Perturbative QCD corrections to processes at the LHC

By

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List of Publications arising from the thesis

Journal

1. "Drell-Yan Production at Threshold to Third Order in QCD"

   Taushif Ahmed, Maguni Mahakhud, ***Narayan Rana*** *and V. Ravindran*;

   Phys. Rev. Lett. **113** (2014) 11, 112002

2. "Two-loop QCD corrections to Higgs $\to b + \overline{b} + g$ amplitude"

   Taushif Ahmed, Maguni Mahakhud, Prakash Mathews, ***Narayan Rana*** *and V. Ravindran*; JHEP **1408** (2014) 075

3. "Spin-2 Form Factors at Three Loop in QCD"

   Taushif Ahmed, Goutam Das, Prakash Mathews, ***Narayan Rana*** *and V. Ravindran*;

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4. "Pseudo-scalar Higgs Boson Production at Threshold N^3LO and N^3LL QCD"

   Taushif Ahmed, M.C. Kumar, Prakash Mathews, ***Narayan Rana*** *and V. Ravindran*;

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List of other Publications, Not included in the thesis

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2. "Rapidity Distributions in Drell-Yan and Higgs Productions at Threshold to Third Order in QCD"

   Taushif Ahmed, M. K. Mandal, ***Narayan Rana*** *and V. Ravindran*;

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3. "Higgs boson production through $b\bar{b}$ annihilation at threshold in N^{3}LO QCD"

   Taushif Ahmed, ***Narayan Rana*** *and V. Ravindran*;

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4. "Higgs Rapidity Distribution in $b\bar{b}$ Annihilation at Threshold in N^{3}LO QCD"

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5. "Pseudo-scalar Form Factors at Three Loops in QCD"

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1. "RG improved Higgs boson production to N^{3}LO in QCD"

   Taushif Ahmed, Goutam Das, M. C. Kumar, ***Narayan Rana*** *and V. Ravindran*;

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2. "Pseudo-scalar Higgs boson production at N^{3}LO$_A$+N^{3}LL$'$"

   Taushif Ahmed, Marco Bonvini, M.C. Kumar, Prakash Mathews, ***Narayan Rana****, V. Ravindran and Luca Rottoli*; arXiv:1606.00837

3. "NNLO QCD Corrections to the Drell-Yan Cross Section in Models of TeV-Scale Gravity"

   Taushif Ahmed, Pulak Banerjee, P. K. Dhani, M.C. Kumar, Prakash Mathews, ***Narayan Rana*** *and V. Ravindran*; arXiv:1606.08454

4. "Two-loop QCD Correction to Massive Spin-2 Resonance $\to q\bar{q}g$"

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1. "Threshold cross sections for Drell-Yan & Higgs productions in N^3LO QCD"
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2. "Pseudo-scalar Higgs form factor at 3 loops and QCD N^3LO threshold corrections to its production at the LHC"
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   based on 'Pseudo-scalar Higgs Boson Production at Threshold N^3LO and N^3LL QCD'
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List of Papers presented at seminars

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List of Corrections requested by the referees

- p.80, line 3: “order of perturbation” is modified to “order in the perturbative expansion”.
- p.83, line 2, the work on NLO QCD corrections to graviton+jet production done in arXiv:1308.2194 is mentioned.
- p.110, Ref. [216] is a book and does not have any journal nor arXiv reference.
- p.110, second paragraph: “One of the interesting possibility” is modified to “One of the interesting possibilities”.
- p.110, “in the small and moderate tanβ, . . . ” is modified to “in the small and moderate tanβ region, . . . ”.
- p.114, after Eq.(6.3): a remark is made about the treatment of γ_5 in d-dimensions.
- p.147, Eq.(7.27): p in the denominator is corrected to p_3.
- p.153: second paragraph “to orders in perturbation theory” is modified to “to all orders in perturbation theory”.
- p.154, 2nd sentence: “the findings of previous Chapter” is modified to “the findings of the previous Chapter”. end of 1st paragraph: “This enable us to” is modified to “This enables us to”.
- p.155: “The detail numerical study” is modified to “The detailed numerical study”.

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Guide: Prof. V. Ravindran

To the ones I love

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Contents

Synopsis

For almost fifty years, the Standard Model (SM) has been providing the modern description of particle physics, consistently. While the symmetry group $SU(3)\times SU(2)\times U(1)$ is the backbone of the finalized version of the SM, Glashow achieved the first step to unify the electromagnetic and weak interactions. In 1967, Weinberg and Salam incorporated the Higgs mechanism into Glashow electroweak theory, successfully explaining the origin of mass in the SM. After the discovery of neutral weak currents by the Gargamelle bubble chamber in 1973, this theory was widely accepted. Later, in 1983, the discovery of the vector bosons (W and Z) by UA1 and UA2 collaborations in CERN, established the electroweak theory thoroughly. In the meantime, the theory of strong interaction, Quantum Chromodynamics (QCD), started to develop with the discovery of color quantum number for quarks and the phenomena of scaling observed in deep inelastic scattering experiments. Finally, it was established firmly when the observed logarithmic scaling violations were found to be in accordance with the predictions of QCD. The only missing particle was the Higgs boson, till July 2012, when ATLAS and CMS in CERN announced its existence. With this monumental breakthrough, the Standard Model is now in its full glory.

Both experimentalists and theorists played important role to achieve this success. While from experimental side, it took tremendous effort to analyze the huge amount of data, on the other hand, precision level achieved by theorists to match that accuracy was also very significant. Specially production channels in hadron colliders are contaminated by initial and final states QCD radiations, which certainly call for inclusion of higher orders in QCD. We noted the necessity of next-to-leading order (NLO) corrections for the first time in case of W and Z discovery. Very soon, the need for higher order contributions became clear and it has now become a natural routine to consider, at least, NLO corrected results for physics studies. However, the large value of the NLO corrections to some of the important processes like Drell-Yan (DY) production or the Higgs boson production in gluon fusion, raise doubt about the reliability of perturbative QCD (pQCD). Also, the formulation of pQCD introduces unphysical mass scales, like renormalization and factorization scales, to deal with the divergences, arising from loop and phase-space integrals. Even though physical quantities are independent of these scales, they appear as logarithmic functions in each order in the perturbative series. Inclusion of higher order terms reduces this dependence. These two key factors motivated to go beyond and compute next-to-next-to-leading order (NNLO) corrections for the before mentioned and many other processes.

In this thesis, we study the above mentioned processes in perturbative QCD taking into account higher order radiative corrections.

For the DY production, NNLO corrections are small, confirming the reliability of perturbation theory, but the scale dependence still was not convincingly small, hinting the need to go beyond. The first step beyond NNLO is to compute threshold approximated cross section at N^{3}LO, as the computation of exact N^{3}LO is very involved and yet to be done. In the threshold limit, the finite partonic cross section factorizes into the overall renormalization constant, the form factor, the splitting kernels and a soft distribution function. Each factorized part, though, contains singularities, in the end they sum up to produce a finite result. Moreover, the structure of the singularities for each part is universal. The fact that the soft distribution function depend only on the initial partons, makes it maximally non-abelian. We exploit the recently available threshold approximated N^{3}LO cross section for the Higgs boson production in gluon fusion and use this universality of soft distribution function to obtain the corresponding one for the DY production. This enable us to obtain the threshold N^{3}LO QCD correction to DY production. We study its numerical impact and find the contribution is small, hence validating the reliability of pQCD.

While we are achieving extraordinary precision to establish the SM, it fails to address issues like providing a consistent quantum field theoretic description of gravity, the origin of neutrino mass, the existence of dark matter and dark energy, the hierarchy problem, to name a few. These shortcomings build a way for physics beyond the SM (BSM), followed by a plethora of ideas. One of such ideas led to TeV scale extra-dimensional models, introduced mainly to address the hierarchy problem. In these models, gravity lives in $d = 4 + n$ dimensions where the SM is constrained to a 3-dimensional brane and n is the extra spatial dimension. The projection of d dimensional graviton to 3 spatial dimensions creates infinite number of Kaluza-Klein (KK) modes, specially containing a massive spin 2 particle for each mode. Considering ADD and RS models, two particular variants of this scenario, this spin 2 particle couples to the SM particles universally through the energy momentum tensor of the SM. While experimental bounds have already put constraints on these models, the recent excess on the di-photon production over the SM predictions revives the chances of its existence. Hence, search for a generic spin 2 particle at the LHC becomes very much necessary and DY process can play an important role for this search. To match the accuracy of the SM predictions, processes involving spin 2 particle also need to be computed to the same level of accuracy. One of the important ingredients to achieve this, is the quark and gluon form factors of the energy momentum tensor of the SM. In the second work in this thesis, we present the details of the computations performed to achieve these form factors. We consider diagrammatic approach to obtain the matrix elements, *i.e.* computing the relevant Feynman diagrams. At third order alone, there are 3374 and 1072 number of Feynman diagrams for gluon and quark form factors, respectively. An in house code using FORM is used to perform Dirac and color algebra and necessary simplifications. The arbitrary distribution of loop momenta in the propagators hides a fact that with proper shifting of the loop momenta, all integrals belong to a few set of propagators, namely three sets for this case. These shifted integrals now are reduced to a few number of integrals, the so-called master integrals, by using the integration-by-parts (IBP) and Lorentz invariant (LI) identities. Finally, the available master integrals (MI) enable us to obtain the three-loop form factors. While the conserved energy momentum tensor of QCD makes the form factors UV finite, there exists singularities of infrared kind in the form factors. We have studied structure of these infrared singularities in these form factors up to three-loop level using Sudakov integro-differential equation and found that the anomalous dimensions originating from soft and collinear regions of the loop integrals coincide with those of the electro-weak vector boson and Higgs form factors confirming the universality of the infrared singularities in QCD amplitudes. These results will be very useful in improving the perturbative predictions of spin-2 resonance production beyond NNLO level at the LHC where searches for such particles are already underway with the upgraded energy and luminosity.

With the availability of the quark and gluon form factors for the energy momentum tensor, the next step is to obtain the threshold approximated cross section. The universality of the soft distribution function, as these depend only on the initial state partons, enable us to obtain the same by exploiting that of the Higgs boson production in gluon fusion. We present the threshold approximated partonic cross section at N^3LO level for quark and gluon initiated processes, respectively, for the first time in this thesis.

Among the several BSM models, supersymmetric theories provide an elegant solution to various phenomena that SM can not explain satisfactorily. In one of its simplest realizations, the minimal supersymmetric extension of the SM (MSSM), the Higgs sector contains two CP-even (scalar), one CP-odd (pseudo-scalar) and two charged Higgs bosons. More generally, the existence of additional scalar and pseudo-scalar bosons which couple to fermions is a prediction of many models which include two Higgs doublets. In the limit of infinite fermion mass, there exists an effective theory that describes the interaction of pseudo-scalar Higgs boson with the gluons. The recently discovered Standard-Model-like

Higgs boson at the LHC prompted the study of the properties of the discovered boson to identify either with lightest scalar or pseudo- scalar Higgs bosons of extended models, indicating precise predictions for their production cross sections. In the fourth work, we present the first results on the production of pseudo-scalar Higgs boson through gluon fusion at the LHC to N^3LO in QCD taking into account only soft gluon effects. One of the crucial ingredients is the three-loop form factors of the effective composite operators that result when top quarks are integrated out and they were computed very recently by some of us. In this work, we obtain the form factors required for the production of a pseudo-scalar through gluon fusion, using the ones for the composite operators. Again, the soft distribution functions, being dependent only on the initial state partons, are same with that of the scalar Higgs boson production in gluon fusion. Hence, we obtain the threshold approximated cross section at N^3LO level. We present a detailed phenomenological study of the pseudo-scalar production at the LHC for various center of mass energies as a function of its mass. While the third order corrections are small, they play an important role in reducing the theoretical uncertainty resulting from renormalization scale. In addition, we have made a detailed comparison against scalar Higgs boson production and found their corrections are very close to each other confirming the universal behavior of the QCD effects even though the operators responsible for their interactions with gluons are very different.

The Higgs boson production channel has indeed comparably large NNLO corrections and going beyond NNLO is very important. It took numerous efforts to finally achieve next-to-next-to-next-to leading order (N^3LO) corrections for the production of the Higgs boson in gluon fusion. The level of precision level that has been achieved in the gluon fusion channel and the need to study the Higgs boson coupling to vector bosons and fermions motivate us to improve the predictions for other production mechanisms for the Higgs boson, namely bottom quark annihilation, vector boson fusion, *etc*. On the other hand, to study the quantum nature of the recently discovered boson in detail, kinematic differential distributions play a major role. Hence, besides inclusive process, production along with a jet is very important. For the gluon fusion channel, recent computation reached the precision at NNLO level. While gluon initiated partonic sub-processes are the dominant one, it is important to include the sub-dominant ones coming from other channels, *e.g*. bottom quark annihilation and the first step in this direction is to obtain the two-loop QCD amplitudes. Also, these amplitudes contribute to N^3LO after performing phase-space integration for one of the partons. With this motivation, in the next work, we obtain the two-loop amplitudes of the Higgs boson decaying to pair of bottom anti-bottom quarks along with a gluon, where the Higgs boson couples to bottom quarks through Yukawa coupling. We consider VFS scheme throughout and the bottom quark is massless except for the Yukawa coupling. We use projection operators to obtain the coefficients for each tensorial structure appearing in this process. We consider the diagrammatic approach and encounter large number of Feynman diagrams with rich Lorentz and gauge structures. Symbolically we simplify the Lorentz, Dirac and color algebra. In addition, the loop integrals become increasingly complicated due to their multiple kinematic dependence. The use of state-of-the-art techniques like IBP and LI identities reduce the large number of loop integrals to few MI's, already available in literature. The Yukawa coupling here needs UV renormalization and the UV finite results contain the universal infrared poles, indicating the correctness of the computation. The results are presented in terms of Harmonic Poly-Logarithms (HPLs). To obtain the relevant amplitude for the production of the Higgs boson along with a jet, we do proper crossing and hence analytic continuation of HPLs.

To summarize, in this thesis, we have systematically computed higher order QCD corrections to some very important processes at the LHC. We have achieved this by using the state-of-the-art modern techniques as well as exploiting the rich universal structure of QCD amplitudes at higher orders in perturbation theory. Our results are very important for precision studies at the LHC.

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1 Introduction

The four fundamental forces in nature: the gravitational, electromagnetic, weak and strong forces, describe all the interactions between matters, known till today. There have been continuous efforts to get a better understanding of all these forces in a single framework. While the classical descriptions of these forces were clearly insufficient, the success of quantum physics and special theory of relativity led to the modern description of each fundamental force in terms of quantum fields. Such a description is the quantum field theory which uses, in addition, the local symmetry. The excitations of the underlying fields define the matter particles. To establish the invariance under local symmetry, it is necessary to introduce gauge fields which, in quantum field theory, represent particles that describe the interaction between the matter particles. The coupling constant defines the strength of the interaction.

The Standard Model (SM) of particle physics collectively describes all the forces, except gravity. Combining electromagnetic and weak interactions with the strong interaction and incorporating the Higgs mechanism, the SM forms a modern theoretical description of elementary particle physics. Till today, it successfully describes most of the experimental phenomena of elementary particles. While the modern version of the SM is based on the local symmetry group $SU(3)\times SU(2)\times U(1)$, the first step was to unify the electromagnetic and weak interaction by Glashow [1]. In 1967, Weinberg [2] and Salam [3] incorporated the Higgs mechanism [4–8] into Glashow's electro-weak theory, successfully explaining the origin of mass of the SM particles, in a gauge invariant way. Finally, the spontaneously broken $SU(2) \times U(1)$ gauge theory describes the electro-weak interaction and the unbroken $SU(3)$ color gauge theory describes the strong interaction.

The SM contains vector bosons that are responsible for interactions:

Force	Electromagnetic	Weak	Strong
Gauge boson	γ	$W^{\pm}, Z$	g

and the matter particles consisting of three generations of leptons and quarks:

Gen I	Gen II	GenIII
e^-	μ^-	τ^-
ν_e	ν_μ	ν_τ
u, d	c, s	b, t

To each fermion, one associate an anti-fermion by charge conjugation. With these particle contents, the SM becomes the holy grail of particle physics.

The SM is not only mathematically profound in the sense that it is renormalizable but also its predictions are consistent with the experimental observations, putting it in firm footing. The first of them was in 1973, the discovery of neutral current predicted by electro-weak theory in the Gargamelle bubble chamber [9]. One had to wait ten more years till 1983 to finally find the vector bosons (W and Z) by UA1 [10] (led by Carlo Rubbia) and UA2 [11] (led by Pierre Darrilat) collaborations in CERN. These discoveries established the electro-weak sector of the SM thoroughly. On the other hand, crucial event for strong interaction was the discovery of the J/ψ resonance, a charmed particle (which contains a charm quark (c)) by a team at the SLAC [12] and one at the BNL [13] in 1974. In 1977, Fermilab E288 experiment [14, 15] produced bottomonium and thus discovering bottom quark. Later, in 1995, the discovery of top quark (t) by CDF [16] and D∅ [17], Tevatron marked another milestone in establishing the SM. Finally, the long-waited and sought after discovery happened in 2012, when ATLAS [18] and CMS [19] in CERN announced the existence of the Higgs boson. With this monumental breakthrough, the SM is now in its full glory.

Behind this success, there lies tremendous effort from experimentalists to achieve precise results from a huge amount of data, as well as from theorists to match that precision by computing contributions from all possible channels and to sufficient orders in perturbation series. It was clear that when experiments became more accurate, there was a need for more precise theoretical predictions and it implied to consider higher order effects. It was first noted in the UA1 and UA2 collaborations when the production rates for the W and Z [20, 21] bosons were in good agreement with the first higher order results in the literature [22–26]. Along with the first order Quantum Chromodynamics (QCD) corrections for the Drell-Yan (DY) process, the same was obtained for Deep Inelastic Scattering (DIS) [27, 28]. Later, the direct searches of the Higgs boson at Large Electron-Positron Collider (LEP) [29] at CERN and Tevatron [30] were crucial in narrowing the search region. Specially direct searches at the LEP excluded the Higgs boson mass below 114.4 GeV and the indirect constraints from electroweak precision measurements [31] bound Higgs boson mass to be less than 152 GeV at 95% confidence level (CL).

With the advent of accelerators in multi-GeV energy range, there is dire need of more precise results from theoretical computations, specially for the hadron colliders, like the Large Hadron Collider (LHC) at CERN. The initial states in such colliders strongly interact and due to the asymptotic freedom of QCD, the QCD corrections play a vital role. While the next-to-next-to leading order (NNLO) corrections to Drell-Yan process [32–35] was extensively necessary to fine-tune the extraction of parton distribution function (PDF), the discovery of the Higgs boson was made possible due to such state-of-the-art computations [36–44].

However, in spite of its significant success, there remain a few questions unanswered. Firstly, a proper quantum field theoretic description of gravitational force still remains missing. The existence of the intermediate boson for gravity, namely graviton, is still a subject of experimental search. There are ample evidences for the existence of dark matter and dark energy, while it can not be explained within the framework of the SM. The SM particle content describes only $\sim 5\%$ of this universe, while the rest are still in dark. Another shortcoming of the SM is the mass of neutrinos. Neutrinos are massless in the model, whereas experimental observation like neutrino oscillation requires them to be massive. All these unexplained scenarios clearly indicates new physics beyond the Standard Model (BSM).

To be able to observe such hints of new physics beyond the SM, it is crucial to have very accurate theoretical predictions of what is expected in the SM, in order to clearly distinguish deviations from the expectations with the maximal significance. The context of this thesis is to acquire results, as precise as possible, for two most important inclusive processes at the LHC, namely the Drell-Yan production and the Higgs boson production.

The physics of massive lepton pair production in hadronic collisions is an interesting subject for last fifty years. In 1970, Christenson *et al.* [45,46] first observed this production in proton nucleus collisions. Many experiments for this were then followed with increasing energies and more precision. The motivation behind this was to find new vector mesons at the beginning and later, it became an important channel to study the structure of hadrons. Specially, the determination of PDFs are done by a fit on large number of cross section data points for channels like DIS or DY. The large size of first order corrections [22–26] raised the doubt about reliability of perturbative QCD (pQCD) and inherently indicating the need for the second order corrections. While the NNLO corrections [32–35] turned out to be small, to check the consistency in the perturbative series, higher order corrections are of much need. Moreover, the factorization of long distance physics from the short distance one via mass factorization and ultraviolet (UV) renormalization introduce two unphysical scales, namely factorization scale and renormalization scale, in the theory. While each order in perturbation series is contaminated with the logarithmic contributions from these scales, they get canceled as we include higher orders and eventually their dependence go away if the entire series is summed. These important factors motivated us to compute Drell-Yan production beyond NNLO.

Physics beyond the SM is dominated by numerous interesting ideas through various models in order to explain the unanswered questions of the SM. One of them was the extra-dimensional models, introduced to remove the hierarchy problem, the large difference between the electro-weak scale and Planck scale which can not be explained within the SM. In these models, the SM is confined in a 3-spatial dimensional brane, while gravity lives in the bulk. The $d = 4 + n$ dimensional graviton, when projected on the 4 space-time dimension, creates a tower of massive spin-2 modes (Arkani-Hamed Dimopoulos Dvali (ADD) model) or large number of massive spin-2 resonances (Randall Sundrum (RS) model). These spin-2 resonances can decay into two leptons and thus mimic the Drell-Yan process. On the other hand, models containing multiple Higgs boson, like Minimal Supersymmetric extension of the SM (MSSM), can provide a Higgs with spin-2. Hence a search for a generic spin-2 in the LHC context is very much necessary. This motivated to compute the first order QCD corrections for spin-2 mediated Drell-Yan like processes [47–51] and the results have been extensively used to constrain the corresponding model parameters. The K-factors are really large and so the scale dependence, doubting the reliability of perturbative series and hence indicating the need for more precision. Recently, in [52], the computation for two-loop quark and gluon form factors was performed, followed by the threshold approximated result in [53] for the production of a generic spin-2 at the LHC. We have obtained the full NNLO contribution very recently in [54]. To go beyond NNLO, we need to compute three-loop quark and gluon form factors. Also, such computations provide a theoretical laboratory to study the universal structure of the UV and infrared (IR) poles in QCD amplitudes.

Other strong candidates of BSM physics are the supersymmetric theories, providing elegant solutions to some of the unexplained phenomena as mentioned earlier. In one of its simplest realizations, the MSSM, the Higgs sector contains multiple Higgs: two CP-even (scalar), one CP-odd (pseudo-scalar) and two charged Higgs bosons. More generally, the existence of additional bosons which couple to fermions of the theory, is a prediction of many BSM models *e.g.* two-Higgs doublet model. An experimental detection of such additional bosons require extensive search at the LHC as well as precise theoretical predictions. We consider one such pseudo-scalar Higgs boson and an effective theory in the limit of infinite fermion mass, which describes the interaction of pseudo-scalar Higgs boson with the gluons. The recent discovery of Standard-Model-like Higgs boson at the LHC prompted the study of the properties of the discovered boson to identify either with lightest scalar or pseudo-scalar Higgs bosons of extended models, indicating the need for precise predictions of their production cross sections. For this production, pQCD predictions are already available at NNLO level [44, 55, 56]. While, for the scalar Higgs boson production, the predictions accuracy have reached through NNLO [42–44] to next-to-next-to-next-to leading order (N^3LO) [57]. This demands similar precision for the pseudo-scalar Higgs boson production and it becomes the key motivation to obtain beyond NNLO cross sections for this production.

As the search of Higgs boson is concerned, the higher order corrections played an important role. The first order corrections [36–39] were large and even the second order corrections [40–44] were not sufficient to stabilize the perturbation, as compare to the Drell-Yan process. Going beyond NNLO was must and a plethora of works appeared [58–70] to finally achieve the third order corrections [57] for the Higgs boson production in the effective theory, where infinite top mass limit gives rise to a gluon-gluon-Higgs effective interaction. While the amount of accuracy obtained through this computation, is incredible, it certainly indicates also to consider the contributions from the sub-dominant channels. One of them is to consider Higgs production in bottom quark annihilation process where bottom quark is treated as massless except in the Yukawa coupling. Second order QCD corrected result [71] is available for this channel, and though the contribution is not large compared to the gluon fusion channel, it certainly competes with its third order correction. On the other hand, studying the production of Higgs boson with an exclusive jet provides a detailed quantum nature of the Higgs boson. And perturbative corrections have the key factor. The next-to leading order (NLO) [72–74] and NNLO [75–79] corrections for the production of a Higgs boson along with a jet in the gluon fusion channel clearly confirms that the corrections are large. This motivates us to compute the same in bottom quark initiated process. As first step, we have computed the two-loop amplitudes that contribute to the process. In the following, we present the outline of this thesis.

Outline

This thesis contains a selection of published works providing a comprehensive picture of the higher order contributions that we have obtained.

In the next Chapter we present a short description on the basics of pQCD. In Chapter 3, following [62, 63], the formalism to obtain the threshold corrected cross section, is discussed in detail. We present a general structure for the threshold approximated partonic cross section here. The universal behavior of QCD amplitudes and factorization properties enable us to unravel the IR and UV structures. A detailed study reveals a Casimir scaling relation between quark and gluon initiated processes for a specific component of the threshold approximated partonic cross section, namely soft distribution function.

In Chapter 4, we use the maximally non-abelian nature of the soft distribution function, showed in Chapter 3, and obtain threshold approximated cross section for the Drell-Yan production at third order [80]. We present a numerical study to find the impact of our new results. We find, the numerical contribution at three-loop level is small, thus confirming the reliability of the perturbation theory. And the dependence on the unphysical scales gets reduced, as expected. Spin offs of this result are the computation of N^{3}LO threshold corrections to the Higgs boson production through bottom quark annihilation [81] and also in association with vector boson [82] at the hadron colliders. Also, rapidity distribution of the Higgs boson in gluon fusion [83], DY [83] and Higgs boson in bottom quark annihilation [84] were obtained at the same level of accuracy.

The first step to obtain beyond NNLO corrections for the production of a massive spin-2 resonance, is to compute the quark and gluon form factors. Chapter 5 contains detail description of the computation of these form factors up to three-loop [85]. We consider diagrammatic approach to obtain the matrix elements, *i.e.* we compute the relevant Feynman diagrams. The huge number of diagrams poses first difficulty and we use QGRAF [86] to deal with. An in-house code using FORM [87, 88] is used to perform Dirac and color algebra and necessary simplifications. We follow the standard technique to reduce numerous Feynman integrals to a few number of integrals, the so-called master integrals (MIs), by using integration-by-parts (IBP) identities [89, 90] through LiteRed [91, 92]. Of course, shifting of loop momenta makes the integrals to belong to few topologies, which is achieved by using Reduze2 [93] before the reduction. Finally, the available MIs enable us to obtain the three-loop form factors. As we use dimensional regularization with space-time dimension $d = 4 + \epsilon$, the poles appear as $\frac{1}{\epsilon^n}$. While, conserve energy momentum tensor of QCD makes this theory UV finite, the IR poles exhibits a universal structure, which in turn acts as a check on the computation. With the availability of the form factors, we present a new result, namely threshold corrected partonic cross section at third order for production of a massive spin-2 resonance.

In Chapter 6 we present the first results [94] on the production of pseudo-scalar Higgs boson through gluon fusion at the LHC to N^{3}LO in QCD taking into account only soft gluon effects. One of the crucial ingredients is the three-loop form factors of the effective composite operators that result when top quarks are integrated out and they were computed very recently by some of us [95]. In [94], we obtain the form factors required for the production of a pseudo-scalar through gluon fusion, using the ones for the composite operators. Again, the soft distribution functions, being dependent only on the initial state partons, are same with that of the scalar Higgs boson production in gluon fusion. Hence, we obtain the threshold approximated cross section at N^{3}LO level. In this chapter, we present a detailed phenomenological study of the pseudo-scalar production at the LHC for various center of mass energies as a function of its mass. While the third order corrections are small, they play an important role in reducing the theoretical uncertainty resulting from renormalization scale. In addition, we have made a detailed comparison against scalar Higgs boson production and found their corrections are very close to each other confirming the universal behavior of the QCD effects even though the operators responsible for their interactions with gluons are very different.

In Chapter 7 we present two-loop amplitude for the Higgs boson production along with a jet in bottom anti-bottom annihilation using VFS scheme. We compute the decay amplitude for $H \to b + \bar{b} + g$, by obtaining first the general tensor structure and then finding appropriate projectors for each coefficient of the tensor basis. The computational procedure follows the earlier one. The Yukawa coupling here needs UV renormalization and finally the UV renormalized result shows a universal structure in IR poles. We present the poles and finite part in ϵ in terms of iterated integrals, namely harmonic polylogarithms (HPLs). To obtain the amplitude for the specific process $b + \bar{b} \to H + g$, we have done the crossing of kinematic variables in the HPL's through analytic continuation.

Finally, in Chapter 8 we make concluding remarks and importance of the computations.

2 Quantum Chromodynamics

In this Chapter we present a very basics concerning QCD, introduce parton model and mass factorization. This chapter is by no means intended for a review on QCD, rather a short introduction to fix the convention and notations.

2.1 Basics of QCD

QCD is the quantum field theory to describe the fundamental strong force. This field theory is a non-Abelian gauge theory based on the symmetry group SU(3). While the formalism to construct the theory is simple, the manifestation is of great complexity. In the very beginning, strong interaction was pictured to center on the general principle of scattering amplitudes without any information on elementary constituents. In this context, theories based on the analytic features of S matrix, such as Regge theory, developed. This was followed first by the technique of current algebra [96] and then came the idea of quarks and partons. The quark model [97–99] successfully accounted for the hadron spectroscopy. In the same time, the idea of parton model [100–102] showed that elementary constituents could explain the experimental results successfully. Soon the concept of color [103, 104] grew and earned a natural extension to gauge theory [105–107]. On the other hand, the study on the renormalization group and operator product expansion [108–111] made it clear that the coupling constant for the quantum field theory describing the strong interaction have to be energy dependent, strong at low energies (long distance physics) and weak at high energies (short distance physics). The concept of asymptotic freedom [112, 113] met this demand remarkably, manifesting the vanishing of the strong coupling constant at high energies. This inherently justified the perturbative expansion of QCD around the free field theory. Yet, the search for free quarks or gluon continued to produce null results. Evidently, the hypothesis of confinement *i.e.* quarks and gluons can not exist as free particles, the only possibility is to form a color-singlet bound hadron state, was born. Finally, these two terms: asymptotic freedom and confinement define the success of QCD in describing the strong interaction. While asymptotic freedom refers to the weakness of the short distance interaction, the confinement of quarks follows from its strength at long distances. However, many predictions of the theory, mainly but not exclusively associated with inclusive processes, do not depend upon its long distance behavior. These short distance predictions are the realm of pQCD.

In the following, we will briefly outline the corresponding Lagrangian, followed by the evolution of the strong coupling constant. We will introduce the parton model and mass factorization in the next.

2.1.1 QCD Lagrangian

The QCD Lagrangian can be formulated with the partonic constituents *i.e.* quark and gluon fields. It reads

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \overline{\Psi}_i (i\gamma^\mu \mathcal{D}^{ij}_\mu - m\delta^{ij}) \Psi_j - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 + \overline{c}^a (-\partial^\mu \mathcal{D}^{ac}_\mu) c^c \,. \tag{2.1}$$

The field strength $F^a_{\mu\nu}$, the covariant derivative in fundamental representation $\mathcal{D}^{ij}_\mu$ and the covariant derivative in adjoint representation $\mathcal{D}^{ac}_\mu$ are defined by

$$F^a_{\mu\nu} = \partial^\mu A^a_\nu - \partial^\nu A^a_\mu + \hat{g}_s f^{abc} A^b_\mu A^c_\nu \,, \tag{2.2}$$

$$\mathcal{D}^{ij}_\mu = \delta^{ij} \partial_\mu - i\hat{g}_s (T^a)^{ij} A^a_\mu \,, \tag{2.3}$$

$$\mathcal{D}_\mu^{ab} = \delta^{ab}\partial_\mu - \hat{g}_s f^{abc} A_\mu^c \tag{2.4}$$

with the following definitions

A_μ^a : the gauge field (gluons)

Ψ_i : the fermionic field (quarks)

c^a : the ghost field

$(T^a)^{ij}$: generators of the group SU(3)

f^{abc} : the structure constants of the group SU(3)

i, j : indices for fundamental representation

a, b, c : indices for adjoint representation

m : mass of the quark field

ξ : the gauge parameter

$\hat{g}_s$: the bare strong coupling constant

The first two terms in Eq. 2.1 are the kinetic ones for the gauge and fermionic fields, respectively. While the third term, the gauge fixing component of the Lagrangian in covariant gauge, is introduced to in order to define the gluon propagator at the time of quantization to obtain a consistent quantum field theory. It originates from the fact that gluon propagator has additional degrees of freedom which gets removed by these constrained gauge fields. Ghost fields cancel the unphysical polarization states of gluon which would appear in the physical measurable quantities. So, the gauge fixing term must be supplemented with the fourth term in Eq. 2.1, the kinetic term for ghost.

2.1.2 Ultraviolet divergences and renormalization

While the exact solution for the QCD Lagrangian is not yet attainable, the asymptotic freedom of QCD, which means the coupling constant decreases for high energies, enables the application of perturbative expansion around the free field theory. The strong coupling constant $\hat{g}_s$ is considered to be the expansion parameter. The standard procedure can be followed starting from computing the Feynman rules to obtaining the physical observables. The leading order (LO) computation is often straightforward. But, the calculation of higher order corrections involves phase space and loop integrals and these integrals yield divergences, of different kinds. To perform the integrals in a well defined manner, we should introduce a regulator. We choose dimensional regularization [114] *i.e.* the space-time dimension to be $d = 4 + \epsilon$ instead of 4. The singularities, then, appear as $\frac{1}{\epsilon^n}$ terms (where n is a positive integer).

According to their origin, the singularities can be classified as follows. Firstly, the singularities which show up in loop integrals, where the loop momentum goes to infinity, are called UV divergences. The IR divergences arise when a parton momentum become soft or collinear to another parton. In this section, we deal with UV divergences.

In dimensional regularization, the UV divergences takes the form $\frac{1}{\epsilon}$. To remove these kind of divergences, the idea of renormalization is introduced. The basic idea is to redefine the fields and coupling constants such that any physical observable is finite as $\epsilon \to 0$. We define the bare fields by a *hat* on the field notation. The bare strong coupling constant $\hat{g}_s$ acquires mass-dimension in dimensional regularization and to make it dimensionless, we introduce the scale μ_0. The divergences in the bare fields are factorized through Z's, the renormalization constants, providing

$$\hat{\phi} = Z_\phi(\mu_R)\phi(\mu_R)\,, \qquad \phi \equiv A, \Psi, c \tag{2.5}$$

$$\hat{g}_s = Z_g(\mu_R)g_s(\mu_R)\,. \tag{2.6}$$

This factorization of UV divergences introduces a new scale dependence, the renormalization scale (μ_R). The Lagrangian with the renormalized fields now contains proper counter terms to cancel the UV divergences at all order. The freedom in absorbing a finite part in addition to the poles in Z is fixed by the scheme of renormalization. In this thesis, we will use $\overline{MS}$ scheme [115], where in addition to the poles, a finite piece $\ln 4\pi - \gamma_E$ is also taken. γ_E is Euler constant with numerical value $\gamma_E = 0.5772$.

2.1.3 Running of the strong coupling constant

Because of renormalization, the strong coupling constant runs. We define $\hat{a}_s = \frac{\hat{\alpha}_s}{4\pi} = \frac{\hat{g}_s^2}{16\pi^2}$ and $a_s = \frac{\alpha_s}{4\pi} = \frac{g_s^2}{16\pi^2}$ to obtain

$$\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon = \frac{a_s}{\mu_R^\epsilon} Z_g^2(\mu_R^2) = \frac{a_s}{\mu_R^\epsilon} Z(\mu_R^2)\,. \tag{2.7}$$

$\hat{a}_s$ does not have explicit dependence on the renormalization scale (μ_R), which leads to the renormalization group equation (RGE)

$$\mu_R^2 \frac{d\hat{a}_s}{d\mu_R^2} = 0\,. \tag{2.8}$$

Differentiating Eq. 2.7 w.r.t μ_R^2 and using RGE, we arrive at

$$\mu_R^2 \frac{d \ln a_s(\mu_R^2)}{d\mu_R^2} = \frac{\epsilon}{2} - \mu_R^2 \frac{d \ln Z(\mu_R^2)}{d\mu_R^2}\,. \tag{2.9}$$

In $\overline{MS}$ scheme, the QCD β-function is defined as

$$\mu_R^2 \frac{d a_s(\mu_R^2)}{d\mu_R^2} = \frac{\epsilon}{2} + \frac{1}{a_s(\mu_R^2)}\, \beta(a_s(\mu_R^2))\,, \tag{2.10}$$

where

$$\beta(a_s(\mu_R^2)) = -\sum_{n=0}^{\infty} \beta_n\, a_s^{n+2}(\mu_R^2)\,. \tag{2.11}$$

Now, Z can be solved in expansion of a_s in terms of β_i as

$$
\begin{aligned}
Z(\mu_R^2) = 1 &+ a_s(\mu_R^2)\Big(\frac{2\beta_0}{\epsilon}\Big) + a_s^2(\mu_R^2)\Big(\frac{4\beta_0^2}{\epsilon^2} + \frac{\beta_1}{\epsilon}\Big) + a_s^3(\mu_R^2)\Big(\frac{8\beta_0^3}{\epsilon^3} + \frac{14\beta_0\beta_1}{3\epsilon^2} + \frac{2\beta_2}{3\epsilon}\Big) \\
&+ a_s^4(\mu_R^2)\Big(\frac{16\beta_0^4}{\epsilon^4} + \frac{46\beta_0^2\beta_1}{3\epsilon^3} + \frac{9\beta_1^2 + 20\beta_0\beta_2}{6\epsilon^2} + \frac{\beta_3}{2\epsilon}\Big) + O(a_s^5(\mu_R^2))\,. \qquad (2.12)
\end{aligned}
$$

We enlist the available [112, 113, 116–121] β_i's in the following

$$
\begin{aligned}
\beta_0 &= \frac{11}{3}C_A - \frac{2}{3}n_f\,, \\
\beta_1 &= \frac{34}{3}C_A^2 - 4n_fT_FC_F - \frac{20}{3}n_fT_FC_A\,, \\
\beta_2 &= \frac{2857}{54}C_A^3 - \frac{1415}{27}C_A^2T_Fn_f + \frac{158}{27}C_AT_F^2n_f^2 \\
&\quad + \frac{44}{9}C_FT_F^2n_f^2 - \frac{205}{9}C_FC_AT_Fn_f + 2C_F^2T_Fn_f\,, \\
\beta_3 &= \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right)C_AC_FT_F^2n_f^2 + \left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right)C_AC_F^2T_Fn_f \\
&\quad + \frac{424}{243}C_AT_F^3n_f^3 + \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right)C_A^2C_FT_Fn_f + \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right)C_A^2T_F^2n_f^2 \\
&\quad + \frac{1232}{243}C_FT_F^3n_f^3 + \left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right)C_A^3T_Fn_f + \left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right)C_A^4 \\
&\quad + \left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right)C_F^2T_F^2n_f^2 + 46C_F^3T_Fn_f + \left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right)n_f\frac{N(N^2+6)}{48} \\
&\quad + \left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right)n_f^2\frac{(N^4 - 6N^2 + 18)}{96N^2} + \left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right)\frac{N^2(N^2+36)}{24}\,, \qquad (2.13)
\end{aligned}
$$

where, C_F and C_A are the Casimirs of the SU(N) gauge theory in fundamental and adjoint representations, respectively. They read

$$
C_A = N, \qquad C_F = \frac{N^2-1}{2N}\,. \qquad (2.14)
$$

n_f denotes the number of active quark flavors. With the β-functions, Z takes care of the UV singularities originating from the strong coupling constant, while the counter terms from the Z_ϕ remove the rest UV divergences, and thus making the observable UV finite.

Note that, for $n_f < 17$ (and so far $n_f = 6$), β_0 is positive. Hence, a_s decreases as μ_R^2

increases, implying the asymptotic freedom. Conversely, at low energies a_s increases and at some point exits the perturbative regime. However, if we continue computing the perturbative solution of the RGE at low energies, the running coupling a_s hits a singularity at some $\mu_R^2 = \Lambda^2$, called the Landau pole. This scale Λ indicates the region of non-perturbative QCD and typically is of the order of MeV. It is clear that at low energies, QCD indeed is strongly coupled, implying confinement. Of course, there exist descriptions relating the short distance behavior to the long distance ones. Such a description, the parton model, will be addressed in the following section.

2.2 The parton model

The parton model is very successful describing the correlation between the low energy confined hadron states to high energy free partons. In this model, hadrons are made of underlying constituents, the so-called partons. A set of distribution functions ($\hat{f}_c(x)$) distinguishes partonic states and such a function $\hat{f}_c(x)$ gives the probability of finding the parton 'c' in the hadron, carrying a momentum fraction x of its parent hadron. The partons are considered to be observed in a frozen state, *i.e.* within a hadron, they do not interact with each other. This makes the scattering process as a weighted, incoherent sum of parton scattering processes, which are calculable within a dynamical theory describing the partons. For example, we consider the following hard-scattering process

$$H_1(P_1) + H_2(P_2) \to F(\{q_i\}) + X \tag{2.15}$$

where the collision of two hadrons H_1 and H_2 with momenta P_1 and P_2 produces the final state F, a system of colorless particles such as lepton pairs, Higgs boson, and so forth. For this process, the idea of parton model is depicted in Figure 2.1 where the followed notations are generic. x_i is the fraction of the parent hadron's momentum, $p_i = x_i P_i$, $i = 1, 2$. $\hat{f}_c^{H_1}(x_1)$, $\hat{f}_d^{H_2}(x_2)$ are the distribution functions of the partons c and d in the hadrons

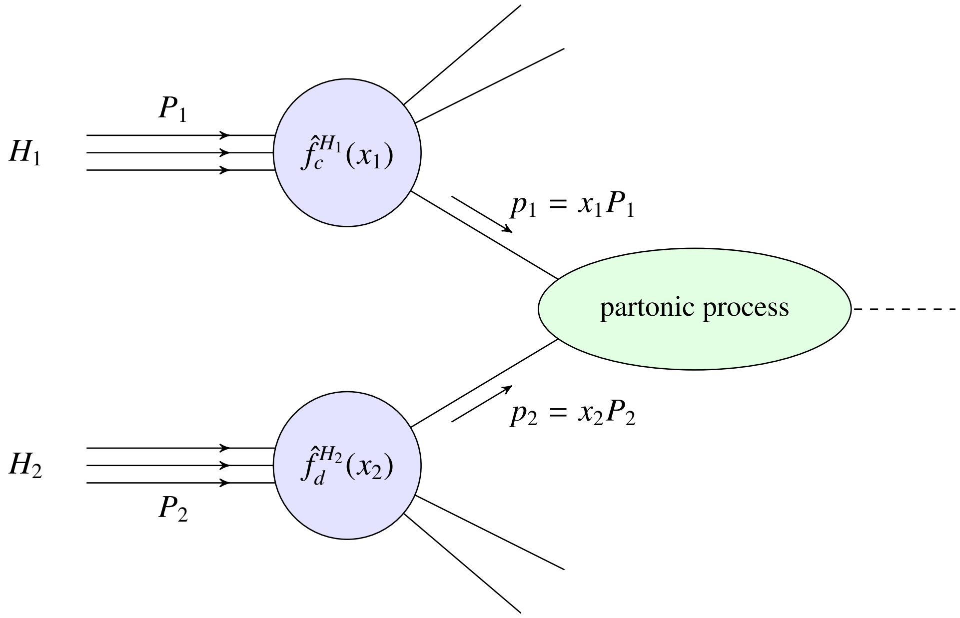


Figure 2.1: A schematic representation of parton model.

H_1 and H_2, respectively. In mathematical terms, the hadronic cross section $\sigma(S, q^2)$ is related to the partonic contributions through the following factorization formula

$$\sigma^{I,i}(\tau, q^2) = \sum_{c,d} \int_\tau^1 dx_1 \int_{\tau/x_1}^1 dx_2 \; \hat{f}_c^{H_1}(x_1) \hat{f}_d^{H_2}(x_2) \; \hat{\sigma}_{cd}^{I,i}\Big(\frac{\tau}{x_1 x_2}, q^2\Big) \tag{2.16}$$

where, I represents the inclusive production of colorless final state/s such as pair of leptons (γ^*), scalar Higgs (H), pseudo-scalar Higgs (A) or a massive spin-2 resonance (T), i indicates the type of initial partons involved at the LO, like quarks (q) or gluons (g). The scaling variable τ is defined as

$$\tau = \frac{q^2}{S} \tag{2.17}$$

with q^2 being the invariant mass of the produced colorless final state/s and $S = (P_1 + P_2)^2$ is the hadronic center of mass energy. We introduce the Mellin convolution with the notation $\otimes$ as

$$f_1(x) \otimes \ldots \otimes f_n(x) = \int_0^1 dx_1 \ldots \int_0^1 dx_n f_1(x_1) \ldots f_n(x_n) \delta(x - x_1 \ldots x_n) \,. \tag{2.18}$$

In this notation, Eq. 2.16 takes the following form

$$\sigma^{I,i}(\tau, q^2) = \tau \sum_{c,d} \hat{f}_c^{H_1} \otimes \hat{f}_d^{H_2} \otimes \hat{\Delta}_{cd}^{I,i}(\tau, q^2)\,, \tag{2.19}$$

where,

$$\hat{\Delta}_{cd}^{I,i}(z, q^2) = \frac{\hat{\sigma}_{cd}^{I,i}(z, q^2)}{z}\,. \tag{2.20}$$

This simple assumption to factorize the long distance physics from the short distance physics via the PDFs, reproduces some important, experimentally verified phenomena, leading the parton model to its success. But, this comes with a price. The bare cross section $\hat{\Delta}_{cd}^{I,i}$ contains divergences, of the IR kind, and that leads to mass factorization. To see how this happens, we will first address the IR behavior of partonic cross section in the next section, followed by another section introducing mass factorization.

2.3 Radiative corrections and infrared safety

The parton model does not account for the rest of the partons which do not take part in the hard process. Hence, to obtain QCD corrections to the partonic process, we have to consider loop as well as real emission contributions. These corrections are UV finite, after renormalization, but still contain another kind of divergences, the IR ones. The IR divergences, also called mass singularities, appear in two forms, called soft and collinear. The soft divergences arise when a loop momentum goes to zero, or there happens a soft real emission. According to Bloch-Nordsieck theorem [122], these singularities cancel when contributions from both virtual and real emission diagrams are added up. Finally, there can be divergences when momenta of two massless particles become parallel. These collinear divergences also cancel owing to the powerful theorem by Kinoshita, Lee and Nauenberg [123, 124]. It states that all divergences arising due to collinear massless partons cancel, after summing up contributions from all possible degenerate states. Now, we consider an inclusive process and hence, this condition is met for the final states only.

So, the left over divergences in the final expression are initial state collinear divergences. These singularities are removed by renormalization of the PDFs, the so-called mass factorization [125–132] procedure.

Similar to the strong coupling constant renormalization, mass factorization introduces a new unphysical mass scale (μ_F). The mass factorization theorem states that after strong coupling constant renormalization and cancellation of all other IR poles, the remaining collinear divergences of $\hat{\Delta}^{I,i}_{cd}(z, q^2)$ can be factorized as singular splitting functions $\Gamma_{cd}(z, \mu_F^2, \epsilon)$ which transforms the bare PDFs $\hat{f}^H_c$ to renormalized ones f^H_c through

$$f_c(\tau, \mu_F^2) = \sum_a \Gamma_{ca}(\mu_F^2, \epsilon) \otimes \hat{f}^H_a\,(\tau)\,. \tag{2.21}$$

The renormalized version of parton model then can be written as

$$\sigma^{I,i}(\tau, q^2) = \tau \sum_{c,d} f^{H_1}_c(\mu_F^2) \otimes f^{H_2}_d(\mu_F^2) \otimes \Delta^{I,i}_{cd}(\tau, \mu_F^2, q^2) \tag{2.22}$$

where, the finite $\Delta^{I,i}_{cd}$ contains the bare partonic cross section $\hat{\Delta}^{I,i}_{cd}(z, q^2)$ convoluted with the universal splitting functions. It is certain that the hadronic cross section or any other physical quantity does not explicitly depend on the mass factorization scale (μ_F). The μ_F-dependence of the partonic cross section is fixed by the choice of the factorization scheme, analogous to the subtraction scheme in renormalization. Once the scheme is fixed, the μ_F-dependence is computable in perturbation theory. Moreover, it does not depend on process or observable, since it is strictly related to the divergent piece. Then the μ_F-independence of a physical quantity leads us to the formulation of renormalization group equation (RGE) for PDFs which we will discuss in the section 3.4 in detail. However, the important facts about mass factorization are

- *Factorization of collinear singularities*: the divergent part of the bare partonic cross section factorizes

- *Universality of collinear divergences*: the factorized part *i.e.* the splitting functions depend only on the initial partons. They are independent of the observable or the process.

Finally, to obtain the hadronic cross section $\sigma^{I,i}(\tau, q^2)$, the renormalized version of parton model is used, as presented in Eq. 2.22. Both, the renormalized partonic cross section $\Delta^{I,i}_{cd}$ and the PDFs $f_c^{H_i}$, are free of divergences. $\Delta^{I,i}_{cd}$'s are obtained considering the perturbative expansion in $a_s(\mu_R^2)$ and this introduces the μ_R-dependence in the obtained result, order by order. While this dependence on μ_R will disappear after summing over the contributions from all orders, terminating at some fixed order exposes an explicit μ_R-dependence. On the other hand, the PDFs are experimentally measured and universal.

3 The Threshold Framework

In this Chapter we briefly present the formalism to obtain the threshold approximated cross sections for a inclusive process at the LHC. We discuss the universality of QCD amplitudes and it's factorization and following [62, 63], we present a method to obtain the threshold approximated results, which are often called soft-and-virtual (SV) cross sections. Finally we obtain a general form for the cross section up to four-loop level in pQCD, in terms of the anomalous dimensions and process dependent components from the form factors and soft contributions.

3.1 Introduction

In the renormalized version of parton model, as in Eq. 2.22, the finite partonic contribution $\Delta^{I,i}_{cd}(\tau, \mu_F)$ modulated by the flux factor $f^{H_1}_c(\mu_F^2) \otimes f^{H_2}_d(\mu_F^2)$, provides the total cross section. The dependency of $\Delta^{I,i}_{cd}(\tau, \mu_F)$ on z comes through the threshold distributions $\delta(1-z)$ and $\mathcal{D}_i$ and functions which are regular in z, where

$$\mathcal{D}_i \equiv \left[\frac{\ln^i(1-z)}{1-z}\right]_+ . \tag{3.1}$$

The emission of soft gluons defines the threshold limit, where the produced colorless boson carries almost all the partonic center of mass energy *i.e.* $q^2 \to \hat{s}$ or $z \to 1$. Near the threshold region, the flux density becomes large. On the other hand, the functions, regular in z, contributes negligibly in this region. Hence, the contributions from the plus-distributions become primary in this limit, and thus threshold approximated results become very important.

We normalize $\Delta^{I,i}$ (we will not carry the parton indices, until required) such that at LO, it is $\delta(1-z)$. For n^{th} order in perturbative expansion, the appearance of $\mathcal{D}_i$ is bound by $0 \leq i \leq 2n-1$. Mathematically,

$$\begin{aligned}\Delta^{I,i} &= \delta(1-z) + a_s[a_{11}\delta(1-z) + a_{12}\mathcal{D}_0 + a_{13}\mathcal{D}_1 + R_1(z)] \\ &\quad + a_s^2[a_{21}\delta(1-z) + \ldots + a_{25}\mathcal{D}_3 + R_2(z)] + \ldots \end{aligned} \tag{3.2}$$

Owing to the famous works by Sterman [133] and Catani *et al.* [134, 135], the contributions from plus-distributions factorize and exponentiate. The threshold only part of $\Delta^{I,i}$ can be separated as

$$\Delta^{I,i} = \Delta^{I,i}_{SV} + \Delta^{I,i}_{reg} \tag{3.3}$$

where, $\Delta^{I,i}_{reg}$ contains all the terms regular in z. Keeping in mind, the factorization of overall renormalization constant, form factors and splitting functions, the threshold only part of $\Delta^{I,i}$ *i.e.* $\Delta^{I,i}_{SV}$ can be given the following structure

$$\Delta^{I,i}_{SV}(z, q^2, \mu_R^2, \mu_F^2) = \mathcal{C} \exp\left(\Psi^{I,i}(z, q^2, \mu_R^2, \mu_F^2, \epsilon)\right)\Big|_{\epsilon=0} \tag{3.4}$$

where the finite function $\Psi^{I,i}$ is

$$\begin{aligned}\Psi^{I,i}(z, q^2, \mu_R^2, \mu_F^2, \epsilon) = \Big(&\ln \left|\hat{F}^{I,i}(\hat{a}_s, Q^2, \mu^2, \epsilon)\right|^2 \delta(1-z) + \ln\left[Z^{I,i}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)\right]^2 \delta(1-z) \\ &- 2m\, \mathcal{C} \ln \Gamma_{ii}(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon) + 2\Phi^{I,i}(\hat{a}_s, q^2, \mu^2, z, \epsilon)\Big) .\end{aligned} \tag{3.5}$$

$\mathcal{C}$ denotes a specific ordered exponentiation as

$$\mathcal{C}e^{f(z)} = \delta(1-z) + \frac{1}{1!}f(z) + \frac{1}{2!}f(z) \otimes f(z) + \cdots \tag{3.6}$$

and m indicates the number of initial QCD legs *e.g.* for DIS, $m = \frac{1}{2}$ and for DY, $m = 1$. The equivalent formalism of the SV approximation is in the Mellin (or N-moment) space, where instead of distributions in z the dominant contributions come from the meromorphic functions of the variable N (see [133,134]) and the threshold limit of $z \to 1$ is translated to $N \to \infty$. In this Chapter we study in detail the universal structures of all the components in $\Psi^{I,i}$, specifically their singularities and how they cancel to provide a finite output.

3.2 Overall renormalization constant

For cases like the Higgs boson production in gluon fusion or a pseudo-scalar Higgs boson production in gluon fusion, it involves an effective Lagrangian which manifests a non-conserved operator. For these cases, in addition to the strong coupling constant renormalization, a overall operator renormalization constant appears to preserve the UV finiteness of the theory. This overall operator renormalization constant $Z^{I,i}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)$ satisfies the following RGE

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z^{I,i}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon) = \sum_{n=1}^{\infty} a_s^n(\mu_R^2) \gamma_{n-1}^{I,i} \,, \tag{3.7}$$

where $\gamma_n^{I,i}$ is the UV anomalous dimension. After performing a perturbative expansion of $Z^{I,i}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)$ in $\hat{a}_s$ as

$$Z^{I,i}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon) = \sum_{n=0}^{\infty} \hat{a}_s^n \left(\frac{\mu_R^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n Z_n^{I,i} \,, \tag{3.8}$$

the l.h.s. of Eq. 3.7 takes the following form

$$\begin{aligned}
\mu_R^2 \frac{d}{d\mu_R^2} \ln Z^{I,i} &= \hat{a}_s \left(\frac{\mu_R^2}{\mu^2}\right)^{\frac{\epsilon}{2}} S_\epsilon Z_1^{I,i} + \hat{a}_s^2 \left(\frac{\mu_R^2}{\mu^2}\right)^{\frac{2\epsilon}{2}} S_\epsilon^2 \left(Z_1^{I,i\,2} - 2Z_2^{I,i}\right) \\
&\quad + \hat{a}_s^3 \left(\frac{\mu_R^2}{\mu^2}\right)^{\frac{3\epsilon}{2}} S_\epsilon^3 \left(Z_1^{I,i\,3} - 3Z_1^{I,i} Z_2^{I,i} + 3Z_3^{I,i}\right)
\end{aligned}$$

$$+ \hat{a}_s^4 \left(\frac{\mu_R^2}{\mu^2}\right)^{\frac{4\epsilon}{2}} S_\epsilon^4 \left(Z_1^{I,i4} - 4Z_1^{I,i2} Z_2^{I,i} + 2Z_2^{I,i2} + 4Z_1^{I,i} Z_3^{I,i} - 4Z_4^{I,i}\right) + O(\hat{a}_s^5) \,. \quad (3.9)$$

Now, using Eq. 2.7 with Eq. 2.12, we solve Eq. 3.7 and obtain $Z_n^{I,i}$ up to 4$^{\text{th}}$ order in $\hat{a}_s$ in terms of so-called UV anomalous dimensions, as presented below

$$\begin{aligned}
Z_0^{I,i} &= 1\,,\\
Z_1^{I,i} &= \frac{1}{\epsilon}\left(2\gamma_0^{I,i}\right),\\
Z_2^{I,i} &= \frac{1}{\epsilon^2}\left(2(\gamma_0^{I,i})^2 - 2\beta_0\gamma_0^{I,i}\right) + \frac{1}{\epsilon}\left(\gamma_1^{I,i}\right),\\
Z_3^{I,i} &= \frac{1}{\epsilon^3}\left(\frac{4}{3}(\gamma_0^{I,i})^3 - 4\beta_0(\gamma_0^{I,i})^2 + \frac{8}{3}\beta_0^2\gamma_0^{I,i}\right) + \frac{1}{\epsilon^2}\left(2\gamma_0^{I,i}\gamma_1^{I,i} - \frac{2}{3}\beta_1\gamma_0^{I,i} - \frac{8}{3}\beta_0\gamma_1^{I,i}\right)\\
&\quad + \frac{1}{\epsilon}\left(\frac{2}{3}\gamma_2^{I,i}\right),\\
Z_4^{I,i} &= \frac{1}{\epsilon^4}\left(\frac{2}{3}(\gamma_0^{I,i})^4 - 4\beta_0(\gamma_0^{I,i})^3 + \frac{22}{3}\beta_0^2(\gamma_0^{I,i})^2 - 4\beta_0^3\gamma_0^{I,i}\right)\\
&\quad + \frac{1}{\epsilon^3}\left(2(\gamma_0^{I,i})^2\gamma_1^{I,i} - \frac{4}{3}\beta_1(\gamma_0^{I,i})^2 - \frac{22}{3}\beta_0\gamma_0^{I,i}\gamma_1^{I,i} + \frac{8}{3}\beta_0\beta_1\gamma_0^{I,i} + 6\beta_0^2\gamma_1^{I,i}\right)\\
&\quad + \frac{1}{\epsilon^2}\left(\frac{1}{2}(\gamma_1^{I,i})^2 + \frac{4}{3}\gamma_0^{I,i}\gamma_2^{I,i} - \frac{1}{3}\beta_2\gamma_0^{I,i} - \beta_1\gamma_1^{I,i} - 3\beta_0\gamma_2^{I,i}\right) + \frac{1}{\epsilon}\left(\frac{1}{2}\gamma_3^{I,i}\right).
\end{aligned} \quad (3.10)$$

Below we present the relevant $\gamma_n^{I,i}$'s for the considered processes.

$$\begin{aligned}
&\gamma_0^{H,g} = \beta_0\,, \gamma_1^{H,g} = 2\beta_1\,, \gamma_2^{H,g} = 3\beta_2\,, \gamma_3^{H,g} = 4\beta_3\,,\\
&\gamma_n^{\gamma^*,q} = 0\,,\\
&\gamma_0^{H,b} = \boldsymbol{C_F}\Big\{3\Big\}\,,\\
&\gamma_1^{H,b} = \boldsymbol{C_F^2}\Big\{\frac{3}{2}\Big\} + \boldsymbol{C_F C_A}\Big\{\frac{97}{6}\Big\} - \boldsymbol{C_F n_f}\Big\{\frac{5}{3}\Big\}\,,\\
&\gamma_2^{H,b} = \boldsymbol{C_F^3}\Big\{\frac{129}{2}\Big\} - \boldsymbol{C_F^2 C_A}\Big\{\frac{129}{4}\Big\} + \boldsymbol{C_F C_A^2}\Big\{\frac{11413}{108}\Big\} + \boldsymbol{C_F^2 n_f}\Big\{-23 + 24\zeta_3\Big\}\\
&\qquad + \boldsymbol{C_F C_A n_f}\Big\{-\frac{278}{27} - 24\zeta_3\Big\} - \boldsymbol{C_F n_f^2}\Big\{\frac{35}{27}\Big\}\,,\\
&\gamma_n^{T,g} = 0\,,\\
&\gamma_n^{T,q} = 0\,,\\
&\gamma_0^{A,g} = \beta_0\,, \gamma_1^{A,g} = \beta_1\,, \gamma_2^{A,g} = \beta_2\,.
\end{aligned} \quad (3.11)$$

3.3 The form factor

Form factors $\hat{F}^{I,i}$ are the matrix elements of local composite operators between physical states, where the operator is color-neutral. $I = H, \gamma^*, A$ and T indicates production of the color-neutral Higgs boson, DY, pseudo-scalar Higgs boson and a massive spin 2 field, respectively. i denotes the initial on-shell states, *i.e.* g, q and b for gluons, quark-antiquark and bottom quark, respectively.

The bare form factors $\hat{F}^{I,i}(\hat{a}_s, Q^2, \mu^2, \epsilon)$ satisfy [136–139] the following integro-differential equation which follows from the gauge invariance

$$Q^2 \frac{d}{dQ^2} \ln \hat{F}^{I,i}(\hat{a}_s, Q^2, \mu^2, \epsilon) = \frac{1}{2}\left[K^i(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon) + G^{I,i}(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon)\right] \tag{3.12}$$

where, all the poles in ϵ are contained in K^i and $G^{I,i}$ is consist of terms finite in ϵ. On the other hand, RG invariance of $\hat{F}^{I,i}$ leads to the following equation

$$\mu_R^2 \frac{d}{d\mu_R^2} K^i(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon) = -\mu_R^2 \frac{d}{d\mu_R^2} G^{I,i}(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon) = -A^i \,, \tag{3.13}$$

where, A^i's are the standard cusp anomalous dimensions and are perturbative series in a_s

$$A^i = \sum_{n=1}^{\infty} a_s^n(\mu_R^2) A_n^i \,. \tag{3.14}$$

The coefficients A_1^i, A_2^i and A_3^i are explicitly known [81, 134, 135, 140–142]. They read

$$\begin{aligned}
A_1^i &= \boldsymbol{C_i}\{4\}\,,\\
A_2^i &= \boldsymbol{C_i C_A}\left\{\frac{268}{9} - 8\zeta_2\right\} + \boldsymbol{C_i n_f}\left\{-\frac{40}{9}\right\}\,,\\
A_3^i &= \boldsymbol{C_i C_A^2}\left\{\frac{490}{3} - \frac{1072\zeta_2}{9} + \frac{88\zeta_3}{3} + \frac{176\zeta_2^2}{5}\right\} + \boldsymbol{C_i C_F n_f}\left\{-\frac{110}{3} + 32\zeta_3\right\}\\
&\quad + \boldsymbol{C_i C_A n_f}\left\{-\frac{836}{27} + \frac{160\zeta_2}{9} - \frac{112\zeta_3}{3}\right\} + \boldsymbol{C_i n_f^2}\left\{-\frac{16}{27}\right\}\,.
\end{aligned} \tag{3.15}$$

These A^i's are maximally non-abelian *i.e.* they are universal except the overall Casimir element C_i, depending on the type of colliding partons. For $i = g$, $C_i = C_A$ and for $i = q$, $C_i = C_F$. This overall dependence on C_i, which is conventionally known as Casimir scaling relation, follows from the soft parton origin of A^i. The validity of this relation beyond $O(a_s^3)$ is a subject of current theoretical study [143].

With the coefficients A_n^i, it is straightforward to obtain the solution for K^i in Eq. 3.13 by performing the following expansion in powers of $\hat{a}_s$

$$K^i\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = \sum_{n=1}^{\infty} \hat{a}_s^n \left(\frac{\mu_R^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n K^{i,n}(\epsilon)\,. \tag{3.16}$$

The solutions [62] for $K^{i,n}(\epsilon)$ consist of poles in ϵ, as considered in the KG equation Eq. 3.12. They are

$$\begin{aligned}
K^{i,1}(\epsilon) &= \frac{1}{\epsilon}\Big(-2A_1^i\Big),\\
K^{i,2}(\epsilon) &= \frac{1}{\epsilon^2}\Big(2\beta_0 A_1^i\Big) + \frac{1}{\epsilon}\Big(-A_2^i\Big),\\
K^{i,3}(\epsilon) &= \frac{1}{\epsilon^3}\Big(-\frac{8}{3}\beta_0^2 A_1^i\Big) + \frac{1}{\epsilon^2}\Big(\frac{2}{3}\beta_1 A_1^i + \frac{8}{3}\beta_0 A_2^i\Big) + \frac{1}{\epsilon}\Big(-\frac{2}{3}A_3^i\Big),\\
K^{i,4}(\epsilon) &= \frac{1}{\epsilon^4}\Big(4\beta_0^3 A_1^i\Big) + \frac{1}{\epsilon^3}\Big(-\frac{8}{3}\beta_0\beta_1 A_1^i - 6\beta_0^2 A_2^i\Big)\\
&\quad + \frac{1}{\epsilon^2}\Big(\frac{1}{3}\beta_2 A_1^i + \beta_1 A_2^i + 3\beta_0 A_3^i\Big) + \frac{1}{\epsilon}\Big(-\frac{1}{2}A_4^i\Big).
\end{aligned} \tag{3.17}$$

We note that all poles at four-loop level can be obtained except the single pole as it involves the fourth order cusp anomalous dimension A_4^i.

To solve the RGE for $G^{I,i}(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon)$, we re-parametrize the integral variable, shifting the scale dependence from the boundary term as follows

$$\begin{aligned}
G^{I,i}\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) &= G^{I,i}\left(a_s(\mu_R^2), \frac{Q^2}{\mu_R^2}, \epsilon\right)\\
&= G^{I,i}\left(a_s(Q^2), 1, \epsilon\right) + \int_{\frac{Q^2}{\mu_R^2}}^{1} \frac{d\lambda^2}{\lambda^2} A^i\left(a_s(\lambda^2\mu_R^2)\right)\,,
\end{aligned} \tag{3.18}$$

while the boundary term can be expressed as perturbative expansion in a_s,

$$G^{I,i}(a_s(Q^2), 1, \epsilon) = \sum_{n=1}^{\infty} a_s^n(Q^2) G_n^{I,i}(\epsilon)\,, \tag{3.19}$$

and performing the integration results in

$$\int_{\frac{Q^2}{\mu_R^2}}^{1} \frac{d\lambda^2}{\lambda^2} A^i\left(a_s(\lambda^2\mu_R^2)\right) = \sum_{n=1}^{\infty} \hat{a}_s^n \left(\frac{\mu_R^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} \left[\left(\frac{Q^2}{\mu_R^2}\right)^{n\frac{\epsilon}{2}} - 1\right] S_\epsilon^n\, K^{i,n}(\epsilon) \tag{3.20}$$

The solutions of RGE 3.13 provides a structure of $\ln \hat{F}^{I,i}(\hat{a}_s, Q^2, \mu^2, \epsilon)$ in terms of $A_n^i, G_n^{I,i}$ and β_n, after the perturbative expansion in $\hat{a}_s$ and solving the KG equation 3.12

$$\ln \hat{F}^{I,i}(\hat{a}_s, Q^2, \mu^2, \epsilon) = \sum_{n=1}^{\infty} \hat{a}_s^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n \hat{\mathcal{L}}_F^{I,i,n}(\epsilon) \tag{3.21}$$

with

$$\begin{aligned}
\hat{\mathcal{L}}_F^{I,i,1}(\epsilon) &= \frac{1}{\epsilon^2}\Big\{-2A_1^i\Big\} + \frac{1}{\epsilon}\Big\{G_1^{I,i}(\epsilon)\Big\}\,,\\
\hat{\mathcal{L}}_F^{I,i,2}(\epsilon) &= \frac{1}{\epsilon^3}\Big\{\beta_0 A_1^i\Big\} + \frac{1}{\epsilon^2}\Big\{-\frac{1}{2}A_2^i - \beta_0 G_1^{I,i}(\epsilon)\Big\} + \frac{1}{\epsilon}\Big\{\frac{1}{2}G_2^{I,i}(\epsilon)\Big\}\,,\\
\hat{\mathcal{L}}_F^{I,i,3}(\epsilon) &= \frac{1}{\epsilon^4}\Big\{-\frac{8}{9}\beta_0^2 A_1^i\Big\} + \frac{1}{\epsilon^3}\Big\{\frac{2}{9}\beta_1 A_1^i + \frac{8}{9}\beta_0 A_2^i + \frac{4}{3}\beta_0^2 G_1^{I,i}(\epsilon)\Big\}\\
&\quad + \frac{1}{\epsilon^2}\Big\{-\frac{2}{9}A_3^i - \frac{1}{3}\beta_1 G_1^{I,i}(\epsilon) - \frac{4}{3}\beta_0 G_2^{I,i}(\epsilon)\Big\} + \frac{1}{\epsilon}\Big\{\frac{1}{3}G_3^{I,i}(\epsilon)\Big\}\,,\\
\hat{\mathcal{L}}_F^{I,i,4}(\epsilon) &= \frac{1}{\epsilon^5}\Big\{\beta_0^3 A_1^i\Big\} + \frac{1}{\epsilon^4}\Big\{-\frac{2}{3}\beta_0\beta_1 A_1^i - \frac{3}{2}\beta_0^2 A_2^i - 2\beta_0^3 G_1^{I,i}(\epsilon)\Big\}\\
&\quad + \frac{1}{\epsilon^3}\Big\{\frac{112}{\beta}{}_{2} A_1^i + \frac{1}{4}\beta_1 A_2^i + \frac{3}{4}\beta_0 A_3^i + \frac{4}{3}\beta_0\beta_1 G_1^{I,i}(\epsilon) + 3\beta_0^2 G_2^{I,i}(\epsilon)\Big\}\\
&\quad + \frac{1}{\epsilon^2}\Big\{-\frac{1}{8}A_4^i - \frac{1}{6}\beta_2 G_1^{I,i}(\epsilon) - \frac{1}{2}\beta_1 G_2^{I,i}(\epsilon) - \frac{3}{2}\beta_0 G_3^{I,i}(\epsilon)\Big\} + \frac{1}{\epsilon}\Big\{\frac{1}{4}G_4^{I,i}(\epsilon)\Big\}\,.
\end{aligned} \tag{3.22}$$

In the above expansion of the form factor, only $G_n^{I,i}$'s contain the information about the process, while A^i's indicates the kind of colliding partons and the rest is universal β-function. Also, to note that, at a given order in a_s, all the poles except the single one can be predicted without explicit computation, upon having sufficient information of the lower orders. Comparing the output of explicit computation of the form factors with the above expansion, $G_n^{I,i}$'s can be obtained. While doing so for the quark and gluon form factors, a universal structure for $G_n^{I,i}$ was observed in [144] for the first time as follows

$$G_n^{I,i}(\epsilon) = 2\left(B_n^i - \gamma_{n-1}^{I,i}\right) + f_n^i + C_n^{I,i} + \sum_{k=1}^{\infty} \epsilon^k g_n^{I,i,k}\,, \tag{3.23}$$

where, the constants $C_n^{I,i}$ read [63]

$$\begin{aligned}
C_1^{I,i} &= 0\,,\\
C_2^{I,i} &= -2\beta_0 g_1^{I,i,1}\,,\\
C_3^{I,i} &= -2\beta_1 g_1^{I,i,1} - 2\beta_0\left(g_2^{I,i,1} + 2\beta_0 g_1^{I,i,2}\right),\\
C_4^{I,i} &= -2\beta_2 g_1^{I,i,1} - 2\beta_1\left(g_2^{I,i,1} + 4\beta_0 g_1^{I,i,2}\right) - 2\beta_0\left(g_3^{I,i,1} + 2\beta_0 g_2^{I,i,2} + 4\beta_0^2 g_1^{I,i,3}\right).
\end{aligned} \tag{3.24}$$

B^i's are the collinear anomalous dimensions which originates from the collinear partons. They are explicitly known [141] up to third order in a_s

$$\begin{aligned}
B_1^g &= C_A\left\{\frac{11}{3}\right\} - n_f\left\{\frac{2}{3}\right\},\\
B_2^g &= C_A^2\left\{\frac{32}{3} + 12\zeta_3\right\} - C_A n_f\left\{\frac{8}{3}\right\} - C_F n_f\{2\},\\
B_3^g &= C_A C_F n_f\left\{-\frac{241}{18}\right\} + C_A n_f^2\left\{\frac{29}{18}\right\} - C_A^2 n_f\left\{\frac{233}{18} + \frac{8}{3}\zeta_2 + \frac{4}{3}\zeta_2^2 + \frac{80}{3}\zeta_3\right\}\\
&\quad + C_A^3\left\{\frac{79}{2} - 16\zeta_2\zeta_3 + \frac{8}{3}\zeta_2 + \frac{22}{3}\zeta_2^2 + \frac{536}{3}\zeta_3 - 80\zeta_5\right\} + C_F n_f^2\left\{\frac{11}{9}\right\} + C_F^2 n_f\{1\},\\
B_1^q &= C_F\{1\},\\
B_2^q &= C_F^2\left\{\frac{3}{2} - 12\zeta_2 + 24\zeta_3\right\} + C_A C_F\left\{\frac{17}{34} + \frac{88}{6}\zeta_2 - 12\zeta_3\right\} - C_F n_f\left\{\frac{1}{3} + \frac{8}{3}\zeta_2\right\},\\
B_3^q &= {C_A}^2 C_F\left\{-2{\zeta_2}^2 + \frac{4496}{27}\zeta_2 - \frac{1552}{9}\zeta_3 + 40\zeta_5 - \frac{1657}{36}\right\}\\
&\quad + C_A {C_F}^2\left\{-\frac{988}{15}{\zeta_2}^2 + 16\zeta_2\zeta_3 - \frac{410}{3}\zeta_2 + \frac{844}{3}\zeta_3 + 120\zeta_5 + \frac{151}{4}\right\}\\
&\quad + C_A C_F n_f\left\{\frac{4}{5}{\zeta_2}^2 - \frac{1336}{27}\zeta_2 + \frac{200}{9}\zeta_3 + 20\right\} + {C_F}^3\left\{\frac{288}{5}{\zeta_2}^2 - 32\zeta_2\zeta_3 + 18\zeta_2\right.\\
&\quad \left. + 68\zeta_3 - 240\zeta_5 + \frac{29}{2}\right\} + {C_F}^2 n_f\left\{\frac{232}{15}{\zeta_2}^2 + \frac{20}{3}\zeta_2 - \frac{136}{3}\zeta_3 - 23\right\}\\
&\quad + C_F {n_f}^2\left\{\frac{80}{27}\zeta_2 - \frac{16}{9}\zeta_3 - \frac{17}{9}\right\}.
\end{aligned} \tag{3.25}$$

f^i's are analogous to the cusp anomalous dimension A^i. f^i have purely soft origin and also demonstrate the maximally non-abelian nature similar to A^i. This was first introduced in the article [144] confirming the property up to two-loop level. Later, in [145], it was verified to be true even at three-loop. The known f^i's are

$$
\begin{aligned}
f_1^i &= 0\,,\\
f_2^i &= \boldsymbol{C_i C_A}\left\{-\frac{22}{3}\zeta_2 - 28\zeta_3 + \frac{808}{27}\right\} + \boldsymbol{C_i n_f}\left\{\frac{4}{3}\zeta_2 - \frac{112}{27}\right\}\,,\\
f_3^i &= \boldsymbol{C_i C_A{}^2}\left\{\frac{352}{5}\zeta_2{}^2 + \frac{176}{3}\zeta_2\zeta_3 - \frac{12650}{81}\zeta_2 - \frac{1316}{3}\zeta_3 + 192\zeta_5 + \frac{136781}{729}\right\}\\
&+ \boldsymbol{C_i C_A n_f}\left\{-\frac{96}{5}\zeta_2{}^2 + \frac{2828}{81}\zeta_2 + \frac{728}{27}\zeta_3 - \frac{11842}{729}\right\}\\
&+ \boldsymbol{C_i C_F n_f}\left\{\frac{32}{5}\zeta_2{}^2 + 4\zeta_2 + \frac{304}{9}\zeta_3 - \frac{1711}{27}\right\}\\
&+ \boldsymbol{C_i n_f{}^2}\left\{-\frac{40}{27}\zeta_2 + \frac{112}{27}\zeta_3 - \frac{2080}{729}\right\}\,.
\end{aligned}
\tag{3.26}
$$

The study of the form factor clearly emphasize the universality in the structure. The anomalous dimensions are only initial state dependent. The constants $g_n^{I,i,k}$ contain the explicit information for the production of a massive particle. The computation of quark form factor for Drell-Yan production $\hat{F}^{\gamma^*,q}$ [32, 33, 145–151] and gluon form factor for the Higgs boson production $\hat{F}^{H,g}$ [145, 148–152] are available up to three-loop level for few years. A recent computation also made $\hat{F}^{H,b}$ [63, 71, 153, 154] available up to three-loop level. The quark and gluon form factors for the production of a massive spin-2 resonance were available [47, 52] up to two-loop level and recently, we have obtained the same at three-loop level in [85]. The technical details of the three-loop computation is part of this thesis and will be discussed in Chapter 5. Later, we also calculated quark and gluon form factors at three-loop level for the operators describing the pseudo-scalar Higgs boson interaction in [95] and the gluon form factor for the pseudo-scalar Higgs boson production in gluon fusion is obtained in [94]. With all the available form factors, we extract $g_n^{I,i,k}$ which are relevant for three-loop and enlist them in Appendix A.

3.4 Splitting functions

The appearance of initial state divergences, which are due to the masslessness of both quarks and gluons, are removed by renormalization of the parton densities, by introducing the mass factorization kernel $\Gamma(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon)$. This mass factorization procedure yields a new scale (μ_F) dependence and a corresponding RGE as follows

$$\mu_F^2 \frac{d}{d\mu_F^2}\Gamma(z, \mu_F^2, \epsilon) = \frac{1}{2} P\left(z, \mu_F^2\right) \otimes \Gamma\left(z, \mu_F^2, \epsilon\right) \tag{3.27}$$

where, $P\left(z, \mu_F^2\right)$ are Altarelli-Parisi splitting functions (matrix valued). To solve the RGE, we perform a perturbative expansion for both $P\left(z, \mu_F^2\right)$ and $\Gamma(z, \mu_F^2, \epsilon)$ in a_s and $\hat{a}_s$, respectively as follows

$$\begin{aligned} P(z, \mu_F^2) &= \sum_{n=1}^{\infty} a_s^n(\mu_F^2) P^{(n-1)}(z) \\ \Gamma(z, \mu_F^2, \epsilon) &= \delta(1-z) + \sum_{n=1}^{\infty} \hat{a}_s^n \left(\frac{\mu_F^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n \Gamma^{(n)}(z, \epsilon)\,. \end{aligned} \tag{3.28}$$

Similar procedure as solving for $Z^{I,i}$ finally provides solution for the RGE of $\Gamma(z, \mu_F^2, \epsilon)$, Eq. 3.27. The solution up to four-loop level are presented in the following

$$\begin{aligned} \Gamma^{(1)}(z, \epsilon) &= \frac{1}{\epsilon} P^{(0)}(z) \\ \Gamma^{(2)}(z, \epsilon) &= \frac{1}{\epsilon^2}\left(\frac{1}{2} P^{(0)}(z) \otimes P^{(0)}(z) - \beta_0 P^{(0)}(z)\right) + \frac{1}{\epsilon}\left(\frac{1}{2} P^{(1)}(z)\right) \\ \Gamma^{(3)}(z, \epsilon) &= \frac{1}{\epsilon^3}\left(\frac{4}{3}\beta_0^2 P^{(0)}(z) - \beta_0 P^{(0)}(z) \otimes P^{(0)}(z)\right. \\ &\quad \left. + \frac{1}{6} P^{(0)}(z) \otimes P^{(0)}(z) \otimes P^{(0)}(z)\right) + \frac{1}{\epsilon^2}\left(\frac{1}{6} P^{(0)}(z) \otimes P^{(1)}(z) - \frac{1}{3}\beta_1 P^{(0)}(z)\right. \\ &\quad \left. + \frac{1}{3} P^{(1)}(z) \otimes P^{(0)}(z) - \frac{4}{3}\beta_0 P^{(1)}(z)\right) + \frac{1}{\epsilon}\left(\frac{1}{3} P^{(2)}(z)\right) \\ \Gamma^{(4)}(z, \epsilon) &= \frac{1}{\epsilon^4}\left(\frac{1}{24} P^{(0)}(z) \otimes P^{(0)}(z) \otimes P^{(0)}(z) \otimes P^{(0)}(z)\right. \\ &\quad - \frac{1}{2}\beta_0 P^{(0)}(z) \otimes P^{(0)}(z) \otimes P^{(0)}(z) + \frac{11}{6}\beta_0^2 P^{(0)}(z) \otimes P^{(0)}(z) \end{aligned}$$

$$
\begin{aligned}
& - 2\beta_0^3 P^{(0)}(z)\Big) + \frac{1}{\epsilon^3}\Big(\frac{1}{24}P^{(0)}(z)\otimes P^{(0)}(z)\otimes P^{(1)}(z) \\
& - \frac{1}{3}\beta_1 P^{(0)}(z)\otimes P^{(0)}(z) + \frac{4}{3}\beta_0\beta_1 P^{(0)}(z) + \frac{1}{12}P^{(0)}(z)\otimes P^{(1)}(z)\otimes P^{(0)}(z) \\
& - \frac{7}{12}\beta_0 P^{(0)}(z)P^{(1)}(z) + \frac{1}{8}P^{(1)}(z)\otimes P^{(0)}(z)\otimes P^{(0)}(z) \\
& - \frac{5}{4}\beta_0 P^{(1)}(z)\otimes P^{(0)}(z) + 3\beta_0^2 P^{(1)}(z)\Big) \\
& + \frac{1}{\epsilon^2}\Big(\frac{1}{12}P^{(0)}(z)\otimes P^{(2)}(z) - \frac{1}{6}\beta_2 P^{(0)}(z) + \frac{1}{8}P^{(1)}(z)\otimes P^{(1)}(z) \\
& - \frac{1}{2}\beta_1 P^{(1)}(z) + \frac{1}{4}P^{(2)}(z)\otimes P^{(0)}(z) - \frac{3}{2}\beta_0 P^{(2)}(z)\Big) + \frac{1}{\epsilon}\Big(\frac{1}{4}P^{(3)}(z)\Big)
\end{aligned}
$$

$P^{(0)}(z)$, $P^{(1)}(z)$ and $P^{(2)}(z)$ are computed in the articles [140, 141]. In the threshold limit $z \to 1$, only the diagonal parts of the splitting functions, $P^{(n)}_{ii}(z)$ and kernels, $\Gamma^{(n)}_{ii}(z,\epsilon)$ contribute, as the non-diagonal terms only contain contributions which are regular in z.

