Soft graviton theorems in higher spacetime dimensions

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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Contents

Synopsis

In the early 1960s Bondi, Van der Burg, Metzner, and Sachs [1,2] discovered that asymptotic symmetry group of four dimensional asymptotically flat spacetimes is not the finite dimensional Poincare group but an infinite dimensional symmetry group which is known as BMS group. In recent years there have been several interesting developments in the studies of asymptotic symmetries in relation to soft theorems following the seminal works by Strominger [3]. Strominger and his collaborators discovered that soft graviton theorem can be realized from asymptotic symmetries associated with large diffeomorphisms of asymptotically flat spacetimes at null infinity. Infrared properties of scattering amplitudes involving gravitons and photons were investigated by Weinberg [4, 5] in 1960s which paved the way for important developments of soft theorems in Quantum Field Theory and String Theory. Since then soft theorems have been studied extensively in various theories of scalars, gauge and also higher spin particles. Contemporary developments of the sophisticated techniques like BCFW recursion relations [6] and CHY formalism [7] for finding scattering amplitudes have led to the discoveries of new soft theorems and have deepened the understanding of many important properties of them. On the other hand understanding the relationships between the newly discovered soft theorems and the associated symmetries is extremely interesting from the perspective of flatspace holography .

In this thesis we probe further into the understanding of soft graviton theorems and asymptotic symmetries of asymptotically flat spacetimes in higher dimensions. In the first part of *the thesis we derive double soft limit to graviton scattering amplitude from the formalism developed by Cachazo, He and Yuan. In the second part of the thesis we study the relation between Weinberg's soft graviton theorem and asymptotic symmetries for asymptotically flat spacetimes of dimensions greater than four.*

0.1 Soft graviton theorem

In a graviton scattering process if the energy of any external graviton becomes infinitesimally small then the scattering amplitude can be expressed as a product of soft factor and amplitude of remaining particles of finite energy. This is the statement of single soft graviton theorem. In 2014 Cachazo and Strominger, using Britto-Cachazo-Feng-Witten recursion relation, found that the soft factor can be perturbatively expanded in powers of energy of the soft graviton. For pure gravity this soft factorization holds to the sub-sub-leading order which can be expressed as

$$\begin{aligned}&\lim_{\tau\to 0}\mathcal{M}_{n+1}\left(k_1,\epsilon_1;k_2,\epsilon_2;\ldots;k_n,\epsilon_n;\tau q,\epsilon\right)\\ =\;&\left(S^{(0)}(q)+S^{(1)}(q)+S^{(2)}(q)\right)\mathcal{M}_n\left(k_1,\epsilon_1;k_2,\epsilon_2;\ldots;k_n,\epsilon_n\right)+O(\tau^2),\end{aligned}$$

where $\mathcal{M}_{n+1}$ is the $(n+1)$-point scattering amplitude including the soft graviton with momentum τq and polarization ϵ and $\mathcal{M}_n$ is the n-point amplitude for finite energy external particles whose momenta and polarizations are denoted by k_a and ϵ_a with $a\in\{1,2,\ldots,n\}$. The soft factors are functions of kinematic variables like momenta and polarizations of both soft particle and particles of finite energy and can be given by

$$S^{(0)}(q)=\frac{1}{\tau}\sum_{a=1}^{n}\frac{\epsilon_{\mu\nu}q^\mu q^\nu}{q\cdot k_a},\qquad S^{(1)}(q)=\sum_{a=1}^{n}\frac{\epsilon_{\mu\nu}k_a^\mu q_\rho\hat{J}_a^{\rho\nu}}{q\cdot k_a},\qquad S^{(2)}(q)=\frac{\tau}{2}\sum_{a=1}^{n}\frac{\epsilon_{\mu\nu}q_\rho q_\sigma\hat{J}_a^{\rho\mu}\hat{J}_a^{\sigma\nu}}{q\cdot k_a}.$$

$\hat{J}_a^{\mu\nu}$ denotes the angular momentum operator of the a-th hard particle and has two parts - orbital angular momentum and a spin part. It should be noted that conservation of total momentum (i.e. sum of all the external hard momenta) of the scattering amplitude implies gauge invariance of leading soft factor, gauge invariance of sub-leading soft factor follows from the conservation of total angular momentum and sub-sub-leading soft factors are gauge invariant in individual finite energy external state.

Sen used covariantization method [8] to prove universality of leading soft graviton theorem with arbitrary number of soft gravitons and sub-leading soft graviton theorem with single soft graviton and any number of finite energy external states with arbitrary masses and spins to all orders in perturbation theory of S-matrix in any generic theory of quantum gravity. Recent work by Laddha and Sen [9] have established that soft graviton theorem is universal to sub-leading order for any generic theory of quantum gravity and the sub-sub-leading factor has a universal part and a non-universal part that depends on details of the specific theory.

0.2 Cachazo-He-Yuan formalism

In 2013 Cachazo, He and Yuan [7] developed an ingenious technique of calculating scattering amplitude for a variety of theories in Quantum Field Theory that include gravity, Yang-Mills, bi-adjoint cubic scalar, non-linear sigma model and many others using integrals over moduli space of punctured Riemann spheres. The punctures on the Riemann sphere correspond to the external scattering states. The underlying principle behind this formalism is to map the singularity structures of the kinematic space to the singularities of an auxiliary space where they can be better understood. This method of studying S-matrix has many advantages over conventional techniques of computing Feynman diagrams and helps to bring out several important features of scattering amplitude which otherwise are not manifested in the Lagrangian descriptions.

In this formalism a tree level scattering amplitude for n massless particles in any arbitrary spacetime dimensions is given in the following way

$$M_n = \int \frac{\mathrm{d}^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a.k_b}{\sigma_a - \sigma_b}\right) I_n(\{k,\epsilon,\sigma\}).$$

where the form of I_n, which is particular function of kinematic variables like momenta and polarizations of external states and also the punctures, depends on the specific theory. σ_is are the locations of punctures and the arguments of delta functions are polynomial equations in σ_is, called *scattering equations*. Because of the $\mathbb{SL}\,(2,\mathbb{C})$ redundancies, only $n-3$ out of n scattering equations are independent, and there are $(n-3)!$ solutions for the σ_is.

The integrand I_n has $\mathbb{SL}\,(2,\mathbb{C})$ weight 4 and the measure transforms accordingly such that the scattering amplitude is $\mathbb{SL}\,(2,\mathbb{C})$ invariant. There are several building blocks formed out of the kinematic variables and the punctures which give rise to I_n.

