\rangle_B\right) + i\frac{1}{2}\sqrt{1-\frac{\mu_1}{a_0}} |\tau\rangle_A \left(\cos\frac{\eta}{2}|0\rangle_B + \sin\frac{\eta}{2}|2\rangle_B\right), \end{split}$$
(3.29c)

$$\begin{split} |\psi_{\alpha,32}^{(4)}\rangle &= \frac{1}{2}\sqrt{1+\frac{\mu_{0}}{a_{0}}} |\chi\rangle_{A} \left(-\sin\frac{\zeta}{2}|0\rangle_{B} + \cos\frac{\zeta}{2}|2\rangle_{B}\right) - \frac{1}{2}\sqrt{1-\frac{\mu_{0}}{a_{0}}} |\kappa\rangle_{A} \left(\cos\frac{\zeta}{2}|0\rangle_{B} + \sin\frac{\zeta}{2}|2\rangle_{B}\right) \\ &+ i\frac{1}{2}\sqrt{1+\frac{\mu_{1}}{a_{0}}} |\omega\rangle_{A} \left(\sin\frac{\eta}{2}|1\rangle_{B} - \cos\frac{\eta}{2}|3\rangle_{B}\right) + i\frac{1}{2}\sqrt{1-\frac{\mu_{1}}{a_{0}}} |\tau\rangle_{A} \left(\cos\frac{\eta}{2}|1\rangle_{B} + \sin\frac{\eta}{2}|3\rangle_{B}\right), \end{split}$$
(3.29d)

where

$$\begin{split} |\chi\rangle_A &= \left(\cos\frac{\zeta}{2}|0\rangle_A + \sin\frac{\zeta}{2}|2\rangle_A\right),\\ |\kappa\rangle_A &= \left(-\sin\frac{\zeta}{2}|0\rangle_A + \cos\frac{\zeta}{2}|2\rangle_A\right),\\ |\omega\rangle_A &= \left(\cos\frac{\eta}{2}|1\rangle_A + \sin\frac{\eta}{2}|3\rangle_A\right),\\ |\tau\rangle_A &= \left(-\sin\frac{\eta}{2}|1\rangle_A + \cos\frac{\eta}{2}|3\rangle_A\right) \end{split}$$

are vectors of an ONB on Alice's subsystem. It is easy to check that the states $|\psi_{\alpha,00}^{(4)}\rangle$, $|\psi_{\alpha,11}^{(4)}\rangle$, $|\psi_{\alpha,31}^{(4)}\rangle$, $|\psi_{\alpha,32}^{(4)}\rangle$ are pairwise orthogonal.

From equation (3.29) it is easily be seen that the spectra of $\rho_{\alpha}^{(A)}$, $\rho_{\alpha,00}^{(A)}$, $\rho_{\alpha,11}^{(A)}$, $\rho_{\alpha,31}^{(A)}$, $\rho_{\alpha,00}^{(A)}$, $\rho_{\alpha,00}^{(B)}$, $\rho_{\alpha,11}^{(B)}$, $\rho_{\alpha,31}^{(B)}$, $\rho_{\alpha,00}^{(B)}$, $\rho_{\alpha,11}^{(B)}$, $\rho_{\alpha,31}^{(B)}$, $\rho_{\alpha,00}^{(B)}$, $\rho_{\alpha,01}^{(B)}$, $\rho_{\alpha,$

 $\rho_{\alpha,32}^{(B)}$ are the same, and the common spectra is of the form $\left\{\frac{1+\frac{\mu_0}{a_0}}{4}, \frac{1-\frac{\mu_1}{a_0}}{4}, \frac{1-\frac{\mu_1}{a_0}}{4}\right\}$. The equality of the spectra of the aforementioned states is at par with the prediction of theorem (3.1.1). The spectrum of $\rho_{\alpha}^{(B)}$ is given by $\left\{\frac{1+\frac{\mu_0}{2a_0}}{4}, \frac{1-\frac{\mu_0}{2a_0}}{4}, \frac{1+\frac{\mu_1}{2a_0}}{4}, \frac{1-\frac{\mu_1}{2a_0}}{4}\right\}$. Substituting these quantities in the LHS of the inequality (3.11) we get that

$$I_{acc}^{LOCC} \le S\left(\rho_{\alpha}^{(B)}\right) = H\left(\frac{1+\frac{\mu_{0}}{2a_{0}}}{4}, \frac{1-\frac{\mu_{0}}{2a_{0}}}{4}, \frac{1+\frac{\mu_{1}}{2a_{0}}}{4}, \frac{1-\frac{\mu_{1}}{2a_{0}}}{4}\right), \tag{3.30}$$

where $H\left(\frac{1+\frac{\mu_0}{2a_0}}{4}, \frac{1-\frac{\mu_0}{2a_0}}{4}, \frac{1+\frac{\mu_1}{2a_0}}{4}, \frac{1-\frac{\mu_1}{2a_0}}{4}\right)$ is the Shannon entropy for $\left\{\frac{1+\frac{\mu_0}{2a_0}}{4}, \frac{1-\frac{\mu_0}{2a_0}}{4}, \frac{1+\frac{\mu_1}{2a_0}}{4}, \frac{1+\frac{\mu_1}{2a_0}}{4}\right\}$. Thus we see that unless $\mu_0 = \mu_1 = 0$, $H(\frac{1}{4}(1+\frac{\mu_0}{2a_0}), \frac{1}{4}(1-\frac{\mu_0}{2a_0}), \frac{1}{4}(1+\frac{\mu_1}{2a_0}), \frac{1}{4}(1-\frac{\mu_1}{2a_0}))$ $< 2 \ bit$. Thus if $\mu_0 \neq 0$ or $\mu_1 \neq 0$, the locally accessible information of the set $\{|\psi_{a,00}\rangle, |\psi_{a,11}\rangle, |\psi_{a,31}\rangle, |\psi_{a,32}\rangle\}$ is lower than 2 *bit*, meaning that there is no LOCC protocol that Alice and Bob can use to perfectly distinguish between states in the set. On the other hand if $\mu_0 = \mu_1 = 0$, Alice's POVM is a trivial one (see equation (3.27)). Thus the states $|\psi_{00}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle$ fail to satisfy the necessary condition. It is significant to note that the local indistinguishability of these states was established *only* after it was demanded that $S(\rho_{\alpha}^{(B)}) = 2 \ bit$, in accordance with theorem (3.1.1). Thus, this also shows that condition (3.15), which is equivalent to theorem (3.1.1) when m = d, is stronger than the OP condition (3.14).

This shows us that there is no LOCC protocol to perfectly distinguish the states in the set, if Alice starts the protocol. Similarly it can be shown that there is no LOCC protocol to perfectly distinguish the states of the set, if Bob starts the protocol; the arguments to establish this follow the same sequence of reasoning as the arguments above.

In the following we list the sets of four Generalized Bell states from $\mathbb{C}^4 \otimes \mathbb{C}^4$ which fail the necessary condition in the same fashion as example (3.1.1); each set listed represents an equivalence class of sets of four Generalized Bell states which are local unitarily

equivalent to it. These are 39 in number.

All sets contain the states
$$|\psi_{00}^{(4)}\rangle$$

and $|\psi_{01}^{(4)}\rangle$; remaining states are listed.
 $\{|\psi_{02}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle\}$ $\{|\psi_{02}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle\}$ $\{|\psi_{10}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle\}$ $\{|\psi_{10}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle\}$
 $\{|\psi_{11}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle\}$ $\{|\psi_{11}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle\}$ $\{|\psi_{11}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle\}$ $\{|\psi_{12}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle\}$ $\{|\psi_{12}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle\}$
 $\{|\psi_{13}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle\}$ $\{|\psi_{13}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle\}$ $\{|\psi_{20}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle\}$ $\{|\psi_{20}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle\}$ $\{|\psi_{20}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle\}$ $\{|\psi_{21}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}$
 $\{|\psi_{22}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle\}$ $\{|\psi_{23}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle\}$ $\{|\psi_{21}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}$ $\{|\psi_{30}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}$ $\{|\psi_{31}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle\}$

Table 3.2: Some sets of four Generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ which aren't distinguishable.

All sets contain the states
$$|\psi_{00}^{(4)}\rangle$$

and $|\psi_{01}^{(2)}\rangle$; remaining states are listed.
 $\{|\psi_{10}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle\}$ $\{|\psi_{10}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle\}$ $\{|\psi_{10}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}$ $\{|\psi_{10}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}$ $\{|\psi_{10}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle\}$
 $\{|\psi_{11}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle\}$ $\{|\psi_{11}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle\}$ $\{|\psi_{11}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}$ $\{|\psi_{11}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}$ $\{|\psi_{11}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}$
 $\{|\psi_{20}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}$ $\{|\psi_{20}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}$ $\{|\psi_{20}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}$ $\{|\psi_{20}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle\}$

Table 3.3: Remaining sets of four Generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ which aren't distinguishable.

The necessary condition is satisfied by all sets in all remaining 83 equivalence classes. That doesn't mean that states in these sets *should* be perfectly locally distinguishable. Therefore it comes as a surprise that states in sets of *all* the remaining 83 equivalence classes are indeed locally distinguishable, and that too by one-way LOCC using only projective measurements. We next list 83 distinct representative sets for all of these equivalence classes and give the LOCC protocols to distinguish the states in each such set.

Among the 122 equivalence classes of sets of 4 Generalized Bell states, we here give a list of 83 equivalence classes which satisfy the necessary condition. Each equivalence class is represented by a set of 4 Generalized Bell states which is contained in it. While satisfying the condition (3.15) doesn't necessarily imply that any of these sets of Generalized Bell states should be perfectly locally distinguishable, surprisingly, we find that that is indeed the case, and that too by one-way LOCC by only projective measurements. Along with each set of states we also give the one-way LOCC protocol for their perfect local distinguishability.

Theorem 3.1.2. Any set of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ of the form

$$\{|\psi_{a0}^{(4)}\rangle, |\psi_{b1}^{(4)}\rangle, |\psi_{c2}^{(4)}\rangle, |\psi_{d3}^{(4)}\rangle\}$$

where $a, b, c, d \in \{0, 1, 2, 3\}$, can be discriminated by one-way LOCC using only projective measurements. Sets of four generalized Bell states representing all the corresponding equivalence classes are listed in table A.1 in the appendix A.1. Similarly, any set of four Generalized bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ which is of the form

$$\{|\psi_{0a}^{(4)}\rangle, |\psi_{1b}^{(4)}\rangle, |\psi_{2c}^{(4)}\rangle, |\psi_{3d}^{(4)}\rangle\},\$$

where $a, b, c, d \in \{0, 1, 2, 3\}$, can be discriminated by one-way LOCC using only projective measurements. Sets of four generalized Bell states representing all the corresponding equivalence classes are listed in table A.2 in the appendix A.1.

Proof. Consider first the set $\{|\psi_{a0}^{(4)}\rangle, |\psi_{b1}^{(4)}\rangle, |\psi_{c1}^{(4)}\rangle, |\psi_{d1}^{(4)}\rangle\}$. Alice starts with a rank-one projective measurement in the following orthonormal basis:

$$|u_0\rangle = |0\rangle, |u_1\rangle = |1\rangle, |u_2\rangle = |2\rangle, |u_3\rangle = |3\rangle.$$

For the k-th outcome of Alice's measurement, the post measurement set will be of the

following form:

$$\{|\psi_{a0}^{(4)}\rangle, |\psi_{b1}^{(4)}\rangle, |\psi_{c1}^{(4)}\rangle, |\psi_{d1}^{(4)}\rangle\} \longrightarrow \{|k\rangle|k\rangle, |k\rangle|k \oplus_4 1\rangle, |k\rangle|k \oplus_4 2\rangle, |k\rangle|k \oplus_4 3\rangle\}.$$

Thus, once Alice tells Bob her measurement's outcome, he needs to perform measurement in the $\{|j\}_{i=0}^4$ basis to perfectly distinguish between the states in the set.

Now consider a set of the form $\{|\psi_{0a}^{(4)}\rangle, |\psi_{1b}^{(4)}\rangle, |\psi_{2c}^{(4)}\rangle, |\psi_{3d}^{(4)}\rangle\}$. Alice starts by performing a rank-one projective measurement corresponding to the following orthonormal basis:

$$|u_{0}\rangle = \frac{1}{2}\sum_{j=0}^{3}|j\rangle, |u_{1}\rangle = \frac{1}{2}\sum_{j=0}^{3}e^{\frac{ij\pi}{2}}|j\rangle, |u_{2}\rangle = \frac{1}{2}\sum_{j=0}^{3}(-1)^{j}|j\rangle, |u_{3}\rangle = \frac{1}{2}\sum_{j=0}^{3}e^{\frac{3ij\pi}{2}}|j\rangle.$$

For the k-th outcome of Alice's measurement, the post measurement set will be of the following form:

$$\{|\psi_{0a}^{(4)}\rangle, |\psi_{1b}^{(4)}\rangle, |\psi_{2c}^{(4)}\rangle, |\psi_{3d}^{(4)}\rangle\} \longrightarrow \{|u_k\rangle|v_k\rangle, |u_k\rangle|v_{k\oplus_41}\rangle, |u_k\rangle|v_{k\oplus_42}\rangle, |u_k\rangle|v_{k\oplus_43}\rangle\},$$

where

$$|v_{0}\rangle = \frac{1}{2} \sum_{j=0}^{3} e^{\frac{(4 \ominus_{4}k)ij\pi}{2}} |j\rangle, |v_{1}\rangle = \frac{1}{2} \sum_{j=0}^{3} e^{\frac{(1 \ominus_{4}k)ij\pi}{2}} |j\rangle, |v_{2}\rangle = \frac{1}{2} \sum_{j=0}^{3} e^{\frac{(2 \ominus_{4}k)ij\pi}{2}} |j\rangle,$$
$$|v_{3}\rangle = \frac{1}{2} \sum_{j=0}^{3} e^{\frac{(3 \ominus_{4}k)ij\pi}{2}} |j\rangle.$$

Thus, once Alice tells Bob her measurement's outcome, he needs to perform measurement in the $\{|v_j\rangle\}_{j=0}^3$ basis to perfectly distinguish between the states in the set. \Box

Theorem 3.1.3. States in each set in the following two tables are perfectly distinguishable by one-way LOCC using only projective measurement:

Proof. Alice performs a rank-one projective measurement in the following orthonormal basis:

Each of the following 10 sets contain the states	
$ \psi_{00}^{(4)}\rangle$ and $ \psi_{01}^{(4)}\rangle$; remaining states are listed below.	
$ \left\{ \left \psi_{02}^{(4)} \rangle, \left \psi_{10}^{(4)} \right\rangle \right\} \left\{ \left \psi_{02}^{(4)} \rangle, \left \psi_{21}^{(4)} \right\rangle \right\} \left\{ \left \psi_{02}^{(4)} \rangle, \left \psi_{32}^{(4)} \right\rangle \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \right\rangle \right\} \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \right\rangle \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \rangle \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \rangle \right\} \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \rangle \right\} \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \rangle \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \rangle \right\} \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \rangle \right\} \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \rangle \right\} \right\} \left\{ \left \psi_{10}^{(4)} \rangle, \left \psi_{20}^{(4)} \rangle \right\} \left \psi_{20}^{(4)} \rangle \right\} \left \psi_{20}^{(4)} \rangle \right \right\} \left \psi_{20}^{(4)} \rangle \right \psi_{20}^{(4)} \rangle \right \left \psi_{20}^{(4)} \rangle \right \left \psi$	
$\left \left\{ \psi_{10}^{(4)}\rangle, \psi_{31}^{(4)}\rangle\right\}\left\{ \psi_{13}^{(4)}\rangle, \psi_{21}^{(4)}\rangle\right\}\left\{ \psi_{20}^{(4)}\rangle, \psi_{21}^{(4)}\rangle\right\}\left\{ \psi_{20}^{(4)}\rangle, \psi_{31}^{(4)}\rangle\right\}\left\{ \psi_{21}^{(4)}\rangle, \psi_{31}^{(4)}\rangle\right\}$	

Table 3.4: Some sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.3.

Each of the following 4 sets contain the states $|\psi_{00}^{(4)}\rangle \text{ and } |\psi_{02}^{(4)}\rangle; \text{ remaining states are listed below.}$ $\left\{|\psi_{10}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle\right\}, \left\{|\psi_{10}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle\right\}, \left\{|\psi_{10}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\right\}, \left\{|\psi_{21}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\right\}.$

Table 3.5: Remaining sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.3.

$$|u_{1}\rangle = \frac{1}{2} \left(-e^{i\frac{\pi}{4}} |0\rangle + |1\rangle + e^{i\frac{\pi}{4}} |2\rangle + |3\rangle \right), |u_{2}\rangle = \frac{1}{2} \left(e^{i\frac{\pi}{4}} |0\rangle + |1\rangle - e^{i\frac{\pi}{4}} |2\rangle + |3\rangle \right), \\ |u_{3}\rangle = \frac{1}{2} \left(e^{i\frac{3\pi}{4}} |0\rangle - |1\rangle + e^{i\frac{3\pi}{4}} |2\rangle + |3\rangle \right), |u_{4}\rangle = -\frac{1}{2} \left(e^{i\frac{3\pi}{4}} |0\rangle - |1\rangle - e^{i\frac{3\pi}{4}} |2\rangle + |3\rangle \right).$$

For each set of states mentioned in tables 3.1.3 and 3.1.3, the remaining part of the LOCC protocol is given in subsection A.2.1 in the appendix.