3.5 Soft distribution functions

The overall renormalization constant ($Z^{I,i}$), the form factor ($\hat{F}^{I,i}$) and the splitting functions (Γ_{ii}) follow a universality to gain the singular structure. A detail study reveals that the UV and collinear part of the single pole in $\hat{F}^{I,i}$ gets canceled by that of from the $Z^{I,i}$ and Γ_{ii}, respectively. Demanding the finiteness of $\Psi^{I,i}$, the soft distribution function $\Phi^{I,i}$ must cancel the remaining singularities which in turn suggests that $\Phi^{I,i}$ should satisfy Sudakov type integro-differential equation similar to $\hat{F}^{I,i}$. Hence, the ansatz is

$$
q^2\frac{d}{dq^2}\Phi^{I,i}\left(\hat{a}_s, q^2, \mu^2, z, \epsilon\right) = \frac{1}{2}\left[\overline{K}^i\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right) + \overline{G}^{I,i}\left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right)\right] \tag{3.29}
$$

where $\overline{K}^i$ and $\overline{G}^{I,i}$ play similar role to K^i and $G^{I,i}$ *i.e.* $\overline{K}^i$ contains all the poles in ϵ and $\overline{G}^{I,i}$ is finite as $\epsilon \to 0$. Also, $\Phi^{I,i}$ does not have explicit dependence on (μ_R), implying a RGE similar to Eq. 3.13

$$
\mu_R^2\frac{d}{d\mu_R^2}\Phi^{I}(\hat{a}_s, q^2, \mu^2, z, \epsilon) = 0\,. \tag{3.30}
$$

These two equations lead to

$$\mu_R^2 \frac{d}{d\mu_R^2} \overline{K}^i \left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \epsilon \right) = -\mu_R^2 \frac{d}{d\mu_R^2} \overline{G}^{I,i} \left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon \right) = -\delta(1-z) \overline{A}^i \tag{3.31}$$

where, to cancel the poles exactly, $\overline{A}^i$ must have to be equal and opposite in sign to A^i. Similar to Eq. 3.16, we perform the following expansion of $\overline{K}^i$ in $\hat{a}_s$

$$\overline{K}^i \left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \epsilon \right) = \delta(1-z) \sum_{n=1}^{\infty} \hat{a}_s^n \left(\frac{\mu_R^2}{\mu^2} \right)^{n\frac{\epsilon}{2}} S_\epsilon^n \overline{K}^{i,n}(\epsilon) \tag{3.32}$$

and the solutions [62] also come out to be same except for a sign, as

$$\begin{aligned}
\overline{K}^{i,1}(\epsilon) &= \frac{1}{\epsilon}\Big(2A_1^i\Big), \\
\overline{K}^{i,2}(\epsilon) &= \frac{1}{\epsilon^2}\Big(-2\beta_0 A_1^i\Big) + \frac{1}{\epsilon}\Big(A_2^i\Big), \\
\overline{K}^{i,3}(\epsilon) &= \frac{1}{\epsilon^3}\Big(\frac{8}{3}\beta_0^2 A_1^i\Big) + \frac{1}{\epsilon^2}\Big(-\frac{2}{3}\beta_1 A_1^i - \frac{8}{3}\beta_0 A_2^i\Big) + \frac{1}{\epsilon}\Big(\frac{2}{3}A_3^i\Big), \\
\overline{K}^{i,4}(\epsilon) &= \frac{1}{\epsilon^4}\Big(-4\beta_0^3 A_1^i\Big) + \frac{1}{\epsilon^3}\Big(\frac{8}{3}\beta_0\beta_1 A_1^i + 6\beta_0^2 A_2^i\Big) \\
&\quad + \frac{1}{\epsilon^2}\Big(-\frac{1}{3}\beta_2 A_1^i - \beta_1 A_2^i - 3\beta_0 A_3^i\Big) + \frac{1}{\epsilon}\Big(\frac{1}{2}A_4^i\Big).
\end{aligned} \tag{3.33}$$

Again, we use the similar idea to solve the RGE for $\overline{G}^{I,i}$, by re-parametrizing the integral variable, shifting the scale dependence from the boundary term as

$$\begin{aligned}
\overline{G}^{I,i} \left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon \right) &= \overline{G}^{I,i} \left(a_s(\mu_R^2), \frac{q^2}{\mu_R^2}, z, \epsilon \right) \\
&= \overline{G}^{I,i} \Big(a_s(q^2), 1, z, \epsilon \Big) - \delta(1-z) \int_{\frac{q^2}{\mu_R^2}}^{1} \frac{d\lambda^2}{\lambda^2} A^i \Big(a_s(\lambda^2 \mu_R^2) \Big) .
\end{aligned}$$

Now, the soft distribution function can be solved as in Eq. 3.21. However, observing the source of plus distribution from the Feynman integrals, we expand $\Phi^{I,i}$ as a perturbative expansion as follows

$$\Phi^{I,i}(\hat{a}_s, q^2, \mu^2, z, \epsilon) = \sum_{n=1}^{\infty} \hat{a}_s^n \left(\frac{q^2(1-z)^2}{\mu^2} \right)^{n\frac{\epsilon}{2}} S_\epsilon^n \left(\frac{n\epsilon}{1-z} \right) \hat{\phi}^{I,i,n}(\epsilon) \tag{3.34}$$

where

$$\hat{\phi}^{I,i,n}(\epsilon) = \frac{1}{n\epsilon}\left(\overline{K}^{i,n}(\epsilon) + \overline{G}_n^{I,i}(\epsilon)\right). \tag{3.35}$$

Note that, $\overline{G}_n^{I,i}(\epsilon)$ is not perturbative coefficient of $\overline{G}^{I,i}(a_s(q^2), 1, \epsilon)$ as the case for form factor, rather is related to it through a combination of plus distributions. Now, for simplicity, instead of $\overline{G}_n^{I,i}(\epsilon)$, we use $\overline{\mathcal{G}}_n^{I,i}(\epsilon)$, which are related via

$$\sum_{n=1}^{\infty} \hat{a}_s^n \left(\frac{q^2(1-z)^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n \overline{G}_n^{I,i}(\epsilon) = \sum_{n=1}^{\infty} a_s^n \left(q^2(1-z)^2\right) \overline{\mathcal{G}}_n^{I,i}(\epsilon) \tag{3.36}$$

and the solution of Eq. 3.29 becomes similar to that of Eq. 3.22 as

$$\hat{\phi}^{I,i,n}(\epsilon) = \hat{\mathcal{L}}_F^{I,i,n}(\epsilon)\Big|_{\left(A^i \to -A^i, G^{I,i}(\epsilon) \to \overline{\mathcal{G}}^{I,i}(\epsilon)\right)}. \tag{3.37}$$

The z-independent constant $\overline{\mathcal{G}}_n^{I,i}(\epsilon)$ can be obtained by comparing the ϵ-coefficients of $\phi^{I,i,(n)}(\epsilon)$ with those arising from the rest in Eq. 3.4 demanding the finiteness at $\epsilon \to 0$. We find $\overline{\mathcal{G}}_n^{I,i}(\epsilon)$ emerges out to have the following universal structure as $G_n^{I,i}(\epsilon)$

$$\overline{\mathcal{G}}_n^{I,i}(\epsilon) = -f_n^i + \overline{C}_n^{I,i} + \sum_{k=1}^{\infty} \epsilon^k \overline{\mathcal{G}}_n^{I,i,k}, \tag{3.38}$$

where

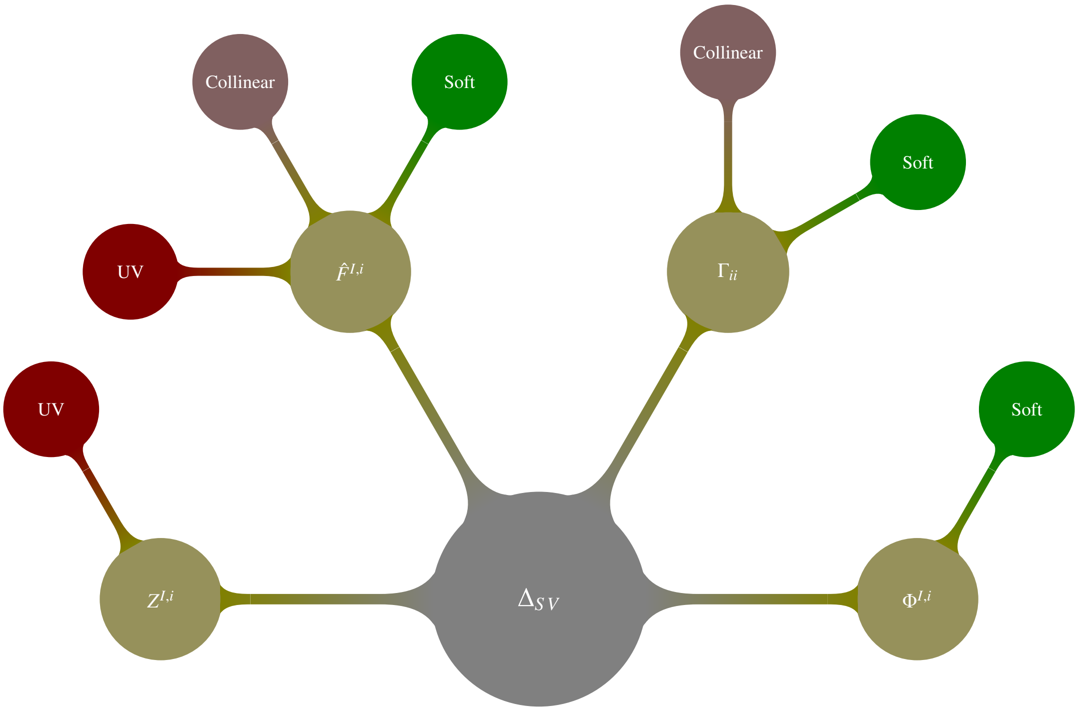
$$\overline{C}_n^{I,i} = C_n^{I,i}\Big|_{\left(g_n^{I,i,k} \to \overline{\mathcal{G}}_n^{I,i,k}\right)} \tag{3.39}$$

The constants $\overline{\mathcal{G}}_n^{I,i,k}$ are input from the explicit computations. For the first time in [62, 63], it was noted that, up to two-loop, these constants also satisfy the Casimir scaling relation *i.e.*

$$\overline{\mathcal{G}}_n^{\gamma^*,q,k} = \frac{C_F}{C_A} \overline{\mathcal{G}}_n^{H,g,k} \tag{3.40}$$

and does not contain any information about the final state I. This fact leads the soft distribution function $\Phi^{I,i}$ to be maximally non-abelian too. The universality of the soft distribution function can be understood by noting that the soft contribution to the cross section is always independent of the spin, color, flavor or any other quantum numbers after factoring out the Born contribution. It solely depends on the gauge group describing the interaction, *e.g*. SU(N). Once we know the soft distribution function for a process *e.g*. the Higgs boson production in gluon fusion, this universality help us to achieve the soft distribution functions for all other process. We present $\overline{\mathcal{G}}_n^{I,i,k}$, relevant for N^{3}LO computation in Appendix B.

3.6 The SV cross section



The detailed study of the overall renormalization constant ($Z^{I,i}$), the form factor ($\hat{F}^{I,i}$), the splitting kernels (Γ_{ii}) and the soft distribution function ($\Phi^{I,i}$) provides a structure for the threshold approximated partonic cross section $\Delta_{SV}^{I,i}$. As shown in the above schematic diagram, all of them contain divergences of any of the three kinds: UV, collinear or soft. But, in the end, they cancel each other and beautifully sum up to provide a finite partonic cross section $\Delta_{SV}^{I,i}$. We compute $\Delta_{SV}^{I,i}$ by expanding it in perturbative series of a_s as

$$\Delta_{SV}^{I,i} = \sum_{n=0}^{\infty} a_s^n(\mu_R^2)\Delta_{SV}^{I,i,n} \,. \tag{3.41}$$

In the following, we present $\Delta_{SV}^{I,i,n}$ up to four-loop level.

$$
\begin{aligned}
\Delta_{SV}^{I,i,1} &= \delta(1-z)(2\overline{\mathcal{G}}_1^{I,i,1} + 2g_1^{I,i,1} + 3\zeta_2 A_1^i + L_{\mu_F}(2B_1^i - 2\gamma_0^{I,i})) \\
&+ \mathcal{D}_0(L_{\mu_F}(2A_1^i) - 2f_1^i) + \mathcal{D}_1(4A_1^i) \qquad (3.42) \\
\Delta_{SV}^{I,i,2} &= \delta(1-z)(\overline{\mathcal{G}}_2^{I,i,1} + 2{\overline{\mathcal{G}}_1^{I,i,1}}^2 + g_2^{I,i,1} + 4g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} + 2{g_1^{I,i,1}}^2 + 2\beta_0\overline{\mathcal{G}}_1^{I,i,2} \\
&+ 2\beta_0 g_1^{I,i,2} - 8\zeta_3 A_1^i f_1^i - 2\zeta_2 {f_1^i}^2 + 3\zeta_2 A_2^i + 6\zeta_2\overline{\mathcal{G}}_1^{I,i,1} A_1^i + 6\zeta_2 g_1^{I,i,1} A_1^i \\
&+ 3\zeta_2\beta_0 f_1^i + 6\zeta_2\beta_0 B_1^i - 6\zeta_2\beta_0\gamma_0^{I,i} + \frac{37}{10}\zeta_2^2 {A_1^i}^2 + L_{\mu_F}(2B_2^i + 4\overline{\mathcal{G}}_1^{I,i,1} B_1^i \\
&+ 4g_1^{I,i,1} B_1^i - 2\gamma_1^{I,i} - 4\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} - 4\gamma_0^{I,i} g_1^{I,i,1} - 2\beta_0\overline{\mathcal{G}}_1^{I,i,1} - 2\beta_0 g_1^{I,i,1} + 8\zeta_3 {A_1^i}^2 \\
&+ 4\zeta_2 A_1^i f_1^i + 6\zeta_2 A_1^i B_1^i - 6\zeta_2\gamma_0^{I,i} A_1^i - 3\zeta_2\beta_0 A_1^i) + L_{\mu_F}^2(2{B_1^i}^2 - 4\gamma_0^{I,i} B_1^i \\
&+ 2{\gamma_0^{I,i}}^2 - \beta_0 B_1^i + \beta_0\gamma_0^{I,i} - 2\zeta_2 {A_1^i}^2)) + \mathcal{D}_0(L_{\mu_F}(-4B_1^i f_1^i + 2A_2^i + 4\overline{\mathcal{G}}_1^{I,i,1} A_1^i \\
&+ 4g_1^{I,i,1} A_1^i + 4\gamma_0^{I,i} f_1^i + 2\beta_0 f_1^i - 2\zeta_2 {A_1^i}^2) + L_{\mu_F}^2(4A_1^i B_1^i - 4\gamma_0^{I,i} A_1^i - \beta_0 A_1^i) \\
&- 2f_2^i - 4\overline{\mathcal{G}}_1^{I,i,1} f_1^i - 4g_1^{I,i,1} f_1^i - 4\beta_0\overline{\mathcal{G}}_1^{I,i,1} + 16\zeta_3 {A_1^i}^2 + 2\zeta_2 A_1^i f_1^i) \\
&+ \mathcal{D}_1(L_{\mu_F}(-8A_1^i f_1^i + 8A_1^i B_1^i - 8\gamma_0^{I,i} A_1^i - 4\beta_0 A_1^i) + L_{\mu_F}^2(4{A_1^i}^2) + 4{f_1^i}^2 \\
&+ 4A_2^i + 8\overline{\mathcal{G}}_1^{I,i,1} A_1^i + 8g_1^{I,i,1} A_1^i + 4\beta_0 f_1^i - 4\zeta_2 {A_1^i}^2) + \mathcal{D}_2(L_{\mu_F}(12{A_1^i}^2) \\
&- 12A_1^i f_1^i - 4\beta_0 A_1^i) + \mathcal{D}_3(8{A_1^i}^2) \qquad (3.43) \\
\Delta_{SV}^{I,i,3} &= \delta(1-z)(\frac{2}{3}\overline{\mathcal{G}}_3^{I,i,1} + 2\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} + \frac{4}{3}{\overline{\mathcal{G}}_1^{I,i,1}}^3 + \frac{2}{3}g_3^{I,i,1} + 2g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} \\
&+ 2g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} + 4g_1^{I,i,1}{\overline{\mathcal{G}}_1^{I,i,1}}^2 + 2g_1^{I,i,1} g_2^{I,i,1} + 4{g_1^{I,i,1}}^2\overline{\mathcal{G}}_1^{I,i,1} + \frac{4}{3}{g_1^{I,i,1}}^3 \\
&+ \frac{4}{3}\beta_1\overline{\mathcal{G}}_1^{I,i,2} + \frac{4}{3}\beta_1 g_1^{I,i,2} + \frac{4}{3}\beta_0\overline{\mathcal{G}}_2^{I,i,2} + 4\beta_0\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} + \frac{4}{3}\beta_0 g_2^{I,i,2} \\
&+ 4\beta_0 g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1} + 4\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} + 4\beta_0 g_1^{I,i,1} g_1^{I,i,2} + \frac{8}{3}\beta_0^2\overline{\mathcal{G}}_1^{I,i,3} + \frac{8}{3}\beta_0^2 g_1^{I,i,3} \\
&- 96\zeta_5 {A_1^i}^2 f_1^i - 64\zeta_5\beta_0 {A_1^i}^2 - \frac{8}{3}\zeta_3 {f_1^i}^3 - 8\zeta_3 A_2^i f_1^i - 8\zeta_3 A_1^i f_2^i - 16\zeta_3\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i \\
&- 16\zeta_3 g_1^{I,i,1} A_1^i f_1^i - 8\zeta_3\beta_0 {f_1^i}^2 - 16\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^i + \frac{160}{3}\zeta_3^2 {A_1^i}^3 - 4\zeta_2 f_1^i f_2^i \\
&+ 3\zeta_2 A_3^i + 3\zeta_2\overline{\mathcal{G}}_2^{I,i,1} A_1^i - 4\zeta_2\overline{\mathcal{G}}_1^{I,i,1} {f_1^i}^2 + 6\zeta_2\overline{\mathcal{G}}_1^{I,i,1} A_2^i + 6\zeta_2{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_1^i \\
&+ 3\zeta_2 g_2^{I,i,1} A_1^i - 4\zeta_2 g_1^{I,i,1} {f_1^i}^2 + 6\zeta_2 g_1^{I,i,1} A_2^i + 12\zeta_2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_1^i + 6\zeta_2 {g_1^{I,i,1}}^2 A_1^i \\
&+ 3\zeta_2\beta_1 f_1^i + 6\zeta_2\beta_1 B_1^i - 6\zeta_2\beta_1\gamma_0^{I,i} + 6\zeta_2\beta_0 f_2^i + 12\zeta_2\beta_0 B_2^i + 6\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,2} A_1^i \\
&- 2\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1} f_1^i + 12\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1} B_1^i + 6\zeta_2\beta_0 g_1^{I,i,2} A_1^i + 6\zeta_2\beta_0 g_1^{I,i,1} f_1^i
\end{aligned}
$$

$$
\begin{aligned}
&+ 12\zeta_2\beta_0 g_1^{I,i,1} B_1^i - 12\zeta_2\beta_0\gamma_1^{I,i} - 12\zeta_2\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} - 12\zeta_2\beta_0\gamma_0^{I,i} g_1^{I,i,1} \\
&- 12\zeta_2\beta_0^2 g_1^{I,i,1} + 40\zeta_2\zeta_3 {A_1^i}^2 f_1^i + 32\zeta_2\zeta_3\beta_0 {A_1^i}^2 - \frac{38}{5}\zeta_2^2 A_1^i {f_1^i}^2 + \frac{37}{5}\zeta_2^2 A_1^i A_2^i \\
&+ \frac{37}{5}\zeta_2^2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 + \frac{37}{5}\zeta_2^2 g_1^{I,i,1}{A_1^i}^2 + \zeta_2^2\beta_0 A_1^i f_1^i + 18\zeta_2^2\beta_0 A_1^i B_1^i - 18\zeta_2^2\beta_0\gamma_0^{I,i}A_1^i \\
&- 3\zeta_2^2\beta_0^2 A_1^i - \frac{283}{42}\zeta_2^3{A_1^i}^3 + L_{\mu_F}(2B_3^i + 2\overline{\mathcal{G}}_2^{I,i,1}B_1^i + 4\overline{\mathcal{G}}_1^{I,i,1}B_2^i + 4{\overline{\mathcal{G}}_1^{I,i,1}}^2 B_1^i \\
&+ 2g_2^{I,i,1}B_1^i + 4g_1^{I,i,1}B_2^i + 8g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}B_1^i + 4{g_1^{I,i,1}}^2B_1^i - 2\gamma_2^{I,i} - 4\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1} \\
&- 4\gamma_1^{I,i}g_1^{I,i,1} - 2\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,1} - 4\gamma_0^{I,i}{\overline{\mathcal{G}}_1^{I,i,1}}^2 - 2\gamma_0^{I,i}g_2^{I,i,1} - 8\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} \\
&- 4\gamma_0^{I,i}{g_1^{I,i,1}}^2 - 2\beta_1\overline{\mathcal{G}}_1^{I,i,1} - 2\beta_1 g_1^{I,i,1} - 2\beta_0\overline{\mathcal{G}}_2^{I,i,1} + 4\beta_0\overline{\mathcal{G}}_1^{I,i,2}B_1^i - 4\beta_0{\overline{\mathcal{G}}_1^{I,i,1}}^2 \\
&- 2\beta_0 g_2^{I,i,1} + 4\beta_0 g_1^{I,i,2}B_1^i - 8\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} - 4\beta_0{g_1^{I,i,1}}^2 - 4\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,2} \\
&- 4\beta_0\gamma_0^{I,i}g_1^{I,i,2} - 4\beta_0^2\overline{\mathcal{G}}_1^{I,i,2} - 4\beta_0^2 g_1^{I,i,2} + 96\zeta_5{A_1^i}^3 + 8\zeta_3 A_1^i{f_1^i}^2 - 16\zeta_3 A_1^i B_1^i f_1^i \\
&+ 16\zeta_3 A_1^i A_2^i + 16\zeta_3\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 + 16\zeta_3 g_1^{I,i,1}{A_1^i}^2 + 16\zeta_3\gamma_0^{I,i}A_1^i f_1^i + 24\zeta_3\beta_0 A_1^i f_1^i \\
&- 4\zeta_2 B_1^i{f_1^i}^2 + 4\zeta_2 A_2^i f_1^i + 6\zeta_2 A_2^i B_1^i + 4\zeta_2 A_1^i f_2^i + 6\zeta_2 A_1^i B_2^i + 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_1^i f_1^i \\
&+ 12\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_1^iB_1^i + 8\zeta_2 g_1^{I,i,1}A_1^i f_1^i + 12\zeta_2 g_1^{I,i,1}A_1^iB_1^i - 6\zeta_2\gamma_1^{I,i}A_1^i + 4\zeta_2\gamma_0^{I,i}{f_1^i}^2 \\
&- 6\zeta_2\gamma_0^{I,i}A_2^i - 12\zeta_2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 12\zeta_2\gamma_0^{I,i}g_1^{I,i,1}A_1^i - 3\zeta_2\beta_1 A_1^i + 4\zeta_2\beta_0{f_1^i}^2 \\
&+ 6\zeta_2\beta_0 B_1^i f_1^i + 12\zeta_2\beta_0{B_1^i}^2 - 6\zeta_2\beta_0 A_2^i - 4\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 12\zeta_2\beta_0 g_1^{I,i,1}A_1^i \\
&- 6\zeta_2\beta_0\gamma_0^{I,i}f_1^i - 24\zeta_2\beta_0\gamma_0^{I,i}B_1^i + 12\zeta_2\beta_0{\gamma_0^{I,i}}^2 - 6\zeta_2\beta_0^2 f_1^i - 12\zeta_2\beta_0^2 B_1^i \\
&+ 12\zeta_2\beta_0^2\gamma_0^{I,i} - 40\zeta_2\zeta_3{A_1^i}^3 + \frac{76}{5}\zeta_2^2{A_1^i}^2 f_1^i + \frac{37}{5}\zeta_2^2{A_1^i}^2B_1^i - \frac{37}{5}\zeta_2^2\gamma_0^{I,i}{A_1^i}^2 \\
&- \zeta_2^2\beta_0{A_1^i}^2) + L_{\mu_F}^2(4B_1^iB_2^i + 4\overline{\mathcal{G}}_1^{I,i,1}{B_1^i}^2 + 4g_1^{I,i,1}{B_1^i}^2 - 4\gamma_1^{I,i}B_1^i - 4\gamma_0^{I,i}B_2^i \\
&- 8\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 8\gamma_0^{I,i}g_1^{I,i,1}B_1^i + 4\gamma_0^{I,i}\gamma_1^{I,i} + 4{\gamma_0^{I,i}}^2\overline{\mathcal{G}}_1^{I,i,1} + 4{\gamma_0^{I,i}}^2 g_1^{I,i,1} - \beta_1 B_1^i \\
&+ \beta_1\gamma_0^{I,i} - 2\beta_0 B_2^i - 6\beta_0\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 6\beta_0 g_1^{I,i,1}B_1^i + 2\beta_0\gamma_1^{I,i} + 6\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} \\
&+ 6\beta_0\gamma_0^{I,i}g_1^{I,i,1} + 2\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} + 2\beta_0^2 g_1^{I,i,1} - 8\zeta_3{A_1^i}^2 f_1^i + 16\zeta_3{A_1^i}^2 B_1^i \\
&- 16\zeta_3\gamma_0^{I,i}{A_1^i}^2 - 12\zeta_3\beta_0{A_1^i}^2 + 8\zeta_2 A_1^iB_1^if_1^i + 6\zeta_2A_1^i{B_1^i}^2 - 4\zeta_2A_1^iA_2^i \\
&- 4\zeta_2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 - 4\zeta_2 g_1^{I,i,1}{A_1^i}^2 - 8\zeta_2\gamma_0^{I,i}A_1^if_1^i - 12\zeta_2\gamma_0^{I,i}A_1^iB_1^i + 6\zeta_2{\gamma_0^{I,i}}^2A_1^i \\
&- 6\zeta_2\beta_0A_1^if_1^i - 9\zeta_2\beta_0A_1^iB_1^i + 9\zeta_2\beta_0\gamma_0^{I,i}A_1^i + 3\zeta_2\beta_0^2A_1^i - \frac{38}{5}\zeta_2^2{A_1^i}^3) \\
&+ L_{\mu_F}^3(\frac{4}{3}{B_1^i}^3 - 4\gamma_0^{I,i}{B_1^i}^2 + 4{\gamma_0^{I,i}}^2B_1^i - \frac{4}{3}{\gamma_0^{I,i}}^3 - 2\beta_0{B_1^i}^2 + 4\beta_0\gamma_0^{I,i}B_1^i - 2\beta_0{\gamma_0^{I,i}}^2
\end{aligned}
$$

$$+ \frac{2}{3}\beta_0^2 B_1^i - \frac{2}{3}\beta_0^2\gamma_0^{I,i} + \frac{8}{3}\zeta_3 A_1^{i\,3} - 4\zeta_2 A_1^{i\,2}B_1^i + 4\zeta_2\gamma_0^{I,i}A_1^{i\,2} + 2\zeta_2\beta_0 A_1^{i\,2}))$$

$$+ \mathcal{D}_0(L_{\mu_F}(-4B_2^i f_1^i - 4B_1^i f_2^i + 2A_3^i + 2\overline{\mathcal{G}}_2^{I,i,1}A_1^i - 8\overline{\mathcal{G}}_1^{I,i,1}B_1^i f_1^i + 4\overline{\mathcal{G}}_1^{I,i,1}A_2^i$$

$$+ 4\overline{\mathcal{G}}_1^{I,i,1\,2}A_1^i + 2g_2^{I,i,1}A_1^i - 8g_1^{I,i,1}B_1^i f_1^i + 4g_1^{I,i,1}A_2^i + 8g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i$$

$$+ 4g_1^{I,i,1\,2}A_1^i + 4\gamma_1^{I,i}f_1^i + 4\gamma_0^{I,i}f_2^i + 8\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}f_1^i + 8\gamma_0^{I,i}g_1^{I,i,1}f_1^i + 2\beta_1 f_1^i$$

$$+ 4\beta_0 f_2^i + 4\beta_0\overline{\mathcal{G}}_1^{I,i,2}A_1^i + 8\beta_0\overline{\mathcal{G}}_1^{I,i,1}f_1^i - 8\beta_0\overline{\mathcal{G}}_1^{I,i,1}B_1^i + 4\beta_0 g_1^{I,i,2}A_1^i + 8\beta_0 g_1^{I,i,1}f_1^i$$

$$+ 8\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} + 8\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} - 64\zeta_3 A_1^{i\,2}f_1^i + 32\zeta_3 A_1^{i\,2}B_1^i - 32\zeta_3\gamma_0^{I,i}A_1^{i\,2}$$

$$- 48\zeta_3\beta_0 A_1^{i\,2} - 12\zeta_2 A_1^i f_1^{i2} + 4\zeta_2 A_1^i B_1^i f_1^i - 4\zeta_2 A_1^i A_2^i - 4\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2}$$

$$- 4\zeta_2 g_1^{I,i,1}A_1^{i\,2} - 4\zeta_2\gamma_0^{I,i}A_1^i f_1^i - 6\zeta_2\beta_0 A_1^i f_1^i + 12\zeta_2\beta_0 A_1^i B_1^i - 12\zeta_2\beta_0\gamma_0^{I,i}A_1^i$$

$$- 23\zeta_2^2 A_1^{i\,3}) + L_{\mu_F}^2(-4B_1^{i\,2}f_1^i + 4A_2^i B_1^i + 4A_1^i B_2^i + 8\overline{\mathcal{G}}_1^{I,i,1}A_1^i B_1^i + 8g_1^{I,i,1}A_1^i B_1^i$$

$$- 4\gamma_1^{I,i}A_1^i + 8\gamma_0^{I,i}B_1^i f_1^i - 4\gamma_0^{I,i}A_2^i - 8\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 8\gamma_0^{I,i}g_1^{I,i,1}A_1^i - 4\gamma_0^{I,i2}f_1^i$$

$$- \beta_1 A_1^i + 6\beta_0 B_1^i f_1^i - 2\beta_0 A_2^i - 6\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 6\beta_0 g_1^{I,i,1}A_1^i - 6\beta_0\gamma_0^{I,i}f_1^i$$

$$- 2\beta_0^2 f_1^i + 32\zeta_3 A_1^{i\,3} + 12\zeta_2 A_1^{i\,2}f_1^i - 4\zeta_2 A_1^{i\,2}B_1^i + 4\zeta_2\gamma_0^{I,i}A_1^{i\,2} + 3\zeta_2\beta_0 A_1^{i\,2})$$

$$+ L_{\mu_F}^3(4A_1^i B_1^{i\,2} - 8\gamma_0^{I,i}A_1^i B_1^i + 4\gamma_0^{I,i2}A_1^i - 4\beta_0 A_1^i B_1^i + 4\beta_0\gamma_0^{I,i}A_1^i + \frac{2}{3}\beta_0^2 A_1^i$$

$$- 4\zeta_2 A_1^{i\,3}) - 2f_3^i - 2\overline{\mathcal{G}}_2^{I,i,1}f_1^i - 4\overline{\mathcal{G}}_1^{I,i,1}f_2^i - 4\overline{\mathcal{G}}_1^{I,i,1\,2}f_1^i - 2g_2^{I,i,1}f_1^i - 4g_1^{I,i,1}f_2^i$$

$$- 8g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}f_1^i - 4g_1^{I,i,1\,2}f_1^i - 4\beta_1\overline{\mathcal{G}}_1^{I,i,1} - 4\beta_0\overline{\mathcal{G}}_2^{I,i,1} - 4\beta_0\overline{\mathcal{G}}_1^{I,i,2}f_1^i - 8\beta_0\overline{\mathcal{G}}_1^{I,i,1\,2}$$

$$- 4\beta_0 g_1^{I,i,2}f_1^i - 8\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} - 8\beta_0^2\overline{\mathcal{G}}_1^{I,i,2} + 192\zeta_5 A_1^{i\,3} + 32\zeta_3 A_1^i f_1^{i2}$$

$$+ 32\zeta_3 A_1^i A_2^i + 32\zeta_3\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2} + 32\zeta_3 g_1^{I,i,1}A_1^{i\,2} + 48\zeta_3\beta_0 A_1^i f_1^i + 4\zeta_2 f_1^{i3}$$

$$+ 2\zeta_2 A_2^i f_1^i + 2\zeta_2 A_1^i f_2^i + 4\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_1^i f_1^i + 4\zeta_2 g_1^{I,i,1}A_1^i f_1^i + 2\zeta_2\beta_0 f_1^{i2}$$

$$- 12\zeta_2\beta_0 B_1^i f_1^i + 4\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^i + 12\zeta_2\beta_0\gamma_0^{I,i}f_1^i - 80\zeta_2\zeta_3 A_1^{i\,3} + 23\zeta_2^2 A_1^{i\,2}f_1^i$$

$$+ 16\zeta_2^2\beta_0 A_1^{i\,2}) + \mathcal{D}_1(L_{\mu_F}(8B_1^i f_1^{i2} - 8A_2^i f_1^i + 8A_2^i B_1^i - 8A_1^i f_2^i + 8A_1^i B_2^i$$

$$- 16\overline{\mathcal{G}}_1^{I,i,1}A_1^i f_1^i + 16\overline{\mathcal{G}}_1^{I,i,1}A_1^i B_1^i - 16g_1^{I,i,1}A_1^i f_1^i + 16g_1^{I,i,1}A_1^i B_1^i - 8\gamma_1^{I,i}A_1^i$$

$$- 8\gamma_0^{I,i}f_1^{i2} - 8\gamma_0^{I,i}A_2^i - 16\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 16\gamma_0^{I,i}g_1^{I,i,1}A_1^i - 4\beta_1 A_1^i - 8\beta_0 f_1^{i2}$$

$$+ 8\beta_0 B_1^i f_1^i - 8\beta_0 A_2^i - 32\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 16\beta_0 g_1^{I,i,1}A_1^i - 8\beta_0\gamma_0^{I,i}f_1^i - 8\beta_0^2 f_1^i$$

$$+ 160\zeta_3 A_1^{i\,3} + 56\zeta_2 A_1^{i\,2}f_1^i - 8\zeta_2 A_1^{i\,2}B_1^i + 8\zeta_2\gamma_0^{I,i}A_1^{i\,2} + 24\zeta_2\beta_0 A_1^{i\,2})$$

$$+ L_{\mu_F}^2(-16A_1^i B_1^i f_1^i + 8A_1^i B_1^{i\,2} + 8A_1^i A_2^i + 8\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2} + 8g_1^{I,i,1}A_1^{i\,2}$$

$$
\begin{aligned}
&+16\gamma_0^{I,i}A_1^if_1^i-16\gamma_0^{I,i}A_1^iB_1^i+8\gamma_0^{I,i\,2}A_1^i+12\beta_0A_1^if_1^i-12\beta_0A_1^iB_1^i\\
&+12\beta_0\gamma_0^{I,i}A_1^i+4\beta_0^2A_1^i-28\zeta_2{A_1^i}^3)+L_{\mu_F}^3(8{A_1^i}^2B_1^i-8\gamma_0^{I,i}{A_1^i}^2-4\beta_0{A_1^i}^2)\\
&+8f_1^if_2^i+4A_3^i+4\overline{\mathcal{G}}_2^{I,i,1}A_1^i+8\overline{\mathcal{G}}_1^{I,i,1}f_1^{i2}+8\overline{\mathcal{G}}_1^{I,i,1}A_2^i+8\overline{\mathcal{G}}_1^{I,i,1\,2}A_1^i+4g_2^{I,i,1}A_1^i\\
&+8g_1^{I,i,1}f_1^{i2}+8g_1^{I,i,1}A_2^i+16g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i+8g_1^{I,i,1\,2}A_1^i+4\beta_1f_1^i+8\beta_0f_2^i\\
&+8\beta_0\overline{\mathcal{G}}_1^{I,i,2}A_1^i+24\beta_0\overline{\mathcal{G}}_1^{I,i,1}f_1^i+8\beta_0g_1^{I,i,2}A_1^i+8\beta_0g_1^{I,i,1}f_1^i+16\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}\\
&-160\zeta_3{A_1^i}^2f_1^i-96\zeta_3\beta_0{A_1^i}^2-28\zeta_2A_1^if_1^{i2}-8\zeta_2A_1^iA_2^i-8\zeta_2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2\\
&-8\zeta_2g_1^{I,i,1}{A_1^i}^2-24\zeta_2\beta_0A_1^if_1^i+24\zeta_2\beta_0A_1^iB_1^i-24\zeta_2\beta_0\gamma_0^{I,i}A_1^i-46\zeta_2^2{A_1^i}^3)\\
&+\mathcal{D}_2(L_{\mu_F}(12A_1^if_1^{i2}-24A_1^iB_1^if_1^i+24A_1^iA_2^i+24\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2+24g_1^{I,i,1}{A_1^i}^2\\
&+24\gamma_0^{I,i}A_1^if_1^i+36\beta_0A_1^if_1^i-8\beta_0A_1^iB_1^i+8\beta_0\gamma_0^{I,i}A_1^i+8\beta_0^2A_1^i-60\zeta_2{A_1^i}^3)\\
&+L_{\mu_F}^2(-12{A_1^i}^2f_1^i+24{A_1^i}^2B_1^i-24\gamma_0^{I,i}{A_1^i}^2-18\beta_0{A_1^i}^2)+L_{\mu_F}^3(4{A_1^i}^3)\\
&-4f_1^{i3}-12A_2^if_1^i-12A_1^if_2^i-24\overline{\mathcal{G}}_1^{I,i,1}A_1^if_1^i-24g_1^{I,i,1}A_1^if_1^i-4\beta_1A_1^i\\
&-12\beta_0f_1^{i2}-8\beta_0A_2^i-32\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^i-8\beta_0g_1^{I,i,1}A_1^i-8\beta_0^2f_1^i+160\zeta_3{A_1^i}^3\\
&+60\zeta_2{A_1^i}^2f_1^i+36\zeta_2\beta_0{A_1^i}^2)+\mathcal{D}_3(L_{\mu_F}(-32{A_1^i}^2f_1^i+16{A_1^i}^2B_1^i-16\gamma_0^{I,i}{A_1^i}^2\\
&-\frac{80}{3}\beta_0{A_1^i}^2)+L_{\mu_F}^2(16{A_1^i}^3)+16A_1^if_1^{i2}+16A_1^iA_2^i+16\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2\\
&+16g_1^{I,i,1}{A_1^i}^2+\frac{80}{3}\beta_0A_1^if_1^i+\frac{16}{3}\beta_0^2A_1^i-40\zeta_2{A_1^i}^3)+\mathcal{D}_4(L_{\mu_F}(20{A_1^i}^3)\\
&-20{A_1^i}^2f_1^i-\frac{40}{3}\beta_0{A_1^i}^2)+\mathcal{D}_5(8{A_1^i}^3)\,,
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
\Delta_{SV}^{I,i,4}&=\delta(1-z)(\frac{1}{2}\overline{\mathcal{G}}_4^{I,i,1}+\frac{1}{2}\overline{\mathcal{G}}_2^{I,i,1\,2}+\frac{4}{3}\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_3^{I,i,1}+2\overline{\mathcal{G}}_1^{I,i,1\,2}\overline{\mathcal{G}}_2^{I,i,1}+\frac{2}{3}\overline{\mathcal{G}}_1^{I,i,1\,4}+\frac{1}{2}g_4^{I,i,1}\\
&+\frac{4}{3}g_3^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}+g_2^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1}+2g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1\,2}+\frac{1}{2}g_2^{I,i,1\,2}+\frac{4}{3}g_1^{I,i,1}\overline{\mathcal{G}}_3^{I,i,1}+4g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1}\\
&+\frac{8}{3}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1\,3}+\frac{4}{3}g_1^{I,i,1}g_3^{I,i,1}+4g_1^{I,i,1}g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}+2g_1^{I,i,1\,2}\overline{\mathcal{G}}_2^{I,i,1}+4g_1^{I,i,1\,2}\overline{\mathcal{G}}_1^{I,i,1\,2}\\
&+2g_1^{I,i,1\,2}g_2^{I,i,1}+\frac{8}{3}g_1^{I,i,1\,3}\overline{\mathcal{G}}_1^{I,i,1}+\frac{2}{3}g_1^{I,i,1\,4}+\beta_2\overline{\mathcal{G}}_1^{I,i,2}+\beta_2g_1^{I,i,2}+\beta_1\overline{\mathcal{G}}_2^{I,i,2}+\frac{8}{3}\beta_1\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2}\\
&+\beta_1g_2^{I,i,2}+\frac{8}{3}\beta_1g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1}+\frac{8}{3}\beta_1g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2}+\frac{8}{3}\beta_1g_1^{I,i,1}g_1^{I,i,2}+\beta_0\overline{\mathcal{G}}_3^{I,i,2}+2\beta_0\overline{\mathcal{G}}_1^{I,i,2}\overline{\mathcal{G}}_2^{I,i,1}\\
&+\frac{8}{3}\beta_0\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,2}+4\beta_0\overline{\mathcal{G}}_1^{I,i,1\,2}\overline{\mathcal{G}}_1^{I,i,2}+\beta_0g_3^{I,i,2}+\frac{8}{3}\beta_0g_2^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1}+2\beta_0g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2}\\
&+2\beta_0g_1^{I,i,2}\overline{\mathcal{G}}_2^{I,i,1}+4\beta_0g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1\,2}+2\beta_0g_1^{I,i,2}g_2^{I,i,1}+\frac{8}{3}\beta_0g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,2}+8\beta_0g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2}\\
&+\frac{8}{3}\beta_0g_1^{I,i,1}g_2^{I,i,2}+8\beta_0g_1^{I,i,1}g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1}+4\beta_0g_1^{I,i,1\,2}\overline{\mathcal{G}}_1^{I,i,2}+4\beta_0g_1^{I,i,1\,2}g_1^{I,i,2}+4\beta_0\beta_1\overline{\mathcal{G}}_1^{I,i,3}
\end{aligned}
$$

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\begin{aligned}
&+ 4\beta_0\beta_1 g_1^{I,i,3} + 2\beta_0^2\overline{\mathcal{G}}_2^{I,i,3} + 2\beta_0^2{\overline{\mathcal{G}}_1^{I,i,2}}^2 + \frac{16}{3}\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,3} + 2\beta_0^2 g_2^{I,i,3} + \frac{16}{3}\beta_0^2 g_1^{I,i,3}\overline{\mathcal{G}}_1^{I,i,1} \\
&+ 4\beta_0^2 g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,2} + 2\beta_0^2 {g_1^{I,i,2}}^2 + \frac{16}{3}\beta_0^2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,3} + \frac{16}{3}\beta_0^2 g_1^{I,i,1} g_1^{I,i,3} + 4\beta_0^3\overline{\mathcal{G}}_1^{I,i,4} + 4\beta_0^3 g_1^{I,i,4} \\
&- 1920\zeta_7 {A_1^i}^3 f_1^i - 1920\zeta_7\beta_0 {A_1^i}^3 - 64\zeta_5 A_1^i {f_1^i}^3 - 192\zeta_5 A_1^i A_2^i f_1^i - 96\zeta_5 {A_1^i}^2 f_2^i \\
&- 192\zeta_5\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 f_1^i - 192\zeta_5 g_1^{I,i,1}{A_1^i}^2 f_1^i - 64\zeta_5\beta_1{A_1^i}^2 - 256\zeta_5\beta_0 A_1^i {f_1^i}^2 - 192\zeta_5\beta_0 A_1^i A_2^i \\
&- 320\zeta_5\beta_0\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 - 128\zeta_5\beta_0 g_1^{I,i,1}{A_1^i}^2 - 256\zeta_5\beta_0^2 A_1^i f_1^i - 8\zeta_3 {f_1^i}^2 f_2^i - 8\zeta_3 A_3^i f_1^i \\
&- 8\zeta_3 A_2^i f_2^i - 8\zeta_3 A_1^i f_3^i - 8\zeta_3\overline{\mathcal{G}}_2^{I,i,1} A_1^i f_1^i - \frac{16}{3}\zeta_3\overline{\mathcal{G}}_1^{I,i,1}{f_1^i}^3 - 16\zeta_3\overline{\mathcal{G}}_1^{I,i,1} A_2^i f_1^i \\
&- 16\zeta_3\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_2^i - 16\zeta_3{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_1^i f_1^i - 8\zeta_3 g_2^{I,i,1} A_1^i f_1^i - \frac{16}{3}\zeta_3 g_1^{I,i,1}{f_1^i}^3 - 16\zeta_3 g_1^{I,i,1} A_2^i f_1^i \\
&- 16\zeta_3 g_1^{I,i,1} A_1^i f_2^i - 32\zeta_3 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i - 16\zeta_3 {g_1^{I,i,1}}^2 A_1^i f_1^i - 8\zeta_3\beta_1 {f_1^i}^2 - 16\zeta_3\beta_1\overline{\mathcal{G}}_1^{I,i,1} A_1^i \\
&- 24\zeta_3\beta_0 f_1^i f_2^i - 16\zeta_3\beta_0\overline{\mathcal{G}}_2^{I,i,1} A_1^i - 16\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,2} A_1^i f_1^i - 32\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,1}{f_1^i}^2 - 16\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_2^i \\
&- 32\zeta_3\beta_0{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_1^i - 16\zeta_3\beta_0 g_1^{I,i,2} A_1^i f_1^i - 16\zeta_3\beta_0 g_1^{I,i,1}{f_1^i}^2 - 32\zeta_3\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_1^i \\
&- 32\zeta_3\beta_0^2\overline{\mathcal{G}}_1^{I,i,2} A_1^i - 48\zeta_3\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} f_1^i + 1792\zeta_3\zeta_5{A_1^i}^4 + 160\zeta_3^2{A_1^i}^2{f_1^i}^2 + 160\zeta_3^2{A_1^i}^2 A_2^i \\
&+ \frac{320}{3}\zeta_3^2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^3 + \frac{320}{3}\zeta_3^2 g_1^{I,i,1}{A_1^i}^3 + 352\zeta_3^2\beta_0{A_1^i}^2 f_1^i + 96\zeta_3^2\beta_0^2{A_1^i}^2 - 2\zeta_2{f_2^i}^2 \\
&- 4\zeta_2 f_1^i f_3^i + 3\zeta_2 A_4^i + 2\zeta_2\overline{\mathcal{G}}_3^{I,i,1} A_1^i - 2\zeta_2\overline{\mathcal{G}}_2^{I,i,1}{f_1^i}^2 + 3\zeta_2\overline{\mathcal{G}}_2^{I,i,1} A_2^i - 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1} f_1^i f_2^i \\
&+ 6\zeta_2\overline{\mathcal{G}}_1^{I,i,1} A_3^i + 6\zeta_2\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} A_1^i - 4\zeta_2{\overline{\mathcal{G}}_1^{I,i,1}}^2{f_1^i}^2 + 6\zeta_2{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_2^i + 4\zeta_2{\overline{\mathcal{G}}_1^{I,i,1}}^3 A_1^i \\
&+ 2\zeta_2 g_3^{I,i,1} A_1^i - 2\zeta_2 g_2^{I,i,1}{f_1^i}^2 + 3\zeta_2 g_2^{I,i,1} A_2^i + 6\zeta_2 g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_1^i - 8\zeta_2 g_1^{I,i,1} f_1^i f_2^i \\
&+ 6\zeta_2 g_1^{I,i,1} A_3^i + 6\zeta_2 g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} A_1^i - 8\zeta_2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}{f_1^i}^2 + 12\zeta_2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_2^i \\
&+ 12\zeta_2 g_1^{I,i,1}{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_1^i + 6\zeta_2 g_1^{I,i,1} g_2^{I,i,1} A_1^i - 4\zeta_2{g_1^{I,i,1}}^2{f_1^i}^2 + 6\zeta_2{g_1^{I,i,1}}^2 A_2^i \\
&+ 12\zeta_2{g_1^{I,i,1}}^2\overline{\mathcal{G}}_1^{I,i,1} A_1^i + 4\zeta_2{g_1^{I,i,1}}^3 A_1^i + 3\zeta_2\beta_2 f_1^i + 6\zeta_2\beta_2 B_1^i - 6\zeta_2\beta_2\gamma_0^{I,i} + 6\zeta_2\beta_1 f_2^i \\
&+ 12\zeta_2\beta_1 B_2^i + 4\zeta_2\beta_1\overline{\mathcal{G}}_1^{I,i,2} A_1^i - 2\zeta_2\beta_1\overline{\mathcal{G}}_1^{I,i,1} f_1^i + 12\zeta_2\beta_1\overline{\mathcal{G}}_1^{I,i,1} B_1^i + 4\zeta_2\beta_1 g_1^{I,i,2} A_1^i \\
&+ 6\zeta_2\beta_1 g_1^{I,i,1} f_1^i + 12\zeta_2\beta_1 g_1^{I,i,1} B_1^i - 12\zeta_2\beta_1\gamma_1^{I,i} - 12\zeta_2\beta_1\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} - 12\zeta_2\beta_1\gamma_0^{I,i} g_1^{I,i,1} \\
&+ 9\zeta_2\beta_0 f_3^i + 18\zeta_2\beta_0 B_3^i + 4\zeta_2\beta_0\overline{\mathcal{G}}_2^{I,i,2} A_1^i - 5\zeta_2\beta_0\overline{\mathcal{G}}_2^{I,i,1} f_1^i + 6\zeta_2\beta_0\overline{\mathcal{G}}_2^{I,i,1} B_1^i \\
&- 4\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,2}{f_1^i}^2 + 6\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,2} A_2^i + 4\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1} f_2^i + 24\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1} B_2^i \\
&+ 12\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} A_1^i - 10\zeta_2\beta_0{\overline{\mathcal{G}}_1^{I,i,1}}^2 f_1^i + 12\zeta_2\beta_0{\overline{\mathcal{G}}_1^{I,i,1}}^2 B_1^i + 4\zeta_2\beta_0 g_2^{I,i,2} A_1^i \\
&+ 3\zeta_2\beta_0 g_2^{I,i,1} f_1^i + 6\zeta_2\beta_0 g_2^{I,i,1} B_1^i - 4\zeta_2\beta_0 g_1^{I,i,2}{f_1^i}^2 + 6\zeta_2\beta_0 g_1^{I,i,2} A_2^i + 12\zeta_2\beta_0 g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1} A_1^i
\end{aligned}
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\begin{aligned}
&+ 12\zeta_2\beta_0 g_1^{I,i,1} f_2^i + 24\zeta_2\beta_0 g_1^{I,i,1} B_2^i + 12\zeta_2\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} A_1^i - 4\zeta_2\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} f_1^i \\
&+ 24\zeta_2\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} B_1^i + 12\zeta_2\beta_0 g_1^{I,i,1} g_1^{I,i,2} A_1^i + 6\zeta_2\beta_0 {g_1^{I,i,1}}^2 f_1^i + 12\zeta_2\beta_0 {g_1^{I,i,1}}^2 B_1^i \\
&- 18\zeta_2\beta_0\gamma_2^{I,i} - 24\zeta_2\beta_0\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1} - 24\zeta_2\beta_0\gamma_1^{I,i} g_1^{I,i,1} - 6\zeta_2\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,1} - 12\zeta_2\beta_0\gamma_0^{I,i}{\overline{\mathcal{G}}_1^{I,i,1}}^2 \\
&- 6\zeta_2\beta_0\gamma_0^{I,i} g_2^{I,i,1} - 24\zeta_2\beta_0\gamma_0^{I,i} g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} - 12\zeta_2\beta_0\gamma_0^{I,i} {g_1^{I,i,1}}^2 - 30\zeta_2\beta_0\beta_1 g_1^{I,i,1} \\
&+ 8\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,3} A_1^i - 10\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,2} f_1^i + 12\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,2} B_1^i - 8\zeta_2\beta_0^2{\overline{\mathcal{G}}_1^{I,i,1}}^2 - 18\zeta_2\beta_0^2 g_2^{I,i,1} \\
&+ 8\zeta_2\beta_0^2 g_1^{I,i,3} A_1^i + 6\zeta_2\beta_0^2 g_1^{I,i,2} f_1^i + 12\zeta_2\beta_0^2 g_1^{I,i,2} B_1^i - 24\zeta_2\beta_0^2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} - 24\zeta_2\beta_0^2 {g_1^{I,i,1}}^2 \\
&- 12\zeta_2\beta_0^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,2} - 12\zeta_2\beta_0^2\gamma_0^{I,i} g_1^{I,i,2} - 36\zeta_2\beta_0^3 g_1^{I,i,2} + 864\zeta_2\zeta_5 {A_1^i}^3 f_1^i + 832\zeta_2\zeta_5\beta_0 {A_1^i}^3 \\
&+ 40\zeta_2\zeta_3 A_1^i {f_1^i}^3 + 80\zeta_2\zeta_3 A_1^i A_2^i f_1^i + 40\zeta_2\zeta_3 {A_1^i}^2 f_2^i + 80\zeta_2\zeta_3\overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 f_1^i \\
&+ 80\zeta_2\zeta_3 g_1^{I,i,1} {A_1^i}^2 f_1^i + 32\zeta_2\zeta_3\beta_1 {A_1^i}^2 + 112\zeta_2\zeta_3\beta_0 A_1^i {f_1^i}^2 - 48\zeta_2\zeta_3\beta_0 A_1^i B_1^i f_1^i \\
&+ 96\zeta_2\zeta_3\beta_0 A_1^i A_2^i + 144\zeta_2\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 + 64\zeta_2\zeta_3\beta_0 g_1^{I,i,1} {A_1^i}^2 + 48\zeta_2\zeta_3\beta_0\gamma_0^{I,i} A_1^i f_1^i \\
&+ 96\zeta_2\zeta_3\beta_0^2 A_1^i f_1^i - 480\zeta_2\zeta_3^2 {A_1^i}^4 + \frac{2}{5}\zeta_2^2 {f_1^i}^4 - \frac{38}{5}\zeta_2^2 A_2^i {f_1^i}^2 + \frac{37}{10}\zeta_2^2 {A_2^i}^2 - \frac{76}{5}\zeta_2^2 A_1^i f_1^i f_2^i \\
&+ \frac{37}{5}\zeta_2^2 A_1^i A_3^i + \frac{37}{10}\zeta_2^2\overline{\mathcal{G}}_2^{I,i,1} {A_1^i}^2 - \frac{76}{5}\zeta_2^2\overline{\mathcal{G}}_1^{I,i,1} A_1^i {f_1^i}^2 + \frac{74}{5}\zeta_2^2\overline{\mathcal{G}}_1^{I,i,1} A_1^i A_2^i + \frac{37}{5}\zeta_2^2{\overline{\mathcal{G}}_1^{I,i,1}}^2 {A_1^i}^2 \\
&+ \frac{37}{10}\zeta_2^2 g_2^{I,i,1} {A_1^i}^2 - \frac{76}{5}\zeta_2^2 g_1^{I,i,1} A_1^i {f_1^i}^2 + \frac{74}{5}\zeta_2^2 g_1^{I,i,1} A_1^i A_2^i + \frac{74}{5}\zeta_2^2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 \\
&+ \frac{37}{5}\zeta_2^2 {g_1^{I,i,1}}^2 {A_1^i}^2 + \zeta_2^2\beta_1 A_1^i f_1^i + 18\zeta_2^2\beta_1 A_1^i B_1^i - 18\zeta_2^2\beta_1\gamma_0^{I,i} A_1^i - \frac{38}{5}\zeta_2^2\beta_0 {f_1^i}^3 \\
&- 12\zeta_2^2\beta_0 B_1^i {f_1^i}^2 - \frac{27}{5}\zeta_2^2\beta_0 A_2^i f_1^i + 18\zeta_2^2\beta_0 A_2^i B_1^i + \frac{42}{5}\zeta_2^2\beta_0 A_1^i f_2^i + 36\zeta_2^2\beta_0 A_1^i B_2^i \\
&+ \frac{37}{5}\zeta_2^2\beta_0\overline{\mathcal{G}}_1^{I,i,2} {A_1^i}^2 - \frac{142}{5}\zeta_2^2\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i + 36\zeta_2^2\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i + \frac{37}{5}\zeta_2^2\beta_0 g_1^{I,i,2} {A_1^i}^2 \\
&+ 2\zeta_2^2\beta_0 g_1^{I,i,1} A_1^i f_1^i + 36\zeta_2^2\beta_0 g_1^{I,i,1} A_1^i B_1^i - 36\zeta_2^2\beta_0\gamma_1^{I,i} A_1^i + 12\zeta_2^2\beta_0\gamma_0^{I,i} {f_1^i}^2 - 18\zeta_2^2\beta_0\gamma_0^{I,i} A_2^i \\
&- 36\zeta_2^2\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} A_1^i - 36\zeta_2^2\beta_0\gamma_0^{I,i} g_1^{I,i,1} A_1^i - \frac{15}{2}\zeta_2^2\beta_0\beta_1 A_1^i - \frac{91}{10}\zeta_2^2\beta_0^2 {f_1^i}^2 + 18\zeta_2^2\beta_0^2 B_1^i f_1^i \\
&+ 18\zeta_2^2\beta_0^2 {B_1^i}^2 - 9\zeta_2^2\beta_0^2 A_2^i - \frac{126}{5}\zeta_2^2\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} A_1^i - 42\zeta_2^2\beta_0^2 g_1^{I,i,1} A_1^i - 18\zeta_2^2\beta_0^2\gamma_0^{I,i} f_1^i \\
&- 36\zeta_2^2\beta_0^2\gamma_0^{I,i} B_1^i + 18\zeta_2^2\beta_0^2{\gamma_0^{I,i}}^2 - 9\zeta_2^2\beta_0^3 f_1^i - 18\zeta_2^2\beta_0^3 B_1^i + 18\zeta_2^2\beta_0^3\gamma_0^{I,i} + 188\zeta_2^2\zeta_3 {A_1^i}^3 f_1^i \\
&+ \frac{1056}{5}\zeta_2^2\zeta_3\beta_0 {A_1^i}^3 - \frac{271}{7}\zeta_2^3 {A_1^i}^2 {f_1^i}^2 - \frac{283}{14}\zeta_2^3 {A_1^i}^2 A_2^i - \frac{283}{21}\zeta_2^3\overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^3 \\
&- \frac{283}{21}\zeta_2^3 g_1^{I,i,1} {A_1^i}^3 - \frac{769}{10}\zeta_2^3\beta_0 {A_1^i}^2 f_1^i + \frac{111}{5}\zeta_2^3\beta_0 {A_1^i}^2 B_1^i - \frac{111}{5}\zeta_2^3\beta_0\gamma_0^{I,i} {A_1^i}^2 \\
&- \frac{1451}{35}\zeta_2^3\beta_0^2 {A_1^i}^2 - \frac{34687}{280}\zeta_2^4 {A_1^i}^4) + \delta(1-z) L_{\mu_F}(2B_4^i + \frac{4}{3}\overline{\mathcal{G}}_3^{I,i,1} B_1^i + 2\overline{\mathcal{G}}_2^{I,i,1} B_2^i \\
&+ 4\overline{\mathcal{G}}_1^{I,i,1} B_3^i + 4\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} B_1^i + 4{\overline{\mathcal{G}}_1^{I,i,1}}^2 B_2^i + \frac{8}{3}{\overline{\mathcal{G}}_1^{I,i,1}}^3 B_1^i + \frac{4}{3} g_3^{I,i,1} B_1^i + 2 g_2^{I,i,1} B_2^i
\end{aligned}
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\begin{aligned}
&+ 4g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}B_1^i + 4g_1^{I,i,1}B_3^i + 4g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1}B_1^i + 8g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}B_2^i + 8g_1^{I,i,1}{\overline{\mathcal{G}}_1^{I,i,1}}^2B_1^i \\
&+ 4g_1^{I,i,1}g_2^{I,i,1}B_1^i + 4{g_1^{I,i,1}}^2B_2^i + 8{g_1^{I,i,1}}^2\overline{\mathcal{G}}_1^{I,i,1}B_1^i + \frac{8}{3}{g_1^{I,i,1}}^3B_1^i - 2\gamma_3^{I,i} - 4\gamma_2^{I,i}\overline{\mathcal{G}}_1^{I,i,1} \\
&- 4\gamma_2^{I,i}g_1^{I,i,1} - 2\gamma_1^{I,i}\overline{\mathcal{G}}_2^{I,i,1} - 4\gamma_1^{I,i}{\overline{\mathcal{G}}_1^{I,i,1}}^2 - 2\gamma_1^{I,i}g_2^{I,i,1} - 8\gamma_1^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} - 4\gamma_1^{I,i}{g_1^{I,i,1}}^2 \\
&- \frac{4}{3}\gamma_0^{I,i}\overline{\mathcal{G}}_3^{I,i,1} - 4\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} - \frac{8}{3}\gamma_0^{I,i}{\overline{\mathcal{G}}_1^{I,i,1}}^3 - \frac{4}{3}\gamma_0^{I,i}g_3^{I,i,1} - 4\gamma_0^{I,i}g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} \\
&- 4\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} - 8\gamma_0^{I,i}g_1^{I,i,1}{\overline{\mathcal{G}}_1^{I,i,1}}^2 - 4\gamma_0^{I,i}g_1^{I,i,1}g_2^{I,i,1} - 8\gamma_0^{I,i}{g_1^{I,i,1}}^2\overline{\mathcal{G}}_1^{I,i,1} - \frac{8}{3}\gamma_0^{I,i}{g_1^{I,i,1}}^3 \\
&- 2\beta_2\overline{\mathcal{G}}_1^{I,i,1} - 2\beta_2g_1^{I,i,1} - 2\beta_1\overline{\mathcal{G}}_2^{I,i,1} + \frac{8}{3}\beta_1\overline{\mathcal{G}}_1^{I,i,2}B_1^i - 4\beta_1{\overline{\mathcal{G}}_1^{I,i,1}}^2 - 2\beta_1g_2^{I,i,1} \\
&+ \frac{8}{3}\beta_1g_1^{I,i,2}B_1^i - 8\beta_1g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} - 4\beta_1{g_1^{I,i,1}}^2 - \frac{8}{3}\beta_1\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,2} - \frac{8}{3}\beta_1\gamma_0^{I,i}g_1^{I,i,2} - 2\beta_0\overline{\mathcal{G}}_3^{I,i,1} \\
&+ \frac{8}{3}\beta_0\overline{\mathcal{G}}_2^{I,i,2}B_1^i + 4\beta_0\overline{\mathcal{G}}_1^{I,i,2}B_2^i - 6\beta_0\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} + 8\beta_0\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2}B_1^i - 4\beta_0{\overline{\mathcal{G}}_1^{I,i,1}}^3 \\
&- 2\beta_0g_3^{I,i,1} + \frac{8}{3}\beta_0g_2^{I,i,2}B_1^i - 6\beta_0g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} + 4\beta_0g_1^{I,i,2}B_2^i + 8\beta_0g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1}B_1^i \\
&- 6\beta_0g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} + 8\beta_0g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2}B_1^i - 12\beta_0g_1^{I,i,1}{\overline{\mathcal{G}}_1^{I,i,1}}^2 - 6\beta_0g_1^{I,i,1}g_2^{I,i,1} + 8\beta_0g_1^{I,i,1}g_1^{I,i,2}B_1^i \\
&- 12\beta_0{g_1^{I,i,1}}^2\overline{\mathcal{G}}_1^{I,i,1} - 4\beta_0{g_1^{I,i,1}}^3 - 4\beta_0\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,2} - 4\beta_0\gamma_1^{I,i}g_1^{I,i,2} - \frac{8}{3}\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,2} \\
&- 8\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} - \frac{8}{3}\beta_0\gamma_0^{I,i}g_2^{I,i,2} - 8\beta_0\gamma_0^{I,i}g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1} - 8\beta_0\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} \\
&- 8\beta_0\gamma_0^{I,i}g_1^{I,i,1}g_1^{I,i,2} - 8\beta_0\beta_1\overline{\mathcal{G}}_1^{I,i,2} - 8\beta_0\beta_1g_1^{I,i,2} - 4\beta_0^2\overline{\mathcal{G}}_2^{I,i,2} + \frac{16}{3}\beta_0^2\overline{\mathcal{G}}_1^{I,i,3}B_1^i \\
&- 12\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} - 4\beta_0^2g_2^{I,i,2} + \frac{16}{3}\beta_0^2g_1^{I,i,3}B_1^i - 12\beta_0^2g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1} - 12\beta_0^2g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} \\
&- 12\beta_0^2g_1^{I,i,1}g_1^{I,i,2} - \frac{16}{3}\beta_0^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,3} - \frac{16}{3}\beta_0^2\gamma_0^{I,i}g_1^{I,i,3} - 8\beta_0^3\overline{\mathcal{G}}_1^{I,i,3} - 8\beta_0^3g_1^{I,i,3} + 1920\zeta_7{A_1^i}^4 \\
&+ 192\zeta_5{A_1^i}^2{f_1^i}^2 - 192\zeta_5{A_1^i}^2B_1^if_1^i + 288\zeta_5{A_1^i}^2A_2^i + 192\zeta_5\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^3 + 192\zeta_5g_1^{I,i,1}{A_1^i}^3 \\
&+ 192\zeta_5\gamma_0^{I,i}{A_1^i}^2f_1^i + 608\zeta_5\beta_0{A_1^i}^2f_1^i - 128\zeta_5\beta_0{A_1^i}^2B_1^i + 128\zeta_5\beta_0\gamma_0^{I,i}{A_1^i}^2 + 256\zeta_5\beta_0^2{A_1^i}^2 \\
&- \frac{16}{3}\zeta_3B_1^i{f_1^i}^3 + 8\zeta_3A_2^i{f_1^i}^2 - 16\zeta_3A_2^iB_1^if_1^i + 8\zeta_3{A_2^i}^2 + 16\zeta_3A_1^if_1^if_2^i - 16\zeta_3A_1^iB_2^if_1^i \\
&- 16\zeta_3A_1^iB_1^if_2^i + 16\zeta_3A_1^iA_3^i + 8\zeta_3\overline{\mathcal{G}}_2^{I,i,1}{A_1^i}^2 + 16\zeta_3\overline{\mathcal{G}}_1^{I,i,1}A_1^i{f_1^i}^2 - 32\zeta_3\overline{\mathcal{G}}_1^{I,i,1}A_1^iB_1^if_1^i \\
&+ 32\zeta_3\overline{\mathcal{G}}_1^{I,i,1}A_1^iA_2^i + 16\zeta_3{\overline{\mathcal{G}}_1^{I,i,1}}^2{A_1^i}^2 + 8\zeta_3g_2^{I,i,1}{A_1^i}^2 + 16\zeta_3g_1^{I,i,1}A_1^i{f_1^i}^2 - 32\zeta_3g_1^{I,i,1}A_1^iB_1^if_1^i \\
&+ 32\zeta_3g_1^{I,i,1}A_1^iA_2^i + 32\zeta_3g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 + 16\zeta_3{g_1^{I,i,1}}^2{A_1^i}^2 + 16\zeta_3\gamma_1^{I,i}A_1^if_1^i + \frac{16}{3}\zeta_3\gamma_0^{I,i}{f_1^i}^3 \\
&+ 16\zeta_3\gamma_0^{I,i}A_2^if_1^i + 16\zeta_3\gamma_0^{I,i}A_1^if_2^i + 32\zeta_3\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^if_1^i + 32\zeta_3\gamma_0^{I,i}g_1^{I,i,1}A_1^if_1^i + 24\zeta_3\beta_1A_1^if_1^i \\
&+ 8\zeta_3\beta_0{f_1^i}^3 - 16\zeta_3\beta_0B_1^i{f_1^i}^2 + 32\zeta_3\beta_0A_2^if_1^i + 40\zeta_3\beta_0A_1^if_2^i + 16\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,2}{A_1^i}^2 \\
&+ 96\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^if_1^i - 32\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^iB_1^i + 16\zeta_3\beta_0g_1^{I,i,2}{A_1^i}^2 + 64\zeta_3\beta_0g_1^{I,i,1}A_1^if_1^i
\end{aligned}
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\begin{aligned}
&+ 16\zeta_3\beta_0\gamma_0^{I,i}{f_1^i}^2 + 32\zeta_3\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i + 24\zeta_3\beta_0^2{f_1^i}^2 + 80\zeta_3\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 320\zeta_3^2{A_1^i}^3 f_1^i \\
&+ \frac{320}{3}\zeta_3^2{A_1^i}^3B_1^i - \frac{320}{3}\zeta_3^2\gamma_0^{I,i}{A_1^i}^3 - 352\zeta_3^2\beta_0{A_1^i}^3 - 4\zeta_2B_2^i{f_1^i}^2 - 8\zeta_2B_1^if_1^if_2^i + 4\zeta_2A_3^if_1^i \\
&+ 6\zeta_2A_3^iB_1^i + 4\zeta_2A_2^if_2^i + 6\zeta_2A_2^iB_2^i + 4\zeta_2A_1^if_3^i + 6\zeta_2A_1^iB_3^i + 4\zeta_2\overline{\mathcal{G}}_2^{I,i,1}A_1^if_1^i \\
&+ 6\zeta_2\overline{\mathcal{G}}_2^{I,i,1}A_1^iB_1^i - 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1}B_1^i{f_1^i}^2 + 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_2^if_1^i + 12\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_2^iB_1^i + 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_1^if_2^i \\
&+ 12\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_1^iB_2^i + 8\zeta_2{\overline{\mathcal{G}}_1^{I,i,1}}^2A_1^if_1^i + 12\zeta_2{\overline{\mathcal{G}}_1^{I,i,1}}^2A_1^iB_1^i + 4\zeta_2g_2^{I,i,1}A_1^if_1^i + 6\zeta_2g_2^{I,i,1}A_1^iB_1^i \\
&- 8\zeta_2g_1^{I,i,1}B_1^i{f_1^i}^2 + 8\zeta_2g_1^{I,i,1}A_2^if_1^i + 12\zeta_2g_1^{I,i,1}A_2^iB_1^i + 8\zeta_2g_1^{I,i,1}A_1^if_2^i + 12\zeta_2g_1^{I,i,1}A_1^iB_2^i \\
&+ 16\zeta_2g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^if_1^i + 24\zeta_2g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^iB_1^i + 8\zeta_2{g_1^{I,i,1}}^2A_1^if_1^i + 12\zeta_2{g_1^{I,i,1}}^2A_1^iB_1^i \\
&- 6\zeta_2\gamma_2^{I,i}A_1^i + 4\zeta_2\gamma_1^{I,i}{f_1^i}^2 - 6\zeta_2\gamma_1^{I,i}A_2^i - 12\zeta_2\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 12\zeta_2\gamma_1^{I,i}g_1^{I,i,1}A_1^i \\
&+ 8\zeta_2\gamma_0^{I,i}f_1^if_2^i - 6\zeta_2\gamma_0^{I,i}A_3^i - 6\zeta_2\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,1}A_1^i + 8\zeta_2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}{f_1^i}^2 - 12\zeta_2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_2^i \\
&- 12\zeta_2\gamma_0^{I,i}{\overline{\mathcal{G}}_1^{I,i,1}}^2A_1^i - 6\zeta_2\gamma_0^{I,i}g_2^{I,i,1}A_1^i + 8\zeta_2\gamma_0^{I,i}g_1^{I,i,1}{f_1^i}^2 - 12\zeta_2\gamma_0^{I,i}g_1^{I,i,1}A_2^i \\
&- 24\zeta_2\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 12\zeta_2\gamma_0^{I,i}{g_1^{I,i,1}}^2A_1^i - 3\zeta_2\beta_2A_1^i + 4\zeta_2\beta_1{f_1^i}^2 + 6\zeta_2\beta_1B_1^if_1^i \\
&+ 12\zeta_2\beta_1{B_1^i}^2 - 6\zeta_2\beta_1A_2^i - 4\zeta_2\beta_1\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 12\zeta_2\beta_1g_1^{I,i,1}A_1^i - 6\zeta_2\beta_1\gamma_0^{I,i}f_1^i \\
&- 24\zeta_2\beta_1\gamma_0^{I,i}B_1^i + 12\zeta_2\beta_1{\gamma_0^{I,i}}^2 + 12\zeta_2\beta_0f_1^if_2^i + 6\zeta_2\beta_0B_2^if_1^i + 12\zeta_2\beta_0B_1^if_2^i \\
&+ 36\zeta_2\beta_0B_1^iB_2^i - 9\zeta_2\beta_0A_3^i - \zeta_2\beta_0\overline{\mathcal{G}}_2^{I,i,1}A_1^i + 8\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,2}A_1^if_1^i + 12\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,2}A_1^iB_1^i \\
&+ 12\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}{f_1^i}^2 - 4\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}B_1^if_1^i + 24\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}{B_1^i}^2 - 10\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_2^i \\
&- 2\zeta_2\beta_0{\overline{\mathcal{G}}_1^{I,i,1}}^2A_1^i - 9\zeta_2\beta_0g_2^{I,i,1}A_1^i + 8\zeta_2\beta_0g_1^{I,i,2}A_1^if_1^i + 12\zeta_2\beta_0g_1^{I,i,2}A_1^iB_1^i \\
&+ 12\zeta_2\beta_0g_1^{I,i,1}{f_1^i}^2 + 12\zeta_2\beta_0g_1^{I,i,1}B_1^if_1^i + 24\zeta_2\beta_0g_1^{I,i,1}{B_1^i}^2 - 18\zeta_2\beta_0g_1^{I,i,1}A_2^i \\
&- 20\zeta_2\beta_0g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 18\zeta_2\beta_0{g_1^{I,i,1}}^2A_1^i - 6\zeta_2\beta_0\gamma_1^{I,i}f_1^i - 36\zeta_2\beta_0\gamma_1^{I,i}B_1^i \\
&- 12\zeta_2\beta_0\gamma_0^{I,i}f_2^i - 36\zeta_2\beta_0\gamma_0^{I,i}B_2^i - 12\zeta_2\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,2}A_1^i + 4\zeta_2\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}f_1^i \\
&- 48\zeta_2\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 12\zeta_2\beta_0\gamma_0^{I,i}g_1^{I,i,2}A_1^i - 12\zeta_2\beta_0\gamma_0^{I,i}g_1^{I,i,1}f_1^i - 48\zeta_2\beta_0\gamma_0^{I,i}g_1^{I,i,1}B_1^i \\
&+ 36\zeta_2\beta_0\gamma_0^{I,i}\gamma_1^{I,i} + 24\zeta_2\beta_0{\gamma_0^{I,i}}^2\overline{\mathcal{G}}_1^{I,i,1} + 24\zeta_2\beta_0{\gamma_0^{I,i}}^2g_1^{I,i,1} - 15\zeta_2\beta_0\beta_1f_1^i - 30\zeta_2\beta_0\beta_1B_1^i \\
&+ 30\zeta_2\beta_0\beta_1\gamma_0^{I,i} - 18\zeta_2\beta_0^2f_2^i - 36\zeta_2\beta_0^2B_2^i - 2\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,2}A_1^i + 6\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}f_1^i \\
&- 36\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 18\zeta_2\beta_0^2g_1^{I,i,2}A_1^i - 18\zeta_2\beta_0^2g_1^{I,i,1}f_1^i - 60\zeta_2\beta_0^2g_1^{I,i,1}B_1^i + 36\zeta_2\beta_0^2\gamma_1^{I,i} \\
&+ 36\zeta_2\beta_0^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} + 60\zeta_2\beta_0^2\gamma_0^{I,i}g_1^{I,i,1} + 36\zeta_2\beta_0^3g_1^{I,i,1} - 864\zeta_2\zeta_5{A_1^i}^4 - 120\zeta_2\zeta_3{A_1^i}^2{f_1^i}^2 \\
&+ 80\zeta_2\zeta_3{A_1^i}^2B_1^if_1^i - 120\zeta_2\zeta_3{A_1^i}^2A_2^i - 80\zeta_2\zeta_3\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^3 - 80\zeta_2\zeta_3g_1^{I,i,1}{A_1^i}^3
\end{aligned}
$$

$$- 80\zeta_2\zeta_3\gamma_0^{I,i}A_1^{i\,2}f_1^i - 264\zeta_2\zeta_3\beta_0 A_1^{i\,2}f_1^i + 112\zeta_2\zeta_3\beta_0 A_1^{i\,2}B_1^i - 112\zeta_2\zeta_3\beta_0\gamma_0^{I,i}A_1^{i\,2}$$
$$- 96\zeta_2\zeta_3\beta_0^2 A_1^{i\,2} - \frac{8}{5}\zeta_2^2 A_1^i f_1^{i3} - \frac{76}{5}\zeta_2^2 A_1^i B_1^i f_1^{i2} + \frac{152}{5}\zeta_2^2 A_1^i A_2^i f_1^i + \frac{74}{5}\zeta_2^2 A_1^i A_2^i B_1^i$$
$$+ \frac{76}{5}\zeta_2^2 A_1^{i\,2} f_2^i + \frac{37}{5}\zeta_2^2 A_1^{i\,2}B_2^i + \frac{152}{5}\zeta_2^2\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2}f_1^i + \frac{74}{5}\zeta_2^2\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2}B_1^i$$
$$+ \frac{152}{5}\zeta_2^2 g_1^{I,i,1}A_1^{i\,2}f_1^i + \frac{74}{5}\zeta_2^2 g_1^{I,i,1}A_1^{i\,2}B_1^i - \frac{37}{5}\zeta_2^2\gamma_1^{I,i}A_1^{i\,2} + \frac{76}{5}\zeta_2^2\gamma_0^{I,i}A_1^i f_1^{i2}$$
$$- \frac{74}{5}\zeta_2^2\gamma_0^{I,i}A_1^i A_2^i - \frac{74}{5}\zeta_2^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2} - \frac{74}{5}\zeta_2^2\gamma_0^{I,i}g_1^{I,i,1}A_1^{i\,2} - \zeta_2^2\beta_1 A_1^{i\,2} + 38\zeta_2^2\beta_0 A_1^i f_1^{i2}$$
$$+ 26\zeta_2^2\beta_0 A_1^i B_1^i f_1^i + 36\zeta_2^2\beta_0 A_1^i B_1^{i\,2} - 3\zeta_2^2\beta_0 A_1^i A_2^i + 21\zeta_2^2\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2} - \frac{47}{5}\zeta_2^2\beta_0 g_1^{I,i,1}A_1^{i\,2}$$
$$- 26\zeta_2^2\beta_0\gamma_0^{I,i}A_1^i f_1^i - 72\zeta_2^2\beta_0\gamma_0^{I,i}A_1^i B_1^i + 36\zeta_2^2\beta_0\gamma_0^{I,i2}A_1^i + \frac{49}{5}\zeta_2^2\beta_0^2 A_1^i f_1^i - 60\zeta_2^2\beta_0^2 A_1^i B_1^i$$
$$+ 60\zeta_2^2\beta_0^2\gamma_0^{I,i}A_1^i + 9\zeta_2^2\beta_0^3 A_1^i - 188\zeta_2^2\zeta_3 A_1^{i\,4} + \frac{542}{7}\zeta_2^3 A_1^{i\,3}f_1^i - \frac{283}{21}\zeta_2^3 A_1^{i\,3}B_1^i$$
$$+ \frac{283}{21}\zeta_2^3\gamma_0^{I,i}A_1^{i\,3} + \frac{769}{10}\zeta_2^3\beta_0 A_1^{i\,3}) + \delta(1-z)L_{\mu_F}^2(+2B_2^{i\,2} + 4B_1^i B_3^i + 2\overline{\mathcal{G}}_2^{I,i,1}B_1^{i\,2}$$
$$+ 8\overline{\mathcal{G}}_1^{I,i,1}B_1^i B_2^i + 4\overline{\mathcal{G}}_1^{I,i,1\,2}B_1^{i\,2} + 2g_2^{I,i,1}B_1^{i\,2} + 8g_1^{I,i,1}B_1^i B_2^i + 8g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}B_1^{i\,2} + 4g_1^{I,i,1\,2}B_1^{i\,2}$$
$$- 4\gamma_2^{I,i}B_1^i - 4\gamma_1^{I,i}B_2^i - 8\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 8\gamma_1^{I,i}g_1^{I,i,1}B_1^i + 2\gamma_1^{I,i2} - 4\gamma_0^{I,i}B_3^i - 4\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,1}B_1^i$$
$$- 8\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}B_2^i - 8\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1\,2}B_1^i - 4\gamma_0^{I,i}g_2^{I,i,1}B_1^i - 8\gamma_0^{I,i}g_1^{I,i,1}B_2^i - 16\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}B_1^i$$
$$- 8\gamma_0^{I,i}g_1^{I,i,1\,2}B_1^i + 4\gamma_0^{I,i}\gamma_2^{I,i} + 8\gamma_0^{I,i}\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1} + 8\gamma_0^{I,i}\gamma_1^{I,i}g_1^{I,i,1} + 2\gamma_0^{I,i2}\overline{\mathcal{G}}_2^{I,i,1} + 4\gamma_0^{I,i2}\overline{\mathcal{G}}_1^{I,i,1\,2}$$
$$+ 2\gamma_0^{I,i2}g_2^{I,i,1} + 8\gamma_0^{I,i2}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} + 4\gamma_0^{I,i2}g_1^{I,i,1\,2} - \beta_2 B_1^i + \beta_2\gamma_0^{I,i} - 2\beta_1 B_2^i - 6\beta_1\overline{\mathcal{G}}_1^{I,i,1}B_1^i$$
$$- 6\beta_1 g_1^{I,i,1}B_1^i + 2\beta_1\gamma_1^{I,i} + 6\beta_1\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} + 6\beta_1\gamma_0^{I,i}g_1^{I,i,1} - 3\beta_0 B_3^i - 5\beta_0\overline{\mathcal{G}}_2^{I,i,1}B_1^i$$
$$+ 4\beta_0\overline{\mathcal{G}}_1^{I,i,2}B_1^{i\,2} - 8\beta_0\overline{\mathcal{G}}_1^{I,i,1}B_2^i - 10\beta_0\overline{\mathcal{G}}_1^{I,i,1\,2}B_1^i - 5\beta_0 g_2^{I,i,1}B_1^i + 4\beta_0 g_1^{I,i,2}B_1^{i\,2}$$
$$- 8\beta_0 g_1^{I,i,1}B_2^i - 20\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 10\beta_0 g_1^{I,i,1\,2}B_1^i + 3\beta_0\gamma_2^{I,i} + 8\beta_0\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1}$$
$$+ 8\beta_0\gamma_1^{I,i}g_1^{I,i,1} + 5\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,1} - 8\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,2}B_1^i + 10\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1\,2} + 5\beta_0\gamma_0^{I,i}g_2^{I,i,1}$$
$$- 8\beta_0\gamma_0^{I,i}g_1^{I,i,2}B_1^i + 20\beta_0\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} + 10\beta_0\gamma_0^{I,i}g_1^{I,i,1\,2} + 4\beta_0\gamma_0^{I,i2}\overline{\mathcal{G}}_1^{I,i,2} + 4\beta_0\gamma_0^{I,i2}g_1^{I,i,2}$$
$$+ 5\beta_0\beta_1\overline{\mathcal{G}}_1^{I,i,1} + 5\beta_0\beta_1 g_1^{I,i,1} + 3\beta_0^2\overline{\mathcal{G}}_2^{I,i,1} - 10\beta_0^2\overline{\mathcal{G}}_1^{I,i,2}B_1^i + 6\beta_0^2\overline{\mathcal{G}}_1^{I,i,1\,2} + 3\beta_0^2 g_2^{I,i,1}$$
$$- 10\beta_0^2 g_1^{I,i,2}B_1^i + 12\beta_0^2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} + 6\beta_0^2 g_1^{I,i,1\,2} + 10\beta_0^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,2} + 10\beta_0^2\gamma_0^{I,i}g_1^{I,i,2}$$
$$+ 6\beta_0^3\overline{\mathcal{G}}_1^{I,i,2} + 6\beta_0^3 g_1^{I,i,2} - 192\zeta_5 A_1^{i\,3}f_1^i + 192\zeta_5 A_1^{i\,3}B_1^i - 192\zeta_5\gamma_0^{I,i}A_1^{i\,3} - 304\zeta_5\beta_0 A_1^{i\,3}$$
$$+ 16\zeta_3 A_1^i B_1^i f_1^{i2} - 16\zeta_3 A_1^i B_1^{i\,2}f_1^i - 16\zeta_3 A_1^i A_2^i f_1^i + 32\zeta_3 A_1^i A_2^i B_1^i - 8\zeta_3 A_1^{i\,2}f_2^i$$
$$+ 16\zeta_3 A_1^{i\,2}B_2^i - 16\zeta_3\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2}f_1^i + 32\zeta_3\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2}B_1^i - 16\zeta_3 g_1^{I,i,1}A_1^{i\,2}f_1^i$$