Factorization properties of scattering amplitude arising from soft and collinear limits can be studied very conveniently in the CHY formalism. Soft graviton theorem for one soft graviton scattering has been derived using CHY formalism in earlier literatures. *In this thesis we derive soft graviton theorem for two soft graviton scattering process from the CHY formalism. In this process we encounter some intricate technicalities that will be explained in detail.*

0.3 Double soft graviton theorem

In the soft limits the moduli space integrations in the CHY formalism become contour integrals. For single soft particle the contour of integration is wrapped around the solutions of the punctures corresponding to finite energy external particles. In case of double soft limits, depending on the separation of the soft punctures contour integrals have to be evaluated. In [10] we calculated the contribution coming from the so called *degener-*

ate solutions, when the two soft punctures come infinitesimally close to each other. In this case we can make a change of variables for the soft punctures, $\sigma_{n+1} = \rho - \frac{\xi}{2}$ and $\sigma_{n+2} = \rho + \frac{\xi}{2}$. Using the scattering equations, solutions for ξ can be expressed in terms of ρ. Now effectively we are left with a single contour integration corresponding to the variable ρ. Applying Cauchy's residue theorem one can compute the value of this contour integration. The result thus obtained was compared with the Feynman diagram computation where it was noticed that the leading order term, which is the product of two single soft factors, was missed in the obtained result. To account for the missing term, contributions from the *non-degenerate solutions* were analyzed in [11]. In this case we have to consider two contour integrals corresponding to σ_{n+1} and σ_{n+2} independently. Residues of these contour integrals give the required result. Along with the leading term, we also obtain two more terms at sub-leading order. However to do this analysis an explicit gauge choice has been made in which polarization tensors $(\epsilon_{p,\mu\nu}, \epsilon_{q,\mu\nu})$ of the two soft particles are orthogonal to momenta $(\tau p_\mu, \tau q_\mu)$ of each other. This result matches with earlier one obtained by Klose et al [12] using BCFW analysis in four dimensions. In later works by Chakrabarti et al [13, 14] using covariantization method developed by Sen, it is shown that the result is valid for any generic theory of quantum gravity.

The next part of the thesis focuses on the supertranslation symmetries for asymptotically flat spacetime in dimensions greater than four.

0.4 Supertranslation symmetries and soft graviton theorem

The asymptotic symmetry group of asymptotically flat spacetimes in four dimension is an infinite dimensional group, popularly known as BMS group which consists of angle dependent translations along the null directions, called supertranslations acting semi-directly on the Lorentz group. Supertranslations form an infinite dimensional Abelian normal subgroup of BMS group. Recently Strominger and his collaborators have shown that Weinberg's soft graviton theorem is related to Ward identity derived from supertranslation invariance of gravitational S-matrix. Subsequent works by Campiglia and Laddha [15] have led to extension of BMS group, which is a semidirect product of supertranslation and Diff($\mathbb{S}^2$), to obtain Ward identities associated with sub-leading soft graviton theorem. With the generalization of soft graviton theorems in higher dimensions a pertinent question is - like in four dimension can the soft graviton theorems be related to asymptotic symmetries of asymptotically flat spacetime in higher dimensions also?

The relation between soft graviton theorem and supertranslation symmetries have been explored in higher even dimensions by Kapec et al in [16]. In this paper it was shown that Weinberg's soft graviton theorem can be recast as a Ward identity at null infinity in higher even dimensions. *In this thesis we try to compute the asymptotic conserved charge from the covariant phase space methods [17] at null infinity and derive the supertranslation Ward identity in higher dimensions.*

We consider perturbative gravity with linearized metric fluctuations around Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, $\kappa = \sqrt{32\pi G_N}$ is the gravitational coupling constant. However we show there are subtleties with the existence of infinite dimensional supertranslation symmetries in higher spacetime dimensions ($D \geq 4$), mainly because of the boundary fall-off conditions of the metric perturbations. The null infinity of the asymptotically flat spacetime is understood as the limit of radial coordinate, r going to infinity while u and angular coordinates are held fixed, such that topologically the space is given by $\mathbb{R} \times \mathbb{S}^{D-2}$. We can also expand the metric perturbations in orders of the radial components off the null infinity. Solving linearized Einstein's equation in vacuum we determine the free radiative data of the gravitational radiation. In D dimension the free radiative data has a fall-off behavior of $r^{-\frac{D-6}{2}}$ and supertranslation acts on the metric perturbation which falls off linear in r in any arbitrary dimensions. Therefore only in four dimension these two powers in r are equal and the radiative data is shifted under supertranslation. In four dimension, because of the presence of supertranslation, there is an enhancement of the asymptotic symmetry group from Poincare to infinite dimensional BMS group. Since the radiative component appears at different order in radial coordinate, the only allowed symmetries are Poincare symmetries. Now we want to explore whether it is possible to have supertranslation symmetries in $D \geq 4$ if we relax the boundary conditions and eventually derive Ward identities relating infrared behavior of graviton amplitude.

The asymptotic conserved charge at null infinity consists of two parts: *soft charge* which is linear in gravitational radiation and the quadratic part, called *hard charge* which also contains matter radiation. We consider scalar field minimally coupled to gravity as the matter contribution. Trace-reversed metric perturbations satisfy the equation

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

Metric perturbations consist of two parts - $h^{(C)}_{\mu\nu}$ which corresponds to the gravitational wave and satisfies homogeneous equation $\Box \bar{h}^{(C)}_{\mu\nu} = 0$ and $h^{(\phi)}_{\mu\nu}$ which is determined by the matter content. Thus we can write $h_{\mu\nu} = h^{(C)}_{\mu\nu} + h^{(\phi)}_{\mu\nu}$. The sourced metric perturbations, $h^{(\phi)}_{\mu\nu}$ typically has faster fall-off conditions than the free metric perturbations, $h^{(C)}_{\mu\nu}$. For the supertranslation charge only free metric perturbations contribute. Now we consider the expansion of angular components of metric perturbations to be $h_{AB} = r h^{(-1)}_{AB} + h^{(0)}_{AB} + \frac{1}{r} h^{(1)}_{AB} + \ldots$ and A, B are the $D-2$ angular coordinates. In four dimension $h^{(-1)}_{AB}$ constitute the free data and the expression of soft charge at null infinity is given by

$$Q_{\xi} = -\frac{1}{16\pi G} \int_{\mathscr{I}^+} du d\Omega_2 \, f(z) \, D^A D^B \dot{h}^{(-1)}_{AB},$$

where dot represents derivative with respect to the retarded time u. $f(z)$ is any unconstrained arbitrary function of angular coordinates. In higher dimensions we consider the vector field

$$\vec{\xi} = f(z) \, \partial_u + \frac{1}{D-2} \Delta f(z) \, \partial_r - \frac{1}{r} D^A f(z) \, \partial_A + \ldots$$

and derive the asymptotic covariant phase charge. In $D \geq 4$, $h^{(-1)}_{AB}$ is non-radiative and gets shifted under $\vec{\xi}$,

$$\delta_{\xi} h^{(-1)}_{AB} = \frac{2}{D-2}\gamma_{AB}\Delta f(z) - (D_A D_B + D_B D_A) f(z).$$