Theorem 3.1.4. The following sets are distinguishable by one-way LOCC using only projective measurement:

Each of the following 8 sets contain the state
$ \psi_{00}^{(4)}\rangle$, the rest are given in the table.
$\{ \psi_{01}^{(4)}\rangle, \psi_{02}^{(4)}\rangle, \psi_{11}^{(4)}\rangle\}, \{ \psi_{01}^{(4)}\rangle, \psi_{02}^{(4)}\rangle, \psi_{31}^{(4)}\rangle\}, \{ \psi_{01}^{(4)}\rangle, \psi_{10}^{(4)}\rangle, \psi_{23}^{(4)}\rangle\}, \{ \psi_{01}^{(4)}\rangle, \psi_{11}^{(4)}\rangle, \psi_{22}^{(4)}\rangle\},$
$\{ \psi_{01}^{(4)}\rangle, \psi_{22}^{(4)}\rangle, \psi_{31}^{(4)}\rangle\}, \{ \psi_{01}^{(4)}\rangle, \psi_{23}^{(4)}\rangle, \psi_{30}^{(4)}\rangle\}, \{ \psi_{02}^{(4)}\rangle, \psi_{11}^{(4)}\rangle, \psi_{21}^{(4)}\rangle\}, \{ \psi_{02}^{(4)}\rangle, \psi_{21}^{(4)}\rangle, \psi_{31}^{(4)}\rangle\}$

Table 3.6: Sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.4.

Proof. Alice performs a rank-one projective measurement in the following orthonormal basis:

3.1. NECESSARY CONDITION FOR LOCC OF MES

$$|u_1\rangle = \frac{1}{\sqrt{2}} (-i|0\rangle + |2\rangle), |u_2\rangle = \frac{1}{\sqrt{2}} (i|0\rangle + |2\rangle), |u_3\rangle = \frac{1}{\sqrt{2}} (-i|1\rangle + |3\rangle), |u_4\rangle = \frac{1}{\sqrt{2}} (i|1\rangle + |3\rangle).$$

For each set of states mentioned in table 3.1.4, the remaining part of the LOCC protocol is given in subsection A.2.2 in the appendix. \Box

Theorem 3.1.5. The following sets are distinguishable by one-way LOCC using only projective measurement:

Each of the following 8 sets contain the state
$ \psi_{00}^{(4)}\rangle$, the rest are given in the table.
$\left\{ \psi_{01}^{(4)}\rangle, \psi_{02}^{(4)}\rangle, \psi_{12}^{(4)}\rangle\right\},\left\{ \psi_{01}^{(4)}\rangle, \psi_{02}^{(4)}\rangle, \psi_{30}^{(4)}\rangle\right\},\left\{ \psi_{01}^{(4)}\rangle, \psi_{11}^{(4)}\rangle, \psi_{21}^{(4)}\rangle\right\},\left\{ \psi_{01}^{(4)}\rangle, \psi_{12}^{(4)}\rangle, \psi_{20}^{(4)}\rangle\right\},\left\{ \psi_{01}^{(4)}\rangle, \psi_{12}^{(4)}\rangle, \psi_{12}^{(4)}\rangle, \psi_{21}^{(4)}\rangle\right\},\left\{ \psi_{01}^{(4)}\rangle, \psi_{12}^{(4)}\rangle, \psi_{12}^{(4)}\rangle, \psi_{12}^{(4)}\rangle, \psi_{12}^{(4)}\rangle\right\},\left\{ \psi_{12}^{(4)}\rangle, \psi_{12}^{($
$\left\{ \psi_{01}^{(4)}\rangle, \psi_{20}^{(4)}\rangle, \psi_{30}^{(4)}\rangle\right\},\left\{ \psi_{01}^{(4)}\rangle, \psi_{21}^{(4)}\rangle, \psi_{33}^{(4)}\rangle\right\},\left\{ \psi_{02}^{(4)}\rangle, \psi_{10}^{(4)}\rangle, \psi_{21}^{(4)}\rangle\right\},\left\{ \psi_{02}^{(4)}\rangle, \psi_{21}^{(4)}\rangle, \psi_{32}^{(4)}\rangle\right\}.$

Table 3.7: Sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.5.

Proof. Alice performs a rank-one projective measurement in the following orthonormal basis:

$$|u_{1}\rangle = \frac{1}{2} \left(-e^{i\frac{3\pi}{4}} |0\rangle + |1\rangle + e^{i\frac{3\pi}{4}} |2\rangle + |3\rangle \right), |u_{2}\rangle = \frac{1}{2} \left(e^{i\frac{3\pi}{4}} |0\rangle + |1\rangle - e^{i\frac{3\pi}{4}} |2\rangle + |3\rangle \right), \\ |u_{3}\rangle = \frac{1}{2} \left(e^{i\frac{\pi}{4}} |0\rangle - |1\rangle + e^{i\frac{\pi}{4}} |2\rangle + |3\rangle \right), |u_{4}\rangle = \frac{1}{2} \left(-e^{i\frac{\pi}{4}} |0\rangle - |1\rangle - e^{i\frac{\pi}{4}} |2\rangle + |3\rangle \right).$$

For each set of states mentioned in table 3.1.5, the remaining part of the LOCC protocol is given in subsection A.2.3 in the appendix.

Theorem 3.1.6. The following sets are distinguishable by one-way LOCC using only projective measurement:

Proof. Alice performs a rank-one projective measurement in the following orthonormal basis:

$$|u_1\rangle = \frac{1}{2}(|0\rangle - |1\rangle - |2\rangle + |3\rangle), |u_2\rangle = \frac{1}{2}(-|0\rangle - |1\rangle + |2\rangle + |3\rangle),$$

Each of the following 10 sets contain the state
$ \psi_{00}^{(4)}\rangle$ the rest are given in the table.
$\left\{ \psi_{01}^{(4)}\rangle, \psi_{10}^{(4)}\rangle, \psi_{11}^{(4)}\rangle \right\}, \left\{ \psi_{01}^{(4)}\rangle, \psi_{10}^{(4)}\rangle, \psi_{30}^{(4)}\rangle \right\}, \left\{ \psi_{01}^{(4)}\rangle, \psi_{10}^{(4)}\rangle, \psi_{32}^{(4)}\rangle \right\}, \left\{ \psi_{01}^{(4)}\rangle, \psi_{11}^{(4)}\rangle, \psi_{12}^{(4)}\rangle \right\},$
$\left \left\{\left \psi_{01}^{(4)}\rangle,\left \psi_{11}^{(4)}\rangle,\left \psi_{31}^{(4)}\rangle\right\},\left\{\left \psi_{01}^{(4)}\rangle,\left \psi_{11}^{(4)}\rangle,\left \psi_{33}^{(4)}\rangle\right\}\right\},\left\{\left \psi_{01}^{(4)}\rangle,\left \psi_{12}^{(4)}\rangle,\left \psi_{12}^{(4)}\rangle,\left \psi_{32}^{(4)}\rangle\right\}\right\},\left\{\left \psi_{01}^{(4)}\rangle,\left \psi_{12}^{(4)}\rangle,\left \psi_{32}^{(4)}\rangle\right\}\right\},\left\{\left \psi_{01}^{(4)}\rangle,\left \psi_{12}^{(4)}\rangle,\left \psi_{32}^{(4)}\rangle\right\}\right\},\left \psi_{32}^{(4)}\rangle\right\},\left \psi_{32}^{(4)}\rangle\right\},\left \psi_{32}^{(4)}\rangle\right\},\left \psi_{32}^{(4)}\rangle\right\},\left \psi_{32}^{(4)}\rangle\right \right\}$
$\left\{ \psi_{01}^{(4)}\rangle, \psi_{13}^{(4)}\rangle, \psi_{31}^{(4)}\rangle \right\}, \left\{ \psi_{01}^{(4)}\rangle, \psi_{13}^{(4)}\rangle, \psi_{33}^{(4)}\rangle \right\}.$

Table 3.8: Sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.6.

 $|u_3\rangle = \frac{1}{2}(-|0\rangle + |1\rangle - |2\rangle + |3\rangle), |u_4\rangle = \frac{i}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle).$

For each set of states mentioned in table 3.1.6, the remaining part of the LOCC protocol is given in subsection A.2.4 in the appendix.

Theorem 3.1.7. The following sets are distinguishable by one-way LOCC using only projective measurement:

Each of the following 5 sets contain the state
$ \psi_{00}^{(4)}\rangle$ the rest are given in the table.
$\left\{ \psi_{01}^{(4)}\rangle, \psi_{10}^{(4)}\rangle, \psi_{33}^{(4)}\rangle \right\}, \left\{ \psi_{01}^{(4)}\rangle, \psi_{11}^{(4)}\rangle, \psi_{32}^{(4)}\rangle \right\}, \left\{ \psi_{01}^{(4)}\rangle, \psi_{12}^{(4)}\rangle, \psi_{31}^{(4)}\rangle \right\}, \left\{ \psi_{01}^{(4)}\rangle, \psi_{13}^{(4)}\rangle, \psi_{30}^{(4)}\rangle \right\},$
$\{ \psi_{01}^{(4)}\rangle, \psi_{11}^{(4)}\rangle, \psi_{30}^{(4)}\rangle\}.$

Table 3.9: Sets of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, which are proved to be one-way locally distinguishable in theorem 3.1.7.

Proof. Alice performs a rank-one projective measurement in the following orthonormal basis:

$$|u_1\rangle = \frac{1}{\sqrt{2}} \left(-|0\rangle + |2\rangle\right), |u_2\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle + |3\rangle\right), |u_3\rangle = \frac{1}{\sqrt{2}} \left(-|1\rangle + |3\rangle\right), |u_4\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |2\rangle\right).$$

For each set of states mentioned in table 3.1.7, the remaining part of the LOCC protocol is given in subsection A.2.5 in the appendix.

Theorem 3.1.8. The following set is distinguishable by one-way LOCC using only projective measurement:

$\{|\psi_{00}^{(4)}\rangle,|\psi_{02}^{(4)}\rangle,|\psi_{20}^{(4)}\rangle,|\psi_{22}^{(4)}\rangle\}.$

Proof. Alice performs a rank-one projective measurement in the following orthonormal basis:

$$|u_1\rangle = \frac{1}{2} (-i|0\rangle - i|1\rangle + |2\rangle + |3\rangle),$$

$$|u_2\rangle = \frac{1}{2} (i|0\rangle + i|1\rangle + |2\rangle + |3\rangle),$$

$$|u_3\rangle = \frac{1}{2} (i|0\rangle - i|1\rangle - |2\rangle + |3\rangle),$$

$$|u_4\rangle = \frac{1}{2} (-i|0\rangle + i|1\rangle - |2\rangle + |3\rangle).$$

The remaining part of the LOCC protocol is given in subsection A.2.6 in the appendix.

With this we prove that the necessary condition is also sufficient to establish the local (in)distinguishability of all sets of four Generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$.

3.1.5 Summary

Based on the upper bound of locally accessible information, we formulated a necessary condition for the perfect distinguishability of a set of MES by LOCC. This necessary condition *genuinely* decreases the complexity of the distinguishability problem, particularly for a set of *n* MES. To illustrate this, we tested the necessary condition for all sets of four Generalized Bell basis states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, and then isolated those sets which failed the test. Surprisingly, we discovered that all the remaining sets are perfectly distinguishable by one-way LOCC using only projective measurements, and to show that we explicitly obtained the LOCC protocol for perfect distinguishability for all of them. That there is no protocol which involves two-way LOCC is interestingly similar to the result in [69], where it was shown that two-way LOCC doesn't play any distinguished role in the perfect distinguishability of a set of four ququad-ququad lattice states in $\mathbb{C}^4 \otimes \mathbb{C}^4$.

In [62] a set of four ququad-ququad lattice states were shown to be indistinguishable by PPT preserving operations. It was recently shown that this is the *only* such set among the sets of ququad-ququad lattice states, which isn't perfectly distinguishable by LOCC [69]. We also tested our necessary condition on the aforementioned set of states and found that they do not satisfy the necessary condition. Thus, our necessary condition is also sufficien to determine the local distinguishability for ququad-ququad lattice states. In fact, our condition is more general since the condition for the local distinguishability for ququad-ququad lattice states given in [69] is particularly specific to ququad-ququad lattice states whereas our necessary condition applies generally to any set of MES.

3.1.6 Future Directions

The central question is to understand how strong the aforementioned necessary condition is. This leads us to potential questions worth exploring for the future.

- To see if the necessary condition is sufficient for the local (in)distinguishability of sets of *n* Generalized Bell states in Cⁿ ⊗ Cⁿ. While it is difficult to prove this for general *n*, I want to know if this holds for *n* = 5 etc. While it is easy to identify those sets (of five Generalized Bell states in C⁵ ⊗ C⁵) which don't satisfy the necessary condition, it is difficult to test if the remaining are locally distinguishable or not.
- More generally, to explore if there are examples of m ≤ n orthogonal MES in Cⁿ ⊗ Cⁿ, which satisfy the necessary condition but are not locally distinguishable. If such examples exist, then it should be possible to construct them through reverse engineering from the necessary condition.
- 3. To characterize those special kinds of MES in $\mathbb{C}^n \otimes \mathbb{C}^n$ for which the aforementioned

necessary condition will turn out to be sufficient in regard to the question of local distinguishability.

3.2 Framework for Distinguishability by 1-LOCC

The work done in this section has been detailed in a paper which has been been published in Physical Review A as a Rapid Communication²[70].

In the topic of perfect local distinguishability of orthogonal multipartite quantum states, most results obtained so far pertain to bipartite systems whose subsystems are of specific dimensions. In contrast very few results for bipartite systems whose subsystems are of arbitrary dimensions, are known. Some prominent results which apply to joint systems, whose subsystems are of arbitrary dimension, are Bennet et al's result [71], which established that members from an unextendible product basis cannot be perfectly distinguished by LOCC, Walgate et al's result [12], which establishes that any two multipartite orthogonal quantum states can be perfectly distinguished using only LOCC, Badziag et al's [8] result, which obtained a Holevo-like upper bound for the locally accessible information for an ensemble of states from a bipartite system, and Cohen's result [13], which established that almost all sets of n + 1 orthogonal states from N n-dimensional multipartite systems are not perfectly distinguishable by LOCC. The reason for there being a few number of such generic results is that a rich variety of (algebraic or geometric) structure is exhibited by different sets of orthogonal states owing to which it is difficult to associate some common property underlying them all, i.e., a common property that would play a crucial role in the local distinguishability of these states. In this work, I propose a framework for the distinguishability by one-way LOCC (1-LOCC) of sets of orthogonal bipartite states in a $n_A \otimes n_B$ bipartite system, where n_A, n_B are the dimensions of both subsytems, labelled as A and B. Firstly, very simple arguments establish that local dis-

Reprinted with permission from Singal, T. Phys. Rev. A 93, 030301(R) – Published 7 March 2016
 Copyright (2016) by the American Physical Society

tinguishability by one-way LOCC requires that the *i*-th party (where i = A, B) perform a rank-one orthogonality preserving measurement. In [10] it was shown that (corresponding to the set of orthogonal states) all orthogonality preserving operators lie in a vector space, which I denote by $\mathcal{T}_{\perp}^{(i)}$. I then give conditions to determine if this vector space contains all the elements of a rank-one measurement, which, if it does, implies that the states are distinguishable by one-way LOCC. The method to extract this information (of the existence of this 1-LOCC protocol) from $\mathcal{T}_{\perp}^{(i)}$ depends on the value of $dim \mathcal{T}_{\perp}^{(i)}$. In this way one can give sweeping results for the 1-LOCC (in)distinguishability of all sets of orthogonal bipartite states corresponding to certain values of $dim \mathcal{T}_{\perp}^{(i)}$. Thus I propose that the value of $dim \mathcal{T}_{\perp}^{(i)}$ gives the common underlying property based on which sweeping results for the 1-LOCC (in)distinguishability of orthogonal bipartite quantum states can be made.

Note that if Alice and Bob represent the two parties of the bipartite system, then to establish whether the given set of states are distinguishable by one-way LOCC or not, one has to extract this (in)distinguishability related information from $\mathcal{T}_{\perp}^{(A)}$ - the subspace of $n_A \times n_A$ hermitian matrices, corresponding to Alice's side - and $\mathcal{T}_{\perp}^{(B)}$ - subspace of $n_B \times n_B$ hermitian matrices corresponding to Bob's side - separately. Since the methods to extract this information apply equally to both sides, there is no loss of generality when I obtain results for the cases where Alice starts the protocol. And since I'm considering the case where Alice starts the protocol, I will only examine how to extract said information from her subspace. This allows me to simplify the notation: $\mathcal{T}_{\perp}^{(A)} \longrightarrow \mathcal{T}_{\perp}$. Also note that if $n_A < n_B$, one can always extend Alice's subsystem A to a larger local system A' whose dimension $n_{A'}$ is equal to n_B . And, similarly, vice versa. This implies that there is also no loss of generality in assuming that the dimensions of both subsystems are equal, thus I will assume that $n_A = n_B = n$. The only reasons for making both these assumptions is to keep the notation simpler.

