QCD RADIATIVE CORRECTIONS TO HIGGS PHYSICS

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Taushif Ahmed

Dedicated to

My Parents and Sister

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3. **Pseudo-scalar Form Factors at Three Loops in QCD**

   Taushif Ahmed, Thomas Gehrmann, Prakash Mathews, Narayan Rana and V. Ravindran

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4. **Pseudo-scalar Higgs Boson Production at Threshold N^{3}LO and N^{3}LL QCD**

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7. **The two-loop QCD correction to massive spin-2 resonance → $q\bar{q}g$**

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CONFERENCE ATTENDED

1. 16 - 19 March 2016 : ***MHV@30: Amplitudes and Modern Applications***, Fermi National Accelerator Laboratory, Chicago, USA.

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3. 11 - 12 October 2015 : ***Recent Trends in AstroParticle and Particle Physics***, Indian Institute of Science, Bangalore, India.

4. 15 - 26 June 2015 : ***ICTP Summer School on Particle Physics***, International Centre for Theoretical Physics, Trieste, Italy.

5. 23 - 28 February 2015 : ***Workshop on LHC and Dark Matter***, Indian Association for the Cultivation of Science, Kolkata, India.

6. 2 - 5 December 2014 ***International Workshop on Frontiers of QCD***, Indian Institute of Technology Bombay, Mumbai, India.

7. 5 - 10 March 2014 ***Discussion Meeting on Radiative Correction***, Institute of Physics, Bhubaneswar, India.

8. 7 - 17 July 2013 ***CTEQ School on QCD and Electroweak Phenomenology***, University of Pittsburgh, Pennsylvania, USA.

9. 7 - 10 January 2013 ***Lecture Workshop in High Energy Physics***, Indian Institute of Technology Bombay, Mumbai, India.

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Taushif Ahmed

ABSTRACT

The central part of this thesis deals with the QCD radiative corrections to some important observables associated with the Drell-Yan, scalar and pseudo scalar Higgs boson productions at three loop or N^3LO order aiming to uplift the accuracy of theoretical results.

The Higgs bosons are produced dominantly at the LHC via gluon fusion through top quark loop, while one of the subdominant ones take place through bottom quark annihilation. However, in some BSM theories, like minimally supersymmetric Standard Model (MSSM), it can contribute substantially. Here, we have computed analytically the inclusive cross section of the Higgs boson produced in this channel under the soft-virtual (SV) approximation at N^3LO QCD following an elegant formalism. This indeed helps to reduce the theoretical uncertainties arising from renormalisation and factorisation scales and consequently improve the reliabilities of the theoretical results. This is the most accurate result for this channel which exists in the literature.

The differential rapidity distribution is among the most important observables, which is expected to be measured in upcoming days at the LHC. This immediately calls for very precise theoretical predictions. The SV corrections to this observable at N^3LO for the Higgs boson, produced through gluon fusion, and leptonic pair in Drell-Yan (DY) production are computed and the numerical impacts of these results are demonstrated. These are the most accurate results for the rapidity distributions of these which exist and undoubtedly, expected to play very important role in the upcoming run at the LHC.

The CP-odd/pseudo-scalar Higgs boson is one of the most prime candidates in BSM which has been studied in great details, taking into account higher order QCD radiative corrections, due to similarities with its CP-even counter part. In this thesis, we have obtained the three loop QCD corrections to the pseudo scalar Higgs boson production and computed the SV correction to the inclusive production cross section.

Contents

SYNOPSIS

The Standard Model (SM) of particle physics is one of the most remarkably successful fundamental theories of all time which got its finishing touch on the eve of July 2012 through the discovery of the long-awaited particle, "the Higgs boson", at the biggest underground particle research amphitheater, the Large Hadron Collider (LHC). It would take a while to make the conclusive remarks about the true identity of the newly-discovered particle. However, after the discovery of this SM-like-Higgs boson, the high energy physics community is standing on the verge of a very crucial era where the new physics may show up as tiny deviations from the predictions of the SM. To exploit this possibility, it is a crying need to make the theoretical predictions, along with the revolutionary experimental progress, to a spectacularly high accuracy within the SM and beyond (BSM).

The most successful and celebrated methodology to perform the theoretical calculations within the SM and BSM are based on the perturbation theory, due to our inability to solve the theory exactly. Under the prescriptions of perturbation theory, all the observables are expanded in powers of the coupling constants present in the underlying Lagrangian. The result obtained from the first term of perturbative series is called the leading order (LO), the next one is called next-to-leading order (NLO) and so on. In most of the cases, the LO results fail miserably to deliver a reliable theoretical prediction of the associated observables, one must go beyond the wall of LO result to achieve a higher accuracy.

Due to the presence of three fundamental forces within the SM, any observable can be expanded in powers of the coupling constants associated with the corresponding forces,

namely, electromagnetic ($\alpha_{\rm EM}$), weak ($\alpha_{\rm EW}$) and strong (α_s) ones and consequently, perturbative calculations can be performed with respect to each of these constants. However, at typical energy scales, at which the hadron colliders undergo operations, the contributions arising from the α_s expansion dominate over the others due to comparatively large values of α_s. Hence, to catch the dominant contributions to any observables, we must concentrate on the α_s expansion and evaluate the terms beyond LO. These are called Quantum Chromo-dynamics (QCD) radiative or perturbative QCD (pQCD) corrections. In addition, the pQCD predictions depend on two unphysical scales, the renormalisation (μ_R) and factorisation (μ_F) scales, which are required to introduced in the process of renormalising the theory. The μ_R arises from the ultraviolet (UV) renormalisation, whereas the mass factorisation (removes collinear singularities) introduces the μ_F. Any fixed order results do depend on these unphysical scales which happens due to the truncation of the perturbative expansion at any finite order. As we include the contributions from higher and higher orders, the dependence of any physical observable on these unphysical scales gradually goes down. Hence, to make a reliable theoretical prediction, it is absolutely necessary to take into account the contributions arising from the higher order QCD corrections to any observable at the hadron colliders.

This thesis arises exactly in this context. The central part of this thesis deals with the QCD radiative corrections to some important observables associated with the Drell-Yan, scalar and pseudo scalar Higgs boson production at three loop or N^{3}LO order. In the subsequent discussions, we will concentrate only on these three processes.

0.1 Soft-Virtual QCD Corrections to Cross Section at N^{3}LO

The Higgs bosons are produced dominantly at the LHC via gluon fusion through top quark loop, while one of the sub-dominant ones take place through bottom quark annihilation. In

the SM, the interaction between the Higgs boson and bottom quarks is controlled through the Yukawa coupling which is reasonably small at typical energy scales. However, in the minimal super symmetric SM (MSSM), this channel can contribute substantially due to enhanced coupling between the Higgs boson and bottom quarks in the large $\tan\beta$ region, where $\tan\beta$ is the ratio of vacuum expectation values of the up and down type Higgs fields. In the present run of LHC, the measurements of the various coupling constants including this one are underway which can shed light on the properties of the newly discovered Higgs boson. Most importantly, for the precision studies we must take into account all the contributions, does not matter how tiny those are, arising from sub-dominant channels along with the dominant ones to reduce the dependence on the unphysical scales and make a reliable prediction.

The computations of the higher order QCD corrections beyond leading order often becomes quite challenging because of the large number of Feynman diagrams and, presence of the complicated loop and phase space integrals. Under this circumstance, when we fail to compute the complete result at certain order, it is quite natural to try an alternative approach to capture the dominant contributions from the missing higher order corrections. It has been observed for many processes that the dominant contributions to an observable often comes from the soft gluon emission diagrams. The contributions arising from the associated soft gluon emission along with the virtual Feynman diagrams are known as the soft-virtual (SV) corrections. *The goal of this section is to discuss the SV QCD corrections to the production cross section of the Higgs boson, produced through bottom quark annihilation.*

The NNLO QCD corrections to this channel are already present in the literature. In addition, the partial result for the N^{3}LO corrections under the SV approximation were also computed long back. *In this work, we have computed the missing part and completed the full SV corrections to the cross section at N^{3}LO.*

The infrared safe contributions from the soft gluons are obtained by adding the soft part

of the cross section with the UV renormalized virtual part and performing mass factorisation using appropriate counter terms. The main ingredients are the form factors, overall operator UV renormalization constant, soft-collinear distribution arising from the real radiations in the partonic subprocesses and mass factorization kernels. The computations of SV cross section at N^3LO QCD require all of these above quantities up to 3-loop order. The relevant form factor becomes available very recently. The soft-collinear distribution at N^3LO was computed by us around the same time. This was calculated from the recent result of N^3LO SV cross section of the Higgs boson productions in gluon fusion by employing a symmetry (maximally non-Abelian property). Prior to this, this symmetry was verified explicitly up to NNLO order. However, neither there was any clear reason to believe that the symmetry would fail nor there was any transparent indication of holding it beyond this order. Nevertheless, we postulate that the relation would hold true even at N^3LO order! This is inspired by the universal properties of the soft gluons which are the underlying reasons behind the existence of this remarkable symmetry. Later, this conjecture is verified by explicit computations performed by two different groups on Drell-Yan process. This symmetry plays the most important role in achieving our goal. With these, along with the existing results of the remaining required ingredients, we obtain the complete analytical expressions of N^3LO SV cross section of the Higgs boson production through bottom quark annihilation. It reduces the scale dependence and provides a more precise result. We demonstrate the impact of this result numerically at the Large Hadron Collider (LHC) briefly. *This is the most accurate result for this channel which exists in the literature till date and it is expected to play an important role in coming days at the LHC.*

0.2 Soft-Virtual QCD Corrections to Rapidity at N^3LO

The productions of the Higgs boson in gluon fusion and leptonic pair in DY are among the most important processes at the LHC which are studied not only to test the SM to

an unprecedented accuracy but also to explore the new physics under BSM. During the present run at the LHC, in addition to the inclusive production cross section, the differential rapidity distribution is among the most important observables, which is expected to be measured in upcoming days. This immediately calls for very precise theoretical predictions.

In the same spirit of the SV corrections to the inclusive production cross section, the dominant contributions to the differential rapidity distributions often arise from the soft gluon emission diagrams. Hence, in the absence of complete fixed order result, the rapidity distribution under SV approximation is the best available alternative in order to capture the dominant contributions from the missing higher orders and stabilise the dependence on unphysical scales. For the Higgs boson production through gluon fusion, we work in the effective theory where the top quark is integrated out. *This section is devoted to demonstrate the SV corrections to this observable at N^{3}LO for the Higgs boson, produced through gluon fusion, and leptonic pair in Drell-Yan (DY) production.*

For the processes under considerations, the NNLO QCD corrections are present, computed long back, and in addition, the partial N^{3}LO SV results are also available. However, due to reasonably large scale uncertainties and crying demand of uplifting the accuracy of theoretical predictions, we must push the boundaries of existing results. *In this work, we have computed the missing part and completed the SV corrections to the rapidity distributions at N^{3}LO QCD.*

The prescription which has been employed to calculate the SV QCD corrections is similar to that of the inclusive cross section, more specifically, it is a generalisation of the other one. The infrared safe contributions under SV approximation can be computed by adding the soft part of the rapidity distribution with the UV renormalised virtual part and performing the mass factorisation using appropriate counter terms. Similar to the inclusive case, the main ingredients to perform this computation are the form factors, overall UV operator renormalisation constant, soft-collinear distribution for rapidity and mass fac-

torisation kernels. These quantities are required up to N^3LO to calculate the rapidity at this order. The three loop quark and gluon form factors were calculated long back. The operator renormalisation constants are also present. For DY, this constant is not required or equivalently equals to unity. The mass factorisation kernels are also available in the literature to the required order. The only missing part was the soft-collinear distribution for rapidity at N^3LO. This was not possible to compute until very recently. Because of the universal behaviour of the soft gluons, the soft-collinear distributions for rapidity and inclusive cross section can be related to all orders in perturbation theory. Employing this beautiful relation, we obtain this quantity at N^3LO from the results of soft-collinear distribution of the inclusive cross section. Using this, along with the existing results of the other relevant quantities, we compute the complete analytical expressions of N^3LO SV correction to the rapidity distributions for the Higgs boson in gluon fusion and leptonic pair in DY. We demonstrate the numerical impact of this correction for the case of Higgs boson at the LHC. This indeed reduces the scale dependence significantly and provides a more reliable theoretical predictions. *These are the most accurate results for the rapidity distributions of the Higgs boson and DY pair which exist in the literature and undoubtedly, expected to play very important role in the upcoming run at the LHC.*

0.3 Pseudo-Scalar Form Factors at Three Loops in QCD

One of the most popular extensions of the SM, namely, the MSSM and two Higgs doublet model have richer Higgs sector containing more than one Higgs boson and there have been intense search strategies to observe them at the LHC. In particular, the production of CP-odd Higgs boson/pseudo-scalar at the LHC has been studied in detail, taking into account higher order QCD radiative corrections, due to similarities with its CP-even counter part. Very recently, the N^3LO QCD corrections to the inclusive production cross section of the CP-even Higgs boson is computed. So, it is very natural to extend the theoretical accuracy for the CP-odd Higgs boson to the same order of N^3LO. This requires the 3-loop quark

and gluon form factors for the pseudo-scalar which are the only missing ingredients to achieve this goal.

Multiloop and multileg computations play a crucial role to achieve the golden task of making precise theoretical predictions. However, the complexity of these computations grows very rapidly with the increase of number of loops and/or external particles. Nevertheless, it has become a reality due to several remarkable developments in due course of time. *This section is devoted to demonstrate the computations of the 3-loop quark and gluon form factors for the pseudo-scalar operators in QCD.*

The coupling of a pseudo-scalar Higgs boson to gluons is mediated through a heavy quark loop. In the limit of large quark mass, it is described by an effective Lagrangian that only admits light degrees of freedom. In this effective theory, we compute the 3-loop massless QCD corrections to the form factor that describes the coupling of a pseudo-scalar Higgs boson to gluons. The evaluation of this 3-loop form factors is truly a non-trivial task not only because of the involvement of a large number of Feynman diagrams but also due to the presence of the axial vector coupling. We work in dimensional regularisation and use the 't Hooft-Veltman prescription for the axial vector current, The state-of-the-art techniques including integration-by-parts (IBP) and Lorentz invariant (LI) identities have been employed to accomplish this task. The UV renormalisation is quite involved since the two operators, present in the Lagrangian, mix under UV renormalization due to the axial anomaly and additionally, a finite renormalisation constant needs to be introduced in order to fulfill the chiral Ward identities. Using the universal infrared (IR) factorization properties, we independently derive the three-loop operator mixing and finite operator renormalisation from the renormalisation group equation for the form factors, thereby confirming recent results, which were computed following a completely different methodology, in the operator product expansion. This form factor is an important ingredient to the precise prediction of the pseudo-scalar Higgs boson production cross section at hadron colliders. We derive the hard matching coefficient in soft-collinear effective

theory (SCET). We also study the form factors in the context of leading transcendentality principle and we find that the diagonal form factors become identical to those of $\mathcal{N} = 4$ upon imposing some identification on the quadratic Casimirs. Later, these form factors are used to calculate the SV corrections to the pseudo-scalar production cross section at N^3LO and N^3LL QCD.

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1 Introduction

Contents

The Standard Model (SM) of Particle Physics is one of most remarkably successful fundamental theories which encapsulates the governing principles of elementary constituents of matter and their interactions. Its development throughout the latter half of the 20th century resulting from an unprecedented collaborative effort of the brightest minds around the world is undoubtedly one of the greatest achievements in human history. Over the duration of many decades around 1970s, the theoretical predictions of the SM were verified one after another with a spectacular accuracy and it got the ultimate credence through the announcement, made on a fine morning of 4th July 2012 at CERN in Geneva:

"If we combine ZZ and $\gamma\gamma$, this is what we get: they line up extremely well in a region of 125 GeV with the combine significance of 5 standard deviation!"

The SM relies on the mathematical framework of quantum field theory (QFT), in which a Lagrangian controls the dynamics and kinematics of the theory. Each kind of particle is described in terms of a dynamical field that pervades space-time. The construction of the SM proceeds through the modern methodology of constructing a QFT, it happens through postulating a set of underlying symmetries of the system and writing down the most general renormalisable Lagrangian from its field content.

The underlying symmetries of the QFT can be largely categorized into global and local ones. The global Poincaré symmetry is postulated for all the relativistic QFT. It consists of the familiar translational symmetry, rotational symmetry and the inertial reference frame invariance central to the special theory of relativity. Being global, its operations must be simultaneously applied to all points of space-time On the other hand, the local gauge symmetry is an internal symmetry that plays the most crucial role in determining the predictions of the underlying QFT. These are the symmetries that act independently at each point in space-time. The SM relies on the local $SU(3)\times SU(2)_L\times U(1)_Y$ gauge symmetry. Each gauge symmetry manifestly gives rise to a fundamental interaction: the electromagnetic interactions are characterized by an U(1), the weak interactions by an SU(2) and the strong interactions by an SU(3) symmetry.

In its current formulation of the SM, it includes three different families of elementary particles. The first ones are called fermions arising from the quantisation of the fermionic fields. These constitute the matter content of the theory. The quanta of the bosonic fields, which form the second family, are the force carriers i.e. the mediators of the strong, weak, and electromagnetic fundamental interactions. In addition to the these, there is a third boson, the Higgs boson resulting from the quantum excitation of the Higgs field. This is the only known scalar particle that was postulated long ago and observed very recently at the Large Hadron Collider (LHC) [1, 2]. The presence of this field, now believed to be confirmed, explains the mechanisms of acquiring mass of some of the fundamental particles when, based on the underlying gauge symmetries controlling their interactions,

they should be massless. This mechanism, which is believed to be one of the most revolutionary ideas of the last century, is known as Brout-Englert-Higgs-Kibble (BEHK) mechanism.

Two of the four known fundamental forces, electromagnetism and weak forces which appear very different at low energies, are actually unified to so called electro-weak force in high energy. The structure of this unified picture is accomplished under the gauge group $SU(2)_L \times U(1)_Y$. The corresponding gauge bosons are the three W bosons of weak isospin from SU(2) and the B boson of weak hypercharge from U(1), all of which are massless. Upon spontaneous symmetry breaking from $SU(2)_L \times U(1)_Y$ to $U(1)_{EM}$, caused by the BEHK mechanism, the three mediators of the electro-weak force, the $W^{\pm}, Z$ bosons acquire mass, leaving the mediator of the electromagnetic force, the photon, as massless. Finally, the theory of strong interactions, Quantum Chromo-Dynamics (QCD) is governed by the unbroken SU(3) gauge group, whose force carriers, the gluons remain massless.

Although the SM is believed to be theoretically self-consistent with a spectacular accuracy and has demonstrated huge and continued successes in providing experimental predictions, it indeed does leave some phenomena unexplained and it falls short of being a complete theory of fundamental interactions. It fails to incorporate the full theory of gravitation as described by general relativity, or account for the accelerating expansion of the universe (as possibly described by dark energy). The model does not contain any viable dark matter particle that possesses all of the required properties deduced from observational cosmology. It also does not incorporate neutrino oscillations (and their non-zero masses).

Currently, the high energy physics community is standing on the verge of a crucial era where the new physics may show up as tiny deviations from the prediction of the SM! To exploit this possibility it is absolutely necessary to make the theoretical predictions, along with the revolutionary experimental progress, to a very high accuracy within the SM and beyond. The relevance of this thesis arises exactly in this context.

The most crucial quantity in the process of accomplishing the job of making any prediction based on QFT is undoubtedly the scattering amplitude. This is the fundamental building block of any observable in QFT. In the upcoming section, we will elaborate on the idea of scattering amplitude which will be followed by a brief description of QCD. We will close the chapter of introduction by introducing the concept of computing the observables under certain approximation, known as soft-virtual approximation.

1.1 Scattering Amplitudes

The fundamental quantity of any QFT which encodes all the underlying symetries of the theory is called the action. This is constructed out of Lagrangian density, which is a functional of the fields present in the theory, and integrating over all space time points:

$$S = \int d^4x \, \mathcal{L}\left[\phi_i(x)\right] . \tag{1.1.1}$$

By construction the QFT is a probabilistic theory and all the observables calculated based on this theory always carry a probabilistic interpretation. For example, an important observable is the total cross section which measures the total probability of any event to happen in colliders. The computation of the cross section, and in fact, almost all the observables in QFT requires the evaluation of scattering matrix (S-matrix) elements which describe the evolution of the system from asymptotic initial to final states due to presence of the interaction. The S-matrix elements are defined as

$$\langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4\delta^{(4)}\left(p_f - p_i\right)\mathcal{M}_{i\to f} \tag{1.1.2}$$

where, the δ_{fi} represents the unscattered forward scattering states, while the other part $\mathcal{M}_{i\to f}$ encapsulates the "actual" interaction (For simplicity, we will call $\mathcal{M}_{i\to f}$ as scattering matrix element.). So, the calculation of all those observables essentially boils down to the

computation of the second quantity. However, the exact computation of this quantity is incredibly difficult in any general field theory. The only viable methodology is provided under the framework of perturbation theory where the matrix elements as well as the observables are expanded in powers of coupling constants, c, present in the theory:

$$\mathcal{M}_{i\to f} = \sum_{n=0}^{\infty} c^n \mathcal{M}^{(n)}_{i\to f} \,. \tag{1.1.3}$$

If the coupling constant is small enough, the evaluation of only the first term of the perturbative expansion often turns out to be a very good approximation that provides a reliable prediction to any observable. However, in QFT, it is a well-known fact that the coupling constants are truly not 'constants', their strength depends on the energy scale at which the interaction takes place. This evolution of the coupling constant may make it comparatively large at some energy scale. In case of Quantum Electro-Dynamics (QED), quantum field theory of electromagnetism, the magnitude of the coupling constant, $c = \alpha_{\rm EM}$, increases with the increase of momentum transfer:

$$\alpha_{\rm EM}(Q^2 \approx 0) \approx \frac{1}{137}\,, \qquad \text{and} \qquad \alpha_{\rm EM}(Q^2 \approx m_W^2) \approx \frac{1}{128} \tag{1.1.4}$$

where, $m_W \approx 80$ GeV is the invariant mass of the W boson. The smallness of $\alpha_{\rm EM}$ at all typical energy scales which can be probed in all collider experiments guarantees very fast convergence of the perturbation series to what we expect to be real non-perturbative result. However, this picture no longer holds true in case of QCD where the coupling constant, $c = \alpha_s$, may become quite large at certain energy scales:

$$\alpha_s(m_p^2) \approx 0.55, \qquad \text{and} \qquad \alpha_s(m_Z^2) \approx 0.1 \tag{1.1.5}$$

where, $m_p \approx 938 MeV$ and $m_Z \approx 90 GeV$ are the masses of the proton and Z boson. Clearly the magnitude 0.55 is far from being small! Hence, computation of only the leading term in perturbative series often turns out to be a very crude approximation which fails to

deliver a reliable prediction. We must take into account the contributions arising beyond leading term.

In perturbation theory, the most acceptable and well known prescription to compute the terms in a perturbative series is provided by Feynman diagrams. Every term of a series is represented through a set of Feynman diagrams and each diagram corresponds to a mathematical expression. Hence, evaluation of a term in any perturbative series boils down to the computation of all the corresponding Feynman diagrams. Given an action of a QFT, one first requires to derive a set of rules, called Feynman rules, which essentially establish the correspondence between the Feynman diagrams and mathematical expressions. With the rules in hand, we just need to draw all the possible Feynman diagrams contributing to the order of our interest and eventually evaluate those using the rules. Needless to say, as the perturbative order increases, the number of Feynman diagrams to be drawn grows so rapidly that after certain order it becomes almost prohibitively large to draw.

In this thesis, we will concentrate only on the aspects of perturbative QCD. We will start our discussion of QCD by introducing the basic aspects of this QFT which will be followed by the writing down the quantum action and corresponding Feynman rules. Then we will discuss how to compute amplitudes beyond leading order in QCD and eventually get reliable numerical predictions at hadron colliders for any process.

1.2 Quantum Chromo-Dynamics

Quantum Chromo-dynamics, familiarly called QCD, is the modern theory of strong interactions, a fundamental force describing the interactions between quarks and gluons which make up hadrons such as the protons, neutrons and pions. QCD is a type of QFT called non-Abelian gauge theory that has underlying SU(3) gauge symmetry. It appears as an expanded version of QED. Whereas in QED there is just one kind of charge, namely electric charge, QCD has three different kinds of charge, labeled by “colour”. Avoiding chauvin-

ism, those are chosen as red, green, and blue. But, of course, the colour charges of QCD have nothing to do with optical colours. Rather, they have properties analogous to electric charges in QED. In particular, the colour charges are conserved in all physical processes. There are also photon-like massless gauge bosons, called gluons, that act as the mediators of the strong interactions between spin-1/2 quarks. Unlike the photons, which mediate the electromagnetic interaction but lacks an electric charge, the gluons themselves carry color charges. Gluons, as a consequence, participate in the strong interactions in addition to mediating it, making QCD substantially harder to analyse than QED.

In sharp contrast to other gauge theories, QCD enjoys two salient features: confinement and asymptotic freedom. The force among quarks/gluons fields does not diminish as they are separated from each others. With the increase in mutual distance between them, the mediating gluon fields gather enough energy to create a pair of quarks/gluons which forbids them to be found as free particles; they are thus forever bound into hadrons such as the protons, neutrons, pions or kaons. Although literature lacks the satisfactory theoretical explanation, confinement is believed to be true as it explains the consistent failure of finding free quarks or gluons. The other interesting property, the asymptotic freedom [3–7], causes bonds between quarks/gluons become asymptotically weaker as energy increases or distance decreases which allows us to perform the calculation in QCD using the technique of perturbation theory. The Nobel prize was awarded for this remarkable discovery of last century.

In perturbative QCD, the basic building blocks of performing any calculation are the Feynman rules, which will be discussed in next subsection.

1.2.1 The QCD Lagrangian and Feynman Rules

The first step in performing perturbative calculations in a QFT is to work out the Feynman rules. The SU(N) gauge invariant classical Lagrangian density encapsulating the

interaction between fermions and non-Abelian gauge fields is

$$\mathcal{L}_{classical} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} + \sum_{f=1}^{n_f}\overline{\psi}^{(f)}_{\alpha,i}\left(i\not{D}_{\alpha\beta,ij} - m_f\delta_{\alpha\beta}\delta_{ij}\right)\psi^{(f)}_{\beta,j}\,. \tag{1.2.1}$$

In the above expression,

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^b_\mu A^c_\nu\,, \\ \not{D}_{\alpha\beta,ij} &\equiv \gamma^\mu_{\alpha\beta}D_{\mu,ij} = \gamma^\mu\left(\delta_{ij}\partial_\mu - ig_s T^a_{ij}A^a_\mu\right) \end{aligned} \tag{1.2.2}$$

where, A^a_μ and $\psi^{(f)}_{\alpha,i}$ are the guage and fermionic quark fields, respectively. The indices represent the following things:

$$\begin{aligned} a, b, \cdots &: \quad \text{color indices in the adjoint representation} \Rightarrow [1, N^2-1]\,, \\ i, j, \cdots &: \quad \text{color indices in the fundamental representation} \Rightarrow [1, N]\,, \\ \alpha, \beta, \cdots &: \quad \text{Dirac spinor indices} \Rightarrow [1, d]\,, \\ \mu, \nu, \cdots &: \quad \text{Lorentz indices} \Rightarrow [1, d]\,. \end{aligned} \tag{1.2.3}$$

Numbers within the '[]' signifies the range of the corresponding indices. d is the space-time dimensions. f is the quark flavour index which runs from 1 to n_f. m_f and g_s are the mass of the quark corresponding to $\psi^{(f)}$ and strong coupling constant, respectively. f^{abc} are the structure constants of SU(N) group. These are related to the Gellmann matrices T^a, generators of the fundamental representations of SU(N), through

$$\left[T^a, T^b\right] = if^{abc}T^c\,. \tag{1.2.4}$$

The T^a are traceless, Hermitian matrices and these are normalised with

$$Tr\left(T^aT^b\right) = T_F\delta^{ab} \tag{1.2.5}$$

where, $T_F = \frac{1}{2}$. They satisfy the following completeness relation

$$\sum_a T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{1.2.6}$$

In addition to the above three parent identities expressed through the Eq. (1.2.4, 1.2.5, 1.2.6), we can have some auxiliary ones which are often useful in simplifying colour algebra:

$$\begin{aligned} \sum_a (T^a T^a)_{ij} &= C_F \delta_{ij}\,, \\ f^{acd} f^{bcd} &= C_A \delta^{ab}\,. \end{aligned} \tag{1.2.7}$$

The $C_A = N$ and $C_F = \frac{N^2-1}{2N}$ are the quadratic Casimirs of the SU(N) group in the adjoint and fundamental representations, respectively. For QCD, the SU(N) group index, $N = 3$ and the flavor number $n_f = 6$.

The quantisation of the non-Abelian gauge theory or the Yang-Mills (YM) theory faces an immediate problem, namely, the propagator of gauge fields cannot be obtained unambiguously. This is directly related to the presence of gauge degrees of freedom inherent into the $\mathcal{L}_{classical}$. We need to perform the gauge fixing in order to get rid of this problem. The gauge fixing in a covariant way, when done through the path integral formalism, generates new particles called Faddeev-Popov (FP) ghosts having spin-0 but obeying fermionic statistics. The absolute necessity of introducing the ghosts in the process of quantising the YM theory is a horrible consequence of the Lagrangian formulation of QFT. There is no observable consequence of these particles, we just need them in order to describe an interacting theory of a massless spin-1 particle using a local manifestly Lorentz invariant Lagrangian. These particles never appear as physical external states but must be included in internal lines to cancel the unphysical degrees of freedom of the gauge fields. Some alternative formulations of non-Abelian gauge theory (such as the lattice) also do not require ghosts. Perturbative gauge theories in certain non-covariant gauges, such as

light-cone or axial gauges, are also ghost free. However, to maintain manifest Lorentz invariance in a perturbative gauge theory, it seems ghosts are unavoidable and in this thesis we will be remained within the regime of covariant gauge and consequently will include ghost fields consistently into our computations.

Upon applying this technique to quantise the YM theory, we end up with getting the following full quantum Lagrangian density:

$$\mathcal{L}_{YM} = \mathcal{L}_{classical} + \mathcal{L}_{gauge-fix} + \mathcal{L}_{ghost} \tag{1.2.8}$$

where, the second and third terms on the right hand side correspond to the gauge fixing and FP contributions, respectively. These are obtained as

$$\begin{aligned} \mathcal{L}_{gauge-fix} &= -\frac{1}{2\xi}\left(\partial^{\mu}A^{a}_{\mu}\right)^{2}, \\ \mathcal{L}_{ghost} &= (\partial^{\mu}\chi^{a*})\, D_{\mu,ab}\,\chi^{b} \end{aligned} \tag{1.2.9}$$

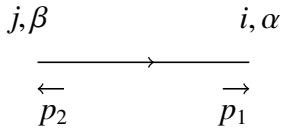
with

$$D_{\mu,ab} \equiv \delta_{ab}\partial_{\mu} - g_{s} f_{abc} A^{c}_{\mu}\,. \tag{1.2.10}$$

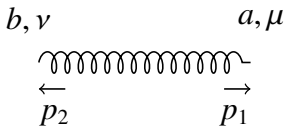
The gauge parameter ξ is arbitrary and it is introduced in order to specify the gauge in a covariant way. This prescription of fixing gauge in a covariant way is known as R_{ξ} gauge. A typical choice which is often used is $\xi = 1$, known as Feynman gauge. We will be working in this Feynman gauge throughout this thesis, unless otherwise mentioned specifically. However, we emphasize that the physical results are independent of the choice of the gauges. The field χ^{a} and χ^{a*} are ghost and anti-ghost fields, respectively.

All the Feynman rules can be read off from the quantized Lagrangian $\mathcal{L}_{YM}$ in Eq. 1.2.8. We will denote the quarks through straight lines, gluons through curly and ghosts through dotted lines. We provide the rules in R_{ξ} gauge.

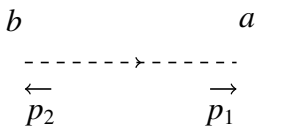
- The propagators for quarks, gluons and ghosts are obtained as respectively:



$$i\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2)\,\delta_{ij}\left(\frac{1}{\not{p}_1-m_f+i\varepsilon}\right)_{\alpha\beta}$$

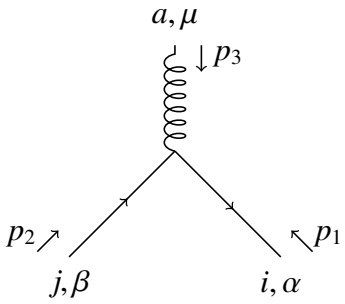


$$i\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2)\,\delta_{ab}\frac{1}{p_1^2}\left[-g_{\mu\nu}+(1-\xi)\frac{p_{1\mu}p_{1\nu}}{p_1^2}\right]$$

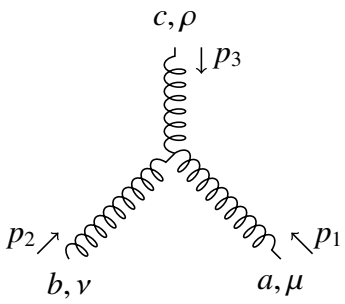


$$i\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2)\,\delta_{ab}\frac{1}{p_1^2}$$

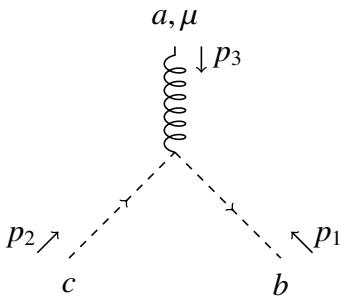
- The interacting vertices are given by:



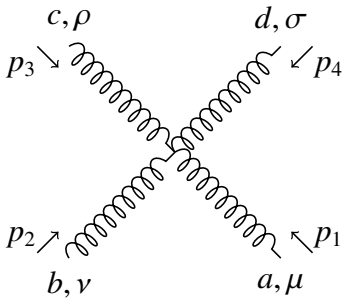
$$ig_s\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,T^a_{ij}\,(\gamma^\mu)_{\alpha\beta}$$



$$\frac{g_s}{3!}\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,f^{abc}$$
$$\times\left[g^{\mu\nu}(p_1-p_2)^\rho+g^{\nu\rho}(p_2-p_3)^\mu+g^{\rho\mu}(p_3-p_1)^\nu\right]$$



$$-g_s\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3)\,f^{abc}\,p_1^{\mu}$$



$$-\frac{g_s^2}{4!}\,(2\pi)^4\,\delta^{(4)}\,(p_1+p_2+p_3+p_4)$$
$$\Big\{\Big(f^{ac,bd}-f^{ad,cb}\Big)\,g^{\mu\nu}g^{\rho\sigma}+\Big(f^{ab,cd}-f^{ad,bc}\Big)\,g^{\mu\rho}g^{\nu\sigma}$$
$$+\Big(f^{ac,db}-f^{ab,cd}\Big)\,g^{\mu\sigma}g^{\nu\rho}\Big\}$$

with

$$f^{ab,cd}\equiv f^{abx}f^{cdx}$$

In addition to these rules, we have keep in mind the following points:

- For any Feynman diagram, the symmetry factor needs to be multiplied appropriately. The symmetry factor is defined as the number of ways one can obtain the topological configuration of the Feynman diagram under consideration.
- For each loop momenta, the integration over the loop momenta, k, needs to be performed with the integration measure $d^dk/(2\pi)^d$ in d-dimensions (in dimensional regularisation).
- For each quark/ghost loop, one has to multiply a factor of (-1).

1.3 Perturbative Calculations in QCD

The asymptotic freedom of the QCD allows us to perform the calculations in high energy regime using the techniques of perturbative QCD (pQCD). In pQCD, we make the

theoretical predictions through the computations of the scattering matrix (S-matrix) elements. The S-matrix elements are directly related to the scattering amplitude which is formally expanded, within the framework of perturbation theory, in powers of coupling constants. This expansion is represented through the set of Feynman diagrams and the Feynman rules encapsulate the connection between these these two. Hence, the theoretical predictions boil down to evaluate the set of Feynman diagrams. Using the Feynman rules presented in the previous Sec. 1.2.1, we can evaluate all the Feynman diagrams.

Achieving precise theoretical predictions demand to go beyond leading order which consists of evaluating the virtual/loop as well as real emission diagrams. However, the contribution arising from the individual one is not finite. The resulting expressions from the evaluation of loop diagrams contain the ultraviolet (UV), soft and collinear divergences. For simplicity, together we call the soft and collinear as infrared (IR) divergence.

The UV divergences arise from the region of large momentum or very high energy (approaching infinity) of the Feynman integrals, or, equivalently, because of the physical phenomena at very short distances. We get rid of this through UV renormalisation. Before performing the UV renormalisation, we need to regulate the Feynman integrals which is essentially required to identify the true nature of divergences. There are several ways to regulate the integrals. The most consistent and beautiful way is the framework of dimensional regularisation [8–10]. Within this, we need to perform the integrals in general d-dimensions which is taken as $4 + \epsilon$ in this thesis. Upon performing the integrals, all the UV singularities arise as poles in ϵ. The UV renormalisation, which is performed through redefining all the quantities present in the Lagrangian, absorbs these poles and gives rise a UV finite result. The UV renormalisation is done at certain energy scale, known as renormalisation scale, μ_R. On the other hand, the soft divergences arise from the low momentum limit (approaching zero) of the loop integrals and the collinear ones arise when any loop momentum becomes collinear to any of the external massless particles. The collinear divergence is a property of theories with massless particles. Hence,

even after performing the UV renormalisation, the resulting expressions obtained through the evaluation the loop integrals are not finite, they contain poles arising from soft and collinear regions of the loop integrals.

To remove the residual IR divergences, we need to add the contributions arising from the real emission diagrams. The latter contains soft as well as collinear divergences which have the same form as that of loop integrals. Once we add the virtual and real emission diagrams and evaluate the phase space integrals, the resulting expressions are guaranteed to be freed from UV, soft and final state collinear singularities, thanks to the Kinoshita-Lee-Nauenberg (KLN) theorem. An analogous result for quantum electrodynamics alone is known as Bloch?Nordsieck cancellation. However, the collinear singularities arising from the collinear configurations involving initial state particles remain. Those are removed at the hadronic level through the techniques, known as mass factorisation, where the residual singularities are absorbed into the bare parton distribution functions (PDF). So, the observables at the hadronic level are finite which are compared with the experimental outcomes at the hadron colliders. Just like UV renormalisation, mass factorisation is done at some energy scale, called factorisation scale, μ_F. The μ_R as well as μ_F are unphysical scales. The dependence of the fixed order results on these scale is an artifact of the truncation of the perturbative series to a finite order. If we can capture the results to all order, then the dependence goes away.

The core part of this thesis deals with the higher order QCD corrections employing the methodology of perturbation theory to some of the very important processes within the SM and beyond. More specifically, the thesis contains

- the soft-virtual QCD corrections to the inclusive cross section of the Higgs boson production through bottom quark annihilation at next-to-next-to-next-to-leading order (N^3LO) [11],
- the soft-virtual QCD corrections at N^3LO to the differential rapidity distributions of the productions of the Higgs boson in gluon fusion and of the leptonic pair in

Drell-Yan [12],

- the three loop QCD corrections to the pseudo-scalar form factors [13, 14].

In the subsequent subsections, we will discuss the above things in brief.

1.3.1 Soft-Virtual Corrections To Cross Section at N^{3}LO QCD

The Higgs bosons are produced dominantly at the LHC via gluon fusion through top quark loop, while one of the sub-dominant ones take place through bottom quark annihilation. In the SM, the interaction between the Higgs boson and bottom quarks is controlled through the Yukawa coupling which is reasonably small at typical energy scales. However, in the minimal super symmetric SM (MSSM), this channel can contribute substantially due to enhanced coupling between the Higgs boson and bottom quarks in the large $\tan\beta$ region, where $\tan\beta$ is the ratio of vacuum expectation values of the up and down type Higgs fields. In the present run of LHC, the measurements of the various coupling constants including this one are underway which can shed light on the properties of the newly discovered Higgs boson [1, 15]. Most importantly, for the precision studies we must take into account all the contributions, does not matter how tiny those are, arising from sub-dominant channels along with the dominant ones to reduce the dependence on the unphysical scales and make a reliable prediction.

The computations of the higher order QCD corrections beyond leading order often become quite challenging because of the large number of Feynman diagrams and, presence of the complicated loop and phase space integrals. Under this circumstance, when we fail to compute the complete result at certain order, it is quite natural to try an alternative approach to capture the dominant contributions from the missing higher order corrections. It has been observed for many processes that the dominant contributions to an observable often comes from the soft gluon emission diagrams. The contributions arising from the associated soft gluon emission along with the virtual Feynman diagrams are known as

the soft-virtual (SV) corrections. *The goal of the works published in the article [11] is to discuss the SV QCD corrections to the production cross section of the Higgs boson, produced through bottom quark annihilation.*

The next-to-next-to-leading order (NNLO) QCD corrections to this channel are already present in the variable flavour scheme (VFS) [16–21], while it is known to NLO in the fixed flavour scheme (FFS) [22–27]. In addition, the partial result for the N^3LO corrections [28–30] under the SV approximation were also computed long back. In both [28,29] and [30], it was not possible to determine the complete contribution at N^3LO due to the lack of information on three loop finite part of bottom anti-bottom Higgs form factor in QCD and the soft gluon radiation at N^3LO level. *In this work [11], we have computed the missing part and completed the full SV corrections to the cross section at N^3LO.*

The infrared safe contributions from the soft gluons are obtained by adding the soft part of the cross section with the UV renormalized virtual part and performing mass factorisation using appropriate counter terms. The main ingredients are the form factors, overall operator UV renormalization constant, soft-collinear distribution arising from the real radiations in the partonic subprocesses and mass factorization kernels. The computations of SV cross section at N^3LO QCD require all of these above quantities up to 3-loop order. The relevant form factor becomes available very recently in [31]. The soft-collinear distribution at N^3LO was computed by us around the same time in [32]. This was calculated from the recent result of N^3LO SV cross section of the Higgs boson productions in gluon fusion [33] by employing a symmetry (maximally non-Abelian property). Prior to this, this symmetry was verified explicitly up to NNLO order. However, neither there was any clear reason to believe that the symmetry would fail nor there was any transparent indication of holding it beyond this order. Nevertheless, we conjecture [32] that the relation would hold true even at N^3LO order! This is inspired by the universal properties of the soft gluons which are the underlying reasons behind the existence of this remarkable symmetry. Later, this conjecture is verified by explicit computations performed by two

different groups on Drell-Yan process [34, 35]. This symmetry plays the most important role in achieving our goal. With these, along with the existing results of the remaining required ingredients, we obtain the complete analytical expressions of N^3LO SV cross section of the Higgs boson production through bottom quark annihilation [11] employing the methodology prescribed in [28, 29]. It reduces the scale dependence and provides a more precise result. We demonstrate the impact of this result numerically at the LHC briefly. *This is the most accurate result for this channel which exists in the literature till date and it is expected to play an important role in coming days at the LHC.*

1.3.2 Soft-Virtual QCD Corrections to Rapidity at N^3LO

The productions of the Higgs boson in gluon fusion and leptonic pair in Drell-Yan (DY) are among the most important processes at the LHC which are studied not only to test the SM to an unprecedented accuracy but also to explore the physics beyond Standard Model (BSM). During the present run at the LHC, in addition to the inclusive production cross section, the differential rapidity distribution is among the most important observables, which is expected to be measured in upcoming days. This immediately calls for very precise theoretical predictions.

In the same spirit of the SV corrections to the inclusive production cross section, the dominant contributions to the differential rapidity distributions often arise from the soft gluon emission diagrams. Hence, in the absence of complete fixed order result, the rapidity distribution under SV approximation is the best available alternative in order to capture the dominant contributions from the missing higher orders and stabilise the dependence on unphysical scales. For the Higgs boson production through gluon fusion, we work in the effective theory where the top quark is integrated out. *This work published in the article [12] is devoted to demonstrate the SV corrections to this observable at N^3LO for the Higgs boson, produced through gluon fusion, and leptonic pair in DY production.*

For the processes under considerations, the NNLO QCD corrections are present [36–38], computed long back, and in addition, the partial N^{3}LO SV results [39] are also available. However, due to reasonably large scale uncertainties and crying demand of uplifting the accuracy of theoretical predictions, we must push the boundaries of existing results. *In this work [12], we have computed the missing part and completed the SV corrections to the rapidity distributions at N^{3}LO QCD.*

The prescription [39] which has been employed to calculate the SV QCD corrections is similar to that of the inclusive cross section, more specifically, it is a generalisation of the other one. The infrared safe contributions under SV approximation can be computed by adding the soft part of the rapidity distribution with the UV renormalised virtual part and performing the mass factorisation using appropriate counter terms. Similar to the inclusive case, the main ingredients to perform this computation are the form factors, overall UV operator renormalisation constant, soft-collinear distribution for rapidity and mass factorisation kernels. These quantities are required up to N^{3}LO to calculate the rapidity at this order. The three loop quark and gluon form factors [40–43] were calculated long back. The operator renormalisation constants are also present. For DY, this constant is not required or equivalently equals to unity. The mass factorisation kernels are also available in the literature to the required order. The only missing part was the soft-collinear distribution for rapidity at N^{3}LO. This was not possible to compute until very recently. Because of the universal behaviour of the soft gluons, the soft-collinear distributions for rapidity and inclusive cross section can be related to all orders in perturbation theory [39]. Employing this beautiful relation, we obtain this quantity at N^{3}LO from the results of soft-collinear distribution of the inclusive cross section [32]. Using this, along with the existing results of the other relevant quantities, we compute the complete analytical expressions of N^{3}LO SV correction to the rapidity distributions for the Higgs boson in gluon fusion and leptonic pair in DY [12]. We demonstrate the numerical impact of this correction for the case of Higgs boson at the LHC. This indeed reduces the scale dependence significantly and provides a more reliable theoretical predictions. *These are the most ac-*

curate results for the rapidity distributions of the Higgs boson and DY pair which exist in the literature and undoubtedly, expected to play very important role in the upcoming run at the LHC.

1.3.3 Pseudo-Scalar Form Factors at Three Loops in QCD

One of the most popular extensions of the SM, namely, the MSSM and two Higgs doublet model have richer Higgs sector containing more than one Higgs boson and there have been intense search strategies to observe them at the LHC. In particular, the production of CP-odd Higgs boson/pseudo-scalar at the LHC has been studied in detail, taking into account higher order QCD radiative corrections, due to similarities with its CP-even counter part. Very recently, the N^3LO QCD corrections to the inclusive production cross section of the CP-even Higgs boson becomes available [44]. So, it is very natural to extend the theoretical accuracy for the CP-odd Higgs boson to the same order of N^3LO. This requires the 3-loop quark and gluon form factors for the pseudo-scalar which are the only missing ingredients to achieve this goal.

Multiloop and multileg computations play a crucial role to achieve the golden task of making precise theoretical predictions. However, the complexity of these computations grows very rapidly with the increase of number of loops and/or external particles. Nevertheless, it has become a reality due to several remarkable developments in due course of time. *These articles [13, 14] are devoted to demonstrate the computations of the 3-loop quark and gluon form factors for the pseudo-scalar operators in QCD.*

The coupling of a pseudo-scalar Higgs boson to gluons is mediated through a heavy quark loop. In the limit of large quark mass, it is described by an effective Lagrangian [45] that only admits light degrees of freedom. In this effective theory, we compute the 3-loop massless QCD corrections to the form factor that describes the coupling of a pseudo-scalar Higgs boson to gluons. The evaluation of this 3-loop form factors is truly a non-trivial

task not only because of the involvement of a large number of Feynman diagrams but also due to the presence of the axial vector coupling. We work in dimensional regularisation and use the 't Hooft-Veltman prescription [8] for the axial vector current, The state-of-the-art techniques including integration-by-parts [46,47] and Lorentz invariant [48] identities have been employed to accomplish this task. The UV renormalisation is quite involved since the two operators, present in the Lagrangian, mix under UV renormalization due to the axial anomaly and additionally, a finite renormalisation constant needs to be introduced in order to fulfill the chiral Ward identities. Using the universal infrared factorization properties, we independently derive [13] the three-loop operator mixing and finite operator renormalisation from the renormalisation group equation for the form factors, thereby confirming recent results [49, 50], which were computed following a completely different methodology, in the operator product expansion. This form factor [13, 14] is an important ingredient to the precise prediction of the pseudo-scalar Higgs boson production cross section at hadron colliders. We derive the hard matching coefficient in soft-collinear effective theory (SCET). We also study the form factors in the context of leading transcendentality principle and we find that the diagonal form factors become identical to those of $\mathcal{N} = 4$ upon imposing some identification on the quadratic Casimirs. Later, these form factors are used to calculate the SV corrections [14] to the pseudo-scalar production cross section at $\mathrm{N^3LO}$ and next-to-next-to-next-to-leading logarithm ($\mathrm{N^3LL}$) QCD.

2 Higgs boson production through $b\bar{b}$ annihilation at threshold in N^{3}LO QCD

The materials presented in this chapter are the result of an original research done in collaboration with Narayan Rana and V. Ravindran, and these are based on the published article [11].

Contents

2.1 Prologue

The discovery of Higgs boson by ATLAS [1] and CMS [2] collaborations of the LHC at CERN has not only shed the light on the dynamics behind the electroweak symmetry breaking but also put the SM of particle physics on a firmer ground. In the SM, the elementary particles such as quarks, leptons and gauge bosons, $Z, W^{\pm}$ acquire their masses through spontaneous symmetry breaking (SSB). The Higgs mechanism provides the framework for SSB. The SM predicts the existence of a Higgs boson whose mass is a parameter of the model. The recent discovery of the SM Higgs boson like particle provides a valuable information on this, namely on its mass which is about 125.5 GeV. The searches for the Higgs boson have been going on for several decades in various experiments. Earlier experiments such as LEP [51] and Tevatron [52] played an important role in the discovery by the LHC collaborations through narrowing down its possible mass range. LEP excluded Higgs boson of mass below 114.4 GeV and their precision electroweak measurements [53] hinted the mass less than 152 GeV at 95% confidence level (CL), while Tevatron excluded Higgs boson of mass in the range 162 – 166 GeV at 95% CL.

Higgs bosons are produced dominantly at the LHC via gluon gluon fusion through top quark loop, while the sub-dominant ones are vector boson fusion, associated production of Higgs boson with vector bosons, with top anti-top pairs and also in bottom anti-bottom annihilation. The inclusive productions of Higgs boson in gluon gluon [53–61], vector boson fusion processes [62] and associated production with vector bosons [63] are known to NNLO accuracy in QCD. Higgs production in bottom anti-bottom annihilation is also known to NNLO accuracy in the variable flavour scheme (VFS) [16–21], while it is known to NLO in the fixed flavour scheme (FFS) [22–27]. In the MSSM, the coupling of bottom quarks to Higgs becomes large in the large $\tan\beta$ region, where $\tan\beta$ is the ratio of vacuum expectation values of up and down type Higgs fields. This can enhance contributions from

bottom anti-bottom annihilation subprocesses.

While the theoretical predictions of NNLO [53–61] and next to next to leading log (NNLL) [64] QCD corrections and of two loop electroweak effects [65, 66] played an important role in the Higgs discovery, the theoretical uncertainties resulting from factorization and renormalization scales are not fully under control. Hence, the efforts to go beyond NNLO are going on intensively. Some of the ingredients to obtain N^{3}LO QCD corrections are already available. For example, quark and gluon form factors [40–42, 67, 68], the mass factorization kernels [69] and the renormalization constant [70] for the effective operator describing the coupling of Higgs boson with the SM fields in the infinite top quark mass limit up to three loop level in dimensional regularization are known for some time. In addition, NNLO soft contributions are known [71] to all orders in ϵ for both DY and Higgs productions using dimensional regularization with space time dimension being $d = 4 + \epsilon$. They were used to obtain the partial N^{3}LO threshold effects [28, 29, 72–74] to Drell-Yan production of di-leptons and inclusive productions of Higgs boson through gluon gluon fusion and in bottom anti-bottom annihilation. Threshold contribution to the inclusive production cross section is expanded in terms of $\delta(1-z)$ and $\mathcal{D}_i(z)$ where

$$\mathcal{D}_i(z) = \left(\frac{\ln^i(1-z)}{1-z}\right)_+ \tag{2.1.1}$$

with the scaling parameter $z = m_H^2/\hat{s}$ for Higgs and $z = m_{l^+l^-}^2/\hat{s}$ for DY. Here m_H, $m_{l^+l^-}$ and $\hat{s}$ are mass of the Higgs boson, invariant mass of the di-leptons and square of the center of mass energy of the partonic reaction responsible for production mechanism respectively. The missing $\delta(1-z)$ terms for the complete N^{3}LO threshold contributions to the Higgs production through gluon gluon fusion are now available due to the seminal work by Anastasiou et al [33] where the relevant soft contributions were obtained from the real radiations at N^{3}LO level. The resummation of threshold effects [75, 76] to infra-red safe observables resulting from their factorization properties as well as Sudakov resummation of soft gluons provides an elegant framework to obtain threshold enhanced

contributions to inclusive and semi inclusive observables order by order in perturbation theory. In [32], using this framework, we exploited the universal structure of the soft radiations to obtain the corresponding soft gluon contributions to DY production, which led to the evaluation of missing $\delta(1-z)$ part of the N^{3}LO threshold corrections. In [35], relevant one loop double real emissions from light-like Wilson lines were computed to obtain threshold corrections to Higgs as well as Drell-Yan productions up to N^{3}LO level providing an independent approach. In [34] the universality of soft gluon contributions near threshold and the results of [33] were used to obtain general expression of the hard-virtual coefficient which contributes to N^{3}LO threshold as well as threshold resummation at next-to-next-to-next-to-leading-logarithmic (N^{3}LL) accuracy for the production cross section of a colourless heavy particle at hadron colliders. For the Higgs production through $b\bar{b}$ annihilation, till date, only partial N^{3}LO threshold corrections are known [28–30] where again the framework of threshold resummation was used. In both [28,29] and [30], it was not possible to determine the $\delta(1-z)$ at N^{3}LO due to the lack of information on three loop finite part of bottom anti-bottom higgs form factor in QCD and the soft gluon radiation at N^{3}LO level. In [30], subleading corrections were also obtained through the method of Mellin moments. The recent results on Higgs form factor with bottom anti-bottom by Gehrmann and Kara [31] and on the universal soft distribution obtained for the Drell-Yan production [32] can now be used to obtain $\delta(1-z)$ part of the threshold N^{3}LO contribution. For the soft gluon radiations in the $b\bar{b}$ annihilation, the results from [32] can be used as they do not depend on the flavour of the incoming quark states. We have set bottom quark mass to be zero throughout except in the Yukawa coupling.

We begin by writing down the relevant interacting Lagrangian in Sec. 2.2. In the Sec. 2.3, we present the formalism of computing threshold QCD corrections to the cross-section and in Sec. 2.4, we present our results for threshold N^{3}LO QCD contributions to Higgs production through $b\bar{b}$ annihilation at hadron colliders and their numerical impact . The numerical impact of threshold enhanced N^{3}LO contributions is demonstrated for the LHC energy $\sqrt{s} = 14$ TeV by studying the stability of the perturbation theory under factor-

ization and renormalization scales. Finally we give a brief summary of our findings in Sec. 2.6.

2.2 The Effective Lagrangian

The interaction of bottom quarks and the scalar Higgs boson is given by the action

$$\mathcal{L}^b = \phi(x)O^b(x) \equiv -\frac{\lambda}{\sqrt{2}}\phi(x)\overline{\psi}_b(x)\psi_b(x) \tag{2.2.1}$$

where, $\psi_b(x)$ and $\phi(x)$ denote the bottom quark and scalar Higgs field, respectively. λ is the Yukawa coupling given by $\sqrt{2}m_b/v$, with the bottom quark mass m_b and the vacuum expectation value $v \approx 246$ GeV. In MSSM, for the pseudo-scalar Higgs boson, we need to replace $\lambda\phi(x)\overline{\psi}_b(x)\psi_b(x)$ by $\tilde{\lambda}\tilde{\phi}(x)\overline{\psi}_b(x)\gamma_5\psi_b(x)$ in the above equation. The MSSM couplings are given by

$$\tilde{\lambda} = \begin{cases} -\frac{\sqrt{2}m_b \sin\alpha}{v\cos\beta}\,, & \tilde{\phi} = h\,, \\ \frac{\sqrt{2}m_b \cos\alpha}{v\cos\beta}\,, & \tilde{\phi} = H\,, \\ \frac{\sqrt{2}m_b \tan\beta}{v}\,, & \tilde{\phi} = A \end{cases}$$

respectively. The angle α measures the mixing of weak and mass eigenstates of neutral Higgs bosons. We use VFS scheme throughout, hence except in the Yukawa coupling, m_b is taken to be zero like other light quarks in the theory.

2.3 Theoretical Framework for Threshold Corrections

The inclusive cross-section for the production of a colorless particle, namely, a Higgs boson through gluon fusion/bottom quark annihilation or a pair of leptons in the Drell-

Yan at the hadron colliders is computed using

$$\sigma^I(\tau, q^2) = \sigma^{I,(0)}(\mu_R^2) \sum_{i,j=q,\bar{q},g} \int_\tau^1 dx\ \Phi_{ij}(x, \mu_F^2)\ \Delta_{ij}^I\left(\frac{\tau}{x}, q^2, \mu_R^2, \mu_F^2\right) \tag{2.3.1}$$

with the partonic flux

$$\Phi_{ij}(x, \mu_F^2) = \int_x^1 \frac{dy}{y} f_i(y, \mu_F^2)\ f_j\left(\frac{x}{y}, \mu_F^2\right). \tag{2.3.2}$$

In the above expressions, $f_i(y, \mu_F^2)$ and $f_j\left(\frac{x}{y}, \mu_F^2\right)$ are the parton distribution functions (PDFs) of the initial state partons i and j with momentum fractions y and x/y, respectively. These are renormalized at the factorization scale μ_F. The dimensionless quantity $\Delta_{ij}^I\left(\frac{\tau}{x}, q^2, \mu_R^2, \mu_F^2\right)$ is called the coefficient function of the partonic cross section for the production of a colorless particle from partons i and j, computed after performing the UV renormalization at scale μ_R and mass factorization at μ_F. The quantity $\sigma^{I,(0)}$ is a pre-factor of the born level cross section. The variable τ is defined as q^2/s, where

$$q^2 = \begin{cases} m_H^2 & \text{for } I = H\,, \\ m_{l^+l^-}^2 & \text{for } I = DY\,. \end{cases} \tag{2.3.3}$$

m_H is the mass of the Higgs boson and $m_{l^+l^-}$is the invariant mass of the final state dilepton pair (l^+l^-), which can be $e^+e^-, \mu^+\mu^-, \tau^+\tau^-$, in the DY production. $\sqrt{s}$ and $\sqrt{\hat{s}}$ stand for the hadronic and partonic center of mass energy, respectively. Throughout this chapter, we denote $I = H$ for the productions of the Higgs boson through gluon (gg) fusion (Fig. 2.1) and bottom quark ($b\bar{b}$) annihilation (Fig. 2.2), whereas we write $I = DY$ for the production of a pair of leptons in the Drell-Yan (Fig. 2.3).

One of the *goals* of this chapter is to study the impact of the contributions arising from the soft gluons to the cross section of a colorless particle production at Hadron colliders. The infrared safe contributions from the soft gluons is obtained by adding the soft part of the

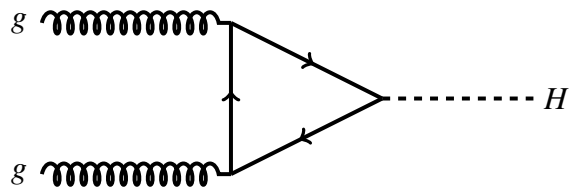


Figure 2.1. Higgs boson production in gluon fusion

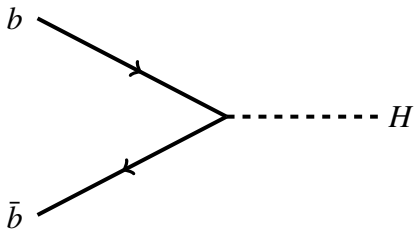


Figure 2.2. Higgs boson production through bottom quark annihilation

cross section with the UV renormalized virtual part and performing mass factorisation using appropriate counter terms. This combination is often called the soft-plus-virtual cross section whereas the remaining portion is known as hard part. Hence, we write the partonic cross section by decomposing into two parts as

$$\Delta_{ij}^{I}(z, q^2, \mu_R^2, \mu_F^2) = \Delta_{ij}^{I,\mathrm{SV}}(z, q^2, \mu_R^2, \mu_F^2) + \Delta_{ij}^{I,\mathrm{hard}}(z, q^2, \mu_R^2, \mu_F^2) \tag{2.3.4}$$

with $z \equiv q^2/\hat{s}$. The SV contributions $\Delta_{ij}^{I,\mathrm{SV}}(z, q^2, \mu_R^2, \mu_F^2)$ contains only the distributions of kind $\delta(1-z)$ and $\mathcal{D}_i$, where the latter one is defined through

$$\mathcal{D}_i \equiv \left[\frac{\ln^i(1-z)}{(1-z)}\right]_+ . \tag{2.3.5}$$

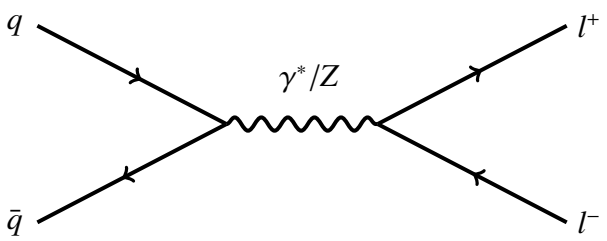


Figure 2.3. Drell-Yan pair production

This is also known as the threshold contributions. On the other hand, the hard part $\Delta_{ij}^{I,\text{hard}}$ contains all the regular terms in z. The SV cross section in z-space is computed in $d = 4+\epsilon$ dimensions, as formulated in [28, 29], using

$$\Delta_{ij}^{I,\text{SV}}(z, q^2, \mu_R^2, \mu_F^2) = C \exp\left(\Psi_{ij}^I\left(z, q^2, \mu_R^2, \mu_F^2, \epsilon\right)\right)\Big|_{\epsilon=0} \tag{2.3.6}$$

where, $\Psi_{ij}^I\left(z, q^2, \mu_R^2, \mu_F^2, \epsilon\right)$ is a finite distribution and C is the convolution defined as

$$Ce^{f(z)} = \delta(1-z) + \frac{1}{1!}f(z) + \frac{1}{2!}f(z) \otimes f(z) + \cdots . \tag{2.3.7}$$

Here $\otimes$ represents Mellin convolution and $f(z)$ is a distribution of the kind $\delta(1-z)$ and $\mathcal{D}_i$. The equivalent formalism of the SV approximation in the Mellin (or N-moment) space, where instead of distributions in z, the dominant contributions come from the meromorphic functions of the variable N (see [75, 76]) and the threshold limit of $z \to 1$ is translated to $N \to \infty$. The $\Psi_{ij}^I\left(z, q^2, \mu_R^2, \mu_F^2, \epsilon\right)$ is constructed from the form factors $\mathcal{F}_{ij}^I(\hat{a}_s, Q^2, \mu^2, \epsilon)$ with $Q^2 = -q^2$, the overall operator UV renormalization constant $Z_{ij}^I(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)$, the soft-collinear distribution $\Phi_{ij}^I(\hat{a}_s, q^2, \mu^2, z, \epsilon)$ arising from the real radiations in the partonic subprocesses and the mass factorization kernels $\Gamma_{ij}^I(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon)$. In terms of the above-mentioned quantities it takes the following form, as presented in [11, 29, 32]

$$\begin{aligned}\Psi_{ij}^I\left(z, q^2, \mu_R^2, \mu_F^2, \epsilon\right) = &\left(\ln\left[Z_{ij}^I(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)\right]^2 + \ln\left|\mathcal{F}_{ij}^I(\hat{a}_s, Q^2, \mu^2, \epsilon)\right|^2\right)\delta(1-z) \\ &+ 2\Phi_{ij}^I(\hat{a}_s, q^2, \mu^2, z, \epsilon) - 2C \ln \Gamma_{ij}^I(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon)\,.\end{aligned} \tag{2.3.8}$$

In this expression, $\hat{a}_s \equiv \hat{g}_s^2/16\pi^2$ is the unrenormalized strong coupling constant which is related to the renormalized one $a_s(\mu_R^2) \equiv a_s$ through the renormalization constant $Z_{a_s}(\mu_R^2) \equiv Z_{a_s}$ as

$$\hat{a}_s S_\epsilon = \left(\frac{\mu^2}{\mu_R^2}\right)^{\epsilon/2} Z_{a_s} a_s\,, \tag{2.3.9}$$

where, $S_\epsilon = \exp\left[(\gamma_E - \ln 4\pi)\epsilon/2\right]$ and μ is the mass scale introduced to keep the $\hat{a}_s$ dimensionless in d-dimensions. $\hat{g}_s$ is the coupling constant appearing in the bare Lagrangian of QCD. Z_{a_s} can be obtained by solving the underlying renormalisation group equation (RGE)

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z_{a_s} = \frac{1}{a_s} \sum_{k=0}^{\infty} a_s^{k+2} \beta_k \tag{2.3.10}$$

where, β_k's are the coefficients of the QCD β-function. The solution of the above RGE in terms of the β_k's and ϵ up to $O(a_s^4)$comes out to be

$$\begin{aligned} Z_{a_s} = 1 &+ a_s \left[\frac{2}{\epsilon}\beta_0\right] + a_s^2 \left[\frac{4}{\epsilon^2}\beta_0^2 + \frac{1}{\epsilon}\beta_1\right] + a_s^3 \left[\frac{8}{\epsilon^3}\beta_0^3 + \frac{14}{3\epsilon^2}\beta_0\beta_1 + \frac{2}{3\epsilon}\beta_2\right] \\ &+ a_s^4 \left[\frac{16}{\epsilon^4}\beta_0^4 + \frac{46}{3\epsilon^3}\beta_0^2\beta_1 + \frac{1}{\epsilon^2}\left(\frac{3}{2}\beta_1^2 + \frac{10}{3}\beta_0\beta_2\right) + \frac{1}{2\epsilon}\beta_3\right]. \end{aligned} \tag{2.3.11}$$

Results beyond this order involve β_4 and higher order β_k's which are not available yet in the literature. The β_k up to $k = 3$ are given by [77]

$$\begin{aligned} \beta_0 &= \frac{11}{3}C_A - \frac{2}{3}n_f\,, \\ \beta_1 &= \frac{34}{3}C_A^2 - 2n_f C_F - \frac{10}{3}n_f C_A\,, \\ \beta_2 &= \frac{2857}{54}C_A^3 - \frac{1415}{54}C_A^2 n_f + \frac{79}{54}C_A n_f^2 + \frac{11}{9}C_F n_f^2 - \frac{205}{18}C_F C_A n_f + C_F^2 n_f\,, \\ \beta_3 &= N^2\left(-\frac{40}{3} + 352\zeta_3\right) + N^4\left(-\frac{10}{27} + \frac{88}{9}\zeta_3\right) + n_f N\left(\frac{64}{9} - \frac{208}{3}\zeta_3\right) \\ &\quad + n_f N^3\left(\frac{32}{27} - \frac{104}{9}\zeta_3\right) + n_f^2 N^{-2}\left(-\frac{44}{3} + 32\zeta_3\right) + n_f^2\left(\frac{44}{9} - \frac{32}{3}\zeta_3\right) \\ &\quad + n_f^2 N^2\left(-\frac{22}{27} + \frac{16}{9}\zeta_3\right) + C_F n_f^3\left(\frac{154}{243}\right) + C_F^2 n_f^2\left(\frac{338}{27} - \frac{176}{9}\zeta_3\right) + C_F^3 n_f\left(23\right) \\ &\quad + C_A n_f^3\left(\frac{53}{243}\right) + C_A C_F n_f^2\left(\frac{4288}{243} + \frac{112}{9}\zeta_3\right) + C_A C_F^2 n_f\left(-\frac{2102}{27} + \frac{176}{9}\zeta_3\right) \\ &\quad + C_A^2 n_f^2\left(\frac{3965}{162} + \frac{56}{9}\zeta_3\right) + C_A^2 C_F n_f\left(\frac{7073}{486} - \frac{328}{9}\zeta_3\right) + C_A^3 n_f\left(-\frac{39143}{162} + \frac{68}{3}\zeta_3\right) \\ &\quad + C_A^4\left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) \end{aligned} \tag{2.3.12}$$

with the SU(N) quadratic casimirs

$$C_A = N, \qquad C_F = \frac{N^2-1}{2N}\,. \tag{2.3.13}$$

n_f is the number of active light quark flavors.

In this chapter, we will confine our discussion on the threshold corrections to the Higgs boson production through bottom quark annihilation and more precisely our main goal is to compute the SV cross section of this process at N^{3}LO QCD. In the subsequent sections, we will demonstrate the methodology to obtain the ingredients, Eq. (2.3.8) for computing the SV cross section of scalar Higgs boson production at N^{3}LO QCD.

2.3.1 The Form Factor

The quark and gluon form factors represent the QCD loop corrections to the transition matrix element from an on-shell quark-antiquark pair or two gluons to a color-neutral operator O. For the scalar Higgs boson production through $b\bar{b}$ annihilation, we consider Yukawa interaction, encapsulated through the operator O^b present in the interacting Lagrangian 2.2.1. For the process under consideration, we need to consider bottom quark form factors. The unrenormalised quark form factors at $O(\hat{a}_s^n)$ are defined through

$$\hat{\mathcal{F}}^{H,(n)}_{b\bar{b}} \equiv \frac{\langle \hat{\mathcal{M}}^{H,(0)}_{b\bar{b}} | \hat{\mathcal{M}}^{H,(n)}_{b\bar{b}} \rangle}{\langle \hat{\mathcal{M}}^{H,(0)}_{b\bar{b}} | \hat{\mathcal{M}}^{H,(0)}_{b\bar{b}} \rangle}\,, \tag{2.3.14}$$

where, $n = 0, 1, 2, 3, \ldots$. In the above expressions $|\hat{\mathcal{M}}^{H,(n)}_{b\bar{b}}\rangle$ is the $O(\hat{a}_s^n)$ contribution to the unrenormalised matrix element for the production of the Higgs boson from on-shell $b\bar{b}$ annihilation. In terms of these quantities, the full matrix element and the full form factors can be written as a series expansion in $\hat{a}_s$ as

$$|\mathcal{M}^H_{b\bar{b}}\rangle \equiv \sum_{n=0}^{\infty} \hat{a}_s^n S_\epsilon^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} |\hat{\mathcal{M}}^{H,(n)}_{b\bar{b}}\rangle\,, \qquad \mathcal{F}^H_{b\bar{b}} \equiv \sum_{n=0}^{\infty} \left[\hat{a}_s^n S_\epsilon^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} \hat{\mathcal{F}}^{H,(n)}_{b\bar{b}}\right], \tag{2.3.15}$$

where $Q^2 = -2\,p_1.p_2 = -q^2$ and p_i ($p_i^2 = 0$) are the momenta of the external on-shell bottom quarks. The results of the form factors up to two loop were present for a long time in [21, 78] and the three loop one was computed recently in [31].

The form factor $\mathcal{F}^H_{b\bar{b}}(\hat{a}_s, Q^2, \mu^2, \epsilon)$ satisfies the KG-differential equation [79–83] which is a direct consequence of the facts that QCD amplitudes exhibit factorisation property, gauge and renormalisation group (RG) invariances:

$$Q^2 \frac{d}{dQ^2} \ln \mathcal{F}^H_{b\bar{b}}(\hat{a}_s, Q^2, \mu^2, \epsilon) = \frac{1}{2}\left[K^H_{b\bar{b}}\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon\right) + G^H_{b\bar{b}}\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right)\right]. \tag{2.3.16}$$

In the above expression, all the poles in dimensional regularisation parameter ϵ are captured in the Q^2 independent function $K^H_{b\bar{b}}$ and the quantities which are finite as $\epsilon \to 0$ are encapsulated in $G^H_{b\bar{b}}$. The solution of the above KG equation can be obtained as [28] (see also [11, 32])

$$\ln \mathcal{F}^H_{b\bar{b}}(\hat{a}_s, Q^2, \mu^2, \epsilon) = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{Q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\mathcal{L}}^H_{b\bar{b},k}(\epsilon) \tag{2.3.17}$$

with

$$\begin{aligned}
\hat{\mathcal{L}}^H_{b\bar{b},1}(\epsilon) =& \frac{1}{\epsilon^2}\left\{-2A^H_{b\bar{b},1}\right\} + \frac{1}{\epsilon}\left\{G^H_{b\bar{b},1}(\epsilon)\right\},\\
\hat{\mathcal{L}}^H_{b\bar{b},2}(\epsilon) =& \frac{1}{\epsilon^3}\left\{\beta_0 A^H_{b\bar{b},1}\right\} + \frac{1}{\epsilon^2}\left\{-\frac{1}{2}A^H_{b\bar{b},2} - \beta_0 G^H_{b\bar{b},1}(\epsilon)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{2}G^H_{b\bar{b},2}(\epsilon)\right\},\\
\hat{\mathcal{L}}^H_{b\bar{b},3}(\epsilon) =& \frac{1}{\epsilon^4}\left\{-\frac{8}{9}\beta_0^2 A^H_{b\bar{b},1}\right\} + \frac{1}{\epsilon^3}\left\{\frac{2}{9}\beta_1 A^H_{b\bar{b},1} + \frac{8}{9}\beta_0 A^H_{b\bar{b},2} + \frac{4}{3}\beta_0^2 G^H_{b\bar{b},1}(\epsilon)\right\}\\
&+ \frac{1}{\epsilon^2}\left\{-\frac{2}{9}A^H_{b\bar{b},3} - \frac{1}{3}\beta_1 G^H_{b\bar{b},1}(\epsilon) - \frac{4}{3}\beta_0 G^H_{b\bar{b},2}(\epsilon)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{3}G^H_{b\bar{b},3}(\epsilon)\right\}.
\end{aligned} \tag{2.3.18}$$

In Appendix D, the derivation of the above solution is discussed in great details. $A^H_{b\bar{b}}$'s are called the cusp anomalous dimensions. The constants $G^H_{b\bar{b},i}$'s are the coefficients of a_s^i in

the following expansions:

$$G^H_{b\bar{b}}\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) = G^H_{b\bar{b}}\left(a_s(Q^2), 1, \epsilon\right) + \int_{\frac{Q^2}{\mu_R^2}}^1 \frac{dx}{x} A^H_{b\bar{b}}(a_s(x\mu_R^2))$$
$$= \sum_{i=1}^{\infty} a_s^i(Q^2) G^H_{b\bar{b},i}(\epsilon) + \int_{\frac{Q^2}{\mu_R^2}}^1 \frac{dx}{x} A^H_{b\bar{b}}(a_s(x\mu_R^2))\,. \tag{2.3.19}$$

However, the solutions of the logarithm of the form factor involves the unknown functions $G^H_{b\bar{b},i}$ which are observed to fulfill [40, 84] the following decomposition in terms of collinear ($B^H_{b\bar{b}}$), soft ($f^H_{b\bar{b}}$) and UV ($\gamma^H_{b\bar{b}}$) anomalous dimensions:

$$G^H_{b\bar{b},i}(\epsilon) = 2\left(B^H_{b\bar{b},i} - \gamma^H_{b\bar{b},i}\right) + f^H_{b\bar{b},i} + C^H_{b\bar{b},i} + \sum_{k=1}^{\infty} \epsilon^k g^{H,k}_{b\bar{b},i}\,, \tag{2.3.20}$$

where, the constants $C^H_{b\bar{b},i}$ are given by [29]

$$\begin{aligned} C^H_{b\bar{b},1} &= 0\,, \\ C^H_{b\bar{b},2} &= -2\beta_0 g^{H,1}_{b\bar{b},1}\,, \\ C^H_{b\bar{b},3} &= -2\beta_1 g^{H,1}_{b\bar{b},1} - 2\beta_0\left(g^{H,1}_{b\bar{b},2} + 2\beta_0 g^{H,2}_{b\bar{b},1}\right)\,. \end{aligned} \tag{2.3.21}$$

In the above expressions, $X^H_{b\bar{b},i}$ with $X = A, B, f$ and $\gamma^H_{b\bar{b},i}$ are defined through the series expansion in powers of a_s:

$$X^H_{b\bar{b}} \equiv \sum_{i=1}^{\infty} a_s^i X^H_{b\bar{b},i}\,, \qquad \text{and} \qquad \gamma^H_{b\bar{b}} \equiv \sum_{i=1}^{\infty} a_s^i \gamma^H_{b\bar{b},i}\,. \tag{2.3.22}$$

$f^I_{i\,\bar{i}}$ are introduced for the first time in the article [84] where it is shown to fulfill the maximally non-Abelian property up to two loop level whose validity is reconfirmed in [40] at three loop level:

$$f^H_{b\bar{b}} = \frac{C_F}{C_A} f^H_{gg}\,. \tag{2.3.23}$$

This identity implies the soft anomalous dimensions for the Higgs boson production in

bottom quark annihilation are related to the same appearing in the Higgs boson production in gluon fusion through a simple ratio of the quadratic casimirs of SU(N) gauge group. The same property is also obeyed by the cusp anomalous dimensions up to three loop level:

$$A^H_{b\bar{b}} = \frac{C_F}{C_A} A^H_{gg} \,. \tag{2.3.24}$$

It is not clear whether this nice property holds true beyond this order of perturbation theory. Moreover, due to universality of the quantities denoted by X, these are independent of the operators insertion. These are only dependent on the initial state partons of any process. Moreover, these are independent of the quark flavors. Hence, being a process of quark annihilation, we can make use of the existing results up to three loop which are employed in case of DY pair productions:

$$X^H_{b\bar{b}} = X^{DY}_{q\bar{q}} = X^I_{q\bar{q}} = X_{q\bar{q}} \,. \tag{2.3.25}$$

Here, q denotes the independence of the quantities on the quark flavors and absence of I represents the independence of the quantities on the nature of colorless particles. $f^H_{b\bar{b}}$ can be found in [40, 84], $A^H_{b\bar{b}}$ in [40, 69, 85, 86] and $B^H_{b\bar{b}}$ in [40, 85] up to three loop level. For readers' convenience we list them all up to three loop level in the Appendix B. Utilising the results of these known quantities and comparing the above expansions of $G^H_{b\bar{b},i}(\epsilon)$, Eq. (2.3.20), with the results of the logarithm of the form factors, we extract the relevant $g^{H,k}_{b\bar{b},i}$ and $\gamma^H_{b\bar{b},i}$'s up to three loop. For soft-virtual cross section at N^{3}LO we need $g^{H,1}_{b\bar{b},3}$ in addition to the quantities arising from one and two loop. The form factors for the Higgs boson production in $b\bar{b}$ annihilation up to two loop can be found in [21, 28, 29] and the three loop one is calculated very recently in the article [31]. These results are employed to extract the required $g^{H,k}_{b\bar{b},i}$'s using Eq. (2.3.17), (2.3.18) and (2.3.20). The relevant one

loop terms are found to be

$$g^{H,1}_{b\bar{b},1} = C_F\left\{-2+\zeta_2\right\}, \qquad g^{H,2}_{b\bar{b},1} = C_F\left\{2-\frac{7}{3}\zeta_3\right\}, \qquad g^{H,3}_{b\bar{b},1} = C_F\left\{-2+\frac{1}{4}\zeta_2+\frac{47}{80}\zeta_2^2\right\}, \tag{2.3.26}$$

the relevant two loop terms are

$$\begin{aligned}
g^{H,1}_{b\bar{b},2} &= C_F n_f\left\{\frac{616}{81}+\frac{10}{9}\zeta_2-\frac{8}{3}\zeta_3\right\}+C_F C_A\left\{-\frac{2122}{81}-\frac{103}{9}\zeta_2+\frac{88}{5}{\zeta_2}^2+\frac{152}{3}\zeta_3\right\}\\
&\quad+C_F^2\left\{8+32\zeta_2-\frac{88}{5}{\zeta_2}^2-60\zeta_3\right\},\\
g^{H,2}_{b\bar{b},2} &= C_F n_f\left\{\frac{7}{12}{\zeta_2}^2-\frac{55}{27}\zeta_2+\frac{130}{27}\zeta_3-\frac{3100}{243}\right\}+C_A C_F\left\{-\frac{365}{24}{\zeta_2}^2+\frac{89}{3}\zeta_2\zeta_3+\frac{1079}{54}\zeta_2\right.\\
&\quad\left.-\frac{2923}{27}\zeta_3-51\zeta_5+\frac{9142}{243}\right\}+C_F^2\left\{\frac{96}{5}{\zeta_2}^2-28\zeta_2\zeta_3-44\zeta_2+116\zeta_3+12\zeta_5-24\right\}
\end{aligned}$$

and the required three loop term is

$$\begin{aligned}
g^{H,1}_{b\bar{b},3} &= C_A^2 C_F\left\{-\frac{6152}{63}{\zeta_2}^3+\frac{2738}{9}{\zeta_2}^2+\frac{976}{9}\zeta_2\zeta_3-\frac{342263}{486}\zeta_2-\frac{1136}{3}{\zeta_3}^2+\frac{19582}{9}\zeta_3\right.\\
&\quad\left.+\frac{1228}{3}\zeta_5+\frac{4095263}{8748}\right\}+C_A C_F^2\left\{-\frac{15448}{105}{\zeta_2}^3-\frac{3634}{45}{\zeta_2}^2-\frac{2584}{3}\zeta_2\zeta_3+\frac{13357}{9}\zeta_2\right.\\
&\quad\left.+296\zeta_3^2-\frac{11570}{9}\zeta_3-\frac{1940}{3}\zeta_5-\frac{613}{3}\right\}+C_A C_F n_f\left\{-\frac{1064}{45}{\zeta_2}^2+\frac{392}{9}\zeta_2\zeta_3+\frac{44551}{243}\zeta_2\right.\\
&\quad\left.-\frac{41552}{81}\zeta_3-72\zeta_5-\frac{6119}{4374}\right\}+C_F^2 n_f\left\{\frac{772}{45}{\zeta_2}^2-\frac{152}{3}\zeta_2\zeta_3-\frac{3173}{18}\zeta_2+\frac{15956}{27}\zeta_3\right.\\
&\quad\left.-\frac{368}{3}\zeta_5+\frac{32899}{324}\right\}+C_F n_f^2\left\{-\frac{40}{9}{\zeta_2}^2-\frac{892}{81}\zeta_2+\frac{320}{81}\zeta_3-\frac{27352}{2187}\right\}\\
&\quad+C_F^3\left\{\frac{21584}{105}{\zeta_2}^3-\frac{1644}{5}{\zeta_2}^2+624\zeta_2\zeta_3-275\zeta_2+48\zeta_3^2-2142\zeta_3+1272\zeta_5+603\right\}.
\end{aligned} \tag{2.3.27}$$

The results up to two loop were present in the literature [28, 29], however the three loop result is the new one which is computed in this thesis for the first time. The other constants

$\gamma^H_{b\bar{b},i}$, appearing in the Eq. (2.3.20), up to three loop ($i = 3$) are obtained as

$$\begin{aligned}
\gamma^H_{b\bar{b},1} &= 3C_F\,,\\
\gamma^H_{b\bar{b},2} &= \frac{3}{2}C_F^2 + \frac{97}{6}C_FC_A - \frac{5}{3}C_Fn_f\,,\\
\gamma^H_{b\bar{b},3} &= \frac{129}{2}C_F^3 - \frac{129}{4}C_F^2C_A + \frac{11413}{108}C_FC_A^2 + \Big(-23 + 24\zeta_3\Big)C_F^2n_f\\
&\quad + \left(-\frac{278}{27} - 24\zeta_3\right)C_FC_An_f - \frac{35}{27}C_Fn_f^2\,.
\end{aligned} \tag{2.3.28}$$

These will be utilised in the next subsection to determine overall operator renormalisation constant.

2.3.2 Operator Renormalisation Constant

The strong coupling constant renormalisation through Z_{a_s} is not sufficient to make the form factor $\mathcal{F}^H_{b\bar{b}}$ completely UV finite, one needs to perform additional renormalisation to remove the residual UV divergences which is reflected through the presence of non-zero $\gamma^H_{b\bar{b}}$ in Eq. (2.3.20). This additional renormalisation is called the overall operator renormalisation which is performed through the constant $Z^H_{b\bar{b}}$. This is determined by solving the underlying RG equation:

$$\mu_R^2\frac{d}{d\mu_R^2}\ln Z^H_{b\bar{b}}\left(\hat{a}_s,\mu_R^2,\mu^2,\epsilon\right) = \sum_{i=1}^{\infty} a_s^i(\mu_R^2)\gamma^H_{b\bar{b},i}\,. \tag{2.3.29}$$

Using the results of $\gamma^H_{b\bar{b},i}$ from Eq. (2.3.28) and solving the above RG equation following the methodology described in the Appendix C, we obtain the following overall renormalisation constant up to three loop level:

$$Z^H_{b\bar{b}} = 1 + \sum_{k=1}^{\infty}\hat{a}_s^kS_\epsilon^k\left(\frac{\mu_R^2}{\mu^2}\right)^{k\frac{\epsilon}{2}}\hat{Z}^{H,(k)}_{b\bar{b}} \tag{2.3.30}$$

where,

$$
\begin{aligned}
\hat{Z}^{H,(1)}_{b\bar{b}} &= \frac{1}{\epsilon}\Bigg\{6C_F\Bigg\},\\
\hat{Z}^{H,(2)}_{b\bar{b}} &= \frac{1}{\epsilon^2}\Bigg\{-22C_FC_A+18C_F^2+4n_fC_F\Bigg\}+\frac{1}{\epsilon}\Bigg\{\frac{97}{6}C_FC_A+\frac{3}{2}C_F^2-\frac{5}{3}n_fC_F\Bigg\},\\
\hat{Z}^{H,(3)}_{b\bar{b}} &= \frac{1}{\epsilon^3}\Bigg\{\frac{968}{9}C_FC_A^2-132C_F^2C_A+36C_F^3-\frac{352}{9}n_fC_FC_A+24n_fC_F^2+\frac{32}{9}n_f^2C_F\Bigg\}\\
&\quad+\frac{1}{\epsilon^2}\Bigg\{-\frac{4880}{27}C_FC_A^2+\frac{247}{3}C_F^2C_A+9C_F^3+\frac{1396}{27}n_fC_FC_A-\frac{10}{3}n_fC_F^2-\frac{80}{27}n_f^2C_F\Bigg\}\\
&\quad+\frac{1}{\epsilon}\Bigg\{\frac{11413}{162}C_FC_A^2-\frac{43}{2}C_F^2C_A+43C_F^3-\frac{556}{81}n_fC_FC_A-\frac{46}{3}n_fC_F^2-\frac{70}{81}n_f^2C_F\\
&\quad-16\zeta_3n_fC_FC_A+16\zeta_3n_fC_F^2\Bigg\}. \qquad (2.3.31)
\end{aligned}
$$

2.3.3 Mass Factorisation Kernel

The UV finite form factor contains additional divergences arising from the soft and collinear regions of the loop momenta. In this section, we address the issue of collinear divergences and describe a prescription to remove them. The collinear singularities that arise in the massless limit of partons are removed by absorbing the divergences in the bare PDF through renormalisation of the PDF. This prescription is called the mass factorisation (MF) and is performed at the factorisation scale μ_F. In the process of performing this, one needs to introduce mass factorisation kernels $\Gamma^I_{ij}(\hat{a}_s,\mu^2,\mu_F^2,z,\epsilon)$ which essentially absorb the collinear singularities. More specifically, MF removes the collinear singularities arising from the collinear configuration associated with the initial state partons. The final state collinear singularities are guaranteed to go away once the phase space integrals are performed after summing over the contributions from virtual and real emission diagrams, thanks to Kinoshita-Lee-Nauenberg (KLN) theorem. The kernels satisfy the following RG equation :

$$
\mu_F^2\frac{d}{d\mu_F^2}\Gamma^I_{ij}(z,\mu_F^2,\epsilon)=\frac{1}{2}\sum_k P^I_{ik}\left(z,\mu_F^2\right)\otimes\Gamma^I_{kj}\left(z,\mu_F^2,\epsilon\right) \qquad (2.3.32)
$$

where, $P^I\left(z,\mu_F^2\right)$ are Altarelli-Parisi splitting functions (matrix valued). Expanding $P^I\left(z,\mu_F^2\right)$ and $\Gamma^I(z,\mu_F^2,\epsilon)$ in powers of the strong coupling constant we get

$$P_{ij}^I(z,\mu_F^2) = \sum_{k=1}^{\infty} a_s^k(\mu_F^2) P_{ij}^{I,(k-1)}(z) \tag{2.3.33}$$

and

$$\Gamma_{ij}^I(z,\mu_F^2,\epsilon) = \delta_{ij}\delta(1-z) + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\mu_F^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Gamma}_{ij}^{I,(k)}(z,\epsilon)\,. \tag{2.3.34}$$

The RG equation of $\Gamma^I(z,\mu_F^2,\epsilon)$, Eq. (2.3.32), can be solved in dimensional regularisation in powers of $\hat{a}_s$. In the $\overline{MS}$ scheme, the kernel contains only the poles in ϵ. The solutions [28] up to the required order $\Gamma^{I,(3)}(z,\epsilon)$ in terms of $P^{I,(k)}(z)$ are presented in the Appendix (C.0.20). The relevant ones up to three loop, $P^{I,(0)}(z)$, $P^{I,(1)}(z)$ and $P^{I,(2)}(z)$ are computed in the articles [69, 85]. For the SV cross section only the diagonal parts of the splitting functions $P_{ij}^{I,(k)}(z)$ and kernels $\Gamma_{ij}^{I,(k)}(z,\epsilon)$ contribute since the diagonal elements of $P_{ij}^{I,(k)}(z)$ contain $\delta(1-z)$ and $\mathcal{D}_0$ whereas the off-diagonal elements are regular in the limit $z \to 1$. The most remarkable fact is that these quantities are universal, independent of the operators insertion. Hence, for the process under consideration, we make use of the existing process independent results of kernels and splitting functions:

$$\Gamma_{ij}^H = \Gamma_{ij}^I = \Gamma_{ij} \qquad \text{and} \qquad P_{ij}^H = P_{ij}^I = P_{ij}\,. \tag{2.3.35}$$

The absence of I represents the independence of these quantities on I. In the next subsection, we discuss the only remaining ingredient, namely, the soft-collinear distribution.