For this reason if we restrict $h^{(-1)}_{AB}$ to be zero then $f(z)$ is constrained and there can not exist supertranslation symmetries in higher dimension. So we have to allow for non-radiative modes to be present and relax the boundary conditions.

Then the problem is the covariant phase space charge at null infinity diverges. The finite part of the charge also does not reproduce the soft graviton theorem. The only resolution seems to be to restrict the boundary conditions. In that case the divergent terms vanish and also the only allowed symmetries are global translations. The asymptotic charge also vanishes. This result is consistent with the analysis of [18].

In odd dimensions there are additional subtleties because of the fractional power fall-off of the metric perturbations in radial coordinate. Mode expansion of gravitons contain half-integer powers of frequency in odd spacetime dimensions. This is another difficulty for understanding soft graviton theorem from asymptotic symmetry analysis for odd dimensions.

We have shown that there is no consistent way of relaxing the boundary conditions such that supertranslation symmetries are allowed in higher spacetime dimensions and at the same time Weinberg's soft graviton theorem can be expressed as Ward identities of gravitational S-matrix following from the supertranslation symmetries. Our results are consistent with the analysis of [19].

1 Introduction

In Quantum Field Theory soft theorems play an important role in understanding deep intricate structures about scattering amplitudes and various other aspects. Soft theorems, as we understand them today, were first studied by Weinberg [4, 5] and by Gross and Jackiw [20, 21] in the context of photon and graviton scattering processes . Since then there have been incredibly huge amount of explorations to understand soft theorems for variety of theories in Quantum Field Theory and String Theory.

What are soft theorems? In any scattering processes if one or more external particles have infinitesimal small momenta as compared to other external states with finite momenta, we call them soft particles and the particles carrying finite energies are known as hard particles. In general this classification of hard and soft particles depend on some energy scales but for our purpose we will refer to the soft particles as having almost vanishing energies and momenta. For many theories scattering amplitudes involving the soft and hard particles can be expressed as product of *soft factor*, which include kinematics of soft and hard particles, and scattering amplitude involving the hard particles only and excluding the soft particles. These are known as soft theorems. Typically the factorization of the amplitudes can be expanded in powers of soft momenta and soft theorems are valid only to certain orders in the expansion.

Scattering amplitudes involving soft gravitons have been shown to factorize beyond leading order. Modern techniques in the scattering amplitudes have been particularly helpful in this context. Using Britto-Cachazo-Feng-Witten (BCFW) recursion relations [6] Cachazo and Strominger [22] found out soft factorization of pure gravity amplitude with one soft graviton to sub-sub-leading order. Soft graviton theorems in specific theories beyond leading order have also been studied in [23–28] from Feynman diagrammatic as well as using modern amplitude methods.

Sen used *covariantization method* [8, 29] to prove universality of leading soft graviton theorem with arbitrary number of soft gravitons and sub-leading soft graviton theorem with single soft graviton and any number of finite energy external states with arbitrary masses and spins to all orders in perturbation theory of S-matrix in any generic theory of quantum gravity. In [9] Laddha and Sen showed that in any theory of quantum gravity scattering amplitude for single soft graviton and any number of finite energy external states with arbitrary spins and masses to sub-sub-leading order in the momentum of the soft graviton can be expressed as sum of two parts: a universal part that depends only on the amplitude without the soft graviton and a non-universal part that depends on the amplitude without the soft graviton as well as details of the theory, in particular the two and three point functions of the theory.

In recent years there has been increasingly growing interest in the studies of various aspects of soft theorems [30–55] following the remarkable observation by Strominger [3] that Weinberg's soft graviton theorem are related to asymptotic symmetry group of asymptotically flat spacetimes. In [3, 56] it was shown that leading soft graviton theorem can be expressed as Ward identities related to *supertranslation symmetries* and there are infinitely many conserved charges corresponding to these symmetries.

What are supertranslations? In early sixties Bondi, Van der Burg, Metzner and Sachs [1, 2] discovered that the asymptotic symmetry group of any four dimensional flat spacetimes is not just the finite dimensional Poincare group but an infinite dimensional group which is now known as BMS group. Using Penrose compactification the asymptotic null infinity of asymptotically flat geometries can be topologically given by $\mathbb{R} \times \mathbb{S}^2$ and there are two such null infinities, one at future which is denoted by $\mathscr{I}^+$ and one at past which is denoted by $\mathscr{I}^-$. BMS group comprises of supertranslations which act semi-directly on the Lorentz group. Supertranslations are angle dependent translations acting at the null infinities and can be arbitrary functions of the coordinates on $\mathbb{S}^2$. Supertranslation vector fields form the infinite dimensional abelian sub-group of the BMS group.

Using the covariant phase space formalism [17] one can calculate conserved charges for the symmetries at asymptotic null infinity. The charges associated to supertranslation symmetries come from the gravitational radiation and contain zero modes of gravitons. These are called *soft charges* and act on the Fock states at future and past null infinities to produce soft graviton insertions in the S-matrix elements. These soft mode insertions give rise to Weinberg's soft graviton theorem. There are also *hard* part of the charges which come from the matter contents. Total asymptotic conserved charges can be given by $Q = Q^{\text{soft}} + Q^{\text{hard}}$. Classically these conserved charges commute with the gravitational S-matrix such that $[Q, S] = 0$. These charges generate supertranslations on the radiative phase space at null infinities. Using the in and out scattering states at $\mathscr{I}^-$ and $\mathscr{I}^+$ respectively we can obtain the Ward identities $\langle \text{out}|Q^+ S - S Q^-|\text{in}\rangle = 0$.