$$
\begin{aligned}
&+ 32\zeta_3 g_1^{I,i,1} A_1^{i\,2} B_1^i - 16\zeta_3 \gamma_1^{I,i} A_1^{i\,2} - 16\zeta_3 \gamma_0^{I,i} A_1^i f_1^{i2} + 32\zeta_3 \gamma_0^{I,i} A_1^i B_1^i f_1^i - 32\zeta_3 \gamma_0^{I,i} A_1^i A_2^i \\
&- 32\zeta_3 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} - 32\zeta_3 \gamma_0^{I,i} g_1^{I,i,1} A_1^{i\,2} - 16\zeta_3 \gamma_0^{I,i2} A_1^i f_1^i - 12\zeta_3 \beta_1 A_1^{i\,2} - 20\zeta_3 \beta_0 A_1^i f_1^{i2} \\
&+ 56\zeta_3 \beta_0 A_1^i B_1^i f_1^i - 36\zeta_3 \beta_0 A_1^i A_2^i - 56\zeta_3 \beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} - 40\zeta_3 \beta_0 g_1^{I,i,1} A_1^{i\,2} \\
&- 56\zeta_3 \beta_0 \gamma_0^{I,i} A_1^i f_1^i - 44\zeta_3 \beta_0^2 A_1^i f_1^i + 160\zeta_3^2 A_1^{i\,4} - 4\zeta_2 B_1^{i\,2} f_1^{i2} + 8\zeta_2 A_2^i B_1^i f_1^i + 6\zeta_2 A_2^i B_1^{i\,2} \\
&- 2\zeta_2 A_2^{i\,2} + 8\zeta_2 A_1^i B_2^i f_1^i + 8\zeta_2 A_1^i B_1^i f_2^i + 12\zeta_2 A_1^i B_1^i B_2^i - 4\zeta_2 A_1^i A_3^i - 2\zeta_2 \overline{\mathcal{G}}_2^{I,i,1} A_1^{i\,2} \\
&+ 16\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i f_1^i + 12\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^{i\,2} - 8\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i A_2^i - 4\zeta_2 \overline{\mathcal{G}}_1^{I,i,1\,2} A_1^{i\,2} - 2\zeta_2 g_2^{I,i,1} A_1^{i\,2} \\
&+ 16\zeta_2 g_1^{I,i,1} A_1^i B_1^i f_1^i + 12\zeta_2 g_1^{I,i,1} A_1^i B_1^{i\,2} - 8\zeta_2 g_1^{I,i,1} A_1^i A_2^i - 8\zeta_2 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} \\
&- 4\zeta_2 g_1^{I,i,1\,2} A_1^{i\,2} - 8\zeta_2 \gamma_1^{I,i} A_1^i f_1^i - 12\zeta_2 \gamma_1^{I,i} A_1^i B_1^i + 8\zeta_2 \gamma_0^{I,i} B_1^i f_1^{i2} - 8\zeta_2 \gamma_0^{I,i} A_2^i f_1^i \\
&- 12\zeta_2 \gamma_0^{I,i} A_2^i B_1^i - 8\zeta_2 \gamma_0^{I,i} A_1^i f_2^i - 12\zeta_2 \gamma_0^{I,i} A_1^i B_2^i - 16\zeta_2 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i \\
&- 24\zeta_2 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i - 16\zeta_2 \gamma_0^{I,i} g_1^{I,i,1} A_1^i f_1^i - 24\zeta_2 \gamma_0^{I,i} g_1^{I,i,1} A_1^i B_1^i + 12\zeta_2 \gamma_0^{I,i} \gamma_1^{I,i} A_1^i \\
&- 4\zeta_2 \gamma_0^{I,i2} f_1^{i2} + 6\zeta_2 \gamma_0^{I,i2} A_2^i + 12\zeta_2 \gamma_0^{I,i2} \overline{\mathcal{G}}_1^{I,i,1} A_1^i + 12\zeta_2 \gamma_0^{I,i2} g_1^{I,i,1} A_1^i - 6\zeta_2 \beta_1 A_1^i f_1^i \\
&- 9\zeta_2 \beta_1 A_1^i B_1^i + 9\zeta_2 \beta_1 \gamma_0^{I,i} A_1^i + 10\zeta_2 \beta_0 B_1^i f_1^{i2} + 6\zeta_2 \beta_0 B_1^{i\,2} f_1^i + 12\zeta_2 \beta_0 B_1^{i\,3} - 8\zeta_2 \beta_0 A_2^i f_1^i \\
&- 15\zeta_2 \beta_0 A_2^i B_1^i - 10\zeta_2 \beta_0 A_1^i f_2^i - 12\zeta_2 \beta_0 A_1^i B_2^i - 4\zeta_2 \beta_0 \overline{\mathcal{G}}_1^{I,i,2} A_1^{i\,2} - 20\zeta_2 \beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i \\
&- 14\zeta_2 \beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i - 4\zeta_2 \beta_0 g_1^{I,i,2} A_1^{i\,2} - 20\zeta_2 \beta_0 g_1^{I,i,1} A_1^i f_1^i - 30\zeta_2 \beta_0 g_1^{I,i,1} A_1^i B_1^i \\
&+ 12\zeta_2 \beta_0 \gamma_1^{I,i} A_1^i - 10\zeta_2 \beta_0 \gamma_0^{I,i} f_1^{i2} - 12\zeta_2 \beta_0 \gamma_0^{I,i} B_1^i f_1^i - 36\zeta_2 \beta_0 \gamma_0^{I,i} B_1^{i\,2} + 15\zeta_2 \beta_0 \gamma_0^{I,i} A_2^i \\
&+ 14\zeta_2 \beta_0 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^i + 30\zeta_2 \beta_0 \gamma_0^{I,i} g_1^{I,i,1} A_1^i + 6\zeta_2 \beta_0 \gamma_0^{I,i2} f_1^i + 36\zeta_2 \beta_0 \gamma_0^{I,i2} B_1^i - 12\zeta_2 \beta_0 \gamma_0^{I,i3} \\
&+ \frac{15}{2}\zeta_2 \beta_0 \beta_1 A_1^i - 6\zeta_2 \beta_0^2 f_1^{i2} - 15\zeta_2 \beta_0^2 B_1^i f_1^i - 30\zeta_2 \beta_0^2 B_1^{i\,2} + 9\zeta_2 \beta_0^2 A_2^i - 2\zeta_2 \beta_0^2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i \\
&+ 18\zeta_2 \beta_0^2 g_1^{I,i,1} A_1^i + 15\zeta_2 \beta_0^2 \gamma_0^{I,i} f_1^i + 60\zeta_2 \beta_0^2 \gamma_0^{I,i} B_1^i - 30\zeta_2 \beta_0^2 \gamma_0^{I,i2} + 9\zeta_2 \beta_0^3 f_1^i + 18\zeta_2 \beta_0^3 B_1^i \\
&- 18\zeta_2 \beta_0^3 \gamma_0^{I,i} + 120\zeta_2 \zeta_3 A_1^{i\,3} f_1^i - 80\zeta_2 \zeta_3 A_1^{i\,3} B_1^i + 80\zeta_2 \zeta_3 \gamma_0^{I,i} A_1^{i\,3} + 132\zeta_2 \zeta_3 \beta_0 A_1^{i\,3} \\
&+ \frac{12}{5}\zeta_2^2 A_1^{i\,2} f_1^{i2} + \frac{152}{5}\zeta_2^2 A_1^{i\,2} B_1^i f_1^i + \frac{37}{5}\zeta_2^2 A_1^{i\,2} B_1^{i\,2} - \frac{114}{5}\zeta_2^2 A_1^{i\,2} A_2^i - \frac{76}{5}\zeta_2^2 \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,3} \\
&- \frac{76}{5}\zeta_2^2 g_1^{I,i,1} A_1^{i\,3} - \frac{152}{5}\zeta_2^2 \gamma_0^{I,i} A_1^{i\,2} f_1^i - \frac{74}{5}\zeta_2^2 \gamma_0^{I,i} A_1^{i\,2} B_1^i + \frac{37}{5}\zeta_2^2 \gamma_0^{I,i2} A_1^{i\,2} - \frac{228}{5}\zeta_2^2 \beta_0 A_1^{i\,2} f_1^i \\
&- \frac{177}{10}\zeta_2^2 \beta_0 A_1^{i\,2} B_1^i + \frac{177}{10}\zeta_2^2 \beta_0 \gamma_0^{I,i} A_1^{i\,2} - \frac{49}{10}\zeta_2^2 \beta_0^2 A_1^{i\,2} - \frac{271}{7}\zeta_2^3 A_1^{i\,4}) \\
&+ \delta(1-z) L_{\mu_F}^3 (4 B_1^{i\,2} B_2^i + \frac{8}{3}\overline{\mathcal{G}}_1^{I,i,1} B_1^{i\,3} + \frac{8}{3} g_1^{I,i,1} B_1^{i\,3} - 4\gamma_1^{I,i} B_1^{i\,2} - 8\gamma_0^{I,i} B_1^i B_2^i \\
&- 8\gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} B_1^{i\,2} - 8\gamma_0^{I,i} g_1^{I,i,1} B_1^{i\,2} + 8\gamma_0^{I,i} \gamma_1^{I,i} B_1^i + 4\gamma_0^{I,i2} B_2^i + 8\gamma_0^{I,i2} \overline{\mathcal{G}}_1^{I,i,1} B_1^i + 8\gamma_0^{I,i2} g_1^{I,i,1} B_1^i
\end{aligned}
$$

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\begin{aligned}
&- 4{\gamma_0^{I,i}}^2\gamma_1^{I,i} - \frac{8}{3}{\gamma_0^{I,i}}^3\overline{\mathcal{G}}_1^{I,i,1} - \frac{8}{3}{\gamma_0^{I,i}}^3 g_1^{I,i,1} - 2\beta_1 {B_1^i}^2 + 4\beta_1\gamma_0^{I,i}B_1^i - 2\beta_1{\gamma_0^{I,i}}^2 - 6\beta_0 B_1^i B_2^i \\
&- 8\beta_0\overline{\mathcal{G}}_1^{I,i,1}{B_1^i}^2 - 8\beta_0 g_1^{I,i,1}{B_1^i}^2 + 6\beta_0\gamma_1^{I,i}B_1^i + 6\beta_0\gamma_0^{I,i}B_2^i + 16\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}B_1^i \\
&+ 16\beta_0\gamma_0^{I,i}g_1^{I,i,1}B_1^i - 6\beta_0\gamma_0^{I,i}\gamma_1^{I,i} - 8\beta_0{\gamma_0^{I,i}}^2\overline{\mathcal{G}}_1^{I,i,1} - 8\beta_0{\gamma_0^{I,i}}^2 g_1^{I,i,1} + \frac{5}{3}\beta_0\beta_1 B_1^i - \frac{5}{3}\beta_0\beta_1\gamma_0^{I,i} \\
&+ 2\beta_0^2 B_2^i + \frac{22}{3}\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}B_1^i + \frac{22}{3}\beta_0^2 g_1^{I,i,1}B_1^i - 2\beta_0^2\gamma_1^{I,i} - \frac{22}{3}\beta_0^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} - \frac{22}{3}\beta_0^2\gamma_0^{I,i}g_1^{I,i,1} \\
&- 2\beta_0^3\overline{\mathcal{G}}_1^{I,i,1} - 2\beta_0^3 g_1^{I,i,1} + 64\zeta_5{A_1^i}^4 - 16\zeta_3{A_1^i}^2B_1^if_1^i + 16\zeta_3{A_1^i}^2{B_1^i}^2 + 8\zeta_3{A_1^i}^2A_2^i \\
&+ \frac{16}{3}\zeta_3\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^3 + \frac{16}{3}\zeta_3 g_1^{I,i,1}{A_1^i}^3 + 16\zeta_3\gamma_0^{I,i}{A_1^i}^2f_1^i - 32\zeta_3\gamma_0^{I,i}{A_1^i}^2B_1^i + 16\zeta_3{\gamma_0^{I,i}}^2{A_1^i}^2 \\
&+ 16\zeta_3\beta_0{A_1^i}^2f_1^i - 32\zeta_3\beta_0{A_1^i}^2B_1^i + 32\zeta_3\beta_0\gamma_0^{I,i}{A_1^i}^2 + \frac{44}{3}\zeta_3\beta_0^2{A_1^i}^2 + 8\zeta_2A_1^i{B_1^i}^2f_1^i \\
&+ 4\zeta_2A_1^i{B_1^i}^3 - 8\zeta_2A_1^iA_2^iB_1^i - 4\zeta_2{A_1^i}^2B_2^i - 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2B_1^i - 8\zeta_2g_1^{I,i,1}{A_1^i}^2B_1^i \\
&+ 4\zeta_2\gamma_1^{I,i}{A_1^i}^2 - 16\zeta_2\gamma_0^{I,i}A_1^iB_1^if_1^i - 12\zeta_2\gamma_0^{I,i}A_1^i{B_1^i}^2 + 8\zeta_2\gamma_0^{I,i}A_1^iA_2^i + 8\zeta_2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 \\
&+ 8\zeta_2\gamma_0^{I,i}g_1^{I,i,1}{A_1^i}^2 + 8\zeta_2{\gamma_0^{I,i}}^2A_1^if_1^i + 12\zeta_2{\gamma_0^{I,i}}^2A_1^iB_1^i - 4\zeta_2{\gamma_0^{I,i}}^3A_1^i + 2\zeta_2\beta_1{A_1^i}^2 \\
&- 16\zeta_2\beta_0A_1^iB_1^if_1^i - 12\zeta_2\beta_0A_1^i{B_1^i}^2 + 6\zeta_2\beta_0A_1^iA_2^i + 8\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 + 8\zeta_2\beta_0g_1^{I,i,1}{A_1^i}^2 \\
&+ 16\zeta_2\beta_0\gamma_0^{I,i}A_1^if_1^i + 24\zeta_2\beta_0\gamma_0^{I,i}A_1^iB_1^i - 12\zeta_2\beta_0{\gamma_0^{I,i}}^2A_1^i + \frac{22}{3}\zeta_2\beta_0^2A_1^if_1^i + 11\zeta_2\beta_0^2A_1^iB_1^i \\
&- 11\zeta_2\beta_0^2\gamma_0^{I,i}A_1^i - 3\zeta_2\beta_0^3A_1^i - 40\zeta_2\zeta_3{A_1^i}^4 - \frac{8}{5}\zeta_2^2{A_1^i}^3f_1^i - \frac{76}{5}\zeta_2^2{A_1^i}^3B_1^i + \frac{76}{5}\zeta_2^2\gamma_0^{I,i}{A_1^i}^3 \\
&+ \frac{76}{5}\zeta_2^2\beta_0{A_1^i}^3) + \delta(1-z)L_{\mu_F}^4(\frac{2}{3}{B_1^i}^4 - \frac{8}{3}\gamma_0^{I,i}{B_1^i}^3 + 4{\gamma_0^{I,i}}^2{B_1^i}^2 - \frac{8}{3}{\gamma_0^{I,i}}^3B_1^i + \frac{2}{3}{\gamma_0^{I,i}}^4 \\
&- 2\beta_0{B_1^i}^3 + 6\beta_0\gamma_0^{I,i}{B_1^i}^2 - 6\beta_0{\gamma_0^{I,i}}^2B_1^i + 2\beta_0{\gamma_0^{I,i}}^3 + \frac{11}{6}\beta_0^2{B_1^i}^2 - \frac{11}{3}\beta_0^2\gamma_0^{I,i}B_1^i + \frac{11}{6}\beta_0^2{\gamma_0^{I,i}}^2 \\
&- \frac{1}{2}\beta_0^3B_1^i + \frac{1}{2}\beta_0^3\gamma_0^{I,i} + \frac{16}{3}\zeta_3{A_1^i}^3B_1^i - \frac{16}{3}\zeta_3\gamma_0^{I,i}{A_1^i}^3 - 4\zeta_3\beta_0{A_1^i}^3 - 4\zeta_2{A_1^i}^2{B_1^i}^2 \\
&+ 8\zeta_2\gamma_0^{I,i}{A_1^i}^2B_1^i - 4\zeta_2{\gamma_0^{I,i}}^2{A_1^i}^2 + 6\zeta_2\beta_0{A_1^i}^2B_1^i - 6\zeta_2\beta_0\gamma_0^{I,i}{A_1^i}^2 - \frac{11}{6}\zeta_2\beta_0^2{A_1^i}^2 + \frac{2}{5}\zeta_2^2{A_1^i}^4) \\
&+ \mathcal{D}_0L_{\mu_F}(-4B_3^if_1^i - 4B_2^if_2^i - 4B_1^if_3^i + 2A_4^i + \frac{4}{3}\overline{\mathcal{G}}_3^{I,i,1}A_1^i - 4\overline{\mathcal{G}}_2^{I,i,1}B_1^if_1^i + 2\overline{\mathcal{G}}_2^{I,i,1}A_2^i \\
&- 8\overline{\mathcal{G}}_1^{I,i,1}B_2^if_1^i - 8\overline{\mathcal{G}}_1^{I,i,1}B_1^if_2^i + 4\overline{\mathcal{G}}_1^{I,i,1}A_3^i + 4\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1}A_1^i - 8{\overline{\mathcal{G}}_1^{I,i,1}}^2B_1^if_1^i + 4{\overline{\mathcal{G}}_1^{I,i,1}}^2A_2^i \\
&+ \frac{8}{3}{\overline{\mathcal{G}}_1^{I,i,1}}^3A_1^i + \frac{4}{3}g_3^{I,i,1}A_1^i - 4g_2^{I,i,1}B_1^if_1^i + 2g_2^{I,i,1}A_2^i + 4g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 8g_1^{I,i,1}B_2^if_1^i \\
&- 8g_1^{I,i,1}B_1^if_2^i + 4g_1^{I,i,1}A_3^i + 4g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1}A_1^i - 16g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}B_1^if_1^i + 8g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_2^i \\
&+ 8g_1^{I,i,1}{\overline{\mathcal{G}}_1^{I,i,1}}^2A_1^i + 4g_1^{I,i,1}g_2^{I,i,1}A_1^i - 8{g_1^{I,i,1}}^2B_1^if_1^i + 4{g_1^{I,i,1}}^2A_2^i + 8{g_1^{I,i,1}}^2\overline{\mathcal{G}}_1^{I,i,1}A_1^i \\
&+ \frac{8}{3}{g_1^{I,i,1}}^3A_1^i + 4\gamma_2^{I,i}f_1^i + 4\gamma_1^{I,i}f_2^i + 8\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1}f_1^i + 8\gamma_1^{I,i}g_1^{I,i,1}f_1^i + 4\gamma_0^{I,i}f_3^i + 4\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,1}f_1^i \\
&+ 8\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}f_2^i + 8\gamma_0^{I,i}{\overline{\mathcal{G}}_1^{I,i,1}}^2f_1^i + 4\gamma_0^{I,i}g_2^{I,i,1}f_1^i + 8\gamma_0^{I,i}g_1^{I,i,1}f_2^i + 16\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}f_1^i
\end{aligned}
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\begin{aligned}
&+8\gamma_0^{I,i} g_1^{I,i,1\,2} f_1^i + 2\beta_2 f_1^i + 4\beta_1 f_2^i + \frac{8}{3}\beta_1 \overline{\mathcal{G}}_1^{I,i,2} A_1^i + 8\beta_1 \overline{\mathcal{G}}_1^{I,i,1} f_1^i - 8\beta_1 \overline{\mathcal{G}}_1^{I,i,1} B_1^i \\
&+\frac{8}{3}\beta_1 g_1^{I,i,2} A_1^i + 8\beta_1 g_1^{I,i,1} f_1^i + 8\beta_1 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} + 6\beta_0 f_3^i + \frac{8}{3}\beta_0 \overline{\mathcal{G}}_2^{I,i,2} A_1^i + 6\beta_0 \overline{\mathcal{G}}_2^{I,i,1} f_1^i \\
&-8\beta_0 \overline{\mathcal{G}}_2^{I,i,1} B_1^i - 8\beta_0 \overline{\mathcal{G}}_1^{I,i,2} B_1^i f_1^i + 4\beta_0 \overline{\mathcal{G}}_1^{I,i,2} A_2^i + 12\beta_0 \overline{\mathcal{G}}_1^{I,i,1} f_2^i - 8\beta_0 \overline{\mathcal{G}}_1^{I,i,1} B_2^i \\
&+8\beta_0 \overline{\mathcal{G}}_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,2} A_1^i + 12\beta_0 \overline{\mathcal{G}}_1^{I,i,1\,2} f_1^i - 16\beta_0 \overline{\mathcal{G}}_1^{I,i,1\,2} B_1^i + \frac{8}{3}\beta_0 g_2^{I,i,2} A_1^i + 6\beta_0 g_2^{I,i,1} f_1^i \\
&-8\beta_0 g_1^{I,i,2} B_1^i f_1^i + 4\beta_0 g_1^{I,i,2} A_2^i + 8\beta_0 g_1^{I,i,2} \overline{\mathcal{G}}_1^{I,i,1} A_1^i + 12\beta_0 g_1^{I,i,1} f_2^i + 8\beta_0 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,2} A_1^i \\
&+24\beta_0 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} f_1^i - 16\beta_0 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} B_1^i + 8\beta_0 g_1^{I,i,1} g_1^{I,i,2} A_1^i + 12\beta_0 g_1^{I,i,1\,2} f_1^i + 8\beta_0 \gamma_1^{I,i} \overline{\mathcal{G}}_1^{I,i,1} \\
&+8\beta_0 \gamma_0^{I,i} \overline{\mathcal{G}}_2^{I,i,1} + 8\beta_0 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,2} f_1^i + 16\beta_0 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1\,2} + 8\beta_0 \gamma_0^{I,i} g_1^{I,i,2} f_1^i + 16\beta_0 \gamma_0^{I,i} g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} \\
&+20\beta_0\beta_1 \overline{\mathcal{G}}_1^{I,i,1} + 12\beta_0^2 \overline{\mathcal{G}}_2^{I,i,1} + \frac{16}{3}\beta_0^2 \overline{\mathcal{G}}_1^{I,i,3} A_1^i + 12\beta_0^2 \overline{\mathcal{G}}_1^{I,i,2} f_1^i - 16\beta_0^2 \overline{\mathcal{G}}_1^{I,i,2} B_1^i \\
&+24\beta_0^2 \overline{\mathcal{G}}_1^{I,i,1\,2} + \frac{16}{3}\beta_0^2 g_1^{I,i,3} A_1^i + 12\beta_0^2 g_1^{I,i,2} f_1^i + 24\beta_0^2 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} + 16\beta_0^2 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,2} \\
&+24\beta_0^3 \overline{\mathcal{G}}_1^{I,i,2} - 1152\zeta_5 A_1^{i\,3} f_1^i + 384\zeta_5 A_1^{i\,3} B_1^i - 384\zeta_5 \gamma_0^{I,i} A_1^{i\,3} - 1344\zeta_5\beta_0 A_1^{i\,3} \\
&-\frac{64}{3}\zeta_3 A_1^i f_1^{i3} + 64\zeta_3 A_1^i B_1^i f_1^{i2} - 128\zeta_3 A_1^i A_2^i f_1^i + 64\zeta_3 A_1^i A_2^i B_1^i - 64\zeta_3 A_1^{i\,2} f_2^i \\
&+32\zeta_3 A_1^{i\,2} B_2^i - 128\zeta_3 \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} f_1^i + 64\zeta_3 \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} B_1^i - 128\zeta_3 g_1^{I,i,1} A_1^{i\,2} f_1^i \\
&+64\zeta_3 g_1^{I,i,1} A_1^{i\,2} B_1^i - 32\zeta_3 \gamma_1^{I,i} A_1^{i\,2} - 64\zeta_3 \gamma_0^{I,i} A_1^i f_1^{i2} - 64\zeta_3 \gamma_0^{I,i} A_1^i A_2^i - 64\zeta_3 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} \\
&-64\zeta_3 \gamma_0^{I,i} g_1^{I,i,1} A_1^{i\,2} - 48\zeta_3\beta_1 A_1^{i\,2} - 160\zeta_3\beta_0 A_1^i f_1^{i2} + 96\zeta_3\beta_0 A_1^i B_1^i f_1^i - 144\zeta_3\beta_0 A_1^i A_2^i \\
&-256\zeta_3\beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} - 128\zeta_3\beta_0 g_1^{I,i,1} A_1^{i\,2} - 96\zeta_3\beta_0 \gamma_0^{I,i} A_1^i f_1^i - 176\zeta_3\beta_0^2 A_1^i f_1^i \\
&+\frac{2240}{3}\zeta_3^2 A_1^{i\,4} + 8\zeta_2 B_1^i f_1^{i3} - 12\zeta_2 A_2^i f_1^{i2} + 4\zeta_2 A_2^i B_1^i f_1^i - 2\zeta_2 A_2^{i\,2} - 24\zeta_2 A_1^i f_1^i f_2^i \\
&+4\zeta_2 A_1^i B_2^i f_1^i + 4\zeta_2 A_1^i B_1^i f_2^i - 4\zeta_2 A_1^i A_3^i - 2\zeta_2 \overline{\mathcal{G}}_2^{I,i,1} A_1^{i\,2} - 24\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^{i2} \\
&+8\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i f_1^i - 8\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i A_2^i - 4\zeta_2 \overline{\mathcal{G}}_1^{I,i,1\,2} A_1^{i\,2} - 2\zeta_2 g_2^{I,i,1} A_1^{i\,2} - 24\zeta_2 g_1^{I,i,1} A_1^i f_1^{i2} \\
&+8\zeta_2 g_1^{I,i,1} A_1^i B_1^i f_1^i - 8\zeta_2 g_1^{I,i,1} A_1^i A_2^i - 8\zeta_2 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} - 4\zeta_2 g_1^{I,i,1\,2} A_1^{i\,2} - 4\zeta_2 \gamma_1^{I,i} A_1^i f_1^i \\
&-8\zeta_2 \gamma_0^{I,i} f_1^{i3} - 4\zeta_2 \gamma_0^{I,i} A_2^i f_1^i - 4\zeta_2 \gamma_0^{I,i} A_1^i f_2^i - 8\zeta_2 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i - 8\zeta_2 \gamma_0^{I,i} g_1^{I,i,1} A_1^i f_1^i \\
&-6\zeta_2\beta_1 A_1^i f_1^i + 12\zeta_2\beta_1 A_1^i B_1^i - 12\zeta_2\beta_1 \gamma_0^{I,i} A_1^i - 12\zeta_2\beta_0 f_1^{i3} + 4\zeta_2\beta_0 B_1^i f_1^{i2} \\
&-24\zeta_2\beta_0 B_1^{i\,2} f_1^i - 8\zeta_2\beta_0 A_2^i f_1^i + 12\zeta_2\beta_0 A_2^i B_1^i - 10\zeta_2\beta_0 A_1^i f_2^i + 24\zeta_2\beta_0 A_1^i B_2^i \\
&-4\zeta_2\beta_0 \overline{\mathcal{G}}_1^{I,i,2} A_1^{i\,2} - 64\zeta_2\beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i + 32\zeta_2\beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i - 4\zeta_2\beta_0 g_1^{I,i,2} A_1^{i\,2} \\
&-16\zeta_2\beta_0 g_1^{I,i,1} A_1^i f_1^i + 24\zeta_2\beta_0 g_1^{I,i,1} A_1^i B_1^i - 24\zeta_2\beta_0 \gamma_1^{I,i} A_1^i - 4\zeta_2\beta_0 \gamma_0^{I,i} f_1^{i2}
\end{aligned}
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\begin{aligned}
&+ 48\zeta_2\beta_0\gamma_0^{I,i}B_1^i f_1^i - 12\zeta_2\beta_0\gamma_0^{I,i}A_2^i - 32\zeta_2\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 24\zeta_2\beta_0\gamma_0^{I,i}g_1^{I,i,1}A_1^i \\
&- 24\zeta_2\beta_0{\gamma_0^{I,i}}^2 f_1^i - 6\zeta_2\beta_0^2{f_1^i}^2 + 36\zeta_2\beta_0^2 B_1^i f_1^i - 44\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 24\zeta_2\beta_0^2 g_1^{I,i,1}A_1^i \\
&- 36\zeta_2\beta_0^2\gamma_0^{I,i}f_1^i + 640\zeta_2\zeta_3{A_1^i}^3 f_1^i - 160\zeta_2\zeta_3{A_1^i}^3 B_1^i + 160\zeta_2\zeta_3\gamma_0^{I,i}{A_1^i}^3 + 640\zeta_2\zeta_3\beta_0{A_1^i}^3 \\
&- 36\zeta_2^2{A_1^i}^2{f_1^i}^2 + 46\zeta_2^2{A_1^i}^2 B_1^i f_1^i - 69\zeta_2^2{A_1^i}^2 A_2^i - 46\zeta_2^2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^3 - 46\zeta_2^2 g_1^{I,i,1}{A_1^i}^3 \\
&- 46\zeta_2^2\gamma_0^{I,i}{A_1^i}^2 f_1^i - 147\zeta_2^2\beta_0{A_1^i}^2 f_1^i + 20\zeta_2^2\beta_0{A_1^i}^2 B_1^i - 20\zeta_2^2\beta_0\gamma_0^{I,i}{A_1^i}^2 - \frac{398}{5}\zeta_2^2\beta_0^2{A_1^i}^2 \\
&- \frac{505}{3}\zeta_2^3{A_1^i}^4) + \mathcal{D}_0 L_{\mu_F}^2(-8B_1^i B_2^i f_1^i - 4{B_1^i}^2 f_2^i + 4A_3^i B_1^i + 4A_2^i B_2^i + 4A_1^i B_3^i \\
&+ 4\overline{\mathcal{G}}_2^{I,i,1}A_1^i B_1^i - 8\overline{\mathcal{G}}_1^{I,i,1}{B_1^i}^2 f_1^i + 8\overline{\mathcal{G}}_1^{I,i,1}A_2^i B_1^i + 8\overline{\mathcal{G}}_1^{I,i,1}A_1^i B_2^i + 8{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_1^i B_1^i \\
&+ 4g_2^{I,i,1}A_1^i B_1^i - 8g_1^{I,i,1}{B_1^i}^2 f_1^i + 8g_1^{I,i,1}A_2^i B_1^i + 8g_1^{I,i,1}A_1^i B_2^i + 16g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i B_1^i \\
&+ 8{g_1^{I,i,1}}^2 A_1^i B_1^i - 4\gamma_2^{I,i}A_1^i + 8\gamma_1^{I,i}B_1^i f_1^i - 4\gamma_1^{I,i}A_2^i - 8\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 8\gamma_1^{I,i}g_1^{I,i,1}A_1^i \\
&+ 8\gamma_0^{I,i}B_2^i f_1^i + 8\gamma_0^{I,i}B_1^i f_2^i - 4\gamma_0^{I,i}A_3^i - 4\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,1}A_1^i + 16\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}B_1^i f_1^i - 8\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_2^i \\
&- 8\gamma_0^{I,i}{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_1^i - 4\gamma_0^{I,i}g_2^{I,i,1}A_1^i + 16\gamma_0^{I,i}g_1^{I,i,1}B_1^i f_1^i - 8\gamma_0^{I,i}g_1^{I,i,1}A_2^i - 16\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i \\
&- 8\gamma_0^{I,i}{g_1^{I,i,1}}^2 A_1^i - 8\gamma_0^{I,i}\gamma_1^{I,i}f_1^i - 4{\gamma_0^{I,i}}^2 f_2^i - 8{\gamma_0^{I,i}}^2\overline{\mathcal{G}}_1^{I,i,1}f_1^i - 8{\gamma_0^{I,i}}^2 g_1^{I,i,1}f_1^i - \beta_2 A_1^i \\
&+ 6\beta_1 B_1^i f_1^i - 2\beta_1 A_2^i - 6\beta_1\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 6\beta_1 g_1^{I,i,1}A_1^i - 6\beta_1\gamma_0^{I,i}f_1^i + 8\beta_0 B_2^i f_1^i + 10\beta_0 B_1^i f_2^i \\
&- 3\beta_0 A_3^i - 5\beta_0\overline{\mathcal{G}}_2^{I,i,1}A_1^i + 8\beta_0\overline{\mathcal{G}}_1^{I,i,2}A_1^i B_1^i + 20\beta_0\overline{\mathcal{G}}_1^{I,i,1}B_1^i f_1^i - 8\beta_0\overline{\mathcal{G}}_1^{I,i,1}{B_1^i}^2 - 8\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_2^i \\
&- 10\beta_0{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_1^i - 5\beta_0 g_2^{I,i,1}A_1^i + 8\beta_0 g_1^{I,i,2}A_1^i B_1^i + 20\beta_0 g_1^{I,i,1}B_1^i f_1^i - 8\beta_0 g_1^{I,i,1}A_2^i \\
&- 20\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 10\beta_0{g_1^{I,i,1}}^2 A_1^i - 8\beta_0\gamma_1^{I,i}f_1^i - 10\beta_0\gamma_0^{I,i}f_2^i - 8\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,2}A_1^i \\
&- 20\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}f_1^i + 16\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 8\beta_0\gamma_0^{I,i}g_1^{I,i,2}A_1^i - 20\beta_0\gamma_0^{I,i}g_1^{I,i,1}f_1^i - 8\beta_0{\gamma_0^{I,i}}^2\overline{\mathcal{G}}_1^{I,i,1} \\
&- 5\beta_0\beta_1 f_1^i - 6\beta_0^2 f_2^i - 10\beta_0^2\overline{\mathcal{G}}_1^{I,i,2}A_1^i - 12\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}f_1^i + 20\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 10\beta_0^2 g_1^{I,i,2}A_1^i \\
&- 12\beta_0^2 g_1^{I,i,1}f_1^i - 20\beta_0^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} - 12\beta_0^3\overline{\mathcal{G}}_1^{I,i,1} + 576\zeta_5{A_1^i}^4 + 32\zeta_3{A_1^i}^2{f_1^i}^2 \\
&- 128\zeta_3{A_1^i}^2 B_1^i f_1^i + 32\zeta_3{A_1^i}^2{B_1^i}^2 + 96\zeta_3{A_1^i}^2 A_2^i + 64\zeta_3\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^3 + 64\zeta_3 g_1^{I,i,1}{A_1^i}^3 \\
&+ 128\zeta_3\gamma_0^{I,i}{A_1^i}^2 f_1^i - 64\zeta_3\gamma_0^{I,i}{A_1^i}^2 B_1^i + 32\zeta_3{\gamma_0^{I,i}}^2{A_1^i}^2 + 192\zeta_3\beta_0{A_1^i}^2 f_1^i - 112\zeta_3\beta_0{A_1^i}^2 B_1^i \\
&+ 112\zeta_3\beta_0\gamma_0^{I,i}{A_1^i}^2 + 88\zeta_3\beta_0^2{A_1^i}^2 - 24\zeta_2 A_1^i B_1^i{f_1^i}^2 + 4\zeta_2 A_1^i{B_1^i}^2 f_1^i + 24\zeta_2 A_1^i A_2^i f_1^i \\
&- 8\zeta_2 A_1^i A_2^i B_1^i + 12\zeta_2{A_1^i}^2 f_2^i - 4\zeta_2{A_1^i}^2 B_2^i + 24\zeta_2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 f_1^i - 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 B_1^i \\
&+ 24\zeta_2 g_1^{I,i,1}{A_1^i}^2 f_1^i - 8\zeta_2 g_1^{I,i,1}{A_1^i}^2 B_1^i + 4\zeta_2\gamma_1^{I,i}{A_1^i}^2 + 24\zeta_2\gamma_0^{I,i}A_1^i{f_1^i}^2 - 8\zeta_2\gamma_0^{I,i}A_1^i B_1^i f_1^i \\
&+ 8\zeta_2\gamma_0^{I,i}A_1^i A_2^i + 8\zeta_2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 + 8\zeta_2\gamma_0^{I,i}g_1^{I,i,1}{A_1^i}^2 + 4\zeta_2{\gamma_0^{I,i}}^2 A_1^i f_1^i + 3\zeta_2\beta_1{A_1^i}^2
\end{aligned}
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\begin{aligned}
&+ 30\zeta_2\beta_0 A_1^i f_1^{i\,2} - 14\zeta_2\beta_0 A_1^i B_1^i f_1^i + 24\zeta_2\beta_0 A_1^i {B_1^i}^2 + 9\zeta_2\beta_0 A_1^i A_2^i + 34\zeta_2\beta_0 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 \\
&+ 10\zeta_2\beta_0 g_1^{I,i,1} {A_1^i}^2 + 14\zeta_2\beta_0\gamma_0^{I,i} A_1^i f_1^i - 48\zeta_2\beta_0\gamma_0^{I,i} A_1^i B_1^i + 24\zeta_2\beta_0 \gamma_0^{I,i\,2} A_1^i + 11\zeta_2\beta_0^2 A_1^i f_1^i \\
&- 30\zeta_2\beta_0^2 A_1^i B_1^i + 30\zeta_2\beta_0^2\gamma_0^{I,i} A_1^i - 320\zeta_2\zeta_3 {A_1^i}^4 + 36\zeta_2^2 {A_1^i}^3 f_1^i - 46\zeta_2^2 {A_1^i}^3 B_1^i \\
&+ 46\zeta_2^2\gamma_0^{I,i} {A_1^i}^3 + \frac{147}{2}\zeta_2^2\beta_0 {A_1^i}^3) + \mathcal{D}_0 L_{\mu_F}^3 (-\frac{8}{3} {B_1^i}^3 f_1^i + 4A_2^i {B_1^i}^2 + 8A_1^i B_1^i B_2^i \\
&+ 8\overline{\mathcal{G}}_1^{I,i,1} A_1^i {B_1^i}^2 + 8g_1^{I,i,1} A_1^i {B_1^i}^2 - 8\gamma_1^{I,i} A_1^i B_1^i + 8\gamma_0^{I,i} {B_1^i}^2 f_1^i - 8\gamma_0^{I,i} A_2^i B_1^i - 8\gamma_0^{I,i} A_1^i B_2^i \\
&- 16\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i - 16\gamma_0^{I,i} g_1^{I,i,1} A_1^i B_1^i + 8\gamma_0^{I,i}\gamma_1^{I,i} A_1^i - 8\gamma_0^{I,i\,2} B_1^i f_1^i + 4\gamma_0^{I,i\,2} A_2^i \\
&+ 8\gamma_0^{I,i\,2}\overline{\mathcal{G}}_1^{I,i,1} A_1^i + 8\gamma_0^{I,i\,2} g_1^{I,i,1} A_1^i + \frac{8}{3}\gamma_0^{I,i\,3} f_1^i - 4\beta_1 A_1^i B_1^i + 4\beta_1\gamma_0^{I,i} A_1^i + 8\beta_0 {B_1^i}^2 f_1^i \\
&- 6\beta_0 A_2^i B_1^i - 6\beta_0 A_1^i B_2^i - 16\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i - 16\beta_0 g_1^{I,i,1} A_1^i B_1^i + 6\beta_0\gamma_1^{I,i} A_1^i \\
&- 16\beta_0\gamma_0^{I,i} B_1^i f_1^i + 6\beta_0\gamma_0^{I,i} A_2^i + 16\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} A_1^i + 16\beta_0\gamma_0^{I,i} g_1^{I,i,1} A_1^i + 8\beta_0\gamma_0^{I,i\,2} f_1^i \\
&+ \frac{5}{3}\beta_0\beta_1 A_1^i - \frac{22}{3}\beta_0^2 B_1^i f_1^i + 2\beta_0^2 A_2^i + \frac{22}{3}\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} A_1^i + \frac{22}{3}\beta_0^2 g_1^{I,i,1} A_1^i + \frac{22}{3}\beta_0^2\gamma_0^{I,i} f_1^i \\
&+ 2\beta_0^3 f_1^i - \frac{64}{3}\zeta_3 {A_1^i}^3 f_1^i + 64\zeta_3 {A_1^i}^3 B_1^i - 64\zeta_3\gamma_0^{I,i} {A_1^i}^3 - 64\zeta_3\beta_0 {A_1^i}^3 + 24\zeta_2 {A_1^i}^2 B_1^i f_1^i \\
&- 4\zeta_2 {A_1^i}^2 {B_1^i}^2 - 12\zeta_2 {A_1^i}^2 A_2^i - 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^3 - 8\zeta_2 g_1^{I,i,1} {A_1^i}^3 - 24\zeta_2\gamma_0^{I,i} {A_1^i}^2 f_1^i \\
&+ 8\zeta_2\gamma_0^{I,i} {A_1^i}^2 B_1^i - 4\zeta_2\gamma_0^{I,i\,2} {A_1^i}^2 - 24\zeta_2\beta_0 {A_1^i}^2 f_1^i + 8\zeta_2\beta_0 {A_1^i}^2 B_1^i - 8\zeta_2\beta_0\gamma_0^{I,i} {A_1^i}^2 \\
&- \frac{11}{3}\zeta_2\beta_0^2 {A_1^i}^2 - 12\zeta_2^2 {A_1^i}^4) + \mathcal{D}_0 L_{\mu_F}^4 (\frac{8}{3} A_1^i {B_1^i}^3 - 8\gamma_0^{I,i} A_1^i {B_1^i}^2 + 8\gamma_0^{I,i\,2} A_1^i B_1^i - \frac{8}{3}\gamma_0^{I,i\,3} A_1^i \\
&- 6\beta_0 A_1^i {B_1^i}^2 + 12\beta_0\gamma_0^{I,i} A_1^i B_1^i - 6\beta_0\gamma_0^{I,i\,2} A_1^i + \frac{11}{3}\beta_0^2 A_1^i B_1^i - \frac{11}{3}\beta_0^2\gamma_0^{I,i} A_1^i - \frac{1}{2}\beta_0^3 A_1^i \\
&+ \frac{16}{3}\zeta_3 {A_1^i}^4 - 8\zeta_2 {A_1^i}^3 B_1^i + 8\zeta_2\gamma_0^{I,i} {A_1^i}^3 + 6\zeta_2\beta_0 {A_1^i}^3) + \mathcal{D}_0(-2f_4^i - \frac{4}{3}\overline{\mathcal{G}}_3^{I,i,1} f_1^i - 2\overline{\mathcal{G}}_2^{I,i,1} f_2^i \\
&- 4\overline{\mathcal{G}}_1^{I,i,1} f_3^i - 4\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} f_1^i - 4\overline{\mathcal{G}}_1^{I,i,1\,2} f_2^i - \frac{8}{3}\overline{\mathcal{G}}_1^{I,i,1\,3} f_1^i - \frac{4}{3} g_3^{I,i,1} f_1^i - 2g_2^{I,i,1} f_2^i \\
&- 4g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} f_1^i - 4g_1^{I,i,1} f_3^i - 4g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} f_1^i - 8g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} f_2^i - 8g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1\,2} f_1^i \\
&- 4g_1^{I,i,1} g_2^{I,i,1} f_1^i - 4g_1^{I,i,1\,2} f_2^i - 8g_1^{I,i,1\,2}\overline{\mathcal{G}}_1^{I,i,1} f_1^i - \frac{8}{3} g_1^{I,i,1\,3} f_1^i - 4\beta_2\overline{\mathcal{G}}_1^{I,i,1} - 4\beta_1\overline{\mathcal{G}}_2^{I,i,1} \\
&- \frac{8}{3}\beta_1\overline{\mathcal{G}}_1^{I,i,2} f_1^i - 8\beta_1\overline{\mathcal{G}}_1^{I,i,1\,2} - \frac{8}{3}\beta_1 g_1^{I,i,2} f_1^i - 8\beta_1 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} - 4\beta_0\overline{\mathcal{G}}_3^{I,i,1} - \frac{8}{3}\beta_0\overline{\mathcal{G}}_2^{I,i,2} f_1^i \\
&- 4\beta_0\overline{\mathcal{G}}_1^{I,i,2} f_2^i - 12\beta_0\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} - 8\beta_0\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} f_1^i - 8\beta_0\overline{\mathcal{G}}_1^{I,i,1\,3} - \frac{8}{3}\beta_0 g_2^{I,i,2} f_1^i \\
&- 4\beta_0 g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} - 4\beta_0 g_1^{I,i,2} f_2^i - 8\beta_0 g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1} f_1^i - 8\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1} - 8\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} f_1^i \\
&- 16\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1\,2} - 8\beta_0 g_1^{I,i,1} g_1^{I,i,2} f_1^i - 8\beta_0 g_1^{I,i,1\,2}\overline{\mathcal{G}}_1^{I,i,1} - 16\beta_0\beta_1\overline{\mathcal{G}}_1^{I,i,2} - 8\beta_0^2\overline{\mathcal{G}}_2^{I,i,2} \\
&- \frac{16}{3}\beta_0^2\overline{\mathcal{G}}_1^{I,i,3} f_1^i - 24\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2} - \frac{16}{3}\beta_0^2 g_1^{I,i,3} f_1^i - 8\beta_0^2 g_1^{I,i,2}\overline{\mathcal{G}}_1^{I,i,1} - 16\beta_0^2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,2}
\end{aligned}
$$

$$
\begin{aligned}
&- 16\beta_0^3\overline{\mathcal{G}}_1^{I,i,3} + 3840\zeta_7 A_1^{i\,4} + 576\zeta_5 A_1^{i\,2} f_1^{i\,2} + 576\zeta_5 A_1^{i\,2} A_2^i + 384\zeta_5\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,3}\\
&+ 384\zeta_5 g_1^{I,i,1} A_1^{i\,3} + 1344\zeta_5\beta_0 A_1^{i\,2} f_1^i + 512\zeta_5\beta_0^2 A_1^{i\,2} + \frac{16}{3}\zeta_3 f_1^{i\,4} + 32\zeta_3 A_2^i f_1^{i\,2}\\
&+ 16\zeta_3 A_2^{i\,2} + 64\zeta_3 A_1^i f_1^i f_2^i + 32\zeta_3 A_1^i A_3^i + 16\zeta_3\overline{\mathcal{G}}_2^{I,i,1} A_1^{i\,2} + 64\zeta_3\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^{i\,2}\\
&+ 64\zeta_3\overline{\mathcal{G}}_1^{I,i,1} A_1^i A_2^i + 32\zeta_3\overline{\mathcal{G}}_1^{I,i,1\,2} A_1^{i\,2} + 16\zeta_3 g_2^{I,i,1} A_1^{i\,2} + 64\zeta_3 g_1^{I,i,1} A_1^i f_1^{i\,2} + 64\zeta_3 g_1^{I,i,1} A_1^i A_2^i\\
&+ 64\zeta_3 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} + 32\zeta_3 g_1^{I,i,1\,2} A_1^{i\,2} + 48\zeta_3\beta_1 A_1^i f_1^i + 32\zeta_3\beta_0 f_1^{i\,3} + 64\zeta_3\beta_0 A_2^i f_1^i\\
&+ 80\zeta_3\beta_0 A_1^i f_2^i + 32\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,2} A_1^{i\,2} + 224\zeta_3\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i + 32\zeta_3\beta_0 g_1^{I,i,2} A_1^{i\,2}\\
&+ 96\zeta_3\beta_0 g_1^{I,i,1} A_1^i f_1^i + 48\zeta_3\beta_0^2 f_1^{i\,2} + 160\zeta_3\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} A_1^i - \frac{2240}{3}\zeta_3^2 A_1^{i\,3} f_1^i - 704\zeta_3^2\beta_0 A_1^{i\,3}\\
&+ 12\zeta_2 f_1^{i\,2} f_2^i + 2\zeta_2 A_3^i f_1^i + 2\zeta_2 A_2^i f_2^i + 2\zeta_2 A_1^i f_3^i + 2\zeta_2\overline{\mathcal{G}}_2^{I,i,1} A_1^i f_1^i + 8\zeta_2\overline{\mathcal{G}}_1^{I,i,1} f_1^{i\,3}\\
&+ 4\zeta_2\overline{\mathcal{G}}_1^{I,i,1} A_2^i f_1^i + 4\zeta_2\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_2^i + 4\zeta_2\overline{\mathcal{G}}_1^{I,i,1\,2} A_1^i f_1^i + 2\zeta_2 g_2^{I,i,1} A_1^i f_1^i + 8\zeta_2 g_1^{I,i,1} f_1^{i\,3}\\
&+ 4\zeta_2 g_1^{I,i,1} A_2^i f_1^i + 4\zeta_2 g_1^{I,i,1} A_1^i f_2^i + 8\zeta_2 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i + 4\zeta_2 g_1^{I,i,1\,2} A_1^i f_1^i + 2\zeta_2\beta_1 f_1^{i\,2}\\
&- 12\zeta_2\beta_1 B_1^i f_1^i + 4\zeta_2\beta_1\overline{\mathcal{G}}_1^{I,i,1} A_1^i + 12\zeta_2\beta_1\gamma_0^{I,i} f_1^i + 6\zeta_2\beta_0 f_1^i f_2^i - 24\zeta_2\beta_0 B_2^i f_1^i\\
&- 12\zeta_2\beta_0 B_1^i f_2^i + 4\zeta_2\beta_0\overline{\mathcal{G}}_2^{I,i,1} A_1^i + 4\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,2} A_1^i f_1^i + 28\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1} f_1^{i\,2}\\
&- 24\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1} B_1^i f_1^i + 4\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_2^i + 8\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1\,2} A_1^i + 4\zeta_2\beta_0 g_1^{I,i,2} A_1^i f_1^i\\
&+ 4\zeta_2\beta_0 g_1^{I,i,1} f_1^{i\,2} - 24\zeta_2\beta_0 g_1^{I,i,1} B_1^i f_1^i + 8\zeta_2\beta_0 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_1^i + 24\zeta_2\beta_0\gamma_1^{I,i} f_1^i + 12\zeta_2\beta_0\gamma_0^{I,i} f_2^i\\
&+ 24\zeta_2\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} f_1^i + 24\zeta_2\beta_0\gamma_0^{I,i} g_1^{I,i,1} f_1^i + 8\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,2} A_1^i + 36\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} f_1^i\\
&- 24\zeta_2\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} B_1^i + 24\zeta_2\beta_0^2 g_1^{I,i,1} f_1^i + 24\zeta_2\beta_0^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} - 1728\zeta_2\zeta_5 A_1^{i\,4} - 320\zeta_2\zeta_3 A_1^{i\,2} f_1^{i\,2}\\
&- 240\zeta_2\zeta_3 A_1^{i\,2} A_2^i - 160\zeta_2\zeta_3\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,3} - 160\zeta_2\zeta_3 g_1^{I,i,1} A_1^{i\,3} - 640\zeta_2\zeta_3\beta_0 A_1^{i\,2} f_1^i\\
&+ 96\zeta_2\zeta_3\beta_0 A_1^{i\,2} B_1^i - 96\zeta_2\zeta_3\beta_0\gamma_0^{I,i} A_1^{i\,2} - 192\zeta_2\zeta_3\beta_0^2 A_1^{i\,2} + 12\zeta_2^2 A_1^i f_1^{i\,3} + 46\zeta_2^2 A_1^i A_2^i f_1^i\\
&+ 23\zeta_2^2 A_1^{i\,2} f_2^i + 46\zeta_2^2\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} f_1^i + 46\zeta_2^2 g_1^{I,i,1} A_1^{i\,2} f_1^i + 16\zeta_2^2\beta_1 A_1^{i\,2} + 62\zeta_2^2\beta_0 A_1^i f_1^{i\,2}\\
&+ 12\zeta_2^2\beta_0 A_1^i B_1^i f_1^i + 48\zeta_2^2\beta_0 A_1^i A_2^i + 78\zeta_2^2\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} + 32\zeta_2^2\beta_0 g_1^{I,i,1} A_1^{i\,2}\\
&- 12\zeta_2^2\beta_0\gamma_0^{I,i} A_1^i f_1^i + \frac{398}{5}\zeta_2^2\beta_0^2 A_1^i f_1^i - 376\zeta_2^2\zeta_3 A_1^{i\,4} + \frac{505}{3}\zeta_2^3 A_1^{i\,3} f_1^i + 176\zeta_2^3\beta_0 A_1^{i\,3})\\
&+ \mathcal{D}_1 L_{\mu_F}(8B_2^i f_1^{i\,2} + 16B_1^i f_1^i f_2^i - 8A_3^i f_1^i + 8A_3^i B_1^i - 8A_2^i f_2^i + 8A_2^i B_2^i - 8A_1^i f_3^i + 8A_1^i B_3^i\\
&- 8\overline{\mathcal{G}}_2^{I,i,1} A_1^i f_1^i + 8\overline{\mathcal{G}}_2^{I,i,1} A_1^i B_1^i + 16\overline{\mathcal{G}}_1^{I,i,1} B_1^i f_1^{i\,2} - 16\overline{\mathcal{G}}_1^{I,i,1} A_2^i f_1^i + 16\overline{\mathcal{G}}_1^{I,i,1} A_2^i B_1^i\\
&- 16\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_2^i + 16\overline{\mathcal{G}}_1^{I,i,1} A_1^i B_2^i - 16\overline{\mathcal{G}}_1^{I,i,1\,2} A_1^i f_1^i + 16\overline{\mathcal{G}}_1^{I,i,1\,2} A_1^i B_1^i - 8g_2^{I,i,1} A_1^i f_1^i\\
&+ 8g_2^{I,i,1} A_1^i B_1^i + 16g_1^{I,i,1} B_1^i f_1^{i\,2} - 16g_1^{I,i,1} A_2^i f_1^i + 16g_1^{I,i,1} A_2^i B_1^i - 16g_1^{I,i,1} A_1^i f_2^i
\end{aligned}
$$

$$+ 16g_1^{I,i,1}A_1^iB_2^i - 32g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^if_1^i + 32g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^iB_1^i - 16g_1^{I,i,1\,2}A_1^if_1^i + 16g_1^{I,i,1\,2}A_1^iB_1^i$$

$$- 8\gamma_2^{I,i}A_1^i - 8\gamma_1^{I,i}f_1^{i\,2} - 8\gamma_1^{I,i}A_2^i - 16\gamma_1^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 16\gamma_1^{I,i}g_1^{I,i,1}A_1^i - 16\gamma_0^{I,i}f_1^if_2^i - 8\gamma_0^{I,i}A_3^i$$

$$- 8\gamma_0^{I,i}\overline{\mathcal{G}}_2^{I,i,1}A_1^i - 16\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}f_1^{i\,2} - 16\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_2^i - 16\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1\,2}A_1^i - 8\gamma_0^{I,i}g_2^{I,i,1}A_1^i$$

$$- 16\gamma_0^{I,i}g_1^{I,i,1}f_1^{i\,2} - 16\gamma_0^{I,i}g_1^{I,i,1}A_2^i - 32\gamma_0^{I,i}g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 16\gamma_0^{I,i}g_1^{I,i,1\,2}A_1^i - 4\beta_2A_1^i$$

$$- 8\beta_1f_1^{i\,2} + 8\beta_1B_1^if_1^i - 8\beta_1A_2^i - 32\beta_1\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 16\beta_1g_1^{I,i,1}A_1^i - 8\beta_1\gamma_0^{I,i}f_1^i - 24\beta_0f_1^if_2^i$$

$$+ 8\beta_0B_2^if_1^i + 16\beta_0B_1^if_2^i - 12\beta_0A_3^i - 28\beta_0\overline{\mathcal{G}}_2^{I,i,1}A_1^i - 16\beta_0\overline{\mathcal{G}}_1^{I,i,2}A_1^if_1^i + 16\beta_0\overline{\mathcal{G}}_1^{I,i,2}A_1^iB_1^i$$

$$- 24\beta_0\overline{\mathcal{G}}_1^{I,i,1}f_1^{i\,2} + 48\beta_0\overline{\mathcal{G}}_1^{I,i,1}B_1^if_1^i - 40\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_2^i - 56\beta_0\overline{\mathcal{G}}_1^{I,i,1\,2}A_1^i - 12\beta_0g_2^{I,i,1}A_1^i$$

$$- 16\beta_0g_1^{I,i,2}A_1^if_1^i + 16\beta_0g_1^{I,i,2}A_1^iB_1^i - 24\beta_0g_1^{I,i,1}f_1^{i\,2} + 16\beta_0g_1^{I,i,1}B_1^if_1^i - 24\beta_0g_1^{I,i,1}A_2^i$$

$$- 80\beta_0g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i - 24\beta_0g_1^{I,i,1\,2}A_1^i - 8\beta_0\gamma_1^{I,i}f_1^i - 16\beta_0\gamma_0^{I,i}f_2^i - 16\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,2}A_1^i$$

$$- 48\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}f_1^i - 16\beta_0\gamma_0^{I,i}g_1^{I,i,2}A_1^i - 16\beta_0\gamma_0^{I,i}g_1^{I,i,1}f_1^i - 20\beta_0\beta_1f_1^i - 24\beta_0^2f_2^i$$

$$- 56\beta_0^2\overline{\mathcal{G}}_1^{I,i,2}A_1^i - 72\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}f_1^i + 32\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}B_1^i - 24\beta_0^2g_1^{I,i,2}A_1^i - 24\beta_0^2g_1^{I,i,1}f_1^i$$

$$- 32\beta_0^2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} - 48\beta_0^3\overline{\mathcal{G}}_1^{I,i,1} + 2688\zeta_5A_1^{i\,4} + 320\zeta_3A_1^{i\,2}f_1^{i\,2} - 320\zeta_3A_1^{i\,2}B_1^if_1^i$$

$$+ 480\zeta_3A_1^{i\,2}A_2^i + 320\zeta_3\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,3} + 320\zeta_3g_1^{I,i,1}A_1^{i\,3} + 320\zeta_3\gamma_0^{I,i}A_1^{i\,2}f_1^i + 992\zeta_3\beta_0A_1^{i\,2}f_1^i$$

$$- 192\zeta_3\beta_0A_1^{i\,2}B_1^i + 192\zeta_3\beta_0\gamma_0^{I,i}A_1^{i\,2} + 352\zeta_3\beta_0^2A_1^{i\,2} + 32\zeta_2A_1^if_1^{i\,3} - 56\zeta_2A_1^iB_1^if_1^{i\,2}$$

$$+ 112\zeta_2A_1^iA_2^if_1^i - 16\zeta_2A_1^iA_2^iB_1^i + 56\zeta_2A_1^{i\,2}f_2^i - 8\zeta_2A_1^{i\,2}B_2^i + 112\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2}f_1^i$$

$$- 16\zeta_2\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2}B_1^i + 112\zeta_2g_1^{I,i,1}A_1^{i\,2}f_1^i - 16\zeta_2g_1^{I,i,1}A_1^{i\,2}B_1^i + 8\zeta_2\gamma_1^{I,i}A_1^{i\,2} + 56\zeta_2\gamma_0^{I,i}A_1^if_1^{i\,2}$$

$$+ 16\zeta_2\gamma_0^{I,i}A_1^iA_2^i + 16\zeta_2\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2} + 16\zeta_2\gamma_0^{I,i}g_1^{I,i,1}A_1^{i\,2} + 24\zeta_2\beta_1A_1^{i\,2} + 140\zeta_2\beta_0A_1^if_1^{i\,2}$$

$$- 96\zeta_2\beta_0A_1^iB_1^if_1^i + 48\zeta_2\beta_0A_1^iB_1^{i\,2} + 72\zeta_2\beta_0A_1^iA_2^i + 168\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2} + 56\zeta_2\beta_0g_1^{I,i,1}A_1^{i\,2}$$

$$+ 96\zeta_2\beta_0\gamma_0^{I,i}A_1^if_1^i - 96\zeta_2\beta_0\gamma_0^{I,i}A_1^iB_1^i + 48\zeta_2\beta_0\gamma_0^{I,i\,2}A_1^i + 104\zeta_2\beta_0^2A_1^if_1^i - 72\zeta_2\beta_0^2A_1^iB_1^i$$

$$+ 72\zeta_2\beta_0^2\gamma_0^{I,i}A_1^i - 1440\zeta_2\zeta_3A_1^{i\,4} + 236\zeta_2^2A_1^{i\,3}f_1^i - 92\zeta_2^2A_1^{i\,3}B_1^i + 92\zeta_2^2\gamma_0^{I,i}A_1^{i\,3}$$

$$+ 314\zeta_2^2\beta_0A_1^{i\,3}) + \mathcal{D}_1L_{\mu_F}^2(8B_1^{i\,2}f_1^{i\,2} - 16A_2^iB_1^if_1^i + 8A_2^iB_1^{i\,2} + 4A_2^{i\,2} - 16A_1^iB_2^if_1^i$$

$$- 16A_1^iB_1^if_2^i + 16A_1^iB_1^iB_2^i + 8A_1^iA_3^i + 4\overline{\mathcal{G}}_2^{I,i,1}A_1^{i\,2} - 32\overline{\mathcal{G}}_1^{I,i,1}A_1^iB_1^if_1^i + 16\overline{\mathcal{G}}_1^{I,i,1}A_1^iB_1^{i\,2}$$

$$+ 16\overline{\mathcal{G}}_1^{I,i,1}A_1^iA_2^i + 8\overline{\mathcal{G}}_1^{I,i,1\,2}A_1^{i\,2} + 4g_2^{I,i,1}A_1^{i\,2} - 32g_1^{I,i,1}A_1^iB_1^if_1^i + 16g_1^{I,i,1}A_1^iB_1^{i\,2}$$

$$+ 16g_1^{I,i,1}A_1^iA_2^i + 16g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^{i\,2} + 8g_1^{I,i,1\,2}A_1^{i\,2} + 16\gamma_1^{I,i}A_1^if_1^i - 16\gamma_1^{I,i}A_1^iB_1^i$$

$$- 16\gamma_0^{I,i}B_1^if_1^{i\,2} + 16\gamma_0^{I,i}A_2^if_1^i - 16\gamma_0^{I,i}A_2^iB_1^i + 16\gamma_0^{I,i}A_1^if_2^i - 16\gamma_0^{I,i}A_1^iB_2^i$$

$$
\begin{aligned}
&+ 32\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i f_1^i - 32\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i B_1^i + 32\gamma_0^{I,i} g_1^{I,i,1}A_1^i f_1^i - 32\gamma_0^{I,i} g_1^{I,i,1}A_1^i B_1^i \\
&+ 16\gamma_0^{I,i}\gamma_1^{I,i}A_1^i + 8{\gamma_0^{I,i}}^2 {f_1^i}^2 + 8{\gamma_0^{I,i}}^2 A_2^i + 16{\gamma_0^{I,i}}^2\overline{\mathcal{G}}_1^{I,i,1}A_1^i + 16{\gamma_0^{I,i}}^2 g_1^{I,i,1}A_1^i + 12\beta_1 A_1^i f_1^i \\
&- 12\beta_1 A_1^i B_1^i + 12\beta_1\gamma_0^{I,i}A_1^i - 20\beta_0 B_1^i {f_1^i}^2 + 8\beta_0 {B_1^i}^2 f_1^i + 16\beta_0 A_2^i f_1^i - 20\beta_0 A_2^i B_1^i \\
&+ 20\beta_0 A_1^i f_2^i - 16\beta_0 A_1^i B_2^i + 8\beta_0\overline{\mathcal{G}}_1^{I,i,2}{A_1^i}^2 + 40\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^i f_1^i - 72\beta_0\overline{\mathcal{G}}_1^{I,i,1}A_1^i B_1^i \\
&+ 8\beta_0 g_1^{I,i,2}{A_1^i}^2 + 40\beta_0 g_1^{I,i,1}A_1^i f_1^i - 40\beta_0 g_1^{I,i,1}A_1^i B_1^i + 16\beta_0\gamma_1^{I,i}A_1^i + 20\beta_0\gamma_0^{I,i}{f_1^i}^2 \\
&- 16\beta_0\gamma_0^{I,i}B_1^i f_1^i + 20\beta_0\gamma_0^{I,i}A_2^i + 72\beta_0\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}A_1^i + 40\beta_0\gamma_0^{I,i}g_1^{I,i,1}A_1^i + 8\beta_0{\gamma_0^{I,i}}^2 f_1^i \\
&+ 10\beta_0\beta_1 A_1^i + 12\beta_0^2{f_1^i}^2 - 20\beta_0^2 B_1^i f_1^i + 12\beta_0^2 A_2^i + 64\beta_0^2\overline{\mathcal{G}}_1^{I,i,1}A_1^i + 24\beta_0^2 g_1^{I,i,1}A_1^i \\
&+ 20\beta_0^2\gamma_0^{I,i}f_1^i + 12\beta_0^3 f_1^i - 320\zeta_3{A_1^i}^3 f_1^i + 320\zeta_3{A_1^i}^3 B_1^i - 320\zeta_3\gamma_0^{I,i}{A_1^i}^3 - 496\zeta_3\beta_0{A_1^i}^3 \\
&- 48\zeta_2{A_1^i}^2{f_1^i}^2 + 112\zeta_2{A_1^i}^2 B_1^i f_1^i - 8\zeta_2{A_1^i}^2{B_1^i}^2 - 84\zeta_2{A_1^i}^2 A_2^i - 56\zeta_2\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^3 \\
&- 56\zeta_2 g_1^{I,i,1}{A_1^i}^3 - 112\zeta_2\gamma_0^{I,i}{A_1^i}^2 f_1^i + 16\zeta_2\gamma_0^{I,i}{A_1^i}^2 B_1^i - 8\zeta_2{\gamma_0^{I,i}}^2{A_1^i}^2 - 168\zeta_2\beta_0{A_1^i}^2 f_1^i \\
&+ 76\zeta_2\beta_0{A_1^i}^2 B_1^i - 76\zeta_2\beta_0\gamma_0^{I,i}{A_1^i}^2 - 52\zeta_2\beta_0^2{A_1^i}^2 - 118\zeta_2^2{A_1^i}^4) + \mathcal{D}_1 L_{\mu_F}^3(-16A_1^i{B_1^i}^2 f_1^i \\
&+ \frac{16}{3}A_1^i{B_1^i}^3 + 16A_1^i A_2^i B_1^i + 8{A_1^i}^2 B_2^i + 16\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 B_1^i + 16g_1^{I,i,1}{A_1^i}^2 B_1^i - 8\gamma_1^{I,i}{A_1^i}^2 \\
&+ 32\gamma_0^{I,i}A_1^i B_1^i f_1^i - 16\gamma_0^{I,i}A_1^i{B_1^i}^2 - 16\gamma_0^{I,i}A_1^i A_2^i - 16\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 - 16\gamma_0^{I,i}g_1^{I,i,1}{A_1^i}^2 \\
&- 16{\gamma_0^{I,i}}^2 A_1^i f_1^i + 16{\gamma_0^{I,i}}^2 A_1^i B_1^i - \frac{16}{3}{\gamma_0^{I,i}}^3 A_1^i - 4\beta_1{A_1^i}^2 + 32\beta_0 A_1^i B_1^i f_1^i - 16\beta_0 A_1^i{B_1^i}^2 \\
&- 12\beta_0 A_1^i A_2^i - 16\beta_0\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^2 - 16\beta_0 g_1^{I,i,1}{A_1^i}^2 - 32\beta_0\gamma_0^{I,i}A_1^i f_1^i + 32\beta_0\gamma_0^{I,i}A_1^i B_1^i \\
&- 16\beta_0{\gamma_0^{I,i}}^2 A_1^i - \frac{44}{3}\beta_0^2 A_1^i f_1^i + \frac{44}{3}\beta_0^2 A_1^i B_1^i - \frac{44}{3}\beta_0^2\gamma_0^{I,i}A_1^i - 4\beta_0^3 A_1^i + \frac{320}{3}\zeta_3{A_1^i}^4 \\
&+ 32\zeta_2{A_1^i}^3 f_1^i - 56\zeta_2{A_1^i}^3 B_1^i + 56\zeta_2\gamma_0^{I,i}{A_1^i}^3 + 56\zeta_2\beta_0{A_1^i}^3) + \mathcal{D}_1 L_{\mu_F}^4(+8{A_1^i}^2{B_1^i}^2 \\
&- 16\gamma_0^{I,i}{A_1^i}^2 B_1^i + 8{\gamma_0^{I,i}}^2{A_1^i}^2 - 12\beta_0{A_1^i}^2 B_1^i + 12\beta_0\gamma_0^{I,i}{A_1^i}^2 + \frac{11}{3}\beta_0^2{A_1^i}^2 - 8\zeta_2{A_1^i}^4) \\
&+ \mathcal{D}_1(4{f_2^i}^2 + 8f_1^i f_3^i + 4A_4^i + \frac{8}{3}\overline{\mathcal{G}}_3^{I,i,1}A_1^i + 4\overline{\mathcal{G}}_2^{I,i,1}{f_1^i}^2 + 4\overline{\mathcal{G}}_2^{I,i,1}A_2^i + 16\overline{\mathcal{G}}_1^{I,i,1}f_1^i f_2^i \\
&+ 8\overline{\mathcal{G}}_1^{I,i,1}A_3^i + 8\overline{\mathcal{G}}_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1}A_1^i + 8{\overline{\mathcal{G}}_1^{I,i,1}}^2{f_1^i}^2 + 8{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_2^i + \frac{16}{3}{\overline{\mathcal{G}}_1^{I,i,1}}^3 A_1^i + \frac{8}{3}g_3^{I,i,1}A_1^i \\
&+ 4g_2^{I,i,1}{f_1^i}^2 + 4g_2^{I,i,1}A_2^i + 8g_2^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_1^i + 16g_1^{I,i,1}f_1^i f_2^i + 8g_1^{I,i,1}A_3^i + 8g_1^{I,i,1}\overline{\mathcal{G}}_2^{I,i,1}A_1^i \\
&+ 16g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}{f_1^i}^2 + 16g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1}A_2^i + 16g_1^{I,i,1}{\overline{\mathcal{G}}_1^{I,i,1}}^2 A_1^i + 8g_1^{I,i,1}g_2^{I,i,1}A_1^i + 8{g_1^{I,i,1}}^2{f_1^i}^2 \\
&+ 8{g_1^{I,i,1}}^2 A_2^i + 16{g_1^{I,i,1}}^2\overline{\mathcal{G}}_1^{I,i,1}A_1^i + \frac{16}{3}{g_1^{I,i,1}}^3 A_1^i + 4\beta_2 f_1^i + 8\beta_1 f_2^i + \frac{16}{3}\beta_1\overline{\mathcal{G}}_1^{I,i,2}A_1^i \\
&+ 24\beta_1\overline{\mathcal{G}}_1^{I,i,1}f_1^i + \frac{16}{3}\beta_1 g_1^{I,i,2}A_1^i + 8\beta_1 g_1^{I,i,1}f_1^i + 12\beta_0 f_3^i + \frac{16}{3}\beta_0\overline{\mathcal{G}}_2^{I,i,2}A_1^i + 20\beta_0\overline{\mathcal{G}}_2^{I,i,1}f_1^i
\end{aligned}
$$