3.2.1 The Framework

Let Alice and Bob have *n* dimensional quantum systems whose corresponding Hilbert spaces are denoted by \mathcal{H}_A and \mathcal{H}_B respectively. Let them share one of *m* orthogonal bipartite states, whose density matrices are $\rho_1^{(AB)}, \rho_2^{(AB)}, \dots, \rho_m^{(AB)}$. They wish to establish which from among the aforementioned *m* states they have in their possession, and they want to do this using 1-LOCC. As mentioned earlier, I assume that Alice starts the protocol. Since $\{\rho_i^{(AB)}\}_{i=1}^m$ are orthogonal, their supports supp $(\rho_i^{(AB)})$ are also orthogonal. Let the spectral decomposition of $\rho_i^{(AB)}$ be given by

$$\rho_i^{(AB)} = \sum_{j=1}^{r_i} \lambda_{ij} |\psi_{ij}\rangle \langle \psi_{ij} |, \qquad (3.31)$$

where r_i is the rank of $\rho_i^{(AB)}$, $\{\lambda_{ij}\}_{j=1}^{r_i}$ are the non-zero eigenvalues of $\rho_i^{(AB)}$ and $\langle \psi_{ij} | \psi_{i'j'} \rangle = \delta_{ii'}\delta_{jj'}$, $\forall 1 \le i \le i' \le m, 1 \le j \le r_i$ and $1 \le j' \le r_{i'}$. Let $\{|j\rangle_A\}_{j=1}^n$ and $\{|j\rangle_B\}_{j=1}^n$ be the standard orthonormal bases (ONB) for \mathcal{H}_A and \mathcal{H}_B respectively. For any $1 \le i \le m$, and any $1 \le j \le r_i$, define $n \times n$ complex matrices W_{ij} by expanding $|\psi_{ij}\rangle_{AB}$ in the product basis $\{|j\rangle_A|k\rangle_B\}_{i,k=1}^n$

$$|\psi_{ij}\rangle_{AB} = \sum_{l,k=1}^{n} \left(W_{ij} \right)_{kl} |l\rangle_{A} |k\rangle_{B}.$$
(3.32)

The orthonormality of the $|\psi_{ij}\rangle_{AB}$ vectors (for *i* and *j* indices) implies that $\operatorname{Tr}(W_{ij}^{\dagger}W_{i'j'}) = \delta_{ii'}\delta_{jj'}, \forall 1 \le i \le i' \le m, 1 \le j \le r_i \text{ and } 1 \le j' \le r_{i'}$. Define the index set $I \equiv \{(i, i', j, j'), | 1 \le i < i' \le m, 1 \le j \le r_i, 1 \le j' \le r_{i'}\}$. The cardinality of *I* is $\sum_{i=1}^{m-1} \sum_{i'=i+1}^{m} r_i r_{i'}$. Let $i = (i, i', j, j') \in I$. Define $W_i \equiv W_{ij}^{\dagger}W_{i'j'}$. Then the W_i 's are $n \times n$ complex matrices with trace zero. Let $H_i \equiv \frac{1}{2} \left(W_i + (W_i)^{\dagger} \right)$ and $A_i \equiv \frac{1}{2i} \left(W_i - (W_i)^{\dagger} \right)$, so that $W_i = H_i + iA_i$. Let S be the real vector space of all $n \times n$ hermitian matrices. $dimS = n^2$. Let \mathcal{T} be a subspace of S, defined by the following equation:

$$\mathcal{T} \equiv \left\{ \sum_{i \in I} a_i H_i + b_i A_i, \ \forall \ a_i, b_i \in \mathbb{R} \right\}.$$
(3.33)

Let \mathcal{T}_{\perp} be the orthogonal complement of \mathcal{T} in \mathcal{S} . Note that since $\operatorname{Tr}(W_i) = 0, \forall i \in I, \mathbb{1}_n \in \mathcal{T}_{\perp}$, where $\mathbb{1}_n$ is the $n \times n$ identity matrix.

For the states to be locally distinguishable it is necessary that Alice's first measurement preserves the orthogonality of all states, i.e., the measurement is orthogonality preserving (OP) for all measurement outcomes. Now consider theorem 3.2.1.

Theorem 3.2.1 (Nathanson [11], Proposition 1). The orthogonal states $\rho_1^{(AB)}, \rho_2^{(AB)}, \cdots, \rho_m^{(AB)}$ are perfectly distinguishable by 1-LOCC if and only if there exists an orthogonality preserving (OP) rank-one POVM on Alice's side which she can use as the starting measurement of the 1-LOCC protocol.

Theorem (3.2.1) implies that if the states are perfectly distinguishable by 1-LOCC then there will always exist a rank-one POVM which is OP which Alice can commence her 1-LOCC protocol with.

Theorem 3.2.2. If the states are perfectly distinguishable by 1-LOCC, Alice can always choose her starting measurement to be an extremal³ rank-one POVM.

Proof. Let $\{|\tilde{l}\rangle\langle\tilde{l}|\}_{l=1}^d$ be a non-extremal OP rank-one POVM which Alice commences the protocol with (where $\sum_{l=1}^d |\tilde{l}\rangle\langle\tilde{l}| = \mathbb{1}_A$). Note that $d \ge n$. Let all POVM elements in $\{|\tilde{l}\rangle\langle\tilde{l}|\}_{l=1}^d$ have a convex decomposition into two distinct extremal rank-one POVMs: $|\tilde{l}\rangle\langle\tilde{l}| = p|\tilde{l}\rangle\langle\tilde{l}'| + (1-p)|\tilde{l}''\rangle\langle\tilde{l}''|$, where $\sum_{l=1}^d |\tilde{l}'\rangle\langle\tilde{l}'| = \sum_{l=1}^d |\tilde{l}''\rangle\langle\tilde{l}''| = \mathbb{1}_A$. The equality $|\tilde{l}\rangle\langle\tilde{l}| = p|\tilde{l}'\rangle\langle\tilde{l}'| + (1-p)|\tilde{l}''\rangle\langle\tilde{l}''|$ is possible if and only if $|\tilde{l}'\rangle_A$ and $|\tilde{l}''\rangle_A$ are linearly dependent, i.e., either one of them is 0 and the other is a scalar multiple of $|\tilde{l}\rangle_A$, or both are scalar multiples of $|\tilde{l}\rangle_A$. This implies that if all elements of $\{|\tilde{l}\rangle\langle\tilde{l}|\}_{l=1}^d$ are OP, then all elements of $\{|\tilde{l}'\rangle\langle\tilde{l}'|\}_{l=1}^d$ and $\{|\tilde{l}''\rangle\langle\tilde{l}''|\}_{l=1}^d$ should also be OP. Hence each extremal rank-one POVM featuring in the convex sum of an OP rank-one POVM is also OP.

An extremal POVM is one which cannot be expressed as a convex sum of other *distinct* POVMs.

Thus theorems 3.2.1 and 3.2.2 imply that if the given states are 1-LOCC distinguishable then there *must* always be an extremal rank-one OP measurement on Alice's side which she can initiate the protocol with. Establishing this is necessary because it helps reduce our search to rank-one extremal POVMs which are OP.

Theorem 3.2.3. The states $\rho_1^{(AB)}, \rho_2^{(AB)}, \cdots, \rho_m^{(AB)}$ are perfectly distinguishable by 1-LOCC if and only if \mathcal{T}_{\perp} contains all elements of an extremal rank-one POVM.

Proof. **ONLY IF:** Here I show that when the states $\rho_1^{(AB)}, \rho_2^{(AB)}, \dots, \rho_m^{(AB)}$ are perfectly distinguishable by 1-LOCC, \mathcal{T}_{\perp} contains all POVM elements of an extremal rank-one POVM. The discussion above suggests that there exists an OP extremal rank-one POVM on Alice's side. Let the elements of this POVM be $\{|\tilde{l}\rangle\langle\tilde{l}|\}_{l=1}^d$ (where $\sum_{l=1}^d |\tilde{l}\rangle\langle\tilde{l}| = \mathbb{1}_A$), and let the Kraus operators of this measurement be $\{|\phi_l\rangle\langle\tilde{l}|\}_{l=1}^d$, where $|\phi_l\rangle_A$ are normalized states in \mathcal{H}_A for all $1 \leq l \leq d$. If the measurement outcome is k, the (unnormalized) *i*-th postmeasurement state is $(|\phi_k\rangle\langle\tilde{k}|\otimes\mathbb{1}_B) \rho_i^{(AB)} (|\tilde{k}\rangle\langle\phi_k|\otimes\mathbb{1}_B)$. Since the *k*-th POVM element is OP, we get the following equations for all $1 \leq i < i' \leq m$, $1 \leq j \leq r_i$ and $1 \leq j' \leq r_{i'}$.

$$\operatorname{Tr}\left(\left(|\tilde{k}\rangle\langle\tilde{k}|\otimes\mathbb{1}_{B}\rangle\rho_{i}^{(AB)}\left(|\tilde{k}\rangle\langle\tilde{k}|\otimes\mathbb{1}_{B}\rangle\rho_{i'}^{(AB)}\right)=0,\right.$$

$$\implies\left.\left(\rho_{i}^{(AB)}\right)^{\frac{1}{2}}\left(|\tilde{k}\rangle\langle\tilde{k}|\otimes\mathbb{1}_{B}\rangle\left(\rho_{i'}^{(AB)}\right)^{\frac{1}{2}}=0,\right.$$

$$\implies\left._{AB}\langle\psi_{ij}\right|\left(|\tilde{k}\rangle\langle\tilde{k}|\otimes\mathbb{1}_{B}\rangle\left|\psi_{i'j'}\rangle_{AB}=0,\right.$$

$$(3.34)$$

where $\mathbb{1}_{B}$ is the identity operator acting on \mathcal{H}_{B} .

Substituting the expressions for $|\psi_{ij}\rangle_{AB}$ from equation (3.32) in equation (3.35) we get

$$\sum_{b,b'=1}^{n} \langle b|\tilde{k}\rangle (W_i)_{bb'} \langle \tilde{k}|b'\rangle = 0.$$
(3.36)

Since $\{|\tilde{l}\rangle\langle \tilde{l}|\}_{l=1}^{d}$ are elements of a POVM, there exists an $d \times n$ isometry matrix U such that $|\tilde{l}\rangle_{A} = \sum_{l'=1}^{n} U_{ll'} |l'\rangle_{A}$. Using the isometry U, define the following d vectors in \mathbb{C}^{n} : $|\tilde{l}^{*}\rangle \equiv (U_{l1}^{*}, U_{l2}^{*}, \cdots, U_{ln}^{*})^{T}$. Then $\langle \tilde{k} | b' \rangle = U_{kb'}^{*}$. Using this in equation (3.36) implies that $\langle \tilde{k}^{*} | W_{i} | \tilde{k}^{*} \rangle = 0$ which implies that $\langle \tilde{k}^{*} | H_{i} | \tilde{k}^{*} \rangle = \langle \tilde{k}^{*} | A_{i} | \tilde{k}^{*} \rangle = 0$, $\forall i \in I$, because $W_{i} =$ $H_i + iA_i$ and the diagonal elements of H_i and A_i are real. This implies that $|\tilde{k}^*\rangle\langle \tilde{k}^*| \in \mathcal{T}_{\perp}$. Since all other POVM elements are also OP, that implies that $|\tilde{l}^*\rangle\langle \tilde{l}^*|$ corresponding to each POVM element lies in \mathcal{T}_{\perp} . That $\{|\tilde{l}\rangle\langle \tilde{l}|\}_{l=1}^d$ is a POVM for \mathcal{H}_A implies that $\sum_{l=1}^d |\tilde{l}\rangle\langle \tilde{l}|$ $= \mathbb{1}_A$, which implies that $\sum_{l=1}^d |\tilde{l}^*\rangle\langle \tilde{l}^*| = \mathbb{1}_n$. Thus $\{|\tilde{l}^*\rangle\langle \tilde{l}^*|\}_{l=1}^d$ are elements of an extremal rank-one POVM lying in \mathcal{T}_{\perp} . IF Let $\{|\tilde{l}^*\rangle\langle \tilde{l}^*|\}_{l=1}^d \subset \mathcal{T}_{\perp}$ be such that $\sum_{l=1}^d |\tilde{l}^*\rangle\langle \tilde{l}^*| = \mathbb{1}_n$, and let this POVM be extremal as well. It is readily seen that the arguments presented in the ONLY IF part can be easily traced backwards, which leads us to conclude that Alice has a corresponding extremal rank-one OP POVM of the form $\{|\tilde{l}\rangle\langle \tilde{l}|\}_{l=1}^d$.

To summarize theorem 3.2.3: consider the equations $\langle \psi_{ij} | (X_A \otimes \mathbb{1}_B) | \psi_{i'j'} \rangle = 0$, for all $1 \leq j \leq r_i$, $1 \leq j' \leq r_{i'}$, and $1 \leq i < i' \leq m$, where *X* is an unknown, with the constraint that *X* is self-adjoint. The solution set for this *X* is a vector space (of self-adjoint operators on acting on \mathcal{H}_A). Note that these *X*'s are all orthogonality preserving. Now, \mathcal{T}_{\perp} is isomorphic to this vector space in the following sense: \mathcal{T}_{\perp} is the complex conjugate of the representation of the *X*'s in the standard ONB { $|j\rangle_A$ }^{*n*}_{*j*=1}. Now, in the vector space of $n \times n$ hermitian matrices, \mathcal{T}_{\perp} is the complement of \mathcal{T} , which can be constructed as mentioned in equation (3.33).

In those cases were Bob starts the protocol we should examine $\mathcal{T}_{\perp}^{(B)}$, instead of $\mathcal{T}_{\perp}^{(A)}$ (which is denoted by \mathcal{T}_{\perp} for ease of notation), to see if $\mathcal{T}_{\perp}^{(B)}$ contains all elements of some rankone POVM. Note that $\mathcal{T}_{\perp}^{(B)}$ is defined to be the complement of $\mathcal{T}^{(B)}$ in S, where $\mathcal{T}^{(B)}$ is defined just such as \mathcal{T} was in equation (3.33), with the difference that W_i takes the form $W_{ij}W_{i'j'}^{\dagger}$, rather than $W_{ij}^{\dagger}W_{i'j'}$.

A subspace of matrices is abelian if any pair of matrices in it commute. All matrices in any abelian subspace of S can be diagonalized in some common eigenbasis. When the dimension of an abelian subspace of S is n, then that abelian subspace is the real space of *all* matrices which are diagonal in the common eigenbasis of the abelian subspace. Since no abelian subspace of dimension greater than n can exist in S, such an abelian subspace is called a maximally abelian subspace (MAS); it has a unique common eigenbasis associated to it.

Corollary 3.2.3.1. The states $\rho_1^{(AB)}, \rho_2^{(AB)}, \cdots, \rho_m^{(AB)}$ are perfectly distinguishable by 1-LOCC using only projective measurements on \mathcal{H}_A and \mathcal{H}_B , if and only if \mathcal{T}_{\perp} contains a MAS.

Proof. **ONLY IF**: Let the states $\rho_1^{(AB)}, \rho_2^{(AB)}, \dots, \rho_m^{(AB)}$ be perfectly distinguishable by 1-LOCC using only projective measurements on \mathcal{H}_A and \mathcal{H}_B . This implies that Alice can perform an OP rank-one projective measurement $\{|k\rangle\langle k|\}_{k=1}^n$. Then (the ONLY IF part of) theorem 3.2.3 implies that \mathcal{T}_{\perp} contains all projectors of a rank-one projective measurement $\{|k^*\rangle\langle k^*|\}_{k=1}^n$. $span(\{|k^*\rangle\langle k^*|\}_{k=1}^n)$ is then a MAS in \mathcal{T}_{\perp} . **IF**: Assume that \mathcal{T}_{\perp} contains a MAS of *S*. One can associate a MAS with a unique orthonormal eigenbasis, which is such that any matrix in said MAS is diagonal when represented in that eigenbasis. Since \mathcal{T}_{\perp} contains the MAS, it will also contain all the rank-one projectors, which project onto the vectors of the MAS' eigenbasis. Collectively, this is the rank-one projective measurement $\{|k^*\rangle\langle k^*|\}_{k=1}^n$, which is a rank-one extremal POVM on \mathbb{C}^n . Then (the IF part of) theorem 3.2.3 implies that a corresponding OP rank-one projective measurement $\{|k\rangle\langle k|\}_{k=1}^n$ exists on Alice's system, hence the states are distinguishable by 1-LOCC using only projective measurements.

Corollary 3.2.3.1 reformulates the question of the existence of a 1-LOCC distinguishability protocol, which employs only rank-one projective measurements, to the question of the existence of a MAS in \mathcal{T}_{\perp} . In fact, there are as many protocols for 1-LOCC distinguishability using only projective measurements, as there are MAS'es in \mathcal{T}_{\perp} . This is demonstrated by the following example. Define the following states in $\mathbb{C}^4 \otimes \mathbb{C}^4$:

$$|\psi_{st}\rangle_{AB} \equiv \sum_{j,k=0}^{3} (W_{st})_{kj} |j\rangle_{A} |k\rangle_{B}, \qquad (3.37)$$

where $(W_{st})_{kj} \equiv \frac{e^{\frac{i\pi js}{2}}}{2} \delta_{j\oplus_4 t,k}, \forall j, k = 0, 1, 2, 3$. Note that any two W_{st} matrices are pairwise orthogonal.

Example 3.2.1. For the 1-LOCC of the set of states $\{|\psi_{00}\rangle_{AB}, |\psi_{02}\rangle_{AB}, |\psi_{20}\rangle_{AB}, |\psi_{22}\rangle_{AB}\}$, it is easy to verify that \mathcal{T} is spanned by the hermitian matrices W_{02}, W_{20} , and W_{22} , all of which commute with each other. These set of matrices share a unique common eigenbasis, which is $S_1 = \{\frac{(1.0, 1.0)^T}{\sqrt{2}}, \frac{(0.1.0, 1)^T}{\sqrt{2}}, \frac{(0.1.0, 1)^T}{\sqrt{2}}\}$. Consider the following ONB for $\mathbb{C}^4 \otimes \mathbb{C}^4$:

$$S_{2} = \{ \frac{(1,1,0,0)^{T}}{\sqrt{2}}, \frac{(1,-1,0,0)^{T}}{\sqrt{2}}, \frac{(0,0,1,1)^{T}}{\sqrt{2}}, \frac{(0,0,1,-1)^{T}}{\sqrt{2}} \},$$

$$S_{3} = \{ \frac{(1,i,-i,1)^{T}}{2}, \frac{(1,-i,i,1)^{T}}{2}, \frac{(1,-i,-i,-1)^{T}}{2}, \frac{(1,i,i,-1)^{T}}{2} \},$$

$$S_{4} = \{ \frac{(1,0,0,i)^{T}}{\sqrt{2}}, \frac{(1,0,0,-i)^{T}}{\sqrt{2}}, \frac{(0,1,i,0)^{T}}{\sqrt{2}}, \frac{(0,1,-i,0)^{T}}{\sqrt{2}} \} \text{ and }$$

$$S_{5} = \{ \frac{(1,i,i,1)^{T}}{2}, \frac{(1,i,-i,-1)^{T}}{2}, \frac{(1,-i,-i,1)^{T}}{2}, \frac{(1,-i,-1,1)^{T}}{2} \}.$$

 $\{S_i\}_{i=1}^5$ are five mutually unbiased bases for $\mathbb{C}^4 \otimes \mathbb{C}^4$. It's easily verified that for i = 2, 3, 4, 5 each of the three matrices W_{02} , W_{20} , W_{22} is orthogonal to any matrix whose eigenbasis is S_i . Thus each S_i corresponds to a MAS in \mathcal{T}_{\perp} , corresponding to which there's a rank-one projective measurement. Using the correspondence⁴: $(a, b, c, d)^T \rightarrow a^* |0\rangle_A + b^* |0\rangle_A + c^* |0\rangle_A + d^* |0\rangle_A$, the ONB corresponding to each of these rank-one projective measurements, can be obtained from S_i , for i = 2, 3, 4, 5. It's easily verified that each rank-one projective measurement is OP, and can be used by Alice to initiate a corresponding 1-LOCC protocol for distinguishing the given set of states.