2.3.4 Soft-Collinear Distribution

The resulting expression from form factor along with the operator renormalisation constant and mass factorisation kernel is not completely finite, it contains some residual di-

vergences which get cancelled against the contribution arising from soft gluon emissions. Hence, the finiteness of $\Delta^{H,\mathrm{SV}}_{b\bar{b}}$ in Eq. (2.3.6) in the limit $\epsilon \to 0$ demands that the soft-collinear distribution, $\Phi^H_{b\bar{b}}(\hat{a}_s, q^2, \mu^2, z, \epsilon)$, has pole structure in ϵ similar to that of residual divergences. In articles [28] and [29], it was shown that $\Phi^H_{b\bar{b}}$ must obey KG type integro-differential equation, which we call $\overline{KG}$ equation, to remove that residual divergences:

$$q^2 \frac{d}{dq^2} \Phi^H_{b\bar{b}} \left(\hat{a}_s, q^2, \mu^2, z, \epsilon\right) = \frac{1}{2} \left[\overline{K}^H_{b\bar{b}} \left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right) + \overline{G}^H_{b\bar{b}} \left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right) \right] . \tag{2.3.36}$$

$\overline{K}^H_{b\bar{b}}$ and $\overline{G}^H_{b\bar{b}}$ play similar roles as those of $K^H_{b\bar{b}}$ and $G^H_{b\bar{b}}$ in Eq. (2.3.16), respectively. Also, $\Phi^H_{b\bar{b}}(\hat{a}_s, q^2, \mu^2, z, \epsilon)$ being independent of μ_R^2 satisfy the RG equation

$$\mu_R^2 \frac{d}{d\mu_R^2} \Phi^H_{b\bar{b}}(\hat{a}_s, q^2, \mu^2, z, \epsilon) = 0 . \tag{2.3.37}$$

This RG invariance and the demand of cancellation of all the residual divergences arising from $\mathcal{F}^H_{b\bar{b}}, Z^H_{b\bar{b}}$ and $\Gamma^H_{b\bar{b}}$ against $\Phi^H_{b\bar{b}}$ implies the solution of the $\overline{KG}$ equation as [28, 29]

$$\begin{aligned} \Phi^H_{b\bar{b}}(\hat{a}_s, q^2, \mu^2, z, \epsilon) &= \Phi^H_{b\bar{b}}(\hat{a}_s, q^2(1-z)^2, \mu^2, \epsilon) \\ &= \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S^{\,i}_{\epsilon}\left(\frac{i\epsilon}{1-z}\right) \hat{\Phi}^H_{b\bar{b},i}(\epsilon) \end{aligned} \tag{2.3.38}$$

with

$$\hat{\Phi}^H_{b\bar{b},i}(\epsilon) = \hat{\mathcal{L}}^H_{b\bar{b},i}(\epsilon) \left(A^H_{b\bar{b},j} \to -A^H_{b\bar{b},j}, G^H_{b\bar{b},j}(\epsilon) \to \overline{\mathcal{G}}^H_{b\bar{b},j}(\epsilon) \right) \tag{2.3.39}$$

where, $\hat{\mathcal{L}}^H_{b\bar{b},i}(\epsilon)$ are defined in Eq. (2.3.18). In Appendix E, the derivation of this solution is depicted in great details. The z-independent constants $\overline{\mathcal{G}}^H_{b\bar{b},i}(\epsilon)$ can be obtained by comparing the poles as well as non-pole terms in ϵ of $\hat{\Phi}^H_{b\bar{b},i}(\epsilon)$ with those arising from form

factor, overall renormalisation constant and splitting functions. We find

$$\overline{\mathcal{G}}^{H}_{b\bar{b},i}(\epsilon) = -f^{H}_{b\bar{b},i} + \overline{C}^{H}_{b\bar{b},i} + \sum_{k=1}^{\infty} \epsilon^{k} \overline{\mathcal{G}}^{H,k}_{b\bar{b},i}\,, \tag{2.3.40}$$

where,

$$\begin{aligned}
\overline{C}^{H}_{b\bar{b},1} &= 0\,,\\
\overline{C}^{H}_{b\bar{b},2} &= -2\beta_0 \overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\,,\\
\overline{C}^{H}_{b\bar{b},3} &= -2\beta_1 \overline{\mathcal{G}}^{H,1}_{b\bar{b},1} - 2\beta_0 \left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},2} + 2\beta_0 \overline{\mathcal{G}}^{H,2}_{b\bar{b},1}\right).
\end{aligned} \tag{2.3.41}$$

However, due to the universality of the soft gluon contribution, $\Phi^{H}_{b\bar{b}}$ must be the same as that of the DY pair production in quark annihilation since this quantity only depends on the initial state partons, it does not depend on the final state colorless particle:

$$\begin{aligned}
\Phi^{H}_{b\bar{b}} &= \Phi^{DY}_{q\bar{q}} = \Phi^{I}_{q\bar{q}}\\
\text{i.e. } \overline{\mathcal{G}}^{H,k}_{b\bar{b},i} &= \overline{\mathcal{G}}^{DY,k}_{q\bar{q},i} = \overline{\mathcal{G}}^{I,k}_{q\bar{q},i}\,.
\end{aligned} \tag{2.3.42}$$

In the above expression, $\Phi^{I}_{q\bar{q}}$ and $\overline{\mathcal{G}}^{I,k}_{q\bar{q},i}$ are written in order to emphasise the universality of these quantities i.e. $\Phi^{I}_{q\bar{q}}$ and $\overline{\mathcal{G}}^{I,k}_{q\bar{q},i}$ can be used for any quark annihilation process, these are independent of the operators insertion. In the beginning, it was observed in [28, 29] that these quantities satisfy the maximally non-Abelian property up to $O(a_s^2)$:

$$\Phi^{I}_{q\bar{q}} = \frac{C_F}{C_A}\Phi^{I}_{gg} \qquad \text{and} \qquad \overline{\mathcal{G}}^{I,k}_{q\bar{q},i} = \frac{C_F}{C_A}\overline{\mathcal{G}}^{I,k}_{gg,i}\,. \tag{2.3.43}$$

Some of the relevant constants, namely, $\overline{\mathcal{G}}^{I,1}_{q\bar{q},1}, \overline{\mathcal{G}}^{I,2}_{q\bar{q},1}, \overline{\mathcal{G}}^{I,1}_{q\bar{q},2}$ are computed [28, 29] from the results of the explicit computations of soft gluon emissions to the DY productions. However, to calculate the SV cross section at N^{3}LO, we need to have the results of $\overline{\mathcal{G}}^{I,3}_{q\bar{q},1}, \overline{\mathcal{G}}^{I,2}_{q\bar{q},2}$. These are obtained by employing the above symmetry (2.3.43). In [71], the soft corrections to the production cross section of the Higgs boson through gluon fusion to $O(a_s^2)$

was computed to all orders in dimensional regularisation parameter ϵ. Utilising this all order result, we extract $\overline{\mathcal{G}}^{H,3}_{gg,1}$ and $\overline{\mathcal{G}}^{H,2}_{gg,2}$. These essentially lead us to obtain the corresponding quantities for DY production by means of the maximally non-Abelian symmetry. The third order constant $\overline{\mathcal{G}}^{H,1}_{gg,3}$ is extracted from the result of SV cross section for the production of the Higgs boson at N^{3}LO [33]. We conjecture that the symmetry relation (2.3.43) holds true even at the three loop level! Therefore, utilising that property we obtain the corresponding quantity for the DY production, $\overline{\mathcal{G}}^{DY,1}_{q\bar{q},3}$ which was presented for the first time in the article [32]. Later the result was reconfirmed through threshold resummation in [34] and explicit computations in [35]. This, in turn, establishes our conjecture of maximally non-Abelian property at N^{3}LO. Being flavor independent, we can employ all these constants to the problem under consideration. Below, we list the relevant ones that contribute up to N^{3}LO level:

$$
\begin{aligned}
\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} &= C_F\left\{-3\zeta_2\right\},\\
\overline{\mathcal{G}}^{H,2}_{b\bar{b},1} &= C_F\left\{\frac{7}{3}\zeta_3\right\},\\
\overline{\mathcal{G}}^{H,3}_{b\bar{b},1} &= C_F\left\{-\frac{3}{16}{\zeta_2}^2\right\},\\
\overline{\mathcal{G}}^{H,1}_{b\bar{b},2} &= C_F n_f\left\{-\frac{328}{81}+\frac{70}{9}\zeta_2+\frac{32}{3}\zeta_3\right\}+C_A C_F\left\{\frac{2428}{81}-\frac{469}{9}\zeta_2+4{\zeta_2}^2-\frac{176}{3}\zeta_3\right\},\\
\overline{\mathcal{G}}^{H,2}_{b\bar{b},2} &= C_A C_F\left\{\frac{11}{40}{\zeta_2}^2-\frac{203}{3}\zeta_2\zeta_3+\frac{1414}{27}\zeta_2+\frac{2077}{27}\zeta_3+43\zeta_5-\frac{7288}{243}\right\}\\
&\quad+C_F n_f\left\{-\frac{1}{20}{\zeta_2}^2-\frac{196}{27}\zeta_2-\frac{310}{27}\zeta_3+\frac{976}{243}\right\}\\
\overline{\mathcal{G}}^{H,1}_{b\bar{b},3} &= C_F {C_A}^2\left\{\frac{152}{63}\,{\zeta_2}^3+\frac{1964}{9}\,{\zeta_2}^2+\frac{11000}{9}\,\zeta_2\zeta_3-\frac{765127}{486}\,\zeta_2+\frac{536}{3}\,{\zeta_3}^2-\frac{59648}{27}\,\zeta_3\right.\\
&\quad\left.-\frac{1430}{3}\,\zeta_5+\frac{7135981}{8748}\right\}+C_F C_A n_f\left\{-\frac{532}{9}\,{\zeta_2}^2-\frac{1208}{9}\,\zeta_2\zeta_3+\frac{105059}{243}\,\zeta_2\right.\\
&\quad\left.+\frac{45956}{81}\,\zeta_3+\frac{148}{3}\,\zeta_5-\frac{716509}{4374}\right\}+C_F^2 n_f\left\{\frac{152}{15}\,{\zeta_2}^2-88\,\zeta_2\zeta_3+\frac{605}{6}\,\zeta_2+\frac{2536}{27}\,\zeta_3\right.\\
&\quad\left.+\frac{112}{3}\,\zeta_5-\frac{42727}{324}\right\}+C_F {n_f}^2\left\{\frac{32}{9}\,{\zeta_2}^2-\frac{1996}{81}\,\zeta_2-\frac{2720}{81}\,\zeta_3+\frac{11584}{2187}\right\}. \qquad (2.3.44)
\end{aligned}
$$

The above $\overline{\mathcal{G}}^{H,k}_{b\bar{b},i}$ enable us to get the $\Phi^H_{b\bar{b}}$ up to three loop level. This completes all the ingredients required to compute the SV cross section up to N^{3}LO that are presented in the next section.

2.4 Results of the SV Cross Sections

In this section, we present our findings of the SV cross section at N^{3}LO along with the results of previous orders. Expanding the SV cross section $\Delta^{H,\mathrm{SV}}_{b\bar{b}}$, Eq. (2.3.6), in powers of $a_s(\mu_F^2)$, we obtain

$$\Delta^{H,\mathrm{SV}}_{b\bar{b}}(z,q^2,\mu_F^2) = \sum_{i=1}^{\infty} a_s^i(\mu_F^2)\Delta^{H,\mathrm{SV}}_{b\bar{b},i}(z,q^2,\mu_F^2) \tag{2.4.1}$$

where,

$$\Delta^{H,\mathrm{SV}}_{b\bar{b},i} = \Delta^{H,\mathrm{SV}}_{b\bar{b},i}|_{\delta}\delta(1-z) + \sum_{j=0}^{2i-1}\Delta^{H,\mathrm{SV}}_{b\bar{b},i}|_{\mathcal{D}_j}\mathcal{D}_j\,.$$

Before presenting the final result, we present the general results of the SV cross section in terms of the anomalous dimensions $A^H_{b\bar{b}}$, $B^H_{b\bar{b}}$, $f^H_{b\bar{b}}$, $\gamma^H_{b\bar{b}}$ and other quantities arising from form factor and soft-collinear distribution below:

$$\begin{aligned}
\Delta^{H,\mathrm{SV}}_{b\bar{b},1} &= \delta(1-z)\left[\left\{2\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} + 2G^{H,1}_{b\bar{b},1}\right\} + \zeta_2\left\{3A^H_{b\bar{b},1}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)\left\{2B^H_{b\bar{b},1} - 2\gamma^H_{b\bar{b},1}\right\}\right] \\
&\quad + \mathcal{D}_0\left[\log\left(\frac{q^2}{\mu_F^2}\right)\left\{2A^H_{b\bar{b},1}\right\} + \left\{-2f^H_{b\bar{b},1}\right\}\right] + \mathcal{D}_1\left[\left\{4A^H_{b\bar{b},1}\right\}\right], \\
\Delta^{H,\mathrm{SV}}_{b\bar{b},2} &= \delta(1-z)\left[\left\{\overline{\mathcal{G}}^{H,1}_{b\bar{b},2} + 2\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2 + g^{H,1}_{b\bar{b},2} + 4g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} + 2\left(g^{H,1}_{b\bar{b},1}\right)^2\right\}\right. \\
&\quad + \beta_0\left\{2\overline{\mathcal{G}}^{H,2}_{b\bar{b},1} + 2g^{H,2}_{b\bar{b},1}\right\} + \zeta_3\left\{-8A^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\} + \zeta_2\left\{-2\left(f^H_{b\bar{b},1}\right)^2 + 3A^H_{b\bar{b},2} + 6\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}\right. \\
&\quad \left.+ 6g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}\right\} + \zeta_2\beta_0\left\{3f^H_{b\bar{b},1} + 6B^H_{b\bar{b},1} - 6\gamma^H_{b\bar{b},1}\right\} + \zeta_2^2\left\{\frac{37}{10}\left(A^H_{b\bar{b},1}\right)^2\right\} \\
&\quad + \log\left(\frac{q^2}{\mu_F^2}\right)\left\{2B^H_{b\bar{b},2} + 4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1} + 4g^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1} - 2\gamma^H_{b\bar{b},2} - 4\gamma^H_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} - 4\gamma^H_{b\bar{b},1}g^{H,1}_{b\bar{b},1}\right\}
\end{aligned}$$

$$
\begin{aligned}
&+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-2\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}-2g^{H,1}_{b\bar{b},1}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\left\{8\left(A^H_{b\bar{b},1}\right)^2\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\left\{4A^H_{b\bar{b},1}f^H_{b\bar{b},1}+6A^H_{b\bar{b},1}B^H_{b\bar{b},1}-6\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0\left\{-3A^H_{b\bar{b},1}\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)\left\{+2\left(B^H_{b\bar{b},1}\right)^2-4\gamma^H_{b\bar{b},1}B^H_{b\bar{b},1}+2\left(\gamma^m m a^H H_{b\bar{b},1}\right)^2\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-B^H_{b\bar{b},1}+\gamma^H_{b\bar{b},1}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\left\{-2\left(A^H_{b\bar{b},1}\right)^2\right\}\Bigg]\\
&+\mathcal{D}_0\Bigg[\beta_0\left\{-4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right\}+\zeta_3\left\{16\left(A^H_{b\bar{b},1}\right)^2\right\}+\zeta_2\left\{2A^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\left\{-4B^H_{b\bar{b},1}f^H_{b\bar{b},1}+2A^H_{b\bar{b},2}+4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}+4g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}+4\gamma^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{2f^H_{b\bar{b},1}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\left\{-2\left(A^H_{b\bar{b},1}\right)^2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\Big\{4A^H_{b\bar{b},1}B^H_{b\bar{b},1}\\
&-4\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-A^H_{b\bar{b},1}\right\}+\left\{-2f^H_{b\bar{b},2}-4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}f^H_{b\bar{b},1}-4g^{H,1}_{b\bar{b},1}f^H_{b\bar{b},1}\right\}\Bigg]\\
&+\mathcal{D}_1\Bigg[\beta_0\left\{4f^H_{b\bar{b},1}\right\}+\zeta_2\left\{-4\left(A^H_{b\bar{b},1}\right)^2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\Big\{-8A^H_{b\bar{b},1}f^H_{b\bar{b},1}+8A^H_{b\bar{b},1}B^H_{b\bar{b},1}\\
&-8\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-4A^H_{b\bar{b},1}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\left\{4\left(A^H_{b\bar{b},1}\right)^2\right\}+\Big\{4\left(f^H_{b\bar{b},1}\right)^2\\
&+4A^H_{b\bar{b},2}+8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}+8g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}\Big\}\Bigg]+\mathcal{D}_2\Bigg[\beta_0\left\{-4A^H_{b\bar{b},1}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\left\{12\left(A^H_{b\bar{b},1}\right)^2\right\}+\left\{-12A^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\}\Bigg]+\mathcal{D}_3\left[\left\{8\left(A^H_{b\bar{b},1}\right)^2\right\}\right],\\
\Delta^{H,\mathrm{SV}}_{b\bar{b},3}&=\delta(1-z)\Bigg[\left\{\frac{2}{3}\overline{\mathcal{G}}^{H,1}_{b\bar{b},3}+2\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},2}+\frac{4}{3}\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^3+\frac{2}{3}g^{H,1}_{b\bar{b},3}+2g^{H,1}_{b\bar{b},2}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\\
&+2g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},2}+4g^{H,1}_{b\bar{b},1}\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2+2g^{H,1}_{b\bar{b},1}g^{H,1}_{b\bar{b},2}+4\left(g^{H,1}_{b\bar{b},1}\right)^2\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}+\frac{4}{3}\left(g^{H,1}_{b\bar{b},1}\right)^3\Bigg\}\\
&+\beta_1\left\{\frac{4}{3}\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}+\frac{4}{3}g^{H,2}_{b\bar{b},1}\right\}+\beta_0\left\{\frac{4}{3}\overline{\mathcal{G}}^{H,2}_{b\bar{b},2}+4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}+\frac{4}{3}g^{H,2}_{b\bar{b},2}+4g^{H,2}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right.\\
&\left.+4g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}+4g^{H,1}_{b\bar{b},1}g^{H,2}_{b\bar{b},1}\right\}+\beta_0^2\left\{\frac{8}{3}\overline{\mathcal{G}}^{H,3}_{b\bar{b},1}+\frac{8}{3}g^{H,3}_{b\bar{b},1}\right\}+\zeta_5\left\{-96\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}\right\}\\
&+\zeta_5\beta_0\left\{-64\left(A^H_{b\bar{b},1}\right)^2\right\}+\zeta_3\left\{-\frac{8}{3}\left(f^H_{b\bar{b},1}\right)^3-8A^H_{b\bar{b},2}f^H_{b\bar{b},1}-8A^H_{b\bar{b},1}f^H_{b\bar{b},2}\right.\\
&\left.-16\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}-16g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\}+\zeta_3\beta_0\left\{-8\left(f^H_{b\bar{b},1}\right)^2-16\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}\right\}\\
&+\zeta_3^2\left\{\frac{160}{3}\left(A^H_{b\bar{b},1}\right)^3\right\}+\zeta_2\left\{-4f^H_{b\bar{b},1}f^H_{b\bar{b},2}+3A^H_{b\bar{b},3}+3\overline{\mathcal{G}}^{H,1}_{b\bar{b},2}A^H_{b\bar{b},1}-4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2\right.
\end{aligned}
$$

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\begin{aligned}
&+ 6\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},2} + 6\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2 A^H_{b\bar{b},1} + 3g^{H,1}_{b\bar{b},2}A^H_{b\bar{b},1} - 4g^{H,1}_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2 + 6g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},2} \\
&+ 12g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1} + 6\left(g^{H,1}_{b\bar{b},1}\right)^2 A^H_{b\bar{b},1}\Big\} + \zeta_2\beta_1\Big\{3f^H_{b\bar{b},1} + 6B^H_{b\bar{b},1} - 6\gamma^H_{b\bar{b},1}\Big\} \\
&+ \zeta_2\beta_0\Big\{6f^H_{b\bar{b},2} + 12B^H_{b\bar{b},2} + 6\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}A^H_{b\bar{b},1} - 2\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}f^H_{b\bar{b},1} + 12\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1} + 6g^{H,2}_{b\bar{b},1}A^H_{b\bar{b},1} \\
&+ 6g^{H,1}_{b\bar{b},1}f^H_{b\bar{b},1} + 12g^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1} - 12\gamma^H_{b\bar{b},2} - 12\gamma^H_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} - 12\gamma^H_{b\bar{b},1}g^{H,1}_{b\bar{b},1}\Big\} \\
&+ \zeta_2\beta_0^2\Big\{-12g^{H,1}_{b\bar{b},1}\Big\} + \zeta_2\zeta_3\Big\{40\left(A^H_{b\bar{b},1}\right)^2 f^H_{b\bar{b},1}\Big\} + \zeta_2\zeta_3\beta_0\Big\{32\left(A^H_{b\bar{b},1}\right)^2\Big\} \\
&+ \zeta_2^2\Big\{-\frac{38}{5}A^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2 + \frac{37}{5}A^H_{b\bar{b},1}A^H_{b\bar{b},2} + \frac{37}{5}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2 + \frac{37}{5}g^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\Big\} \\
&+ \zeta_2^2\beta_0\Big\{A^H_{b\bar{b},1}f^H_{b\bar{b},1} + 18A^H_{b\bar{b},1}B^H_{b\bar{b},1} - 18\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\Big\} + \zeta_2^2\beta_0^2\Big\{-3A^H_{b\bar{b},1}\Big\} \\
&+ \zeta_2^3\Big\{-\frac{283}{42}\left(A^H_{b\bar{b},1}\right)^3\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{2B^H_{b\bar{b},3} + 2\overline{\mathcal{G}}^{H,1}_{b\bar{b},2}B^H_{b\bar{b},1} + 4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}B^H_{b\bar{b},2} \\
&+ 4\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2 B^H_{b\bar{b},1} + 2g^{H,1}_{b\bar{b},2}B^H_{b\bar{b},1} + 4g^{H,1}_{b\bar{b},1}B^H_{b\bar{b},2} + 8g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1} + 4\left(g^{H,1}_{b\bar{b},1}\right)^2 B^H_{b\bar{b},1} \\
&- 2\gamma^H_{b\bar{b},3} - 4\gamma^H_{b\bar{b},2}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} - 4\gamma^H_{b\bar{b},2}g^{H,1}_{b\bar{b},1} - 2\gamma^H_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},2} - 4\gamma^H_{b\bar{b},1}\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2 - 2\gamma^H_{b\bar{b},1}g^{H,1}_{b\bar{b},2} \\
&- 8\gamma^H_{b\bar{b},1}g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} - 4\gamma^H_{b\bar{b},1}\left(g^{H,1}_{b\bar{b},1}\right)^2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\beta_1\Big\{-2\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} - 2g^{H,1}_{b\bar{b},1}\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\beta_0\Big\{-2\overline{\mathcal{G}}^{H,1}_{b\bar{b},2} + 4\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}B^H_{b\bar{b},1} - 4\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2 - 2g^{H,1}_{b\bar{b},2} + 4g^{H,2}_{b\bar{b},1}B^H_{b\bar{b},1} \\
&- 8g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1} - 4\left(g^{H,1}_{b\bar{b},1}\right)^2 - 4\gamma^H_{b\bar{b},1}\overline{\mathcal{G}}^{H,2}_{b\bar{b},1} - 4\gamma^H_{b\bar{b},1}g^{H,2}_{b\bar{b},1}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\Big\{-4\overline{\mathcal{G}}^{H,2}_{b\bar{b},1} \\
&- 4g^{H,2}_{b\bar{b},1}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_5\Big\{+96\left(A^H_{b\bar{b},1}\right)^3\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\Big\{8A^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2 \\
&- 16A^H_{b\bar{b},1}B^H_{b\bar{b},1}f^H_{b\bar{b},1} + 16A^H_{b\bar{b},1}A^H_{b\bar{b},2} + 16\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2 + 16g^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2 \\
&+ 16\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\beta_0\Big\{24A^H_{b\bar{b},1}f^H_{b\bar{b},1}\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\Big\{-4B^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2 + 4A^H_{b\bar{b},2}f^H_{b\bar{b},1} + 6A^H_{b\bar{b},2}B^H_{b\bar{b},1} + 4A^H_{b\bar{b},1}f^H_{b\bar{b},2} \\
&+ 6A^H_{b\bar{b},1}B^H_{b\bar{b},2} + 8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1} + 12\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}B^H_{b\bar{b},1} + 8g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1} \\
&+ 12g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}B^H_{b\bar{b},1} - 6\gamma^H_{b\bar{b},2}A^H_{b\bar{b},1} + 4\gamma^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2 - 6\gamma^H_{b\bar{b},1}A^H_{b\bar{b},2} \\
&- 12\gamma^H_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1} - 12\gamma^H_{b\bar{b},1}g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_1\Big\{-3A^H_{b\bar{b},1}\Big\}
\end{aligned}
$$

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\begin{aligned}
&+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0\Big\{4\left(f^H_{b\bar{b},1}\right)^2+6B^H_{b\bar{b},1}f^H_{b\bar{b},1}+12\left(B^H_{b\bar{b},1}\right)^2-6A^H_{b\bar{b},2}-4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}\\
&-12g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}-6\gamma^H_{b\bar{b},1}f^H_{b\bar{b},1}-24\gamma^H_{b\bar{b},1}B^H_{b\bar{b},1}+12\left(\gamma^H_{b\bar{b},1}\right)^2\Big\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0^2\Big\{-6f^H_{b\bar{b},1}-12B^H_{b\bar{b},1}+12\gamma^H_{b\bar{b},1}\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\zeta_3\Big\{-40\left(A^H_{b\bar{b},1}\right)^3\Big\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2^2\Big\{+\frac{76}{5}\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}+\frac{37}{5}\left(A^H_{b\bar{b},1}\right)^2B^H_{b\bar{b},1}-\frac{37}{5}\gamma^H_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\Big\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2^2\beta_0\Big\{-\left(A^H_{b\bar{b},1}\right)^2\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\Big\{4B^H_{b\bar{b},1}B^H_{b\bar{b},2}+4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(B^H_{b\bar{b},1}\right)^2\\
&+4g^{H,1}_{b\bar{b},1}\left(B^H_{b\bar{b},1}\right)^2-4\gamma^H_{b\bar{b},2}B^H_{b\bar{b},1}-4\gamma^H_{b\bar{b},1}B^H_{b\bar{b},2}-8\gamma^H_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1}-8\gamma^H_{b\bar{b},1}g^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1}\\
&+4\gamma^H_{b\bar{b},1}\gamma^H_{b\bar{b},2}+4\left(\gamma^H_{b\bar{b},1}\right)^2\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}+4\left(\gamma^H_{b\bar{b},1}\right)^2g^{H,1}_{b\bar{b},1}\Big\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_1\Big\{-B^H_{b\bar{b},1}+\gamma^H_{b\bar{b},1}\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0\Big\{-2B^H_{b\bar{b},2}-6\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1}-6g^{H,1}_{b\bar{b},1}B^H_{b\bar{b},1}\\
&+2\gamma^H_{b\bar{b},2}+6\gamma^H_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}+6\gamma^H_{b\bar{b},1}g^{H,1}_{b\bar{b},1}\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\Big\{+2\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}+2g^{H,1}_{b\bar{b},1}\Big\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\Big\{-8\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}+16\left(A^H_{b\bar{b},1}\right)^2B^H_{b\bar{b},1}-16\gamma^H_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\Big\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\beta_0\Big\{-12\left(A^H_{b\bar{b},1}\right)^2\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\Big\{8A^H_{b\bar{b},1}B^H_{b\bar{b},1}f^H_{b\bar{b},1}\\
&+6A^H_{b\bar{b},1}\left(B^H_{b\bar{b},1}\right)^2-4A^H_{b\bar{b},1}A^H_{b\bar{b},2}-4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2-4g^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2-8\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}\\
&-12\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}B^H_{b\bar{b},1}+6\left(\gamma^H_{b\bar{b},1}\right)^2A^H_{b\bar{b},1}\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0\Big\{-6A^H_{b\bar{b},1}f^H_{b\bar{b},1}-9A^H_{b\bar{b},1}B^H_{b\bar{b},1}\\
&+9\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0^2\Big\{3A^H_{b\bar{b},1}\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_2^2\Big\{-\frac{38}{5}\left(A^H_{b\bar{b},1}\right)^3\Big\}\\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)\Big\{\frac{4}{3}\left(B^H_{b\bar{b},1}\right)^3-4\gamma^H_{b\bar{b},1}\left(B^H_{b\bar{b},1}\right)^2+4\left(\gamma^H_{b\bar{b},1}\right)^2B^H_{b\bar{b},1}-\frac{4}{3}\left(\gamma^H_{b\bar{b},1}\right)^3\Big\}\\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)\beta_0\Big\{-2\left(B^H_{b\bar{b},1}\right)^2+4\gamma^H_{b\bar{b},1}B^H_{b\bar{b},1}-2\left(\gamma^H_{b\bar{b},1}\right)^2\Big\}\\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\Big\{\frac{2}{3}B^H_{b\bar{b},1}-\frac{2}{3}\gamma^H_{b\bar{b},1}\Big\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\Big\{\frac{8}{3}\left(A^H_{b\bar{b},1}\right)^3\Big\}\\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\Big\{-4\left(A^H_{b\bar{b},1}\right)^2B^H_{b\bar{b},1}+4\gamma^H_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\Big\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0\Big\{2\left(A^H_{b\bar{b},1}\right)^2\Big\}\Big]\\
&+\mathcal{D}_0\Big[\beta_1\Big\{-4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\Big\}+\beta_0\Big\{-4\overline{\mathcal{G}}^{H,1}_{b\bar{b},2}-4\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}f^H_{b\bar{b},1}-8\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2-4g^{H,2}_{b\bar{b},1}f^H_{b\bar{b},1}\\
&-8g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\Big\}+\beta_0^2\Big\{-8\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}\Big\}+\zeta_5\Big\{192\left(A^H_{b\bar{b},1}\right)^3\Big\}+\zeta_3\Big\{32A^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 32A^{H}_{b\bar{b},1}A^{H}_{b\bar{b},2} + 32\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^{H}_{b\bar{b},1}\right)^2 + 32g^{H,1}_{b\bar{b},1}\left(A^{H}_{b\bar{b},1}\right)^2\Big\} + \zeta_3\beta_0\Big\{48A^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1}\Big\}\\
&+ \zeta_2\Big\{4\left(f^{H}_{b\bar{b},1}\right)^3 + 2A^{H}_{b\bar{b},2}f^{H}_{b\bar{b},1} + 2A^{H}_{b\bar{b},1}f^{H}_{b\bar{b},2} + 4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1} + 4g^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1}\Big\}\\
&+ \zeta_2\beta_0\Big\{2\left(f^{H}_{b\bar{b},1}\right)^2 - 12B^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1} + 4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1} + 12\gamma^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1}\Big\}\\
&+ \zeta_2\zeta_3\Big\{-80\left(A^{H}_{b\bar{b},1}\right)^3\Big\} + \zeta_2^2\Big\{23\left(A^{H}_{b\bar{b},1}\right)^2 f^{H}_{b\bar{b},1}\Big\} + \zeta_2^2\beta_0\Big\{16\left(A^{H}_{b\bar{b},1}\right)^2\Big\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\Big\{-4B^{H}_{b\bar{b},2}f^{H}_{b\bar{b},1} - 4B^{H}_{b\bar{b},1}f^{H}_{b\bar{b},2} + 2A^{H}_{b\bar{b},3} + 2\overline{\mathcal{G}}^{H,1}_{b\bar{b},2}A^{H}_{b\bar{b},1} - 8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}B^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1}\\
&+ 4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},2} + 4\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2 A^{H}_{b\bar{b},1} + 2g^{H,1}_{b\bar{b},2}A^{H}_{b\bar{b},1} - 8g^{H,1}_{b\bar{b},1}B^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1} + 4g^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},2}\\
&+ 8g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1} + 4\left(g^{H,1}_{b\bar{b},1}\right)^2 A^{H}_{b\bar{b},1} + 4\gamma^{H}_{b\bar{b},2}f^{H}_{b\bar{b},1} + 4\gamma^{H}_{b\bar{b},1}f^{H}_{b\bar{b},2} + 8\gamma^{H}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}f^{H}_{b\bar{b},1}\\
&+ 8\gamma^{H}_{b\bar{b},1}g^{H,1}_{b\bar{b},1}f^{H}_{b\bar{b},1}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\beta_1\Big\{2f^{H}_{b\bar{b},1}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\beta_0\Big\{4f^{H}_{b\bar{b},2} + 4\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}A^{H}_{b\bar{b},1}\\
&+ 8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}f^{H}_{b\bar{b},1} - 8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}B^{H}_{b\bar{b},1} + 4g^{H,2}_{b\bar{b},1}A^{H}_{b\bar{b},1} + 8g^{H,1}_{b\bar{b},1}f^{H}_{b\bar{b},1} + 8\gamma^{H}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\Big\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\Big\{8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\Big\{-64\left(A^{H}_{b\bar{b},1}\right)^2 f^{H}_{b\bar{b},1} + 32\left(A^{H}_{b\bar{b},1}\right)^2 B^{H}_{b\bar{b},1}\\
&- 32\gamma^{H}_{b\bar{b},1}\left(A^{H}_{b\bar{b},1}\right)^2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\beta_0\Big\{-48\left(A^{H}_{b\bar{b},1}\right)^2\Big\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\Big\{-12A^{H}_{b\bar{b},1}\left(f^{H}_{b\bar{b},1}\right)^2 + 4A^{H}_{b\bar{b},1}B^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1} - 4A^{H}_{b\bar{b},1}A^{H}_{b\bar{b},2}\\
&- 4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^{H}_{b\bar{b},1}\right)^2 - 4g^{H,1}_{b\bar{b},1}\left(A^{H}_{b\bar{b},1}\right)^2 - 4\gamma^{H}_{b\bar{b},1}A^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1}\Big\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0\Big\{-6A^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1} + 12A^{H}_{b\bar{b},1}B^{H}_{b\bar{b},1} - 12\gamma^{H}_{b\bar{b},1}A^{H}_{b\bar{b},1}\Big\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2^2\Big\{-23\left(A^{H}_{b\bar{b},1}\right)^3\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)\Big\{-4\left(B^{H}_{b\bar{b},1}\right)^2 f^{H}_{b\bar{b},1} + 4A^{H}_{b\bar{b},2}B^{H}_{b\bar{b},1}\\
&+ 4A^{H}_{b\bar{b},1}B^{H}_{b\bar{b},2} + 8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1}B^{H}_{b\bar{b},1} + 8g^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1}B^{H}_{b\bar{b},1} - 4\gamma^{H}_{b\bar{b},2}A^{H}_{b\bar{b},1} + 8\gamma^{H}_{b\bar{b},1}B^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1}\\
&- 4\gamma^{H}_{b\bar{b},1}A^{H}_{b\bar{b},2} - 8\gamma^{H}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1} - 8\gamma^{H}_{b\bar{b},1}g^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1} - 4\left(\gamma^{H}_{b\bar{b},1}\right)^2 f^{H}_{b\bar{b},1}\Big\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_1\Big\{-A^{H}_{b\bar{b},1}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0\Big\{6B^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1} - 2A^{H}_{b\bar{b},2} - 6\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1}\\
&- 6g^{H,1}_{b\bar{b},1}A^{H}_{b\bar{b},1} - 6\gamma^{H}_{b\bar{b},1}f^{H}_{b\bar{b},1}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\Big\{-2f^{H}_{b\bar{b},1}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\Big\{32\left(A^{H}_{b\bar{b},1}\right)^3\Big\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\Big\{12\left(A^{H}_{b\bar{b},1}\right)^2 f^{H}_{b\bar{b},1} - 4\left(A^{H}_{b\bar{b},1}\right)^2 B^{H}_{b\bar{b},1} + 4\gamma^{H}_{b\bar{b},1}\left(A^{H}_{b\bar{b},1}\right)^2\Big\}
\end{aligned}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0\left\{3\left(A^H_{b\bar{b},1}\right)^2\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)\left\{4A^H_{b\bar{b},1}\left(B^H_{b\bar{b},1}\right)^2-8\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}B^H_{b\bar{b},1}\right.
$$
$$
\left.+4\left(\gamma^H_{b\bar{b},1}\right)^2A^H_{b\bar{b},1}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-4A^H_{b\bar{b},1}B^H_{b\bar{b},1}+4\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\right\}
$$
$$
+\log^3\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\left\{\frac{2}{3}A^H_{b\bar{b},1}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\left\{-4\left(A^H_{b\bar{b},1}\right)^3\right\}+\left\{-2f^H_{b\bar{b},3}-2\overline{\mathcal{G}}^{H,1}_{b\bar{b},2}f^H_{b\bar{b},1}\right.
$$
$$
-4\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}f^H_{b\bar{b},2}-4\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}-2g^{H,1}_{b\bar{b},2}f^H_{b\bar{b},1}-4g^{H,1}_{b\bar{b},1}f^H_{b\bar{b},2}-8g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}f^H_{b\bar{b},1}
$$
$$
\left.-4\left(g^{H,1}_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}\right\}\Bigg]
$$
$$
+\mathcal{D}_1\Bigg[\beta_1\left\{4f^H_{b\bar{b},1}\right\}+\beta_0\left\{8f^H_{b\bar{b},2}+8\overline{\mathcal{G}}^{H,2}_{b\bar{b},1}A^H_{b\bar{b},1}+24\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}f^H_{b\bar{b},1}+8g^{H,2}_{b\bar{b},1}A^H_{b\bar{b},1}\right.
$$
$$
\left.+8g^{H,1}_{b\bar{b},1}f^H_{b\bar{b},1}\right\}+\beta_0^2\left\{16\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right\}+\zeta_3\left\{-160\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}\right\}+\zeta_3\beta_0\left\{-96\left(A^H_{b\bar{b},1}\right)^2\right\}
$$
$$
+\zeta_2\left\{-28A^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2-8A^H_{b\bar{b},1}A^H_{b\bar{b},2}-8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2-8g^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\right\}
$$
$$
+\zeta_2\beta_0\left\{-24A^H_{b\bar{b},1}f^H_{b\bar{b},1}+24A^H_{b\bar{b},1}B^H_{b\bar{b},1}-24\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\right\}+\zeta_2^2\left\{-46\left(A^H_{b\bar{b},1}\right)^3\right\}
$$
$$
+\log\left(\frac{q^2}{\mu_F^2}\right)\left\{8B^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2-8A^H_{b\bar{b},2}f^H_{b\bar{b},1}+8A^H_{b\bar{b},2}B^H_{b\bar{b},1}-8A^H_{b\bar{b},1}f^H_{b\bar{b},2}+8A^H_{b\bar{b},1}B^H_{b\bar{b},2}\right.
$$
$$
-16\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}+16\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}B^H_{b\bar{b},1}-16g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}+16g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}B^H_{b\bar{b},1}
$$
$$
\left.-8\gamma^H_{b\bar{b},2}A^H_{b\bar{b},1}-8\gamma^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2-8\gamma^H_{b\bar{b},1}A^H_{b\bar{b},2}-16\gamma^H_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}-16\gamma^H_{b\bar{b},1}g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}\right\}
$$
$$
+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_1\left\{-4A^H_{b\bar{b},1}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-8\left(f^H_{b\bar{b},1}\right)^2+8B^H_{b\bar{b},1}f^H_{b\bar{b},1}-8A^H_{b\bar{b},2}\right.
$$
$$
\left.-32\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}-16g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}-8\gamma^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\left\{-8f^H_{b\bar{b},1}\right\}
$$
$$
+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_3\left\{160\left(A^H_{b\bar{b},1}\right)^3\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\left\{56\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}-8\left(A^H_{b\bar{b},1}\right)^2B^H_{b\bar{b},1}\right.
$$
$$
\left.+8\gamma^H_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\beta_0\left\{+24\left(A^H_{b\bar{b},1}\right)^2\right\}
$$
$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)\left\{-16A^H_{b\bar{b},1}B^H_{b\bar{b},1}f^H_{b\bar{b},1}+8A^H_{b\bar{b},1}\left(B^H_{b\bar{b},1}\right)^2+8A^H_{b\bar{b},1}A^H_{b\bar{b},2}+8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\right.
$$
$$
\left.+8g^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2+16\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}-16\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}B^H_{b\bar{b},1}+8\left(\gamma^H_{b\bar{b},1}\right)^2A^H_{b\bar{b},1}\right\}
$$
$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{12A^H_{b\bar{b},1}f^H_{b\bar{b},1}-12A^H_{b\bar{b},1}B^H_{b\bar{b},1}+12\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\left\{4A^H_{b\bar{b},1}\right\}
$$
$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\left\{-28\left(A^H_{b\bar{b},1}\right)^3\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)\left\{8\left(A^H_{b\bar{b},1}\right)^2B^H_{b\bar{b},1}-8\gamma^H_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\right\}
$$

$$
\begin{aligned}
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-4\left(A^H_{b\bar{b},1}\right)^2\right\}+\left\{8f^H_{b\bar{b},1}f^H_{b\bar{b},2}+4A^H_{b\bar{b},3}+4\overline{\mathcal{G}}^{H,1}_{b\bar{b},2}A^H_{b\bar{b},1}+8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2\right.\\
&\left.+8\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},2}+8\left(\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\right)^2A^H_{b\bar{b},1}+4g^{H,1}_{b\bar{b},2}A^H_{b\bar{b},1}+8g^{H,1}_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2+8g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},2}\right.\\
&\left.+16g^{H,1}_{b\bar{b},1}\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}+8\left(g^{H,1}_{b\bar{b},1}\right)^2A^H_{b\bar{b},1}\right\}\Bigg]\\
&+\mathcal{D}_2\Bigg[\beta_1\left\{-4A^H_{b\bar{b},1}\right\}+\beta_0\left\{-12\left(f^H_{b\bar{b},1}\right)^2-8A^H_{b\bar{b},2}-32\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}-8g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}\right\}\\
&+\beta_0^2\left\{-8f^H_{b\bar{b},1}\right\}+\zeta_3\left\{160\left(A^H_{b\bar{b},1}\right)^3\right\}+\zeta_2\left\{60\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}\right\}+\zeta_2\beta_0\left\{36\left(A^H_{b\bar{b},1}\right)^2\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\left\{12A^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2-24A^H_{b\bar{b},1}B^H_{b\bar{b},1}f^H_{b\bar{b},1}+24A^H_{b\bar{b},1}A^H_{b\bar{b},2}+24\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\right.\\
&\left.+24g^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2+24\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{36A^H_{b\bar{b},1}f^H_{b\bar{b},1}-8A^H_{b\bar{b},1}B^H_{b\bar{b},1}\right.\\
&\left.+8\gamma^H_{b\bar{b},1}A^H_{b\bar{b},1}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_0^2\left\{+8A^H_{b\bar{b},1}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\zeta_2\left\{-60\left(A^H_{b\bar{b},1}\right)^3\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)\left\{-12\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}+24\left(A^H_{b\bar{b},1}\right)^2B^H_{b\bar{b},1}-24\gamma^H_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-18\left(A^H_{b\bar{b},1}\right)^2\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)\left\{4\left(A^H_{b\bar{b},1}\right)^3\right\}+\left\{-4\left(f^H_{b\bar{b},1}\right)^3\right.\\
&\left.-12A^H_{b\bar{b},2}f^H_{b\bar{b},1}-12A^H_{b\bar{b},1}f^H_{b\bar{b},2}-24\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}-24g^{H,1}_{b\bar{b},1}A^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\}\Bigg]\\
&+\mathcal{D}_3\Bigg[\beta_0\left\{\frac{80}{3}A^H_{b\bar{b},1}f^H_{b\bar{b},1}\right\}+\beta_0^2\left\{+\frac{16}{3}A^H_{b\bar{b},1}\right\}+\zeta_2\left\{-40\left(A^H_{b\bar{b},1}\right)^3\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\left\{-32\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}+16\left(A^H_{b\bar{b},1}\right)^2B^H_{b\bar{b},1}-16\gamma^H_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)\beta_0\left\{-\frac{80}{3}\left(A^H_{b\bar{b},1}\right)^2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)\left\{16\left(A^H_{b\bar{b},1}\right)^3\right\}+\left\{16A^H_{b\bar{b},1}\left(f^H_{b\bar{b},1}\right)^2\right.\\
&\left.+16A^H_{b\bar{b},1}A^H_{b\bar{b},2}+16\overline{\mathcal{G}}^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2+16g^{H,1}_{b\bar{b},1}\left(A^H_{b\bar{b},1}\right)^2\right\}\Bigg]\\
&+\mathcal{D}_4\Bigg[\beta_0\left\{-\frac{40}{3}\left(A^H_{b\bar{b},1}\right)^2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)\left\{20\left(A^H_{b\bar{b},1}\right)^3\right\}+\left\{-20\left(A^H_{b\bar{b},1}\right)^2f^H_{b\bar{b},1}\right\}\Bigg]\\
&+\mathcal{D}_5\Bigg[\left\{+8\left(A^H_{b\bar{b},1}\right)^3\right\}\Bigg]. \qquad (2.4.2)
\end{aligned}
$$

Upon substituting the values of all the anomalous dimensions, beta functions and $g^{H,k}_{b\bar{b},i}$, $\overline{\mathcal{G}}^{H,k}_{b\bar{b},i}$, we obtain the results of the scalar Higgs boson production cross section at threshold

in $b\bar{b}$ annihilation up to N^{3}LO for the choices of the scale $\mu_R^2 = \mu_F^2$:

$$
\begin{aligned}
\Delta^{H,\mathrm{SV}}_{b\bar{b},1} &= \delta(1-z)\left[C_F\left\{-4+8\zeta_2\right\}\right] + \mathcal{D}_0\left[C_F\left\{8\log\left(\frac{q^2}{\mu_F^2}\right)\right\} + \mathcal{D}_1\left[C_F\left\{16\right\}\right]\right],\\
\Delta^{H,\mathrm{SV}}_{b\bar{b},2} &= \delta(1-z)\left[C_F C_A\left\{\frac{166}{9} - 8\zeta_3 + \frac{232}{9}\zeta_2 - \frac{12}{5}\zeta_2^2\right\} + C_F^2\left\{16 - 60\zeta_3 + \frac{8}{5}\zeta_2^2\right\}\right.\\
&\quad + n_f C_F\left\{\frac{8}{9} + 8\zeta_3 - \frac{40}{9}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A\left\{-12 - 24\zeta_3\right\}\\
&\quad \left. + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{176\zeta_3 - 24\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{-32\zeta_2\right\}\right]\\
&\quad + \mathcal{D}_0\left[C_F C_A\left\{-\frac{1616}{27} + 56\zeta_3 + \frac{176}{3}\zeta_2\right\} + C_F^2\left\{256\zeta_3\right\}\right.\\
&\quad + n_f C_F\left\{\frac{224}{27} - \frac{32}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A\left\{\frac{536}{9} - 16\zeta_2\right\}\\
&\quad + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{-32 - 64\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F\left\{-\frac{80}{9}\right\}\\
&\quad \left. + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A\left\{-\frac{44}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F\left\{\frac{8}{3}\right\}\right]\\
&\quad + \mathcal{D}_1\left[C_F C_A\left\{\frac{1072}{9} - 32\zeta_2\right\} + C_F^2\left\{-64 - 128\zeta_2\right\} + n_f C_F\left\{-\frac{160}{9}\right\}\right.\\
&\quad \left. + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A\left\{-\frac{176}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F\left\{+\frac{32}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{64\right\}\right]\\
&\quad + \mathcal{D}_2\left[C_F C_A\left\{-\frac{176}{3}\right\} + n_f C_F\left\{\frac{32}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{192\right\}\right]\\
&\quad + \mathcal{D}_3\left[C_F^2\left\{128\right\}\right],\\
\Delta^{H,\mathrm{SV}}_{b\bar{b},3} &= \delta(1-z)\left[C_F C_A^2\left\{\frac{68990}{81} - 84\zeta_5 - \frac{14212}{81}\zeta_3 - \frac{400}{3}\zeta_3^2 - 272\zeta_2 - \frac{1064}{3}\zeta_2\zeta_3\right.\right.\\
&\quad \left. + \frac{2528}{27}\zeta_2^2 + \frac{13264}{315}\zeta_2^3\right\} + C_F^2 C_A\left\{-\frac{982}{3} - \frac{37144}{9}\zeta_5 - \frac{10940}{9}\zeta_3 + \frac{3280}{3}\zeta_3^2\right.\\
&\quad \left. + \frac{22106}{27}\zeta_2 + \frac{27872}{9}\zeta_2\zeta_3 - \frac{62468}{135}\zeta_2^2 - \frac{20816}{315}\zeta_2^3\right\} + C_F^3\left\{\frac{1078}{3} + 848\zeta_5\right.\\
&\quad \left. - 1188\zeta_3 + \frac{10336}{3}\zeta_3^2 - \frac{550}{3}\zeta_2 - 64\zeta_2\zeta_3 + \frac{152}{5}\zeta_2^2 - \frac{184736}{315}\zeta_2^3\right\}\\
&\quad + n_f C_F C_A\left\{-\frac{11540}{81} - 8\zeta_5 + \frac{2552}{81}\zeta_3 + \frac{3368}{81}\zeta_2 + \frac{208}{3}\zeta_2\zeta_3 - \frac{6728}{135}\zeta_2^2\right\}\\
&\quad + n_f C_F^2\left\{-\frac{70}{9} + \frac{5536}{9}\zeta_5 + \frac{4088}{9}\zeta_3 - \frac{2600}{27}\zeta_2 - \frac{5504}{9}\zeta_2\zeta_3 + \frac{12152}{135}\zeta_2^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ n_f^2 C_F \left\{ \frac{16}{27} - \frac{1120}{81}\zeta_3 - \frac{32}{81}\zeta_2 + \frac{128}{27}\zeta_2^2 \right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{ - \frac{1180}{3} + 80\zeta_5 \right.\\
&\left. - \frac{2576}{9}\zeta_3 + \frac{160}{3}\zeta_2 + \frac{68}{5}\zeta_2^2 \right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{ \frac{388}{3} + 240\zeta_5 + \frac{27040}{9}\zeta_3 \right.\\
&\left. - \frac{3380}{27}\zeta_2 - 1120\zeta_2\zeta_3 - \frac{8}{3}\zeta_2^2 \right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{ - 100 + 5664\zeta_5 - 568\zeta_3 + 132\zeta_2 \right.\\
&\left. - 2752\zeta_2\zeta_3 - \frac{384}{5}\zeta_2^2 \right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{ \frac{196}{3} + \frac{208}{9}\zeta_3 - \frac{16}{3}\zeta_2 - \frac{8}{5}\zeta_2^2 \right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{ \frac{8}{3} - \frac{4528}{9}\zeta_3 + \frac{152}{27}\zeta_2 + \frac{112}{15}\zeta_2^2 \right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{ \frac{64}{9}\zeta_3 \right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{ 44 + 88\zeta_3 \right\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{ - 880\zeta_3 - \frac{3496}{9}\zeta_2 + 128\zeta_2^2 \right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{ 128\zeta_2 \right.\\
&\left. - \frac{1792}{5}\zeta_2^2 \right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{ - 8 - 16\zeta_3 \right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{ 160\zeta_3 \right.\\
&\left. + \frac{496}{9}\zeta_2 \right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{ \frac{352}{3}\zeta_2 \right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{ \frac{512}{3}\zeta_3 \right\}\\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{ - \frac{64}{3}\zeta_2 \right\} \Big] + \mathcal{D}_0 \Big[C_F C_A^2 \left\{ - \frac{594058}{729} - 384\zeta_5 + \frac{40144}{27}\zeta_3 \right.\\
&\left. + \frac{98224}{81}\zeta_2 - \frac{352}{3}\zeta_2\zeta_3 - \frac{2992}{15}\zeta_2^2 \right\} + C_F^2 C_A \left\{ \frac{6464}{27} + \frac{32288}{9}\zeta_3 + \frac{6592}{27}\zeta_2 \right.\\
&\left. - 1472\zeta_2\zeta_3 + \frac{1408}{3}\zeta_2^2 \right\} + C_F^3 \left\{ 12288\zeta_5 - 1024\zeta_3 - 6144\zeta_2\zeta_3 \right\}\\
&+ n_f C_F C_A \left\{ \frac{125252}{729} - \frac{2480}{9}\zeta_3 - \frac{29392}{81}\zeta_2 + \frac{736}{15}\zeta_2^2 \right\} + n_f C_F^2 \left\{ \frac{842}{9} \right.\\
&\left. - \frac{5728}{9}\zeta_3 - \frac{1504}{27}\zeta_2 - \frac{1472}{15}\zeta_2^2 \right\} + n_f^2 C_F \left\{ - \frac{3712}{729} + \frac{320}{27}\zeta_3 + \frac{640}{27}\zeta_2 \right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{ \frac{62012}{81} - 352\zeta_3 - \frac{6016}{9}\zeta_2 + \frac{352}{5}\zeta_2^2 \right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{ - \frac{272}{3} - 2880\zeta_3 - \frac{10432}{9}\zeta_2 + \frac{1824}{5}\zeta_2^2 \right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{ 128 - 480\zeta_3 + 512\zeta_2 - \frac{7104}{5}\zeta_2^2 \right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{ - \frac{16408}{81} + 192\zeta_2 \right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{ - \frac{92}{3} + 640\zeta_3 \right.\\
&\left. + \frac{1600}{9}\zeta_2 \right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{ \frac{800}{81} - \frac{128}{9}\zeta_2 \right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{ - \frac{7120}{27} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{176}{3}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \Big\{ - \frac{112}{3} - 192\zeta_3 + \frac{1760}{3}\zeta_2 \Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \Big\{ 2432\zeta_3 - 192\zeta_2 \Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{ \frac{2312}{27} - \frac{32}{3}\zeta_2 \Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \Big\{ - \frac{8}{3} - \frac{320}{3}\zeta_2 \Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \Big\{ - \frac{160}{27} \Big\} \\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \Big\{ \frac{968}{27} \Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \Big\{ - 256\zeta_2 \Big\} \\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{ - \frac{352}{27} \Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \Big\{ \frac{32}{27} \Big\} \Big] \\
&+ \mathcal{D}_1 \Big[C_F C_A^2 \Big\{ \frac{124024}{81} - 704\zeta_3 - \frac{12032}{9}\zeta_2 + \frac{704}{5}\zeta_2^2 \Big\} + C_F^2 C_A \Big\{ - \frac{544}{3} - 5760\zeta_3 \\
&- \frac{20864}{9}\zeta_2 + \frac{3648}{5}\zeta_2^2 \Big\} + C_F^3 \Big\{ 256 - 960\zeta_3 + 1024\zeta_2 - \frac{14208}{5}\zeta_2^2 \Big\} \\
&+ n_f C_F C_A \Big\{ - \frac{32816}{81} + 384\zeta_2 \Big\} + n_f C_F^2 \Big\{ - \frac{184}{3} + 1280\zeta_3 + \frac{3200}{9}\zeta_2 \Big\} \\
&+ n_f^2 C_F \Big\{ \frac{1600}{81} - \frac{256}{9}\zeta_2 \Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \Big\{ - \frac{28480}{27} + \frac{704}{3}\zeta_2 \Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \Big\{ - \frac{24704}{27} + 512\zeta_3 + \frac{9856}{3}\zeta_2 \Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \Big\{ 11008\zeta_3 \\
&- 384\zeta_2 \Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{ + \frac{9248}{27} - \frac{128}{3}\zeta_2 \Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \Big\{ \frac{3296}{27} \\
&- \frac{1792}{3}\zeta_2 \Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \Big\{ - \frac{640}{27} \Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \Big\{ \frac{1936}{9} \Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \Big\{ \frac{8576}{9} - 256\zeta_2 \Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \Big\{ - 256 - 2048\zeta_2 \Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{ - \frac{704}{9} \Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \Big\{ - \frac{1280}{9} \Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \Big\{ \frac{64}{9} \Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \Big\{ - \frac{704}{3} \Big\} \\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \Big\{ \frac{128}{3} \Big\} \Big] + \mathcal{D}_2 \Big[C_F C_A^2 \Big\{ - \frac{28480}{27} + \frac{704}{3}\zeta_2 \Big\} \\
&+ C_F^2 C_A \Big\{ - \frac{10816}{9} + 1344\zeta_3 + \frac{11264}{3}\zeta_2 \Big\} + C_F^3 \Big\{ 10240\zeta_3 \Big\} \\
&+ n_f C_F C_A \Big\{ \frac{9248}{27} - \frac{128}{3}\zeta_2 \Big\} + n_f C_F^2 \Big\{ \frac{1696}{9} - \frac{2048}{3}\zeta_2 \Big\} + n_f^2 C_F \Big\{ - \frac{640}{27} \Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \Big\{ \frac{3872}{9} \Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \Big\{ \frac{8576}{3} - 768\zeta_2 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{-768-4608\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{-\frac{1408}{9}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F^2\left\{-\frac{1280}{3}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_F\left\{\frac{128}{9}\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2 C_A\left\{-1056\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_F^2\left\{192\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{256\right\}\Big]\\
&+\mathcal{D}_3\Big[C_F C_A^2\left\{\frac{7744}{27}\right\}+C_F^2 C_A\left\{\frac{17152}{9}-512\zeta_2\right\}+C_F^3\left\{-512-3072\zeta_2\right\}\\
&+n_f C_F C_A\left\{-\frac{2816}{27}\right\}+n_f C_F^2\left\{-\frac{2560}{9}\right\}+n_f^2 C_F\left\{\frac{256}{27}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_F^2 C_A\left\{-\frac{14080}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F^2\left\{\frac{2560}{9}\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{1024\right\}\Big]+\mathcal{D}_4\Big[C_F^2 C_A\left\{-\frac{7040}{9}\right\}+n_f C_F^2\left\{\frac{1280}{9}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{1280\right\}\Big]+\mathcal{D}_5\Big[C_F^3\left\{512\right\}\Big]. \qquad (2.4.3)
\end{aligned}
$$

The results at NLO $\left(\Delta_{b\bar{b},1}^{H,\mathrm{SV}}\right)$ and NNLO $\left(\Delta_{b\bar{b},2}^{H,\mathrm{SV}}\right)$ match with the existing ones [21]. At N^{3}LO level, only $\Delta_{b\bar{b},3}^{H,\mathrm{SV}}|_{\mathcal{D}_j}$ were known [28, 29], remaining terms were not available due to absence of the required quantities $g_{b\bar{b},2}^{H,2}$, $g_{b\bar{b},3}^{H,1}$ from form factors and $\overline{\mathcal{G}}_{b\bar{b},2}^{H,2}$, $\overline{\mathcal{G}}_{b\bar{b},3}^{H,1}$ from soft-collinear distributions. The recent results of $g_{b\bar{b},2}^{H,2}$, $g_{b\bar{b},3}^{H,1}$ from [31], $\overline{\mathcal{G}}_{b\bar{b},2}^{H,2}$ from [71] and $\overline{\mathcal{G}}_{b\bar{b},3}^{H,1}$ from [32] are being employed to compute the missing $\delta(1-z)$ part i.e. $\Delta_{b\bar{b},3}^{H,\mathrm{SV}}|_{\delta}$ which completes the full evaluation of the SV cross section at N^{3}LO $\left(\Delta_{b\bar{b},3}^{H,\mathrm{SV}}\right)$ and is presented for the first time in [11] by us. For the sake of completeness, we mention the leading order contribution which is

$$
\Delta_{b\bar{b},0}^{H}=\delta(1-z) \qquad (2.4.4)
$$

and the overall factor in Eq. (2.3.1) comes out to be

$$
\sigma_{b\bar{b}}^{H,(0)}\left(\mu_F^2\right)=\frac{\pi\lambda^2\left(\mu_F^2\right)}{12m_H^2}. \qquad (2.4.5)
$$

The above results are presented for the choice $\mu_R = \mu_F$. The dependence of the SV cross section on renormalisation scale μ_R can be easily restored by employing the RG evolution of a_s from μ_F to μ_R [87]:

$$a_s\left(\mu_R^2\right) = a_s\left(\mu_F^2\right)\frac{1}{\omega} + a_s^2\left(\mu_F^2\right)\left\{\frac{1}{\omega^2}\left(-\eta_1\log\omega\right)\right\} + a_s^3\left(\mu_F^2\right)\left\{\frac{1}{\omega^2}\left(\eta_1^2-\eta_2\right)\right.$$
$$\left. + \frac{1}{\omega^3}\left(-\eta_1^2+\eta_2-\eta_1^2\log\omega+\eta_1^2\log^2\omega\right)\right\} \tag{2.4.6}$$

where

$$\omega \equiv 1 - \beta_0 a_s\left(\mu_F^2\right)\log\left(\frac{\mu_F^2}{\mu_R^2}\right),$$
$$\eta_i \equiv \frac{\beta_i}{\beta_0}. \tag{2.4.7}$$

The above result of the evolution of the a_s is a resummed one and the fixed order result can be easily obtained by performing the series expansion of this equation (2.4.6).

2.5 Numerical Impact of SV Cross Sections

The numerical impact of our results can be studied using the exact LO, NLO, NNLO $\Delta^H_{b\bar{b},i}$, $i = 0,1,2$ and the threshold N^{3}LO result $\Delta^{H,\mathrm{SV}}_{b\bar{b},3}$. We have used $\sqrt{s} = 14$ TeV for the LHC, the Z boson mass $M_Z = 91.1876$ GeV and Higgs boson mass $m_H = 125.5$ GeV throughout. The strong coupling constant $\alpha_s(\mu_R^2)$ ($a_s = \alpha_s/4\pi$) is evolved using the 4-loop RG equations with $\alpha_s^{\mathrm{N^3LO}}(m_Z) = 0.117$ and for parton density sets we use MSTW 2008NNLO [88]. The Yukawa coupling is evolved using 4 loop RG with $\lambda(m_b) = \sqrt{2}m_b(m_b)/v$ and $m_b(m_b) = 4.3$ GeV.

The renormalization scale dependence is studied by varying μ_R between 0.1 m_H and 10 m_H keeping $\mu_F = m_H/4$ fixed. For the factorization scale, we have fixed $\mu_R = m_H$ and varied μ_F between 0.1 m_H and 10 m_H. We find that the perturbation theory behaves better if we

include more and more higher order terms (see Fig.2.4).

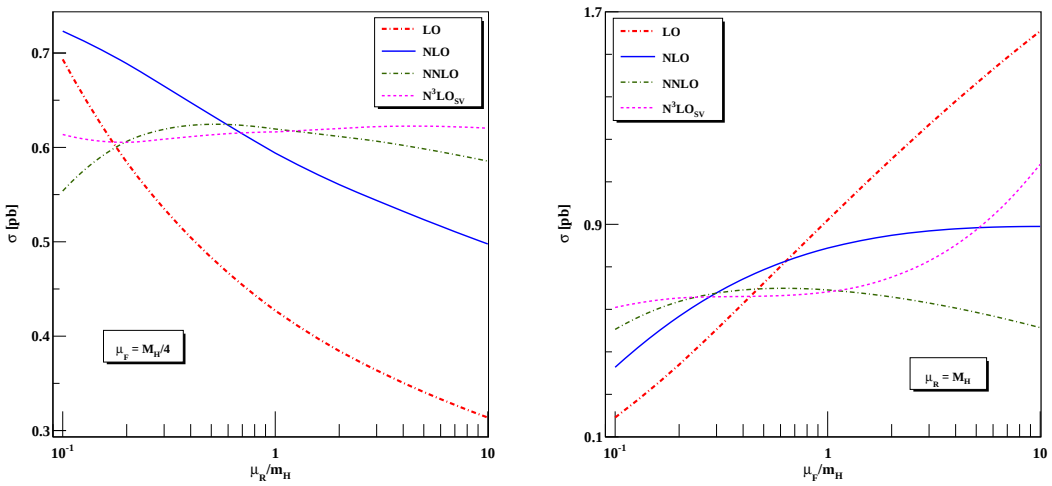


Figure 2.4. Total cross section for Higgs production in $b\bar{b}$ annihilation at various orders in a_s as a function of μ_R/m_H (left panel) and of μ_F/m_H (right panel) at the LHC with $\sqrt{s} = 14$ TeV.

2.6 Summary

To summarize, we have systematically developed a framework to compute threshold contributions in QCD to the production of Higgs boson in bottom anti-bottom annihilation subprocesses at the hadron colliders. This formalism is applicable for any colorless particle. Factorization of UV, soft and collinear singularities and exponentiation of their sum allow us to obtain threshold corrections order by order in perturbation theory. Using the recently obtained N^{3}LO soft distribution function for Drell-Yan production and the three loop Higgs form factor with bottom anti-bottom quarks, we have obtained threshold N^{3}LO corrections to Higgs production through bottom anti-bottom annihilation. We have also studied the stability of our result under renormalization and factorization scales.

3 Rapidity Distributions of Drell-Yan and Higgs Boson at Threshold in N^3LO QCD

The materials presented in this chapter are the result of an original research done in collaboration with Manoj K. Mandal, Narayan Rana and V. Ravindran, and these are based on the published article [12].

Contents

3.1 Prologue

The Drell-Yan production [89] of a pair of leptons at the LHC is one of the cleanest processes that can be studied not only to test the SM to an unprecedented accuracy but also to probe physics beyond the SM (BSM) scenarios in a very clear environment. Rapidity distributions of Z boson [90] and charge asymmetries of leptons in W boson decays [91] constrain various parton densities and, in addition, possible excess events can provide hints to BSM physics, namely R-parity violating supersymmetric models, models with Z' or with contact interactions and large extra-dimension models. One of the production mechanisms responsible for discovering the Higgs boson of the SM at the LHC [1, 2] is the gluon-gluon fusion through top quark loop. Being a dominant one, it will continue to play a major role in studying the properties of the Higgs boson and its coupling to other SM particles. Distributions of transverse momentum and rapidity of the Higgs boson are going to be very useful tools to achieve this task. Like the inclusive rates [55–61, 64, 92–98], the rapidity distribution of dileptons in DY production and of the Higgs boson in gluon-gluon fusion are also known to NNLO level in perturbative QCD due to seminal works by Anastasiou *et al.* [36]. The quark and gluon form factors [40–42, 67], the mass factorization kernels [69], and the renormalization constant [70, 99, 100] for the effective operator describing the coupling of the Higgs boson with the SM fields in the infinite top quark mass limit up to three loop level in dimensional regularization with space-time dimensions $n = 4 + \epsilon$ were found to be useful to obtain the N^{3}LO threshold effects [28, 29, 72–74] to the inclusive Higgs boson and DY productions at the LHC, excluding $\delta(1-z)$ terms, where the scaling parameter is $z = m^2_{l^+l^-}/\hat{s}$ for the DY process and $z = m^2_H/\hat{s}$ for the Higgs boson. Here, $m_{l^+l^-}$, m_H and $\hat{s}$ are the invariant mass of the dileptons, the mass of the Higgs boson, and square of the center of mass energy of the partonic reaction

responsible for the production mechanism, respectively. Recently, Anastasiou *et al.* [33] made an important contribution in computing the total rate for the Higgs boson production at N^{3}LO resulting from the threshold region including the $\delta(1-z)$ term. Their result, along with three loop quark form factors and mass factorization kernels, was used to compute the DY cross section at N^{3}LO at threshold in [32].

In this thesis, we will apply the formalism developed in [39] to obtain rapidity distributions of the dilepton pair and of the Higgs boson at N^{3}LO in the threshold region using the available information that led to the computation of the N^{3}LO threshold corrections to the inclusive Higgs boson [33] and DY productions [32].

We begin by writing down the relevant interacting Lagrangian in Sec. 3.2. In the Sec. 3.3, we present the formalism of computing threshold QCD corrections to the differential rapidity distribution and in Sec. 3.4, we present our results for the threshold N^{3}LO QCD corrections to the rapidity distributions of the dilepton pairs in DY and Higgs boson. The numerical impact in case of Higgs boson is discussed in brief in Sec. 3.5. The numerical impact of threshold enhanced N^{3}LO contributions is demonstrated for the LHC energy $\sqrt{s} = 14$ TeV by studying the stability of the perturbation theory under factorization and renormalization scales. Finally we give a brief summary of our findings in Sec. 3.6.

3.2 The Lagrangian

In the SM, the scalar Higgs boson couples to gluons only indirectly through a virtual heavy quark loop. This loop can be integrated out in the limit of infinite quark mass. The resulting effective Lagrangian encapsulates the interaction between a scalar ϕ and QCD particles and reads:

$$\mathcal{L}^{H}_{\text{eff}} = G_H \phi(x) O^H(x) \tag{3.2.1}$$

with

$$
\begin{aligned}
O^H(x) &\equiv -\frac{1}{4} G^a_{\mu\nu}(x) G^{a,\mu\nu}(x)\,, \\
G_H &\equiv -\frac{2^{5/4}}{3} a_s(\mu_R^2) G_F^{\frac{1}{2}} C_H\left(a_s(\mu_R^2), \frac{\mu_R^2}{m_t^2}\right).
\end{aligned} \tag{3.2.2}
$$

$C_H(\mu_R^2)$ is the Wilson coefficient, given as a perturbative expansion in the $\overline{MS}$ renormalised strong coupling constant $a_s \equiv a_s(\mu_R^2)$, evaluated at the renormalisation scale μ_R. This is given by [70, 101, 102]

$$
\begin{aligned}
C_H\left(a_s(\mu_R^2), \frac{\mu_R^2}{m_t^2}\right) = 1 &+ a_s\left\{11\right\} + a_s^2\left\{\frac{2777}{18} + 19L_t + n_f\left(-\frac{67}{6} + \frac{16}{3}L_t\right)\right\} \\
&+ a_s^3\left\{-\frac{2892659}{648} + \frac{897943}{144}\zeta_3 + \frac{3466}{9}L_t + 209L_t^2\right. \\
&+ n_f\left(\frac{40291}{324} - \frac{110779}{216}\zeta_3 + \frac{1760}{27}L_t + 46L_t^2\right) \\
&\left. + n_f^2\left(-\frac{6865}{486} + \frac{77}{27}L_t - \frac{32}{9}L_t^2\right)\right\}
\end{aligned} \tag{3.2.3}
$$

up to $O(a_s^3)$ with $L_t = \log\left(\mu_R^2/m_t^2\right)$ and n_f is the number of active light quark flavors. For the DY process, we work in the framework of exact SM with $n_f = 5$ number of active light quark flavors.

3.3 Theoretical Framework for Threshold Corrections to Rapidity

The differential rapidity distribution for the production of a colorless particle, namely, a Higgs boson through gluon fusion/bottom quark annihilation or a pair of leptons in the DY at the hadron colliders can be computed using

$$
\frac{d}{dY}\sigma_Y^I\left(\tau, q^2, Y\right) = \sigma_Y^{I,(0)}\left(\tau, q^2, \mu_R^2\right) W^I\left(\tau, q^2, Y, \mu_R^2\right). \tag{3.3.1}
$$

In the above expression, Y stands for the rapidity which is defined as

$$Y \equiv \frac{1}{2}\log\left(\frac{P_2.q}{P_1.q}\right) \tag{3.3.2}$$

where, P_i and q are the momentum of the incoming hadrons and the colorless particle, respectively. The variable τ equals q^2/s with

$$q^2 = \begin{cases} m_H^2 & \text{for } I = H\,, \\ m_{l^+l^-}^2 & \text{for } I = \text{DY}\,. \end{cases} \tag{3.3.3}$$

m_H is the mass of the Higgs boson and $m_{l^+l^-}$is the invariant mass of the final state dilepton pair (l^+l^-), which can be $e^+e^-, \mu^+\mu^-, \tau^+\tau^-$, in the DY production. $\sqrt{s}$ and $\sqrt{\hat{s}}$ stand for the hadronic and partonic center of mass energy, respectively. Throughout this chapter, we denote $I = H$ for the productions of the Higgs boson through gluon (gg) fusion (Fig. 3.1) and bottom quark ($b\bar{b}$) annihilation (Fig. 3.2), whereas we write I =DY for the production of a pair of leptons in the DY (Fig. 3.3). In Eq. (3.3.2), σ_Y^I is defined through

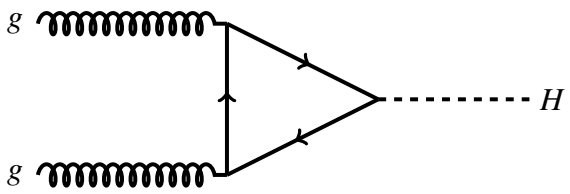


Figure 3.1. Higgs boson production in gluon fusion

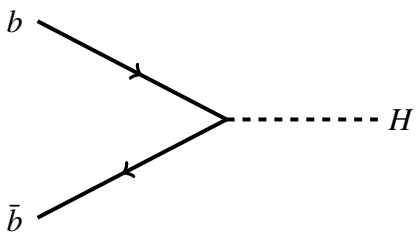


Figure 3.2. Higgs boson production through bottom quark annihilation

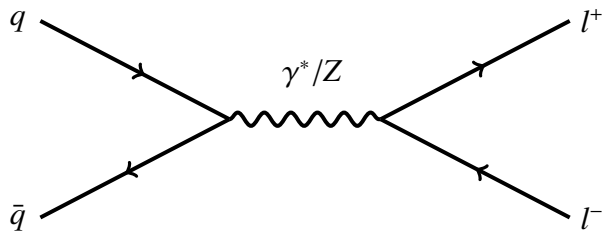


Figure 3.3. Drell-Yan pair production

$$\sigma_Y^I\left(\tau, q^2, Y\right) = \begin{cases} \sigma^I\left(\tau, q^2, Y\right) & \text{for } I = H\,, \\ \frac{d}{dq^2}\sigma^I\left(\tau, q^2, Y\right) & \text{for } I = \text{DY}\,. \end{cases} \tag{3.3.4}$$

where, $\sigma^I\left(\tau, q^2\right)$ is the inclusive production cross section. $\sigma_Y^{I,(0)}$ is an overall prefactor extracted from the leading order contribution. The other quantity W^I is given by

$$\begin{aligned} W^I\left(\tau, q^2, Y, \mu_R^2\right) &= \frac{\left(Z^I(\mu_R^2)\right)^2}{\sigma_Y^{I,(0)}} \sum_{i,j=q,\bar{q},g} \int_0^1 dx_1 \int_0^1 dx_2 \hat{\mathcal{H}}_{ij}^I(x_1, x_2) \int_0^1 dz \delta(\tau - z x_1 x_2) \\ &\quad \times \int dPS_{1+X} |\hat{\mathcal{M}}_{ij}^I|^2 \delta\left(Y - \frac{1}{2}\log\left(\frac{P_2.q}{P_1.q}\right)\right), \\ &\equiv \sum_{i,j=q,\bar{q},g} \int_0^1 dx_1 \int_0^1 dx_2 \hat{\mathcal{H}}_{ij}^I(x_1, x_2) \frac{1}{x_1 x_2} \hat{\Delta}_{Y,ij}^I\left(\tau, Y, \hat{a}_s, \mu^2, q^2, \mu_R^2, \epsilon\right) \end{aligned} \tag{3.3.5}$$

where, we have introduced the dimensionless differential partonic cross section $\hat{\Delta}_{Y,ij}^I$. Z^I is the overall operator UV renormalisation constant, x_k $(k = 1, 2)$ are the momentum fractions of the initial state partons i.e. $p_k = x_k P_k$ and $\hat{\mathcal{H}}_{ij}^I$ stands for

$$\hat{\mathcal{H}}_{ij}^I(x_1, x_2) \equiv \begin{cases} \hat{f}_i(x_1)\, \hat{f}_j(x_2) & \text{for } I = \text{DY}\,, \\ \hat{f}_i(x_1)\, \hat{f}_j(x_2) & \text{for } I = H \text{ through } b\bar{b} \text{ annihilation}\,, \\ x_1 \hat{f}_i(x_1)\, x_2 \hat{f}_j(x_2) & \text{for } I = H \text{ in } gg \text{ fusion}\,. \end{cases}$$

$\hat{f}_i(x_k)$ is the unrenormalised PDF of the initial state partons i with momentum fractions x_k. X is the remnants other than the colorless particle I, dPS_{1+X} is the phase space element

for the $I + X$ system and $\hat{\mathcal{M}}^I_{ij}$ represents the partonic level scattering matrix element for the process $ij \to I$. The renormalised PDF, $f_i\left(x_1, \mu_F^2\right)$, renormalised at the factorisation scale μ_F, is related to the unrenormalised ones through Altarelli-Parisi (AP) kernel:

$$f_i\left(x_k, \mu_F^2\right) = \sum_{j=q,\bar{q},g} \int_{x_k}^{1} \frac{dz}{z} \Gamma_{ij}\left(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon\right) \hat{f}_j\left(\frac{x_k}{z}\right) \tag{3.3.6}$$

where the scale μ is introduced to keep the unrenormalised strong coupling constant $\hat{a}_s$ dimensionless in space-time dimensions $d = 4 + \epsilon$. $\hat{a}_s \equiv \hat{g}_s^2/16\pi^2$ is the unrenormalized strong coupling constant which is related to the renormalized one $a_s(\mu_R^2) \equiv a_s$ through the renormalization constant $Z_{a_s}(\mu_R^2) \equiv Z_{a_s}$, Eq. (2.3.9). The form of e Z_{a_s} is presented in Eq. (2.3.11). Expanding the AP kernel in powers of $\hat{a}_s$, we get

$$\Gamma^I_{ij}(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon) = \delta_{ij}\delta(1-z) + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\mu_F^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Gamma}^{I,(k)}_{ij}(z, \epsilon)\,. \tag{3.3.7}$$

$\hat{\Gamma}^{I,(k)}_{ij}(z, \epsilon)$ in terms of the Altarelli-Parisi splitting functions $P^{I,(k)}_{ij}\left(z, \mu_F^2\right)$ are presented in the Appendix (C.0.20). Employing the Eq. (3.3.6), we can write the renormalised $\mathcal{H}^I_{ij}\left(x_1, x_2, \mu_F^2\right)$ as

$$\mathcal{H}^I_{ij}\left(x_1, x_2, \mu_F^2\right) = \sum_{k,l} \int_{x_1}^{1} \frac{dy_1}{y_1} \int_{x_2}^{1} \frac{dy_2}{y_2} \Gamma^I_{ik}(\hat{a}_s, \mu^2, \mu_F^2, y_1, \epsilon) \hat{\mathcal{H}}^I_{kl}\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right) \Gamma^I_{jl}(\hat{a}_s, \mu^2, \mu_F^2, y_2, \epsilon)\,. \tag{3.3.8}$$

In addition to renormalising the PDF, the AP kernels absorb the initial state collinear singularities present in the $\hat{\Delta}^I_{Y,ij}$ through

$$\begin{aligned}\Delta^I_{Y,ij}\left(\tau, Y, q^2, \mu_R^2, \mu_F^2\right) = \int \frac{dy_1}{y_1} \int \frac{dy_2}{y_2} \left(\Gamma^I\left(\hat{a}_s, \mu^2, \mu_F^2, y_1, \epsilon\right)\right)^{-1}_{ik} \hat{\Delta}^I_{Y,kl}\left(\tau, Y, \hat{a}_s, \mu^2, q^2, \mu_R^2, \epsilon\right) \\ \times \left(\Gamma^I\left(\hat{a}_s, \mu^2, \mu_F^2, y_2, \epsilon\right)\right)^{-1}_{jl}\,.\end{aligned} \tag{3.3.9}$$

The $\Delta^I_{Y,ij}$ is free of UV, soft and collinear singularities. With these we can express W^I in terms of the renormalised quantities. Before writing down the renormalised version of the Eq. (3.3.5), we introduce two symmetric variables x_1^0 and x_2^0 instead of Y and τ through

$$Y \equiv \frac{1}{2}\log\left(\frac{x_1^0}{x_2^0}\right), \qquad \tau \equiv x_1^0 x_2^0 . \tag{3.3.10}$$

In terms of these new variables, the contributions arising from partonic subprocesses can be shown to depend on the ratios $z_j = x_j^0/x_j$ which take the role of scaling variables at the partonic level. In terms of these newly introduced variables, we get the renormalised W^I as

$$W^I\left(x_1^0, x_2^0, q^2, \mu_R^2\right) = \sum_{i,j=q,\bar{q},g} \int_{x_1^0}^1 \frac{dz_1}{z_1} \int_{x_2^0}^1 \frac{dz_2}{z_2} \mathcal{H}^I_{ij}\left(\frac{x_1^0}{z_1}\frac{x_2^0}{z_2}, \mu_F^2\right) \Delta^I_{Y,ij}\left(z_1, z_2, q^2, \mu_R^2, \mu_F^2\right) . \tag{3.3.11}$$

The *goal* of this chapter is to study the impact of the contributions arising from the soft gluons to the differential rapidity distributions of a colorless particle production at Hadron colliders. The infrared safe contributions from the soft gluons is obtained by adding the soft part of the distribution with the UV renormalized virtual part and performing mass factorisation using appropriate counter terms. This combination is often called the soft-plus-virtual (SV) rapidity distribution whereas the remaining portion is known as hard part. Hence, we write the rapidity distribution by decomposing into two parts as

$$\Delta^I_{Y,ij}(z_1, z_2, q^2, \mu_R^2, \mu_F^2) = \Delta^{I,\mathrm{SV}}_{Y,ij}(z_1, z_2, q^2, \mu_R^2, \mu_F^2) + \Delta^{I,\mathrm{hard}}_{Y,ij}(z_1, z_2, q^2, \mu_R^2, \mu_F^2) . \tag{3.3.12}$$

The SV contributions $\Delta^{I,\mathrm{SV}}_{Y,ij}(z_1, z_2, q^2, \mu_R^2, \mu_F^2)$ contains only the distributions of kind $\delta(1-z_1)$, $\delta(1-z_2)$ and $\mathcal{D}_i$, $\overline{\mathcal{D}}_i$ where the latter ones are defined through

$$\mathcal{D}_i \equiv \left[\frac{\ln^i(1-z_1)}{(1-z_1)}\right]_+ , \qquad \overline{\mathcal{D}}_i \equiv \left[\frac{\ln^i(1-z_2)}{(1-z_2)}\right]_+ \quad \text{with} \quad i = 0, 1, 2, \ldots . \tag{3.3.13}$$

This is also known as the threshold contributions. On the other hand, the hard part $\Delta^{I,\text{hard}}_{Y,ij}$ contains all the regular terms in z_1 and z_2. The SV rapidity distribution in z-space is computed in d-dimensions, as formulated in [39], using

$$\Delta^{I,\text{SV}}_{Y,ij}(z_1,z_2,q^2,\mu_R^2,\mu_F^2) = C\exp\left(\Psi^I_{Y,ij}\left(z_1,z_2,q^2,\mu_R^2,\mu_F^2,\epsilon\right)\right)\Big|_{\epsilon=0} \tag{3.3.14}$$

where, $\Psi^I_{Y,ij}\left(z_1,z_2,q^2,\mu_R^2,\mu_F^2,\epsilon\right)$ is a finite distribution and C is the convolution defined as

$$Ce^{f(z_1,z_2)} = \delta(1-z_1)\delta(1-z_2) + \frac{1}{1!}f(z_1,z_2) + \frac{1}{2!}f(z_1,z_2)\otimes f(z_1,z_2) + \cdots . \tag{3.3.15}$$

Here, $\otimes$ represents the double Mellin convolution with respect to the pair of variables z_1, z_2 and $f(z_1,z_2)$ is a distribution of the kind $\delta(1-z_j)$, $\mathcal{D}_i$ and $\overline{\mathcal{D}}_i$. The $\Psi^I_{Y,ij}\left(z_1,z_2,q^2,\mu_R^2,\mu_F^2,\epsilon\right)$ is constructed from the form factors $\mathcal{F}^I_{ij}(\hat{a}_s,Q^2,\mu^2,\epsilon)$ with $Q^2=-q^2$, the overall operator UV renormalization constant $Z^I_{ij}(\hat{a}_s,\mu_R^2,\mu^2,\epsilon)$, the soft-collinear distribution $\Phi^I_{Y,ij}(\hat{a}_s,\, q^2,\, \mu^2,z_1,z_2,\epsilon)$ arising from the real radiations in the partonic subprocesses and the mass factorization kernels $\Gamma^I_{ij}(\hat{a}_s,\mu^2,\mu_F^2,z_j,\epsilon)$. In terms of the above-mentioned quantities it takes the following form, as presented in [12, 39, 103]

$$\begin{aligned}\Psi^I_{Y,ij}\left(z_1,z_2,q^2,\mu_R^2,\mu_F^2,\epsilon\right) =& \left(\ln\left[Z^I_{ij}(\hat{a}_s,\mu_R^2,\mu^2,\epsilon)\right]^2 + \ln\left|\mathcal{F}^I_{ij}(\hat{a}_s,Q^2,\mu^2,\epsilon)\right|^2\right)\delta(1-z_1)\delta(1-z_2)\\ &+ 2\Phi^I_{Y,ij}(\hat{a}_s,q^2,\mu^2,z_1,z_2,\epsilon) - C\ln\Gamma^I_{ij}(\hat{a}_s,\mu^2,\mu_F^2,z_1,\epsilon)\delta(1-z_2)\\ &- C\ln\Gamma^I_{ij}(\hat{a}_s,\mu^2,\mu_F^2,z_2,\epsilon)\delta(1-z_1)\,.\end{aligned} \tag{3.3.16}$$

In this chapter, we will confine our discussion on the threshold corrections to the Higgs boson production through gluon fusion and DY pair productions. More precisely, our main *goal* is to compute the SV corrections to the rapidity distributions of these two processes at N^{3}LO QCD. In the subsequent sections, we will demonstrate the methodology to get the ingredients, Eq. (3.3.16) to compute the SV rapidity distributions at N^{3}LO QCD.