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\begin{aligned}
&+ 8\beta_0 \overline{\mathcal{G}}_1^{I,i,2} {f_1^i}^2 + 8\beta_0 \overline{\mathcal{G}}_1^{I,i,2} A_2^i + 32\beta_0 \overline{\mathcal{G}}_1^{I,i,1} f_2^i + 16\beta_0 \overline{\mathcal{G}}_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,2} A_1^i + 40\beta_0 {\overline{\mathcal{G}}_1^{I,i,1}}^2 f_1^i \\
&+ \frac{16}{3}\beta_0 g_2^{I,i,2} A_1^i + 4\beta_0 g_2^{I,i,1} f_1^i + 8\beta_0 g_1^{I,i,2} {f_1^i}^2 + 8\beta_0 g_1^{I,i,2} A_2^i + 16\beta_0 g_1^{I,i,2} \overline{\mathcal{G}}_1^{I,i,1} A_1^i \\
&+ 16\beta_0 g_1^{I,i,1} f_2^i + 16\beta_0 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,2} A_1^i + 48\beta_0 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} f_1^i + 16\beta_0 g_1^{I,i,1} g_1^{I,i,2} A_1^i \\
&+ 8\beta_0 {g_1^{I,i,1}}^2 f_1^i + 40\beta_0\beta_1 \overline{\mathcal{G}}_1^{I,i,1} + 24\beta_0^2 \overline{\mathcal{G}}_2^{I,i,1} + \frac{32}{3}\beta_0^2 \overline{\mathcal{G}}_1^{I,i,3} A_1^i + 40\beta_0^2 \overline{\mathcal{G}}_1^{I,i,2} f_1^i + 48\beta_0^2 {\overline{\mathcal{G}}_1^{I,i,1}}^2 \\
&+ \frac{32}{3}\beta_0^2 g_1^{I,i,3} A_1^i + 8\beta_0^2 g_1^{I,i,2} f_1^i + 32\beta_0^2 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} + 48\beta_0^3 \overline{\mathcal{G}}_1^{I,i,2} - 2688\zeta_5 {A_1^i}^3 f_1^i \\
&- 2688\zeta_5\beta_0 {A_1^i}^3 - \frac{320}{3}\zeta_3 A_1^i {f_1^i}^3 - 320\zeta_3 A_1^i A_2^i f_1^i - 160\zeta_3 {A_1^i}^2 f_2^i - 320\zeta_3 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 f_1^i \\
&- 320\zeta_3 g_1^{I,i,1} {A_1^i}^2 f_1^i - 96\zeta_3\beta_1 {A_1^i}^2 - 416\zeta_3\beta_0 A_1^i {f_1^i}^2 - 288\zeta_3\beta_0 A_1^i A_2^i - 512\zeta_3\beta_0 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 \\
&- 192\zeta_3\beta_0 g_1^{I,i,1} {A_1^i}^2 - 352\zeta_3\beta_0^2 A_1^i f_1^i + \frac{4480}{3}\zeta_3^2 {A_1^i}^4 - 8\zeta_2 {f_1^i}^4 - 28\zeta_2 A_2^i {f_1^i}^2 - 4\zeta_2 {A_2^i}^2 \\
&- 56\zeta_2 A_1^i f_1^i f_2^i - 8\zeta_2 A_1^i A_3^i - 4\zeta_2 \overline{\mathcal{G}}_2^{I,i,1} {A_1^i}^2 - 56\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i {f_1^i}^2 - 16\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i A_2^i \\
&- 8\zeta_2 {\overline{\mathcal{G}}_1^{I,i,1}}^2 {A_1^i}^2 - 4\zeta_2 g_2^{I,i,1} {A_1^i}^2 - 56\zeta_2 g_1^{I,i,1} A_1^i {f_1^i}^2 - 16\zeta_2 g_1^{I,i,1} A_1^i A_2^i - 16\zeta_2 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 \\
&- 8\zeta_2 {g_1^{I,i,1}}^2 {A_1^i}^2 - 24\zeta_2\beta_1 A_1^i f_1^i + 24\zeta_2\beta_1 A_1^i B_1^i - 24\zeta_2\beta_1 \gamma_0^{I,i} A_1^i - 28\zeta_2\beta_0 {f_1^i}^3 \\
&+ 24\zeta_2\beta_0 B_1^i {f_1^i}^2 - 40\zeta_2\beta_0 A_2^i f_1^i + 24\zeta_2\beta_0 A_2^i B_1^i - 32\zeta_2\beta_0 A_1^i f_2^i + 48\zeta_2\beta_0 A_1^i B_2^i \\
&- 8\zeta_2\beta_0 \overline{\mathcal{G}}_1^{I,i,2} {A_1^i}^2 - 160\zeta_2\beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i + 48\zeta_2\beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i - 8\zeta_2\beta_0 g_1^{I,i,2} {A_1^i}^2 \\
&- 48\zeta_2\beta_0 g_1^{I,i,1} A_1^i f_1^i + 48\zeta_2\beta_0 g_1^{I,i,1} A_1^i B_1^i - 48\zeta_2\beta_0 \gamma_1^{I,i} A_1^i - 24\zeta_2\beta_0 \gamma_0^{I,i} {f_1^i}^2 \\
&- 24\zeta_2\beta_0 \gamma_0^{I,i} A_2^i - 48\zeta_2\beta_0 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^i - 48\zeta_2\beta_0 \gamma_0^{I,i} g_1^{I,i,1} A_1^i - 36\zeta_2\beta_0^2 {f_1^i}^2 + 24\zeta_2\beta_0^2 B_1^i f_1^i \\
&- 112\zeta_2\beta_0^2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i - 48\zeta_2\beta_0^2 g_1^{I,i,1} A_1^i - 24\zeta_2\beta_0^2 \gamma_0^{I,i} f_1^i + 1440\zeta_2\zeta_3 {A_1^i}^3 f_1^i \\
&+ 1376\zeta_2\zeta_3\beta_0 {A_1^i}^3 - 118\zeta_2^2 {A_1^i}^2 {f_1^i}^2 - 138\zeta_2^2 {A_1^i}^2 A_2^i - 92\zeta_2^2 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^3 - 92\zeta_2^2 g_1^{I,i,1} {A_1^i}^3 \\
&- 314\zeta_2^2\beta_0 {A_1^i}^2 f_1^i - 24\zeta_2^2\beta_0 {A_1^i}^2 B_1^i + 24\zeta_2^2\beta_0 \gamma_0^{I,i} {A_1^i}^2 - \frac{796}{5}\zeta_2^2\beta_0^2 {A_1^i}^2 - \frac{1010}{3}\zeta_2^3 {A_1^i}^4) \\
&+ \mathcal{D}_2 L_{\mu_F}(-8B_1^i {f_1^i}^3 + 12A_2^i {f_1^i}^2 - 24A_2^i B_1^i f_1^i + 12{A_2^i}^2 + 24A_1^i f_1^i f_2^i - 24A_1^i B_2^i f_1^i \\
&- 24A_1^i B_1^i f_2^i + 24A_1^i A_3^i + 12\overline{\mathcal{G}}_2^{I,i,1} {A_1^i}^2 + 24\overline{\mathcal{G}}_1^{I,i,1} A_1^i {f_1^i}^2 - 48\overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i f_1^i \\
&+ 48\overline{\mathcal{G}}_1^{I,i,1} A_1^i A_2^i + 24{\overline{\mathcal{G}}_1^{I,i,1}}^2 {A_1^i}^2 + 12g_2^{I,i,1} {A_1^i}^2 + 24g_1^{I,i,1} A_1^i {f_1^i}^2 - 48g_1^{I,i,1} A_1^i B_1^i f_1^i \\
&+ 48g_1^{I,i,1} A_1^i A_2^i + 48g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 + 24{g_1^{I,i,1}}^2 {A_1^i}^2 + 24\gamma_1^{I,i} A_1^i f_1^i + 8\gamma_0^{I,i} {f_1^i}^3 + 24\gamma_0^{I,i} A_2^i f_1^i \\
&+ 24\gamma_0^{I,i} A_1^i f_2^i + 48\gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i + 48\gamma_0^{I,i} g_1^{I,i,1} A_1^i f_1^i + 36\beta_1 A_1^i f_1^i - 8\beta_1 A_1^i B_1^i \\
&+ 8\beta_1 \gamma_0^{I,i} A_1^i + 12\beta_0 {f_1^i}^3 - 24\beta_0 B_1^i {f_1^i}^2 + 48\beta_0 A_2^i f_1^i - 16\beta_0 A_2^i B_1^i + 60\beta_0 A_1^i f_2^i
\end{aligned}
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\begin{aligned}
&- 8\beta_0 A_1^i B_2^i + 24\beta_0 \overline{\mathcal{G}}_1^{I,i,2} {A_1^i}^2 + 144\beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i - 64\beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_1^i B_1^i + 24\beta_0 g_1^{I,i,2} {A_1^i}^2 \\
&+ 96\beta_0 g_1^{I,i,1} A_1^i f_1^i - 16\beta_0 g_1^{I,i,1} A_1^i B_1^i + 8\beta_0 \gamma_1^{I,i} A_1^i + 24\beta_0 \gamma_0^{I,i} f_1^{i2} + 16\beta_0 \gamma_0^{I,i} A_2^i \\
&+ 64\beta_0 \gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} A_1^i + 16\beta_0 \gamma_0^{I,i} g_1^{I,i,1} A_1^i + 20\beta_0 \beta_1 A_1^i + 36\beta_0^2 f_1^{i2} - 16\beta_0^2 B_1^i f_1^i + 24\beta_0^2 A_2^i \\
&+ 144\beta_0^2 \overline{\mathcal{G}}_1^{I,i,1} A_1^i + 24\beta_0^2 g_1^{I,i,1} A_1^i + 16\beta_0^2 \gamma_0^{I,i} f_1^i + 24\beta_0^3 f_1^i - 960\zeta_3 {A_1^i}^3 f_1^i + 320\zeta_3 {A_1^i}^3 B_1^i \\
&- 320\zeta_3 \gamma_0^{I,i} {A_1^i}^3 - 1088\zeta_3 \beta_0 {A_1^i}^3 - 180\zeta_2 {A_1^i}^2 f_1^{i2} + 120\zeta_2 {A_1^i}^2 B_1^i f_1^i - 180\zeta_2 {A_1^i}^2 A_2^i \\
&- 120\zeta_2 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^3 - 120\zeta_2 g_1^{I,i,1} {A_1^i}^3 - 120\zeta_2 \gamma_0^{I,i} {A_1^i}^2 f_1^i - 412\zeta_2 \beta_0 {A_1^i}^2 f_1^i \\
&+ 144\zeta_2 \beta_0 {A_1^i}^2 B_1^i - 144\zeta_2 \beta_0 \gamma_0^{I,i} {A_1^i}^2 - 140\zeta_2 \beta_0^2 {A_1^i}^2 - 282\zeta_2^2 {A_1^i}^4) + \mathcal{D}_2 L_{\mu_F}^2 (24 A_1^i B_1^i f_1^{i2} \\
&- 24 A_1^i {B_1^i}^2 f_1^i - 24 A_1^i A_2^i f_1^i + 48 A_1^i A_2^i B_1^i - 12 {A_1^i}^2 f_2^i + 24 {A_1^i}^2 B_2^i - 24 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 f_1^i \\
&+ 48 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 B_1^i - 24 g_1^{I,i,1} {A_1^i}^2 f_1^i + 48 g_1^{I,i,1} {A_1^i}^2 B_1^i - 24\gamma_1^{I,i} {A_1^i}^2 - 24\gamma_0^{I,i} A_1^i f_1^{i2} \\
&+ 48\gamma_0^{I,i} A_1^i B_1^i f_1^i - 48\gamma_0^{I,i} A_1^i A_2^i - 48\gamma_0^{I,i} \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 - 48\gamma_0^{I,i} g_1^{I,i,1} {A_1^i}^2 - 24\gamma_0^{I,i2} A_1^i f_1^i \\
&- 18\beta_1 {A_1^i}^2 - 30\beta_0 A_1^i f_1^{i2} + 84\beta_0 A_1^i B_1^i f_1^i - 8\beta_0 A_1^i {B_1^i}^2 - 54\beta_0 A_1^i A_2^i - 84\beta_0 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 \\
&- 60\beta_0 g_1^{I,i,1} {A_1^i}^2 - 84\beta_0 \gamma_0^{I,i} A_1^i f_1^i + 16\beta_0 \gamma_0^{I,i} A_1^i B_1^i - 8\beta_0 \gamma_0^{I,i2} A_1^i - 66\beta_0^2 A_1^i f_1^i \\
&+ 20\beta_0^2 A_1^i B_1^i - 20\beta_0^2 \gamma_0^{I,i} A_1^i - 12\beta_0^3 A_1^i + 480\zeta_3 {A_1^i}^4 + 180\zeta_2 {A_1^i}^3 f_1^i - 120\zeta_2 {A_1^i}^3 B_1^i \\
&+ 120\zeta_2 \gamma_0^{I,i} {A_1^i}^3 + 206\zeta_2 \beta_0 {A_1^i}^3) + \mathcal{D}_2 L_{\mu_F}^3 (-24 {A_1^i}^2 B_1^i f_1^i + 24 {A_1^i}^2 {B_1^i}^2 + 12 {A_1^i}^2 A_2^i \\
&+ 8 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^3 + 8 g_1^{I,i,1} {A_1^i}^3 + 24\gamma_0^{I,i} {A_1^i}^2 f_1^i - 48\gamma_0^{I,i} {A_1^i}^2 B_1^i + 24\gamma_0^{I,i2} {A_1^i}^2 + 24\beta_0 {A_1^i}^2 f_1^i \\
&- 48\beta_0 {A_1^i}^2 B_1^i + 48\beta_0 \gamma_0^{I,i} {A_1^i}^2 + 22\beta_0^2 {A_1^i}^2 - 60\zeta_2 {A_1^i}^4) + \mathcal{D}_2 L_{\mu_F}^4 (8 {A_1^i}^3 B_1^i - 8\gamma_0^{I,i} {A_1^i}^3 \\
&- 6\beta_0 {A_1^i}^3) + \mathcal{D}_2 (-12 f_1^{i2} f_2^i - 12 A_3^i f_1^i - 12 A_2^i f_2^i - 12 A_1^i f_3^i - 12 \overline{\mathcal{G}}_2^{I,i,1} A_1^i f_1^i - 8 \overline{\mathcal{G}}_1^{I,i,1} f_1^{i3} \\
&- 24 \overline{\mathcal{G}}_1^{I,i,1} A_2^i f_1^i - 24 \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_2^i - 24 \overline{\mathcal{G}}_1^{I,i,1\,2} A_1^i f_1^i - 12 g_2^{I,i,1} A_1^i f_1^i - 8 g_1^{I,i,1} f_1^{i3} \\
&- 24 g_1^{I,i,1} A_2^i f_1^i - 24 g_1^{I,i,1} A_1^i f_2^i - 48 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i - 24 g_1^{I,i,1\,2} A_1^i f_1^i - 4\beta_2 A_1^i - 12\beta_1 f_1^{i2} \\
&- 8\beta_1 A_2^i - 32\beta_1 \overline{\mathcal{G}}_1^{I,i,1} A_1^i - 8\beta_1 g_1^{I,i,1} A_1^i - 36\beta_0 f_1^i f_2^i - 12\beta_0 A_3^i - 28\beta_0 \overline{\mathcal{G}}_2^{I,i,1} A_1^i \\
&- 24\beta_0 \overline{\mathcal{G}}_1^{I,i,2} A_1^i f_1^i - 48\beta_0 \overline{\mathcal{G}}_1^{I,i,1} f_1^{i2} - 40\beta_0 \overline{\mathcal{G}}_1^{I,i,1} A_2^i - 56\beta_0 \overline{\mathcal{G}}_1^{I,i,1\,2} A_1^i - 4\beta_0 g_2^{I,i,1} A_1^i \\
&- 24\beta_0 g_1^{I,i,2} A_1^i f_1^i - 24\beta_0 g_1^{I,i,1} f_1^{i2} - 16\beta_0 g_1^{I,i,1} A_2^i - 64\beta_0 g_1^{I,i,1} \overline{\mathcal{G}}_1^{I,i,1} A_1^i - 8\beta_0 g_1^{I,i,1\,2} A_1^i \\
&- 20\beta_0 \beta_1 f_1^i - 24\beta_0^2 f_2^i - 56\beta_0^2 \overline{\mathcal{G}}_1^{I,i,2} A_1^i - 88\beta_0^2 \overline{\mathcal{G}}_1^{I,i,1} f_1^i - 8\beta_0^2 g_1^{I,i,2} A_1^i - 16\beta_0^2 g_1^{I,i,1} f_1^i \\
&- 48\beta_0^3 \overline{\mathcal{G}}_1^{I,i,1} + 2688\zeta_5 {A_1^i}^4 + 480\zeta_3 {A_1^i}^2 f_1^{i2} + 480\zeta_3 {A_1^i}^2 A_2^i + 320\zeta_3 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^3 \\
&+ 320\zeta_3 g_1^{I,i,1} {A_1^i}^3 + 1088\zeta_3 \beta_0 {A_1^i}^2 f_1^i + 352\zeta_3 \beta_0^2 {A_1^i}^2 + 60\zeta_2 A_1^i f_1^{i3} + 120\zeta_2 A_1^i A_2^i f_1^i
\end{aligned}
$$

$$
\begin{aligned}
&+ 60\zeta_2 A_1^{i\,2} f_2^i + 120\zeta_2\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} f_1^i + 120\zeta_2 g_1^{I,i,1} A_1^{i\,2} f_1^i + 36\zeta_2\beta_1 A_1^{i\,2} + 176\zeta_2\beta_0 A_1^i f_1^{i2} \\
&- 72\zeta_2\beta_0 A_1^i B_1^i f_1^i + 108\zeta_2\beta_0 A_1^i A_2^i + 192\zeta_2\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} + 72\zeta_2\beta_0 g_1^{I,i,1} A_1^{i\,2} \\
&+ 72\zeta_2\beta_0\gamma_0^{I,i} A_1^i f_1^i + 140\zeta_2\beta_0^2 A_1^i f_1^i - 24\zeta_2\beta_0^2 A_1^i B_1^i + 24\zeta_2\beta_0^2\gamma_0^{I,i} A_1^i - 1440\zeta_2\zeta_3 A_1^{i\,4} \\
&+ 282\zeta_2^2 A_1^{i\,3} f_1^i + 302\zeta_2^2\beta_0 A_1^{i\,3}) + \mathcal{D}_3 L_{\mu_F}(-\frac{32}{3} A_1^i f_1^{i3} + 32 A_1^i B_1^i f_1^{i2} - 64 A_1^i A_2^i f_1^i \\
&+ 32 A_1^i A_2^i B_1^i - 32 A_1^{i\,2} f_2^i + 16 A_1^{i\,2} B_2^i - 64\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} f_1^i + 32\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} B_1^i - 64 g_1^{I,i,1} A_1^{i\,2} f_1^i \\
&+ 32 g_1^{I,i,1} A_1^{i\,2} B_1^i - 16\gamma_1^{I,i} A_1^{i\,2} - 32\gamma_0^{I,i} A_1^i f_1^{i2} - 32\gamma_0^{I,i} A_1^i A_2^i - 32\gamma_0^{I,i}\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} \\
&- 32\gamma_0^{I,i} g_1^{I,i,1} A_1^{i\,2} - \frac{80}{3}\beta_1 A_1^{i\,2} - 80\beta_0 A_1^i f_1^{i2} + \frac{160}{3}\beta_0 A_1^i B_1^i f_1^i - 80\beta_0 A_1^i A_2^i \\
&- \frac{400}{3}\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} - \frac{208}{3}\beta_0 g_1^{I,i,1} A_1^{i\,2} - \frac{160}{3}\beta_0\gamma_0^{I,i} A_1^i f_1^i - \frac{304}{3}\beta_0^2 A_1^i f_1^i + \frac{32}{3}\beta_0^2 A_1^i B_1^i \\
&- \frac{32}{3}\beta_0^2\gamma_0^{I,i} A_1^i - 16\beta_0^3 A_1^i + \frac{2240}{3}\zeta_3 A_1^{i\,4} + 320\zeta_2 A_1^{i\,3} f_1^i - 80\zeta_2 A_1^{i\,3} B_1^i + 80\zeta_2\gamma_0^{I,i} A_1^{i\,3} \\
&+ \frac{968}{3}\zeta_2\beta_0 A_1^{i\,3}) + \mathcal{D}_3 L_{\mu_F}^2(16 A_1^{i\,2} f_1^{i2} - 64 A_1^{i\,2} B_1^i f_1^i + 16 A_1^{i\,2} B_1^{i\,2} + 48 A_1^{i\,2} A_2^i \\
&+ 32\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,3} + 32 g_1^{I,i,1} A_1^{i\,3} + 64\gamma_0^{I,i} A_1^{i\,2} f_1^i - 32\gamma_0^{I,i} A_1^{i\,2} B_1^i + 16\gamma_0^{I,i2} A_1^{i\,2} + 96\beta_0 A_1^{i\,2} f_1^i \\
&- \frac{184}{3}\beta_0 A_1^{i\,2} B_1^i + \frac{184}{3}\beta_0\gamma_0^{I,i} A_1^{i\,2} + \frac{152}{3}\beta_0^2 A_1^{i\,2} - 160\zeta_2 A_1^{i\,4}) + \mathcal{D}_3 L_{\mu_F}^3(-\frac{32}{3} A_1^{i\,3} f_1^i \\
&+ 32 A_1^{i\,3} B_1^i - 32\gamma_0^{I,i} A_1^{i\,3} - 32\beta_0 A_1^{i\,3}) + \mathcal{D}_3 L_{\mu_F}^4(\frac{8}{3} A_1^{i\,4}) + \mathcal{D}_3(\frac{8}{3} f_1^{i4} + 16 A_2^i f_1^{i2} + 8 A_2^{i\,2} \\
&+ 32 A_1^i f_1^i f_2^i + 16 A_1^i A_3^i + 8\overline{\mathcal{G}}_2^{I,i,1} A_1^{i\,2} + 32\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^{i2} + 32\overline{\mathcal{G}}_1^{I,i,1} A_1^i A_2^i + 16\overline{\mathcal{G}}_1^{I,i,1\,2} A_1^{i\,2} \\
&+ 8 g_2^{I,i,1} A_1^{i\,2} + 32 g_1^{I,i,1} A_1^i f_1^{i2} + 32 g_1^{I,i,1} A_1^i A_2^i + 32 g_1^{I,i,1}\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} + 16 g_1^{I,i,1\,2} A_1^{i\,2} \\
&+ \frac{80}{3}\beta_1 A_1^i f_1^i + 16\beta_0 f_1^{i3} + \frac{112}{3}\beta_0 A_2^i f_1^i + \frac{128}{3}\beta_0 A_1^i f_2^i + 16\beta_0\overline{\mathcal{G}}_1^{I,i,2} A_1^{i\,2} \\
&+ \frac{352}{3}\beta_0\overline{\mathcal{G}}_1^{I,i,1} A_1^i f_1^i + 16\beta_0 g_1^{I,i,2} A_1^{i\,2} + \frac{160}{3}\beta_0 g_1^{I,i,1} A_1^i f_1^i + \frac{40}{3}\beta_0\beta_1 A_1^i + \frac{88}{3}\beta_0^2 f_1^{i2} \\
&+ 16\beta_0^2 A_2^i + 96\beta_0^2\overline{\mathcal{G}}_1^{I,i,1} A_1^i + \frac{32}{3}\beta_0^2 g_1^{I,i,1} A_1^i + 16\beta_0^3 f_1^i - \frac{2240}{3}\zeta_3 A_1^{i\,3} f_1^i - \frac{2176}{3}\zeta_3\beta_0 A_1^{i\,3} \\
&- 160\zeta_2 A_1^{i\,2} f_1^{i2} - 120\zeta_2 A_1^{i\,2} A_2^i - 80\zeta_2\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,3} - 80\zeta_2 g_1^{I,i,1} A_1^{i\,3} - \frac{968}{3}\zeta_2\beta_0 A_1^{i\,2} f_1^i \\
&+ 48\zeta_2\beta_0 A_1^{i\,2} B_1^i - 48\zeta_2\beta_0\gamma_0^{I,i} A_1^{i\,2} - \frac{304}{3}\zeta_2\beta_0^2 A_1^{i\,2} - 188\zeta_2^2 A_1^{i\,4}) + \mathcal{D}_4 L_{\mu_F}(40 A_1^{i\,2} f_1^{i2} \\
&- 40 A_1^{i\,2} B_1^i f_1^i + 60 A_1^{i\,2} A_2^i + 40\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,3} + 40 g_1^{I,i,1} A_1^{i\,3} + 40\gamma_0^{I,i} A_1^{i\,2} f_1^i + \frac{380}{3}\beta_0 A_1^{i\,2} f_1^i \\
&- \frac{80}{3}\beta_0 A_1^{i\,2} B_1^i + \frac{80}{3}\beta_0\gamma_0^{I,i} A_1^{i\,2} + \frac{160}{3}\beta_0^2 A_1^{i\,2} - 180\zeta_2 A_1^{i\,4}) + \mathcal{D}_4 L_{\mu_F}^2(-40 A_1^{i\,3} f_1^i \\
&+ 40 A_1^{i\,3} B_1^i - 40\gamma_0^{I,i} A_1^{i\,3} - \frac{190}{3}\beta_0 A_1^{i\,3}) + \mathcal{D}_4 L_{\mu_F}^3(\frac{40}{3} A_1^{i\,4}) + \mathcal{D}_4(-\frac{40}{3} A_1^i f_1^{i3} \\
&- 40 A_1^i A_2^i f_1^i - 20 A_1^{i\,2} f_2^i - 40\overline{\mathcal{G}}_1^{I,i,1} A_1^{i\,2} f_1^i - 40 g_1^{I,i,1} A_1^{i\,2} f_1^i - \frac{40}{3}\beta_1 A_1^{i\,2} - \frac{160}{3}\beta_0 A_1^i f_1^{i2}
\end{aligned}
$$

$$
\begin{aligned}
&- 40\beta_0 A_1^i A_2^i - \frac{200}{3}\beta_0 \overline{\mathcal{G}}_1^{I,i,1} {A_1^i}^2 - \frac{80}{3}\beta_0 g_1^{I,i,1} {A_1^i}^2 - \frac{160}{3}\beta_0^2 A_1^i f_1^i - 8\beta_0^3 A_1^i + \frac{1120}{3}\zeta_3 {A_1^i}^4 \\
&+ 180\zeta_2 {A_1^i}^3 f_1^i + \frac{520}{3}\zeta_2\beta_0 {A_1^i}^3) + \mathcal{D}_5 L_{\mu_F}(-48{A_1^i}^3 f_1^i + 16{A_1^i}^3 B_1^i - 16\gamma_0^{I,i}{A_1^i}^3 \\
&- 56\beta_0 {A_1^i}^3) + \mathcal{D}_5 L_{\mu_F}^2(24{A_1^i}^4) + \mathcal{D}_5(24{A_1^i}^2 f_1^{i2} + 24{A_1^i}^2 A_2^i + 16\overline{\mathcal{G}}_1^{I,i,1}{A_1^i}^3 + 16 g_1^{I,i,1}{A_1^i}^3 \\
&+ 56\beta_0 {A_1^i}^2 f_1^i + \frac{64}{3}\beta_0^2 {A_1^i}^2 - 72\zeta_2 {A_1^i}^4) + \mathcal{D}_6 L_{\mu_F}(\frac{56}{3}{A_1^i}^4) + \mathcal{D}_6(-\frac{56}{3}{A_1^i}^3 f_1^i - \frac{56}{3}\beta_0 {A_1^i}^3) \\
&+ \mathcal{D}_7(\frac{16}{3}{A_1^i}^4)\,,
\end{aligned}
\tag{3.45}
$$

where,

$$
L_{\mu_F} = \log\left(\frac{q^2}{\mu_F^2}\right). \tag{3.46}
$$

Each $\Delta_{SV}^{I,i,n}$ contains the process independent anomalous dimensions A_n^i, B_n^i and f_n^i and process dependent terms like $\gamma_n^{I,i}$, $g_n^{I,i,k}$ and $\overline{\mathcal{G}}_n^{I,i,k}$.

4 DY Production at Threshold to Third Order in QCD

In this Chapter we apply the technique to obtain threshold approximated partonic cross section, presented in the previous chapter, to the Drell-Yan pair production. We present the missing $\delta(1-z)$ part at N^{3}LO. The numerical impact of our findings establish the reliability of pQCD and also reduce the dependence of the scale uncertainty.

4.1 Introduction

From determining electro-weak model parameters, constraining the PDFs to calibrating the detector, the Drell-Yan process works as a standard candle. High production rate with clean signature makes this process an important one. As discussed earlier, the physics of massive lepton pair production in hadronic collisions gained interest almost fifty years before, when in 1970, Christenson *et al.* [45, 46] first observed this production in proton nucleus collisions. Slowly, it became one of the most important process from both experimental and theoretical perspectives. The need to multiply a large K-factor with the LO result to match the experimental data, urged to compute QCD corrections. While the first order corrections [22–26] successfully addressed the issue, but the largeness of the correction raised the doubt about reliability of pQCD and inherently indicating the need for the second order corrections. A plethora of marvelous works put the current accuracy at NNLO level, both for the integrated cross section [32–35] and rapidity distribution [155, 156]. While the NNLO corrections turned out to be small, to check the consistency in the perturbative series, higher order corrections are of much need. A large abundance of events collected at the LHC RUN-II, combined with a very precise theoretical determination of the process, can be a very powerful test of perturbative QCD. While computing exact N^3LO is still non-achievable, to go beyond NNLO, there have been several attempts, starting with resumming the large threshold logarithms to obtain accuracy at next-to-next-to leading log (NNLL) level in [142]. Partial results were obtained for threshold approximated cross section at N^3LO in [59–63], where the term proportional to $\delta(1-z)$ was missing.

Alike the DY process, one other process with such importance is the production of Higgs boson, the recent discovery of which by the ATLAS [18] and CMS [19] collaborations has put the SM in firm footing. Theoretical advances played an important role in this discovery. While first order corrections [36–39] were known for long, it took almost ten years to finally achieve the next order [40–44] corrections. Later, the resummed result at NNLL QCD corrections [58] supplemented with two-loop electroweak effects [157, 158] also were crucial. In [59–63], partial threshold approximated cross section was obtained also for the Higgs boson production in gluon fusion, excluding the $\delta(1-z)$ contribution.

Note that the finite, mass factorized threshold contribution to the inclusive production cross section is expanded in terms of $\delta(1-z)$ and $\mathcal{D}_i(z)$. The $\delta(1-z)$ part of N^3LO threshold contribution for both the processes was not known until recently because of the unavailability of the complete soft contributions arising from real emissions, while the exact two [148] and three-loop [145, 147, 149, 151] quark and gluon form factors and NNLO soft contributions [159] to all orders in ϵ are already known. The recent computation by Anastasiou et al [66] considering all these soft effects from gluon radiations, provides the full threshold N^3LO result for the Higgs boson production in gluon fusion. In this chapter, we investigate the impact of these soft gluon contributions on $\delta(1-z)$ part of the N^3LO

to DY production. This enables to compute the full threshold approximated N^{3}LO cross section for DY production.

4.2 The Drell-Yan formalism

In this section, we present the Drell-Yan formalism. We will start with defining the process as

$$H_1 + H_2 \rightarrow V + X \rightarrow l_1 + l_2 + X \tag{4.1}$$

where H_1 and H_2 are the incoming hadrons and V is a on-shell vector boson (γ^* or Z), which decays into a pair of lepton (l_1, l_2). In Figure 4.1, we have depicted the relevant kinematics. The corresponding equation for the parton model Eq. 2.19 takes the following

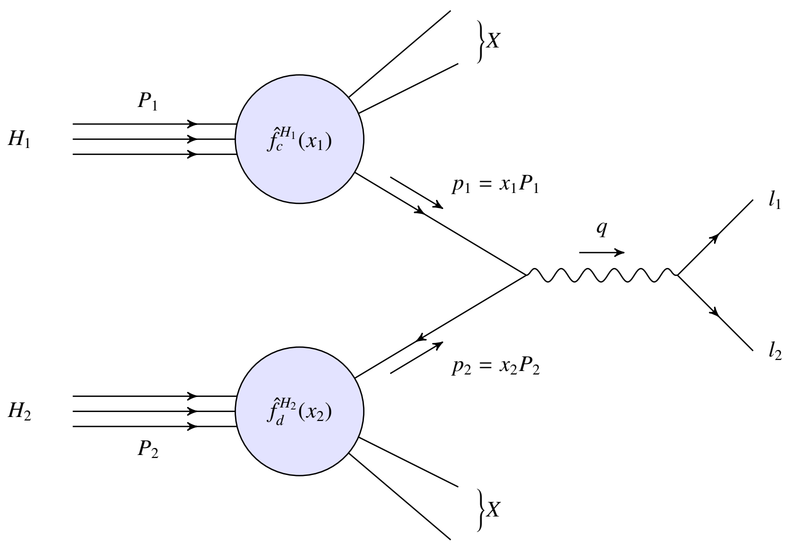


Figure 4.1: A schematic representation for Drell-Yan process.

form for DY production

$$2S \, \frac{d\sigma^{\gamma^*,q}}{dQ^2}(\tau, Q^2) = \sum_{cd} \hat{f}_c^{H_1} \otimes \hat{f}_d^{H_2} \otimes 2\hat{s} \, \frac{d\hat{\sigma}_{cd}^{\gamma^*,q}}{dQ^2} \tag{4.2}$$

where the notation followed are mentioned earlier. q is the momentum of the intermediate vector boson and $Q^2 = q^2$. $2\hat{s} \, \frac{d\hat{\sigma}_{cd}^{\gamma^*,q}}{dQ^2}$ denotes the partonic cross section. As the pair of leptons come only from the decay of the intermediate vector boson, the leptonic contribution to $2\hat{s}\,\frac{d\hat{\sigma}^{\gamma^*,q}_{cd}}{dQ^2}$ can be distinguished with the vector boson propagator as

$$2\hat{s}\,\frac{d\sigma^{\gamma^*,q}_{cd}}{dQ^2} = \frac{1}{2\pi}\sum_{jj'=\gamma,Z}\int dPS_{m+1}|M_H^{cd\to jj'}|^2 .\, P_j(q) \,.\, P^*_{j'}(q) \,.\, L^{jj'\to l^+l^-}(q) \tag{4.3}$$

In the above equation $|M_H^{cd\to jj'}|^2$ represents contribution from hadrons while L indicates the leptonic part. The “dot product" symbol in the above equation represents sum over Lorentz indices between the propagators and matrix element squared. The $m+1$ body phase space is defined as

$$\int dPS_{m+1} = \int\prod_i^m\left(\frac{d^np_i}{(2\pi)^n}2\pi\delta^+(p_i^2)\right)\left(\frac{d^nq}{(2\pi)^n}2\pi\delta^+(q^2-Q^2)\right) \times(2\pi)^n\delta^{(n)}\left(p_1+p_2-q-\sum_i^m k_i\right) \tag{4.4}$$

k_i's are momenta, carried by initial state parton emissions. j/j' denotes the vector boson and specifically $jj'=\gamma Z$ indicates the interference term. The propagators read

$$P_\gamma(q) = -\frac{i}{Q^2}g_{\mu\nu} \equiv g_{\mu\nu}\tilde{P}_\gamma(Q^2), \tag{4.5}$$

$$P_Z(q) = -\frac{i}{(Q^2-M_Z^2-iM_Z\Gamma_Z)}g_{\mu\nu} \equiv g_{\mu\nu}\tilde{P}_Z(Q^2), \tag{4.6}$$

where M_Z is mass of the Z boson. The leptonic tensor $L^{jj'\to l^+l^-}(q)$ involves the decay $\gamma/Z \to l^+ + l^-$, and as a result $L^{jj'\to l^+l^-}(q) = (-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2})L^{jj'}(Q^2)$. Below, we enlist $L^{jj'}(Q^2)$ for $jj'=\gamma\gamma,\gamma Z, ZZ$

$$L^{\gamma\gamma}(Q^2) = Q^2\frac{2\alpha}{3} \tag{4.7}$$

$$L^{\gamma Z}(Q^2) = -Q^2\frac{2\alpha g_e^V}{3c_Ws_W} \tag{4.8}$$

$$L^{ZZ}(Q^2) = Q^2\frac{2\alpha}{3c_W^2s_W^2}({g_e^V}^2+{g_e^A}^2) \tag{4.9}$$

In this above set of equations, α is fine structure constant, $c_W \equiv \cos\theta_W$, $s_W \equiv \sin\theta_W$ with θ_W being the weak mixing angle. g_f^V and g_f^A are defined as

$$g_f^V = \frac{1}{2}T_f^3 - s_W^2 Q_f\,, \qquad g_f^A = -\frac{1}{2}T_f^3 \tag{4.10}$$

with Q_f being the charge and T_f^3 being the isospin of the fermion. Finally, Eq. 4.2 takes the form

$$2S\,\frac{d\sigma^{\gamma^*,q}}{dQ^2}(\tau,Q^2) = \frac{1}{2\pi}\sum_{jj'=\gamma,Z}\tilde{P}_j(Q^2)\,\tilde{P}^*_{j'}(Q^2)\,L^{jj'}(q)\sum_{cd}\hat{f}_c^{H_1}\otimes\hat{f}_d^{H_2}\otimes\hat{\sigma}_{cd}^{jj'} \tag{4.11}$$

where

$$\hat{\sigma}_{cd}^{jj'} = \int dPS_{m+1}|M_H^{cd\to jj'}|^2\Big(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\Big)\,. \tag{4.12}$$

We will casually ignore the leptonic part from now and apply the mass factorization to the above equation to obtain

$$2S\,\frac{d\sigma^{\gamma^*,q}}{dQ^2}(\tau,Q^2) = \sum_{cd} f_c^{H_1}\otimes f_d^{H_2}\otimes\Delta_{cd}^{\gamma^*,q} \tag{4.13}$$

Our aim is to compute the full threshold only component of $\Delta_{cd}^{\gamma^*,q}$ *i.e.* $\Delta_{SV}^{\gamma^*,q}$

4.3 Computation of complete threshold contribution

As discussed in chapter 3, the threshold component $\Delta_{SV}^{\gamma^*,q}$ for DY can be expanded in a perturbative series of a_s as

$$\Delta_{SV}^{\gamma^*,q}(z) = \sum_{n=0}^{\infty} a_s^n(\mu_R^2)\Delta_{SV}^{\gamma^*,q,n}(z,\mu_R^2) \quad \text{where}$$
$$\Delta_{SV}^{\gamma^*,q,n}(z,\mu_R^2) = \Delta_{SV}^{\gamma^*,q,n}(\mu_R^2)|_\delta\,\delta(1-z) + \sum_{j=0}^{2n-1}\Delta_{SV}^{\gamma^*,q,n}(\mu_R^2)|_{\mathcal{D}_j}\mathcal{D}_j\,. \tag{4.14}$$

The availability of all the anomalous dimensions: UV ($\gamma_n^{\gamma^*,q}$), cusp (A_n^q), collinear (B_n^q) and soft (f_n^q), components of the form factors ($g_n^{\gamma^*,q,k}$) up to three-loop level *i.e.* $n = 3$, and components of the soft distribution functions ($\overline{\mathcal{G}}_n^{\gamma^*,q,k}$) up to two-loop level *i.e.* $n = 2$, enabled the authors to obtain $\Delta_{SV}^{\gamma^*,q,3}|_{\mathcal{D}_i}$ at N^{3}LO level in [59–63]. As from Eq. 3.44, evidently, the missing ingredient to compute $\Delta_{SV}^{\gamma^*,q,n}(\mu_R^2)|_\delta$ is $\overline{\mathcal{G}}_3^{\gamma^*,q,1}$. In [66], Anastasiou *et al.* have presented the cross section for the threshold production of the Higgs boson in gluon fusion *i.e.* specifically they have obtained $\Delta_{SV}^{H,g,3}|_\delta$ through exact computation. Moreover, the corresponding elements $\gamma_n^{H,g}$), A_n^g, B_n^g, f_n^g and $g_n^{H,g,k}$ were known up to required accuracy. We use them along with Eq. 3.44 to extract the three-loop component, $\overline{\mathcal{G}}_3^{H,g,1}$, of the soft distribution functions. We find that $\overline{\mathcal{G}}_3^{H,g,1}$, as expected, has a overall Casimir of adjoint representation of SU(N). In following, we present the result for $\overline{\mathcal{G}}_3^{I,i,1}$.

$$
\begin{aligned}
\overline{\mathcal{G}}_3^{I,i,1} &= \boldsymbol{C_i C_A}^2 \Big(\frac{152}{63}\,\zeta_2{}^3 + \frac{1964}{9}\,\zeta_2{}^2 + \frac{11000}{9}\,\zeta_2\zeta_3 - \frac{765127}{486}\,\zeta_2 + \frac{536}{3}\,\zeta_3{}^2 \\
&- \frac{59648}{27}\,\zeta_3 - \frac{1430}{3}\,\zeta_5 + \frac{7135981}{8748}\Big) + \boldsymbol{C_i C_A n_f}\Big(-\frac{532}{9}\,\zeta_2{}^2 - \frac{1208}{9}\,\zeta_2\zeta_3 \\
&+ \frac{105059}{243}\,\zeta_2 + \frac{45956}{81}\,\zeta_3 + \frac{148}{3}\,\zeta_5 - \frac{716509}{4374}\Big) \\
&+ \boldsymbol{C_i C_F n_f}\Big(\frac{152}{15}\,\zeta_2{}^2 - 88\,\zeta_2\zeta_3 + \frac{605}{6}\,\zeta_2 + \frac{2536}{27}\,\zeta_3 + \frac{112}{3}\,\zeta_5 - \frac{42727}{324}\Big) \\
&+ \boldsymbol{C_i n_f}^2\Big(\frac{32}{9}\,\zeta_2{}^2 - \frac{1996}{81}\,\zeta_2 - \frac{2720}{81}\,\zeta_3 + \frac{11584}{2187}\Big)
\end{aligned}
\tag{4.15}
$$

and again, n_f is the number of light flavors and $\boldsymbol{C_i} \equiv \boldsymbol{C_A}, \boldsymbol{C_F}$ for $i = g, q$ respectively. Finally, supplemented with the findings of Chapter 3, the following equation

$$
\overline{\mathcal{G}}_n^{\gamma^*,q,k} = \frac{\boldsymbol{C_F}}{\boldsymbol{C_A}} \overline{\mathcal{G}}_n^{H,g,k}, \tag{4.16}
$$

we determine the corresponding three-loop component, $\overline{\mathcal{G}}_3^{\gamma^*,q,1}$, of the soft distribution functions for the DY production and thus $\Delta_{SV}^{\gamma^*,q,3}|_\delta$. This completes the evaluation of full DY SV contributions at N^{3}LO level. The $\Delta_{SV}^{\gamma^*,q,3}|_\delta$ is presented below

$$
\Delta_{SV}^{\gamma^*,q,3}|_\delta = \boldsymbol{C_A^2 C_F}\bigg(\frac{13264}{315}\,\zeta_2{}^3 + \frac{14611}{135}\,\zeta_2{}^2 - \frac{884}{3}\,\zeta_2\zeta_3 + 843\,\zeta_2 - \frac{400}{3}\,\zeta_3{}^2
$$

$$
\begin{aligned}
&+ \frac{82385}{81}\,\zeta_3 - 204\,\zeta_5 - \frac{1505881}{972}\Big) + \boldsymbol{C_A}C_F^2\Big(-\frac{20816}{315}\,{\zeta_2}^3 - \frac{1664}{135}\,{\zeta_2}^2 \\
&+ \frac{28736}{9}\,\zeta_2\zeta_3 - \frac{13186}{27}\,\zeta_2 + \frac{3280}{3}\,{\zeta_3}^2 - \frac{20156}{9}\,\zeta_3 - \frac{39304}{9}\,\zeta_5 + \frac{74321}{36}\Big) \\
&+ \boldsymbol{C_A}C_F n_f\Big(-\frac{5756}{135}\,{\zeta_2}^2 + \frac{208}{3}\,\zeta_2\zeta_3 - \frac{28132}{81}\,\zeta_2 - \frac{6016}{81}\,\zeta_3 - 8\,\zeta_5 \\
&+ \frac{110651}{243}\Big) + \boldsymbol{C_F^3}\Big(-\frac{184736}{315}\,{\zeta_2}^3 + \frac{412}{5}\,{\zeta_2}^2 + 80\,\zeta_2\zeta_3 - \frac{130}{3}\,\zeta_2 \\
&+ \frac{10336}{3}\,{\zeta_3}^2 - 460\,\zeta_3 + 1328\,\zeta_5 - \frac{5599}{6}\Big) \\
&+ \boldsymbol{C_F^2}n_f\Big(\frac{272}{135}\,{\zeta_2}^2 - \frac{5504}{9}\,\zeta_2\zeta_3 + \frac{2632}{27}\,\zeta_2 + \frac{3512}{9}\,\zeta_3 + \frac{5536}{9}\,\zeta_5 - \frac{421}{3}\Big) \\
&+ \boldsymbol{C_F}n_{f,v}\Big(\frac{\boldsymbol{N}^2-4}{\boldsymbol{N}}\Big)\Big(-\frac{4}{5}\,{\zeta_2}^2 + 20\,\zeta_2 + \frac{28}{3}\,\zeta_3 - \frac{160}{3}\,\zeta_5 + 8\Big) \\
&+ \boldsymbol{C_F}n_f^2\Big(\frac{128}{27}\,{\zeta_2}^2 + \frac{2416}{81}\,\zeta_2 - \frac{1264}{81}\,\zeta_3 - \frac{7081}{243}\Big)
\end{aligned}
\tag{4.17}
$$

where, $\boldsymbol{n}_{f,v}$ is proportional to the charge weighted sum of the quark flavors [151]. With the newly obtained result, we present the complete threshold approximated N^{3}LO result in the following

$$
\begin{aligned}
\Delta_{SV}^{\gamma^*,q,1} &= \delta(1-z)\Big\{\boldsymbol{C}_F(-16+8\zeta_2)\Big\} + \mathcal{D}_1\Big\{\boldsymbol{C}_F(16)\Big\} \\
\Delta_{SV}^{\gamma^*,q,2} &= \delta(1-z)\Big\{\boldsymbol{C}_F C_A\Big(-\frac{1535}{12} + 28\zeta_3 + \frac{592}{9}\zeta_2 - \frac{12}{5}\zeta_2^2\Big) + \boldsymbol{C}_F^2\Big(\frac{511}{4} - 60\zeta_3 \\
&\quad - 70\zeta_2 + \frac{8}{5}\zeta_2^2\Big) + \boldsymbol{n}_f C_F\Big(\frac{127}{6} + 8\zeta_3 - \frac{112}{9}\zeta_2\Big)\Big\} + \mathcal{D}_0\Big\{\boldsymbol{C}_F C_A\Big(-\frac{1616}{27} \\
&\quad + 56\zeta_3 + \frac{176}{3}\zeta_2\Big) + \boldsymbol{C}_F^2\Big(256\zeta_3\Big) + \boldsymbol{n}_f C_F\Big(\frac{224}{27} - \frac{32}{3}\zeta_2\Big)\Big\} \\
&\quad + \mathcal{D}_1\Big\{\boldsymbol{C}_F C_A\Big(\frac{1072}{9} - 32\zeta_2\Big) + \boldsymbol{C}_F^2\Big(-256 - 128\zeta_2\Big) + \boldsymbol{n}_f C_F\Big(-\frac{160}{9}\Big)\Big\} \\
&\quad + \mathcal{D}_2\Big\{\boldsymbol{C}_F C_A\Big(-\frac{176}{3}\Big) + \boldsymbol{n}_f C_F\Big(\frac{32}{3}\Big)\Big\} + \mathcal{D}_3 \boldsymbol{C}_F^2\Big(128\Big) \\
\Delta_{SV}^{\gamma^*,q,3} &= \delta(1-z)\Big\{\boldsymbol{C}_F n_{f,v}\Big(\frac{\boldsymbol{N}^2-4}{\boldsymbol{N}}\Big)\Big(8 - \frac{160}{3}\zeta_5 + \frac{28}{3}\zeta_3 + 20\zeta_2 - \frac{4}{5}\zeta_2^2\Big) \\
&\quad + \boldsymbol{C}_F C_A^2\Big(-\frac{1505881}{972} - 204\zeta_5 + \frac{82385}{81}\zeta_3 - \frac{400}{3}\zeta_3^2 + 843\zeta_2 - \frac{884}{3}\zeta_2\zeta_3 \\
&\quad + \frac{14611}{135}\zeta_2^2 + \frac{13264}{315}\zeta_2^3\Big) + \boldsymbol{C}_F^2 C_A\Big(\frac{74321}{36} - \frac{39304}{9}\zeta_5 - \frac{20156}{9}\zeta_3 + \frac{3280}{3}\zeta_3^2 \\
&\quad - \frac{13186}{27}\zeta_2 + \frac{28736}{9}\zeta_2\zeta_3 - \frac{1664}{135}\zeta_2^2 - \frac{20816}{315}\zeta_2^3\Big) + \boldsymbol{C}_F^3\Big(-\frac{5599}{6} + 1328\zeta_5 \\
&\quad - 460\zeta_3 + \frac{10336}{3}\zeta_3^2 - \frac{130}{3}\zeta_2 + 80\zeta_2\zeta_3 + \frac{412}{5}\zeta_2^2 - \frac{184736}{315}\zeta_2^3\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ n_f C_F C_A\Big(\frac{110651}{243} - 8\zeta_5 - \frac{6016}{81}\zeta_3 - \frac{28132}{81}\zeta_2 + \frac{208}{3}\zeta_2\zeta_3 - \frac{5756}{135}\zeta_2^2\Big)\\
&+ n_f C_F^2\Big(- \frac{421}{3} + \frac{5536}{9}\zeta_5 + \frac{3512}{9}\zeta_3 + \frac{2632}{27}\zeta_2 - \frac{5504}{9}\zeta_2\zeta_3 + \frac{272}{135}\zeta_2^2\Big)\\
&+ n_f^2 C_F\Big(- \frac{7081}{243} - \frac{1264}{81}\zeta_3 + \frac{2416}{81}\zeta_2 + \frac{128}{27}\zeta_2^2\Big)\Big\}\\
&+ \mathcal{D}_0\Big\{C_F C_A^2\Big(- \frac{594058}{729} - 384\zeta_5 + \frac{40144}{27}\zeta_3 + \frac{98224}{81}\zeta_2 - \frac{352}{3}\zeta_2\zeta_3\\
&- \frac{2992}{15}\zeta_2^2\Big) + C_F^2 C_A\Big(\frac{25856}{27} + \frac{26240}{9}\zeta_3 - \frac{12416}{27}\zeta_2 - 1472\zeta_2\zeta_3 + \frac{1408}{3}\zeta_2^2\Big)\\
&+ C_F^3\Big(12288\zeta_5 - 4096\zeta_3 - 6144\zeta_2\zeta_3\Big) + n_f C_F C_A\Big(\frac{125252}{729} - \frac{2480}{9}\zeta_3\\
&- \frac{29392}{81}\zeta_2 + \frac{736}{15}\zeta_2^2\Big) + n_f C_F^2\Big(- 6 - \frac{5728}{9}\zeta_3 + \frac{1952}{27}\zeta_2 - \frac{1472}{15}\zeta_2^2\Big)\\
&+ n_f^2 C_F\Big(- \frac{3712}{729} + \frac{320}{27}\zeta_3 + \frac{640}{27}\zeta_2\Big)\Big\} + \mathcal{D}_1\Big\{C_F C_A^2\Big(\frac{124024}{81} - 704\zeta_3\\
&- \frac{12032}{9}\zeta_2 + \frac{704}{5}\zeta_2^2\Big) + C_F^2 C_A\Big(- \frac{35572}{9} - 5184\zeta_3 - \frac{11648}{9}\zeta_2 + \frac{3648}{5}\zeta_2^2\Big)\\
&+ C_F^3\Big(2044 - 960\zeta_3 + 2976\zeta_2 - \frac{14208}{5}\zeta_2^2\Big) + n_f C_F C_A\Big(- \frac{32816}{81} + 384\zeta_2\Big)\\
&+ n_f C_F^2\Big(\frac{4288}{9} + 1280\zeta_3 + \frac{2048}{9}\zeta_2\Big) + n_f^2 C_F\Big(\frac{1600}{81} - \frac{256}{9}\zeta_2\Big)\Big\}\\
&+ \mathcal{D}_2\Big\{C_F C_A^2\Big(- \frac{28480}{27} + \frac{704}{3}\zeta_2\Big) + C_F^2 C_A\Big(- \frac{4480}{9} + 1344\zeta_3 + \frac{11264}{3}\zeta_2\Big)\\
&+ C_F^3\Big(10240\zeta_3\Big) + n_f C_F C_A\Big(\frac{9248}{27} - \frac{128}{3}\zeta_2\Big) + n_f C_F^2\Big(\frac{544}{9} - \frac{2048}{3}\zeta_2\Big)\\
&+ n_f^2 C_F\Big(- \frac{640}{27}\Big)\Big\} + \mathcal{D}_3\Big\{C_F C_A^2\Big(\frac{7744}{27}\Big) + C_F^2 C_A\Big(\frac{17152}{9} - 512\zeta_2\Big)\\
&+ C_F^3\Big(- 2048 - 3072\zeta_2\Big) + n_f C_F C_A\Big(- \frac{2816}{27}\Big) + n_f C_F^2\Big(- \frac{2560}{9}\Big)\\
&+ n_f^2 C_F\Big(\frac{256}{27}\Big)\Big\} + \mathcal{D}_4\Big\{C_F^2 C_A\Big(- \frac{7040}{9}\Big) + n_f C_F^2\Big(\frac{1280}{9}\Big)\Big\}\\
&+ \mathcal{D}_5\Big\{C_F^3\Big(512\Big)\Big\}
\end{aligned}
\tag{4.18}
$$

The new information on $\overline{\mathcal{G}}_3^{I,i,1}$ also enables us to obtain coefficients of $\mathcal{D}_1$ term at N^{4}LO for the Higgs boson production in gluon fusion as well as DY production

$$
\begin{aligned}
\Delta_{SV}^{H,g,4}|_{\mathcal{D}_1} = 4A_4^g &+ C_A{}^4\Big(- \frac{925312}{15}\zeta_2{}^3 - \frac{6818912}{135}\zeta_2{}^2 + 504768\,\zeta_2\zeta_3 - \frac{8201024}{81}\zeta_2\\
&+ \frac{1375936}{3}\zeta_3{}^2 - \frac{2768224}{9}\zeta_3 - \frac{5545408}{9}\zeta_5 + \frac{56712956}{729}\Big)\\
&+ C_A{}^3 n_f\Big(\frac{1063424}{135}\zeta_2{}^2 - 87168\,\zeta_2\zeta_3 + \frac{3016096}{81}\zeta_2 + \frac{227296}{3}\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1031296}{9}\,\zeta_5 - \frac{24352976}{729}\Big) + \boldsymbol{C_A}^2 C_F n_f \Big(\frac{15488}{45}\,{\zeta_2}^2 - 4608\,\zeta_2\zeta_3 \\
&+ \frac{81952}{9}\,\zeta_2 + 13568\,\zeta_3 + 2560\,\zeta_5 - \frac{1503964}{81}\Big) + \boldsymbol{C_A}^2 {n_f}^2 \Big(- \frac{128}{3}\,{\zeta_2}^2 \\
&- \frac{285248}{81}\,\zeta_2 - \frac{116672}{27}\,\zeta_3 + \frac{2145568}{729}\Big) + \boldsymbol{C_A} {C_F}^2 n_f \Big(\frac{9472}{3}\,\zeta_3 - 5120\,\zeta_5 \\
&+ \frac{9728}{9}\Big) + \boldsymbol{C_A} C_F {n_f}^2 \Big(- \frac{2816}{45}\,{\zeta_2}^2 - \frac{4864}{9}\,\zeta_2 - \frac{5248}{3}\,\zeta_3 + \frac{221992}{81}\Big) \\
&+ \boldsymbol{C_A} {n_f}^3 \Big(\frac{2560}{27}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{14848}{729}\Big)
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
\Delta^{\gamma^*,q,4}_{SV}|_{\mathcal{D}_1} &= 4A^q_4 + \boldsymbol{C_A}^3 C_F \Big(\frac{65824}{15}\,{\zeta_2}^2 + \frac{7744}{3}\,\zeta_2\zeta_3 - \frac{2376352}{81}\,\zeta_2 - \frac{951712}{27}\,\zeta_3 \\
&+ 8448\,\zeta_5 + \frac{15047260}{729}\Big) + \boldsymbol{C_A}^2 {C_F}^2 \Big(- \frac{1150592}{315}\,{\zeta_2}^3 + \frac{2086768}{135}\,{\zeta_2}^2 \\
&+ \frac{174656}{3}\,\zeta_2\zeta_3 - \frac{1472104}{81}\,\zeta_2 + \frac{3008}{3}\,{\zeta_3}^2 - \frac{9640624}{81}\,\zeta_3 - 3264\,\zeta_5 \\
&- \frac{44425832}{729}\Big) + \boldsymbol{C_A}^2 C_F n_f \Big(- \frac{5632}{3}\,{\zeta_2}^2 - \frac{1408}{3}\,\zeta_2\zeta_3 + \frac{1142048}{81}\,\zeta_2 \\
&+ \frac{344416}{27}\,\zeta_3 - 1536\,\zeta_5 - \frac{1995904}{243}\Big) + \boldsymbol{C_A} {C_F}^3 \Big(\frac{794624}{63}\,{\zeta_2}^3 \\
&- \frac{10769984}{135}\,{\zeta_2}^2 + \frac{3791744}{9}\,\zeta_2\zeta_3 + \frac{1984280}{27}\,\zeta_2 + \frac{224512}{3}\,{\zeta_3}^2 \\
&- \frac{382208}{27}\,\zeta_3 - \frac{5765248}{9}\,\zeta_5 + 48248\Big) + \boldsymbol{C_A} {C_F}^2 n_f \Big(- \frac{344768}{135}\,{\zeta_2}^2 \\
&- 7680\,\zeta_2\zeta_3 + \frac{372784}{81}\,\zeta_2 + \frac{3960320}{81}\,\zeta_3 - 128\,\zeta_5 + \frac{10426804}{729}\Big) \\
&+ \boldsymbol{C_A} C_F {n_f}^2 \Big(\frac{2944}{15}\,{\zeta_2}^2 - \frac{171328}{81}\,\zeta_2 - \frac{36800}{27}\,\zeta_3 + \frac{221104}{243}\Big) \\
&+ \boldsymbol{C_F}^4 \Big(- \frac{4450816}{63}\,{\zeta_2}^3 + \frac{161728}{5}\,{\zeta_2}^2 + 16640\,\zeta_2\zeta_3 - \frac{100192}{3}\,\zeta_2 \\
&+ \frac{1148416}{3}\,{\zeta_3}^2 - 7360\,\zeta_3 + 21248\,\zeta_5 - \frac{44792}{3}\Big) + \boldsymbol{C_F}^3 n_f \Big(\frac{1413632}{135}\,{\zeta_2}^2 \\
&- \frac{748544}{9}\,\zeta_2\zeta_3 - \frac{221312}{27}\,\zeta_2 + \frac{12032}{27}\,\zeta_3 + \frac{1022464}{9}\,\zeta_5 - \frac{19528}{9}\Big) \\
&+ \boldsymbol{C_F}^2 n_{f,v} \Big(\frac{\boldsymbol{N}^2 - 4}{\boldsymbol{N}}\Big) \Big(- \frac{64}{5}\,{\zeta_2}^2 + 320\,\zeta_2 + \frac{448}{3}\,\zeta_3 - \frac{2560}{3}\,\zeta_5 + 128\Big) \\
&+ \boldsymbol{C_F}^2 {n_f}^2 \Big(\frac{256}{135}\,{\zeta_2}^2 - \frac{5504}{27}\,\zeta_2 - \frac{330112}{81}\,\zeta_3 - \frac{376472}{729}\Big) \\
&+ \boldsymbol{C_F} {n_f}^3 \Big(\frac{2560}{27}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{14848}{729}\Big).
\end{aligned}
\tag{4.20}
$$

We have set $\mu_R^2 = \mu_F^2 = q^2$ and their dependence can be retrieved using appropriate renormalization group equation. The four-loop cusp anomalous dimension A_4^q has been obtained numerically by employing the [1/1] Padé estimate by Moch et al in [160]

$$A_4^q = 7849, 4313, 1553 \qquad \text{for } n_f = 3, 4, 5 \tag{4.21}$$

Considering the validity of Casimir scaling relation at four loop level for the cusp anomalous dimension

$$A_4^g = 17660, 9704, 3494 \qquad \text{for } n_f = 3, 4, 5\,. \tag{4.22}$$

4.4 Numerical analysis

We present the contribution from $\Delta^{sv}_{q,3}|_\delta$ to pure N^{3}LO$_{sv}$ as $\delta_{\text{N}^3\text{LO}}$ and the contributions from $\Delta^{sv}_{q,3}|_{\mathcal{D}_i}$s to pure N^{3}LO$_{sv}$ as $\mathcal{D}_{\text{N}^3\text{LO}}$ in Table 4.1 for different invariant masses ($m_{l^+l^-} \equiv Q$) of the di-leptons. The corresponding parameters, throughout we have used for this numerical study, are $\sqrt{S} = 14$ TeV for the LHC, number of light quark flavors $n_f = 5$, Fermi constant $G_F = 4541.68$ pb, the Z boson mass $m_Z = 91.1876$ GeV and top quark mass $m_t = 173.4$ GeV. The strong coupling constant $\alpha_s(\mu_R^2)$ is evolved using the 4-loop renormalization group equations with $\alpha_s^{\text{N}^3\text{LO}}(m_Z) = 0.117$ and for PDFs we have used MSTW 2008NNLO [161].

Q	$10^3\delta_{\text{N}^3\text{LO}}$	$10^3\mathcal{D}_{\text{N}^3\text{LO}}$	NNLO (sv)	NNLO	N^{3}LO (sv)	N^{3}LO$_{sv}$
30	11.386	-8.397	0.497	0.543	0.500	0.546
50	2.561	-2.053	0.147	0.158	0.148	0.158
70	1.724	-1.466	0.117	0.124	0.118	0.124
90	140.114	-124.493	10.749	11.296	10.765	11.311
100	5.410	-4.865	0.436	0.458	0.436	0.459
200	$4.567\ 10^{-2}$	$-4.421\ 10^{-2}$	$4.917\ 10^{-3}$	$5.233\ 10^{-3}$	$4.918\ 10^{-3}$	$5.234\ 10^{-3}$
400	$3.153\ 10^{-3}$	$-3.368\ 10^{-3}$	$4.364\ 10^{-4}$	$4.694\ 10^{-4}$	$4.362\ 10^{-4}$	$4.692\ 10^{-4}$
600	$6.473\ 10^{-4}$	$-7.455\ 10^{-4}$	$1.032\ 10^{-4}$	$1.116\ 10^{-4}$	$1.032\ 10^{-4}$	$1.116\ 10^{-4}$
800	$2.006\ 10^{-4}$	$-2.456\ 10^{-4}$	$3.538\ 10^{-5}$	$3.836\ 10^{-5}$	$3.534\ 10^{-5}$	$3.832\ 10^{-5}$
10^3	$7.755\ 10^{-5}$	$-9.959\ 10^{-5}$	$1.480\ 10^{-5}$	$1.607\ 10^{-5}$	$1.478\ 10^{-5}$	$1.605\ 10^{-5}$

Table 4.1: Contributions of $\delta_{\text{N}^3\text{LO}}, \mathcal{D}_{\text{N}^3\text{LO}}$, NNLO ($sv$), exact NNLO, N^{3}LO ($sv$) and N^{3}LO$_{sv}$

We find that the δ contribution is almost equal and opposite in sign to sum of the contributions from the $\mathcal{D}_i$'s. Hence adding the δ part reduces the pure $\mathrm{N^3LO}_{sv}$ term to one order in magnitude, establishing the dominance of the δ term. We have studied the effect of threshold corrections resulting from distributions such as $\delta(1-z)$ and $\mathcal{D}_i$ both at NLO as well as NNLO levels. In the following, we report our findings based on the numerical analysis presented in Table 4.1 for two different ranges of Q, namely $Q = 200-1000$ GeV (above m_Z) and $30-100$ GeV (below m_Z). At NLO, if we keep only the distributions and drop contributions from hard radiations coming from $q\overline{q}$ and $q(\overline{q})g$ initiated processes, we find that the resulting NLO corrected cross section is about 95% of the exact result at NLO level. Similarly, if we keep the distributions and drop all the hard radiations both in NLO as well as in NNLO terms, we find that resulting NNLO corrected result (NNLO(sv)) is about 95% of the exact one at NNLO level. Hence, it is expected that the sum ($\mathrm{N^3LO}_{sv}$) of threshold contributions of $\mathrm{N^3LO}$ terms and the exact NNLO corrected result would constitute the dominant contribution at $\mathrm{N^3LO}$ level. Like NNLO, the threshold contributions in $\mathrm{N^3LO}$ terms are also moderate and hence the perturbation theory behaves well.

In Figure 4.2, we have plotted $Q^2 \frac{d\sigma}{dQ^2}$ against Q resulting from LO, NLO, NNLO and $\mathrm{N^3LO}_{sv}$. We note that like NNLO, $\mathrm{N^3LO}_{sv}$ does not change cross section significantly.

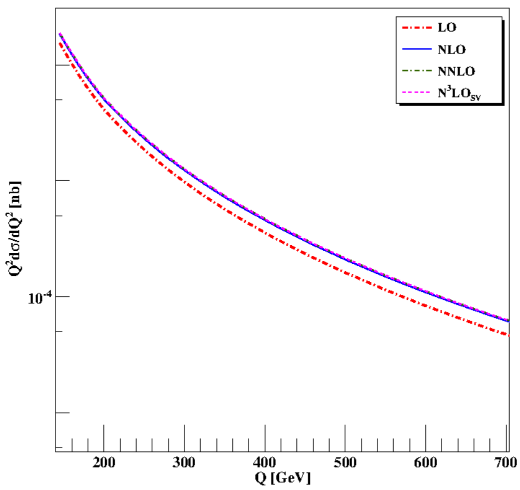


Figure 4.2: Total cross section for DY production at LHC.

In Figure 4.3, we have shown the dependence of our result on renormalization scale at various orders in perturbation theory. We have plotted $R^{(i)} = \sigma^{i(\mu_R)}/\sigma^{i(Q)}$ where i = NLO, NNLO, N^3LO_{sv} versus $\frac{\mu_R}{Q}$ and the reduction in the scale dependence is evident as we increase the order in the perturbative expansion.

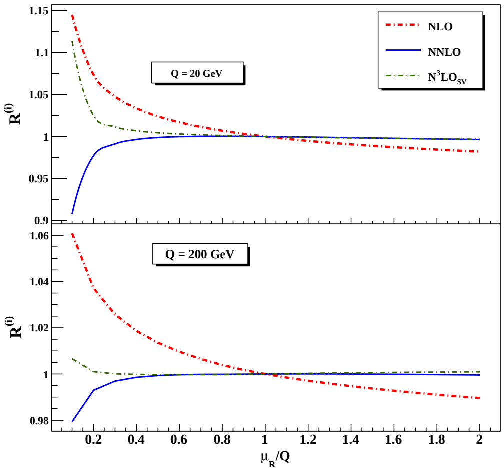


Figure 4.3: Renormalization scale (μ_R) variation for DY production at LHC.

4.5 Conclusions

To summarize, systematically we have obtained threshold corrected inclusive cross sections for DY production up to N^3LO level in pQCD. In process, we have exploited the universality of QCD amplitudes, namely factorization of soft and collinear divergences, renormalization group invariance and resummation of threshold contributions. We use the recent N^3LO SV contribution to production cross section of the Higgs boson in gluon fusion to obtain corresponding $\delta(1-z)$ part of DY production at N^3LO. The numerical study of the acquired results establish the importance of the δ term. We find that the impact of the δ contribution is quite large to the pure N^3LO_{sv} corrections. We have also demonstrated the dominance of threshold corrections at every order in perturbation theory. The results presented in this Chapter will not only be a benchmark for the full N^3LO contribution but also an important step in the precision study with Drell-Yan process at the LHC.

5 Spin-2 Form Factors at Three Loop in QCD

In this Chapter we present the three-loop QCD corrections to the spin-2 quark-antiquark and spin-2 gluon-gluon form factors in SU(N) gauge theory with n_f light flavors. These form factors contribute to both quark-antiquark and gluon-gluon initiated processes involving spin-2 particle in the hadronic reactions at the LHC. The detailed study of the structure of IR singularities in these form factors up to three-loop level using Sudakov integro-differential equation reveals that the anomalous dimensions originating from soft and collinear regions of the loop integrals coincide with those of the electro-weak vector boson and Higgs form factors confirming the universality of the IR singularities in QCD amplitudes.

5.1 Introduction

In the context of the recent discovery of the new boson at the LHC, with mass of about 125 GeV [18, 19], there has been renewed interest in massive spin-2 resonance which could also lead to similar final states [162]. The massive spin-2 could be a Kaluza-Klein (KK) graviton of the TeV scale gravity models [163–166] as a result of gravity propagating in the extra dimensional bulk or any generic spin-2 resonance in some other new physics scenarios. It was noted in [167] that gauge symmetry and Lorentz invariance forbid operators of dimension four that could lead to a coupling of a massive spin-2 resonance to a pair of the SM particles. Further, if the flavor and CP symmetries of the SM are respected by these new physics scenarios, the leading dimension five operator is none other than the energy momentum tensor $T_{\mu\nu}$ of the SM particles. The structure of the operator coupling thus being identical to the KK graviton, though the constant coefficients could be different for the KK graviton or any generic spin-2 imposter. Nonetheless, methods to distinguish KK graviton from the imposter have been proposed [167] and will be of importance for BSM searches at the LHC which is now operational at higher energies and luminosity. The increasing accuracy of the experimental data at the LHC Run-II, demands an equally precise theoretical predictions.

It is imperative that the search for a signal from BSM scenarios at the LHC requires same level of accuracy as the background processes. While there are significant amount of works to obtain results, as precise as possible, in the processes like the Drell-Yan production [34, 80, 168], the Higgs boson production in gluon fusion [42–44, 57, 66, 67, 69], the Higgs boson production in bottom quark annihilation [71, 81] and associated production of the Higgs with a vector boson [82, 169] at the LHC, it is imperative that competing BSM models are also available to the same accuracy. To this direction, one of the essential ingredients for QCD corrections are the form factors. The $q\bar{q} \to$ spin-2 and $gg \to$ spin-2 form factors act as a principal building block for phenomenological study and is at present available up to two-loop in QCD [52], while for many processes of interest they are now available to the three-loop order [145,147,149,151,154,170]. On the other hand, precise theoretical predictions for various observables up to NLO in QCD, put stringent bounds [171–173] on the parameters of ADD and RS models. Also, inclusion of higher order terms to the Born contributions, reduced the large uncertainties arising from renormalization and factorization scales, evidently. The NLO QCD predictions based on fixed order as well as parton shower improved in the MadGraph5_aMC@NLO [174] framework for di-final states [47, 49, 50, 175–180] productions in the gravity mediated models have already played crucial role in constraining the parameters of ADD and RS models. Also production of di-photon along with a jet through gravity mediated model was presented in [181] at NLO accuracy. In [53], NNLO corrections for the graviton production were obtained in the threshold limit. Recently some of us have computed the full NNLO QCD contributions [54] to the production of a graviton which decays to a pair of leptons. Graviton production along with a jet in gluon fusion has been computed in [182] to improve these predictions beyond NNLO. As these corrections are only sensitive to the tensorial interaction and not sensitive to the details of the model, these results are applicable to production of any generic spin-2 resonance. Hence, in this chapter, we take the first step towards going beyond NNLO for the resonant production of a generic spin-2 particle at the LHC, namely the computation of quark and gluon form factors at three-loop level in pQCD with n_f light flavors. We report the first results on the threshold effects at N^{3}LO in the next chapter and demonstrate the importance of such corrections at the LHC in the context of spin-2 resonance searches.

In addition to the phenomenological importance with respect to precise predictions of some observable, form factors in QCD are of considerable theoretical interest in terms of the factorization and universal nature of the singular structure. Studying the IR pole structure and factorization properties of these IR singularities in multi-loop QCD amplitudes with tensorial coupling to 3-loop order and to confirm the standard expectation of QCD amplitudes [183–186] is an essential prerequisite. The spin-2 field being a tensor of rank-2 is coupled to the energy-momentum tensor $T_{\mu\nu}$, which is a symmetric and conserved quantity. The operator $T_{\mu\nu}$ of QCD is finite [187], which would imply no UV renormalization is required. Further $T_{\mu\nu}$ consists of gauge invariant terms and in addition gauge dependent and ghost terms, we explicitly observe that to the three-loop order these form factors are independent of the gauge dependent and ghost terms [187, 188], which is an important check of the calculation. From a computational point of view three-loop amplitudes with higher tensorial coupling is being attempted for the first time. At the intermediate stages of the computation this leads to higher rank tensorial integrals resulting from more than 3000 three-loop Feynman amplitudes contributing to the gluon form factor alone. This computation again establishes the power of several state-of-the-art techniques namely IBP and LI identities.

In the next section 5.3, we describe the effective Lagrangian. In section 5.4, after defining the quark and gluon form factors, we present the computational details at three-loop level followed by the results. The details of UV renormalization and universal structure of IR poles are given in section 5.5 and section 5.6 respectively. In section 5.8, we study the universality of the leading transcendental (LT) terms in both the quark and gluon form factors by setting $C_A = C_F = N$ and $n_f = N$. Finally we conclude with our findings in section 5.9.

5.2 The LED models

A generic spin-2 can arise from extension of supersymmetric models or models in TeV scale gravity. We will concentrate in the second one, the gravity models with extra dimensions. These models were introduced mainly to address the hierarchy problem, the large difference between the electro-weak scale and Planck scale.

Based on the string theory ideas, the first models of its kind was introduced in 1998 by Arkani-Hamed, Dimopolous and Dvali, later named as ADD model [163–165]. This consists of the idea that gravity propagates in all dimensions while the SM is confined to a 3-dimensional brane. Another proposal came from Randall and Sundrum, namely the RS model [166], where a single extra-dimension was considered in an Anti-deSitter (AdS_5) metric.

5.2.1 ADD model

We start by reviewing the conventional Kaluza-Klein (KK) scenario. To begin with, we consider the space-time dimension $d = 4 + n$, where n, the extra spatial dimensions, are compactified over the scale R. The fundamental scale M_S, now, becomes different from the 4-dimensional Planck scale M_P and can be closer to the electro-weak scale, thus removing the hierarchy. M_P is no longer a relevant scale, rather can be related to M_S through

$$M_P^2 \sim M_S^{n+2} R^n \,. \tag{5.1}$$

On the other hand, the extra dimensions effects the basic electromagnetic theory, which is already tested to high precision. To save this, one confines the SM to a 3-dimensional brane, while gravity propagates in the d dimensions.

We will follow the formalism presented by Han, Lykken and Zhang (HLZ) [189]. The $d = 4 + n$ gravity is compactified on a n-dimensional torii with the scales set to R. To obtain the self-interaction terms, Fierz-Pauli Lagrangian is considered. Since, our interest lies in the phenomenological appearance of a generic spin 2, we do not discuss the details regarding self-interaction. The information to collect, is that from the four-dimensional perspective, the zero modes of the $(4 + n)$-dimensional graviton become the graviton, n massless $U(1)$ gauge bosons and $n(n + 1)/2$ massless scalar bosons, while the KK modes in each level re-organize themselves to a massive spin-2 particle, $(n - 1)$ massive vector bosons and $n(n - 1)/2$ massive scalar bosons. We denote the KK modes by $h^{\mu\nu}_{\vec{n}}$ where $\vec{n}$ is a d-dimensional vector with all positive components. The mass corresponding to each mode is given by

$$m^2_{\vec{n}} = \frac{4\pi^2 \vec{n}^2}{R^2} \,. \tag{5.2}$$

The corresponding interaction Lagrangian then takes the following form

$$\mathcal{L} = -\frac{\kappa}{2} \sum_{\vec{n}=0}^{\infty} T_{\mu\nu}(x)\, h^{\mu\nu}_{\vec{n}}(x) \,. \tag{5.3}$$

where $\kappa = \sqrt{16\pi G_N^{(4+n)}}$ with $G_N^{(4+n)}$ the Newton constant in $d = 4 + n$ dimension. $T_{\mu\nu}$ is the energy momentum tensor for the localized SM fields. Following HLZ [189], the propagator for the massive spin-2 states $h^{\mu\nu}_{\vec{n}}$ is

$$i\Delta^{\mu\nu;\rho\sigma}(q) = \frac{iB^{\mu\nu;\rho\sigma}(q)}{q^2 - m_{\vec{n}}^2 + i\epsilon} \tag{5.4}$$

where

$$\begin{aligned} B^{\mu\nu;\rho\sigma}(q) \quad = \quad & \left(g^{\mu\rho} - \frac{q^\mu q^\rho}{q.q}\right)\left(g^{\nu\sigma} - \frac{q^\nu q^\sigma}{q.q}\right) + \left(g^{\mu\sigma} - \frac{q^\mu q^\sigma}{q.q}\right)\left(g^{\nu\rho} - \frac{q^\nu q^\rho}{q.q}\right) \\ & - \frac{2}{d-1}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q.q}\right)\left(g^{\rho\sigma} - \frac{q^\rho q^\sigma}{q.q}\right) . \end{aligned} \tag{5.5}$$

with q being the spin-2 momentum. One can construct five polarization tensors and the polarization sum can be obtained as

$$\sum_s \varepsilon^s_{\mu\nu} \varepsilon^{s*}_{\rho\sigma} = B_{\mu\nu;\rho\sigma} . \tag{5.6}$$

Since the KK states are almost degenerate in mass, the effective propagator can be obtained after summing over the propagator with the following KK state density

$$\rho(m_{\vec{n}}) = \frac{R^n m_{\vec{n}}^{n-2}}{(4\pi)^{n/2}\Gamma(n/2)} , \tag{5.7}$$

and the effective graviton operator can be expressed as

$$i\Delta^{\mu\nu;\rho\sigma}(q) = \frac{i\mathcal{D}(Q^2) B^{\mu\nu;\rho\sigma}(q)}{q^2 - m_{\vec{n}}^2 + i\epsilon} \tag{5.8}$$

where

$$\begin{aligned} \mathcal{D}(Q^2) &= \sum_{\vec{n}} \frac{1}{Q^2 - m_{\vec{n}}^2 + i\epsilon} \\ &= \frac{Q^{n-2} R^n}{(4\pi)^{n/2}\Gamma(n/2)} \Big[- i\pi + 2I(\frac{M_S}{Q}) \Big] \end{aligned} \tag{5.9}$$

with

$$
\begin{aligned}
I(\frac{M_S}{Q}) &= -\sum_{k=1}^{n/2-1} \frac{1}{2k}\left(\frac{M_S}{Q}\right)^{2k} - \frac{1}{2}\log\left(\frac{M_S^2}{Q^2} - 1\right) \qquad n = \text{even} \\
&= -\sum_{k=1}^{(n-1)/2} \frac{1}{2k-1}\left(\frac{M_S}{Q}\right)^{2k-1} + \frac{1}{2}\log\left(\frac{M_S + Q}{M_S - Q}\right) \qquad n = \text{odd}\,.
\end{aligned} \tag{5.10}
$$

The coupling κ suppresses the interaction of KK modes to the SM fields. However, the collective effect of summing over large number of accessible KK modes compensates the suppression, making the effective coupling large enough to have significant observable effects.

5.2.2 RS model

The RS model contains only one spatial extra dimension (y) which is compactified on a S^1/Z^2 orbifold with radius R. Two fixed points of the orbifold holds two 3-dimensional branes: the one at $y = 0$ with positive tension is called the Planck brane and the other one at $y = \pi R$ with negative tension is called the TeV brane. The 5-dimensional Einstein equation provides a warped solution with the metric

$$
ds^2 = e^{-2\mathcal{K}Ry} g_{\mu\nu} dx^\mu dx^\nu + R^2 dy^2\,. \tag{5.11}
$$

This space is non-factorizable and have a constant negative curvature. $\mathcal{K}$ is the curvature of this AdS$_5$ space-time and $g_{\mu\nu}$ is the usual Minkowski metric. The exponential warp factor can generate the hierarchical difference between Planck and electro-weak scales for $\mathcal{K}R \sim O(10)$, thus avoiding the hierarchy problem. Further, the value of $\mathcal{K}R$ can be stabilized without fine tuning by introducing [190, 191] a modulus field in the bulk and minimizing its potential. As earlier, the graviton and the modulus field propagate in the bulk, while the SM is constrained to the TeV brane and the projection of 5-dimensional graviton in the 4-dimensional bulk generates the spectrum of KK modes. The zero mode is M_P suppressed and the excited modes are massive and only TeV suppressed. We denote the massive KK modes by $\tilde{h}_n^{\mu\nu}$ which interacts with the SM fields as follows

$$\mathcal{L}_{int} \sim -\frac{1}{M_P} T_{\mu\nu}(x)\, \tilde{h}_0^{\mu\nu}(x) + \frac{e^{\pi\mathcal{K}R}}{M_P} \sum_{n=1}^{\infty} T_{\mu\nu}(x)\, \tilde{h}_n^{\mu\nu}(x)\,. \tag{5.12}$$

The mass of each $\tilde{h}_n^{\mu\nu}$ is given by

$$m_n = x_n \mathcal{K} e^{\pi\mathcal{K}R} \tag{5.13}$$

with x_n being the zeroes of the Bessel function $J_1(x)$. Evidently, except for the overall warp factor, the Feynman rules [189, 192] are same for both ADD and RS model.

5.3 The effective Lagrangian

To describe the interaction of a generic spin-2 field with the SM fields, we consider the following effective Lagrangian, written down in a gauge invariant way with denoting the spin-2 field by $h^{\mu\nu}$ and the SM energy momentum tensor by $T_{\mu\nu}$. Since we are interested only in the QCD corrections to processes involving spin-2 fields, we restrict ourselves to the QCD part of $T_{\mu\nu}$ and the corresponding action reads [163–166] as

$$\mathcal{S} = \mathcal{S}_{SM} + \mathcal{S}_h - \frac{\kappa}{2} \int d^4x\, T_{\mu\nu}^{QCD}(x)\, h^{\mu\nu}(x)\,, \tag{5.14}$$

where $\mathcal{S}_{SM}$ is the SM action, $\mathcal{S}_h$ is the kinetic energy part of the action corresponding to spin-2 fields, κ is a dimensionful coupling and $T_{\mu\nu}^{QCD}$ is the energy momentum tensor of QCD given by

$$\begin{aligned} T_{\mu\nu}^{QCD} \;=\;& -g_{\mu\nu}\mathcal{L}_{QCD} - F_{\mu\rho}^a F_\nu^{a\rho} - \frac{1}{\xi} g_{\mu\nu}\partial^\rho(A_\rho^a \partial^\sigma A_\sigma^a) + \frac{1}{\xi}(A_\nu^a \partial_\mu(\partial^\sigma A_\sigma^a) + A_\mu^a \partial_\nu(\partial^\sigma A_\sigma^a)) \\ & + \frac{i}{4}[\overline{\psi}\gamma_\mu(\overrightarrow{\partial}_\nu - ig_s T^a A_\nu^a)\psi - \overline{\psi}(\overleftarrow{\partial}_\nu + ig_s T^a A_\nu^a)\gamma_\mu\psi + \overline{\psi}\gamma_\nu(\overrightarrow{\partial}_\mu - ig_s T^a A_\mu^a)\psi \end{aligned}$$

$$-\overline{\psi}(\overleftarrow{\partial}_\mu + ig_s T^a A^a_\mu)\gamma_\nu\psi] + \partial_\mu\overline{c}^a(\partial_\nu c^a - g_s f^{abc} A^c_\nu c^b)$$

$$+\partial_\nu\overline{c}^a(\partial_\mu c^a - g_s f^{abc} A^c_\mu c^b). \tag{5.15}$$

g_s is the strong coupling constant and ξ is the gauge fixing parameter. The T^a are generators and f^{abc} are the structure constants of $SU(3)$. Note that spin-2 fields couple to ghost fields (c^a) [193] as well in order to cancel unphysical degrees of freedom of gluon fields (A^a_μ).

5.4 The form factors

The form factors are basic vertex functions, and are as such fundamental ingredients for many precision calculations in QCD. They couple an external, color-neutral off-shell current to a pair of partons. For the spin-2 case, the quark and gluon form factors can be defined by sandwiching the energy-momentum tensor between on-shell quark and gluon states respectively normalized by their respective Born amplitudes:

$$\begin{aligned}\hat{\mathcal{F}}^{T,i}(Q^2,\epsilon) &= \frac{\hat{\mathcal{M}}_i^{(0)*}.M_i}{\hat{\mathcal{M}}_i^{(0)*}.\hat{\mathcal{M}}_i^{(0)}} \\ &= \sum_{n=0}^{\infty} \hat{a}_s^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n \hat{\mathcal{F}}^{T,i,n}(\epsilon), \qquad i = q, g\end{aligned} \tag{5.16}$$

where M_i are the unrenormalized amplitudes computed in powers of the bare strong coupling constant $\hat{a}_s$ using dimensional regularization in $d = 4 + \epsilon$ dimensions, that is

$$M_i(Q^2,\epsilon) = \sum_{n=0}^{\infty} \hat{a}_s^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n \hat{\mathcal{M}}_i^{(n)}(\epsilon), \tag{5.17}$$

where $Q^2 = -2p_1 \cdot p_2$ and p_1, p_2 are the momenta of external quark or gluon on-shell states.

In [52], both one and two-loop form factors were presented in dimensional regularization and later on, they were used in [53] to compute the threshold corrections to Drell-Yan production at the LHC in ADD and RS models to second order in strong coupling constant. In the following, we present the third order correction to the form factors in QCD.

5.4.1 Computational procedure

This section is devoted to the description of the computational procedure that we follow to obtain both the quark and gluon form factors of the energy momentum tensor to third order in strong coupling constant using dimensional regularization. The relevant Feynman diagrams are generated using QGRAF [86]. At third order alone, there are 3374 and 1072 number of Feynman diagrams for gluon and quark form factors respectively. The QGRAF generated amplitudes are then converted into a suitable format using routines developed using the symbolic manipulation program FORM [87]. Both group as well as Lorentz indices are carefully handled to express the form factors in a suitable color basis involving Casimir operators of $SU(N)$ with the coefficients containing three-loop scalar integrals. For the gluon form factor we have summed only the physical polarizations of the external gluons using

$$\sum_s \varepsilon^\mu(p_i, s)\varepsilon^{\nu *}(p_i, s) = -g^{\mu\nu} + \frac{p_i^\mu q_i^\nu + q_i^\mu p_i^\nu}{p_i.q_i} \tag{5.18}$$

where, p_i is the i^{th}-gluon momentum and q_i is the corresponding light-like momentum. We choose $q_1 = p_2$ and $q_2 = p_1$ for simplicity. For the external spin-2 fields, we have used the d dimensional polarization sum given in Eq. 5.6. We have used Feynman gauge throughout.

At three-loop level, we find that the diagrams contributing to form factors can have at most 9 independent propagators involving two external momenta p_1, p_2 and three internal loop momenta k_1, k_2, k_3, while the maximum number of scalar products that can appear in the numerator of each diagram can be 12. Hence we need to increase the number of propagators to 12 which allow us to classify all the three-loop diagrams into three different auxiliary topologies. We take the help of Reduze2 [93] for this purpose. The topologies [151] that are used in our computation are given below

$$\begin{aligned}
&\mathrm{A}_1: \quad \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{23}, \mathcal{D}_{1;1}, \mathcal{D}_{1;12}, \mathcal{D}_{2;1}, \mathcal{D}_{2;12}, \mathcal{D}_{3;1}, \mathcal{D}_{3;12} \\
&\mathrm{A}_2: \quad \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{23}, \mathcal{D}_{13;2}, \mathcal{D}_{1;12}, \mathcal{D}_{2;1}, \mathcal{D}_{12;2}, \mathcal{D}_{3;1}, \mathcal{D}_{3;12} \\
&\mathrm{A}_3: \quad \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{123}, \mathcal{D}_{1;1}, \mathcal{D}_{1;12}, \mathcal{D}_{2;1}, \mathcal{D}_{2;12}, \mathcal{D}_{3;1}, \mathcal{D}_{3;12}
\end{aligned} \tag{5.19}$$

where,

$$\begin{aligned}
\mathcal{D}_i = k_i^2,\ \mathcal{D}_{ij} = (k_i - k_j)^2,\ \mathcal{D}_{ijl} = (k_i - k_j - k_l)^2, \\
\mathcal{D}_{i;j} = (k_i - p_j)^2,\ \mathcal{D}_{i;jl} = (k_i - p_j - p_l)^2,\ \mathcal{D}_{ij;l} = (k_i - k_j - p_l)^2 \,.
\end{aligned} \tag{5.20}$$

The resulting integrals classified in terms of three topologies, are then reduced to a set of master integrals by using a systematic approach that uses Integration by parts (IBP) [89,90] and Lorentz invariant (LI) [194] identities. The IBP identities follow from the fact that within dimensional regularization, the integrals are finite and well-behaved and hence any integrand at the boundary must be zero. Following this, the generalization of Gauss theorem implies the integral of the total derivative with respect to any loop momenta to be zero, that is

$$\int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_3}{(2\pi)^d} \frac{\partial}{\partial k_i} \cdot \left(v_j \frac{1}{\prod_l D_l^{n_l}} \right) = 0 \,, \tag{5.21}$$

where n_l is an element of $\vec{n} = (n_1, \cdots, n_{12})$ with $n_l \in Z$ and D_ls are propagators which depend on the loop and external momenta. The four vector v_j^μ can be both loop and external momenta. Performing differentiation on the left hand side and expressing the scalar products of k_i and p_j linearly in terms of $\mathcal{D}_l$'s, one obtains the IBP identities as

$$\sum_i a_i J(b_{i,1} + n_1, ..., b_{i,12} + n_{12}) = 0 \tag{5.22}$$

where

$$J(\vec{m}) = J(m_1, \cdots, m_{12}) = \int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_3}{(2\pi)^d} \frac{1}{\prod_l D_l^{m_l}} \tag{5.23}$$

with $b_{i,j} \in \{-1, 0, 1\}$ and a_i are polynomial in n_j. The LI identities follow from the fact that the loop integrals are invariant under Lorentz transformations of the external momenta, that is

$$p_i^{\mu} p_j^{\nu} \left(\sum_k p_{k[\nu} \frac{\partial}{\partial p_k^{\mu]}} \right) J(\vec{n}) = 0. \tag{5.24}$$

For the case of three-loop form factor, there are 15 IBP identities and 1 LI identity for each integrand, and hence there are large number of equations for the whole system. These equations can be solved to relate the large number of scalar integrals and express them in terms of a set of fewer integrals which are the so called master integrals. To solve this large system of equations, there are dedicated computer algebra tools like AIR [195], FIRE [196], REDUZE [93, 197], LiteRed [91, 92] etc. We use the Mathematica based package LiteRedV1.82 along with MintV1.1 [198].