The significance of corollary 3.2.3.1 is that for certain values of $\dim \mathcal{T}_{\perp}$, it is easy to check if \mathcal{T}_{\perp} contains a MAS or not, which immediately indicates the existence or non-existence of a 1-LOCC protocol (which employs only rank-one projective measurements). It will be seen that subsequent corollaries and theorems depend on corollary 3.2.3.1.

That said, the non-existence of a MAS in \mathcal{T}_{\perp} does not rule out the existence of a nonprojective extremal rank-one POVM $\{|\tilde{l}^*\rangle\langle\tilde{l}^*|\}_{l=1}^d$ in \mathcal{T}_{\perp} , where d > n. Theorem (3.2.3) tells us that if such a non-projective extremal rank-one POVM exists in \mathcal{T}_{\perp} , then there exists a 1-LOCC distinguishability protocol which commences with an OP non-projective

 $[\]overline{4}$ To understand this correspondence, refer to equation (3.36), and the paragraph after equation (3.36) in theorem 3.2.3.

extremal rank-one POVM with POVM elements $\{|\tilde{l}\rangle\langle \tilde{l}|\}_{l=1}^d$. In such a case one can consider \mathcal{H}_A to be a *n*-dimensional subspace of an extended *d*-dimensional space $\mathcal{H}_{A'}$, so that the states $|\psi_i\rangle_{AB} \longrightarrow |\psi_i\rangle_{A'B}$ lie in $\mathcal{H}_{A'} \otimes \mathcal{H}_B$. Then S', \mathcal{T}' and \mathcal{T}'_{\perp} are spaces of $d \times d$ hermitian matrices corresponding to Alice's extended space $\mathcal{H}_{A'}$, and \mathcal{T}'_{\perp} will contain an *d*-dimensional MAS, which corresponds to an *d*-element rank one projective measurement on $\mathcal{H}_{A'}$. This *d*-element projective measurement takes the form of the aforementioned non-projective OP extremal rank-one POVM on \mathcal{H}_A , i.e., $\{|\tilde{l}\rangle\langle \tilde{l}|\}_{l=1}^d$. Note that since the POVM elements of any extremal rank-one POVM are linearly independent [72], $d \le n^2$. It is sensible to search for an *d*-dimensional MAS in \mathcal{T}'_{\perp} only after establishing that \mathcal{T}_{\perp} doesn't contain a *n*-dimensional MAS. Often the value of $dim\mathcal{T}_{\perp}$ itself gives us information about OP rank-one POVMs which Alice can perform. For e.g., Walgate et al's result in [12], which says that any two orthogonal bipartite pure states are 1-LOCC distinguishable, corresponds to the case m = 2 which correspond to the classes $dim\mathcal{T}_{\perp} \ge n^2 - 2$. I now give an alternative proof of Walgate et al's result for the case m = 2.

Theorem 3.2.4. When $dim \mathcal{T}_{\perp} \ge n^2 - 2$, \mathcal{T}_{\perp} always contains a MAS.

Proof. This proof is by induction. Assume that $\dim \mathcal{T}_{\perp} = n^2 - 2$. This implies that $\dim \mathcal{T} = 2$. Let *A* and *H* be two linearly independent $n \times n$ matrices in \mathcal{T} . *Proposition* P(n): For any two $n \times n$ hermitian matrices *H* and *A*, there exists a $n \times n$ unitary *U*, so that the diagonals of $U^{\dagger}HU$ and $U^{\dagger}AU$ are multiples of $\mathbb{1}_n$. It's known that P(2) is true [12]. The goal is to prove that P(n + 1) is true assuming that P(n) is true. Let *H* and *A* be two $n + 1 \times n + 1$ traceless hermitian matrices. Let H_n and A_n be their $n \times n$ upper diagonal block matrices. Since P(n) is true, there is a $n \times n$ unitary V_n , so that diagonals of $V_n^{\dagger}H_nV_n$ and $V_n^{\dagger}A_nV_n$ are multiples of $\mathbb{1}_n$. Embedd V_n as the $n \times n$ upper diagonal block of a $n + 1 \times n + 1$ unitary *V* whose n + 1-th diagonal element is 1. Then it is easy to see that the diagonals of the $n \times n$ upper diagonal block of $V^{\dagger}HV$ and $V^{\dagger}AV$ are scalar multiples of $\mathbb{1}_n$. Since $V^{\dagger}HV$ and $V^{\dagger}AV$ are traceless, their diagonals are scalar multiples of matrix $D_{\lambda} \equiv \frac{1}{\sqrt{n(n+1)}}Diag(1, 1, \dots, 1, -n)$, which is traceless. Let $V^{\dagger}HV$ and $V^{\dagger}AV$ have components α and $\beta \in \mathbb{R}$ along D_{λ} . Then $A' \equiv \frac{1}{\sqrt{n^2+\beta^2}}(-\beta V^{\dagger}HV + \alpha V^{\dagger}AV)$ has a

zero diagonal, and component of D_{λ} along $H' \equiv \frac{1}{\sqrt{a^2 + \beta^2}} (\alpha V^{\dagger} HV + \beta V^{\dagger} AV)$ is 1. Let the (n, n + 1)-th matrix element of A' be $ae^{-i\phi}$. Define $D_u \equiv Diag(1, 1, \dots, 1, e^{\frac{-i(n+2\phi)}{4}}, e^{\frac{i(n+2\phi)}{4}})$, then the 2 × 2 lower diagonal block of $A'' \equiv D_u^{\dagger} A' D_u$ is a scalar multiple of σ_y . The diagonal of $H'' \equiv D_u^{\dagger} H' D_u$ remains invariant. Let the real part of the (n, n + 1)-th matrix element of H'' be h. Using an SO(2) transformation, rotate between the n-th and n + 1-th matrix elements of H'' to obtain H''', while keeping all other elements fixed. A'' will remain invariant. Thus the real part of the 2×2 lower diagonal block of H''' will undergo the transformation

$$\begin{pmatrix} 1 & h \\ h & -n \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & h \\ h & -n \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1-n}{2} + \frac{1+n}{2}\cos\theta + h\sin\theta & h\cos\theta - \frac{1+n}{2}\sin\theta \\ h\cos\theta - \frac{1+n}{2}\sin\theta & \frac{1-n}{2} - \frac{1+n}{2}\cos\theta - h\sin\theta \end{pmatrix}$$

I want to solve for θ in the equation: $\frac{1-n}{2} - \frac{1+n}{2}\cos\theta - h\sin\theta = 0$. When $\theta = 0$, the LHS is -n and when $\theta = \pi$, the LHS is 1. Since the LHS is a continuous function of θ , there must be some $\theta \in (0, \pi)$ for which the LHS is zero. Choose θ to be this value. Then H''' and A'' are matrices whose n + 1-th diagonal elements are both zero. Using P(n) on the $n \times n$ upper diagonal blocks of H''' and A'', H''' and A'' can be rotated to obtain corresponding matrices whose diagonals are zero and which span the correspondingly rotated \mathcal{T} . Then the correspondingly rotated \mathcal{T}_{\perp} contains all diagonal matrices which span a MAS.

Another example: when $dim \mathcal{T}_{\perp} = 1$, Ye et al [10] showed that the states aren't distinguishable by LOCC at all. When limiting protocols to 1-LOCC, the theorem 3.2.5 makes a stronger statement.

Theorem 3.2.5. If $dim \mathcal{T}_{\perp} \leq n - 1$, there is no 1-LOCC protocol which Alice can initiate to distinguish the states.

Proof. Theorem 3.2.3 implies if the states are distinguishable by a 1-LOCC protocol, then \mathcal{T}_{\perp} contains an extremal rank-one POVM. The number of POVM elements in such

a POVM will always be $\ge n$ and these POVM elements are always linearly independent [72]. Hence $dim\mathcal{T}_{\perp}$ must be $\ge n$ for it to contain all POVM elements of an extremal rank one POVM.

Having covered $dim \mathcal{T}_{\perp} < n$, I now move onto the case for $dim \mathcal{T}_{\perp} = n$.

Theorem 3.2.6. When $dim \mathcal{T}_{\perp} = n$, the states are distinguishable by 1-LOCC if and only if \mathcal{T}_{\perp} is a MAS of S.

Proof. The IF part is already covered in corollary 3.2.3.1. **ONLY IF:** Given that $\dim \mathcal{T}_{\perp} = n$. Suppose that the states are distinguishable by 1-LOCC. Theorem 3.2.3 then implies that \mathcal{T}_{\perp} contains all POVM elements of an extremal rank-one POVM $\{|\tilde{k}^*\rangle\langle \tilde{k}^*|\}_{k=1}^d$. That the POVM elements of a rank-one extremal POVM are linearly independent [73] implies that d = n (since $|\tilde{k}^*\rangle\langle \tilde{k}^*| \in \mathcal{T}_{\perp}$, and $\dim \mathcal{T}_{\perp} = n$). This implies that the isometric matrix relating $\{|\tilde{k}^*\rangle\}_{k=1}^n$ to any ONB of \mathbb{C}^n has to be a $n \times n$ unitary matrix, which implies that $\{|\tilde{k}^*\rangle\langle \tilde{k}^*|\}_{k=1}^n \longrightarrow \{|k^*\rangle\langle k^*|\}_{k=1}^n$ is a rank-one projective measurement. Since $span(\{|k^*\rangle\langle k^*|\}_{k=1}^n) = \mathcal{T}_{\perp}, \mathcal{T}_{\perp}$ is a MAS of S.

Consider the special case where m = n and when the states are pure: $\rho_i^{(AB)} \longrightarrow |\psi_i\rangle_{AB}$. Also then W_{ij} matrices change: $W_{ij} \longrightarrow W_i$, and the index set I is $\{(i, i'), \forall 1 \le i < i' \le n\}$. The cardinality of I now is $\frac{n(n-1)}{2}$. One can generally expect $\{H_i, A_i\}_{i \in I}$ to be a linearly independent set, which, in the case of m = n implies that $dim\mathcal{T} = n(n-1)$ and $dim\mathcal{T}_{\perp} = n$ for almost all sets of n orthogonal states in $\mathcal{H}_A \otimes \mathcal{H}_B$. This is indeed the case; consider corollary 3.2.6.1.

Corollary 3.2.6.1. Theorem 3.2.6 gives the necessary and sufficient condition for the 1-LOCC distinguishability of almost all sets of *n* orthogonal pure states from $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e., it gives an algorithm to compute whether or not almost any set of *n* orthogonal bipartite pure states from $\mathcal{H}_A \otimes \mathcal{H}_B$ is distinguishable by 1-LOCC.

The proof for this is essentially the same as the proof which Cohen gave [13] for stating that almost all sets of $n \ge n+1$ orthogonal multipartite qudit states in $n^{\otimes N}$ systems ($N \ge 2$) are locally indistinguishable, but for the sake of completeness I give a rigorous proof for this case in the appendix B. I now give an example for theorem 3.2.6.

Example 3.2.2. For the 1-LOCC of the set of states $\{|\psi_{00}\rangle_{AB}, |\psi_{01}\rangle_{AB}, |\psi_{10}\rangle_{AB}, |\psi_{33}\rangle_{AB}\}, \mathcal{T}$ is spanned by the following hermitian matrices: $\frac{W_{01}+W_{03}}{2}, \frac{W_{01}-W_{03}}{2i}, \frac{W_{10}-W_{30}}{2}, \frac{W_{10}-W_{30}}{2i}, \frac{W_{33}+iW_{11}}{2}, \frac{W_{01}-W_{30}}{2i}, \frac{W_{13}-iW_{31}}{2i}, \frac{W_{32}+W_{12}}{2}, \frac{W_{32}-W_{12}}{2}, \frac{W_{23}-W_{21}}{2}, \frac{W_{23}+W_{21}}{2i}$. Hence $dim\mathcal{T} = 12$. This implies that $dim\mathcal{T}_{\perp} = 4$, where \mathcal{T}_{\perp} is spanned by the hermitian matrices $\mathbb{1}_4, W_{22}, W_{02}$, and W_{20} . Note that all these matrices commute with each other (see example 3.2.1). Thus \mathcal{T}_{\perp} is a MAS. The common eigenbasis, which diagonalizes any matrix in \mathcal{T}_{\perp} is $\{\frac{1}{\sqrt{2}}(1, 0, 1, 0)^T, \frac{1}{\sqrt{2}}(0, 1, 0, 1)^T, \frac{1}{\sqrt{2}}(0, 1, 0, -1)^T\}$. It's then seen that Alice can initiate a 1-LOCC protocol to distinguish the given set of states by performing rank-one projective measurement in the ONB $\{\frac{|0|A+|2|A}{\sqrt{2}}, \frac{|0|A-|2|A}{\sqrt{2}}, \frac{|1|A+|3|A}{\sqrt{2}}, \frac{|1|A-|3|A}{\sqrt{2}}\}$.

From the discussion so far it has been established that when $\dim \mathcal{T}_{\perp} \leq n$ or when $\dim \mathcal{T}_{\perp} \geq n^2 - 2$, it can conclusively be said if \mathcal{T}_{\perp} contains a MAS or not. In the cases when $n + 1 \leq \dim \mathcal{T}_{\perp} \leq n^2 - 3$, it is difficult to obtain a general algorithm which conclusively establishes if \mathcal{T}_{\perp} contains a MAS or not. For certain values of $\dim \mathcal{T}_{\perp}$ greater than n (and smaller than n^2), I will give a necessary condition for \mathcal{T}_{\perp} to not contain a MAS. For that consider the following: let $\dim \mathcal{T}_{\perp} = n + t$, where $t \geq 1$. Let $\{T_i\}_{i=0}^{n+t-1}$ be an ONB for \mathcal{T}_{\perp} , with $T_0 = \frac{1}{\sqrt{n}} \mathbb{1}_n$. Let C be the real vector space, spanned by the matrices in $\{i[T_j, T_k] = 0 \leq j < k \leq n + t - 1\}$, where $[T_j, T_k] \equiv T_j T_k - T_k T_j$.

Theorem 3.2.7. When $1 \le t \le \sqrt{3n^2 - 3n + \frac{1}{4}} - (n - \frac{3}{2})$, \mathcal{T}_{\perp} contains no MAS if *dimC* > $tn + \frac{t(t-3)}{2}$.

Proof. Since $T_0 = \frac{1}{\sqrt{n}} \mathbb{1}_n$, varying over the indices $0 \le j < k \le n + t - 1$, one obtains $\frac{(n+t-1)(n+t-2)}{2}$ commutators $i[T_j, T_k]$. If \mathcal{T}_{\perp} contains a MAS, choose the ONB $\{T_i\}_{i=0}^{n+t-1}$ such that $\{T_i\}_{i=0}^{n-1}$ is an ONB for this MAS, where again $T_0 = \frac{1}{\sqrt{n}} \mathbb{1}_n$. Then $i[T_j, T_k] = 0, \forall 0 \le 1$
$j < k \le n-1$, which implies that $\frac{(n-1)(n-2)}{2}$ of the $\frac{(n+t-1)(n+t-2)}{2}$ aforementioned commutators are zero, which implies that *dimC* can be at most $Min\left\{tn + \frac{t(t-3)}{2}, n^2 - 1\right\}$.

Now assume that one doesn't know if \mathcal{T}_{\perp} contains a MAS or not. When $1 \le t \le \sqrt{3n^2 - 3n + \frac{1}{4}} - (n - \frac{3}{2})$ then $n \le tn + \frac{t(t-3)}{2} \le n^2 - 1$. In such a case if $dimC > tn + \frac{t(t-3)}{2}$, then that implies that \mathcal{T}_{\perp} contains no MAS.

For $\dim \mathcal{T}_{\perp} = n + 1$, one can give a necessary and a sufficient condition for \mathcal{T}_{\perp} to contain a MAS. For that consider the following: let $\{G_i\}_{i=1}^{\dim C}$ be an ONB for *C*. Then for each $j \in \{1, 2, \dots, \dim C\}$, define the $n + 1 \times n + 1$ real antisymmetric matrix Γ_j , whose matrix elements are given by $(\Gamma_j)_{kl} = i \operatorname{Tr}(G_j[T_k, T_l])$. Let \mathcal{G} be the real vector space obtained by spanning the set $\{\Gamma_j\}_{j=1}^{\dim C}$ on the field of real numbers. Let $\{\Omega_j\}_{j=1}^{\dim \mathcal{G}}$ be an ONB for \mathcal{G} . Theorem 2.4.2 implies that if $\dim C > n$, then \mathcal{T}_{\perp} doesn't contain a MAS. Hence assume that $\dim C \leq n$. This implies that $\dim \mathcal{G} \leq n$ too.

Theorem 3.2.8. When $dim\mathcal{T}_{\perp} = n + 1$, \mathcal{T}_{\perp} contains a MAS if and only if Ω_j is rank 2 for all $j = 1, 2, \cdots, dim\mathcal{G}$ and $\bigcap_{i=1}^{dim\mathcal{G}} Supp(\Omega_j)$ is one dimensional.