3.3.1 The Form Factor

The quark and gluon form factors represent the QCD loop corrections to the transition matrix element from an on-shell quark-antiquark pair or two gluons to a color-neutral particle. For the processes under consideration, we require gluon form factors in case of scalar Higgs boson production in gg fusion and quark form factors for DY pair productions from $q\bar{q}$ annihilation (happens through intermediate off-shell photon, γ^* or Z-boson). The unrenormalised quark form factors at $O(\hat{a}_s^n)$ are defined through

$$\hat{\mathcal{F}}^{I,(n)}_{i\,\bar{i}} \equiv \frac{\langle \hat{\mathcal{M}}^{I,(0)}_{i\,\bar{i}} | \hat{\mathcal{M}}^{I,(n)}_{i\,\bar{i}} \rangle}{\langle \hat{\mathcal{M}}^{I,(0)}_{i\,\bar{i}} | \hat{\mathcal{M}}^{I,(0)}_{i\,\bar{i}} \rangle}, \qquad n = 0, 1, 2, 3, \cdots \tag{3.3.17}$$

with

$$i\,\bar{i} = \begin{cases} gg & \text{for H,} \\ q\bar{q} & \text{for DY .} \end{cases}$$

In the above expressions $|\hat{\mathcal{M}}^{I,(n)}_{i\,\bar{i}}\rangle$ is the $O(\hat{a}_s^n)$ contribution to the unrenormalised matrix element for the production of the particle I from on-shell $i\,\bar{i}$ annihilation. In terms of these quantities, the full matrix element and the full form factors can be written as a series expansion in $\hat{a}_s$ as

$$|\mathcal{M}^{I}_{i\,\bar{i}}\rangle \equiv \sum_{n=0}^{\infty} \hat{a}_s^n S_\epsilon^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} |\hat{\mathcal{M}}^{I,(n)}_{i\,\bar{i}}\rangle\,, \qquad \mathcal{F}^{I}_{i\,\bar{i}} \equiv \sum_{n=0}^{\infty} \left[\hat{a}_s^n S_\epsilon^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} \hat{\mathcal{F}}^{I,(n)}_{i\,\bar{i}}\right], \tag{3.3.18}$$

where $Q^2 = -2\,p_1.p_2 = -q^2$ and p_i ($p_i^2 = 0$) are the momenta of the external on-shell quarks or gluons. Gluon form factors $\mathcal{F}^H_{gg}$ up to three loops in QCD were computed in [40–42, 68, 104, 105]. The quark form factors $\mathcal{F}^{\rm DY}_{q\bar{q}}$ up to three loops in QCD are available from [40–42, 67, 68, 95, 96, 105, 106].

The form factor $\mathcal{F}^I_{i\,\bar{i}}(\hat{a}_s, Q^2, \mu^2, \epsilon)$ satisfies the KG-differential equation [79–83] which is a direct consequence of the facts that QCD amplitudes exhibit factorisation property, gauge

and renormalisation group (RG) invariances:

$$Q^2\frac{d}{dQ^2}\ln\mathcal{F}^I_{i\,\bar{i}}(\hat{a}_s,Q^2,\mu^2,\epsilon)=\frac{1}{2}\left[K^I_{i\,\bar{i}}\left(\hat{a}_s,\frac{\mu_R^2}{\mu^2},\epsilon\right)+G^I_{i\,\bar{i}}\left(\hat{a}_s,\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\epsilon\right)\right]. \tag{3.3.19}$$

In the above expression, all the poles in dimensional regularisation parameter ϵ are captured in the Q^2 independent function $K^I_{i\,\bar{i}}$ and the quantities which are finite as $\epsilon\to 0$ are encapsulated in $G^I_{i\,\bar{i}}$. The solution of the above KG equation can be obtained as [28] (see also [11, 32])

$$\ln\mathcal{F}^I_{i\,\bar{i}}(\hat{a}_s,Q^2,\mu^2,\epsilon)=\sum_{k=1}^{\infty}\hat{a}_s^k S_\epsilon^k\left(\frac{Q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}}\hat{\mathcal{L}}^I_{i\,\bar{i},k}(\epsilon) \tag{3.3.20}$$

with

$$\begin{aligned}
\hat{\mathcal{L}}^I_{i\,\bar{i},1}(\epsilon)=&\frac{1}{\epsilon^2}\left\{-2A^I_{i\,\bar{i},1}\right\}+\frac{1}{\epsilon}\left\{G^I_{i\,\bar{i},1}(\epsilon)\right\},\\
\hat{\mathcal{L}}^I_{i\,\bar{i},2}(\epsilon)=&\frac{1}{\epsilon^3}\left\{\beta_0A^I_{i\,\bar{i},1}\right\}+\frac{1}{\epsilon^2}\left\{-\frac{1}{2}A^I_{i\,\bar{i},2}-\beta_0G^I_{i\,\bar{i},1}(\epsilon)\right\}+\frac{1}{\epsilon}\left\{\frac{1}{2}G^I_{i\,\bar{i},2}(\epsilon)\right\},\\
\hat{\mathcal{L}}^I_{i\,\bar{i},3}(\epsilon)=&\frac{1}{\epsilon^4}\left\{-\frac{8}{9}\beta_0^2A^I_{i\,\bar{i},1}\right\}+\frac{1}{\epsilon^3}\left\{\frac{2}{9}\beta_1A^I_{i\,\bar{i},1}+\frac{8}{9}\beta_0A^I_{i\,\bar{i},2}+\frac{4}{3}\beta_0^2G^I_{i\,\bar{i},1}(\epsilon)\right\}\\
&+\frac{1}{\epsilon^2}\left\{-\frac{2}{9}A^I_{i\bar{i},3}-\frac{1}{3}\beta_1G^I_{i\,\bar{i},1}(\epsilon)-\frac{4}{3}\beta_0G^I_{i\,\bar{i},2}(\epsilon)\right\}+\frac{1}{\epsilon}\left\{\frac{1}{3}G^I_{i\,\bar{i},3}(\epsilon)\right\}.
\end{aligned} \tag{3.3.21}$$

In Appendix D, the derivation of the above solution is discussed in great details. $A^I_{i\,\bar{i}}$'s are called the cusp anomalous dimensions. The constants $G^I_{i\,\bar{i},i}$'s are the coefficients of a_s^i in the following expansions:

$$\begin{aligned}
G^I_{i\,\bar{i}}\left(\hat{a}_s,\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\epsilon\right)&=G^I_{i\,\bar{i}}\left(a_s(Q^2),1,\epsilon\right)+\int_{\frac{Q^2}{\mu_R^2}}^1\frac{dx}{x}A^I_{i\,\bar{i}}(a_s(x\mu_R^2))\\
&=\sum_{k=1}^{\infty}a_s^k(Q^2)G^I_{i\,\bar{i},k}(\epsilon)+\int_{\frac{Q^2}{\mu_R^2}}^1\frac{dx}{x}A^I_{i\,\bar{i}}(a_s(x\mu_R^2)).
\end{aligned} \tag{3.3.22}$$

However, the solutions of the logarithm of the form factor involves the unknown functions $G^I_{i\,\bar{i},k}$ which are observed to fulfill [40, 84] the following decomposition in terms of

collinear ($B^I_{i\,\bar{i}}$), soft ($f^I_{i\,\bar{i}}$) and UV ($\gamma^I_{i\,\bar{i}}$) anomalous dimensions:

$$G^I_{i\,\bar{i},k}(\epsilon) = 2\left(B^I_{i\,\bar{i},k} - \gamma^I_{i\,\bar{i},k}\right) + f^I_{i\,\bar{i},k} + C^I_{i\,\bar{i},k} + \sum_{l=1}^{\infty} \epsilon^l g^{I,l}_{i\,\bar{i},k}\,, \tag{3.3.23}$$

where, the constants $C^I_{i\,\bar{i},k}$ are given by [29]

$$\begin{aligned} C^I_{i\,\bar{i},1} &= 0\,, \\ C^I_{i\,\bar{i},2} &= -2\beta_0 g^{I,1}_{i\,\bar{i},1}\,, \\ C^I_{i\,\bar{i},3} &= -2\beta_1 g^{I,1}_{i\,\bar{i},1} - 2\beta_0\left(g^{I,1}_{i\,\bar{i},2} + 2\beta_0 g^{I,2}_{i\,\bar{i},1}\right). \end{aligned} \tag{3.3.24}$$

In the above expressions, $X^I_{i\,\bar{i},k}$ with $X = A, B, f$ and $\gamma^I_{i\,\bar{i},k}$ are defined through the series expansion in powers of a_s:

$$X^I_{i\,\bar{i}} \equiv \sum_{k=1}^{\infty} a_s^k X^I_{i\,\bar{i},k}\,, \qquad \text{and} \qquad \gamma^I_{i\,\bar{i}} \equiv \sum_{k=1}^{\infty} a_s^k \gamma^I_{i\,\bar{i},k}\,. \tag{3.3.25}$$

$f^I_{i\,\bar{i}}$ are introduced for the first time in the article [84] where it is shown to fulfill the maximally non-Abelian property up to two loop level whose validity is reconfirmed in [40] at three loop:

$$f^I_{q\bar{q}} = \frac{C_F}{C_A} f^I_{gg}\,. \tag{3.3.26}$$

This identity implies the soft anomalous dimensions for the production of a colorless particle in quark annihilation are related to the same appearing in the gluon fusion through a simple ratio of quadratic Casimirs of SU(N) gauge group. The same property is also obeyed by the cusp anomalous dimensions up to three loop level:

$$A^I_{q\bar{q}} = \frac{C_F}{C_A} A^I_{gg}\,. \tag{3.3.27}$$

It is not clear whether this nice property holds true beyond this order of perturbation

theory. Moreover, due to universality of the quantities denoted by X, these are independent of the operators insertion. These are only dependent on the initial state partons of any process:

$$X^I_{i\,\bar{i}} = X_{i\,\bar{i}}\,. \tag{3.3.28}$$

Moreover, these are independent of the quark flavors. Here, absence of I represents the independence of the quantities on the nature of colorless particles. $f^I_{i\,\bar{i}}$ can be found in [40, 84], $A^H_{i\,\bar{i}}$ in [40, 69, 85, 86] and $B^H_{i\,\bar{i}}$ in [40, 85] up to three loop level. For readers' convenience we list them all up to three loop level in the Appendix B. Utilising the results of these known quantities and comparing the above expansions of $G^I_{i\,\bar{i},k}(\epsilon)$ with the results of the logarithm of the form factors, we extract the relevant $g^{I,l}_{i\,\bar{i},k}$ and $\gamma^I_{i\,\bar{i},k}$'s up to three loop level using Eq. (3.3.20), (3.3.21) and (3.3.23). The relevant one loop terms for $I = H$ and $i\,\bar{i} = gg$ are found to be

$$g^{H,1}_{gg,1} = C_A\zeta_2\,, \qquad g^{H,2}_{gg,1} = C_A\left\{1 - \frac{7}{3}\zeta_3\right\}, \qquad g^{H,3}_{gg,1} = C_A\left\{\frac{47}{80}{\zeta_2}^2 - \frac{3}{2}\right\}, \tag{3.3.29}$$

the relevant two loop terms are

$$\begin{aligned}
g^{H,1}_{gg,2} &= {C_A}^2\left\{\frac{67}{3}\zeta_2 - \frac{44}{3}\zeta_3 + \frac{4511}{81}\right\} + C_A n_f\left\{-\frac{10}{3}\zeta_2 - \frac{40}{3}\zeta_3 - \frac{1724}{81}\right\} \\
&\quad + C_F n_f\left\{16\zeta_3 - \frac{67}{3}\right\}, \\
g^{H,2}_{gg,2} &= {C_A}^2\left\{\frac{671}{120}{\zeta_2}^2 + \frac{5}{3}\zeta_2\zeta_3 - \frac{142}{9}\zeta_2 + \frac{1139}{27}\zeta_3 - 39\zeta_5 - \frac{141677}{972}\right\} + C_A n_f\left\{\frac{259}{60}{\zeta_2}^2\right. \\
&\quad \left. + \frac{16}{9}\zeta_2 + \frac{604}{27}\zeta_3 + \frac{24103}{486}\right\} + C_F n_f\left\{-\frac{16}{3}{\zeta_2}^2 - \frac{7}{3}\zeta_2 - \frac{92}{3}\zeta_3 + \frac{2027}{36}\right\},
\end{aligned} \tag{3.3.30}$$

and the required three loop term is

$$g^{H,1}_{gg,3} = {C_A}^2 n_f\left\{-\frac{128}{45}{\zeta_2}^2 - \frac{88}{9}\zeta_2\zeta_3 - \frac{14225}{243}\zeta_2 - \frac{11372}{81}\zeta_3 + \frac{272}{3}\zeta_5 - \frac{5035009}{2187}\right\}$$

$$+ C_F {n_f}^2 \left\{ -\frac{368}{45}{\zeta_2}^2 - \frac{88}{9}\zeta_2 - \frac{1376}{9}\zeta_3 + \frac{6508}{27} \right\} + C_A C_F n_f \left\{ \frac{1568}{45}{\zeta_2}^2 + 40\zeta_2\zeta_3 + \frac{503}{18}\zeta_2 + \frac{20384}{27}\zeta_3 + \frac{608}{3}\zeta_5 - \frac{473705}{324} \right\} + C_A {n_f}^2 \left\{ \frac{232}{45}{\zeta_2}^2 + \frac{100}{27}\zeta_2 + \frac{6992}{81}\zeta_3 + \frac{912301}{4374} \right\} + C_A^3 \left\{ -\frac{12352}{315}{\zeta_2}^3 - \frac{5744}{45}{\zeta_2}^2 - \frac{1496}{9}\zeta_2\zeta_3 + \frac{221521}{486}\zeta_2 - \frac{104}{3}{\zeta_3}^2 - \frac{57830}{27}\zeta_3 + \frac{3080}{3}\zeta_5 + \frac{39497339}{8748} \right\} + {C_F}^2 n_f \left\{ 296\zeta_3 - 480\zeta_5 + \frac{304}{3} \right\}. \quad (3.3.31)$$

Similarly for $I = \mathrm{DY}$ and $i\,\bar{i} = q\bar{q}$, we have for one loop

$$g^{\mathrm{DY},1}_{q\bar{q},1} = C_F \{\zeta_2 - 8\}\,, \qquad g^{\mathrm{DY},2}_{q\bar{q},1} = C_F \left\{ -\frac{3}{4}\zeta_2 - \frac{7}{3}\zeta_3 + 8 \right\},$$
$$g^{\mathrm{DY},3}_{q\bar{q},1} = C_F \left\{ \frac{47}{80}{\zeta_2}^2 + \zeta_2 + \frac{7}{4}\zeta_3 - 8 \right\}, \quad (3.3.32)$$

for two loop we require

$$g^{\mathrm{DY},1}_{q\bar{q},2} = {C_F}^2 \left\{ -\frac{88}{5}{\zeta_2}^2 + 58\zeta_2 - 60\zeta_3 - \frac{1}{4} \right\} + C_A C_F \left\{ \frac{88}{5}{\zeta_2}^2 - \frac{575}{18}\zeta_2 + \frac{260}{3}\zeta_3 - \frac{70165}{324} \right\} + C_F n_f \left\{ \frac{37}{9}\zeta_2 - \frac{8}{3}\zeta_3 + \frac{5813}{162} \right\},$$
$$g^{\mathrm{DY},2}_{q\bar{q},2} = {C_F}^2 \left\{ \frac{108}{5}{\zeta_2}^2 - 28\zeta_2\zeta_3 - \frac{437}{4}\zeta_2 + 184\zeta_3 + 12\zeta_5 - \frac{109}{16} \right\} + C_A C_F \left\{ -\frac{653}{24}{\zeta_2}^2 + \frac{89}{3}\zeta_2\zeta_3 + \frac{7297}{108}\zeta_2 - \frac{12479}{54}\zeta_3 - 51\zeta_5 + \frac{1547797}{3888} \right\} + C_F n_f \left\{ \frac{7}{12}{\zeta_2}^2 - \frac{425}{54}\zeta_2 + \frac{301}{27}\zeta_3 - \frac{129389}{1944} \right\} \quad (3.3.33)$$

and the only required three loop term is

$$g^{\mathrm{DY},1}_{q\bar{q},3} = {C_F}^3 \left\{ \frac{21584}{105}{\zeta_2}^3 - 534{\zeta_2}^2 + 840\zeta_2\zeta_3 - 206\zeta_2 + 48{\zeta_3}^2 - 2130\zeta_3 + 1992\zeta_5 - \frac{1527}{4} \right\} + {C_A}{C_F}^2 \left\{ -\frac{15448}{105}{\zeta_2}^3 + \frac{2432}{45}{\zeta_2}^2 - \frac{3448}{3}\zeta_2\zeta_3 + \frac{55499}{18}\zeta_2 + 296{\zeta_3}^2 - \frac{23402}{9}\zeta_3 - \frac{3020}{3}\zeta_5 + \frac{230}{3} \right\} + {C_F}^2 n_f \left\{ -\frac{704}{45}{\zeta_2}^2 - \frac{152}{3}\zeta_2\zeta_3 - \frac{7541}{18}\zeta_2 + \frac{19700}{27}\zeta_3 - \frac{368}{3}\zeta_5 + \frac{73271}{162} \right\} + {C_A}^2 C_F \left\{ -\frac{6152}{63}{\zeta_2}^3 + \frac{37271}{90}{\zeta_2}^2 + \frac{1786}{9}\zeta_2\zeta_3 - \frac{1083305}{486}\zeta_2 - \frac{1136}{3}{\zeta_3}^2 \right.$$

$$+\frac{85883}{18}\zeta_3+\frac{688}{3}\zeta_5-\frac{48902713}{8748}\Big\}+C_F{n_f}^2\Big\{-\frac{40}{9}{\zeta_2}^2-\frac{3466}{81}\zeta_2+\frac{536}{81}\zeta_3$$
$$-\frac{258445}{2187}\Big\}+C_AC_Fn_f\Big\{-\frac{1298}{45}{\zeta_2}^2+\frac{392}{9}\zeta_2\zeta_3+\frac{155008}{243}\zeta_2-\frac{68660}{81}\zeta_3-72\zeta_5$$
$$+\frac{3702974}{2187}\Big\}+C_Fn_{f,v}\left(\frac{N^2-4}{N}\right)\Big\{-\frac{6}{5}{\zeta_2}^2+30\zeta_2+14\zeta_3-80\zeta_5+12\Big\}. \tag{3.3.34}$$

$n_{f,v}$ is proportional to the charge weighted sum of the quark flavors [42]. The other constants $\gamma^I_{i\,\bar{i},k}$, appearing in the Eq. (3.3.23), up to three loop ($k=3$) are obtained as

$$\gamma^H_{gg,1}=\beta_0\,,\qquad \gamma^H_{gg,2}=2\beta_1\,,\qquad \gamma^H_{gg,3}=3\beta_2$$
$$\text{and}\qquad \gamma^{\rm DY}_{q\bar{q}}=0\,. \tag{3.3.35}$$

β_i are the coefficient of QCD-β function, presented in Eq. (2.3.12). These will be utilised in the next subsection to determine the overall operator renormalisation constants.

3.3.2 Operator Renormalisation Constant

The strong coupling constant renormalisation through Z_{a_s} may not be sufficient to make the form factor $\mathcal{F}^I_{i\,\bar{i}}$ completely UV finite, one needs to perform additional renormalisation to remove the residual UV divergences which is reflected through the presence of non-zero $\gamma^I_{i\,\bar{i}}$. Due to non-zero γ^H_{gg} in Eq. (3.3.35), overall UV renormalisation is required for the Higgs boson production in gluon fusion. However, for DY this is not required. This additional renormalisation is called the overall operator renormalisation which is performed through the constant $Z^I_{i\,\bar{i}}$. This is determined by solving the underlying RG equation:

$$\mu_R^2\frac{d}{d\mu_R^2}\ln Z^I_{i\,\bar{i}}\left(\hat{a}_s,\mu_R^2,\mu^2,\epsilon\right)=\sum_{k=1}^{\infty}a_s^k(\mu_R^2)\gamma^I_{i\,\bar{i},k}\,. \tag{3.3.36}$$

Using the results of $\gamma^I_{i\,\bar{i},k}$ from Eq. (3.3.35) and solving the above RG equation following the methodology described in the Appendix C, we obtain the following overall renormal-

isation constant up to three loop level:

$$Z^I_{i\,\bar{i}} = 1 + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\mu_R^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{Z}^{I,(k)}_{i\,\bar{i}} \tag{3.3.37}$$

where,

$$\begin{aligned}
\hat{Z}^{H,(1)}_{gg} &= \frac{1}{\epsilon}\Big\{2\beta_0\Big\}, \\
\hat{Z}^{H,(2)}_{gg} &= \frac{1}{\epsilon}\Big\{2\beta_1\Big\}, \\
\hat{Z}^{H,(3)}_{gg} &= \frac{1}{\epsilon^2}\Big\{-2\beta_0\beta_1\Big\} + \frac{1}{\epsilon}\Big\{2\beta_2\Big\} \\
\text{and} \qquad \hat{Z}^{\rm DY}_{q\bar{q}} &= 1\,.
\end{aligned} \tag{3.3.38}$$

3.3.3 Mass Factorisation Kernel

The UV finite form factor contains additional divergences arising from the soft and collinear regions of the loop momenta. In this section, we address the issue of collinear divergences and describe a prescription to remove them. The collinear singularities that arise in the massless limit of partons are removed by absorbing the divergences in the bare PDF through renormalisation of the PDF. This prescription is called the mass factorisation (MF) and is performed at the factorisation scale μ_F. In the process of performing this, one needs to introduce mass factorisation kernels $\Gamma^I_{ij}(\hat{a}_s, \mu^2, \mu_F^2, z_j, \epsilon)$ which essentially absorb the collinear singularities. More specifically, MF removes the collinear singularities arising from the collinear configuration associated with the initial state partons. The final state collinear singularities are guaranteed to go away once the phase space integrals are performed after summing over the contributions from virtual and real emission diagrams, thanks to Kinoshita-Lee-Nauenberg theorem. The kernels satisfy the following

RG equation :

$$\mu_F^2 \frac{d}{d\mu_F^2} \Gamma_{ij}^I(z_j, \mu_F^2, \epsilon) = \frac{1}{2} \sum_k P_{ik}^I \left(z_j, \mu_F^2\right) \otimes \Gamma_{kj}^I \left(z_j, \mu_F^2, \epsilon\right) \tag{3.3.39}$$

where, $P^I \left(z_j, \mu_F^2\right)$ are Altarelli-Parisi splitting functions (matrix valued). Expanding $P^I \left(z_j, \mu_F^2\right)$ and $\Gamma^I(z_j, \mu_F^2, \epsilon)$ in powers of the strong coupling constant we get

$$P_{ij}^I(z_j, \mu_F^2) = \sum_{k=1}^{\infty} a_s^k(\mu_F^2) P_{ij}^{I,(k-1)}(z) \tag{3.3.40}$$

and

$$\Gamma_{ij}^I(z, \mu_F^2, \epsilon) = \delta_{ij}\delta(1-z) + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\mu_F^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Gamma}_{ij}^{I,(k)}(z, \epsilon)\,. \tag{3.3.41}$$

The RG equation of $\Gamma^I(z, \mu_F^2, \epsilon)$, Eq. (3.3.39), can be solved in dimensional regularisation in powers of $\hat{a}_s$. In the $\overline{MS}$ scheme, the kernel contains only the poles in ϵ. The solutions [28] up to the required order $\Gamma^{I,(3)}(z, \epsilon)$ in terms of $P^{I,(k)}(z)$ are presented in the Appendix (C.0.20). The relevant ones up to three loop, $P^{I,(0)}(z)$, $P^{I,(1)}(z)$ and $P^{I,(2)}(z)$ are computed in the articles [69, 85]. For the SV cross section only the diagonal parts of the splitting functions $P_{ij}^{I,(k)}(z)$ and kernels $\Gamma_{ij}^{I,(k)}(z, \epsilon)$ contribute since the diagonal elements of $P_{ij}^{I,(k)}(z)$ contain $\delta(1-z)$ and $\mathcal{D}_0$ whereas the off-diagonal elements are regular in the limit $z \to 1$. The most remarkable fact is that these quantities are universal, independent of the operators insertion. Hence, for the processes under consideration, we make use of the existing process independent results of kernels and splitting functions:

$$\Gamma_{ij}^H = \Gamma_{ij}^{\rm DY} = \Gamma_{ij}^I = \Gamma_{ij} \qquad \text{and} \qquad P_{ij}^H = P_{ij}^{\rm DY} = P_{ij}^I = P_{ij}\,. \tag{3.3.42}$$

The absence of I represents the independence of these quantities on I. In the next subsection, we discuss the only remaining ingredient, namely, the soft-collinear distribution.

3.3.4 Soft-Collinear Distribution for Rapidity

The resulting expression from form factor along with the operator renormalisation constant and mass factorisation kernel is not completely finite, it contains some residual divergences which get cancelled against the contribution arising from soft gluon emissions. Hence, the finiteness of $\Delta^{I,\mathrm{SV}}_{Y,i\,\bar{i}}$ in Eq. (3.3.14) in the limit $\epsilon \to 0$ demands that the soft-collinear distribution, $\Phi^{I}_{Y,i\,\bar{i}}(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon)$, has pole structure in ϵ similar to that of residual divergences. In article [39], it was shown that $\Phi^{I}_{Y,i\,\bar{i}}$ must obey KG type integro-differential equation, which we call $\overline{KG}_Y$ equation, to remove that residual divergences:

$$q^2\frac{d}{dq^2}\Phi^{I}_{Y,i\,\bar{i}}\left(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon\right) = \frac{1}{2}\left[\overline{K}^{I}_{Y,i\,\bar{i}}\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z_1, z_2, \epsilon\right) + \overline{G}^{I}_{Y,i\,\bar{i}}\left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z_1, z_2, \epsilon\right)\right]. \tag{3.3.43}$$

$\overline{K}^{I}_{Y,i\,\bar{i}}$ and $\overline{G}^{I}_{Y,i\,\bar{i}}$ play similar roles as those of $K^{I}_{i\,\bar{i}}$ and $G^{I}_{i\,\bar{i}}$ in Eq. (3.3.19), respectively. Also, $\Phi^{I}_{Y,i\,\bar{i}}(\hat{a}_s, q^2, \mu^2, z, \epsilon)$ being independent of μ_R^2 satisfy the RG equation

$$\mu_R^2\frac{d}{d\mu_R^2}\Phi^{I}_{Y,i\,\bar{i}}(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) = 0\,. \tag{3.3.44}$$

This RG invariance and the demand of cancellation of all the residual divergences arising from $\mathcal{F}^{I}_{i\,\bar{i}}, Z^{I}_{i\,\bar{i}}$ and $\Gamma^{I}_{i\,\bar{i}}$ against $\Phi^{I}_{Y,i\,\bar{i}}$ implies the solution of the $\overline{KG}_Y$ equation as [39]

$$\Phi^{I}_{Y,i\,\bar{i}}(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) = \sum_{k=1}^{\infty}\hat{a}_s^k S_\epsilon^k\left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}}\hat{\Phi}^{I}_{Y,i\,\bar{i},k}(z_1, z_2, \epsilon) \tag{3.3.45}$$

with

$$\begin{aligned}
\hat{\Phi}^{I}_{Y,i\,\bar{i},k}(z_1, z_2, \epsilon) &= \left\{(k\epsilon)^2\frac{1}{4(1-z_1)(1-z_2)}\left[(1-z_1)(1-z_2)\right]^{k\frac{\epsilon}{2}}\right\}\hat{\Phi}^{I}_{Y,i\,\bar{i},k}(\epsilon)\,,\\
\hat{\Phi}^{I}_{Y,i\,\bar{i},k}(\epsilon) &= \hat{\mathcal{L}}^{I}_{i\bar{i},k}\left(A^I_l \to -A^I_l, G^I_l \to \overline{\mathcal{G}}^{I}_{Y,i\,\bar{i},l}(\epsilon)\right).
\end{aligned} \tag{3.3.46}$$

where, $\hat{\mathcal{L}}^{I}_{i\,\bar{i},k}(\epsilon)$ are defined in Eq. (3.3.21). In Appendix F, the derivation of this solution is depicted in great details. The z_j-independent constants $\overline{\mathcal{G}}^{I}_{Y,i\,\bar{i},l}(\epsilon)$ can be obtained by comparing the poles as well as non-pole terms in ϵ of $\hat{\Phi}^{I}_{Y,i\,\bar{i},k}(\epsilon)$ with those arising from form factor, overall renormalisation constant and splitting functions. We find

$$\overline{\mathcal{G}}^{I}_{Y,i\,\bar{i},k}(\epsilon) = -f^{I}_{i\,\bar{i},k} + \overline{C}^{I}_{Y,i\,\bar{i},k} + \sum_{l=1}^{\infty} \epsilon^{l} \overline{\mathcal{G}}^{I,l}_{Y,i\,\bar{i},k} \tag{3.3.47}$$

where

$$\begin{aligned}
\overline{C}^{I}_{Y,i\,\bar{i},1} &= 0\,, \\
\overline{C}^{I}_{Y,i\,\bar{i},2} &= -2\beta_0 \overline{\mathcal{G}}^{I,1}_{Y,i\,\bar{i},1}\,, \\
\overline{C}^{I}_{Y,i\,\bar{i},3} &= -2\beta_1 \overline{\mathcal{G}}^{I,1}_{Y,i\,\bar{i},1} - 2\beta_0 \left(\overline{\mathcal{G}}^{I,1}_{Y,i\,\bar{i},2} + 2\beta_0 \overline{\mathcal{G}}^{I,2}_{Y,i\,\bar{i},1} \right).
\end{aligned} \tag{3.3.48}$$

In-depth understanding about the pole structures including the single pole [84] of the form factors, overall operator renormalisation constants and mass factorisation kernels helps us to predict all the poles of the soft-collinear distribution. However, to determine the finite part of the SV corrections to the rapidity distribution, we need the coefficients of ϵ^k ($k \geq 1$), $\overline{\mathcal{G}}^{I,k}_{Y,i\,\bar{i},l}$. Now, we address the question of determining those constants. This is achieved with the help of an identity which has been found:

$$\int_0^1 dx_1^0 \int_0^1 dx_2^0 \left(x_1^0 x_2^0\right)^{\mathcal{N}-1} \frac{d\sigma^{I}_{ij}}{dY} = \int_0^1 d\tau\, \tau^{\mathcal{N}-1} \sigma^{I}_{ij}\,. \tag{3.3.49}$$

In the large $\mathcal{N}$ limit i.e. $\mathcal{N} \to \infty$ the above identity relates [39] the $\hat{\Phi}^{I}_{Y,i\,\bar{i},k}(\epsilon)$ with the corresponding $\hat{\Phi}^{I}_{i\,\bar{i},k}(\epsilon)$ appearing in the computation of SV cross section, Eq. (2.3.38):

$$\hat{\Phi}^{I}_{Y,i\,\bar{i},k}(\epsilon) = \frac{\Gamma(1+k\epsilon)}{\Gamma^2(1+k\frac{\epsilon}{2})} \hat{\Phi}^{I}_{i\,\bar{i},k}(\epsilon)\,. \tag{3.3.50}$$

Hence, the computation of soft-collinear distribution for the inclusive production cross

section is sufficient to determine the corresponding one for the rapidity distribution. All the properties satisfied by $\hat{\Phi}^I_{i\,\bar{i},k}(\epsilon)$ are obeyed by $\hat{\Phi}^I_{Y,i\,\bar{i},k}(\epsilon)$ too, see Sec. 2.3.4 for all the details. Utilising the relation (3.3.50), the relevant constants $\overline{\mathcal{G}}^{I,l}_{Y,i\,\bar{i},k}$ to determine $\hat{\Phi}^I_{Y,i\,\bar{i},k}(\epsilon)$ up to N^{3}LO level are found to be

$$
\begin{aligned}
\overline{\mathcal{G}}^{\mathrm{DY},1}_{Y,q\bar{q},1} &= C_F\left\{-\zeta_2\right\},\\
\overline{\mathcal{G}}^{\mathrm{DY},2}_{Y,q\bar{q},1} &= C_F\left\{\frac{1}{3}\zeta_3\right\},\\
\overline{\mathcal{G}}^{\mathrm{DY},3}_{Y,q\bar{q},1} &= C_F\left\{\frac{1}{80}\zeta_2^2\right\},\\
\overline{\mathcal{G}}^{\mathrm{DY},1}_{Y,q\bar{q},2} &= C_AC_F\left\{-4\zeta_2{}^2-\frac{67}{3}\zeta_2-\frac{44}{3}\zeta_3+\frac{2428}{81}\right\}+C_Fn_f\left\{\frac{8}{3}\zeta_3+\frac{10}{3}\zeta_2-\frac{328}{81}\right\},\\
\overline{\mathcal{G}}^{\mathrm{DY},2}_{Y,q\bar{q},2} &= C_AC_F\left\{-\frac{319}{120}\zeta_2{}^2-\frac{71}{3}\zeta_2\zeta_3+\frac{202}{9}\zeta_2+\frac{469}{27}\zeta_3+43\zeta_5-\frac{7288}{243}\right\}\\
&\quad+C_Fn_f\left\{\frac{29}{60}\zeta_2{}^2-\frac{28}{9}\zeta_2-\frac{70}{27}\zeta_3+\frac{976}{243}\right\},\\
\overline{\mathcal{G}}^{\mathrm{DY},1}_{Y,q\bar{q},3} &= C_A{}^2C_F\left\{\frac{17392}{315}\zeta_2{}^3+\frac{1538}{45}\zeta_2{}^2+\frac{4136}{9}\zeta_2\zeta_3-\frac{379417}{486}\zeta_2+\frac{536}{3}\zeta_3{}^2-936\zeta_3\right.\\
&\quad\left.-\frac{1430}{3}\zeta_5+\frac{7135981}{8748}\right\}+C_AC_Fn_f\left\{-\frac{1372}{45}\zeta_2{}^2-\frac{392}{9}\zeta_2\zeta_3+\frac{51053}{243}\zeta_2\right.\\
&\quad\left.+\frac{12356}{81}\zeta_3+\frac{148}{3}\zeta_5-\frac{716509}{4374}\right\}+C_Fn_f{}^2\left\{\frac{152}{45}\zeta_2{}^2-\frac{316}{27}\zeta_2-\frac{320}{81}\zeta_3+\frac{11584}{2187}\right\}\\
&\quad+C_F{}^2n_f\left\{\frac{152}{15}\zeta_2{}^2-40\zeta_2\,\zeta_3+\frac{275}{6}\zeta_2+\frac{1672}{27}\zeta_3+\frac{112}{3}\zeta_5-\frac{42727}{324}\right\}. \qquad (3.3.51)
\end{aligned}
$$

The corresponding constants for the Higgs boson production in gluon fusion can be obtained by employing the identity

$$
\overline{\mathcal{G}}^{H,k}_{Y,gg,i}=\frac{C_A}{C_F}\overline{\mathcal{G}}^{\mathrm{DY},k}_{Y,q\bar{q},i}\,. \qquad (3.3.52)
$$

The results up to $O(a_s^2)$ were present in the literature [39] and the term at $O(a_s^3)$ is computed for the first time by us in the article [12]. Using these, the $\Phi^I_{Y,i\,\bar{i}}$ can be obtained which are presented up to three loops in the Appendix F.0.1. This completes all the ingredients required to compute the SV correction to the rapidity distributions up to N^{3}LO that

are provided in the next section.

3.4 Results of the SV Rapidity Distributions

In this section, we present our findings of the SV rapidity distributions at N^{3}LO along with the results of the previous orders. Expanding the SV rapidity distribution , Eq. (3.3.14), in powers of $a_s(\mu_F^2)$, we obtain

$$\Delta^{I,\mathrm{SV}}_{Y,i\,\bar{i}}\left(z_1,z_2,q^2,\mu_F^2\right)=\sum_{k=1}^{\infty}a_s^k(\mu_F^2)\Delta^{I,\mathrm{SV}}_{Y,i\,\bar{i},k}\left(z_1,z_2,q^2,\mu_F^2\right) \tag{3.4.1}$$

where,

$$\begin{aligned}\Delta^{I,\mathrm{SV}}_{Y,i\,\bar{i},k}&=\Delta^{I,\mathrm{SV}}_{Y,i\,\bar{i},k}|_{\delta\delta}\delta(1-z_1)\delta(1-z_2)+\sum_{j=0}^{\infty}\Delta^{I,\mathrm{SV}}_{Y,i\,\bar{i},k}|_{\delta\mathcal{D}_j}\delta(1-z_2)\mathcal{D}_j\\&+\sum_{j=0}^{\infty}\Delta^{I,\mathrm{SV}}_{Y,i\,\bar{i},k}|_{\delta\overline{\mathcal{D}}_j}\delta(1-z_2)\overline{\mathcal{D}}_j+\sum_{j\circledS l}\Delta^{I,\mathrm{SV}}_{Y,i\,\bar{i},k}|_{\mathcal{D}_j\overline{\mathcal{D}}_l}\mathcal{D}_j\overline{\mathcal{D}}_l\,.\end{aligned} \tag{3.4.2}$$

The symbol $j\circledS l$ implies $j,l\geq 0$ and $j+l\leq(2k-2)$. Terms proportional to $\mathcal{D}_j$ and/or $\overline{\mathcal{D}}_j$ in Eq. (3.4.2) were obtained in [39] and the first term is possible to calculate as the results for the threshold N^{3}LO QCD corrections to the production cross section are now available for DY [32] and the Higgs boson [33] productions. By setting $\mu_R^2=\mu_F^2$ we present the results. For $I=H$ and $i\,\bar{i}=gg$, we obtain [12]

$$\begin{aligned}\Delta^{H,\mathrm{SV}}_{Y,gg,1}&=\delta(1-z_1)\delta(1-z_2)\left[C_A\left\{12\zeta_2\right\}\right]+\mathcal{D}_0\delta(1-z_2)\left[\log\left(\frac{q^2}{\mu_F^2}\right)C_A\left\{4\right\}\right]\\&+\mathcal{D}_0\overline{\mathcal{D}}_0\left[C_A\left\{4\right\}\right]+\mathcal{D}_1\delta(1-z_2)\left[C_A\left\{4\right\}\right]+\overline{\mathcal{D}}_0\delta(1-z_1)\left[\log\left(\frac{q^2}{\mu_F^2}\right)C_A\left\{4\right\}\right]\\&+\overline{\mathcal{D}}_1\delta(1-z_1)\left[C_A\left\{4\right\}\right],\\\Delta^{H,\mathrm{SV}}_{Y,gg,2}&=\delta(1-z_1)\delta(1-z_2)\left[C_A^2\left\{93-44\zeta_3+\frac{268}{3}\zeta_2+\frac{252}{5}\zeta_2^2\right\}+n_fC_A\left\{-\frac{80}{3}-8\zeta_3\right.\right.\\&\left.\left.-\frac{40}{3}\zeta_2\right\}+n_fC_F\left\{-\frac{67}{3}+16\zeta_3\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{-24+56\zeta_3-44\zeta_2\right\}\right.\end{aligned}$$

$$
\begin{aligned}
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A \left\{8 + 8\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F \left\{4\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^2 \left\{-16\zeta_2\right\}\Big] \\
&+ \mathcal{D}_0\delta(1-z_2)\Big[C_A^2\left\{-\frac{808}{27} + 60\zeta_3 + \frac{44}{3}\zeta_2\right\} + n_f C_A\left\{\frac{112}{27} - \frac{8}{3}\zeta_2\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{\frac{268}{9} + 8\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A\left\{-\frac{40}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{-\frac{22}{3}\right\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A\left\{\frac{4}{3}\right\}\Big] + \mathcal{D}_0\overline{\mathcal{D}}_0\Big[C_A^2\left\{\frac{268}{9} + 8\zeta_2\right\} + n_f C_A\left\{-\frac{40}{9}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{-\frac{44}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A\left\{\frac{8}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{16\right\}\Big] \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_1\Big[C_A^2\left\{-\frac{44}{3}\right\} + n_f C_A\left\{\frac{8}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{48\right\}\Big] + \mathcal{D}_0\overline{\mathcal{D}}_2\Big[C_A^2\left\{24\right\}\Big] \\
&+ \mathcal{D}_1\delta(1-z_2)\Big[C_A^2\left\{\frac{268}{9} + 8\zeta_2\right\} + n_f C_A\left\{-\frac{40}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{-\frac{44}{3}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A\left\{\frac{8}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{16\right\}\Big] + \mathcal{D}_1\overline{\mathcal{D}}_0\Big[C_A^2\left\{-\frac{44}{3}\right\} + n_f C_A\left\{\frac{8}{3}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{48\right\}\Big] + \mathcal{D}_1\overline{\mathcal{D}}_1\Big[C_A^2\left\{48\right\}\Big] + \mathcal{D}_2\delta(1-z_2)\Big[C_A^2\left\{-\frac{22}{3}\right\} + n_f C_A\left\{\frac{4}{3}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{24\right\}\Big] + \mathcal{D}_2\overline{\mathcal{D}}_0\Big[C_A^2\left\{24\right\}\Big] + \mathcal{D}_3\delta(1-z_2)\Big[C_A^2\left\{8\right\}\Big] \\
&+ \overline{\mathcal{D}}_0\delta(1-z_1)\Big[C_A^2\left\{-\frac{808}{27} + 60\zeta_3 + \frac{44}{3}\zeta_2\right\} + n_f C_A\left\{\frac{112}{27} - \frac{8}{3}\zeta_2\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{\frac{268}{9} + 8\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A\left\{-\frac{40}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{-\frac{22}{3}\right\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A\left\{\frac{4}{3}\right\}\Big] + \overline{\mathcal{D}}_1\delta(1-z_1)\Big[C_A^2\left\{\frac{268}{9} + 8\zeta_2\right\} + n_f C_A\left\{-\frac{40}{9}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{-\frac{44}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A\left\{\frac{8}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{16\right\}\Big] \\
&+ \overline{\mathcal{D}}_2\delta(1-z_1)\Big[C_A^2\left\{-\frac{22}{3}\right\} + n_f C_A\left\{\frac{4}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_A^2\left\{24\right\}\Big] \\
&+ \overline{\mathcal{D}}_3\delta(1-z_1)\Big[C_A^2\left\{8\right\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{Y,gg,3}^{H,\mathrm{SV}} &= \delta(1-z_1)\delta(1-z_2)\Big[C_A^3\left\{\frac{215131}{81} + \frac{1364}{9}\zeta_5 - \frac{54820}{27}\zeta_3 + \frac{1600}{3}\zeta_3^2 + \frac{41914}{27}\zeta_2\right. \\
&\left. - 88\zeta_2\zeta_3 + \frac{40432}{135}\zeta_2^2 + \frac{12032}{105}\zeta_2^3\right\} + n_f C_A^2\left\{-\frac{98059}{81} + \frac{1192}{9}\zeta_5 + \frac{2536}{27}\zeta_3\right. \\
&\left. - \frac{7108}{27}\zeta_2 - 272\zeta_2\zeta_3 + \frac{1240}{27}\zeta_2^2\right\} + n_f C_F C_A\left\{-\frac{63991}{81} + 160\zeta_5 + 400\zeta_3 - \frac{2270}{9}\zeta_2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ 288\zeta_2\zeta_3 + \frac{176}{45}\zeta_2^2\Big\} + n_f C_F^2\Big\{\frac{608}{9} - 320\zeta_5 + \frac{592}{3}\zeta_3\Big\} + n_f^2 C_A\Big\{\frac{2515}{27} + \frac{112}{3}\zeta_3 \\
&- \frac{64}{3}\zeta_2 - \frac{208}{15}\zeta_2^2\Big\} + n_f^2 C_F\Big\{\frac{8962}{81} - \frac{224}{3}\zeta_3 - \frac{184}{9}\zeta_2 - \frac{32}{45}\zeta_2^2\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{ - \frac{8284}{9} + 224\zeta_5 + \frac{10408}{9}\zeta_3 - \frac{22528}{27}\zeta_2 - 224\zeta_2\zeta_3 - \frac{1276}{3}\zeta_2^2\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{\frac{4058}{9} - \frac{1120}{9}\zeta_3 + \frac{8488}{27}\zeta_2 + \frac{232}{3}\zeta_2^2\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{\frac{616}{3} - \frac{352}{3}\zeta_3 + 72\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\Big\{ - 4\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{ - \frac{370}{9} - \frac{32}{3}\zeta_3 - \frac{160}{9}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F\Big\{ - \frac{104}{3} + \frac{64}{3}\zeta_3\Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{88 - 264\zeta_3 - \frac{692}{9}\zeta_2 - \frac{384}{5}\zeta_2^2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{ - \frac{136}{3} + 48\zeta_3 \\
&- \frac{208}{9}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{ - \frac{44}{3}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{\frac{16}{3} + \frac{16}{3}\zeta_2\Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F\Big\{\frac{8}{3}\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{\frac{128}{3}\zeta_3 + \frac{176}{3}\zeta_2\Big\} \\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{ - \frac{32}{3}\zeta_2\Big\}\Big] + \mathcal{D}_0\delta(1-z_2)\Big[C_A^3\Big\{ - \frac{297029}{729} + 192\zeta_5 + \frac{27128}{27}\zeta_3 \\
&+ \frac{18056}{81}\zeta_2 - \frac{608}{3}\zeta_2\zeta_3 + \frac{88}{5}\zeta_2^2\Big\} + n_f C_A^2\Big\{\frac{62626}{729} - \frac{392}{3}\zeta_3 - \frac{6416}{81}\zeta_2 + \frac{16}{5}\zeta_2^2\Big\} \\
&+ n_f C_F C_A\Big\{\frac{1711}{27} - \frac{304}{9}\zeta_3 - 8\zeta_2 - \frac{32}{5}\zeta_2^2\Big\} + n_f^2 C_A\Big\{ - \frac{1856}{729} - \frac{32}{27}\zeta_3 + \frac{160}{27}\zeta_2\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{\frac{61138}{81} - 704\zeta_3 + \frac{104}{9}\zeta_2 - \frac{64}{5}\zeta_2^2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{ - \frac{16844}{81} \\
&+ 32\zeta_3 + \frac{64}{3}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{ - 126 + 96\zeta_3\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{\frac{400}{81} \\
&- \frac{32}{9}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{ - \frac{6152}{27} + 352\zeta_3 - \frac{176}{3}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{\frac{2020}{27} \\
&+ \frac{32}{3}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{20\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{ - \frac{80}{27}\Big\} \\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{\frac{484}{27} - 64\zeta_2\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{ - \frac{176}{27}\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{\frac{16}{27}\Big\}\Big] \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_0\Big[C_A^3\Big\{\frac{61138}{81} - 704\zeta_3 + \frac{104}{9}\zeta_2 - \frac{64}{5}\zeta_2^2\Big\} + n_f C_A^2\Big\{ - \frac{16844}{81} + 32\zeta_3 + \frac{64}{3}\zeta_2\Big\} \\
&+ n_f C_F C_A\Big\{ - 126 + 96\zeta_3\Big\} + n_f^2 C_A\Big\{\frac{400}{81} - \frac{32}{9}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{ - \frac{5392}{9}
\end{aligned}
$$

$$+ 960\zeta_3 + 176\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2 \Big\{\frac{4072}{27} - 32\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{24\Big\}$$
$$+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A \Big\{-\frac{160}{27}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^3 \Big\{292 - 192\zeta_2\Big\}$$
$$+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2 \Big\{-\frac{496}{9}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A \Big\{\frac{16}{9}\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_A^3 \Big\{-\frac{176}{3}\Big\}$$
$$+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2 \Big\{\frac{32}{3}\Big\}\Big] + \mathcal{D}_0\overline{\mathcal{D}}_1\Big[C_A^3\Big\{-\frac{16816}{27} + 976\zeta_3 + \frac{1232}{3}\zeta_2\Big\}$$
$$+ n_f C_A^2\Big\{\frac{3656}{27} - \frac{224}{3}\zeta_2\Big\} + n_f C_F C_A\Big\{8\Big\} + n_f^2 C_A\Big\{-\frac{160}{27}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{\frac{7400}{9}$$
$$- 384\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{-\frac{1312}{9}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{\frac{32}{9}\Big\}$$
$$+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{-264\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{48\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{64\Big\}\Big]$$
$$+ \mathcal{D}_0\overline{\mathcal{D}}_2\Big[C_A^3\Big\{\frac{3700}{9} - 192\zeta_2\Big\} + n_f C_A^2\Big\{-\frac{656}{9}\Big\} + n_f^2 C_A\Big\{\frac{16}{9}\Big\}$$
$$+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{-\frac{880}{3}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{\frac{160}{3}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{192\Big\}\Big]$$
$$+ \mathcal{D}_0\overline{\mathcal{D}}_3\Big[C_A^3\Big\{-\frac{880}{9}\Big\} + n_f C_A^2\Big\{\frac{160}{9}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{160\Big\}\Big] + \mathcal{D}_0\overline{\mathcal{D}}_4\Big[C_A^3\Big\{40\Big\}\Big]$$
$$+ \mathcal{D}_1\delta(1-z_2)\Big[C_A^3\Big\{\frac{61138}{81} - 704\zeta_3 + \frac{104}{9}\zeta_2 - \frac{64}{5}\zeta_2^2\Big\} + n_f C_A^2\Big\{-\frac{16844}{81} + 32\zeta_3$$
$$+ \frac{64}{3}\zeta_2\Big\} + n_f C_F C_A\Big\{-126 + 96\zeta_3\Big\} + n_f^2 C_A\Big\{\frac{400}{81} - \frac{32}{9}\zeta_2\Big\}$$
$$+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{-\frac{5392}{9} + 960\zeta_3 + 176\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{\frac{4072}{27} - 32\zeta_2\Big\}$$
$$+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{24\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{-\frac{160}{27}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{292 - 192\zeta_2\Big\}$$
$$+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{-\frac{496}{9}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{\frac{16}{9}\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{-\frac{176}{3}\Big\}$$
$$+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{\frac{32}{3}\Big\}\Big] + \mathcal{D}_1\overline{\mathcal{D}}_0\Big[C_A^3\Big\{-\frac{16816}{27} + 976\zeta_3 + \frac{1232}{3}\zeta_2\Big\}$$
$$+ n_f C_A^2\Big\{\frac{3656}{27} - \frac{224}{3}\zeta_2\Big\} + n_f C_F C_A\Big\{8\Big\} + n_f^2 C_A\Big\{-\frac{160}{27}\Big\}$$
$$+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{\frac{7400}{9} - 384\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{-\frac{1312}{9}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A\Big\{\frac{32}{9}\Big\}$$
$$+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{-264\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2\Big\{48\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_A^3\Big\{64\Big\}\Big]$$

$$
\begin{aligned}
&+ \mathcal{D}_1\overline{\mathcal{D}}_1\Bigg[C_A^3\left\{\frac{7400}{9} - 384\zeta_2\right\} + n_f C_A^2\left\{-\frac{1312}{9}\right\} + n_f^2 C_A\left\{\frac{32}{9}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{1760}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{320}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{384\right\}\Bigg] \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_2\Bigg[C_A^3\left\{-\frac{880}{3}\right\} + n_f C_A^2\left\{\frac{160}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{480\right\}\Bigg] \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_3\Bigg[C_A^3\left\{160\right\}\Bigg] + \mathcal{D}_2\delta(1-z_2)\Bigg[C_A^3\left\{-\frac{8408}{27} + 488\zeta_3 + \frac{616}{3}\zeta_2\right\} \\
&+ n_f C_A^2\left\{\frac{1828}{27} - \frac{112}{3}\zeta_2\right\} + n_f C_F C_A\left\{4\right\} + n_f^2 C_A\left\{-\frac{80}{27}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{3700}{9}\right. \\
&\left. - 192\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{656}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{16}{9}\right\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-132\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{24\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{32\right\}\Bigg] \\
&+ \mathcal{D}_2\overline{\mathcal{D}}_0\Bigg[C_A^3\left\{\frac{3700}{9} - 192\zeta_2\right\} + n_f C_A^2\left\{-\frac{656}{9}\right\} + n_f^2 C_A\left\{\frac{16}{9}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{880}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{160}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{192\right\}\Bigg] \\
&+ \mathcal{D}_2\overline{\mathcal{D}}_1\Bigg[C_A^3\left\{-\frac{880}{3}\right\} + n_f C_A^2\left\{\frac{160}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{480\right\}\Bigg] \\
&+ \mathcal{D}_2\overline{\mathcal{D}}_2\Bigg[C_A^3\left\{240\right\}\Bigg] + \mathcal{D}_3\delta(1-z_2)\Bigg[C_A^3\left\{\frac{3700}{27} - 64\zeta_2\right\} + n_f C_A^2\left\{-\frac{656}{27}\right\} \\
&+ n_f^2 C_A\left\{\frac{16}{27}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{880}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{160}{9}\right\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{64\right\}\Bigg] + \mathcal{D}_3\overline{\mathcal{D}}_0\Bigg[C_A^3\left\{-\frac{880}{9}\right\} + n_f C_A^2\left\{\frac{160}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{160\right\}\Bigg] \\
&+ \mathcal{D}_3\overline{\mathcal{D}}_1\Bigg[C_A^3\left\{160\right\}\Bigg] + \mathcal{D}_4\delta(1-z_2)\Bigg[C_A^3\left\{-\frac{220}{9}\right\} + n_f C_A^2\left\{\frac{40}{9}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{40\right\}\Bigg] + \mathcal{D}_4\overline{\mathcal{D}}_0\Bigg[C_A^3\left\{40\right\}\Bigg] + \mathcal{D}_5\delta(1-z_2)\Bigg[C_A^3\left\{8\right\}\Bigg] \\
&+ \overline{\mathcal{D}}_0\delta(1-z_1)\Bigg[C_A^3\left\{-\frac{297029}{729} + 192\zeta_5 + \frac{27128}{27}\zeta_3 + \frac{18056}{81}\zeta_2 - \frac{608}{3}\zeta_2\zeta_3 + \frac{88}{5}\zeta_2^2\right\} \\
&+ n_f C_A^2\left\{\frac{62626}{729} - \frac{392}{3}\zeta_3 - \frac{6416}{81}\zeta_2 + \frac{16}{5}\zeta_2^2\right\} + n_f C_F C_A\left\{\frac{1711}{27} - \frac{304}{9}\zeta_3 - 8\zeta_2\right. \\
&\left. - \frac{32}{5}\zeta_2^2\right\} + n_f^2 C_A\left\{-\frac{1856}{729} - \frac{32}{27}\zeta_3 + \frac{160}{27}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{61138}{81} - 704\zeta_3\right. \\
&\left. + \frac{104}{9}\zeta_2 - \frac{64}{5}\zeta_2^2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{16844}{81} + 32\zeta_3 + \frac{64}{3}\zeta_2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{-126+96\zeta_3\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{400}{81}-\frac{32}{9}\zeta_2\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{6152}{27}+352\zeta_3-\frac{176}{3}\zeta_2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{2020}{27}\right.\\
&\left.+\frac{32}{3}\zeta_2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{20\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{-\frac{80}{27}\right\}\\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{484}{27}-64\zeta_2\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{176}{27}\right\}\\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{16}{27}\right\}\Bigg]+\overline{\mathcal{D}}_1\delta(1-z_1)\Bigg[C_A^3\left\{\frac{61138}{81}-704\zeta_3+\frac{104}{9}\zeta_2-\frac{64}{5}\zeta_2^2\right\}\\
&+n_f C_A^2\left\{-\frac{16844}{81}+32\zeta_3+\frac{64}{3}\zeta_2\right\}+n_f C_F C_A\left\{-126+96\zeta_3\right\}+n_f^2 C_A\left\{\frac{400}{81}\right.\\
&\left.-\frac{32}{9}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{5392}{9}+960\zeta_3+176\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{4072}{27}\right.\\
&\left.-32\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{24\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{-\frac{160}{27}\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{292-192\zeta_2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{496}{9}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{16}{9}\right\}\\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{176}{3}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{32}{3}\right\}\Bigg]+\overline{\mathcal{D}}_2\delta(1-z_1)\Bigg[C_A^3\left\{-\frac{8408}{27}\right.\\
&\left.+488\zeta_3+\frac{616}{3}\zeta_2\right\}+n_f C_A^2\left\{\frac{1828}{27}-\frac{112}{3}\zeta_2\right\}+n_f C_F C_A\left\{4\right\}+n_f^2 C_A\left\{-\frac{80}{27}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{3700}{9}-192\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{656}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{16}{9}\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-132\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{24\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{32\right\}\Bigg]\\
&+\overline{\mathcal{D}}_3\delta(1-z_1)\Bigg[C_A^3\left\{\frac{3700}{27}-64\zeta_2\right\}+n_f C_A^2\left\{-\frac{656}{27}\right\}+n_f^2 C_A\left\{\frac{16}{27}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{880}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{160}{9}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{64\right\}\Bigg]\\
&+\overline{\mathcal{D}}_4\delta(1-z_1)\Bigg[C_A^3\left\{-\frac{220}{9}\right\}+n_f C_A^2\left\{\frac{40}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{40\right\}\Bigg]\\
&+\overline{\mathcal{D}}_5\delta(1-z_1)\Bigg[C_A^3\left\{8\right\}\Bigg] \qquad (3.4.3)
\end{aligned}
$$

and for $I = \mathrm{DY}$ and $i\,\bar{i} = q\bar{q}$, we get [12]

$$
\Delta^{\mathrm{DY,SV}}_{Y,q\bar{q},1} = \delta(1-z_1)\delta(1-z_2)\Bigg[C_F\left\{-16+12\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_F\left\{6\right\}\Bigg]
$$

$$
\begin{aligned}
&+ \mathcal{D}_0\delta(1-z_2)\left[\log\left(\frac{q^2}{\mu_F^2}\right)C_F\left\{4\right\}\right] + \mathcal{D}_0\overline{\mathcal{D}}_0\left[C_F\left\{4\right\}\right] + \mathcal{D}_1\delta(1-z_2)\left[C_F\left\{4\right\}\right]\\
&+ \overline{\mathcal{D}}_0\delta(1-z_1)\left[\log\left(\frac{q^2}{\mu_F^2}\right)C_F\left\{4\right\}\right] + \overline{\mathcal{D}}_1\delta(1-z_1)\left[C_F\left\{4\right\}\right],\\
\Delta^{\mathrm{DY,SV}}_{Y,q\bar{q},2} &= \delta(1-z_1)\delta(1-z_2)\left[C_FC_A\left\{-\frac{1535}{12}+\frac{172}{3}\zeta_3+\frac{860}{9}\zeta_2-\frac{52}{5}\zeta_2^2\right\}\right.\\
&+ C_F^2\left\{\frac{511}{4}-60\zeta_3-134\zeta_2+\frac{304}{5}\zeta_2^2\right\} + n_fC_F\left\{\frac{127}{6}+\frac{8}{3}\zeta_3-\frac{152}{9}\zeta_2\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{\frac{193}{3}-24\zeta_3-\frac{44}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{-93+80\zeta_3+48\zeta_2\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{-\frac{34}{3}+\frac{8}{3}\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{-11\right\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{18-16\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{2\right\}\Big] + \mathcal{D}_0\delta(1-z_2)\Big[C_FC_A\Big\{-\frac{808}{27}\\
&+ 28\zeta_3+\frac{44}{3}\zeta_2\Big\} + C_F^2\left\{32\zeta_3\right\} + n_fC_F\left\{\frac{112}{27}-\frac{8}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{\frac{268}{9}-8\zeta_2\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{-64+16\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{-\frac{40}{9}\right\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{-\frac{22}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{24\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{\frac{4}{3}\right\}\Big]\\
&+ \mathcal{D}_0\overline{\mathcal{D}}_0\Big[C_FC_A\Big\{\frac{268}{9}-8\zeta_2\Big\} + C_F^2\Big\{-64+16\zeta_2\Big\} + n_fC_F\Big\{-\frac{40}{9}\Big\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\Big\{-\frac{44}{3}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{24\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{\frac{8}{3}\right\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{16\right\}\Big] + \mathcal{D}_0\overline{\mathcal{D}}_1\Big[C_FC_A\Big\{-\frac{44}{3}\Big\} + n_fC_F\left\{\frac{8}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{48\right\}\Big]\\
&+ \mathcal{D}_0\overline{\mathcal{D}}_2\Big[C_F^2\left\{24\right\}\Big] + \mathcal{D}_1\delta(1-z_2)\Big[C_FC_A\Big\{\frac{268}{9}-8\zeta_2\Big\} + C_F^2\Big\{-64+16\zeta_2\Big\}\\
&+ n_fC_F\Big\{-\frac{40}{9}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\Big\{-\frac{44}{3}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{24\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{\frac{8}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{16\right\}\Big] + \mathcal{D}_1\overline{\mathcal{D}}_0\Big[C_FC_A\Big\{-\frac{44}{3}\Big\}\\
&+ n_fC_F\left\{\frac{8}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{48\right\}\Big] + \mathcal{D}_1\overline{\mathcal{D}}_1\Big[C_F^2\left\{48\right\}\Big]\\
&+ \mathcal{D}_2\delta(1-z_2)\Big[C_FC_A\Big\{-\frac{22}{3}\Big\} + n_fC_F\left\{\frac{4}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2\left\{24\right\}\Big] + \mathcal{D}_2\overline{\mathcal{D}}_0\Big[C_F^2\left\{24\right\}\Big]\\
&+ \mathcal{D}_3\delta(1-z_2)\Big[C_F^2\left\{8\right\}\Big] + \overline{\mathcal{D}}_0\delta(1-z_1)\Big[C_FC_A\Big\{-\frac{808}{27}+28\zeta_3+\frac{44}{3}\zeta_2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ C_F^2\left\{32\zeta_3\right\} + n_f C_F\left\{\frac{112}{27} - \frac{8}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A\left\{\frac{268}{9} - 8\zeta_2\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{-64 + 16\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F\left\{-\frac{40}{9}\right\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A\left\{-\frac{22}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{24\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F\left\{\frac{4}{3}\right\}\Bigg]\\
&+ \overline{\mathcal{D}}_1\delta(1-z_1)\Bigg[C_F C_A\left\{\frac{268}{9} - 8\zeta_2\right\} + C_F^2\left\{-64 + 16\zeta_2\right\} + n_f C_F\left\{-\frac{40}{9}\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A\left\{-\frac{44}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{24\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F\left\{\frac{8}{3}\right\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{16\right\}\Bigg] + \overline{\mathcal{D}}_2\delta(1-z_1)\Bigg[C_F C_A\left\{-\frac{22}{3}\right\} + n_f C_F\left\{\frac{4}{3}\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2\left\{24\right\}\Bigg] + \overline{\mathcal{D}}_3\delta(1-z_1)\Bigg[C_F^2\left\{8\right\}\Bigg],\\
\Delta^{\mathrm{DY,SV}}_{Y,q\bar{q},3} &= \delta(1-z_1)\delta(1-z_2)\Bigg[C_F\left(\frac{N^2-4}{N}\right) n_{f,v}\left\{8 - \frac{160}{3}\zeta_5 + \frac{28}{3}\zeta_3 + 20\zeta_2 - \frac{4}{5}\zeta_2^2\right\}\\
&+ C_F C_A^2\left\{-\frac{1505881}{972} - 204\zeta_5 + \frac{125105}{81}\zeta_3 - \frac{400}{3}\zeta_3^2 + \frac{99289}{81}\zeta_2 - 588\zeta_2\zeta_3\right.\\
&\left.- \frac{2921}{135}\zeta_2^2 + \frac{24352}{315}\zeta_2^3\right\} + C_F^2 C_A\left\{\frac{74321}{36} - \frac{7624}{9}\zeta_5 - \frac{5972}{3}\zeta_3 + \frac{1264}{3}\zeta_3^2\right.\\
&\left.- \frac{39865}{27}\zeta_2 + \frac{10736}{9}\zeta_2\zeta_3 + \frac{137968}{135}\zeta_2^2 - \frac{78272}{315}\zeta_2^3\right\} + C_F^3\left\{-\frac{5599}{6} + 1328\zeta_5\right.\\
&\left.- 460\zeta_3 + \frac{736}{3}\zeta_3^2 + \frac{1403}{3}\zeta_2 - 160\zeta_2\zeta_3 - \frac{3164}{5}\zeta_2^2 + \frac{90016}{315}\zeta_2^3\right\}\\
&+ n_f C_F C_A\left\{\frac{110651}{243} - 8\zeta_5 - \frac{19888}{81}\zeta_3 - \frac{12112}{27}\zeta_2 + \frac{272}{3}\zeta_2\zeta_3 - \frac{2828}{135}\zeta_2^2\right\}\\
&+ n_f C_F^2\left\{-\frac{421}{3} - \frac{224}{9}\zeta_5 + 360\zeta_3 + \frac{5848}{27}\zeta_2 - \frac{1472}{9}\zeta_2\zeta_3 - \frac{19408}{135}\zeta_2^2\right\}\\
&+ n_f^2 C_F\left\{-\frac{7081}{243} - \frac{304}{81}\zeta_3 + \frac{2816}{81}\zeta_2 + \frac{592}{135}\zeta_2^2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2\left\{\frac{3082}{3} + 80\zeta_5\right.\\
&\left.- \frac{2296}{3}\zeta_3 - \frac{13600}{27}\zeta_2 + \frac{1084}{15}\zeta_2^2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\left\{-\frac{3439}{2} + 240\zeta_5 + \frac{19864}{9}\zeta_3\right.\\
&\left.+ \frac{30260}{27}\zeta_2 - 608\zeta_2\zeta_3 - \frac{1792}{5}\zeta_2^2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{\frac{1495}{2} - 96\zeta_5 - 1504\zeta_3\right.\\
&\left.- 348\zeta_2 + 384\zeta_2\zeta_3 + 192\zeta_2^2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\left\{-\frac{3052}{9} + \frac{304}{3}\zeta_3 + \frac{5192}{27}\zeta_2\right.\\
&\left.- \frac{184}{15}\zeta_2^2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\left\{230 - \frac{2032}{9}\zeta_3 - \frac{5360}{27}\zeta_2 + \frac{304}{5}\zeta_2^2\right\}
\end{aligned}
$$

$$
+\log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{\frac{220}{9} - \frac{448}{27}\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{ - \frac{2429}{9} + 88\zeta_3 + \frac{484}{9}\zeta_2\right\}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{551 - 496\zeta_3 - \frac{3332}{9}\zeta_2 + 64\zeta_2^2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{ - 270\right.
$$

$$
\left. + 480\zeta_3 + 328\zeta_2 - \frac{704}{5}\zeta_2^2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{\frac{850}{9} - 16\zeta_3 - \frac{176}{9}\zeta_2\right\}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{ - 92 + 64\zeta_3 + \frac{536}{9}\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{ - \frac{68}{9} + \frac{16}{9}\zeta_2\right\}
$$

$$
+\log^3\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{\frac{242}{9}\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{ - 66 + \frac{176}{3}\zeta_2\right\}
$$

$$
+\log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{36 + \frac{128}{3}\zeta_3 - 96\zeta_2\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{ - \frac{88}{9}\right\}
$$

$$
\left. + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{12 - \frac{32}{3}\zeta_2\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{\frac{8}{9}\right\}\right]
$$

$$
+ \mathcal{D}_0\delta(1-z_2)\left[C_F C_A^2 \left\{ - \frac{297029}{729} - 192\zeta_5 + \frac{14264}{27}\zeta_3 + \frac{27752}{81}\zeta_2 - \frac{176}{3}\zeta_2\zeta_3\right.\right.
$$

$$
\left. - \frac{616}{15}\zeta_2^2\right\} + C_F^2 C_A \left\{\frac{12928}{27} + \frac{256}{9}\zeta_3 - \frac{9568}{27}\zeta_2 - 16\zeta_2\zeta_3 + \frac{176}{3}\zeta_2^2\right\} + C_F^3 \left\{384\zeta_5\right.
$$

$$
\left. - 512\zeta_3 - 128\zeta_2\zeta_3\right\} + n_f C_F C_A \left\{\frac{62626}{729} - \frac{536}{9}\zeta_3 - \frac{7760}{81}\zeta_2 + \frac{208}{15}\zeta_2^2\right\}
$$

$$
+ n_f C_F^2 \left\{ - 3 - \frac{944}{9}\zeta_3 + \frac{1384}{27}\zeta_2 - \frac{256}{15}\zeta_2^2\right\} + n_f^2 C_F \left\{ - \frac{1856}{729} - \frac{32}{27}\zeta_3 + \frac{160}{27}\zeta_2\right\}
$$

$$
+\log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{\frac{31006}{81} - 176\zeta_3 - \frac{680}{3}\zeta_2 + \frac{176}{5}\zeta_2^2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{ - \frac{3503}{3}\right.
$$

$$
\left. + \frac{136}{3}\zeta_3 + \frac{4312}{9}\zeta_2 - \frac{48}{5}\zeta_2^2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{511 - 48\zeta_3 - 24\zeta_2 - \frac{192}{5}\zeta_2^2\right\}
$$

$$
+\log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{ - \frac{8204}{81} + \frac{512}{9}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{144 + \frac{320}{3}\zeta_3 - \frac{592}{9}\zeta_2\right\}
$$

$$
+\log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{\frac{400}{81} - \frac{32}{9}\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{ - \frac{3560}{27} + \frac{88}{3}\zeta_2\right\}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{\frac{1660}{3} - 96\zeta_3 - \frac{56}{3}\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{ - 372 + 448\zeta_3\right\}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{\frac{1156}{27} - \frac{16}{3}\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{ - \frac{268}{3} - \frac{16}{3}\zeta_2\right\}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{ - \frac{80}{27}\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{\frac{484}{27}\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{ - 88\right\}
$$

$$
+\log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{72 - 64\zeta_2\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{ - \frac{176}{27}\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{16\right\}
$$

$$
+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{\frac{16}{27}\right\}\Big] + \mathcal{D}_0\overline{\mathcal{D}}_0\Big[C_FC_A^2\left\{\frac{31006}{81} - 176\zeta_3 - \frac{680}{3}\zeta_2 + \frac{176}{5}\zeta_2^2\right\}
$$

$$
+ C_F^2C_A\left\{-\frac{8893}{9} - \frac{368}{3}\zeta_3 + \frac{3520}{9}\zeta_2 - \frac{48}{5}\zeta_2^2\right\} + C_F^3\left\{511 - 240\zeta_3 - 24\zeta_2\right.
$$

$$
\left. - \frac{192}{5}\zeta_2^2\right\} + n_fC_FC_A\left\{-\frac{8204}{81} + \frac{512}{9}\zeta_2\right\} + n_fC_F^2\left\{\frac{1072}{9} + \frac{320}{3}\zeta_3 - \frac{448}{9}\zeta_2\right\}
$$

$$
+ n_f^2C_F\left\{\frac{400}{81} - \frac{32}{9}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{-\frac{7120}{27} + \frac{176}{3}\zeta_2\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{\frac{11644}{27} + 128\zeta_3 + \frac{560}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{-372 + 832\zeta_3\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{\frac{2312}{27} - \frac{32}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{-\frac{1984}{27} - \frac{128}{3}\zeta_2\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{-\frac{160}{27}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{\frac{484}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{\frac{956}{9}\right.
$$

$$
\left. - 64\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{-184 - 128\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{176}{9}\right\}
$$

$$
+ \log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{-\frac{104}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{\frac{16}{9}\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{-\frac{176}{3}\right\}
$$

$$
+ \log^3\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{96\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{\frac{32}{3}\right\}\Big]
$$

$$
+ \mathcal{D}_0\overline{\mathcal{D}}_1\Big[C_FC_A^2\left\{-\frac{7120}{27} + \frac{176}{3}\zeta_2\right\} + C_F^2C_A\left\{-\frac{1120}{9} + 336\zeta_3 + 352\zeta_2\right\}
$$

$$
+ C_F^3\left\{640\zeta_3\right\} + n_fC_FC_A\left\{\frac{2312}{27} - \frac{32}{3}\zeta_2\right\} + n_fC_F^2\left\{\frac{136}{9} - 64\zeta_2\right\}
$$

$$
+ n_f^2C_F\left\{-\frac{160}{27}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{\frac{968}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{\frac{1880}{3} - 192\zeta_2\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{-768 - 192\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{352}{9}\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{-\frac{272}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{\frac{32}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{-264\right\}
$$

$$
+ \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{288\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{48\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{64\right\}\Big]
$$

$$
+ \mathcal{D}_0\overline{\mathcal{D}}_2\Big[C_FC_A^2\left\{\frac{484}{9}\right\} + C_F^2C_A\left\{\frac{1072}{3} - 96\zeta_2\right\} + C_F^3\left\{-384 - 96\zeta_2\right\}
$$

$$
+ n_fC_FC_A\left\{-\frac{176}{9}\right\} + n_fC_F^2\left\{-\frac{160}{3}\right\} + n_f^2C_F\left\{\frac{16}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{-\frac{880}{3}\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{144\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{\frac{160}{3}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{192\right\}\Big]
$$

$$
+ \mathcal{D}_0\overline{\mathcal{D}}_3\left[C_F^2C_A\left\{-\frac{880}{9}\right\} + n_fC_F^2\left\{\frac{160}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{160\right\}\right]
$$

$$
+ \mathcal{D}_0\overline{\mathcal{D}}_4\left[C_F^3\left\{+40\right\}\right] + \mathcal{D}_1\delta(1-z_2)\left[C_FC_A^2\left\{\frac{31006}{81} - 176\zeta_3 - \frac{680}{3}\zeta_2 + \frac{176}{5}\zeta_2^2\right\}\right.
$$

$$
+ C_F^2C_A\left\{-\frac{8893}{9} - \frac{368}{3}\zeta_3 + \frac{3520}{9}\zeta_2 - \frac{48}{5}\zeta_2^2\right\} + C_F^3\left\{511 - 240\zeta_3 - 24\zeta_2\right.
$$

$$
\left.\left. - \frac{192}{5}\zeta_2^2\right\} + n_fC_FC_A\left\{-\frac{8204}{81} + \frac{512}{9}\zeta_2\right\} + n_fC_F^2\left\{\frac{1072}{9} + \frac{320}{3}\zeta_3 - \frac{448}{9}\zeta_2\right\}\right.
$$

$$
+ n_f^2C_F\left\{\frac{400}{81} - \frac{32}{9}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{-\frac{7120}{27} + \frac{176}{3}\zeta_2\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{\frac{11644}{27} + 128\zeta_3 + \frac{560}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{-372 + 832\zeta_3\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{\frac{2312}{27} - \frac{32}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{-\frac{1984}{27} - \frac{128}{3}\zeta_2\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{-\frac{160}{27}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{\frac{484}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{\frac{956}{9}\right.
$$

$$
\left. - 64\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{-184 - 128\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{176}{9}\right\}
$$

$$
+ \log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{-\frac{104}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{\frac{16}{9}\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{-\frac{176}{3}\right\}
$$

$$
+ \log^3\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{96\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{\frac{32}{3}\right\}\left.\right] + \mathcal{D}_1\overline{\mathcal{D}}_0\left[C_FC_A^2\left\{-\frac{7120}{27} + \frac{176}{3}\zeta_2\right\}\right.
$$

$$
+ C_F^2C_A\left\{-\frac{1120}{9} + 336\zeta_3 + 352\zeta_2\right\} + C_F^3\left\{640\zeta_3\right\} + n_fC_FC_A\left\{\frac{2312}{27} - \frac{32}{3}\zeta_2\right\}
$$

$$
+ n_fC_F^2\left\{\frac{136}{9} - 64\zeta_2\right\} + n_f^2C_F\left\{-\frac{160}{27}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{\frac{968}{9}\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{\frac{1880}{3} - 192\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{-768 - 192\zeta_2\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{352}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{-\frac{272}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{\frac{32}{9}\right\}
$$

$$
+ \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{-264\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{288\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{48\right\}
$$

$$
+ \log^3\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{64\right\}\left.\right] + \mathcal{D}_1\overline{\mathcal{D}}_1\left[C_FC_A^2\left\{\frac{968}{9}\right\} + C_F^2C_A\left\{\frac{2144}{3} - 192\zeta_2\right\}\right.
$$

$$
+ C_F^3\left\{-768 - 192\zeta_2\right\} + n_fC_FC_A\left\{-\frac{352}{9}\right\} + n_fC_F^2\left\{-\frac{320}{3}\right\} + n_f^2C_F\left\{\frac{32}{9}\right\}
$$

$$
+ \log\left(\frac{q^2}{\mu_F^2}\right)C_F^2C_A\left\{-\frac{1760}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F^3\left\{288\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{\frac{320}{3}\right\}
$$

$$
\begin{aligned}
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{384\right\}\Big] + \mathcal{D}_1\overline{\mathcal{D}}_2\Big[C_F^2 C_A\left\{-\frac{880}{3}\right\} + n_f C_F^2\left\{\frac{160}{3}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{480\right\}\Big] + \mathcal{D}_1\overline{\mathcal{D}}_3\Big[C_F^3\left\{160\right\}\Big] + \mathcal{D}_2\delta(1-z_2)\Big[C_F C_A^2\left\{-\frac{3560}{27} + \frac{88}{3}\zeta_2\right\} \\
&+ C_F^2 C_A\left\{-\frac{560}{9} + 168\zeta_3 + 176\zeta_2\right\} + C_F^3\left\{320\zeta_3\right\} + n_f C_F C_A\left\{\frac{1156}{27} - \frac{16}{3}\zeta_2\right\} \\
&+ n_f C_F^2\left\{\frac{68}{9} - 32\zeta_2\right\} + n_f^2 C_F\left\{-\frac{80}{27}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2\left\{\frac{484}{9}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\left\{\frac{940}{3} - 96\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{-384 - 96\zeta_2\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\left\{-\frac{176}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\left\{-\frac{136}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F\left\{\frac{16}{9}\right\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\left\{-132\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{144\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\left\{24\right\} \\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{32\right\}\Big] + \mathcal{D}_2\overline{\mathcal{D}}_0\Big[C_F C_A^2\left\{\frac{484}{9}\right\} + C_F^2 C_A\left\{\frac{1072}{3} - 96\zeta_2\right\} \\
&+ C_F^3\left\{-384 - 96\zeta_2\right\} + n_f C_F C_A\left\{-\frac{176}{9}\right\} + n_f C_F^2\left\{-\frac{160}{3}\right\} + n_f^2 C_F\left\{\frac{16}{9}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\left\{-\frac{880}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{144\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\left\{\frac{160}{3}\right\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{192\right\}\Big] + \mathcal{D}_2\overline{\mathcal{D}}_1\Big[C_F^2 C_A\left\{-\frac{880}{3}\right\} + n_f C_F^2\left\{\frac{160}{3}\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{480\right\}\Big] + \mathcal{D}_2\overline{\mathcal{D}}_2\Big[C_F^3\left\{240\right\}\Big] + \mathcal{D}_3\delta(1-z_2)\Big[C_F C_A^2\left\{\frac{484}{27}\right\} \\
&+ C_F^2 C_A\left\{\frac{1072}{9} - 32\zeta_2\right\} + C_F^3\left\{-128 - 32\zeta_2\right\} + n_f C_F C_A\left\{-\frac{176}{27}\right\} \\
&+ n_f C_F^2\left\{-\frac{160}{9}\right\} + n_f^2 C_F\left\{\frac{16}{27}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\left\{-\frac{880}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{48\right\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\left\{\frac{160}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{64\right\}\Big] + \mathcal{D}_3\overline{\mathcal{D}}_0\Big[C_F^2 C_A\left\{-\frac{880}{9}\right\} \\
&+ n_f C_F^2\left\{\frac{160}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{160\right\}\Big] + \mathcal{D}_3\overline{\mathcal{D}}_1 C_F^3\Big[\left\{160\right\}\Big] \\
&+ \mathcal{D}_4\delta(1-z_2)\Big[C_F^2 C_A\left\{-\frac{220}{9}\right\} + n_f C_F^2\left\{\frac{40}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\left\{40\right\}\Big] \\
&+ \mathcal{D}_4\overline{\mathcal{D}}_0\Big[C_F^3\left\{40\right\}\Big] + \mathcal{D}_5\delta(1-z_2)\Big[C_F^3\left\{8\right\}\Big] + \overline{\mathcal{D}}_0\delta(1-z_1)\Big[C_F C_A^2\left\{-\frac{297029}{729}\right. \\
&\left. - 192\zeta_5 + \frac{14264}{27}\zeta_3 + \frac{27752}{81}\zeta_2 - \frac{176}{3}\zeta_2\zeta_3 - \frac{616}{15}\zeta_2^2\right\} + C_F^2 C_A\left\{\frac{12928}{27} + \frac{256}{9}\zeta_3\right.
\end{aligned}
$$

$$
- \frac{9568}{27}\zeta_2 - 16\zeta_2\zeta_3 + \frac{176}{3}\zeta_2^2\Big\} + C_F^3\Big\{384\zeta_5 - 512\zeta_3 - 128\zeta_2\zeta_3\Big\}
$$
$$
+ n_f C_F C_A\Big\{\frac{62626}{729} - \frac{536}{9}\zeta_3 - \frac{7760}{81}\zeta_2 + \frac{208}{15}\zeta_2^2\Big\} + n_f C_F^2\Big\{-3 - \frac{944}{9}\zeta_3 + \frac{1384}{27}\zeta_2
$$
$$
- \frac{256}{15}\zeta_2^2\Big\} + n_f^2 C_F\Big\{-\frac{1856}{729} - \frac{32}{27}\zeta_3 + \frac{160}{27}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2\Big\{\frac{31006}{81} - 176\zeta_3
$$
$$
- \frac{680}{3}\zeta_2 + \frac{176}{5}\zeta_2^2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\Big\{-\frac{3503}{3} + \frac{136}{3}\zeta_3 + \frac{4312}{9}\zeta_2 - \frac{48}{5}\zeta_2^2\Big\}
$$
$$
+ \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\Big\{511 - 48\zeta_3 - 24\zeta_2 - \frac{192}{5}\zeta_2^2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{-\frac{8204}{81}
$$
$$
+ \frac{512}{9}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\Big\{144 + \frac{320}{3}\zeta_3 - \frac{592}{9}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F\Big\{\frac{400}{81}
$$
$$
- \frac{32}{9}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2\Big\{-\frac{3560}{27} + \frac{88}{3}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\Big\{\frac{1660}{3} - 96\zeta_3
$$
$$
- \frac{56}{3}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3\Big\{-372 + 448\zeta_3\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{\frac{1156}{27} - \frac{16}{3}\zeta_2\Big\}
$$
$$
+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\Big\{-\frac{268}{3} - \frac{16}{3}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F\Big\{-\frac{80}{27}\Big\}
$$
$$
+ \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2\Big\{\frac{484}{27}\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\Big\{-88\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^3\Big\{72 - 64\zeta_2\Big\}
$$
$$
+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{-\frac{176}{27}\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\Big\{16\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F\Big\{\frac{16}{27}\Big\}\Big]
$$
$$
+ \overline{\mathcal{D}}_1\delta(1-z_1)\Big[C_F C_A^2\Big\{\frac{31006}{81} - 176\zeta_3 - \frac{680}{3}\zeta_2 + \frac{176}{5}\zeta_2^2\Big\} + C_F^2 C_A\Big\{-\frac{8893}{9}
$$
$$
- \frac{368}{3}\zeta_3 + \frac{3520}{9}\zeta_2 - \frac{48}{5}\zeta_2^2\Big\} + C_F^3\Big\{511 - 240\zeta_3 - 24\zeta_2 - \frac{192}{5}\zeta_2^2\Big\}
$$
$$
+ n_f C_F C_A\Big\{-\frac{8204}{81} + \frac{512}{9}\zeta_2\Big\} + n_f C_F^2\Big\{\frac{1072}{9} + \frac{320}{3}\zeta_3 - \frac{448}{9}\zeta_2\Big\} + n_f^2 C_F\Big\{\frac{400}{81}
$$
$$
- \frac{32}{9}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2\Big\{-\frac{7120}{27} + \frac{176}{3}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\Big\{\frac{11644}{27} + 128\zeta_3
$$
$$
+ \frac{560}{3}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3\Big\{-372 + 832\zeta_3\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{\frac{2312}{27} - \frac{32}{3}\zeta_2\Big\}
$$
$$
+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2\Big\{-\frac{1984}{27} - \frac{128}{3}\zeta_2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F\Big\{-\frac{160}{27}\Big\}
$$
$$
+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2\Big\{\frac{484}{9}\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A\Big\{\frac{956}{9} - 64\zeta_2\Big\}
$$
$$
+ \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3\Big\{-184 - 128\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A\Big\{-\frac{176}{9}\Big\}
$$

$$
\begin{aligned}
&+\log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{-\frac{104}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{\frac{16}{9}\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{-\frac{176}{3}\right\} \\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{96\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{\frac{32}{3}\right\}\Bigg] + \overline{\mathcal{D}}_2 \delta(1-z_1)\Bigg[C_F C_A^2 \left\{-\frac{3560}{27}\right. \\
&\left.+\frac{88}{3}\zeta_2\right\} + C_F^2 C_A \left\{-\frac{560}{9} + 168\zeta_3 + 176\zeta_2\right\} + C_F^3 \left\{320\zeta_3\right\} + n_f C_F C_A \left\{\frac{1156}{27}\right. \\
&\left.-\frac{16}{3}\zeta_2\right\} + n_f C_F^2 \left\{\frac{68}{9} - 32\zeta_2\right\} + n_f^2 C_F \left\{-\frac{80}{27}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F C_A^2 \left\{\frac{484}{9}\right\} \\
&+\log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{\frac{940}{3} - 96\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{-384 - 96\zeta_2\right\} \\
&+\log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \left\{-\frac{176}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{-\frac{136}{3}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_F \left\{\frac{16}{9}\right\} \\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{-132\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{144\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{24\right\} \\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{32\right\}\Bigg] + \overline{\mathcal{D}}_3 \delta(1-z_1)\Bigg[C_F C_A^2 \left\{\frac{484}{27}\right\} + C_F^2 C_A \left\{\frac{1072}{9} - 32\zeta_2\right\} \\
&+C_F^3 \left\{-128 - 32\zeta_2\right\} + n_f C_F C_A \left\{-\frac{176}{27}\right\} + n_f C_F^2 \left\{-\frac{160}{9}\right\} + n_f^2 C_F \left\{\frac{16}{27}\right\} \\
&+\log\left(\frac{q^2}{\mu_F^2}\right) C_F^2 C_A \left\{-\frac{880}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{48\right\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F^2 \left\{\frac{160}{9}\right\} \\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{64\right\}\Bigg] + \overline{\mathcal{D}}_4 \delta(1-z_1)\Bigg[C_F^2 C_A \left\{-\frac{220}{9}\right\} + n_f C_F^2 \left\{\frac{40}{9}\right\} \\
&+\log\left(\frac{q^2}{\mu_F^2}\right) C_F^3 \left\{40\right\}\Bigg] + \overline{\mathcal{D}}_5 \delta(1-z_1)\Bigg[C_F^3 \left\{8\right\}\Bigg]. \qquad (3.4.4)
\end{aligned}
$$

For sake of completeness, we mention the leading order contribution which is

$$
\Delta^I_{Y,i\,\bar{i},0} = \delta(1-z_1)\delta(1-z_2)\,. \qquad (3.4.5)
$$

The above results are presented for the choice $\mu_R = \mu_F$. The dependence of the SV rapidity distributions on renormalisation scale μ_R can be easily restored by employing the RG evolution of a_s from μ_F to μ_R [87] using Eq. (2.4.6). In the next Sec. 3.5, we discuss the numerical impact of the N^{3}LO SV correction to the Higgs rapidity distribution at the LHC.