We find that the form factors at three-loop level can be expressed in terms of 22 master integrals. Following the same notation as of [151], the master integrals can be distinguished into three topological types: genuine three-loop integrals with vertex functions ($A_{t,i}$), three-loop propagator integrals ($B_{t,i}$) and integrals which are product of one and two-loop integrals ($C_{t,i}$). Defining a generic three-loop master integral as

$$A_{i,m_1^i m_2^i \cdots m_{12}^i} = \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \int \frac{d^d k_3}{(2\pi)^d} \frac{1}{\prod_j D_j^{m_j^i}}, \qquad i = 1, 2, 3 \tag{5.25}$$

where D_j is the j^{th} element of the set A_i, we identify the resulting master integrals in our computation with those given in [151] and they are listed[1] in Figure 5.1 and Figure 5.2.

The master integrals were computed in [90, 148, 150, 199–203] to relevant orders in ϵ and we have used them to complete our computation of the form factors up to three-loop level. The electronic version of the results of both quark and gluon form factors in terms of the master integrals $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ for arbitrary d is attached with the arXiv version. In the next section, we present the three-loop results for both the form factors expanded in

[1] These figure have been taken from [151].

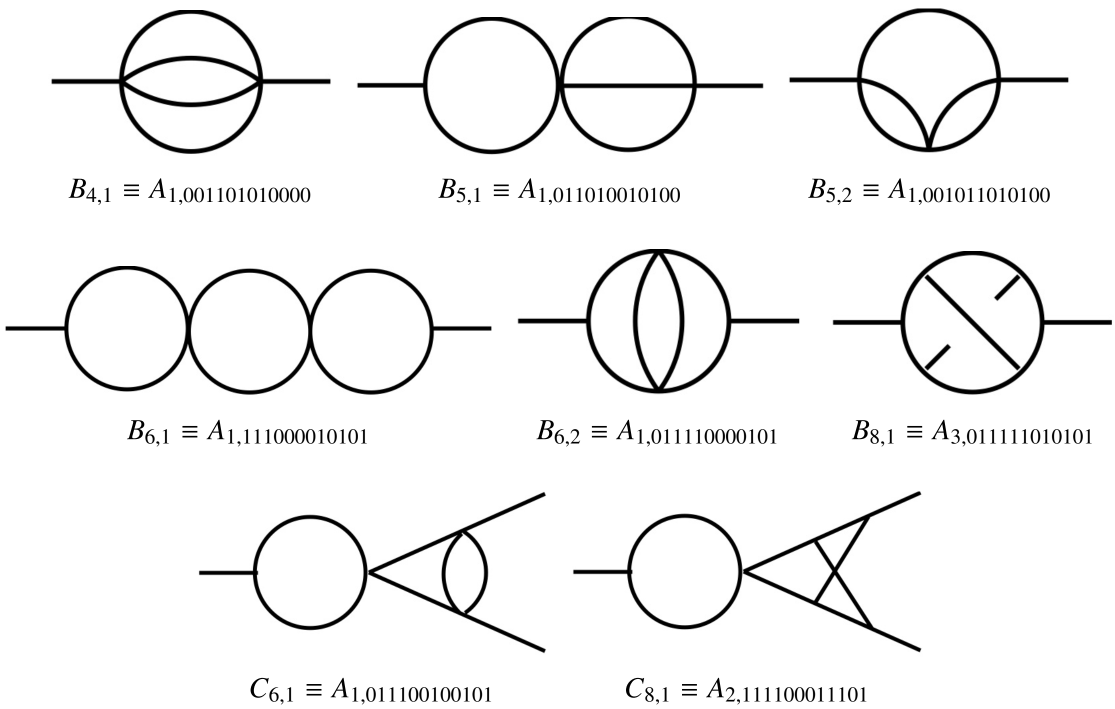


Figure 5.1: Two-point and factorizable three-point three-loop master integrals.

powers of ϵ along with already known one and two-loop results.

5.4.2 Results

In this section we present one, two and three-loop quark and gluon form factors after expanding in powers of ϵ to relevant order. The one and two-loop results completely agree with [52] and the three-loop ones are new.

$$
\begin{aligned}
\hat{\mathcal{F}}^{T,g,1} = C_A\Big\{ & -\frac{8}{\epsilon^2} + \frac{1}{\epsilon}\frac{22}{3} - \frac{203}{18} + \zeta_2 + \epsilon\Big(\frac{2879}{216} - \frac{7}{3}\zeta_3 - \frac{11}{12}\zeta_2\Big) + \epsilon^2\Big(-\frac{37307}{2592} + \frac{77}{36}\zeta_3 \\
& + \frac{203}{144}\zeta_2 + \frac{47}{80}\zeta_2^2\Big) + \epsilon^3\Big(\frac{465143}{31104} - \frac{31}{20}\zeta_5 - \frac{1421}{432}\zeta_3 - \frac{2879}{1728}\zeta_2 + \frac{7}{24}\zeta_2\zeta_3 - \frac{517}{960}\zeta_2^2\Big) \\
& + \epsilon^4\Big(-\frac{5695811}{373248} + \frac{341}{240}\zeta_5 + \frac{20153}{5184}\zeta_3 - \frac{49}{144}\zeta_3^2 + \frac{37307}{20736}\zeta_2 - \frac{77}{288}\zeta_2\zeta_3 + \frac{9541}{11520}\zeta_2^2 \\
& + \frac{949}{4480}\zeta_2^3\Big)\Big\} + n_f\Big\{-\frac{4}{3}\frac{1}{\epsilon} + \frac{35}{18} + \epsilon\Big(-\frac{497}{216} + \frac{1}{6}\zeta_2\Big) + \epsilon^2\Big(\frac{6593}{2592} - \frac{7}{18}\zeta_3 - \frac{35}{144}\zeta_2\Big) \\
& + \epsilon^3\Big(-\frac{84797}{31104} + \frac{245}{432}\zeta_3 + \frac{497}{1728}\zeta_2 + \frac{47}{480}\zeta_2^2\Big) + \epsilon^4\Big(\frac{1072433}{373248} - \frac{31}{120}\zeta_5 - \frac{3479}{5184}\zeta_3 \\
& - \frac{6593}{20736}\zeta_2 + \frac{7}{144}\zeta_2\zeta_3 - \frac{329}{2304}\zeta_2^2\Big)\Big\}, \qquad (5.26)
\end{aligned}
$$

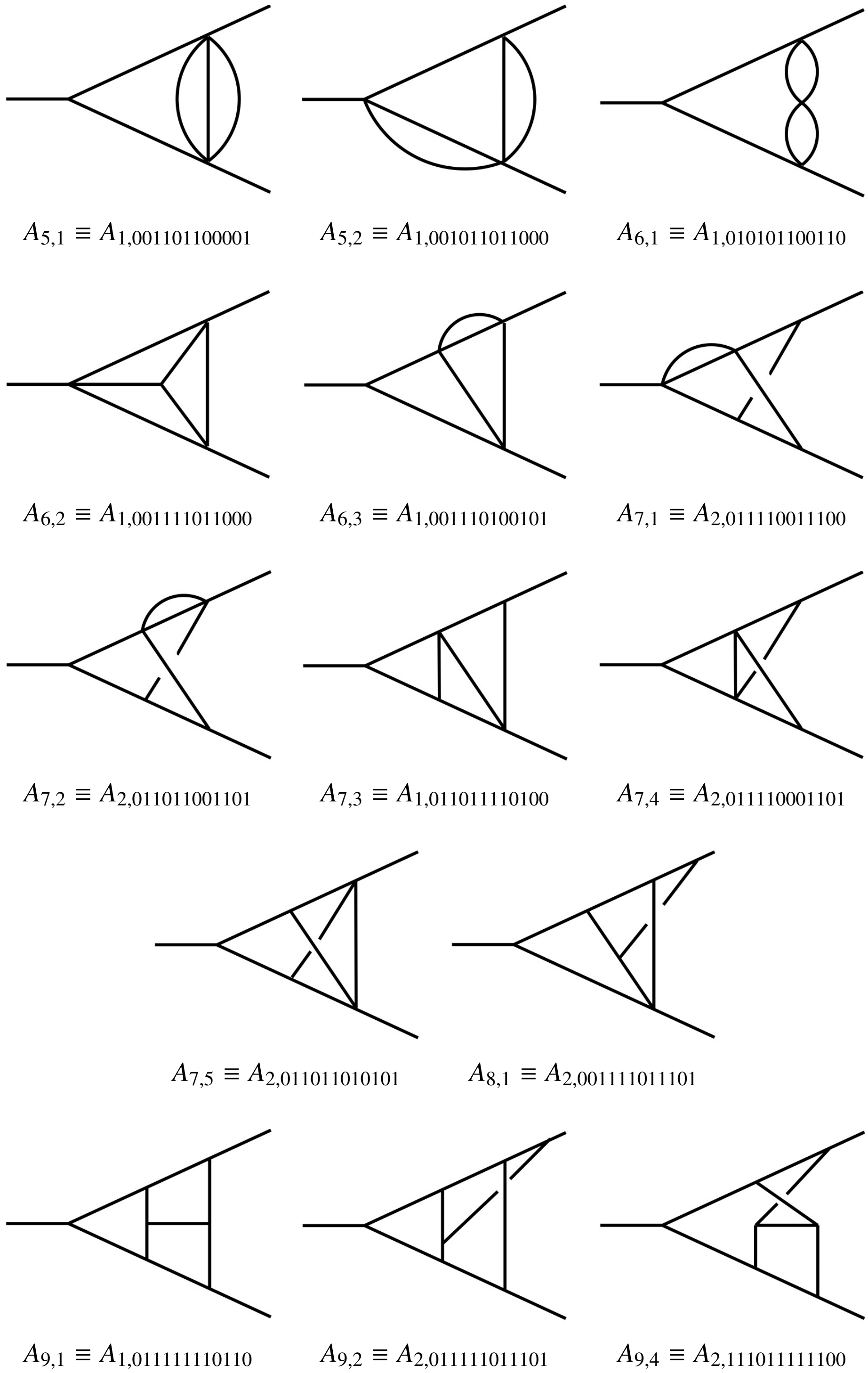


Figure 5.2: Genuine three-loop master integrals with vertex function.

$$
\begin{aligned}
\hat{\mathcal{F}}^{T,q,1} = C_F\Big\{ & -\frac{8}{\epsilon^2} + \frac{6}{\epsilon} - 10 + \zeta_2 + \epsilon\Big(12 - \frac{7}{3}\zeta_3 - \frac{3}{4}\zeta_2\Big) + \epsilon^2\Big(- 13 + \frac{7}{4}\zeta_3 + \frac{5}{4}\zeta_2 + \frac{47}{80}\zeta_2^2\Big) \\
& + \epsilon^3\Big(\frac{27}{2} - \frac{31}{20}\zeta_5 - \frac{35}{12}\zeta_3 - \frac{3}{2}\zeta_2 + \frac{7}{24}\zeta_2\zeta_3 - \frac{141}{320}\zeta_2^2\Big) + \epsilon^4\Big(- \frac{55}{4} + \frac{93}{80}\zeta_5 + \frac{7}{2}\zeta_3 \\
& - \frac{49}{144}\zeta_3^2 + \frac{13}{8}\zeta_2 - \frac{7}{32}\zeta_2\zeta_3 + \frac{47}{64}\zeta_2^2 + \frac{949}{4480}\zeta_2^3\Big)\Big\}, \qquad (5.27)
\end{aligned}
$$

$$\begin{aligned}
\hat{\mathcal{F}}^{T,g,2} = {} & C_A^2\Big\{\frac{32}{\epsilon^4} - \frac{44}{\epsilon^3} + \frac{1}{\epsilon^2}\Big(\frac{226}{3} - 4\zeta_2\Big) + \frac{1}{\epsilon}\Big(- 81 + \frac{50}{3}\zeta_3 + \frac{11}{3}\zeta_2\Big) + \frac{5249}{108} - 11\zeta_3 \\
& - \frac{67}{18}\zeta_2 - \frac{21}{5}\zeta_2^2 + \epsilon\Big(\frac{59009}{1296} - \frac{71}{10}\zeta_5 + \frac{433}{18}\zeta_3 - \frac{337}{108}\zeta_2 - \frac{23}{6}\zeta_2\zeta_3 + \frac{99}{40}\zeta_2^2\Big) \\
& + \epsilon^2\Big(- \frac{1233397}{5184} + \frac{759}{20}\zeta_5 - \frac{8855}{216}\zeta_3 + \frac{901}{36}\zeta_3^2 + \frac{12551}{648}\zeta_2 + \frac{77}{36}\zeta_2\zeta_3 - \frac{4843}{720}\zeta_2^2 \\
& + \frac{2313}{280}\zeta_2^3\Big)\Big\} + C_A n_f\Big\{\frac{8}{\epsilon^3} + \frac{1}{\epsilon^2}\Big(- \frac{40}{3}\Big) + \frac{1}{\epsilon}\Big(\frac{41}{3} - \frac{2}{3}\zeta_2\Big) - \frac{605}{108} + 10\zeta_3 + \frac{5}{9}\zeta_2 \\
& + \epsilon\Big(- \frac{21557}{1296} - \frac{182}{9}\zeta_3 + \frac{145}{108}\zeta_2 - \frac{57}{20}\zeta_2^2\Big) + \epsilon^2\Big(\frac{320813}{5184} + \frac{71}{10}\zeta_5 + \frac{6407}{216}\zeta_3 - \frac{3617}{648}\zeta_2 \\
& - \frac{43}{18}\zeta_2\zeta_3 + \frac{1099}{180}\zeta_2^2\Big)\Big\} + C_F n_f\Big\{ - \frac{2}{\epsilon} + \frac{61}{6} - 8\zeta_3 + \epsilon\Big(- \frac{2245}{72} + \frac{59}{3}\zeta_3 + \frac{1}{2}\zeta_2 \\
& + \frac{12}{5}\zeta_2^2\Big) + \epsilon^2\Big(\frac{64177}{864} - 14\zeta_5 - \frac{335}{9}\zeta_3 - \frac{83}{24}\zeta_2 + 2\zeta_2\zeta_3 - \frac{179}{30}\zeta_2^2\Big)\Big\},
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
\hat{\mathcal{F}}^{T,q,2} = {} & C_F^2\Big\{\frac{32}{\epsilon^4} - \frac{48}{\epsilon^3} + \frac{1}{\epsilon^2}\Big(98 - 8\zeta_2\Big) + \frac{1}{\epsilon}\Big(- \frac{309}{2} + \frac{128}{3}\zeta_3\Big) + \frac{5317}{24} - 90\zeta_3 + \frac{41}{2}\zeta_2 \\
& - 13\zeta_2^2 + \epsilon\Big(- \frac{28127}{96} + \frac{92}{5}\zeta_5 + \frac{1327}{6}\zeta_3 - \frac{1495}{24}\zeta_2 - \frac{56}{3}\zeta_2\zeta_3 + \frac{173}{6}\zeta_2^2\Big) \\
& + \epsilon^2\Big(\frac{1244293}{3456} - \frac{311}{10}\zeta_5 - \frac{34735}{72}\zeta_3 + \frac{652}{9}\zeta_3^2 + \frac{38543}{288}\zeta_2 + \frac{193}{6}\zeta_2\zeta_3 - \frac{10085}{144}\zeta_2^2 \\
& + \frac{223}{20}\zeta_2^3\Big)\Big\} + C_A C_F\Big\{\frac{44}{3}\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}\Big(- \frac{332}{9} + 4\zeta_2\Big) + \frac{1}{\epsilon}\Big(\frac{4921}{54} - 26\zeta_3 + \frac{11}{3}\zeta_2\Big) \\
& - \frac{120205}{648} + \frac{755}{9}\zeta_3 - \frac{251}{9}\zeta_2 + \frac{44}{5}\zeta_2^2 + \epsilon\Big(\frac{2562925}{7776} - \frac{51}{2}\zeta_5 - \frac{5273}{27}\zeta_3 + \frac{14761}{216}\zeta_2 \\
& + \frac{89}{6}\zeta_2\zeta_3 - \frac{3299}{120}\zeta_2^2\Big) + \epsilon^2\Big(- \frac{50471413}{93312} + \frac{3971}{60}\zeta_5 + \frac{282817}{648}\zeta_3 - \frac{569}{12}\zeta_3^2 - \frac{351733}{2592}\zeta_2 \\
& - \frac{1069}{36}\zeta_2\zeta_3 + \frac{7481}{120}\zeta_2^2 - \frac{809}{280}\zeta_2^3\Big)\Big\} + C_F n_f\Big\{ - \frac{8}{3}\frac{1}{\epsilon^3} + \frac{56}{9}\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\Big(- \frac{425}{27} - \frac{2}{3}\zeta_2\Big) \\
& + \frac{9989}{324} - \frac{26}{9}\zeta_3 + \frac{38}{9}\zeta_2 + \epsilon\Big(- \frac{202253}{3888} + \frac{2}{27}\zeta_3 - \frac{989}{108}\zeta_2 + \frac{41}{60}\zeta_2^2\Big) \\
& + \epsilon^2\Big(\frac{3788165}{46656} - \frac{121}{30}\zeta_5 - \frac{935}{324}\zeta_3 + \frac{22937}{1296}\zeta_2 - \frac{13}{18}\zeta_2\zeta_3 + \frac{97}{180}\zeta_2^2\Big)\Big\},
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
\hat{\mathcal{F}}^{T,g,3} = {} & C_A^3\Big\{\frac{1}{\epsilon^6}\Big(- \frac{256}{3}\Big) + \frac{1}{\epsilon^5}\Big(\frac{352}{3}\Big) + \frac{1}{\epsilon^4}\Big(- \frac{14744}{81}\Big) + \frac{1}{\epsilon^3}\Big(\frac{13126}{243} - \frac{176}{3}\zeta_3 + \frac{484}{27}\zeta_2\Big) \\
& + \frac{1}{\epsilon^2}\Big(\frac{149939}{486} - \frac{440}{27}\zeta_3 - \frac{4321}{81}\zeta_2 + \frac{494}{45}\zeta_2^2\Big) + \frac{1}{\epsilon}\Big(- \frac{14639165}{17496} + \frac{1756}{15}\zeta_5 - \frac{634}{9}\zeta_3 \\
& + \frac{112633}{972}\zeta_2 + \frac{170}{9}\zeta_2\zeta_3 + \frac{4213}{180}\zeta_2^2\Big) + \frac{1056263429}{1049760} + \frac{5014}{45}\zeta_5 + \frac{539}{2430}\zeta_3 - \frac{1766}{9}\zeta_3^2 \\
& - \frac{1988293}{11664}\zeta_2 - \frac{92}{9}\zeta_2\zeta_3 - \frac{64997}{2160}\zeta_2^2 - \frac{22523}{270}\zeta_2^3\Big\} + C_A^2 n_f\Big\{\frac{1}{\epsilon^5}\Big(- \frac{64}{3}\Big) + \frac{1}{\epsilon^4}\Big(\frac{1840}{81}\Big) \\
& + \frac{1}{\epsilon^3}\Big(\frac{5818}{243} - \frac{88}{27}\zeta_2\Big) + \frac{1}{\epsilon^2}\Big(- \frac{56783}{486} - \frac{1456}{27}\zeta_3 + \frac{892}{81}\zeta_2\Big) + \frac{1}{\epsilon}\Big(\frac{3370273}{17496} + \frac{5831}{81}\zeta_3 \\
& - \frac{26173}{972}\zeta_2 + \frac{1153}{90}\zeta_2^2\Big) + \frac{5797271}{1049760} - \frac{11528}{45}\zeta_5 + \frac{5401}{30}\zeta_3 + \frac{489781}{11664}\zeta_2 + \frac{2}{9}\zeta_2\zeta_3 \\
& - \frac{923}{72}\zeta_2^2\Big\} + C_A n_f^2\Big\{\frac{1}{\epsilon^4}\Big(\frac{160}{81}\Big) + \frac{1}{\epsilon^3}\Big(- \frac{1340}{243}\Big) + \frac{1}{\epsilon^2}\Big(7 - \frac{4}{27}\zeta_2\Big) + \frac{1}{\epsilon}\Big(\frac{45077}{8748}
\end{aligned}$$

$$+ \frac{940}{81}\zeta_3 - \frac{5}{162}\zeta_2\Big) - \frac{32220173}{524880} - \frac{122141}{2430}\zeta_3 + \frac{661}{216}\zeta_2 - \frac{1777}{540}\zeta_2^2\Big\}$$
$$+ C_A C_F n_f \Big\{ \frac{1}{\epsilon^3}\Big(\frac{128}{9}\Big) + \frac{1}{\epsilon^2}\Big(- \frac{1712}{27} + \frac{512}{9}\zeta_3\Big) + \frac{1}{\epsilon}\Big(\frac{11732}{81} - \frac{2360}{27}\zeta_3 - \frac{8}{3}\zeta_2$$
$$- \frac{256}{15}\zeta_2^2\Big) - \frac{152656}{1215} + \frac{1256}{9}\zeta_5 - \frac{10754}{405}\zeta_3 + \frac{34}{3}\zeta_2 + \frac{132}{5}\zeta_2^2\Big\} + C_F^2 n_f \Big\{ \frac{1}{\epsilon}\Big(\frac{2}{3}\Big)$$
$$- \frac{241}{18} + 80\zeta_5 - \frac{148}{3}\zeta_3\Big\} + C_F n_f^2 \Big\{ \frac{1}{\epsilon^2}\Big(- \frac{16}{9}\Big) + \frac{1}{\epsilon}\Big(\frac{388}{27} - \frac{32}{3}\zeta_3\Big) - \frac{5623}{81}$$
$$+ \frac{412}{9}\zeta_3 + \frac{2}{3}\zeta_2 + \frac{16}{5}\zeta_2^2\Big\}, \tag{5.30}$$

$$\hat{\mathcal{F}}^{T,q,3} = C_F^3 \Big\{ \frac{1}{\epsilon^6}\Big(- \frac{256}{3}\Big) + \frac{1}{\epsilon^5}\Big(192\Big) + \frac{1}{\epsilon^4}\Big(- 464 + 32\zeta_2\Big) + \frac{1}{\epsilon^3}\Big(888 - \frac{800}{3}\zeta_3 + 24\zeta_2\Big)$$
$$+ \frac{1}{\epsilon^2}\Big(- \frac{4582}{3} + 808\zeta_3 - 258\zeta_2 + \frac{426}{5}\zeta_2^2\Big) + \frac{1}{\epsilon}\Big(\frac{14375}{6} - \frac{1288}{5}\zeta_5 - \frac{6854}{3}\zeta_3$$
$$+ \frac{2629}{3}\zeta_2 + \frac{428}{3}\zeta_2\zeta_3 - \frac{7199}{30}\zeta_2^2\Big) - \frac{765629}{216} + \frac{12074}{15}\zeta_5 + \frac{47557}{9}\zeta_3 - \frac{1826}{3}\zeta_3^2$$
$$- \frac{78665}{36}\zeta_2 - \frac{361}{3}\zeta_2\zeta_3 + \frac{201691}{360}\zeta_2^2 - \frac{9095}{252}\zeta_2^3\Big\} + C_A^2 C_F \Big\{ \frac{1}{\epsilon^4}\Big(- \frac{3872}{81}\Big)$$
$$+ \frac{1}{\epsilon^3}\Big(\frac{52168}{243} - \frac{704}{27}\zeta_2\Big) + \frac{1}{\epsilon^2}\Big(- \frac{187292}{243} + \frac{6688}{27}\zeta_3 - \frac{2212}{81}\zeta_2 - \frac{352}{45}\zeta_2^2\Big)$$
$$+ \frac{1}{\epsilon}\Big(\frac{4856336}{2187} + \frac{272}{3}\zeta_5 - \frac{36884}{27}\zeta_3 + \frac{120769}{243}\zeta_2 + \frac{176}{9}\zeta_2\zeta_3 - \frac{1604}{15}\zeta_2^2\Big) - \frac{71947001}{13122}$$
$$- \frac{2588}{9}\zeta_5 + \frac{2464213}{486}\zeta_3 - \frac{1136}{9}\zeta_3^2 - \frac{1479931}{729}\zeta_2 - \frac{926}{9}\zeta_2\zeta_3 + \frac{54071}{108}\zeta_2^2 - \frac{6152}{189}\zeta_2^3\Big\}$$
$$+ C_A C_F^2 \Big\{ \frac{1}{\epsilon^5}\Big(- \frac{352}{3}\Big) + \frac{1}{\epsilon^4}\Big(\frac{3448}{9} - 32\zeta_2\Big) + \frac{1}{\epsilon^3}\Big(- \frac{29620}{27} + 208\zeta_3 + \frac{28}{3}\zeta_2\Big)$$
$$+ \frac{1}{\epsilon^2}\Big(\frac{207442}{81} - 1096\zeta_3 + \frac{2471}{9}\zeta_2 - \frac{332}{5}\zeta_2^2\Big) + \frac{1}{\epsilon}\Big(- \frac{2529065}{486} + 284\zeta_5 + \frac{10603}{3}\zeta_3$$
$$- \frac{71101}{54}\zeta_2 - \frac{430}{3}\zeta_2\zeta_3 + \frac{66091}{180}\zeta_2^2\Big) + \frac{56048957}{5832} - \frac{18274}{45}\zeta_5 - \frac{185921}{18}\zeta_3 + \frac{1616}{3}\zeta_3^2$$
$$+ \frac{1324001}{324}\zeta_2 + \frac{1870}{9}\zeta_2\zeta_3 - \frac{2254603}{2160}\zeta_2^2 - \frac{18619}{1260}\zeta_2^3\Big\} + C_A C_F n_f \Big\{ \frac{1}{\epsilon^4}\Big(\frac{1408}{81}\Big)$$
$$+ \frac{1}{\epsilon^3}\Big(- \frac{18032}{243} + \frac{128}{27}\zeta_2\Big) + \frac{1}{\epsilon^2}\Big(\frac{64220}{243} - \frac{1024}{27}\zeta_3 + \frac{1264}{81}\zeta_2\Big) + \frac{1}{\epsilon}\Big(- \frac{1613122}{2187}$$
$$+ \frac{19784}{81}\zeta_3 - \frac{41062}{243}\zeta_2 + \frac{88}{5}\zeta_2^2\Big) + \frac{11339972}{6561} - \frac{128}{3}\zeta_5 - \frac{64730}{81}\zeta_3 + \frac{916217}{1458}\zeta_2$$
$$+ \frac{392}{9}\zeta_2\zeta_3 - \frac{2161}{27}\zeta_2^2\Big\} + C_F^2 n_f \Big\{ \frac{1}{\epsilon^5}\Big(\frac{64}{3}\Big) + \frac{1}{\epsilon^4}\Big(- \frac{592}{9}\Big) + \frac{1}{\epsilon^3}\Big(\frac{5080}{27} + \frac{8}{3}\zeta_2\Big)$$
$$+ \frac{1}{\epsilon^2}\Big(- \frac{34060}{81} + \frac{584}{9}\zeta_3 - \frac{458}{9}\zeta_2\Big) + \frac{1}{\epsilon}\Big(\frac{194287}{243} - \frac{5978}{27}\zeta_3 + \frac{5651}{27}\zeta_2 - \frac{337}{18}\zeta_2^2\Big)$$
$$- \frac{3887467}{2916} + \frac{278}{45}\zeta_5 + \frac{70499}{81}\zeta_3 - \frac{99691}{162}\zeta_2 - \frac{343}{9}\zeta_2\zeta_3 + \frac{49001}{1080}\zeta_2^2\Big\}$$
$$+ C_F n_f^2 \Big\{ \frac{1}{\epsilon^4}\Big(- \frac{128}{81}\Big) + \frac{1}{\epsilon^3}\Big(\frac{1504}{243}\Big) + \frac{1}{\epsilon^2}\Big(- \frac{592}{27} - \frac{16}{9}\zeta_2\Big) + \frac{1}{\epsilon}\Big(\frac{128312}{2187} - \frac{272}{81}\zeta_3$$
$$+ \frac{380}{27}\zeta_2\Big) - \frac{857536}{6561} - \frac{2852}{243}\zeta_3 - \frac{1250}{27}\zeta_2 - \frac{83}{135}\zeta_2^2\Big\}, \tag{5.31}$$

where $C_A = N$, $C_F = (N^2-1)/2N$ and n_f is the number of active quark flavors. Note that the color factors that appear in the above form factors are the same ones that one finds in the quark and gluon form factors [151] except in the $\hat{F}_g^{T,(1)}$ where there is an additional term proportional to n_f. This is simply because of the fact that the energy momentum tensor contains both quark as well as gluon fields and its matrix element between gluon states at one-loop level can have quark loop contribution giving rise to explicit n_f dependence. Beyond one-loop level, no new color structure is expected. In the next section, we describe how these form factors can be renormalized up to three-loop level through coupling constant renormalization. We then study the universal structure of the IR poles in ϵ through Sudakov's KG equation up to three-loop level. It provides a crucial check for our new results on the form factors.

5.5 Ultraviolet renormalization

Using the Eq. 2.7 and Eq. 2.12, we can express M_i (Eq. 5.16) in powers of renormalized a_s with UV finite matrix elements $\mathcal{M}_i^{(n)}$

$$M_i = \left(\mathcal{M}_i^{(0)} + a_s \mathcal{M}_i^{(1)} + a_s^2 \mathcal{M}_i^{(2)} + a_s^3 \mathcal{M}_i^{(3)} + \mathcal{O}(a_s^4)\right) \tag{5.32}$$

where,

$$\begin{aligned}
\mathcal{M}_i^{(0)} &= \hat{\mathcal{M}}_i^{(0)}, \\
\mathcal{M}_i^{(1)} &= \left(\frac{Q^2}{\mu_R^2}\right)^{\frac{\epsilon}{2}} \hat{\mathcal{M}}_i^{(1)}, \\
\mathcal{M}_i^{(2)} &= \left(\frac{Q^2}{\mu_R^2}\right)^{\epsilon} \hat{\mathcal{M}}_i^{(2)} + r_1 \left(\frac{Q^2}{\mu_R^2}\right)^{\frac{\epsilon}{2}} \hat{\mathcal{M}}_i^{(1)}, \\
\mathcal{M}_i^{(3)} &= \left(\frac{Q^2}{\mu_R^2}\right)^{\frac{3\epsilon}{2}} \hat{\mathcal{M}}_i^{(3)} + 2r_1 \left(\frac{Q^2}{\mu_R^2}\right)^{\epsilon} \hat{\mathcal{M}}_i^{(2)} + r_2 \left(\frac{Q^2}{\mu_R^2}\right)^{\frac{\epsilon}{2}} \hat{\mathcal{M}}_i^{(1)}.
\end{aligned} \tag{5.33}$$

Using above equations, we can obtain the renormalized form factors $F^{T,i}$ in terms of a_s.

5.6 Infrared singularities and universal pole structure

The results on multi-parton amplitudes beyond LO in pQCD have not only played an important role in understanding the IR structure of the theory but also allowed us to successfully carry out various resummation programs for physical observables in the kinematic regions where the fixed order perturbation theory breaks down. The most important one along this line was the very successful proposal by Catani [183] (see also [184]) on one and two-loop QCD amplitudes using the universal subtraction operators. In [144], for the first time, the structure of single pole term in both quark and gluon form factors up to two-loop level was unraveled. It was shown explicitly that the single pole can be written as a linear combination of UV, collinear and soft anomalous dimensions. The fact that this feature continues to hold even at three-loop level for the same form factors was observed in [145]. The structure of the single pole term for the multiparton amplitudes was studied in detail in [204,205]. Later on, the generalization of the proposal by Catani was achieved by Becher and Neubert [185] and also by Gardi and Magnea [186] beyond two-loop.

We consider the generic solution to the KG integro-differential equation (Eq. 3.12) satisfied by the form factors and extract the cusp anomalous dimensions by comparing Eq. 3.22 with the form factors presented in the previous section. We find

$$
\begin{aligned}
A_1^{T,i} &= C_i\{4\}\,,\\
A_2^{T,i} &= C_iC_A\left\{\frac{268}{9} - 8\zeta_2\right\} + C_in_f\left\{-\frac{40}{9}\right\}\,,\\
A_3^{T,i} &= C_iC_A^2\left\{\frac{490}{3} - \frac{1072\zeta_2}{9} + \frac{88\zeta_3}{3} + \frac{176\zeta_2^2}{5}\right\} + C_iC_Fn_f\left\{-\frac{110}{3} + 32\zeta_3\right\}\\
&\quad + C_iC_An_f\left\{-\frac{836}{27} + \frac{160\zeta_2}{9} - \frac{112\zeta_3}{3}\right\} + C_in_f^2\left\{-\frac{16}{27}\right\}\,.
\end{aligned}
$$

where $C_i = C_F$ for $i = q$ and $C_i = C_A$ for $i = g$. We find that they not only are maximally non-abelian but also coincide with those that appear in the quark and gluon form factors which are available up to three-loop level in the literature [140, 141].

$$A_n^{T,g} = \frac{C_A}{C_F} A_n^{T,q} \qquad \text{and} \qquad A_n^{T,i} = A_n^i \qquad i = q, g\,. \tag{5.34}$$

Following Eq. 3.23, the same structure for $G_n^{T,i}(\epsilon)$ is also observed, and we extract $B_n^{T,i}$ and $f_n^{T,i}$ from the form factors computed up to three-loop level. They read

$$\begin{aligned}
B_1^{T,g} &= C_A\left\{\frac{11}{3}\right\} - n_f\left\{\frac{2}{3}\right\},\\
B_2^{T,g} &= C_A^2\left\{\frac{32}{3} + 12\zeta_3\right\} - n_f C_A\left\{\frac{8}{3}\right\} - n_f C_F\{2\},\\
B_3^{T,g} &= C_A C_F n_f\left\{-\frac{241}{18}\right\} + C_A n_f^2\left\{\frac{29}{18}\right\} - C_A^2 n_f\left\{\frac{233}{18} + \frac{8}{3}\zeta_2 + \frac{4}{3}\zeta_2^2 + \frac{80}{3}\zeta_3\right\}\\
&\quad + C_A^3\left\{\frac{79}{2} - 16\zeta_2\zeta_3 + \frac{8}{3}\zeta_2 + \frac{22}{3}\zeta_2^2 + \frac{536}{3}\zeta_3 - 80\zeta_5\right\} + C_F n_f^2\left\{\frac{11}{9}\right\} + C_F^2 n_f\{1\},\\
B_1^{T,q} &= C_F\{3\},\\
B_2^{T,q} &= C_F^2\left\{\frac{3}{2} - 12\zeta_2 + 24\zeta_3\right\} + C_A C_F\left\{\frac{17}{34} + \frac{88}{6}\zeta_2 - 12\zeta_3\right\} + n_f C_F T_F\left\{-\frac{2}{3} - \frac{16}{3}\zeta_2\right\},\\
B_3^{T,q} &= C_A^2 C_F\left\{-2{\zeta_2}^2 + \frac{4496}{27}\zeta_2 - \frac{1552}{9}\zeta_3 + 40\zeta_5 - \frac{1657}{36}\right\} + C_A C_F^2\left\{-\frac{988}{15}{\zeta_2}^2 + 16\zeta_2\zeta_3\right.\\
&\quad \left. - \frac{410}{3}\zeta_2 + \frac{844}{3}\zeta_3 + 120\zeta_5 + \frac{151}{4}\right\} + C_A C_F n_f\left\{\frac{4}{5}{\zeta_2}^2 - \frac{1336}{27}\zeta_2 + \frac{200}{9}\zeta_3 + 20\right\}\\
&\quad + C_F^3\left\{\frac{288}{5}{\zeta_2}^2 - 32\zeta_2\zeta_3 + 18\zeta_2 + 68\zeta_3 - 240\zeta_5 + \frac{29}{2}\right\}\\
&\quad + C_F^2 n_f\left\{\frac{232}{15}{\zeta_2}^2 + \frac{20}{3}\zeta_2 - \frac{136}{3}\zeta_3 - 23\right\} + C_F n_f^2\left\{\frac{80}{27}\zeta_2 - \frac{16}{9}\zeta_3 - \frac{17}{9}\right\},
\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}
f_1^{T,i} &= 0,\\
f_2^{T,i} &= C_i C_A\left\{-\frac{22}{3}\zeta_2 - 28\zeta_3 + \frac{808}{27}\right\} + C_i n_f\left\{\frac{4}{3}\zeta_2 - \frac{112}{27}\right\},\\
f_3^{T,i} &= C_i C_A^2\left\{\frac{352}{5}{\zeta_2}^2 + \frac{176}{3}\zeta_2\zeta_3 - \frac{12650}{81}\zeta_2 - \frac{1316}{3}\zeta_3 + 192\zeta_5 + \frac{136781}{729}\right\}\\
&\quad + C_i C_A n_f\left\{-\frac{96}{5}{\zeta_2}^2 + \frac{2828}{81}\zeta_2 + \frac{728}{27}\zeta_3 - \frac{11842}{729}\right\}\\
&\quad + C_i C_F n_f\left\{\frac{32}{5}{\zeta_2}^2 + 4\zeta_2 + \frac{304}{9}\zeta_3 - \frac{1711}{27}\right\}
\end{aligned}$$

$$+ C_i {n_f}^2 \left\{ -\frac{40}{27}\zeta_2 + \frac{112}{27}\zeta_3 - \frac{2080}{729} \right\} . \tag{5.36}$$

We find that the above $B_n^{T,i}$ are identical to the ones that appear in quark and gluon form factors of [145]:

$$B_n^{T,i} = B_n^i \,, \qquad i = q, g, \; n = 1, 2, 3 \,. \tag{5.37}$$

Similar to cusp anomalous dimensions, $f_i^{T,i}$ also satisfy the property of maximally non-abelian and in addition, they coincide with those that appear in the quark and gluon form factors which are available up to three-loop level in the literature [144],

$$f_n^{T,g} = \frac{C_A}{C_F} f_n^{T,q} \qquad \text{and} \qquad f_n^{T,i} = f_n^i \qquad i = q, g \qquad n = 1, 2, 3. \tag{5.38}$$

The UV anomalous dimensions are found to be identically zero due to the conservation of QCD energy momentum tensor, i.e.,

$$\gamma_n^{T,i} = 0. \tag{5.39}$$

The universal behavior of IR poles in terms of the cusp (A^i), collinear (B^i) and soft (f^i) anomalous dimensions provides a crucial check on our computation. The remaining terms namely $g_n^{T,i,k}$'s in Eq. 3.23 can be extracted from the form factors and they are listed in Appendix A.

5.7 Threshold approximated cross sections at N^{3}LO

The availability of $g_n^{T,i,k}$ enable us to obtain the threshold approximated partonic cross sections for the production of a massive spin-2 resonance in quark and gluon initiated process. Following the method described in chapter 3, we expand $\Delta_{SV}^{T,i}$ in a perturbative series of a_s as

$$\Delta^{T,i}_{SV}(z) = \sum_{n=0}^{\infty} a_s^n(\mu_R^2)\Delta^{T,i,n}_{SV}(z,\mu_R^2)\,. \tag{5.40}$$

To obtain $\Delta^{T,i}_{SV}$ at N^{3}LO, the only missing pieces were $g_3^{T,i,1}$ and $\overline{\mathcal{G}}_3^{I,i,1}$. We obtained $\overline{\mathcal{G}}_3^{I,i,1}$ in [80] and this recent computation provides us the first one. Below, we present the threshold approximated partonic cross sections up to three-loop level

$$\Delta^{T,g,0}_{SV} = \delta(1-z)\,, \tag{5.41}$$

$$\begin{aligned}\Delta^{T,g,1}_{SV} &= \delta(1-z)\Big[C_A\Big(-\frac{203}{9}+8\zeta_2\Big)+n_f\Big(\frac{35}{9}\Big)+\log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A\Big(\frac{22}{3}\Big)+n_f\Big(-\frac{4}{3}\Big)\Big\}\Big]\\&\quad+\mathcal{D}_0\Big[\log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A\Big(8\Big)\Big\}\Big]+\mathcal{D}_1\Big[C_A\Big(16\Big)\Big],\end{aligned} \tag{5.42}$$

$$\begin{aligned}\Delta^{T,g,2}_{SV} &= \delta(1-z)\Big[C_A^2\Big(\frac{7801}{324}-\frac{88}{3}\zeta_3-\frac{224}{9}\zeta_2-\frac{4}{5}\zeta_2^2\Big)+n_fC_A\Big(-\frac{2983}{162}+\frac{64}{3}\zeta_3-\frac{94}{9}\zeta_2\Big)\\&\quad+n_fC_F\Big(\frac{61}{3}-16\zeta_3\Big)+n_f^2\Big(\frac{1225}{324}+\frac{8}{3}\zeta_2\Big)+\log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A^2\Big(-\frac{1657}{27}+152\zeta_3+\frac{88}{3}\zeta_2\Big)\\&\quad+n_fC_A\Big(+\frac{647}{27}-\frac{16}{3}\zeta_2\Big)+n_fC_F\Big(-4\Big)+n_f^2\Big(-\frac{70}{27}\Big)\Big\}\\&\quad+\log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{C_A^2\Big(\frac{121}{9}-32\zeta_2\Big)+n_fC_A\Big(-\frac{44}{9}\Big)+n_f^2\Big(\frac{4}{9}\Big)\Big\}\Big]\\&\quad+\mathcal{D}_0\Big[C_A^2\Big(-\frac{1616}{27}+312\zeta_3+\frac{176}{3}\zeta_2\Big)+n_fC_A\Big(\frac{224}{27}-\frac{32}{3}\zeta_2\Big)\\&\quad+\log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A^2\Big(-\frac{1088}{9}-80\zeta_2\Big)+n_fC_A\Big(\frac{200}{9}\Big)\Big\}+\log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{C_A^2\Big(44\Big)\\&\quad+n_fC_A\Big(-8\Big)\Big\}\Big]+\mathcal{D}_1\Big[C_A^2\Big(-\frac{2176}{9}-160\zeta_2\Big)+n_fC_A\Big(\frac{400}{9}\Big)\\&\quad+\log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A^2\Big(\frac{176}{3}\Big)+n_fC_A\Big(-\frac{32}{3}\Big)\Big\}+\log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{C_A^2\Big(64\Big)\Big\}\Big]\\&\quad+\mathcal{D}_2\Big[C_A^2\Big(-\frac{176}{3}\Big)+n_fC_A\Big(\frac{32}{3}\Big)+\log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A^2\Big(192\Big)\Big\}\Big]\\&\quad+\mathcal{D}_3\Big[C_A^2\Big(128\Big)\Big],\end{aligned} \tag{5.43}$$

$$\Delta^{T,g,3}_{SV} = \delta(1-z)\Big[C_A^3\Big(-\frac{303707}{810}-\frac{28636}{9}\zeta_5-\frac{186194}{135}\zeta_3+\frac{13216}{3}\zeta_3^2+\frac{923}{3}\zeta_2$$

$$
\begin{aligned}
&+ \frac{8944}{3}\zeta_2\zeta_3 + \frac{29416}{135}\zeta_2^2 - \frac{64096}{105}\zeta_2^3\Big) + n_f C_A^2\Big(\frac{55546}{405} + \frac{2752}{9}\zeta_5 + \frac{55229}{135}\zeta_3 \\
&- \frac{5369}{27}\zeta_2 - \frac{1508}{3}\zeta_2\zeta_3 - \frac{10393}{135}\zeta_2^2\Big) + n_f C_F C_A\Big(\frac{65866}{405} + 80\zeta_5 - \frac{5552}{45}\zeta_3 - \frac{520}{9}\zeta_2 \\
&- \frac{128}{3}\zeta_2\zeta_3 - \frac{32}{45}\zeta_2^2\Big) + n_f C_F^2\Big(-\frac{241}{9} + 160\zeta_5 - \frac{296}{3}\zeta_3\Big) \\
&+ n_f^2 C_A\Big(-\frac{1487}{135} - \frac{448}{45}\zeta_3 + \frac{1019}{27}\zeta_2 + \frac{304}{45}\zeta_2^2\Big) + n_f^2 C_F\Big(-\frac{2617}{162} + 8\zeta_3 + 24\zeta_2\Big) \\
&+ \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A^3\Big(-\frac{1319}{9} + 5984\zeta_5 - \frac{3496}{3}\zeta_3 + \frac{15232}{27}\zeta_2 - 3872\zeta_2\zeta_3 - \frac{396}{5}\zeta_2^2\Big) \\
&+ n_f C_A^2\Big(\frac{523}{9} + \frac{760}{3}\zeta_3 - \frac{2368}{27}\zeta_2 + \frac{72}{5}\zeta_2^2\Big) + n_f C_F C_A\Big(\frac{55}{3} - 16\zeta_2\Big) + n_f C_F^2\Big(2\Big) \\
&+ n_f^2 C_A\Big(-\frac{41}{9}\Big) + n_f^2 C_F\Big(-\frac{16}{3}\Big)\Big\} + \log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{C_A^3\Big(\frac{110}{3} + \frac{968}{3}\zeta_3 + \frac{736}{3}\zeta_2 \\
&- \frac{1152}{5}\zeta_2^2\Big) + n_f C_A^2\Big(-14 - \frac{176}{3}\zeta_3 - \frac{160}{3}\zeta_2\Big) + n_f C_F C_A\Big(-\frac{22}{3}\Big) + n_f^2 C_A\Big(\frac{4}{3}\Big) \\
&+ n_f^2 C_F\Big(\frac{4}{3}\Big)\Big\} + \log\Big(\frac{q^2}{\mu_F^2}\Big)^3\Big\{C_A^3\Big(\frac{512}{3}\zeta_3 - \frac{352}{3}\zeta_2\Big) + n_f C_A^2\Big(\frac{64}{3}\zeta_2\Big)\Big\}\Big] \\
&+ \mathcal{D}_0\Big[C_A^3\Big(\frac{390086}{729} + 11904\zeta_5 - \frac{46952}{27}\zeta_3 + \frac{29824}{81}\zeta_2 - \frac{23200}{3}\zeta_2\zeta_3 + \frac{4048}{15}\zeta_2^2\Big) \\
&+ n_f C_A^2\Big(-\frac{180844}{729} + \frac{3320}{9}\zeta_3 + \frac{3200}{81}\zeta_2 - \frac{544}{15}\zeta_2^2\Big) + n_f C_F C_A\Big(\frac{3422}{27} - \frac{608}{9}\zeta_3 \\
&- 32\zeta_2 - \frac{64}{5}\zeta_2^2\Big) + n_f^2 C_A\Big(\frac{19808}{729} + \frac{320}{27}\zeta_3 - \frac{160}{9}\zeta_2\Big) + \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A^3\Big(-\frac{66746}{81} \\
&- \frac{3344}{3}\zeta_3 + \frac{4144}{3}\zeta_2 - \frac{4928}{5}\zeta_2^2\Big) + n_f C_A^2\Big(\frac{18052}{81} + \frac{800}{3}\zeta_3 - \frac{1184}{3}\zeta_2\Big) \\
&+ n_f C_F C_A\Big(\frac{268}{3} - 64\zeta_3\Big) + n_f^2 C_A\Big(-\frac{446}{81} + \frac{64}{3}\zeta_2\Big)\Big\} + \log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{C_A^3\Big(\frac{116}{9} \\
&+ 2240\zeta_3 - 176\zeta_2\Big) + n_f C_A^2\Big(\frac{140}{9} + 32\zeta_2\Big) + n_f C_F C_A\Big(-24\Big) + n_f^2 C_A\Big(-\frac{40}{9}\Big)\Big\} \\
&+ \log\Big(\frac{q^2}{\mu_F^2}\Big)^3\Big\{C_A^3\Big(\frac{968}{27} - 256\zeta_2\Big) + n_f C_A^2\Big(-\frac{352}{27}\Big) + n_f^2 C_A\Big(\frac{32}{27}\Big)\Big\}\Big] \\
&+ \mathcal{D}_1\Big[C_A^3\Big(-\frac{6932}{9} - \frac{20416}{3}\zeta_3 + \frac{17120}{9}\zeta_2 - \frac{9856}{5}\zeta_2^2\Big) + n_f C_A^2\Big(\frac{1480}{9} + \frac{4096}{3}\zeta_3 \\
&- \frac{4288}{9}\zeta_2\Big) + n_f C_F C_A\Big(\frac{536}{3} - 128\zeta_3\Big) + n_f^2 C_A\Big(\frac{100}{9} + \frac{128}{9}\zeta_2\Big) \\
&+ \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A^3\Big(-\frac{21536}{27} + 11520\zeta_3 + \frac{5632}{3}\zeta_2\Big) + n_f C_A^2\Big(\frac{2720}{27} - \frac{1024}{3}\zeta_2\Big) \\
&+ n_f C_F C_A\Big(-32\Big)\Big\} + \log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{C_A^3\Big(-\frac{1472}{3} - 2304\zeta_2\Big) + n_f C_A^2\Big(\frac{320}{3}\Big)\Big\} \\
&+ \log\Big(\frac{q^2}{\mu_F^2}\Big)^3\Big\{C_A^3\Big(\frac{704}{3}\Big) + n_f C_A^2\Big(-\frac{128}{3}\Big)\Big\}\Big] + \mathcal{D}_2\Big[C_A^3\Big(-1168 + 11584\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{11968}{3}\zeta_2\Big) + n_f C_A^2\Big(\frac{656}{9} - \frac{2176}{3}\zeta_2\Big) + n_f C_F C_A\Big(32\Big) + n_f^2 C_A\Big(\frac{160}{9}\Big) \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_A^3\Big(-1472 - 5376\zeta_2\Big) + n_f C_A^2\Big(320\Big)\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^2\Big\{C_A^3\Big(352\Big) \\
&+ n_f C_A^2\Big(-64\Big)\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^3\Big\{C_A^3\Big(256\Big)\Big\}\Big] + \mathcal{D}_3\Big[C_A^3\Big(-\frac{18752}{27} - 3584\zeta_2\Big) \\
&+ n_f C_A^2\Big(\frac{2944}{27}\Big) + n_f^2 C_A\Big(\frac{256}{27}\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_A^3\Big(-\frac{5632}{9}\Big) + n_f C_A^2\Big(\frac{1024}{9}\Big)\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^2\Big\{C_A^3\Big(1024\Big)\Big\}\Big] + \mathcal{D}_4\Big[C_A^3\Big(-\frac{7040}{9}\Big) + n_f C_A^2\Big(\frac{1280}{9}\Big) \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_A^3\Big(1280\Big)\Big\}\Big] + \mathcal{D}_5\Big[C_A^3\Big(512\Big)\Big], \qquad (5.44)
\end{aligned}
$$

$$
\Delta_{SV}^{T,q,0} = \delta(1-z)\,, \qquad (5.45)
$$

$$
\begin{aligned}
\Delta_{SV}^{T,q,1} &= \delta(1-z)\Big[C_F\Big(-20 + 8\zeta_2\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F\Big(6\Big)\Big\}\Big] + \mathcal{D}_0\Big[\log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F\Big(8\Big)\Big\}\Big] \\
&+ \mathcal{D}_1\Big[C_F\Big(16\Big)\Big] \qquad (5.46)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{SV}^{T,q,2} &= \delta(1-z)\Big[C_F C_A\Big(-\frac{5941}{36} + 92\zeta_3 + \frac{328}{9}\zeta_2 - \frac{12}{5}\zeta_2^2\Big) + C_F^2\Big(\frac{2293}{12} - 124\zeta_3 \\
&- 70\zeta_2 + \frac{8}{5}\zeta_2^2\Big) + n_f C_F\Big(\frac{461}{18} + 8\zeta_3 - \frac{64}{9}\zeta_2\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F C_A\Big(79 - 24\zeta_3\Big) \\
&+ C_F^2\Big(-117 + 176\zeta_3 + 24\zeta_2\Big) + n_f C_F\Big(-14\Big)\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^2\Big\{C_F C_A\Big(-11\Big) \\
&+ C_F^2\Big(18 - 32\zeta_2\Big) + n_f C_F\Big(2\Big)\Big\}\Big] + \mathcal{D}_0\Big[C_F C_A\Big(-\frac{1616}{27} + 56\zeta_3 + \frac{176}{3}\zeta_2\Big) \\
&+ C_F^2\Big(256\zeta_3\Big) + n_f C_F\Big(\frac{224}{27} - \frac{32}{3}\zeta_2\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F C_A\Big(\frac{536}{9} - 16\zeta_2\Big) \\
&+ C_F^2\Big(-160 - 64\zeta_2\Big) + n_f C_F\Big(-\frac{80}{9}\Big)\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^2\Big\{C_F C_A\Big(-\frac{44}{3}\Big) \\
&+ C_F^2\Big(48\Big) + n_f C_F\Big(\frac{8}{3}\Big)\Big\}\Big] + \mathcal{D}_1\Big[C_F C_A\Big(\frac{1072}{9} - 32\zeta_2\Big) + C_F^2\Big(-320 - 128\zeta_2\Big) \\
&+ n_f C_F\Big(-\frac{160}{9}\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F C_A\Big(-\frac{176}{3}\Big) + C_F^2\Big(96\Big) + n_f C_F\Big(\frac{32}{3}\Big)\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^2\Big\{C_F^2\Big(64\Big)\Big\}\Big] + \mathcal{D}_2\Big[C_F C_A\Big(-\frac{176}{3}\Big) + n_f C_F\Big(\frac{32}{3}\Big) \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F^2\Big(192\Big)\Big]\Big] + \mathcal{D}_3\Big[C_F^2\Big(128\Big)\Big], \qquad (5.47)
\end{aligned}
$$

$$
\Delta_{SV}^{T,q,3} = \delta(1-z)\Big[C_F C_A^2\Big(-\frac{1970041}{972} - \frac{4292}{3}\zeta_5 + \frac{282365}{81}\zeta_3 - \frac{400}{3}\zeta_3^2 + \frac{9685}{27}\zeta_2
$$

$$
\begin{aligned}
&- \frac{1028}{3}\zeta_2\zeta_3 + \frac{5459}{27}\zeta_2^2 + \frac{13264}{315}\zeta_2^3\Big) + C_F^2 C_A\Big(\frac{328511}{108} - \frac{26824}{9}\zeta_5 - \frac{57388}{9}\zeta_3 \\
&+ \frac{3280}{3}\zeta_3^2 + \frac{8894}{27}\zeta_2 + \frac{33728}{9}\zeta_2\zeta_3 - \frac{26288}{135}\zeta_2^2 - \frac{20816}{315}\zeta_2^3\Big) + C_F^3\Big(- \frac{2807}{2} \\
&+ \frac{3344}{3}\zeta_5 + \frac{4300}{3}\zeta_3 + \frac{10336}{3}\zeta_3^2 - \frac{1082}{3}\zeta_2 - 432\zeta_2\zeta_3 + \frac{892}{5}\zeta_2^2 - \frac{184736}{315}\zeta_2^3\Big) \\
&+ n_f C_F C_A\Big(\frac{130871}{243} + \frac{136}{3}\zeta_5 - \frac{29488}{81}\zeta_3 - \frac{16348}{81}\zeta_2 + \frac{224}{3}\zeta_2\zeta_3 - \frac{7988}{135}\zeta_2^2\Big) \\
&+ n_f C_F^2\Big(- \frac{697}{3} + \frac{5536}{9}\zeta_5 + \frac{6200}{9}\zeta_3 - \frac{1256}{27}\zeta_2 - \frac{5504}{9}\zeta_2\zeta_3 + \frac{3536}{135}\zeta_2^2\Big) \\
&+ n_f^2 C_F\Big(- \frac{7345}{243} - \frac{2416}{81}\zeta_3 + \frac{1744}{81}\zeta_2 + \frac{128}{27}\zeta_2^2\Big) \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F C_A^2\Big(\frac{36310}{27} + 80\zeta_5 - \frac{9176}{9}\zeta_3 - \frac{224}{9}\zeta_2 + \frac{68}{5}\zeta_2^2\Big) \\
&+ C_F^2 C_A\Big(- \frac{43727}{18} + 240\zeta_5 + \frac{39688}{9}\zeta_3 + \frac{10696}{27}\zeta_2 - 1120\zeta_2\zeta_3 - \frac{256}{15}\zeta_2^2\Big) \\
&+ C_F^3\Big(\frac{2231}{2} + 5664\zeta_5 - 4128\zeta_3 + 120\zeta_2 - 2752\zeta_2\zeta_3 - \frac{336}{5}\zeta_2^2\Big) \\
&+ n_f C_F C_A\Big(- \frac{11732}{27} + \frac{976}{9}\zeta_3 + \frac{256}{9}\zeta_2 - \frac{8}{5}\zeta_2^2\Big) + n_f C_F^2\Big(\frac{3022}{9} - \frac{4432}{9}\zeta_3 \\
&- \frac{1936}{27}\zeta_2 + \frac{112}{15}\zeta_2^2\Big) + n_f^2 C_F\Big(\frac{820}{27} + \frac{64}{9}\zeta_3 - \frac{32}{9}\zeta_2\Big)\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^2 \Big\{C_F C_A^2\Big(- \frac{971}{3} + 88\zeta_3\Big) + C_F^2 C_A\Big(683 - 1024\zeta_3 - \frac{4288}{9}\zeta_2 + 128\zeta_2^2\Big) \\
&+ C_F^3\Big(- 342 + 1056\zeta_3 + 640\zeta_2 - \frac{1792}{5}\zeta_2^2\Big) + n_f C_F C_A\Big(114 - 16\zeta_3\Big) \\
&+ n_f C_F^2\Big(- 116 + 160\zeta_3 + \frac{640}{9}\zeta_2\Big) + n_f^2 C_F\Big(- \frac{28}{3}\Big)\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^3 \Big\{C_F C_A^2\Big(\frac{242}{9}\Big) \\
&+ C_F^2 C_A\Big(- 66 + \frac{352}{3}\zeta_2\Big) + C_F^3\Big(36 + \frac{512}{3}\zeta_3 - 192\zeta_2\Big) + n_f C_F C_A\Big(- \frac{88}{9}\Big) \\
&+ n_f C_F^2\Big(12 - \frac{64}{3}\zeta_2\Big) + n_f^2 C_F\Big(\frac{8}{9}\Big)\Big\}\Big] \\
&+ \mathcal{D}_0\Big[C_F C_A^2\Big(- \frac{594058}{729} - 384\zeta_5 + \frac{40144}{27}\zeta_3 + \frac{98224}{81}\zeta_2 - \frac{352}{3}\zeta_2\zeta_3 - \frac{2992}{15}\zeta_2^2\Big) \\
&+ C_F^2 C_A\Big(\frac{32320}{27} + \frac{24224}{9}\zeta_3 - \frac{18752}{27}\zeta_2 - 1472\zeta_2\zeta_3 + \frac{1408}{3}\zeta_2^2\Big) + C_F^3\Big(12288\zeta_5 \\
&- 5120\zeta_3 - 6144\zeta_2\zeta_3\Big) + n_f C_F C_A\Big(\frac{125252}{729} - \frac{2480}{9}\zeta_3 - \frac{29392}{81}\zeta_2 + \frac{736}{15}\zeta_2^2\Big) \\
&+ n_f C_F^2\Big(- \frac{1058}{27} - \frac{5728}{9}\zeta_3 + \frac{3104}{27}\zeta_2 - \frac{1472}{15}\zeta_2^2\Big) + n_f^2 C_F\Big(- \frac{3712}{729} + \frac{320}{27}\zeta_3 \\
&+ \frac{640}{27}\zeta_2\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F C_A^2\Big(\frac{62012}{81} - 352\zeta_3 - \frac{6016}{9}\zeta_2 + \frac{352}{5}\zeta_2^2\Big) \\
&+ C_F^2 C_A\Big(- \frac{25834}{9} - 1744\zeta_3 - \frac{4192}{9}\zeta_2 + \frac{1824}{5}\zeta_2^2\Big) + C_F^3\Big(\frac{4586}{3} + 544\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&+2000\zeta_2-\frac{7104}{5}\zeta_2^2\Big)+\boldsymbol{n_fC_FC_A}\Big(-\frac{16408}{81}+192\zeta_2\Big)+\boldsymbol{n_fC_F^2}\Big(\frac{3232}{9}+640\zeta_3\\
&+\frac{832}{9}\zeta_2\Big)+\boldsymbol{n_f^2C_F}\Big(\frac{800}{81}-\frac{128}{9}\zeta_2\Big)\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)^2\Big\{\boldsymbol{C_FC_A^2}\Big(-\frac{7120}{27}+\frac{176}{3}\zeta_2\Big)\\
&+\boldsymbol{C_F^2C_A}\Big(\frac{3848}{3}-192\zeta_3+\frac{1472}{3}\zeta_2\Big)+\boldsymbol{C_F^3}\Big(-936+2432\zeta_3-576\zeta_2\Big)\\
&+\boldsymbol{n_fC_FC_A}\Big(\frac{2312}{27}-\frac{32}{3}\zeta_2\Big)+\boldsymbol{n_fC_F^2}\Big(-\frac{632}{3}-\frac{320}{3}\zeta_2\Big)+\boldsymbol{n_f^2C_F}\Big(-\frac{160}{27}\Big)\Big\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)^3\Big\{\boldsymbol{C_FC_A^2}\Big(\frac{968}{27}\Big)+\boldsymbol{C_F^2C_A}\Big(-176\Big)+\boldsymbol{C_F^3}\Big(144-256\zeta_2\Big)\\
&+\boldsymbol{n_fC_FC_A}\Big(-\frac{352}{27}\Big)+\boldsymbol{n_fC_F^2}\Big(32\Big)+\boldsymbol{n_f^2C_F}\Big(\frac{32}{27}\Big)\Big\}\Big]+\mathcal{D}_1\Big[\boldsymbol{C_FC_A^2}\Big(\frac{124024}{81}\\
&-704\zeta_3-\frac{12032}{9}\zeta_2+\frac{704}{5}\zeta_2^2\Big)+\boldsymbol{C_F^2C_A}\Big(-\frac{15068}{3}-4160\zeta_3-\frac{14720}{9}\zeta_2+\frac{3648}{5}\zeta_2^2\Big)\\
&+\boldsymbol{C_F^3}\Big(\frac{9172}{3}-1984\zeta_3+4000\zeta_2-\frac{14208}{5}\zeta_2^2\Big)+\boldsymbol{n_fC_FC_A}\Big(-\frac{32816}{81}+384\zeta_2\Big)\\
&+\boldsymbol{n_fC_F^2}\Big(\frac{1856}{3}+1280\zeta_3+\frac{2816}{9}\zeta_2\Big)+\boldsymbol{n_f^2C_F}\Big(\frac{1600}{81}-\frac{256}{9}\zeta_2\Big)\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\Big\{\boldsymbol{C_FC_A^2}\Big(-\frac{28480}{27}+\frac{704}{3}\zeta_2\Big)+\boldsymbol{C_F^2C_A}\Big(\frac{59248}{27}+512\zeta_3+\frac{9280}{3}\zeta_2\Big)\\
&+\boldsymbol{C_F^3}\Big(-1872+11008\zeta_3-1152\zeta_2\Big)+\boldsymbol{n_fC_FC_A}\Big(\frac{9248}{27}-\frac{128}{3}\zeta_2\Big)\\
&+\boldsymbol{n_fC_F^2}\Big(-\frac{10240}{27}-\frac{1792}{3}\zeta_2\Big)+\boldsymbol{n_f^2C_F}\Big(-\frac{640}{27}\Big)\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)^2\Big\{\boldsymbol{C_FC_A^2}\Big(\frac{1936}{9}\Big)\\
&+\boldsymbol{C_F^2C_A}\Big(\frac{3824}{9}-256\zeta_2\Big)+\boldsymbol{C_F^3}\Big(-992-2048\zeta_2\Big)+\boldsymbol{n_fC_FC_A}\Big(-\frac{704}{9}\Big)\\
&+\boldsymbol{n_fC_F^2}\Big(-\frac{416}{9}\Big)+\boldsymbol{n_f^2C_F}\Big(\frac{64}{9}\Big)\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)^3\Big\{\boldsymbol{C_F^2C_A}\Big(-\frac{704}{3}\Big)+\boldsymbol{C_F^3}\Big(384\Big)\\
&+\boldsymbol{n_fC_F^2}\Big(\frac{128}{3}\Big)\Big\}\Big]+\mathcal{D}_2\Big[\boldsymbol{C_FC_A^2}\Big(-\frac{28480}{27}+\frac{704}{3}\zeta_2\Big)+\boldsymbol{C_F^2C_A}\Big(-\frac{2368}{9}+1344\zeta_3\\
&+\frac{11264}{3}\zeta_2\Big)+\boldsymbol{C_F^3}\Big(10240\zeta_3\Big)+\boldsymbol{n_fC_FC_A}\Big(\frac{9248}{27}-\frac{128}{3}\zeta_2\Big)\\
&+\boldsymbol{n_fC_F^2}\Big(\frac{160}{9}-\frac{2048}{3}\zeta_2\Big)+\boldsymbol{n_f^2C_F}\Big(-\frac{640}{27}\Big)+\log\left(\frac{q^2}{\mu_F^2}\right)\Big\{\boldsymbol{C_FC_A^2}\Big(\frac{3872}{9}\Big)\\
&+\boldsymbol{C_F^2C_A}\Big(\frac{7520}{3}-768\zeta_2\Big)+\boldsymbol{C_F^3}\Big(-3840-4608\zeta_2\Big)+\boldsymbol{n_fC_FC_A}\Big(-\frac{1408}{9}\Big)\\
&+\boldsymbol{n_fC_F^2}\Big(-\frac{1088}{3}\Big)+\boldsymbol{n_f^2C_F}\Big(\frac{128}{9}\Big)\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)^2\Big\{\boldsymbol{C_F^2C_A}\Big(-1056\Big)+\boldsymbol{C_F^3}\Big(1152\Big)\\
&+\boldsymbol{n_fC_F^2}\Big(192\Big)\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)^3\Big\{\boldsymbol{C_F^3}\Big(256\Big)\Big\}\Big]+\mathcal{D}_3\Big[\boldsymbol{C_FC_A^2}\Big(\frac{7744}{27}\Big)\\
&+\boldsymbol{C_F^2C_A}\Big(\frac{17152}{9}-512\zeta_2\Big)+\boldsymbol{C_F^3}\Big(-2560-3072\zeta_2\Big)+\boldsymbol{n_fC_FC_A}\Big(-\frac{2816}{27}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ n_f C_F^2\Big(-\frac{2560}{9}\Big) + n_f^2 C_F\Big(\frac{256}{27}\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F^2 C_A\Big(-\frac{14080}{9}\Big) + C_F^3\Big(768\Big) \\
&+ n_f C_F^2\Big(\frac{2560}{9}\Big)\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^2 \Big\{C_F^3\Big(1024\Big)\Big\}\Big] + \mathcal{D}_4\Big[C_F^2 C_A\Big(-\frac{7040}{9}\Big) \\
&+ n_f C_F^2\Big(\frac{1280}{9}\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_F^3\Big(1280\Big)\Big\}\Big] + \mathcal{D}_5\Big[C_F^3\Big(512\Big)\Big]. \qquad (5.48)
\end{aligned}
$$

The numerical impact of these findings will be studied somewhere else.

5.8 Universal behavior of Leading Transcendental contribution

The form factor of a scalar composite operator belonging to the stress-energy tensor super-multiplet of conserved currents of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with gauge group SU(N) was studied in the article [206] to three-loop level. In this theory, observation shows that scattering amplitudes can be expressed as a linear combinations of polylogarithmic functions of uniform degree $2l$ with constant coefficients, where l is order of the loop. In other words, the scattering amplitudes in $\mathcal{N} = 4$ SYM exhibit uniform transcendentality unlike QCD loop amplitudes.

In addition to the above-mentioned interesting property exhibited by the scattering amplitudes in $\mathcal{N} = 4$ SYM, the authors of [206] have made an interesting observation in the context of form factors, which is, the results of the quark and gluon form factors in QCD can be related to the form factors of scalar composite operator in $\mathcal{N}$-extended SYM upon employing the identification [207] for the SU(N) color factors as $C_A = C_F = N$ and $n_f = \mathcal{N}N$. For $\mathcal{N} = 1$ the LT part of the quark and gluon form factors in QCD not only coincide with each other but also become identical to the form factors of scalar composite operator computed in $\mathcal{N} = 4$ SYM, up to a normalization factor of 2^l. This holds true even for terms proportional to positive powers of ϵ up to transcendentality 8 (which is the highest order for which all three-loop master integrals are available [208]). In the above-mentioned article [206], this observation has been made up to three-loop level. This correspondence between the QCD form factors and that of the $\mathcal{N} = 4$ SYM is inspired by the well known leading transcendentality principle [207, 209, 210] which relates anomalous dimensions of the twist two operators in $\mathcal{N} = 4$ SYM to the LT terms of such operators computed in QCD. However, unlike the case for $\mathcal{N} = 1$, the quark and gluon form factors in QCD get additional contributions arising from diagrams with scalar particles in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM [210]. Having learned these interesting behavior of the form factors and anomalous dimensions, we are led to examine the LT terms of the form factors $\hat{\mathcal{F}}_g^T$ and $\hat{\mathcal{F}}_q^T$ appearing in the context of spin-2. To our surprise, we find the similar behavior, namely, upon employing the same substitution of the color factors for $\mathcal{N} = 1$, the LT terms of these form factors are not only identical to each other but also coincide with the LT terms of the QCD form factors as well as with the LT terms of the scalar form factors in $\mathcal{N} = 4$ SYM [206]. We find that this is indeed true even for positive powers of ϵ up to three-loop level providing another evidence for the leading transcendentality principle.

5.9 Conclusions

We have presented both quark-antiquark and gluon-gluon form factors of the spin-2 fields that couple to fields of SU(N) gauge theory with n_f light flavors. We have used state-of-the-art methods to perform this computation efficiently as the number of Feynman diagrams involved is quite large compared to other known form factors. We have used IBP and LI identities to express the form factors in terms of 22 master integrals. We have presented the form factors in terms of these master integrals for arbitrary d as well as in powers of $\epsilon = d - 4$ to appropriate order, thanks to the availability of the master integrals to relevant orders in ϵ for further study. These form factors are important components to the scattering cross sections involving spin-2 fields beyond LO in QCD. We have shown that these form factors do satisfy Sudakov integro-differential equation and hence exhibit identical IR structure of other form factors such as those appearing in electro-weak vector boson and Higgs productions up to three-loop level. We have also shown these factors do not require overall renormalization due to the conservation property of the energy momentum tensor. We also find striking similarities between the LT terms of the form factors presented here and those of the quark and gluon form factors in QCD as well as the scalar form factor in $\mathcal{N} = 4$ SYM upon employing appropriate substitution of the SU(N) color factors. Our results will be useful in improving the perturbative predictions of spin-2 resonance production beyond NNLO level at the LHC where searches for such particles are already underway with the upgraded energy and luminosity.

6 Pseudo-scalar Higgs Boson Production at Threshold N^{3}LO and N^{3}LL QCD

In this Chapter we present the first results on the production of pseudo-scalar Higgs boson through gluon fusion at the LHC to N^{3}LO in QCD taking into account only soft gluon effects. We have considered the effective theory where the pseudo-scalar Higgs boson couples with the gluons in the large top quark mass limit. The recent computation of the three-loop pseudo-scalar Higgs boson form factor and the third order universal soft distribution function in QCD enable us to achieve this. Along with the fixed order results, we also present the process dependent resummation coefficient for threshold resummation to N^{3}LL in QCD. Finally, we study the phenomenological impact of these threshold N^{3}LO corrections to pseudo-scalar Higgs boson production at the LHC and their role to reduce the renormalization scale dependence.

6.1 Introduction

The spectacular discovery of the Higgs boson [18, 19] at the LHC has put the SM of elementary particles as well as the theory of electro-weak symmetry breaking [4–8] in the firm footing. The consistency of the measured decay rates of the Higgs boson to a pair of vector bosons namely W^+W^-, ZZ and fermions $b\overline{b}, \tau\overline{\tau}$ with the precise predictions of the SM for the measured Higgs boson mass of 125 GeV within the experimental uncertainty [211, 212] makes this discovery very robust. In addition, there is a strong evidence that the discovered Higgs boson has spin zero and even parity [213,214]. The ongoing 13 TeV run at LHC will indeed provide further scope to study the properties of the Higgs boson in great detail.

While all the predictions of the SM has been tested experimentally, making it a complete theory, the model fails to explain phenomena like baryon asymmetry in the Universe, existence of dark matter, neutrino mass *etc*. There are several extensions of the SM, motivated to address these issues and the minimal version of Supersymmetric Standard Model (MSSM) [215] is one of the most elegant one. The Higgs sector of it is comprised of a pair of Higgs doublets which after symmetry breaking gives two CP even Higgs bosons h, H, one CP odd (pseudo-scalar) Higgs boson (A) and two charged Higgs bosons H$^\pm$ [216–223] . The predicted upper bound on the mass of the lightest Higgs boson (h) up to three-loop level is consistent [224–226] with the recently observed Higgs boson at the LHC. There are continuous efforts to test the predictions of MSSM or its variants and the second phase of the LHC will surely shed more light on them. One of the interesting possibilities is to look for the CP odd Higgs boson in the gluon fusion through heavy fermions as its coupling is appreciable in the small and moderate $\tan\beta$ region, the ratio of vacuum expectation values $v_i, i = 1, 2$. In addition, large gluon flux can boost the cross section.

Since, the production mechanism of the pseudo-scalar Higgs boson of mass m_A in gluon fusion is through heavy quarks, besides $\tan\beta$, the cross section is proportional to square of the strong coupling constant. Similar to the scalar Higgs boson in the SM, the LO prediction of the pseudo-scalar Higgs boson production at the LHC suffers from theoretical uncertainties arising from the unphysical scales of the theory. In particular, large uncertainties come due to the renormalization scale μ_R, introduced in the strong coupling constant renormalization and mild uncertainties arise due to the factorization scale μ_F in the PDFs. Predictions based on one-loop pQCD corrections [39, 227–229] reduce these uncertainties (in the conventional range with the central scale $\mu = m_A/2$ and $m_A = 200$ GeV) from about 48% to 35% while increasing the LO cross section substantially, by as large as 67%. To go beyond NLO, following the approach introduced for the scalar Higgs boson in the SM, one introduces an effective theory considering large top quark mass limit. Such an approach [38, 39], in the case of scalar Higgs boson production turned out to be the most successful one as the finite quark mass effects were found to be within 1% [230–234] of the effective theory output [42–44] at NNLO level. For the production of pseudo-scalar Higgs boson at the hadron colliders, up to NNLO pQCD predictions are available both in the effective theory [44, 55, 56], as well as considering the finite mass effect of the top quark [235, 236]. The small contributions from the finite mass effect of the top quark, again, justifies the validity of the effective theory approach. In the effective theory, for the pseudo-scalar Higgs boson, the NNLO correction increases the NLO cross section by about 15% and reduces the scale uncertainties to about 15%. Due to large gluon flux at the threshold, namely when m_A approaches to the partonic center of mass energy, the cross section is dominated by the presence of soft gluon contributions. These contributions often can spoil the reliability of the predictions based on fixed order perturbative computations. Resummation of large logarithms, resulting from soft gluons, to all orders in the perturbation theory solves this problem. The systematic predictions based on the NNLL resummed result [58–65, 70] demonstrate the reliability of the approach and also reduce the scale uncertainties.

A complete calculation at NNLO [42–44], leading logarithms at N^{3}LO in the threshold limit [59–63] and NNLL soft gluon resummation [58] for production of the scalar Higgs boson in gluon fusion are known for more than a decade. Recently there have been series of works on predicting inclusive scalar Higgs boson production in gluon fusion beyond this level in pQCD. The computation to obtain the full threshold approximated cross section [66] *i.e.* the missing $\delta(1-z)$ contribution at N^{3}LO level, was the first among them.

This was confirmed independently in [67]. Later, the sub-leading collinear logarithms were computed in [68, 69]. A milestone in this direction was recently achieved by Anastasiou *et. al.* who have accomplished the complete N^{3}LO prediction [57] of the scalar Higgs boson production through gluon fusion at the hadron colliders in the effective theory. These third order corrections increase the cross section by a few percent, about 2% and reduce the scale uncertainty by about 2%. Using these predictions, the soft gluon resummed results are obtained at N^{3}LL level in [168, 237]. In [238], the authors have performed resummation in soft-collinear effective theory (SCET) approach for the first time for the scalar Higgs boson production at N^{3}LL. We also note that the three-loop virtual corrections are already available in [239] considering the finite top mass effects and the full result is yet to be computed.

While the next step in the wish list is to obtain the complete N^{3}LO predictions for production of the pseudo-scalar Higgs boson in gluon fusion, the first task in this direction is to obtain the threshold enhanced cross section at N^{3}LO level. One of the crucial ingredients is the form factor of the effective composite operators that couple to the pseudo-scalar Higgs boson, computed between partonic states. One and two-loop results for them between gluon states were computed for NNLO production cross section in [55, 56, 144]. The analytic expressions up to two-loop level can be found in [144], computed considering the space-time dimension $d = 4 + \epsilon$. Threshold corrections to production of the pseudo-scalar Higgs boson at N^{3}LO level requires the knowledge of the form factors up to three-loop level. Also, one and two-loop corrections are required to desired accuracy in ϵ, namely up to ϵ^2 for one-loop and up to ϵ at two-loop. In [95], we obtained the three-loop form factors of the effective composite operators between quark and gluon states along with the lower order ones to desired accuracy in ϵ. In the present Chapter we will describe how threshold corrections at N^{3}LO level can be obtained from the formalism developed in [62, 63] and discussed in Chapter 3 using the available information on recently computed three-loop form factor of the pseudo-scalar Higgs boson [95], the universal soft-collinear distribution [80] and operator renormalization constant [95, 240, 241] and the mass factorization kernels [140, 141] known to three-loop level. In addition, we compute third order corrections to the N-independent part of the resummed cross section [133, 134] using our formalism [62, 63]. We also present the numerical impact of our findings with a brief conclusion.

The underlying effective theory is discussed in Section 6.2. This is followed by a brief description of the framework of the computation in Section 6.3. In Section 6.4 we present the analytic expressions of our findings up to N^{3}LO level in pQCD. In Section 6.5, the N-independent parts of the threshold resummed cross section in Mellin space have been presented up to third order in pQCD. Before making concluding remarks, in Section 6.6 we demonstrate the numerical implications of the new results.

6.2 The effective Lagrangian

The pseudo-scalar Higgs boson couples to gluons only indirectly through a virtual heavy quark loop which can be integrated out in the infinite quark mass limit. The effective Lagrangian [242] describing the interaction between the pseudo-scalar Higgs boson χ^A and QCD particles in the limit of infinite top quark mass, is given by

$$\mathcal{L}^A_{\rm eff} = \chi^A(x)\Big[-\frac{1}{8} C_G O_G(x) - \frac{1}{2} C_J O_J(x)\Big] \tag{6.1}$$

where the composite operators are defined as

$$O_G(x) = G^{\mu\nu}_a \tilde{G}_{a,\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma} G^{\mu\nu}_a G^{\rho\sigma}_a \,, \qquad O_J(x) = \partial_\mu \left(\bar{\psi}\gamma^\mu\gamma_5\psi\right) . \tag{6.2}$$

$G^{\mu\nu}_a$ and ψ represent gluonic field strength tensor and quark field, respectively. Similar to the scalar Higgs boson in the SM, the Wilson coefficients C_G and C_J of these two operators appear as a result of integrating out the heavy quark loop. The Adler-Bardeen theorem [243] prevents C_G to receive any QCD corrections beyond one-loop, whereas C_J

starts only at second order in a_s. These Wilson coefficients [242] are given by

$$\begin{aligned} C_G &= -a_s 2^{\frac{5}{4}} G_F^{\frac{1}{2}} \cot\beta \,, \\ C_J &= -\left[a_s C_F \left(\frac{3}{2} - 3 \ln \frac{\mu_R^2}{m_t^2} \right) + a_s^2 C_J^{(2)} + \cdots \right] C_G \,. \end{aligned} \tag{6.3}$$

G_F is the Fermi constant and $\cot\beta$ is the mixing angle in the Two-Higgs-Doublet model. m_t symbolizes the mass of of the top quark. To define γ_5, we have followed the prescription in dimensional regularization, introduced by 't Hooft and Veltman through [114]

$$\gamma_5 = i \frac{1}{4!} \varepsilon_{\nu_1 \nu_2 \nu_3 \nu_4} \gamma^{\nu_1} \gamma^{\nu_2} \gamma^{\nu_3} \gamma^{\nu_4} \,. \tag{6.4}$$

Here, $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor and all the Lorentz indices are d-dimensional [240]. The renormalization of the strong coupling constant, as in Eq. 2.7, is through

$$\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon = \frac{a_s}{\mu_R^\epsilon} Z(a_s(\mu_R^2)) \tag{6.5}$$

The renormalization constant $Z(a_s(\mu_R^2))$ up to $O(a_s^3)$ is presented in Eq. 2.12.