Proof. **IF** Assume that Ω_j is rank 2, $\forall 1 \leq j \leq dim\mathcal{G}$, and $\bigcap_{j=1}^{dim\mathcal{G}} Supp(\Omega_j)$ is one dimensional, spanned by the real (n + 1)-tuple $\underline{e}_{n+1} \equiv (e_{1n+1}, e_{2n+1}, \cdots, e_{n+1n+1})^T$. Since Ω_j is anti-symmetric and real, and since it is rank 2, there exists a real (n + 1)-tuple $\underline{e}_j \equiv (e_{1j}, e_{2j}, \cdots, e_{n+1j})^T$ so that $\Omega_j = \underline{e}_{n+1} \underline{e}_j^T - \underline{e}_j \underline{e}_{n+1}^T$. In fact, there is a degree of freedom in choosing \underline{e}_j : Ω_j is invariant for any arbitrary value of the inner product $\underline{e}_j^T \cdot \underline{e}_{n+1}$. Choose \underline{e}_j to be orthogonal to \underline{e}_{n+1} . In that case, for Ω_j to be orthogonal to $\Omega_{j'}$ (Ω_j and $\Omega_{j'}$ belong to an ONB for \mathcal{G}), we require the inner product $\underline{e}_j^T \cdot \underline{e}_{j'} = 0$, for $j \neq j'$. Let $\Gamma_j = \sum_{k=1}^{dim\mathcal{G}} \alpha_{kj} \Omega_k = \underline{e}_{n+1} \underline{g}_j^T - \underline{g}_j \underline{e}_{n+1}^T$, where $\underline{g}_j \equiv \sum_{k=1}^{dim\mathcal{G}} \alpha_{kj} \underline{e}_k$. Hence Γ_j are also rank 2 matrices. If $dim\mathcal{G} < n$, then complete the basis $\{\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_{dim\mathcal{G}} \in \underline{e}_{dim\mathcal{G}+1}, \cdots, \underline{e}_{n+1}\}$. One can normalize Ω_j to be such that $\{\underline{e}_j\}_{j=1}^{n+1}$ is an ONB for \mathbb{C}^{n+1} . Arrange \underline{e}_j^T as rows of a $n + 1 \times n + 1$ orthogonal matrix O in ascending order of j from 0 to n+1. Then $O\Gamma_jO^T$ is such that its $n \times n$ upper diagonal block is

zero, i.e., for all $1 \le j \le dimC$ and for $1 \le k, l \le n$

$$\left(O\Gamma_j O^T \right)_{kl} = 0 \implies \sum_{s,t=1}^{n+1} O_{ks} \operatorname{Tr} \left(G_j [T_s, T_t] \right) O_{tl}^T = 0,$$

$$\implies \operatorname{Tr} \left(G_j [T'_k, T'_l] \right) = 0,$$

$$(3.38)$$

where $T'_k \equiv \sum_{l=1}^{n+1} O_{kl}T_l$. Since $T'_k \in \mathcal{T}_{\perp}$, $[T'_k, T'_l] \in C$. But since $\{G_j\}_{j=1}^{\dim C}$ is an ONB for *C*, equation (3.38) implies that $[T'_k, T'_l] = 0$ when $1 \le k, l \le n$. This implies that $\{T'_j\}_{j=1}^n$ is a MAS in \mathcal{T}_{\perp} . **ONLY IF** Assume that \mathcal{T}_{\perp} contains a MAS and let $\{T_j\}_{j=1}^n$ be an ONB for this MAS. Then $\operatorname{Tr}(G_j[T_k, T_l]) = 0$ when $1 \le k, l \le n$. This implies that the $n \times n$ upper diagonal block of Γ_j is zero, which makes it a rank 2 matrix. The same is true for $\{\Omega_j\}_{j=1}^{\dim \mathcal{G}}$, since it is an ONB for $\operatorname{span}(\{\Gamma_j\}_{j=1}^{\dim \mathcal{C}})$. Now the only non-zero entries in Ω_j are along the n + 1-th column and the n + 1-th row. For Ω_j and $\Omega_{j'}$ to be orthogonal one requires their corresponding n + 1-th columns (and n + 1-th rows) to be orthogonal as well. This implies $\bigcap_{j=1}^{\dim \mathcal{G}} \operatorname{Supp}(\Omega_j)$ is spanned by the vector $(0, 0, \dots, 0, 1)^T$, and is hence one dimensional.

Finally, I give an example of the utility of theorem (3.2.8).

Example 3.2.3. For set of states $\{|\psi_{00}\rangle_{AB}, |\psi_{01}\rangle_{AB}, |\psi_{12}\rangle_{AB}, |\psi_{30}\rangle_{AB}\}, \mathcal{T}_{\perp}$ is spanned by $\{T_1 = \mathbb{1}_4, T_2 = W_{02}, T_3 = \frac{W_{21} - W_{23}}{2}, T_4 = \frac{W_{21} + W_{23}}{2i}, T_5 = W_{20}\}$. *C* is spanned by

$$G_{1} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, G_{2} = \frac{i}{2\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$
 (3.39)

 \mathcal{G} is spanned by the following matrices

 Ω_1 and Ω_2 are rank 2 and $Supp(\Omega_1) \cap Supp(\Omega_2)$ is spanned by $(0, 0, 0, 0, 1)^T$. Since $(\Gamma_j)_{kl} = i \operatorname{Tr}(G_j[T_k, T_l])$, the fact that the 4 × 4 upper diagonal block of Γ_1 are zero implies that T_1, T_2, T_3, T_4 span a MAS in \mathcal{T}_\perp . Upon computing the common eigenbasis of this MAS, and using the correspondence given in example 3.2.1, we obtained the ONB: $\left\{\frac{|0\rangle_A + |1\rangle_A + |2\rangle_A + |3\rangle_A}{2}, \frac{|0\rangle_A - |1\rangle_A - |2\rangle_A + |3\rangle_A}{2}, \frac{|0\rangle_A + |1\rangle_A - |2\rangle_A - |3\rangle_A}{2}\right\}$. Then Alice can initiate a 1-LOCC protocol to distinguish the given states by measuring in this ONB.

3.2.2 Summary

As mentioned earlier, I made two assumptions in the beginning of this section to keep the notation simpler: that Alice always starts the 1-LOCC protocol and that the dimensions of Alice's and Bob's subsystems are equal. The results derived under these assumptions actually hold for the more general scenarios where both Alice or Bob start the 1-LOCC protocol and when the dimensions of Alice's and Bob's subsystems are unequal. A broad summary of the results in this section can then be given as follows: for the *i*-th party of the $n_A \otimes n_B$ dimensional bipartite system, the set of all sets of orthogonal bipartite states can be partitioned into different classes, based on the value of $dim \mathcal{T}_{\perp}^{(i)}$ of each set of orthogonal bipartite states. Sweeping results about the existence of 1-LOCC distinguishability protocols, which the *i*-th party can initiate, can be made about all sets of orthogonal bipartite states, which lie in certain classes for example: (i) if $dim \mathcal{T}_{\perp}^{(i)} < n$ there is no rank-one POVM which the *i*-th party can initiate the protocol with, (ii) if $dim \mathcal{T}_{\perp} = n$, such a protocol exists if and only if $\mathcal{T}_{\perp}^{(i)}$ is a MAS, (iii) when $\dim \mathcal{T}_{\perp}^{(i)} \ge n_i^2 - 2$, such a protocol will always exist. Hence, in one sweep, once can establish results for all sets of orthogonal states which fall in a class, by proving said result for that class. To add a final comment on the usefulness of this framework: note that in [13], Cohen used the same structure to show that almost all sets of $\geq n + 1$ orthogonal *N*-qudit multipartite states (in $(\mathbb{C}^n)^{\otimes N}$) are not distinguishable by LOCC. Putting all this together, I hence argue that a deeper study of this structure will be a rewarding experience for studying problems of distinguishability

of orthogonal states by LOCC.

3.2.3 Future Directions

The significance noted above implies that it would be rewarding to study more deeply the framework proposed. I now give a list of tentative questions that I wish to pursue.

- 1. In [70] I gave necessary and sufficient conditions for the existence of 1-LOCC distinguishability protocols, which are initiated with projective measurements, for classes corresponding to the range $2 \le dim T_{\perp} \le n + 1$. It is natural to ask how one can extract similar information from classes corresponding to the range $n + 2 \le dim T_{\perp} \le n^2 3$.
 - (a) I anticipate that the necessary and sufficient condition for the existence of such a 1-LOCC protocol (which starts with a projective measurement) for the local distinguishability of sets in classes corresponding to dimT_⊥ ≥ n + 2 will be similar to the necessary and sufficient conditions obtained for the class corresponding to dimT_⊥ = n + 1.
 - (b) Walgate et al's result [12] tells us that all sets in the classes corresponding to dimT_⊥ ≥ n² - 2, are locally distinguishable by 1-LOCC. This begs the following question: is there some value of k > 2, so that sets in classes corresponding to dimT_⊥ ≥ n² - k are always distinguishable by 1-LOCC? More generally, for each value of n can one find some k > 2 such that sets in classes corresponding to dimT_⊥ ≥ n² - k are always 1-LOCC distinguishable?
- 2. Generalization of the framework to two-way LOCC (2-LOCC) and beyond: just as the information content of all 1-LOCC local distinguishability protocols is contained in a single subspace of hermitian matrices \mathcal{T}_{\perp} , the information content of all 2-LOCC protocols is distributed across different subspaces of hermitian matrices.

These subspaces can be 'arranged' as a heirarchy: the first one in the heirarchy contains information of all orthogonality preserving measurements that the first party can initiate, and for each outcome of that measurement, there is a different postmeasurement subspace of hermitian matrices which contains the information for the remaining part of the LOCC protocol. While this heirarchy of subspaces of hermitian matrices makes the structure for 2-LOCC more difficult to grasp, simple examples like the ones given by Nathanson in [11], could give us perspective on finer details of the structure.

Chapter 4

Discussion

In this chapter I will summarize the contents of this thesis and discuss the reasons why the contents of this thesis will be of interest to the quantum information community, particularly the community interested in studying quantum state discrimination. The next part of the chapter will be devoted to enunciating anticipated future directions which this thesis leads me to.

4.1 Summary and Significance

In this thesis were studies two different kinds of problems in quantum state discrimination: minimum error discrimination and local distinguishability of quantum states. It is seen that while the task underpinning these problems is the same - to discriminate among different states in an ensemble - there is little else that both problems share in common. In particular, the mathematical structure of both problems bear little resemblance to each other. This is also in part due to the fact that the nature of discrimination in both problems is different: in the MED problems, I sought to obtain the optimal probability of success for discriminating among non-orthogonal states, whereas in the local distinguishability problem one I sought to given conditions for the perfect local distinguishability of orthogonal bipartite states. That is, in the former case, one seeks to minimize error, while in the latter case one seeks a yes or no answer to whether one can perfectly distinguish the given states locally or not. Both problems are fairly challenging.

The MED problem: The algebraic structure of the MED of LI pure state ensembles is known, and I exploited this structure, to give a technique to compute the optimal POVM and optimal success probability for all LI pure state ensembles. And I generalized this structure to ensembles of LI mixed states and showed how to compute the optimal POVM and success probability for all LI mixed state ensembles as well. I now give some salient features of the studied problem which maybe of interest for future directions.

- i. While the structure of the MED problem for LI pure states was discovered a long time back [1, 2, 3], it was never used to obtain the optimal POVM for the MED of said ensembles. In this work, said structure was used to arrive at the solution. This *completes* the work done in [1, 2, 3] by using the derived structure to arrive at the solution. Similarly, said structure has been generalized to mixed state ensembles, for which solutions can also be obtained.
- ii. The algorithm is simple to implement. In particular, Newton-Raphson's method is the only prerequisite to apply the technique.
- iii. In recent years, there has been a marked shift in adopting a geometric approach to solving the MED problem. This is evident by the amount of work which has been done in just the last three years [49, 74, 75, 48, 50]. The content of my work in MED is based on the algebraic structure of the problem and hence stands out by contrast to contemporary work on the topic.
- iv. The relation between the MED for LI mixed state ensembles and MED for LI pure state ensembles as given by theorem 2.5.12, tells us that for each LI mixed state ensemble, there is a corresponding pure state decomposition, such that the optimal POVM for the MED of the former decomposes into the optimal POVM for the

MED of the latter. This relation between MED of LI mixed state ensembles and LI pure state ensembles bears a resemblance with the composition law of Shannon entropy [64, 76], i.e., when the events A, B, etc in the sample space can be divided into more events A1, A2, A3, etc, B1, B2, B3, etc, then the Shannon entropy becomes a weighted average over the Shannon entropies of $H(A1, A2, A3, \cdots)$, $H(B1, B2, B3, \cdots)$, etc.

Local Distinguishability of Quantum States: The motivation underlying the decision to take up problems in the topic of perfect local distinguishability of quantum states was the desire to obtain certain general conditions which sets of orthogonal bipartite states need to satisfy for states in them to be locally distinguishable.

The first of these results gives a powerful necessary condition for the local distinguishability of MES in $\mathbb{C}^n \otimes \mathbb{C}^n$ systems. Given the elevated status which MES enjoy in quantum information theory, the local distinguishability of MES has gained significant traction over the past few years [61, 60, 9, 68, 62, 11, 77, 78, 79]. In [9], Nathanson used an upper bound to the locally accessible information to derive the condition that no more than *n* orthogonal MES in $\mathbb{C}^n \otimes \mathbb{C}^n$ systems can be locally discriminated. Me and my colleagues (Ramij Rahaman, Sibasish Ghosh and Guruprasad Kar) anticipated that this upper bound could be used extract more information about the local distinguishability of MES, particularly for *n* MES in $\mathbb{C}^d \otimes \mathbb{C}^d$ bipartite quantum systems. It was this hunch that eventually lead to the necessary condition (3.15). I now list the following reasons which explain why this result will be of interest to the quantum information community.

- i. The necessary condition (3.15) is simple to test for any set of *n* MES in $\mathbb{C}^n \otimes \mathbb{C}^n$ systems.
- ii. The condition is very powerful as is demonstrated in subsection 3.1.4, where it was shown that this condition is sufficient for the local distinguishability of some sets of MES.

The second of the results gives a framework for local distinguishability of orthogonal bipartite states by one-way LOCC. In particular, the existence of a one-way LOCC protocol to distinguish a given set of states in $n \otimes n$ quantum systems is associated with the existence of a maximally abelian subspace in a subspace of $n \times n$ hermitian matrices. In an earlier work, Ye et al. [10] introduced these subspaces, which Cohen also used in [13] to show that almost all sets of n + 1 multipartite states in $n^{\otimes N}$ quantum systems are not distinguishable by LOCC. Thus the existence of these subspaces was already known before, but despite the significance of Cohen's result, there was little mention of these subspaces in the literature. I undertook the exercise to show that these subspaces contain information of all one-way LOCC protocol which can be initiated to discriminate the set of states. Hitherto results on local distinguishability of orthogonal bipartite states were based on the geometric and algebraic properties of the sets of states under study. For instance, there has been a significant amount of study of the local distinguishability of generalized Bell states, which are lattice states [61, 60, 9, 68, 62, 11, 77, 78, 79]. One encounters a rich diversity of geometric and algebraic structure exhibited by different sets of orthogonal bipartite states, owing to which it is difficult to identify some signature property which determines if a given set of orthogonal bipartite states are locally distinguishable or not. This result is a first step towards such a result, in the sense that it gives the necessary and sufficient condition for the one-way LOCC distinguishability of the set of states, regardless of its geometric or algebraic structure. This marks a significant shift in approach to the problem of local distinguishability of orthogonal bipartite states, which has mostly hitherto been done on a case-bycase basis.

Another feature of the work is the departure from employing separable measurements or PPT preserving measurements to arrive at the results. The intractable nature of LOCC makes it difficult to study the local (in)distinguishability of quantum states under the LOCC constraint, owing to which there has been a growing trend to study distinguishability of bipartite and multipartite quantum states with separable operations or PPT preserving operations. The results in this thesis are developed based solely on LOCC considerations, and will therefore attract the attention of the quantum information community, particularly, the smaller community interested in the local distinguishability of quantum states.

4.2 General Future Directions

I saliently outline anticipated future directions which arise from the content of this thesis. The difference between the future directions mentioned here and those in section 2.6 and subsections 3.1.6 and 3.2.2, is that the anticipated future directions here are of a more general and less technical nature.

- i. Theorem 2.5.12 states that if $\{\Pi_i\}_{i=1}^m$ is the optimal POVM for the MED of $\widetilde{P}_{\underline{r}} = \{p_i, \rho_i\}_{i=1}^m$, then there exists a spectral decomposition of the Π_i 's, of the form $\Pi_i = \sum_{\underline{i} \in I_i} |w_{\underline{i}}^*\rangle \langle w_{\underline{i}}^*|$ for all $1 \leq i \leq m$, where $\langle w_{\underline{i}}^*|w_{\underline{j}}^*\rangle = \delta_{\underline{i}\underline{j}}$, and a pure state decomposition of the $p_i\rho_i$'s, i.e., $p_i\rho_i = \sum_{\underline{i} \in I_i} |\widetilde{\psi}_{\underline{i}}^*\rangle \langle \widetilde{\psi}_{\underline{j}}^*|$ for $1 \leq i \leq m$, such that $\{|w_{\underline{i}}^*\rangle \langle w_{\underline{i}}^*|\}_{\underline{i} \in I}$ is the optimal POVM for the MED of $\widetilde{P}_{\underline{1}} = \{\langle \widetilde{\psi}_{\underline{i}}^*|\widetilde{\psi}_{\underline{j}}^*\rangle, |\psi_{\underline{i}}^*\rangle \langle \psi_{\underline{j}}^*|\}_{\underline{i} \in I}$, where $|\psi_{\underline{i}}^*\rangle = \frac{1}{\sqrt{|\widetilde{\psi}_{w_i}^*|\widetilde{\psi}_{\underline{j}}^*}}|\widetilde{\psi}_{\underline{j}}^*\rangle$. Thus the MED of an ensemble of LI mixed states is directly related to the MED of an ensemble which comprises of a LI pure state decomposition of the former mixed states. An interesting question to pursue would be to know how this theorem can be generalized. An obvious choice of generalization is to see if such a phenomenon also holds for ensembles of states which are not LI. But one can also ask if such a phenomenon also holds for other quantum state discrimination problems, for instance, for unambiguous state discrimination problem, or even, the locally accessible information problem.
- ii. The Holevo-Schumacher-Westermoreland theorem says if an ensemble \widetilde{P} is used as a quantum sources of classical information, then the asymptotic rate of informa-

tion transmission over a noiceless channel is given by the Holevo bound, defined by equation (3.6). It is desired to know if the Holevo-like upper bound for the locally accessible information for an ensemble of bipartite quantum states, defined by equation (3.8), is also asymptotically attainable, when the decoding operations are restricted to LOCC.

iii. Is there any connection between the necessary condition (3.15) and theorem 3.2.3 for sets of MES?