3.5 Numerical Impact of SV Rapidity Distributions

In this section, we confine ourselves to the numerical impact of the SV rapidity distributions of the Higgs boson production through gluon fusion. We present the relative contributions in percentage of the pure N^{3}LO terms in Eq. (3.4.2) with respect to $\Delta^{H,\mathrm{SV}}_{Y,gg,3}$, for rapidity $Y = 0$ in Table 3.1 and 3.2. The notation $\mathcal{D}_i\overline{\mathcal{D}}_j$ corresponds to the sum of

	$\delta\delta$	$\delta\overline{\mathcal{D}}_0$	$\delta\overline{\mathcal{D}}_1$	$\delta\overline{\mathcal{D}}_2$	$\delta\overline{\mathcal{D}}_3$	$\delta\overline{\mathcal{D}}_4$	$\delta\overline{\mathcal{D}}_5$	$\mathcal{D}_0\overline{\mathcal{D}}_0$	$\mathcal{D}_0\overline{\mathcal{D}}_1$
%	73.3	16.0	9.1	31.4	1.0	-9.9	-23.1	-13.7	-10.7

Table 3.1. Relative contributions of pure N^{3}LO terms.

	$\mathcal{D}_0\overline{\mathcal{D}}_2$	$\mathcal{D}_0\overline{\mathcal{D}}_3$	$\mathcal{D}_0\overline{\mathcal{D}}_4$	$\mathcal{D}_1\overline{\mathcal{D}}_1$	$\mathcal{D}_1\overline{\mathcal{D}}_2$	$\mathcal{D}_1\overline{\mathcal{D}}_3$	$\mathcal{D}_2\overline{\mathcal{D}}_2$
%	-0.3	3.1	7.3	-0.2	3.8	8.6	4.2

Table 3.2. Relative contributions of pure N^{3}LO terms.

the contributions coming from $\mathcal{D}_i\overline{\mathcal{D}}_j$ and $\mathcal{D}_j\overline{\mathcal{D}}_i$. We have used $\sqrt{s} = 14$ TeV for the LHC, $G_F = 4541.68$ pb, the Z boson mass $m_Z = 91.1876$ GeV, top quark mass $m_t = 173.4$ GeV and the Higgs boson mass $m_H = 125.5$ GeV throughout. For the Higgs boson production, we use the effective theory where top quark is integrated out in the large m_t limit. The strong coupling constant $\alpha_s(\mu_R^2)$ is evolved using the 4-loop RG equations with $\alpha_s^{\mathrm{N^3LO}}(m_Z) = 0.117$ and for parton density sets we use MSTW 2008NNLO [88], as N^{3}LO evolution kernels are not yet available. In [107], Forte *et al.* pointed out that the Higgs boson cross sections will remain unaffected with this shortcoming. However, for the DY process, it is not clear whether the same will be true. We find that the contribution from the $\delta(1-z_1)\delta(1-z_2)$ part is the largest. Impact of the threshold NNLO and N^{3}LO contributions to the Higgs boson rapidity distribution at the LHC is presented in Fig. 3.4. The dependence on the renormalization and factorization scales can by studied by varying them in the range $m_H/2 < \mu_R, \mu_F < 2m_H$. We find that the inclusion of the threshold

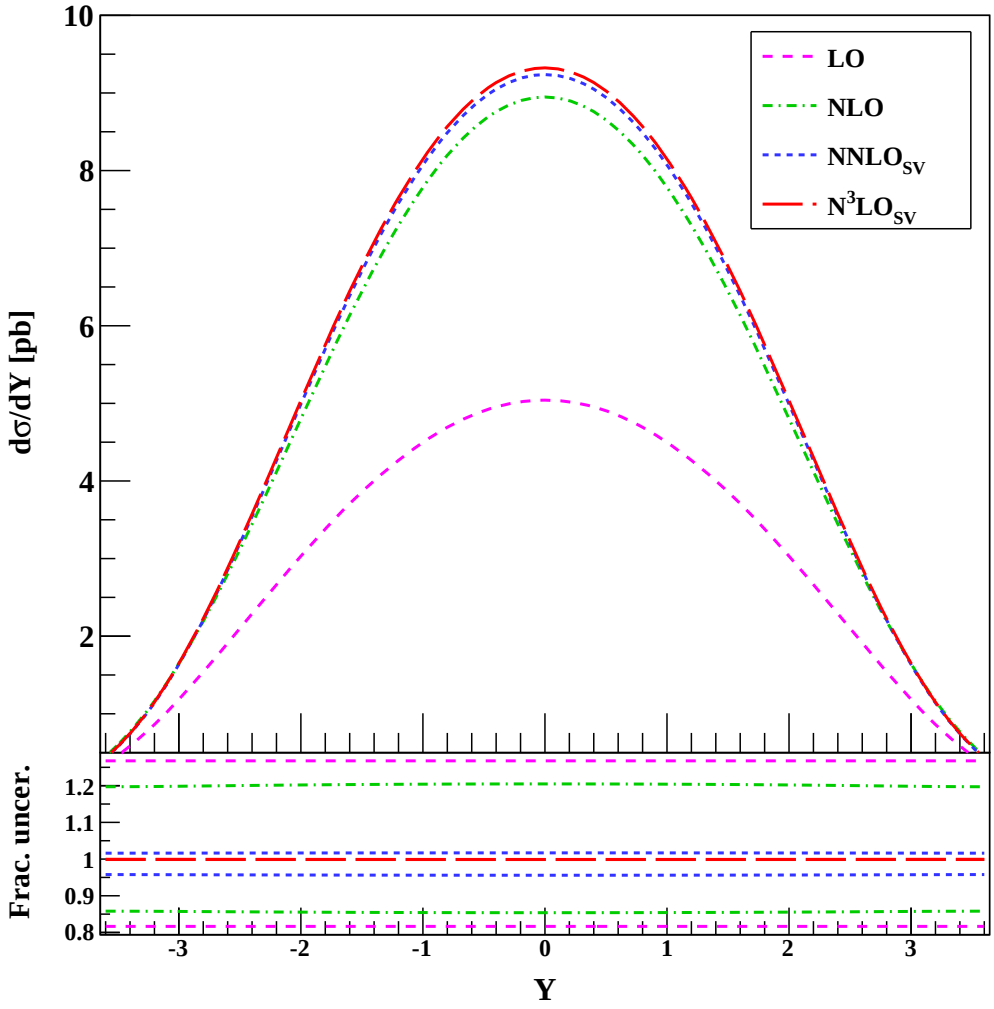


Figure 3.4. Rapidity distribution of Higgs boson

correction at N^{3}LO further reduces their dependence. For the inclusive Higgs boson production, we find that about 50% of exact NNLO contribution comes from threshold NLO and NNLO terms. It increases to 80% if we use exact NLO and threshold NNLO terms. Hence, it is expected that the rapidity distribution of the Higgs boson will receive a significant contribution from the threshold region compared to inclusive rate due to the soft emission over the entire range of Y. Our numerical study with threshold enhanced NNLO rapidity distribution confirms our expectation. Comparing our threshold NNLO results against exact NNLO distribution using the FEHiP [38] code , we find that about 90% of exact NNLO distribution comes from the threshold region as can be seen from Table 3.3 and 3.4, in accordance with [108], where it was shown that for low τ ($m_H^2/s \approx 10^{-5}$) values

the threshold terms are dominant, thanks to the inherent property of the matrix element, which receives the largest radiative corrections from the phase-space points corresponding to Born kinematics. Here we have used the exact results up to the NLO level. Because

Y	0.0	0.4	0.8	1.2	1.6
NNLO	11.21	10.96	10.70	9.13	7.80
NNLO_{SV}	9.81	9.61	8.99	8.00	6.71
NNLO_{SV}(A)	10.67	10.46	9.84	8.82	7.48
$\text{N}^3\text{LO}_{\text{SV}}$	11.62	11.36	11.07	9.44	8.04
$\text{N}^3\text{LO}_{\text{SV}}$(A)	11.88	11.62	11.33	9.70	8.30
$K3$	2.31	2.29	2.36	2.21	2.17

Table 3.3. Contributions of exact NNLO, NNLO_{SV}, $\text{N}^3\text{LO}_{\text{SV}}$, and $K3$.

Y	2.0	2.4	2.8	3.2	3.6
NNLO	6.10	4.23	2.66	1.40	0.54
NNLO_{SV}	5.21	3.66	2.25	1.14	0.42
NNLO_{SV}(A)	5.90	4.24	2.69	1.42	0.56
$\text{N}^3\text{LO}_{\text{SV}}$	6.27	4.33	2.70	1.40	0.53
$\text{N}^3\text{LO}_{\text{SV}}$(A)	6.51	4.54	2.88	1.53	0.60
$K3$	2.07	1.89	1.70	1.63	1.51

Table 3.4. Contributions of exact NNLO, NNLO_{SV}, $\text{N}^3\text{LO}_{\text{SV}}$, and $K3$.

of an inherent ambiguity in the definition of the partonic cross section at threshold one can multiply a factor $zg(z)$, where $z = \tau/x_1x_2$ and $\lim_{z\to 1} g(z) = 1$, with the partonic flux and divide the same in the partonic cross section for an inclusive rate. In [64, 109] this was exploited to take into account the subleading collinear logs also, thereby making the threshold approximation a better one. Recently, Anastasiou *et al.* used this in [33] to modify the partonic flux keeping the partonic cross section unaltered to improve the threshold effects. Following [33, 110], we introduce $G(z_1, z_2)$ such that $\lim_{z_1,z_2\to 1} G = 1$ in Eq. (3.5.1):

$$W^I\left(x_1^0, x_2^0, q^2, \mu_R^2\right) = \sum_{i,j=q,\bar{q},g} \int_{x_1^0}^1 \frac{dz_1}{z_1} \int_{x_2^0}^1 \frac{dz_2}{z_2} \mathcal{H}_{ij}^I\left(\frac{x_1^0}{z_1}\frac{x_2^0}{z_2}, \mu_F^2\right) G(z_1, z_2)$$

$$\times \lim_{z_1,z_2\to 1}\left[\frac{\Delta^I_{Y,ij}\left(z_1,z_2,q^2,\mu_R^2,\mu_F^2\right)}{G(z_1,z_2)}\right]. \tag{3.5.1}$$

We also find that with the choice $G(z_1,z_2) = z_1^2 z_2^2$, the threshold NNLO results are remarkably close to the exact ones for the entire range of Y [see Table 3.3 and 3.4, denoted by (A)]. This clearly demonstrates the dominance of threshold contributions to rapidity distribution of the Higgs boson production at the NNLO level. Assuming that the trend will not change drastically beyond NNLO, we present numerical values for N^{3}LO distributions for $G(z_1,z_2) = 1, z_1^2 z_2^2$, respectively, as N^{3}LO$_{\rm SV}$ and N^{3}LO$_{\rm SV}(A)$ in Table 3.3 and 3.3. The threshold N^{3}LO terms give $6\%(Y=0)$ to $12\%(Y=3.6)$ additional correction over the NNLO contribution to the inclusive DY production. Finally, in Table 3.3 and 3.4, we have presented $K3 = $ N^{3}LO$_{\rm SV}$/LO as a function of Y in order to demonstrate the sensitivity of higher order effects to the rapidity Y.

3.6 Summary

To summarize, we present the full threshold enhanced N^{3}LO QCD corrections to rapidity distributions of the dilepton pair in the DY process and of the Higgs boson in gluon fusion at the LHC. These are the most accurate results for these observables available in the literature. We show that the infrared structure of QCD amplitudes, in particular, their factorization properties, along with Sudakov resummation of soft gluons and renormalization group invariance provide an elegant framework to compute these threshold corrections systematically for rapidity distributions order by order in QCD perturbation theory. The recent N^{3}LO results for inclusive DY and Higgs boson production cross sections at the threshold provide crucial ingredients to obtain $\delta(1-z_1)\delta(1-z_2)$ contribution of their rapidity distributions for the first time. We find that this contribution numerically dominates over the rest of the terms in $\Delta^{H,\rm SV}_{Y,gg,3}$ at the LHC. Inclusion of N^{3}LO contributions reduces the scale dependence further. We also demonstrate the dominance of the thresh-

old contribution to rapidity distributions by comparing it against the exact NNLO for two different choices of $G(z_1, z_2)$. Finally, we find that threshold N^{3}LO rapidity distribution with $G(z_1, z_2) = 1, z_1^2 z_2^2$ shows a moderate effect over NNLO distribution.

3.7 Outlook-Beyond N^{3}LO

The results presented above is the most accurate one existing in the literature. However, in coming future, we may need to go beyond this threshold N^{3}LO in hope of making more precise theoretical predictions. The immediate step would be to compute the complete N^{3}LO QCD corrections to the differential rapidity distributions. No doubt, this is an extremely challenging goal! Presently, though we are incapable of computing this result, we can obtain the general form of the threshold N^{4}LO QCD corrections to the rapidity distributions! However, due to unavailability of the quantities, namely, form factors, anomalous dimensions, soft-collinear distributions at 4-loop level, we are unable to estimate the contributions arising from this. Nevertheless, the general form of this contribution is presented in the Appendix G which can be utilised to make the predictions once the missing ingredients become available in future. This result is the new one which is presented in this thesis for the first time, this was not presented in the article [12].

4 A Diagrammatic Approach To Compute Multiloop Amplitude

Contents

4.1 Prologue

The scattering amplitudes play the most crucial role in any quantum field theory. These are the gateway to unveil the elegant structures associated with the quantum world. At the phenomenological level, they are the main ingredients in predicting the observables at high energy colliders for the processes within and beyond the SM. Hence, the efficient evaluation of the scattering amplitudes is of prime importance at theoretical as well as experimental level. However, in perturbative QFT, the theoretical predictions based on the leading order calculation happens to be unreliable. One must go beyond the leading order

to make the predictions more accurate and reliable. While considering the effects arising from the higher orders, the contributions coming from the QCD radiations dominate substantially, in particular, at high energy colliders like Tevatron or LHC. In this thesis, we are concentrating only on the corrections arising from the QCD sector. In the process of computing these higher order QCD corrections, one has to carry out three different types of contributions, namely, virtual, real and real-virtual processes. Upon clubbing together all the three contributions appropriately, finite result for any observable is obtained. As very much expected, the complexity involved in the calculations grows very rapidly as we go towards higher and higher orders in perturbation theory, where more and more different pieces interfere with each other that eventually contribute to the final physical observables.

In this Chapter, we will confine our discussion only to the higher order QCD virtual or loop corrections. There exists at least two different formalisms to compute these.

1. *Diagrammatic approach:* one directly evaluates all the relevant Feynman diagrams appearing at the perturbative order under consideration.

2. *Unitary-based approach:* the unitary properties of the scattering amplitudes are employed extensively to avoid the direct evaluation of all the Feynman diagrams.

Despite the spectacular beauty of the unitary based approach, its applicability to the computation of the amplitudes remains confined mostly to one loop or only few multiloop problems. Its generalisation to any multiloop computation is still unavailable in the literature. In these more complicated scenarios, the first methodology of directly evaluating the Feynman diagrams is more effective and is therefore employed more often.

4.2 Feynman Diagrams and Simplifications

For any generic scattering process in QFT, we can expand any observable in powers of all the coupling constants present in the underlying Lagrangian. Feynman diagrams are the diagrammatic representations of this expansion. In this thesis, we confine our discussion into QCD. Let us consider a scattering process involving E external particles with momenta $p_1, p_2, \cdots, p_E$. Without loss of generality, we consider the cross-section which can be expanded in powers of strong coupling constant:

$$\sigma_E\left(p_1, p_2, \cdots, p_E\right) = \sum_{l=0}^{\infty} a_s^l \sigma_E^{(l)}\left(p_1, p_2, \cdots, p_E\right) . \tag{4.2.1}$$

For the sake of simplicity, we suppress all the dependence on quantum numbers of the external particles. The index l denotes the order of perturbative expansion. The cross section for $l = 0$ is called the leading order (LO), $l = 1$ is next-to-leading order (NLO) and so on. The cross section at at each perturbative order, $\sigma_E^{(l)}$, is related to the scattering matrix elements through

$$\begin{aligned}
\sigma_E^{(0)} &= K \int |\,|\mathcal{M}_E^{(0)}\rangle\,|^2 \, d\Phi_E \,, \\
\sigma_E^{(1)} &= K \int 2\,\mathrm{Re}\left(\langle \mathcal{M}_E^{(0)} | \mathcal{M}_E^{(1)} \rangle\right) d\Phi_E + K \int |\,|\mathcal{M}_{E+1}^{(0)}\rangle\,|^2 \, d\Phi_{E+1} \,, \\
\sigma_E^{(2)} &= K \int 2\,\mathrm{Re}\left(\langle \mathcal{M}_E^{(0)} | \mathcal{M}_E^{(2)} \rangle\right) d\Phi_E + K \int 2\,\mathrm{Re}\left(\langle \mathcal{M}_{E+1}^{(0)} | \mathcal{M}_{E+1}^{(1)} \rangle\right) d\Phi_{E+1} \\
&\quad + K \int |\,|\mathcal{M}_{E+2}^{(0)}\rangle\,|^2 \, d\Phi_{E+2} \\
&\text{and so on.}
\end{aligned} \tag{4.2.2}$$

In the above set of equations, $|\mathcal{M}_E^{(l)}\rangle$ is the scattering amplitude at l^{th} order in perturbation theory involving E number of external particles. The quantity $d\Phi_E$ is the phase space element. "Re" denotes the real part of the amplitude and K is an overall constant containing various factors. The amplitudes with E number of external particles and $l \geq 1$ represent

the contributions arising from the virtual Feynman diagrams, whereas amplitudes with more than E number of external particles come from the real emission diagrams. In this chapter, we address the issue of evaluating the virtual diagrams. The scattering matrix element can also be expanded perturbatively in powers of a_s as

$$|\mathcal{M}_E\rangle = \sum_{l=0}^{\infty} a_s^l |\mathcal{M}_E^{(l)}\rangle \,. \tag{4.2.3}$$

Each term in the right hand side can be represented through a set of Feynman diagrams of same perturbative order. In this chapter, we will explain the prescription to evaluate the contribution to the matrix element arising from the virtual diagrams.

The evaluation of the scattering matrix element at any particular order begins with the generation of associated Feynman diagrams. We make use of a package, named, QGRAF [111] to accomplish this job. QGRAF does not provide the graphical representation of the Feynman diagrams, rather it generates those symbolically. We use our in-house codes written in FORM [112] to convert the raw output into a format for further computation. Employing the Feynman rules derived from the underlying Lagrangian, which are the languages establishing the connection between the diagrams and the corresponding formal mathematical expressions, we obtain the amplitude. The raw amplitude contains series of Dirac gamma matrices, QCD color factors, Dirac and Lorentz indices. We simplify those using our in-house codes. We perform the color simplification in SU(N) gauge theory and follow dimensional regularisation where the space-time dimension is considered to be $d = 4 + \epsilon$. The amplitude, beyond leading order, consists of a set of tensorial Feynman integrals. Instead of handling the tensorial integrals, we multiply the amplitude with appropriate projectors to convert those to scalar integrals. Hence, the problem essentially boils down to solving those scalar integrals. Often, at any typical order in perturbation theory, this involves hundreds or thousands of different scalar loop integrals. Of course, start evaluating all of these integrals is not a practical way of dealing with the problem. Remarkably, it has been found that the appeared integrals are not independent of each

other, they can be related through some set of identities! This drastically reduces the independent integrals which ultimately need to be computed. In the next section, we will elaborate this procedure.

4.3 Reduction to Master Integrals

The dimensionally regularised Feynman loop integrals do satisfy a large number of relations, which allow one to express most of those integrals in terms of a much smaller subset of independent integrals (where "independent" is to be understood in the sense of the identities introduced below), which are now commonly referred to as the Master Integrals (MIs). For a detailed review on this, see [48, 113]. These identities are known as **integration-by-parts** and **Lorentz invariance** identities.

4.3.1 Integration-by-Parts Identities (IBP)

The integration-by-parts identities [46,47] are the most important class of identities which establish the relations among the dimensionally regularised scalar Feynman integrals. These can be seen as a generalisation of Gauss' divergence theorem in d-dimensions. They are based on the fact that, given a Feynman integral which is a function of space-time dimensions d, there always exists a value of d in the complex plane where the integral is well defined and consequently convergent. The necessary condition for the convergence of an integral is the integrand be zero at the boundaries. This condition can be rephrased as, the integral of the total derivative with respect to any loop momenta vanishes, that is

$$\int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{\partial}{\partial k_i^\mu} \left(v_s^\mu \frac{1}{\mathcal{D}_1^{b_1} \dots \mathcal{D}_\beta^{b_\beta}} \right) = 0 \,. \tag{4.3.1}$$

In the expression, k_j are the loop momenta, v_s^μ can be loop or external momenta, $v_s^\mu = \{k_1^\mu, \cdots, k_l^\mu; p_1^\mu, \cdots, p_E^\mu\}$. $\mathcal{D}_i$ are the propagators that depend on the masses, loop and

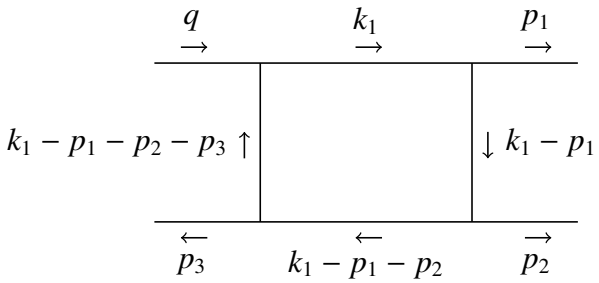


Figure 4.1. One loop box

external momenta. To begin with, a diagram contains a set of propagators as well as scalar products involving the loop and external momenta. However, we can express all the scalar products involving loop momenta in terms of propagators. This is possible since any Lorentz scalar can be written either in terms of scalar products or propagators. Both of the representations are equivalent. For our convenience, we choose to work in the propagator representation. Performing the differentiation on the left hand side of the above Eq. (4.3.1), one obtains set of IBP identities. Let us demonstrate the role of IBP identities through an one-loop example.

- **Example:** We consider an one loop box diagram, depicted through Fig. 4.1 where, all the external legs are taken to be massless, for simplicity, and the momentum $q = p_1 + p_2 + p_3$. The corresponding dimensionally regularised Feynman integral can be cast into the following form

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathcal{D}_1^{b_1}\,\mathcal{D}_2^{b_2}\,\mathcal{D}_3^{b_3}\,\mathcal{D}_4^{b_4}} \equiv I\,[b_1, b_2, b_3, b_4] \tag{4.3.2}$$

with

$$\begin{aligned}
\mathcal{D}_1 &\equiv k_1\,,\\
\mathcal{D}_2 &\equiv (k_1 - p_1)\,,\\
\mathcal{D}_3 &\equiv (k_1 - p_1 - p_2)\,,\\
\mathcal{D}_4 &\equiv (k_1 - p_1 - p_2 - p_3)\,.
\end{aligned} \tag{4.3.3}$$

We can obtain 4-set of IBP identities for each choice of the set $\{b_1, b_2, b_3, b_4\}$. For a choice of $v_s^\mu = p_1^\mu$ in Eq. (4.3.1), we obtain the corresponding IBP identities as

$$0 = \int \frac{d^d k}{(2\pi)^d}\Big[b_1\left(-1+\frac{\mathcal{D}_2}{\mathcal{D}_1}\right)+b_2\left(1-\frac{\mathcal{D}_1}{\mathcal{D}_2}\right)-b_3\left(\frac{\mathcal{D}_1}{\mathcal{D}_3}-\frac{\mathcal{D}_2}{\mathcal{D}_3}-\frac{s}{\mathcal{D}_3}\right) - b_4\left(\frac{\mathcal{D}_1}{\mathcal{D}_4}-\frac{\mathcal{D}_2}{\mathcal{D}_4}-\frac{s}{\mathcal{D}_4}-\frac{u}{\mathcal{D}_4}\right)\Big]\frac{1}{\mathcal{D}_1^{b_1}\mathcal{D}_2^{b_2}\mathcal{D}_3^{b_3}\mathcal{D}_4^{b_4}}\,. \tag{4.3.4}$$

It can be symbolically expressed as

$$\begin{aligned} 0 &= b_1(-1+1^+2^-)+b_2(1-2^+1^-)-b_3(3^+1^- - 3^+2^- - s\,3^+) \\ &\quad - b_4(4^+1^- - 4^+2^- - s\,4^+ - u\,4^+) \end{aligned} \tag{4.3.5}$$

where, we have made use of the convention as $1^+2^- I[b_1, b_2, b_3, b_4] = I[b_1+1, b_2-1, b_3, b_4]$ and the associated Mandelstam variables are defined as $s \equiv (p_1+p_2)^2 = 2p_1.p_2$, $t \equiv (p_2+p_3)^2 = 2p_2.p_3$, $u \equiv (p_1+p_3)^2 = 2p_1.p_3$. From the Eq. (4.3.5), it is clear that the IBP identities provide recursion relations among the integrals of a topology and/or its sub-topologies. Similarly, we can get the IBP identities corresponding to other external as well as internal momenta. Upon employing all of these identities, it can be shown that there exists only three MIs, which are $I[1,0,1,0]$, $I[1,0,0,1]$ and $I[1,1,1,1]$. Hence, at the end we need to evaluate only three independent integrals corresponding to the problem under consideration. For higher loop and more number of external legs, the IBP identities often become too clumsy to handle manually. Hence, these identities are generated systematically with the help of some computer algorithms in some packages, such as AIR [114], FIRE [115], REDUZE [116, 117], LiteRed [118, 119].

4.3.2 Lorentz Invariant Identities (LI)

The Lorentz invariance of the scalar Feynman integrals can be used in order to obtain more set of identities among the integrals, which are known as Lorentz invariant identities [48]:

$$p_j^{[\mu} p_k^{\nu]} \sum_i p_{i,[\mu} \frac{\partial}{\partial p_i^{\nu]}} I(p_i) = 0\,. \tag{4.3.6}$$

It has been recently found [120] that the LI identities are not independent from IBP ones, since these can be reproduced generating and solving larger systems of IBPs. However, use of LI identities along with the IBP helps to speed up the solution. Hence, in almost all of the computer codes for performing automated reduction to MIs, LI identities are therefore extensively used.

Employing the IBP and LI identities, we obtain a set of MIs which ultimately need to be evaluated. Upon evaluation of the MIs, we can obtain the final unrenormalised result of the virtual corrections. Often these contain UV as well as soft and collinear divergences. The UV divergences are removed through UV renormalisation. The UV renormalised result of the virtual corrections exhibit a universal infrared pole structures which serve a crucial check on the correctness of the computation. In the next chapter, we employ this methodology to compute the three loop quark and gluon form factors in QCD for the production of a pseudo-scalar.

4.4 Summary

We have discussed the techniques largely used for the computations of the multiloop amplitudes which is mostly based on the IBP and LI identities. These are employed in the computer codes to automatise the reduction process. Among some packages, we utilise LiteRed [118, 119] for our computations. In these articles [13, 121–123], we have applied this methodology successfully to compute the 2- and 3-loop QCD corrections. In the next

chapter, we will present the computation of 3-loop QCD form factors for the pseudo-scalar production where we have essentially made use of the methodology discussed in this chapter.

5 Pseudo-Scalar Form Factors at Three Loops in QCD

The materials presented in this chapter are the result of an original research done in collaboration with Thomas Gehrmann, M. C. Kumar, Prakash Mathews, Narayan Rana and V. Ravindran, and these are based on the published articles [13, 14].

Contents

5.1 Prologue

Form factors are the matrix elements of local composite operators between physical states. In the calculation of scattering cross sections, they provide the purely virtual corrections. For example, in the context of hard scattering processes such as Drell-Yan [97, 124] and the Higgs boson production in gluon fusion [44, 54–57, 59–61, 84, 125–128], the form factors corresponding to the vector current operator $\overline{\psi}\gamma_\mu\psi$ and the gluonic operator $G^a_{\mu\nu}G^{a,\mu\nu}$ contribute, respectively. Here ψ is the fermionic field operator and $G^a_{\mu\nu}$ is the field tensor of the non-Abelian gauge field A^a_μ corresponding to the gauge group SU(N). In QCD the form factors can be computed order by order in the strong coupling constant using perturbation theory. Beyond leading order, the UV renormalisation of the form factors involves the renormalisation of the composite operator itself, besides the standard procedure for coupling constant and external fields.

The resulting UV finite form factors still contain divergences of infrared origin, namely, soft and collinear divergences due to the presence of massless gluons and quarks/ anti-quarks in the theory. The inclusive hard scattering cross sections require, in addition to the form factor, the real-emission partonic subprocesses as well as suitable mass factorisation kernels for incoming partons. The soft divergences in the form factor resulting from the gluons cancel against those present in the real emission processes and the mass factorisation kernels remove the remaining collinear divergences rendering the hadronic inclusive cross section IR finite. While these IR divergences cancel among various parts in the perturbative computations, they can give rise to logarithms involving physical scales and kinematic scaling variables of the processes under study. In kinematical regions where these logarithms become large, they may affect the convergence and reliability of the perturbation series expansion in powers of the coupling constant. The solution for this problem goes back to the pioneering work by Sudakov [79] on the asymptotic behaviour of the form factor in Quantum Electrodynamics: all leading logarithms can be summed up to

all orders in perturbation theory. Later on, this resummation was extended to non-leading logarithms [81] and systematised for non-Abelian gauge theories [82]. Ever since, form factors have been central to understand the underlying structure of amplitudes in gauge theories.

The infrared origin of universal logarithmic corrections to form factors [83] and scattering amplitudes results in a close interplay between resummation and infrared pole structure. Working in dimensional regularisation in $d = 4 + \epsilon$ dimensions, these poles appear as inverse powers in the Laurent expansion in ϵ. In a seminal paper, Catani [129] proposed a universal formula for the IR pole structure of massless two-loop QCD amplitudes of arbitrary multiplicity (valid through to double pole terms). This formula was later on justified systematically from infrared factorization [130], thereby also revealing the structure of the single poles in terms of the anomalous dimensions for the soft radiation. In [84], it was shown that the single pole term in quark and gluon form factors up to two loop level can be shown to decompose into UV ($\gamma_I, I = q, g$) and universal collinear (B_I), color singlet soft (f_I) anomalous dimensions, later on observed to hold even at three loop level in [40]. An all loop conjecture for the pole structure of the on-shell multi-loop multi-leg amplitudes in SU(N) gauge theory with n_f light flavors in terms of cusp (A_I), collinear (B_I) and soft anomalous dimensions (Γ_{IJ}, f_I - colour non-singlet as well as singlet) was proposed by Becher and Neubert [131] and Gardi and Magnea [132], generalising the earlier results [129, 130]. The validity of this conjecture beyond three loops depends on the presence/absence of non-trivial colour correlations and crossing ratios involving kinematical invariants [133] and there exists no all-order proof at present. The three-loop expressions for cusp, collinear and colour singlet soft anomalous dimensions were extracted [73, 134] from the three loop flavour singlet [85] and non-singlet [69] splitting functions, thereby also predicting [40] the full pole structure of the three-loop form factors.

The three-loop quark and gluon form factors through to finite terms were computed in [31, 41, 42, 135] and subsequently extended to higher powers in the ϵ expansion [43]. These

results were enabled by modern techniques for multi-loop calculations in quantum field theory, in particular integral reduction methods. These are based on IBP [46, 47] and LI [48] identities which reduce the set of thousands of multi-loop integrals to the one with few integrals, so called MIs in dimensional regularisation. To solve these large systems of IBP and LI identities, the Laporta algorithm [136], which is based on lexicographic ordering of the integrals, is the main tool of choice. It has been implemented in several computer algebra codes [114–119]. The MIs relevant to the form factors are single-scale three-loop vertex functions, for which analytical expressions were derived in Refs. [42, 68, 105, 137–139].

Recently, some of us have applied these state-of-the-art methods to accomplish the task of computing spin-2 quark and gluon form factors up to three loops [123] level in SU(N) gauge theory with n_f light flavours. These form factors are ingredients to the precise description of production cross sections for graviton production, that are predicted in extensions of the SM. In addition, the spin-2 form factors relate to operators with higher tensorial structure and thus provide the opportunity to test the versatility and robustness of calculational techniques for the vertex functions at three loop level. The results [123] also confirm the universality of the UV and IR structure of the gauge theory amplitudes in dimensional regularisation.

In the present work, we derive the three-loop corrections to the quark and gluon form factors for pseudo-scalar operators. These operators appear frequently in effective field theory descriptions of extensions of the SM. Most notably, a pseudo-scalar state coupling to massive fermions is an inherent prediction of any two-Higgs doublet model [140–147]. In the limit of infinite fermion mass, this gives rise to the operator insertions considered here. The recent discovery of a Standard-Model-like Higgs boson at the LHC [1, 15] has not only revived the interest in such Higgs bosons but also prompted the study of the properties of the discovered boson to identify either with lightest scalar or pseudo-scalar Higgs bosons of extended models. Such a study requires precise predictions for their production

cross sections. In the context of a CP-even scalar Higgs boson, results for the inclusive production cross section in the gluon fusion are available up to N^{3}LO QCD [44, 59–61], based on an effective scalar coupling that results from integration of massive quark loops that mediate the coupling of the Higgs boson to gluons [148–150]. On the other hand for the CP-odd pseudo-scalar, only NNLO QCD results [61, 151–154] in the effective theory [45] are known. The exact quark mass dependence for scalar and pseudo-scalar production is known to NLO QCD [56, 155], and is usually included through a re-weighting of the effective theory results. Soft gluon resummation of the gluon fusion cross section has been performed to N^{3}LL for the scalar case [28, 29, 34, 64, 72, 74, 156–158] and to NNLL for the pseudo-scalar case [159]. A generic threshold resummation formula valid to N^{3}LL accuracy for colour-neutral final states was derived in [34], requiring only the virtual three-loop amplitudes as process-dependent input. The numerical impact of soft gluon resummation in scalar and pseudo-scalar Higgs boson production and its combination with mass corrections is reviewed comprehensively in [160]. The three-loop corrections to the pseudo-scalar form factors computed in this thesis are an important ingredient to the N^{3}LO and N^{3}LL gluon fusion cross sections [14] for pseudo-scalar Higgs boson production, thereby enabling predictions at the same level of precision that is attained in the scalar case.

The framework of the calculation is outlined in Section 5.2, where we describe the effective theory [45]. Due to the pseudo-scalar coupling, one is left with two effective operators with same quantum number and mass dimensions, which mix under renormalisation. Since these operators contain the Levi-Civita tensor as well as γ_5, the computation of the matrix elements requires additional care in $4 + \epsilon$ dimensions where neither Levi-Civita tensor nor γ_5 can be defined unambiguously. We use the prescription by 't Hooft and Veltman [8, 49] to define γ_5. We describe the calculation in Section 5.3, putting particular emphasis on the UV renormalisation. Exploiting the universal IR pole structure of the form factors, we determine the UV renormalisation constants and mixing of the effective operators up to three loop level. We also show that the finite renormalisation constant,

known up to three loops [49], required to preserve one loop nature of the chiral anomaly, is consistent with anomalous dimensions of the overall renormalisation constants. As a first application of our form factors, we compute the hard matching functions for N^{3}LL resummation in soft-collinear effective theory (SCET) in Section 5.5. Section 5.6 summarises our results and contains an outlook on future applications to precision phenomenology of pseudo-scalar Higgs production.

5.2 Framework of the Calculation

5.2.1 The Effective Lagrangian

A pseudo-scalar Higgs boson couples to gluons only indirectly through a virtual heavy quark loop. This loop can be integrated out in the limit of infinite quark mass. The resulting effective Lagrangian [45] encapsulates the interaction between a pseudo-scalar Φ^A and QCD particles and reads:

$$\mathcal{L}^A_{\text{eff}} = \Phi^A(x)\Big[-\frac{1}{8}C_G O_G(x) - \frac{1}{2}C_J O_J(x)\Big] \tag{5.2.1}$$

where the operators are defined as

$$O_G(x) = G^{\mu\nu}_a \tilde{G}_{a,\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma} G^{\mu\nu}_a G^{\rho\sigma}_a\,, \qquad O_J(x) = \partial_\mu\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right)\,. \tag{5.2.2}$$

The Wilson coefficients C_G and C_J are obtained by integrating out the heavy quark loop, and C_G does not receive any QCD corrections beyond one loop due to the Adler-Bardeen theorem [161], while C_J starts only at second order in the strong coupling constant. Expanded in $a_s \equiv g_s^2/(16\pi^2) = \alpha_s/(4\pi)$, they read

$$C_G = -a_s 2^{\frac{5}{4}} G_F^{\frac{1}{2}} \cot\beta$$

$$C_J = -\left[a_s C_F \left(\frac{3}{2} - 3\ln\frac{\mu_R^2}{m_t^2}\right) + a_s^2 C_J^{(2)} + \cdots\right] C_G \,. \tag{5.2.3}$$

In the above expressions, $G_a^{\mu\nu}$ and ψ represent gluonic field strength tensor and light quark fields, respectively and G_F is the Fermi constant and $\cot\beta$ is the mixing angle in a generic Two-Higgs-Doublet model. $a_s \equiv a_s\left(\mu_R^2\right)$ is the strong coupling constant renormalised at the scale μ_R which is related to the unrenormalised one, $\hat{a}_s \equiv \hat{g}_s^2/(16\pi^2)$ through

$$\hat{a}_s S_\epsilon = \left(\frac{\mu^2}{\mu_R^2}\right)^{\epsilon/2} Z_{a_s} a_s \tag{5.2.4}$$

with $S_\epsilon = \exp\left[(\gamma_E - \ln 4\pi)\epsilon/2\right]$ and μ is the scale introduced to keep the strong coupling constant dimensionless in $d = 4 + \epsilon$ space-time dimensions. The renormalisation constant Z_{a_s} [77] is given by

$$Z_{a_s} = 1 + a_s\left[\frac{2}{\epsilon}\beta_0\right] + a_s^2\left[\frac{4}{\epsilon^2}\beta_0^2 + \frac{1}{\epsilon}\beta_1\right] + a_s^3\left[\frac{8}{\epsilon^3}\beta_0^3 + \frac{14}{3\epsilon^2}\beta_0\beta_1 + \frac{2}{3\epsilon}\beta_2\right] \tag{5.2.5}$$

up to $O(a_s^3)$. β_i are the coefficients of the QCD β functions which are given by [77] and presented in Eq. (2.3.12).

5.2.2 Treatment of γ_5 in Dimensional Regularization

Higher order calculations of chiral quantities in dimensional regularization face the problem of defining a generalization of the inherently four-dimensional objects γ_5 and $\varepsilon^{\mu\nu\rho\sigma}$ to values of $d \neq 4$. In this thesis, we have followed the most practical and self-consistent definition of γ_5 for multiloop calculations in dimensional regularization which was introduced by 't Hooft and Veltman through [8]

$$\gamma_5 = i\frac{1}{4!}\varepsilon_{\nu_1\nu_2\nu_3\nu_4}\gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\nu_3}\gamma^{\nu_4} \,. \tag{5.2.6}$$

Here, $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor which is contracted as

$$\varepsilon_{\mu_1\nu_1\lambda_1\sigma_1}\,\varepsilon^{\mu_2\nu_2\lambda_2\sigma_2} = \begin{vmatrix} \delta_{\mu_1}^{\mu_2} & \delta_{\mu_1}^{\nu_2} & \delta_{\mu_1}^{\lambda_2} & \delta_{\mu_1}^{\sigma_2} \\ \delta_{\nu_1}^{\mu_2} & \delta_{\nu_1}^{\nu_2} & \delta_{\nu_1}^{\lambda_2} & \delta_{\nu_1}^{\sigma_2} \\ \delta_{\lambda_1}^{\mu_2} & \delta_{\lambda_1}^{\nu_2} & \delta_{\lambda_1}^{\lambda_2} & \delta_{\lambda_1}^{\sigma_2} \\ \delta_{\sigma_1}^{\mu_2} & \delta_{\sigma_1}^{\nu_2} & \delta_{\sigma_1}^{\lambda_2} & \delta_{\sigma_1}^{\sigma_2} \end{vmatrix} \tag{5.2.7}$$

and all the Lorentz indices are considered to be d-dimensional [49]. In this scheme, a finite renormalisation of the axial vector current is required in order to fulfill chiral Ward identities and the Adler-Bardeen theorem. We discuss this in detail in Section 5.3.2 below.

5.3 Pseudo-scalar Quark and Gluon Form Factors

The quark and gluon form factors describe the QCD loop corrections to the transition matrix element from a color-neutral operator O to an on-shell quark-antiquark pair or to two gluons. For the pseudo-scalar interaction, we need to consider the two operators O_G and O_J, defined in Eq. (5.2.2), thus yielding in total four form factors. We define the unrenormalised gluon form factors at $O(\hat{a}_s^n)$ as

$$\hat{\mathcal{F}}_g^{G,(n)} \equiv \frac{\langle \hat{\mathcal{M}}_g^{G,(0)} | \hat{\mathcal{M}}_g^{G,(n)} \rangle}{\langle \hat{\mathcal{M}}_g^{G,(0)} | \hat{\mathcal{M}}_g^{G,(0)} \rangle}, \qquad \hat{\mathcal{F}}_g^{J,(n)} \equiv \frac{\langle \hat{\mathcal{M}}_g^{G,(0)} | \hat{\mathcal{M}}_g^{J,(n+1)} \rangle}{\langle \hat{\mathcal{M}}_g^{G,(0)} | \hat{\mathcal{M}}_g^{J,(1)} \rangle} \tag{5.3.1}$$

and similarly the unrenormalised quark form factors through

$$\hat{\mathcal{F}}_q^{G,(n)} \equiv \frac{\langle \hat{\mathcal{M}}_q^{J,(0)} | \hat{\mathcal{M}}_q^{G,(n+1)} \rangle}{\langle \hat{\mathcal{M}}_q^{J,(0)} | \hat{\mathcal{M}}_q^{G,(1)} \rangle}, \qquad \hat{\mathcal{F}}_q^{J,(n)} \equiv \frac{\langle \hat{\mathcal{M}}_q^{J,(0)} | \hat{\mathcal{M}}_q^{J,(n)} \rangle}{\langle \hat{\mathcal{M}}_q^{J,(0)} | \hat{\mathcal{M}}_q^{J,(0)} \rangle} \tag{5.3.2}$$

where, $n = 0, 1, 2, 3, \ldots$. In the above expressions $|\hat{\mathcal{M}}_\beta^{\lambda,(n)}\rangle$ is the $O(\hat{a}_s^n)$ contribution to the unrenormalised matrix element for the transition from the bare operator $[O_\lambda]_B$ ($\lambda = G, J$)

to a quark-antiquark pair ($\beta = q$) or to two gluons ($\beta = g$). The expansion of these quantities in powers of $\hat{a}_s$ is performed through

$$|\mathcal{M}_\beta^\lambda\rangle \equiv \sum_{n=0}^{\infty} \hat{a}_s^n S_\epsilon^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} |\hat{\mathcal{M}}_\beta^{\lambda,(n)}\rangle \quad \text{and} \quad \mathcal{F}_\beta^\lambda \equiv \sum_{n=0}^{\infty} \left[\hat{a}_s^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n \hat{\mathcal{F}}_\beta^{\lambda,(n)}\right]. \tag{5.3.3}$$

where, $Q^2 = -2\, p_1.p_2$ and $p_i's$ ($p_i^2 = 0$) are the momenta of the external quarks and gluons. Note that $|\hat{\mathcal{M}}_q^{G,(n)}\rangle$ and $|\hat{\mathcal{M}}_g^{J,(n)}\rangle$ start from $n = 1$ i.e. from one loop level.

5.3.1 Calculation of the Unrenormalised Form Factors

The calculation of the unrenormalised pseudo-scalar form factors up to three loops follows closely the steps used in the derivation of the three-loop scalar and vector form factors [31, 42]. The Feynman diagrams for all transition matrix elements (Eq. (5.3.1), Eq. (5.3.2)) are generated using QGRAF [111]. The numbers of diagrams contributing to three loop amplitudes are 1586 for $|\hat{\mathcal{M}}_g^{G,(3)}\rangle$, 447 for $|\hat{\mathcal{M}}_g^{J,(3)}\rangle$, 400 for $|\hat{\mathcal{M}}_q^{G,(3)}\rangle$ and 244 for $|\hat{\mathcal{M}}_q^{J,(3)}\rangle$ where all the external particles are considered to be on-shell. The raw output of QGRAF is converted to a format suitable for further manipulation. A set of in-house routines written in the symbolic manipulating program FORM [112] is utilized to perform the simplification of the matrix elements involving Lorentz and color indices. Contributions arising from ghost loops are taken into account as well since we use Feynman gauge for internal gluons. For the external on-shell gluons, we ensure the summing over only transverse polarization states by employing an axial polarization sum:

$$\sum_s \varepsilon^{\mu\,*}(p_i, s)\varepsilon^\nu(p_i, s) = -\eta^{\mu\nu} + \frac{p_i^\mu q_i^\nu + q_i^\mu p_i^\nu}{p_i.q_i} \quad , \tag{5.3.4}$$

where p_i is the i^{th}-gluon momentum, q_i is the corresponding reference momentum which is an arbitrary light like 4-vector and s stands for spin (polarization) of gluons. We choose

$q_1 = p_2$ and $q_2 = p_1$ for our calculation. Finally, traces over the Dirac matrices are carried out in d dimensions.

The expressions involve thousands of three-loop scalar integrals. However, they are expressible in terms of a much smaller set of scalar integrals, called master integrals (MIs), by use of IBP [46, 47] and LI [48] identities. These identities follow from the Poincare invariance of the integrands, they result in a large linear system of equations for the integrals relevant to given external kinematics at a fixed loop-order. The LI identities are not linearly independent from the IBP identities [120], their inclusion does however help to accelerate the solution of the system of equations. By employing lexicographic ordering of these integrals (Laporta algorithm, [136]), a reduction to MIs is accomplished. Several implementations of the Laporta algorithm exist in the literature: AIR [114], FIRE [115], Reduze2 [116, 117] and LiteRed [118, 119]. In the context of the present calculation, we used LiteRed [118, 119] to perform the reductions of all the integrals to MIs.

Each three-loop Feynman integral is expressed in terms of a list of propagators involving loop momenta that can be attributed to one of the following three sets (auxiliary topologies, [42])

$$\begin{aligned}
&\mathrm{A}_1 : \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{23}, \mathcal{D}_{1;1}, \mathcal{D}_{1;12}, \mathcal{D}_{2;1}, \mathcal{D}_{2;12}, \mathcal{D}_{3;1}, \mathcal{D}_{3;12}\} \\
&\mathrm{A}_2 : \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{23}, \mathcal{D}_{13;2}, \mathcal{D}_{1;12}, \mathcal{D}_{2;1}, \mathcal{D}_{12;2}, \mathcal{D}_{3;1}, \mathcal{D}_{3;12}\} \\
&\mathrm{A}_3 : \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{123}, \mathcal{D}_{1;1}, \mathcal{D}_{1;12}, \mathcal{D}_{2;1}, \mathcal{D}_{2;12}, \mathcal{D}_{3;1}, \mathcal{D}_{3;12}\} .
\end{aligned} \tag{5.3.5}$$

In the above sets

$$\mathcal{D}_i = k_i^2, \mathcal{D}_{ij} = (k_i - k_j)^2, \mathcal{D}_{ijl} = (k_i - k_j - k_l)^2,$$

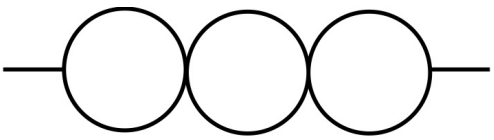
$$\mathcal{D}_{i;j} = (k_i - p_j)^2, \mathcal{D}_{i;jl} = (k_i - p_j - p_l)^2, \mathcal{D}_{ij;l} = (k_i - k_j - p_l)^2$$

To accomplish this, we have used the package Reduze2 [116,117]. In each set in Eq. (5.3.5), $\mathcal{D}'$s are linearly independent and form a complete basis in a sense that any Lorentz-invariant scalar product involving loop momenta and external momenta can be expressed uniquely in terms of $\mathcal{D}'$s from that set.

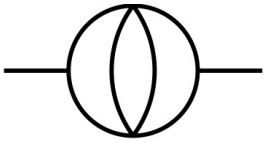
As a result, we can express the unrenormalised form factors in terms of 22 topologically different master integrals (MIs) which can be broadly classified into three different types: genuine three-loop integrals with vertex functions ($A_{t,i}$), three-loop propagator integrals ($B_{t,i}$) and integrals which are product of one- and two-loop integrals ($C_{t,i}$). Defining a generic three loop master integral through

$$A_{i,m_1^i m_2^i \cdots m_{12}^i} = \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \int \frac{d^d k_3}{(2\pi)^d} \frac{1}{\prod_j D_j^{m_j^i}}, \qquad i = 1,2,3 \tag{5.3.6}$$

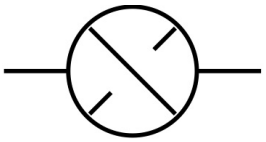
where D_j is the j^{th} element of the basis set A_i. We identify the resulting master integrals appeared in our computation to those given in [42] and they are listed in the figures below.



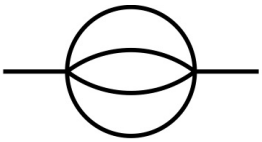
$B_{6,1} \equiv A_{1,111000010101}$



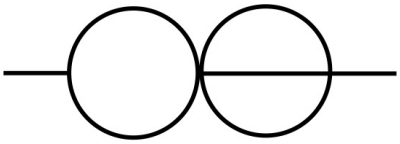
$B_{6,2} \equiv A_{1,011110000101}$



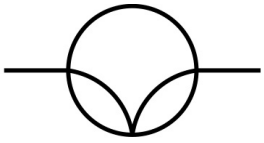
$B_{8,1} \equiv A_{3,011111010101}$



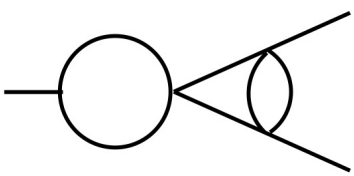
$B_{4,1} \equiv A_{1,001101010000}$



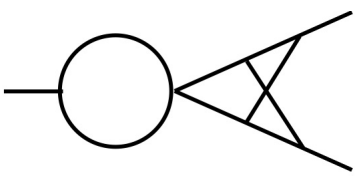
$B_{5,1} \equiv A_{1,011010010100}$



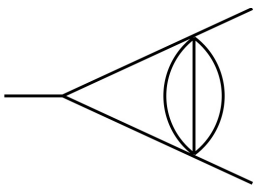
$B_{5,2} \equiv A_{1,001011010100}$



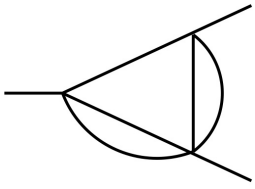
$C_{6,1} \equiv A_{1,011100100101}$



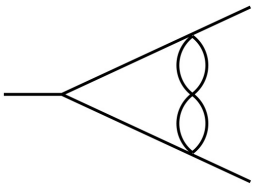
$C_{8,1} \equiv A_{2,111100011101}$



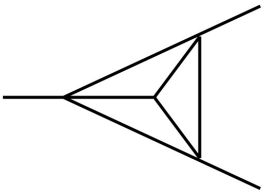
$A_{5,1} \equiv A_{1,001101100001}$



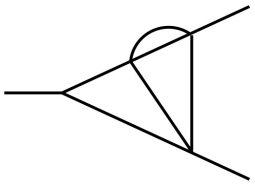
$A_{5,2} \equiv A_{1,001011011000}$



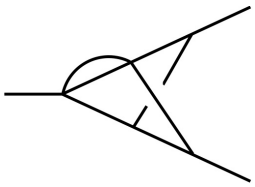
$A_{6,1} \equiv A_{1,010101100110}$



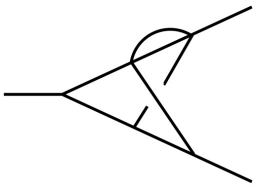
$A_{6,2} \equiv A_{1,001111011000}$



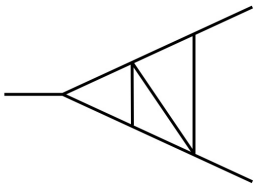
$A_{6,3} \equiv A_{1,001110100101}$



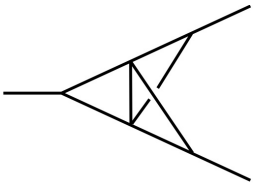
$A_{7,1} \equiv A_{2,011110011100}$



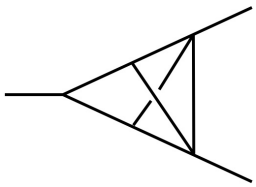
$A_{7,2} \equiv A_{2,011011001101}$



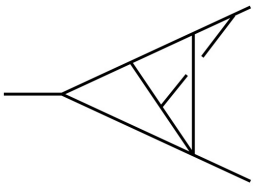
$A_{7,3} \equiv A_{1,011011110100}$



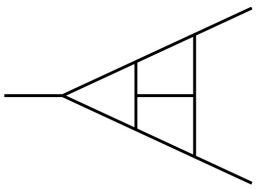
$A_{7,4} \equiv A_{2,011110001101}$



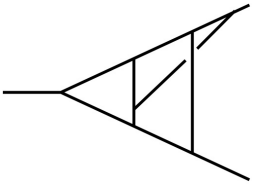
$A_{7,5} \equiv A_{2,011011010101}$



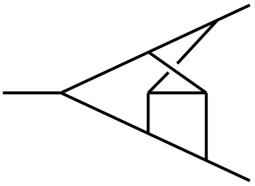
$A_{8,1} \equiv A_{2,001111011101}$



$A_{9,1} \equiv A_{1,011111110110}$



$A_{9,2} \equiv A_{2,011111011101}$



$A_{9,4} \equiv A_{2,111011111100}$

These integrals were computed analytically as Laurent series in ϵ in [68, 105, 137–139] and are collected in the appendix of [42]. Inserting those, we obtain the final expressions for the unrenormalised (bare) form factors that are listed in Appendix H.

5.3.2 UV Renormalisation

To obtain ultraviolet-finite expressions for the form factors, a renormalisation of the coupling constant and of the operators is required. The UV renormalisation of the operators $[O_G]_B$ and $[O_J]_B$ involves some non-trivial prescriptions. These are in part related to the formalism used for the γ_5 matrix, section 5.2.2 above.

This formalism fails to preserve the anti-commutativity of γ_5 with γ^μ in d dimensions. In addition, the standard properties of the axial current and Ward identities, which are valid in a basic regularization scheme like the one of Pauli-Villars, are violated as well. As a consequence, one fails to restore the correct renormalised axial current, which is defined as [49, 162]

$$J_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi = i\frac{1}{3!}\varepsilon^{\mu\nu_1\nu_2\nu_3}\bar{\psi}\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}\psi \tag{5.3.7}$$

in dimensional regularization. To rectify this, one needs to introduce a finite renormalisation constant Z_5^s [161, 163] in addition to the standard overall ultraviolet renormalisation constant $Z_{\overline{MS}}^s$ within the $\overline{MS}$-scheme:

$$\left[J_5^\mu\right]_R = Z_5^s Z_{\overline{MS}}^s \left[J_5^\mu\right]_B \,. \tag{5.3.8}$$

By evaluating the appropriate Feynman diagrams explicitly, $Z_{\overline{MS}}^s$ can be computed, however the finite renormalisation constant is not fixed through this calculation. To determine Z_5^s one has to demand the conservation of the one loop character [164] of the operator relation of the axial anomaly in dimensional regularization:

$$\begin{aligned} \left[\partial_\mu J_5^\mu\right]_R &= a_s \frac{n_f}{2}\left[G\tilde{G}\right]_R \\ \text{i.e.} \quad [O_J]_R &= a_s \frac{n_f}{2}[O_G]_R \,. \end{aligned} \tag{5.3.9}$$

The bare operator $[O_J]_B$ is renormalised multiplicatively exactly in the same way as the

axial current J_5^μ through

$$[O_J]_R = Z_5^s Z_{\overline{MS}}^s [O_J]_B \,, \tag{5.3.10}$$

whereas the other one $[O_G]_B$ mixes under the renormalisation through

$$[O_G]_R = Z_{GG} [O_G]_B + Z_{GJ} [O_J]_B \tag{5.3.11}$$

with the corresponding renormalisation constants Z_{GG} and Z_{GJ}. The above two equations can be combined to express them through the matrix equation

$$[O_i]_R = Z_{ij} \left[O_j\right]_B \tag{5.3.12}$$

with

$$\begin{aligned} &i, j = \{G, J\}\,, \\ O \equiv \begin{bmatrix} O_G \\ O_J \end{bmatrix} \qquad &\text{and} \qquad Z \equiv \begin{bmatrix} Z_{GG} & Z_{GJ} \\ Z_{JG} & Z_{JJ} \end{bmatrix}. \end{aligned} \tag{5.3.13}$$

In the above expressions

$$\begin{aligned} &Z_{JG} = 0 \qquad \text{to all orders in perturbation theory}\,, \\ &Z_{JJ} \equiv Z_5^s Z_{\overline{MS}}^s \,. \end{aligned} \tag{5.3.14}$$

We determine the above-mentioned renormalisation constants $Z_{\overline{MS}}^s, Z_{GG}, Z_{GJ}$ up to $O\left(a_s^3\right)$ from our calculation of the bare on-shell pseudo-scalar form factors described in the previous subsection. This procedure provides a completely independent approach to their original computation, which was done in the operator product expansion [50].

Our approach to compute those Z_{ij} is based on the infrared evolution equation for the form

factor, and will be detailed in Section 5.3.3 below. Moreover, we can fix Z_5^s up to $O(a_s^2)$ by demanding the operator relation of the axial anomaly (Eq. (5.3.9)). Using these overall operator renormalisation constants along with strong coupling constant renormalisation through Z_{a_s}, Eq. (5.2.5), we obtain the UV finite on-shell quark and gluon form factors.

To define the UV renormalised form factors, we introduce a quantity $\mathcal{S}_\beta^\lambda$, constructed out of bare matrix elements, through

$$\mathcal{S}_g^G \equiv Z_{GG}\langle\hat{\mathcal{M}}_g^{G,(0)}|\mathcal{M}_g^G\rangle + Z_{GJ}\langle\hat{\mathcal{M}}_g^{G,(0)}|\mathcal{M}_g^J\rangle$$

and

$$\mathcal{S}_q^G \equiv Z_{GG}\langle\hat{\mathcal{M}}_q^{J,(0)}|\mathcal{M}_q^G\rangle + Z_{GJ}\langle\hat{\mathcal{M}}_q^{J,(0)}|\mathcal{M}_q^J\rangle\,. \tag{5.3.15}$$

Expanding the quantities appearing on the right hand side of the above equation in powers of a_s :

$$\begin{aligned} |\mathcal{M}_\beta^\lambda\rangle &= \sum_{n=0}^{\infty} a_s^n |\mathcal{M}_\beta^{\lambda,(n)}\rangle\,, \\ Z_I &= \sum_{n=0}^{\infty} a_s^n Z_I^{(n)} \qquad \text{with} \qquad I = GG, GJ\ , \end{aligned} \tag{5.3.16}$$

we can write

$$\mathcal{S}_g^G = \sum_{n=0}^{\infty} a_s^n \mathcal{S}_g^{G,(n)} \qquad \text{and} \qquad \mathcal{S}_q^G = \sum_{n=1}^{\infty} a_s^n \mathcal{S}_q^{G,(n)}\,. \tag{5.3.17}$$

Then the UV renormalised form factors corresponding to O_G are defined as

$$\begin{aligned} \left[\mathcal{F}_g^G\right]_R \equiv \frac{\mathcal{S}_g^G}{\mathcal{S}_g^{G,(0)}} &= Z_{GG}\mathcal{F}_g^G + Z_{GJ}\mathcal{F}_g^J \frac{\langle\mathcal{M}_g^{G,(0)}|\mathcal{M}_g^{J,(1)}\rangle}{\langle\mathcal{M}_g^{G,(0)}|\mathcal{M}_g^{G,(0)}\rangle} \\ &\equiv 1 + \sum_{n=1}^{\infty} a_s^n \left[\mathcal{F}_g^{G,(n)}\right]_R\,, \end{aligned}$$

$$\left[\mathcal{F}_q^G\right]_R \equiv \frac{\mathcal{S}_q^G}{a_s \mathcal{S}_q^{G,(1)}} = \frac{Z_{GG}\mathcal{F}_q^G\langle\mathcal{M}_q^{J,(0)}|\mathcal{M}_q^{G,(1)}\rangle + Z_{GJ}\mathcal{F}_q^J\langle\mathcal{M}_q^{J,(0)}|\mathcal{M}_q^{J,(0)}\rangle}{a_s\left[\langle\mathcal{M}_q^{J,(0)}|\mathcal{M}_q^{G,(1)}\rangle + Z_{GJ}^{(1)}\langle\mathcal{M}_q^{J,(0)}|\mathcal{M}_q^{J,(0)}\rangle\right]}$$
$$\equiv 1 + \sum_{n=1}^{\infty} a_s^n \left[\mathcal{F}_q^{G,(n)}\right]_R \tag{5.3.18}$$

where

$$\mathcal{S}_g^{G,(0)} = \langle\mathcal{M}_g^{G,(0)}|\mathcal{M}_g^{G,(0)}\rangle\,,$$
$$\mathcal{S}_q^{G,(1)} = \langle\mathcal{M}_q^{J,(0)}|\mathcal{M}_q^{G,(1)}\rangle + Z_{GJ}^{(1)}\langle\mathcal{M}_q^{J,(0)}|\mathcal{M}_q^{J,(0)}\rangle\,. \tag{5.3.19}$$

Similarly, for defining the UV finite form factors for the other operator O_J we introduce

$$\mathcal{S}_g^J \equiv Z_5^s Z_{\overline{MS}}^s \langle\hat{\mathcal{M}}_g^{G,(0)}|\mathcal{M}_g^J\rangle$$

and

$$\mathcal{S}_q^J \equiv Z_5^s Z_{\overline{MS}}^s \langle\hat{\mathcal{M}}_q^{J,(0)}|\mathcal{M}_q^J\rangle\,. \tag{5.3.20}$$

Expanding $Z_{\overline{MS}}^s$ and $|\mathcal{M}_\beta^\lambda\rangle$ in powers of a_s, following Eq. (5.3.16), we get

$$\mathcal{S}_g^J = \sum_{n=1}^{\infty} a_s^n \mathcal{S}_g^{J,(n)} \qquad \text{and} \qquad \mathcal{S}_q^J = \sum_{n=0}^{\infty} a_s^n \mathcal{S}_q^{J,(n)}\,. \tag{5.3.21}$$

With these we define the UV renormalised form factors corresponding to O_J through

$$\left[\mathcal{F}_g^J\right]_R \equiv \frac{\mathcal{S}_g^J}{a_s \mathcal{S}_g^{J,(1)}} = Z_5^s Z_{\overline{MS}}^s \mathcal{F}_g^J \equiv 1 + \sum_{n=1}^{\infty} a_s^n \left[\mathcal{F}_g^{J,(n)}\right]_R\,,$$
$$\left[\mathcal{F}_q^J\right]_R \equiv \frac{\mathcal{S}_q^J}{\mathcal{S}_q^{J,(0)}} = Z_5^s Z_{\overline{MS}}^s \mathcal{F}_q^J = 1 + \sum_{n=1}^{\infty} a_s^n \left[\mathcal{F}_q^{J,(n)}\right]_R \tag{5.3.22}$$

where

$$\mathcal{S}_g^{J,(1)} = \langle \mathcal{M}_g^{G,(0)} | \mathcal{M}_g^{J,(1)} \rangle \,,$$
$$\mathcal{S}_q^{J,(0)} = \langle \mathcal{M}_q^{J,(0)} | \mathcal{M}_q^{J,(0)} \rangle \,. \tag{5.3.23}$$

The finite renormalisation constant Z_5^s is multiplied in Eq. (5.3.20) to restore the axial anomaly equation in dimensional regularisation. We determine all required renormalisation constants from consistency conditions on the universal structure of the infrared poles of the renormalised form factors in the next section, and use these constants to derive the UV-finite form factors in Section 5.3.4.

5.3.3 Infrared Singularities and Universal Pole Structure

The renormalised form factors are ultraviolet-finite, but still contain divergences of infrared origin. In the calculation of physical quantities (which fulfill certain infrared-safety criteria [165]), these infrared singularities are cancelled by contributions from real radiation processes that yield the same observable final state, and by mass factorization contributions associated with initial-state partons. The pole structures of these infrared divergences arising in QCD form factors exhibit some universal behaviour. The very first successful proposal along this direction was presented by Catani [129] (see also [130]) for one and two-loop QCD amplitudes using the universal subtraction operators. The factorization of the single pole in quark and gluon form factors in terms of soft and collinear anomalous dimensions was first revealed in [84] up to two loop level whose validity at three loop was later established in the article [40]. The proposal by Catani was generalized beyond two loops by Becher and Neubert [131] and by Gardi and Magnea [132]. Below, we outline this behaviour in the context of pseudo-scalar form factors up to three loop level, following closely the notation used in [28].

The unrenormalised form factors $\mathcal{F}_\beta^\lambda(\hat{a}_s, Q^2, \mu^2, \epsilon)$ satisfy the so-called KG-differential

equation [79–82] which is dictated by the factorization property, gauge and renormalisation group (RG) invariances:

$$Q^2\frac{d}{dQ^2}\ln\mathcal{F}_\beta^\lambda(\hat{a}_s,Q^2,\mu^2,\epsilon)=\frac{1}{2}\left[K_\beta^\lambda(\hat{a}_s,\frac{\mu_R^2}{\mu^2},\epsilon)+G_\beta^\lambda(\hat{a}_s,\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\epsilon)\right] \tag{5.3.24}$$

where all poles in the dimensional regulator ϵ are contained in the Q^2 independent function K_β^λ and the finite terms in $\epsilon\to 0$ are encapsulated in G_β^λ. RG invariance of the form factor implies

$$\mu_R^2\frac{d}{d\mu_R^2}K_\beta^\lambda(\hat{a}_s,\frac{\mu_R^2}{\mu^2},\epsilon)=-\mu_R^2\frac{d}{d\mu_R^2}G_\beta^\lambda(\hat{a}_s,\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\epsilon)=-A_\beta^\lambda(a_s(\mu_R^2))=-\sum_{i=1}^{\infty}a_s^i(\mu_R^2)A_{\beta,i}^\lambda \tag{5.3.25}$$

where, $A_{\beta,i}^\lambda$ on the right hand side are the i-loop cusp anomalous dimensions. It is straightforward to solve for K_β^λ in Eq. (5.3.25) in powers of bare strong coupling constant $\hat{a}_s$ by performing the following expansion

$$K_\beta^\lambda\left(\hat{a}_s,\frac{\mu_R^2}{\mu^2},\epsilon\right)=\sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{\mu_R^2}{\mu^2}\right)^{i\frac{\epsilon}{2}}S_\epsilon^i K_{\beta,i}^\lambda(\epsilon)\,. \tag{5.3.26}$$

The solutions $K_{\beta,i}^\lambda(\epsilon)$ consist of simple poles in ϵ with the coefficients consisting of $A_{\beta,i}^\lambda$ and β_i. These can be found in [28, 29]. On the other hand, the RGE of $G_{\beta,i}^\lambda(\hat{a}_s,\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\epsilon)$ can be solved. The solution contains two parts, one is dependent on μ_R^2 whereas the other part depends only the boundary point $\mu_R^2=Q^2$. The μ_R^2 dependent part can eventually be expressed in terms of A_β^λ:

$$G_\beta^\lambda(\hat{a}_s,\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\epsilon)=G_\beta^\lambda(a_s(Q^2),1,\epsilon)+\int_{\frac{Q^2}{\mu_R^2}}^1\frac{dx}{x}A_\beta^\lambda(a_s\left(x\mu_R^2\right))\,. \tag{5.3.27}$$

The boundary term can be expanded in powers of a_s as

$$G_\beta^\lambda(a_s(Q^2),1,\epsilon)=\sum_{i=1}^{\infty}a_s^i(Q^2)G_{\beta,i}^\lambda(\epsilon)\,. \tag{5.3.28}$$

The solutions of K_β^λ and G_β^λ enable us to solve the KG equation (Eq. (5.3.24)) and thereby facilitate to obtain the $\ln \mathcal{F}_\beta^\lambda(\hat{a}_s, Q^2, \mu^2, \epsilon)$ in terms of $A_{\beta,i}^\lambda, G_{\beta,i}^\lambda$ and β_i which is given by [28]

$$\ln \mathcal{F}_\beta^\lambda(\hat{a}_s, Q^2, \mu^2, \epsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \hat{\mathcal{L}}_{\beta,i}^\lambda(\epsilon) \tag{5.3.29}$$

with

$$\begin{aligned}
\hat{\mathcal{L}}_{\beta,1}^\lambda(\epsilon) =& \frac{1}{\epsilon^2}\left\{-2A_{\beta,1}^\lambda\right\} + \frac{1}{\epsilon}\left\{G_{\beta,1}^\lambda(\epsilon)\right\},\\
\hat{\mathcal{L}}_{\beta,2}^\lambda(\epsilon) =& \frac{1}{\epsilon^3}\left\{\beta_0 A_{\beta,1}^\lambda\right\} + \frac{1}{\epsilon^2}\left\{-\frac{1}{2}A_{\beta,2}^\lambda - \beta_0 G_{\beta,1}^\lambda(\epsilon)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{2}G_{\beta,2}^\lambda(\epsilon)\right\},\\
\hat{\mathcal{L}}_{\beta,3}^\lambda(\epsilon) =& \frac{1}{\epsilon^4}\left\{-\frac{8}{9}\beta_0^2 A_{\beta,1}^\lambda\right\} + \frac{1}{\epsilon^3}\left\{\frac{2}{9}\beta_1 A_{\beta,1}^\lambda + \frac{8}{9}\beta_0 A_{\beta,2}^\lambda + \frac{4}{3}\beta_0^2 G_{\beta,1}^\lambda(\epsilon)\right\}\\
&+ \frac{1}{\epsilon^2}\left\{-\frac{2}{9}A_{\beta,3}^\lambda - \frac{1}{3}\beta_1 G_{\beta,1}^\lambda(\epsilon) - \frac{4}{3}\beta_0 G_{\beta,2}^\lambda(\epsilon)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{3}G_{\beta,3}^\lambda(\epsilon)\right\}.
\end{aligned} \tag{5.3.30}$$

All these form factors are observed to satisfy [40, 84] the following decomposition in terms of collinear (B_β^λ), soft (f_β^λ) and UV (γ_β^λ) anomalous dimensions:

$$G_{\beta,i}^\lambda(\epsilon) = 2\left(B_{\beta,i}^\lambda - \gamma_{\beta,i}^\lambda\right) + f_{\beta,i}^\lambda + C_{\beta,i}^\lambda + \sum_{k=1}^{\infty} \epsilon^k g_{\beta,i}^{\lambda,k}\,, \tag{5.3.31}$$

where the constants $C_{\beta,i}^\lambda$ are given by [29]

$$\begin{aligned}
C_{\beta,1}^\lambda &= 0\,,\\
C_{\beta,2}^\lambda &= -2\beta_0 g_{\beta,1}^{\lambda,1}\,,\\
C_{\beta,3}^\lambda &= -2\beta_1 g_{\beta,1}^{\lambda,1} - 2\beta_0\left(g_{\beta,2}^{\lambda,1} + 2\beta_0 g_{\beta,1}^{\lambda,2}\right).
\end{aligned} \tag{5.3.32}$$

In the above expressions, $X_{\beta,i}^\lambda$ with $X = A, B, f$ and $\gamma_{\beta,i}^\lambda$ are defined through

$$X_\beta^\lambda \equiv \sum_{i=1}^{\infty} a_s^i X_{\beta,i}^\lambda\,, \qquad \text{and} \qquad \gamma_\beta^\lambda \equiv \sum_{i=1}^{\infty} a_s^i \gamma_{\beta,i}^\lambda\,. \tag{5.3.33}$$

Within this framework, we will now determine this universal structure of IR singular-

ities of the pseudo-scalar form factors. This prescription will be used subsequently to determine the overall operator renormalisation constants.

We begin with the discussion of form factors corresponding to O_J. The results of the form factors $\mathcal{F}_\beta^J$ for $\beta = q, g$, which have been computed up to three loop level in this article are being used to extract the unknown factors, $\gamma_{\beta,i}^J$ and $g_{\beta,i}^{J,k}$, by employing the KG equation. Since the $\mathcal{F}_\beta^J$ satisfy KG equation, we can obtain the solutions Eq. (5.3.29) along with Eq. (5.3.30) and Eq. (5.3.31) to examine our results against the well known decomposition of the form factors in terms of the quantities X_β^J. These are universal, and appear also in the vector and scalar quark and gluon form factors [40, 84]. They are known [11, 84–86, 166] up to three loop level in the literature. Using these in the above decomposition, we obtain $\gamma_{\beta,i}^J$. The other process dependent constants, namely, $g_{\beta,i}^{J,k}$ can be obtained by comparing the coefficients of ϵ^k in Eq. (5.3.30) at every order in $\hat{a}_s$. We can get the quantities $\gamma_{g,i}^J$ and $g_{g,i}^{J,k}$ up to two loop level, since this process starts at one loop. From gluon form factors we get

$$\begin{aligned}
\gamma_{g,1}^J &= 0\,, \\
\gamma_{g,2}^J &= C_A C_F \left\{ -\frac{44}{3} \right\} + C_F n_f \left\{ -\frac{10}{3} \right\}.
\end{aligned} \tag{5.3.34}$$

Similarly, from the quark form factors we obtain

$$\begin{aligned}
\gamma_{q,1}^J &= 0\,, \\
\gamma_{q,2}^J &= C_A C_F \left\{ -\frac{44}{3} \right\} + C_F n_f \left\{ -\frac{10}{3} \right\}, \\
\gamma_{q,3}^J &= C_A^2 C_F \left\{ -\frac{3578}{27} \right\} + C_F^2 n_f \left\{ \frac{22}{3} \right\} - C_F n_f^2 \left\{ \frac{26}{27} \right\} + C_A C_F^2 \left\{ \frac{308}{3} \right\} \\
&\quad + C_A C_F n_f \left\{ -\frac{149}{27} \right\}.
\end{aligned} \tag{5.3.35}$$

Note that $\gamma_{q,i}^J = \gamma_{g,i}^J$ which is expected since these are the UV anomalous dimensions associated with the same operator $[O_J]_B$. The $\gamma_{\beta,i}^J$ are further used to obtain the overall

operator renormalisation constant $Z^s_{\overline{MS}}$ through the RGE:

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z^\lambda(a_s, \mu_R^2, \epsilon) = \sum_{i=1}^{\infty} a_s^i \gamma_i^\lambda. \tag{5.3.36}$$

The general solution of the RGE is obtained as

$$\begin{aligned} Z^\lambda = 1 + a_s\left[\frac{1}{\epsilon}2\gamma_1^\lambda\right] + a_s^2\left[\frac{1}{\epsilon^2}\left\{2\beta_0\gamma_1^\lambda + 2(\gamma_1^\lambda)^2\right\} + \frac{1}{\epsilon}\gamma_2^\lambda\right] + a_s^3\left[\frac{1}{\epsilon^3}\left\{8\beta_0^2\gamma_1^\lambda + 4\beta_0(\gamma_1^\lambda)^2\right.\right. \\ \left.\left. + \frac{4(\gamma_1^\lambda)^3}{3}\right\} + \frac{1}{\epsilon^2}\left\{\frac{4\beta_1\gamma_1^\lambda}{3} + \frac{4\beta_0\gamma_2^\lambda}{3} + 2\gamma_1^\lambda\gamma_2^\lambda\right\} + \frac{1}{\epsilon}\left\{\frac{2\gamma_3^\lambda}{3}\right\}\right]. \end{aligned} \tag{5.3.37}$$

By substituting the results of $\gamma^J_{\beta,i}$ in the above solution we get $Z^s_{\overline{MS}}$ up to $O(a_s^3)$:

$$\begin{aligned} Z^s_{\overline{MS}} = 1 + a_s^2\left[C_A C_F\left\{-\frac{44}{3\epsilon}\right\} + C_F n_f\left\{-\frac{10}{3\epsilon}\right\}\right] + a_s^3\left[C_A^2 C_F\left\{-\frac{1936}{27\epsilon^2} - \frac{7156}{81\epsilon}\right\}\right. \\ \left. + C_F^2 n_f\left\{\frac{44}{9\epsilon}\right\} + C_F n_f^2\left\{\frac{80}{27\epsilon^2} - \frac{52}{81\epsilon}\right\} + C_A C_F^2\left\{\frac{616}{9\epsilon}\right\} + C_A C_F n_f\left\{-\frac{88}{27\epsilon^2} - \frac{298}{81\epsilon}\right\}\right], \end{aligned} \tag{5.3.38}$$

which agrees completely with the known result in [49]. In order to restore the axial anomaly equation in dimensional regularization (see Section 5.3.2 above), we must multiply the $Z^s_{\overline{MS}}[O_J]_B$ by a finite renormalisation constant Z_5^s, which reads [49]

$$Z_5^s = 1 + a_s\{-4C_F\} + a_s^2\left\{22C_F^2 - \frac{107}{9}C_A C_F + \frac{31}{18}C_F n_f\right\}. \tag{5.3.39}$$

Following the computation of the operator mixing constants below, we will be able to verify explicitly that this expression yields the correct expression for the axial anomaly.

Now, we move towards the discussion of O_G form factors. Similar to previous case, we consider the form factors $Z^{-1}_{GG}[\mathcal{F}^G_\beta]_R$, defined through Eq. (5.3.18), to extract the unknown constants, $\gamma^G_{\beta,i}$ and $g^{G,k}_{\beta,i}$, by utilizing the KG differential equation. Since, $[\mathcal{F}^G_\beta]_R$ is UV

finite, the product of Z_{GG}^{-1} with $[\mathcal{F}_\beta^G]_R$ can effectively be treated as unrenormalised form factor and hence we can demand that $Z_{GG}^{-1}[\mathcal{F}_\beta^G]_R$ satisfy KG equation. Further we make use of the solutions Eq. (5.3.29) in conjunction with Eq. (5.3.30) and Eq. (5.3.31) to compare our results against the universal decomposition of the form factors in terms of the constants X_β^G. Upon substituting the existing results of the quantities $A_{\beta,i}^G$, $B_{\beta,i}^G$ and $f_{\beta,i}^G$ up to three loops, which are obtained in case of quark and gluon form factors, we determine the anomalous dimensions $\gamma_{\beta,i}^G$ and the constants $g_{\beta,i}^{G,k}$. However, it is only possible to get the factors $\gamma_{q,i}^G$ and $g_{q,i}^{G,k}$ up to two loops because of the absence of a tree level amplitude in the quark initiated process for the operator O_G. Since $[\mathcal{F}_\beta^G]_R$ are UV finite, the anomalous dimensions $\gamma_{\beta,i}^G$ must be equal to the anomalous dimension corresponding to the renormalisation constant Z_{GG}. This fact is being used to determine the overall renormalisation constants Z_{GG} and Z_{GJ} up to three loop level where these quantities are parameterized in terms of the newly introduced anomalous dimensions γ_{ij} through the matrix equation

$$\mu_R^2 \frac{d}{d\mu_R^2} Z_{ij} \equiv \gamma_{ik} Z_{kj} \qquad \text{with} \qquad i,j,k = G,J \tag{5.3.40}$$

This can be equivalently written as

$$\gamma_{ij} = \left(\mu_R^2 \frac{d}{d\mu_R^2} Z_{ik}\right) \left(Z^{-1}\right)_{kj} . \tag{5.3.41}$$

The general solution (See Example 2 in Appendix C) of the RGE up to a_s^3 is obtained as

$$\begin{aligned} Z_{ij} = \delta_{ij} &+ a_s \left[\frac{2}{\epsilon}\gamma_{ij,1}\right] + a_s^2 \left[\frac{1}{\epsilon^2}\left\{2\beta_0\gamma_{ij,1} + 2\gamma_{ik,1}\gamma_{kj,1}\right\} + \frac{1}{\epsilon}\left\{\gamma_{ij,2}\right\}\right] + a_s^3 \left[\frac{1}{\epsilon^3}\left\{\frac{8}{3}\beta_0^2\gamma_{ij,1}\right.\right. \\ &+ 4\beta_0\gamma_{ik,1}\gamma_{kj,1} + \frac{4}{3}\gamma_{ik,1}\gamma_{kl,1}\gamma_{lj,1}\Big\} + \frac{1}{\epsilon^2}\left\{\frac{4}{3}\beta_1\gamma_{ij,1} + \frac{4}{3}\beta_0\gamma_{ij,2} + \frac{2}{3}\gamma_{ik,1}\gamma_{kj,2}\right. \\ &\left.\left. + \frac{4}{3}\gamma_{ik,2}\gamma_{kj,1}\right\} + \frac{1}{\epsilon}\left\{\frac{2}{3}\gamma_{ij,3}\right\}\right] \end{aligned} \tag{5.3.42}$$

where, γ_{ij} is expanded in powers of a_s as

$$\gamma_{ij} = \sum_{n=1}^{\infty} a_s^n \gamma_{ij,n} \,. \tag{5.3.43}$$

Demanding the vanishing of $\gamma^G_{\beta,i}$, we get

$$\begin{aligned}
\gamma_{GG} &= a_s\left[\frac{11}{3}C_A - \frac{2}{3}n_f\right] + a_s^2\left[\frac{34}{3}C_A^2 - \frac{10}{3}C_A n_f - 2C_F n_f\right] + a_s^3\left[\frac{2857}{54}C_A^3 - \frac{1415}{54}C_A^2 n_f\right. \\
&\quad \left. - \frac{205}{18}C_A C_F n_f + C_F^2 n_f + \frac{79}{54}C_A n_f^2 + \frac{11}{9}C_F n_f^2\right], \\
\gamma_{GJ} &= a_s\left[-12C_F\right] + a_s^2\left[-\frac{284}{3}C_A C_F + 36C_F^2 + \frac{8}{3}C_F n_f\right] + a_s^3\left[-\frac{1607}{3}C_A^2 C_F\right. \\
&\quad + 461 C_A C_F^2 - 126 C_F^3 - \frac{164}{3}C_A C_F n_f + 214 C_F^2 n_f + \frac{52}{3}C_F n_f^2 + 288 C_A C_F n_f \zeta_3 \\
&\quad \left. - 288 C_F^2 n_f \zeta_3\right].
\end{aligned} \tag{5.3.44}$$

In addition to the demand of vanishing $\gamma^G_{\beta,i}$, it is required to use the results of γ_{JJ} and γ_{JG}, which are implied by the definition, Eq. (5.3.40), up to $O(a_s^2)$ to determine the above-mentioned γ_{GG} and γ_{GJ} up to the given order. This is a consequence of the fact that the operators mix under UV renormalisation. Following Eq. (5.3.40) along with Eq. (5.3.14), Eq. (5.3.38) and Eq. (5.3.39), we obtain

$$\gamma_{JJ} = a_s\left[-\epsilon 2C_F\right] + a_s^2\left[\epsilon\left\{-\frac{107}{9}C_A C_F + 14 C_F^2 + \frac{31}{18}C_F n_f\right\} - 6 C_F n_f\right] \tag{5.3.45}$$

and

$$\gamma_{JG} = 0 \,. \tag{5.3.46}$$

As it happens, we note that γ_{JJ}'s are ϵ-dependent and in fact, this plays a crucial role in determining the other quantities. Our results are in accordance with the existing ones, γ_{GG} and γ_{GJ}, which are available up to $O(a_s^2)$ [49] and $O(a_s^3)$ [50], respectively. In addition to the existing ones, here we compute the new result of γ_{GG} at $O(a_s^3)$. It was observed through

explicit computation in the article [49] that

$$\gamma_{GG} = -\frac{\beta}{a_s} \tag{5.3.47}$$

holds true up to two loop level but there was no statement on the validity of this relation beyond that order. In [50], it was demonstrated in the operator product expansion that the relation holds even at three loop. Here, through explicit calculation, we arrive at the same conclusion that the relation is still valid at three loop level which can be seen if we look at the $\gamma_{GG,3}$ in Eq. (5.3.44) which is equal to the β_2.

Before ending the discussion of γ_{ij}, we examine our results against the axial anomaly relation. The renormalisation group invariance of the anomaly equation (Eq. (5.3.9)), see [49], gives

$$\gamma_{JJ} = \frac{\beta}{a_s} + \gamma_{GG} + a_s \frac{n_f}{2} \gamma_{GJ} \,. \tag{5.3.48}$$

Through our calculation up to three loop level we find that our results are in complete agreement with the above anomaly equation through

$$\gamma_{GG} = -\frac{\beta}{a_s} \qquad \text{and} \qquad \gamma_{GJ} = \left(a_s \frac{n_f}{2}\right)^{-1} \gamma_{JJ} \tag{5.3.49}$$

in the limit of $\epsilon \to 0$. This serves as one of the most crucial checks on our computation.

Additionally, if we conjecture the above relations to hold beyond three loops (which could be doubted in light of recent findings [133]), then we can even predict the ϵ-independent part of the γ_{JJ} at $O(a_s^3)$:

$$\gamma_{JJ}|_{\epsilon\to 0} = a_s^2 \left[-6 C_F n_f \right] + a_s^3 \left[-\frac{142}{3} C_A C_F n_f + 18 C_F^2 n_f + \frac{4}{3} C_F n_f^2 \right]. \tag{5.3.50}$$

The results of γ_{ij} uniquely specify Z_{ij}, through Eq. (5.3.42). We summarize the resulting

expressions of Z_{ij} below:

$$
\begin{aligned}
Z_{GG} = 1 &+ a_s\left[\frac{22}{3\epsilon}C_A - \frac{4}{3\epsilon}n_f\right] + a_s^2\left[\frac{1}{\epsilon^2}\left\{\frac{484}{9}C_A^2 - \frac{176}{9}C_An_f + \frac{16}{9}n_f^2\right\} + \frac{1}{\epsilon}\left\{\frac{34}{3}C_A^2\right.\right. \\
&\left.\left. - \frac{10}{3}C_An_f - 2C_Fn_f\right\}\right] + a_s^3\left[\frac{1}{\epsilon^3}\left\{\frac{10648}{27}C_A^3 - \frac{1936}{9}C_A^2n_f + \frac{352}{9}C_An_f^2 - \frac{64}{27}n_f^3\right\}\right. \\
&+ \frac{1}{\epsilon^2}\left\{\frac{5236}{27}C_A^3 - \frac{2492}{27}C_A^2n_f - \frac{308}{9}C_AC_Fn_f + \frac{280}{27}C_An_f^2 + \frac{56}{9}C_Fn_f^2\right\} \\
&\left.+ \frac{1}{\epsilon}\left\{\frac{2857}{81}C_A^3 - \frac{1415}{81}C_A^2n_f - \frac{205}{27}C_AC_Fn_f + \frac{2}{3}C_F^2n_f + \frac{79}{81}C_An_f^2 + \frac{22}{27}C_Fn_f^2\right\}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
Z_{GJ} = &\, a_s\left[-\frac{24}{\epsilon}C_F\right] + a_s^2\left[\frac{1}{\epsilon^2}\left\{-176C_AC_F + 32C_Fn_f\right\} + \frac{1}{\epsilon}\left\{-\frac{284}{3}C_AC_F + 84C_F^2\right.\right. \\
&\left.\left. + \frac{8}{3}C_Fn_f\right\}\right] + a_s^3\left[\frac{1}{\epsilon^3}\left\{-\frac{3872}{3}C_A^2C_F + \frac{1408}{3}C_AC_Fn_f - \frac{128}{3}C_Fn_f^2\right\}\right. \\
&+ \frac{1}{\epsilon^2}\left\{-\frac{9512}{9}C_A^2C_F + \frac{2200}{3}C_AC_F^2 + \frac{2272}{9}C_AC_Fn_f - \frac{64}{3}C_F^2n_f - \frac{32}{9}C_Fn_f^2\right\} \\
&+ \frac{1}{\epsilon}\left\{-\frac{3214}{9}C_A^2C_F + \frac{5894}{9}C_AC_F^2 - 356C_F^3 - \frac{328}{9}C_AC_Fn_f + \frac{1096}{9}C_F^2n_f + \frac{104}{9}C_Fn_f^2\right. \\
&\left.\left.+ 192C_AC_Fn_f\zeta_3 - 192C_F^2n_f\zeta_3\right\}\right]. \qquad (5.3.51)
\end{aligned}
$$

Z_{GG} and Z_{GJ} are in agreement with the results already available in the literature up to $O(a_s^2)$ [49] and $O(a_s^3)$ [50], where a completely different approach and methodology was used.