6.3 Threshold corrections

The inclusive cross section for the production of a colorless pseudo-scalar in gluon fusion at the hadron colliders can be computed using

$$\sigma^A(\tau, m_A^2) = \sigma^{A,(0)}(\mu_R^2) \sum_{a,b=q,\bar{q},g} \int_\tau^1 dy \; \Phi_{ab}(y, \mu_F^2) \Delta_{ab}^A \left(\frac{\tau}{y}, m_A^2, \mu_R^2, \mu_F^2 \right) \tag{6.6}$$

where, the Born cross section at the parton level including the finite top quark mass dependence is given by

$$\sigma^{A,(0)}(\mu_R^2) = \frac{\pi \sqrt{2} G_F}{16} a_s^2 \cot^2\beta \, |\tau_A f(\tau_A)|^2 . \tag{6.7}$$

Here

$$\tau_A = \frac{4m_t^2}{m_A^2} \tag{6.8}$$

and the function $f(\tau_A)$ is given by

$$f(\tau_A) = \begin{cases} \arcsin^2 \frac{1}{\sqrt{\tau_A}} & \tau_A \geq 1\,, \\ -\frac{1}{4}\left(\ln \frac{1-\sqrt{1-\tau_A}}{1+\sqrt{1-\tau_A}} + i\pi\right)^2 & \tau_A < 1\,. \end{cases} \tag{6.9}$$

Φ_{ab} is the parton flux and it reads

$$\Phi_{ab}(y, \mu_F^2) = \int_y^1 \frac{dx}{x} f_a(x, \mu_F^2) f_b\left(\frac{y}{x}, \mu_F^2\right)\,, \tag{6.10}$$

where, f_a and f_b are the PDFs of the initial state partons a and b, renormalized at the factorization scale μ_F. $\Delta_{ab}^A\left(\frac{\tau}{y}, m_A^2, \mu_R^2, \mu_F^2\right)$ is the partonic cross section, for the sub-process initiated by the partons a and b, computed after performing the overall operator UV renormalization at scale μ_R and mass factorization at a scale μ_F. As earlier, the scaling variable τ is defined as q^2/s with $q^2 = m_A^2$.

The aim of this chapter is to study in detail the impact of the soft gluon contributions to the pseudo-scalar Higgs boson production cross section at hadron colliders. We obtain the IR safe contribution by adding the soft part of the cross section to the UV renormalized virtual part and performing mass factorization using appropriate counter terms. This combination is often called the SV cross section whereas the rest is known as hard part. Following the formalism presented in Chapter 3, specially Eq. 3.3, we find that the SV cross section $\Delta_{SV}^{A,g}$ is constructed from the form factors $\mathcal{F}^{A,g}(\hat{a}_s, Q^2, \mu^2, \epsilon)$ with $Q^2 = -q^2$, the overall operator UV renormalization constant $Z^{A,g}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)$, the soft distribution functions $\Phi^{A,g}(\hat{a}_s, q^2, \mu^2, z, \epsilon)$ arising from the real radiations in the partonic sub-processes and the mass factorization kernels $\Gamma_{gg}(\hat{a}_s, \mu_F^2, \mu^2, z, \epsilon)$, as presented in Eq. 3.4. The corresponding form factors and overall operator renormalization constant up to three-loop level were presented in [94], while the mass factorization kernels were known [140, 141] up to three-loop level and the soft distribution functions were obtained to required accuracy in [80]. With the availability of the required components, we present the threshold approximated cross section at third order in pQCD in the next section.

6.4 SV cross sections

In this section, we present our findings of the SV cross section at N^{3}LO along with the results of previous orders. Expanding the SV cross section $\Delta_{SV}^{A,g}$ in powers of a_s, we obtain

$$\Delta_g^{A,\mathrm{SV}}(z,q^2,\mu_R^2,\mu_F^2) = \sum_{i=0}^{\infty} a_s^i \Delta_{g,i}^{A,\mathrm{SV}}(z,q^2,\mu_R^2,\mu_F^2) \tag{6.11}$$

where,

$$\Delta_{g,i}^{A,\mathrm{SV}} = \Delta_{g,i}^{A,\mathrm{SV}}|_\delta \delta(1-z) + \sum_{j=0}^{2i-1} \Delta_{g,i}^{A,\mathrm{SV}}|_{\mathcal{D}_j}\mathcal{D}_j\,.$$

Here, we present the results of the pseudo-scalar Higgs boson production cross section up to N^{3}LO level for the choices of the scale $\mu_R^2 = \mu_F^2$ for which the Eq. 6.11 reads

$$\Delta_g^{A,\mathrm{SV}}(z,q^2) = \sum_{i=0}^{\infty} a_s^i(q^2)\Delta_{g,i}^{A,\mathrm{SV}}(z,q^2,\mu_F^2)\,. \tag{6.12}$$

with the following $\Delta_{g,i}^{A,\mathrm{SV}}(z,q^2,\mu_F^2)$:

$$\Delta_{SV}^{A,g,0} = \delta(1-z)\,, \tag{6.13}$$

$$\Delta_{SV}^{A,g,1} = \delta(1-z)\Big[\boldsymbol{C_A}\Big(8+8\zeta_2\Big)\Big] + \mathcal{D}_0\Big[\log\Big(\frac{q^2}{\mu_F^2}\Big)\boldsymbol{C_A}\Big(8\Big)\Big] + \mathcal{D}_1\Big[\boldsymbol{C_A}\Big(16\Big)\Big] \tag{6.14}$$

$$\begin{aligned}\Delta_{SV}^{A,g,2} = {}& \delta(1-z)\Big[\boldsymbol{C_A^2}\Big(\frac{494}{3} - \frac{220}{3}\zeta_3 + \frac{1112}{9}\zeta_2 - \frac{4}{5}\zeta_2^2\Big) + \boldsymbol{n_f C_A}\Big(-\frac{82}{3} - \frac{8}{3}\zeta_3 - \frac{80}{9}\zeta_2\Big)\\ &+ \boldsymbol{n_f C_F}\Big(-\frac{160}{3} + 16\zeta_3 + 12\log\Big(\frac{q^2}{m_t^2}\Big)\Big) + \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{\boldsymbol{C_A^2}\Big(-\frac{92}{3} + 152\zeta_3 - \frac{88}{3}\zeta_2\Big)\\ &+ \boldsymbol{n_f C_A}\Big(\frac{20}{3} + \frac{16}{3}\zeta_2\Big)\Big\} + \log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{\boldsymbol{C_A^2}\Big(-32\zeta_2\Big)\Big\}\Big] + \mathcal{D}_0\Big[\boldsymbol{C_A^2}\Big(-\frac{1616}{27} + 312\zeta_3\end{aligned}$$

$$
\begin{aligned}
&+ \frac{176}{3}\zeta_2\Big) + \boldsymbol{n_f C_A}\Big(\frac{224}{27} - \frac{32}{3}\zeta_2\Big) + \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{\boldsymbol{C_A^2}\Big(\frac{1112}{9} - 80\zeta_2\Big) + \boldsymbol{n_f C_A}\Big(-\frac{80}{9}\Big)\Big\} \\
&+ \log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{\boldsymbol{C_A^2}\Big(-\frac{44}{3}\Big) + \boldsymbol{n_f C_A}\Big(\frac{8}{3}\Big)\Big\}\Big] + \mathcal{D}_1\Big[\boldsymbol{C_A^2}\Big(\frac{2224}{9} - 160\zeta_2\Big) \\
&+ \boldsymbol{n_f C_A}\Big(-\frac{160}{9}\Big) + \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{\boldsymbol{C_A^2}\Big(-\frac{176}{3}\Big) + \boldsymbol{n_f C_A}\Big(\frac{32}{3}\Big)\Big\} + \log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{\boldsymbol{C_A^2}(64)\Big\}\Big] \\
&+ \mathcal{D}_2\Big[\boldsymbol{C_A^2}\Big(-\frac{176}{3}\Big) + \boldsymbol{n_f C_A}\Big(\frac{32}{3}\Big) + \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{\boldsymbol{C_A^2}\Big(192\Big)\Big\}\Big] \\
&+ \mathcal{D}_3\Big[\boldsymbol{C_A^2}\Big(128\Big)\Big], \qquad (6.15)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{SV}^{A,g,3} &= \delta(1-z)\Big[\boldsymbol{n_f C_J^{(2)}}\Big(-4\Big) + \boldsymbol{C_A^3}\Big(\frac{114568}{27} - \frac{30316}{9}\zeta_5 - 3932\zeta_3 + \frac{13216}{3}\zeta_3^2 \\
&+ \frac{137756}{81}\zeta_2 + \frac{7832}{3}\zeta_2\zeta_3 - \frac{61892}{135}\zeta_2^2 - \frac{64096}{105}\zeta_2^3\Big) + \boldsymbol{n_f C_A^2}\Big(-\frac{113366}{81} + \frac{6952}{9}\zeta_5 \\
&+ \frac{8840}{27}\zeta_3 - \frac{10888}{81}\zeta_2 - \frac{2000}{3}\zeta_2\zeta_3 + \frac{21032}{135}\zeta_2^2\Big) + \boldsymbol{n_f C_F C_A}\Big(-1797 + 160\zeta_5 \\
&+ \frac{1856}{3}\zeta_3 - \frac{4160}{9}\zeta_2 + 192\zeta_2\zeta_3 + \frac{176}{45}\zeta_2^2 + 96\log\Big(\frac{q^2}{m_t^2}\Big)(1+\zeta_2)\Big) \\
&+ \boldsymbol{n_f C_F^2}\Big(\frac{457}{3} - 320\zeta_5 + 208\zeta_3\Big) + \boldsymbol{n_f^2 C_A}\Big(\frac{6914}{81} + \frac{688}{27}\zeta_3 - \frac{1696}{81}\zeta_2 - \frac{608}{45}\zeta_2^2\Big) \\
&+ \boldsymbol{n_f^2 C_F}\Big(\frac{1498}{9} - \frac{224}{3}\zeta_3 - \frac{40}{9}\zeta_2 - \frac{32}{45}\zeta_2^2\Big) + \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{C_A^3\Big(-\frac{36064}{27} + 5984\zeta_5 \\
&+ \frac{36152}{9}\zeta_3 - \frac{14128}{27}\zeta_2 - 3872\zeta_2\zeta_3 - \frac{220}{3}\zeta_2^2\Big) + \boldsymbol{n_f C_A^2}\Big(\frac{13064}{27} - 416\zeta_3 + \frac{1760}{9}\zeta_2 \\
&+ \frac{40}{3}\zeta_2^2\Big) + \boldsymbol{n_f C_F C_A}\Big(\frac{3628}{9} - \frac{352}{3}\zeta_3 + 16\zeta_2 - 88\log\Big(\frac{q^2}{m_t^2}\Big)\Big) \\
&+ \boldsymbol{n_f^2 C_A}\Big(-\frac{976}{27} - \frac{32}{9}\zeta_3 - \frac{320}{27}\zeta_2\Big) + \boldsymbol{n_f^2 C_F}\Big(-\frac{640}{9} + \frac{64}{3}\zeta_3 + 16\log\Big(\frac{q^2}{m_t^2}\Big)\Big)\Big\} \\
&+ \log\Big(\frac{q^2}{\mu_F^2}\Big)^2\Big\{\boldsymbol{C_A^3}\Big(\frac{1012}{9} - 792\zeta_3 - \frac{5624}{9}\zeta_2 - \frac{1152}{5}\zeta_2^2\Big) + \boldsymbol{n_f C_A^2}\Big(-\frac{404}{9} + 144\zeta_3 \\
&+ 32\zeta_2\Big) + \boldsymbol{n_f^2 C_A}\Big(\frac{40}{9} + \frac{32}{9}\zeta_2\Big)\Big\} + \log\Big(\frac{q^2}{\mu_F^2}\Big)^3\Big\{\boldsymbol{C_A^3}\Big(\frac{512}{3}\zeta_3 + \frac{352}{3}\zeta_2\Big) \\
&+ \boldsymbol{n_f C_A^2}\Big(-\frac{64}{3}\zeta_2\Big)\Big\}\Big] + \mathcal{D}_0\Big[\boldsymbol{C_A^3}\Big(-\frac{943114}{729} + 11904\zeta_5 + \frac{210448}{27}\zeta_3 + \frac{175024}{81}\zeta_2 \\
&- \frac{23200}{3}\zeta_2\zeta_3 + \frac{4048}{15}\zeta_2^2\Big) + \boldsymbol{n_f C_A^2}\Big(\frac{173636}{729} - \frac{7600}{9}\zeta_3 - \frac{41680}{81}\zeta_2 - \frac{544}{15}\zeta_2^2\Big) \\
&+ \boldsymbol{n_f C_F C_A}\Big(\frac{3422}{27} - \frac{608}{9}\zeta_3 - 32\zeta_2 - \frac{64}{5}\zeta_2^2\Big) + \boldsymbol{n_f^2 C_A}\Big(-\frac{3712}{729} + \frac{320}{27}\zeta_3 + \frac{640}{27}\zeta_2\Big) \\
&+ \log\Big(\frac{q^2}{\mu_F^2}\Big)\Big\{\boldsymbol{C_A^3}\Big(\frac{207308}{81} - \frac{11264}{3}\zeta_3 - \frac{6784}{3}\zeta_2 - \frac{4928}{5}\zeta_2^2\Big) + \boldsymbol{n_f C_A^2}\Big(-\frac{39880}{81} \\
&+ \frac{1472}{3}\zeta_3 + \frac{3008}{9}\zeta_2\Big) + \boldsymbol{n_f C_F C_A}\Big(-500 + 192\zeta_3 + 96\log\Big(\frac{q^2}{m_t^2}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ n_f^2 C_A\Big(\frac{800}{81} - \frac{128}{9}\zeta_2\Big)\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^2 \Big\{C_A^3\Big(-\frac{16912}{27} + 2240\zeta_3 + \frac{1232}{3}\zeta_2\Big) \\
&+ n_f C_A^2\Big(\frac{4328}{27} - \frac{224}{3}\zeta_2\Big) + n_f C_F C_A\Big(8\Big) + n_f^2 C_A\Big(-\frac{160}{27}\Big)\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^3 \Big\{C_A^3\Big(\frac{968}{27} - 256\zeta_2\Big) + n_f C_A^2\Big(-\frac{352}{27}\Big) + n_f^2 C_A\Big(\frac{32}{27}\Big)\Big\}\Big] \\
&+ \mathcal{D}_1\Big[C_A^3\Big(\frac{414616}{81} - \frac{22528}{3}\zeta_3 - \frac{13568}{3}\zeta_2 - \frac{9856}{5}\zeta_2^2\Big) + n_f C_A^2\Big(-\frac{79760}{81} \\
&+ \frac{2944}{3}\zeta_3 + \frac{6016}{9}\zeta_2\Big) + n_f C_F C_A\Big(-1000 + 384\zeta_3 + 192\log\left(\frac{q^2}{m_t^2}\right)\Big) \\
&+ n_f^2 C_A\Big(\frac{1600}{81} - \frac{256}{9}\zeta_2\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_A^3\Big(-\frac{26752}{9} + 11520\zeta_3 + \frac{9152}{3}\zeta_2\Big) \\
&+ n_f C_A^2\Big(\frac{18016}{27} - \frac{1664}{3}\zeta_2\Big) + n_f C_F C_A\Big(32\Big) + n_f^2 C_A\Big(-\frac{640}{27}\Big)\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^2 \Big\{C_A^3\Big(1680 - 2304\zeta_2\Big) + n_f C_A^2\Big(-\frac{1984}{9}\Big) + n_f^2 C_A\Big(\frac{64}{9}\Big)\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^3 \Big\{C_A^3\Big(-\frac{704}{3}\Big) + n_f C_A^2\Big(\frac{128}{3}\Big)\Big\}\Big] \\
&+ \mathcal{D}_2\Big[C_A^3\Big(-\frac{79936}{27} + 11584\zeta_3 + \frac{11968}{3}\zeta_2\Big) + n_f C_A^2\Big(\frac{16928}{27} - \frac{2176}{3}\zeta_2\Big) \\
&+ n_f C_F C_A\Big(32\Big) + n_f^2 C_A\Big(-\frac{640}{27}\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_A^3\Big(\frac{43424}{9} - 5376\zeta_2\Big) \\
&+ n_f C_A^2\Big(-\frac{5248}{9}\Big) + n_f^2 C_A\Big(\frac{128}{9}\Big)\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^2 \Big\{C_A^3\Big(-1056\Big) + n_f C_A^2\Big(192\Big)\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^3 \Big\{C_A^3\Big(256\Big)\Big\}\Big] + \mathcal{D}_3\Big[C_A^3\Big(\frac{86848}{27} - 3584\zeta_2\Big) + n_f C_A^2\Big(-\frac{10496}{27}\Big) \\
&+ n_f^2 C_A\Big(\frac{256}{27}\Big) + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_A^3\Big(-\frac{14080}{9}\Big) + n_f C_A^2\Big(\frac{2560}{9}\Big)\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^2 \Big\{C_A^3\Big(1024\Big)\Big\}\Big] + \mathcal{D}_4\Big[C_A^3\Big(-\frac{7040}{9}\Big) + n_f C_A^2\Big(\frac{1280}{9}\Big) \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{C_A^3\Big(1280\Big)\Big\}\Big] + \mathcal{D}_5\Big[C_A^3\Big(512\Big)\Big]. \qquad (6.16)
\end{aligned}
$$

The SV cross section up to NNLO are in agreement with the existing ones, computed in the article [44, 55, 56]. The result at N^{3}LO i.e. $\Delta_{SV}^{A,g,3}$ is the new one.

6.5 Threshold resummation

The computation of the fixed order results, in power series expansion of the strong coupling constant a_s, has already reached a milestone with spectacular accuracy. However, in certain cases, it becomes necessary to resum the dominant contributions to all orders in a_s to achieve more reliable predictions and to reduce the scale uncertainties significantly. In case of threshold approximations, due to soft gluon emission, the fixed order corrections in pQCD may yield large threshold logarithms of the kind $\mathcal{D}_i$, defined in Eq. 3.1. Hence we must resum these logarithmic contributions to all orders in a_s. The resummation of these so-called Sudakov logarithms is usually pursued in Mellin space using the formalism developed in [133, 134, 244, 245]. Another approach is to utilize the framework of SCET [246–252]. Here, we consider the Mellin space formalism.

6.5.1 Mellin space prescription

Under this prescription, the threshold resummation is performed in Mellin-N space where the N-th order Mellin moment is defined with respect to the partonic scaling variable z. In Mellin space, the threshold limit $z \to 1$ corresponds to $N \to \infty$ and the plus distributions $\mathcal{D}_i$, Eq. 3.1, take the form $\ln^{i-1} N$. These logarithmic contributions are evaluated to all orders by performing the threshold resummation through [133, 134, 244, 245]

$$\Delta^{A,\mathrm{res}}_{g,N}(q^2, \mu_R^2, \mu_F^2) = C^{A,\mathrm{th}}_g(q^2, \mu_R^2, \mu_F^2)\Delta_{g,N}(q^2)\,. \tag{6.17}$$

The component $C^{A,\mathrm{th}}_g$ depends on both the initial as well as final state particles, though it is independent of the variable N. On the other hand, the remaining part $\Delta_{g,N}$ does not care about the details of the final state particle, it only depends on the initial state partons and the variable N. Being independent of the nature of the final state, $\Delta_{g,N}$ can be considered as a universal quantity which is same for any operator. In addition, it is investigated in the articles [133, 134] that it arises solely from the soft parton radiation and it resums all the perturbative contributions $a_s^n \ln^m N$ ($m \geq 0$) to all orders. Our goal is to calculate the threshold resummation factor $C_g^{A,\rm th}$ which encapsulates all the remaining N-independent contributions to the resummed partonic cross section 6.17. Below, we demonstrate the prescription based on our formalism to calculate this quantity $C_g^{A,\rm th}$ order by order in perturbation theory.

In the article [63], it was shown how the soft distribution functions $\Phi^{A,g}(=\Phi^g)$ captures all the features of the N-space resummation. In this section, we discuss that prescription briefly in the present context. Using the well known identity

$$\frac{1}{1-z}\left[(1-z)^2\right]^{j\frac{\epsilon}{2}} = \frac{1}{j\epsilon}\delta(1-z) + \left(\frac{1}{1-z}\left[(1-z)^2\right]^{j\frac{\epsilon}{2}}\right)_+ , \tag{6.18}$$

we can express the soft distribution function as

$$\begin{aligned}\Phi^{A,g} = &\left(\frac{1}{1-z}\left\{\int_{\mu_R^2}^{q^2(1-z)^2}\frac{d\lambda^2}{\lambda^2}A^g\left(a_s(\lambda^2)\right) + \overline{G}^{A,g}\left(a_s(q^2(1-z)^2),\epsilon\right)\right\}\right)_+\\ &+\delta(1-z)\sum_{n=1}^{\infty}\hat{a}_s^n\left(\frac{q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}}S_\epsilon^n\hat{\phi}^{A,g,n}(\epsilon) + \left(\frac{1}{1-z}\right)_+\sum_{n=1}^{\infty}\hat{a}_s^n\left(\frac{\mu_R^2}{\mu^2}\right)^{n\frac{\epsilon}{2}}S_\epsilon^n\overline{K}^{g,n}(\epsilon)\end{aligned} \tag{6.19}$$

where, A^g is the cusp anomalous dimension corresponding to gluonic operator. All the poles in ϵ are contained within $\overline{K}^g$ and the finite terms are dumped into $\overline{G}^{A,g}$. The components $\overline{K}^{g,n}(\epsilon)$ are defined through the expansion of $\overline{K}^g$ in powers of $\hat{a}_s$ as in Eq. 3.32. The identification of the first plus distribution part of $\Phi^{A,g}$, Eq. 6.19, with the factor contributing to the process independent $\Delta_{g,N}(q^2)$ has been discussed in [63] which reads

$$\Delta_{g,N} = \exp\left[\int_0^1 dz\frac{z^{N-1}-1}{1-z}\left\{2\int_{q^2}^{q^2(1-z)^2}\frac{d\lambda^2}{\lambda^2}A^g\left(a_s(\lambda^2)\right) + D^g\left(a_s(q^2(1-z)^2)\right)\right\}\right] \tag{6.20}$$

with

$$D^g\left(a_s(q^2(1-z)^2)\right) = 2\overline{G}^g\left(a_s(q^2(1-z)^2),\epsilon\right)|_{\epsilon=0} \, . \tag{6.21}$$

In the above expression, the superscript A has been omitted to emphasize the universal nature of these quantities. The remaining part of the Eq. 6.19 along with the other parts, namely, form factor, operator renormalization constant and mass factorization kernel in Eq. 3.5 contribute to $C_g^{A,\text{th}}$. Expanding this in powers of a_s as

$$C_g^{A,\text{th}} = 1 + \sum_{j=1}^{\infty} a_s^j \, C_{g,j}^{A,\text{th}} \,, \tag{6.22}$$

we determine $C_{g,j}^{A,\text{th}}$ up to three-loop ($j = 3$) order which are provided below (with the choice $\mu_R^2 = \mu_F^2 = q^2$):

$$\begin{aligned}
C_{g,1}^{A,\text{th}} &= \boldsymbol{C_A}\Big\{8 + 8\zeta_2\Big\}\,, \\
C_{g,2}^{A,\text{th}} &= \boldsymbol{C_A^2}\Big\{\frac{494}{3} + \frac{1112}{9}\zeta_2 + 12\zeta_2^2 - \frac{220}{3}\zeta_3\Big\} + \boldsymbol{C_A n_f}\Big\{-\frac{82}{3} - \frac{80}{9}\zeta_2 - \frac{8}{3}\zeta_3\Big\} \\
&\quad + \boldsymbol{C_F n_f}\Big\{-\frac{160}{3} + 12\ln\left(\frac{q^2}{m_t^2}\right) + 16\zeta_3\Big\}\,, \\
C_{g,3}^{A,\text{th}} &= \boldsymbol{n_f C_J^{(2)}}\Big\{-4\Big\} + \boldsymbol{C_F n_f^2}\Big\{\frac{1498}{9} - \frac{40}{9}\zeta_2 - \frac{32}{45}\zeta_2^2 - \frac{224}{3}\zeta_3\Big\} + \boldsymbol{C_F^2 n_f}\Big\{\frac{457}{3} + 208\zeta_3 \\
&\quad - 320\zeta_5\Big\} + \boldsymbol{C_A^2 n_f}\Big\{-\frac{113366}{81} - \frac{10888}{81}\zeta_2 + \frac{17192}{135}\zeta_2^2 + \frac{584}{3}\zeta_3 - \frac{464}{3}\zeta_2\zeta_3 + \frac{808}{9}\zeta_5\Big\} \\
&\quad + \boldsymbol{C_A^3}\Big\{\frac{114568}{27} + \frac{137756}{81}\zeta_2 - \frac{4468}{27}\zeta_2^2 - \frac{32}{5}\zeta_2^3 - \frac{80308}{27}\zeta_3 - \frac{616}{3}\zeta_2\zeta_3 + 96\zeta_3^2 \\
&\quad + \frac{3476}{9}\zeta_5\Big\} + \boldsymbol{C_A n_f^2}\Big\{\frac{6914}{81} - \frac{1696}{81}\zeta_2 - \frac{608}{45}\zeta_2^2 + \frac{688}{27}\zeta_3\Big\} + \boldsymbol{C_A C_F n_f}\Big\{-1797 \\
&\quad + 96\ln\left(\frac{q^2}{m_t^2}\right) - \frac{4160}{9}\zeta_2 + 96\ln\left(\frac{q^2}{m_t^2}\right)\zeta_2 + \frac{176}{45}\zeta_2^2 + \frac{1856}{3}\zeta_3 + 192\zeta_2\zeta_3 \\
&\quad + 160\zeta_5\Big\}\,.
\end{aligned} \tag{6.23}$$

The above new result of $C_{g,3}^{A,\text{th}}$ along with the universal factor $\Delta_{g,N}$ provide the threshold resummed cross section of the pseudo-scalar Higgs boson production at N^{3}LL accuracy. The more elaborate discussion on this prescription to perform threshold resummation is presented later in [253].

6.6 Numerical impact of SV cross section

In this section, we present our findings on the numerical impact of threshold N^{3}LO predictions in QCD for the production of a pseudo-scalar Higgs boson at the LHC and also make comparison with the corresponding results for the SM Higgs boson. While we have all the ingredients up to three-loop level, the one of the Wilson coefficients, namely, $C_J^{(2)}$ is known only to two-loop level. Note that C_G is exact due to Adler-Bardeen theorem [243]. Due to the unavailability of $C_J^{(2)}$ in the literature, we discuss the impact of missing three-loop contribution later on in this section by varying this quantity. As we are interested in quantifying the QCD effects, we assume that pseudo-scalar Higgs boson couples only to top quarks. Hence, the dominant contribution resulting from bottom quark initiated processes can be included in a systematic way in our numerical study but we do not perform it here. Moreover, our predictions are based on the effective theory approach where the top quarks are integrated out and we have only light quarks. Like in the case of predictions for the scalar Higgs boson production in the effective theory, for the pseudo-scalar Higgs boson production we multiply the born cross section computed using the finite top mass (m_t = 172.5 GeV) with higher orders which are obtained in the effective theory. Without loss of generality, we normalize the cross section by $\cot^2\beta$. The mass of the pseudo-scalar Higgs boson is taken to be m_A = 200 GeV. We use `MSTW2008` [161] PDFs throughout where the LO, NLO and NNLO parton level cross sections are convoluted with the corresponding `MSTW2208lo, MSTW2008nlo` and `MSTW2008nnlo` PDFs while for N^{3}LO$_{\rm SV}$ cross sections we use `MSTW2008nnlo` PDFs. The strong coupling constant is provided by the respective PDFs from `LHAPDF` with $\alpha_s(m_Z)$ = 0.1394(LO), 0.12018(NLO) and 0.11707(NNLO).

To estimate the impact of QCD corrections, we define the K-factors as

$$\mathrm{K}^{(1)} = \frac{\sigma^{\rm NLO}}{\sigma^{\rm LO}}, \qquad \mathrm{K}^{(2)} = \frac{\sigma^{\rm NNLO}}{\sigma^{\rm LO}}, \qquad \mathrm{K}^{(3)} = \frac{\sigma^{\rm N^3LO_{SV}}}{\sigma^{\rm LO}} \tag{6.24}$$

In Figure 6.1, for LHC13, we plot the pseudo-scalar Higgs boson production cross section

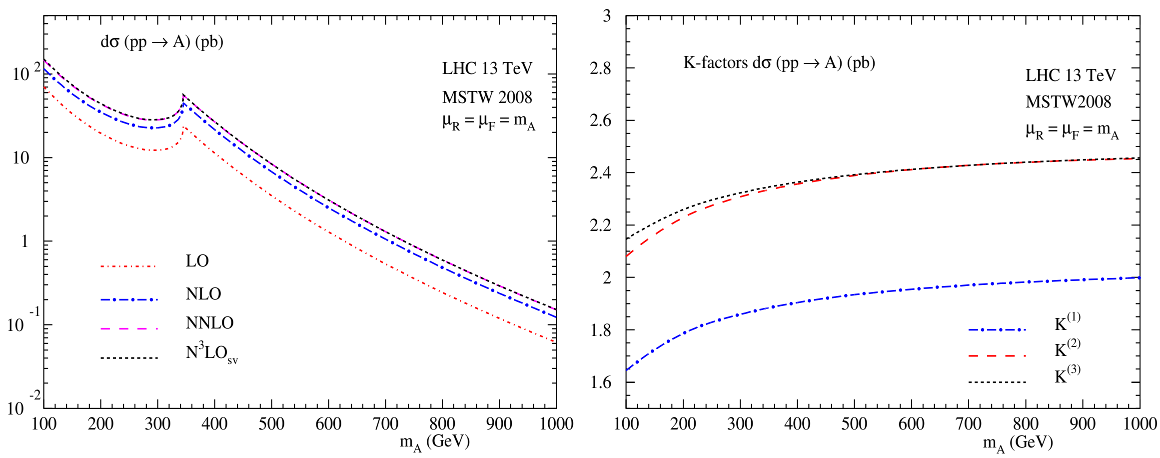


Figure 6.1: Pseudo-scalar Higgs boson production cross section (left panel) for LHC13 and the corresponding K-factors (right panel). The observed spike at 345 GeV indicates the top quark pair threshold region.

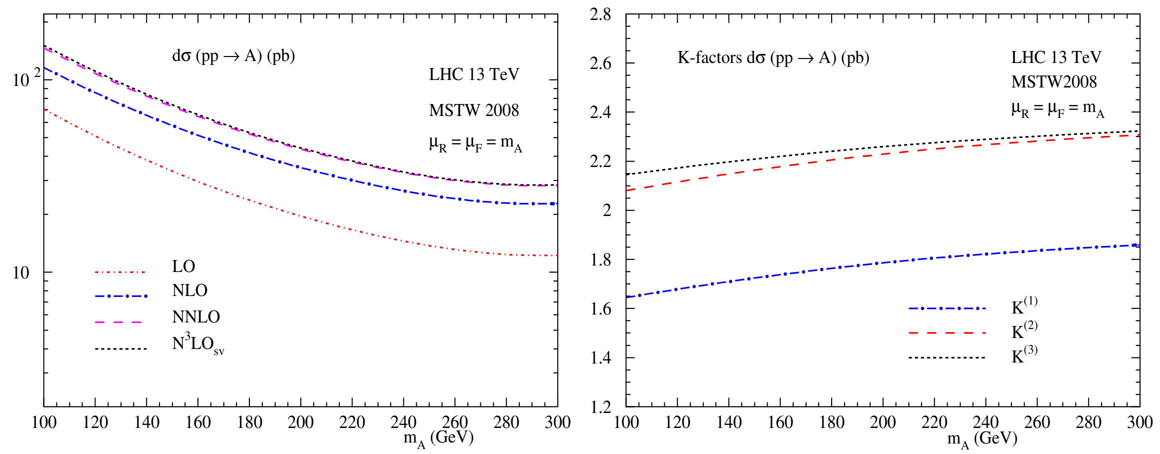


Figure 6.2: Same as Figure 6.1 but for smaller values of m_A.

as a function of its mass m_A. Since we retain the dependence on the m_t at the born level, beyond the top pair threshold ($\tau_A > 1$), due to change in the functional dependence of τ_A one finds a spike at $2m_t$ (left panel). We note here that the effective theory (EFT) formalism formally holds only for the pseudo scalar masses up to top pair threshold. However, from the knowledge of the QCD corrections to Higgs boson production up to NNLO, we notice that below the top pair threshold, the difference between the results of EFT and finite top contributions is about 5% and is even smaller at NNLO, about 1%. We assumed the same to hold in the QCD corrections for pseudo scalar production for scalar masses above the top pair threshold. The corresponding K-factors are given in the right panel and are in general found to increase with m_A. The NLO correction enhances the LO predictions by as much as 100% for m_A = 1 TeV, whereas the NNLO correction adds about an additional 45%. On the other hand the N^{3}LO$_{SV}$ correction is found to be about 1.5% of LO for small mass region m_A < 300 GeV and for higher m_A values the correction at the N^{3}LO$_{SV}$ level becomes even smaller, about 0.3% for m_A = 1 TeV. In either case, these N^{3}LO$_{SV}$ effects show a convergence of the perturbation series. In Figure 6.2, we present similar results but only for pseudo-scalar masses below the top pair threshold where the effective theory approximation works very well.

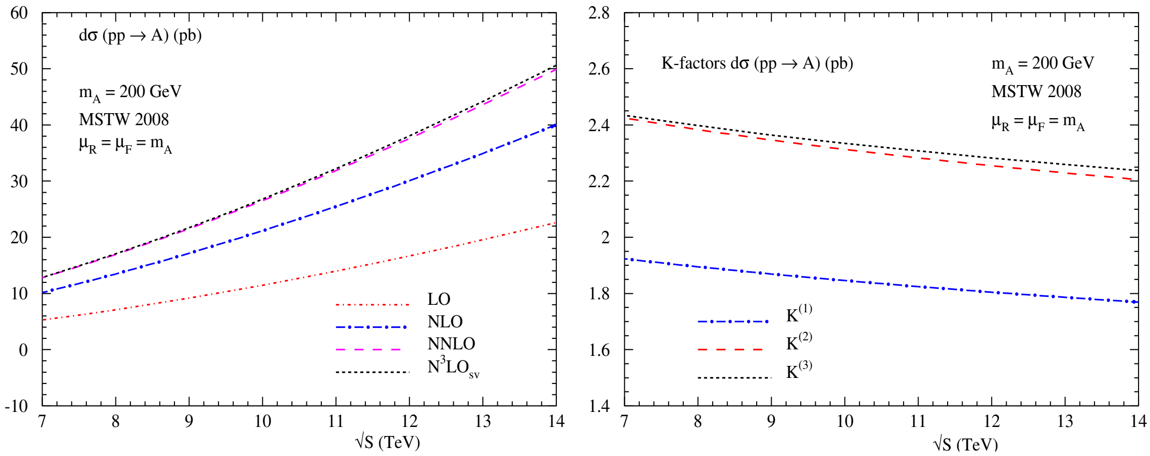


Figure 6.3: Pseudo-scalar Higgs boson production cross sections as a function of $\sqrt{S}$ (left panel) and the corresponding K-factors (right panel).

In Figure 6.3, we present the cross sections as a function of the center of mass energy $\sqrt{S}$ of the incoming protons at the LHC. The increase in the cross sections (left panel) with $\sqrt{S}$ is simply because of the increase in the corresponding parton fluxes for any given m_A. On the contrary, the corresponding K-factors (right panel) increase with decreasing $\sqrt{S}$ for fixed m_A. A similar pattern is shown both in Figure 6.1 & Figure 6.2 where the K-factors increase with m_A for a given $\sqrt{S}$. The guiding principle for the behavior of the K-factors in these two cases is the same, namely, as m_A approaches $\sqrt{S}$, the cross sections are dominated by large soft gluon effects.

$\sqrt{S}$ TeV	SM Higgs			Pseudo-scalar		
	$K^{(1)}$	$K^{(2)}$	$K^{(3)}$	$K^{(1)}$	$K^{(2)}$	$K^{(3)}$
7	1.83	2.31	2.44	1.84	2.34	2.37
8	1.79	2.27	2.40	1.81	2.29	2.33
10	1.74	2.19	2.33	1.76	2.22	2.26
13	1.68	2.10	2.24	1.69	2.13	2.18
14	1.66	2.08	2.22	1.67	2.10	2.16

Table 6.1: K-factors for the scalar and pseudo-scalar Higgs boson production cross sections up to N^3LO_{SV} for different energies at LHC. Here, $m_H = m_A = 125$GeV.

The QCD corrections to pseudo-scalar Higgs boson production are found to be similar to those of the SM Higgs production due to universal IR structure of the gluon initiated processes. We give a numerical comparison between their K-factors at various orders. We take $m_H = m_A = 125$ GeV and ignore bottom as well as other light quarks and electroweak effects for both the cases. Although the full N^3LO QCD corrections are already available for the SM Higgs boson, for comparison we take into account only the N^3LO_{SV}. Table 6.1 contains the K-factors, defined in Eq. 6.24 up to N^3LO_{SV} in QCD for both Higgs and pseudo-scalar Higgs boson as a function of $\sqrt{S}$. For this mass region, the QCD corrections are positive and hence the K-factors increase with the order in the perturbation theory. Moreover, these K-factors, following the line of argument given before, are found to decrease with $\sqrt{S}$ but they are identical in both the cases. The difference between the Higgs and the pseudo-scalar Higgs boson cross sections in their respective K-factors is noticed at the second decimal place only. At three-loop level, $K^{(3)}$ is found to be around 2.4(2.2) for 7(14) TeV case.

Mass	SM Higgs				Pseudo-scalar			
	LO	NLO	NNLO	N^3LO_{SV}	LO	NLO	NNLO	N^3LO_{SV}
124	20.32	34.08	42.76	45.60	47.02	79.46	100.03	102.54
125	20.01	33.58	42.13	44.92	46.32	78.35	98.61	101.06
126	19.70	33.10	41.51	44.26	45.63	77.26	97.22	99.62

Table 6.2: Scalar and pseudo-scalar Higgs boson cross sections up to N^3LO_{SV} for LHC13.

The tiny difference between them can be attributed to the presence of an additional operator present in the effective interaction, namely O_J which along with the matching coefficient formally enters from NNLO onwards for the gluon initiated processes. For quark anti-quark initiated processes, this contribution vanishes as the quark flavors are massless. The gluon initiated processes involving only O_J can contribute at N^{4}LO and beyond. However, the interference effects of O_G and O_J will show up in the gluon initiated processes at NNLO. Thus, the operator O_J has non-zero contributions at the lowest order namely at two-loop level. However, the presence of such an interference contribution is found to be very small and is the main difference between the SM Higgs and the pseudo-scalar Higgs boson contribution. The QCD corrections through soft and collinear gluon emissions for this interference contribution will be of even higher order and hence will contribute at the three-loop level and beyond. In Table 6.2, we present the Higgs and pseudo-scalar Higgs boson production cross sections up to N^{3}LO$_{\rm SV}$ as a function of the scalar mass around 125 GeV. The pseudo-scalar Higgs boson cross section is about twice as big as that of the Higgs boson and the convergence of perturbation series is good and the K-factors are roughly the same for both the cases.

We have also studied the impact of missing three-loop contribution to C_J i.e. $C_J^{(2)}$. At higher orders starting from N^{3}LO$_{\rm SV}$ onwards, the second term $C_J^{(2)}$ in the Wilson coefficient C_J can given non-zero contribution. To estimate the numerical impact of this term, we assume the following form of $C_J^{(2)}$ as

$$C_J^{(2)} = \left[a \ln^2 \frac{\mu_R^2}{m_t^2} + b \ln \frac{\mu_R^2}{m_t^2} + c \right], \tag{6.25}$$

and vary the parameters a, b and c in the range $[-10, 10]$. We found that the contribution of such a $C_J^{(2)}$ term changes the cross sections only at the third decimal place and hence we ignore its contribution in the rest of our phenomenological study.

Since the predictions are sensitive to the choice of parton density functions, we have estimated the uncertainty resulting from them by choosing the central fit for various well known PDF sets such ABM11 [254] , CT10 [255], MSTW2008 [161] and NNPDF23 [256]. For N^{3}LO$_{\rm SV}$ cross sections, however, we use NNLO PDF sets. The corresponding strong

PDF set	SM Higgs			Pseudo-scalar		
	NLO	NNLO	N^3LO_{SV}	NLO	NNLO	N^3LO_{SV}
ABM11	33.19	39.59	41.99	77.42	92.66	94.64
CT10	31.79	41.84	44.67	74.15	97.94	100.44
MSTW2008	33.59	42.13	44.92	78.35	98.61	101.06
NNPDF 23	33.55	43.01	45.87	78.26	100.70	103.19

Table 6.3: PDF uncertainties in the scalar and pseudo-scalar Higgs boson production cross sections up to N^3LO_{SV} for LHC13 and for $m_H = m_A = 125$GeV.

coupling constant is directly taken from the `LHAPDF` [257]. In Table 6.3, we present the SM Higgs boson and pseudo-scalar Higgs boson production cross sections at NLO, NNLO and N^3LO_{SV} for LHC13. We find that for NLO, CT10 gives lowest cross section while MSTW2008 gives highest, whereas for NNLO and N^3LO_{SV}, ABM11 gives lowest and NNPDF23 gives highest. The percentage uncertainty arising from PDF sets at any order is defined as $(\sigma^A_{\max} - \sigma^A_{\min})/\sigma^A_{\min} \times 100$ where, $\sigma^A_{\max}$ and $\sigma^A_{\min}$ are the highest and lowest cross sections at any order obtained from the PDFs considered, respectively. This PDF uncertainties in the case of Higgs boson cross sections are about 5.7% at NLO, 8.6% at NNLO and 9.2% at N^3LO_{SV}. For pseudo-scalar Higgs boson production the cross sections are approximately twice the Higgs cross sections, but the percentage of PDF uncertainties are almost the same.

The SV corrections give a rough estimate of the fixed order (FO) QCD corrections and are often useful in absence of the latter. However, the relative contribution of these SV corrections to the full FO results crucially depends on the kinematic region and in some cases on the process under study. For the SM Higgs or pseudo-scalar Higgs boson with a mass of about 125 GeV, it is far from the threshold region $\tau = m_H^2/S \to 1$ for $\sqrt{S} = 13$ TeV. Since, the parton fluxes corresponding to this mass region are very high, apart from the threshold logarithms the contributions of the regular terms as well as of other subprocesses present in the FO corrections are expected to be reasonably very high. For Higgs or pseudo-scalar Higgs boson, the prediction at NLO_{SV} level differs from the LO by only a few percent whereas the regular terms at NLO contribute significantly and increase

LO prediction by about 70%. Similar is the case even at NNLO. Thus the SV corrections poorly estimate the FO ones, however, if we redefine the hadron level cross sections without affecting the total cross sections in such a way that the parton fluxes peak near the threshold region [66, 83, 258], then the SV contributions can be shown to dominate over the regular ones. This is due to arbitrariness involved in splitting the parton level cross section in terms of threshold enhanced and regular ones. Using a regular function $G(z)$, we can write the hadronic cross section as

$$\sigma^A(\tau) = \sigma^{A,(0)} \sum_{a,b=q,\bar{q},g} \int_\tau^1 dy\, G\left(\frac{\tau}{y}\right) \Phi_{ab}(y) \left(\frac{\Delta^A_{ab}\left(\frac{\tau}{y}\right)}{G\left(\frac{\tau}{y}\right)} \right) \tag{6.26}$$

where $\Delta(z)/G(z)$ can be decomposed as

$$\Delta(z)/G(z) = \Delta^{\rm SV}(z) + \tilde{\Delta}^{\rm hard}(z) \tag{6.27}$$

In the above equation the $\Delta^{\rm SV}$ is independent of $G(z)$ (if $\lim_{z\to 1} G(z) \to 1$) and contains only distributions, whereas the hard part $\tilde{\Delta}^{\rm hard}$ is modified due to $G(z)$. Hence the SV part of the cross section at the hadron level depends on the choice of $G(z)$. For the peculiar choice $G(z) = z^2$, the $\Delta^{\rm SV}$ dominates over $\tilde{\Delta}^{\rm hard}$ in such a way that almost the entire NLO and NNLO corrections (Eq. 6.26) results from $\Delta^{\rm SV}$ alone. As was noted earlier $G(z) = 1$ corresponds to the standard SV contribution. Note that the flux Φ_{ab} is modified to $\Phi^{\rm mod}_{ab}(y) = \Phi_{ab}(y)G(\tau/y)$ which is responsible for this behavior. We may denote the SV cross sections thus obtained with these modified fluxes as NLO$_{\rm (sv)}$, NNLO$_{\rm (sv)}$ and N^{3}LO$_{\rm (sv)}$ while those obtained with the normal fluxes as NLO$_{\rm sv}$, NNLO$_{\rm sv}$ and N^{3}LO$_{\rm sv}$. In Figure 6.4, we depict the comparison between the SV cross sections obtained from the modified parton fluxes using $G(z) = z^2$ and the normal fixed order results that are obtained from the standard parton fluxes, for both the SM Higgs boson (left panel) and the pseudo-scalar Higgs boson (right panel). We notice that the SV results are significantly closer to the corresponding fixed order ones. Incidentally, this agreement is good for NLO as well as for NNLO where different sub-processes appear, and also for several values of $\sqrt{S}$

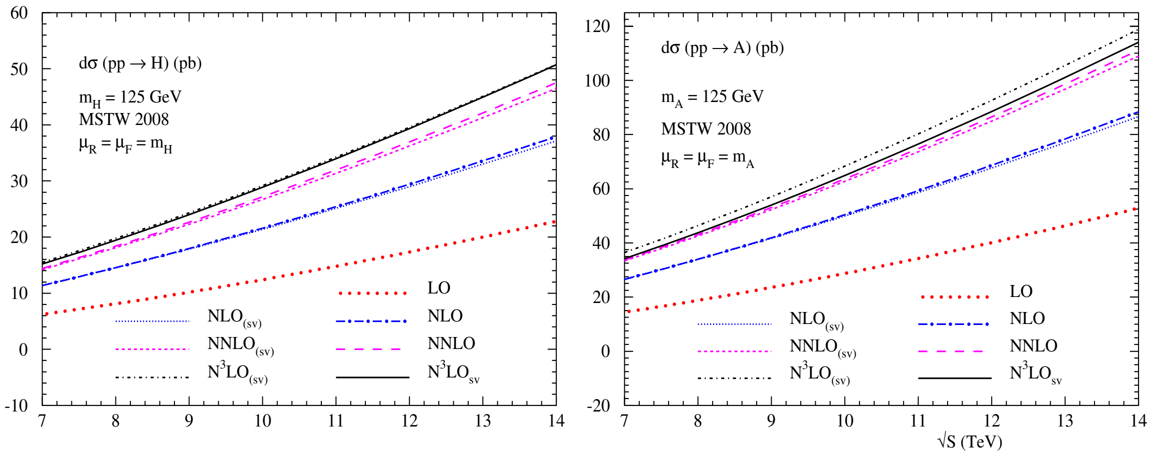


Figure 6.4: Modified soft-plus-virtual vs fixed order results for scalar and pseudo-scalar Higgs boson production cross sections for different energies at LHC.

where the integration range over the parton fluxes is different. While this could be purely accidental, this good agreement might hint some subtle aspect hidden and might be useful in the phenomenology.

Motivated by the above observation, one can convolute the perturbative coefficients $\Delta^{(3)}_{\rm SV}$ with the modified parton fluxes $\Phi^{\rm mod}_{ab}(y)$ for the choice of $G(z) = z^2$ to get $\rm N^3LO_{(sv)}$ which could approximate the full $\rm N^3LO$ result. This way, we present in Figure 6.4, the SV corrections obtained using $G(z) = 1$ and $G(z) = z^2$ for Higgs as well as pseudo-scalar Higgs boson productions.

Next, we present the scale (μ_R, μ_F) uncertainties up to $\rm N^3LO_{SV}$ in Figure 6.5 for the choice of $m_A = 200$ GeV. In the rest of our numerical analysis for studying the scale uncertainties, we simply use the modified parton fluxes for the choice $G(z) = z^2$ at three-loop level. In the left panel, we vary the renormalization scale μ_R between $m_A/4$ and $4m_A$, keeping $\mu_F = m_A$ fixed. Unlike the Drell-Yan process, for the pseudo-scalar Higgs boson production the renormalization scale μ_R enters even at LO through the strong coupling constant a_s. This is identical to the SM Higgs boson production in the gluon fusion channel. This is the main source of large scale uncertainty at LO. It gets significantly reduced when we include NLO and NNLO corrections as expected and it continues to do so at $\rm N^3LO$ level.

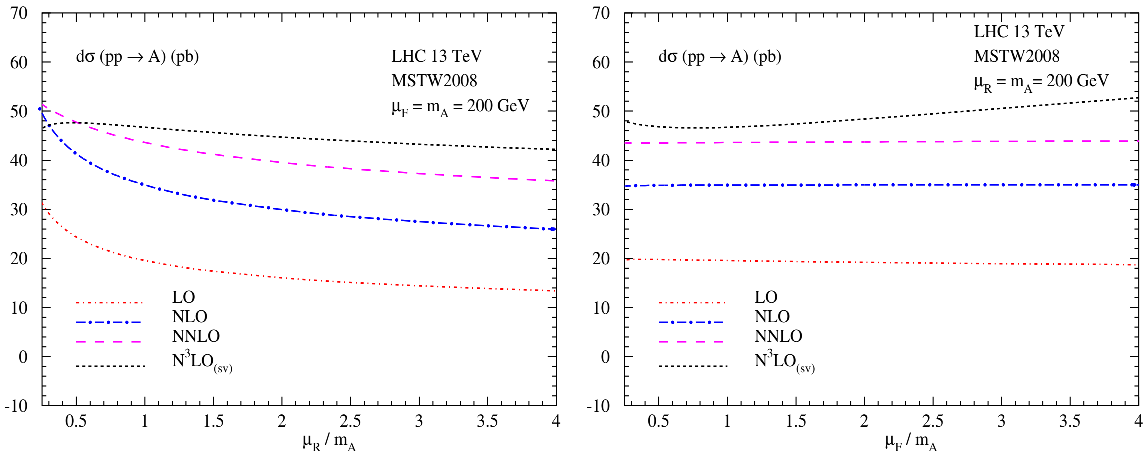


Figure 6.5: Scale uncertainties associated with the pseudo-scalar Higgs boson production cross sections for LHC13. Variation with μ_R keeping $\mu_F = m_A$ fixed (left panel). Variation with μ_F keeping $\mu_R = m_A$ fixed (right panel).

In the right panel, we show the factorization scale uncertainties by varying μ_F from $m_A/4$ to $4m_A$ and fixing $\mu_R = m_A$. Here, the fixed order results show improvement in the reduction of factorization scale uncertainty from NLO to NNLO. However, due to the lack of PDFs at N^{3}LO level and also due to the missing regular contributions from the parton level cross sections, the N^{3}LO$_{\text{SV}}$ cross sections do not show any improvement of the factorization scale uncertainties. However, we observe that with the modified parton fluxes, the factorization scale uncertainties in N^{3}LO$_{(\text{SV})}$ get significantly reduced compared to N^{3}LO$_{\text{SV}}$. In Figure 6.6, we show the combined effect of μ_R and μ_F scale uncertainties by

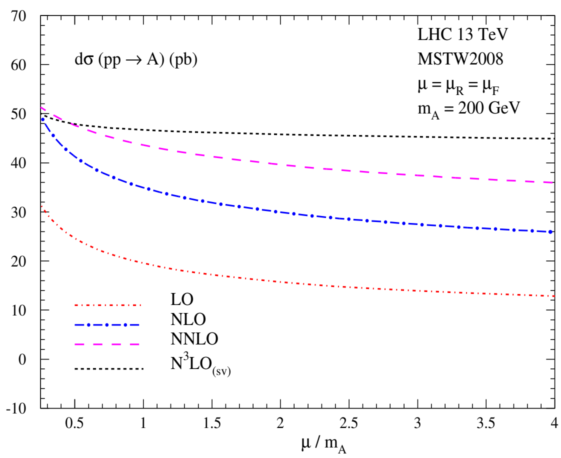


Figure 6.6: Scale uncertainties associated with the pseudo-scalar Higgs boson production cross sections for LHC13 with $\mu = \mu_R = \mu_F$.

varying the scale μ between $m_A/4$ and $4m_A$, where $\mu = \mu_R = \mu_F$. Here, the NNLO cross

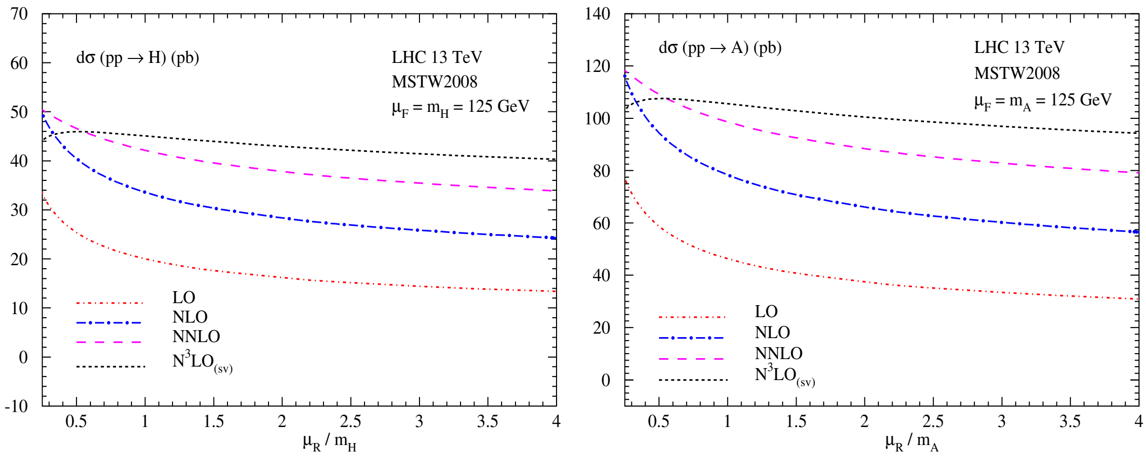


Figure 6.7: Renormalization scale (μ_R) uncertainties associated with the scalar (left panel) and pseudo-scalar (right panel) Higgs boson production cross sections for LHC13, keeping $\mu_F = m_H = m_A$ fixed.

sections show a good improvement over the NLO ones against the scale variations, while the $N^3LO_{(SV)}$ cross sections are found to be more stable than the NNLO ones.

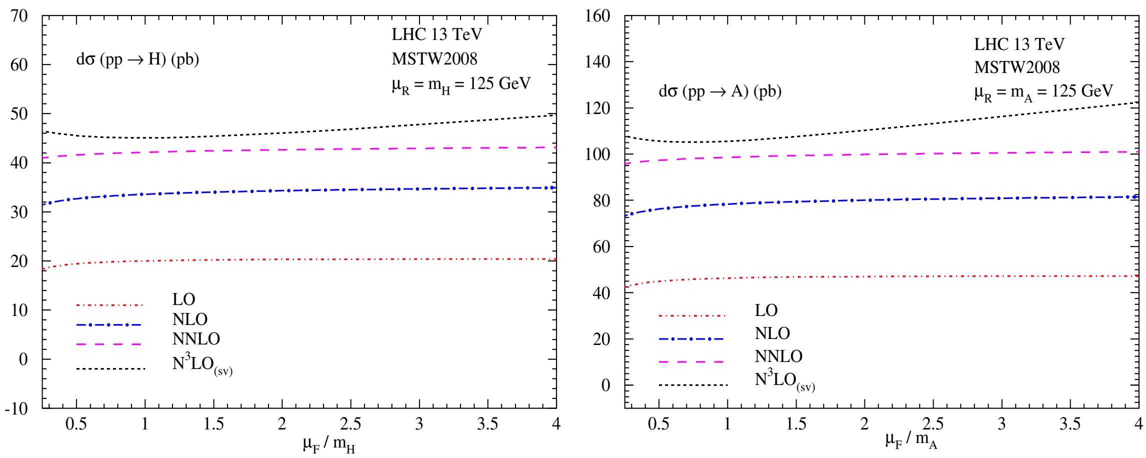


Figure 6.8: Factorization scale (μ_R) uncertainties associated with the scalar (left panel) and pseudo-scalar (right panel) Higgs boson production cross sections for LHC13, keeping $\mu_R = m_H = m_A$ fixed.

Further, we also study the renormalization and factorization scale variations of both the cross sections for the production of SM Higgs boson and pseudo-scalar Higgs boson for $m_H = m_A = 125$ GeV by varying them between $m_A/4$ and $4m_A$. In Figure 6.7, the

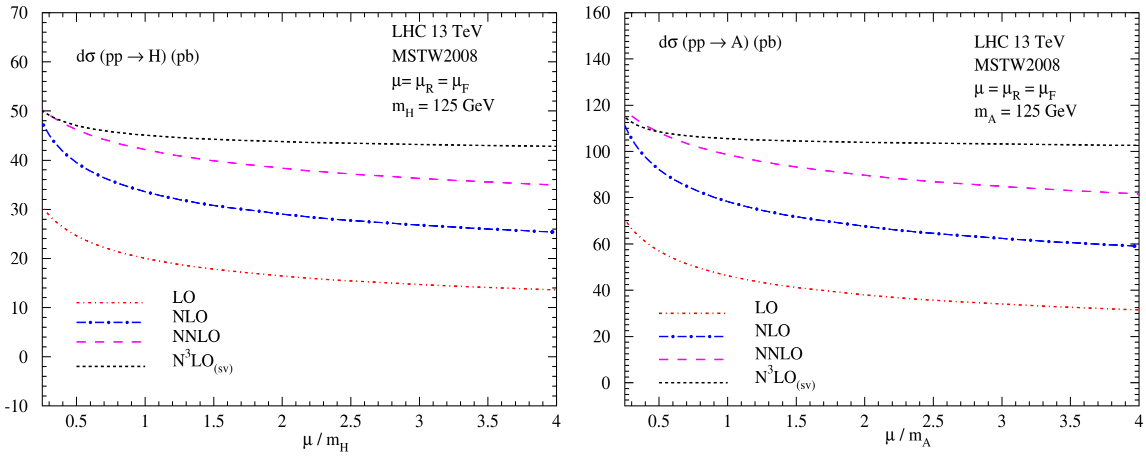


Figure 6.9: Scale ($\mu = \mu_R = \mu_F$) uncertainties associated with the scalar (left panel) and pseudo-scalar (right panel) Higgs boson production cross sections for LHC13.

renormalization scale uncertainties are given for Higgs boson (left panel) and for pseudo-scalar Higgs boson (right panel), for $\mu_F = m_A = m_H$. In Figure 6.8, we present similar results but for the factorization scale uncertainties keeping $\mu_R = m_H = m_A$. Moreover, in Figure 6.9, we present the combined effect by varying $\mu = \mu_R = \mu_F$. The pattern of the results for the μ_R, μ_F and the combined variations are similar to the earlier analysis for $m_A = 200$ GeV where the renormalization scale uncertainties get stabilized further after including the third order threshold corrections while the scale uncertainties due to $\mu = \mu_R = \mu_F$ variation get significantly improved at N^{3}LO$_{(SV)}$ than at NNLO.

6.7 Conclusions

In this Chapter, using the recently available pseudo-scalar Higgs boson form factors up to three-loop and the third order soft distribution function from the real radiations, a complete N^{3}LO threshold correction to the production of pseudo-scalar Higgs boson at the LHC has been obtained. The computation is performed using z space representation of resummed cross section. We have exploited the universal structure of soft function that appears in scalar Higgs boson production at the LHC. We found that the singularities resulting from soft and collinear regions in the virtual diagrams cancel against those from the universal soft functions as well as from mass factorization kernels. Using our approach, we have also computed the process dependent coefficient that appears in the threshold resummed cross section. This will be useful for resummed predictions at N^{3}LL in QCD. Using threshold corrected N^{3}LO results, we have presented a detailed phenomenological study of the pseudo-scalar Higgs boson production at the LHC for various center of mass energies as a function of its mass. While the third order corrections are small, they play an important role in reducing the theoretical uncertainty resulting from renormalization scale. In addition, we have made a detailed comparison against scalar Higgs boson production and found their corrections are very close to each other confirming the universal behaviour of the QCD effects even though the operators responsible for their interactions with gluons are very different.

7 Two-loop QCD Corrections to Higgs $\rightarrow b + \bar{b} + g$ Amplitude

To study the properties of the Higgs boson, the production channels of the Higgs boson in association with jets are important. Exclusive observables like production rates involving cuts on the final state jets or differential distributions of rapidity, transverse momentum of the observed Higgs boson *etc*. are often well suited. While the dominant contributions come from the gluon initiated partonic sub-processes, it is important to include the sub-dominant ones coming from other channels. In this chapter, we study one such channel namely the Higgs boson production in association with a jet in bottom anti-bottom annihilation process. We compute the relevant amplitudes for $H \to b + \overline{b} + g$ up to two-loop level in QCD where Higgs couples to bottom quark through Yukawa coupling. After obtaining the general tensor structure of this process, we use projection operators to find the coefficients for each tensorial structure. We have demonstrated that the UV renormalized amplitudes exhibit the universal IR structure predicted by the QCD factorization in dimensional regularization. The finite parts of the one and two-loop amplitudes are presented in Appendix D after subtracting the IR poles using Catani's subtraction operators.

7.1 Introduction

The tests of the SM have been going on for several decades in various experiments and most of its predictions have been tested in an unprecedented accuracy. The recent discovery of Higgs boson by ATLAS [259] and CMS [260] collaborations at the LHC puts the SM on firm footing. The Higgs boson results from Higgs mechanism that provides a framework for electro-weak symmetry breaking. Elementary particles such as leptons, quarks, gauge bosons and Higgs boson acquire masses through the Higgs mechanism. The mass of the Higgs boson being a parameter of the theory can not be predicted by the SM and hence its discovery provides a valuable information on this. Results from Higgs searches at LEP [29] and Tevatron [30] were crucial ingredients to the recent discovery in narrowing down the search regions for the LHC collaborations. The direct searches at the LEP excluded Higgs of mass below 114.4 GeV and the precision electro-weak measurements [31] hinted for Higgs boson in the mass less than 152 GeV at 95% confidence level (CL). Tevatron on the other hand excluded Higgs of mass in the range 162 – 166 GeV at 95% CL.

The dominant production mechanism for the Higgs production at the LHC is gluon gluon fusion through top quark loop. The sub-dominant ones come from vector boson fusion, associated production of Higgs with vector bosons and top anti-top pairs and bottom anti-bottom annihilation. The inclusive production cross section for the Higgs production is known to an unprecedented accuracy due to many breakthroughs in the computation of amplitudes, loop and phase space integrals. For gluon-gluon [37–44, 58], vector boson fusion processes [261], and associated production with vector bosons [169, 262], the inclusive rates are known to NNLO accuracy in QCD. There are also studies related to the Higgs production in association with bottom quarks which were also motivated to study Higgs boson in certain SUSY models, namely MSSM. The coupling of bottom quarks become large in the large $\tan\beta$ region, where $\tan\beta$ is the vacuum expectation values of up and down type Higgs fields in the Higgs sector of MSSM. Such large couplings can enhance gluon fusion as well as bottom quark fusion sub-processes. Fully inclusive cross section for Higgs production in association with bottom quark to NNLO level accuracy is also known in the variable flavor scheme (VFS) [71, 263–267], while it is known only up to NLO level in the fixed flavor scheme (FFS) [268–273]. In the VFS, one assumes the initial state bottom quarks inside the proton. They are there as a result of emission of collinear bottom anti-bottom states from the gluons intrinsically present inside the proton. They being collinear give large logs which need to be resummed. The resummed contribution is the source for non-vanishing bottom and anti-bottom PDFs inside the proton in the VFS scheme.

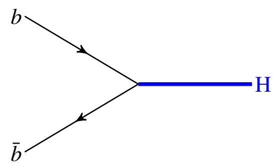
The differential distributions for Higgs production and its decay to pair of photons [274] or massive vector bosons [275, 276] have also been known at NNLO level in QCD in the infinite top quark mass limit. Such exclusive observables allow direct comparison of theoretical predictions with experimental results which include kinematical cuts on the final state particles. In particular, observables with jet vetos enhance the significance of the signal considerably allowing us to study the properties of Higgs boson and its coupling to other SM particles. NNLO QCD prediction [75] for production of Higgs with one jet through effective gluon-gluon-Higgs vertex in the infinite top quark mass limit is available, thanks to various ingredients that are computed to the required accuracy by different groups [277–280]. As the experimental accuracy improves, it will be important to include other sub-dominant production mechanisms. In this chapter, we provide the relevant one and two-loop amplitudes for the process $H \to b + \overline{b} + g$ which is analytically continued also to obtain the production of Higgs boson with one jet in bottom anti-bottom annihilation, i.e., $b + \overline{b} \to H + g$, where Higgs couples to bottom quark through Yukawa coupling denoted by λ. We use VFS scheme throughout. This will be an important supplement to the Higgs boson with one jet at NNLO level as it includes the bottom quark effects in VFS scheme.

Beyond LO in perturbation theory, one encounters large number of Feynman amplitudes with rich Lorentz and gauge structures. In addition, the loop integrals become increasingly complicated due to their multiple kinematic dependence. Generation of diagrams, simplification of Lorentz, Dirac and color indices can be done symbolically. Using integration by parts (IBP) and Lorentz invariant (LI) identities the large number of loop integrals can be reduced in a rather straight forward way to few master integrals (MI). The two-loop MIs for four legs processes where all fields but one external leg are massless were solved by Gehrmann and Remiddi [281] using an elegant method of differential equations.

In this chapter we present one and two-loop QCD amplitudes for the process $H \to b+\overline{b}+g$ treating both bottom and other four light quarks massless. We do not include top quark in our analysis. To obtain IR safe observables, we require, in addition to these two-loop amplitudes, one-loop corrected $H \to b + \overline{b} + 2$ partons and tree level $H \to b + \overline{b} + 3$ partons amplitudes. Note that they are individually IR singular due to the presence of massless partons in the amplitudes. There exist several equally efficient frameworks which use these IR sensitive contributions to combine them to obtain IR safe observables. They go by the names sector decomposition [282–288], q_T-subtraction [289] and antenna subtraction [290–296] methods. More recently the method developed by Czakon using sector decomposition and FKS [297] phase space slicing, was applied to obtain top quark pair production [298–300] at NNLO level and NNLO QED corrections [301] to $Z \to e^+e^-$. Antenna subtraction was used to obtain NNLO QCD corrections to di-jet production at the LHC. The NNLO corrections to Higgs plus one jet resulting from only gluon-gluon-Higgs effective interaction are obtained recently in [280] making best use of the subtraction methods in an efficient way. The amplitudes presented in this chapter, will constitute contributions coming from bottom-antibottom-Higgs interactions to Higgs plus one jet observable at NNLO level. We have presented the amplitudes in the form suitable for easier implementation to study IR safe hadron level observables involving Higgs plus one jet at NNLO in QCD.

In the next section, we discuss the Lagrangian that describes coupling of Higgs boson with bottom quark, explain how the projector technique can be used to obtain the amplitudes and describe the renormalization and factorization properties of the amplitudes. Section 7.3 is dedicated to the computational details. Final results in compact form are given in Section 7.4 and corresponding coefficients are given in the Appendix. In section 7.5, we conclude with our findings.

7.2 Theory



We consider the Yukawa interaction of the Higgs boson with the bottom quark in the VFS scheme, as shown in the above figure. The interaction part of the action is given by

$$S_I^b = -\frac{\lambda}{\sqrt{2}} \int d^4x \, \phi(x) \overline{\psi}_b(x) \psi_b(x) \tag{7.1}$$

where, $\psi_b(x)$ denotes the bottom quark field and $\phi(x)$ the Higgs field. λ is the Yukawa coupling given by $\sqrt{2}m_b/v$, with m_b being the bottom quark mass and the vacuum expectation value $v \approx 246$ GeV. For the pseudo-scalar Higgs of MSSM, the interaction part $\lambda\phi(x)\overline{\psi}_b(x)\psi_b(x)$ of the action gets replaced by $\tilde{\lambda}\tilde{\phi}(x)\overline{\psi}_b(x)\gamma_5\psi_b(x)$ in the above equation where, the MSSM couplings are

$$\tilde{\lambda} = \begin{cases} -\frac{\sqrt{2}m_b \sin\alpha}{v\cos\beta}, & \tilde{\phi} = h\,, \\ \frac{\sqrt{2}m_b \cos\alpha}{v\cos\beta}, & \tilde{\phi} = H\,, \\ \frac{\sqrt{2}m_b \tan\beta}{v}, & \tilde{\phi} = A \end{cases}$$

respectively. The angle α is the measure of mixing of weak and mass eigenstates of neutral

Higgs bosons. In the VFS scheme, except in the Yukawa coupling, m_b is zero elsewhere like other light quarks in the theory. The initial state bottom quarks in this approach arise from gluon splitting in the proton, parametrized in terms of bottom quark PDFs. We work in Feynman gauge throughout.

7.2.1 Notation and kinematics

We consider the decay of Higgs boson to a bottom quark, anti-bottom quark and a gluon

$$H(q) \longrightarrow b(p_1) + \bar{b}(p_2) + g(p_3)\,, \tag{7.2}$$

with the on-shell conditions $p_i^2 = 0,\ i = 1, 2, 3$. The associated Mandelstam variables are defined as

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv (p_2 + p_3)^2, \qquad u \equiv (p_1 + p_3)^2 \tag{7.3}$$

which satisfy

$$s > 0,\ t > 0,\ u > 0,\ \ s + t + u = M_H^2 \equiv Q^2 > 0 \tag{7.4}$$

where, M_H is the mass of the Higgs boson. Unlike the two-loop four-point functions with all legs on-shell, which can be expressed in terms of Nielsen's polylogarithms [302], the closed analytic expressions for two-loop four-point functions [281, 303] with one leg off-shell contain two new classes of functions: HPLs [304, 305] and two-dimensional harmonic polylogarithms (2dHPLs) [306], which will be discussed in detail in Appendix C. We define the following dimensionless invariants which appear in HPLs and 2dHPLs as

$$x \equiv s/Q^2, \qquad y \equiv u/Q^2, \qquad z \equiv t/Q^2 \tag{7.5}$$

satisfying $0 < x, y, z < 1$ with the constraint $x + y + z = 1$. We will use Q^2, y and z as independent variables.

Analytical continuation

In order to obtain the amplitudes, as listed below, relevant for production of the Higgs boson along with a jet, we need to perform proper crossing on the decay amplitudes.

$$
\begin{aligned}
&1. \quad \overline{b}(-p_1) + b(-p_2) \to g(p_3) + H(p_4) \\
&2. \quad b(-p_2) + g(-p_3) \to b(p_1) + H(p_4) \\
&3. \quad \overline{b}(-p_1) + g(-p_3) \to \overline{b}(p_2) + H(p_4)
\end{aligned}
\tag{7.6}
$$

We follow the prescription presented in [307] for this purpose. In the decay process, Q^2 is time-like and all the Mandelstam variables are positive. The relevant region is the inner triangle of the kinematical plane, the shaded region as shown in the *l.h.s* of Figure 7.1. Note that, in this region, y and z are both real and positive and hence all the HPLs and 2dHPLs are analytic and real. While, for the crossed processes, Q^2 is time-like too, but not all the Mandelstam variables are positive. For the process 1, $Q^2 = M_H^2 > 0$,

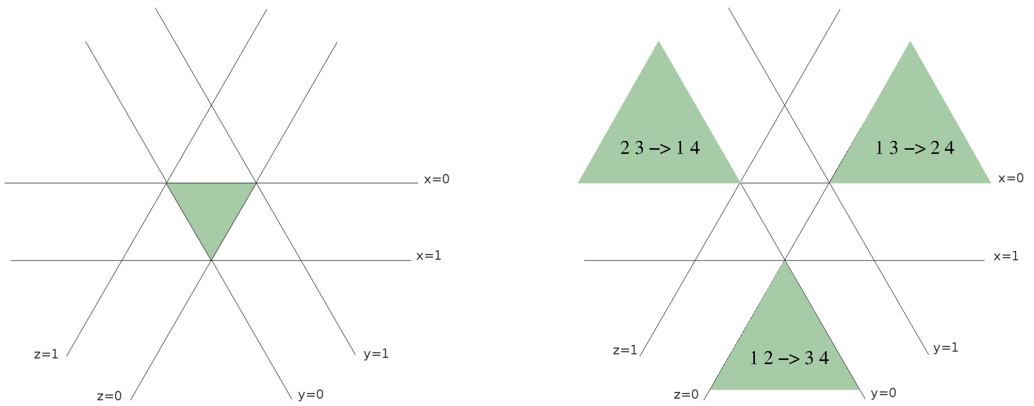


Figure 7.1: Regions of the kinematic plane relevant to the processes.

$s > 0$, $t < 0$ and $u < 0$, with the region $12 \to 34$ as shown in the *r.h.s* of Figure 7.1. In this region, y and z being negative, we introduce the dimensionless parameters u_1 and v_1 with the following definitions

$$
u_1 \equiv -\frac{u}{s}, \qquad v_1 \equiv \frac{Q^2}{s}
\tag{7.7}
$$

such that $0 < u_1 < 1$ and $0 < v_1 < 1$. In terms of these new variables, HPLs and 2dHPLs are analytic, thus these new variables enable us to map the region $12 \to 34$ to the inner triangle. Similarly, for the process 2, shown by the region $23 \to 14$ in the *r.h.s* of Figure 7.1, $Q^2 = M_H^2 > 0$, $s < 0$, $t > 0$ and $u < 0$ and the required dimensionless parameters are u_2 and v_2 with the following definitions

$$u_2 \equiv -\frac{u}{t}, \qquad v_2 \equiv \frac{Q^2}{t} \tag{7.8}$$

such that $0 < u_2 < 1$ and $0 < v_2 < 1$. The last one, as shown by the region $13 \to 24$ in the *r.h.s* of Figure 7.1, is trivially related to the second one.

7.2.2 The general structure of the amplitude

In this section, we describe the projector technique which we used to obtain the coefficients for each tensorial structure of the amplitude for $H \to b + \overline{b} + g$. Since the amplitude contains one external gluon, *i.e.* a single Lorentz index, it can be expressed as

$$|\mathcal{M}\rangle = \mathcal{S}_\mu(b, \bar{b}; g)\epsilon^\mu \tag{7.9}$$

where, ϵ^μ is the gluon polarization vector. Three independent momenta p_1, p_2, p_3 and the Dirac matrices only can carry the Lorentz index μ. On the other hand, Dirac equation for the quarks needs to be satisfied and also $p_3.\epsilon = 0$. These provide the amplitude the following general structure in terms of the coefficients A', A'' and A_2:

$$\mathcal{S}_\mu(b, \bar{b}; g) = \bar{u}(p_1)\{A'\ p_{1\mu} + A''\ p_{2\mu} + A_2\ \not{p}_3\gamma_\mu\}v(p_2)\,. \tag{7.10}$$

Meanwhile, QCD Ward identity gives

$$A'\ p_1.p_3 + A''\ p_2.p_3 = 0 \ \Rightarrow\ A' = -A''\ \frac{p_2.p_3}{p_1.p_3} \equiv A_1\ p_2.p_3\,. \tag{7.11}$$

Hence, the amplitude takes the following form

$$
\begin{aligned}
\mathcal{S}_\mu(b,\bar{b};g)\,\epsilon^\mu &= \bar{u}(p_1)\{A_1\,(p_2.p_3\;p_{1\mu} - p_1.p_3\;p_{2\mu}) + A_2\;\not{p}_3\gamma_\mu\}v(p_2)\,\epsilon^\mu \\
&\equiv A_1\,\mathrm{T}_1 + A_2\,\mathrm{T}_2\;.
\end{aligned} \tag{7.12}
$$

We define the tensor structures as

$$
\begin{aligned}
\mathrm{T}_1 &= \bar{u}(p_1)\{(p_2.p_3\;p_{1\mu} - p_1.p_3\;p_{2\mu})\}v(p_2)\,\epsilon^\mu\,, \\
\mathrm{T}_2 &= \bar{u}(p_1)\{\not{p}_3\gamma_\mu\}v(p_2)\,\epsilon^\mu\,.
\end{aligned} \tag{7.13}
$$

The coefficients A_m $(m = 1, 2)$ can be obtained from the amplitude $|\mathcal{M}\rangle$ using appropriate projectors $\mathcal{P}(A_m)$

$$
A_m = \sum_{\text{spins}} \mathcal{P}(A_m)\mathcal{S}_\mu(b,\bar{b};g)\,\epsilon^\mu = \sum_{\text{spins}} \mathcal{P}(A_m)|\mathcal{M}\rangle \tag{7.14}
$$

where, in d space-time dimensions, the projectors are found to be

$$
\begin{aligned}
\mathcal{P}(A_1) &= \frac{2(d-2)}{s^2\,t\,u\,(d-3)}\mathrm{T}_1{}^\dagger + \frac{1}{s\,t\,u\,(d-3)}\mathrm{T}_2{}^\dagger\,, \\
\mathcal{P}(A_2) &= \frac{1}{s\,t\,u\,(d-3)}\mathrm{T}_1{}^\dagger + \frac{1}{2\,t\,u\,(d-3)}\mathrm{T}_2{}^\dagger\,.
\end{aligned} \tag{7.15}
$$

Expanding the coefficients A_m in powers of strong coupling constant a_s, we obtain

$$
A_m = \frac{\lambda}{\mu_R^\epsilon}\,4\pi\,\sqrt{a_s}T^a_{ij}\{A_m^{(0)} + a_s A_m^{(1)} + a_s^2 A_m^{(2)} + O(a_s^3)\} \tag{7.16}
$$

where, T^a are the Gell-Mann matrices, a is adjoint and i, j are fundamental indices of SU(3) and μ_R is the renormalization scale. These coefficients $A_m^{(l)}$ completely specify the amplitude order by order in perturbation theory. As described in section 7.2.1, for Higgs + 1 jet production, the above amplitudes have to be suitably crossed and the coefficients A_m will be expressed in terms of corresponding u_i and v_i.

7.2.3 Ultraviolet renormalization

The Feynman amplitudes for the process $H \to b + \overline{b} + g$ beyond LO develop UV divergences in QCD. We have used dimensional regularization to regulate them taking space-time dimension to be $d = 4 + \epsilon$. The scale μ_0 is introduced to scale the mass dimension of the dimension-full strong coupling constant in d dimensions. As usual, the dimensionless strong coupling constant is denoted by $\hat{g}_s$ in d dimensions, and the unrenormalized amplitude can be expanded in terms of $\hat{a}_s = \hat{g}_s^2/16\pi^2$ as

$$|\mathcal{M}\rangle = \frac{\hat{\lambda}}{\mu_0^{\epsilon}} S_\epsilon \left(\frac{\hat{a}_s}{\mu_0^{\epsilon}} S_\epsilon\right)^{\frac{1}{2}} \left\{ |\hat{\mathcal{M}}^{(0)}\rangle + \left(\frac{\hat{a}_s}{\mu_0^{\epsilon}} S_\epsilon\right) |\hat{\mathcal{M}}^{(1)}\rangle + \left(\frac{\hat{a}_s}{\mu_0^{\epsilon}} S_\epsilon\right)^2 |\hat{\mathcal{M}}^{(2)}\rangle + O(\hat{a}_s^3) \right\} \tag{7.17}$$

where, $S_\epsilon = \exp[\frac{\epsilon}{2}(\gamma_E - \ln 4\pi)]$ with Euler constant $\gamma_E = 0.5772\ldots$, results from loop integrals beyond LO. $|\hat{\mathcal{M}}^{(i)}\rangle$ is the unrenormalized color-space vector which represents the i^{th} loop amplitude. The UV renormalization of the strong coupling constant follows the prescription presented in the section 2.1.3. Specifically, in $\overline{MS}$ scheme, the renormalized coupling constant $a_s \equiv a_s(\mu_R^2)$ at the renormalization scale μ_R is related to unrenormalized coupling constant $\hat{a}_s$ by Eq. 2.7. The bare Yukawa coupling constant $\hat{\lambda}$ is renormalized by

$$\begin{aligned} \frac{\hat{\lambda}}{\mu_0^{\epsilon}} S_\epsilon &= \frac{\lambda}{\mu_R^{\epsilon}} Z_\lambda(\mu_R^2) \\ &= \frac{\lambda}{\mu_R^{\epsilon}} \left[1 + a_s \left(\frac{1}{\epsilon} r_{\lambda_{1;1}} \right) + a_s^2 \left(\frac{1}{\epsilon^2} r_{\lambda_{2;2}} + \frac{1}{\epsilon} r_{\lambda_{2;1}} \right) + O(a_s^3) \right], \end{aligned} \tag{7.18}$$

with $\lambda = \lambda(\mu_R^2)$ and

$$r_{\lambda_{1;1}} = 6C_F\,,\ r_{\lambda_{2;2}} = (18C_F^2 + 6\beta_0 C_F)\,,\ r_{\lambda_{2,1}} = \left(\frac{3}{2}C_F^2 + \frac{97}{6}C_F C_A - \frac{10}{3}C_F T_F n_f\right). \tag{7.19}$$

Using Eq. 2.7, Eq. 2.12 and Eq. 7.18, we now can express $|\mathcal{M}\rangle$ (Eq. 7.17), as perturbative expansion in a_s with UV finite matrix elements $|\mathcal{M}^{(i)}\rangle$

$$|\mathcal{M}\rangle = \frac{\lambda}{\mu_R^{\epsilon}} (a_s)^{\frac{1}{2}} \left(|\mathcal{M}^{(0)}\rangle + a_s |\mathcal{M}^{(1)}\rangle + a_s^2 |\mathcal{M}^{(2)}\rangle + O(a_s^3) \right). \tag{7.20}$$

The UV finite matrix elements $|\mathcal{M}^{(i)}\rangle$ are related to the unrenormalized ones through

$$
\begin{aligned}
|\mathcal{M}^{(0)}\rangle &= \left(\frac{1}{\mu_R^{\epsilon}}\right)^{\frac{1}{2}} |\hat{\mathcal{M}}^{(0)}\rangle \,, \\
|\mathcal{M}^{(1)}\rangle &= \left(\frac{1}{\mu_R^{\epsilon}}\right)^{\frac{3}{2}} \left[\, |\hat{\mathcal{M}}^{(1)}\rangle + \mu_R^{\epsilon}(\frac{r_{a_1}}{2} + r_{\lambda_1})|\hat{\mathcal{M}}^{(0)}\rangle \,\right] \,, \\
|\mathcal{M}^{(2)}\rangle &= \left(\frac{1}{\mu_R^{\epsilon}}\right)^{\frac{5}{2}} \left[\, |\hat{\mathcal{M}}^{(2)}\rangle + \mu_R^{\epsilon}\Big(\frac{3r_{a_1}}{2} + r_{\lambda_1}\Big)|\hat{\mathcal{M}}^{(1)}\rangle + \mu_R^{2\epsilon}\left(\frac{r_{a_2}}{2} - \frac{r_{a_1}^2}{8} + \frac{r_{a_1}\, r_{\lambda_1}}{2} + r_{\lambda_2}\right)|\hat{\mathcal{M}}^{(0)}\rangle \,\right]
\end{aligned}
\tag{7.21}
$$

with

$$
\begin{aligned}
r_{a_1} &= \left(\frac{1}{\epsilon}2\beta_0\right) \,, \qquad & r_{a_2} &= \left(\frac{1}{\epsilon^2}4\beta_0^2 + \frac{1}{\epsilon}\beta_1\right) \,, \\
r_{\lambda_1} &= \left(\frac{1}{\epsilon}r_{\lambda_{1;1}}\right) \,, \qquad & r_{\lambda_2} &= \left(\frac{1}{\epsilon^2}r_{\lambda_{2;2}} + \frac{1}{\epsilon}r_{\lambda_{2;1}}\right) \,.
\end{aligned}
\tag{7.22}
$$

In section 7.3, we present the computational details to obtain the unrenormalized amplitudes $|\hat{\mathcal{M}}^{(l)}\rangle,\ l = 0, 1, 2$.