Appendices

Appendix A

LOCC of Four Generalized Bell States in $\mathbb{C}^4\otimes\mathbb{C}^4$

A.1 Sets of states in theorem 3.1.2

Each set contains the state $|\psi_{00}^{(4)}\rangle$; the remaining states for each set are listed below:

$$\begin{cases} |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{03}^{(4)}\rangle \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{03}^{(4)}\rangle \right\} \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \\ \left\{ |\psi_{01}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle \\ \left\{ |\psi_{02}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle \\ \left\{ |\psi_{02}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle \\ \left\{ |\psi_{02}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \\ \left\{ |\psi_{02}^{(4)}\rangle,$$

Table A.1: Sets of states of the form $\{|\psi_{a0}^{(4)}\rangle, |\psi_{b1}^{(4)}\rangle, |\psi_{c2}^{(4)}\rangle, |\psi_{d3}^{(4)}\rangle\}$, where $a, b, c, d \in \{0, 1, 2, 3\}$.

$$\begin{cases} |\psi_{10}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle \right\} \\ \left\{ |\psi_{10}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle \right\} \\ \left\{ |\psi_{11}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle \right\} \\ \left\{ |\psi_{11}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle \right\} \\ \left\{ |\psi_{11}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{11}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle \right\} \\ \left\{ |\psi_{11}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{11}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle \right\} \\ \left\{ |\psi_{11}^{$$

Table A.2: Sets of states of the form $\{|\psi_{0a}^{(4)}\rangle, |\psi_{1b}^{(4)}\rangle, |\psi_{2c}^{(4)}\rangle, |\psi_{3d}^{(4)}\rangle, \}$, where $a, b, c, d \in \mathbb{Z}_4$

A.2 LOCC protocols for states in theorems 3.1.3 to 3.1.8

In theorems 3.1.3, 3.1.4, 3.1.5, 3.1.6, 3.1.7 and 3.1.8 are listed sets of four Generalized Bell States in $\mathbb{C}^4 \otimes \mathbb{C}^4$, other than those listed in theorem 3.1.2, which are distinguishable by one-way LOCC. It is assumed that Alice always initiates the one-way LOCC protocol, and the measurement she initiates the LOCC protocol with is given in the proofs of theorems 3.1.3, 3.1.4, 3.1.5, 3.1.6 and 3.1.8. In this section we present the remaining part of these LOCC protocols, i.e., we list the ONB in which Bob will perform measurement.

A.2.1 PMRS for each set in theorem 3.1.3

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{\overline{1}}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)} angle$	$\frac{1}{2}\tilde{e}^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2}\tilde{ 1\rangle} + \frac{1}{2}\tilde{e}^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2}\tilde{ 3\rangle}$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(\tilde{4})}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{\overline{n}}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{\overline{n}}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{10}^{(\tilde{4})}\rangle$	$\frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\tilde{i}}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{21}^{(\tilde{4})}\rangle$	$-\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{21}^{(\tilde{4})}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0\rangle - \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 1\rangle + \frac{\tilde{1}}{2} 2\rangle - \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{\tilde{i}\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{\tilde{i}\frac{\pi}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle+\frac{1}{2} 1\rangle+\frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle+\frac{1}{2} 3\rangle$
$ \psi_{32}^{(\tilde{4})}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$\frac{1}{2}\tilde{e}^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2}\tilde{ 1\rangle} - \frac{1}{2}\tilde{e}^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2}\tilde{ 3\rangle}$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The set is $\{ |\psi_{00}^{(4)} \rangle, |\psi_{01}^{(4)} \rangle, |\psi_{10}^{(4)} \rangle, |\psi_{13}^{(4)} \rangle \}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{\overline{i}}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$\frac{i}{2} 0 \rangle - \frac{1}{2} e^{i \frac{3\pi}{4}} 1 \rangle - \frac{i}{2} 2 \rangle - \frac{1}{2} e^{i \frac{3\pi}{4}} 3 \rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{\tilde{i}}{2} 0\rangle + \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 1\rangle - \frac{\tilde{i}}{2} 2\rangle - \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{\tilde{i}}{2} 0\rangle - \frac{1}{2}e^{\tilde{i}\frac{\pi}{4}} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2}e^{\tilde{i}\frac{\pi}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0 angle - \frac{1}{2} 1 angle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2 angle - \frac{1}{2} 3 angle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{\tilde{1}}{2} 1\rangle + \frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{\tilde{1}}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0 angle - \frac{1}{2} 1 angle - \frac{1}{2}e^{i\frac{\pi}{4}} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{\tilde{1}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\tilde{1}}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{\overline{1}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{1}}{2} 3\rangle$

The set is $\{ |\psi_{00}^{(4)} \rangle, |\psi_{01}^{(4)} \rangle, |\psi_{10}^{(4)} \rangle, |\psi_{31}^{(4)} \rangle \}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{\overline{i}}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} 0 \rangle - \frac{1}{2} e^{i \frac{3\pi}{4}} 1 \rangle - \frac{i}{2} 2 \rangle - \frac{1}{2} e^{i \frac{3\pi}{4}} 3 \rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} \tilde{ 0\rangle} - \frac{1}{2} e^{i\frac{\pi}{4}} 1\rangle + \frac{i}{2} \tilde{ 2\rangle} + \frac{1}{2} e^{i\frac{\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{\tilde{i}}{2} 0\rangle + \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 1\rangle + \frac{\tilde{i}}{2} 2\rangle - \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle - \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{\overline{i}}{2} 2\rangle - \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0 angle - \frac{1}{2} 1 angle - \frac{1}{2}e^{i\frac{\pi}{4}} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{i}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{\overline{1}}{2} 0\rangle - \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{\overline{1}}{2} 2\rangle - \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{\overline{1}}{2} 0\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{\overline{1}}{2} 2\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle$, $|\psi_{01}^{(4)}\rangle$, $|\psi_{20}^{(4)}\rangle$, $|\psi_{21}^{(4)}\rangle$ }.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{\tilde{1}}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\tilde{\pi}}{4}} 1\rangle - \frac{\tilde{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\tilde{\pi}}{4}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0\rangle - \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 1\rangle + \frac{\tilde{1}}{2} 2\rangle - \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$\frac{1}{2}\tilde{e}^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2}\tilde{ 1\rangle} - \frac{1}{2}\tilde{e}^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2}\tilde{ 3\rangle}$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} 0 \rangle - \frac{1}{2} \overline{e}^{i\frac{3\pi}{4}} 1 \rangle - \frac{i}{2} 2 \rangle - \frac{1}{2} \overline{e}^{i\frac{3\pi}{4}} 3 \rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2}\tilde{ 0\rangle} - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{i}{2}\tilde{ 2\rangle} + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{i}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(\tilde{4})}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle+\frac{\overline{1}}{2} 1\rangle-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle-\frac{\overline{1}}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{1}{2}\tilde{e}^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2}\tilde{ 1\rangle} - \frac{1}{2}\tilde{e}^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2}\tilde{ 3\rangle}$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{10}^{(\tilde{4})}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\overline{i}}{2} 1\rangle - \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle$, $|\psi_{02}^{(4)}\rangle$, $|\psi_{10}^{(4)}\rangle$, $|\psi_{23}^{(4)}\rangle$ }.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{23}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(\tilde{4})}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{23}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\tilde{\pi}}{4}} 1\rangle - \frac{\tilde{1}}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\tilde{\pi}}{4}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{10}^{(\bar{4})}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{23}^{(4)}\rangle$	$\frac{1}{2} \tilde{ 0\rangle} - \frac{1}{2} e^{i\frac{\pi}{4}} \tilde{ 1\rangle} - \frac{1}{2} \tilde{ 2\rangle} - \frac{1}{2} e^{i\frac{\pi}{4}} \tilde{ 3\rangle}$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{\overline{1}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{1}}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\tilde{i}}{2} 3\rangle$
$ \psi_{23}^{(4)}\rangle$	$\frac{\ddot{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{\ddot{\pi}}{4}} 1\rangle - \frac{\ddot{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{\ddot{\pi}}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(\tilde{4})}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle+\frac{\overline{1}}{2} 1\rangle-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle-\frac{\overline{1}}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 0\rangle+\frac{\tilde{i}}{2} 1\rangle+\frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 2\rangle+\frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{\overline{1}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{1}}{2} 3\rangle$
$ \psi_{10}^{(\tilde{4})}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\overline{i}}{2} 1\rangle - \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The set is $\{ |\psi_{00}^{(4)} \rangle, |\psi_{02}^{(4)} \rangle, |\psi_{21}^{(4)} \rangle, |\psi_{30}^{(4)} \rangle \}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle+\frac{1}{2} 1\rangle+\frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle+\frac{1}{2} 3\rangle$
$ \psi_{21}^{(\bar{4})}\rangle$	$-\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{21}^{(\bar{4})}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle+\frac{\overline{1}}{2} 1\rangle-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle-\frac{\overline{1}}{2} 3\rangle$
$ \psi_{21}^{(\bar{4})}\rangle$	$-\frac{\overline{1}}{2} 0\rangle - \frac{1}{2}e^{i\frac{\overline{n}}{4}} 1\rangle + \frac{\overline{1}}{2} 2\rangle - \frac{1}{2}e^{i\frac{\overline{n}}{4}} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{\overline{1}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{1}}{2} 3\rangle$
$ \psi_{21}^{(\tilde{4})}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

A.2.2 PMRS for each set in theorem 3.1.4

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = -\frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$\frac{i}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 0 angle + \frac{i}{\sqrt{2}} 2 angle$
$ \psi_{11}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1 angle+\frac{1}{\sqrt{2}} 3 angle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$\frac{i}{\sqrt{2}} 1 angle + \frac{1}{\sqrt{2}} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 0 angle + \frac{i}{\sqrt{2}} 2 angle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle-\frac{1}{\sqrt{2}} 2 angle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1 angle+\frac{1}{\sqrt{2}} 3 angle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{31}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{31}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{31}^{(4)} angle$	$\frac{\sqrt{2}}{\sqrt{2}} 0\rangle + \frac{\sqrt{2}}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{\sqrt{2}}{\sqrt{2}} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{\sqrt{2}}{\sqrt{2}} 0 angle - \frac{\sqrt{2}}{\sqrt{2}} 2 angle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0 angle + \frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{10}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{23}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{01}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{10}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{23}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{10}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{23}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle - \frac{1}{\sqrt{2}} 2 angle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{10}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{23}^{(4)}\rangle$	$\frac{\dot{v}^2}{\sqrt{2}} 0\rangle - \frac{\dot{v}^2}{\sqrt{2}} 2\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1 angle + \frac{1}{\sqrt{2}} 3 angle$
$ \psi_{11}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{22}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle + \frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{22}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{22}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{22}^{(4)} angle$	$-\frac{\sqrt{2}}{\sqrt{2}} 1\rangle+\frac{i}{\sqrt{2}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0 angle + \frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{22}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{31}^{(4)} angle$	$\frac{\sqrt{2}}{\sqrt{2}} 1\rangle - \frac{\sqrt{2}}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{22}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{31}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{22}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{31}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{22}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{23}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{23}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{30}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{23}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{30}^{(4)} angle$	$-\frac{\tilde{i}}{\sqrt{2}} 0 angle -\frac{1}{\sqrt{2}} 2 angle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{23}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{30}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{23}^{(4)}\rangle$	$\frac{i}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{30}^{23}\rangle$	$-\frac{\sqrt{2}}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$
The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0 angle + \frac{1}{\sqrt{2}} 2 angle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{21}^{(4)} angle$	$\frac{\sqrt{2}}{\sqrt{2}} 1\rangle + \frac{\sqrt{2}}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle+\frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0 angle-\frac{i}{\sqrt{2}} 2 angle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{\sqrt{2}}{\sqrt{2}} 0\rangle + \frac{\sqrt{2}}{\sqrt{2}} 2\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{\sqrt{2}}{\sqrt{2}} 0\rangle+\frac{i}{\sqrt{2}} 2\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0 angle + \frac{i}{\sqrt{2}} 2 angle$
$ \psi_{21}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{31}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{31}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{31}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{02}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$

A.2.3 PMRS for each set in theorem 3.1.5

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = -\frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{12}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0 angle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1 angle - \frac{1}{2} 2 angle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3 angle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle+\frac{1}{2} 1\rangle-\frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle-\frac{1}{2} 3\rangle$
$ \psi_{12}^{(\bar{4})}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{12}^{(\tilde{4})}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{30}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{\overline{1}}{2} 0\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle - \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{i}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{\tilde{1}}{2} 0\rangle - \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{\tilde{1}}{2} 2\rangle + \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{i}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\overline{1}}{2} 1\rangle - \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{1}}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\overline{1}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{\overline{1}}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle - \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{20}^{(\bar{4})}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{20}^{(\tilde{4})}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{\overline{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 0\rangle - \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0 angle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1 angle - \frac{1}{2} 2 angle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3 angle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{\overline{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{33}^{(4)} angle$	$-\frac{\overline{i}}{2} 0\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{\overline{i}}{2} 2\rangle + \frac{\overline{1}}{2}e^{i\frac{\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{33}^{(4)}\rangle$	$-\frac{\tilde{i}}{2} 0\rangle - \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 1\rangle + \frac{\tilde{i}}{2} 2\rangle - \frac{\tilde{1}}{2}e^{i\frac{\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{33}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{i}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 1\rangle - \frac{\overline{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\overline{n}}{4}} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{33}^{(4)}\rangle$	$\frac{i}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0\rangle - \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 1\rangle - \frac{\tilde{1}}{2} 2\rangle + \frac{1}{2}e^{i\frac{\tilde{\pi}}{4}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{\overline{1}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{\overline{1}}{2} 3\rangle$
$ \psi_{10}^{(\tilde{4})}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{\overline{i}}{2} 1\rangle - \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{5\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{5\pi}{4}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{21}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{2}e^{i\frac{3\pi}{4}}|0\rangle + \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{\pi}{4}} 3\rangle$
$ \psi_{32}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}} 2\rangle - \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = \frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)} angle$	$-\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{21}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{32}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{1}{2}e^{i\frac{\pi}{4}}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}e^{i\frac{\pi}{4}}|2\rangle + \frac{1}{2}|3\rangle.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2}e^{i\frac{3\pi}{4}} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{\overline{1}}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle - \frac{\overline{1}}{2} 3\rangle$
$ \psi_{21}^{(\tilde{4})}\rangle$	$-\frac{1}{2} 0\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2}e^{i\frac{3\pi}{4}} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 0\rangle + \frac{i}{2} 1\rangle + \frac{\overline{1}}{2}e^{i\frac{3\pi}{4}} 2\rangle + \frac{i}{2} 3\rangle$

A.2.4 PMRS for each set in theorem 3.1.6

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0 angle + \frac{1}{2} 1 angle - \frac{1}{2} 2 angle - \frac{1}{2} 3 angle$
$ \psi_{10}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0 angle - \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle - \frac{\overline{i}}{2} 3 angle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{11}^{(4)} angle$	$-\overline{\frac{i}{2}} 0 angle - \overline{\frac{1}{2}} 1 angle - \overline{\frac{i}{2}} 2 angle - \overline{\frac{1}{2}} 3 angle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{i}{2} 0 angle + \frac{i}{2} 1 angle - \frac{i}{2} 2 angle - \frac{i}{2} 3 angle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{\overline{i}}{2} 0\rangle - \frac{\overline{i}}{2} 1\rangle + \frac{\overline{i}}{2} 2\rangle + \frac{\overline{i}}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2} 2\rangle + \frac{\overline{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{\overline{1}}{2} 1\rangle + \frac{\overline{1}}{2} 2\rangle + \frac{\overline{1}}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle + \frac{\overline{i}}{2} 1\rangle - \frac{\overline{i}}{2} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle - \frac{\overline{i}}{2} 1\rangle - \frac{\overline{i}}{2} 2\rangle + \frac{\overline{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{i}{2} 1 angle + \frac{i}{2} 2 angle - \frac{i}{2} 3 angle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\overline{i}}{2} 0 angle - \frac{\overline{i}}{2} 1 angle + \frac{\overline{i}}{2} 2 angle + \frac{\overline{i}}{2} 3 angle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0 angle + \frac{1}{2} 1 angle - \frac{1}{2} 2 angle - \frac{1}{2} 3 angle$
$ \psi_{10}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0 angle - \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle - \frac{\overline{i}}{2} 3 angle$
$ \psi_{32}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2} 2\rangle + \frac{\overline{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0 angle+\frac{\tilde{i}}{2} 1 angle-\frac{\tilde{1}}{2} 2 angle+\frac{\tilde{i}}{2} 3 angle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0 angle + \frac{\overline{i}}{2} 1 angle - \frac{\overline{i}}{2} 2 angle - \frac{\overline{i}}{2} 3 angle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{10}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$\frac{\dot{i}}{2} 0 angle + \frac{\dot{i}}{2} 1 angle - \frac{\dot{i}}{2} 2 angle - \frac{\dot{i}}{2} 3 angle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{12}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0 angle - \frac{\tilde{i}}{2} 1 angle - \frac{\tilde{1}}{2} 2 angle - \frac{\tilde{i}}{2} 3 angle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0 angle + \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle + \frac{\overline{i}}{2} 3 angle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{\tilde{i}}{2} 0\rangle - \frac{\tilde{i}}{2} 1\rangle + \frac{\tilde{i}}{2} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0 angle + \frac{1}{2} 1 angle - \frac{1}{2} 2 angle - \frac{1}{2} 3 angle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2}\tilde{ }0\rangle - \frac{1}{2}\tilde{ }1\rangle + \frac{i}{2}\tilde{ }2\rangle - \frac{1}{2}\tilde{ }3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle - \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{33}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{33}^{(4)}\rangle$	$\frac{i}{2}\tilde{ }0\rangle - \frac{1}{2}\tilde{ }1\rangle + \frac{i}{2}\tilde{ }2\rangle - \frac{1}{2}\tilde{ }3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0 angle + \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle + \frac{\overline{i}}{2} 3 angle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{33}^{(4)}\rangle$	$-\frac{\tilde{i}}{2} 0 angle - \frac{\tilde{i}}{2} 1 angle + \frac{\tilde{i}}{2} 2 angle + \frac{\tilde{i}}{2} 3 angle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{33}^{(4)}\rangle$	$-\frac{\overline{i}}{2} 0\rangle + \frac{\overline{i}}{2} 1\rangle + \frac{\overline{i}}{2} 2\rangle - \frac{\overline{i}}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0 angle - \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle - \frac{\overline{i}}{2} 3 angle$
$ \psi_{30}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2} 2\rangle + \frac{\overline{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0 angle - \frac{\overline{i}}{2} 1 angle - \frac{\overline{i}}{2} 2 angle + \frac{\overline{i}}{2} 3 angle$
$ \psi_{30}^{(4)}\rangle$	$-\frac{\tilde{i}}{2} 0\rangle - \frac{\tilde{i}}{2} 1\rangle + \frac{\tilde{i}}{2} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{1}{2} 0 angle - \frac{i}{2} 1 angle + \frac{1}{2} 2 angle - \frac{i}{2} 3 angle$
$ \psi_{32}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle + \frac{\overline{i}}{2} 1\rangle + \frac{\overline{1}}{2} 2\rangle + \frac{\overline{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{\tilde{1}}{2} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{1}}{2} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0 angle + \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle + \frac{\overline{i}}{2} 3 angle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{\tilde{i}}{2} 0\rangle + \frac{\tilde{i}}{2} 1\rangle + \frac{\tilde{i}}{2} 2\rangle - \frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0 angle - \frac{\overline{i}}{2} 1 angle - \frac{\overline{i}}{2} 2 angle + \frac{\overline{i}}{2} 3 angle$
$ \psi_{32}^{(4)}\rangle$	$\frac{\tilde{i}}{2} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{i}}{2} 2\rangle - \frac{\tilde{i}}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{i}{2}\tilde{ }0\rangle - \frac{1}{2}\tilde{ }1\rangle + \frac{i}{2}\tilde{ }2\rangle - \frac{1}{2}\tilde{ }3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0 angle + \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle + \frac{\overline{i}}{2} 3 angle$
$ \psi_{13}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle - \frac{\overline{i}}{2} 1\rangle - \frac{\overline{i}}{2} 2\rangle + \frac{\overline{i}}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{\tilde{i}}{2} 0\rangle + \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{i}}{2} 2\rangle - \frac{\tilde{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle + \frac{\overline{i}}{2} 1\rangle - \frac{\overline{i}}{2} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{\tilde{i}}{2} 0\rangle - \frac{\tilde{i}}{2} 1\rangle - \frac{\tilde{i}}{2} 2\rangle + \frac{\tilde{i}}{2} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{33}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = -\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{33}^{(4)}\rangle$	$\frac{i}{2}\tilde{ }0\rangle - \frac{1}{2}\tilde{ }1\rangle + \frac{i}{2}\tilde{ }2\rangle - \frac{1}{2}\tilde{ }3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{i}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0 angle + \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle + \frac{\overline{i}}{2} 3 angle$
$ \psi_{13}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle - \frac{\overline{i}}{2} 1\rangle - \frac{\overline{i}}{2} 2\rangle + \frac{\overline{i}}{2} 3\rangle$
$ \psi_{33}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)}\rangle$	$-\frac{i}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0\rangle + \frac{\overline{i}}{2} 1\rangle - \frac{\overline{i}}{2} 2\rangle - \frac{\overline{i}}{2} 3\rangle$
$ \psi_{33}^{(4)}\rangle$	$-\frac{1}{2} 0 angle+\frac{1}{2} 1 angle+\frac{1}{2} 2 angle-\frac{1}{2} 3 angle$