5.3.4 Results of UV Renormalised Form Factors

Using the renormalisation constants obtained in the previous section, we get all the UV renormalised form factors $\left[\mathcal{F}_\beta^\lambda\right]_R$, defined in Eq. (5.3.18) and Eq. (5.3.22), up to three loops. In this section we present the results for the choice of the scales $\mu_R^2 = \mu_F^2 = q^2$.

$$
\left[\mathcal{F}_g^{G,(1)}\right]_R = 2n_fT_F\left\{-\frac{4}{3\epsilon}\right\} + C_A\left\{-\frac{8}{\epsilon^2} + \frac{22}{3\epsilon} + 4 + \zeta_2 + \epsilon\left(-6 - \frac{7}{3}\zeta_3\right) + \epsilon^2\left(7 - \frac{\zeta_2}{2}\right.\right.
$$

$$+\frac{47}{80}\zeta_2^2\Big)+\epsilon^3\Big(-\frac{15}{2}+\frac{3}{4}\zeta_2+\frac{7}{6}\zeta_3+\frac{7}{24}\zeta_2\zeta_3-\frac{31}{20}\zeta_5\Big)\Big\}, \tag{5.3.52}$$

$$\begin{aligned}\left[\mathcal{F}_g^{G,(2)}\right]_R &= 4n_f^2T_F^2\left\{\frac{16}{9\epsilon^2}\right\}+C_A^2\left\{\frac{32}{\epsilon^4}-\frac{308}{3\epsilon^3}+\left(\frac{62}{9}-4\zeta_2\right)\frac{1}{\epsilon^2}+\left(\frac{2780}{27}+\frac{11}{3}\zeta_2+\frac{50}{3}\zeta_3\right)\frac{1}{\epsilon}\right.\\
&-\frac{3293}{81}+\frac{115}{6}\zeta_2-\frac{21}{5}\zeta_2^2-33\zeta_3+\epsilon\Big(-\frac{114025}{972}-\frac{235}{18}\zeta_2+\frac{1111}{120}\zeta_2^2+\frac{1103}{54}\zeta_3\\
&-\frac{23}{6}\zeta_2\zeta_3-\frac{71}{10}\zeta_5\Big)+\epsilon^2\Big(\frac{4819705}{11664}-\frac{694}{27}\zeta_2-\frac{2183}{240}\zeta_2^2+\frac{2313}{280}\zeta_2^3-\frac{7450}{81}\zeta_3\\
&\left.-\frac{11}{36}\zeta_2\zeta_3+\frac{901}{36}\zeta_3^2-\frac{341}{20}\zeta_5\Big)\right\}+2C_An_fT_F\left\{\frac{56}{3\epsilon^3}-\frac{52}{3\epsilon^2}+\left(-\frac{272}{27}-\frac{2}{3}\zeta_2\right)\frac{1}{\epsilon}\right.\\
&-\frac{295}{81}-\frac{5}{3}\zeta_2-2\zeta_3+\epsilon\Big(\frac{15035}{486}+\frac{\zeta_2}{18}+\frac{59}{60}\zeta_2^2+\frac{383}{27}\zeta_3\Big)+\epsilon^2\Big(-\frac{116987}{1458}+\frac{583}{108}\zeta_2\\
&\left.-\frac{329}{72}\zeta_2^2-\frac{1688}{81}\zeta_3+\frac{61}{18}\zeta_2\zeta_3-\frac{49}{10}\zeta_5\Big)\right\}+2C_Fn_fT_F\Big\{-\frac{2}{\epsilon}-\frac{71}{3}+8\zeta_3+\epsilon\Big(\frac{2665}{36}\\
&-\frac{19}{6}\zeta_2-\frac{8}{3}\zeta_2^2-\frac{64}{3}\zeta_3\Big)+\epsilon^2\Big(-\frac{68309}{432}+\frac{505}{36}\zeta_2+\frac{64}{9}\zeta_2^2+\frac{455}{9}\zeta_3-\frac{10}{3}\zeta_2\zeta_3\\
&+8\zeta_5\Big)\Big\},\end{aligned} \tag{5.3.53}$$

$$\begin{aligned}\left[\mathcal{F}_g^{G,(3)}\right]_R &= 8n_f^3T_F^3\Big\{-\frac{64}{27\epsilon^3}\Big\}+4C_Fn_f^2T_F^2\left\{\frac{56}{9\epsilon^2}+\left(\frac{874}{27}-\frac{32}{3}\zeta_3\right)\frac{1}{\epsilon}-\frac{418}{27}+2\zeta_2+\frac{16}{5}\zeta_2^2\right.\\
&\left.-\frac{80}{9}\zeta_3\right\}+2C_F^2n_fT_F\left\{\frac{2}{3\epsilon}+\frac{457}{6}+104\zeta_3-160\zeta_5\right\}+2C_A^2n_fT_F\Big\{-\frac{320}{3\epsilon^5}\\
&+\frac{28480}{81\epsilon^4}+\Big(-\frac{608}{243}+\frac{56}{27}\zeta_2\Big)\frac{1}{\epsilon^3}+\Big(-\frac{54088}{243}+\frac{676}{81}\zeta_2+\frac{272}{27}\zeta_3\Big)\frac{1}{\epsilon^2}\\
&+\Big(-\frac{623293}{2187}-\frac{7072}{243}\zeta_2-\frac{941}{90}\zeta_2^2-\frac{7948}{81}\zeta_3\Big)\frac{1}{\epsilon}+\frac{6345979}{13122}-\frac{42971}{729}\zeta_2+\frac{687}{20}\zeta_2^2\\
&+\frac{652}{3}\zeta_3-\frac{301}{9}\zeta_2\zeta_3+\frac{4516}{45}\zeta_5\Big\}+4C_An_f^2T_F^2\Big\{-\frac{2720}{81\epsilon^4}+\frac{7984}{243\epsilon^3}+\Big(\frac{560}{27}\\
&+\frac{8}{27}\zeta_2\Big)\frac{1}{\epsilon^2}+\Big(\frac{10889}{2187}+\frac{140}{81}\zeta_2+\frac{328}{81}\zeta_3\Big)\frac{1}{\epsilon}+\frac{9515}{6561}+\frac{10}{27}\zeta_2-\frac{157}{135}\zeta_2^2-\frac{20}{243}\zeta_3\Big\}\\
&+2C_AC_Fn_fT_F\Big\{\frac{272}{9\epsilon^3}+\Big(\frac{4408}{27}-\frac{640}{9}\zeta_3\Big)\frac{1}{\epsilon^2}+\Big(-\frac{65110}{81}+\frac{74}{3}\zeta_2+\frac{352}{15}\zeta_2^2\\
&+\frac{6496}{27}\zeta_3\Big)\frac{1}{\epsilon}+\frac{1053625}{972}-\frac{311}{2}\zeta_2-\frac{1168}{15}\zeta_2^2-\frac{24874}{81}\zeta_3+48\zeta_2\zeta_3+\frac{32}{9}\zeta_5\Big\}\\
&+C_A^3\Big\{-\frac{256}{3\epsilon^6}+\frac{1760}{3\epsilon^5}-\frac{62264}{81\epsilon^4}+\Big(-\frac{176036}{243}-\frac{308}{27}\zeta_2-\frac{176}{3}\zeta_3\Big)\frac{1}{\epsilon^3}+\Big(\frac{207316}{243}\\
&-\frac{8164}{81}\zeta_2+\frac{494}{45}\zeta_2^2+\frac{9064}{27}\zeta_3\Big)\frac{1}{\epsilon^2}+\Big(\frac{2763800}{2187}+\frac{36535}{243}\zeta_2-\frac{12881}{180}\zeta_2^2-\frac{3988}{9}\zeta_3\end{aligned}$$

$$+\frac{170}{9}\zeta_2\zeta_3+\frac{1756}{15}\zeta_5\Big)\frac{1}{\epsilon}-\frac{84406405}{26244}+\frac{617773}{1458}\zeta_2+\frac{144863}{1080}\zeta_2^2-\frac{22523}{270}\zeta_2^3$$
$$+\frac{44765}{243}\zeta_3-\frac{1441}{18}\zeta_2\zeta_3-\frac{1766}{9}\zeta_3^2+\frac{13882}{45}\zeta_5\Big\}, \tag{5.3.54}$$

$$\left[\mathcal{F}_q^{G,(1)}\right]_R = C_F\Big\{-\frac{8}{\epsilon^2}+\frac{6}{\epsilon}-\frac{33}{4}+\zeta_2+\epsilon\Big(\frac{29}{16}+\frac{25}{48}\zeta_2-\frac{7}{3}\zeta_3\Big)+\epsilon^2\Big(\frac{299}{192}-\frac{1327}{576}\zeta_2$$
$$+\frac{1387}{2880}\zeta_2^2+\frac{143}{48}\zeta_3\Big)+\epsilon^3\Big(-\frac{13763}{2304}+\frac{32095}{6912}\zeta_2-\frac{1559}{3456}\zeta_2^2+\frac{61}{6912}\zeta_2^3-\frac{1625}{576}\zeta_3$$
$$+\frac{377}{864}\zeta_2\zeta_3-\frac{31}{20}\zeta_5\Big)\Big\}+2n_fT_F\Big\{-\frac{445}{162}+\epsilon\Big(\frac{8231}{1944}-\frac{239}{1944}\zeta_2-\frac{2}{3}\zeta_3\Big)$$
$$+\epsilon^2\Big(-\frac{50533}{7776}+\frac{1835}{7776}\zeta_2+\frac{22903}{116640}\zeta_2^2+\frac{9125}{5832}\zeta_3+\frac{1}{18}\zeta_2\zeta_3\Big)+\epsilon^3\Big(\frac{2754151}{279936}$$
$$-\frac{35083}{93312}\zeta_2-\frac{316343}{699840}\zeta_2^2-\frac{22903}{1399680}\zeta_2^3-\frac{61121}{23328}\zeta_3+\frac{2053}{34992}\zeta_2\zeta_3-\frac{1}{216}\zeta_2^2\zeta_3$$
$$-\frac{7}{54}\zeta_3^2-\frac{7}{6}\zeta_5\Big)\Big\}+C_A\Big\{\frac{7115}{324}-\frac{2}{3}\zeta_2-2\zeta_3+\epsilon\Big(-\frac{114241}{3888}+\frac{7321}{3888}\zeta_2+\frac{53}{90}\zeta_2^2$$
$$+\frac{13}{3}\zeta_3+\frac{1}{6}\zeta_2\zeta_3\Big)+\epsilon^2\Big(\frac{692435}{15552}-\frac{55117}{15552}\zeta_2-\frac{326369}{233280}\zeta_2^2-\frac{53}{1080}\zeta_2^3-\frac{90235}{11664}\zeta_3$$
$$-\frac{41}{108}\zeta_2\zeta_3-\frac{1}{72}\zeta_2^2\zeta_3-\frac{7}{18}\zeta_3^2-5\zeta_5\Big)+\epsilon^3\Big(-\frac{37171073}{559872}+\frac{1013165}{186624}\zeta_2$$
$$+\frac{3399073}{1399680}\zeta_2^2+\frac{34037663}{19595520}\zeta_2^3+\frac{53}{12960}\zeta_2^4+\frac{585439}{46656}\zeta_3-\frac{56159}{69984}\zeta_2\zeta_3+\frac{3223}{12960}\zeta_2^2\zeta_3$$
$$+\frac{1}{864}\zeta_2^3\zeta_3+\frac{8}{9}\zeta_3^2+\frac{7}{108}\zeta_2\zeta_3^2+8\zeta_5+\frac{5}{12}\zeta_2\zeta_5\Big)\Big\}, \tag{5.3.55}$$

$$\left[\mathcal{F}_q^{G,(2)}\right]_R = 4n_f^2T_F^2\Big\{\frac{9505}{1458}+\epsilon\Big(-\frac{146177}{5832}+\frac{12419}{17496}\zeta_2+\frac{38}{9}\zeta_3\Big)\Big\}+2C_Fn_fT_F\Big\{\frac{8}{\epsilon^3}+\frac{1636}{81\epsilon^2}$$
$$+\Big(-\frac{12821}{243}-\frac{247}{243}\zeta_2+\frac{16}{3}\zeta_3\Big)\frac{1}{\epsilon}+\frac{20765}{324}+\frac{35}{486}\zeta_2+\frac{85}{2916}\zeta_2^2+\frac{6265}{729}\zeta_3-\frac{4}{9}\zeta_2\zeta_3$$
$$+\epsilon\Big(-\frac{1457425}{34992}-\frac{11146}{729}\zeta_2-\frac{232457}{174960}\zeta_2^2-\frac{85}{34992}\zeta_2^3+\frac{9907}{1458}\zeta_3-\frac{7723}{4374}\zeta_2\zeta_3$$
$$+\frac{1}{27}\zeta_2^2\zeta_3+\frac{28}{27}\zeta_3^2-\frac{20}{9}\zeta_5\Big)\Big\}+C_A^2\Big\{\frac{2796445}{5832}-\frac{587}{18}\zeta_2+\frac{53}{30}\zeta_2^2-\frac{185}{2}\zeta_3-\frac{10}{3}\zeta_2\zeta_3$$
$$+20\zeta_5+\epsilon\Big(-\frac{34321157}{23328}+\frac{10420379}{69984}\zeta_2+\frac{589}{20}\zeta_2^2+\frac{7921}{2520}\zeta_2^3+\frac{8411}{24}\zeta_3-\frac{329}{72}\zeta_2\zeta_3$$
$$+\frac{5}{18}\zeta_2^2\zeta_3+13\zeta_3^2-\frac{757}{18}\zeta_5-\frac{5}{3}\zeta_2\zeta_5\Big)\Big\}+2C_An_fT_F\Big\{-\frac{178361}{1458}+\frac{44}{9}\zeta_2-\frac{76}{45}\zeta_2^2$$
$$-\frac{44}{9}\zeta_3+\epsilon\Big(\frac{2357551}{5832}-\frac{478171}{17496}\zeta_2-\frac{137}{135}\zeta_2^2+\frac{19}{135}\zeta_2^3-\frac{1621}{27}\zeta_3-\frac{40}{27}\zeta_2\zeta_3$$
$$+\frac{22}{3}\zeta_5\Big)\Big\}+C_AC_F\Big\{-\frac{44}{\epsilon^3}+\Big(-\frac{13654}{81}+\frac{28}{3}\zeta_2+16\zeta_3\Big)\frac{1}{\epsilon^2}+\Big(\frac{186925}{486}$$

$$
\begin{aligned}
&- \frac{3919}{486}\zeta_2 - \frac{212}{45}\zeta_2^2 - \frac{218}{3}\zeta_3 - \frac{4}{3}\zeta_2\zeta_3\Big)\frac{1}{\epsilon} - \frac{61613}{81} + \frac{59399}{972}\zeta_2 + \frac{749513}{29160}\zeta_2^2 \\
&+ \frac{53}{135}\zeta_2^3 + \frac{213517}{1458}\zeta_3 + \frac{91}{27}\zeta_2\zeta_3 + \frac{1}{9}\zeta_2^2\zeta_3 + \frac{28}{9}\zeta_3^2 + \epsilon\Big(\frac{35327209}{34992} - \frac{2158003}{23328}\zeta_2 \\
&- \frac{3532645}{69984}\zeta_2^2 - \frac{11307767}{2449440}\zeta_2^3 - \frac{53}{1620}\zeta_2^4 - \frac{1030169}{2916}\zeta_3 + \frac{191915}{8748}\zeta_2\zeta_3 - \frac{817}{405}\zeta_2^2\zeta_3 \\
&- \frac{1}{108}\zeta_2^3\zeta_3 - \frac{121}{9}\zeta_3^2 - \frac{14}{27}\zeta_2\zeta_3^2 - \frac{43}{6}\zeta_5\Big)\Big\} + C_F^2\Big\{\frac{32}{\epsilon^4} - \frac{48}{\epsilon^3} + \Big(84 - 8\zeta_2\Big)\frac{1}{\epsilon^2} \\
&+ \Big(-\frac{125}{2} - \frac{61}{6}\zeta_2 + \frac{128}{3}\zeta_3\Big)\frac{1}{\epsilon} + \frac{6881}{216} + \frac{193}{12}\zeta_2 - \frac{281}{24}\zeta_2^2 - \frac{1037}{18}\zeta_3 \\
&+ \epsilon\Big(\frac{166499}{2592} - \frac{3761}{648}\zeta_2 + \frac{3451}{480}\zeta_2^2 - \frac{31}{288}\zeta_2^3 + \frac{10607}{108}\zeta_3 - \frac{1081}{108}\zeta_2\zeta_3 + \frac{328}{45}\zeta_5\Big)\Big\},
\end{aligned}
\tag{5.3.56}
$$

$$
\begin{aligned}
\left[\mathcal{F}_g^{J,(1)}\right]_R &= 2n_f T_F\Big\{-\frac{4}{3\epsilon}\Big\} + C_A\Big\{-\frac{8}{\epsilon^2} + \frac{22}{3\epsilon} + 4 + \zeta_2 + \epsilon\Big(-\frac{15}{2} + \zeta_2 - \frac{16}{3}\zeta_3\Big) \\
&+ \epsilon^2\Big(\frac{287}{24} - 2\zeta_2 + \frac{127}{80}\zeta_2^2\Big) + \epsilon^3\Big(-\frac{5239}{288} + \frac{151}{48}\zeta_2 + \frac{19}{120}\zeta_2^2 + \frac{\zeta_3}{12} + \frac{7}{6}\zeta_2\zeta_3 \\
&- \frac{91}{20}\zeta_5\Big)\Big\} + C_F\Big\{\epsilon\Big(-\frac{21}{2} + 6\zeta_3\Big) + \epsilon^2\Big(\frac{155}{8} - \frac{5}{2}\zeta_2 - \frac{9}{5}\zeta_2^2 - \frac{9}{2}\zeta_3\Big) \\
&+ \epsilon^3\Big(-\frac{1025}{32} + \frac{83}{16}\zeta_2 + \frac{27}{20}\zeta_2^2 + \frac{20}{3}\zeta_3 - \frac{3}{4}\zeta_2\zeta_3 + \frac{21}{2}\zeta_5\Big)\Big\},
\end{aligned}
\tag{5.3.57}
$$

$$
\begin{aligned}
\left[\mathcal{F}_g^{J,(2)}\right]_R &= 4n_f^2 T_F^2\Big\{\frac{16}{9\epsilon^2}\Big\} + C_A C_F\Big\{\Big(84 - 48\zeta_3\Big)\frac{1}{\epsilon} - 232 + 20\zeta_2 + \frac{72}{5}\zeta_2^2 + 80\zeta_3 \\
&+ \epsilon\Big(\frac{17545}{108} - 58\zeta_2 - 24\zeta_2^2 - \frac{38}{3}\zeta_3 + 10\zeta_2\zeta_3 - 14\zeta_5\Big) + \epsilon^2\Big(\frac{402635}{1296} - \frac{233}{36}\zeta_2 \\
&+ \frac{72}{5}\zeta_2^2 + \frac{17}{70}\zeta_2^3 + \frac{535}{12}\zeta_3 - 2\zeta_2\zeta_3 - 34\zeta_3^2 - \frac{1355}{6}\zeta_5\Big)\Big\} + 2C_A n_f T_F\Big\{\frac{56}{3\epsilon^3} - \frac{52}{3\epsilon^2} \\
&+ \Big(-\frac{272}{27} - \frac{2}{3}\zeta_2\Big)\frac{1}{\epsilon} - \frac{133}{81} - 3\zeta_2 + 2\zeta_3 + \epsilon\Big(\frac{7153}{243} - \frac{7}{18}\zeta_2 - \frac{13}{60}\zeta_2^2 + \frac{599}{27}\zeta_3\Big) \\
&+ \epsilon^2\Big(-\frac{135239}{1458} + \frac{1139}{108}\zeta_2 - \frac{167}{24}\zeta_2^2 - \frac{3146}{81}\zeta_3 + \frac{73}{18}\zeta_2\zeta_3 - \frac{137}{30}\zeta_5\Big)\Big\} \\
&+ 2C_F n_f T_F\Big\{-\frac{2}{\epsilon} - \frac{29}{3} + \epsilon\Big(\frac{14989}{216} - \frac{25}{6}\zeta_2 - \frac{4}{15}\zeta_2^2 - 32\zeta_3\Big) + \epsilon^2\Big(-\frac{606661}{2592} \\
&+ \frac{2233}{72}\zeta_2 + \frac{158}{15}\zeta_2^2 + \frac{1409}{18}\zeta_3 - 2\zeta_2\zeta_3 + \frac{82}{3}\zeta_5\Big)\Big\} + C_A^2\Big\{+\frac{32}{\epsilon^4} - \frac{308}{3\epsilon^3} \\
&+ \Big(\frac{62}{9} - 4\zeta_2\Big)\frac{1}{\epsilon^2} + \Big(\frac{3104}{27} - \frac{13}{3}\zeta_2 + \frac{122}{3}\zeta_3\Big)\frac{1}{\epsilon} - \frac{7397}{81} + \frac{77}{2}\zeta_2 - \frac{61}{5}\zeta_2^2 - 55\zeta_3 \\
&+ \epsilon\Big(-\frac{32269}{972} - \frac{997}{36}\zeta_2 + \frac{1049}{120}\zeta_2^2 - \frac{2393}{108}\zeta_3 - \frac{53}{6}\zeta_2\zeta_3 + \frac{369}{10}\zeta_5\Big)
\end{aligned}
$$

$$+\epsilon^2\left(\frac{4569955}{11664}-\frac{15323}{432}\zeta_2+\frac{2129}{180}\zeta_2^2-\frac{7591}{840}\zeta_2^3-\frac{4099}{1296}\zeta_3-\frac{605}{36}\zeta_2\zeta_3+\frac{775}{36}\zeta_3^2+\frac{2011}{30}\zeta_5\right)\Bigg\}+C_F^2\Bigg\{\epsilon\left(\frac{763}{12}+17\zeta_3-60\zeta_5\right)+\epsilon^2\left(-\frac{18857}{144}+\frac{31}{3}\zeta_2-\frac{76}{15}\zeta_2^2+\frac{120}{7}\zeta_2^3-145\zeta_3+4\zeta_2\zeta_3+30\zeta_3^2+\frac{470}{3}\zeta_5\right)\Bigg\}, \tag{5.3.58}$$

$$\left[\mathcal{F}_q^{J,(1)}\right]_R = C_F\Bigg\{-\frac{8}{\epsilon^2}+\frac{6}{\epsilon}-6+\zeta_2+\epsilon\left(-1-\frac{3}{4}\zeta_2\frac{7}{3}\zeta_3\right)+\epsilon^2\left(\frac{5}{2}+\frac{\zeta_2}{4}+\frac{47}{80}\zeta_2^2+\frac{7}{4}\zeta_3\right)+\epsilon^3\left(-\frac{13}{4}+\frac{\zeta_2}{8}-\frac{141}{320}\zeta_2^2-\frac{7}{12}\zeta_3+\frac{7}{24}\zeta_2\zeta_3-\frac{31}{20}\zeta_5\right)\Bigg\}, \tag{5.3.59}$$

$$\left[\mathcal{F}_q^{J,(2)}\right]_R = 2C_F n_f T_F\Bigg\{\frac{8}{\epsilon^3}-\frac{16}{9\epsilon^2}+\left(-\frac{65}{27}-2\zeta_2\right)\frac{1}{\epsilon}-\frac{3115}{324}+\frac{23}{9}\zeta_2+\frac{2}{9}\zeta_3+\epsilon\left(\frac{129577}{3888}-\frac{731}{108}\zeta_2-\frac{\zeta_2^2}{10}+\frac{119}{27}\zeta_3\right)+\epsilon^2\left(-\frac{3054337}{46656}+\frac{20951}{1296}\zeta_2-\frac{145}{144}\zeta_2^2-\frac{2303}{324}\zeta_3-\frac{10}{9}\zeta_2\zeta_3-\frac{59}{30}\zeta_5\right)\Bigg\}+C_F^2\Bigg\{\frac{32}{\epsilon^4}-\frac{48}{\epsilon^3}+\left(66-8\zeta_2\right)\frac{1}{\epsilon^2}+\left(-\frac{53}{2}+\frac{128}{3}\zeta_3\right)\frac{1}{\epsilon}-\frac{121}{8}+\frac{\zeta_2}{2}-13\zeta_2^2-58\zeta_3+\epsilon\left(\frac{3403}{32}+\frac{27}{8}\zeta_2+\frac{171}{10}\zeta_2^2+\frac{559}{6}\zeta_3-\frac{56}{3}\zeta_2\zeta_3+\frac{92}{5}\zeta_5\right)+\epsilon^2\left(-\frac{21537}{128}-\frac{825}{32}\zeta_2-\frac{457}{16}\zeta_2^2+\frac{223}{20}\zeta_2^3-\frac{4205}{24}\zeta_3+\frac{27}{2}\zeta_2\zeta_3+\frac{652}{9}\zeta_3^2-\frac{231}{10}\zeta_5\right)\Bigg\}+C_A C_F\Bigg\{-\frac{44}{\epsilon^3}+\left(\frac{64}{9}+4\zeta_2\right)\frac{1}{\epsilon^2}+\left(\frac{961}{54}+11\zeta_2-26\zeta_3\right)\frac{1}{\epsilon}-\frac{30493}{648}-\frac{193}{18}\zeta_2+\frac{44}{5}\zeta_2^2+\frac{313}{9}\zeta_3+\epsilon\left(-\frac{79403}{7776}+\frac{133}{216}\zeta_2-\frac{229}{20}\zeta_2^2-\frac{4165}{54}\zeta_3+\frac{89}{6}\zeta_2\zeta_3-\frac{51}{2}\zeta_5\right)+\epsilon^2\left(\frac{9732323}{93312}+\frac{41363}{2592}\zeta_2+\frac{33151}{1440}\zeta_2^2-\frac{809}{280}\zeta_2^3+\frac{89929}{648}\zeta_3-\frac{80}{9}\zeta_2\zeta_3-\frac{569}{12}\zeta_3^2+\frac{2809}{60}\zeta_5\right)\Bigg\}, \tag{5.3.60}$$

$$\left[\mathcal{F}_q^{J,(3)}\right]_R = Z_5^{s,(3)}+4C_F n_f^2 T_F^2\Bigg\{-\frac{704}{81\epsilon^4}+\frac{64}{243\epsilon^3}+\left(\frac{184}{81}+\frac{16}{9}\zeta_2\right)\frac{1}{\epsilon^2}+\left(-\frac{4834}{2187}+\frac{40}{27}\zeta_2+\frac{16}{81}\zeta_3\right)\frac{1}{\epsilon}+\frac{538231}{13122}-\frac{680}{81}\zeta_2-\frac{188}{135}\zeta_2^2-\frac{416}{243}\zeta_3\Bigg\}+C_F^3\Bigg\{-\frac{256}{3\epsilon^6}+\frac{192}{\epsilon^5}+\left(-336+32\zeta_2\right)\frac{1}{\epsilon^4}+\left(280+24\zeta_2-\frac{800}{3}\zeta_3\right)\frac{1}{\epsilon^3}+\left(-58-66\zeta_2+\frac{426}{5}\zeta_2^2+552\zeta_3\right)\frac{1}{\epsilon^2}+\left(-\frac{4193}{6}+83\zeta_2-\frac{1461}{10}\zeta_2^2-\frac{3142}{3}\zeta_3+\frac{428}{3}\zeta_2\zeta_3-\frac{1288}{5}\zeta_5\right)\frac{1}{\epsilon}+\frac{41395}{24}+\frac{1933}{12}\zeta_2+\frac{10739}{40}\zeta_2^2-\frac{9095}{252}\zeta_2^3+1385\zeta_3-35\zeta_2\zeta_3-\frac{1826}{3}\zeta_3^2-\frac{562}{5}\zeta_5\Bigg\}$$

$$
\begin{aligned}
&+ 2C_F^2 n_f T_F \Bigg\{ - \frac{64}{\epsilon^5} + \frac{560}{9\epsilon^4} + \left(- \frac{680}{27} + 24\zeta_2 \right) \frac{1}{\epsilon^3} + \left(\frac{5180}{81} - \frac{266}{9}\zeta_2 - \frac{440}{9}\zeta_3 \right) \frac{1}{\epsilon^2} \\
&+ \left(- \frac{78863}{243} + \frac{2381}{27}\zeta_2 + \frac{287}{18}\zeta_2^2 - \frac{938}{27}\zeta_3 \right) \frac{1}{\epsilon} + \frac{1369027}{1458} - \frac{16610}{81}\zeta_2 - \frac{8503}{1080}\zeta_2^2 \\
&+ \frac{22601}{81}\zeta_3 + \frac{35}{3}\zeta_2\zeta_3 - \frac{386}{9}\zeta_5 \Bigg\} + C_A^2 C_F \Bigg\{ - \frac{21296}{81\epsilon^4} + \left(- \frac{22928}{243} + \frac{880}{27}\zeta_2 \right) \frac{1}{\epsilon^3} \\
&+ \left(\frac{23338}{243} + \frac{6500}{81}\zeta_2 - \frac{352}{45}\zeta_2^2 - \frac{3608}{27}\zeta_3 \right) \frac{1}{\epsilon^2} + \left(\frac{139345}{4374} + \frac{14326}{243}\zeta_2 + \frac{332}{15}\zeta_2^2 \right. \\
&\left. - \frac{7052}{27}\zeta_3 + \frac{176}{9}\zeta_2\zeta_3 + \frac{272}{3}\zeta_5 \right) \frac{1}{\epsilon} - \frac{10659797}{52488} - \frac{207547}{729}\zeta_2 + \frac{19349}{270}\zeta_2^2 - \frac{6152}{189}\zeta_2^3 \\
&+ \frac{361879}{486}\zeta_3 + \frac{344}{3}\zeta_2\zeta_3 - \frac{1136}{9}\zeta_3^2 - \frac{2594}{9}\zeta_5 \Bigg\} + 2C_A C_F n_f T_F \Bigg\{ + \frac{7744}{81\epsilon^4} + \left(\frac{6016}{243} \right. \\
&\left. - \frac{160}{27}\zeta_2 \right) \frac{1}{\epsilon^3} + \left(- \frac{8272}{243} - \frac{1904}{81}\zeta_2 + \frac{848}{27}\zeta_3 \right) \frac{1}{\epsilon^2} + \left(\frac{17318}{2187} - \frac{5188}{243}\zeta_2 - \frac{88}{15}\zeta_2^2 \right. \\
&\left. + \frac{1928}{81}\zeta_3 \right) \frac{1}{\epsilon} - \frac{4158659}{13122} + \frac{81778}{729}\zeta_2 - \frac{17}{135}\zeta_2^2 - \frac{5881}{27}\zeta_3 + \frac{22}{3}\zeta_2\zeta_3 + \frac{176}{3}\zeta_5 \Bigg\} \\
&+ C_A C_F^2 \Bigg\{ \frac{352}{\epsilon^5} + \left(- \frac{2888}{9} - 32\zeta_2 \right) \frac{1}{\epsilon^4} + \left(\frac{4436}{27} - 108\zeta_2 + 208\zeta_3 \right) \frac{1}{\epsilon^3} \\
&+ \left(\frac{39844}{81} + \frac{983}{9}\zeta_2 - \frac{332}{5}\zeta_2^2 - \frac{1928}{9}\zeta_3 \right) \frac{1}{\epsilon^2} + \left(- \frac{97048}{243} - \frac{12361}{54}\zeta_2 + \frac{2975}{36}\zeta_2^2 \right. \\
&\left. + \frac{3227}{3}\zeta_3 - \frac{430}{3}\zeta_2\zeta_3 + 284\zeta_5 \right) \frac{1}{\epsilon} - \frac{709847}{729} + \frac{36845}{324}\zeta_2 - \frac{536683}{2160}\zeta_2^2 - \frac{18619}{1260}\zeta_2^3 \\
&- \frac{31537}{18}\zeta_3 - \frac{518}{3}\zeta_2\zeta_3 + \frac{1616}{3}\zeta_3^2 + \frac{1750}{9}\zeta_5 \Bigg\} . \qquad (5.3.61)
\end{aligned}
$$

5.3.5 Universal Behaviour of Leading Transcendentality Contribution

In [135], the form factor of a scalar composite operator belonging to the stress-energy tensor super-multiplet of conserved currents of $\mathcal{N} = 4$ super Yang-Mills (SYM) with gauge group SU(N) was studied to three-loop level. Since the theory is UV finite in $d = 4$ space-time dimensions, it is an ideal framework to study the IR structures of amplitudes in perturbation theory. In this theory, one observes that scattering amplitudes can be expressed as a linear combinations of polylogarithmic functions of uniform degree $2l$, where l is the order of the loop, with constant coefficients. In other words, the scattering

amplitudes in $\mathcal{N} = 4$ SYM exhibit uniform transcendentality, in contrast to QCD loop amplitudes, which receive contributions from all degrees of transcendentality up to $2l$.

The three-loop QCD quark and gluon form factors [42] display an interesting relation to the SYM form factor. Upon replacement [167] of the color factors $C_A = C_F = N$ and $T_f n_f = N/2$, the leading transcendental (LT) parts of the quark and gluon form factors in QCD not only coincide with each other but also become identical, up to a normalization factor of 2^l, to the form factors of scalar composite operator computed in $\mathcal{N} = 4$ SYM [135].

This correspondence between the QCD form factors and that of the $\mathcal{N} = 4$ SYM can be motivated by the leading transcendentality principle [167–169] which relates anomalous dimensions of the twist two operators in $\mathcal{N} = 4$ SYM to the LT terms of such operators computed in QCD. Examining the diagonal pseudo-scalar form factors $\mathcal{F}_g^G$ and $\mathcal{F}_q^J$, we find a similar behaviour: the LT terms of these form factors with replacement $C_A = C_F = N$ and $T_f n_f = N/2$ are not only identical to each other but also coincide with the LT terms of the QCD form factors [42] with the same replacement as well as with the LT terms of the scalar form factors in $\mathcal{N} = 4$ SYM [135], up to a normalization factor of 2^l. This observation holds true for the finite terms in ϵ, and could equally be validated for higher-order terms up to transcendentality 8 (which is the highest order for which all three-loop master integrals are available [170]). In addition to checking the diagonal form factors, we also examined the off-diagonal ones namely, $\mathcal{F}_q^G$, $\mathcal{F}_g^J$, where we find that the LT terms these two form factors are identical to each other after the replacement of colour factors. However, the LT terms of these do not coincide with those of the diagonal ones.

5.4 Gluon Form Factors for the Pseudo-scalar Higgs Boson Production

The complete form factor for the production of a pseudo-scalar Higgs boson through gluon fusion, $\hat{\mathcal{F}}_g^{A,(n)}$, can be written in terms of the individual gluon form factors, Eq. (5.3.3), as follows:

$$\mathcal{F}_g^A = \mathcal{F}_g^G + \left(\frac{Z_{GJ}}{Z_{GG}} + \frac{4C_J}{C_G}\frac{Z_{JJ}}{Z_{GG}}\right)\mathcal{F}_g^J \frac{\langle \hat{\mathcal{M}}_g^{G,(0)} | \hat{\mathcal{M}}_g^{J,(1)} \rangle}{\langle \hat{\mathcal{M}}_g^{G,(0)} | \hat{\mathcal{M}}_g^{G,(0)} \rangle}\,. \tag{5.4.1}$$

In the above expression, the quantities $Z_{ij}(i, j = G, J)$ are the overall operator renormalization constants which are required to introduce in the context of UV renormalization. These are discussed in Sec. 5.3.2 in great detail. The ingredients of the form factor $\mathcal{F}_g^A$, namely, $\mathcal{F}_g^G$ and $\mathcal{F}_g^J$ have been calculated up to three loop level by us [13] and are presented in the Appendix H. Using those results we obtain the three loop form factor for the pseudo-scalar Higgs boson production through gluon fusion. In this section, we present the unrenormalized form factors $\hat{\mathcal{F}}_g^{A,(n)}$ up to three loop where the components are defined through the expansion

$$\mathcal{F}_g^A \equiv \sum_{n=0}^{\infty}\left[\hat{a}_s^n \left(\frac{Q^2}{\mu^2}\right)^{n\frac{\epsilon}{2}} S_\epsilon^n \hat{\mathcal{F}}_g^{A,(n)}\right]. \tag{5.4.2}$$

We present the unrenormalized results for the choice of the scale $\mu_R^2 = \mu_F^2 = q^2$ as follows:

$$\begin{aligned}
\hat{\mathcal{F}}_g^{A,(1)} = C_A\Bigg\{ & -\frac{8}{\epsilon^2} + 4 + \zeta_2 + \epsilon\left(-6 - \frac{7}{3}\zeta_3\right) + \epsilon^2\left(7 - \frac{\zeta_2}{2} + \frac{47}{80}\zeta_2^2\right) + \epsilon^3\Bigg(-\frac{15}{2} \\
& + \frac{3}{4}\zeta_2 + \frac{7}{6}\zeta_3 + \frac{7}{24}\zeta_2\zeta_3 - \frac{31}{20}\zeta_5\Bigg) + \epsilon^4\left(\frac{31}{4} - \frac{7}{8}\zeta_2 - \frac{47}{160}\zeta_2^2 + \frac{949}{4480}\zeta_2^3 - \frac{7}{4}\zeta_3 - \frac{49}{144}\zeta_3^2\right) \\
& + \epsilon^5\Bigg(-\frac{63}{8} + \frac{15}{16}\zeta_2 + \frac{141}{320}\zeta_2^2 + \frac{49}{24}\zeta_3 - \frac{7}{48}\zeta_2\zeta_3 + \frac{329}{1920}\zeta_2^2\zeta_3 + \frac{31}{40}\zeta_5 + \frac{31}{160}\zeta_2\zeta_5 \\
& - \frac{127}{112}\zeta_7\Bigg) + \epsilon^6\Bigg(\frac{127}{16} - \frac{31}{32}\zeta_2 - \frac{329}{640}\zeta_2^2 - \frac{949}{8960}\zeta_2^3 + \frac{55779}{716800}\zeta_2^4 - \frac{35}{16}\zeta_3 + \frac{7}{32}\zeta_2\zeta_3
\end{aligned}$$

$$
\begin{aligned}
&+\frac{49}{288}\zeta_3^2+\frac{49}{1152}\zeta_2\zeta_3^2-\frac{93}{80}\zeta_5-\frac{217}{480}\zeta_3\zeta_5\Big)+\epsilon^7\Big(-\frac{255}{32}+\frac{63}{64}\zeta_2+\frac{141}{256}\zeta_2^2+\frac{2847}{17920}\zeta_2^3\\
&+\frac{217}{96}\zeta_3-\frac{49}{192}\zeta_2\zeta_3-\frac{329}{3840}\zeta_2^2\zeta_3+\frac{949}{15360}\zeta_2^3\zeta_3-\frac{49}{192}\zeta_3^2-\frac{343}{10368}\zeta_3^3+\frac{217}{160}\zeta_5\\
&-\frac{31}{320}\zeta_2\zeta_5+\frac{1457}{12800}\zeta_2^2\zeta_5+\frac{127}{224}\zeta_7+\frac{127}{896}\zeta_2\zeta_7-\frac{511}{576}\zeta_9\Big)\Big\},\\
\hat{\mathcal{F}}_g^{A,(2)} &= C_F n_f\Big\{-\frac{80}{3}+6\ln\Big(\frac{q^2}{m_t^2}\Big)+8\zeta_3+\epsilon\Big(\frac{2827}{36}-9\ln\Big(\frac{q^2}{m_t^2}\Big)-\frac{19}{6}\zeta_2-\frac{8}{3}\zeta_2^2\\
&-\frac{64}{3}\zeta_3\Big)+\epsilon^2\Big(-\frac{70577}{432}+\frac{21}{2}\ln\Big(\frac{q^2}{m_t^2}\Big)+\frac{1037}{72}\zeta_2-\frac{3}{4}\ln\Big(\frac{q^2}{m_t^2}\Big)\zeta_2+\frac{64}{9}\zeta_2^2+\frac{455}{9}\zeta_3\\
&-\frac{10}{3}\zeta_2\zeta_3+8\zeta_5\Big)+\epsilon^3\Big(\frac{1523629}{5184}-\frac{45}{4}\ln\Big(\frac{q^2}{m_t^2}\Big)-\frac{14975}{432}\zeta_2+\frac{9}{8}\ln\Big(\frac{q^2}{m_t^2}\Big)\zeta_2\\
&-\frac{70997}{4320}\zeta_2^2+\frac{22}{35}\zeta_2^3-\frac{3292}{27}\zeta_3+\frac{7}{4}\ln\Big(\frac{q^2}{m_t^2}\Big)\zeta_3+\frac{80}{9}\zeta_2\zeta_3+15\zeta_3^2-\frac{64}{3}\zeta_5\Big)\\
&+\epsilon^4\Big(-\frac{30487661}{62208}+\frac{93}{8}\ln\Big(\frac{q^2}{m_t^2}\Big)+\frac{43217}{648}\zeta_2-\frac{21}{16}\ln\Big(\frac{q^2}{m_t^2}\Big)\zeta_2+\frac{1991659}{51840}\zeta_2^2\\
&-\frac{141}{320}\ln\Big(\frac{q^2}{m_t^2}\Big)\zeta_2^2-\frac{176}{105}\zeta_2^3+\frac{694231}{2592}\zeta_3-\frac{21}{8}\ln\Big(\frac{q^2}{m_t^2}\Big)\zeta_3-\frac{9757}{432}\zeta_2\zeta_3-\frac{1681}{180}\zeta_2^2\zeta_3\\
&-40\zeta_3^2+\frac{8851}{180}\zeta_5-2\zeta_2\zeta_5-\frac{127}{8}\zeta_7\Big)\Big\}+C_A n_f\Big\{-\frac{8}{3\epsilon^3}+\frac{20}{9\epsilon^2}+\Big(\frac{106}{27}+2\zeta_2\Big)\frac{1}{\epsilon}\\
&-\frac{1591}{81}-\frac{5}{3}\zeta_2-\frac{74}{9}\zeta_3+\epsilon\Big(\frac{24107}{486}-\frac{23}{18}\zeta_2+\frac{51}{20}\zeta_2^2+\frac{383}{27}\zeta_3\Big)+\epsilon^2\Big(-\frac{146147}{1458}\\
&+\frac{799}{108}\zeta_2-\frac{329}{72}\zeta_2^2-\frac{1436}{81}\zeta_3+\frac{25}{6}\zeta_2\zeta_3-\frac{271}{30}\zeta_5\Big)+\epsilon^3\Big(\frac{6333061}{34992}-\frac{11531}{648}\zeta_2\\
&++\frac{1499}{240}\zeta_2^2\frac{253}{1680}\zeta_2^3+\frac{19415}{972}\zeta_3-\frac{235}{36}\zeta_2\zeta_3-\frac{1153}{108}\zeta_3^2+\frac{535}{36}\zeta_5\Big)+\epsilon^4\Big(-\frac{128493871}{419904}\\
&+\frac{133237}{3888}\zeta_2-\frac{21533}{2592}\zeta_2^2+\frac{649}{1440}\zeta_2^3-\frac{156127}{5832}\zeta_3+\frac{215}{27}\zeta_2\zeta_3+\frac{517}{80}\zeta_2^2\zeta_3+\frac{14675}{648}\zeta_3^2\\
&-\frac{2204}{135}\zeta_5+\frac{171}{40}\zeta_2\zeta_5+\frac{229}{336}\zeta_7\Big)\Big\}+C_A^2\Big\{\frac{32}{\epsilon^4}+\frac{44}{3\epsilon^3}+\Big(-\frac{422}{9}-4\zeta_2\Big)\frac{1}{\epsilon^2}+\Big(\frac{890}{27}\\
&-11\zeta_2+\frac{50}{3}\zeta_3\Big)\frac{1}{\epsilon}+\frac{3835}{81}+\frac{115}{6}\zeta_2-\frac{21}{5}\zeta_2^2+\frac{11}{9}\zeta_3+\epsilon\Big(-\frac{213817}{972}-\frac{103}{18}\zeta_2+\frac{77}{120}\zeta_2^2\\
&+\frac{1103}{54}\zeta_3-\frac{23}{6}\zeta_2\zeta_3-\frac{71}{10}\zeta_5\Big)+\epsilon^2\Big(\frac{6102745}{11664}-\frac{991}{27}\zeta_2-\frac{2183}{240}\zeta_2^2+\frac{2313}{280}\zeta_2^3-\frac{8836}{81}\zeta_3\\
&-\frac{55}{12}\zeta_2\zeta_3+\frac{901}{36}\zeta_3^2+\frac{341}{60}\zeta_5\Big)+\epsilon^3\Big(-\frac{142142401}{139968}+\frac{75881}{648}\zeta_2+\frac{79819}{2160}\zeta_2^2-\frac{2057}{480}\zeta_2^3\\
&+\frac{606035}{1944}\zeta_3-\frac{251}{72}\zeta_2\zeta_3-\frac{1291}{80}\zeta_2^2\zeta_3-\frac{5137}{216}\zeta_3^2+\frac{14459}{360}\zeta_5+\frac{313}{40}\zeta_2\zeta_5-\frac{3169}{28}\zeta_7\Big)\\
&+\epsilon^4\Big(\frac{2999987401}{1679616}-\frac{1943429}{7776}\zeta_2-\frac{15707}{160}\zeta_2^2-\frac{35177}{20160}\zeta_2^3+\frac{50419}{1600}\zeta_2^4-\frac{16593479}{23328}\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1169}{27}\zeta_2\zeta_3+\frac{22781}{1440}\zeta_2^2\zeta_3+\frac{93731}{1296}\zeta_3^2-\frac{1547}{144}\zeta_2\zeta_3^2-\frac{8137}{54}\zeta_5-\frac{1001}{80}\zeta_2\zeta_5+\frac{845}{24}\zeta_3\zeta_5\\
&-\frac{33}{2}\zeta_{5,3}+\frac{56155}{672}\zeta_7\Big)\Big\},\\
\hat{\mathcal{F}}_g^{A,(3)} &= n_f C_J^{(2)}\Big\{-2+3\epsilon\Big\}+C_F n_f^2\Big\{\Big(-\frac{640}{9}+16\ln\Big(\frac{q^2}{m_t^2}\Big)+\frac{64}{3}\zeta_3\Big)\frac{1}{\epsilon}+\frac{7901}{27}\\
&-24\ln\Big(\frac{q^2}{m_t^2}\Big)-\frac{32}{3}\zeta_2-\frac{112}{15}\zeta_2^2-\frac{848}{9}\zeta_3\Big\}+C_F^2 n_f\Big\{\frac{457}{6}+104\zeta_3-160\zeta_5\Big\}\\
&+C_A^2 n_f\Big\{\frac{64}{3\epsilon^5}-\frac{32}{81\epsilon^4}+\Big(-\frac{18752}{243}-\frac{376}{27}\zeta_2\Big)\frac{1}{\epsilon^3}+\Big(\frac{36416}{243}-\frac{1700}{81}\zeta_2+\frac{2072}{27}\zeta_3\Big)\frac{1}{\epsilon^2}\\
&+\Big(\frac{62642}{2187}+\frac{22088}{243}\zeta_2-\frac{2453}{90}\zeta_2^2-\frac{3988}{81}\zeta_3\Big)\frac{1}{\epsilon}-\frac{14655809}{13122}-\frac{60548}{729}\zeta_2+\frac{917}{60}\zeta_2^2\\
&-\frac{772}{27}\zeta_3-\frac{439}{9}\zeta_2\zeta_3+\frac{3238}{45}\zeta_5\Big\}+C_A n_f^2\Big\{-\frac{128}{81\epsilon^4}+\frac{640}{243\epsilon^3}+\Big(\frac{128}{27}+\frac{80}{27}\zeta_2\Big)\frac{1}{\epsilon^2}\\
&+\Big(-\frac{93088}{2187}-\frac{400}{81}\zeta_2-\frac{1328}{81}\zeta_3\Big)\frac{1}{\epsilon}+\frac{1066349}{6561}-\frac{56}{27}\zeta_2+\frac{797}{135}\zeta_2^2+\frac{13768}{243}\zeta_3\Big\}\\
&+C_A C_F n_f\Big\{-\frac{16}{9\epsilon^3}+\Big(\frac{5980}{27}-48\ln\Big(\frac{q^2}{m_t^2}\Big)-\frac{640}{9}\zeta_3\Big)\frac{1}{\epsilon^2}+\Big(-\frac{20377}{81}\\
&-16\ln\Big(\frac{q^2}{m_t^2}\Big)+\frac{86}{3}\zeta_2+\frac{352}{15}\zeta_2^2+\frac{1744}{27}\zeta_3\Big)\frac{1}{\epsilon}+72\ln\Big(\frac{q^2}{m_t^2}\Big)-\frac{587705}{972}-\frac{551}{6}\zeta_2\\
&+12\ln\Big(\frac{q^2}{m_t^2}\Big)\zeta_2-\frac{96}{5}\zeta_2^2+\frac{12386}{81}\zeta_3+48\zeta_2\zeta_3+\frac{32}{9}\zeta_5\Big\}+C_A^3\Big\{-\frac{256}{3\epsilon^6}-\frac{352}{3\epsilon^5}\\
&+\frac{16144}{81\epsilon^4}+\Big(\frac{22864}{243}+\frac{2068}{27}\zeta_2-\frac{176}{3}\zeta_3\Big)\frac{1}{\epsilon^3}+\Big(-\frac{172844}{243}-\frac{1630}{81}\zeta_2+\frac{494}{45}\zeta_2^2\\
&-\frac{836}{27}\zeta_3\Big)\frac{1}{\epsilon^2}+\Big(\frac{2327399}{2187}-\frac{71438}{243}\zeta_2+\frac{3751}{180}\zeta_2^2-\frac{842}{9}\zeta_3+\frac{170}{9}\zeta_2\zeta_3+\frac{1756}{15}\zeta_5\Big)\frac{1}{\epsilon}\\
&+\frac{16531853}{26244}+\frac{918931}{1458}\zeta_2+\frac{27251}{1080}\zeta_2^2-\frac{22523}{270}\zeta_2^3-\frac{51580}{243}\zeta_3+\frac{77}{18}\zeta_2\zeta_3-\frac{1766}{9}\zeta_3^2\\
&+\frac{20911}{45}\zeta_5\Big\}. \qquad (5.4.3)
\end{aligned}
$$

The results up to two loop level is consistent with the existing ones [84] and the three loop result is the new one. These are later used to compute the SV cross-section for the production of a pseudo-scalar particle through gluon fusion at N^{3}LO QCD [14]. This is an essential ingredient to compute all the other associated observables.

5.5 Hard Matching Coefficients in SCET

Soft-collinear effective theory (SCET, [171–177]) is a systematic expansion of the full QCD theory in terms of particle modes with different infrared scaling behaviour. It provides a framework to perform threshold resummation. In the effective theory, the infrared poles of the full high energy QCD theory manifest themselves as ultraviolet poles [178–180], which then can be resummed by employing the renormalisation group evolution from larger scales to the smaller ones. To ensure matching of SCET and full QCD, one computes the matrix elements in both theories and adjusts the Wilson coefficients of SCET accordingly. For the on-shell matching of these two theories, the matching coefficients relevant to pseudo-scalar production in gluon fusion can be obtained directly from the gluon form factors.

The UV renormalised form factors in QCD contain IR divergences. Since the IR poles in QCD turn into UV ones in SCET, we can remove the IR divergences with the help of a renormalisation constant $Z_g^{A,h}$, which essentially absorbs all residual IR poles and produces finite results. The result is the matching coefficient $C_g^{A,\text{eff}}$, which is defined through the following factorisation relation:

$$C_g^{A,\text{eff}}\left(Q^2,\mu_h^2\right) \equiv \lim_{\epsilon\to 0}(Z_g^{A,h})^{-1}(\epsilon,Q^2,\mu_h^2)\left[\mathcal{F}_g^A\right]_R(\epsilon,Q^2) \tag{5.5.1}$$

where, the UV renormalised form factor $\left[\mathcal{F}_g^A\right]_R$, is defined as

$$\left[\mathcal{F}_g^A\right]_R = \left[\mathcal{F}_g^G\right]_R + \frac{4C_J}{C_G}\left[\mathcal{F}_g^J\right]_R\left(a_s\frac{S_g^{J,(1)}}{S_g^{G,(0)}}\right). \tag{5.5.2}$$

The parameter μ_h is the newly introduced mass scale at which the above factorisation is carried out. For the UV renormalised form factors $[\mathcal{F}_g^A]_R$ in Eq. (5.5.1), we fixed the other

scales as $\mu_R^2 = \mu_F^2 = \mu_h^2$. Upon expanding the $Z_g^{A,h}$ and $C_g^{A,\text{eff}}$ in powers of a_s as

$$Z_g^{A,h}(\epsilon, Q^2, \mu_h^2) = 1 + \sum_{i=1}^{\infty} a_s^i(\mu_h^2) Z_{g,i}^{A,h}(\epsilon, Q^2, \mu_h^2)\,,$$
$$C_g^{A,\text{eff}}\left(Q^2, \mu_h^2\right) = 1 + \sum_{i=1}^{\infty} a_s^i(\mu_h^2) C_{g,i}^{A,\text{eff}}\left(Q^2, \mu_h^2\right) \tag{5.5.3}$$

and utilising the above Eq. (5.5.1), we compute the $Z_{g,i}^{A,h}$ as well as $C_{g,i}^{A,\text{eff}}$ up to three loops ($i = 3$). Demanding the cancellation of the residual IR poles of $\left[\mathcal{F}_g^A\right]_R$ against the poles of $(Z_{g,i}^{A,h})^{-1}$, we compute $Z_{g,i}^{A,h}$ which comes out to be

$$\begin{aligned}
Z_{g,1}^{A,h} &= C_A\left\{-\frac{8}{\epsilon^2} + \left(-4L + \frac{22}{3}\right)\frac{1}{\epsilon}\right\} - n_f\left\{\frac{4}{3\epsilon}\right\},\\
Z_{g,2}^{A,h} &= C_F n_f\left\{-\frac{2}{\epsilon}\right\} + n_f^2\left\{\frac{16}{9\epsilon^2}\right\} + C_A n_f\left\{\frac{56}{3\epsilon^3} + \left(-\frac{52}{3} + 8L\right)\frac{1}{\epsilon^2} + \left(-\frac{128}{27} + \frac{20}{9}L\right.\right.\\
&\quad \left.\left. + \frac{2}{3}\zeta_2\right)\frac{1}{\epsilon}\right\} + C_A^2\left\{\frac{32}{\epsilon^4} + \left(-\frac{308}{3} + 32L\right)\frac{1}{\epsilon^3} + \left(\frac{350}{9} - 44L + 8L^2 + 4\zeta_2\right)\frac{1}{\epsilon^2} + \left(\frac{692}{27}\right.\right.\\
&\quad \left.\left. - \frac{134}{9}L - \frac{11}{3}\zeta_2 + 4L\zeta_2 - 2\zeta_3\right)\frac{1}{\epsilon}\right\},\\
Z_{g,3}^{A,h} &= C_F^2 n_f\left\{\frac{2}{3\epsilon}\right\} + C_F n_f^2\left\{\frac{56}{9\epsilon^2} + \frac{22}{27\epsilon}\right\} - n_f^3\left\{\frac{64}{27\epsilon^3}\right\} + C_A^2 n_f\left\{-\frac{320}{3\epsilon^5} + \left(\frac{28480}{81}\right.\right.\\
&\quad \left.\left. - 96L\right)\frac{1}{\epsilon^4} + \left(-\frac{18752}{243} + \frac{3152}{27}L - \frac{64}{3}L^2 - \frac{448}{27}\zeta_2\right)\frac{1}{\epsilon^3} + \left(-\frac{32656}{243} + \frac{7136}{81}L\right.\right.\\
&\quad \left.\left. - \frac{80}{9}L^2 + \frac{1000}{81}\zeta_2 - \frac{104}{9}L\zeta_2 + \frac{344}{27}\zeta_3\right)\frac{1}{\epsilon^2} + \left(-\frac{30715}{2187} + \frac{836}{81}L + \frac{2396}{243}\zeta_2 - \frac{160}{27}L\zeta_2\right.\right.\\
&\quad \left.\left. - \frac{328}{45}\zeta_2^2 - \frac{712}{81}\zeta_3 + \frac{112}{9}L\zeta_3\right)\frac{1}{\epsilon}\right\} + C_A n_f^2\left\{-\frac{2720}{81\epsilon^4} + \left(\frac{7984}{243} - \frac{352}{27}L\right)\frac{1}{\epsilon^3} + \left(\frac{368}{27}\right.\right.\\
&\quad \left.\left. - \frac{400}{81}L - \frac{40}{27}\zeta_2\right)\frac{1}{\epsilon^2} + \left(\frac{269}{2187} + \frac{16}{81}L - \frac{40}{81}\zeta_2 + \frac{112}{81}\zeta_3\right)\frac{1}{\epsilon}\right\} + C_A C_F n_f\left\{\frac{272}{9\epsilon^3}\right.\\
&\quad + \left(-\frac{704}{27} + \frac{40}{3}L - \frac{64}{9}\zeta_3\right)\frac{1}{\epsilon^2} + \left(-\frac{2434}{81} + \frac{110}{9}L + \frac{4}{3}\zeta_2 + \frac{32}{15}\zeta_2^2 + \frac{304}{27}\zeta_3\right.\\
&\quad \left.\left. - \frac{32}{3}L\zeta_3\right)\frac{1}{\epsilon}\right\} + C_A^3\left\{-\frac{256}{3\epsilon^6} + \left(\frac{1760}{3} - 128L\right)\frac{1}{\epsilon^5} + \left(-\frac{72632}{81} + 528L - 64L^2\right.\right.\\
&\quad \left.\left. - 32\zeta_2\right)\frac{1}{\epsilon^4} + \left(-\frac{29588}{243} - \frac{5824}{27}L + \frac{352}{3}L^2 - \frac{32}{3}L^3 + \frac{2464}{27}\zeta_2 - 48L\zeta_2 + 16\zeta_3\right)\frac{1}{\epsilon^3}\right.\\
&\quad + \left(\frac{80764}{243} - \frac{25492}{81}L + \frac{536}{9}L^2 - \frac{1486}{81}\zeta_2 + \frac{572}{9}L\zeta_2 - 16L^2\zeta_2 - \frac{352}{45}\zeta_2^2 - \frac{836}{27}\zeta_3\right.
\end{aligned}$$

$$+ 8L\zeta_3\Big)\frac{1}{\epsilon^2} + \Big(\frac{194372}{2187} - \frac{490}{9}L - \frac{12218}{243}\zeta_2 + \frac{1072}{27}L\zeta_2 + \frac{1276}{45}\zeta_2^2 - \frac{176}{15}L\zeta_2^2 - \frac{244}{9}\zeta_3$$
$$- \frac{88}{9}L\zeta_3 + \frac{80}{9}\zeta_2\zeta_3 + \frac{32}{3}\zeta_5\Big)\frac{1}{\epsilon}\Big\}. \tag{5.5.4}$$

After cancellation of the IR poles, we are left with the following finite matching coefficients:

$$C_{g,1}^{A,\text{eff}} = C_A\Big\{ - L^2 + 4 + \zeta_2\Big\},$$
$$C_{g,2}^{A,\text{eff}} = C_A^2\Big\{\frac{1}{2}L^4 + \frac{11}{9}L^3 + L^2\Big(- \frac{103}{9} + \zeta_2\Big) + L\Big(- \frac{10}{27} - \frac{22}{3}\zeta_2 - 2\zeta_3\Big) + \frac{4807}{81} + \frac{91}{6}\zeta_2$$
$$+ \frac{1}{2}\zeta_2^2 - \frac{143}{9}\zeta_3\Big\} + C_A n_f\Big\{ - \frac{2}{9}L^3 + \frac{10}{9}L^2 + L\Big(\frac{34}{27} + \frac{4}{3}\zeta_2\Big) - \frac{943}{81} - \frac{5}{3}\zeta_2 - \frac{46}{9}\zeta_3\Big\}$$
$$+ C_F n_f\Big\{ - \frac{80}{3} + 6\ln\Big(\frac{\mu_h^2}{m_t^2}\Big) + 8\zeta_3\Big\},$$
$$C_{g,3}^{A,\text{eff}} = n_f C_J^{(2)}\Big\{ - 2\Big\} + C_F n_f^2\Big\{L\Big(- \frac{320}{9} + 8\ln\Big(\frac{\mu_h^2}{m_t^2}\Big) + \frac{32}{3}\zeta_3\Big) + \frac{749}{9} - \frac{20}{9}\zeta_2 - \frac{16}{45}\zeta_2^2$$
$$- \frac{112}{3}\zeta_3\Big\} + C_F^2 n_f\Big\{\frac{457}{6} + 104\zeta_3 - 160\zeta_5\Big\} + C_A^2 n_f\Big\{\frac{2}{9}L^5 - \frac{8}{27}L^4 + L^3\Big(- \frac{752}{81}$$
$$- \frac{2}{3}\zeta_2\Big) + L^2\Big(\frac{512}{27} - \frac{103}{9}\zeta_2 + \frac{118}{9}\zeta_3\Big) + L\Big(\frac{129283}{729} + \frac{4198}{81}\zeta_2 - \frac{48}{5}\zeta_2^2 + \frac{28}{9}\zeta_3\Big)$$
$$- \frac{7946273}{13122} - \frac{19292}{729}\zeta_2 + \frac{73}{45}\zeta_2^2 - \frac{2764}{81}\zeta_3 - \frac{82}{9}\zeta_2\zeta_3 + \frac{428}{9}\zeta_5\Big\} + C_A^3\Big\{ - \frac{1}{6}L^6 - \frac{11}{9}L^5$$
$$+ L^4\Big(\frac{389}{54} - \frac{3}{2}\zeta_2\Big) + L^3\Big(\frac{2206}{81} + \frac{11}{3}\zeta_2 + 2\zeta_3\Big) + L^2\Big(- \frac{20833}{162} + \frac{757}{18}\zeta_2 - \frac{73}{10}\zeta_2^2$$
$$+ \frac{143}{9}\zeta_3\Big) + \frac{2222}{9}\zeta_5 + L\Big(- \frac{500011}{1458} - \frac{16066}{81}\zeta_2 + \frac{176}{5}\zeta_2^2 + \frac{1832}{27}\zeta_3 + \frac{34}{3}\zeta_2\zeta_3$$
$$+ 16\zeta_5\Big) + \frac{41091539}{26244} + \frac{316939}{1458}\zeta_2 - \frac{1399}{270}\zeta_2^2 - \frac{24389}{1890}\zeta_2^3 - \frac{176584}{243}\zeta_3 - \frac{605}{9}\zeta_2\zeta_3$$
$$- \frac{104}{9}\zeta_3^2\Big\} + C_A n_f^2\Big\{ - \frac{2}{27}L^4 + \frac{40}{81}L^3 + L^2\Big(\frac{80}{81} + \frac{8}{9}\zeta_2\Big) + L\Big(- \frac{12248}{729} - \frac{80}{27}\zeta_2$$
$$- \frac{128}{27}\zeta_3\Big) + \frac{280145}{6561} + \frac{4}{9}\zeta_2 + \frac{4}{27}\zeta_2^2 + \frac{4576}{243}\zeta_3\Big\} + C_A C_F n_f\Big\{ - \frac{2}{3}L^3 + L^2\Big(\frac{215}{6}$$
$$- 6\ln\Big(\frac{\mu_h^2}{m_t^2}\Big) - 16\zeta_3\Big) + L\Big(\frac{9173}{54} - 44\ln\Big(\frac{\mu_h^2}{m_t^2}\Big) + 4\zeta_2 + \frac{16}{5}\zeta_2^2 - \frac{376}{9}\zeta_3\Big)$$
$$+ 24\ln\Big(\frac{\mu_h^2}{m_t^2}\Big) - \frac{726935}{972} - \frac{415}{18}\zeta_2 + 6\ln\Big(\frac{\mu_h^2}{m_t^2}\Big)\zeta_2 - \frac{64}{45}\zeta_2^2 + \frac{20180}{81}\zeta_3 + \frac{64}{3}\zeta_2\zeta_3$$

$$+\frac{608}{9}\zeta_5\Bigg\}. \tag{5.5.5}$$

In the above expressions, $L = \ln\left(Q^2/\mu_h^2\right) = \ln\left(-q^2/\mu_h^2\right)$. These matching coefficients allow to perform the matching of the SCET-based resummation onto the full QCD calculation up to three-loop order.

Before ending the discussion of this section, we demonstrate the universal factorisation property fulfilled by the anomalous dimension of the $Z_g^{A,h}$ which is defined through the RG equation

$$\mu_h^2\frac{d}{d\mu_h^2}\ln Z_g^{A,h}(\epsilon,Q^2,\mu_h^2) \equiv \gamma_g^{A,h}(Q^2,\mu_h^2) = \sum_{i=1}^{\infty} a_s^i(\mu_h^2)\gamma_{g,i}^{A,h}(Q^2,\mu_h^2)\,. \tag{5.5.6}$$

The renormalisation group invariance of the UV renormalised $[F_g^A]_R(\epsilon,Q^2)$ with respect to the scale μ_h implies

$$\mu_h^2\frac{d}{d\mu_h^2}\ln Z_g^{A,h} + \mu_h^2\frac{d}{d\mu_h^2}\ln C_g^{A,\text{eff}} = 0\,. \tag{5.5.7}$$

By explicitly evaluating the $\gamma_{g,i}^{A,h}$ using the results of $Z_g^{A,h}$ (Eq. (5.5.4)) up to three loops ($i = 3$), we find that these satisfy the following decomposition in terms of the universal factors $A_{g,i}$, $B_{g,i}$ and $f_{g.i}$:

$$\gamma_{g,i}^{A,h} = -\frac{1}{2}A_{g,i}L + \left(B_{g,i} + \frac{1}{2}f_{g,i}\right). \tag{5.5.8}$$

This in turn implies the evolution equation of the matching coefficients as

$$\mu_h^2\frac{d}{d\mu_h^2}\ln C_{g,i}^{A,\text{eff}} = \frac{1}{2}A_{g,i}L - \left(B_{g,i} + \frac{1}{2}f_{g,i}\right) \tag{5.5.9}$$

which is in complete agreement with the existing results [181] upon identifying

$$\gamma^V = B_{g.i} + \frac{1}{2}f_{g,i}\,. \tag{5.5.10}$$

5.6 Summary

In this part of the thesis, we derived the three-loop massless QCD corrections to the quark and gluon form factors of pseudo-scalar operators. Working in dimensional regularisation, we used the 't Hooft-Veltman prescription for γ_5 and the Levi-Civita tensor, which requires non-trivial finite renormalisation to maintain the symmetries of the theory. By exploiting the universal behaviour of the infrared pole structure at three loops in QCD, we were able to independently determine the renormalisation constants and operator mixing, in agreement with earlier results that were obtained in a completely different approach [49, 50].

The three-loop corrections to the pseudo-scalar form factors are an important ingredient to precision Higgs phenomenology. They will ultimately allow to bring the gluon fusion cross section for pseudo-scalar Higgs production to the same level of accuracy that has been accomplished most recently for scalar Higgs production with fixed order $\mathrm{N^3LO}$ [44] and soft-gluon resummation at $\mathrm{N^3LL}$ [34, 156, 158, 160].

With our new results, the soft-gluon resummation for pseudo-scalar Higgs production [159, 160] can be extended imminently to $\mathrm{N^3LL}$ accuracy [14], given the established formalisms at this order [34, 156]. With the derivation of the three-loop pseudo-scalar form factors presented here, all ingredients to this calculation are now available. Another imminent application is the threshold approximation to the $\mathrm{N^3LO}$ cross section [14]. By exploiting the universal infrared structure [34], one can use the result of an explicit computation of the threshold contribution to the $\mathrm{N^3LO}$ cross section for scalar Higgs production [33] to derive threshold results for other processes essentially through the ratios of the respective form factors (which is no longer possible beyond threshold [44, 182], where the corrections become process-specific), as was done for the Drell-Yan process [32] and for Higgs production from bottom quark annihilation [11].

6 Conclusions and Outlooks

No doubt, the whole particle physics community is standing on the verge of a crucial era, where the main tasks can be largely categorized into two parts: testing the SM with unprecedented accuracy and searching for the physics beyond SM. In achieving these golden tasks, precise theoretical predictions play a very crucial role. The field of precision studies at theoretical level is mostly controlled by the higher order corrections to the scattering amplitudes, that are the basic building blocks of constructing any observable in QFT. Among all the higher order corrections, the QCD ones contribute substantially to any typical observable. This thesis deals with this higher order QCD radiative corrections to the observables associated with the Drell-Yan, scalar and pseudo-scalar Higgs boson.

The Higgs boson is among the best candidates at hadron collider, and hence it is of utmost importance to make the theoretical prediction as precise as possible to the associated observables. In the first part of the thesis, Chapter 2, we have computed the N^3LO QCD radiative corrections, arising from the soft gluons, to the inclusive production cross section of the Higgs boson produced through bottom quark annihilation [11]. Of course, this is not the dominant production channel of the scalar Higgs boson in the SM, nonetheless its contribution must also be taken into account in this spectacular precision studies. In order to achieve this, we have systematically employed an elegant prescription [28, 29]. The factorisation of QCD amplitudes, gauge invariance, renormalisation group invariance and the Sudakov resummation of soft gluons are at the heart of this formalism. The recently

available three loop $Hb\bar{b}$ QCD form factors [31] and the soft gluon contributions calculated [32] from the threshold QCD corrections to the Higgs boson at N^{3}LO [33], enable us to compute the full N^{3}LO soft-virtual QCD corrections to the production cross section of the Higgs boson produced through bottom quark annihilation. One of the most beautiful parts of this calculation is that even without evaluating all the hundreds or thousands of Feynman diagrams contributing to the real emissions, we have obtained the required contribution arising from the soft gluons! The universal nature of the soft gluons are the underlying reasons behind this remarkable feature. We have also demonstrated the numerical impact of this result at the LHC. This is the most accurate result for this production channel existing in the literature till date and it is expected to play an important role in coming days.

In the second part of the thesis, Chapter 3, we have dealt with an another very important observable, namely, the rapidity distributions of the Higgs boson produced through gluon fusion and the leptonic pair in Drell-Yan. The importance of these two processes are quite self-evident! We have computed the threshold enhanced N^{3}LO QCD corrections [12] to these observables employing the formalism developed in the article [39]. The skeleton of this elegant prescription which has been employed is also based on the properties, like, the factorisation of QCD amplitudes, gauge invariance, renormalisation group invariance and the Sudakov resummation of soft gluons. With the help of recently computed inclusive production cross section of the Higgs boson [33] and Drell-Yan [32] at threshold N^{3}LO QCD, we have computed the contributions arising from the soft gluons to the processes under consideration. These were the only missing ingredients to achieve our goal. Our newly calculated part of this distribution is found to be the most dominant one compared to the other contributions. We have demonstrated numerically the impact of this result for the Higgs boson at the LHC. Indeed, inclusion of this N^{3}LO contributions does reduce the dependence on the unphysical renormalisation and factorisation scales. It is worth mentioning that, this beautiful formalism not only helps us to compute the rapidity distribution at threshold, but also enhance our understanding about the underlying structures

of the QCD amplitudes.

In the third part of the thesis, Chapter 4, we have discussed the relatively modern techniques of the multiloop computations which have been employed to get some of the results calculated in this thesis. The backbone of this methodology is the integration-by-parts [46, 47] and Lorentz invariant [48] identities. The successful implementation of these in computer codes revolutionizes the area of multiloop computations.

The last part, Chapter 5, is dealt with a particle, pseudo-scalar, which is not included in particle spectrum of the SM, but is believed to be present in the nature. Intensive search for this particle has been going on for past several years, although nothing conclusive evidence has been found. However, to make conclusive remark about the existence of this particle, we need to revamp the understanding about this particle and improve the precision of the theoretical predictions. This work arises exactly at this context. In these articles [13,14], we have computed one of the important ingredients to calculate the inclusive production cross section or the differential distributions for the pseudo-scalar at N^3LO QCD which is presently the level of accuracy for the scalar Higgs boson, achieved very recently [44]. In particular, we have derived the three loop massless QCD corrections to the quark and gluon form factors of the pseudo-scalar. Unlike the scalar Higgs boson, this problem involves the γ_5 which makes the life interesting as well as challenging. We have handled them under the 't Hooft-Veltman prescription for the γ_5 and Levi-Civita tensor in dimensional regularisation. Employing this prescription, however, brings some additional complication, namely, it violates the chiral Ward identity. In order to rectify this, we need to perform an additional and non-trivial finite renormalisation. By exploiting the universal behaviour of the infrared pole structure at three loops in QCD, we were able to independently determine the renormalisation constants and operator mixing, in agreement with the earlier results that were obtained in a completely different approach [49,50]. We must emphasize the approach which we have employed here for the first time is exactly opposite to the usual one: the infrared pole structures of the form factors have been taken

to be universal that dictates us to obtain the UV operator renormalisation constants upon imposing the demand of UV finiteness. With our new results, the threshold approximation to the N^{3}LO inclusive production cross section for the pseudo-scalar through gluon fusion are obtained [14] by us. This is also extended to the N^{3}LL resummed accuracy in [14]. We have also computed the hard matching coefficients in the context of soft-collinear effective theory which are later employed to obtain the N^{3}LL' resummed cross section [183]. We have also found some interesting facts about the form factors in the context of Leading Transcendentality principle [167–169]: the LT terms of the diagonal form factors with replacement $C_A = C_F = N$ and $T_f n_f = N/2$ are not only identical to each other but also coincide with the LT terms of the QCD form factors [42] with the same replacement as well as with the LT terms of the scalar form factors in $\mathcal{N} = 4$ SYM [135], up to a normalization factor of 2^l. This observation holds true for the finite terms in ϵ, and could equally be validated for higher-order terms up to transcendentality 8 (which is the highest order for which all three-loop master integrals are available [170]). In addition to checking the diagonal form factors, we also examined the off-diagonal ones, where we find that the LT terms these two form factors are identical to each other after the replacement of colour factors. However, the LT terms of these do not coincide with those of the diagonal ones.

The state-of-the-art techniques, which mostly use our in-house codes, have been employed extensively to carry out all the computations presented in this thesis. The prescription of computing the threshold correction is applicable for any colorless final state particle. We are in the process of extending this formalism to the case of threshold resummation of differential rapidity distributions. The methodology of calculating the pseudo-scalar form factors can be generalized to the cases involving any number of operators which can mix among each others under UV renormalisation.

In conclusion, it has been a while the Higgs-like particle has been discovered at the LHC and finally, we are very close to having enough statistics for precision measurements of

the Higgs quantum numbers and coupling constants to fermions and gauge bosons. This, along with the precise results from theoreticians like us, hopefully, would help to explore the underlying nature of the electroweak symmetry breaking and possibly open the door of new physics.

Appendices

A Inclusive Production Cross Section

In QCD improved parton model, the inclusive cross-section for the production of a colorless particle can be computed using

$$\sigma^I(\tau, q^2) = \sum_{a,b=q,\bar{q},g} \int_0^1 dx_1 \int_0^1 dx_2 f_a(x_1, \mu_F^2) f_b(x_2, \mu_F^2) \sigma^I_{ab}\left(z, q^2, \mu_R^2, \mu_F^2\right) \tag{A.0.1}$$

where, f's are the partonic distribution functions factorised at the mass scale μ_F. σ^I_{ab} is the partonic cross section for the production of colorless particle I from the partons a and b. This is UV renormalised at renormalisation scale μ_R and mass factorised at μ_F. The other quantities are defined as

$$\begin{aligned} q^2 &= m_I^2 \,, \\ \tau &= \frac{q^2}{S} \,, \\ z &= \frac{q^2}{\hat{s}} \,. \end{aligned} \tag{A.0.2}$$

In the above expression, S and $\hat{s}$ are square of the hadronic and partonic center of mass energies, respectively, and they are related by

$$\hat{s} = x_1 x_2 S \,. \tag{A.0.3}$$

By introducing the identity

$$\int dz\delta(\tau - x_1 x_2 z) = \frac{1}{x_1 x_2} = \frac{S}{\hat{s}} \tag{A.0.4}$$

in Eq. (A.0.1), we can rewrite the Eq. (A.0.1) as

$$\sigma^I(\tau, q^2) = \sigma^{I,(0)}(\mu_R^2) \sum_{ab=q,\bar{q},g} \int_\tau^1 dx\ \Phi_{ab}(x, \mu_F^2)\ \Delta_{ab}^I\left(\frac{\tau}{x}, q^2, \mu_R^2, \mu_F^2\right). \tag{A.0.5}$$

The partonic flux Φ_{ab} is defined through

$$\Phi_{ab}(x, \mu_F^2) = \int_x^1 \frac{dy}{y} f_a(y, \mu_F^2)\ f_b\left(\frac{x}{y}, \mu_F^2\right) \tag{A.0.6}$$

and the dimensionless quantity Δ_{ab}^I is called the coefficient function of the partonic level cross section. Upon normalising the partonic level cross section by the born one, we obtain Δ_{ab}^I i.e.

$$\Delta_{ab}^I \equiv \frac{\sigma_{ab}^I}{\sigma^{I,(0)}}. \tag{A.0.7}$$

B Anomalous Dimensions

Here we present A [40, 69, 85, 86], f [40, 84], and B [40, 85] up to three loop level. The A's are given by

$$
\begin{aligned}
A_{gg,1} &= C_A\left\{4\right\},\\
A_{gg,2} &= C_A^2\left\{\frac{268}{9}-8\zeta_2\right\}+C_A n_f\left\{-\frac{40}{9}\right\},\\
A_{gg,3} &= C_A^3\left\{\frac{490}{3}-\frac{1072\zeta_2}{9}+\frac{88\zeta_3}{3}+\frac{176\zeta_2^2}{5}\right\}+C_A C_F n_f\left\{-\frac{110}{3}+32\zeta_3\right\}\\
&\quad + C_A^2 n_f\left\{-\frac{836}{27}+\frac{160\zeta_2}{9}-\frac{112\zeta_3}{3}\right\}+C_A n_f^2\left\{-\frac{16}{27}\right\}
\end{aligned}
$$

and

$$
A_{q\bar{q},i} = A_{b\bar{b},i} = \frac{C_F}{C_A}A_{gg,i}\,. \tag{B.0.1}
$$

The f's are obtained as

$$
\begin{aligned}
f_{gg,1} &= 0\,,\\
f_{gg,2} &= C_A^2\left\{-\frac{22}{3}\zeta_2-28\zeta_3+\frac{808}{27}\right\}+C_A n_f\left\{\frac{4}{3}\zeta_2-\frac{112}{27}\right\},\\
f_{gg,3} &= {C_A}^3\left\{\frac{352}{5}{\zeta_2}^2+\frac{176}{3}\zeta_2\zeta_3-\frac{12650}{81}\zeta_2-\frac{1316}{3}\zeta_3+192\zeta_5+\frac{136781}{729}\right\}\\
&\quad + C_A^2 n_f\left\{-\frac{96}{5}{\zeta_2}^2+\frac{2828}{81}\zeta_2+\frac{728}{27}\zeta_3-\frac{11842}{729}\right\}\\
&\quad + C_A C_F n_f\left\{\frac{32}{5}{\zeta_2}^2+4\zeta_2+\frac{304}{9}\zeta_3-\frac{1711}{27}\right\}+C_A {n_f}^2\left\{-\frac{40}{27}\zeta_2+\frac{112}{27}\zeta_3-\frac{2080}{729}\right\}
\end{aligned}
$$

and

$$f_{q\bar{q},i} = f_{b\bar{b},i} = \frac{C_F}{C_A} f_{gg,i}\,. \tag{B.0.2}$$

Similarly the B's are given by

$$\begin{aligned}
B_{gg,1} &= C_A \left\{\frac{11}{3}\right\} - n_f \left\{\frac{2}{3}\right\}, \\
B_{gg,2} &= C_A^2 \left\{\frac{32}{3} + 12\zeta_3\right\} - n_f C_A \left\{\frac{8}{3}\right\} - n_f C_F \Big\{2\Big\}, \\
B_{gg,3} &= C_A C_F n_f \left\{-\frac{241}{18}\right\} + C_A n_f^2 \left\{\frac{29}{18}\right\} - C_A^2 n_f \left\{\frac{233}{18} + \frac{8}{3}\zeta_2 + \frac{4}{3}\zeta_2^2 + \frac{80}{3}\zeta_3\right\} \\
&\quad + C_A^3 \left\{\frac{79}{2} - 16\zeta_2\zeta_3 + \frac{8}{3}\zeta_2 + \frac{22}{3}\zeta_2^2 + \frac{536}{3}\zeta_3 - 80\zeta_5\right\} + C_F n_f^2 \left\{\frac{11}{9}\right\} + C_F^2 n_f \Big\{1\Big\}, \\
B_{q\bar{q},1} &= C_F \Big\{3\Big\}, \\
B_{q\bar{q},2} &= C_F^2 \left\{\frac{3}{2} - 12\zeta_2 + 24\zeta_3\right\} + C_A C_F \left\{\frac{17}{34} + \frac{88}{6}\zeta_2 - 12\zeta_3\right\} + n_f C_F T_F \left\{-\frac{2}{3} - \frac{16}{3}\zeta_2\right\}, \\
B_{q\bar{q},3} &= {C_A}^2 C_F \left\{-2{\zeta_2}^2 + \frac{4496}{27}\zeta_2 - \frac{1552}{9}\zeta_3 + 40\zeta_5 - \frac{1657}{36}\right\} + C_A {C_F}^2 \left\{-\frac{988}{15}{\zeta_2}^2\right. \\
&\quad \left. + 16\zeta_2\zeta_3 - \frac{410}{3}\zeta_2 + \frac{844}{3}\zeta_3 + 120\zeta_5 + \frac{151}{4}\right\} + C_A C_F n_f \left\{\frac{4}{5}{\zeta_2}^2 - \frac{1336}{27}\zeta_2 + \frac{200}{9}\zeta_3\right. \\
&\quad \left. + 20\right\} + {C_F}^3 \left\{\frac{288}{5}{\zeta_2}^2 - 32\zeta_2\zeta_3 + 18\zeta_2 + 68\zeta_3 - 240\zeta_5 + \frac{29}{2}\right\} \\
&\quad + {C_F}^2 n_f \left\{\frac{232}{15}{\zeta_2}^2 + \frac{20}{3}\zeta_2 - \frac{136}{3}\zeta_3 - 23\right\} + C_F {n_f}^2 \left\{\frac{80}{27}\zeta_2 - \frac{16}{9}\zeta_3 - \frac{17}{9}\right\},
\end{aligned}$$

and

$$B_{q\bar{q},i} = B_{b\bar{b},i}\,. \tag{B.0.3}$$

C Solving Renormalisation Group Equation

To demonstrate the methodoogy of solving RGE, let us consider a general form of an RGE with respect to the renormalisation scale μ_R:

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln M = N \tag{C.0.1}$$

where, M and N are functions of μ_R. We need to solve for M in terms of N. Our goal is to solve it order by order in perturbation theory. We start by expanding the quantities in powers of $a_s \equiv a_s(\mu_R^2)$:

$$\begin{aligned} M &= 1 + \sum_{k=1}^{\infty} a_s^k M^{(k)} , \\ N &= \sum_{k=1}^{\infty} a_s^k N^{(k)} . \end{aligned} \tag{C.0.2}$$

The μ_R dependence of M and N on μ_R enters through a_s. All the coefficients $M^{(k)}$ and $N^{(k)}$ are independent of μ_R. Hence, $\ln M$ can be written as

$$\ln M = \sum_{k=1}^{\infty} a_s^k M_k \tag{C.0.3}$$

with

$$
\begin{aligned}
M_1 &= M^{(1)}\,,\\
M_2 &= -\frac{1}{2}(M^{(1)})^2 + M^{(2)}\,,\\
M_3 &= \frac{1}{3}(M^{(1)})^3 - M^{(1)}M^{(2)} + M^{(3)}\,,\\
M_4 &= -\frac{1}{4}(M^{(1)})^4 + (M^{(1)})^2M^{(2)} - \frac{1}{2}(M^{(2)})^2 - M^{(1)}M^{(3)} + M^{(4)}\,.
\end{aligned}
\tag{C.0.4}
$$

Using the above expansions in RGE. (C.0.1) and using the RGE of a_s

$$
\mu_R^2\frac{d}{d\mu_R^2}a_s = \frac{\epsilon}{2}a_s - \sum_{k=0}^{\infty}\beta_k\, a_s^{k+2}
\tag{C.0.5}
$$

we get M_k's by comparing the coefficients of a_s as

$$
\begin{aligned}
M_1 &= \frac{2}{\epsilon}N^{(1)}\,,\\
M_2 &= \frac{2}{\epsilon^2}\beta_0 N^{(1)} + \frac{1}{\epsilon}N^{(2)}\,,\\
M_3 &= \frac{8}{3\epsilon^3}\beta_0^2 N^{(1)} + \frac{1}{\epsilon^2}\left\{\frac{4}{3}\beta_1 N^{(1)} + \frac{4}{3}\beta_0 N^{(2)}\right\} + \frac{2}{3\epsilon}N^{(3)}\,,\\
M_4 &= \frac{4}{\epsilon^4}\beta_0^3 N^{(1)} + \frac{1}{\epsilon^3}\left\{4\beta_0\beta_1 N^{(1)} + 2\beta_0^2 N^{(2)}\right\} + \frac{1}{\epsilon^2}\left\{\beta_2 N^{(1)} + \beta_1 N^{(2)} + \beta_0 N^{(3)}\right\} + \frac{1}{2\epsilon}N^{(4)}\,.
\end{aligned}
\tag{C.0.6}
$$

By equating the Eq. (C.0.3) and (C.0.6) we obtain,

$$
\begin{aligned}
M^{(1)} &= \frac{2}{\epsilon}N^{(1)}\,,\\
M^{(2)} &= \frac{1}{\epsilon^2}\left\{2\beta_0 N^{(1)} + 2(N^{(1)})^2\right\} + \frac{1}{\epsilon}N^{(2)}\,,\\
M^{(3)} &= \frac{1}{\epsilon^3}\left\{\frac{8}{3}\beta_0^2 N^{(1)} + 4\beta_0(N^{(1)})^2 + \frac{4}{3}(N^{(1)})^3\right\} + \frac{1}{\epsilon^2}\left\{\frac{4}{3}\beta_1 N^{(1)} + \frac{4}{3}\beta_0 N^{(2)} + 2N^{(1)}N^{(2)}\right\}\\
&\quad + \frac{2}{3\epsilon}N^{(3)}\,,\\
M^{(4)} &= \frac{1}{\epsilon^4}\left\{4\beta_0^3 N^{(1)} + \frac{22}{3}\beta_0^2(N^{(1)})^2 + 4\beta_0(N^{(1)})^3 + \frac{2}{3}(N^{(1)})^4\right\} + \frac{1}{\epsilon^3}\left\{4\beta_0\beta_1 N^{(1)} + \frac{8}{3}\beta_1(N^{(1)})^2\right.
\end{aligned}
$$

$$+ 2\beta_0^2 N^{(2)} + \frac{14}{3}\beta_0 N^{(1)} N^{(2)} + 2(N^{(1)})^2 N^{(2)}\Big\} + \frac{1}{\epsilon^2}\Big\{\beta_2 N^{(1)} + \beta_1 N^{(2)} + \frac{1}{2}(N^{(2)})^2 + \beta_0 N^{(3)} + \frac{4}{3}N^{(1)} N^{(3)}\Big\} + \frac{1}{2\epsilon} N^{(4)} \,. \tag{C.0.7}$$

We have presented the solution up to $O(a_s^4)$. However, this procedure can be easily generalised to all orders in a_s.