7.2.4 Infrared factorization

In addition to UV divergences, the amplitudes beyond LO suffer from divergences of soft and collinear types, due to the presence of soft gluons and collinear massless partons. According to the KLN theorem [123, 124], to obtain IR safe observables, we need to include appropriate contributions coming from real emission processes along with mass factorization counter terms and to perform sum over degenerate configurations. Thanks to factorization properties of QCD amplitudes, the IR divergence structure of the amplitudes is well understood. The earliest account on two-loop QCD amplitudes was by Catani [183], who predicted the IR poles in ϵ of multi-parton QCD amplitudes in dimensional regularization excluding two-loop single pole. In [184], Sterman and Tejeda-Yeomans demonstrated the connection of single pole in ϵ to a soft anomalous dimension matrix, later computed in [204, 205] using factorization properties of the scattering amplitudes along with IR evolution equations. The decomposition of single pole term into universal collinear and soft anomalous dimensions at two-loop level in QCD was first observed in electromagnetic and Higgs form factors [144]. Becher and Neubert [185], using soft collinear effective theory, derived the exact formula for the IR divergences of scattering amplitudes with an arbitrary number of loops and legs in massless QCD including single pole in dimensional regularization. Gardi and Magnea also arrived at, a similar all order result [186] using Wilson lines for hard partons and soft and eikonal jet functions in dimensional regularization. Following Catani [183], we express the renormalized amplitudes $|\mathcal{M}^{(i)}\rangle$ in terms of the universal subtraction operators $\mathbf{I}_b^{(i)}(\epsilon)$ as follows[1]

$$
\begin{aligned}
|\mathcal{M}^{(1)}\rangle &= 2\,\mathbf{I}_b^{(1)}(\epsilon)\,|\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(1)fin}\rangle\,,\\
|\mathcal{M}^{(2)}\rangle &= 2\,\mathbf{I}_b^{(1)}(\epsilon)\,|\mathcal{M}^{(1)}\rangle + 4\,\mathbf{I}_b^{(2)}(\epsilon)\,|\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(2)fin}\rangle\,.
\end{aligned}
\tag{7.23}
$$

The corresponding subtraction operators for this process are

$$
\begin{aligned}
\mathbf{I}_b^{(1)}(\epsilon) &= \frac{1}{2}\frac{e^{-\frac{\epsilon}{2}\gamma_E}}{\Gamma(1+\frac{\epsilon}{2})}\Bigg[\Bigg(\frac{4}{\epsilon^2}-\frac{3}{\epsilon}\Bigg)(C_A-2C_F)\Bigg\{\Bigg(-\frac{s}{\mu_R^2}\Bigg)^{\frac{\epsilon}{2}}\Bigg\}\\
&\quad+\Bigg(-\frac{4C_A}{\epsilon^2}+\frac{3C_A}{2\epsilon}+\frac{\beta_0}{2\epsilon}\Bigg)\Bigg\{\Bigg(-\frac{t}{\mu_R^2}\Bigg)^{\frac{\epsilon}{2}}+\Bigg(-\frac{u}{\mu_R^2}\Bigg)^{\frac{\epsilon}{2}}\Bigg\}\Bigg]\,,\\
\mathbf{I}_b^{(2)}(\epsilon) &= -\frac{1}{2}\mathbf{I}_b^{(1)}(\epsilon)\Bigg[\mathbf{I}_b^{(1)}(\epsilon)-\frac{2\beta_0}{\epsilon}\Bigg]+\frac{e^{\frac{\epsilon}{2}\gamma_E}\,\Gamma(1+\epsilon)}{\Gamma(1+\frac{\epsilon}{2})}\Bigg[-\frac{\beta_0}{\epsilon}+K\Bigg]\,\mathbf{I}_b^{(1)}(2\epsilon)\\
&\quad+\Big(2\mathbf{H}_q^{(2)}(\epsilon)+\mathbf{H}_g^{(2)}(\epsilon)\Big)
\end{aligned}
\tag{7.24}
$$

with

$$
K = \Bigg(\frac{67}{18}-\frac{\pi^2}{6}\Bigg)C_A-\frac{10}{9}T_F n_f\,,
\tag{7.25}
$$

$$
\begin{aligned}
\mathbf{H}_q^{(2)}(\epsilon) &= \frac{1}{\epsilon}\Bigg\{C_A C_F\Bigg(-\frac{245}{432}+\frac{23}{16}\zeta_2-\frac{13}{4}\zeta_3\Bigg)+C_F^2\Bigg(\frac{3}{16}-\frac{3}{2}\zeta_2+3\zeta_3\Bigg)\\
&\quad+C_F n_f\Bigg(\frac{25}{216}-\frac{1}{8}\zeta_2\Bigg)\Bigg\}\,,
\end{aligned}
$$

[1] The numerical coefficients 2 and 4 with $\mathbf{I}^{(i)}$ come due to the different definition of a_s between ours and Catani.

$$\mathbf{H}_g^{(2)}(\epsilon) = \frac{1}{\epsilon}\Big\{C_A^2\Big(-\frac{5}{24}-\frac{11}{48}\zeta_2-\frac{1}{4}\zeta_3\Big)+C_A n_f\Big(\frac{29}{54}+\frac{1}{24}\zeta_2\Big)-\frac{1}{4}C_F n_f-\frac{5}{54}n_f^2\Big\}. \quad (7.26)$$

The process dependent parts are the Born amplitude $|\mathcal{M}^{(0)}\rangle$ and the finite parts $|\mathcal{M}^{(l)fin}\rangle, l = 1, 2$ and explicit computation is required for their determination.

7.3 Calculation of the amplitudes

We now describe the computational details of the coefficients A_m from the amplitudes $|\hat{\mathcal{M}}^{(l)}\rangle$ for the process $H \to b + \bar{b} + g$ up to two-loop level in QCD perturbation theory. We have used QGRAF [86] to generate the Feynman amplitudes for this process. The numbers of diagrams are 2, 13 and 251 at tree level, one-loop level and two-loop level, respectively, excluding tadpole and self energy corrections to the external legs.

Using FORM [87, 88] and Mathematica, output of the QGRAF is converted to a form, suitable for further symbolic manipulation. Using the projectors as in Eq. 7.15, we have computed unrenormalized $\hat{A}_i$ from these amplitudes. They contain only scalar products among internal and external momenta. The physical polarization sum for the external on-shell gluon leg is taken as

$$\sum_s \varepsilon^\mu(p_3, s)\varepsilon^{\nu *}(p_3, s) = -g^{\mu\nu} + \frac{p_3^\mu q^\nu + q^\mu p_3^\nu}{p_3.q} \quad (7.27)$$

where, p_3 is the gluon momentum and q is an arbitrary light-like four-vector for which we choose $q = p_1$. The Lorentz contractions and Dirac algebra are performed in $d = 4+\epsilon$ dimensions. Evaluating the one and two-loop tensor integrals, makes it a challenge, as it involves a large number of integrals as well as complicated kinematics. We overcome this by using the technique of reduction *i.e.* first reducing them to an irreducible set of MIs using IBP identities and LI identities and substituting the MIs evaluated to desired accuracy in ϵ. We have used a Mathematica package LiteRed [91, 92] to use IBP [89, 90] and LI identities [194] in an efficient manner. The relevant MIs for the kinematic

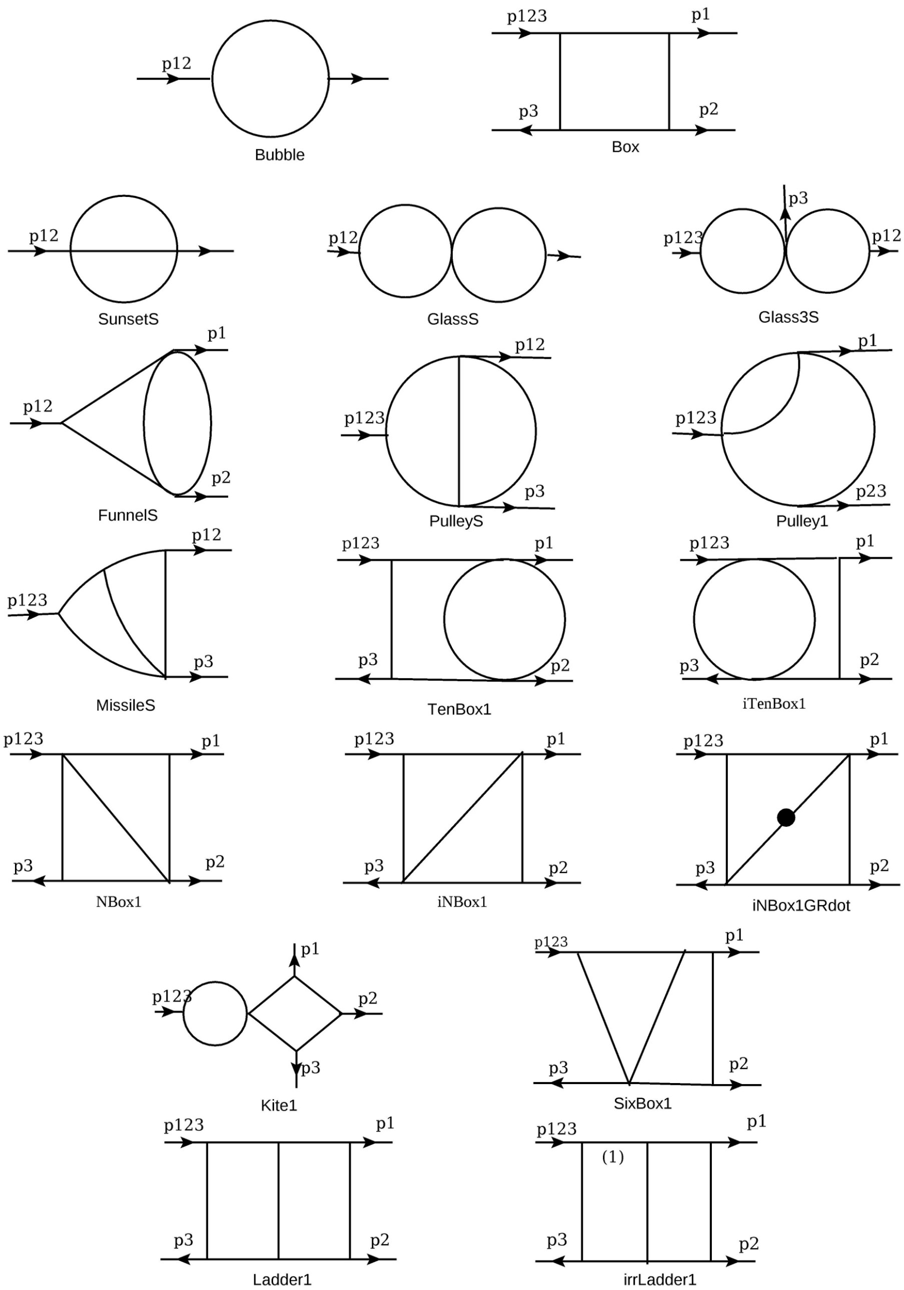


Figure 7.2: Planar topologies of master integrals.

configuration of the problem at hand are analytically known from the seminal works of Gehrmann and Remiddi [281]. We use them to obtain the unrenormalized coefficients in a Laurent series in ϵ. In order to optimize the use of LiteRed, we have reduced all the one and two-loop integrals to belong to few integral sets. This is done by shifting the loop momenta suitably using an in-house and FORM based algorithm. We find that the sets for both one and two-loop integrals are exactly same as those given in [182] for the case of massive spin-2 resonance → 3 gluons. The topologies of the appearing planar and non-planar master integrals are shown in Figure 7.2 and Figure 7.3 respectively. For

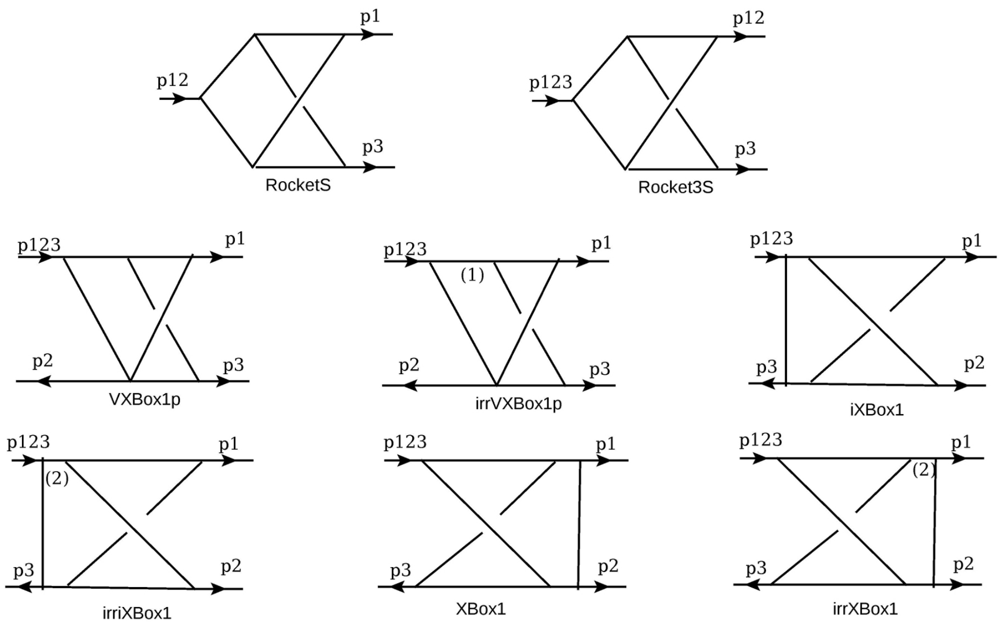


Figure 7.3: Non-planar topologies of master integrals.

one-loop diagrams, the integral belongs to one of the following sets:

$$\{\mathcal{D}_1,\ \mathcal{D}_{1;1},\ \mathcal{D}_{1;12},\ \mathcal{D}_{1;123}\}\,,\{\mathcal{D}_1,\ \mathcal{D}_{1;2},\ \mathcal{D}_{1;23},\ \mathcal{D}_{1;123}\}\,,\{\mathcal{D}_1,\ \mathcal{D}_{1;3},\ \mathcal{D}_{1;31},\ \mathcal{D}_{1;123}\}\,. \tag{7.28}$$

At two-loop, there are nine independent Lorentz invariants, namely $\{(k_\alpha{\cdot}k_\beta), (k_\alpha{\cdot}p_i)\}, \alpha,\beta = 1,2;\ i = 1,...,3$. Shifting of loop momenta guarantees each two-loop Feynman integral to contain terms belonging to one of the following six sets:

$$\begin{aligned}
&\{\mathcal{D}_0,\ \mathcal{D}_1,\ \mathcal{D}_2,\ \mathcal{D}_{1;1},\ \mathcal{D}_{2;1},\ \mathcal{D}_{1;12},\ \mathcal{D}_{2;12},\ \mathcal{D}_{1;123},\ \mathcal{D}_{2;123}\}\,,\\
&\{\mathcal{D}_0,\ \mathcal{D}_1,\ \mathcal{D}_2,\ \mathcal{D}_{1;2},\ \mathcal{D}_{2;2},\ \mathcal{D}_{1;23},\ \mathcal{D}_{2;23},\ \mathcal{D}_{1;123},\ \mathcal{D}_{2;123}\}\,,\\
&\{\mathcal{D}_0,\ \mathcal{D}_1,\ \mathcal{D}_2,\ \mathcal{D}_{1;3},\ \mathcal{D}_{2;3},\ \mathcal{D}_{1;31},\ \mathcal{D}_{2;31},\ \mathcal{D}_{1;123},\ \mathcal{D}_{2;123}\}\,,\\
&\{\mathcal{D}_0,\ \mathcal{D}_1,\ \mathcal{D}_2,\ \mathcal{D}_{1;1},\ \mathcal{D}_{2;1},\ \mathcal{D}_{0;3},\ \mathcal{D}_{1;12},\ \mathcal{D}_{2;12},\ \mathcal{D}_{1;123}\}\,,\\
&\{\mathcal{D}_0,\ \mathcal{D}_1,\ \mathcal{D}_2,\ \mathcal{D}_{1;2},\ \mathcal{D}_{2;2},\ \mathcal{D}_{0;1},\ \mathcal{D}_{1;23},\ \mathcal{D}_{2;23},\ \mathcal{D}_{1;123}\}\,,\\
&\{\mathcal{D}_0,\ \mathcal{D}_1,\ \mathcal{D}_2,\ \mathcal{D}_{1;3},\ \mathcal{D}_{2;3},\ \mathcal{D}_{0;2},\ \mathcal{D}_{1;31},\ \mathcal{D}_{2;31},\ \mathcal{D}_{1;123}\}
\end{aligned} \tag{7.29}$$

where,

$$\mathcal{D}_0 = (k_1 - k_2)^2,\ \mathcal{D}_\alpha = k_\alpha^2,\ \mathcal{D}_{\alpha;i} = (k_\alpha - p_i)^2,\ \mathcal{D}_{\alpha;ij} = (k_\alpha - p_i - p_j)^2,$$
$$\mathcal{D}_{0;i} = (k_1 - k_2 - p_i)^2,\ \mathcal{D}_{\alpha;ijk} = (k_\alpha - p_i - p_j - p_k)^2\ . \tag{7.30}$$

The UV singularities present in the bare coefficients are systematically removed using Eq. 2.7 & Eq. 7.18. The resulting UV finite coefficients do contain divergences from soft and collinear partons. In the next section, we will demonstrate that our results correctly reproduce divergences described in the section 7.2.4 at one and two-loop level. We will also present the finite parts of the coefficients A_m up to two-loop level.

7.4 Results

In this section, we present the results up to two-loop level in QCD for the amplitude $H \to b + \bar{b} + g$ in the $\overline{MS}$ scheme. The results are presented after subtracting the one and two-loop universal subtraction operators $I_b^{(i)}(\epsilon),\ i = 1, 2$ as described in the section 7.2.4. Following Eq. 7.12, Eq. 7.16 & Eq. 7.20, the l^{th} loop amplitude can be written as

$$|\mathcal{M}^{(l)}\rangle = 4\pi\ T^a_{ij}\{A_1^{(l)}\mathrm{T}_1 + A_2^{(l)}\mathrm{T}_2\}\ . \tag{7.31}$$

Following Eq. 7.21, the renormalized coefficients $A_m^{(l)}$ are related to their bare counterparts $\hat{A}_m^{(l)}$ through

$$A_m^{(0)} = \left(\frac{1}{\mu_R^\epsilon}\right)^{\frac{1}{2}} \hat{A}_m^{(0)}\ ,$$
$$A_m^{(1)} = \left(\frac{1}{\mu_R^\epsilon}\right)^{\frac{3}{2}} \left[\ \hat{A}_m^{(1)} + \mu_R^\epsilon\Big(\frac{r_{a_1}}{2} + r_{\lambda_1}\Big)\hat{A}_m^{(0)}\ \right]\ ,$$
$$A_m^{(2)} = \left(\frac{1}{\mu_R^\epsilon}\right)^{\frac{5}{2}} \left[\ \hat{A}_m^{(2)} + \mu_R^\epsilon\Big(\frac{3r_{a_1}}{2} + r_{\lambda_1}\Big)\hat{A}_m^{(1)} + \mu_R^{2\epsilon}\left(\frac{r_{a_2}}{2} - \frac{r_{a_1}^2}{8} + \frac{r_{a_1}}{2}r_{\lambda_1} + r_{\lambda_2}\right)\hat{A}_m^{(0)}\ \right]. \tag{7.32}$$

As discussed in the previous section, we follow the procedure to compute the bare coefficients $\hat{A}_m^{(l)}$ and Eq. 7.32 give the renormalized coefficients. The finite parts of the coefficients $A_m^{(l)}$ are defined after subtracting terms proportional to universal subtraction terms $I_b^{(l)}$ as follows

$$A_m^{(1)} = 2\,\mathbf{I}_b^{(1)}(\epsilon)\,A_m^{(0)} + A_m^{(1)fin}\,,$$

$$A_m^{(2)} = 2\,\mathbf{I}_b^{(1)}(\epsilon)\,A_m^{(1)} + 4\,\mathbf{I}_b^{(2)}(\epsilon)\,A_m^{(0)} + A_m^{(2)fin} \tag{7.33}$$

where, we have used Eq. 7.12 & Eq. 7.23.

Expanding the *r.h.s.* Eq. 7.32 and Eq. 7.33 in powers of ϵ, we find exact agreement of all the poles in ϵ, providing a crucial test on the correctness of our computation. We organize the finite parts of the coefficients with the following expansions

$$A_m^{(l)fin} = \sum_{n=0}^{l} A_m^{(0)}\,\mathcal{B}_{m;n}^{(l)}\,\ln^n\Big(-\frac{Q^2}{\mu^2}\Big) \tag{7.34}$$

where,

$$A_1^{(0)} = -\frac{4i}{t\,u} \qquad \text{and} \qquad A_2^{(0)} = i\Big(\frac{1}{t} + \frac{1}{u}\Big) \tag{7.35}$$

and the remaining coefficients $\mathcal{B}_{m;n}^{(l)}$ are presented in Appendix D. We also performed an independent computation of $\langle\mathcal{M}^{(0)}|\mathcal{M}^{(l)}\rangle$ for $l = 1, 2$ without using any projectors and then compared against one obtained using the projectors, i.e using the coefficients $A_m^{(l)}$. We find both give the same result, providing an independent check on our computation.

Following [307] [2], we have obtained results for the crossed reactions given in Eq. 7.6 relevant for the production of Higgs boson along with a jet at hadron colliders. The corresponding finite coefficients $A_m^{(l)fin}$ are attached with the arXiv submission of [308].

[2] We thank Thomas Gehrmann for providing relevant analytically continued HPLs and 2d HPLs.

7.5 Conclusions

We have presented the amplitudes for the partonic sub-process $H \rightarrow b + \overline{b} + g$ and other sub-processes related by crossing, up to two-loop level in QCD that contribute to exclusive observables involving the Higgs boson and a jet. The dominant one is from gluon fusion which is already known to this accuracy. We have used dimensional regularization to perform our computation. Using appropriate projectors, the amplitude is expressed in terms of two scalar coefficients A_m. We have found that the IR structure of the amplitude is according to Catani's prediction on QCD amplitudes up to two-loop level. Also, the coefficient of single pole term is found to be in agreement with predictions based on the observation of the universal behavior of poles in the multi-parton QCD amplitudes.

8 Conclusion

The physics program at the LHC depends not only on the precise measurements but also predictions from the theory with same level of precision in order to unravel the mysteries of nature. Higher order QCD as well as electro-weak corrections play important role first to confirm the predictions of SM to an unprecedented level of accuracy and then to hint physics beyond the SM, if any, through deviations. To set bounds on the parameters of the BSM, unambiguous predictions taking into radiative corrections become indispensable. In this context, we have systematically computed higher order QCD corrections to some of the important processes at the LHC, namely Drell-Yan, Higgs production in bottom quark fusion, pseudo-scalar boson production. We have achieved them by using the state-of-the-art modern techniques as well as exploiting the rich universal structure of QCD amplitudes at higher orders in perturbation theory.

In Chapter 2 we have presented a brief introduction to perturbative QCD. Chapter 3 contains the detailed description of the formalism to obtain threshold contributions to inclusive production of a colorless particle in hadron collisions to all orders in perturbation theory. We exploit the universality of the QCD amplitudes, namely the factorization theorems and renormalization group equations, to achieve this. Demanding the finiteness of hadronic cross section, a physical observable, we find that the soft distribution function, which contains distributions of the kind $\delta(1-z)$ and $\mathcal{D}_i$, depends only on the initial state partons *i.e.* except a overall Casimir depending on the initial quark or gluon. In other words, but for overall Casimir, the soft distribution functions of quark and gluon are found to be same.

In Chapter 4 we implement the findings of the previous Chapter to obtain the threshold approximated cross section to N^{3}LO in pQCD for the Drell-Yan process at the LHC. For DY, NNLO QCD corrected predictions were already available for long and these corrections are small. Still, to confirm the reliability of perturbation theory and to reduce the dependence on the unphysical scales, which still was not convincingly small, one needs to go beyond NNLO. While the computation of exact N^{3}LO is very involved and yet to be done, the first step is to obtain threshold approximated cross section at this order. In this Chapter we exploit the recently available threshold approximated N^{3}LO cross section for the Higgs boson production in gluon fusion and use the universality of soft distribution function, as described in Chapter 3, to obtain the corresponding one for the DY production. This enables us to obtain the threshold N^{3}LO QCD correction to DY production. We study in detail its numerical impact and find the contribution is small validating the reliability of pQCD.

Next, in Chapter 5 we present the quark and gluon form factors of a massive spin-2 resonance that couples to the fields of SU(N) gauge theory with nf light flavors through the energy momentum tensor. It involves the large number of Feynman diagrams and the complicated Lorentz and Dirac algebra resulting from spin-2 interaction. Hence the computation is extremely challenging. We have used state-of-the-art techniques to overcome the difficulties. QGRAF is used to generate the corresponding Feynman graphs and an in-house code is used to convert the output into a suitable for for further manipulation. We have extensively used FORM for Lorentz and Dirac algebra. These simplifications result into numerous numbers of scalar integrals which have been reduced to 22 Master Integrals using IBP and LI identities. Thanks to the availability of the MIs, we finally present the form factors in powers of $\epsilon = d - 4$. The form factors do contain divergences, of IR kind, as the conservation of energy momentum tensor ensures the absence of UV ones. Moreover, these form factors do satisfy Sudakov integro-differential equation and hence exhibit identical IR structure of other form factors such as those appearing in electro-weak vector boson and Higgs productions up to three-loop level. Additionally, with the newly available form factors, we compute the threshold approximated partonic cross section for production of a massive spin-2 resonance at N^{3}LO QCD and present in this thesis for the first time. The detailed numerical study will be presented elsewhere. We also find striking similarities between the LT terms of the form factors presented here and those of the quark and gluon form factors in QCD as well as the scalar form factor in $\mathcal{N} = 4$ SYM upon employing appropriate substitution of the SU(N) color factors. Our results will be useful in improving the perturbative predictions of spin-2 resonance production beyond NNLO level at the LHC where searches for such particles are already underway with the upgraded energy and luminosity.

Chapter 6 is dedicated to achieve the first results on the production of pseudo-scalar Higgs boson through gluon fusion at the LHC to N^{3}LO in QCD taking into account only soft gluon effects. The crucial ingredients are the three loop form factors of the effective composite operators, resulting from the fact that top quarks are integrated out and the soft distribution functions. Very recently we have computed the required form factors and presented in [94] along with this work. Again, the soft distribution functions, being dependent only on the initial state partons, are same with that of the scalar Higgs boson production in gluon fusion. Using these, we obtain the threshold approximated cross section at N^{3}LO level. We present the phenomenological impact of the threshold corrected N^{3}LO cross section, studying the pseudo-scalar production cross section at the LHC for various center of mass energies as a function of its mass. While the third order corrections are small, they play an important role in reducing the theoretical uncertainty arising from renormalization scale. Additionally, we have made a relative study of these results against production of the scalar Higgs boson at the LHC, considering equivalent parameters and found the behavior of the corrections is very close to each other confirming the universal behavior of the QCD effects even though the operators responsible for their interactions with gluons are very different.

In Chapter 7 we compute two-loop amplitudes of the Higgs boson decaying to a pair of bottom anti-bottom quarks along with a gluon, where the Higgs boson couples to bottom quarks through Yukawa coupling. We consider VFS scheme throughout and the bottom quark is massless except for the Yukawa coupling. To start with, we study the general tensor structure of the amplitude and then we use projection operators to obtain the coefficients for each tensorial structure appearing in this process. As the computation is performed considering the diagrammatic approach, large number of Feynman diagrams appear with rich Lorentz and gauge structures. Symbolically we perform the Lorentz, Dirac and color algebra. Moreover, due to multiple kinematic dependence, the loop integrals become increasingly complicated. Using IBP and LI identities, we reduce the large number of loop integrals to few MI's, already available in literature. The Yukawa coupling here needs UV renormalization and the UV finite results contain the universal IR poles *a la* Catani, indicating the correctness of the computation. The results are presented in terms of Harmonic Poly-Logarithms (HPLs). To obtain the relevant amplitude for the production of the Higgs boson along with a jet, we have done a proper crossing and hence analytic continuation of HPLs.

A Components of the form factor

In this Appendix we present the process dependent component of the form factor $g_n^{I,i,k}$, as defined in Section 3.3, for the inclusive production of a massive colorless particle (I) like the Higgs boson (H), a pair of leptons (γ^*), a massive spin-2 resonance (T) or a pseudo-scalar Higgs boson (A) in either gluon ($i = g$) or quark ($i = q$) or specifically bottom quark ($i = b$) initiated process, up to three-loop.

For production of the Higgs boson in gluon fusion considering the effective theory, the corresponding $I = H$ and $i = g$ and $g_n^{I,i,k}$, are presented as follows

$$
\begin{aligned}
g_1^{H,g,1} &= \boldsymbol{C_A}\zeta_2\,, \\
g_1^{H,g,2} &= \boldsymbol{C_A}\Big\{1 - \frac{7}{3}\zeta_3\Big\}\,, \\
g_1^{H,g,3} &= \boldsymbol{C_A}\Big\{\frac{47}{80}\zeta_2{}^2 - \frac{3}{2}\Big\}\,, \\
g_2^{H,g,1} &= \boldsymbol{C_A}^2\Big\{\frac{67}{3}\zeta_2 - \frac{44}{3}\zeta_3 + \frac{4511}{81}\Big\} + \boldsymbol{C_A n_f}\Big\{-\frac{10}{3}\zeta_2 - \frac{40}{3}\zeta_3 - \frac{1724}{81}\Big\} \\
&\quad + \boldsymbol{C_F n_f}\Big\{16\zeta_3 - \frac{67}{3}\Big\}\,, \\
g_2^{H,g,2} &= \boldsymbol{C_A}^2\Big\{\frac{671}{120}\zeta_2{}^2 + \frac{5}{3}\zeta_2\zeta_3 - \frac{142}{9}\zeta_2 + \frac{1139}{27}\zeta_3 - 39\zeta_5 - \frac{141677}{972}\Big\} \\
&\quad + \boldsymbol{C_A n_f}\Big\{\frac{259}{60}\zeta_2{}^2 + \frac{16}{9}\zeta_2 + \frac{604}{27}\zeta_3 + \frac{24103}{486}\Big\} \\
&\quad + \boldsymbol{C_F n_f}\Big\{-\frac{16}{3}\zeta_2{}^2 - \frac{7}{3}\zeta_2 - \frac{92}{3}\zeta_3 + \frac{2027}{36}\Big\}\,, \\
g_3^{H,g,1} &= \boldsymbol{C_A}^2\boldsymbol{n_f}\Big\{-\frac{128}{45}\zeta_2{}^2 - \frac{88}{9}\zeta_2\zeta_3 - \frac{14225}{243}\zeta_2 - \frac{11372}{81}\zeta_3 + \frac{272}{3}\zeta_5 - \frac{5035009}{2187}\Big\}
\end{aligned}
$$

$$+ C_F n_f^2\Big\{ -\frac{368}{45}\zeta_2^2 - \frac{88}{9}\zeta_2 - \frac{1376}{9}\zeta_3 + \frac{6508}{27}\Big\} + C_A C_F n_f\Big\{\frac{1568}{45}\zeta_2^2 + 40\zeta_2\zeta_3 + \frac{503}{18}\zeta_2 + \frac{20384}{27}\zeta_3 + \frac{608}{3}\zeta_5 - \frac{473705}{324}\Big\} + C_A n_f^2\Big\{\frac{232}{45}\zeta_2^2 + \frac{100}{27}\zeta_2 + \frac{6992}{81}\zeta_3 + \frac{912301}{4374}\Big\} + C_A^3\Big\{ -\frac{12352}{315}\zeta_2^3 - \frac{5744}{45}\zeta_2^2 - \frac{1496}{9}\zeta_2\zeta_3 + \frac{221521}{486}\zeta_2 - \frac{104}{3}\zeta_3^2 - \frac{57830}{27}\zeta_3 + \frac{3080}{3}\zeta_5 + \frac{39497339}{8748}\Big\} + C_F^2 n_f\Big\{296\zeta_3 - 480\zeta_5 + \frac{304}{3}\Big\}. \tag{A.1}$$

For the Drell-Yan production, $I = \gamma^*$ and $i = q$ and the corresponding $g_n^{I,i,k}$ up to three-loop level are presented in the following

$$\begin{aligned}
g_1^{\gamma^*,q,1} &= C_F \{\zeta_2 - 8\}\ ,\\
g_1^{\gamma^*,q,2} &= C_F\Big\{ -\frac{3}{4}\zeta_2 - \frac{7}{3}\zeta_3 + 8\Big\},\\
g_1^{\gamma^*,q,3} &= C_F\Big\{\frac{47}{80}\zeta_2^2 + \zeta_2 + \frac{7}{4}\zeta_3 - 8\Big\},\\
g_2^{\gamma^*,q,1} &= C_F^2\Big\{ -\frac{88}{5}\zeta_2^2 + 58\zeta_2 - 60\zeta_3 - \frac{1}{4}\Big\} + C_A C_F\Big\{\frac{88}{5}\zeta_2^2 - \frac{575}{18}\zeta_2 + \frac{260}{3}\zeta_3\\
&\quad - \frac{70165}{324}\Big\} + C_F n_f\Big\{\frac{37}{9}\zeta_2 - \frac{8}{3}\zeta_3 + \frac{5813}{162}\Big\},\\
g_2^{\gamma^*,q,2} &= C_F^2\Big\{\frac{108}{5}\zeta_2^2 - 28\zeta_2\zeta_3 - \frac{437}{4}\zeta_2 + 184\zeta_3 + 12\zeta_5 - \frac{109}{16}\Big\}\\
&\quad + C_A C_F\Big\{ -\frac{653}{24}\zeta_2^2 + \frac{89}{3}\zeta_2\zeta_3 + \frac{7297}{108}\zeta_2 - \frac{12479}{54}\zeta_3 - 51\zeta_5 + \frac{1547797}{3888}\Big\}\\
&\quad + C_F n_f\Big\{\frac{7}{12}\zeta_2^2 - \frac{425}{54}\zeta_2 + \frac{301}{27}\zeta_3 - \frac{129389}{1944}\Big\},\\
g_3^{\gamma^*,q,1} &= C_F^3\Big\{\frac{21584}{105}\zeta_2^3 - 534\zeta_2^2 + 840\zeta_2\zeta_3 - 206\zeta_2 + 48\zeta_3^2 - 2130\zeta_3 + 1992\zeta_5\\
&\quad - \frac{1527}{4}\Big\} + C_A C_F^2\Big\{ -\frac{15448}{105}\zeta_2^3 + \frac{2432}{45}\zeta_2^2 - \frac{3448}{3}\zeta_2\zeta_3 + \frac{55499}{18}\zeta_2\\
&\quad + 296\zeta_3^2 - \frac{23402}{9}\zeta_3 - \frac{3020}{3}\zeta_5 + \frac{230}{3}\Big\} + C_F^2 n_f\Big\{ -\frac{704}{45}\zeta_2^2 - \frac{152}{3}\zeta_2\zeta_3\\
&\quad - \frac{7541}{18}\zeta_2 + \frac{19700}{27}\zeta_3 - \frac{368}{3}\zeta_5 + \frac{73271}{162}\Big\} + C_A^2 C_F\Big\{ -\frac{6152}{63}\zeta_2^3 + \frac{37271}{90}\zeta_2^2\\
&\quad + \frac{1786}{9}\zeta_2\zeta_3 - \frac{1083305}{486}\zeta_2 - \frac{1136}{3}\zeta_3^2 + \frac{85883}{18}\zeta_3 + \frac{688}{3}\zeta_5 - \frac{48902713}{8748}\Big\}\\
&\quad + C_F n_f^2\Big\{ -\frac{40}{9}\zeta_2^2 - \frac{3466}{81}\zeta_2 + \frac{536}{81}\zeta_3 - \frac{258445}{2187}\Big\} + C_A C_F n_f\Big\{ -\frac{1298}{45}\zeta_2^2\\
&\quad + \frac{392}{9}\zeta_2\zeta_3 + \frac{155008}{243}\zeta_2 - \frac{68660}{81}\zeta_3 - 72\zeta_5 + \frac{3702974}{2187}\Big\}
\end{aligned}$$

$$+ \boldsymbol{C_F n_{f,v}} \left(\frac{\boldsymbol{N^2 - 4}}{\boldsymbol{N}}\right)\left\{ - \frac{6}{5}\zeta_2{}^2 + 30\zeta_2 + 14\zeta_3 - 80\zeta_5 + 12\right\}. \tag{A.2}$$

For the Higgs boson production in bottom quark annihilation, considering the VFS scheme, we denote $I = H$ and $i = b$, and the corresponding $g_n^{I,i,k}$ are

$$
\begin{aligned}
g_1^{H,b,1} &= \boldsymbol{C_F} \{-2 + \zeta_2\} \ , \\
g_1^{H,b,2} &= \boldsymbol{C_F}\left\{2 - \frac{7}{3}\zeta_3\right\}, \\
g_1^{H,b,3} &= \boldsymbol{C_F}\left\{ - 2 + \frac{1}{4}\zeta_2 + \frac{47}{80}\zeta_2^2\right\}, \\
g_2^{H,b,1} &= \boldsymbol{C_F n_f}\left\{\frac{616}{81} + \frac{10}{9}\zeta_2 - \frac{8}{3}\zeta_3\right\} + \boldsymbol{C_F C_A}\left\{ - \frac{2122}{81} - \frac{103}{9}\zeta_2 + \frac{88}{5}\zeta_2{}^2 + \frac{152}{3}\zeta_3\right\} \\
&\quad + \boldsymbol{C_F^2}\left\{8 + 32\zeta_2 - \frac{88}{5}\zeta_2{}^2 - 60\zeta_3\right\}, \\
g_2^{H,b,2} &= \boldsymbol{C_F n_f}\left\{\frac{7}{12}\zeta_2{}^2 - \frac{55}{27}\zeta_2 + \frac{130}{27}\zeta_3 - \frac{3100}{243}\right\} + \boldsymbol{C_A C_F}\left\{ - \frac{365}{24}\zeta_2{}^2 + \frac{89}{3}\zeta_2\zeta_3\right. \\
&\quad \left. + \frac{1079}{54}\zeta_2 - \frac{2923}{27}\zeta_3 - 51\zeta_5 + \frac{9142}{243}\right\} + \boldsymbol{C_F^2}\left\{\frac{96}{5}\zeta_2{}^2 - 28\zeta_2\zeta_3 - 44\zeta_2\right. \\
&\quad \left. + 116\zeta_3 + 12\zeta_5 - 24\right\}, \\
g_3^{H,b,1} &= \boldsymbol{C_A^2 C_F}\left\{ - \frac{6152}{63}\zeta_2{}^3 + \frac{2738}{9}\zeta_2{}^2 + \frac{976}{9}\zeta_2\zeta_3 - \frac{342263}{486}\zeta_2 - \frac{1136}{3}\zeta_3{}^2\right. \\
&\quad \left. + \frac{19582}{9}\zeta_3 + \frac{1228}{3}\zeta_5 + \frac{4095263}{8748}\right\} + \boldsymbol{C_A C_F^2}\left\{ - \frac{15448}{105}\zeta_2{}^3 - \frac{3634}{45}\zeta_2{}^2\right. \\
&\quad \left. - \frac{2584}{3}\zeta_2\zeta_3 + \frac{13357}{9}\zeta_2 + 296\zeta_3^2 - \frac{11570}{9}\zeta_3 - \frac{1940}{3}\zeta_5 - \frac{613}{3}\right\} \\
&\quad + \boldsymbol{C_A C_F n_f}\left\{ - \frac{1064}{45}\zeta_2{}^2 + \frac{392}{9}\zeta_2\zeta_3 + \frac{44551}{243}\zeta_2 - \frac{41552}{81}\zeta_3 - 72\zeta_5 - \frac{6119}{4374}\right\} \\
&\quad + \boldsymbol{C_F^2 n_f}\left\{\frac{772}{45}\zeta_2{}^2 - \frac{152}{3}\zeta_2\zeta_3 - \frac{3173}{18}\zeta_2 + \frac{15956}{27}\zeta_3 - \frac{368}{3}\zeta_5 + \frac{32899}{324}\right\} \\
&\quad + \boldsymbol{C_F n_f^2}\left\{ - \frac{40}{9}\zeta_2{}^2 - \frac{892}{81}\zeta_2 + \frac{320}{81}\zeta_3 - \frac{27352}{2187}\right\} + \boldsymbol{C_F^3}\left\{\frac{21584}{105}\zeta_2{}^3 - \frac{1644}{5}\zeta_2{}^2\right. \\
&\quad \left. + 624\zeta_2\zeta_3 - 275\zeta_2 + 48\zeta_3^2 - 2142\zeta_3 + 1272\zeta_5 + 603\right\}.
\end{aligned} \tag{A.3}
$$

For the production of a massive spin-2 resonance in gluon fusion, $I = T$, $i = g$ and correspondingly $g_n^{I,i,k}$ are

$$
\begin{aligned}
g_1^{T,g,1} &= \boldsymbol{C_A}\Big\{-\frac{203}{18}+\zeta_2\Big\}+\boldsymbol{n_f}\Big\{\frac{35}{18}\Big\}\,,\\
g_1^{T,g,2} &= \boldsymbol{C_A}\Big\{\frac{2879}{216}-\frac{7}{3}\zeta_3-\frac{11}{12}\zeta_2\Big\}+\boldsymbol{n_f}\Big\{-\frac{497}{216}+\frac{1}{6}\zeta_2\Big\}\,,\\
g_1^{T,g,3} &= \boldsymbol{C_A}\Big\{-\frac{37307}{2592}+\frac{77}{36}\zeta_3+\frac{203}{144}\zeta_2+\frac{47}{80}\zeta_2^2\Big\}+\boldsymbol{n_f}\Big\{\frac{6593}{2592}-\frac{7}{18}\zeta_3-\frac{35}{144}\zeta_2\Big\}\,,\\
g_2^{T,g,1} &= \boldsymbol{C_A^2}\Big\{-\frac{19333}{54}+\frac{88}{3}\zeta_3+\frac{799}{18}\zeta_2\Big\}+\boldsymbol{C_A n_f}\Big\{\frac{34991}{324}+\frac{32}{3}\zeta_3-\frac{82}{9}\zeta_2\Big\}\\
&\quad+\boldsymbol{C_F n_f}\Big\{\frac{61}{3}-16\zeta_3\Big\}+\boldsymbol{n_f^2}\Big\{-\frac{2219}{324}+\frac{2}{9}\zeta_2\Big\}\,,\\
g_2^{T,g,2} &= \boldsymbol{C_A^2}\Big\{\frac{2863591}{3888}-39\zeta_5-\frac{437}{6}\zeta_3-\frac{6521}{72}\zeta_2+\frac{5}{3}\zeta_2\zeta_3-\frac{737}{120}\zeta_2^2\Big\}\\
&\quad+\boldsymbol{C_A n_f}\Big\{-\frac{849385}{3888}-\frac{448}{27}\zeta_3+\frac{183}{8}\zeta_2-\frac{221}{60}\zeta_2^2\Big\}+\boldsymbol{C_F n_f}\Big\{-\frac{2245}{36}\\
&\quad+\frac{118}{3}\zeta_3+\zeta_2+\frac{24}{5}\zeta_2^2\Big\}+\boldsymbol{n_f^2}\Big\{\frac{1999}{162}-\frac{14}{27}\zeta_3-\frac{35}{36}\zeta_2\Big\}\,,\\
g_3^{T,g,1} &= \boldsymbol{C_A^3}\Big\{-\frac{963193039}{87480}+\frac{828721}{486}\zeta_2-\frac{3236}{15}\zeta_2^2-\frac{12352}{315}\zeta_2^3+\frac{59309}{135}\zeta_3\\
&\quad-\frac{1244}{9}\zeta_2\zeta_3-\frac{104}{3}\zeta_3^2+\frac{3920}{3}\zeta_5\Big\}+\boldsymbol{C_A^2 n_f}\Big\{\frac{420886019}{87480}-\frac{135338}{243}\zeta_2+\frac{7891}{90}\zeta_2^2\\
&\quad+\frac{821461}{810}\zeta_3-\frac{466}{9}\zeta_2\zeta_3-\frac{1828}{3}\zeta_5\Big\}+\boldsymbol{C_A C_F n_f}\Big\{\frac{2550521}{1620}-\frac{159}{2}\zeta_2-\frac{232}{5}\zeta_2^2\\
&\quad-\frac{149684}{135}\zeta_3+72\zeta_2\zeta_3+\frac{248}{3}\zeta_5\Big\}+\boldsymbol{C_A n_f^2}\Big\{-\frac{53764609}{87480}+\frac{2957}{54}\zeta_2-\frac{64}{9}\zeta_2^2\\
&\quad-\frac{61088}{405}\zeta_3\Big\}+\boldsymbol{C_F^2 n_f}\Big\{-\frac{241}{6}-148\zeta_3+240\zeta_5\Big\}+\boldsymbol{C_F n_f^2}\Big\{-\frac{8467}{36}+2\zeta_2\\
&\quad+\frac{32}{5}\zeta_2^2+\frac{1420}{9}\zeta_3\Big\}+\boldsymbol{n_f^3}\Big\{\frac{4165}{216}-\frac{70}{81}\zeta_2\Big\}\,.
\end{aligned}
\tag{A.4}
$$

For the production of a massive spin-2 resonance in quark anti-quark annihilation, $I = T$, $i = q$ and correspondingly $g_n^{I,i,k}$ are as follows

$$
\begin{aligned}
g_1^{T,q,1} &= \boldsymbol{C_F}\Big\{-10+\zeta_2\Big\}\,,\\
g_1^{T,q,2} &= \boldsymbol{C_F}\Big\{12-\frac{7}{3}\zeta_3-\frac{3}{4}\zeta_2\Big\}\,,\\
g_1^{T,q,3} &= \boldsymbol{C_F}\Big\{-13+\frac{7}{4}\zeta_3+\frac{5}{4}\zeta_2+\frac{47}{80}\zeta_2^2\Big\}\,,\\
g_2^{T,q,1} &= \boldsymbol{C_F^2}\Big\{-\frac{107}{12}-124\zeta_3+90\zeta_2-\frac{88}{5}\zeta_2^2\Big\}+\boldsymbol{C_A C_F}\Big\{-\frac{91693}{324}+\frac{452}{3}\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{1103}{18}\zeta_2 + \frac{88}{5}\zeta_2^2\Big\} + C_F n_f\Big\{\frac{7397}{162} - \frac{8}{3}\zeta_3 + \frac{85}{9}\zeta_2\Big\}, \\
g_2^{T,q,2} &= C_F^2\Big\{\frac{1249}{48} + 12\zeta_5 + 328\zeta_3 - \frac{2431}{12}\zeta_2 - 28\zeta_2\zeta_3 + \frac{676}{15}\zeta_2^2\Big\} \\
& + C_A C_F\Big\{\frac{2192269}{3888} - 51\zeta_5 - \frac{20399}{54}\zeta_3 + \frac{15751}{108}\zeta_2 + \frac{89}{3}\zeta_2\zeta_3 - \frac{2027}{40}\zeta_2^2\Big\} \\
& + C_F n_f\Big\{ - \frac{168557}{1944} - \frac{59}{27}\zeta_3 - \frac{1079}{54}\zeta_2 + \frac{7}{12}\zeta_2^2\Big\}, \\
g_3^{T,q,1} &= C_F^3\Big\{ - \frac{1491}{4} - 134\zeta_2 - \frac{3822}{5}\zeta_2^2 + \frac{21584}{105}\zeta_2^3 - 1570\zeta_3 + 840\zeta_2\zeta_3 + 48\zeta_3^2 \\
& + 1672\zeta_5\Big\} + C_A C_F^2\Big\{ - 579 + \frac{89179}{18}\zeta_2 - \frac{2432}{45}\zeta_2^2 - \frac{15448}{105}\zeta_2^3 - \frac{69962}{9}\zeta_3 \\
& - \frac{3256}{3}\zeta_2\zeta_3 + 296\zeta_3^2 + \frac{3220}{3}\zeta_5\Big\} + C_A^2 C_F\Big\{ - \frac{64243573}{8748} - \frac{1957097}{486}\zeta_2 \\
& + \frac{65443}{90}\zeta_2^2 - \frac{6152}{63}\zeta_2^3 + \frac{57301}{6}\zeta_3 + \frac{1138}{9}\zeta_2\zeta_3 - \frac{1136}{3}\zeta_3^2 - \frac{4832}{3}\zeta_5\Big\} \\
& + C_A C_F n_f\Big\{\frac{4626914}{2187} + \frac{298936}{243}\zeta_2 - \frac{1274}{15}\zeta_2^2 - \frac{111788}{81}\zeta_3 + \frac{464}{9}\zeta_2\zeta_3 + 8\zeta_5\Big\} \\
& + C_F^2 n_f\Big\{\frac{102773}{162} - \frac{13093}{18}\zeta_2 - \frac{544}{45}\zeta_2^2 + \frac{38276}{27}\zeta_3 - \frac{152}{3}\zeta_2\zeta_3 - \frac{368}{3}\zeta_5\Big\} \\
& + C_F n_f^2\Big\{ - \frac{301321}{2187} - \frac{7114}{81}\zeta_2 - \frac{40}{9}\zeta_2^2 - \frac{2632}{81}\zeta_3\Big\}. \qquad \text{(A.5)}
\end{aligned}
$$

For the production of a pseudo-scalar Higgs boson in gluon fusion in the effective theory, we denote $I = A$, $i = g$ and correspondingly $g_n^{I,i,k}$ are

$$
\begin{aligned}
g_1^{A,g,1} &= C_A\Big\{4 + \zeta_2\Big\}, \\
g_1^{A,g,2} &= C_A\Big\{ - 6 - \frac{7}{3}\zeta_3\Big\}, \\
g_1^{A,g,3} &= C_A\Big\{7 - \frac{1}{2}\zeta_2 + \frac{47}{80}\zeta_2^2\Big\}, \\
g_2^{A,g,1} &= C_A^2\Big\{\frac{11882}{81} + \frac{67}{3}\zeta_2 - \frac{44}{3}\zeta_3\Big\} + C_A n_f\Big\{ - \frac{2534}{81} - \frac{10}{3}\zeta_2 - \frac{40}{3}\zeta_3\Big\} \\
& + C_F n_f\Big\{ - \frac{160}{3} + 12\ln\left(\frac{\mu_R^2}{m_t^2}\right) + 16\zeta_3\Big\}, \\
g_2^{A,g,2} &= C_F n_f\Big\{\frac{2827}{18} - 18\ln\left(\frac{\mu_R^2}{m_t^2}\right) - \frac{19}{3}\zeta_2 - \frac{16}{3}\zeta_2^2 - \frac{128}{3}\zeta_3\Big\} + C_A n_f\Big\{\frac{21839}{243} \\
& - \frac{17}{9}\zeta_2 + \frac{259}{60}\zeta_2^2 + \frac{766}{27}\zeta_3\Big\} + C_A^2\Big\{ - \frac{223861}{486} + \frac{80}{9}\zeta_2 + \frac{671}{120}\zeta_2^2 + \frac{2111}{27}\zeta_3 \\
& + \frac{5}{3}\zeta_2\zeta_3 - 39\zeta_5\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g_3^{A,g,1} =\; & \boldsymbol{n_f C_J^{(2)}}\Big\{-6\Big\} + \boldsymbol{C_F n_f^2}\Big\{\frac{12395}{27} - \frac{136}{9}\zeta_2 - \frac{368}{45}\zeta_2^2 - \frac{1520}{9}\zeta_3 \\
& - 24\ln\left(\frac{\mu_R^2}{m_t^2}\right)\Big\} + \boldsymbol{C_F^2 n_f}\Big\{\frac{457}{2} + 312\zeta_3 - 480\zeta_5\Big\} + \boldsymbol{C_A^2 n_f}\Big\{-\frac{12480497}{4374} \\
& - \frac{2075}{243}\zeta_2 - \frac{128}{45}\zeta_2^2 - \frac{12992}{81}\zeta_3 - \frac{88}{9}\zeta_2\zeta_3 + \frac{272}{3}\zeta_5\Big\} + \boldsymbol{C_A^3}\Big\{\frac{62867783}{8748} \\
& + \frac{146677}{486}\zeta_2 - \frac{5744}{45}\zeta_2^2 - \frac{12352}{315}\zeta_2^3 - \frac{67766}{27}\zeta_3 - \frac{1496}{9}\zeta_2\zeta_3 - \frac{104}{3}\zeta_3^2 \\
& + \frac{3080}{3}\zeta_5\Big\} + \boldsymbol{C_A n_f^2}\Big\{\frac{514997}{2187} - \frac{8}{27}\zeta_2 + \frac{232}{45}\zeta_2^2 + \frac{7640}{81}\zeta_3\Big\} \\
& + \boldsymbol{C_A C_F n_f}\Big\{-\frac{1004195}{324} + \frac{1031}{18}\zeta_2 + \frac{1568}{45}\zeta_2^2 + \frac{25784}{27}\zeta_3 \\
& + 40\zeta_2\zeta_3 + \frac{608}{3}\zeta_5 + 132\ln\left(\frac{\mu_R^2}{m_t^2}\right)\Big\}. \qquad \text{(A.6)}
\end{aligned}
$$

B Components of the soft part

In this Appendix we present a component of the soft distribution function $\overline{\mathcal{G}}_n^{I,i,k}$, which depends only on the initial state partons, as discussed in Section 3.5. Depending on the initial state partons, each $\overline{\mathcal{G}}_n^{I,i,k}$ acquires an overall Casimir C_i, where for gluon initiated process ($i = g$), $\boldsymbol{C_i} = \boldsymbol{C_A}$ and for quark initiated process ($i = q$), $\boldsymbol{C_i} = \boldsymbol{C_F}$. Note that, $\overline{\mathcal{G}}_n^{I,i,k}$ contains no information about the process (I).

$$
\begin{aligned}
\overline{\mathcal{G}}_1^{I,i,1} &= \boldsymbol{C_i}\Big\{-3\zeta_2\Big\},\\
\overline{\mathcal{G}}_1^{I,i,2} &= \boldsymbol{C_i}\Big\{\frac{7}{3}\zeta_3\Big\},\\
\overline{\mathcal{G}}_1^{I,i,3} &= \boldsymbol{C_i}\Big\{-\frac{3}{16}\zeta_2^2\Big\},\\
\overline{\mathcal{G}}_2^{I,i,1} &= \boldsymbol{C_i n_f}\Big\{-\frac{328}{81}+\frac{70}{9}\zeta_2+\frac{32}{3}\zeta_3\Big\}+\boldsymbol{C_i C_A}\Big\{\frac{2428}{81}-\frac{469}{9}\zeta_2+4\zeta_2^2-\frac{176}{3}\zeta_3\Big\},\\
\overline{\mathcal{G}}_2^{I,i,2} &= \boldsymbol{C_i C_A}\Big\{\frac{11}{40}\zeta_2^2-\frac{203}{3}\zeta_2\zeta_3+\frac{1414}{27}\zeta_2+\frac{2077}{27}\zeta_3+43\zeta_5-\frac{7288}{243}\Big\}\\
&\quad+\boldsymbol{C_i n_f}\Big\{-\frac{1}{20}\zeta_2^2-\frac{196}{27}\zeta_2-\frac{310}{27}\zeta_3+\frac{976}{243}\Big\},\\
\overline{\mathcal{G}}_3^{I,i,1} &= \boldsymbol{C_i C_A^2}\Big\{\frac{152}{63}\,\zeta_2^3+\frac{1964}{9}\,\zeta_2^2+\frac{11000}{9}\,\zeta_2\zeta_3-\frac{765127}{486}\,\zeta_2+\frac{536}{3}\,\zeta_3^2\\
&\quad-\frac{59648}{27}\,\zeta_3-\frac{1430}{3}\,\zeta_5+\frac{7135981}{8748}\Big\}+\boldsymbol{C_i C_A n_f}\Big\{-\frac{532}{9}\,\zeta_2^2-\frac{1208}{9}\,\zeta_2\zeta_3\\
&\quad+\frac{105059}{243}\,\zeta_2+\frac{45956}{81}\,\zeta_3+\frac{148}{3}\,\zeta_5-\frac{716509}{4374}\Big\}+\boldsymbol{C_i C_F n_f}\Big\{\frac{152}{15}\,\zeta_2^2\\
&\quad-88\,\zeta_2\zeta_3+\frac{605}{6}\,\zeta_2+\frac{2536}{27}\,\zeta_3+\frac{112}{3}\,\zeta_5-\frac{42727}{324}\Big\}\\
&\quad+\boldsymbol{C_i n_f^2}\Big\{\frac{32}{9}\,\zeta_2^2-\frac{1996}{81}\,\zeta_2-\frac{2720}{81}\,\zeta_3+\frac{11584}{2187}\Big\}.
\end{aligned}
\tag{B.1}
$$

C Harmonic polylogarithms

The generalized polylogarithms $S_{n,p}(x)$ of Nielsen [302] turn out to be insufficient for the computation two-loop integrals involving multiple scales. To overcome this limitation, HPLs was introduced [281, 303, 304]. HPLs are some kind of iterated integrals *i.e*. obtained by the repeated integration of rational factors, the so-called kernels. When the kernels contain only the integration variable, the resulting functions are called one-dimensional HPLs or simply HPLs [304, 305]. If the kernels depend on a further variable, besides the integration variable, it results to 2dHPLs [306]. In the following, we recall their definitions and summarize their properties.

C.1 One-dimensional harmonic polylogarithms

HPL, represented by $H(\vec{m}_w; y)$, is one variable function of say y, which is called it's argument. Another dependency is on the set of indices,represented by a w-dimensional vector $\vec{m}_w$ whose elements are $\{1, 0, -1\}$ and whose number indicates the weight w of the HPL. The elements $\{1, 0, -1\}$ represent three kernels as follows

$$f(1; y) \equiv \frac{1}{1-y}, \qquad f(0; y) \equiv \frac{1}{y}, \qquad f(-1; y) \equiv \frac{1}{1+y}\,. \tag{C.1}$$

The weight 1 ($w = 1$) HPLs are defined as

$$H(1, y) \equiv -\ln(1-y), \qquad H(0, y) \equiv \ln y, \qquad H(-1, y) \equiv \ln(1+y)\,. \tag{C.2}$$

For $w > 1$, the definition of $H(m, \vec{m}_w; y)$ reads

$$H(m, \vec{m}_w; y) \equiv \int_0^y dx\, f(m, x)\, H(\vec{m}_w; x), \qquad m \in 0, \pm 1\,. \tag{C.3}$$

The known polylogarithms, *i.e*. the Nielsen polylogarithms and Euler polylogarithms can be expressed in terms of HPLs as follows where $\vec{0}_m$ and $\vec{1}_m$ imply all the m components are 0 and 1, respectively.

$$\begin{aligned} S_{n,p}(y) &= H(\vec{0}_n, \vec{1}_p; y) \\ Li_n(y) &= H(\vec{0}_{n-1}, 1; y) \end{aligned} \tag{C.4}$$

Properties of the HPLs

• The HPLs are linearly independent.

• *Product algebra*: The product of two HPLs with weights w_1 and w_2 of the same argument y is a combination of HPLs of argument y with weight $w = w_1 + w_2$,

$$H(\vec{m}_{w_1}; y) H(\vec{m}_{w_2}; y) = \sum_{\vec{m}_w = \vec{m}_{w_1} \uplus \vec{m}_{w_2}} H(\vec{m}_w; y) \tag{C.5}$$

where $\vec{m}_{w_1} \uplus \vec{m}_{w_2}$ represents all mergers of $\vec{m}_{w_1}$ and $\vec{m}_{w_2}$ preserving the relative order of the elements of both sets.

• *Integration-by-parts identities*: HPLs fulfill the following identity

$$\begin{aligned} H(m_1, m_2, \ldots, m_w; y) &= H(m_1; y) H(m_2, \ldots, m_w; y) - H(m_2, m_1; y) H(m_3, \ldots, m_w; y) \\ &\quad + \cdots + (-1)^{w+1} H(m_w, \ldots, m_2, m_1; y)\,. \end{aligned} \tag{C.6}$$

Both the identities relates a HPL of weight w to other HPLs of same weight and product of HPLs of lower weights. Hence, we can choose a 'minimal' set for a weight w and express all other HPLs of same weight in terms of the elements of that set and lower weight HPLs. We note that, in our context, the element '-1' does not appear.

C.2 Two-dimensional harmonic polylogarithms

The generalization of the HPLs to 2dHPLs involve introducing one more variable in the kernels and to mark these new rational function, the new elements $\{2, 3\}$ appear in $\vec{m}_w$. They represent the following kernels

$$f(2; y) \equiv f(1-z; y) \equiv \frac{1}{1-y-z}, \qquad f(3; y) \equiv f(z; y) \equiv \frac{1}{y+z} \tag{C.7}$$

and correspondingly the weight 1 ($w = 1$) 2dHPLs read

$$H(2, y) \equiv -\ln\left(1 - \frac{y}{1-z}\right), \qquad H(3, y) \equiv \ln\left(\frac{y+z}{z}\right). \tag{C.8}$$

We do not consider any kernel corresponding to '-1', as it does not appear for our case. Below we present few necessary relations used in our computation

$$\begin{aligned}
H(0,2,0,y) &= H(0,y)H(0,2,y) - 2H(0,0,2,y),\\
H(1,0,2,y) &= H(1,y)H(0,2,y) - H(0,1,2,y) - H(0,2,1,y),\\
H(1,2,0,y) &= H(1,y)H(2,0,y) - H(2,1,y)H(0,y) + H(0,2,1,y),\\
H(2,0,0,y) &= H(2,y)H(0,0,y) - H(0,2,y)H(0,y) + H(0,0,2,y),\\
H(2,0,2,y) &= H(2,y)H(0,2,y) - 2H(0,2,2,y),\\
H(2,2,2,y) &= \frac{1}{3}H(2,y)H(2,2,y),\\
H(2,2,0,y) &= H(2,y)H(2,0,y) - H(2,2,y)H(0,y) + H(0,2,2,y),\\
H(3,0,2,y) &= H(3,y)H(0,2,y) - H(0,3,2,y) - H(0,2,3,y),\\
H(3,2,0,y) &= H(3,y)H(2,0,y) - H(2,3,y)H(0,y) + H(0,2,3,y),\\
H(3,3,2,y) &= H(3,y)H(3,2,y) - H(3,3,y)H(2,y) + H(2,3,3,y),\\
H(2,1,0,y) &= H(2,y)H(1,0,y) - H(1,2,y)H(0,y) + H(0,1,2,y),\\
H(2,3,2,y) &= H(2,y)H(3,2,y) - 2H(3,2,2,y).
\end{aligned} \tag{C.9}$$

D Results for H $\to b + \bar{b} + g$

In this Appendix we present the coefficients $\mathcal{B}^{(l)}_{m;n}$ as noted in Section 7.4.

D.1 One-loop coefficients

$$\mathcal{B}^{(1)}_{1;1} = \frac{1}{6}(-11C_A - 18C_F + 2n_f)$$

$$\begin{aligned}\mathcal{B}^{(1)}_{1;0} = \frac{C_A}{6}\big(& -6H(0,y)H(0,z) - 6H(0,y)H(1,z) - 6H(2,y)H(0,z) + 12H(3,y)\ H(1,z) \\ & - 10H(0,y) - 9H(2,y) - 6H(0,2,y) - 6H(2,0,y) + 12H(3,2,y) - 10H(0,z) \\ & - 9\ H(1,z) + 6H(0,1,z) - 6H(1,0,z) - 6\zeta_2\big) + C_F\big(2H(0,y)H(1,z) \\ & - 4H(3,y)H(1,z) + 2H(2,y)H(0,z) + 3H(2,y) + 12\ H(0,2,y) - 2H(1,0,y) \\ & + 2H(2,0,y) - 4H(3,2,y) + 3H(1,z) - 2H(0,1,z) - 2\big) + n_f\frac{1}{6}(H(0,y) + H(0,z))\end{aligned}$$

$$\mathcal{B}^{(1)}_{2;1} = \frac{1}{6}(-11C_A - 18C_F + 2n_f)$$

$$\begin{aligned}\mathcal{B}^{(1)}_{2;0} = \frac{C_A}{6}\big(& -6H(0,y)H(0,z) - 6H(0,y)H(1,z) - 6H(2,y)H(0,z) + 12H(3,y)\ H(1,z) \\ & - 10H(0,y) - 9H(2,y) - 6H(0,2,y) - 6H(2,0,y) + 12H(3,2,y) - 10H(0,z) \\ & - 9\ H(1,z) + 6H(0,1,z) - 6H(1,0,z) - 6\zeta_2 + 6\big) + C_F\big(2H(0,y)H(1,z) \\ & - 4H(3,y)H(1,z) + 2H(2,y)H(0,z) + 3H(2,y) + 2H(0,2,y) - 2\ H(1,0,y) \\ & + 2H(2,0,y) - 4H(3,2,y) + 3H(1,z) - 2H(0,1,z) - 3\big) + n_f\frac{1}{6}(H(0,y) + H(0,z))\end{aligned}$$

D.2 Two-loop coefficients

$$\mathcal{B}^{(2)}_{1;2} = \frac{1}{24}(121C_A^2 + 44C_A(6C_F - n_f) + 4(27C_F^2 - 12C_F n_f + n_f^2))$$

$$\begin{aligned}
\mathcal{B}^{(2)}_{1;1} = {} & C_A^2 \frac{1}{108}(594H(0,y)H(0,z) + 594H(0,y)H(1,z) + 594H(2,y)H(0,z) \\
& - 1188\ H(3,y)H(1,z) + 990H(0,y) + 891H(2,y) + 594H(0,2,y) \\
& + 594H(2,0,y) - 1188\ H(3,2,y) + 990H(0,z) + 891H(1,z) - 594H(0,1,z) \\
& + 594H(1,0,z) + 495\ \zeta_2 - 108\zeta_3 - 702) + C_A C_F \frac{1}{108}(- 1188H(0,y)H(1,z) \\
& + 54H(0,y)(6H(0,z) + 6H(1,z) + 10) - 1188\ H(2,y)H(0,z) \\
& + 54(6H(2,y) + 10)H(0,z) + 1728H(3,y)H(1,z) - 1296H(2,y) \\
& - 864\ H(0,2,y) + 1188H(1,0,y) - 864H(2,0,y) + 1728H(3,2,y) \\
& - 1296H(1,z) + 864\ H(0,1,z) + 324H(1,0,z) + 1566\zeta_2 - 2808\zeta_3 \\
& - 1048) + C_F^2 \frac{1}{108}(- 648H(0,y)H(1,z) - 648H(2,y)H(0,z) \\
& + 1296H(3,y)H(1,z) - 972\ H(2,y) - 648H(0,2,y) + 648H(1,0,y) \\
& - 648H(2,0,y) + 1296H(3,2,y) - 972H(1,z) + 648\ H(0,1,z) \\
& - 1296\zeta_2 + 2592\zeta_3 + 648) + C_A n_f \frac{1}{108}(- 2(- 27(4H(3,y) - 3)H(1,z) \\
& + 81H(2,y) + 54H(0,2,y) + 54\ H(2,0,y) - 108H(3,2,y) - 54H(0,1,z) \\
& + 54H(1,0,z) + 45\zeta_2 - 206) - 9H(0,y)\ (12H(0,z) + 12H(1,z) + 31) \\
& - 9(12H(2,y) + 31)H(0,z)) + C_F n_f \frac{1}{54}(108H(0,y)H(1,z) + 108H(2,y)H(0,z) \\
& - 216H(3,y)H(1,z) - 27H(0,y) + 162\ H(2,y) + 108H(0,2,y) \\
& - 108H(1,0,y) + 108H(2,0,y) - 216H(3,2,y) - 27H(0,z) + 162\ H(1,z) \\
& - 108H(0,1,z) - 54\zeta_2 + 32) + n_f^2 \frac{1}{54}(9H(0,y) + 9H(0,z) - 20)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}^{(2)}_{1;0} = C_A^2 \Big\{ & \zeta_4(39/8) + \zeta_3\Big(- H(0,y) - 6H(1,y) + 11H(2,y) - H(0,z) + 5H(1,z) \\
& + 6(t+u)/s + 407/36\Big) + \zeta_2\Big(108 s^2 H(0,y)H(0,z)
\end{aligned}$$

$$+ 72s^2H(1,y)H(0,z) + 108s^2H(0,y)H(1,z) + 72s^2\ H(1,y)H(1,z)$$

$$+ 36s^2H(2,y)H(0,z) - 72s^2H(3,y)H(1,z) + 147s^2H(0,y) + 453\ s^2H(2,y)$$

$$+ 108s^2H(0,2,y) + 72s^2H(1,2,y) + 36s^2H(2,0,y) - 72s^2\ H(3,2,y)$$

$$+ 147s^2H(0,z) + 237s^2H(1,z) + 36s^2H(0,1,z) + 36s^2H(1,0,z) + 72\ s^2H(1,1,z)$$

$$- 216s(s+t)H(1,y) - 108stH(0,z) - 108suH(0,y) - 216su\ H(1,z) - 40s^2$$

$$+ 96tu\Big)/(36s^2) + \Big(- \frac{13}{4}H(2,y)H(1,z) + \frac{134}{9}H(3,y)H(1,z) + 7H(0,0,y)\ H(1,z)$$

$$- \frac{2}{3}H(0,2,y)H(1,z) - 7H(0,3,y)H(1,z) - \frac{2}{3}H(2,0,y)\ H(1,z) + \frac{13}{3}H(2,3,y)H(1,z)$$

$$- 7H(3,0,y)H(1,z) + \frac{4}{3}H(3,2,y)\ H(1,z) + \frac{22}{3}H(3,3,y)H(1,z)$$

$$+ 2H(0,0,2,y)H(1,z) + 2H(0,2,0,y)\ H(1,z) + 4H(0,3,0,y)H(1,z)$$

$$+ 4H(0,3,3,y)H(1,z) + 2H(1,0,3,y)H(1,z) - 2\ H(1,2,3,y)H(1,z)$$

$$+ 2H(2,0,0,y)H(1,z) + 2H(2,0,3,y)H(1,z) + 2H(2,1,0,y)\ H(1,z)$$

$$+ 4H(2,2,3,y)H(1,z) + 2H(2,3,0,y)H(1,z) - 8H(2,3,3,y)H(1,z)$$

$$+ 4\ H(3,0,3,y)H(1,z) + 4H(3,3,0,y)H(1,z) - 16H(3,3,3,y)H(1,z)$$

$$- \frac{361}{54}\ H(1,z) - \frac{361}{54}H(2,y) + \frac{80}{9}H(0,0,y) + 7H(2,y)H(0,0,z)$$

$$+ 2\ H(0,0,y)H(0,0,z) + \frac{80}{9}H(0,0,z) + \frac{11}{3}H(2,y)\ H(0,1,z) + \frac{1}{3}H(3,y)H(0,1,z)$$

$$+ 2H(0,0,y)H(0,1,z) + \frac{179}{18}\ H(0,1,z) + 2H(0,0,z)H(0,2,y)$$

$$- 2H(0,1,z)H(0,2,y) - \frac{89}{18}H(0,2,y) - 2\ H(0,1,z)H(0,3,y)$$

$$+ \frac{8tuH(1,0,y)}{3s^2} + \frac{8tuH(1,0,z)}{3\ s^2} + 6H(2,y)H(1,0,z) - 7H(3,y)H(1,0,z)$$

$$+ 2H(0,0,y)H(1,0,z) - 6H(0,3,y)\ H(1,0,z) - \frac{89}{18}H(1,0,z) + \frac{4}{3}H(3,y)H(1,1,z)$$

$$+ 2H(0,0,y)\ H(1,1,z) - \frac{13}{4}H(1,1,z) - 2H(0,1,z)H(1,2,y)$$

$$
\begin{aligned}
&+2H(1,0,z)H(1,2,y)+2\ H(0,0,z)H(2,0,y)+2H(0,1,z)H(2,0,y)\\
&+2H(1,0,z)H(2,0,y)-\frac{89}{18}\ H(2,0,y)+2H(0,0,z)H(2,2,y)+4H(0,1,z)H(2,2,y)\\
&+4H(1,0,z)\ H(2,2,y)-\frac{13}{4}H(2,2,y)-6H(0,1,z)H(2,3,y)+2H(1,0,z)H(2,3,y)\\
&+2\ H(0,1,z)H(3,0,y)-2H(1,0,z)H(3,0,y)+\frac{134}{9}H(3,2,y)-12H(0,1,z)\ H(3,3,y)\\
&+4H(1,0,z)H(3,3,y)-2H(1,y)H(0,0,1,z)-2H(2,y)H(0,0,1,z)\\
&-8\ H(3,y)H(0,0,1,z)+\frac{1}{3}H(0,0,1,z)+7H(0,0,2,y)+\frac{3u\ H(0,1,0,y)}{s}+\frac{3tH(0,1,0,z)}{s}\\
&+2H(1,y)H(0,1,0,z)+6H(2,y)\ H(0,1,0,z)+4H(3,y)H(0,1,0,z)+\frac{2}{3}H(0,1,1,z)\\
&+7\ H(0,2,0,y)-\frac{2}{3}H(0,2,2,y)-7H(0,3,2,y)+2H(2,y)H(1,0,0,z)+7\ H(1,0,0,z)\\
&-2H(1,y)H(1,0,1,z)+6H(2,y)H(1,0,1,z)+\frac{11}{3}\ H(1,0,1,z)-\frac{6(s+t)H(1,1,0,y)}{s}\\
&-\frac{6uH(1,1,0,z)}{s}+2\ H(1,y)H(1,1,0,z)+6H(2,y)H(1,1,0,z)+H(0,y)(-\frac{3H(1,0,z)\ u}{s}\\
&+\frac{8u}{3s}+(\frac{8tu}{3s^2}-\frac{14}{3})\ H(0,z)-\frac{89}{18}H(1,z)+7H(0,0,z)-\frac{13}{6}H(0,1,z)+\frac{17}{2}\ H(1,0,z)\\
&-\frac{2}{3}H(1,1,z)-2H(0,0,1,z)-2H(0,1,1,z)+2H(1,0,0,z)+2\ H(1,1,0,z)-\frac{142}{27})\\
&+7H(2,0,0,y)-\frac{2}{3}\ H(2,0,2,y)+\frac{20}{3}H(2,1,0,y)-\frac{2}{3}H(2,2,0,y)+\frac{13}{3}\ H(2,3,2,y)\\
&-7H(3,0,2,y)-7H(3,2,0,y)+\frac{4}{3}H(3,2,2,y)+\frac{22}{3}\ H(3,3,2,y)+H(0,z)(\frac{8t}{3s}\\
&-\frac{89}{18}H(2,y)+7\ H(0,0,y)+\frac{29}{6}H(0,2,y)-\frac{3(s+t)H(1,0,y)}{s}+\frac{23}{2}\ H(2,0,y)\\
&-\frac{2}{3}H(2,2,y)-7H(3,2,y)+2H(0,0,2,y)+2H(0,2,0,y)-2\ H(0,2,2,y)-6H(0,3,2,y)\\
&-2H(1,0,2,y)+2H(2,0,0,y)+2H(2,2,0,y)+2\ H(2,3,2,y)-2H(3,0,2,y)\\
&-2H(3,2,0,y)+4H(3,3,2,y)-\frac{142}{27})+8\ H(0,0,1,0,z)+2H(0,0,1,1,z)\\
&+2H(0,0,2,2,y)+2H(0,1,0,1,z)+6H(0,1,1,0,z)+2\ H(0,2,0,2,y)\\
&+4H(0,2,1,0,y)+2H(0,2,2,0,y)+4H(0,3,0,2,y)+4H(0,3,2,0,y)+4\ H(0,3,3,2,y)\\
&+6H(1,0,1,0,z)+2H(1,0,3,2,y)+2H(1,1,0,0,z)+4H(1,1,0,1,z)+6\ H(1,1,1,0,z)\\
&-2H(1,2,3,2,y)+2H(2,0,0,2,y)+2H(2,0,1,0,y)+2H(2,0,2,0,y)+2\ H(2,0,3,2,y)\\
&+2H(2,1,0,2,y)+2H(2,1,2,0,y)+2H(2,2,0,0,y)+4H(2,2,1,0,y)\\
&+4\ H(2,2,3,2,y)+2H(2,3,0,2,y)+2H(2,3,2,0,y)-8H(2,3,3,2,y)\\
&+4H(3,0,3,2,y)+4\ H(3,3,0,2,y)+4H(3,3,2,0,y)-16H(3,3,3,2,y)-\frac{571}{81}\Big)\Big\}
\end{aligned}
$$

$$+C_AC_F\Big\{\zeta_4(\frac{93}{4})+\zeta_3(22H(1,y)-18H(2,y)+4H(1,z)+\frac{4t^2}{s^2}+\frac{4u^2}{s^2}-\frac{8\,(t+u)}{s}+\frac{335}{18})+\zeta_2($$

$$-\frac{4t^2H(1,y)}{s^2}-\frac{2t^2H(0,z)}{s^2}-\frac{2u^2\,H(0,y)}{s^2}-\frac{4u^2H(1,z)}{s^2}+\frac{8tH(1,y)}{s}+\frac{4t\,H(0,z)}{s}+\frac{4uH(0,y)}{s}$$

$$+\frac{8uH(1,z)}{s}-\frac{107t^2H(2,y)}{6\,(t+u)^2}-\frac{79t^2H(1,z)}{6(t+u)^2}-\frac{119tuH(2,y)}{3\,(t+u)^2}-\frac{91tuH(1,z)}{3(t+u)^2}-\frac{107u^2H(2,y)}{6\,(t+u)^2}$$

$$-\frac{79u^2H(1,z)}{6(t+u)^2}-2H(1,y)H(0,z)-6H(0,y)\,H(1,z)-2H(1,y)H(1,z)-4H(2,y)H(0,z)$$

$$+\frac{14}{3}H(1,y)+2H(0,2,y)+6\;H(1,0,y)-2H(1,2,y)-4H(2,0,y)-8H(2,1,y)$$

$$+8H(2,2,y)+2H(0,1,z)+2H(1,0,z)-2\;H(1,1,z)-\frac{18tu}{s^2}-\frac{2t^2}{s(t+u)}-\frac{2u^2}{s\,(t+u)}$$

$$-\frac{17}{6})+\Big(-\frac{2H(1,0,y)t^2}{s(t+u)}-\frac{5H(2,y)H(1,0,z)t^2}{3\,(t+u)^2}+\frac{2H(1,0,z)t^2}{s(t+u)}+\frac{10H(0,1,0,y)t^2}{3\,(t+u)^2}$$

$$+\frac{2H(0,1,0,z)t^2}{s^2}+\frac{28H(0,1,0,z)t^2}{3\,(t+u)^2}-\frac{4H(1,1,0,y)t^2}{s^2}+\frac{6H(1,1,0,z)\,t^2}{(t+u)^2}$$

$$+\frac{3H(2,1,0,y)t^2}{(t+u)^2}-\frac{3H(1,z)\,H(3,y)t}{u}+\frac{4uH(1,0,y)t}{s(t+u)}-\frac{18uH(1,0,y)\,t}{s^2}-\frac{152H(1,0,y)t}{9(t+u)}$$

$$+\frac{4uH(1,0,z)t}{s\,(t+u)}-\frac{18uH(1,0,z)t}{s^2}-\frac{22uH(2,y)H(1,0,z)t}{3\,(t+u)^2}-\frac{H(1,0,z)t}{(t+u)}-\frac{3H(3,2,y)t}{u}$$

$$+\frac{32\,uH(0,1,0,y)t}{3(t+u)^2}+\frac{68uH(0,1,0,z)t}{3\,(t+u)^2}-\frac{4H(0,1,0,z)t}{s}+\frac{8H(1,1,0,y)t}{s}$$

$$+\frac{8u\,H(1,1,0,z)t}{(t+u)^2}+\frac{2uH(2,1,0,y)\,t}{(t+u)^2}+\frac{604}{27}H(1,z)+2H(1,z)H(2,y)+\frac{604}{27}\,H(2,y)$$

$$-\frac{304}{9}H(1,z)H(3,y)-14H(1,z)H(0,0,y)-14H(2,y)\,H(0,0,z)-\frac{22}{3}H(2,y)H(0,1,z)$$

$$-\frac{20}{3}H(3,y)H(0,1,z)-4\,H(0,0,y)H(0,1,z)-\frac{197}{9}H(0,1,z)-\frac{14}{3}H(1,z)H(0,2,y)$$

$$-4\,H(0,0,z)H(0,2,y)+6H(0,1,z)H(0,2,y)+\frac{107}{9}H(0,2,y)+20H(1,z)\,H(0,3,y)$$

$$+4H(0,1,z)H(0,3,y)+\frac{2u^2H(1,0,y)}{s(t+u)}-\frac{143u\,H(1,0,y)}{9(t+u)}+3H(1,z)H(1,0,y)$$

$$
\begin{aligned}
&- 2H(0,1,z)H(1,0,y) - \frac{2u^2\, H(1,0,z)}{s(t+u)} - \frac{5uH(1,0,z)}{(t+u)} - \frac{5u^2H(2,y)\, H(1,0,z)}{3(t+u)^2}\\
&+ 20H(3,y)H(1,0,z) - 4H(0,0,y)H(1,0,z) + 8H(0,2,y)\ H(1,0,z)\\
&+ 16H(0,3,y)H(1,0,z) + 2H(1,0,y)H(1,0,z) + \frac{28}{3}H(3,y)\ H(1,1,z)\\
&- 8H(0,0,y)H(1,1,z) + 4H(0,3,y)H(1,1,z) + 2H(1,1,z) + 6H(0,1,z)\ H(1,2,y)\\
&- 2H(1,0,z)H(1,2,y) - \frac{14}{3}H(1,z)H(2,0,y) - 4H(0,0,z)\ H(2,0,y)\\
&- 2H(0,1,z)H(2,0,y) - 6H(1,0,z)H(2,0,y) + \frac{107}{9}H(2,0,y) - 8\ H(0,0,z)H(2,2,y)\\
&- 16H(0,1,z)H(2,2,y) - 8H(1,0,z)H(2,2,y) + 2\ H(2,2,y) - \frac{8}{3}H(1,z)H(2,3,y)\\
&+ 12H(0,1,z)H(2,3,y) - 4H(1,0,z)\ H(2,3,y) + 20H(1,z)H(3,0,y)\\
&- 4H(0,1,z)H(3,0,y) + 4H(1,1,z)\ H(3,0,y) + \frac{28}{3}H(1,z)H(3,2,y)\\
&- 4H(0,1,z)H(3,2,y) - 4H(1,0,z)\ H(3,2,y) - \frac{304}{9}H(3,2,y) - \frac{80}{3}H(1,z)H(3,3,y)\\
&+ 28H(0,1,z)\ H(3,3,y) - 12H(1,0,z)H(3,3,y) - 8H(1,1,z)H(3,3,y)\\
&+ 6H(1,y)H(0,0,1,z) + 16\ H(3,y)H(0,0,1,z) - \frac{2}{3}H(0,0,1,z)\\
&- 8H(1,z)H(0,0,2,y) - 14H(0,0,2,y) + 4\ H(1,z)H(0,0,3,y) + \frac{2u^2H(0,1,0,y)}{s^2}\\
&+ \frac{10u^2H(0,1,0,y)}{3\,(t+u)^2} - \frac{4uH(0,1,0,y)}{s} + 2H(1,z)H(0,1,0,y) + \frac{28u^2\, H(0,1,0,z)}{3(t+u)^2}\\
&- 2H(1,y)H(0,1,0,z) - 12H(2,y)H(0,1,0,z) - 4\ H(3,y)H(0,1,0,z)\\
&- 4H(3,y)H(0,1,1,z) + \frac{14}{3}H(0,1,1,z) - 8H(1,z)\ H(0,2,0,y) - 14H(0,2,0,y)\\
&- \frac{14}{3}H(0,2,2,y) + 2H(1,z)H(0,2,3,y) - 8\ H(1,z)H(0,3,0,y) + 4H(1,z)H(0,3,2,y)\\
&+ 20H(0,3,2,y) - 12H(1,z)H(0,3,3,y) + 4\ H(1,z)H(1,0,0,y) + 14H(1,0,0,y)\\
&- 4H(2,y)H(1,0,0,z) + 6H(1,y)H(1,0,1,z) - 18\ H(2,y)H(1,0,1,z)\\
&- 4H(3,y)H(1,0,1,z) - \frac{13}{3}H(1,0,1,z) + 3H(1,0,2,y) - 6\ H(1,z)H(1,0,3,y)\\
&+ \frac{14}{3}H(1,1,0,y) + H(0,y)((-\frac{18t\,u}{s^2} + \frac{4tu}{s(t+u)} + 2)H(0,z) + (\frac{3\,u}{t} + \frac{107}{9})H(1,z)\\
&+ \frac{1}{3}(-\frac{19H(1,0,z)\,t^2}{(t+u)^2} - \frac{54ut}{s(s+t)} - \frac{50uH(1,0,z)\,t}{(t+u)^2} + \frac{10t}{(s+t)} - \frac{36u}{(s+t)} + 13\ H(0,1,z)\\
&- \frac{6u^2H(1,0,z)}{s^2} - \frac{19u^2\, H(1,0,z)}{(t+u)^2} + \frac{12uH(1,0,z)}{s} - 14H(1,1,z) + 12H(0,0,1,z)\\
&+ 6\ H(0,1,0,z) + 24H(0,1,1,z) + 6H(1,0,1,z) - 12H(1,1,0,z) + \frac{10\ s}{(s+t)}))\\
&- \frac{4u^2H(1,1,0,z)}{s^2} + \frac{6u^2\, H(1,1,0,z)}{(t+u)^2} + \frac{8uH(1,1,0,z)}{s} - 2H(1,y)H(1,1,0,z)\\
&- 18\ H(2,y)H(1,1,0,z) - 4H(3,y)H(1,1,0,z) + 3H(1,2,0,y) + 6H(1,z)H(1,2,3,y)\\
&- 8\ H(1,z)H(2,0,0,y) - 14H(2,0,0,y) - \frac{14}{3}H(2,0,2,y) - 4H(1,z)\ H(2,0,3,y)
\end{aligned}
$$