A.2.5 PMRS for each set in theorem 3.1.7

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{10}^{(4)}\rangle, |\psi_{33}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1 angle+\frac{1}{\sqrt{2}} 3 angle$
$ \psi_{10}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0 angle - \frac{1}{\sqrt{2}} 2 angle$
$ \psi_{33}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1 angle+\frac{1}{\sqrt{2}} 3 angle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{10}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{33}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 0 angle+\frac{i}{\sqrt{2}} 2 angle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\left -\frac{1}{\sqrt{2}} \right 1 \rangle + \frac{1}{\sqrt{2}} 3 \rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0 angle - \frac{1}{\sqrt{2}} 2 angle$
$ \psi_{10}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{33}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{10}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{33}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{32}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{32}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{32}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle - \frac{i}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{32}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{32}^{(4)}\rangle$	$-\frac{\sqrt{2}}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{12}^{(4)}\rangle, |\psi_{31}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{12}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{31}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{12}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$
$ \psi_{31}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\left -\frac{1}{\sqrt{2}} 1 ight angle+\frac{1}{\sqrt{2}} 3 ight angle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{12}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{31}^{(4)} angle$	$\frac{i}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{12}^{(4)}\rangle$	$-\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{31}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{13}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{13}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{30}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{13}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{30}^{(4)} angle$	$-\frac{i}{\sqrt{2}} 1 angle+\frac{i}{\sqrt{2}} 3 angle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{13}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{30}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{13}^{(4)}\rangle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{30}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{01}^{(4)}\rangle, |\psi_{11}^{(4)}\rangle, |\psi_{30}^{(4)}\rangle\}.$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0 angle+\frac{1}{\sqrt{2}} 2 angle$
$ \psi_{01}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle - \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{30}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{\sqrt{i}}{\sqrt{2}} 0\rangle + \frac{\sqrt{i}}{\sqrt{2}} 2\rangle$
$ \psi_{30}^{(4)} angle$	$-\frac{\sqrt{2}}{\sqrt{2}} 1\rangle+\frac{\sqrt{2}}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_3\rangle = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{01}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{11}^{(4)} angle$	$-\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{30}^{(4)} angle$	$\frac{i}{\sqrt{2}} 1\rangle + \frac{i}{\sqrt{2}} 3\rangle$

The PMRS on Alice's side is $|u_4\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 2\rangle$
$ \psi_{01}^{(4)}\rangle$	$\frac{1}{\sqrt{2}} 1\rangle + \frac{1}{\sqrt{2}} 3\rangle$
$ \psi_{11}^{(4)}\rangle$	$\frac{\sqrt{2}}{\frac{1}{\sqrt{2}}} 1\rangle - \frac{\sqrt{2}}{\sqrt{2}} 3\rangle$
$ \mu_{(4)}^{(4)}\rangle$	$\frac{\sqrt{2}}{1} \frac{\sqrt{2}}{1} \frac{\sqrt{2}}{1$
$ \psi_{30} $	$\sqrt{2}^{107}$ $\sqrt{2}^{127}$

A.2.6 PMRS for each set in theorem 3.1.8

The set is $\{|\psi_{00}^{(4)}\rangle, |\psi_{02}^{(4)}\rangle, |\psi_{20}^{(4)}\rangle, |\psi_{22}^{(4)}\rangle\}.$

The PMRS on Alice's side is $|u_1\rangle = -\frac{i}{2}|0\rangle - \frac{i}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$\frac{\overline{i}}{2} 0 angle - \frac{\overline{i}}{2} 1 angle + \frac{\overline{1}}{2} 2 angle - \frac{\overline{1}}{2} 3 angle$
$ \psi_{22}^{(4)}\rangle$	$\frac{\overline{1}}{2} 0\rangle - \frac{\overline{1}}{2} 1\rangle + \frac{\overline{i}}{2} 2\rangle - \frac{\overline{i}}{2} 3\rangle$

The PMRS on Alice's side is $|u_2\rangle = \frac{i}{2}|0\rangle + \frac{i}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$\frac{1}{2} 0\rangle + \frac{1}{2} 1\rangle - \frac{i}{2} 2\rangle - \frac{i}{2} 3\rangle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle + \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{22}^{(4)}\rangle$	$\frac{1}{2} \tilde{ 0\rangle} - \frac{1}{2} \tilde{ 1\rangle} - \frac{i}{2} \tilde{ 2\rangle} + \frac{i}{2} \tilde{ 3\rangle}$

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$-\frac{i}{2} 0 angle + \frac{i}{2} 1 angle - \frac{1}{2} 2 angle + \frac{1}{2} 3 angle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2} 0 angle + \frac{1}{2} 1 angle - \frac{i}{2} 2 angle + \frac{i}{2} 3 angle$
$ \psi_{20}^{(4)}\rangle$	$-\frac{\overline{i}}{2} 0 angle - \frac{\overline{i}}{2} 1 angle - \frac{\overline{1}}{2} 2 angle - \frac{\overline{1}}{2} 3 angle$
$ \psi_{22}^{(4)}\rangle$	$-\frac{\overline{1}}{2} 0 angle - \frac{\overline{1}}{2} 1 angle - \frac{\overline{i}}{2} 2 angle - \frac{\overline{i}}{2} 3 angle$

The PMRS on Alice's side is $|u_4\rangle = -\frac{i}{2}|0\rangle + \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$.

Pre-Measurement State	Reduced Post-Measurement State on Bob's Side
$ \psi_{00}^{(4)} angle$	$\frac{i}{2} 0\rangle - \frac{i}{2} 1\rangle - \frac{1}{2} 2\rangle + \frac{1}{2} 3\rangle$
$ \psi_{02}^{(4)}\rangle$	$-\frac{1}{2} 0 angle + \frac{1}{2} 1 angle + \frac{i}{2} 2 angle - \frac{i}{2} 3 angle$
$ \psi_{20}^{(4)}\rangle$	$\frac{i}{2} 0\rangle + \frac{i}{2} 1\rangle - \frac{1}{2} 2\rangle - \frac{1}{2} 3\rangle$
$ \psi_{22}^{(4)}\rangle$	$-\frac{1}{2} 0\rangle - \frac{1}{2} 1\rangle + \frac{i}{2} 2\rangle + \frac{i}{2} 3\rangle$

Appendix B

Proof of Corollary 3.2.6.1

This proof is similar to the Cohen's proof of theorem 1 in [13].

Denote by $\mathcal{G}(m, n)$ the manifold of all sets of *m* orthogonal bipartite pure states $\{|\psi_i\rangle_{AB}\}_{i=1}^m \subset \mathcal{H}_A \otimes \mathcal{H}_B$, where $\langle \psi_i | \psi_{i'} \rangle = \delta_{ii'}, \forall 1 \leq i < i' \leq m$. Hence every point in $\mathcal{G}(m, n)$ is associated with a set of $n \times n$ orthonormal complex matrices $\{W_i\}_{i=1}^m$ (see equation (2) in main text), i.e., $\operatorname{Tr}(W_i^{\dagger}W_{i'}) = \delta_{ii'}, \forall 1 \leq i < i' \leq m$. Let's represent the rows of W_i as $\psi_{i^n}, \psi_{i^2}, \dots, \psi_{id}$. Vectorize the W_i matrices by arranging these rows $\{\psi_{ij}\}_{j=1}^n$ as complex n^2 -tuples, i.e., $(\psi_{i^1}, \psi_{i^2}, \dots, \psi_{id}) \in \mathbb{C}^{n^2}$, and arrange these vectorized W_i 's as the first upper *m* rows of a $n^2 \times n^2$ unitary matrix *U*, whose remaining rows are arbitary (insofar as the matrix remains unitary). Hence any point of $\mathcal{G}(m, n)$ can be associated with the first upper *m* columns of a $n^2 \times n^2$ unitary matrix $U \in U(n^2)$. In fact, since the overall phases of these *m* columns, the permutation of the order of their appearance in the set of first *n* columns of *U* and the rest of the $n^2 - m$ columns in *U* are insignificant to describe the corresponding set of orthogonal pure states from $\mathcal{H}_A \otimes \mathcal{H}_B$, the manifold $\mathcal{G}(m, n)$ is given by $U(n^2)/(U(1)^{\times m} \times S_n \times U(n^2 - m))$. This is a real manifold.

Let $u(n^2)$ be the space of all $n^2 \times n^2$ hermitian matrices, then it is the space of generators for $n^2 \times n^2$ unitary matrices, i.e., if $G \in u(n^2)$, then e^{-iG} is a $n^2 \times n^2$ unitary matrix. Associate the ordered set of the first *m* rows of e^{-iG} with the set of *m* vectorized W_i 's. Then the

set $\{W_i\}_{i=1}^m$ corresponds to some set of *m* orthonormal states $\{|\psi_i\rangle_{AB}\}_{i=1}^m$. This maps any $G \in \mathfrak{u}(n^2)$ to a point in $\mathcal{G}(m, n)$ unambiguously. Let's denote this map by $\mathscr{R} : \mathfrak{u}(n^2) \longrightarrow \mathcal{G}(m, n)$. So $\mathscr{R}(G)$ is a point in $\mathcal{G}(m, n)$ corresponding to $\{|\psi_i\rangle_{AB}\}_{i=1}^m$. In the following I specify norm-induced-metric for various spaces.

- 1. Metric for all $n^2 \times n^2$ matrices is given by the standard Hilbert Schmidt norm.
- 2. Let $\{A_i\}_{i=1}^m$ be an arbitrary set of *m* complex $n \times n$ matrices, then

$$||\{A_i\}_{i=1}^m|| = \left(\sum_{i=1}^m \operatorname{Tr}(A_i^{\dagger}A_i)\right)^{\frac{1}{2}}.$$

3. Let $\{|\eta_i\rangle_{AB}\}_{i=1}^m$ be a set of *m* arbitrary vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$, then $\|\{|\eta_i\rangle_{AB}\}_{i=1}^m\| = (\sum_{i=1}^m |\eta_i\rangle_{AB})^{\frac{1}{2}}$.

Then $G \longrightarrow e^{-iG}$ is continuous, $e^{-iG} \longrightarrow \{W_i\}_{i=1}^m$ is continuous and $\{W_i\}_{i=1}^m \longrightarrow \{|\psi_i\rangle_{AB}\}_{i=1}^m$ is continuous. This implies that \mathscr{R} is continuous. It is easy to see that \mathscr{R} is onto but not one-to-one.

For any set of *m* orthonormal states $\{|\psi_i\rangle_{AB}\}_{i=1}^m$, one can obtain the n(n-1) matrices $\{H_i, A_i\}_{i \in I}$. Vectorize each of these matrices and arrange them as rows of a $m(m-1) \times n^2$ matrix *M*. Define $\mathscr{D} : \mathscr{G}(m, n) \longrightarrow \mathbb{R}$ by $\mathscr{D}(\{|\psi_i\rangle_{AB}\}_{i=1}^m) \equiv Det(MM^{\dagger})$. The goal is to establish that for no point in $\mathscr{G}(m, n)$ is there an open neighbourhood \mathcal{N} containing said point such that \mathscr{D} vanishes entirely in \mathcal{N} . Since \mathscr{D} is continuous on $\mathscr{G}(m, n)$ and \mathscr{R} is continuous on $u(n^2)$, $\mathscr{D} \circ \mathscr{R}$ is continuous on $u(n^2)$. Hence, if \mathscr{D} vanishes entirely in some open neighbourhood \mathcal{N} of $\{|\psi_i\rangle_{AB}\}_{i=1}^m$ in $\mathscr{G}(m, n)$, and if $\mathscr{R}(G) = \{|\psi_i\rangle_{AB}\}_{i=1}^m$, then there is some open neighbourhood m of $G \in u(n^2)$ where $\mathscr{D} \circ \mathscr{R}$ vanishes entirely too. Hence one needs to show that $\mathscr{D} \circ \mathscr{R}$ doesn't vanish entirely in any open neighbourhood of any point *G* in $u(n^2)$.

Let $\{\lambda_i\}_{i=1}^{n^2}$ be an ONB for $\mathfrak{u}(n^2)$. Let $G = \underset{\rightarrow}{\alpha} \underset{\rightarrow}{\lambda}$ be a point in $\mathfrak{u}(n^2)$ which has an open neighbourhood \mathfrak{m} in which $\mathscr{D} \circ \mathscr{R}$ vanishes entirely. Then there exists some $\epsilon_s \in \mathbb{R}$ be such that $(\alpha + \epsilon_s \hat{m}) \cdot \lambda \in \mathfrak{m}$ for all unit vectors \hat{m} lying on S^{n^2-1} . Then

$$e^{-i(\underline{\alpha}+\epsilon\hat{m}).\underline{\lambda}} = e^{-i\underline{\alpha}.\underline{\lambda}} + \epsilon \begin{pmatrix} -i\hat{m}.\underline{\lambda} - \frac{1}{2!}(\hat{m}.\underline{\lambda})(\underline{\alpha}.\underline{\lambda}) + (\underline{\alpha}.\underline{\lambda})(\hat{m}.\underline{\lambda}) \\ + \frac{i}{3!}(\underline{\alpha}.\underline{\lambda})^{2}(\hat{m}.\underline{\lambda}) + (\underline{\alpha}.\underline{\lambda})(\hat{m}.\underline{\lambda})(\underline{\alpha}.\underline{\lambda}) + (\hat{m}.\underline{\lambda})(\underline{\alpha}.\underline{\lambda})^{2} \end{pmatrix} + \cdots \end{pmatrix}$$

$$+ \epsilon^{2} \left(-\frac{(\hat{m}.\underline{\lambda})^{2}}{2} + i\frac{(\hat{m}.\underline{\lambda})^{2}(\underline{\alpha}.\underline{\lambda}) + (\hat{m}.\underline{\lambda})(\underline{\alpha}.\underline{\lambda})(\hat{m}.\underline{\lambda}) + (\underline{\alpha}.\underline{\lambda})(\hat{m}.\underline{\lambda})^{2}}{3!} + \cdots \right) + O(\epsilon^{3}).$$
(B.1)

Hence it is easy to see that as $G \longrightarrow G + \epsilon \hat{m} . \lambda_{q}$, the W_i matrices transform as $W_i \longrightarrow W_i + \epsilon W_i^{(1)}(\hat{m}) + \epsilon^2 W_i^{(2)}(\hat{m}) + O(\epsilon^3)$, where $\epsilon W_i^{(1)}(\hat{m})$ is the first order change in ϵ , $\epsilon^2 W_i^{(2)}(\hat{m})$ is the second order change in ϵ and so on. Since equation (B.1) gives the Taylor series expansion of $e^{-i(q+\epsilon \hat{m}).4}$ about $\epsilon = 0$, $W_i + \sum_{k=1}^{\infty} \epsilon^k W_i^{(k)}(\hat{m})$ is the Taylor series expansion of about $\epsilon = 0$. In fact the radius of convergence for the latter is determined by the former, and since the expression in (B.1) converges for all $\epsilon \in \mathbb{R}$ for the former, it does so too for the latter. Now $\mathcal{D}(\{|\psi_i\rangle_{AB}\}_{i=1}^m) \equiv Det(MM^{\dagger})$ is a polynomial of the matrix elements of W_i . So when W_i goes to $W_i + \sum_{k=1}^{\infty} \epsilon^k W_i^{(k)}(\hat{m}), (\mathcal{D} \circ \mathcal{R})(G) \longrightarrow (\mathcal{D} \circ \mathcal{R})(G) + \epsilon(\mathcal{D} \circ \mathcal{R})^{(1)}(\hat{m}) + \epsilon^2(\mathcal{D} \circ \mathcal{R})^{(2)}(\hat{m}) + O(\epsilon^3)$, where $\epsilon(\mathcal{D} \circ \mathcal{R})^{(1)}(\hat{m})$ is the first order change in ϵ , $\epsilon^2(\mathcal{D} \circ \mathcal{R})^{(2)}(\hat{m})$ is the second order change in ϵ and so on. Note that $(\mathcal{D} \circ \mathcal{R})(G) + \sum_{k=1}^{\infty} \epsilon^k (\mathcal{D} \circ \mathcal{R})^{(k)}(\hat{m})$ is the Taylor series of $\mathcal{D} \circ \mathcal{R}$ about G in the direction \hat{m} . Since the Taylor series $W_i + \sum_{k=1}^{\infty} \epsilon^k W_i^{(k)}(\hat{m})$ convergences for all $\epsilon \in \mathbb{R}$, and since \mathcal{D} is a polynomial in the matrix elements of W_i .