- **Example 1: RGE of Z_{a_s}**

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z_{a_s} = \frac{1}{a_s} \sum_{k=0}^{\infty} a_s^{k+2} \beta_k \tag{C.0.8}$$

Comparing this RGE of Z_{a_s} with Eq. (C.0.1) we get

$$N^{(k)} = \beta_{k-1} \qquad k \in [1, \infty) \,. \tag{C.0.9}$$

By putting the values of $N^{(k)}$ in the general solution (C.0.7), we get the corresponding solutions of Z_{a_s} as

$$Z_{a_s} = 1 + \sum_{k=1}^{\infty} a_s^k Z_{a_s}^{(k)} \tag{C.0.10}$$

where

$$\begin{aligned}
Z_{a_s}^{(1)} &= \frac{2}{\epsilon}\beta_0 \,, \\
Z_{a_s}^{(2)} &= \frac{4}{\epsilon^2}\beta_0^2 + \frac{1}{\epsilon}\beta_1 \,, \\
Z_{a_s}^{(3)} &= \frac{8}{\epsilon^3}\beta_0^3 + \frac{14}{3\epsilon^2}\beta_0\beta_1 + \frac{2}{3\epsilon}\beta_2 \,, \\
Z_{a_s}^{(4)} &= \frac{16}{\epsilon^4}\beta_0^4 + \frac{46}{3\epsilon^3}\beta_0^2\beta_1 + \frac{1}{\epsilon^2}\left(\frac{3}{2}\beta_1^2 + \frac{10}{3}\beta_0\beta_2\right) + \frac{1}{2\epsilon}\beta_3 \,.
\end{aligned} \tag{C.0.11}$$

The Z_{a_s} can also be expressed in powers of $\hat{a}_s$ by utilising the

$$a_s = \hat{a}_s S_\epsilon \left(\frac{\mu_R^2}{\mu^2}\right)^{\epsilon/2} Z_{a_s}^{-1} \tag{C.0.12}$$

iteratively. We get

$$Z_{a_s} = 1 + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\mu_R^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{Z}_{a_s}^{(k)} \tag{C.0.13}$$

where

$$\begin{aligned}
\hat{Z}_{a_s}^{(1)} &= \frac{2}{\epsilon}\beta_0\,, \\
\hat{Z}_{a_s}^{(2)} &= \frac{1}{\epsilon}\beta_1\,, \\
\hat{Z}_{a_s}^{(3)} &= -\frac{4}{3\epsilon^2}\beta_0\beta_1 + \frac{2}{3\epsilon}\beta_2\,, \\
\hat{Z}_{a_s}^{(4)} &= \frac{2}{\epsilon^3}\beta_0^2\beta_1 + \frac{1}{\epsilon^2}\left(-\frac{1}{2}\beta_1^2 - 2\beta_0\beta_2\right) + \frac{1}{2\epsilon}\beta_3\,.
\end{aligned} \tag{C.0.14}$$

To arrive at the above result, we need to use the $Z_{a_s}^{-1}$ in powers of $\hat{a}_s$:

$$Z_{a_s}^{-1} = 1 + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\mu_R^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{Z}_{a_s}^{-1,(k)} \tag{C.0.15}$$

where

$$\begin{aligned}
\hat{Z}_{a_s}^{-1,(1)} &= -\frac{2}{\epsilon}\beta_0\,, \\
\hat{Z}_{a_s}^{-1,(2)} &= \frac{4}{\epsilon^2}\beta_0^2 - \frac{1}{\epsilon}\beta_1\,, \\
\hat{Z}_{a_s}^{-1,(3)} &= -\frac{8}{\epsilon^3}\beta_0^3 + \frac{16}{3\epsilon^2}\beta_0\beta_1 - \frac{2}{3\epsilon}\beta_2\,, \\
\hat{Z}_{a_s}^{-1,(4)} &= \frac{16}{\epsilon^4}\beta_0^4 - \frac{58}{3\epsilon^3}\beta_0^2\beta_1 + \frac{1}{\epsilon^2}\left(\frac{3}{2}\beta_1^2 + \frac{14}{3}\beta_0\beta_2\right) - \frac{1}{2\epsilon}\beta_3\,.
\end{aligned} \tag{C.0.16}$$

- **Example 2: Solution of the Mass Factorisation Kernel**

The mass factorisation kernel satisfies the RG equation (2.3.32)

$$\mu_F^2 \frac{d}{d\mu_F^2}\Gamma_{ij}^I(z,\mu_F^2,\epsilon) = \frac{1}{2}\sum_k P_{ik}^I\left(z,\mu_F^2\right) \otimes \Gamma_{kj}^I\left(z,\mu_F^2,\epsilon\right) \tag{C.0.17}$$

where, $P^I\left(z,\mu_F^2\right)$ are Altarelli-Parisi splitting functions (matrix valued). Expanding

$P^I\left(z,\mu_F^2\right)$ and $\Gamma^I(z,\mu_F^2,\epsilon)$ in powers of the strong coupling constant we get

$$P^I(z,\mu_F^2) = \sum_{k=1}^{\infty} a_s^k(\mu_F^2) P^{I,(k-1)}(z) \tag{C.0.18}$$

and

$$\Gamma^I(z,\mu_F^2,\epsilon) = \delta(1-z) + \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\mu_F^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \Gamma^{I,(k)}(z,\epsilon)\,. \tag{C.0.19}$$

Following the techniques prescribed above, it can be solved. However, unlike the previous cases here we have to take care of the fact that P^I and Γ^I are matrix valued quantities i.e. they are non-commutative. Upon solving we obtain the general solution as

$$\begin{aligned}
\Gamma^{I,(1)}(z,\epsilon) &= \frac{1}{\epsilon}\left\{P^{I,(0)}(z)\right\},\\
\Gamma^{I,(2)}(z,\epsilon) &= \frac{1}{\epsilon^2}\left\{-\beta_0 P^{I,(0)}(z) + \frac{1}{2}P^{I,(0)}(z)\otimes P^{I,(0)}(z)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{2}P^{I,(1)}(z)\right\},\\
\Gamma^{I,(3)}(z,\epsilon) &= \frac{1}{\epsilon^3}\left\{\frac{4}{3}\beta_0^2 P^{I,(0)} - \beta_0 P^{I,(0)}\otimes P^{I,(0)} + \frac{1}{6}P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(0)}\right\}\\
&\quad + \frac{1}{\epsilon^2}\left\{-\frac{1}{3}\beta_1 P^{I,(0)} + \frac{1}{6}P^{I,(0)}\otimes P^{I,(1)} - \frac{4}{3}\beta_0 P^{I,(1)} + \frac{1}{3}P^{I,(1)}\otimes P^{I,(0)}\right\}\\
&\quad + \frac{1}{\epsilon}\left\{\frac{1}{3}P^{I,(2)}\right\},\\
\Gamma^{I,(4)}(z,\epsilon) &= \frac{1}{\epsilon^4}\left\{-2\beta_0^3 P^{I,(0)} + \frac{11}{6}\beta_0^2 P^{I,(0)}\otimes P^{I,(0)} - \frac{1}{2}\beta_0 P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(0)}\right.\\
&\quad \left. + \frac{1}{24}P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(0)}\right\} + \frac{1}{\epsilon^3}\left\{\frac{4}{3}\beta_0\beta_1 P^{I,(0)} - \frac{1}{3}\beta_1 P^{I,(0)}\otimes P^{I,(0)}\right.\\
&\quad + \frac{1}{24}P^{I,(0)}\otimes P^{I,(0)}\otimes P^{I,(1)} - \frac{7}{12}\beta_0 P^{I,(0)}\otimes P^{I,(1)} + \frac{1}{12}P^{I,(0)}\otimes P^{I,(1)}\otimes P^{I,(0)}\\
&\quad \left. + 3\beta_0^2 P^{I,(1)} - \frac{5}{4}\beta_0 P^{I,(1)}\otimes P^{I,(0)} + \frac{1}{8}P^{I,(1)}\otimes P^{I,(0)}\otimes P^{I,(0)}\right\}\\
&\quad + \frac{1}{\epsilon^2}\left\{-\frac{1}{6}\beta_2 P^{I,(0)} + \frac{1}{12}P^{I,(0)}\otimes P^{I,(2)} - \frac{1}{2}\beta_1 P^{I,(1)} + \frac{1}{8}P^{I,(1)}\otimes P^{I,(1)}\right.\\
&\quad \left. - \frac{3}{2}\beta_0 P^{I,(2)} + \frac{1}{4}P^{I,(2)}\otimes P^{I,(0)}\right\} + \frac{1}{\epsilon}\left\{\frac{1}{4}P^{I,(3)}\right\}.
\end{aligned} \tag{C.0.20}$$

In the soft-virtual limit, only the diagonal parts of the kernels contribute. Our findings are consistent with the existing diagonal solutions which can be found in the article [28].

D Solving KG Equation

The form factor satisfies the KG differential equation (See Sec. 2.3.1):

$$Q^2\frac{d}{dQ^2}\ln\mathcal{F}^I_{ij}(\hat{a}_s,Q^2,\mu^2,\epsilon)=\frac{1}{2}\left[K^I_{ij}\left(\hat{a}_s,\frac{\mu_R^2}{\mu^2},\epsilon\right)+G^I_{ij}\left(\hat{a}_s,\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\epsilon\right)\right]. \tag{D.0.1}$$

In this appendix we demonstrate the procedure to solve the KG equation. RG invariance of the $\mathcal{F}$ with respect to the renormalisation scale μ_R implies

$$\mu_R^2\frac{d}{d\mu_R^2}K^I_{ij}\left(\hat{a}_s,\frac{\mu_R^2}{\mu^2},\epsilon\right)=-\mu_R^2\frac{d}{d\mu_R^2}G^I_{ij}\left(\hat{a}_s,\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\epsilon\right)=-A^I_{ij}\left(a_s(\mu_R^2)\right) \tag{D.0.2}$$

where, A^I_{ij}'s are the cusp anomalous dimensions. Unlike the previous cases, we expand K^I_{ij} in powers of unrenormalised $\hat{a}_s$ as

$$K^I_{ij}\left(\hat{a}_s,\frac{\mu_R^2}{\mu^2},\epsilon\right)=\sum_{k=1}^{\infty}\hat{a}_s^k S_\epsilon^k\left(\frac{\mu_R^2}{\mu^2}\right)^{k\frac{\epsilon}{2}}\hat{K}^I_{ij,k}(\epsilon) \tag{D.0.3}$$

whereas we define the components $A^I_{ij,k}$ through

$$A^I_{ij}=\sum_{k=1}^{\infty}a_s^k\left(\mu_R^2\right)A^I_{ij,k}\,. \tag{D.0.4}$$

Following the methodology discussed in Appendix C, we can solve for $\hat{K}^I_{ij,k}(\epsilon)$

$$\hat{K}^I_{ij,1}(\epsilon)=\frac{1}{\epsilon}\Big\{-2A^I_{ij,1}\Big\},$$

$$\hat{K}^I_{ij,2}(\epsilon) = \frac{1}{\epsilon^2}\left\{2\beta_0 A^I_{ij,1}\right\} + \frac{1}{\epsilon}\left\{-A^I_{ij,2}\right\},$$
$$\hat{K}^I_{ij,3}(\epsilon) = \frac{1}{\epsilon^3}\left\{-\frac{8}{3}\beta_0^2 A^I_{ij,1}\right\} + \frac{1}{\epsilon^2}\left\{\frac{2}{3}\beta_1 A^I_{ij,1} + \frac{8}{3}\beta_0 A^I_{ij,2}\right\} + \frac{1}{\epsilon}\left\{-\frac{2}{3}A^I_{ij,3}\right\},$$
$$\hat{K}^I_{ij,4}(\epsilon) = \frac{1}{\epsilon^4}\left\{4\beta_0^3 A^I_{ij,1}\right\} + \frac{1}{\epsilon^3}\left\{-\frac{8}{3}\beta_0\beta_1 A^I_{ij,1} - 6\beta_0^2 A^I_{ij,2}\right\} + \frac{1}{\epsilon^2}\left\{\frac{1}{3}\beta_2 A^I_{ij,1} + \beta_1 A^I_{ij,2} + 3\beta_0 A^I_{ij,3}\right\}$$
$$+ \frac{1}{\epsilon}\left\{-\frac{1}{2}A^I_{ij,4}\right\} \tag{D.0.5}$$

Due to dependence of G^I_{ij} on Q^2, we need to handle it differently. Integrating the RGE of G^I_{ij}, (D.0.2), we get

$$G^I_{ij}\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon\right) - G^I_{ij}\left(\hat{a}_s, 1, \frac{Q^2}{\mu^2}, \epsilon\right) = \int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A^I_{ij}$$
$$\Rightarrow \quad G^I_{ij}\left(a_s(\mu_R^2), \frac{Q^2}{\mu_R^2}, \epsilon\right) = G^I_{ij}\left(a_s(Q^2), 1, \epsilon\right) + \int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A^I_{ij} \tag{D.0.6}$$

Consider the second part of the above Eq. (D.0.6)

$$\int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A^I_{ij} = \int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} \sum_{k=1}^{\infty} a_s^k A^I_{ij,k}$$
$$= \sum_{k=1}^{\infty} \int_{\frac{Q^2}{\mu^2}}^{\frac{\mu_R^2}{\mu^2}} \frac{dX^2}{X^2} \hat{a}_s^k S_\epsilon^k \left(X^2\right)^{k\frac{\epsilon}{2}} \left(Z^{-1}_{a_s}(X^2)\right)^k A^I_{ij,k} \tag{D.0.7}$$

where we have made the change of integration variable from μ_R to X by $\mu_R^2 = X^2\mu^2$. By using the $Z^{-1}_{a_s}(X^2)$ from Eq. (C.0.15) and evaluating the integral we obtain

$$\int_{Q^2}^{\mu_R^2} \frac{d\mu_R^2}{\mu_R^2} A^I_{ij} = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{\mu_R^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \left[\left(\frac{Q^2}{\mu_R^2}\right)^{k\frac{\epsilon}{2}} - 1\right] \hat{K}^I_{ij,k}(\epsilon)\,. \tag{D.0.8}$$

The first part of G^I_{ij} in Eq. (D.0.6) can be expanded in powers of $a_s(Q^2)$ as

$$G^I_{ij}\left(a_s(Q^2),1,\epsilon\right) = \sum_{k=1}^{\infty} a_s^k(Q^2) G^I_{ij}(\epsilon)\,. \tag{D.0.9}$$

By putting back the Eq. (D.0.3), (D.0.7) and (D.0.8) in the original KG equation (D.0.1), we solve for $\ln \mathcal{F}^I_{ij}(\hat{a}_s, Q^2, \mu^2, \epsilon)$:

$$\ln \mathcal{F}^I_{ij}(\hat{a}_s, Q^2, \mu^2, \epsilon) = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{Q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\mathcal{L}}^I_{ij,k}(\epsilon) \tag{D.0.10}$$

with

$$\begin{aligned}
\hat{\mathcal{L}}^I_{ij,1}(\epsilon) &= \frac{1}{\epsilon^2}\left\{-2A^I_{ij,1}\right\} + \frac{1}{\epsilon}\left\{G^I_{ij,1}(\epsilon)\right\},\\
\hat{\mathcal{L}}^I_{ij,2}(\epsilon) &= \frac{1}{\epsilon^3}\left\{\beta_0 A^I_{ij,1}\right\} + \frac{1}{\epsilon^2}\left\{-\frac{1}{2}A^I_{ij,2} - \beta_0 G^I_{ij,1}(\epsilon)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{2}G^I_{ij,2}(\epsilon)\right\},\\
\hat{\mathcal{L}}^I_{ij,3}(\epsilon) &= \frac{1}{\epsilon^4}\left\{-\frac{8}{9}\beta_0^2 A^I_{ij,1}\right\} + \frac{1}{\epsilon^3}\left\{\frac{2}{9}\beta_1 A^I_{ij,1} + \frac{8}{9}\beta_0 A^I_{ij,2} + \frac{4}{3}\beta_0^2 G^I_{ij,1}(\epsilon)\right\}\\
&\quad + \frac{1}{\epsilon^2}\left\{-\frac{2}{9}A^I_{ij,3} - \frac{1}{3}\beta_1 G^I_{ij,1}(\epsilon) - \frac{4}{3}\beta_0 G^I_{ij,2}(\epsilon)\right\} + \frac{1}{\epsilon}\left\{\frac{1}{3}G^I_{ij,3}(\epsilon)\right\},\\
\hat{\mathcal{L}}^I_{ij,4}(\epsilon) &= \frac{1}{\epsilon^5}\left\{A^I_{ij,1}\beta_0^3\right\} + \frac{1}{\epsilon^4}\left\{-\frac{3}{2}A^I_{ij,2}\beta_0^2 - \frac{2}{3}A^I_{ij,1}\beta_0\beta_1 - 2\beta_0^3 G^I_{ij,1}(\epsilon))\right\}\\
&\quad + \frac{1}{\epsilon^3}\left\{\frac{3}{4}A^I_{ij,3}\beta_0 + \frac{1}{4}A^I_{ij,2}\beta_1 + \frac{1}{12}A^I_{ij,1}\beta_2 + \frac{4}{3}\beta_0\beta_1 G^I_{ij,1}(\epsilon) + 3\beta_0^2 G^I_{ij,2}(\epsilon))\right\}\\
&\quad + \frac{1}{\epsilon^2}\left\{-\frac{1}{8}A^I_{ij,4} - \frac{1}{6}\beta_2 G^I_{ij,1}(\epsilon) - \frac{1}{2}\beta_1 G^I_{ij,2}(\epsilon) - \frac{3}{2}\beta_0 G^I_{ij,3}(\epsilon)\right\}\\
&\quad + \frac{1}{\epsilon}\left\{\frac{1}{4}G^I_{ij,4}(\epsilon)\right\}.
\end{aligned} \tag{D.0.11}$$

This methodology can easily be generalised to all orders in perturbation theory.

E Soft-Collinear Distribution

In Sec. 2.3.4, we introduced the soft-collinear distribution $\Phi^H_{b\bar{b}}$ in the context of computing SV cross section of the Higgs boson production in $b\bar{b}$ annihilation. In this appendix, we intend to elaborate the methodology of finding this distribution. For the sake of generalisation, we use I instead of H and omit the partonic indices. To understand the underlying logics behind finding Φ^I, let us consider an example at one loop level. The generalisation to higher loop is straightforward.

As discussed in the Sec. 2.3, the SV cross section in z-space can be computed in $d = 4+\epsilon$ dimensions using

$$\Delta^{I,\mathrm{SV}}(z, q^2, \mu_R^2, \mu_F^2) = C \exp\left(\Psi^I\left(z, q^2, \mu_R^2, \mu_F^2, \epsilon\right)\right)\Big|_{\epsilon=0} \tag{E.0.1}$$

where, $\Psi^I\left(z, q^2, \mu_R^2, \mu_F^2, \epsilon\right)$ is a finite distribution and C is the convolution defined through Eq. (3.3.15). The Ψ^I is given by, Eq. (2.3.8)

$$\begin{aligned}\Psi^I\left(z, q^2, \mu_R^2, \mu_F^2, \epsilon\right) = &\left(\ln\left[Z^I(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)\right]^2 + \ln\left|\mathcal{F}^I(\hat{a}_s, Q^2, \mu^2, \epsilon)\right|^2\right)\delta(1-z) \\ &+ 2\Phi^I(\hat{a}_s, q^2, \mu^2, z, \epsilon) - 2C \ln \Gamma^I(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon)\,.\end{aligned} \tag{E.0.2}$$

For all the details about the notations, see Sec. 2.3. Considering only the poles at $O(a_s)$ with $\mu_R = \mu_F$ we obtain,

$$\ln\left(Z^{I,(1)}\right)^2 = a_s(\mu_F^2)\frac{4\gamma_1^I}{\epsilon}\,,$$

$$\ln|\mathcal{F}^{I,(1)}|^2 = a_s(\mu_F^2)\left(\frac{q^2}{\mu_F^2}\right)^{\frac{\epsilon}{2}}\left[-\frac{4A_1^I}{\epsilon^2}+\frac{1}{\epsilon}\left(2f_1^I+4B_1^I-4\gamma_1^I\right)\right],$$
$$2C\ln\Gamma^{I,(1)} = 2a_s(\mu_F^2)\left[\frac{2B_1^I}{\epsilon}\delta(1-z)+\frac{2A_1^I}{\epsilon}\mathcal{D}_0\right] \tag{E.0.3}$$

where, the components are defined through the expansion of these quantities in powers of $a_s(\mu_F^2)$

$$\begin{aligned}
\Psi^I &= \sum_{k=1}^{\infty} a_s^k\left(\mu_F^2\right)\Psi^{I,(k)}\,,\\
\ln(Z^I)^2 &= \sum_{k=1}^{\infty} a_s^k\left(\mu_F^2\right)Z^{I,(k)}\,,\\
\ln|\mathcal{F}^I|^2 &= \sum_{k=1}^{\infty} a_s^k\left(\mu_F^2\right)\left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}}\ln|\mathcal{F}^{I,(k)}|^2\,,\\
\Phi^I &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2)\Phi_k^I\,,\\
\ln\Gamma^I &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2)\ln\Gamma^{I,(k)}
\end{aligned} \tag{E.0.4}$$

and

$$\mathcal{D}_i \equiv \left[\frac{\ln^i(1-z)}{1-z}\right]_+\,. \tag{E.0.5}$$

Collecting the coefficients of $a_s(\mu_F^2)$, we get

$$\Psi^{I,(1)}|_{\text{poles}} = \left[\left\{-\frac{4A_1^I}{\epsilon^2}+\frac{2f_1^I}{\epsilon}\right\}\delta(1-z)-\frac{4A_1^I}{\epsilon}\mathcal{D}_0\right]+2\Phi_1^I \tag{E.0.6}$$

where, we have not shown the $\ln(q^2/\mu_F^2)$ terms. To cancel the remaining divergences appearing in the above Eq. (E.0.6) for obtaining a finite cross section, we must demand that Φ_1^I have exactly the same poles with opposite sign:

$$2\Phi_1^I|_{\text{poles}} = -\left[\left\{-\frac{4A_1^I}{\epsilon^2}+\frac{2f_1^I}{\epsilon}\right\}\delta(1-z)-\frac{4A_1^I}{\epsilon}\mathcal{D}_0\right] \tag{E.0.7}$$

In addition, Φ^I also should be RG invariant with respect to μ_R:

$$\mu_R^2 \frac{d}{d\mu_R^2} \Phi^I = 0\,. \tag{E.0.8}$$

We make an *ansatz*, the above two demands, Eq. (E.0.7) and (E.0.8) can be accomplished if Φ^I satisfies the KG-type integro-differential equation which we call $\overline{KG}$:

$$q^2 \frac{d}{dq^2} \Phi^I \left(\hat{a}_s, q^2, \mu^2, z, \epsilon\right) = \frac{1}{2}\left[\overline{K}^I\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right) + \overline{G}^I\left(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon\right)\right]. \tag{E.0.9}$$

$\overline{K}^I$ contains all the poles whereas $\overline{G}^I$ consists of only the finite terms in ϵ. RG invariance (E.0.8) of Φ^I dictates

$$\mu_R^2 \frac{d}{d\mu_R^2} \overline{K}^I = -\mu_R^2 \frac{d}{d\mu_R^2} \overline{G}^I \equiv Y^I \tag{E.0.10}$$

where, we introduce a quantity Y^I. Following the methodology of solving the KG equation discussed in the Appendix D, we can write the solution of Φ^I as

$$\Phi^I(\hat{a}_s, q^2, \mu^2, z, \epsilon) = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Phi}_k^I(z, \epsilon) \tag{E.0.11}$$

with

$$\hat{\Phi}_k^I(z, \epsilon) = \hat{\mathcal{L}}_k^I\left(A_i^I \to Y_i^I, G_i^I \to \overline{G}_i^I(z, \epsilon)\right). \tag{E.0.12}$$

where we define the components through the expansions

$$\begin{aligned} Y^I &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2) Y_k^I\,, \\ \overline{G}^I(z, \epsilon) &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2) \overline{G}_k^I(z, \epsilon). \end{aligned} \tag{E.0.13}$$

This solution directly follows from the Eq. (D.0.10). Hence we get

$$2\hat{\Phi}_1^I(z,\epsilon) = \frac{1}{\epsilon^2}\Big\{-4Y_1^I\Big\} + \frac{2}{\epsilon}\Big\{\overline{G}_1^I(z,\epsilon)\Big\}. \tag{E.0.14}$$

By expressing the components of Φ^I in powers of $a_s(\mu_F^2)$, we obtain

$$\begin{aligned}\Phi^I(\hat{a}_s,q^2,\mu^2,z,\epsilon) &= \sum_{k=1}^{\infty}\hat{a}_s^k S_\epsilon^k\left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}}\hat{\Phi}_k^I(z,\epsilon)\\ &= \sum_{k=1}^{\infty}a_s^k(\mu_F^2)\left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}}Z_{a_s}^k\hat{\Phi}_k^I(z,\epsilon)\\ &\equiv \sum_{k=1}^{\infty}a_s^k(\mu_F^2)\left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}}\Phi_k^I(z,\epsilon)\end{aligned} \tag{E.0.15}$$

and at $\mathcal{O}(a_s(\mu_F^2))$, $\hat{\Phi}_1^I(z,\epsilon) = \Phi_1^I(z,\epsilon)$ upon suppressing the terms like $\log(q^2/\mu_F^2)$. Hence, by comparing the Eq. (E.0.7) and (E.0.14), we conclude

$$\begin{aligned}Y_1^I &= -A_1^I\delta(1-z)\,,\\ \overline{G}_1^I(z,\epsilon) &= -f_1^I\delta(1-z) + 2A_1^I\mathcal{D}_0 + \sum_{k=1}^{\infty}\epsilon^k\overline{g}_1^{I,k}(z)\,.\end{aligned} \tag{E.0.16}$$

The coefficients of $\epsilon^k, \overline{g}_1^{I,k}(z)$ can only be determined through explicit computations. These do not contribute to the infrared poles associated with Φ^I. This uniquely fixes the unknown soft-collinear distribution Φ^I at one loop order. This prescription can easily be generalised to higher orders in a_s. In our calculation of the SV cross section, instead of solving in this way, we follow a bit different methodology which is presented below.

Keeping the demands (E.0.7) and (E.0.8) in mind, we propose the solution of the $\overline{KG}$ equation as (See Eq. (E.0.11))

$$\begin{aligned}\hat{\Phi}_k^I(z,\epsilon) &\equiv \left\{k\epsilon\frac{1}{1-z}\left[(1-z)^2\right]^{k\frac{\epsilon}{2}}\right\}\hat{\Phi}_k^I(\epsilon)\\ &= \left\{\delta(1-z) + \sum_{j=0}^{\infty}\frac{(k\epsilon)^{j+1}}{j!}\mathcal{D}_j\right\}\hat{\Phi}_k^I(\epsilon)\,.\end{aligned} \tag{E.0.17}$$

The RG invariance of Φ^I, Eq. (E.0.8), implies

$$\mu_R^2 \frac{d}{d\mu_R^2} \overline{K}^I = -\mu_R^2 \frac{d}{d\mu_R^2} \overline{G}^I \equiv Y'^I \tag{E.0.18}$$

where, we introduce a quantity Y', analogous to Y. Hence, the solution can be obtained as

$$\hat{\Phi}_k^I(\epsilon) = \hat{\mathcal{L}}_k^I \left(A_i^I \to Y_i'^I, G_i^I \to \overline{\mathcal{G}}_i^I(\epsilon) \right). \tag{E.0.19}$$

Hence, according to the Eq. (D.0.11), for $k = 1$ we get

$$2\Phi_1^I(z,\epsilon) = \left\{ \frac{1}{\epsilon^2} \left(-4Y_1'^I \right) + \frac{2}{\epsilon} \overline{\mathcal{G}}_1^I(\epsilon)) \right\} \delta(1-z) + \left\{ -\frac{4Y_1'^I}{\epsilon^2} + \frac{2}{\epsilon} \overline{\mathcal{G}}_1^I(\epsilon) \right\} \sum_{j=0}^{\infty} \frac{\epsilon^{j+1}}{j!} \mathcal{D}_j \tag{E.0.20}$$

where, Y'^I and $\overline{\mathcal{G}}^I$ are expanded similar to Eq. (E.0.13). Comparison between the two solutions depicted in Eq. (E.0.7) and (E.0.20), we can write

$$\begin{aligned} Y_1'^I &= -A_1^I \\ \overline{\mathcal{G}}_1^I(\epsilon) &= -f_1^I + \sum_{k=1}^{\infty} \epsilon^k \overline{\mathcal{G}}_1^{I,k}. \end{aligned} \tag{E.0.21}$$

Explicit computation is required to determine the coefficients of ϵ^k, $\overline{\mathcal{G}}_1^{I,k}$. This solution is used in Eq. (2.3.38) in the context of SV cross section of Higgs boson production. The method is generalised to higher orders in a_s to obtain the results of the soft-collinear distribution. In the next subsection, we present the results of the soft-collinear distribution up to three loops.

E.0.1 Results

We define the renormalised components of the $\Phi^I_{q\bar{q},k}$ through

$$\begin{aligned}\Phi^I_{q\bar{q}}(\hat{a}_s, q^2, \mu^2, z\epsilon) &= \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Phi}^I_{q\bar{q},k}(z,\epsilon) \\ &= \sum_{k=1}^{\infty} a_s^k\left(\mu_F^2\right) \Phi^I_{q\bar{q},k}\left(z,\epsilon,q^2,\mu_F^2\right) \end{aligned} \tag{E.0.22}$$

where, we make the choice of the renormalisation scale $\mu_R = \mu_F$. The μ_R dependence can be easily restored by using the evolution equation of strong coupling constant, Eq. (2.4.6). Below, we present the $\Phi^I_{i\,\bar{i},k}$ for $i\,\bar{i} = q\bar{q}$ up to three loops and the corresponding components for $i\,\bar{i} = gg$ can be obtained using maximally non-Abelian property fulfilled by this distribution:

$$\Phi^I_{gg,k} = \frac{C_A}{C_F}\Phi^I_{q\bar{q},k}\,. \tag{E.0.23}$$

The results are given by

$$\begin{aligned}
\Phi^I_{q\bar{q},1} &= \delta(1-z)\left[\frac{1}{\epsilon^2}C_F\left\{8\right\} + \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)C_F\left\{4\right\} + C_F\left\{-3\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_F\left\{1\right\}\right] \\
&\quad + \mathcal{D}_0\left[\frac{1}{\epsilon}C_F\left\{8\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_F\left\{4\right\}\right] + \mathcal{D}_1\left[C_F\left\{8\right\}\right], \\
\Phi^I_{q\bar{q},2} &= \delta(1-z)\left[\frac{1}{\epsilon^3}C_FC_A\left\{44\right\} + \frac{1}{\epsilon^3}n_fC_F\left\{-8\right\} + \frac{1}{\epsilon^2}C_FC_A\left\{\frac{134}{9} - 4\zeta_2\right\}\right. \\
&\quad + \frac{1}{\epsilon^2}n_fC_F\left\{-\frac{20}{9}\right\} + \frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{\frac{44}{3}\right\} + \frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{-\frac{8}{3}\right\} \\
&\quad + \frac{1}{\epsilon}C_FC_A\left\{-\frac{404}{27} + 14\zeta_3 + \frac{11}{3}\zeta_2\right\} + \frac{1}{\epsilon}n_fC_F\left\{\frac{56}{27} - \frac{2}{3}\zeta_2\right\} \\
&\quad + \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{\frac{134}{9} - 4\zeta_2\right\} + \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{-\frac{20}{9}\right\} + C_FC_A\left\{\frac{1214}{81}\right. \\
&\quad \left.\left. - \frac{187}{9}\zeta_3 - \frac{469}{18}\zeta_2 + 2\zeta_2^2\right\} + n_fC_F\left\{-\frac{164}{81} + \frac{34}{9}\zeta_3 + \frac{35}{9}\zeta_2\right\}\right. \\
&\quad + \log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{-\frac{404}{27} + 14\zeta_3 + \frac{44}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{\frac{56}{27} - \frac{8}{3}\zeta_2\right\}
\end{aligned}$$

$$
\begin{aligned}
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{\frac{67}{9}-2\zeta_2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{-\frac{10}{9}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{-\frac{11}{9}\right\}\\
&+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{\frac{2}{9}\right\}\Bigg]+\mathcal{D}_0\Bigg[\frac{1}{\epsilon^2}C_FC_A\left\{\frac{88}{3}\right\}+\frac{1}{\epsilon^2}n_fC_F\left\{-\frac{16}{3}\right\}+\frac{1}{\epsilon}C_FC_A\left\{\frac{268}{9}\right.\\
&\left.-8\zeta_2\right\}+\frac{1}{\epsilon}n_fC_F\left\{-\frac{40}{9}\right\}+C_FC_A\left\{-\frac{808}{27}+28\zeta_3+\frac{88}{3}\zeta_2\right\}+n_fC_F\left\{\frac{112}{27}-\frac{16}{3}\zeta_2\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{\frac{268}{9}-8\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{-\frac{40}{9}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{-\frac{22}{3}\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{\frac{4}{3}\right\}\Bigg]+\mathcal{D}_1\Bigg[C_FC_A\left\{\frac{536}{9}-16\zeta_2\right\}+n_fC_F\left\{-\frac{80}{9}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A\left\{-\frac{88}{3}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F\left\{\frac{16}{3}\right\}\Bigg]\\
&+\mathcal{D}_2\Bigg[C_FC_A\left\{-\frac{88}{3}\right\}+n_fC_F\left\{\frac{16}{3}\right\}\Bigg],\\
\Phi^I_{q\bar{q},3}=\;&\delta(1-z)\Bigg[\frac{1}{\epsilon^4}C_FC_A^2\left\{\frac{21296}{81}\right\}+\frac{1}{\epsilon^4}n_fC_FC_A\left\{-\frac{7744}{81}\right\}+\frac{1}{\epsilon^4}n_f^2C_F\left\{\frac{704}{81}\right\}\\
&+\frac{1}{\epsilon^3}C_FC_A^2\left\{\frac{49064}{243}-\frac{880}{27}\zeta_2\right\}+\frac{1}{\epsilon^3}n_fC_FC_A\left\{-\frac{15520}{243}+\frac{160}{27}\zeta_2\right\}\\
&+\frac{1}{\epsilon^3}n_fC_F^2\left\{-\frac{128}{9}\right\}+\frac{1}{\epsilon^3}n_f^2C_F\left\{\frac{800}{243}\right\}+\frac{1}{\epsilon^3}\log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{\frac{1936}{27}\right\}\\
&+\frac{1}{\epsilon^3}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{704}{27}\right\}+\frac{1}{\epsilon^3}\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{+\frac{64}{27}\right\}\\
&+\frac{1}{\epsilon^2}C_FC_A^2\left\{-\frac{8956}{243}+\frac{2024}{27}\zeta_3-\frac{692}{81}\zeta_2+\frac{352}{45}\zeta_2^2\right\}+\frac{1}{\epsilon^2}n_fC_FC_A\left\{\frac{4024}{243}-\frac{560}{27}\zeta_3\right.\\
&\left.-\frac{208}{81}\zeta_2\right\}+\frac{1}{\epsilon^2}n_fC_F^2\left\{-\frac{220}{27}+\frac{64}{9}\zeta_3\right\}+\frac{1}{\epsilon^2}n_f^2C_F\left\{-\frac{160}{81}+\frac{16}{27}\zeta_2\right\}\\
&+\frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{\frac{8344}{81}-\frac{176}{9}\zeta_2\right\}+\frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{2672}{81}+\frac{32}{9}\zeta_2\right\}\\
&+\frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{-\frac{16}{3}\right\}+\frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\left\{\frac{160}{81}\right\}+\frac{1}{\epsilon}C_FC_A^2\left\{-\frac{136781}{2187}\right.\\
&\left.-64\zeta_5+\frac{1316}{9}\zeta_3+\frac{12650}{243}\zeta_2-\frac{176}{9}\zeta_2\zeta_3-\frac{352}{15}\zeta_2^2\right\}+\frac{1}{\epsilon}n_fC_FC_A\left\{\frac{11842}{2187}-\frac{728}{81}\zeta_3\right.\\
&\left.-\frac{2828}{243}\zeta_2+\frac{32}{5}\zeta_2^2\right\}+\frac{1}{\epsilon}n_fC_F^2\left\{\frac{1711}{81}-\frac{304}{27}\zeta_3-\frac{4}{3}\zeta_2-\frac{32}{15}\zeta_2^2\right\}+\frac{1}{\epsilon}n_f^2C_F\left\{\frac{2080}{2187}\right.\\
&\left.-\frac{112}{81}\zeta_3+\frac{40}{81}\zeta_2\right\}+\frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\left\{\frac{490}{9}+\frac{88}{9}\zeta_3-\frac{1072}{27}\zeta_2+\frac{176}{15}\zeta_2^2\right\}\\
&+\frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{836}{81}-\frac{112}{9}\zeta_3+\frac{160}{27}\zeta_2\right\}+\frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\left\{-\frac{110}{9}\right.
\end{aligned}
$$

$$+\frac{32}{3}\zeta_3\Big\}+\frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\Big\{-\frac{16}{81}\Big\}+C_FC_A^2\Big\{\frac{5211949}{26244}-\frac{484}{9}\zeta_5-\frac{128966}{243}\zeta_3$$
$$+\frac{536}{9}\zeta_3^2-\frac{578479}{1458}\zeta_2+242\zeta_2\zeta_3+\frac{9457}{135}\zeta_2^2+\frac{152}{189}\zeta_2^3\Big\}+n_fC_FC_A\Big\{-\frac{412765}{13122}$$
$$-\frac{8}{3}\zeta_5+\frac{9856}{81}\zeta_3+\frac{75155}{729}\zeta_2-\frac{44}{3}\zeta_2\zeta_3-\frac{2528}{135}\zeta_2^2\Big\}+n_fC_F^2\Big\{-\frac{42727}{972}+\frac{112}{9}\zeta_5$$
$$+\frac{2284}{81}\zeta_3+\frac{605}{18}\zeta_2-\frac{88}{3}\zeta_2\zeta_3+\frac{152}{45}\zeta_2^2\Big\}+n_f^2C_F\Big\{-\frac{128}{6561}-\frac{1480}{243}\zeta_3-\frac{404}{81}\zeta_2$$
$$+\frac{148}{135}\zeta_2^2\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\Big\{-\frac{297029}{1458}-96\zeta_5+\frac{10036}{27}\zeta_3+\frac{24556}{81}\zeta_2-\frac{88}{3}\zeta_2\zeta_3$$
$$-\frac{748}{15}\zeta_2^2\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\Big\{\frac{31313}{729}-\frac{620}{9}\zeta_3-\frac{7348}{81}\zeta_2+\frac{184}{15}\zeta_2^2\Big\}$$
$$+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\Big\{\frac{1711}{54}-\frac{152}{9}\zeta_3-8\zeta_2-\frac{16}{5}\zeta_2^2\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\Big\{-\frac{928}{729}+\frac{80}{27}\zeta_3$$
$$+\frac{160}{27}\zeta_2\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\Big\{\frac{15503}{162}-44\zeta_3-\frac{752}{9}\zeta_2+\frac{44}{5}\zeta_2^2\Big\}$$
$$+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\Big\{-\frac{2051}{81}+24\zeta_2\Big\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\Big\{-\frac{55}{6}+8\zeta_3\Big\}$$
$$+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\Big\{\frac{100}{81}-\frac{16}{9}\zeta_2\Big\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_FC_A^2\Big\{-\frac{1780}{81}+\frac{44}{9}\zeta_2\Big\}$$
$$+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\Big\{\frac{578}{81}-\frac{8}{9}\zeta_2\Big\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_fC_F^2\Big\{\frac{2}{3}\Big\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_F\Big\{-\frac{40}{81}\Big\}$$
$$+\log\left(\frac{q^2}{\mu_F^2}\right)^4C_FC_A^2\Big\{\frac{121}{54}\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)^4n_fC_FC_A\Big\{-\frac{22}{27}\Big\}+\log\left(\frac{q^2}{\mu_F^2}\right)^4n_f^2C_F\Big\{\frac{2}{27}\Big\}\Big]$$
$$+\mathcal{D}_0\Big[\frac{1}{\epsilon^3}C_FC_A^2\Big\{\frac{3872}{27}\Big\}+\frac{1}{\epsilon^3}n_fC_FC_A\Big\{-\frac{1408}{27}\Big\}+\frac{1}{\epsilon^3}n_f^2C_F\Big\{\frac{128}{27}\Big\}$$
$$+\frac{1}{\epsilon^2}C_FC_A^2\Big\{\frac{16688}{81}-\frac{352}{9}\zeta_2\Big\}+\frac{1}{\epsilon^2}n_fC_FC_A\Big\{-\frac{5344}{81}+\frac{64}{9}\zeta_2\Big\}+\frac{1}{\epsilon^2}n_fC_F^2\Big\{-\frac{32}{3}\Big\}$$
$$+\frac{1}{\epsilon^2}n_f^2C_F\Big\{\frac{320}{81}\Big\}+\frac{1}{\epsilon}C_FC_A^2\Big\{\frac{980}{9}+\frac{176}{9}\zeta_3-\frac{2144}{27}\zeta_2+\frac{352}{15}\zeta_2^2\Big\}$$
$$+\frac{1}{\epsilon}n_fC_FC_A\Big\{-\frac{1672}{81}-\frac{224}{9}\zeta_3+\frac{320}{27}\zeta_2\Big\}+\frac{1}{\epsilon}n_fC_F^2\Big\{-\frac{220}{9}+\frac{64}{3}\zeta_3\Big\}$$
$$+\frac{1}{\epsilon}n_f^2C_F\Big\{-\frac{32}{81}\Big\}+C_FC_A^2\Big\{-\frac{297029}{729}-192\zeta_5+\frac{20072}{27}\zeta_3+\frac{49112}{81}\zeta_2-\frac{176}{3}\zeta_2\zeta_3$$
$$-\frac{1496}{15}\zeta_2^2\Big\}+n_fC_FC_A\Big\{\frac{62626}{729}-\frac{1240}{9}\zeta_3-\frac{14696}{81}\zeta_2+\frac{368}{15}\zeta_2^2\Big\}+n_fC_F^2\Big\{\frac{1711}{27}$$
$$-\frac{304}{9}\zeta_3-16\zeta_2-\frac{32}{5}\zeta_2^2\Big\}+n_f^2C_F\Big\{-\frac{1856}{729}+\frac{160}{27}\zeta_3+\frac{320}{27}\zeta_2\Big\}$$

$$
+\log\left(\frac{q^2}{\mu_F^2}\right)C_F C_A^2\left\{\frac{31006}{81}-176\zeta_3-\frac{3008}{9}\zeta_2+\frac{176}{5}\zeta_2^2\right\}
$$
$$
+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{-\frac{8204}{81}+96\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F^2\left\{-\frac{110}{3}+32\zeta_3\right\}
$$
$$
+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_F\left\{\frac{400}{81}-\frac{64}{9}\zeta_2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_F C_A^2\left\{-\frac{3560}{27}+\frac{88}{3}\zeta_2\right\}
$$
$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{\frac{1156}{27}-\frac{16}{3}\zeta_2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_F^2\left\{4\right\}
$$
$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_F\left\{-\frac{80}{27}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_F C_A^2\left\{\frac{484}{27}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{-\frac{176}{27}\right\}
$$
$$
+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_F\left\{\frac{16}{27}\right\}\Bigg]+\mathcal{D}_1\Bigg[C_F C_A^2\left\{\frac{62012}{81}-352\zeta_3-\frac{6016}{9}\zeta_2+\frac{352}{5}\zeta_2^2\right\}
$$
$$
+n_f C_F C_A\left\{-\frac{16408}{81}+192\zeta_2\right\}+n_f C_F^2\left\{-\frac{220}{3}+64\zeta_3\right\}+n_f^2 C_F\left\{\frac{800}{81}-\frac{128}{9}\zeta_2\right\}
$$
$$
+\log\left(\frac{q^2}{\mu_F^2}\right)C_F C_A^2\left\{-\frac{14240}{27}+\frac{352}{3}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{\frac{4624}{27}-\frac{64}{3}\zeta_2\right\}
$$
$$
+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F^2\left\{16\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_F\left\{-\frac{320}{27}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_F C_A^2\left\{\frac{968}{9}\right\}
$$
$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{-\frac{352}{9}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_F\left\{\frac{32}{9}\right\}\Bigg]+\mathcal{D}_2\Bigg[C_F C_A^2\left\{-\frac{14240}{27}\right.
$$
$$
\left.+\frac{352}{3}\zeta_2\right\}+n_f C_F C_A\left\{\frac{4624}{27}-\frac{64}{3}\zeta_2\right\}+n_f C_F^2\left\{16\right\}+n_f^2 C_F\left\{-\frac{320}{27}\right\}
$$
$$
+\log\left(\frac{q^2}{\mu_F^2}\right)C_F C_A^2\left\{\frac{1936}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{-\frac{704}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_F\left\{\frac{64}{9}\right\}\Bigg]
$$
$$
+\mathcal{D}_3\Bigg[C_F C_A^2\left\{\frac{3872}{27}\right\}+n_f C_F C_A\left\{-\frac{1408}{27}\right\}+n_f^2 C_F\left\{\frac{128}{27}\right\}\Bigg]. \tag{E.0.24}
$$

F Soft-Collinear Distribution for Rapidity

In Sec. 3.3.4, we introduced the soft-collinear distribution $\Phi^I_{i\,\bar{i}}$ in the context of computing SV correction to the differential rapidity distribution of a colorless particle at Hadron collider. In this appendix, we intend to elaborate the methodology of finding this distribution. For simplicity, we will omit the partonic indices for our further calculation. To understand the underlying logics behind finding Φ^I, let us consider an example at one loop level. The generalisation to higher loop is straightforward. The whole discussion of this appendix is closely related to the Appendix E where we discussed the soft-collinear distribution for inclusive production cross section for a colorless particle.

As discussed in the Sec. 3.3, the SV cross section in z-space can be computed in $d = 4+\epsilon$ dimensions using

$$\Delta^{I,\mathrm{SV}}_Y(z_1, z_2, q^2, \mu_R^2, \mu_F^2) = C \exp\left(\Psi^I_Y\left(z_1, z_2, q^2, \mu_R^2, \mu_F^2, \epsilon\right)\right)\Big|_{\epsilon=0} \tag{F.0.1}$$

where, $\Psi^I_Y\left(z_1, z_2, q^2, \mu_R^2, \mu_F^2, \epsilon\right)$ is a finite distribution and C is the double Mellin convolution defined through Eq. (**??**). The Ψ^I is given by, Eq. (3.3.16)

$$\begin{aligned}\Psi^I_{Y,ij}\left(z_1, z_2, q^2, \mu_R^2, \mu_F^2, \epsilon\right) = &\left(\ln\left[Z^I_{ij}(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)\right]^2 + \ln\left|\mathcal{F}^I_{ij}(\hat{a}_s, Q^2, \mu^2, \epsilon)\right|^2\right)\delta(1-z_1)\delta(1-z_2)\\ &+ 2\Phi^I_{Y,ij}(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) - C \ln \Gamma^I_{ij}(\hat{a}_s, \mu^2, \mu_F^2, z_1, \epsilon)\delta(1-z_2)\end{aligned}$$

$$- C \ln \Gamma^I_{ij}(\hat{a}_s, \mu^2, \mu_F^2, z_2, \epsilon)\delta(1-z_1)\,. \tag{F.0.2}$$

For all the details about the notations, see Sec. 3.3. Considering only the poles at $O(a_s)$ with $\mu_R = \mu_F$ we obtain,

$$\begin{aligned}
\ln\left(Z^{I,(1)}\right)^2 &= a_s(\mu_F^2)\frac{4\gamma_1^I}{\epsilon}\,,\\
\ln|\mathcal{F}^{I,(1)}|^2 &= a_s(\mu_F^2)\left(\frac{q^2}{\mu_F^2}\right)^{\frac{\epsilon}{2}}\left[-\frac{4A_1^I}{\epsilon^2}+\frac{1}{\epsilon}\left(2f_1^I+4B_1^I-4\gamma_1^I\right)\right],\\
C\ln\Gamma^{I,(1)}(z_1) &= a_s(\mu_F^2)\left[\frac{2B_1^I}{\epsilon}\delta(1-z_1)+\frac{2A_1^I}{\epsilon}\mathcal{D}_0\right],\\
C\ln\Gamma^{I,(1)}(z_2) &= a_s(\mu_F^2)\left[\frac{2B_1^I}{\epsilon}\delta(1-z_2)+\frac{2A_1^I}{\epsilon}\overline{\mathcal{D}}_0\right]
\end{aligned} \tag{F.0.3}$$

where, the components are defined through the expansion of these quantities in powers of $a_s(\mu_F^2)$

$$\begin{aligned}
\Psi_Y^I &= \sum_{k=1}^{\infty} a_s^k\left(\mu_F^2\right)\Psi_Y^{I,(k)}\,,\\
\ln(Z^I)^2 &= \sum_{k=1}^{\infty} a_s^k\left(\mu_F^2\right)Z^{I,(k)}\,,\\
\ln|\mathcal{F}^I|^2 &= \sum_{k=1}^{\infty} a_s^k\left(\mu_F^2\right)\left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}}\ln|\mathcal{F}^{I,(k)}|^2\,,\\
\Phi_Y^I &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2)\Phi_{Y,k}^I\,,\\
\ln\Gamma^I &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2)\ln\Gamma^{I,(k)}
\end{aligned} \tag{F.0.4}$$

and

$$\begin{aligned}
\mathcal{D}_i &\equiv \left[\frac{\ln^i(1-z_1)}{1-z_1}\right]_+\,,\\
\overline{\mathcal{D}}_i &\equiv \left[\frac{\ln^i(1-z_2)}{1-z_2}\right]_+\,.
\end{aligned} \tag{F.0.5}$$

Collecting the coefficients of $a_s(\mu_F^2)$, we get

$$\Psi^{I,(1)}|_{Y,\text{poles}} = \left[\left\{-\frac{4A_1^I}{\epsilon^2}+\frac{2f_1^I}{\epsilon}\right\}\delta(1-z_1)\delta(1-z_2) - \frac{2A_1^I}{\epsilon}\left\{\delta(1-z_1)\overline{\mathcal{D}}_0+\delta(1-z_2)\mathcal{D}_0\right\}\right] + 2\Phi_{Y,1}^I \tag{F.0.6}$$

where, we have suppressed the $\ln(q^2/\mu_F^2)$ terms. To cancel the remaining divergences appearing in the above Eq. (F.0.6) for obtaining a finite rapidity distribution, we must demand that $\Phi_{Y,1}^I$ has exactly the same poles with opposite sign:

$$2\Phi_{Y,1}^I|_{\text{poles}} = -\left[\left\{-\frac{4A_1^I}{\epsilon^2}+\frac{2f_1^I}{\epsilon}\right\}\delta(1-z_1)\delta(1-z_2) - \frac{2A_1^I}{\epsilon}\left\{\delta(1-z_1)\overline{\mathcal{D}}_0+\delta(1-z_2)\mathcal{D}_0\right\}\right] \tag{F.0.7}$$

In addition, Φ_Y^I also should be RG invariant with respect to μ_R:

$$\mu_R^2\frac{d}{d\mu_R^2}\Phi_Y^I = 0\,. \tag{F.0.8}$$

We make an *ansatz*, the above two demands, Eq. (E.0.7) and (E.0.8) can be accomplished if Φ_Y^I satisfies the KG-type integro-differential equation which we call $\overline{KG}_Y$:

$$q^2\frac{d}{dq^2}\Phi_Y^I\left(\hat{a}_s,q^2,\mu^2,z_1,z_2,\epsilon\right) = \frac{1}{2}\left[\overline{K}_Y^I\left(\hat{a}_s,\frac{\mu_R^2}{\mu^2},z_1,z_2,\epsilon\right)+\overline{G}_Y^I\left(\hat{a}_s,\frac{q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},z_1,z_2,\epsilon\right)\right]. \tag{F.0.9}$$

$\overline{K}_Y^I$ contains all the poles whereas $\overline{G}_Y^I$ consists of only the finite terms in ϵ. RG invariance (F.0.8) of Φ_Y^I dictates

$$\mu_R^2\frac{d}{d\mu_R^2}\overline{K}_Y^I = -\mu_R^2\frac{d}{d\mu_R^2}\overline{G}_Y^I \equiv X_Y^I \tag{F.0.10}$$

where, we introduce a quantity X_Y^I. Following the methodology of solving the KG equa-

tion discussed in the Appendix D, we can write the solution of Φ_Y^I as

$$\Phi_Y^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) = \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Phi}_{Y,k}^I(z_1, z_2, \epsilon) \tag{F.0.11}$$

with

$$\hat{\Phi}_{Y,k}^I(z_1, z_2, \epsilon) = \hat{\mathcal{L}}_k^I \left(A_i^I \to X_{Y,i}^I, G_i^I \to \overline{G}_{Y,i}^I(z, \epsilon)\right). \tag{F.0.12}$$

where we define the components through the expansions

$$\begin{aligned} X_Y^I &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2) X_{Y,k}^I \,, \\ \overline{G}_Y^I(z_1, z_2, \epsilon) &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2) \overline{G}_{Y,k}^I(z_1, z_2, \epsilon). \end{aligned} \tag{F.0.13}$$

This solution directly follows from the Eq. (D.0.10). Hence we get

$$2\hat{\Phi}_{Y,1}^I(z, \epsilon) = \frac{1}{\epsilon^2}\left\{-4X_{Y,1}^I\right\} + \frac{2}{\epsilon}\left\{\overline{G}_{Y,1}^I(z_1, z_2, \epsilon)\right\}. \tag{F.0.14}$$

By expressing the components of Φ_Y^I in powers of $a_s(\mu_F^2)$, we obtain

$$\begin{aligned} \Phi^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) &= \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Phi}_{Y,k}^I(z_1, z_2, \epsilon) \\ &= \sum_{k=1}^{\infty} a_s^k(\mu_F^2) \left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}} Z_{a_s}^k \hat{\Phi}_{Y,k}^I(z_1, z_2, \epsilon) \\ &\equiv \sum_{k=1}^{\infty} a_s^k(\mu_F^2) \left(\frac{q^2}{\mu_F^2}\right)^{k\frac{\epsilon}{2}} \Phi_{Y,k}^I(z, \epsilon) \end{aligned} \tag{F.0.15}$$

and at $O(a_s(\mu_F^2))$, $\hat{\Phi}_{Y,1}^I(z, \epsilon) = \Phi_{Y,1}^I(z, \epsilon)$ upon suppressing the terms like $\log(q^2/\mu_F^2)$. Hence, by comparing the Eq. (F.0.7) and (F.0.14), we conclude

$$X_{Y,1}^I = -A_1^I \delta(1 - z_1)\delta(1 - z_2)\,,$$

$$\overline{G}^I_{Y,1}(z_1,z_2,\epsilon) = -f_1^I\delta(1-z_1)\delta(1-z_2) + A_1^I\Big\{\delta(1-z_1)\overline{\mathcal{D}}_0 + \delta(1-z_2)\mathcal{D}_0\Big\}$$
$$+\sum_{k=1}^{\infty}\epsilon^k\overline{g}^{I,k}_{Y,1}(z)\,. \tag{F.0.16}$$

The coefficients of $\epsilon^k, \overline{g}^{I,k}_{Y,1}(z)$ can only be determined through explicit computations. These do not contribute to the infrared poles associated with Φ^I_Y. This uniquely fixes the unknown soft-collinear distribution Φ^I_Y at one loop order. This prescription can easily be generalised to higher orders in a_s. In our calculation of the SV correction to rapidity distribution, instead of solving in this way, we follow a bit different methodology which is presented below.

Keeping the demands (F.0.7) and (F.0.8) in mind, we propose the solution of the $\overline{KG}_Y$ equation as (See Eq. (F.0.11)) (which is just the extension of the Eq. (E.0.17) from one variable z to a case of two variables z_1 and z_2)

$$\begin{aligned}\hat{\Phi}^I_{Y,k}(z_1,z_2,\epsilon) &\equiv \left\{(k\epsilon)^2\frac{1}{4(1-z_1)(1-z_2)}\left[(1-z_1)(1-z_2)\right]^{k\frac{\epsilon}{2}}\right\}\hat{\Phi}^I_{Y,k}(\epsilon)\\ &= \left\{\frac{k\epsilon}{2}\frac{1}{(1-z_1)}\left[(1-z_1)^2\right]^{k\frac{\epsilon}{4}}\right\}\left\{\frac{k\epsilon}{2}\frac{1}{(1-z_2)}\left[(1-z_2)^2\right]^{k\frac{\epsilon}{4}}\right\}\hat{\Phi}^I_{Y,k}(\epsilon)\\ &= \left\{\delta(1-z_1)+\sum_{j=0}^{\infty}\frac{(k\epsilon/2)^{j+1}}{j!}\mathcal{D}_j\right\}\left\{\delta(1-z_2)+\sum_{l=0}^{\infty}\frac{(k\epsilon/2)^{l+1}}{l!}\overline{\mathcal{D}}_l\right\}\hat{\Phi}^I_{Y,k}(\epsilon)\,.\end{aligned} \tag{F.0.17}$$

The RG invariance of Φ^I_Y, Eq. (F.0.8), implies

$$\mu_R^2\frac{d}{d\mu_R^2}\overline{K}^I_Y = -\mu_R^2\frac{d}{d\mu_R^2}\overline{G}^I_Y \equiv X'^I_Y \tag{F.0.18}$$

where, we introduce a quantity X'^I_Y, analogous to X^I_Y. Hence, the solution can be obtained as

$$\hat{\Phi}^I_{Y,k}(\epsilon) = \hat{\mathcal{L}}^I_k\left(A^I_i \to X'^I_{Y,i}, G^I_{Y,i} \to \overline{\mathcal{G}}^I_{Y,i}(\epsilon)\right). \tag{F.0.19}$$

Hence, according to the Eq. (D.0.11), for $k = 1$ we get

$$
\begin{aligned}
2\Phi^I_{Y,1}(z_1, z_2, \epsilon) &= 2\hat{\Phi}^I_{Y,1}(z_1, z_2, \epsilon) \\
&= \left\{\delta(1-z_1) + \sum_{j=0}^{\infty} \frac{(k\epsilon/2)^{j+1}}{j!}\mathcal{D}_j\right\}\left\{\delta(1-z_2) + \sum_{l=0}^{\infty} \frac{(k\epsilon/2)^{l+1}}{l!}\overline{\mathcal{D}}_l\right\} \\
&\quad \left\{\frac{1}{\epsilon^2}\left(-4X'^I_{Y,1}\right) + \frac{2}{\epsilon}\overline{\mathcal{G}}^I_{Y,1}(\epsilon)\right\}
\end{aligned} \tag{F.0.20}
$$

where, X'^I_Y and $\overline{\mathcal{G}}^I_Y$ are expanded similar to Eq. (F.0.13). Comparison between the two solutions depicted in Eq. (F.0.7) and (F.0.20), we can write

$$
\begin{aligned}
X'^I_{Y,1} &= -A^I_1 \\
\overline{\mathcal{G}}^I_{Y,1}(\epsilon) &= -f^I_1 + \sum_{k=1}^{\infty} \epsilon^k \overline{\mathcal{G}}^{I,k}_{Y,1}\,.
\end{aligned} \tag{F.0.21}
$$

Explicit computation is required to determine the coefficients of ϵ^k, $\overline{\mathcal{G}}^{I,k}_{Y,1}$. This solution is used in Eq. (3.3.45) in the context of SV correction to differential rapidity distribution of Higgs boson production or leptonic pair in DY production. The method is generalised to higher orders in a_s to obtain the results of the soft-collinear distribution. Hence, the all order solution of Φ^I_Y is

$$
\Phi^I(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) = \sum_{k=1}^{\infty} \hat{a}_s^k S^k_\epsilon \left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Phi}^I_{Y,k}(z_1, z_2, \epsilon) \tag{F.0.22}
$$

with

$$
\begin{aligned}
\hat{\Phi}^I_{Y,k}(z_1, z_2, \epsilon) &= \left\{(k\epsilon)^2 \frac{1}{4(1-z_1)(1-z_2)}\left[(1-z_1)(1-z_2)\right]^{k\frac{\epsilon}{2}}\right\}\hat{\Phi}^I_{Y,k}(\epsilon)\,, \\
\hat{\Phi}^I_{Y,k}(\epsilon) &= \hat{\mathcal{L}}^I_k\left(A^I_i \to -A^I_i, G^I_{Y,i} \to \overline{\mathcal{G}}^I_{Y,i}(\epsilon)\right)\,.
\end{aligned} \tag{F.0.23}
$$

Up to three loop, $\overline{\mathcal{G}}^I_{Y,i}(\epsilon)$ are found to be

$$\overline{\mathcal{G}}^I_{Y,i}(\epsilon) = -f^I_i + \overline{C}^I_{Y,i} + \sum_{k=1}^{\infty} \epsilon^k \overline{\mathcal{G}}^{I,k}_{Y,i} \tag{F.0.24}$$

where

$$\begin{aligned}
\overline{C}^I_{Y,1} &= 0\,, \\
\overline{C}^I_{Y,2} &= -2\beta_0 \overline{\mathcal{G}}^{I,1}_{Y,1}\,, \\
\overline{C}^I_{Y,3} &= -2\beta_1 \overline{\mathcal{G}}^{I,1}_{Y,1} - 2\beta_0 \left(\overline{\mathcal{G}}^{I,1}_{Y,2} + 2\beta_0 \overline{\mathcal{G}}^{I,2}_{Y,1}\right).
\end{aligned} \tag{F.0.25}$$

These are employed in the computation of rapidity distributions in Chapter 3. In the next subsection, we present the results of the soft-collinear distribution up to three loops.

F.0.1 Results

We define the renormalised components of the $\Phi^I_{Y,i\,\bar{i},k}$ through

$$\begin{aligned}
\Phi^I_{Y,i\,\bar{i}}(\hat{a}_s, q^2, \mu^2, z_1, z_2, \epsilon) &= \sum_{k=1}^{\infty} \hat{a}_s^k S_\epsilon^k \left(\frac{q^2}{\mu^2}\right)^{k\frac{\epsilon}{2}} \hat{\Phi}^I_{Y,i\,\bar{i},k}(z_1, z_2, \epsilon) \\
&= \sum_{k=1}^{\infty} a_s^k\left(\mu_F^2\right) \Phi^I_{Y,i\,\bar{i},k}\left(z_1, z_2, \epsilon, q^2, \mu_F^2\right)
\end{aligned} \tag{F.0.26}$$

where, we make the choice of the renormalisation scale $\mu_R = \mu_F$. The μ_R dependence can be easily restored by using the evolution equation of strong coupling constant, Eq. (2.4.6). Below, we present the $\Phi^I_{Y,i\,\bar{i},k}$ for $I = H$ and $i\,\bar{i} = gg$ up to three loops and the corresponding components for $I = \mathrm{DY}$ and $i\,\bar{i} = q\bar{q}$ can be obtained using maximally non-Abelian property fulfilled by this distribution:

$$\Phi^H_{Y,gg,k} = \frac{C_A}{C_F} \Phi^{\mathrm{DY}}_{Y,q\,\bar{q},k}\,. \tag{F.0.27}$$

The results are given by

$$
\begin{aligned}
\Phi^H_{Y,gg,1} &= \delta(1-z_1)\delta(1-z_2)\left[\frac{1}{\epsilon^2}C_A\left\{8\right\} + \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)C_A\left\{4\right\} + C_A\left\{-\zeta_2\right\}\right.\\
&\left. + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_A\left\{1\right\}\right] + \mathcal{D}_0\delta(1-z_2)\left[\frac{1}{\epsilon}C_A\left\{4\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_A\left\{2\right\}\right]\\
&+ \mathcal{D}_0\overline{\mathcal{D}}_0\left[C_A\left\{2\right\}\right] + \mathcal{D}_1\delta(1-z_2)\left[C_A\left\{2\right\}\right] + \overline{\mathcal{D}}_0\delta(1-z_1)\left[\frac{1}{\epsilon}C_A\left\{4\right\}\right.\\
&\left. + \log\left(\frac{q^2}{\mu_F^2}\right)C_A\left\{2\right\}\right] + \overline{\mathcal{D}}_1\delta(1-z_1)\left[C_A\left\{2\right\}\right],\\
\Phi^H_{Y,gg,2} &= \delta(1-z_1)\delta(1-z_2)\left[\frac{1}{\epsilon^3}C_A^2\left\{44\right\} + \frac{1}{\epsilon^3}n_f C_A\left\{-8\right\} + \frac{1}{\epsilon^2}C_A^2\left\{\frac{134}{9} - 4\zeta_2\right\}\right.\\
&+ \frac{1}{\epsilon^2}n_f C_A\left\{-\frac{20}{9}\right\} + \frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{\frac{44}{3}\right\} + \frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A\left\{-\frac{8}{3}\right\}\\
&+ \frac{1}{\epsilon}C_A^2\left\{-\frac{404}{27} + 14\zeta_3 + \frac{11}{3}\zeta_2\right\} + \frac{1}{\epsilon}n_f C_A\left\{\frac{56}{27} - \frac{2}{3}\zeta_2\right\}\\
&+ \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{\frac{134}{9} - 4\zeta_2\right\} + \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A\left\{-\frac{20}{9}\right\}\\
&+ C_A^2\left\{\frac{1214}{81} - \frac{55}{9}\zeta_3 - \frac{67}{6}\zeta_2 - 2\zeta_2^2\right\} + n_f C_A\left\{-\frac{164}{81} + \frac{10}{9}\zeta_3 + \frac{5}{3}\zeta_2\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{-\frac{404}{27} + 14\zeta_3 + \frac{22}{3}\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A\left\{\frac{56}{27} - \frac{4}{3}\zeta_2\right\}\\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{+\frac{67}{9} - 2\zeta_2\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A\left\{-\frac{10}{9}\right\}\\
&\left. + \log^3\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{-\frac{11}{9}\right\} + \log^3\left(\frac{q^2}{\mu_F^2}\right)n_f C_A\left\{\frac{2}{9}\right\}\right]\\
&+ \delta(1-z_2)\mathcal{D}_0\left[\frac{1}{\epsilon^2}C_A^2\left\{\frac{44}{3}\right\} + \frac{1}{\epsilon^2}n_f C_A\left\{-\frac{8}{3}\right\} + \frac{1}{\epsilon}C_A^2\left\{\frac{134}{9} - 4\zeta_2\right\}\right.\\
&+ \frac{1}{\epsilon}n_f C_A\left\{-\frac{20}{9}\right\} + C_A^2\left\{-\frac{404}{27} + 14\zeta_3 + \frac{22}{3}\zeta_2\right\} + n_f C_A\left\{\frac{56}{27} - \frac{4}{3}\zeta_2\right\}\\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{\frac{134}{9} - 4\zeta_2\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A\left\{-\frac{20}{9}\right\} + \log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{-\frac{11}{3}\right\}\\
&\left. + \log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A\left\{\frac{2}{3}\right\}\right]\\
&+ \mathcal{D}_0\overline{\mathcal{D}}_0\left[C_A^2\left\{\frac{134}{9} - 4\zeta_2\right\} + n_f C_A\left\{-\frac{20}{9}\right\} + \log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{-\frac{22}{3}\right\}\right.\\
&\left. + \log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A\left\{\frac{4}{3}\right\}\right] + \mathcal{D}_0\overline{\mathcal{D}}_1\left[C_A^2\left\{-\frac{22}{3}\right\} + n_f C_A\left\{\frac{4}{3}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathcal{D}_1\delta(1-z_2)\left[C_A^2\left\{\frac{134}{9}-4\zeta_2\right\}+n_fC_A\left\{-\frac{20}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{-\frac{22}{3}\right\}\right.\\
&\left.+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_A\left\{\frac{4}{3}\right\}\right]+\mathcal{D}_1\overline{\mathcal{D}}_0\left[C_A^2\left\{-\frac{22}{3}\right\}+n_fC_A\left\{+\frac{4}{3}\right\}\right]\\
&+ \mathcal{D}_2\delta(1-z_2)\left[C_A^2\left\{-\frac{11}{3}\right\}+n_fC_A\left\{\frac{2}{3}\right\}\right]\\
&+ \overline{\mathcal{D}}_0\delta(1-z_1)\left[\frac{1}{\epsilon^2}C_A^2\left\{\frac{44}{3}\right\}+\frac{1}{\epsilon^2}n_fC_A\left\{-\frac{8}{3}\right\}+\frac{1}{\epsilon}C_A^2\left\{\frac{134}{9}-4\zeta_2\right\}\right.\\
&+\frac{1}{\epsilon}n_fC_A\left\{-\frac{20}{9}\right\}+C_A^2\left\{-\frac{404}{27}+14\zeta_3+\frac{22}{3}\zeta_2\right\}+n_fC_A\left\{\frac{56}{27}-\frac{4}{3}\zeta_2\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{\frac{134}{9}-4\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_A\left\{-\frac{20}{9}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{-\frac{11}{3}\right\}\\
&\left.+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_A\left\{\frac{2}{3}\right\}\right]+\overline{\mathcal{D}}_1\delta(1-z_1)\left[C_A^2\left\{\frac{134}{9}-4\zeta_2\right\}+n_fC_A\left\{-\frac{20}{9}\right\}\right.\\
&\left.+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^2\left\{-\frac{22}{3}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_A\left\{\frac{4}{3}\right\}\right]\\
&+ \overline{\mathcal{D}}_2\delta(1-z_1)\left[C_A^2\left\{-\frac{11}{3}\right\}+n_fC_A\left\{\frac{2}{3}\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
\Phi^H_{Y,gg,3} &= \delta(1-z_1)\delta(1-z_2)\left[\frac{1}{\epsilon^4}C_A^3\left\{\frac{21296}{81}\right\}+\frac{1}{\epsilon^4}n_fC_A^2\left\{-\frac{7744}{81}\right\}+\frac{1}{\epsilon^4}n_f^2C_A\left\{\frac{704}{81}\right\}\right.\\
&+\frac{1}{\epsilon^3}C_A^3\left\{\frac{49064}{243}-\frac{880}{27}\zeta_2\right\}+\frac{1}{\epsilon^3}n_fC_A^2\left\{-\frac{15520}{243}+\frac{160}{27}\zeta_2\right\}\\
&+\frac{1}{\epsilon^3}n_fC_FC_A\left\{-\frac{128}{9}\right\}+\frac{1}{\epsilon^3}n_f^2C_A\left\{\frac{800}{243}\right\}+\frac{1}{\epsilon^3}\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{1936}{27}\right\}\\
&+\frac{1}{\epsilon^3}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_A^2\left\{-\frac{704}{27}\right\}+\frac{1}{\epsilon^3}\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_A\left\{\frac{64}{27}\right\}\\
&+\frac{1}{\epsilon^2}C_A^3\left\{-\frac{8956}{243}+\frac{2024}{27}\zeta_3-\frac{692}{81}\zeta_2+\frac{352}{45}\zeta_2^2\right\}+\frac{1}{\epsilon^2}n_fC_A^2\left\{\frac{4024}{243}-\frac{560}{27}\zeta_3\right.\\
&\left.-\frac{208}{81}\zeta_2\right\}+\frac{1}{\epsilon^2}n_fC_FC_A\left\{-\frac{220}{27}+\frac{64}{9}\zeta_3\right\}+\frac{1}{\epsilon^2}n_f^2C_A\left\{-\frac{160}{81}+\frac{16}{27}\zeta_2\right\}\\
&+\frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{8344}{81}-\frac{176}{9}\zeta_2\right\}+\frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_A^2\left\{-\frac{2672}{81}+\frac{32}{9}\zeta_2\right\}\\
&+\frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{16}{3}\right\}+\frac{1}{\epsilon^2}\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_A\left\{\frac{160}{81}\right\}\\
&+\frac{1}{\epsilon}C_A^3\left\{-\frac{136781}{2187}-64\zeta_5+\frac{1316}{9}\zeta_3+\frac{12650}{243}\zeta_2-\frac{176}{9}\zeta_2\zeta_3-\frac{352}{15}\zeta_2^2\right\}\\
&+\frac{1}{\epsilon}n_fC_A^2\left\{\frac{11842}{2187}-\frac{728}{81}\zeta_3-\frac{2828}{243}\zeta_2+\frac{32}{5}\zeta_2^2\right\}+\frac{1}{\epsilon}n_fC_FC_A\left\{\frac{1711}{81}-\frac{304}{27}\zeta_3\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4}{3}\zeta_2 - \frac{32}{15}\zeta_2^2\Big\} + \frac{1}{\epsilon} n_f^2 C_A \Big\{\frac{2080}{2187} - \frac{112}{81}\zeta_3 + \frac{40}{81}\zeta_2\Big\} + \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right) C_A^3 \Big\{\frac{490}{9} \\
&+ \frac{88}{9}\zeta_3 - \frac{1072}{27}\zeta_2 + \frac{176}{15}\zeta_2^2\Big\} + \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2 \Big\{ - \frac{836}{81} - \frac{112}{9}\zeta_3 + \frac{160}{27}\zeta_2\Big\} \\
&+ \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{ - \frac{110}{9} + \frac{32}{3}\zeta_3\Big\} + \frac{1}{\epsilon}\log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A \Big\{ - \frac{16}{81}\Big\} \\
&+ C_A^3 \Big\{\frac{5211949}{26244} - \frac{484}{9}\zeta_5 - \frac{64886}{243}\zeta_3 + \frac{536}{9}\zeta_3^2 - \frac{299425}{1458}\zeta_2 + \frac{286}{3}\zeta_2\zeta_3 + \frac{691}{135}\zeta_2^2 \\
&+ \frac{17392}{945}\zeta_2^3\Big\} + n_f C_A^2 \Big\{ - \frac{412765}{13122} - \frac{8}{3}\zeta_5 + \frac{2920}{81}\zeta_3 + \frac{38237}{729}\zeta_2 - 4\zeta_2\zeta_3 \\
&- \frac{1064}{135}\zeta_2^2\Big\} + n_f C_F C_A \Big\{ - \frac{42727}{972} + \frac{112}{9}\zeta_5 + \frac{1636}{81}\zeta_3 + \frac{275}{18}\zeta_2 - \frac{40}{3}\zeta_2\zeta_3 \\
&+ \frac{152}{45}\zeta_2^2\Big\} + n_f^2 C_A \Big\{ - \frac{128}{6561} - \frac{40}{243}\zeta_3 - \frac{68}{27}\zeta_2 + \frac{124}{135}\zeta_2^2\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) C_A^3 \Big\{ - \frac{297029}{1458} - 96\zeta_5 + \frac{7132}{27}\zeta_3 + \frac{13876}{81}\zeta_2 - \frac{88}{3}\zeta_2\zeta_3 - \frac{308}{15}\zeta_2^2\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2 \Big\{\frac{31313}{729} - \frac{268}{9}\zeta_3 - \frac{3880}{81}\zeta_2 + \frac{104}{15}\zeta_2^2\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{\frac{1711}{54} - \frac{152}{9}\zeta_3 - 4\zeta_2 - \frac{16}{5}\zeta_2^2\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A \Big\{ - \frac{928}{729} \\
&- \frac{16}{27}\zeta_3 + \frac{80}{27}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) C_A^3 \Big\{\frac{15503}{162} - 44\zeta_3 - \frac{170}{3}\zeta_2 + \frac{44}{5}\zeta_2^2\Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2 \Big\{ - \frac{2051}{81} + \frac{128}{9}\zeta_2\Big\} + \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{ - \frac{55}{6} + 8\zeta_3\Big\} \\
&+ \log^2\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A \Big\{\frac{100}{81} - \frac{8}{9}\zeta_2\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) C_A^3 \Big\{ - \frac{1780}{81} + \frac{44}{9}\zeta_2\Big\} \\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_A^2 \Big\{\frac{578}{81} - \frac{8}{9}\zeta_2\Big\} + \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f C_F C_A \Big\{\frac{2}{3}\Big\} \\
&+ \log^3\left(\frac{q^2}{\mu_F^2}\right) n_f^2 C_A \Big\{ - \frac{40}{81}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^4 C_A^3 \Big\{\frac{121}{54}\Big\} + \log\left(\frac{q^2}{\mu_F^2}\right)^4 n_f C_A^2 \Big\{ - \frac{22}{27}\Big\} \\
&+ \log\left(\frac{q^2}{\mu_F^2}\right)^4 n_f^2 C_A \Big\{\frac{2}{27}\Big\}\Big] \\
&+ \mathcal{D}_0\delta(1-z_2)\Big[\frac{1}{\epsilon^3} C_A^3 \Big\{\frac{1936}{27}\Big\} + \frac{1}{\epsilon^3} n_f C_A^2 \Big\{ - \frac{704}{27}\Big\} + \frac{1}{\epsilon^3} n_f^2 C_A \Big\{\frac{64}{27}\Big\} \\
&+ \frac{1}{\epsilon^2} C_A^3 \Big\{\frac{8344}{81} - \frac{176}{9}\zeta_2\Big\} + \frac{1}{\epsilon^2} n_f C_A^2 \Big\{ - \frac{2672}{81} + \frac{32}{9}\zeta_2\Big\} + \frac{1}{\epsilon^2} n_f C_F C_A \Big\{ - \frac{16}{3}\Big\} \\
&+ \frac{1}{\epsilon^2} n_f^2 C_A \Big\{\frac{160}{81}\Big\} + \frac{1}{\epsilon} C_A^3 \Big\{\frac{490}{9} + \frac{88}{9}\zeta_3 - \frac{1072}{27}\zeta_2 + \frac{176}{15}\zeta_2^2\Big\}
\end{aligned}
$$

$$
+\frac{1}{\epsilon}n_f C_A^2\left\{-\frac{836}{81}-\frac{112}{9}\zeta_3+\frac{160}{27}\zeta_2\right\}+\frac{1}{\epsilon}n_f C_F C_A\left\{-\frac{110}{9}+\frac{32}{3}\zeta_3\right\}
$$

$$
+\frac{1}{\epsilon}n_f^2 C_A\left\{-\frac{16}{81}\right\}+C_A^3\left\{-\frac{297029}{1458}-96\zeta_5+\frac{7132}{27}\zeta_3+\frac{13876}{81}\zeta_2-\frac{88}{3}\zeta_2\zeta_3\right.
$$

$$
\left.-\frac{308}{15}\zeta_2^2\right\}+n_f C_A^2\left\{\frac{31313}{729}-\frac{268}{9}\zeta_3-\frac{3880}{81}\zeta_2+\frac{104}{15}\zeta_2^2\right\}+n_f C_F C_A\left\{\frac{1711}{54}\right.
$$

$$
\left.-\frac{152}{9}\zeta_3-4\zeta_2-\frac{16}{5}\zeta_2^2\right\}+n_f^2 C_A\left\{-\frac{928}{729}-\frac{16}{27}\zeta_3+\frac{80}{27}\zeta_2\right\}
$$

$$
+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{15503}{81}-88\zeta_3-\frac{340}{3}\zeta_2+\frac{88}{5}\zeta_2^2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{4102}{81}\right.
$$

$$
\left.+\frac{256}{9}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{-\frac{55}{3}+16\zeta_3\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{200}{81}-\frac{16}{9}\zeta_2\right\}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{1780}{27}+\frac{44}{3}\zeta_2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{578}{27}-\frac{8}{3}\zeta_2\right\}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{-\frac{40}{27}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{242}{27}\right\}
$$

$$
+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{88}{27}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{8}{27}\right\}\Bigg]
$$

$$
+\mathcal{D}_0\overline{\mathcal{D}}_0\Bigg[C_A^3\left\{+\frac{15503}{81}-88\zeta_3-\frac{340}{3}\zeta_2+\frac{88}{5}\zeta_2^2\right\}+n_f C_A^2\left\{-\frac{4102}{81}+\frac{256}{9}\zeta_2\right\}
$$

$$
+n_f C_F C_A\left\{-\frac{55}{3}+16\zeta_3\right\}+n_f^2 C_A\left\{\frac{200}{81}-\frac{16}{9}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{3560}{27}\right.
$$

$$
\left.+\frac{88}{3}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{1156}{27}-\frac{16}{3}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{4\right\}
$$

$$
+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{-\frac{80}{27}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{242}{9}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{88}{9}\right\}
$$

$$
+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{8}{9}\right\}\Bigg]+\mathcal{D}_0\overline{\mathcal{D}}_1\Bigg[C_A^3\left\{-\frac{3560}{27}+\frac{88}{3}\zeta_2\right\}+n_f C_A^2\left\{\frac{1156}{27}-\frac{16}{3}\zeta_2\right\}
$$

$$
+n_f C_F C_A\left\{4\right\}+n_f^2 C_A\left\{-\frac{80}{27}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{484}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{176}{9}\right\}
$$

$$
+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{16}{9}\right\}\Bigg]+\mathcal{D}_0\overline{\mathcal{D}}_2\Bigg[C_A^3\left\{\frac{242}{9}\right\}+n_f C_A^2\left\{-\frac{88}{9}\right\}+n_f^2 C_A\left\{\frac{8}{9}\right\}\Bigg]
$$

$$
+\mathcal{D}_1\delta(1-z_2)\Bigg[C_A^3\left\{\frac{15503}{81}-88\zeta_3-\frac{340}{3}\zeta_2+\frac{88}{5}\zeta_2^2\right\}+n_f C_A^2\left\{-\frac{4102}{81}+\frac{256}{9}\zeta_2\right\}
$$

$$
+n_f C_F C_A\left\{-\frac{55}{3}+16\zeta_3\right\}+n_f^2 C_A\left\{\frac{200}{81}-\frac{16}{9}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{3560}{27}\right.
$$

$$
\left.+\frac{88}{3}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{1156}{27}-\frac{16}{3}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{4\right\}
$$

$$
\begin{aligned}
&+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_A\left\{-\frac{80}{27}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{242}{9}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_A^2\left\{-\frac{88}{9}\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_A\left\{+\frac{8}{9}\right\}\Bigg]+\mathcal{D}_1\overline{\mathcal{D}}_0\Bigg[C_A^3\left\{-\frac{3560}{27}+\frac{88}{3}\zeta_2\right\}+n_fC_A^2\left\{\frac{1156}{27}\right.\\
&\left.-\frac{16}{3}\zeta_2\right\}+n_fC_FC_A\left\{4\right\}+n_f^2C_A\left\{-\frac{80}{27}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{484}{9}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_A^2\left\{-\frac{176}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_A\left\{\frac{16}{9}\right\}\Bigg]+\mathcal{D}_1\overline{\mathcal{D}}_1\Bigg[C_A^3\left\{\frac{484}{9}\right\}\\
&+n_fC_A^2\left\{-\frac{176}{9}\right\}+n_f^2C_A\left\{\frac{16}{9}\right\}\Bigg]+\mathcal{D}_2\delta(1-z_2)\Bigg[C_A^3\left\{-\frac{1780}{27}+\frac{44}{3}\zeta_2\right\}\\
&+n_fC_A^2\left\{\frac{578}{27}-\frac{8}{3}\zeta_2\right\}+n_fC_FC_A\left\{2\right\}+n_f^2C_A\left\{-\frac{40}{27}\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{242}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_A^2\left\{-\frac{88}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_A\left\{\frac{8}{9}\right\}\Bigg]\\
&+\mathcal{D}_2\overline{\mathcal{D}}_0\Bigg[C_A^3\left\{\frac{242}{9}\right\}+n_fC_A^2\left\{-\frac{88}{9}\right\}+n_f^2C_A\left\{\frac{8}{9}\right\}\Bigg]\\
&+\mathcal{D}_3\delta(1-z_2)\Bigg[C_A^3\left\{\frac{242}{27}\right\}+n_fC_A^2\left\{-\frac{88}{27}\right\}+n_f^2C_A\left\{\frac{8}{27}\right\}\Bigg]\\
&+\overline{\mathcal{D}}_0\delta(1-z_1)\Bigg[\frac{1}{\epsilon^3}C_A^3\left\{\frac{1936}{27}\right\}+\frac{1}{\epsilon^3}n_fC_A^2\left\{-\frac{704}{27}\right\}+\frac{1}{\epsilon^3}n_f^2C_A\left\{\frac{64}{27}\right\}\\
&+\frac{1}{\epsilon^2}C_A^3\left\{\frac{8344}{81}-\frac{176}{9}\zeta_2\right\}+\frac{1}{\epsilon^2}n_fC_A^2\left\{-\frac{2672}{81}+\frac{32}{9}\zeta_2\right\}+\frac{1}{\epsilon^2}n_fC_FC_A\left\{-\frac{16}{3}\right\}\\
&+\frac{1}{\epsilon^2}n_f^2C_A\left\{\frac{160}{81}\right\}+\frac{1}{\epsilon}C_A^3\left\{\frac{490}{9}+\frac{88}{9}\zeta_3-\frac{1072}{27}\zeta_2+\frac{176}{15}\zeta_2^2\right\}\\
&+\frac{1}{\epsilon}n_fC_A^2\left\{-\frac{836}{81}-\frac{112}{9}\zeta_3+\frac{160}{27}\zeta_2\right\}+\frac{1}{\epsilon}n_fC_FC_A\left\{-\frac{110}{9}+\frac{32}{3}\zeta_3\right\}\\
&+\frac{1}{\epsilon}n_f^2C_A\left\{-\frac{16}{81}\right\}+C_A^3\left\{-\frac{297029}{1458}-96\zeta_5+\frac{7132}{27}\zeta_3+\frac{13876}{81}\zeta_2-\frac{88}{3}\zeta_2\zeta_3\right.\\
&\left.-\frac{308}{15}\zeta_2^2\right\}+n_fC_A^2\left\{\frac{31313}{729}-\frac{268}{9}\zeta_3-\frac{3880}{81}\zeta_2+\frac{104}{15}\zeta_2^2\right\}+n_fC_FC_A\left\{\frac{1711}{54}\right.\\
&\left.-\frac{152}{9}\zeta_3-4\zeta_2-\frac{16}{5}\zeta_2^2\right\}+n_f^2C_A\left\{-\frac{928}{729}-\frac{16}{27}\zeta_3+\frac{80}{27}\zeta_2\right\}\\
&+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{15503}{81}-88\zeta_3-\frac{340}{3}\zeta_2+\frac{88}{5}\zeta_2^2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_A^2\left\{-\frac{4102}{81}\right.\\
&\left.+\frac{256}{9}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_fC_FC_A\left\{-\frac{55}{3}+16\zeta_3\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2C_A\left\{\frac{200}{81}-\frac{16}{9}\zeta_2\right\}\\
&+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{1780}{27}+\frac{44}{3}\zeta_2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_fC_A^2\left\{\frac{578}{27}-\frac{8}{3}\zeta_2\right\}
\end{aligned}
$$

$$+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{2\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{-\frac{40}{27}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{242}{27}\right\}$$
$$+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{88}{27}\right\}+\log^3\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{8}{27}\right\}\Bigg]$$
$$+\overline{\mathcal{D}}_1\delta(1-z_1)\Bigg[C_A^3\left\{\frac{15503}{81}-88\zeta_3-\frac{340}{3}\zeta_2+\frac{88}{5}\zeta_2^2\right\}+n_f C_A^2\left\{-\frac{4102}{81}+\frac{256}{9}\zeta_2\right\}$$
$$+n_f C_F C_A\left\{-\frac{55}{3}+16\zeta_3\right\}+n_f^2 C_A\left\{\frac{200}{81}-\frac{16}{9}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{-\frac{3560}{27}$$
$$+\frac{88}{3}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{\frac{1156}{27}-\frac{16}{3}\zeta_2\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_F C_A\left\{4\right\}$$
$$+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{-\frac{80}{27}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{242}{9}\right\}+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{88}{9}\right\}$$
$$+\log^2\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{8}{9}\right\}\Bigg]+\overline{\mathcal{D}}_2\delta(1-z_1)\Bigg[C_A^3\left\{-\frac{1780}{27}+\frac{44}{3}\zeta_2\right\}+n_f C_A^2\left\{\frac{578}{27}$$
$$-\frac{8}{3}\zeta_2\right\}+n_f C_F C_A\left\{2\right\}+n_f^2 C_A\left\{-\frac{40}{27}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)C_A^3\left\{\frac{242}{9}\right\}$$
$$+\log\left(\frac{q^2}{\mu_F^2}\right)n_f C_A^2\left\{-\frac{88}{9}\right\}+\log\left(\frac{q^2}{\mu_F^2}\right)n_f^2 C_A\left\{\frac{8}{9}\right\}\Bigg]+\overline{\mathcal{D}}_3\delta(1-z_1)\Bigg[C_A^3\left\{\frac{242}{27}\right\}$$
$$+n_f C_A^2\left\{-\frac{88}{27}\right\}+n_f^2 C_A\left\{\frac{8}{27}\right\}\Bigg]. \tag{F.0.28}$$