Asymptotic symmetries for soft graviton theorems when external finite energy states are massive have been studied in [57]. In [15, 58–60] Campiglia and Laddha found out Ward identities related to the asymptotic symmetries corresponding to sub-leading soft graviton theorem by considering a particular extension of BMS group which is called *generalized BMS group*. They considered this generalized BMS group as semi direct product of super-translations with smooth diffeomorphisms of the conformal sphere $\left[\text{Diff}\left(\mathbb{S}^2\right)\right]$. Charges associated with the $\left[\text{Diff}\left(\mathbb{S}^2\right)\right]$ symmetries give rise to Ward identities which are in one to one correspondence with sub-leading soft graviton theorem.

1.1 Outline

Discovery of relations between soft graviton theorems and asymptotic symmetries of asymptotically flat spacetimes have made the studies of soft graviton theorems particularly interesting from the perspective of both Field Theories and General Relativity. Motivated by the contemporary developments we have explored a particular case of soft graviton theorems when two of the scattered gravitons are taken to be soft simultaneously. As mentioned earlier, inventions of new techniques for amplitudes have paved the way for studying these soft theorems much more conveniently.

Here we have used *Cachazo-He-Yuan formalism* [7, 61–63] to find double soft limit to graviton amplitude for pure gravity. Cachazo, He and Yuan (CHY) discovered an outstanding formalism of calculating many Quantum Field Theory scattering amplitudes using integrations over moduli space of punctured complex spheres. In CHY formalism various field theory amplitudes involving massless particles can be computed at tree level in arbitrary spacetime dimensions without requiring Feynman diagrammatic. From this method many underlying features and relations between different theories like Kawai-Lewellen-Tye orthogonality, color-kinematic duality, double-copy relations etc. emerge which are otherwise not very manifest in the conventional Lagrangian descriptions. Studies of soft theorems and factorization properties of scattering amplitudes become extremely simpler in CHY formalism. One primary objective of this thesis is to study simultaneous double soft limit of graviton scattering amplitude to sub-leading order in powers of the momenta of soft gravitons [10, 11]. In the simulataneous soft limit scaling of both the soft momenta are taken to be same (τp and τq with $\tau \to 0$). We find that at leading order double soft factor is given as product of two leading single soft factors ($S^{(0)}(p)S^{(0)}(q)$). Sub-leading double soft factor has three terms - two of which are symmetric combination of products of leading and sub-leading soft factors ($S^{(0)}(p)S^{(1)}(q) + S^{(0)}(q)S^{(1)}(p)$) and the third term is a contact term which can not be expressed as product form of single soft factors. We have also verified the result obtained from CHY formalism with that from Feynman diagrammatic.

In the second part of the thesis we study supertranslation symmetries and Weinberg's soft graviton theorem for asymptotically flat spacetimes in higher dimensions. In four dimension leading soft graviton theorem can be interpreted as Ward identities of gravitational S-matrix related to supertranslation symmetries, our goal is to find out how much of such correspondence between asymptotic symmetries and leading soft graviton theorem can be achieved. Due to the fall-off conditions of the metric perturbations, in [19, 64, 65] it is claimed that supertranslation symmetries can not exist in spacetime dimensions greater than four. This is because of the fact that supertranslations always shift the components of metric perturbations falling off with inverse power in radial coordinate in all spacetime dimensions and radiative modes appear in higher powers of radial coordinate in higher dimensions. Only in four dimension the two powers happen to be the same which leads to the enhancement of the asymptotic symmetry group. Therefore in order to preserve supertranslation symmetries in dimensions greater than four we have to relax the boundary conditions. However there is no straightforward way to set the boundary conditions compatible with the existence of supertranslations in higher dimensions, so we have to find out the appropriate fall-off conditions of the metric perturbations by trial and error and check their consistencies using linearized Einstein's equations and gauge conditions. Relaxing the boundary conditions lead to the presence of *non-propagating* modes in the metric perturbations in addition to the propagating modes. The non-propagating components do not have time dependence and therefore can not be expanded in terms of plane waves. Because of these extra modes which have slower fall-off behavior than the graviton modes, computation of the conserved charge at asymptotic null infinity from the covariant phase space formalism [17] is complicated. The main problem is the appearance of divergent terms in the charge calculation when the limit of radial coordinate is taken to infinity. It remains a challenge to show that the divergent terms are canceled and the finite contribution to the asymptotic charge produces Weinberg's soft graviton theorem.

This thesis is organized as follows: in chapter 2 we review soft graviton theorem for single soft graviton scattering. We will mainly review the analysis of Cachazo and Strominger to derive soft graviton theorem to sub-sub-leading order in the momentum of the soft graviton. In chapter 3 we give an elementary introduction to Cachazo-He-Yuan formalism. We will see some basic examples of scalar, gluon and graviton amplitudes and methods of taking soft limits for these scattering amplitudes. We present the main results of this thesis in chapter 4. We study double soft limit of graviton scattering amplitude for pure gravity using both Cachazo-He-Yuan formalism and Feynman diagrammatic. In this chapter we also analyze double soft limit of pure Yang-Mills theory. In chapter 5 we study asymptotic symmetries related to Weinberg's soft graviton theorem in higher spacetime dimensions. We first review the supertranslation Ward identities and leading soft graviton theorem in four dimension and then comment on the difficulties involved in deriving such Ward identities in higher dimensions.

2 Soft graviton theorem

Soft graviton theorems in Quantum Field Theory have been studied over several decades since early sixties by Weinberg followed by many authors in recent years [4,5,8,9,20–29]. In this chapter we will review soft limit of graviton scattering amplitude when energy of one of the external particles in the scattering process, which is a graviton, is taken to be extremely small compared to the energies of other scattered particles. In Sec.(2.1) we will present the result for leading soft graviton theorem. In Sec.(2.2) we will show soft graviton theorems beyond leading order for pure graviton amplitude.

2.1 Weinberg's soft graviton theorem

In a scattering process involving graviton, when energy of one external graviton state goes to significantly small compared to that of other external states, then the scattering amplitude can be expressed as a product of *soft factor*, which involves momenta and polarizations of both soft and hard external particles and amplitude consisting of only hard particles to leading order in the energy of the soft graviton. This is the statement of Weinberg's soft graviton theorem.