Now let $\mathcal{D} \circ \mathcal{R}$ vanish in m. This implies that $(\mathcal{D} \circ \mathcal{R})(G + \epsilon \hat{m}) = 0$, for all $\hat{m} \in S^{n^{2}-1}$ and $\epsilon \in [0, \epsilon_{s}]$, where ϵ_{s} was chosen so that $(\alpha + \epsilon_{s} \hat{m}) : \lambda \in m$. The Taylor series of $\mathcal{D} \circ \mathcal{R}$ about *G* is a summation of monomials in ϵ , i.e., $(\mathcal{D} \circ \mathcal{R})^{(k)} \epsilon^{k}$, which are linearly independent in the range $\epsilon \in [0, \epsilon_{s}]$. Hence the only way that such a summation vanishes for all $\epsilon \in [0, \epsilon_{s}]$ is if $\mathcal{D}^{(k)}(\hat{m}) = 0$ for all $k \in \mathbb{N}$ and $\hat{m} \in S^{n^{2}-1}$, and if $(\mathcal{D} \circ \mathcal{R})(G) = 0$. But note that the radius of

convergence for ϵ in this Taylor series is ∞ . Hence $\mathscr{D} \circ \mathscr{R}$ vanishes all over $\mathfrak{u}(n^2)$. And that implies that \mathscr{D} vanishes all over $\mathscr{G}(m, n)$. The following counter-example will disprove this: let $|\psi_i\rangle_{AB} \equiv |s_i\rangle_A |0\rangle_B$, where $|0\rangle_B \in \mathcal{H}_B$. Then $Tr_B(|\psi_i\rangle\langle\psi_{i'}|) = |s_i\rangle\langle s_{i'}|$ when $i \neq i'$, so \mathcal{T} is spanned by the complex congugate of matrices representing $\frac{1}{2}(|s_i\rangle\langle s_{i'}| + |s_{i'}\rangle\langle s_i|)$ and $\frac{1}{2i}(|s_i\rangle\langle s_{i'}| - |s_{i'}\rangle\langle s_i|)$, for all $1 \leq i < i' \leq n$, in the standard basis. All these matrices are linearly independent, so $dim\mathcal{T}_{\perp} = n$ and $\mathscr{D}(\{|s_i\rangle_A|0\rangle_B\}_{i=1}^m) \neq 0$. Hence it is not possible for \mathscr{D} to vanish entirely in any open neighbourhood of any point in $\mathscr{G}(m, n)$. This also holds true for the particular case when m = n.

References

- V. P. Belavkin. Optimal multiple quantum statistical hypothesis testing. *Stochastics*, I:315–345, 1975.
- [2] V.P. Belavkin and V. Maslov. *Mathematical Aspects of Computer Engineering*. Mir Publishers, 1988.
- [3] Carlos Mochon. Family of generalized "pretty good" measurements and the minimal-error pure-state discrimination problems for which they are optimal. *Phys. Rev. A*, 73:032328, Mar 2006.
- [4] R. S Kennedy. On the optimum receiver for the m-ary linearly independent pure state problem. *M. I. T. Res. Lab.: Electron. Quart. Progr. Rep.*, 110:142–146, July 1973.
- [5] Carl W. Helstrom. Quantum Detection and Estimation. Academic Press, 1976.
- [6] Yonina C. Eldar. von neumann measurement is optimal for detecting linearly independent mixed quantum states. *Phys. Rev. A*, 68:052303, Nov 2003.
- [7] E. Chitambar, Runyao Duan, and Min-Hsiu Hsieh. When do local operations and classical communication suffice for two-qubit state discrimination? *Information Theory, IEEE Transactions on*, 60(3):1549–1561, March 2014.

- [8] Piotr Badziag, Michał Horodecki, Aditi Sen(De), and Ujjwal Sen. Locally accessible information: How much can the parties gain by cooperating? *Phys. Rev. Lett.*, 91:117901, Sep 2003.
- [9] Michael Nathanson. Distinguishing bipartitite orthogonal states using locc: Best and worst cases. *Journal of Mathematical Physics*, 46(6):–, 2005.
- [10] Ming-Yong Ye, Wei Jiang, Ping-Xing Chen, Yong-Sheng Zhang, Zheng-Wei Zhou, and Guang-Can Guo. Local distinguishability of orthogonal quantum states and generators of su(*N*). *Phys. Rev. A*, 76:032329, Sep 2007.
- [11] Michael Nathanson. Three maximally entangled states can require two-way local operations and classical communication for local discrimination. *Phys. Rev. A*, 88:062316, Dec 2013.
- [12] Jonathan Walgate, Anthony J. Short, Lucien Hardy, and Vlatko Vedral. Local distinguishability of multipartite orthogonal quantum states. *Phys. Rev. Lett.*, 85:4972– 4975, Dec 2000.
- [13] Scott M. Cohen. Almost every set of $n \ge d + 1$ orthogonal states on $d^{\bigotimes n}$ is locally indistinguishable. *Phys. Rev. A*, 77:060309, Jun 2008.
- [14] Horace P. Yuen, R.S. Kennedy, and M. Lax. Optimum testing of multiple hypotheses in quantum detection theory. *Information Theory, IEEE Transactions on*, 21(2):125– 134, Mar 1975.
- [15] A. S. Holevo. Statistical decision theory for quantum systems. *Journal of Multivariate Analysis*, 3(4):337 – 394, 1973.
- [16] Masashi Ban, Keiko Kurokawa, Rei Momose, and Osamu Hirota. Optimum measurements for discrimination among symmetric quantum states and parameter estimation. *International Journal of Theoretical Physics*, 36(6):1269–1288, 1997.

- [17] I.D. Ivanovic. How to differentiate between non-orthogonal states. *Physics Letters* A, 123(6):257 – 259, 1987.
- [18] D. Dieks. Overlap and distinguishability of quantum states. *Physics Letters A*, 126(5–6):303 306, 1988.
- [19] Asher Peres. How to differentiate between non-orthogonal states. *Physics Letters A*, 128(1–2):19 –, 1988.
- [20] Gregg Jaeger and Abner Shimony. Optimal distinction between two non-orthogonal quantum states. *Physics Letters A*, 197(2):83 – 87, 1995.
- [21] Anthony Chefles. Unambiguous discrimination between linearly independent quantum states. *Physics Letters A*, 239(6):339 – 347, 1998.
- [22] Anthony Chefles and Stephen M. Barnett. Optimum unambiguous discrimination between linearly independent symmetric states. *Physics Letters A*, 250(4–6):223 – 229, 1998.
- [23] Asher Peres and Daniel R. Terno. Optimal distinction between non-orthogonal quantum states. *Journal of Physics A: Mathematical and General*, 31(34):7105, 1998.
- [24] Yuqing Sun, Mark Hillery, and János A. Bergou. Optimum unambiguous discrimination between linearly independent nonorthogonal quantum states and its optical realization. *Phys. Rev. A*, 64:022311, Jul 2001.
- [25] János A. Bergou, Ulrike Herzog, and Mark Hillery. Quantum filtering and discrimination between sets of boolean functions. *Phys. Rev. Lett.*, 90:257901, Jun 2003.
- [26] Yonina C. Eldar, Mihailo Stojnic, and Babak Hassibi. Optimal quantum detectors for unambiguous detection of mixed states. *Phys. Rev. A*, 69:062318, Jun 2004.
- [27] Ulrike Herzog and János A. Bergou. Optimum unambiguous discrimination of two mixed quantum states. *Phys. Rev. A*, 71:050301, May 2005.

- [28] Philippe Raynal and Norbert Lütkenhaus. Optimal unambiguous state discrimination of two density matrices: A second class of exact solutions. *Phys. Rev. A*, 76:052322, Nov 2007.
- [29] Ulrike Herzog and János A. Bergou. Optimum unambiguous identification of d unknown pure qudit states. *Phys. Rev. A*, 78:032320, Sep 2008.
- [30] Matthias Kleinmann, Hermann Kampermann, and Dagmar Bruß. Unambiguous discrimination of mixed quantum states: Optimal solution and case study. *Phys. Rev. A*, 81:020304, Feb 2010.
- [31] M. Kleinmann, H. Kampermann, and D. Bruß. Structural approach to unambiguous discrimination of two mixed quantum states. *Journal of Mathematical Physics*, 51(3), 2010.
- [32] János A. Bergou, Ulrike Futschik, and Edgar Feldman. Optimal unambiguous discrimination of pure quantum states. *Phys. Rev. Lett.*, 108:250502, Jun 2012.
- [33] Somshubhro Bandyopadhyay. Unambiguous discrimination of linearly independent pure quantum states: Optimal average probability of success. *Phys. Rev. A*, 90:030301, Sep 2014.
- [34] Donghoon Ha and Younghun Kwon. Analysis of optimal unambiguous discrimination of three pure quantum states. *Phys. Rev. A*, 91:062312, Jun 2015.
- [35] Reinhard F. Werner. Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, 40:4277–4281, Oct 1989.
- [36] Asher Peres and William K. Wootters. Optimal detection of quantum information. *Phys. Rev. Lett.*, 66:1119–1122, Mar 1991.
- [37] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, Mar 1993.

- [38] Charles H. Bennett, David P. DiVincenzo, Christopher A. Fuchs, Tal Mor, Eric Rains, Peter W. Shor, John A. Smolin, and William K. Wootters. Quantum nonlocality without entanglement. *Phys. Rev. A*, 59:1070–1091, Feb 1999.
- [39] Anthony Chefles. Quantum state discrimination. *Contemporary Physics*, 41(6):401–424, 2000, http://dx.doi.org/10.1080/00107510010002599.
- [40] János A. Bergou, Ulrike Herzog, and Mark Hillery. *Quantum State Estimation*, chapter 11 Discrimination of Quantum States, pages 417–465. Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.
- [41] Stephen M. Barnett and Sarah Croke. Quantum state discrimination. Adv. Opt. Photon., 1(2):238–278, Apr 2009.
- [42] Y.C. Eldar, A. Megretski, and George C. Verghese. Optimal detection of symmetric mixed quantum states. *Information Theory, IEEE Transactions on*, 50(6):1198– 1207, June 2004.
- [43] M. Ježek, J. Řeháček, and J. Fiurášek. Finding optimal strategies for minimum-error quantum-state discrimination. *Phys. Rev. A*, 65:060301, Jun 2002.
- [44] C.W. Helstrom. Bayes-cost reduction algorithm in quantum hypothesis testing (corresp.). *Information Theory, IEEE Transactions on*, 28(2):359–366, Mar 1982.
- [45] Jon Tyson. Estimates of non-optimality of quantum measurements and a simple iterative method for computing optimal measurements. quant-ph/09020395.
- [46] C.-L. Chou and L. Y. Hsu. Minimum-error discrimination between symmetric mixed quantum states. *Phys. Rev. A*, 68:042305, Oct 2003.
- [47] Stephen M. Barnett. Minimum-error discrimination between multiply symmetric states. *Phys. Rev. A*, 64:030303, Aug 2001.
- [48] Joonwoo Bae and Won-Young Hwang. Minimum-error discrimination of qubit states: Methods, solutions, and properties. *Phys. Rev. A*, 87:012334, Jan 2013.

- [49] Donghoon Ha and Younghun Kwon. Complete analysis for three-qubit mixed-state discrimination. *Phys. Rev. A*, 87:062302, Jun 2013.
- [50] Donghoon Ha and Younghun Kwon. Discriminating *n*-qudit states using geometric structure. *Phys. Rev. A*, 90:022330, Aug 2014.
- [51] Tanmay Singal and Sibasish Ghosh. Minimum error discrimination for an ensemble of linearly independent pure states. *Journal of Physics A: Mathematical and Theoretical*, 49(16):165304, 2016.
- [52] Spivak M. A Comprehensive Introduction to Differential Geometry, Vol. 1. Publish or Perish Inc., 1999.
- [53] Jon Tyson. Error rates of belavkin weighted quantum measurements and a converse to holevo's asymptotic optimality theorem. *Phys. Rev. A*, 79:032343, Mar 2009.
- [54] Steven G. Krantz and Harold R. Parks. *The Implicit Function Theorem: History, Theorem and Applications.* Birkhauser, 2002.
- [55] Reinhold A Bertlmann and Philipp Krammer. Bloch vectors for qudits. Journal of Physics A: Mathematical and Theoretical, 41(23):235303, 2008.
- [56] S. Boyd and L. Vandenberghe. *Convex Optimization*, chapter 11, page 575. Cambridge University Press, 2004.
- [57] Tanmay Singal and Sibasish Ghosh. Algebraic structure of the minimum error discrimination problem for linearly independent density matrices, http://arxiv.org/abs/1412.7174v1.
- [58] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher. Concentrating partial entanglement by local operations. *Phys. Rev. A*, 53:2046–2052, Apr 1996.
- [59] Charles Bennett and Gilles Brassard. Quantum cryptography: Public key distribution and coin tossing. pages 175–179. IEEE press, 1984.
- [60] Heng Fan. Distinguishability and indistinguishability by local operations and classical communication. *Phys. Rev. Lett.*, 92:177905, Apr 2004.
- [61] Sibasish Ghosh, Guruprasad Kar, Anirban Roy, and Debasis Sarkar. Distinguishability of maximally entangled states. *Phys. Rev. A*, 70:022304, Aug 2004.
- [62] Nengkun Yu, Runyao Duan, and Mingsheng Ying. Four locally indistinguishable ququad-ququad orthogonal maximally entangled states. *Phys. Rev. Lett.*, 109:020506, Jul 2012.
- [63] Charles H. Bennett and Stephen J. Wiesner. Communication via one- and twoparticle operators on einstein-podolsky-rosen states. *Phys. Rev. Lett.*, 69:2881–2884, Nov 1992.
- [64] C. E. Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27(3):379–423, July 1948.
- [65] C. E. Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27(4):623–656, Oct 1948.
- [66] Michael A. Nielsen and Isaac Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.
- [67] Tanmay Singal, Ramij Rahaman, Sibasish Ghosh, and Guruprasad Kar. Complete analysis of perfect local distinguishability of ensemble of four generalized bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, http://arxiv.org/abs/1506.03667.
- [68] Somshubhro Bandyopadhyay, Sibasish Ghosh, and Guruprasad Kar. Locc distinguishability of unilaterally transformable quantum states. *New Journal of Physics*, 13(12):123013, 2011.
- [69] Guojing Tian, Sixia Yu, Fei Gao, Qiaoyan Wen, and C. H. Oh. Local discrimination of four or more maximally entangled states. *Phys. Rev. A*, 91:052314, May 2015.

- [70] Tanmay Singal. Framework for distinguishability of orthogonal bipartite states by one-way local operations and classical communication. *Phys. Rev. A*, 93:030301, Mar 2016.
- [71] Charles H. Bennett, David P. DiVincenzo, Tal Mor, Peter W. Shor, John A. Smolin, and Barbara M. Terhal. Unextendible product bases and bound entanglement. *Phys. Rev. Lett.*, 82:5385–5388, Jun 1999.
- [72] Giacomo Mauro D'Ariano, Paoloplacido Lo Presti, and Paolo Perinotti. Classical randomness in quantum measurements. J. Phys. A: Math. Gen., 38(26):5979, 2005.
- [73] Anthony Chefles. Condition for unambiguous state discrimination using local operations and classical communication. *Phys. Rev. A*, 69:050307, May 2004.
- [74] Joonwoo Bae. Structure of minimum-error quantum state discrimination. New Journal of Physics, 15(7):073037, 2013.
- [75] Joonwoo Bae, Won-Young Hwang, and Yeong-Deok Han. No-signaling principle can determine optimal quantum state discrimination. *Phys. Rev. Lett.*, 107:170403, Oct 2011.
- [76] E. T. Jaynes. Information theory and statistical mechanics. *Phys. Rev.*, 106:620–630, May 1957.
- [77] Zhi-Chao Zhang, Ke-Qin Feng, Fei Gao, and Qiao-Yan Wen. Distinguishing maximally entangled states by one-way local operations and classical communication. *Phys. Rev. A*, 91:012329, Jan 2015.
- [78] Yan-Ling Wang, Mao-Sheng Li, Zhu-Jun Zheng, and Shao-Ming Fei. On small set of one-way locc indistinguishability of maximally entangled states. *Quantum Information Processing*, pages 1–8, 2016.
- [79] Guojing Tian, Sixia Yu, Fei Gao, Qiaoyan Wen, and C. H. Oh. Local discrimination of four or more maximally entangled states. *Phys. Rev. A*, 91:052314, May 2015.