G Rapidity Distributions at Threshold N^{4}LO QCD

In this appendix, we present the general form of the QCD corrections to the differential rapidity distribuions of the Drell-Yan pair or Higgs boson at threshold N^{4}LO. Due to unavailability of the involved quantities at 4-loop level, we are unable to calculate this contributions. The result for the choice of the scales $\mu_R^2 = \mu_F^2 = q^2$ reads:

$$
\begin{aligned}
\Delta^{I,\mathrm{SV}}_{Y,i\bar{i},4} = \delta(1-z_1)\delta(1-z_2)\Bigg\{ & \frac{1}{2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},4} + \frac{1}{2}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\right)^2 + \frac{4}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3} + 2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} + \frac{2}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^4 \\
& + \frac{1}{2}g^{I,1}_{i\bar{i},4} + \frac{4}{3}g^{I,1}_{i\bar{i},3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} + 2g^{I,1}_{i\bar{i},2}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 + \frac{1}{2}\left(g^{I,1}_{i\bar{i},2}\right)^2 + \frac{4}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3} \\
& + 4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} + \frac{8}{3}g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3 + \frac{4}{3}g^{I,1}_{i\bar{i},1}g^{I,1}_{i\bar{i},3} + 4g^{I,1}_{i\bar{i},1}g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + 2\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} \\
& + 4\left(g^{I,1}_{i\bar{i},1}\right)^2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 + 2\left(g^{I,1}_{i\bar{i},1}\right)^2 g^{I,1}_{i\bar{i},2} + \frac{8}{3}\left(g^{I,1}_{i\bar{i},1}\right)^3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + \frac{2}{3}\left(g^{I,1}_{i\bar{i},1}\right)^4 - 30\zeta_7\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} \\
& - 4\zeta_5 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 12\zeta_5 A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 6\zeta_5\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - 12\zeta_5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
& - 12\zeta_5 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 2\zeta_3\left(f^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - 2\zeta_3 A^I_{i\bar{i},3}f^I_{i\bar{i},1} - 2\zeta_3 A^I_{i\bar{i},2}f^I_{i\bar{i},2} \\
& - 2\zeta_3 A^I_{i\bar{i},1}f^I_{i\bar{i},3} - 2\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{4}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 4\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} \\
& - 4\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2} - 4\zeta_3\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 2\zeta_3 g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{4}{3}\zeta_3 g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 \\
& - 4\zeta_3 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 4\zeta_3 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2} - 8\zeta_3 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 4\zeta_3\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} \\
& + 28\zeta_3\zeta_5\left(A^I_{i\bar{i},1}\right)^4 + 10\zeta_3^2\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + 10\zeta_3^2\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + \frac{20}{3}\zeta_3^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{20}{3}\zeta_3^2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3-\zeta_2\left(f^I_{i\bar{i},2}\right)^2-2\zeta_2 f^I_{i\bar{i},1}f^I_{i\bar{i},3}+3\zeta_2 A^I_{i\bar{i},4}+2\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},3}A^I_{i\bar{i},1}\\
&-\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2+3\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},2}-4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2}+6\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},3}\\
&+6\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}-2\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2+6\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},2}+4\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3A^I_{i\bar{i},1}\\
&+2\zeta_2 g^{I,1}_{i\bar{i},3}A^I_{i\bar{i},1}-\zeta_2 g^{I,1}_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2+3\zeta_2 g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},2}+6\zeta_2 g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-4\zeta_2 g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2}\\
&+6\zeta_2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},3}+6\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}-4\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+12\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}\\
&+12\zeta_2 g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}+6\zeta_2 g^{I,1}_{i\bar{i},1}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}-2\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2+6\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},2}\\
&+12\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+4\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^3A^I_{i\bar{i},1}+18\zeta_2\zeta_5\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}+4\zeta_2\zeta_3A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3\\
&+4\zeta_2\zeta_3A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}+2\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},2}+4\zeta_2\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\\
&+4\zeta_2\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}-10\zeta_2\zeta_3^2\left(A^I_{i\bar{i},1}\right)^4+\frac{3}{10}\zeta_2^2\left(f^I_{i\bar{i},1}\right)^4-\frac{11}{5}\zeta_2^2A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2\\
&+\frac{49}{10}\zeta_2^2\left(A^I_{i\bar{i},2}\right)^2-\frac{22}{5}\zeta_2^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2}+\frac{49}{5}\zeta_2^2A^I_{i\bar{i},1}A^I_{i\bar{i},3}+\frac{49}{10}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2\\
&-\frac{22}{5}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+\frac{98}{5}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2}+\frac{49}{5}\zeta_2^2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2\\
&+\frac{49}{10}\zeta_2^2 g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2-\frac{22}{5}\zeta_2^2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+\frac{98}{5}\zeta_2^2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2}\\
&+\frac{98}{5}\zeta_2^2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+\frac{49}{5}\zeta_2^2\left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2+3\zeta_2^2\zeta_3\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}\\
&-\frac{191}{70}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2+\frac{1181}{70}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2A^I_{i\bar{i},2}+\frac{1181}{105}\zeta_2^3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3\\
&+\frac{1181}{105}\zeta_2^3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3+\frac{1401}{280}\zeta_2^4\left(A^I_{i\bar{i},1}\right)^4\Bigg\}\\
&+\delta(1-z_1)\delta(1-z_2)C_A\Bigg\{\frac{11}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},3}+\frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{88}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}+\frac{44}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\\
&+\frac{11}{3}g^{I,2}_{i\bar{i},3}+\frac{88}{9}g^{I,2}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{22}{3}g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{22}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{44}{3}g^{I,2}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{22}{3}g^{I,2}_{i\bar{i},1}g^{I,1}_{i\bar{i},2}\\
&+\frac{88}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}+\frac{88}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{88}{9}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},2}+\frac{88}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{44}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\\
&+\frac{44}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2g^{I,2}_{i\bar{i},1}-110\zeta_7\left(A^I_{i\bar{i},1}\right)^3-\frac{176}{3}\zeta_5A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-44\zeta_5A^I_{i\bar{i},1}A^I_{i\bar{i},2}\\
&-\frac{220}{3}\zeta_5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-\frac{88}{3}\zeta_5 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-22\zeta_3f^I_{i\bar{i},1}f^I_{i\bar{i},2}-\frac{44}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}\\
&-\frac{44}{3}\zeta_3\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{88}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-\frac{44}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}-\frac{88}{3}\zeta_3\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}\\
&-\frac{44}{3}\zeta_3 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{44}{3}\zeta_3 g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-\frac{88}{3}\zeta_3 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{242}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\\
&+33\zeta_2f^I_{i\bar{i},3}+66\zeta_2B^I_{i\bar{i},3}+\frac{44}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1}-\frac{11}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}+22\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}B^I_{i\bar{i},1}
\end{aligned}
$$

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\begin{aligned}
&- \frac{22}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 22\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{88}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2} + 88\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},2} \\
&+ 44\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{22}{3}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 44\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + \frac{44}{3}\zeta_2 g^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1} \\
&+ 11\zeta_2 g^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1} + 22\zeta_2 g^{I,1}_{i\bar{i},2}B^I_{i\bar{i},1} - \frac{22}{3}\zeta_2 g^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 22\zeta_2 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2} + 44\zeta_2 g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&+ 44\zeta_2 g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2} + 88\zeta_2 g^{I,1}_{i\bar{i},1}B^I_{i\bar{i},2} + 44\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{44}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} \\
&+ 88\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1} + 44\zeta_2 g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + 22\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 44\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} \\
&- 66\zeta_2\gamma^I_{i\bar{i},3} - 88\zeta_2\gamma^I_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - 88\zeta_2\gamma^I_{i\bar{i},2}g^{I,1}_{i\bar{i},1} - 22\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} - 44\zeta_2\gamma^I_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 \\
&- 22\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},2} - 88\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - 44\zeta_2\gamma^I_{i\bar{i},1}\left(g^{I,1}_{i\bar{i},1}\right)^2 + \frac{220}{3}\zeta_2\zeta_5\left(A^I_{i\bar{i},1}\right)^3 \\
&+ \frac{88}{3}\zeta_2\zeta_3 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 44\zeta_2\zeta_3 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} + 44\zeta_2\zeta_3 A^I_{i\bar{i},1}A^I_{i\bar{i},2} + 44\zeta_2\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&+ \frac{88}{3}\zeta_2\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 44\zeta_2\zeta_3\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{121}{15}\zeta_2^2\left(f^I_{i\bar{i},1}\right)^3 - 22\zeta_2^2 B^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 \\
&+ \frac{451}{15}\zeta_2^2 A^I_{i\bar{i},2}f^I_{i\bar{i},1} + 66\zeta_2^2 A^I_{i\bar{i},2}B^I_{i\bar{i},1} + \frac{1034}{15}\zeta_2^2 A^I_{i\bar{i},1}f^I_{i\bar{i},2} + 132\zeta_2^2 A^I_{i\bar{i},1}B^I_{i\bar{i},2} \\
&+ \frac{539}{15}\zeta_2^2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{506}{15}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 132\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1} \\
&+ \frac{539}{15}\zeta_2^2 g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 66\zeta_2^2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 132\zeta_2^2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1} - 132\zeta_2^2\gamma^I_{i\bar{i},2}A^I_{i\bar{i},1} \\
&+ 22\zeta_2^2\gamma^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 66\zeta_2^2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},2} - 132\zeta_2^2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - 132\zeta_2^2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&+ \frac{308}{15}\zeta_2^2\zeta_3\left(A^I_{i\bar{i},1}\right)^3 + \frac{539}{10}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{539}{5}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} - \frac{539}{5}\zeta_2^3\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Bigg\} \\
&+ \delta(1-z_1)\delta(1-z_2)C_A^2\Bigg\{ + \frac{242}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},2} + \frac{34}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2} + \frac{242}{9}\left(\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\right)^2 + \frac{1936}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} \\
&+ \frac{272}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + \frac{242}{9}g^{I,3}_{i\bar{i},2} + \frac{34}{3}g^{I,2}_{i\bar{i},2} + \frac{1936}{27}g^{I,3}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + \frac{484}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} \\
&+ \frac{272}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + \frac{242}{9}\left(g^{I,2}_{i\bar{i},1}\right)^2 + \frac{1936}{27}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} \\
&+ \frac{272}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + \frac{1936}{27}g^{I,1}_{i\bar{i},1}g^{I,3}_{i\bar{i},1} + \frac{272}{9}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1} - \frac{1936}{9}\zeta_5 A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{136}{3}\zeta_5\left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{68}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^2 - \frac{968}{9}\zeta_3\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{484}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{136}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&+ \frac{242}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^2 + 68\zeta_2 f^I_{i\bar{i},2} + 136\zeta_2 B^I_{i\bar{i},2} + \frac{968}{9}\zeta_2\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{242}{9}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1} \\
&+ \frac{484}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}B^I_{i\bar{i},1} + \frac{136}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{68}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + 136\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1} \\
&- \frac{484}{9}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 - 242\zeta_2 g^{I,1}_{i\bar{i},2} + \frac{968}{9}\zeta_2 g^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{242}{3}\zeta_2 g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{484}{3}\zeta_2 g^{I,2}_{i\bar{i},1}B^I_{i\bar{i},1} \\
&+ \frac{136}{3}\zeta_2 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + 68\zeta_2 g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + 136\zeta_2 g^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1} - \frac{968}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}
\end{aligned}
$$

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\begin{aligned}
&- \frac{968}{3}\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2 - 136\zeta_2\gamma^I_{i\bar{i},2} - \frac{484}{3}\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} - 136\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{484}{3}\zeta_2\gamma^I_{i\bar{i},1}g^{I,2}_{i\bar{i},1}\\
&- 136\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1} + \frac{484}{3}\zeta_2\zeta_3 A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{136}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2 + \frac{1331}{30}\zeta_2^2\left(f^I_{i\bar{i},1}\right)^2\\
&+ 242\zeta_2^2 B^I_{i\bar{i},1}f^I_{i\bar{i},1} + 242\zeta_2^2\left(B^I_{i\bar{i},1}\right)^2 - 121\zeta_2^2 A^I_{i\bar{i},2} + 102\zeta_2^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 204\zeta_2^2 A^I_{i\bar{i},1}B^I_{i\bar{i},1}\\
&- \frac{2662}{45}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{1694}{3}\zeta_2^2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - 242\zeta_2^2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1} - 484\zeta_2^2\gamma^I_{i\bar{i},1}B^I_{i\bar{i},1}\\
&- 204\zeta_2^2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1} + 242\zeta_2^2\left(\gamma I_{i\bar{i},1}\right)^2 - \frac{11737}{105}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2\Bigg\}\\
&+ \delta(1-z_1)\delta(1-z_2)C_A^3\Bigg\{\frac{5324}{27}\overline{\mathcal{G}}^{I,4}_{i\bar{i},1} + \frac{1496}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} + \frac{2857}{54}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + \frac{5324}{27}g^{I,4}_{i\bar{i},1}\\
&+ \frac{1496}{9}g^{I,3}_{i\bar{i},1} + \frac{2857}{54}g^{I,2}_{i\bar{i},1} + \frac{2857}{18}\zeta_2 f^I_{i\bar{i},1} + \frac{2857}{9}\zeta_2 B^I_{i\bar{i},1} - \frac{5324}{3}\zeta_2 g^{I,2}_{i\bar{i},1} - \frac{3740}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\\
&- \frac{2857}{9}\zeta_2\gamma^I_{i\bar{i},1} - \frac{1331}{3}\zeta_2^2 f^I_{i\bar{i},1} - \frac{2662}{3}\zeta_2^2 B^I_{i\bar{i},1} - \frac{935}{3}\zeta_2^2 A^I_{i\bar{i},1} + \frac{2662}{3}\zeta_2^2\gamma^I_{i\bar{i},1}\Bigg\}\\
&+ \delta(1-z_1)\delta(1-z_2)n_f\Bigg\{ - \frac{2}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},3} - \frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} - \frac{16}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2} - \frac{8}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\\
&- \frac{2}{3}g^{I,2}_{i\bar{i},3} - \frac{16}{9}g^{I,2}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{4}{3}g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} - \frac{4}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} - \frac{8}{3}g^{I,2}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 - \frac{4}{3}g^{I,2}_{i\bar{i},1}g^{I,1}_{i\bar{i},2}\\
&- \frac{16}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2} - \frac{16}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} - \frac{16}{9}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},2} - \frac{16}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{8}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\\
&- \frac{8}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 g^{I,2}_{i\bar{i},1} + 20\zeta_7\left(A^I_{i\bar{i},1}\right)^3 + \frac{32}{3}\zeta_5 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 8\zeta_5 A^I_{i\bar{i},1}A^I_{i\bar{i},2}\\
&+ \frac{40}{3}\zeta_5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{16}{3}\zeta_5 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 4\zeta_3 f^I_{i\bar{i},1}f^I_{i\bar{i},2} + \frac{8}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}\\
&+ \frac{8}{3}\zeta_3\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{16}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{8}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{16}{3}\zeta_3\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}\\
&+ \frac{8}{3}\zeta_3 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{8}{3}\zeta_3 g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{16}{3}\zeta_3 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{44}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\\
&- 6\zeta_2 f^I_{i\bar{i},3} - 12\zeta_2 B^I_{i\bar{i},3} - \frac{8}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1} + \frac{2}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1} - 4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}B^I_{i\bar{i},1}\\
&+ \frac{4}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 4\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{16}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2} - 16\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},2} - 8\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+ \frac{4}{3}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 8\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} - \frac{8}{3}\zeta_2 g^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1} - 2\zeta_2 g^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1} - 4\zeta_2 g^{I,1}_{i\bar{i},2}B^I_{i\bar{i},1}\\
&+ \frac{4}{3}\zeta_2 g^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 4\zeta_2 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2} - 8\zeta_2 g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - 8\zeta_2 g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2} - 16\zeta_2 g^{I,1}_{i\bar{i},1}B^I_{i\bar{i},2}\\
&- 8\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{8}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - 16\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1} - 8\zeta_2 g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&- 4\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 8\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + 12\zeta_2\gamma^I_{i\bar{i},3} + 16\zeta_2\gamma^I_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + 16\zeta_2\gamma^I_{i\bar{i},2}g^{I,1}_{i\bar{i},1}\\
&+ 4\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} + 8\zeta_2\gamma^I_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 + 4\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},2} + 16\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + 8\zeta_2\gamma^I_{i\bar{i},1}\left(g^{I,1}_{i\bar{i},1}\right)^2
\end{aligned}
$$

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\begin{aligned}
&- \frac{40}{3}\zeta_2\zeta_5\left(A^I_{i\bar{i},1}\right)^3 - \frac{16}{3}\zeta_2\zeta_3 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 8\zeta_2\zeta_3 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} - 8\zeta_2\zeta_3 A^I_{i\bar{i},1}A^I_{i\bar{i},2} \\
&- 8\zeta_2\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{16}{3}\zeta_2\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - 8\zeta_2\zeta_3\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{22}{15}\zeta_2^2\left(f^I_{i\bar{i},1}\right)^3 \\
&+ 4\zeta_2^2 B^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{82}{15}\zeta_2^2 A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 12\zeta_2^2 A^I_{i\bar{i},2}B^I_{i\bar{i},1} - \frac{188}{15}\zeta_2^2 A^I_{i\bar{i},1}f^I_{i\bar{i},2} - 24\zeta_2^2 A^I_{i\bar{i},1}B^I_{i\bar{i},2} \\
&- \frac{98}{15}\zeta_2^2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{92}{15}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 24\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1} - \frac{98}{15}\zeta_2^2 g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&- 12\zeta_2^2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 24\zeta_2^2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1} + 24\zeta_2^2\gamma^I_{i\bar{i},2}A^I_{i\bar{i},1} - 4\zeta_2^2\gamma^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 \\
&+ 12\zeta_2^2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},2} + 24\zeta_2^2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + 24\zeta_2^2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{56}{15}\zeta_2^2\zeta_3\left(A^I_{i\bar{i},1}\right)^3 \\
&- \frac{49}{5}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - \frac{98}{5}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + \frac{98}{5}\zeta_2^3\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Bigg\} \\
&+ \delta(1-z_1)\delta(1-z_2)n_f C_A\Bigg\{ - \frac{88}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},2} - \frac{10}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2} - \frac{88}{9}\left(\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\right)^2 - \frac{704}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} \\
&- \frac{80}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} - \frac{88}{9}g^{I,3}_{i\bar{i},2} - \frac{10}{3}g^{I,2}_{i\bar{i},2} - \frac{704}{27}g^{I,3}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{176}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} - \frac{80}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \\
&- \frac{88}{9}\left(g^{I,2}_{i\bar{i},1}\right)^2 - \frac{704}{27}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} - \frac{80}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} - \frac{704}{27}g^{I,1}_{i\bar{i},1}g^{I,3}_{i\bar{i},1} - \frac{80}{9}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1} \\
&+ \frac{704}{9}\zeta_5 A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{40}{3}\zeta_5\left(A^I_{i\bar{i},1}\right)^2 + \frac{20}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^2 + \frac{352}{9}\zeta_3\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&+ \frac{176}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{40}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{88}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^2 - 20\zeta_2 f^I_{i\bar{i},2} - 40\zeta_2 B^I_{i\bar{i},2} \\
&- \frac{352}{9}\zeta_2\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{88}{9}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{176}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}B^I_{i\bar{i},1} - \frac{40}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&- \frac{20}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - 40\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1} + \frac{176}{9}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 + 88\zeta_2 g^{I,1}_{i\bar{i},2} - \frac{352}{9}\zeta_2 g^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&- \frac{88}{3}\zeta_2 g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{176}{3}\zeta_2 g^{I,2}_{i\bar{i},1}B^I_{i\bar{i},1} - \frac{40}{3}\zeta_2 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - 20\zeta_2 g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - 40\zeta_2 g^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1} \\
&+ \frac{352}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + \frac{352}{3}\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2 + 40\zeta_2\gamma^I_{i\bar{i},2} + \frac{176}{3}\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + 40\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \\
&+ \frac{176}{3}\zeta_2\gamma^I_{i\bar{i},1}g^{I,2}_{i\bar{i},1} + 40\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1} - \frac{176}{3}\zeta_2\zeta_3 A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{40}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{242}{15}\zeta_2^2\left(f^I_{i\bar{i},1}\right)^2 - 88\zeta_2^2 B^I_{i\bar{i},1}f^I_{i\bar{i},1} - 88\zeta_2^2\left(B^I_{i\bar{i},1}\right)^2 + 44\zeta_2^2 A^I_{i\bar{i},2} - 30\zeta_2^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} \\
&- 60\zeta_2^2 A^I_{i\bar{i},1}B^I_{i\bar{i},1} + \frac{968}{45}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{616}{3}\zeta_2^2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + 88\zeta_2^2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1} + 176\zeta_2^2\gamma^I_{i\bar{i},1}B^I_{i\bar{i},1} \\
&+ 60\zeta_2^2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1} - 88\zeta_2^2\left(\gamma I_{i\bar{i},1}\right)^2 + \frac{4268}{105}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2\Bigg\} \\
&+ \delta(1-z_1)\delta(1-z_2)n_f C_A^2\Bigg\{ - \frac{968}{9}\overline{\mathcal{G}}^{I,4}_{i\bar{i},1} - \frac{712}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} - \frac{1415}{54}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} - \frac{968}{9}g^{I,4}_{i\bar{i},1} \\
&- \frac{712}{9}g^{I,3}_{i\bar{i},1} - \frac{1415}{54}g^{I,2}_{i\bar{i},1} - \frac{1415}{18}\zeta_2 f^I_{i\bar{i},1} - \frac{1415}{9}\zeta_2 B^I_{i\bar{i},1} + 968\zeta_2 g^{I,2}_{i\bar{i},1} + \frac{1780}{3}\zeta_2 g^{I,1}_{i\bar{i},1}
\end{aligned}
$$

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\begin{aligned}
&+\frac{1415}{9}\zeta_2\gamma^I_{i\bar{i},1}+242\zeta_2^2 f^I_{i\bar{i},1}+484\zeta_2^2 B^I_{i\bar{i},1}+\frac{445}{3}\zeta_2^2 A^I_{i\bar{i},1}-484\zeta_2^2\gamma^I_{i\bar{i},1}\Big\}\\
&+\delta(1-z_1)\delta(1-z_2)n_f C_F\Big\{-2\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}-\frac{16}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-2g^{I,2}_{i\bar{i},2}-\frac{16}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-\frac{16}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{16}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}+8\zeta_5\left(A^I_{i\bar{i},1}\right)^2+4\zeta_3\left(f^I_{i\bar{i},1}\right)^2+8\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-12\zeta_2 f^I_{i\bar{i},2}\\
&-24\zeta_2 B^I_{i\bar{i},2}-8\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-24\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}-8\zeta_2 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-12\zeta_2 g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-24\zeta_2 g^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}+24\zeta_2\gamma^I_{i\bar{i},2}+24\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+24\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}\\
&-8\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2-18\zeta_2^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}-36\zeta_2^2A^I_{i\bar{i},1}B^I_{i\bar{i},1}+36\zeta_2^2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Big\}\\
&+\delta(1-z_1)\delta(1-z_2)n_f C_F C_A\Big\{-\frac{88}{3}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}-\frac{205}{18}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{88}{3}g^{I,3}_{i\bar{i},1}-\frac{205}{18}g^{I,2}_{i\bar{i},1}\\
&-\frac{205}{6}\zeta_2 f^I_{i\bar{i},1}-\frac{205}{3}\zeta_2 B^I_{i\bar{i},1}+220\zeta_2 g^{I,1}_{i\bar{i},1}+\frac{205}{3}\zeta_2\gamma^I_{i\bar{i},1}+55\zeta_2^2A^I_{i\bar{i},1}\Big\}\\
&+\delta(1-z_1)\delta(1-z_2)n_f C_F^2\Big\{\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+g^{I,2}_{i\bar{i},1}+3\zeta_2 f^I_{i\bar{i},1}+6\zeta_2 B^I_{i\bar{i},1}-6\zeta_2\gamma^I_{i\bar{i},1}\Big\}\\
&+\delta(1-z_1)\delta(1-z_2)n_f^2\Big\{+\frac{8}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},2}+\frac{8}{9}\left(\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\right)^2+\frac{64}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}\\
&+\frac{8}{9}g^{I,3}_{i\bar{i},2}+\frac{64}{27}g^{I,3}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{16}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{8}{9}\left(g^{I,2}_{i\bar{i},1}\right)^2+\frac{64}{27}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}+\frac{64}{27}g^{I,1}_{i\bar{i},1}g^{I,3}_{i\bar{i},1}\\
&-\frac{64}{9}\zeta_5A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{32}{9}\zeta_3\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{16}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{8}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^2+\frac{32}{9}\zeta_2\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-\frac{8}{9}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{16}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}B^I_{i\bar{i},1}-\frac{16}{9}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2-8\zeta_2 g^{I,1}_{i\bar{i},2}+\frac{32}{9}\zeta_2 g^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+\frac{8}{3}\zeta_2 g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{16}{3}\zeta_2 g^{I,2}_{i\bar{i},1}B^I_{i\bar{i},1}-\frac{32}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{32}{3}\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2-\frac{16}{3}\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\\
&-\frac{16}{3}\zeta_2\gamma^I_{i\bar{i},1}g^{I,2}_{i\bar{i},1}+\frac{16}{3}\zeta_2\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{22}{15}\zeta_2^2\left(f^I_{i\bar{i},1}\right)^2+8\zeta_2^2B^I_{i\bar{i},1}f^I_{i\bar{i},1}+8\zeta_2^2\left(B^I_{i\bar{i},1}\right)^2\\
&-4\zeta_2^2A^I_{i\bar{i},2}-\frac{88}{45}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{56}{3}\zeta_2^2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-8\zeta_2^2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}-16\zeta_2^2\gamma^I_{i\bar{i},1}B^I_{i\bar{i},1}\\
&+8\zeta_2^2\left(\gamma I_{i\bar{i},1}\right)^2-\frac{388}{105}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\delta(1-z_1)\delta(1-z_2)n_f^2C_A\Big\{\frac{176}{9}\overline{\mathcal{G}}^{I,4}_{i\bar{i},1}+\frac{80}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}+\frac{79}{54}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{176}{9}g^{I,4}_{i\bar{i},1}+\frac{80}{9}g^{I,3}_{i\bar{i},1}\\
&+\frac{79}{54}g^{I,2}_{i\bar{i},1}+\frac{79}{18}\zeta_2 f^I_{i\bar{i},1}+\frac{79}{9}\zeta_2 B^I_{i\bar{i},1}-176\zeta_2 g^{I,2}_{i\bar{i},1}-\frac{200}{3}\zeta_2 g^{I,1}_{i\bar{i},1}-\frac{79}{9}\zeta_2\gamma^I_{i\bar{i},1}-44\zeta_2^2f^I_{i\bar{i},1}\\
&-88\zeta_2^2B^I_{i\bar{i},1}-\frac{50}{3}\zeta_2^2A^I_{i\bar{i},1}+88\zeta_2^2\gamma^I_{i\bar{i},1}\Big\}\\
&+\delta(1-z_1)\delta(1-z_2)n_f^2C_F\Big\{\frac{16}{3}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}+\frac{11}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{16}{3}g^{I,3}_{i\bar{i},1}+\frac{11}{9}g^{I,2}_{i\bar{i},1}+\frac{11}{3}\zeta_2 f^I_{i\bar{i},1}
\end{aligned}
$$

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\begin{aligned}
&+\frac{22}{3}\zeta_2 B^I_{i\bar{i},1}-40\zeta_2 g^{I,1}_{i\bar{i},1}-\frac{22}{3}\zeta_2\gamma^I_{i\bar{i},1}-10\zeta_2^2 A^I_{i\bar{i},1}\Big\}\\
&+\delta(1-z_1)\delta(1-z_2)n_f^3\Big\{-\frac{32}{27}\overline{\mathcal{G}}^{I,4}_{i\bar{i},1}-\frac{32}{27}g^{I,4}_{i\bar{i},1}+\frac{32}{3}\zeta_2 g^{I,2}_{i\bar{i},1}+\frac{8}{3}\zeta_2^2 f^I_{i\bar{i},1}+\frac{16}{3}\zeta_2^2 B^I_{i\bar{i},1}\\
&-\frac{16}{3}\zeta_2^2\gamma^I_{i\bar{i},1}\Big\}\\
&+\mathcal{D}_0\delta(1-z_2)\Big\{-f^I_{i\bar{i},4}-\frac{2}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3}f^I_{i\bar{i},1}-\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},2}-2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},3}-2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}\\
&-2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2}-\frac{4}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}-\frac{2}{3}g^{I,1}_{i\bar{i},3}f^I_{i\bar{i},1}-g^{I,1}_{i\bar{i},2}f^I_{i\bar{i},2}-2g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-2g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},3}\\
&-2g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}-4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2}-4g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}-2g^{I,1}_{i\bar{i},1}g^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}-2\left(g^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2}\\
&-4\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}+30\zeta_7\left(A^I_{i\bar{i},1}\right)^4+18\zeta_5\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2\\
&+18\zeta_5\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2}+12\zeta_5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3+12\zeta_5 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3+\frac{2}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^4\\
&+4\zeta_3 A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2+2\zeta_3\left(A^I_{i\bar{i},2}\right)^2+8\zeta_3 A^I_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2}+4\zeta_3 A^I_{i\bar{i},1}A^I_{i\bar{i},3}+2\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2\\
&+8\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+8\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2}+4\zeta_3\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2+2\zeta_3 g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2\\
&+8\zeta_3 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+8\zeta_3 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2}+8\zeta_3 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+4\zeta_3\left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2\\
&-\frac{70}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}+3\zeta_2\left(f^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2}-\zeta_2 A^I_{i\bar{i},3}f^I_{i\bar{i},1}-\zeta_2 A^I_{i\bar{i},2}f^I_{i\bar{i},2}-\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},3}\\
&-\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1}+2\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3-2\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}-2\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2}\\
&-2\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\zeta_2 g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1}+2\zeta_2 g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3-2\zeta_2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}\\
&-2\zeta_2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2}-4\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-2\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1}-18\zeta_2\zeta_5\left(A^I_{i\bar{i},1}\right)^4\\
&-14\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2-6\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2}-4\zeta_2\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3\\
&-4\zeta_2\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3+\zeta_2^2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3-\zeta_2^2 A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}-\frac{1}{2}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2}\\
&-\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}-\zeta_2^2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}-3\zeta_2^2\zeta_3\left(A^I_{i\bar{i},1}\right)^4-\frac{1}{6}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\}\\
&+\mathcal{D}_0\delta(1-z_2)C_A\Big\{-\frac{22}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3}-\frac{44}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}f^I_{i\bar{i},1}-\frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},2}-22\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\\
&-\frac{44}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{44}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3-\frac{44}{9}g^{I,2}_{i\bar{i},2}f^I_{i\bar{i},1}-\frac{22}{3}g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{22}{3}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},2}\\
&-\frac{44}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{44}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}-\frac{44}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{88}{3}g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\\
&-\frac{44}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{44}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+154\zeta_5\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}+\frac{44}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^3
\end{aligned}
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\begin{aligned}
&+\frac{88}{3}\zeta_3 A^I_{i\bar{i},2} f^I_{i\bar{i},1}+\frac{110}{3}\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},2}+\frac{44}{3}\zeta_3 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+\frac{308}{3}\zeta_3 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}\\
&+\frac{44}{3}\zeta_3 g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+44\zeta_3 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{242}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^3-11\zeta_2 f^I_{i\bar{i},1} f^I_{i\bar{i},2}\\
&-44\zeta_2 B^I_{i\bar{i},2} f^I_{i\bar{i},1}-22\zeta_2 B^I_{i\bar{i},1} f^I_{i\bar{i},2}-\frac{22}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},2} A^I_{i\bar{i},1}-\frac{22}{3}\zeta_2 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}\\
&+\frac{44}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-44\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} B^I_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{22}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},2}-\frac{44}{3}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}\\
&-\frac{22}{3}\zeta_2 g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{22}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-44\zeta_2 g^{I,1}_{i\bar{i},1} B^I_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{44}{3}\zeta_2 g^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\\
&+44\zeta_2 \gamma^I_{i\bar{i},2} f^I_{i\bar{i},1}+22\zeta_2 \gamma^I_{i\bar{i},1} f^I_{i\bar{i},2}+44\zeta_2 \gamma^I_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1}+44\zeta_2 \gamma^I_{i\bar{i},1} g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1}\\
&-\frac{308}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}+44\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1}-44\zeta_2\zeta_3 \gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&-\frac{11}{3}\zeta_2^2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-22\zeta_2^2 A^I_{i\bar{i},1} B^I_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{11}{3}\zeta_2^2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+22\zeta_2^2 \gamma^I_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)C_A^2\Bigg\{-\frac{484}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}-\frac{68}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}-\frac{968}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{136}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1}\\
&-\frac{484}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{136}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2-\frac{968}{27} g^{I,3}_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{136}{9} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{484}{9} g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-\frac{968}{9} g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{136}{3} g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{1936}{9}\zeta_5\left(A^I_{i\bar{i},1}\right)^2+\frac{242}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^2+68\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},1}\\
&+\frac{2420}{9}\zeta_3 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}-\frac{34}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2-68\zeta_2 B^I_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{484}{9}\zeta_2 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1}+\frac{242}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1}\\
&-\frac{484}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} B^I_{i\bar{i},1}-\frac{68}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}+\frac{484}{3}\zeta_2 g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1}+68\zeta_2 \gamma^I_{i\bar{i},1} f^I_{i\bar{i},1}+\frac{484}{3}\zeta_2 \gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-\frac{484}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2+\frac{2783}{45}\zeta_2^2 A^I_{i\bar{i},1} f^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)C_A^3\Bigg\{-\frac{10648}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}-\frac{2992}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{2857}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_f\Bigg\{+\frac{4}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3}+\frac{8}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2} f^I_{i\bar{i},1}+\frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},2}+4\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1}\\
&+\frac{8}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3+\frac{8}{9} g^{I,2}_{i\bar{i},2} f^I_{i\bar{i},1}+\frac{4}{3} g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{4}{3} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},2}+\frac{8}{3} g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1}+\frac{8}{3} g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\\
&+\frac{8}{3} g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1}+\frac{16}{3} g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{8}{3} g^{I,1}_{i\bar{i},1} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1}+\frac{8}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-28\zeta_5\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}-\frac{8}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^3-\frac{16}{3}\zeta_3 A^I_{i\bar{i},2} f^I_{i\bar{i},1}-\frac{20}{3}\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},2}-\frac{8}{3}\zeta_3 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&-\frac{56}{3}\zeta_3 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}-\frac{8}{3}\zeta_3 g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-8\zeta_3 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}+\frac{44}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^3\\
&+2\zeta_2 f^I_{i\bar{i},1} f^I_{i\bar{i},2}+8\zeta_2 B^I_{i\bar{i},2} f^I_{i\bar{i},1}+4\zeta_2 B^I_{i\bar{i},1} f^I_{i\bar{i},2}+\frac{4}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},2} A^I_{i\bar{i},1}+\frac{4}{3}\zeta_2 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}\\
&-\frac{8}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+8\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} B^I_{i\bar{i},1} f^I_{i\bar{i},1}+\frac{4}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},2}+\frac{8}{3}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}
\end{aligned}
$$

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\begin{aligned}
&+\frac{4}{3}\zeta_2 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{4}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+8\zeta_2 g^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{8}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-8\zeta_2\gamma^I_{i\bar{i},2}f^I_{i\bar{i},1}-4\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},2}-8\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-8\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+\frac{56}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}-8\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2B^I_{i\bar{i},1}+8\zeta_2\zeta_3\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+\frac{2}{3}\zeta_2^2A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+4\zeta_2^2A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{2}{3}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-4\zeta_2^2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_fC_A\Bigg\{\frac{176}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}+\frac{20}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{352}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{40}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+\frac{176}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{40}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{352}{27}g^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{40}{9}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{176}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&+\frac{352}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{40}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{704}{9}\zeta_5\left(A^I_{i\bar{i},1}\right)^2-\frac{88}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^2-20\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{880}{9}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{10}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2+20\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{176}{9}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{88}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+\frac{176}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}+\frac{20}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{176}{3}\zeta_2g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-20\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{176}{3}\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&+\frac{176}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2-\frac{1012}{45}\zeta_2^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_fC_A^2\Bigg\{\frac{1936}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}+\frac{1424}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{1415}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_fC_F\Bigg\{4\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{8}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+8\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{8}{3}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+8g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-12\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}+2\zeta_2\left(f^I_{i\bar{i},1}\right)^2+12\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}+4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-12\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_fC_FC_A\Bigg\{\frac{176}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{205}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_fC_F^2\Bigg\{-2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_f^2\Bigg\{-\frac{16}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}-\frac{32}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{16}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{32}{27}g^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{16}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{32}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{64}{9}\zeta_5\left(A^I_{i\bar{i},1}\right)^2+\frac{8}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^2+\frac{80}{9}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-\frac{16}{9}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{8}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{16}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}+\frac{16}{3}\zeta_2g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{16}{3}\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-\frac{16}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2+\frac{92}{45}\zeta_2^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_f^2C_A\Bigg\{-\frac{352}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}-\frac{160}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{79}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\delta(1-z_2)n_f^2C_F\Bigg\{-\frac{32}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{22}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}
\end{aligned}
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\begin{aligned}
&+ \mathcal{D}_0\delta(1-z_2)n_f^3\Bigg\{ + \frac{64}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}\Bigg\}\\
&+ \mathcal{D}_0\overline{\mathcal{D}}_0C_A\Bigg\{ + 11f^I_{i\bar{i},3} + \frac{44}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1} + \frac{55}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1} + \frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2}\\
&+ \frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2} + \frac{44}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{110}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{44}{9}g^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1} + \frac{11}{3}g^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}\\
&+ \frac{22}{3}g^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{22}{3}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{44}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{44}{3}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2} + \frac{44}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+ 44g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{44}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{22}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 154\zeta_5\left(A^I_{i\bar{i},1}\right)^3\\
&- \frac{286}{3}\zeta_3A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 66\zeta_3A^I_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{352}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - 44\zeta_3g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&- \frac{22}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^3 + 22\zeta_2B^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{22}{3}\zeta_2A^I_{i\bar{i},2}f^I_{i\bar{i},1} + 22\zeta_2A^I_{i\bar{i},2}B^I_{i\bar{i},1} + \frac{22}{3}\zeta_2A^I_{i\bar{i},1}f^I_{i\bar{i},2}\\
&+ 44\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},2} + \frac{22}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{88}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 44\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1}\\
&+ \frac{22}{3}\zeta_2g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 44\zeta_2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1} - 44\zeta_2\gamma^I_{i\bar{i},2}A^I_{i\bar{i},1} - 22\zeta_2\gamma^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&- 22\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},2} - 44\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - 44\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{374}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^3\\
&- \frac{11}{6}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 22\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} - 22\zeta_2^2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Bigg\}\\
&+ \mathcal{D}_0\overline{\mathcal{D}}_0C_A^2\Bigg\{ + \frac{68}{3}f^I_{i\bar{i},2} + \frac{242}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} + \frac{968}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{1210}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{136}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+ 68\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{484}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 + \frac{968}{27}g^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{242}{9}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{136}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+ \frac{68}{3}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{968}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{2662}{9}\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 68\zeta_3\left(A^I_{i\bar{i},1}\right)^2 - \frac{121}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2\\
&+ \frac{242}{3}\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1} + 68\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1} - \frac{968}{9}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{484}{3}\zeta_2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&- \frac{242}{3}\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1} - 68\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{2783}{45}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2\Bigg\}\\
&+ \mathcal{D}_0\overline{\mathcal{D}}_0C_A^3\Bigg\{\frac{2857}{54}f^I_{i\bar{i},1} + \frac{5324}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + \frac{3740}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+ \mathcal{D}_0\overline{\mathcal{D}}_0n_f\Bigg\{ - 2f^I_{i\bar{i},3} - \frac{8}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1} - \frac{10}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1} - \frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2}\\
&- \frac{16}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2} - \frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{20}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - \frac{8}{9}g^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1} - \frac{2}{3}g^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}\\
&- \frac{4}{3}g^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{4}{3}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{8}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{8}{3}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2} - \frac{8}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&- 8g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{8}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 28\zeta_5\left(A^I_{i\bar{i},1}\right)^3 + \frac{52}{3}\zeta_3A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+ 12\zeta_3A^I_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{64}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 8\zeta_3g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{4}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^3 - 4\zeta_2B^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2
\end{aligned}
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$$
\begin{aligned}
&+\frac{4}{3}\zeta_2 A^I_{i\bar{i},2} f^I_{i\bar{i},1} - 4\zeta_2 A^I_{i\bar{i},2} B^I_{i\bar{i},1} - \frac{4}{3}\zeta_2 A^I_{i\bar{i},1} f^I_{i\bar{i},2} - 8\zeta_2 A^I_{i\bar{i},1} B^I_{i\bar{i},2} - \frac{4}{3}\zeta_2 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \\
&+\frac{16}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 8\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} B^I_{i\bar{i},1} - \frac{4}{3}\zeta_2 g^{I,2}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 - 8\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} B^I_{i\bar{i},1} \\
&+8\zeta_2 \gamma^I_{i\bar{i},2} A^I_{i\bar{i},1} + 4\zeta_2 \gamma^I_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 + 4\zeta_2 \gamma^I_{i\bar{i},1} A^I_{i\bar{i},2} + 8\zeta_2 \gamma^I_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + 8\zeta_2 \gamma^I_{i\bar{i},1} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&-\frac{68}{3}\zeta_2\zeta_3 \left(A^I_{i\bar{i},1}\right)^3 + \frac{1}{3}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 4\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + 4\zeta_2^2 \gamma^I_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f C_A \Bigg\{ -\frac{20}{3} f^I_{i\bar{i},2} - \frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} - \frac{352}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{440}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{40}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&-20\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{176}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 - \frac{352}{27} g^{I,3}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{88}{9} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{40}{9} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&-\frac{20}{3} g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{352}{9} g^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + \frac{968}{9}\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 20\zeta_3 \left(A^I_{i\bar{i},1}\right)^2 + \frac{44}{3}\zeta_2 \left(f^I_{i\bar{i},1}\right)^2 \\
&-\frac{88}{3}\zeta_2 B^I_{i\bar{i},1} f^I_{i\bar{i},1} - 20\zeta_2 A^I_{i\bar{i},1} B^I_{i\bar{i},1} + \frac{352}{9}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{176}{3}\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{88}{3}\zeta_2 \gamma^I_{i\bar{i},1} f^I_{i\bar{i},1} \\
&+20\zeta_2 \gamma^I_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{1012}{45}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f C_A^2 \Bigg\{ -\frac{1415}{54} f^I_{i\bar{i},1} - \frac{968}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} - \frac{1780}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f C_F \Bigg\{ -4 f^I_{i\bar{i},2} - \frac{8}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} - 12\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{8}{3} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} - 4 g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} \\
&+12\zeta_3 \left(A^I_{i\bar{i},1}\right)^2 - 12\zeta_2 A^I_{i\bar{i},1} B^I_{i\bar{i},1} + 12\zeta_2 \gamma^I_{i\bar{i},1} A^I_{i\bar{i},1} \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f C_F C_A \Bigg\{ -\frac{205}{18} f^I_{i\bar{i},1} - \frac{220}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f C_F^2 \Bigg\{ f^I_{i\bar{i},1} \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f^2 \Bigg\{ \frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} + \frac{32}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{40}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{16}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 + \frac{32}{27} g^{I,3}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&+\frac{8}{9} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{32}{9} g^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{88}{9}\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{4}{3}\zeta_2 \left(f^I_{i\bar{i},1}\right)^2 + \frac{8}{3}\zeta_2 B^I_{i\bar{i},1} f^I_{i\bar{i},1} \\
&-\frac{32}{9}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{16}{3}\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{8}{3}\zeta_2 \gamma^I_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{92}{45}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f^2 C_A \Bigg\{ \frac{79}{54} f^I_{i\bar{i},1} + \frac{176}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + \frac{200}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f^2 C_F \Bigg\{ \frac{11}{9} f^I_{i\bar{i},1} + \frac{40}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \Bigg\} \\
&+\mathcal{D}_0\overline{\mathcal{D}}_0 n_f^3 \Bigg\{ -\frac{32}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathcal{D}_0\overline{\mathcal{D}}_0\Big\{ \left(f^I_{i\bar{i},2}\right)^2 + 2f^I_{i\bar{i},1}f^I_{i\bar{i},3} + A^I_{i\bar{i},4} + \frac{2}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3}A^I_{i\bar{i},1} + \overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 + \overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},2} \\
&+ 4\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2} + 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},3} + 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + 2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} \\
&+ \frac{4}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3 A^I_{i\bar{i},1} + \frac{2}{3}g^{I,1}_{i\bar{i},3}A^I_{i\bar{i},1} + g^{I,1}_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 + g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},2} + 2g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&+ 4g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2} + 2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},3} + 2g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} \\
&+ 4g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} + 2g^{I,1}_{i\bar{i},1}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 2\left(g^{I,1}_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + 2\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} \\
&+ 4\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^3 A^I_{i\bar{i},1} - 42\zeta_5\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} - \frac{20}{3}\zeta_3 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 \\
&- 20\zeta_3 A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 10\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - 20\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 20\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
&+ \frac{70}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^4 - \zeta_2\left(f^I_{i\bar{i},1}\right)^4 - 2\zeta_2 A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 + \zeta_2\left(A^I_{i\bar{i},2}\right)^2 - 4\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2} \\
&+ 2\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},3} + \zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 - 4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2} \\
&+ 2\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 + \zeta_2 g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 - 4\zeta_2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 4\zeta_2 g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2} \\
&+ 4\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 2\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 + 30\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} - \frac{5}{2}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 \\
&+ \frac{3}{2}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + \zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \zeta_2^2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \frac{1}{6}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^4\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_1 C_A\Big\{ - 33f^I_{i\bar{i},1}f^I_{i\bar{i},2} - 11A^I_{i\bar{i},3} - \frac{77}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} - 22\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 44\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 \\
&- \frac{110}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{154}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} - \frac{11}{3}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} - 22g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 22g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 \\
&- \frac{44}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{176}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{22}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} + \frac{748}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
&+ \frac{143}{3}\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 66\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} + 33\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2} + 44\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&+ 22\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 66\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{77}{6}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_1 C_A^2\Big\{ - \frac{242}{3}f^I_{i\bar{i},2} - 34\left(f^I_{i\bar{i},1}\right)^2 - \frac{68}{3}A^I_{i\bar{i},2} - \frac{1694}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{2662}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} \\
&- \frac{272}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{242}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{484}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{68}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{2662}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 \\
&+ \frac{1573}{9}\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{242}{3}\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1} + 34\zeta_2\left(A^I_{i\bar{i},1}\right)^2 + \frac{242}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_1 C_A^3\Big\{ - \frac{1870}{9}f^I_{i\bar{i},1} - \frac{2857}{54}A^I_{i\bar{i},1} - \frac{5324}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_1 n_f\Big\{ + 6f^I_{i\bar{i},1}f^I_{i\bar{i},2} + 2A^I_{i\bar{i},3} + \frac{14}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 4\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 8\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2
\end{aligned}
$$

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\begin{aligned}
&+\frac{20}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}+\frac{28}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}+\frac{2}{3}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}+4g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}+4g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+\frac{8}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}+\frac{32}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}-\frac{136}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\\
&-\frac{26}{3}\zeta_2A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+12\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}-6\zeta_2A^I_{i\bar{i},1}A^I_{i\bar{i},2}-8\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&-4\zeta_2g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-12\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{7}{3}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_fC_A\Bigg\{\frac{88}{3}f^I_{i\bar{i},2}+10\left(f^I_{i\bar{i},1}\right)^2+\frac{20}{3}A^I_{i\bar{i},2}+\frac{616}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{968}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+\frac{80}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{88}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{176}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{20}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{968}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2\\
&-\frac{572}{9}\zeta_2A^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{88}{3}\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}-10\zeta_2\left(A^I_{i\bar{i},1}\right)^2-\frac{88}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_fC_A^2\Bigg\{\frac{890}{9}f^I_{i\bar{i},1}+\frac{1415}{54}A^I_{i\bar{i},1}+\frac{968}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_fC_F\Bigg\{6\left(f^I_{i\bar{i},1}\right)^2+4A^I_{i\bar{i},2}+16\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+4g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-6\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_fC_FC_A\Bigg\{\frac{110}{3}f^I_{i\bar{i},1}+\frac{205}{18}A^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_fC_F^2\Bigg\{-A^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_f^2\Bigg\{-\frac{8}{3}f^I_{i\bar{i},2}-\frac{56}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{88}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{8}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{16}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+\frac{88}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2+\frac{52}{9}\zeta_2A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{8}{3}\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}+\frac{8}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_f^2C_A\Bigg\{-\frac{100}{9}f^I_{i\bar{i},1}-\frac{79}{54}A^I_{i\bar{i},1}-\frac{176}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_f^2C_F\Bigg\{-\frac{20}{3}f^I_{i\bar{i},1}-\frac{11}{9}A^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1n_f^3\Bigg\{+\frac{32}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_0\overline{\mathcal{D}}_1\Bigg\{-3\left(f^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},2}-3A^I_{i\bar{i},3}f^I_{i\bar{i},1}-3A^I_{i\bar{i},2}f^I_{i\bar{i},2}-3A^I_{i\bar{i},1}f^I_{i\bar{i},3}-3\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3-6\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}-6\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2}-6\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}-3g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-2g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3-6g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}-6g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2}-12g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-6\left(g^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}+42\zeta_5\left(A^I_{i\bar{i},1}\right)^4+30\zeta_3\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2+30\zeta_3\left(A^I_{i\bar{i},1}\right)^2A^I_{i\bar{i},2}\\
&+20\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3+20\zeta_3g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3+6\zeta_2A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3+6\zeta_2A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}
\end{aligned}
$$

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\begin{aligned}
&+ 3\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} + 6\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 6\zeta_2 g^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 30\zeta_2\zeta_3 \left(A^I_{i\bar{i},1}\right)^4 \\
&+ \frac{9}{2}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} \Big\} \\
&+ \mathcal{D}_0 \overline{\mathcal{D}}_2 C_A \Big\{ 11 \left(f^I_{i\bar{i},1}\right)^3 + \frac{77}{3} A^I_{i\bar{i},2} f^I_{i\bar{i},1} + \frac{88}{3} A^I_{i\bar{i},1} f^I_{i\bar{i},2} + 11 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \\
&+ \frac{242}{3} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 11 g^{I,2}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 + \frac{110}{3} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{374}{3}\zeta_3 \left(A^I_{i\bar{i},1}\right)^3 \\
&- \frac{451}{6}\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 33\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} - 33\zeta_2 \gamma^I_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \Big\} \\
&+ \mathcal{D}_0 \overline{\mathcal{D}}_2 C_A^2 \Big\{ \frac{1331}{18} \left(f^I_{i\bar{i},1}\right)^2 + \frac{121}{3} A^I_{i\bar{i},2} + \frac{170}{3} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 242 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{242}{9} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&- \frac{968}{9}\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 \Big\} + \mathcal{D}_0 \overline{\mathcal{D}}_2 C_A^3 \Big\{ \frac{1331}{9} f^I_{i\bar{i},1} + \frac{935}{9} A^I_{i\bar{i},1} \Big\} \\
&+ \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f \Big\{ - 2 \left(f^I_{i\bar{i},1}\right)^3 - \frac{14}{3} A^I_{i\bar{i},2} f^I_{i\bar{i},1} - \frac{16}{3} A^I_{i\bar{i},1} f^I_{i\bar{i},2} - 2 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{44}{3} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 2 g^{I,2}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 - \frac{20}{3} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{68}{3}\zeta_3 \left(A^I_{i\bar{i},1}\right)^3 \\
&+ \frac{41}{3}\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 6\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + 6\zeta_2 \gamma^I_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \Big\} \\
&+ \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f C_A \Big\{ - \frac{242}{9} \left(f^I_{i\bar{i},1}\right)^2 - \frac{44}{3} A^I_{i\bar{i},2} - \frac{50}{3} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 88 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{88}{9} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&+ \frac{352}{9}\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 \Big\} + \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f C_A^2 \Big\{ - \frac{242}{3} f^I_{i\bar{i},1} - \frac{445}{9} A^I_{i\bar{i},1} \Big\} \\
&+ \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f C_F \Big\{ - 10 A^I_{i\bar{i},1} f^I_{i\bar{i},1} \Big\} + \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f C_F C_A \Big\{ - \frac{55}{3} A^I_{i\bar{i},1} \Big\} \\
&+ \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f^2 \Big\{ \frac{22}{9} \left(f^I_{i\bar{i},1}\right)^2 + \frac{4}{3} A^I_{i\bar{i},2} + 8 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{8}{9} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{32}{9}\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 \Big\} \\
&+ \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f^2 C_A \Big\{ \frac{44}{3} f^I_{i\bar{i},1} + \frac{50}{9} A^I_{i\bar{i},1} \Big\} + \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f^2 C_F \Big\{ \frac{10}{3} A^I_{i\bar{i},1} \Big\} \\
&+ \mathcal{D}_0 \overline{\mathcal{D}}_2 n_f^3 \Big\{ - \frac{8}{9} f^I_{i\bar{i},1} \Big\} + \mathcal{D}_0 \overline{\mathcal{D}}_2 \Big\{ \frac{1}{2} \left(f^I_{i\bar{i},1}\right)^4 + 3 A^I_{i\bar{i},2} \left(f^I_{i\bar{i},1}\right)^2 + \frac{3}{2} \left(A^I_{i\bar{i},2}\right)^2 \\
&+ 6 A^I_{i\bar{i},1} f^I_{i\bar{i},1} f^I_{i\bar{i},2} + 3 A^I_{i\bar{i},1} A^I_{i\bar{i},3} + \frac{3}{2} \overline{\mathcal{G}}^{I,1}_{i\bar{i},2} \left(A^I_{i\bar{i},1}\right)^2 + 6 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 + 6 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} \\
&+ 3 \left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 \left(A^I_{i\bar{i},1}\right)^2 + \frac{3}{2} g^{I,1}_{i\bar{i},2} \left(A^I_{i\bar{i},1}\right)^2 + 6 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 + 6 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} \\
&+ 6 g^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 + 3 \left(g^{I,1}_{i\bar{i},1}\right)^2 \left(A^I_{i\bar{i},1}\right)^2 - 35\zeta_3 \left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} - \frac{21}{2}\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 \left(f^I_{i\bar{i},1}\right)^2 \\
&- \frac{9}{2}\zeta_2 \left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} - 3\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^3 - 3\zeta_2 g^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^3 - \frac{9}{4}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^4 \Big\}
\end{aligned}
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\begin{aligned}
&+ \mathcal{D}_0\overline{\mathcal{D}}_3 C_A\Big\{ - \frac{220}{9}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{55}{3}A^I_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{275}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{110}{9}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&+ \frac{275}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_3 C_A^2\Big\{ - \frac{2420}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{170}{9}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_3 C_A^3\Big\{ - \frac{1331}{27}A^I_{i\bar{i},1}\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_3 n_f\Big\{\frac{40}{9}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{10}{3}A^I_{i\bar{i},1}A^I_{i\bar{i},2} \\
&+ \frac{50}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{20}{9}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{50}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_3 n_f C_A\Big\{\frac{880}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{50}{9}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_3 n_f C_A^2\Big\{ + \frac{242}{9}A^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_3 n_f C_F\Big\{\frac{10}{3}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_3 n_f^2\Big\{ - \frac{80}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_3 n_f^2 C_A\Big\{ - \frac{44}{9}A^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_3 n_f^3\Big\{ + \frac{8}{27}A^I_{i\bar{i},1}\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_3\Big\{ - \frac{5}{3}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 5A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - \frac{5}{2}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} \\
&- 5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 5g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{35}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^4 + \frac{15}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_4 C_A\Big\{ + \frac{385}{24}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_4 C_A^2\Big\{ + \frac{605}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_4 n_f\Big\{ - \frac{35}{12}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_4 n_f C_A\Big\{ - \frac{220}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_4 n_f^2\Big\{\frac{20}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_4\Big\{\frac{15}{8}\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + \frac{15}{8}\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} \\
&+ \frac{5}{4}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \frac{5}{4}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - \frac{15}{8}\zeta_2\left(A^I_{i\bar{i},1}\right)^4\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_5 C_A\Big\{ - \frac{77}{24}\left(A^I_{i\bar{i},1}\right)^3\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_5 n_f\Big\{\frac{7}{12}\left(A^I_{i\bar{i},1}\right)^3\Big\} + \mathcal{D}_0\overline{\mathcal{D}}_5\Big\{ - \frac{7}{8}\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_0\overline{\mathcal{D}}_6\Big\{\frac{7}{48}\left(A^I_{i\bar{i},1}\right)^4\Big\} + \mathcal{D}_1\delta(1-z_2)\Big\{ + \left(f^I_{i\bar{i},2}\right)^2 + 2f^I_{i\bar{i},1}f^I_{i\bar{i},3} + A^I_{i\bar{i},4} + \frac{2}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3}A^I_{i\bar{i},1} \\
&+ \overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 + \overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},2} + 4\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2} + 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},3} + 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} \\
&+ 2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + 2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + \frac{4}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3 A^I_{i\bar{i},1} + \frac{2}{3}g^{I,1}_{i\bar{i},3}A^I_{i\bar{i},1} + g^{I,1}_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 \\
&+ g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},2} + 2g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + 4g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2} + 2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},3} + 2g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} \\
&+ 4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} + 4g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} + 2g^{I,1}_{i\bar{i},1}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} \\
&+ 2\left(g^{I,1}_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + 2\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + 4\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^3 A^I_{i\bar{i},1} \\
&- 42\zeta_5\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} - \frac{20}{3}\zeta_3 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 20\zeta_3 A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 10\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2}
\end{aligned}
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\begin{aligned}
&- 20\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 20\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{70}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^4 - \zeta_2\left(f^I_{i\bar{i},1}\right)^4 \\
&- 2\zeta_2 A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 + \zeta_2\left(A^I_{i\bar{i},2}\right)^2 - 4\zeta_2 A^I_{i\bar{i},1} f^I_{i\bar{i},1} f^I_{i\bar{i},2} + 2\zeta_2 A^I_{i\bar{i},1} A^I_{i\bar{i},3} + \zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 \\
&- 4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} + 2\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 + \zeta_2 g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 \\
&- 4\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 4\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} + 4\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 2\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 \\
&+ 30\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} - \frac{5}{2}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + \frac{3}{2}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + \zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 \\
&+ \zeta_2^2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \frac{1}{6}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^4\Bigg\} \\
&+ \mathcal{D}_1\delta(1-z_2)C_A\Bigg\{11 f^I_{i\bar{i},3} + \frac{44}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2} A^I_{i\bar{i},1} + \frac{55}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} f^I_{i\bar{i},1} + \frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 \\
&+ \frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},2} + \frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},2} + \frac{44}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{110}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{44}{9} g^{I,2}_{i\bar{i},2} A^I_{i\bar{i},1} \\
&+ \frac{11}{3} g^{I,1}_{i\bar{i},2} f^I_{i\bar{i},1} + \frac{22}{3} g^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{22}{3} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},2} + \frac{44}{3} g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{44}{3} g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},2} \\
&+ \frac{44}{3} g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + 44 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{44}{3} g^{I,1}_{i\bar{i},1} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{22}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
&- 154\zeta_5\left(A^I_{i\bar{i},1}\right)^3 - \frac{286}{3}\zeta_3 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 66\zeta_3 A^I_{i\bar{i},1} A^I_{i\bar{i},2} - \frac{352}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&- 44\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{22}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^3 + 22\zeta_2 B^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{22}{3}\zeta_2 A^I_{i\bar{i},2} f^I_{i\bar{i},1} \\
&+ 22\zeta_2 A^I_{i\bar{i},2} B^I_{i\bar{i},1} + \frac{22}{3}\zeta_2 A^I_{i\bar{i},1} f^I_{i\bar{i},2} + 44\zeta_2 A^I_{i\bar{i},1} B^I_{i\bar{i},2} + \frac{22}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{88}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 44\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} B^I_{i\bar{i},1} + \frac{22}{3}\zeta_2 g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 44\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} B^I_{i\bar{i},1} \\
&- 44\zeta_2\gamma^I_{i\bar{i},2} A^I_{i\bar{i},1} - 22\zeta_2\gamma^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 22\zeta_2\gamma^I_{i\bar{i},1} A^I_{i\bar{i},2} - 44\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&- 44\zeta_2\gamma^I_{i\bar{i},1} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{374}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^3 - \frac{11}{6}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 22\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} \\
&- 22\zeta_2^2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Bigg\} + \mathcal{D}_1\delta(1-z_2)C_A^2\Bigg\{\frac{68}{3} f^I_{i\bar{i},2} + \frac{242}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} + \frac{968}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&+ \frac{1210}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{136}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + 68\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{484}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 + \frac{968}{27} g^{I,3}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&+ \frac{242}{9} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{136}{9} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{68}{3} g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{968}{9} g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{2662}{9}\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},1} \\
&- 68\zeta_3\left(A^I_{i\bar{i},1}\right)^2 - \frac{121}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2 + \frac{242}{3}\zeta_2 B^I_{i\bar{i},1} f^I_{i\bar{i},1} + 68\zeta_2 A^I_{i\bar{i},1} B^I_{i\bar{i},1} - \frac{968}{9}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&- \frac{484}{3}\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{242}{3}\zeta_2\gamma^I_{i\bar{i},1} f^I_{i\bar{i},1} - 68\zeta_2\gamma^I_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{2783}{45}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2\Bigg\} \\
&+ \mathcal{D}_1\delta(1-z_2)C_A^3\Bigg\{\frac{2857}{54} f^I_{i\bar{i},1} + \frac{5324}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + \frac{3740}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}
\end{aligned}
$$

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\begin{aligned}
&+\mathcal{D}_1\delta(1-z_2)n_f\Bigg\{-2f^I_{i\bar{i},3}-\frac{8}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1}-\frac{10}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}-\frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-\frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2}\\
&-\frac{16}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2}-\frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{20}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}-\frac{8}{9}g^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1}-\frac{2}{3}g^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}\\
&-\frac{4}{3}g^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-\frac{4}{3}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2}-\frac{8}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{8}{3}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2}-\frac{8}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-8g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{8}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}+28\zeta_5\left(A^I_{i\bar{i},1}\right)^3+\frac{52}{3}\zeta_3A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+12\zeta_3A^I_{i\bar{i},1}A^I_{i\bar{i},2}+\frac{64}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+8\zeta_3g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+\frac{4}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^3-4\zeta_2B^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+\frac{4}{3}\zeta_2A^I_{i\bar{i},2}f^I_{i\bar{i},1}-4\zeta_2A^I_{i\bar{i},2}B^I_{i\bar{i},1}-\frac{4}{3}\zeta_2A^I_{i\bar{i},1}f^I_{i\bar{i},2}-8\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},2}-\frac{4}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&+\frac{16}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-8\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1}-\frac{4}{3}\zeta_2g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-8\zeta_2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1}\\
&+8\zeta_2\gamma^I_{i\bar{i},2}A^I_{i\bar{i},1}+4\zeta_2\gamma^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+4\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},2}+8\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+8\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-\frac{68}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^3+\frac{1}{3}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}-4\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2B^I_{i\bar{i},1}+4\zeta_2^2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Bigg\}\\
&+\mathcal{D}_1\delta(1-z_2)n_fC_A\Bigg\{-\frac{20}{3}f^I_{i\bar{i},2}-\frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}-\frac{352}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{440}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{40}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-20\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{176}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2-\frac{352}{27}g^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{88}{9}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{40}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{20}{3}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{352}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{968}{9}\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}+20\zeta_3\left(A^I_{i\bar{i},1}\right)^2\\
&+\frac{44}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2-\frac{88}{3}\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}-20\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}+\frac{352}{9}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{176}{3}\zeta_2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+\frac{88}{3}\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}+20\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{1012}{45}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2\Bigg\}\\
&+\mathcal{D}_1\delta(1-z_2)n_fC_A^2\Bigg\{-\frac{1415}{54}f^I_{i\bar{i},1}-\frac{968}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{1780}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_1\delta(1-z_2)n_fC_F\Bigg\{-4f^I_{i\bar{i},2}-\frac{8}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-12\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{8}{3}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-4g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+12\zeta_3\left(A^I_{i\bar{i},1}\right)^2-12\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}+12\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_1\delta(1-z_2)n_fC_FC_A\Bigg\{-\frac{205}{18}f^I_{i\bar{i},1}-\frac{220}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}+\mathcal{D}_1\delta(1-z_2)n_fC_F^2\Bigg\{f^I_{i\bar{i},1}\Bigg\}\\
&+\mathcal{D}_1\delta(1-z_2)n_f^2\Bigg\{+\frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{32}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{40}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{16}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{32}{27}g^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+\frac{8}{9}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{32}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{88}{9}\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{4}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2+\frac{8}{3}\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{32}{9}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{16}{3}\zeta_2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{8}{3}\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{92}{45}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2\Bigg\}
\end{aligned}
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\begin{aligned}
&+ \mathcal{D}_1\delta(1-z_2)n_f^2 C_A\Big\{ + \frac{79}{54}f^I_{i\bar{i},1} + \frac{176}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + \frac{200}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_1\delta(1-z_2)n_f^2 C_F\Big\{ + \frac{11}{9}f^I_{i\bar{i},1} + \frac{40}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\} + \mathcal{D}_1\delta(1-z_2)n_f^3\Big\{ - \frac{32}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 C_A\Big\{ - 33f^I_{i\bar{i},1}f^I_{i\bar{i},2} - 11A^I_{i\bar{i},3} - \frac{77}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} - 22\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 44\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&- \frac{110}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{154}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} - \frac{11}{3}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} - 22g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 22g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&- \frac{44}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{176}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{22}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} + \frac{748}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\\
&+ \frac{143}{3}\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 66\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} + 33\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2} + 44\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&+ 22\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 66\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{77}{6}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3\Big\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 C_A^2\Big\{ - \frac{242}{3}f^I_{i\bar{i},2} - 34\left(f^I_{i\bar{i},1}\right)^2 - \frac{68}{3}A^I_{i\bar{i},2} - \frac{1694}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{2662}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&- \frac{272}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{242}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{484}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{68}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{2662}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2\\
&+ \frac{1573}{9}\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{242}{3}\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1} + 34\zeta_2\left(A^I_{i\bar{i},1}\right)^2 + \frac{242}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 C_A^3\Big\{ - \frac{1870}{9}f^I_{i\bar{i},1} - \frac{2857}{54}A^I_{i\bar{i},1} - \frac{5324}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 n_f\Big\{ + 6f^I_{i\bar{i},1}f^I_{i\bar{i},2} + 2A^I_{i\bar{i},3} + \frac{14}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 4\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 8\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+ \frac{20}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{28}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} + \frac{2}{3}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 4g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 4g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+ \frac{8}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{32}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} - \frac{136}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\\
&- \frac{26}{3}\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 12\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} - 6\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2} - 8\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&- 4\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - 12\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{7}{3}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3\Big\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 n_f C_A\Big\{ + \frac{88}{3}f^I_{i\bar{i},2} + 10\left(f^I_{i\bar{i},1}\right)^2 + \frac{20}{3}A^I_{i\bar{i},2} + \frac{616}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{968}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+ \frac{80}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{88}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{176}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{20}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{968}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2\\
&- \frac{572}{9}\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{88}{3}\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1} - 10\zeta_2\left(A^I_{i\bar{i},1}\right)^2 - \frac{88}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 n_f C_A^2\Big\{ + \frac{890}{9}f^I_{i\bar{i},1} + \frac{1415}{54}A^I_{i\bar{i},1} + \frac{968}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 n_f C_F\Big\{ + 6\left(f^I_{i\bar{i},1}\right)^2 + 4A^I_{i\bar{i},2} + 16\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + 4g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - 6\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Big\}
\end{aligned}
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\begin{aligned}
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 n_f C_F C_A\left\{ + \frac{110}{3} f^I_{i\bar{i},1} + \frac{205}{18} A^I_{i\bar{i},1}\right\} + \mathcal{D}_1\overline{\mathcal{D}}_0 n_f C_F^2\left\{ - A^I_{i\bar{i},1}\right\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 n_f^2\left\{ - \frac{8}{3} f^I_{i\bar{i},2} - \frac{56}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{88}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{8}{9} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{16}{9} g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1}\right.\\
&\left. + \frac{88}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 + \frac{52}{9}\zeta_2 A^I_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{8}{3}\zeta_2 A^I_{i\bar{i},1} B^I_{i\bar{i},1} + \frac{8}{3}\zeta_2\gamma^I_{i\bar{i},1} A^I_{i\bar{i},1}\right\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_0 n_f^2 C_A\left\{ - \frac{100}{9} f^I_{i\bar{i},1} - \frac{79}{54} A^I_{i\bar{i},1} - \frac{176}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right\} + \mathcal{D}_1\overline{\mathcal{D}}_0 n_f^2 C_F\left\{ - \frac{20}{3} f^I_{i\bar{i},1}\right.\\
&\left. - \frac{11}{9} A^I_{i\bar{i},1}\right\} + \mathcal{D}_1\overline{\mathcal{D}}_0 n_f^3\left\{ + \frac{32}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right\} + \mathcal{D}_1\overline{\mathcal{D}}_0\left\{ - 3\left(f^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - 3A^I_{i\bar{i},3} f^I_{i\bar{i},1}\right.\\
&\left. - 3A^I_{i\bar{i},2} f^I_{i\bar{i},2} - 3A^I_{i\bar{i},1} f^I_{i\bar{i},3} - 3\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 6\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},2} f^I_{i\bar{i},1}\right.\\
&\left. - 6\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},2} - 6\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 3g^{I,1}_{i\bar{i},2} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 2g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 6g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},2} f^I_{i\bar{i},1}\right.\\
&\left. - 6g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},2} - 12 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 6\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 42\zeta_5\left(A^I_{i\bar{i},1}\right)^4\right.\\
&\left. + 30\zeta_3\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + 30\zeta_3\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + 20\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + 20\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3\right.\\
&\left. + 6\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 + 6\zeta_2 A^I_{i\bar{i},1} A^I_{i\bar{i},2} f^I_{i\bar{i},1} + 3\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} + 6\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\right.\\
&\left. + 6\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 30\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^4 + \frac{9}{2}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\right\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_1 C_A\left\{ + 22\left(f^I_{i\bar{i},1}\right)^3 + \frac{154}{3} A^I_{i\bar{i},2} f^I_{i\bar{i},1} + \frac{176}{3} A^I_{i\bar{i},1} f^I_{i\bar{i},2} + 22\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\right.\\
&\left. + \frac{484}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 22 g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{220}{3} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{748}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^3\right.\\
&\left. - \frac{451}{3}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 66\zeta_2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} - 66\zeta_2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\right\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_1 C_A^2\left\{ + \frac{1331}{9}\left(f^I_{i\bar{i},1}\right)^2 + \frac{242}{3} A^I_{i\bar{i},2} + \frac{340}{3} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 484\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\right.\\
&\left. + \frac{484}{9} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{1936}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\right\} + \mathcal{D}_1\overline{\mathcal{D}}_1 C_A^3\left\{ + \frac{2662}{9} f^I_{i\bar{i},1} + \frac{1870}{9} A^I_{i\bar{i},1}\right\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_1 n_f\left\{ - 4\left(f^I_{i\bar{i},1}\right)^3 - \frac{28}{3} A^I_{i\bar{i},2} f^I_{i\bar{i},1} - \frac{32}{3} A^I_{i\bar{i},1} f^I_{i\bar{i},2} - 4\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\right.\\
&\left. - \frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 4 g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{40}{3} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{136}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^3\right.\\
&\left. + \frac{82}{3}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 12\zeta_2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + 12\zeta_2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\right\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_1 n_f C_A\left\{ - \frac{484}{9}\left(f^I_{i\bar{i},1}\right)^2 - \frac{88}{3} A^I_{i\bar{i},2} - \frac{100}{3} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 176\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\right.\\
&\left. - \frac{176}{9} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{704}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\right\} + \mathcal{D}_1\overline{\mathcal{D}}_1 n_f C_A^2\left\{ - \frac{484}{3} f^I_{i\bar{i},1} - \frac{890}{9} A^I_{i\bar{i},1}\right\}
\end{aligned}
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\begin{aligned}
&+ \mathcal{D}_1\overline{\mathcal{D}}_1 n_f C_F\Big\{ - 20 A^I_{i\bar{i},1} f^I_{i\bar{i},1}\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_1 n_f C_F C_A\Big\{ - \frac{110}{3} A^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_1 n_f^2\Big\{ + \frac{44}{9}\left(f^I_{i\bar{i},1}\right)^2 + \frac{8}{3} A^I_{i\bar{i},2} + 16\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{16}{9} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{64}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_1 n_f^2 C_A\Big\{ + \frac{88}{3} f^I_{i\bar{i},1} + \frac{100}{9} A^I_{i\bar{i},1}\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_1 n_f^2 C_F\Big\{ + \frac{20}{3} A^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_1 n_f^3\Big\{ - \frac{16}{9} f^I_{i\bar{i},1}\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_1\Big\{ + \left(f^I_{i\bar{i},1}\right)^4 + 6 A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 + 3\left(A^I_{i\bar{i},2}\right)^2 \\
&+ 12 A^I_{i\bar{i},1} f^I_{i\bar{i},1} f^I_{i\bar{i},2} + 6 A^I_{i\bar{i},1} A^I_{i\bar{i},3} + 3\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 + 12\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 \\
&+ 12\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} + 6\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 + 3 g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 + 12 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 \\
&+ 12 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} + 12 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 6\left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 - 70\zeta_3\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} \\
&- 21\zeta_2\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 - 9\zeta_2\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} - 6\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - 6\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 \\
&- \frac{9}{2}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^4\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_2 C_A\Big\{ - \frac{220}{3} A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 55 A^I_{i\bar{i},1} A^I_{i\bar{i},2} - \frac{275}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{110}{3} g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{275}{3}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_2 C_A^2\Big\{ - \frac{2420}{9} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{170}{3}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_2 C_A^3\Big\{ - \frac{1331}{9} A^I_{i\bar{i},1}\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_2 n_f\Big\{ + \frac{40}{3} A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 10 A^I_{i\bar{i},1} A^I_{i\bar{i},2} \\
&+ \frac{50}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{20}{3} g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{50}{3}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_2 n_f C_A\Big\{ + \frac{880}{9} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{50}{3}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_2 n_f C_A^2\Big\{ + \frac{242}{3} A^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_2 n_f C_F\Big\{ + 10\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_2 n_f^2\Big\{ - \frac{80}{9} A^I_{i\bar{i},1} f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_2 n_f^2 C_A\Big\{ - \frac{44}{3} A^I_{i\bar{i},1}\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_2 n_f^3\Big\{ + \frac{8}{9} A^I_{i\bar{i},1}\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_2\Big\{ - 5 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 \\
&- 15 A^I_{i\bar{i},1} A^I_{i\bar{i},2} f^I_{i\bar{i},1} - \frac{15}{2}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - 15\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 15 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
&+ 35\zeta_3\left(A^I_{i\bar{i},1}\right)^4 + \frac{45}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_3 C_A\Big\{ + \frac{385}{6}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_3 C_A^2\Big\{ + \frac{2420}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_3 n_f\Big\{ - \frac{35}{3}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_3 n_f C_A\Big\{ - \frac{880}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_3 n_f^2\Big\{ + \frac{80}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_1\overline{\mathcal{D}}_3\Big\{ + \frac{15}{2}\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + \frac{15}{2}\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + 5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + 5 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{15}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^4\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_4 C_A\Big\{ - \frac{385}{24}\left(A^I_{i\bar{i},1}\right)^3\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_4 n_f\Big\{ + \frac{35}{12}\left(A^I_{i\bar{i},1}\right)^3\Big\}\\
&+ \mathcal{D}_1\overline{\mathcal{D}}_4\Big\{ - \frac{35}{8}\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\} + \mathcal{D}_1\overline{\mathcal{D}}_5\Big\{ + \frac{7}{8}\left(A^I_{i\bar{i},1}\right)^4\Big\}\\
&+ \mathcal{D}_2\delta(1-z_2)\Big\{ - \frac{3}{2}\left(f^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - \frac{3}{2}A^I_{i\bar{i},3}f^I_{i\bar{i},1} - \frac{3}{2}A^I_{i\bar{i},2}f^I_{i\bar{i},2} - \frac{3}{2}A^I_{i\bar{i},1}f^I_{i\bar{i},3}\\
&- \frac{3}{2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2}\\
&- 3\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{3}{2}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 3g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 3g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2}\\
&- 6g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 3\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 21\zeta_5\left(A^I_{i\bar{i},1}\right)^4 + 15\zeta_3\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2\\
&+ 15\zeta_3\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + 10\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + 10\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + 3\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3\\
&+ 3\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} + \frac{3}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} + 3\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 3\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\\
&- 15\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^4 + \frac{9}{4}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_2\delta(1-z_2)C_A\Big\{ - \frac{33}{2}f^I_{i\bar{i},1}f^I_{i\bar{i},2} - \frac{11}{2}A^I_{i\bar{i},3} - \frac{77}{6}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} - 11\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&- 22\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{55}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{77}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} - \frac{11}{6}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} - 11g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&- 11g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{22}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{88}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{11}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}\\
&+ \frac{374}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{143}{6}\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 33\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{33}{2}\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2}\\
&+ 22\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 11\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 33\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{77}{12}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3\Big\}\\
&+ \mathcal{D}_2\delta(1-z_2)C_A^2\Big\{ - \frac{121}{3}f^I_{i\bar{i},2} - 17\left(f^I_{i\bar{i},1}\right)^2 - \frac{34}{3}A^I_{i\bar{i},2} - \frac{847}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&- \frac{1331}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{136}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{121}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{242}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{34}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+ \frac{1331}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 + \frac{1573}{18}\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{121}{3}\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1} + 17\zeta_2\left(A^I_{i\bar{i},1}\right)^2\\
&+ \frac{121}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Big\} + \mathcal{D}_2\delta(1-z_2)C_A^3\Big\{ - \frac{935}{9}f^I_{i\bar{i},1} - \frac{2857}{108}A^I_{i\bar{i},1} - \frac{2662}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_2\delta(1-z_2)n_f\Big\{ + 3f^I_{i\bar{i},1}f^I_{i\bar{i},2} + A^I_{i\bar{i},3} + \frac{7}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 4\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+ \frac{10}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{14}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} + \frac{1}{3}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 2g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 2g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+ \frac{4}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{16}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{2}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} - \frac{68}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{13}{3}\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 6\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} - 3\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2} - 4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&- 2\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - 6\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{7}{6}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3\Bigg\} \\
&+ \mathcal{D}_2\delta(1-z_2)n_f C_A\Bigg\{ + \frac{44}{3}f^I_{i\bar{i},2} + 5\left(f^I_{i\bar{i},1}\right)^2 + \frac{10}{3}A^I_{i\bar{i},2} + \frac{308}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{484}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} \\
&+ \frac{40}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{44}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{88}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{10}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{484}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{286}{9}\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{44}{3}\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1} - 5\zeta_2\left(A^I_{i\bar{i},1}\right)^2 - \frac{44}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Bigg\} \\
&+ \mathcal{D}_2\delta(1-z_2)n_f C_A^2\Bigg\{ + \frac{445}{9}f^I_{i\bar{i},1} + \frac{1415}{108}A^I_{i\bar{i},1} + \frac{484}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\} \\
&+ \mathcal{D}_2\delta(1-z_2)n_f C_F\Bigg\{ + 3\left(f^I_{i\bar{i},1}\right)^2 + 2A^I_{i\bar{i},2} + 8\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + 2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - 3\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Bigg\} \\
&+ \mathcal{D}_2\delta(1-z_2)n_f C_F C_A\Bigg\{ + \frac{55}{3}f^I_{i\bar{i},1} + \frac{205}{36}A^I_{i\bar{i},1}\Bigg\} \\
&+ \mathcal{D}_2\delta(1-z_2)n_f C_F^2\Bigg\{ - \frac{1}{2}A^I_{i\bar{i},1}\Bigg\} + \mathcal{D}_2\delta(1-z_2)n_f^2\Bigg\{ - \frac{4}{3}f^I_{i\bar{i},2} - \frac{28}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&- \frac{44}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{4}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{8}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{44}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 + \frac{26}{9}\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} \\
&- \frac{4}{3}\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1} + \frac{4}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Bigg\} + \mathcal{D}_2\delta(1-z_2)n_f^2 C_A\Bigg\{ - \frac{50}{9}f^I_{i\bar{i},1} - \frac{79}{108}A^I_{i\bar{i},1} \\
&- \frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\} + \mathcal{D}_2\delta(1-z_2)n_f^2 C_F\Bigg\{ - \frac{10}{3}f^I_{i\bar{i},1} - \frac{11}{18}A^I_{i\bar{i},1}\Bigg\} + \mathcal{D}_2\delta(1-z_2)n_f^3\Bigg\{\frac{16}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\} \\
&+ \mathcal{D}_2\overline{\mathcal{D}}_0 C_A\Bigg\{ + 11\left(f^I_{i\bar{i},1}\right)^3 + \frac{77}{3}A^I_{i\bar{i},2}f^I_{i\bar{i},1} + \frac{88}{3}A^I_{i\bar{i},1}f^I_{i\bar{i},2} + 11\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&+ \frac{242}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 11g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{110}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{374}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^3 \\
&- \frac{451}{6}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 33\zeta_2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} - 33\zeta_2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Bigg\} \\
&+ \mathcal{D}_2\overline{\mathcal{D}}_0 C_A^2\Bigg\{ + \frac{1331}{18}\left(f^I_{i\bar{i},1}\right)^2 + \frac{121}{3}A^I_{i\bar{i},2} + \frac{170}{3}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 242\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} \\
&+ \frac{242}{9}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{968}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Bigg\} + \mathcal{D}_2\overline{\mathcal{D}}_0 C_A^3\Bigg\{ + \frac{1331}{9}f^I_{i\bar{i},1} + \frac{935}{9}A^I_{i\bar{i},1}\Bigg\} \\
&+ \mathcal{D}_2\overline{\mathcal{D}}_0 n_f\Bigg\{ - 2\left(f^I_{i\bar{i},1}\right)^3 - \frac{14}{3}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - \frac{16}{3}A^I_{i\bar{i},1}f^I_{i\bar{i},2} - 2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{44}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 2g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{20}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{68}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^3 \\
&+ \frac{41}{3}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 6\zeta_2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + 6\zeta_2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Bigg\}
\end{aligned}
$$

$$+ \mathcal{D}_2\overline{\mathcal{D}}_0 n_f C_A\left\{ - \frac{242}{9}\left(f^I_{i\bar{i},1}\right)^2 - \frac{44}{3}A^I_{i\bar{i},2} - \frac{50}{3}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 88\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{88}{9}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\right.$$
$$\left. + \frac{352}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\right\} + \mathcal{D}_2\overline{\mathcal{D}}_0 n_f C_A^2\left\{ - \frac{242}{3}f^I_{i\bar{i},1} - \frac{445}{9}A^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_0 n_f C_F\left\{ - 10A^I_{i\bar{i},1}f^I_{i\bar{i},1}\right\} + \mathcal{D}_2\overline{\mathcal{D}}_0 n_f C_F C_A\left\{ - \frac{55}{3}A^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_0 n_f^2\left\{ + \frac{22}{9}\left(f^I_{i\bar{i},1}\right)^2 + \frac{4}{3}A^I_{i\bar{i},2} + 8\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{8}{9}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{32}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_0 n_f^2 C_A\left\{ + \frac{44}{3}f^I_{i\bar{i},1} + \frac{50}{9}A^I_{i\bar{i},1}\right\} + \mathcal{D}_2\overline{\mathcal{D}}_0 n_f^2 C_F\left\{ + \frac{10}{3}A^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_0 n_f^3\left\{ - \frac{8}{9}f^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_0\left\{ + \frac{1}{2}\left(f^I_{i\bar{i},1}\right)^4 + 3A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 + \frac{3}{2}\left(A^I_{i\bar{i},2}\right)^2 + 6A^I_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2} + 3A^I_{i\bar{i},1}A^I_{i\bar{i},3}\right.$$
$$+ \frac{3}{2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 + 6\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 6\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2} + 3\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2$$
$$+ \frac{3}{2}g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 + 6g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 6g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2} + 6g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2$$
$$+ 3\left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 - 35\zeta_3\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} - \frac{21}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 - \frac{9}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2}$$
$$\left. - 3\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - 3\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - \frac{9}{4}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^4\right\} + \mathcal{D}_2\overline{\mathcal{D}}_1 C_A\left\{ - \frac{220}{3}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\right.$$
$$\left. - 55A^I_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{275}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{110}{3}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{275}{3}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_1 C_A^2\left\{ - \frac{2420}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{170}{3}\left(A^I_{i\bar{i},1}\right)^2\right\} + \mathcal{D}_2\overline{\mathcal{D}}_1 C_A^3\left\{ - \frac{1331}{9}A^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_1 n_f\left\{ + \frac{40}{3}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 10A^I_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{50}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{20}{3}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\right.$$
$$\left. - \frac{50}{3}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\right\} + \mathcal{D}_2\overline{\mathcal{D}}_1 n_f C_A\left\{ + \frac{880}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{50}{3}\left(A^I_{i\bar{i},1}\right)^2\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_1 n_f C_A^2\left\{ + \frac{242}{3}A^I_{i\bar{i},1}\right\} + \mathcal{D}_2\overline{\mathcal{D}}_1 n_f C_F\left\{ + 10\left(A^I_{i\bar{i},1}\right)^2\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_1 n_f^2\left\{ - \frac{80}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\right\} + \mathcal{D}_2\overline{\mathcal{D}}_1 n_f^2 C_A\left\{ - \frac{44}{3}A^I_{i\bar{i},1}\right\} + \mathcal{D}_2\overline{\mathcal{D}}_1 n_f^3\left\{ + \frac{8}{9}A^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_1\left\{ - 5A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 15A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - \frac{15}{2}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - 15\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\right.$$
$$\left. - 15g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 35\zeta_3\left(A^I_{i\bar{i},1}\right)^4 + \frac{45}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_2\overline{\mathcal{D}}_2 C_A\left\{ + \frac{385}{4}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\right\} + \mathcal{D}_2\overline{\mathcal{D}}_2 C_A^2\left\{ + \frac{1210}{9}\left(A^I_{i\bar{i},1}\right)^2\right\}$$