If there are n external hard particles of momenta denoted by k_i, $\forall\ i \in \{1, \ldots n\}$ and a soft graviton of momentum τq with $\tau \to 0$ then Weinberg's soft graviton theorem can be written as

$$\lim_{\tau \to 0} \mathcal{M}_{n+1}(\tau q, \{k_i\}) = \frac{1}{\tau} S^{(0)}(q) \mathcal{M}_n(\{k_i\}) + O\left(\tau^0\right). \tag{2.1.1}$$

Here $S^{(0)}(q)$ is the soft graviton factor. The finite energy external particles can be massless or massive and of arbitrary spins. From the Feynman diagrams leading order factorization can be understood when the soft graviton is attached to one of the finite energy external legs through a three-point vertex. Leading soft factor contains a pole of the form $(\tau\, q \cdot k_i)^{-1}$ which comes from the propagator of the external line to which the soft graviton is attached.

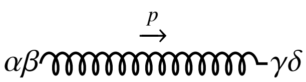
We will give a pedagogical derivation of the leading soft graviton theorem for a simple case of scattering amplitude when the external finite energy states are massless scalars [56]. The action for Einstein gravity coupled to free massless scalar is given by

$$S = - \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{2}{\kappa^2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right], \tag{2.1.2}$$

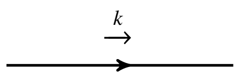
where the coupling constant κ is given by $\kappa^2 = 32\pi G$. In the weak field approximation $g_{\mu\nu}$ can be perturbatively expanded around flat metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + \ldots \tag{2.1.3}$$

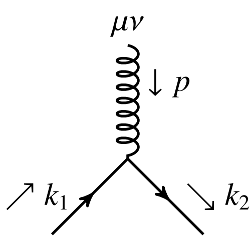
Feynman rules for pure gravity are given in Sec.(B.1.2). In the harmonic gauge the Feynman rules are



$$= -\frac{i}{2} \frac{\left(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta} \right)}{p^2 - i\epsilon},$$

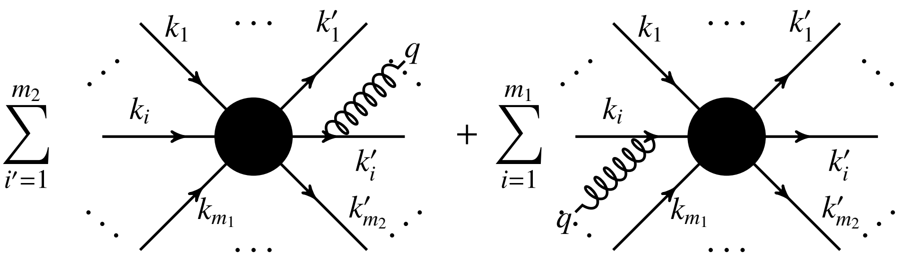


$$= -\frac{i}{k^2 - i\epsilon},$$



$$= \frac{i\kappa}{2} \left(k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - \eta_{\mu\nu} k_1 \cdot k_2 \right). \tag{2.1.4}$$

We will consider the scattering amplitude involving m_1 incoming scalars with momenta $k_1, \dots k_{m_1}$ and m_2 outgoing scalars with momenta $k'_1, \dots k'_{m_2}$ such that $m_1 + m_2 = n$. The soft graviton of momentum τq can be attached either to the incoming hard external line or to an outgoing hard line. The polarization tensor of the soft graviton is denoted by $\epsilon_{\mu\nu}(q)$ which satisfies the constraint $q^\mu \epsilon_{\mu\nu} - \frac{1}{2} q_\nu \epsilon^\mu{}_\mu = 0$. In the limit $\tau \to 0$ at leading order we get the following Feynman diagrams



Feynman diagrams with external soft graviton attached to any internal lines contribute at sub-leading order in τ. Hence evaluation of the above Feynman diagrams yield

$$
\begin{aligned}
& \mathcal{M}_{\mu\nu}\left(\tau q, k_1, \dots k_{m_1}; k'_1, \dots, k'_{m_2}\right) \\
= \quad & \sum_{i=1}^{m_2} \mathcal{M}\left(k'_1, \dots, k'_i + \tau q, \dots, k'_{m_2}; k_1, \dots, k_{m_1}\right) \frac{-i}{\left(k'_i + \tau q\right)^2 - i\epsilon} \\
& \qquad \frac{i\kappa}{2}\left[k'_{i\mu}\left(k'_i + \tau q\right)_\nu + k'_{i\nu}\left(k'_i + \tau q\right)_\mu - \eta_{\mu\nu} k'_i \cdot \left(k'_i + \tau q\right)\right] \\
+ \quad & \sum_{i=1}^{m_1} \mathcal{M}\left(k'_1, \dots, k'_{m_2}; k_1, \dots k_i + \tau q, \dots, k_{m_1}\right) \frac{-i}{\left(k'_i - \tau q\right)^2 - i\epsilon} \\
& \qquad \frac{i\kappa}{2}\left[k_{i\mu}\left(k_i - \tau q\right)_\nu + k_{i\nu}\left(k_i + \tau q\right)_\mu - \eta_{\mu\nu} k_i \cdot \left(k_i - \tau q\right)\right] \\
\overset{\tau\to 0}{=} \quad & \frac{\kappa}{2\tau}\left[\sum_{i=1}^{m_2} \frac{k'_{i\mu} k'_{i\nu}}{k'_i \cdot q} - \sum_{i=1}^{m_1} \frac{k_{i\mu} k_{i\nu}}{k_i \cdot q}\right] \mathcal{M}\left(k_1, \dots k_{m_1}; k'_1, \dots, k'_{m_2}\right) \\
& \qquad + \mathcal{O}\left(\tau^0\right). \qquad (2.1.5)
\end{aligned}
$$

Therefore contracting the above expression with polarization tensor of the soft graviton

Weinberg's soft graviton theorem (modulo the constant numerical factor) follows

$$\mathcal{M}(\tau q, k_1, \ldots k_n) = \frac{1}{\tau} \sum_{i=1}^{n} \eta_i \frac{\epsilon_{\mu\nu} k_i^{\mu} k_i^{\nu}}{q \cdot k_i} \mathcal{M}(k_1, \ldots k_n) + O\left(\tau^0\right), \tag{2.1.6}$$

where $\eta_i = \pm 1$ depending on whether the soft graviton is attached to outgoing state or incoming state respectively.