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\begin{aligned}
&+\frac{3u^2H(2,1,0,y)}{(t+u)^2}-2H(1,z)\,H(2,1,0,y)-\frac{14}{3}H(2,2,0,y)-16H(1,z)H(2,2,3,y)\\
&-4H(1,z)\,H(2,3,0,y)-\frac{8}{3}H(2,3,2,y)+16H(1,z)H(2,3,3,y)+4H(1,z)\,H(3,0,2,y)\\
&+20H(3,0,2,y)-12H(1,z)H(3,0,3,y)+4H(1,z)H(3,2,0,y)+20\,H(3,2,0,y)\\
&+\frac{28}{3}H(3,2,2,y)-8H(1,z)H(3,2,3,y)-12H(1,z)\,H(3,3,0,y)+H(0,z)((\frac{3t}{u}\\
&+\frac{107}{9})\,H(2,y)+\frac{1}{3}(-\frac{47H(2,0,y)t^2}{(t+u)^2}-\frac{54u\,t}{s(s+u)}-\frac{106uH(2,0,y)t}{(t+u)^2}-\frac{36\,t}{(s+u)}\\
&+\frac{10u}{(s+u)}-47H(0,2,y)+(-\frac{6\,t^2}{s^2}+\frac{12t}{s}+28)H(1,0,y)-\frac{47u^2\,H(2,0,y)}{(t+u)^2}\\
&-14H(2,2,y)+60H(3,2,y)-24H(0,0,2,y)+6\,H(0,1,0,y)+12H(0,2,2,y)\\
&+48H(0,3,2,y)+12H(1,0,0,y)+12H(1,0,2,y)+6\,H(1,2,0,y)-12H(2,0,0,y)\\
&+6H(2,0,2,y)-6H(2,1,0,y)-12H(2,2,0,y)-12\,H(2,3,2,y)+24H(3,0,2,y)\\
&+12H(3,2,2,y)-36H(3,3,2,y)+\frac{10\,s}{(s+u)}))-8H(1,z)H(3,3,2,y)-\frac{80}{3}H(3,3,2,y)\\
&+40\,H(1,z)H(3,3,3,y)+4H(0,0,1,0,y)-8H(0,0,1,0,z)-8H(0,0,1,1,z)\\
&-8\,H(0,0,2,2,y)+4H(0,0,3,2,y)-8H(0,1,0,1,z)+2H(0,1,0,2,y)\\
&-12H(0,1,1,0,z)+2\,H(0,1,2,0,y)-8H(0,2,0,2,y)-2H(0,2,1,0,y)\\
&-8H(0,2,2,0,y)+2H(0,2,3,2,y)-8\,H(0,3,0,2,y)-8H(0,3,2,0,y)\\
&+4H(0,3,2,2,y)-12H(0,3,3,2,y)-2H(1,0,0,1,z)+4\,H(1,0,0,2,y)\\
&+4H(1,0,1,0,y)-4H(1,0,1,0,z)+4H(1,0,2,0,y)-6H(1,0,3,2,y)\\
&-12\,H(1,1,0,1,z)-14H(1,1,1,0,z)+4H(1,2,0,0,y)+4H(1,2,1,0,y)\\
&+6H(1,2,3,2,y)-8\,H(2,0,0,2,y)-8H(2,0,2,0,y)-4H(2,0,3,2,y)+4H(2,1,0,0,y)\\
&-2H(2,1,0,2,y)-8\,H(2,1,1,0,y)-2H(2,1,2,0,y)-8H(2,2,0,0,y)-8H(2,2,1,0,y)\\
&-16H(2,2,3,2,y)-4\,H(2,3,0,2,y)-4H(2,3,2,0,y)+16H(2,3,3,2,y)\\
&+8H(3,0,1,0,y)+4H(3,0,2,2,y)-12\,H(3,0,3,2,y)+4H(3,2,0,2,y)\\
&-8H(3,2,1,0,y)+4H(3,2,2,0,y)-8H(3,2,3,2,y)-12\,H(3,3,0,2,y)\\
&-12H(3,3,2,0,y)-8H(3,3,2,2,y)+40\,H(3,3,3,2,y)-\frac{467}{81}-\frac{3uH(1,z)H(3,y)}{t}\\
&-\frac{3u\,H(0,1,z)}{t}+\frac{3uH(0,2,y)}{t}-\frac{3u^2H(1,0,y)}{(t+u)\,t}+\frac{3uH(2,0,y)}{t}-\frac{3uH(3,2,y)}{t}\Big)\Big\}
\end{aligned}
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\begin{aligned}
&+ C_F^2\Big\{\zeta_4(-22) + \zeta_3(-16H(1,y) + 8H(2,y) - 8H(1,z) - \frac{4t^2}{s^2} - \frac{4u^2}{s^2} + \frac{8\,(t+u)}{s} \\
&- 30) + \zeta_2(\frac{4t^2H(1,y)}{s^2} + \frac{2t^2H(0,z)}{s^2} + \frac{2u^2\,H(0,y)}{s^2} + \frac{4u^2H(1,z)}{s^2} - \frac{8tH(1,y)}{s} \\
&- \frac{4t\,H(0,z)}{s} - \frac{4uH(0,y)}{s} - \frac{8uH(1,z)}{s} + \frac{6t^2\,H(2,y)}{(t+u)^2} + \frac{6t^2H(1,z)}{(t+u)^2} + \frac{16tu\,H(2,y)}{(t+u)^2} \\
&+ \frac{16tuH(1,z)}{(t+u)^2} + \frac{6u^2\,H(2,y)}{(t+u)^2} + \frac{6u^2H(1,z)}{(t+u)^2} - 8H(2,y)H(1,z) + 8\,H(0,1,y) \\
&- 8H(0,2,y) + 8H(1,1,y) + 8H(2,1,y) - 16H(2,2,y) + \frac{12t\,u}{s^2} + \frac{2t^2}{s(t+u)} + \frac{2u^2}{s(t+u)} \\
&+ 16) + \Big(\frac{2H(1,0,y)t^2}{s(t+u)} - \frac{2H(1,0,z)t^2}{s\,(t+u)} + \frac{12H(0,1,0,y)t^2}{(t+u)^2} - \frac{2H(0,1,0,z)\,t^2}{s^2} \\
&+ \frac{4H(1,1,0,y)t^2}{s^2} - \frac{6H(1,1,0,z)\,t^2}{(t+u)^2} - \frac{12H(2,1,0,y)t^2}{(t+u)^2} + \frac{6H(1,z)\,H(3,y)t}{u} \\
&- \frac{4uH(1,0,y)t}{s(t+u)} + \frac{12uH(1,0,y)\,t}{s^2} + \frac{6H(1,0,y)t}{(t+u)} - \frac{4uH(1,0,z)t}{s\,(t+u)} + \frac{12uH(1,0,z)t}{s^2} \\
&+ \frac{4uH(2,y)H(1,0,z)\,t}{(t+u)^2} - \frac{2H(1,0,z)t}{(t+u)} + \frac{6H(3,2,y)\,t}{u} + \frac{20uH(0,1,0,y)t}{(t+u)^2} \\
&- \frac{4uH(0,1,0,z)\,t}{(t+u)^2} + \frac{4H(0,1,0,z)t}{s} - \frac{8H(1,1,0,y)t}{s} - \frac{8u\,H(1,1,0,z)t}{(t+u)^2} - \frac{20uH(2,1,0,y)t}{(t+u)^2} \\
&- 18\,H(1,z) + 9H(1,z)H(2,y) - 18H(2,y) + 8H(1,z)H(3,y) + 12H(3,y)H(0,1,z) \\
&+ 4\,H(0,1,z) + 12H(1,z)H(0,2,y) - 4H(0,1,z)H(0,2,y) - 4H(0,2,y) \\
&- 12H(1,z)\,H(0,3,y) - \frac{2u^2H(1,0,y)}{s(t+u)} + \frac{8u\,H(1,0,y)}{(t+u)} - 6H(1,z)H(1,0,y) \\
&+ 4H(0,1,z)H(1,0,y) + \frac{2u^2\,H(1,0,z)}{s(t+u)} + \frac{2uH(1,0,z)}{(t+u)} - 12H(3,y)H(1,0,z) \\
&- 8\,H(0,2,y)H(1,0,z) - 8H(0,3,y)H(1,0,z) - 24H(3,y)H(1,1,z) \\
&+ 8H(0,0,y)\,H(1,1,z) - 8H(0,3,y)H(1,1,z) + 9H(1,1,z) - 4H(0,1,z)H(1,2,y) \\
&+ 12H(1,z)\,H(2,0,y) - 4H(0,1,z)H(2,0,y) - 4H(2,0,y) + 8H(0,0,z)H(2,2,y) \\
&+ 16H(0,1,z)\,H(2,2,y) + 9H(2,2,y) - 12H(1,z)H(2,3,y) - 12H(1,z)H(3,0,y) \\
&- 8H(1,1,z)\,H(3,0,y) - 24H(1,z)H(3,2,y) + 8H(0,1,z)H(3,2,y) \\
&+ 8H(3,2,y) + 24H(1,z)\,H(3,3,y) - 8H(0,1,z)H(3,3,y) + 8H(1,0,z)H(3,3,y) \\
&+ 16H(1,1,z)H(3,3,y) - 4\,H(1,y)H(0,0,1,z) + 8H(2,y)H(0,0,1,z) \\
&+ 8H(1,z)H(0,0,2,y) - 8H(1,z)\,H(0,0,3,y) - \frac{2u^2H(0,1,0,y)}{s^2} + \frac{12u^2\,H(0,1,0,y)}{(t+u)^2} \\
&+ \frac{4uH(0,1,0,y)}{s} - 4H(1,z)H(0,1,0,y) + 4\,H(2,y)H(0,1,0,z) + 8H(3,y)H(0,1,1,z) \\
&- 12H(0,1,1,z) + 8H(1,z)H(0,2,0,y) + 12\,H(0,2,2,y) - 4H(1,z)H(0,2,3,y)
\end{aligned}
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\begin{aligned}
&- 8H(1,z)H(0,3,2,y) - 12H(0,3,2,y) + 8H(1,z)\,H(0,3,3,y) - 8H(1,z)H(1,0,0,y)\\
&+ 2H(0,y)\Big(\frac{H(1,0,z)\,u^2}{s^2} - \frac{2t(s-3(t+u))H(0,z)u}{s^2(t+u)} - \frac{2\,H(1,0,z)u}{s} + \frac{2tH(1,0,z)u}{(t+u)^2}\\
&+ \frac{6\,u}{s} + \Big(-\frac{3u}{t} - 2\Big)H(1,z) + 6H(1,1,z) - 4H(0,1,1,z) - 2\,H(1,0,1,z)\Big)\\
&- 4H(1,y)H(1,0,1,z) + 12H(2,y)H(1,0,1,z) + 8H(3,y)\,H(1,0,1,z)\\
&- 6H(1,0,1,z) - 6H(1,0,2,y) + 4H(1,z)H(1,0,3,y) + \frac{4u^2\,H(1,1,0,z)}{s^2}\\
&- \frac{6u^2H(1,1,0,z)}{(t+u)^2} - \frac{8u\,H(1,1,0,z)}{s} + 4H(2,y)H(1,1,0,z) - 6H(1,2,0,y)\\
&- 4H(1,z)H(1,2,3,y) + 8H(1,z)\,H(2,0,0,y) + 12H(2,0,2,y) - \frac{12u^2H(2,1,0,y)}{(t+u)^2}\\
&- 4H(1,z)\,H(2,1,0,y) + 12H(2,2,0,y) + 16H(1,z)H(2,2,3,y) - 12H(2,3,2,y)\\
&- 8H(1,z)\,H(3,0,2,y) - 12H(3,0,2,y) + 8H(1,z)H(3,0,3,y) - 8H(1,z)H(3,2,0,y)\\
&- 12\,H(3,2,0,y) - 24H(3,2,2,y) + 16H(1,z)H(3,2,3,y) + 8H(1,z)H(3,3,0,y)\\
&+ 16H(1,z)\,H(3,3,2,y) + 24H(3,3,2,y) + 2H(0,z)\Big(\frac{3H(2,0,y)\,t^2}{(t+u)^2} + \frac{(s+t)H(1,0,y)t}{s^2}\\
&+ \frac{8uH(2,0,y)\,t}{(t+u)^2} + \frac{6t}{s} + \Big(-\frac{3t}{u} - 2\Big)H(2,y) + 6\,H(0,2,y) - \frac{3(s+t)H(1,0,y)}{s}\\
&+ \frac{3u^2\,H(2,0,y)}{(t+u)^2} + 6H(2,2,y) - 6H(3,2,y) + 4H(0,0,2,y) - 4H(0,3,2,y) - 2\,H(2,0,2,y)\\
&- 4H(3,0,2,y) - 4H(3,2,2,y) + 4H(3,3,2,y)\Big) - 16H(1,z)\,H(3,3,3,y) + 8H(0,0,1,1,z)\\
&+ 8H(0,0,2,2,y) - 8H(0,0,3,2,y) + 8H(0,1,0,1,z) - 4\,H(0,1,0,2,y) + 8H(0,1,1,0,y)\\
&+ 8H(0,1,1,0,z) - 4H(0,1,2,0,y) + 8H(0,2,0,2,y) - 4\,H(0,2,1,0,y) + 8H(0,2,2,0,y)\\
&- 4H(0,2,3,2,y) - 8H(0,3,2,2,y) + 8H(0,3,3,2,y) + 4\,H(1,0,0,1,z) - 8H(1,0,0,2,y)\\
&+ 4H(1,0,1,0,y) - 8H(1,0,2,0,y) + 4H(1,0,3,2,y) + 8\,H(1,1,0,0,y) + 8H(1,1,0,1,z)\\
&+ 8H(1,1,1,0,y) + 8H(1,1,1,0,z) - 8H(1,2,0,0,y) - 4\,H(1,2,1,0,y) - 4H(1,2,3,2,y)\\
&+ 8H(2,0,0,2,y) - 4H(2,0,1,0,y) + 8H(2,0,2,0,y) - 8\,H(2,1,0,0,y) - 4H(2,1,0,2,y)\\
&+ 8H(2,1,1,0,y) - 4H(2,1,2,0,y) + 8H(2,2,0,0,y) + 16\,H(2,2,3,2,y)\\
&- 8H(3,0,1,0,y) - 8H(3,0,2,2,y) + 8H(3,0,3,2,y) - 8H(3,2,0,2,y)\\
&+ 8\,H(3,2,1,0,y) - 8H(3,2,2,0,y) + 16H(3,2,3,2,y) + 8H(3,3,0,2,y)\\
&+ 8H(3,3,2,0,y) + 16\,H(3,3,2,2,y) - 16H(3,3,3,2,y) + 6 + \frac{6uH(1,z)H(3,y)}{t}\\
&+ \frac{6u\,H(0,1,z)}{t} - \frac{6uH(0,2,y)}{t} + \frac{6u^2H(1,0,y)}{(t+u)\,t} - \frac{6uH(2,0,y)}{t} + \frac{6uH(3,2,y)}{t}\Big)\Big\}
\end{aligned}
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\begin{aligned}
&+ C_A n_f\Big\{\zeta_3(-\frac{37}{18}) + \zeta_2(-\frac{1}{3}H(0,y) + \frac{1}{6}H(2,y) - \frac{1}{3}H(0,z) + \frac{1}{6}\ H(1,z) + \frac{tu}{3s^2} - \frac{7}{36})\\
&+\Big(\frac{1}{36}H(0,y)(H(0,z)(\frac{12tu}{s^2}+20)+31\ H(1,z)-36H(0,0,z)+24H(0,1,z)-36H(1,0,z)\\
&+ 24H(1,1,z) + \frac{12\ u}{s} + 86) + \frac{tuH(1,0,y)}{3s^2} + \frac{tuH(1,0,z)}{3s^2} + H(0,z)\ (\frac{31}{36}H(2,y)\\
&- H(0,0,y) - \frac{1}{3}\ H(0,2,y) - H(2,0,y) + \frac{2}{3}H(2,2,y) + H(3,2,y) + \frac{t}{3\ s} + \frac{43}{18})\\
&+ H(2,y)H(1,z) - \frac{20}{9}H(3,y)H(1,z) - H(0,0,y)\ H(1,z) + \frac{2}{3}H(0,2,y)H(1,z)\\
&+ H(0,3,y)H(1,z) + \frac{2}{3}H(2,0,y)\ H(1,z) - \frac{4}{3}H(2,3,y)H(1,z) + H(3,0,y)H(1,z)\\
&- \frac{4}{3}H(3,2,y)\ H(1,z) - \frac{4}{3}H(3,3,y)H(1,z) - H(2,y)H(0,0,z) - \frac{2}{3}H(2,y)\ H(0,1,z)\\
&-\frac{1}{3}H(3,y)H(0,1,z)+H(3,y)H(1,0,z)-\frac{4}{3}H(3,y)\ H(1,1,z)+\frac{17}{27}H(2,y)-\frac{41}{18}H(0,0,y)\\
&+ \frac{31}{36}\ H(0,2,y) + \frac{31}{36}H(2,0,y) + H(2,2,y) - \frac{20}{9}\ H(3,2,y) - H(0,0,2,y) - H(0,2,0,y)\\
&+ \frac{2}{3}\ H(0,2,2,y) + H(0,3,2,y) - H(2,0,0,y) + \frac{2}{3}H(2,0,2,y) - \frac{2}{3}\ H(2,1,0,y)\\
&+ \frac{2}{3}H(2,2,0,y) - \frac{4}{3}\ H(2,3,2,y) + H(3,0,2,y) + H(3,2,0,y) - \frac{4}{3}H(3,2,2,y)\\
&- \frac{4}{3}\ H(3,3,2,y) + \frac{17}{27}H(1,z) - \frac{41}{18}H(0,0,z) - \frac{49}{36}\ H(0,1,z) + \frac{31}{36}H(1,0,z)\\
&+ H(1,1,z) - \frac{1}{3}H(0,0,1,z) - \frac{2}{3}\ H(0,1,1,z) - H(1,0,0,z) - \frac{2}{3}H(1,0,1,z) + \frac{65}{162}\Big)\Big\}\\
&+ C_F n_f\Big\{\zeta_3(-\frac{1}{9}) + \zeta_2(\frac{1}{3}(4H(1,y) - 5H(2,y) - H(1,z) + 1)) + \Big(-\frac{279}{2}H(0,y)H(1,z)\\
&-108H(0,y)H(0,1,z)+27H(0,y)H(1,0,z)-108\ H(0,y)H(1,1,z)-\frac{279}{2}H(2,y)H(0,z)\\
&- 162H(2,y)H(1,z) + 360H(3,y)\ H(1,z) + 162H(0,0,y)H(1,z) + 162H(2,y)H(0,0,z)\\
&+108H(2,y)H(0,1,z)+54H(3,y)\ H(0,1,z)+54H(0,2,y)H(0,z)-108H(0,2,y)H(1,z)\\
&- 162H(0,3,y)H(1,z) - 27\ H(1,0,y)H(0,z) - 108H(2,y)H(1,0,z)\\
&- 162H(3,y)H(1,0,z) + 216H(3,y)\ H(1,1,z) + 54H(2,0,y)H(0,z)\\
&- 108H(2,0,y)H(1,z) - 108H(2,2,y)H(0,z) + 216\ H(2,3,y)H(1,z)\\
&- 162H(3,0,y)H(1,z) - 162H(3,2,y)H(0,z) + 216H(3,2,y)\ H(1,z)\\
&+ 216H(3,3,y)H(1,z) - 27H(0,y) - 102H(2,y) - \frac{279}{2}H(0,2,y) + 180\ H(1,0,y)\\
&- \frac{279}{2}H(2,0,y) - 162H(2,2,y) + 360H(3,2,y) + 162H(0,0,2,y) - 27\ H(0,1,0,y)\\
&+ 162H(0,2,0,y) - 108H(0,2,2,y) - 162H(0,3,2,y) - 162H(1,0,0,y)\\
&+ 108\ H(1,1,0,y) + 162H(2,0,0,y) - 108H(2,0,2,y) - 108H(2,2,0,y)\\
&+ 216H(2,3,2,y) - 162\ H(3,0,2,y) - 162H(3,2,0,y) + 216H(3,2,2,y)\\
&+216H(3,3,2,y)-27H(0,z)-102\ H(1,z)+\frac{441}{2}H(0,1,z)+\frac{81}{2}H(1,0,z)-162H(1,1,z)\\
&+ 54\ H(0,0,1,z) - 27H(0,1,0,z) + 108H(0,1,1,z) + 108H(1,0,1,z) + 200\Big)/81\Big\}
\end{aligned}
$$

$$+ n_f^2\Big\{\frac{1}{108}(H(0,y)(3H(0,z)-20)+15H(0,0,y)-20H(0,z)+15H(0,0,z)+6\ \zeta_2)\Big\}$$

$$\mathcal{B}^{(2)}_{2;2} = \frac{1}{24}(121C_A{}^2+44C_A(6C_F-n_f)+4\ (27C_F{}^2-12C_Fn_f+n_f{}^2))$$

$$\begin{aligned}
\mathcal{B}^{(2)}_{2;1} = C_A^2\Big\{&\frac{1}{108}(594H(0,y)H(0,z)+594H(0,y)H(1,z)+594H(2,y)H(0,z)\\
&-1188\ H(3,y)H(1,z)+990H(0,y)+891H(2,y)+594H(0,2,y)\\
&+594H(2,0,y)-1188\ H(3,2,y)+990H(0,z)+891H(1,z)-594H(0,1,z)\\
&+594H(1,0,z)+495\ \zeta_2-108\zeta_3-1296)\Big\}+C_AC_F\Big\{\frac{1}{108}(-1188H(0,y)H(1,z)\\
&+54H(0,y)(6H(0,z)+6H(1,z)+10)-1188\ H(2,y)H(0,z)\\
&+54(6H(2,y)+10)H(0,z)+1728H(3,y)H(1,z)-1296H(2,y)-864\ H(0,2,y)\\
&+1188H(1,0,y)-864H(2,0,y)+1728H(3,2,y)-1296H(1,z)+864\ H(0,1,z)\\
&+324H(1,0,z)+1566\zeta_2-2808\zeta_3-778)\Big\}+C_F^2\Big\{\frac{1}{108}(-648H(0,y)H(1,z)\\
&-648H(2,y)H(0,z)+1296H(3,y)H(1,z)-972\ H(2,y)-648H(0,2,y)\\
&+648H(1,0,y)-648H(2,0,y)+1296H(3,2,y)-972H(1,z)+648\ H(0,1,z)\\
&-1296\zeta_2+2592\zeta_3+972)\Big\}+C_An_f\Big\{\frac{1}{108}(-2(-27(4H(3,y)-3)H(1,z)\\
&+81H(2,y)+54H(0,2,y)+54\ H(2,0,y)-108H(3,2,y)-54H(0,1,z)\\
&+54H(1,0,z)+45\zeta_2-260)-9H(0,y)\ (12H(0,z)+12H(1,z)+31)\\
&-9(12H(2,y)+31)H(0,z))\Big\}+C_Fn_f\Big\{\frac{1}{108}(-2(54(4H(3,y)-3)H(1,z)\\
&-162H(2,y)-108H(0,2,y)+108\ H(1,0,y)-108H(2,0,y)\\
&+216H(3,2,y)+108H(0,1,z)+54\zeta_2+22)-9H(0,y)\ (6-24H(1,z))\\
&-9(6-24H(2,y))H(0,z))\Big\}+n_f^2\Big\{\frac{1}{6}H(0,y)+\frac{1}{6}H(0,z)-\frac{10}{27}\Big\}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{B}^{(2)}_{2;0} = C_A^2\Big\{ & \zeta_4\Big(\frac{39}{8}\Big) + \zeta_3\Big(- H(0,y) - 6H(1,y) + 11H(2,y) - H(0,z) + 5H(1,z) - (25s^4(t+u) \\
& + 2s^3\,(79t^2 + 266tu + 79u^2) + s^2(133t^3 + 1097t^2u + 1097t\,u^2 + 133u^3) \\
& + 2stu(241t^2 + 806tu + 241u^2) + 457t^2\,(t+u)u^2)/(36(s+t)^2(s+u)^2(t+u))\Big) \\
& + \zeta_2\Big((H(0,z)(s^2(109t+49u) + 2st(115t+61u) + t^2(121t+85\,u)))/(12(s+t)^2(t+u)) \\
& + (H(0,y)(s^2(49t+109u) + 2\,su(61t+115u) + u^2(85t+121u)))/(12(s+u)^2(t+u)) \\
& + (H(2,y)(- 144s^5 - 257s^4(t+u) - 2s^3(35\,t^2 + 202tu + 35u^2) \\
& + s^2(43t^3 - 25t^2u - 25tu^2 + 43u^3) + 2\,stu(55t^2 + 74tu + 55u^2) \\
& + 79t^2(t+u)u^2))/(12(s+t)^2(s+u)^2(t+u)) + (H(1,z)(- 72s^5 - s^4(113\,t + 41u) \\
& + 2s^3(t^2 + 14tu + 73u^2) + s^2(43t^3 + 191t^2u + 407\,tu^2 + 115u^3) + 2stu(55t^2 \\
& + 182tu + 127u^2) + t^2u^2(79\,t + 151u)))/(12(s+t)^2(s+u)^2(t+u)) \\
& + \frac{6(s+t)\,H(1,y)}{(t+u)} + 3H(0,y)H(0,z) + 2H(1,y)H(0,z) + 3H(0,y)H(1,z) \\
& + 2H(1,y)\,H(1,z) + H(2,y)H(0,z) - 2H(3,y)H(1,z) + 3H(0,2,y) + 2H(1,2,y) \\
& + H(2,0,y) - 2\,H(3,2,y) + H(0,1,z) + H(1,0,z) + 2H(1,1,z) - (28s^3(t+u) \\
& + s^2(28\,t^2 + 71tu + 28u^2) + 34st(t+u)u - 3t^2u^2)/(9s(s+t)(s+u)(t+u))\Big) \\
& + \Big(- \frac{2(143t + 89u)H(0,y)}{27(t+u)} + \frac{1}{3}(\frac{tu}{s\,(t+u)} - 17)H(0,z)H(0,y) - ((179u^2 \\
& + 161su + 107tu + 107\,st)H(1,z)H(0,y))/(18(s+u)(t+u)) + 7H(0,0,z)H(0,y) \\
& - (((13t+25u)s^2 + 2u(19t+28u)s + 31(t+u)\,u^2)H(0,1,z)H(0,y))/(6(s+u)^2(t+u)) \\
& + ((17t + 23u)\,H(1,0,z)H(0,y))/(2(t+u)) - \frac{2}{3}H(1,1,z)H(0,y) \\
& - 2H(0,0,1,z)\,H(0,y) - 2H(0,1,1,z)H(0,y) + 2H(1,0,0,z)H(0,y) \\
& + 2H(1,1,0,z)H(0,y) - \frac{2(89t + 143u)H(0,z)}{27(t+u)} - \frac{140}{27}H(1,z) \\
& - ((179t^2 + 161st + 107ut + 107su)H(0,z)H(2,y))/(18(s+t)\,(t+u)) \\
& - \frac{13}{4}H(1,z)H(2,y) - \frac{140}{27}H(2,y) + ((188\,s^2 + 197(t+u)s \\
& + 206tu)H(1,z)H(3,y))/(9(s+t)\,(s+u)) + 7H(0,z)H(0,0,y) + 7H(1,z)H(0,0,y)
\end{aligned}
$$

$$+\frac{80}{9}H(0,0,y)+7\ H(2,y)H(0,0,z)+2H(0,0,y)H(0,0,z)$$

$$+\frac{80}{9}H(0,0,z)+(((215\ t+269u)s^2+(215t^2+520ut+287u^2)s$$

$$+tu(233t+305u))\ H(0,1,z))/(18(s+t)(s+u)(t+u))+((18s^5$$

$$+59\ (t+u)s^4+(61t^2+152ut+61u^2)s^3+4(5t^3+31u\ t^2+31u^2t+5u^3)s^2$$

$$+34t(t+u)^2us+11t^2(t+u)\ u^2)H(2,y)H(0,1,z))/(3(s+t)^2(s+u)^2\ (t+u))$$

$$+\frac{1}{3}H(3,y)H(0,1,z)+2H(0,0,y)H(0,1,z)-((179\ u^2+161su+107tu$$

$$+107st)H(0,2,y))/(18(s+u)\ (t+u))+(((17t+29u)s^2+2t(14t+23u)s$$

$$+11t^2\ (t+u))H(0,z)H(0,2,y))/(6(s+t)^2(t+u))-\frac{2}{3}\ H(1,z)H(0,2,y)$$

$$+2H(0,0,z)H(0,2,y)-2H(0,1,z)H(0,2,y)-(((5t+9\ u)s^4+(9t^2+28ut$$

$$+19u^2)s^3+2(2t^3+14ut^2+21u^2\ t+5u^3)s^2+2tu(5t^2+14ut+9u^2)s$$

$$+7t^2(t+u)\ u^2)H(1,z)H(0,3,y))/((s+t)^2(s+u)^2(t+u))-2H(0,1,z)\ H(0,3,y)$$

$$+(u(6s^2-2ts+9us+tu)H(1,0,y))/(3s(s+u)\ (t+u))-\frac{3uH(0,z)H(1,0,y)}{(t+u)}$$

$$-(3(s+t+u)\ H(1,z)H(1,0,y))/((t+u))-(((125t+107u)s^2$$

$$+t(125t+119u)\ s-6t^2u)H(1,0,z))/(18s(s+t)(t+u))+((-3\ s^3$$

$$-(t-3u)s^2+8t(t+u)s+6t^2(t+u))H(2,y)\ H(1,0,z))/((s+t)^2(t+u))$$

$$-7H(3,y)H(1,0,z)+2H(0,0,y)H(1,0,z)-6\ H(0,3,y)H(1,0,z)$$

$$+\frac{4}{3}H(3,y)H(1,1,z)+2H(0,0,y)\,H(1,1,z)-\frac{13}{4}H(1,1,z)-2H(0,1,z)H(1,2,y)$$

$$+2H(1,0,z)\,H(1,2,y)-((179u^2+161su+107tu+107st)H(2,0,y))/(18\,(s+u)(t+u))$$

$$+\frac{23}{2}H(0,z)H(2,0,y)-\frac{2}{3}H(1,z)\,H(2,0,y)+2H(0,0,z)H(2,0,y)+2H(0,1,z)H(2,0,y)$$

$$+2H(1,0,z)\;H(2,0,y)-\frac{2}{3}H(0,z)H(2,2,y)+2H(0,0,z)H(2,2,y)$$

$$+4H(0,1,z)\,H(2,2,y)+4H(1,0,z)H(2,2,y)-\frac{13}{4}H(2,2,y)+((18s^5+61\,(t+u)s^4$$

$$+5(13t^2+32ut+13u^2)s^3+2(11t^3+67u\,t^2+67u^2t+11u^3)s^2+38t(t+u)^2us$$

$$+13t^2(t+u)\,u^2)H(1,z)H(2,3,y))/(3(s+t)^2(s+u)^2(t+u))-6\,H(0,1,z)H(2,3,y)$$

$$+2H(1,0,z)H(2,3,y)-7H(1,z)H(3,0,y)+2H(0,1,z)\,H(3,0,y)-2H(1,0,z)H(3,0,y)$$

$$+((188s^2+197(t+u)s+206t\,u)H(3,2,y))/(9(s+t)(s+u))-7H(0,z)H(3,2,y)$$

$$+\frac{4}{3}\;H(1,z)H(3,2,y)+\frac{22}{3}H(1,z)H(3,3,y)-12H(0,1,z)H(3,3,y)$$

$$+4H(1,0,z)\,H(3,3,y)+(((7u-5t)s^4+(-13t^2+4ut+17u^2)s^3-2\,(4t^3+8ut^2-13u^2t$$

$$-5u^3)s^2+2tu(-5t^2+2ut+7\,u^2)s+t^2(t+u)u^2)H(0,0,1,z))/(3(s+t)^2\,(s+u)^2(t+u))$$

$$-2H(1,y)H(0,0,1,z)-2H(2,y)H(0,0,1,z)-8H(3,y)\;H(0,0,1,z)$$

$$+2H(0,z)H(0,0,2,y)+2H(1,z)H(0,0,2,y)+7H(0,0,2,y)-(u(s^2$$

$$+(u-2t)s-3tu)H(0,1,0,y))/((s+u)^2\;(t+u))-(t(s^2+(t-2u)s$$

$$-3tu)\;H(0,1,0,z))/((s+t)^2(t+u))+2H(1,y)H(0,1,0,z)+6H(2,y)\;H(0,1,0,z)$$

$$+4H(3,y)H(0,1,0,z)+\frac{2}{3}H(0,1,1,z)+2H(0,z)H(0,2,0,y)+2\;H(1,z)H(0,2,0,y)$$

$$+7H(0,2,0,y)-2H(0,z)H(0,2,2,y)-\frac{2}{3}H(0,2,2,y)+4\;H(1,z)H(0,3,0,y)$$

$$-(((5t+9u)s^4+(9t^2+28ut+19u^2)\;s^3+2(2t^3+14ut^2+21u^2t+5u^3)s^2$$

$$+2tu(5t^2+14u\,t+9u^2)s+7t^2(t+u)u^2)H(0,3,2,y))/((s+t)^2\;(s+u)^2(t+u))$$

$$-6H(0,z)H(0,3,2,y)+4H(1,z)H(0,3,3,y)+2H(2,y)\;H(1,0,0,z)$$

$$+ 7H(1,0,0,z) + ((9s^5 + 32(t+u)s^4 + (34t^2 + 80\,ut + 34u^2)s^3 + (11t^3 + 61ut^2 + 61u^2t$$

$$+ 11u^3)s^2 + tu\,(16t^2 + 23ut + 16u^2)s + 2t^2(t+u)u^2)\,H(1,0,1,z))/(3(s+t)^2(s+u)^2(t+u))$$

$$- 2H(1,y)H(1,0,1,z) + 6\,H(2,y)H(1,0,1,z) - (3(s+t+u)H(1,0,2,y))/((t+u))$$

$$- 2H(0,z)\,H(1,0,2,y) + 2H(1,z)H(1,0,3,y) + (6(s+t)\,H(1,1,0,y))/((t+u))$$

$$+ ((2(t+3u)s^2 + t(5t+14u)s + 3t^2(t+3\,u))H(1,1,0,z))/((s+t)^2(t+u))$$

$$+ 2H(1,y)H(1,1,0,z) + 6H(2,y)\,H(1,1,0,z) - \frac{3(s+t+u)H(1,2,0,y)}{(t+u)}$$

$$- 2H(1,z)\,H(1,2,3,y) + 2H(0,z)H(2,0,0,y) + 2H(1,z)H(2,0,0,y)$$

$$+ 7H(2,0,0,y) - \frac{2}{3}\,H(2,0,2,y) + 2H(1,z)H(2,0,3,y) + ((-9s^3 + (11t-u)s^2$$

$$+ 28(t+u)\,us + 20(t+u)u^2)H(2,1,0,y))/(3(s+u)^2(t+u)) + 2H(1,z)\,H(2,1,0,y)$$

$$+ 2H(0,z)H(2,2,0,y) - \frac{2}{3}H(2,2,0,y) + 4H(1,z)H(2,2,3,y) + 2\,H(1,z)H(2,3,0,y)$$

$$+ ((18s^5 + 61(t+u)s^4 + 5(13t^2 + 32u\,t + 13u^2)s^3 + 2(11t^3 + 67ut^2 + 67u^2t$$

$$+ 11u^3)s^2 + 38t\,(t+u)^2us + 13t^2(t+u)u^2)H(2,3,2,y))/(3(s+t)^2\,(s+u)^2(t+u))$$

$$+ 2H(0,z)H(2,3,2,y) - 8H(1,z)H(2,3,3,y) - 2H(0,z)\,H(3,0,2,y) - 7H(3,0,2,y)$$

$$+ 4H(1,z)H(3,0,3,y) - 2H(0,z)H(3,2,0,y) - 7\,H(3,2,0,y) + \frac{4}{3}H(3,2,2,y)$$

$$+ 4H(1,z)H(3,3,0,y) + 4H(0,z)\,H(3,3,2,y) + \frac{22}{3}H(3,3,2,y) - 16H(1,z)H(3,3,3,y)$$

$$+ 8H(0,0,1,0,z) + 2\,H(0,0,1,1,z) + 2H(0,0,2,2,y) + 2H(0,1,0,1,z) + 6H(0,1,1,0,z)$$

$$+ 2H(0,2,0,2,y) + 4\,H(0,2,1,0,y) + 2H(0,2,2,0,y) + 4H(0,3,0,2,y) + 4H(0,3,2,0,y)$$

$$+ 4H(0,3,3,2,y) + 6\,H(1,0,1,0,z) + 2H(1,0,3,2,y) + 2H(1,1,0,0,z) + 4H(1,1,0,1,z)$$

$$+ 6H(1,1,1,0,z) - 2\,H(1,2,3,2,y) + 2H(2,0,0,2,y) + 2H(2,0,1,0,y) + 2H(2,0,2,0,y)$$

$$+ 2H(2,0,3,2,y) + 2\,H(2,1,0,2,y) + 2H(2,1,2,0,y) + 2H(2,2,0,0,y) + 4H(2,2,1,0,y)$$

$$+ 4H(2,2,3,2,y) + 2\,H(2,3,0,2,y) + 2H(2,3,2,0,y) - 8H(2,3,3,2,y)$$

$$+ 4H(3,0,3,2,y) + 4H(3,3,0,2,y) + 4\,H(3,3,2,0,y) - 16H(3,3,3,2,y) + \frac{761}{81}\Big)\Big\}$$

$$+ C_A C_F\Big\{\zeta_4\Big(\frac{93}{4}\Big) + \zeta_3\Big(22H(1,y) - 18H(2,y) + 4H(1,z) + (803s^5(t+u) + 4s^4(424\,t^2$$
$$+ 965tu + 424u^2) + s^3(821t^3 + 5329t^2u + 5329tu^2 + 821\,u^3) + s^2(-72t^4 + 1966t^3u$$
$$+ 5336t^2u^2 + 1966tu^3 - 72\,u^4) + stu(-144t^3 + 1307t^2u + 1307tu^2 - 144u^3)$$
$$- 72t^2\,u^2(t^2+u^2))/(18s(s+t)^2(s+u)^2(t+u))\Big) + \zeta_2\Big(-(2H(1,y)(18s^2 + 23st + 11su$$
$$-6t^2))/(3s\,(t+u)) + (tH(0,z)(-10s^3 - 3s^2(7t+2u) - 9st(t+u) + 2\,t^3))/(s(s+t)^2(t+u))$$
$$+ (uH(0,y)(-10s^3 - 3s^2(2\,t+7u) - 9s(t+u)u + 2u^3))/(s(s+u)^2\,(t+u))$$
$$+ (H(2,y)(144s^5(t+u) + s^4(289t^2 + 554t\,u + 289u^2) + 4s^3(32t^3 + 147t^2u + 147tu^2$$
$$+ 32u^3) + s^2\,(-17t^4 + 90t^3u + 166t^2u^2 + 90tu^3 - 17u^4) - 2stu(35\,t^3 + 111t^2u$$
$$+ 111tu^2 + 35u^3) - t^2u^2(71t^2 + 166tu + 71\,u^2)))/(6(s+t)^2(s+u)^2(t+u)^2) - (H(1,z)\,($$
$$-72s^6(t+u)^2 - s^5(101t^3 + 231t^2u + 183tu^2 + 53\,u^3) + 4s^4(t+u)^2(8t^2 + 25tu + 26u^2)$$
$$+ s^3(61\,t^5 + 427t^4u + 1120t^3u^2 + 1216t^2u^3 + 523tu^4 + 61u^5) + 2s^2\,(t+u)^2u(79t^3$$
$$+ 236t^2u + 79tu^2 - 12u^3) + stu^2\,(115t^4 + 369t^3u + 321t^2u^2 + 19tu^3 - 48u^4)$$
$$- 24t^2\,(t+u)^2u^4))/(6s(s+t)^2(s+u)^2(t+u)^3) - 2\,H(1,y)H(0,z) - 6H(0,y)H(1,z)$$
$$-2H(1,y)H(1,z) - 4H(2,y)H(0,z) + 2H(0,2,y) + 6\,H(1,0,y) - 2H(1,2,y) - 4H(2,0,y)$$
$$- 8H(2,1,y) + 8H(2,2,y) + 2H(0,1,z) + 2H(1,0,z) - 2\,H(1,1,z) + (19s^3(t+u)$$
$$+ s^2(19t^2 + 182tu + 19u^2) + 145\,st(t+u)u + 108t^2u^2)/(6s(s+t)(s+u)(t+u))\Big)$$

$$+ \Big(((26t^2 + 26st + 33ut + 15su)H(0,y))/(3(s+t)\,(t+u)) + ((18tu)/(s(t+u))$$
$$+ 3)H(0,z)\,H(0,y) + ((242u^2 + 215su + 134tu + 134st)H(1,z)H(0,y))/(9\,(s+u)(t+u))$$
$$+ (((13t + 31u)s^2 + u(44t + 71u)s + 40\,(t+u)u^2)H(0,1,z)H(0,y))/(3(s+u)^2\,(t+u))$$
$$- ((19st^2 + 62sut + 31su^2 - 6(t+u)u^2)\,H(1,0,z)H(0,y))/(3s(t+u)^2)$$
$$- \frac{14}{3}H(1,1,z)H(0,y) + 4H(0,0,1,z)\,H(0,y) + 2H(0,1,0,z)H(0,y)$$
$$+ 8H(0,1,1,z)H(0,y) + 2H(1,0,1,z)H(0,y) - 4\,H(1,1,0,z)H(0,y) + ((26u^2 + 26su$$
$$+ 33tu + 15st)H(0,z))/(3\,(s+u)(t+u)) + \frac{803}{54}H(1,z) + ((242t^2 + 215st + 134\,ut$$
$$+ 134su)H(0,z)H(2,y))/(9(s+t)(t+u)) + 2H(1,z)\,H(2,y) + \frac{803}{54}H(2,y) - ((466s^2$$
$$+ 493(t+u)s + 520t\,u)H(1,z)H(3,y))/(9(s+t)(s+u)) - 14H(1,z)H(0,0,y)$$
$$- 14\,H(2,y)H(0,0,z) - (((251t + 332u)s^2 + (251t^2 + 637ut + 359\,u^2)s$$
$$+ 2tu(139t + 193u))H(0,1,z))/(9(s+t)(s+u)\,(t+u)) - ((36s^5 + 112(t+u)s^4 + (107t^2$$
$$+ 268ut + 107\,u^2)s^3 + (31t^3 + 191ut^2 + 191u^2t + 31u^3)s^2 + 2tu\,(22t^2 + 35ut + 22u^2)s$$
$$+ 4t^2(t+u)u^2)H(2,y)\,H(0,1,z))/(3(s+t)^2(s+u)^2(t+u)) - \frac{20}{3}H(3,y)\,H(0,1,z)$$
$$- 4H(0,0,y)H(0,1,z) + ((242u^2 + 215su + 134tu + 134s\,t)H(0,2,y))/(9(s+u)(t+u))$$
$$- (((29t + 47u)s^2 + t\,(49t + 76u)s + 20t^2(t+u))H(0,z)H(0,2,y))/(3(s+t)^2\,(t+u))$$
$$- \frac{14}{3}H(1,z)H(0,2,y) - 4H(0,0,z)H(0,2,y) + 6H(0,1,z)\,H(0,2,y) + ((2(7t + 13u)s^4$$
$$+ 5(5t^2 + 16ut + 11u^2)\,s^3 + (11t^3 + 79ut^2 + 121u^2t + 29u^3)s^2 + 4tu(7t^2$$
$$+ 20u\,t + 13u^2)s + 20t^2(t+u)u^2)H(1,z)H(0,3,y))/((s+t)^2\,(s+u)^2(t+u))$$

$$+ 4H(0,1,z)H(0,3,y) + ((-2(76t + 103u)\ s^2 + (37t - 233u)us$$

$$+ 162tu^2)H(1,0,y))/(9s(s+u)\ (t+u)) + (2(3t^2 + 8st$$

$$+ 14su)H(0,z)H(1,0,y))/(3s\ (t+u)) + (\frac{6s}{(t+u)} + 9)H(1,z)H(1,0,y)$$

$$- 2H(0,1,z)\ H(1,0,y) + (((t-2u)s^2 + t(t+19u)s + 18t^2u)H(1,0,z))/(s\ (s+t)(t+u))$$

$$+ ((18(t+u)s^3 + (31t^2 + 32ut + 13\ u^2)s^2 - t(t^2 + 17ut - 8u^2)s - 2t^2(7t^2$$

$$+ 20ut + 7\ u^2))H(2,y)H(1,0,z))/(3(s+t)^2(t+u)^2) + 20H(3,y)\ H(1,0,z)$$

$$- 4H(0,0,y)H(1,0,z) + 8H(0,2,y)H(1,0,z) + 16H(0,3,y)H(1,0,z)$$

$$+ 2\ H(1,0,y)H(1,0,z) + \frac{28}{3}H(3,y)H(1,1,z) - 8H(0,0,y)H(1,1,z)$$

$$+ 4\ H(0,3,y)H(1,1,z) + 2H(1,1,z) + 6H(0,1,z)H(1,2,y) - 2H(1,0,z)\ H(1,2,y)$$

$$+ ((242u^2 + 215su + 134tu + 134st)H(2,0,y))/(9\ (s+u)(t+u)) - ((47t^3$$

$$+ 153ut^2 + 153u^2t + 47u^3)\ H(0,z)H(2,0,y))/(3(t+u)^3) - \frac{14}{3}H(1,z)H(2,0,y)$$

$$- 4H(0,0,z)\ H(2,0,y) - 2H(0,1,z)H(2,0,y) - 6H(1,0,z)H(2,0,y)$$

$$- \frac{14}{3}H(0,z)\ H(2,2,y) - 8H(0,0,z)H(2,2,y) - 16H(0,1,z)H(2,2,y)$$

$$- 8H(1,0,z)H(2,2,y) + 2\ H(2,2,y) - ((36s^5 + 98(t+u)s^4 + (79t^2 + 212ut$$

$$+ 79\ u^2)s^3 + (17t^3 + 121ut^2 + 121u^2t + 17u^3)s^2 + 2tu\ (8t^2 + 7ut + 8u^2)s$$

$$- 10t^2(t+u)u^2)H(1,z)\ H(2,3,y))/(3(s+t)^2(s+u)^2(t+u)) + 12H(0,1,z)H(2,3,y)$$

$$- 4\ H(1,0,z)H(2,3,y) + 20H(1,z)H(3,0,y) - 4H(0,1,z)H(3,0,y)$$

$$+ 4H(1,1,z)\ H(3,0,y) - ((466s^2 + 493(t+u)s + 520tu)H(3,2,y))/(9\ (s+t)(s+u))$$

$$+ 20H(0,z)H(3,2,y) + \frac{28}{3}H(1,z)H(3,2,y) - 4\ H(0,1,z)H(3,2,y) - 4H(1,0,z)H(3,2,y)$$

$$- \frac{80}{3}H(1,z)H(3,3,y) + 28\ H(0,1,z)H(3,3,y) - 12H(1,0,z)H(3,3,y)$$

$$- 8H(1,1,z)H(3,3,y) + ((4(4\ t - 5u)s^4 + (41t^2 - 8ut - 49u^2)s^3 + (25t^3 + 53ut^2 - 73u^2\ t$$

$$- 29u^3)s^2 + 8tu(4t^2 - ut - 5u^2)s - 2t^2(t+u)\ u^2)H(0,0,1,z))/(3(s+t)^2(s+u)^2(t+u))$$

$$+ 6H(1,y)\, H(0,0,1,z) + 16H(3,y)H(0,0,1,z) - 8H(0,z)H(0,0,2,y)$$

$$- 8H(1,z)H(0,0,2,y) - 14\, H(0,0,2,y) + 4H(1,z)H(0,0,3,y) + ((-6(t+u)u^4$$

$$- s(17\, t^2 + 22ut + 17u^2)u^2 + s^2(2t^2 + 19ut - 7u^2)u + 2s^3\, (5t^2 + 13ut$$

$$+ 2u^2))H(0,1,0,y))/(3s(s+u)^2\, (t+u)^2) + 2H(0,z)H(0,1,0,y) + 2H(1,z)H(0,1,0,y)$$

$$+ ((-6\,(t+u)t^4 + s(t^2 + 14ut + u^2)t^2 + s^2(29t^2 + 91ut + 38\, u^2)t + s^3(22t^2 + 62ut$$

$$+ 28u^2))H(0,1,0,z))/(3s\,(s+t)^2(t+u)^2) - 2H(1,y)H(0,1,0,z) - 12H(2,y)H(0,1,0,z)$$

$$- 4H(3,y)\, H(0,1,0,z) - 4H(3,y)H(0,1,1,z) + \frac{14}{3}H(0,1,1,z) - 8H(1,z)\, H(0,2,0,y)$$

$$- 14H(0,2,0,y) + 4H(0,z)H(0,2,2,y) - \frac{14}{3}H(0,2,2,y) + 2\, H(1,z)H(0,2,3,y)$$

$$- 8H(1,z)H(0,3,0,y) + ((2(7t+13u)s^4 + 5(5\, t^2 + 16ut + 11u^2)s^3 + (11t^3 + 79ut^2 + 121u^2t$$

$$+ 29u^3)\, s^2 + 4tu(7t^2 + 20ut + 13u^2)s + 20t^2(t+u)u^2)\, H(0,3,2,y))/((s+t)^2(s+u)^2(t+u))$$

$$+ 16H(0,z)H(0,3,2,y) + 4\, H(1,z)H(0,3,2,y) - 12H(1,z)H(0,3,3,y)$$

$$+ 4H(0,z)H(1,0,0,y) + 4H(1,z)\, H(1,0,0,y) + 14H(1,0,0,y) - 4H(2,y)H(1,0,0,z)$$

$$- ((18s^5 + 49\,(t+u)s^4 + (35t^2 + 88ut + 35u^2)s^3 + 4(t^3 + 5ut^2 + 5\, u^2t + u^3)s^2$$

$$- 2tu(5t^2 + 28ut + 5u^2)s - 23t^2\, (t+u)u^2)H(1,0,1,z))/(3(s+t)^2(s+u)^2(t+u))$$

$$+ 6\, H(1,y)H(1,0,1,z) - 18H(2,y)H(1,0,1,z) - 4H(3,y)H(1,0,1,z) + (\frac{6s}{(t+u)}$$

$$+ 9)H(1,0,2,y) + 4H(0,z)H(1,0,2,y) - 6H(1,z)\, H(1,0,3,y) - (2(18s^2 + 23ts + 11us$$

$$- 6t^2)H(1,1,0,y))/(3s\,(t+u)) - ((2u(3t+u)s^3 + (3t^3 + 21ut^2 + 6u^2t - 4\, u^3)s^2$$

$$+ t(3t^3 + 18ut^2 + 3u^2t - 8u^3)s - 4t^2\, (t+u)u^2)H(1,1,0,z))/(s(s+t)^2(t+u)^2)$$

$$- 2H(1,y)\, H(1,1,0,z) - 18H(2,y)H(1,1,0,z) - 4H(3,y)H(1,1,0,z)$$

$$+ (\frac{6s}{(t+u)} + 9)H(1,2,0,y) + 2H(0,z)H(1,2,0,y)$$

$$
\begin{aligned}
&+6H(1,z)H(1,2,3,y)-4\,H(0,z)H(2,0,0,y)-8H(1,z)H(2,0,0,y)-14H(2,0,0,y)\\
&+2H(0,z)\,H(2,0,2,y)-\frac{14}{3}H(2,0,2,y)-4H(1,z)H(2,0,3,y)+((6\,(t+u)s^3+(9t^2\\
&+20ut+15u^2)s^2+u(12t^2+13ut+9\,u^2)s-4tu^3)H(2,1,0,y))/((s+u)^2(t+u)^2)\\
&-2H(0,z)\,H(2,1,0,y)-2H(1,z)H(2,1,0,y)-4H(0,z)H(2,2,0,y)-\frac{14}{3}\,H(2,2,0,y)\\
&-16H(1,z)H(2,2,3,y)-4H(1,z)H(2,3,0,y)-((36s^5+98\,(t+u)s^4+(79t^2\\
&+212ut+79u^2)s^3+(17t^3+121u\,t^2+121u^2t+17u^3)s^2+2tu(8t^2+7ut+8u^2)s\\
&-10\,t^2(t+u)u^2)H(2,3,2,y))/(3(s+t)^2(s+u)^2\,(t+u))-4H(0,z)H(2,3,2,y)\\
&+16H(1,z)H(2,3,3,y)+8H(0,z)H(3,0,2,y)+4\,H(1,z)H(3,0,2,y)+20H(3,0,2,y)\\
&-12H(1,z)H(3,0,3,y)+4H(1,z)H(3,2,0,y)+20\,H(3,2,0,y)+4H(0,z)H(3,2,2,y)\\
&+\frac{28}{3}H(3,2,2,y)-8H(1,z)\,H(3,2,3,y)-12H(1,z)H(3,3,0,y)-12H(0,z)H(3,3,2,y)\\
&-8H(1,z)\,H(3,3,2,y)-\frac{80}{3}H(3,3,2,y)+40H(1,z)H(3,3,3,y)+4H(0,0,1,0,y)\\
&-8\,H(0,0,1,0,z)-8H(0,0,1,1,z)-8H(0,0,2,2,y)+4H(0,0,3,2,y)-8H(0,1,0,1,z)\\
&+2\,\,H(0,1,0,2,y)-12H(0,1,1,0,z)+2H(0,1,2,0,y)-8H(0,2,0,2,y)\\
&-2H(0,2,1,0,y)-8\,\,H(0,2,2,0,y)+2H(0,2,3,2,y)-8H(0,3,0,2,y)\\
&-8H(0,3,2,0,y)+4H(0,3,2,2,y)-12\,\,H(0,3,3,2,y)-2H(1,0,0,1,z)\\
&+4H(1,0,0,2,y)+4H(1,0,1,0,y)-4H(1,0,1,0,z)+4\,\,H(1,0,2,0,y)\\
&-6H(1,0,3,2,y)-12H(1,1,0,1,z)-14H(1,1,1,0,z)+4H(1,2,0,0,y)\\
&+4\,H(1,2,1,0,y)+6H(1,2,3,2,y)-8H(2,0,0,2,y)-8H(2,0,2,0,y)-4H(2,0,3,2,y)\\
&+4\,H(2,1,0,0,y)-2H(2,1,0,2,y)-8H(2,1,1,0,y)-2H(2,1,2,0,y)-8H(2,2,0,0,y)\\
&-8\,\,H(2,2,1,0,y)-16H(2,2,3,2,y)-4H(2,3,0,2,y)-4H(2,3,2,0,y)\\
&+16H(2,3,3,2,y)+8\,\,H(3,0,1,0,y)+4H(3,0,2,2,y)-12H(3,0,3,2,y)\\
&+4H(3,2,0,2,y)-8H(3,2,1,0,y)+4\,\,H(3,2,2,0,y)-8H(3,2,3,2,y)\\
&-12H(3,3,0,2,y)-12H(3,3,2,0,y)-8\,H(3,3,2,2,y)+40H(3,3,3,2,y)-\frac{4003}{162}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+C_F^2\Big\{\zeta_4\Big(-22\Big)+\zeta_3\Big(-16H(1,y)+8H(2,y)-8H(1,z)-(2(25s^5(t+u)+s^4(51\,t^2\\
&+112tu+51u^2)+4s^3(6t^3+37t^2u+37tu^2+6u^3)-2\,s^2(t^4-27t^3u-69t^2u^2-27tu^3\\
&+u^4)+stu(-4t^3+33\,t^2u+33tu^2-4u^3)-2t^2u^2(t^2+u^2)))/(s\,(s+t)^2(s+u)^2(t+u))\Big)\\
&+\zeta_2\Big((2tH(0,z)(4s^3+2s^2(4t+u)+3st(t+u)-t^3))/(s\,(s+t)^2(t+u))+(2uH(0,y)(4s^3\\
&+2s^2(t+4u)+3s\,(t+u)u-u^3))/(s(s+u)^2(t+u))+(2H(2,y)(s^4\,(5t^2+12tu\\
&+5u^2)+s^3(11t^3+39t^2u+39tu^2+11\,u^3)+s^2(6t^4+42t^3u+76t^2u^2+42tu^3+6u^4)\\
&+2stu\,(7t^3+26t^2u+26tu^2+7u^3)+t^2u^2(9t^2+20tu+9\,u^2)))/((s+t)^2(s+u)^2(t+u)^2)\\
&+(2H(1,z)\,(s^5(5t^2+16tu+9u^2)+s^4(11t^3+47t^2u+53t\,u^2+17u^3)+s^3(6t^4+46t^3u\\
&+92t^2u^2+54tu^3+6\,u^4)+2s^2u(7t^4+29t^3u+29t^2u^2+6tu^3-u^4)+st\,u^2(9t^3+20t^2u\\
&+5tu^2-4u^3)-2t^2(t+u)u^4))/(s\,(s+t)^2(s+u)^2(t+u)^2)+(4t(2s-t)H(1,y))/(s\,(t+u))\\
&-8H(2,y)H(1,z)+8H(0,1,y)-8H(0,2,y)+8H(1,1,y)+8H(2,1,y)-16\,H(2,2,y)\\
&+(2(6s^3(t+u)+s^2(6t^2+4tu+6u^2)-s\,t(t+u)u-6t^2u^2))/(s(s+t)(s+u)(t+u))\Big)
\end{aligned}
$$

$$+\Big(-(12tuH(0,y)H(0,z))/(s(t+u))$$
$$-(2(3s(2t+u)+t(7t+3u))\ H(2,y)H(0,z))/((s+t)(t+u))$$
$$+(2((4t+6u)s^2+t(7t+10\,u)s+3t^2(t+u))H(0,2,y)H(0,z))/((s+t)^2\ (t+u))$$
$$-(2(t^2+s(t+3u))H(1,0,y)H(0,z))/(s\ (t+u))+((6t^3+22ut^2+22u^2t$$
$$+6u^3)H(2,0,y)\ H(0,z))/((t+u)^3)+12H(2,2,y)H(0,z)-12H(3,2,y)H(0,z)$$
$$+8H(0,0,2,y)\ H(0,z)-8H(0,3,2,y)H(0,z)-4H(2,0,2,y)H(0,z)$$
$$-8H(3,0,2,y)H(0,z)-8\ H(3,2,2,y)H(0,z)+8H(3,3,2,y)H(0,z)$$
$$-(2(3s(t+2u)+u(3t+7u))\ H(0,y)H(1,z))/((s+u)(t+u))-9H(1,z)+9H(1,z)H(2,y)$$
$$-9\ H(2,y)+((20s^2+22(t+u)s+24tu)H(1,z)\ H(3,y))/((s+t)(s+u))$$
$$+(2((4t+7u)s^2+(4t^2+13u\,t+8u^2)s+tu(5t+9u))H(0,1,z))/((s+t)(s+u)\ (t+u))$$
$$-(2u((s+t)+u)(2s+3u)H(0,y)H(0,1,z))/((s+u)^2\ (t+u))-(2(2(t+u)s^4$$
$$+(5t^2+12ut+5u^2)\ s^3+(3t^3+19ut^2+19u^2t+3u^3)s^2+2tu(4t^2+11u\,t$$
$$+4u^2)s+6t^2(t+u)u^2)H(2,y)H(0,1,z))/((s+t)^2\ (s+u)^2(t+u))$$
$$+12H(3,y)H(0,1,z)-(2(3s(t+2u)+u(3t+7\,u))H(0,2,y))/((s+u)(t+u))$$
$$+12H(1,z)H(0,2,y)-4H(0,1,z)\ H(0,2,y)-(2(4(t+2u)s^4+(7t^2$$
$$+24ut+17u^2)\ s^3+(3t^3+23ut^2+37u^2t+9u^3)s^2+8tu(t^2+3ut+2\,u^2)s$$
$$+6t^2(t+u)u^2)H(1,z)H(0,3,y))/((s+t)^2\ (s+u)^2(t+u))+(2((3t+5u)s^2+(6u^2$$
$$-4t\,u)s-6tu^2)H(1,0,y))/(s(s+u)(t+u))-6H(1,z)\ H(1,0,y)+4H(0,1,z)H(1,0,y)$$
$$-(2t(s^2+(t+7u)s+6tu)\ H(1,0,z))/(s(s+t)(t+u))$$
$$+(2u(2s(2t+u)-(t+u)u)\ H(0,y)H(1,0,z))/(s(t+u)^2)+(2t(2(t+2u)s^2+(5\,t^2+11ut$$
$$+2u^2)s+t(3t^2+8ut+3u^2))H(2,y)\ H(1,0,z))/((s+t)^2(t+u)^2)-12H(3,y)H(1,0,z)$$
$$-8H(0,2,y)\ H(1,0,z)-8H(0,3,y)H(1,0,z)+12H(0,y)H(1,1,z)-24H(3,y)H(1,1,z)$$
$$+8\ H(0,0,y)H(1,1,z)-8H(0,3,y)H(1,1,z)+9H(1,1,z)-4H(0,1,z)H(1,2,y)$$
$$-(2\ (3s(t+2u)+u(3t+7u))H(2,0,y))/((s+u)(t+u))+12H(1,z)\ H(2,0,y)$$
$$-4H(0,1,z)H(2,0,y)+8H(0,0,z)H(2,2,y)+16H(0,1,z)H(2,2,y)+9\ H(2,2,y)$$

$$
\begin{aligned}
&- (2(8(t+u)s^4 + (17t^2 + 36ut + 17u^2)\, s^3 + (9t^3 + 49ut^2 + 49u^2t + 9u^3)s^2 + 2tu(10t^2 + 23u\, t \\
&+ 10u^2)s + 12t^2(t+u)u^2)H(1,z)H(2,3,y))/((s+t)^2\,(s+u)^2(t+u)) - 12H(1,z)H(3,0,y) \\
&- 8H(1,1,z)\, H(3,0,y) + ((20s^2 + 22(t+u)s + 24tu)\, H(3,2,y))/((s+t)(s+u)) \\
&- 24H(1,z)H(3,2,y) + 8H(0,1,z)H(3,2,y) + 24\, H(1,z)H(3,3,y) \\
&- 8H(0,1,z)H(3,3,y) + 8H(1,0,z)H(3,3,y) + 16H(1,1,z)\, H(3,3,y) - (2s(t-u)(2s^3 \\
&+ 5(t+u)s^2 + (3t^2 + 10ut + 3\,u^2)s + 4t(t+u)u)H(0,0,1,z))/((s+t)^2(s+u)^2\,(t+u)) \\
&- 4H(1,y)H(0,0,1,z) + 8H(2,y)H(0,0,1,z) + 8H(1,z)H(0,0,2,y) \\
&- 8\, H(1,z)H(0,0,3,y) + (2((t+u)u^4 + s(9t^2 + 16ut + 9\,u^2)u^2 + 2s^2(7t^2 + 12ut + 7u^2)u \\
&+ 2s^3(3t^2 + 5u\,t + 3u^2))H(0,1,0,y))/(s(s+u)^2(t+u)^2) - 4H(1,z)\, H(0,1,0,y) + (2t(\\
&- 2us^3 + 2(t^2 + u^2)s^2 + t(3\,t^2 + 4ut + 3u^2)s + t^3(t+u))H(0,1,0,z))/(s(s+t)^2\,(t+u)^2) \\
&+ 4H(2,y)H(0,1,0,z) - 8H(0,y)H(0,1,1,z) + 8H(3,y)H(0,1,1,z) \\
&- 12\, H(0,1,1,z) + 8H(1,z)H(0,2,0,y) + 12H(0,2,2,y) - 4H(1,z)H(0,2,3,y) \\
&- (2\,(4(t+2u)s^4 + (7t^2 + 24ut + 17u^2)s^3 + (3t^3 + 23u\,t^2 + 37u^2t + 9u^3)s^2 \\
&+ 8tu(t^2 + 3ut + 2u^2)s + 6t^2\,(t+u)u^2)H(0,3,2,y))/((s+t)^2(s+u)^2(t+u)) \\
&- 8\, H(1,z)H(0,3,2,y) + 8H(1,z)H(0,3,3,y) - 8H(1,z)H(1,0,0,y) \\
&- (2(5\,(t+u)s^4 + (11t^2 + 24ut + 11u^2)s^3 + (6t^3 + 34ut^2 + 34\,u^2t + 6u^3)s^2 \\
&+ 2tu(7t^2 + 17ut + 7u^2)s + 9t^2\,(t+u)u^2)H(1,0,1,z))/((s+t)^2(s+u)^2(t+u)) \\
&- 4\, H(0,y)H(1,0,1,z) - 4H(1,y)H(1,0,1,z)
\end{aligned}
$$

$$
\begin{aligned}
&+ 12H(2,y)H(1,0,1,z) + 8H(3,y)\, H(1,0,1,z) - 6H(1,0,2,y) + 4H(1,z)H(1,0,3,y) \\
&+ (4(2s-t)tH(1,1,0,y))/(s\,(t+u)) - (2((t^2 - 2ut - u^2)s^3 + (t^3 - 7ut^2 - 2\,u^2t \\
&+ 2u^3)s^2 + (4tu^3 - 6t^3u)s + 2t^2(t+u)\,u^2)H(1,1,0,z))/(s(s+t)^2(t+u)^2) \\
&+ 4H(2,y)H(1,1,0,z) - 6\, H(1,2,0,y) - 4H(1,z)H(1,2,3,y) + 8H(1,z)H(2,0,0,y) \\
&+ 12H(2,0,2,y) - (2\,((6t^2 + 8ut + 4u^2)s^2 + u(10t^2 + 13ut + 7u^2)\, s + u^2(3t^2 + 4ut \\
&+ 3u^2))H(2,1,0,y))/((s+u)^2\,(t+u)^2) - 4H(1,z)H(2,1,0,y) + 12H(2,2,0,y) \\
&+ 16H(1,z)H(2,2,3,y) - (2\,(8(t+u)s^4 + (17t^2 + 36ut + 17u^2)s^3 + (9t^3 + 49\,ut^2 + 49u^2t \\
&+ 9u^3)s^2 + 2tu(10t^2 + 23ut + 10u^2)s + 12\,t^2(t+u)u^2)H(2,3,2,y))/((s+t)^2(s+u)^2\,(t+u)) \\
&- 8H(1,z)H(3,0,2,y) - 12H(3,0,2,y) + 8H(1,z)H(3,0,3,y) - 8H(1,z)\, H(3,2,0,y) \\
&- 12H(3,2,0,y) - 24H(3,2,2,y) + 16H(1,z)H(3,2,3,y) + 8H(1,z)\, H(3,3,0,y) \\
&+ 16H(1,z)H(3,3,2,y) + 24H(3,3,2,y) - 16H(1,z)H(3,3,3,y) + 8\, H(0,0,1,1,z) \\
&+ 8H(0,0,2,2,y) - 8H(0,0,3,2,y) + 8H(0,1,0,1,z) - 4H(0,1,0,2,y) + 8\, H(0,1,1,0,y) \\
&+ 8H(0,1,1,0,z) - 4H(0,1,2,0,y) + 8H(0,2,0,2,y) - 4H(0,2,1,0,y) + 8\, H(0,2,2,0,y) \\
&- 4H(0,2,3,2,y) - 8H(0,3,2,2,y) + 8H(0,3,3,2,y) + 4H(1,0,0,1,z) - 8\, H(1,0,0,2,y) \\
&+ 4H(1,0,1,0,y) - 8H(1,0,2,0,y) + 4H(1,0,3,2,y) + 8H(1,1,0,0,y) + 8\, H(1,1,0,1,z) \\
&+ 8H(1,1,1,0,y) + 8H(1,1,1,0,z) - 8H(1,2,0,0,y) - 4H(1,2,1,0,y) - 4\, H(1,2,3,2,y) \\
&+ 8H(2,0,0,2,y) - 4H(2,0,1,0,y) + 8H(2,0,2,0,y) - 8H(2,1,0,0,y) \\
&- 4\, H(2,1,0,2,y) + 8H(2,1,1,0,y) - 4H(2,1,2,0,y) + 8H(2,2,0,0,y) \\
&+ 16H(2,2,3,2,y) - 8\, H(3,0,1,0,y) - 8H(3,0,2,2,y) + 8H(3,0,3,2,y) \\
&- 8H(3,2,0,2,y) + 8H(3,2,1,0,y) - 8\, H(3,2,2,0,y) + 16H(3,2,3,2,y) \\
&+ 8H(3,3,0,2,y) + 8H(3,3,2,0,y) + 16\, H(3,3,2,2,y) - 16H(3,3,3,2,y) + \frac{19}{2}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ C_A n_f \Big\{ \zeta_3 \Big(-\frac{37}{18} \Big) + \zeta_2 \Big(-\frac{1}{3} H(0,y) + \frac{1}{6} H(2,y) - \frac{1}{3} H(0,z) + \frac{1}{6}\, H(1,z) - \frac{tu}{3s(t+u)} \\
&- \frac{7}{36} \Big) + \Big(H(0,y)H(0,z)(\frac{5}{9} - \frac{tu}{3s(t+u)}) - \frac{tu\, H(1,0,y)}{3s(t+u)} + H(1,0,z)(\frac{31}{36} - \frac{tu}{3s\,(t+u)}) \\
&+ \frac{(29t+20u)H(0,y)}{9(t+u)} + \frac{(20t+29u)\, H(0,z)}{9(t+u)} + \frac{31}{36} H(0,y)H(1,z) - H(0,y)\, H(0,0,z) \\
&+ \frac{2}{3} H(0,y)H(0,1,z) - H(0,y)H(1,0,z) + \frac{2}{3} H(0,y)\, H(1,1,z) + \frac{31}{36} H(2,y)H(0,z) \\
&+ H(2,y)H(1,z) - \frac{20}{9} H(3,y)\, H(1,z) - H(0,0,y)H(0,z) - H(0,0,y)H(1,z) \\
&- H(2,y)H(0,0,z) - \frac{2}{3} H(2,y)\, H(0,1,z) - \frac{1}{3} H(3,y)H(0,1,z) - \frac{1}{3} H(0,2,y)H(0,z) \\
&+ \frac{2}{3}\, H(0,2,y)H(1,z) + H(0,3,y)H(1,z) + H(3,y)H(1,0,z) - \frac{4}{3} H(3,y)\, H(1,1,z) \\
&- H(2,0,y)H(0,z) + \frac{2}{3} H(2,0,y)H(1,z) + \frac{2}{3} H(2,2,y)\, H(0,z) - \frac{4}{3} H(2,3,y)H(1,z) \\
&+ H(3,0,y)H(1,z) + H(3,2,y)\, H(0,z) - \frac{4}{3} H(3,2,y)H(1,z) - \frac{4}{3} H(3,3,y)H(1,z) \\
&+ \frac{17}{27}\, H(2,y) - \frac{41}{18} H(0,0,y) + \frac{31}{36} H(0,2,y) + \frac{31}{36}\, H(2,0,y) + H(2,2,y) - \frac{20}{9} H(3,2,y) \\
&- H(0,0,2,y) - H(0,2,0,y) + \frac{2}{3}\, H(0,2,2,y) + H(0,3,2,y) - H(2,0,0,y) + \frac{2}{3} H(2,0,2,y) \\
&- \frac{2}{3}\, H(2,1,0,y) + \frac{2}{3} H(2,2,0,y) - \frac{4}{3}\, H(2,3,2,y) + H(3,0,2,y) + H(3,2,0,y) \\
&- \frac{4}{3} H(3,2,2,y) - \frac{4}{3}\, H(3,3,2,y) + \frac{17}{27} H(1,z) - \frac{41}{18} H(0,0,z) - \frac{49}{36}\, H(0,1,z) \\
&+ H(1,1,z) - \frac{1}{3} H(0,0,1,z) - \frac{2}{3}\, H(0,1,1,z) - H(1,0,0,z) - \frac{2}{3} H(1,0,1,z) - \frac{439}{162} \Big) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+C_F n_f\Big\{\zeta_3\Big(-\frac{1}{9}\Big)+\zeta_2\Big(\frac{4}{3}H(1,y)-\frac{5}{3}H(2,y)-\frac{1}{3}H(1,z)+\frac{1}{3}\Big)+\Big(-\frac{(7t+3u)H(0,y)}{6(t+u)}\\
&-\frac{(3t+7u)H(0,z)}{6\,(t+u)}-\frac{31}{18}H(0,y)H(1,z)-\frac{4}{3}H(0,y)\;H(0,1,z)+\frac{1}{3}H(0,y)H(1,0,z)\\
&-\frac{4}{3}H(0,y)\;H(1,1,z)-\frac{31}{18}H(2,y)H(0,z)-2H(2,y)H(1,z)+\frac{40}{9}H(3,y)\;H(1,z)\\
&+2H(0,0,y)H(1,z)+2H(2,y)H(0,0,z)+\frac{4}{3}H(2,y)\;H(0,1,z)+\frac{2}{3}H(3,y)H(0,1,z)\\
&+\frac{2}{3}H(0,2,y)H(0,z)-\frac{4}{3}\;H(0,2,y)H(1,z)-2H(0,3,y)H(1,z)-\frac{1}{3}H(1,0,y)H(0,z)\\
&-\frac{4}{3}\;H(2,y)H(1,0,z)-2H(3,y)H(1,0,z)+\frac{8}{3}H(3,y)H(1,1,z)+\frac{2}{3}\;H(2,0,y)H(0,z)\\
&-\frac{4}{3}H(2,0,y)H(1,z)-\frac{4}{3}H(2,2,y)\;H(0,z)+\frac{8}{3}H(2,3,y)H(1,z)-2H(3,0,y)H(1,z)\\
&-2H(3,2,y)\;H(0,z)+\frac{8}{3}H(3,2,y)H(1,z)+\frac{8}{3}H(3,3,y)H(1,z)-\frac{34}{27}\;H(2,y)-\frac{31}{18}H(0,2,y)\\
&+\frac{20}{9}H(1,0,y)-\frac{31}{18}H(2,0,y)-2\;H(2,2,y)+\frac{40}{9}H(3,2,y)+2H(0,0,2,y)-\frac{1}{3}H(0,1,0,y)\\
&+2\;H(0,2,0,y)-\frac{4}{3}H(0,2,2,y)-2H(0,3,2,y)-2H(1,0,0,y)+\frac{4}{3}\;H(1,1,0,y)\\
&+2H(2,0,0,y)-\frac{4}{3}H(2,0,2,y)-\frac{4}{3}\;H(2,2,0,y)+\frac{8}{3}H(2,3,2,y)-2H(3,0,2,y)\\
&-2H(3,2,0,y)+\frac{8}{3}\;H(3,2,2,y)+\frac{8}{3}H(3,3,2,y)-\frac{34}{27}H(1,z)+\frac{49}{18}\;H(0,1,z)+\frac{1}{2}H(1,0,z)\\
&-2H(1,1,z)+\frac{2}{3}H(0,0,1,z)-\frac{1}{3}\;H(0,1,0,z)+\frac{4}{3}H(0,1,1,z)+\frac{4}{3}H(1,0,1,z)+\frac{371}{81}\Big)\Big\}\\
&+n_f^2\Big\{\frac{1}{36}H(0,y)H(0,z)-\frac{5}{27}H(0,y)+\frac{5}{36}\;H(0,0,y)-\frac{5}{27}H(0,z)+\frac{5}{36}H(0,0,z)+\frac{\zeta_2}{18}\Big\}
\end{aligned}
$$

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