$$
\begin{aligned}
&+ \mathcal{D}_2\overline{\mathcal{D}}_2 n_f\Big\{ - \frac{35}{2}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\Big\} + \mathcal{D}_2\overline{\mathcal{D}}_2 n_f C_A\Big\{ - \frac{440}{9}\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+ \mathcal{D}_2\overline{\mathcal{D}}_2 n_f^2\Big\{ + \frac{40}{9}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_2\overline{\mathcal{D}}_2\Big\{ + \frac{45}{4}\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + \frac{45}{4}\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2}\\
&+ \frac{15}{2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \frac{15}{2}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - \frac{45}{4}\zeta_2\left(A^I_{i\bar{i},1}\right)^4\Big\} + \mathcal{D}_2\overline{\mathcal{D}}_3 C_A\Big\{ - \frac{385}{12}\left(A^I_{i\bar{i},1}\right)^3\Big\}\\
&+ \mathcal{D}_2\overline{\mathcal{D}}_3 n_f\Big\{ + \frac{35}{6}\left(A^I_{i\bar{i},1}\right)^3\Big\} + \mathcal{D}_2\overline{\mathcal{D}}_3\Big\{ - \frac{35}{4}\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\} + \mathcal{D}_2\overline{\mathcal{D}}_4\Big\{ + \frac{35}{16}\left(A^I_{i\bar{i},1}\right)^4\Big\}\\
&+ \mathcal{D}_3\delta(1-z_2)\Big\{ + \frac{1}{6}\left(f^I_{i\bar{i},1}\right)^4 + A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 + \frac{1}{2}\left(A^I_{i\bar{i},2}\right)^2 + 2A^I_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2} + A^I_{i\bar{i},1}A^I_{i\bar{i},3}\\
&+ \frac{1}{2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 + 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2} + \left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2\\
&+ \frac{1}{2}g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 + 2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2} + 2g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&+ \left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 - \frac{35}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} - \frac{7}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 - \frac{3}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2}\\
&- \zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - \zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - \frac{3}{4}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^4\Big\} + \mathcal{D}_3\delta(1-z_2)C_A\Big\{ + \frac{11}{3}\left(f^I_{i\bar{i},1}\right)^3\\
&+ \frac{77}{9}A^I_{i\bar{i},2}f^I_{i\bar{i},1} + \frac{88}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},2} + \frac{11}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{242}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{11}{3}g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&+ \frac{110}{9}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{374}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^3 - \frac{451}{18}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 11\zeta_2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1}\\
&- 11\zeta_2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_3\delta(1-z_2)C_A^2\Big\{ + \frac{1331}{54}\left(f^I_{i\bar{i},1}\right)^2 + \frac{121}{9}A^I_{i\bar{i},2} + \frac{170}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+ \frac{242}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{242}{27}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{968}{27}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_3\delta(1-z_2)C_A^3\Big\{ + \frac{1331}{27}f^I_{i\bar{i},1}\\
&+ \frac{935}{27}A^I_{i\bar{i},1}\Big\} + \mathcal{D}_3\delta(1-z_2)n_f\Big\{ - \frac{2}{3}\left(f^I_{i\bar{i},1}\right)^3 - \frac{14}{9}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - \frac{16}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},2}\\
&- \frac{2}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{44}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{2}{3}g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{20}{9}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{68}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^3\\
&+ \frac{41}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 2\zeta_2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + 2\zeta_2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+ \mathcal{D}_3\delta(1-z_2)n_f C_A\Big\{ - \frac{242}{27}\left(f^I_{i\bar{i},1}\right)^2 - \frac{44}{9}A^I_{i\bar{i},2} - \frac{50}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&- \frac{88}{27}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{352}{27}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_3\delta(1-z_2)n_f C_A^2\Big\{ - \frac{242}{9}f^I_{i\bar{i},1} - \frac{445}{27}A^I_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_3\delta(1-z_2)n_f C_F\Big\{ - \frac{10}{3}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Big\} + \mathcal{D}_3\delta(1-z_2)n_f C_F C_A\Big\{ - \frac{55}{9}A^I_{i\bar{i},1}\Big\}\\
&+ \mathcal{D}_3\delta(1-z_2)n_f^2\Big\{ + \frac{22}{27}\left(f^I_{i\bar{i},1}\right)^2 + \frac{4}{9}A^I_{i\bar{i},2} + \frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{8}{27}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{32}{27}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Big\}
\end{aligned}
$$

$$+ \mathcal{D}_3\delta(1-z_2)n_f^2C_A\left\{ + \frac{44}{9}f^I_{i\bar{i},1} + \frac{50}{27}A^I_{i\bar{i},1}\right\} + \mathcal{D}_3\delta(1-z_2)n_f^2C_F\left\{ + \frac{10}{9}A^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_3\delta(1-z_2)n_f^3\left\{ - \frac{8}{27}f^I_{i\bar{i},1}\right\} + \mathcal{D}_3\overline{\mathcal{D}}_0C_A\left\{ - \frac{220}{9}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{55}{3}A^I_{i\bar{i},1}A^I_{i\bar{i},2}\right.$$
$$\left. - \frac{275}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{110}{9}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{275}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_0C_A^2\left\{ - \frac{2420}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{170}{9}\left(A^I_{i\bar{i},1}\right)^2\right\} + \mathcal{D}_3\overline{\mathcal{D}}_0C_A^3\left\{ - \frac{1331}{27}A^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_0n_f\left\{ + \frac{40}{9}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{10}{3}A^I_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{50}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{20}{9}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\right.$$
$$\left. - \frac{50}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\right\} + \mathcal{D}_3\overline{\mathcal{D}}_0n_fC_A\left\{ + \frac{880}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{50}{9}\left(A^I_{i\bar{i},1}\right)^2\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_0n_fC_A^2\left\{ + \frac{242}{9}A^I_{i\bar{i},1}\right\} + \mathcal{D}_3\overline{\mathcal{D}}_0n_fC_F\left\{ + \frac{10}{3}\left(A^I_{i\bar{i},1}\right)^2\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_0n_f^2\left\{ - \frac{80}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\right\} + \mathcal{D}_3\overline{\mathcal{D}}_0n_f^2C_A\left\{ - \frac{44}{9}A^I_{i\bar{i},1}\right\} + \mathcal{D}_3\overline{\mathcal{D}}_0n_f^3\left\{ + \frac{8}{27}A^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_0\left\{ - \frac{5}{3}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 5A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - \frac{5}{2}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},2} - 5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\right.$$
$$\left. - 5g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1} + \frac{35}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^4 + \frac{15}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_1C_A\left\{ + \frac{385}{6}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\right\} + \mathcal{D}_3\overline{\mathcal{D}}_1C_A^2\left\{ + \frac{2420}{27}\left(A^I_{i\bar{i},1}\right)^2\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_1n_f\left\{ - \frac{35}{3}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\right\} + \mathcal{D}_3\overline{\mathcal{D}}_1n_fC_A\left\{ - \frac{880}{27}\left(A^I_{i\bar{i},1}\right)^2\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_1n_f^2\left\{ + \frac{80}{27}\left(A^I_{i\bar{i},1}\right)^2\right\} + \mathcal{D}_3\overline{\mathcal{D}}_1\left\{ + \frac{15}{2}\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + \frac{15}{2}\left(A^I_{i\bar{i},1}\right)^2A^I_{i\bar{i},2}\right.$$
$$\left. + 5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + 5g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - \frac{15}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^4\right\} + \mathcal{D}_3\overline{\mathcal{D}}_2C_A\left\{ - \frac{385}{12}\left(A^I_{i\bar{i},1}\right)^3\right\}$$
$$+ \mathcal{D}_3\overline{\mathcal{D}}_2n_f\left\{ + \frac{35}{6}\left(A^I_{i\bar{i},1}\right)^3\right\} + \mathcal{D}_3\overline{\mathcal{D}}_2\left\{ - \frac{35}{4}\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}\right\} + \mathcal{D}_3\overline{\mathcal{D}}_3\left\{ + \frac{35}{12}\left(A^I_{i\bar{i},1}\right)^4\right\}$$
$$+ \mathcal{D}_4\delta(1-z_2)\left\{ - \frac{5}{12}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - \frac{5}{4}A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - \frac{5}{8}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},2}\right.$$
$$\left. - \frac{5}{4}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1} - \frac{5}{4}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1} + \frac{35}{12}\zeta_3\left(A^I_{i\bar{i},1}\right)^4 + \frac{15}{8}\zeta_2\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}\right\}$$
$$+ \mathcal{D}_4\delta(1-z_2)C_A\left\{ - \frac{55}{9}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{55}{12}A^I_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{275}{36}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\right.$$
$$\left. - \frac{55}{18}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{275}{36}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\right\} + \mathcal{D}_4\delta(1-z_2)C_A^2\left\{ - \frac{605}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{85}{18}\left(A^I_{i\bar{i},1}\right)^2\right\}$$

$$
\begin{aligned}
&+ \mathcal{D}_4\delta(1-z_2)C_A^3\Big\{ - \frac{1331}{108}A^I_{i\bar{i},1}\Big\} + \mathcal{D}_4\delta(1-z_2)n_f\Big\{ + \frac{10}{9}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{5}{6}A^I_{i\bar{i},1}A^I_{i\bar{i},2} \\
&+ \frac{25}{18}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{5}{9}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{25}{18}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\Big\} \\
&+ \mathcal{D}_4\delta(1-z_2)n_fC_A\Big\{ + \frac{220}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{25}{18}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_4\delta(1-z_2)n_fC_A^2\Big\{ + \frac{121}{18}A^I_{i\bar{i},1}\Big\} + \mathcal{D}_4\delta(1-z_2)n_fC_F\Big\{ + \frac{5}{6}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_4\delta(1-z_2)n_f^2\Big\{ - \frac{20}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Big\} + \mathcal{D}_4\delta(1-z_2)n_f^2C_A\Big\{ - \frac{11}{9}A^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_4\delta(1-z_2)n_f^3\Big\{ + \frac{2}{27}A^I_{i\bar{i},1}\Big\} + \mathcal{D}_4\overline{\mathcal{D}}_0C_A\Big\{ + \frac{385}{24}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_4\overline{\mathcal{D}}_0C_A^2\Big\{ + \frac{605}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_4\overline{\mathcal{D}}_0n_f\Big\{ - \frac{35}{12}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_4\overline{\mathcal{D}}_0n_fC_A\Big\{ - \frac{220}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_4\overline{\mathcal{D}}_0n_f^2\Big\{ + \frac{20}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_4\overline{\mathcal{D}}_0\Big\{ + \frac{15}{8}\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + \frac{15}{8}\left(A^I_{i\bar{i},1}\right)^2A^I_{i\bar{i},2} + \frac{5}{4}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \frac{5}{4}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 \\
&- \frac{15}{8}\zeta_2\left(A^I_{i\bar{i},1}\right)^4\Big\} + \mathcal{D}_4\overline{\mathcal{D}}_1C_A\Big\{ - \frac{385}{24}\left(A^I_{i\bar{i},1}\right)^3\Big\} + \mathcal{D}_4\overline{\mathcal{D}}_1n_f\Big\{ + \frac{35}{12}\left(A^I_{i\bar{i},1}\right)^3\Big\} \\
&+ \mathcal{D}_4\overline{\mathcal{D}}_1\Big\{ - \frac{35}{8}\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}\Big\} + \mathcal{D}_4\overline{\mathcal{D}}_2\Big\{ + \frac{35}{16}\left(A^I_{i\bar{i},1}\right)^4\Big\} \\
&+ \mathcal{D}_5\delta(1-z_2)\Big\{ + \frac{3}{8}\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + \frac{3}{8}\left(A^I_{i\bar{i},1}\right)^2A^I_{i\bar{i},2} + \frac{1}{4}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \frac{1}{4}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 \\
&- \frac{3}{8}\zeta_2\left(A^I_{i\bar{i},1}\right)^4\Big\} + \mathcal{D}_5\delta(1-z_2)C_A\Big\{ + \frac{77}{24}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_5\delta(1-z_2)C_A^2\Big\{ + \frac{121}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_5\delta(1-z_2)n_f\Big\{ - \frac{7}{12}\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\Big\} \\
&+ \mathcal{D}_5\delta(1-z_2)n_fC_A\Big\{ - \frac{44}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \mathcal{D}_5\delta(1-z_2)n_f^2\Big\{ + \frac{4}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} \\
&+ \mathcal{D}_5\overline{\mathcal{D}}_0C_A\Big\{ - \frac{77}{24}\left(A^I_{i\bar{i},1}\right)^3\Big\} + \mathcal{D}_5\overline{\mathcal{D}}_0n_f\Big\{ + \frac{7}{12}\left(A^I_{i\bar{i},1}\right)^3\Big\} \\
&+ \mathcal{D}_5\overline{\mathcal{D}}_0\Big\{ - \frac{7}{8}\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}\Big\} + \mathcal{D}_5\overline{\mathcal{D}}_1\Big\{ + \frac{7}{8}\left(A^I_{i\bar{i},1}\right)^4\Big\} \\
&+ \mathcal{D}_6\delta(1-z_2)\Big\{ - \frac{7}{48}\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}\Big\} + \mathcal{D}_6\delta(1-z_2)C_A\Big\{ - \frac{77}{144}\left(A^I_{i\bar{i},1}\right)^3\Big\} \\
&+ \mathcal{D}_6\delta(1-z_2)n_f\Big\{ + \frac{7}{72}\left(A^I_{i\bar{i},1}\right)^3\Big\} + \mathcal{D}_6\overline{\mathcal{D}}_0\Big\{ + \frac{7}{48}\left(A^I_{i\bar{i},1}\right)^4\Big\}
\end{aligned}
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\begin{aligned}
&+ \mathcal{D}_7\delta(1-z_2)\Big\{ + \frac{1}{48}\left(A^I_{i\bar{i},1}\right)^4 \Big\} + \overline{\mathcal{D}}_0\delta(1-z_1)\Big\{ - f^I_{i\bar{i},4} - \frac{2}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3} f^I_{i\bar{i},1} - \overline{\mathcal{G}}^{I,1}_{i\bar{i},2} f^I_{i\bar{i},2} \\
&- 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},3} - 2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} f^I_{i\bar{i},1} - 2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - \frac{4}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} - \frac{2}{3} g^{I,1}_{i\bar{i},3} f^I_{i\bar{i},1} \\
&- g^{I,1}_{i\bar{i},2} f^I_{i\bar{i},2} - 2g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} - 2g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},3} - 2g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} f^I_{i\bar{i},1} - 4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},2} \\
&- 4g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 2g^{I,1}_{i\bar{i},1} g^{I,1}_{i\bar{i},2} f^I_{i\bar{i},1} - 2\left(g^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - 4\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} \\
&- \frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} + 30\zeta_7\left(A^I_{i\bar{i},1}\right)^4 + 18\zeta_5\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + 18\zeta_5\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} \\
&+ 12\zeta_5\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + 12\zeta_5 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \frac{2}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^4 + 4\zeta_3 A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2 \\
&+ 2\zeta_3\left(A^I_{i\bar{i},2}\right)^2 + 8\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},1} f^I_{i\bar{i},2} + 4\zeta_3 A^I_{i\bar{i},1} A^I_{i\bar{i},3} + 2\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 \\
&+ 8\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 8\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} + 4\zeta_3\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 + 2\zeta_3 g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2 \\
&+ 8\zeta_3 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + 8\zeta_3 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} + 8\zeta_3 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 4\zeta_3\left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{70}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} + 3\zeta_2\left(f^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - \zeta_2 A^I_{i\bar{i},3} f^I_{i\bar{i},1} - \zeta_2 A^I_{i\bar{i},2} f^I_{i\bar{i},2} - \zeta_2 A^I_{i\bar{i},1} f^I_{i\bar{i},3} \\
&- \zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 2\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 2\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},2} f^I_{i\bar{i},1} - 2\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},2} \\
&- 2\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} f^I_{i\bar{i},1} - \zeta_2 g^{I,1}_{i\bar{i},2} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 2\zeta_2 g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 2\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},2} f^I_{i\bar{i},1} \\
&- 2\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},2} - 4\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 2\zeta_2\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} f^I_{i\bar{i},1} - 18\zeta_2\zeta_5\left(A^I_{i\bar{i},1}\right)^4 \\
&- 14\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 - 6\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} - 4\zeta_2\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - 4\zeta_2\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 \\
&+ \zeta_2^2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - \zeta_2^2 A^I_{i\bar{i},1} A^I_{i\bar{i},2} f^I_{i\bar{i},1} - \frac{1}{2}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - \zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
&- \zeta_2^2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 3\zeta_2^2\zeta_3\left(A^I_{i\bar{i},1}\right)^4 - \frac{1}{6}\zeta_2^3\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\} \\
&+ \overline{\mathcal{D}}_0\delta(1-z_1)C_A\Big\{ - \frac{22}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3} - \frac{44}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2} f^I_{i\bar{i},1} - \frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},2} - 22\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} \\
&- \frac{44}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{44}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3 - \frac{44}{9} g^{I,2}_{i\bar{i},2} f^I_{i\bar{i},1} - \frac{22}{3} g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{22}{3} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},2} \\
&- \frac{44}{3} g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{44}{3} g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} - \frac{44}{3} g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{88}{3} g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 \\
&- \frac{44}{3} g^{I,1}_{i\bar{i},1} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{44}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} + 154\zeta_5\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{44}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^3 \\
&+ \frac{88}{3}\zeta_3 A^I_{i\bar{i},2} f^I_{i\bar{i},1} + \frac{110}{3}\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},2} + \frac{44}{3}\zeta_3\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{308}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} \\
&+ \frac{44}{3}\zeta_3 g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 44\zeta_3 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} - \frac{242}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^3 - 11\zeta_2 f^I_{i\bar{i},1} f^I_{i\bar{i},2} \\
&- 44\zeta_2 B^I_{i\bar{i},2} f^I_{i\bar{i},1} - 22\zeta_2 B^I_{i\bar{i},1} f^I_{i\bar{i},2} - \frac{22}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} A^I_{i\bar{i},1} - \frac{22}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1}
\end{aligned}
$$

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\begin{aligned}
&+\frac{44}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-44\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{22}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}-\frac{44}{3}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}\\
&-\frac{22}{3}\zeta_2 g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{22}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-44\zeta_2 g^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{44}{3}\zeta_2 g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+44\zeta_2\gamma^I_{i\bar{i},2}f^I_{i\bar{i},1}+22\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},2}+44\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+44\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{308}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}+44\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2B^I_{i\bar{i},1}-44\zeta_2\zeta_3\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&-\frac{11}{3}\zeta_2^2A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-22\zeta_2^2A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{11}{3}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+22\zeta_2^2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}\\
&+\overline{\mathcal{D}}_0\delta(1-z_1)C_A^2\Bigg\{-\frac{484}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}-\frac{68}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}-\frac{968}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{136}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{484}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{136}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2-\frac{968}{27}g^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{136}{9}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{484}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-\frac{968}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{136}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{1936}{9}\zeta_5\left(A^I_{i\bar{i},1}\right)^2+\frac{242}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^2+68\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+\frac{2420}{9}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{34}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2-68\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{484}{9}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{242}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{484}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}-\frac{68}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{484}{3}\zeta_2 g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+68\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{484}{3}\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-\frac{484}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2+\frac{2783}{45}\zeta_2^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}+\overline{\mathcal{D}}_0\delta(1-z_1)C_A^3\Bigg\{-\frac{10648}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}-\frac{2992}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\\
&-\frac{2857}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}+\overline{\mathcal{D}}_0\delta(1-z_1)n_f\Bigg\{+\frac{4}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3}+\frac{8}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}f^I_{i\bar{i},1}+\frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},2}+4\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\\
&+\frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{8}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3+\frac{8}{9}g^{I,2}_{i\bar{i},2}f^I_{i\bar{i},1}+\frac{4}{3}g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{4}{3}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},2}+\frac{8}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+\frac{8}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{8}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{16}{3}g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{8}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{8}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\\
&-28\zeta_5\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}-\frac{8}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^3-\frac{16}{3}\zeta_3A^I_{i\bar{i},2}f^I_{i\bar{i},1}-\frac{20}{3}\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},2}-\frac{8}{3}\zeta_3\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&-\frac{56}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{8}{3}\zeta_3g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-8\zeta_3g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{44}{3}\zeta_3^2\left(A^I_{i\bar{i},1}\right)^3\\
&+2\zeta_2f^I_{i\bar{i},1}f^I_{i\bar{i},2}+8\zeta_2B^I_{i\bar{i},2}f^I_{i\bar{i},1}+4\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},2}+\frac{4}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}+\frac{4}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{8}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+8\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{4}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}+\frac{8}{3}\zeta_2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}\\
&+\frac{4}{3}\zeta_2g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{4}{3}\zeta_2g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+8\zeta_2g^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{8}{3}\zeta_2g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-8\zeta_2\gamma^I_{i\bar{i},2}f^I_{i\bar{i},1}-4\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},2}-8\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-8\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{56}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}\\
&-8\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2B^I_{i\bar{i},1}+8\zeta_2\zeta_3\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+\frac{2}{3}\zeta_2^2A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+4\zeta_2^2A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+\frac{2}{3}\zeta_2^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-4\zeta_2^2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}+\overline{\mathcal{D}}_0\delta(1-z_1)n_fC_A\Bigg\{+\frac{176}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}+\frac{20}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}
\end{aligned}
$$

$$+\frac{352}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{40}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{176}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{40}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{352}{27}g^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}$$
$$+\frac{40}{9}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{176}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{352}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{40}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{704}{9}\zeta_5\left(A^I_{i\bar{i},1}\right)^2$$
$$-\frac{88}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^2-20\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{880}{9}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{10}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2+20\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}$$
$$+\frac{176}{9}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{88}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{176}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}+\frac{20}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{176}{3}\zeta_2g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}$$
$$-20\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{176}{3}\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{176}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2-\frac{1012}{45}\zeta_2^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_0\delta(1-z_1)n_fC_A^2\Bigg\{+\frac{1936}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}+\frac{1424}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{1415}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_0\delta(1-z_1)n_fC_F\Bigg\{+4\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{8}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+8\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{8}{3}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+8g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}$$
$$-12\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}+2\zeta_2\left(f^I_{i\bar{i},1}\right)^2+12\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}+4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-12\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_0\delta(1-z_1)n_fC_FC_A\Bigg\{+\frac{176}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{205}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_0\delta(1-z_1)n_fC_F^2\Bigg\{-2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}+\overline{\mathcal{D}}_0\delta(1-z_1)n_f^2\Bigg\{-\frac{16}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}-\frac{32}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}$$
$$-\frac{16}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{32}{27}g^{I,3}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{16}{9}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{32}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{64}{9}\zeta_5\left(A^I_{i\bar{i},1}\right)^2$$
$$+\frac{8}{3}\zeta_3\left(f^I_{i\bar{i},1}\right)^2+\frac{80}{9}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{16}{9}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{8}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{16}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}B^I_{i\bar{i},1}$$
$$+\frac{16}{3}\zeta_2g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{16}{3}\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{16}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^2+\frac{92}{45}\zeta_2^2A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_0\delta(1-z_1)n_f^2C_A\Bigg\{-\frac{352}{9}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}-\frac{160}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{79}{27}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_0\delta(1-z_1)n_f^2C_F\Bigg\{-\frac{32}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{22}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}+\overline{\mathcal{D}}_0\delta(1-z_1)n_f^3\Bigg\{+\frac{64}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_1\delta(1-z_1)\Bigg\{+\left(f^I_{i\bar{i},2}\right)^2+2f^I_{i\bar{i},1}f^I_{i\bar{i},3}+A^I_{i\bar{i},4}+\frac{2}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},3}A^I_{i\bar{i},1}+\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2+\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},2}$$
$$+4\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2}+2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},3}+2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}+2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2+2\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},2}$$
$$+\frac{4}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^3A^I_{i\bar{i},1}+\frac{2}{3}g^{I,1}_{i\bar{i},3}A^I_{i\bar{i},1}+g^{I,1}_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2+g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},2}+2g^{I,1}_{i\bar{i},2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}$$
$$+4g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2}+2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},3}+2g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}+4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+4g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}$$
$$+4g^{I,1}_{i\bar{i},1}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},1}+2g^{I,1}_{i\bar{i},1}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}+2\left(g^{I,1}_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2+2\left(g^{I,1}_{i\bar{i},1}\right)^2A^I_{i\bar{i},2}$$
$$+4\left(g^{I,1}_{i\bar{i},1}\right)^2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^3A^I_{i\bar{i},1}-42\zeta_5\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}-\frac{20}{3}\zeta_3A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3$$

$$
\begin{aligned}
&- 20\zeta_3 A^I_{i\bar{i},1} A^I_{i\bar{i},2} f^I_{i\bar{i},1} - 10\zeta_3 \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - 20\zeta_3 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
&- 20\zeta_3 g^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{70}{3}\zeta_3^2 \left(A^I_{i\bar{i},1}\right)^4 - \zeta_2 \left(f^I_{i\bar{i},1}\right)^4 - 2\zeta_2 A^I_{i\bar{i},2} \left(f^I_{i\bar{i},1}\right)^2 + \zeta_2 \left(A^I_{i\bar{i},2}\right)^2 \\
&- 4\zeta_2 A^I_{i\bar{i},1} f^I_{i\bar{i},1} f^I_{i\bar{i},2} + 2\zeta_2 A^I_{i\bar{i},1} A^I_{i\bar{i},3} + \zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},2} \left(A^I_{i\bar{i},1}\right)^2 - 4\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 \\
&+ 4\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} + 2\zeta_2 \left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 \left(A^I_{i\bar{i},1}\right)^2 + \zeta_2 g^{I,1}_{i\bar{i},2} \left(A^I_{i\bar{i},1}\right)^2 - 4\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 \\
&+ 4\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} A^I_{i\bar{i},2} + 4\zeta_2 g^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 + 2\zeta_2 \left(g^{I,1}_{i\bar{i},1}\right)^2 \left(A^I_{i\bar{i},1}\right)^2 + 30\zeta_2\zeta_3 \left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1} \\
&- \frac{5}{2}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 \left(f^I_{i\bar{i},1}\right)^2 + \frac{3}{2}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2} + \zeta_2^2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^3 + \zeta_2^2 g^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^3 \\
&+ \frac{1}{6}\zeta_2^3 \left(A^I_{i\bar{i},1}\right)^4 \Big\} + \overline{\mathcal{D}}_1 \delta(1-z_1) C_A \Big\{ + 11 f^I_{i\bar{i},3} + \frac{44}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2} A^I_{i\bar{i},1} + \frac{55}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} f^I_{i\bar{i},1} \\
&+ \frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 + \frac{22}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},2} + \frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},2} + \frac{44}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{110}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
&+ \frac{44}{9} g^{I,2}_{i\bar{i},2} A^I_{i\bar{i},1} + \frac{11}{3} g^{I,1}_{i\bar{i},2} f^I_{i\bar{i},1} + \frac{22}{3} g^{I,2}_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 + \frac{22}{3} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},2} + \frac{44}{3} g^{I,2}_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&+ \frac{44}{3} g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},2} + \frac{44}{3} g^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + 44 g^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{44}{3} g^{I,1}_{i\bar{i},1} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{22}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} \\
&- 154\zeta_5 \left(A^I_{i\bar{i},1}\right)^3 - \frac{286}{3}\zeta_3 A^I_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 - 66\zeta_3 A^I_{i\bar{i},1} A^I_{i\bar{i},2} - \frac{352}{3}\zeta_3 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \\
&- 44\zeta_3 g^{I,1}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 - \frac{22}{3}\zeta_2 \left(f^I_{i\bar{i},1}\right)^3 + 22\zeta_2 B^I_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 - \frac{22}{3}\zeta_2 A^I_{i\bar{i},2} f^I_{i\bar{i},1} \\
&+ 22\zeta_2 A^I_{i\bar{i},2} B^I_{i\bar{i},1} + \frac{22}{3}\zeta_2 A^I_{i\bar{i},1} f^I_{i\bar{i},2} + 44\zeta_2 A^I_{i\bar{i},1} B^I_{i\bar{i},2} + \frac{22}{3}\zeta_2 \overline{\mathcal{G}}^{I,2}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \\
&- \frac{88}{3}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} f^I_{i\bar{i},1} + 44\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} B^I_{i\bar{i},1} + \frac{22}{3}\zeta_2 g^{I,2}_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 + 44\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} B^I_{i\bar{i},1} \\
&- 44\zeta_2 \gamma^I_{i\bar{i},2} A^I_{i\bar{i},1} - 22\zeta_2 \gamma^I_{i\bar{i},1} \left(f^I_{i\bar{i},1}\right)^2 - 22\zeta_2 \gamma^I_{i\bar{i},1} A^I_{i\bar{i},2} - 44\zeta_2 \gamma^I_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&- 44\zeta_2 \gamma^I_{i\bar{i},1} g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{374}{3}\zeta_2\zeta_3 \left(A^I_{i\bar{i},1}\right)^3 - \frac{11}{6}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 22\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} \\
&- 22\zeta_2^2 \gamma^I_{i\bar{i},1} \left(A^I_{i\bar{i},1}\right)^2 \Big\} + \overline{\mathcal{D}}_1 \delta(1-z_1) C_A^2 \Big\{ + \frac{68}{3} f^I_{i\bar{i},2} + \frac{242}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2} + \frac{968}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&+ \frac{1210}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{136}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + 68\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{484}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 + \frac{968}{27} g^{I,3}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&+ \frac{242}{9} g^{I,2}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{136}{9} g^{I,2}_{i\bar{i},1} A^I_{i\bar{i},1} + \frac{68}{3} g^{I,1}_{i\bar{i},1} f^I_{i\bar{i},1} + \frac{968}{9} g^{I,1}_{i\bar{i},1} \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} - \frac{2662}{9}\zeta_3 A^I_{i\bar{i},1} f^I_{i\bar{i},1} \\
&- 68\zeta_3 \left(A^I_{i\bar{i},1}\right)^2 - \frac{121}{3}\zeta_2 \left(f^I_{i\bar{i},1}\right)^2 + \frac{242}{3}\zeta_2 B^I_{i\bar{i},1} f^I_{i\bar{i},1} + 68\zeta_2 A^I_{i\bar{i},1} B^I_{i\bar{i},1} - \frac{968}{9}\zeta_2 \overline{\mathcal{G}}^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} \\
&- \frac{484}{3}\zeta_2 g^{I,1}_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{242}{3}\zeta_2 \gamma^I_{i\bar{i},1} f^I_{i\bar{i},1} - 68\zeta_2 \gamma^I_{i\bar{i},1} A^I_{i\bar{i},1} - \frac{2783}{45}\zeta_2^2 \left(A^I_{i\bar{i},1}\right)^2 \Big\} \\
&+ \overline{\mathcal{D}}_1 \delta(1-z_1) C_A^3 \Big\{ + \frac{2857}{54} f^I_{i\bar{i},1} + \frac{5324}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1} + \frac{3740}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{\mathcal{D}}_1\delta(1-z_1)n_f\Big\{-2f^I_{i\bar{i},3}-\frac{8}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1}-\frac{10}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}-\frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-\frac{4}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2}\\
&-\frac{16}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2}-\frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{20}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}-\frac{8}{9}g^{I,2}_{i\bar{i},2}A^I_{i\bar{i},1}-\frac{2}{3}g^{I,1}_{i\bar{i},2}f^I_{i\bar{i},1}\\
&-\frac{4}{3}g^{I,2}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2-\frac{4}{3}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},2}-\frac{8}{3}g^{I,2}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{8}{3}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},2}-\frac{8}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-8g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{8}{3}g^{I,1}_{i\bar{i},1}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{4}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}+28\zeta_5\left(A^I_{i\bar{i},1}\right)^3+\frac{52}{3}\zeta_3A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+12\zeta_3A^I_{i\bar{i},1}A^I_{i\bar{i},2}+\frac{64}{3}\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+8\zeta_3g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+\frac{4}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^3-4\zeta_2B^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+\frac{4}{3}\zeta_2A^I_{i\bar{i},2}f^I_{i\bar{i},1}-4\zeta_2A^I_{i\bar{i},2}B^I_{i\bar{i},1}-\frac{4}{3}\zeta_2A^I_{i\bar{i},1}f^I_{i\bar{i},2}-8\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},2}-\frac{4}{3}\zeta_2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&+\frac{16}{3}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-8\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1}-\frac{4}{3}\zeta_2g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-8\zeta_2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}B^I_{i\bar{i},1}\\
&+8\zeta_2\gamma^I_{i\bar{i},2}A^I_{i\bar{i},1}+4\zeta_2\gamma^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+4\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},2}+8\zeta_2\gamma^I_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+8\zeta_2\gamma^I_{i\bar{i},1}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-\frac{68}{3}\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^3+\frac{1}{3}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}-4\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2B^I_{i\bar{i},1}+4\zeta_2^2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\overline{\mathcal{D}}_1\delta(1-z_1)n_fC_A\Big\{-\frac{20}{3}f^I_{i\bar{i},2}-\frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}-\frac{352}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{440}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{40}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-20\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{176}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2-\frac{352}{27}g^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{88}{9}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{40}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{20}{3}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{352}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}+\frac{968}{9}\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}+20\zeta_3\left(A^I_{i\bar{i},1}\right)^2\\
&+\frac{44}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2-\frac{88}{3}\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}-20\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}+\frac{352}{9}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{176}{3}\zeta_2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+\frac{88}{3}\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}+20\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{1012}{45}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\overline{\mathcal{D}}_1\delta(1-z_1)n_fC_A^2\Big\{-\frac{1415}{54}f^I_{i\bar{i},1}-\frac{968}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}-\frac{1780}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\}\\
&+\overline{\mathcal{D}}_1\delta(1-z_1)n_fC_F\Big\{-4f^I_{i\bar{i},2}-\frac{8}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-12\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{8}{3}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-4g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&+12\zeta_3\left(A^I_{i\bar{i},1}\right)^2-12\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}+12\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Big\}+\overline{\mathcal{D}}_1\delta(1-z_1)n_fC_FC_A\Big\{-\frac{205}{18}f^I_{i\bar{i},1}\\
&-\frac{220}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\}+\overline{\mathcal{D}}_1\delta(1-z_1)n_fC_F^2\Big\{+f^I_{i\bar{i},1}\Big\}+\overline{\mathcal{D}}_1\delta(1-z_1)n_f^2\Big\{+\frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}+\frac{32}{27}\overline{\mathcal{G}}^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&+\frac{40}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{16}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2+\frac{32}{27}g^{I,3}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{8}{9}g^{I,2}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{32}{9}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}-\frac{88}{9}\zeta_3A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{4}{3}\zeta_2\left(f^I_{i\bar{i},1}\right)^2+\frac{8}{3}\zeta_2B^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{32}{9}\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{16}{3}\zeta_2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{8}{3}\zeta_2\gamma^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&-\frac{92}{45}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^2\Big\}+\overline{\mathcal{D}}_1\delta(1-z_1)n_f^2C_A\Big\{+\frac{79}{54}f^I_{i\bar{i},1}+\frac{176}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}+\frac{200}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\}
\end{aligned}
$$

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\begin{aligned}
&+ \overline{\mathcal{D}}_1\delta(1-z_1)n_f^2C_F\Big\{ + \frac{11}{9}f^I_{i\bar{i},1} + \frac{40}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_1\delta(1-z_1)n_f^3\Big\{ - \frac{32}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\Big\}\\
&+ \overline{\mathcal{D}}_2\delta(1-z_1)\Big\{ - \frac{3}{2}\left(f^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - \frac{3}{2}A^I_{i\bar{i},3}f^I_{i\bar{i},1} - \frac{3}{2}A^I_{i\bar{i},2}f^I_{i\bar{i},2} - \frac{3}{2}A^I_{i\bar{i},1}f^I_{i\bar{i},3}\\
&- \frac{3}{2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2} - 3\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&- \frac{3}{2}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - 3g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1} - 3g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},2} - 6g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&- 3\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 21\zeta_5\left(A^I_{i\bar{i},1}\right)^4 + 15\zeta_3\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + 15\zeta_3\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2}\\
&+ 10\zeta_3\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + 10\zeta_3 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + 3\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 + 3\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}\\
&+ \frac{3}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} + 3\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + 3\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 15\zeta_2\zeta_3\left(A^I_{i\bar{i},1}\right)^4\\
&+ \frac{9}{4}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_2\delta(1-z_1)C_A\Big\{ - \frac{33}{2}f^I_{i\bar{i},1}f^I_{i\bar{i},2} - \frac{11}{2}A^I_{i\bar{i},3} - \frac{77}{6}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}\\
&- 11\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 22\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{55}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{77}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} - \frac{11}{6}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1}\\
&- 11g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - 11g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{22}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} - \frac{88}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{11}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1}\\
&+ \frac{374}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{143}{6}\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - 33\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{33}{2}\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2}\\
&+ 22\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 11\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + 33\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{77}{12}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3\Big\}\\
&+ \overline{\mathcal{D}}_2\delta(1-z_1)C_A^2\Big\{ - \frac{121}{3}f^I_{i\bar{i},2} - 17\left(f^I_{i\bar{i},1}\right)^2 - \frac{34}{3}A^I_{i\bar{i},2} - \frac{847}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{1331}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}\\
&- \frac{136}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{121}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{242}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{34}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{1331}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2\\
&+ \frac{1573}{18}\zeta_2 A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{121}{3}\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1} + 17\zeta_2\left(A^I_{i\bar{i},1}\right)^2 + \frac{121}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Big\}\\
&+ \overline{\mathcal{D}}_2\delta(1-z_1)C_A^3\Big\{ - \frac{935}{9}f^I_{i\bar{i},1} - \frac{2857}{108}A^I_{i\bar{i},1} - \frac{2662}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Big\}\\
&+ \overline{\mathcal{D}}_2\delta(1-z_1)n_f\Big\{ + 3f^I_{i\bar{i},1}f^I_{i\bar{i},2} + A^I_{i\bar{i},3} + \frac{7}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 2\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 4\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+ \frac{10}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{14}{3}\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} + \frac{1}{3}g^{I,1}_{i\bar{i},2}A^I_{i\bar{i},1} + 2g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + 2g^{I,1}_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+ \frac{4}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{16}{3}g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{2}{3}\left(g^{I,1}_{i\bar{i},1}\right)^2 A^I_{i\bar{i},1} - \frac{68}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - \frac{13}{3}\zeta_2 A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2\\
&+ 6\zeta_2 A^I_{i\bar{i},1}B^I_{i\bar{i},1}f^I_{i\bar{i},1} - 3\zeta_2 A^I_{i\bar{i},1}A^I_{i\bar{i},2} - 4\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - 2\zeta_2 g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - 6\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\\
&- \frac{7}{6}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^3\Big\} + \overline{\mathcal{D}}_2\delta(1-z_1)n_fC_A\Big\{ + \frac{44}{3}f^I_{i\bar{i},2} + 5\left(f^I_{i\bar{i},1}\right)^2 + \frac{10}{3}A^I_{i\bar{i},2} + \frac{308}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}
\end{aligned}
$$

$$+\frac{484}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{40}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{44}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{88}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{10}{3}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}$$
$$-\frac{484}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2-\frac{286}{9}\zeta_2A^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{44}{3}\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}-5\zeta_2\left(A^I_{i\bar{i},1}\right)^2-\frac{44}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_2\delta(1-z_1)n_fC_A^2\Bigg\{+\frac{445}{9}f^I_{i\bar{i},1}+\frac{1415}{108}A^I_{i\bar{i},1}+\frac{484}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_2\delta(1-z_1)n_fC_F\Bigg\{+3\left(f^I_{i\bar{i},1}\right)^2+2A^I_{i\bar{i},2}+8\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-3\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Bigg\}$$
$$+\overline{\mathcal{D}}_2\delta(1-z_1)n_fC_FC_A\Bigg\{+\frac{55}{3}f^I_{i\bar{i},1}+\frac{205}{36}A^I_{i\bar{i},1}\Bigg\}+\overline{\mathcal{D}}_2\delta(1-z_1)n_fC_F^2\Bigg\{-\frac{1}{2}A^I_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_2\delta(1-z_1)n_f^2\Bigg\{-\frac{4}{3}f^I_{i\bar{i},2}-\frac{28}{9}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{44}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{4}{9}g^{I,2}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{8}{9}g^{I,1}_{i\bar{i},1}f^I_{i\bar{i},1}$$
$$+\frac{44}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^2+\frac{26}{9}\zeta_2A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{4}{3}\zeta_2A^I_{i\bar{i},1}B^I_{i\bar{i},1}+\frac{4}{3}\zeta_2\gamma^I_{i\bar{i},1}A^I_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_2\delta(1-z_1)n_f^2C_A\Bigg\{-\frac{50}{9}f^I_{i\bar{i},1}-\frac{79}{108}A^I_{i\bar{i},1}-\frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_2\delta(1-z_1)n_f^2C_F\Bigg\{-\frac{10}{3}f^I_{i\bar{i},1}-\frac{11}{18}A^I_{i\bar{i},1}\Bigg\}+\overline{\mathcal{D}}_2\delta(1-z_1)n_f^3\Bigg\{+\frac{16}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\Bigg\}$$
$$+\overline{\mathcal{D}}_3\delta(1-z_1)\Bigg\{+\frac{1}{6}\left(f^I_{i\bar{i},1}\right)^4+A^I_{i\bar{i},2}\left(f^I_{i\bar{i},1}\right)^2+\frac{1}{2}\left(A^I_{i\bar{i},2}\right)^2+2A^I_{i\bar{i},1}f^I_{i\bar{i},1}f^I_{i\bar{i},2}+A^I_{i\bar{i},1}A^I_{i\bar{i},3}$$
$$+\frac{1}{2}\overline{\mathcal{G}}^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2+2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2}+\left(\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2$$
$$+\frac{1}{2}g^{I,1}_{i\bar{i},2}\left(A^I_{i\bar{i},1}\right)^2+2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2+2g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}A^I_{i\bar{i},2}+2g^{I,1}_{i\bar{i},1}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2$$
$$+\left(g^{I,1}_{i\bar{i},1}\right)^2\left(A^I_{i\bar{i},1}\right)^2-\frac{35}{3}\zeta_3\left(A^I_{i\bar{i},1}\right)^3f^I_{i\bar{i},1}-\frac{7}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2-\frac{3}{2}\zeta_2\left(A^I_{i\bar{i},1}\right)^2A^I_{i\bar{i},2}$$
$$-\zeta_2\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3-\zeta_2g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3-\frac{3}{4}\zeta_2^2\left(A^I_{i\bar{i},1}\right)^4\Bigg\}+\overline{\mathcal{D}}_3\delta(1-z_1)C_A\Bigg\{+\frac{11}{3}\left(f^I_{i\bar{i},1}\right)^3$$
$$+\frac{77}{9}A^I_{i\bar{i},2}f^I_{i\bar{i},1}+\frac{88}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},2}+\frac{11}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2+\frac{242}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{11}{3}g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2$$
$$+\frac{110}{9}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{374}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^3-\frac{451}{18}\zeta_2\left(A^I_{i\bar{i},1}\right)^2f^I_{i\bar{i},1}+11\zeta_2\left(A^I_{i\bar{i},1}\right)^2B^I_{i\bar{i},1}$$
$$-11\zeta_2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Bigg\}+\overline{\mathcal{D}}_3\delta(1-z_1)C_A^2\Bigg\{+\frac{1331}{54}\left(f^I_{i\bar{i},1}\right)^2+\frac{121}{9}A^I_{i\bar{i},2}+\frac{170}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},1}$$
$$+\frac{242}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}+\frac{242}{27}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}-\frac{968}{27}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Bigg\}+\overline{\mathcal{D}}_3\delta(1-z_1)C_A^3\Bigg\{+\frac{1331}{27}f^I_{i\bar{i},1}$$
$$+\frac{935}{27}A^I_{i\bar{i},1}\Bigg\}+\overline{\mathcal{D}}_3\delta(1-z_1)n_f\Bigg\{-\frac{2}{3}\left(f^I_{i\bar{i},1}\right)^3-\frac{14}{9}A^I_{i\bar{i},2}f^I_{i\bar{i},1}-\frac{16}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},2}$$
$$-\frac{2}{3}\overline{\mathcal{G}}^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-\frac{44}{9}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}-\frac{2}{3}g^{I,2}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2-\frac{20}{9}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}f^I_{i\bar{i},1}+\frac{68}{9}\zeta_3\left(A^I_{i\bar{i},1}\right)^3$$

$$
\begin{aligned}
&+\frac{41}{9}\zeta_2\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - 2\zeta_2\left(A^I_{i\bar{i},1}\right)^2 B^I_{i\bar{i},1} + 2\zeta_2\gamma^I_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\overline{\mathcal{D}}_3\delta(1-z_1)n_f C_A\Big\{-\frac{242}{27}\left(f^I_{i\bar{i},1}\right)^2 - \frac{44}{9}A^I_{i\bar{i},2} - \frac{50}{9}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{88}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1}\\
&-\frac{88}{27}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{352}{27}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Big\} + \overline{\mathcal{D}}_3\delta(1-z_1)n_f C_A^2\Big\{-\frac{242}{9}f^I_{i\bar{i},1} - \frac{445}{27}A^I_{i\bar{i},1}\Big\}\\
&+\overline{\mathcal{D}}_3\delta(1-z_1)n_f C_F\Big\{-\frac{10}{3}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_3\delta(1-z_1)n_f C_F C_A\Big\{-\frac{55}{9}A^I_{i\bar{i},1}\Big\}\\
&+\overline{\mathcal{D}}_3\delta(1-z_1)n_f^2\Big\{+\frac{22}{27}\left(f^I_{i\bar{i},1}\right)^2 + \frac{4}{9}A^I_{i\bar{i},2} + \frac{8}{3}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} + \frac{8}{27}g^{I,1}_{i\bar{i},1}A^I_{i\bar{i},1} - \frac{32}{27}\zeta_2\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\overline{\mathcal{D}}_3\delta(1-z_1)n_f^2 C_A\Big\{+\frac{44}{9}f^I_{i\bar{i},1} + \frac{50}{27}A^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_3\delta(1-z_1)n_f^2 C_F\Big\{+\frac{10}{9}A^I_{i\bar{i},1}\Big\}\\
&+\overline{\mathcal{D}}_3\delta(1-z_1)n_f^3\Big\{-\frac{8}{27}f^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_4\delta(1-z_1)\Big\{-\frac{5}{12}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^3 - \frac{5}{4}A^I_{i\bar{i},1}A^I_{i\bar{i},2}f^I_{i\bar{i},1}\\
&-\frac{5}{8}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},2} - \frac{5}{4}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} - \frac{5}{4}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1} + \frac{35}{12}\zeta_3\left(A^I_{i\bar{i},1}\right)^4\\
&+\frac{15}{8}\zeta_2\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_4\delta(1-z_1)C_A\Big\{-\frac{55}{9}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 - \frac{55}{12}A^I_{i\bar{i},1}A^I_{i\bar{i},2}\\
&-\frac{275}{36}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 - \frac{55}{18}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{275}{36}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\Big\}\\
&+\overline{\mathcal{D}}_4\delta(1-z_1)C_A^2\Big\{-\frac{605}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1} - \frac{85}{18}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \overline{\mathcal{D}}_4\delta(1-z_1)C_A^3\Big\{-\frac{1331}{108}A^I_{i\bar{i},1}\Big\}\\
&+\overline{\mathcal{D}}_4\delta(1-z_1)n_f\Big\{+\frac{10}{9}A^I_{i\bar{i},1}\left(f^I_{i\bar{i},1}\right)^2 + \frac{5}{6}A^I_{i\bar{i},1}A^I_{i\bar{i},2} + \frac{25}{18}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2 + \frac{5}{9}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^2\\
&-\frac{25}{18}\zeta_2\left(A^I_{i\bar{i},1}\right)^3\Big\} + \overline{\mathcal{D}}_4\delta(1-z_1)n_f C_A\Big\{+\frac{220}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1} + \frac{25}{18}\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\overline{\mathcal{D}}_4\delta(1-z_1)n_f C_A^2\Big\{+\frac{121}{18}A^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_4\delta(1-z_1)n_f C_F\Big\{+\frac{5}{6}\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\overline{\mathcal{D}}_4\delta(1-z_1)n_f^2\Big\{-\frac{20}{27}A^I_{i\bar{i},1}f^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_4\delta(1-z_1)n_f^2 C_A\Big\{-\frac{11}{9}A^I_{i\bar{i},1}\Big\}\\
&+\overline{\mathcal{D}}_4\delta(1-z_1)n_f^3\Big\{+\frac{2}{27}A^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_5\delta(1-z_1)\Big\{+\frac{3}{8}\left(A^I_{i\bar{i},1}\right)^2\left(f^I_{i\bar{i},1}\right)^2 + \frac{3}{8}\left(A^I_{i\bar{i},1}\right)^2 A^I_{i\bar{i},2}\\
&+\frac{1}{4}\overline{\mathcal{G}}^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 + \frac{1}{4}g^{I,1}_{i\bar{i},1}\left(A^I_{i\bar{i},1}\right)^3 - \frac{3}{8}\zeta_2\left(A^I_{i\bar{i},1}\right)^4\Big\}\\
&+\overline{\mathcal{D}}_5\delta(1-z_1)C_A\Big\{+\frac{77}{24}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_5\delta(1-z_1)C_A^2\Big\{+\frac{121}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\overline{\mathcal{D}}_5\delta(1-z_1)n_f\Big\{-\frac{7}{12}\left(A^I_{i\bar{i},1}\right)^2 f^I_{i\bar{i},1}\Big\} + \overline{\mathcal{D}}_5\delta(1-z_1)n_f C_A\Big\{-\frac{44}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\}\\
&+\overline{\mathcal{D}}_5\delta(1-z_1)n_f^2\Big\{+\frac{4}{27}\left(A^I_{i\bar{i},1}\right)^2\Big\} + \overline{\mathcal{D}}_6\delta(1-z_1)\Big\{-\frac{7}{48}\left(A^I_{i\bar{i},1}\right)^3 f^I_{i\bar{i},1}\Big\}
\end{aligned}
$$

$$+\overline{\mathcal{D}}_6\delta(1-z_1)C_A\left\{-\frac{77}{144}\left(A^I_{i\bar{i},1}\right)^3\right\}+\overline{\mathcal{D}}_6\delta(1-z_1)n_f\left\{+\frac{7}{72}\left(A^I_{i\bar{i},1}\right)^3\right\}$$
$$+\overline{\mathcal{D}}_7\delta(1-z_1)\left\{+\frac{1}{48}\left(A^I_{i\bar{i},1}\right)^4\right\}. \tag{G.0.1}$$

H Results of the Unrenormalised Three Loop Form Factors for the Pseudo-Scalar

In this appendix, we present the unrenormalised quark and gluon form factors for the pseudo-scalar production up to three loops for the operators $[O_G]_B$ and $[O_J]_B$. Specifically, we present $\hat{\mathcal{F}}_\beta^{G,(n)}$ and $\hat{\mathcal{F}}_\beta^{J,(n)}$ for $\beta = q, g$ up to $n = 3$ which are defined in Sec. 5.3. One and two loop results completely agree with the existing literature [84]. It should be noted that the form factors at $n = 2$ for $\hat{\mathcal{F}}_q^{G,(n)}$ and $\hat{\mathcal{F}}_g^{J,(n)}$ correspond to the contributions arising from three loop diagrams since these processes start at one loop order.

$$\hat{\mathcal{F}}_g^{G,(1)} = C_A\Bigg\{ -\frac{8}{\epsilon^2} + 4 + \zeta_2 + \epsilon\left(-6 - \frac{7}{3}\zeta_3 \right) + \epsilon^2\left(7 - \frac{\zeta_2}{2} + \frac{47}{80}\zeta_2^2 \right) + \epsilon^3\Bigg(-\frac{15}{2} + \frac{3}{4}\zeta_2 + \frac{7}{6}\zeta_3 + \frac{7}{24}\zeta_2\zeta_3 - \frac{31}{20}\zeta_5 \Bigg)\Bigg\}, \tag{H.0.1}$$

$$\hat{\mathcal{F}}_g^{G,(2)} = 2C_A n_f T_F\Bigg\{ -\frac{8}{3\epsilon^3} + \frac{20}{9\epsilon^2} + \left(\frac{106}{27} + 2\zeta_2 \right)\frac{1}{\epsilon} - \frac{1591}{81} - \frac{5}{3}\zeta_2 - \frac{74}{9}\zeta_3 + \epsilon\Bigg(\frac{24107}{486} - \frac{23}{18}\zeta_2 + \frac{51}{20}\zeta_2^2 + \frac{383}{27}\zeta_3 \Bigg) + \epsilon^2\Bigg(-\frac{146147}{1458} + \frac{799}{108}\zeta_2 - \frac{329}{72}\zeta_2^2 - \frac{1436}{81}\zeta_3 + \frac{25}{6}\zeta_2\zeta_3$$

$$
- \frac{271}{30}\zeta_5\Big)\Big\} + C_A^2\Big\{\frac{32}{\epsilon^4} + \frac{44}{3\epsilon^3} + \Big(-\frac{422}{9} - 4\zeta_2\Big)\frac{1}{\epsilon^2} + \Big(\frac{890}{27} - 11\zeta_2 + \frac{50}{3}\zeta_3\Big)\frac{1}{\epsilon}
$$
$$
+ \frac{3835}{81} + \frac{115}{6}\zeta_2 - \frac{21}{5}\zeta_2^2 + \frac{11}{9}\zeta_3 + \epsilon\Big(-\frac{213817}{972} - \frac{103}{18}\zeta_2 + \frac{77}{120}\zeta_2^2 + \frac{1103}{54}\zeta_3
$$
$$
- \frac{23}{6}\zeta_2\zeta_3 - \frac{71}{10}\zeta_5\Big) + \epsilon^2\Big(\frac{6102745}{11664} - \frac{991}{27}\zeta_2 - \frac{2183}{240}\zeta_2^2 + \frac{2313}{280}\zeta_2^3 - \frac{8836}{81}\zeta_3 - \frac{55}{12}\zeta_2\zeta_3
$$
$$
+ \frac{901}{36}\zeta_3^2 + \frac{341}{60}\zeta_5\Big)\Big\} + 2C_F n_f T_F\Big\{\frac{12}{\epsilon} - \frac{125}{3} + 8\zeta_3 + \epsilon\Big(\frac{3421}{36} - \frac{14}{3}\zeta_2 - \frac{8}{3}\zeta_2^2
$$
$$
- \frac{64}{3}\zeta_3\Big) + \epsilon^2\Big(-\frac{78029}{432} + \frac{293}{18}\zeta_2 + \frac{64}{9}\zeta_2^2 + \frac{973}{18}\zeta_3 - \frac{10}{3}\zeta_2\zeta_3 + 8\zeta_5\Big)\Big\}, \tag{H.0.2}
$$

$$
\hat{\mathcal{F}}_g^{G,(3)} = 4C_F n_f^2 T_F^2\Big\{\frac{16}{\epsilon^2} + \Big(-\frac{796}{9} + \frac{64}{3}\zeta_3\Big)\frac{1}{\epsilon} + \frac{8387}{27} - \frac{38}{3}\zeta_2 - \frac{112}{15}\zeta_2^2 - \frac{848}{9}\zeta_3\Big\}
$$
$$
+ 2C_F^2 n_f T_F\Big\{\frac{6}{\epsilon} - \frac{353}{6} + 176\zeta_3 - 160\zeta_5\Big\} + 2C_A^2 n_f T_F\Big\{\frac{64}{3\epsilon^5} - \frac{32}{81\epsilon^4} + \Big(-\frac{18752}{243}
$$
$$
- \frac{376}{27}\zeta_2\Big)\frac{1}{\epsilon^3} + \Big(\frac{36416}{243} - \frac{1700}{81}\zeta_2 + \frac{2072}{27}\zeta_3\Big)\frac{1}{\epsilon^2} + \Big(\frac{62642}{2187} + \frac{22088}{243}\zeta_2 - \frac{2453}{90}\zeta_2^2
$$
$$
- \frac{3988}{81}\zeta_3\Big)\frac{1}{\epsilon} - \frac{14655809}{13122} - \frac{60548}{729}\zeta_2 + \frac{917}{60}\zeta_2^2 - \frac{772}{27}\zeta_3 - \frac{439}{9}\zeta_2\zeta_3 + \frac{3238}{45}\zeta_5\Big\}
$$
$$
+ 4C_A n_f^2 T_F^2\Big\{-\frac{128}{81\epsilon^4} + \frac{640}{243\epsilon^3} + \Big(\frac{128}{27} + \frac{80}{27}\zeta_2\Big)\frac{1}{\epsilon^2} + \Big(-\frac{93088}{2187} - \frac{400}{81}\zeta_2
$$
$$
- \frac{1328}{81}\zeta_3\Big)\frac{1}{\epsilon} + \frac{1066349}{6561} - \frac{56}{27}\zeta_2 + \frac{797}{135}\zeta_2^2 + \frac{13768}{243}\zeta_3\Big\} + 2C_A C_F n_f T_F\Big\{-\frac{880}{9\epsilon^3}
$$
$$
+ \Big(\frac{6844}{27} - \frac{640}{9}\zeta_3\Big)\frac{1}{\epsilon^2} + \Big(-\frac{16219}{81} + \frac{158}{3}\zeta_2 + \frac{352}{15}\zeta_2^2 + \frac{1744}{27}\zeta_3\Big)\frac{1}{\epsilon} - \frac{753917}{972}
$$
$$
- \frac{593}{6}\zeta_2 - \frac{96}{5}\zeta_2^2 + \frac{4934}{81}\zeta_3 + 48\zeta_2\zeta_3 + \frac{32}{9}\zeta_5\Big\} + C_A^3\Big\{-\frac{256}{3\epsilon^6} - \frac{352}{3\epsilon^5} + \frac{16144}{81\epsilon^4}
$$
$$
+ \Big(\frac{22864}{243} + \frac{2068}{27}\zeta_2 - \frac{176}{3}\zeta_3\Big)\frac{1}{\epsilon^3} + \Big(-\frac{172844}{243} - \frac{1630}{81}\zeta_2 + \frac{494}{45}\zeta_2^2 - \frac{836}{27}\zeta_3\Big)\frac{1}{\epsilon^2}
$$
$$
+ \Big(\frac{2327399}{2187} - \frac{71438}{243}\zeta_2 + \frac{3751}{180}\zeta_2^2 - \frac{842}{9}\zeta_3 + \frac{170}{9}\zeta_2\zeta_3 + \frac{1756}{15}\zeta_5\Big)\frac{1}{\epsilon} + \frac{16531853}{26244}
$$
$$
+ \frac{918931}{1458}\zeta_2 + \frac{27251}{1080}\zeta_2^2 - \frac{22523}{270}\zeta_2^3 - \frac{51580}{243}\zeta_3 + \frac{77}{18}\zeta_2\zeta_3 - \frac{1766}{9}\zeta_3^2 + \frac{20911}{45}\zeta_5\Big\}, \tag{H.0.3}
$$

$$\hat{\mathcal{F}}_g^{J,(1)} = C_A\Bigg\{ -\frac{8}{\epsilon^2} + 4 + \zeta_2 + \epsilon\Bigg(-\frac{15}{2} + \zeta_2 - \frac{16}{3}\zeta_3\Bigg) + \epsilon^2\Bigg(\frac{287}{24} - 2\zeta_2 + \frac{127}{80}\zeta_2^2\Bigg) + \epsilon^3\Bigg(-\frac{5239}{288} + \frac{151}{48}\zeta_2 + \frac{19}{120}\zeta_2^2 + \frac{\zeta_3}{12} + \frac{7}{6}\zeta_2\zeta_3 - \frac{91}{20}\zeta_5\Bigg)\Bigg\} + C_F\Bigg\{4 + \epsilon\Bigg(-\frac{21}{2} + 6\zeta_3\Bigg) + \epsilon^2\Bigg(\frac{155}{8} - \frac{5}{2}\zeta_2 - \frac{9}{5}\zeta_2^2 - \frac{9}{2}\zeta_3\Bigg) + \epsilon^3\Bigg(-\frac{1025}{32} + \frac{83}{16}\zeta_2 + \frac{27}{20}\zeta_2^2 + \frac{20}{3}\zeta_3 - \frac{3}{4}\zeta_2\zeta_3 + \frac{21}{2}\zeta_5\Bigg)\Bigg\}, \tag{H.0.4}$$

$$\hat{\mathcal{F}}_g^{J,(2)} = 2C_A n_f T_F\Bigg\{ -\frac{8}{3\epsilon^3} + \frac{20}{9\epsilon^2} + \Bigg(\frac{106}{27} + 2\zeta_2\Bigg)\frac{1}{\epsilon} - \frac{1753}{81} - \frac{\zeta_2}{3} - \frac{110}{9}\zeta_3 + \epsilon\Bigg(\frac{14902}{243} - \frac{103}{18}\zeta_2 + \frac{241}{60}\zeta_2^2 + \frac{599}{27}\zeta_3\Bigg) + \epsilon^2\Bigg(-\frac{411931}{2916} + \frac{2045}{108}\zeta_2 - \frac{2353}{360}\zeta_2^2 - \frac{3128}{81}\zeta_3 + \frac{43}{6}\zeta_2\zeta_3 - \frac{167}{10}\zeta_5\Bigg)\Bigg\} + C_A C_F\Bigg\{ -\frac{32}{\epsilon^2} + \Bigg(\frac{208}{3} - 48\zeta_3\Bigg)\frac{1}{\epsilon} - \frac{451}{9} + 24\zeta_2 + \frac{72}{5}\zeta_2^2 - 8\zeta_3 + \epsilon\Bigg(-\frac{16385}{108} - \frac{52}{3}\zeta_2 + \frac{12}{5}\zeta_2^2 + 32\zeta_3 + 10\zeta_2\zeta_3 - 14\zeta_5\Bigg) + \epsilon^2\Bigg(\frac{1073477}{1296} - \frac{815}{9}\zeta_2 + \frac{19}{20}\zeta_2^2 + \frac{17}{70}\zeta_2^3 - \frac{1915}{36}\zeta_3 + 9\zeta_2\zeta_3 - 34\zeta_3^2 - \frac{2279}{6}\zeta_5\Bigg)\Bigg\} + 2C_F n_f T_F\Bigg\{\frac{26}{3\epsilon} - \frac{709}{18} + 16\zeta_3 + \epsilon\Bigg(\frac{26149}{216} - \frac{65}{6}\zeta_2 - \frac{76}{15}\zeta_2^2 - 44\zeta_3\Bigg) + \epsilon^2\Bigg(-\frac{828061}{2592} + \frac{3229}{72}\zeta_2 + \frac{212}{15}\zeta_2^2 + \frac{1729}{18}\zeta_3 - 4\zeta_2\zeta_3 + \frac{166}{3}\zeta_5\Bigg)\Bigg\} + C_A^2\Bigg\{\frac{32}{\epsilon^4} + \frac{44}{3\epsilon^3} + \Bigg(-\frac{422}{9} - 4\zeta_2\Bigg)\frac{1}{\epsilon^2} + \Bigg(\frac{1214}{27} - 19\zeta_2 + \frac{122}{3}\zeta_3\Bigg)\frac{1}{\epsilon} + \frac{1513}{81} + \frac{143}{6}\zeta_2 - \frac{61}{5}\zeta_2^2 + \frac{209}{9}\zeta_3 + \epsilon\Bigg(-\frac{202747}{972} + \frac{59}{36}\zeta_2 - \frac{349}{24}\zeta_2^2 - \frac{2393}{108}\zeta_3 - \frac{53}{6}\zeta_2\zeta_3 + \frac{369}{10}\zeta_5\Bigg) + \epsilon^2\Bigg(\frac{7681921}{11664} - \frac{35255}{432}\zeta_2 + \frac{1711}{180}\zeta_2^2 - \frac{7591}{840}\zeta_2^3 - \frac{5683}{1296}\zeta_3 - \frac{407}{12}\zeta_2\zeta_3 + \frac{775}{36}\zeta_3^2 + \frac{4013}{30}\zeta_5\Bigg)\Bigg\} + C_F^2\Bigg\{ -6 + \epsilon\Bigg(\frac{259}{12} + 41\zeta_3 - 60\zeta_5\Bigg) + \epsilon^2\Bigg(-\frac{7697}{144} + \frac{\zeta_2}{3} - \frac{184}{15}\zeta_2^2 + \frac{120}{7}\zeta_2^3 - 163\zeta_3 + 4\zeta_2\zeta_3 + 30\zeta_3^2 + \frac{470}{3}\zeta_5\Bigg)\Bigg\}, \tag{H.0.5}$$

$$\hat{\mathcal{F}}_q^{G,(1)} = 2n_f T_F\left\{\frac{4}{3\epsilon} - \frac{19}{9} + \epsilon\left(\frac{355}{108} - \frac{\zeta_2}{6}\right) + \epsilon^2\left(-\frac{6523}{1296} + \frac{19}{72}\zeta_2 + \frac{25}{18}\zeta_3\right) + \epsilon^3\left(\frac{118675}{15552} - \frac{355}{864}\zeta_2 - \frac{191}{480}\zeta_2^2 - \frac{475}{216}\zeta_3\right)\right\} + C_F\left\{-\frac{8}{\epsilon^2} + \frac{6}{\epsilon} - \frac{11}{2} + \zeta_2 + \epsilon\left(\frac{25}{8} - \frac{3}{4}\zeta_2 - \frac{7}{3}\zeta_3\right) + \epsilon^2\left(-\frac{11}{32} - \frac{21}{16}\zeta_2 + \frac{47}{80}\zeta_2^2 + \frac{7}{4}\zeta_3\right) + \epsilon^3\left(-\frac{415}{128} + \frac{223}{64}\zeta_2 - \frac{141}{320}\zeta_2^2 - \frac{155}{48}\zeta_3 + \frac{7}{24}\zeta_2\zeta_3 - \frac{31}{20}\zeta_5\right)\right\} + C_A\left\{-\frac{22}{3\epsilon} + \frac{269}{18} + \epsilon\left(-\frac{5045}{216} + \frac{23}{12}\zeta_2 + 3\zeta_3\right) + \epsilon^2\left(\frac{90893}{2592} - \frac{485}{144}\zeta_2 - \frac{4}{5}\zeta_2^2 - \frac{275}{36}\zeta_3\right) + \epsilon^3\left(-\frac{1620341}{31104} + \frac{8861}{1728}\zeta_2 + \frac{751}{320}\zeta_2^2 + \frac{4961}{432}\zeta_3 + \frac{\zeta_2\zeta_3}{8} + \frac{15}{2}\zeta_5\right)\right\}, \tag{H.0.6}$$

$$\hat{\mathcal{F}}_q^{G,(2)} = 4n_f^2 T_F^2\left\{\frac{16}{9\epsilon^2} - \frac{152}{27\epsilon} + \frac{124}{9} - \frac{4}{9}\zeta_2 + \epsilon\left(-\frac{7426}{243} + \frac{38}{27}\zeta_2 + \frac{136}{27}\zeta_3\right) + \epsilon^2\left(\frac{47108}{729} - \frac{31}{9}\zeta_2 - \frac{43}{30}\zeta_2^2 - \frac{1292}{81}\zeta_3\right)\right\} + C_A^2\left\{\frac{484}{9\epsilon^2} - \frac{6122}{27\epsilon} + \frac{1865}{3} - \frac{319}{9}\zeta_2 - 66\zeta_3 + \epsilon\left(-\frac{702941}{486} + \frac{14969}{108}\zeta_2 + \frac{299}{20}\zeta_2^2 + \frac{31441}{108}\zeta_3 + 5\zeta_2\zeta_3 - 30\zeta_5\right) + \epsilon^2\left(\frac{18199507}{5832} - \frac{5861}{16}\zeta_2 - \frac{63233}{720}\zeta_2^2 - \frac{691}{140}\zeta_2^3 - \frac{995915}{1296}\zeta_3 + \frac{52}{3}\zeta_2\zeta_3 - \frac{39}{2}\zeta_3^2 - \frac{1343}{12}\zeta_5\right)\right\} + 2C_F n_f T_F\left\{-\frac{40}{3\epsilon^3} + \frac{280}{9\epsilon^2} + \left(-\frac{1417}{27} + 2\zeta_2\right)\frac{1}{\epsilon} + \frac{22021}{324} - \frac{14}{3}\zeta_2 - \frac{82}{9}\zeta_3 + \epsilon\left(-\frac{238717}{3888} - \frac{73}{12}\zeta_2 + \frac{25}{12}\zeta_2^2 + \frac{394}{27}\zeta_3\right) + \epsilon^2\left(-\frac{290075}{46656} + \frac{6181}{144}\zeta_2 - \frac{499}{180}\zeta_2^2 - \frac{9751}{324}\zeta_3 + \frac{13}{6}\zeta_2\zeta_3 - \frac{29}{6}\zeta_5\right)\right\} + C_F^2\left\{\frac{32}{\epsilon^4} - \frac{48}{\epsilon^3} + \left(62 - 8\zeta_2\right)\frac{1}{\epsilon^2} + \left(-\frac{113}{2} + \frac{128}{3}\zeta_3\right)\frac{1}{\epsilon} + \frac{581}{24} + \frac{27}{2}\zeta_2 - 13\zeta_2^2 - 58\zeta_3 + \epsilon\left(\frac{12275}{288} - \frac{331}{24}\zeta_2 + \frac{493}{30}\zeta_2^2 + \frac{587}{6}\zeta_3 - \frac{56}{3}\zeta_2\zeta_3 + \frac{92}{5}\zeta_5\right) + \epsilon^2\left(-\frac{456779}{3456} - \frac{2011}{96}\zeta_2 - \frac{1279}{80}\zeta_2^2 + \frac{223}{20}\zeta_2^3 - \frac{13363}{72}\zeta_3 - \frac{5}{2}\zeta_2\zeta_3 + \frac{652}{9}\zeta_3^2 - \frac{193}{30}\zeta_5\right)\right\} + 2C_A n_f T_F\left\{-\frac{176}{9\epsilon^2} + \frac{1972}{27\epsilon} - \frac{1708}{9} + \frac{80}{9}\zeta_2 + 4\zeta_3\right.$$

$$+\epsilon\left(\frac{104858}{243}-\frac{853}{27}\zeta_2-\frac{2}{3}\zeta_2^2-\frac{1622}{27}\zeta_3\right)+\epsilon^2\left(-\frac{5369501}{5832}+\frac{1447}{18}\zeta_2+\frac{817}{45}\zeta_2^2\right.$$
$$\left.+\frac{31499}{162}\zeta_3+\frac{7}{3}\zeta_2\zeta_3+19\zeta_5\right)\Bigg\}+C_AC_F\Bigg\{\frac{220}{3\epsilon^3}+\left(-\frac{1804}{9}+4\zeta_2\right)\frac{1}{\epsilon^2}+\left(\frac{20777}{54}$$
$$-19\zeta_2-50\zeta_3\right)\frac{1}{\epsilon}-\frac{397181}{648}+\frac{161}{3}\zeta_2+\frac{76}{5}\zeta_2^2+\frac{1333}{9}\zeta_3+\epsilon\left(\frac{6604541}{7776}-\frac{669}{8}\zeta_2\right.$$
$$\left.-\frac{5519}{120}\zeta_2^2-\frac{8398}{27}\zeta_3+\frac{89}{6}\zeta_2\zeta_3-\frac{51}{2}\zeta_5\right)+\epsilon^2\left(-\frac{93774821}{93312}+\frac{20035}{288}\zeta_2+\frac{33377}{360}\zeta_2^2\right.$$
$$\left.+\frac{1793}{840}\zeta_2^3+\frac{390731}{648}\zeta_3-\frac{445}{12}\zeta_2\zeta_3-\frac{425}{12}\zeta_3^2+\frac{641}{12}\zeta_5\right)\Bigg\}, \tag{H.0.7}$$

$$\hat{\mathcal{F}}_q^{J,(1)}=C_F\Bigg\{-\frac{8}{\epsilon^2}+\frac{6}{\epsilon}-2+\zeta_2+\epsilon\left(-1-\frac{3}{4}\zeta_2-\frac{7}{3}\zeta_3\right)+\epsilon^2\left(\frac{5}{2}+\frac{\zeta_2}{4}+\frac{47}{80}\zeta_2^2+\frac{7}{4}\zeta_3\right)$$
$$+\epsilon^3\left(-\frac{13}{4}+\frac{\zeta_2}{8}-\frac{141}{320}\zeta_2^2-\frac{7}{12}\zeta_3+\frac{7}{24}\zeta_2\zeta_3-\frac{31}{20}\zeta_5\right)\Bigg\}, \tag{H.0.8}$$

$$\hat{\mathcal{F}}_q^{J,(2)}=2C_Fn_fT_F\Bigg\{-\frac{8}{3\epsilon^3}+\frac{56}{9\epsilon^2}+\left(-\frac{47}{27}-\frac{2}{3}\zeta_2\right)\frac{1}{\epsilon}-\frac{4105}{324}+\frac{14}{9}\zeta_2-\frac{26}{9}\zeta_3+\epsilon\left(\frac{142537}{3888}\right.$$
$$\left.-\frac{695}{108}\zeta_2+\frac{41}{60}\zeta_2^2+\frac{182}{27}\zeta_3\right)+\epsilon^2\left(-\frac{3256513}{46656}+\frac{21167}{1296}\zeta_2-\frac{287}{180}\zeta_2^2-\frac{2555}{324}\zeta_3\right.$$
$$\left.-\frac{13}{18}\zeta_2\zeta_3-\frac{121}{30}\zeta_5\right)\Bigg\}+C_F^2\Bigg\{\frac{32}{\epsilon^4}-\frac{48}{\epsilon^3}+\left(34-8\zeta_2\right)\frac{1}{\epsilon^2}+\left(-\frac{5}{2}+\frac{128}{3}\zeta_3\right)\frac{1}{\epsilon}-\frac{361}{8}$$
$$+\frac{9}{2}\zeta_2-13\zeta_2^2-58\zeta_3+\epsilon\left(\frac{3275}{32}+\frac{3}{8}\zeta_2+\frac{171}{10}\zeta_2^2+\frac{503}{6}\zeta_3-\frac{56}{3}\zeta_2\zeta_3+\frac{92}{5}\zeta_5\right)$$
$$+\epsilon^2\left(-\frac{20257}{128}-\frac{793}{32}\zeta_2-\frac{2097}{80}\zeta_2^2+\frac{223}{20}\zeta_2^3-\frac{4037}{24}\zeta_3+\frac{27}{2}\zeta_2\zeta_3+\frac{652}{9}\zeta_3^2-\frac{231}{10}\zeta_5\right)\Bigg\}$$
$$+C_AC_F\Bigg\{\frac{44}{3\epsilon^3}+\left(-\frac{332}{9}+4\zeta_2\right)\frac{1}{\epsilon^2}+\left(\frac{2545}{54}+\frac{11}{3}\zeta_2-26\zeta_3\right)\frac{1}{\epsilon}-\frac{18037}{648}-\frac{47}{9}\zeta_2$$
$$+\frac{44}{5}\zeta_2^2+\frac{467}{9}\zeta_3+\epsilon\left(-\frac{221963}{7776}-\frac{263}{216}\zeta_2-\frac{1891}{120}\zeta_2^2-\frac{2429}{27}\zeta_3+\frac{89}{6}\zeta_2\zeta_3-\frac{51}{2}\zeta_5\right)$$
$$+\epsilon^2\left(\frac{11956259}{93312}+\frac{38987}{2592}\zeta_2+\frac{9451}{360}\zeta_2^2-\frac{809}{280}\zeta_2^3+\frac{92701}{648}\zeta_3-\frac{397}{36}\zeta_2\zeta_3-\frac{569}{12}\zeta_3^2\right.$$
$$\left.+\frac{3491}{60}\zeta_5\right)\Bigg\}, \tag{H.0.9}$$

$$
\begin{aligned}
\hat{\mathcal{F}}_q^{J,(3)} = {} & 4C_F n_f^2 T_F^2 \Bigg\{ - \frac{128}{81\epsilon^4} + \frac{1504}{243\epsilon^3} + \left(- \frac{16}{9} - \frac{16}{9}\zeta_2 \right) \frac{1}{\epsilon^2} + \Bigg(- \frac{73432}{2187} + \frac{188}{27}\zeta_2 \\
& - \frac{272}{81}\zeta_3 \Bigg) \frac{1}{\epsilon} + \frac{881372}{6561} - 26\zeta_2 - \frac{83}{135}\zeta_2^2 + \frac{3196}{243}\zeta_3 \Bigg\} + C_F^3 \Bigg\{ - \frac{256}{3\epsilon^6} + \frac{192}{\epsilon^5} + \Bigg(- 208 \\
& + 32\zeta_2 \Bigg) \frac{1}{\epsilon^4} + \left(88 + 24\zeta_2 - \frac{800}{3}\zeta_3 \right) \frac{1}{\epsilon^3} + \left(254 - 98\zeta_2 + \frac{426}{5}\zeta_2^2 + 552\zeta_3 \right) \frac{1}{\epsilon^2} \\
& + \left(- \frac{5045}{6} + 83\zeta_2 - \frac{1461}{10}\zeta_2^2 - \frac{2630}{3}\zeta_3 + \frac{428}{3}\zeta_2\zeta_3 - \frac{1288}{5}\zeta_5 \right) \frac{1}{\epsilon} + \frac{38119}{24} + \frac{1885}{12}\zeta_2 \\
& + \frac{8659}{40}\zeta_2^2 - \frac{9095}{252}\zeta_2^3 + 1153\zeta_3 - 35\zeta_2\zeta_3 - \frac{1826}{3}\zeta_3^2 - \frac{562}{5}\zeta_5 \Bigg\} + 2C_F^2 n_f T_F \Bigg\{ \frac{64}{3\epsilon^5} \\
& - \frac{592}{9\epsilon^4} + \left(\frac{1480}{27} + \frac{8}{3}\zeta_2 \right) \frac{1}{\epsilon^3} + \left(\frac{7772}{81} - \frac{266}{9}\zeta_2 + \frac{584}{9}\zeta_3 \right) \frac{1}{\epsilon^2} + \Bigg(- \frac{116735}{243} + \frac{2633}{27}\zeta_2 \\
& - \frac{337}{18}\zeta_2^2 - \frac{5114}{27}\zeta_3 \Bigg) \frac{1}{\epsilon} + \frac{3396143}{2916} - \frac{32329}{162}\zeta_2 + \frac{8149}{216}\zeta_2^2 + \frac{39773}{81}\zeta_3 - \frac{343}{9}\zeta_2\zeta_3 \\
& + \frac{278}{45}\zeta_5 \Bigg\} + C_A^2 C_F \Bigg\{ - \frac{3872}{81\epsilon^4} + \left(\frac{52168}{243} - \frac{704}{27}\zeta_2 \right) \frac{1}{\epsilon^3} + \Bigg(- \frac{117596}{243} - \frac{2212}{81}\zeta_2 \\
& - \frac{352}{45}\zeta_2^2 + \frac{6688}{27}\zeta_3 \Bigg) \frac{1}{\epsilon^2} + \left(\frac{1322900}{2187} + \frac{39985}{243}\zeta_2 - \frac{1604}{15}\zeta_2^2 - \frac{24212}{27}\zeta_3 + \frac{176}{9}\zeta_2\zeta_3 \right. \\
& \left. + \frac{272}{3}\zeta_5 \right) \frac{1}{\epsilon} + \frac{1213171}{13122} - \frac{198133}{729}\zeta_2 + \frac{146443}{540}\zeta_2^2 - \frac{6152}{189}\zeta_2^3 + \frac{970249}{486}\zeta_3 - \frac{926}{9}\zeta_2\zeta_3 \\
& - \frac{1136}{9}\zeta_3^2 + \frac{772}{9}\zeta_5 \Bigg\} + 2C_A C_F n_f T_F \Bigg\{ \frac{1408}{81\epsilon^4} + \Bigg(- \frac{18032}{243} + \frac{128}{27}\zeta_2 \Bigg) \frac{1}{\epsilon^3} + \left(\frac{24620}{243} \right. \\
& \left. + \frac{1264}{81}\zeta_2 - \frac{1024}{27}\zeta_3 \right) \frac{1}{\epsilon^2} + \left(\frac{212078}{2187} - \frac{16870}{243}\zeta_2 + \frac{88}{5}\zeta_2^2 + \frac{12872}{81}\zeta_3 \right) \frac{1}{\epsilon} - \frac{5807647}{6561} \\
& + \frac{299915}{1458}\zeta_2 - \frac{5492}{135}\zeta_2^2 - \frac{42941}{81}\zeta_3 + \frac{422}{9}\zeta_2\zeta_3 - \frac{28}{3}\zeta_5 \Bigg\} + C_A C_F^2 \Bigg\{ - \frac{352}{3\epsilon^5} + \Bigg(\frac{3448}{9} \\
& - 32\zeta_2 \Bigg) \frac{1}{\epsilon^4} + \Bigg(- \frac{16948}{27} + \frac{28}{3}\zeta_2 + 208\zeta_3 \Bigg) \frac{1}{\epsilon^3} + \left(\frac{44542}{81} + \frac{1127}{9}\zeta_2 - \frac{332}{5}\zeta_2^2 \right. \\
& \left. - 840\zeta_3 \right) \frac{1}{\epsilon^2} + \left(\frac{149299}{486} - \frac{12757}{54}\zeta_2 + \frac{9839}{36}\zeta_2^2 + \frac{5467}{3}\zeta_3 - \frac{430}{3}\zeta_2\zeta_3 + 284\zeta_5 \right) \frac{1}{\epsilon} \\
& - \frac{15477463}{5832} + \frac{21455}{324}\zeta_2 - \frac{1002379}{2160}\zeta_2^2 - \frac{18619}{1260}\zeta_2^3 - \frac{51781}{18}\zeta_3 + \frac{910}{9}\zeta_2\zeta_3 + \frac{1616}{3}\zeta_3^2 \\
& - \frac{3394}{45}\zeta_5 \Bigg\} . \qquad \text{(H.0.10)}
\end{aligned}
$$

I Harmonic Polylogarithms

The logarithms, polylogarithms ($\mathrm{Li}_n(x)$) and Nielsen's polylogarithm ($\mathrm{S}_{n,p}(x)$) appear naturally in the analytical expressions of radiative corrections in pQCD which are defined through

$$\begin{aligned}
&\ln(x) = \int_1^x \frac{dt}{t}\,,\\
&\mathrm{Li}_n(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^n} = \int_0^x \frac{dt}{t}\mathrm{Li}_{n-1}(t)\,, \qquad \text{e.g.} \quad \mathrm{Li}_1(x) = -\ln(1-x)\,,\\
&S_{n,p}(x) \equiv \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{dt}{t}[\ln(t)]^{n-1}[\ln(1-xt)]^p\,,\\
&\quad \text{e.g.} \quad S_{n-1,1}(x) = \mathrm{Li}_n(x)\,.
\end{aligned} \tag{I.0.1}$$

However, for higher order radiative corrections (2-loops and beyond), these functions are not sufficient to evaluate all the loop integrals appearing in the Feynman graphs. This is overcome by introducing a new set of functions which are called **Harmonic Polylogarithms (HPLs)**. These are essentially a generalisation of Nielsen's polylogarithms. In this appendix, we briefly describe the definition and properties of HPL [184] and 2dHPL. HPL is represented by $H(\vec{m}_w; y)$ with a w-dimensional vector $\vec{m}_w$ of parameters and its argument y. w is called the weight of the HPL. The elements of $\vec{m}_w$ belong to $\{1, 0, -1\}$ through which the following rational functions are represented

$$f(1;y) \equiv \frac{1}{1-y}, \qquad f(0;y) \equiv \frac{1}{y}, \qquad f(-1;y) \equiv \frac{1}{1+y}\,. \tag{I.0.2}$$

The weight 1 ($w = 1$) HPLs are defined by

$$H(1,y) \equiv -\ln(1-y), \qquad H(0,y) \equiv \ln y, \qquad H(-1,y) \equiv \ln(1+y)\,. \tag{I.0.3}$$

For $w > 1$, $H(m,\vec{m}_w;y)$ is defined by

$$H(m,\vec{m}_w;y) \equiv \int_0^y dx\, f(m,x)\, H(\vec{m}_w;x), \qquad m \in 0,\pm 1\,. \tag{I.0.4}$$

The 2dHPLs are defined in the same way as Eq. (I.0.4) with the new elements $\{2,3\}$ in $\vec{m}_w$ representing a new class of rational functions

$$f(2;y) \equiv f(1-z;y) \equiv \frac{1}{1-y-z}, \qquad f(3;y) \equiv f(z;y) \equiv \frac{1}{y+z} \tag{I.0.5}$$

and correspondingly with the weight 1 ($w = 1$) 2dHPLs

$$H(2,y) \equiv -\ln\Big(1-\frac{y}{1-z}\Big), \qquad H(3,y) \equiv \ln\Big(\frac{y+z}{z}\Big)\,. \tag{I.0.6}$$

I.0.1 Properties

Shuffle algebra : A product of two HPL with weights w_1 and w_2 of the same argument y is a combination of HPLs with weight $(w_1 + w_2)$ and argument y, such that all possible permutations of the elements of $\vec{m}_{w_1}$ and $\vec{m}_{w_2}$ are considered preserving the relative orders of the elements of $\vec{m}_{w_1}$ and $\vec{m}_{w_2}$,

$$H(\vec{m}_{w_1};y)H(\vec{m}_{w_2};y) = \sum_{\vec{m}_w\,=\,\vec{m}_{w_1}\uplus\vec{m}_{w_2}} H(\vec{m}_w;y). \tag{I.0.7}$$

Integration-by-parts identities : The ordering of the elements of $\vec{m}_w$ in an HPL with weight w and argument y can be reversed using integration-by-parts and in the process, some

products of two HPLs are generated in the following way

$$\begin{aligned} H(\vec{m}_w; y) \equiv H(m_1, m_2, ..., m_w; y) &= H(m_1, y) H(m_2, ..., m_w; y) \\ &- H(m_2, m_1, y) H(m_3, ..., m_w; y) \\ &+ ... + (-1)^{w+1} H(m_w, ..., m_2, m_1; y) \,. \end{aligned} \tag{I.0.8}$$

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