Weinberg's soft graviton theorem holds for any generic quantum theory of gravity. The leading soft factor is universal and is valid for arbitrary spins and masses of the external finite energy states. *Equivalence principle* says at low energies gravity couples to all forms of momenta with the same strength, implying the fact that the gravitational coupling constant is same for all particles.

Gauge invariance Under gauge transformation polarization tensor of graviton changes as

$$\epsilon_{\mu\nu}(q) \to \epsilon_{\mu\nu} + q_\mu \Lambda_\nu + q_\nu \Lambda_\mu \tag{2.1.7}$$

where Λ is any vector field satisfying $\Lambda^\mu q_\mu = 0$.

Then from Eq.(2.1.6) we obtain

$$\delta\epsilon^{\mu\nu} \mathcal{M}_{\mu\nu} = \Lambda^\mu \left(\sum_{i=1}^{n} \eta_i k_{i\mu} \right) \mathcal{M} = 0. \tag{2.1.8}$$

The last equality follows from the conservation of total momentum of the scattered particles.

2.2 Subleading soft graviton theorems

Almost fifty years after the discovery of Weinberg's soft graviton theorem, Cachazo and Strominger [22] found that factorization of the graviton amplitude in the soft limit can be extended beyond the leading order to sub-sub-leading order in the the energy of the soft graviton. Cachazo and Strominger used spinor helicity variables to derive the factorization of an $(n+1)$-point graviton amplitude for pure gravity from Britto-Cachazo-Feng-Witten recursion relations [6]. Factorization of the $(n+1)$-point amplitude can be given by

$$\mathcal{M}_{n+1}(k_1,\ldots,k_n,q) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right)\mathcal{M}_n + O\left(q^2\right). \tag{2.2.1}$$

In this section we review the analysis of Cachazo and Strominger [22] to derive the soft factorization of pure graviton amplitude at tree level. In spinor helicity variables we will use $\{\lambda_{i\alpha}, \tilde{\lambda}_{i\dot{\alpha}}\}$ for hard particles' momenta k_i and $\left(\lambda_{s\alpha}, \tilde{\lambda}_{s\dot{\alpha}}\right)$ for soft momentum q with $\alpha, \dot{\alpha} = 1, 2$. Therefore we can write

$$\begin{aligned} k_{i\,\alpha\dot{\alpha}} &= \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}}, \qquad i \in \{1, 2, \ldots, n\} \\ q_{\alpha\dot{\alpha}} &= \lambda_{s\alpha}\tilde{\lambda}_{s\dot{\alpha}}. \end{aligned} \tag{2.2.2}$$

Momentum conservation implies

$$\sum_{i=1}^{n} \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}} + \lambda_{s\alpha}\tilde{\lambda}_{s\dot{\alpha}} = 0. \tag{2.2.3}$$

Since we are considering gravitons only, helicity of i-th particle is given by $h_i = \pm 2$. Then $(n+1)$-point full amplitude can be expressed as

$$\mathcal{M}_{n+1} = M_{n+1}\left(\{\lambda_1, \tilde{\lambda}_i, h_i\}, \ldots, \{\lambda_n, \tilde{\lambda}_n, h_n\}, \{\lambda_s, \tilde{\lambda}_s, h_s\}\right)\delta^4\left(\sum_{i=1}^{n} \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}} + \lambda_{s\alpha}\tilde{\lambda}_{s\dot{\alpha}}\right). \tag{2.2.4}$$

Here M_{n+1} is the momentum stripped amplitude and delta function imposes the momentum conservation constraint.

To take the soft limit to $\mathcal{M}_{n+1}$ we will use a small parameter ϵ to rescale the soft momentum as

$$\lambda_s \to \sqrt{\epsilon}\lambda_s, \qquad \tilde{\lambda}_s \to \sqrt{\epsilon}\tilde{\lambda}_s \tag{2.2.5}$$

so that

$$q_{\alpha\dot{\alpha}} = \epsilon\lambda_{s\alpha}\tilde{\lambda}_{s\dot{\alpha}}. \tag{2.2.6}$$

We can make an expansion of the $(n+1)$-point amplitude in ϵ in the limit $\epsilon \to 0$.

Little group for massless particle is $SO(2)$ which allows the spinor variables to scale as $\lambda_\alpha \to t^{-1}\lambda_\alpha$ and $\tilde{\lambda}_{\dot{\alpha}} \to t\tilde{\lambda}_{\dot{\alpha}}$. Under this scaling amplitude should transform with the little group weights as

$$M\left(t^{-1}\lambda, t\tilde{\lambda}\right) = t^{2h} M\left(\lambda, \tilde{\lambda}\right). \tag{2.2.7}$$

Taking $t = \epsilon^{\frac{1}{2}}$ for a positive helicity soft graviton we can write $\mathcal{M}_{n+1}$ as

$$\mathcal{M}_{n+1}\left(\sqrt{\epsilon}\lambda_s, \sqrt{\epsilon}\tilde{\lambda}_s, +2\right) = \epsilon^2 \mathcal{M}_{n+1}\left(\epsilon\lambda_s, \tilde{\lambda}_s, +2\right). \tag{2.2.8}$$

Eq.(2.2.8)implies that we can take $\lambda_s(\epsilon) \to 0$ keeping the antiholomorphic one $\tilde{\lambda}_s$ remains finite. We will derive the factorization of $\mathcal{M}_{n+1}$ under this holomorphic soft limit.

Now let us consider the following complex deformations in the soft momenta and n-th hard momentum

$$\lambda_s(z) = \lambda_s + z\lambda_n, \qquad \tilde{\lambda}_n(z) = \tilde{\lambda}_n - z\tilde{\lambda}_s. \tag{2.2.9}$$

Then the stripped $(n+1)$-point is a rational function of the deformation parameter z given by $M_{n+1}(z)$. The original amplitude can be obtained by calculating the residue at $z = 0$ in the following contour integration

$$M_{n+1} = \frac{1}{2\pi i} \oint_{|z|\to 0} \mathrm{d}z \frac{M_{n+1}(z)}{z}. \tag{2.2.10}$$

We can evaluate the above integration by deforming the contour away from the original one to infinity, then there will be other poles which have to be considered. In general $M_{n+1}(z) \to \frac{1}{z}$ as $z \to \infty$, therefore there are no poles at infinity. Hence the only poles will come from the poles of the integrand arising when the propagators go on-shell

$$k_I^2 = (q(z) + k_{a_1} + \ldots + k_{a_m})^2 = 0, \tag{2.2.11}$$

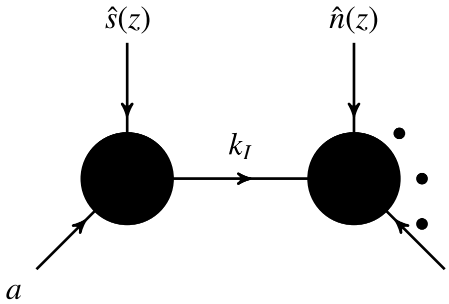
for any non-empty subset $\{a_1, a_2, \ldots, a_m\} \in \{1, 2, \ldots, n-1\}$.

From Eq.(2.2.11) we can obtain location of the poles at $z = z^*$ and the amplitude factorizes on these residues. Therefore we obtain

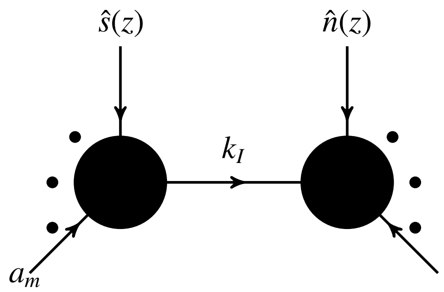
$$\begin{aligned} M_{n+1} &= \frac{1}{2\pi i} \oint_{|z|\to 0} \mathrm{d}z \frac{M_{n+1}(z)}{z} \\ &= -\frac{1}{2\pi i} \sum_{\{a_1,\ldots,a_m\}} \oint_{z\to z^*} \frac{\mathrm{d}z}{z} \frac{M_L\left(s(z), a_1, \ldots, a_m, \{k_I, h_I\}\right) M_R\left(\{-k_I, -h_I\}, \overline{a_1, \ldots, a_m}, a_n(z)\right)}{K_I(z)^2} \\ &= -\sum_{\{a_1,\ldots,a_m\}} M_L\left(s(z^*), a_1, \ldots, a_m, \{k_I, h_I\}\right) \frac{1}{\left(q + k_{a_1} + \ldots + k_{a_m}\right)^2} \\ &\qquad M_R\left(\{-k_I, -h_I\}, \overline{a_1, \ldots, a_m}, a_n(z^*)\right), \end{aligned} \tag{2.2.12}$$

where $z^* = -\frac{(q+k_{a_1}+\ldots+k_{a_m})^2}{2(q+k_{a_1}+\ldots+k_{a_m})^2\cdot r}$ and r is a null momentum orthogonal to both q and k_n. Here $\{\overline{a_1, \ldots, a_m}\}$ denotes the complementary set of $\{a_1, \ldots, a_m\}$ but does not contain the n-th particle. We have to consider opposite signs for the momentum of the internal propagator because if it is incoming for left amplitude it will be outgoing for the right amplitude and vice versa. Also we have to take sum over all the helicities, h_I of the virtual particle corresponding to the internal momentum.

We will have to consider the following two types of diagram:



(a) Diagram: 1



(b) Diagram: 2

In the first type of diagram there is a single hard particle on the left amplitude which is labeled by $a \in \{1, \ldots, n-1\}$. M_L can have either MHV or anti-MHV configuration. We will assume the soft graviton is of positive helicity, then the helicities of the virtual particle will be opposite to that of the hard external particle. Contribution from this diagram can be given by

$$A_{n+1} = \sum_{a=1}^{n-1} \sum_{h_I=\pm 2} M_L\left(\{\hat{s}(z^*)\}, a, \{k_I, h_I\}\right) \frac{1}{(q+k_a)^2} M_R\left(\{-k_I, -h_I\}, \bar{a}, \hat{n}(z^*)\right). \tag{2.2.13}$$

From Eq.(2.2.11) we obtain

$$z^* = -\frac{\langle as \rangle}{\langle an \rangle}, \qquad \lambda_I = \lambda_a, \qquad \tilde{\lambda}_I = \tilde{\lambda}_a + \frac{\langle sn \rangle}{\langle an \rangle}\tilde{\lambda}_s. \tag{2.2.14}$$

If the helicity of the hard particle a is positive then

$$M_L = \left(\frac{[\hat{s}a]^3}{[aI][I\hat{s}]}\right)^2 = \frac{[sa]^2 \langle an \rangle^2}{\langle sn \rangle^2}. \tag{2.2.15}$$

It can be checked that above expression for M_L holds for negative helicity of a-th particle also. Therefore Eq.(2.2.13) becomes

$$A_{n+1} = \sum_{a=1}^{n-1} \frac{[sa]\langle na \rangle^2}{\langle sa \rangle \langle ns \rangle^2} M_n\left(\{\lambda_1, \tilde{\lambda}_1, h_1\}, \ldots, \{\lambda_a, \tilde{\lambda}_a + \frac{\langle sn \rangle}{\langle an \rangle}\tilde{\lambda}_s, h_a\}, \ldots, \{\lambda_n, \tilde{\lambda}_n + \frac{\langle as \rangle}{\langle an \rangle}\tilde{\lambda}_s, h_n\}\right), \tag{2.2.16}$$

where we have used the expression of the propagator as $\frac{1}{(q+k_a)^2} = \frac{1}{\langle sa \rangle [sa]}$.

We can write the momentum conserving delta function as

$$\delta^4\left(\sum_{a=1}^{n} \lambda_a \tilde{\lambda}_a + \lambda_s \tilde{\lambda}_s\right) = \delta^4\left(\sum_{\substack{b=1 \\ b \neq a}}^{n-1} \lambda_b \tilde{\lambda}_b + \lambda_a\left(\tilde{\lambda}_a + \frac{\langle sn \rangle}{\langle an \rangle}\tilde{\lambda}_s\right) + \lambda_n\left(\tilde{\lambda}_n + \frac{\langle as \rangle}{\langle an \rangle}\tilde{\lambda}_s\right)\right). \tag{2.2.17}$$

Then combining Eq.(2.2.16) and Eq.(2.2.17) we obtain

$$\mathcal{A}_{n+1} = \sum_{a=1}^{n-1} \frac{[sa]\langle na\rangle^2}{\langle sa\rangle\langle ns\rangle^2} \mathcal{M}_n \left(\{\lambda_1, \tilde{\lambda}_1, h_1\}, \ldots, \{\lambda_a, \tilde{\lambda}_a + \frac{\langle sn\rangle}{\langle an\rangle}\tilde{\lambda}_s, h_a\}, \ldots, \{\lambda_n, \tilde{\lambda}_n + \frac{\langle as\rangle}{\langle an\rangle}\tilde{\lambda}_s, h_n\} \right). \tag{2.2.18}$$

Now we can take a holomorphic soft limit by employing the scaling $\lambda_s \to \epsilon \lambda_s$ for small ϵ. From Taylor series expansion of $\mathcal{M}_n$ around $\epsilon = 0$ we get

$$\begin{aligned} \mathcal{A}_{n+1} &= \sum_{a=1}^{n-1} \frac{1}{\epsilon^3} \frac{[sa]\langle na\rangle^2}{\langle sa\rangle\langle ns\rangle^2} \mathcal{M}_n \left(\{\lambda_1, \tilde{\lambda}_1, h_1\}, \ldots, \{\lambda_a, \tilde{\lambda}_a + \epsilon\frac{\langle sn\rangle}{\langle an\rangle}\tilde{\lambda}_s, h_a\}, \ldots, \{\lambda_n, \tilde{\lambda}_n + \epsilon\frac{\langle as\rangle}{\langle an\rangle}\tilde{\lambda}_s, h_n\} \right) \\ &= \sum_{a=1}^{n-1} \frac{1}{\epsilon^3} \frac{[sa]\langle na\rangle^2}{\langle sa\rangle\langle ns\rangle^2} \left[1 + \epsilon \left(\frac{\langle ns\rangle}{\langle na\rangle} \tilde{\lambda}_s^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\alpha}}} + \frac{\langle as\rangle}{\langle an\rangle} \tilde{\lambda}_s^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_n^{\dot{\alpha}}} \right) + \frac{\epsilon^2}{2} \left(\frac{\langle ns\rangle}{\langle na\rangle} \tilde{\lambda}_s^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\alpha}}} + \frac{\langle as\rangle}{\langle an\rangle} \tilde{\lambda}_s^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_n^{\dot{\alpha}}} \right)^2 \right. \\ &\qquad \left. + \ldots \right] \mathcal{M}_n \left(\{\lambda_1, \tilde{\lambda}_1, h_1\}, \ldots, \{\lambda_a, \tilde{\lambda}_a, h_a\}, \ldots, \{\lambda_n, \tilde{\lambda}_n, h_n\} \right). \end{aligned} \tag{2.2.19}$$

Let us now calculate each of these terms in Eq.(2.2.19) separately.

First term We consider the expression $\sum_{a=1}^{n-1} \frac{[sa]\langle na\rangle^2}{\langle sa\rangle\langle ns\rangle^2}$. In spinor helicity notations polarization tensor of the soft graviton with positive helicity is given by $\epsilon_s^{\mu\nu} = \epsilon_{s\ \alpha\dot{\alpha}\beta\dot{\beta}} \left(\sigma^{\alpha\dot{\alpha}}\right)^\mu \left(\sigma^{\beta\dot{\beta}}\right)^\nu$, where

$$\epsilon_{s\ \alpha\dot{\alpha}\beta\dot{\beta}} = \frac{1}{2} \left[\frac{\lambda_{x\alpha}\tilde{\lambda}_{s\dot{\alpha}}}{\langle xs\rangle\langle ys\rangle} + (x \leftrightarrow y) \right]. \tag{2.2.20}$$

Here x and y are the reference spinors such that $\epsilon_{s\ \alpha\dot{\alpha}\beta\dot{\beta}}\lambda_x^\alpha = 0$ and $\epsilon_{s\ \alpha\dot{\alpha}\beta\dot{\beta}}\lambda_y^\alpha = 0$. To make our computation simpler we can choose these reference spinors to be

$$x = y = n. \tag{2.2.21}$$

This means the momentum of the n-th particle is orthogonal to the polarization tensor of the soft graviton. With this choice it can be seen

$$\frac{\epsilon_s^{\mu\nu} k_{a\mu} k_{a\nu}}{q \cdot k_a} = \frac{\langle na\rangle^2 [sa]}{\langle ns\rangle^2 \langle sa\rangle}. \tag{2.2.22}$$

Then we can immediately recover leading soft graviton factor

$$S^{(0)}(q) = \sum_{a=1}^{n} \frac{\epsilon_s^{\mu\nu} k_{a\mu} k_{a\nu}}{q \cdot k_a} = \sum_{a=1}^{n} \frac{\langle na\rangle^2 [sa]}{\langle ns\rangle^2 \langle sa\rangle}. \tag{2.2.23}$$

The term with $a = n$ in the sum vanishes because of the particular choice of the reference spinors in Eq.(2.2.21).

Second term The angular momentum operator acting on the a-th hard particle is given by

$$\hat{J}_a^{\mu\nu} = k_a^\mu \frac{\partial}{\partial k_{a\nu}} - k_a^\nu \frac{\partial}{\partial k_{a\mu}} + J_a^{\mu\nu}, \tag{2.2.24}$$

where $J_a^{\mu\nu}$ is the spin angular momentum part defined as

$$(J^{\mu\nu})_{\rho\sigma}{}^{\alpha\beta}\, \epsilon_{\alpha\beta} = \delta^\mu_\rho \epsilon^\nu{}_\sigma - \delta^\nu_\rho \epsilon^\mu{}_\sigma + \delta^\mu_\sigma \epsilon_\rho{}^\nu - \delta^\nu_\sigma \epsilon_\rho{}^\mu. \tag{2.2.25}$$

Using spinor helicity variables we can write

$$\hat{J}_{a\mu\nu} \sigma^\mu_{\alpha\dot{\alpha}} \sigma^\nu_{\beta\dot{\beta}} = -2 J_{a\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} - 2\epsilon_{\alpha\beta} \tilde{J}_{a\dot{\alpha}\dot{\beta}}, \tag{2.2.26}$$

where holomorphic and anti-holomorphic parts of the angular momentum operator can be given by

$$J_{a\alpha\beta} = \frac{i}{2}\left(\lambda_{a\alpha} \frac{\partial}{\partial \lambda_a^\beta} + \lambda_{a\beta} \frac{\partial}{\partial \lambda_a^\alpha}\right), \qquad \tilde{J}_{a\dot{\alpha}\dot{\beta}} = \frac{i}{2}\left(\tilde{\lambda}_{a\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\beta}}} + \tilde{\lambda}_{a\dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\alpha}}}\right). \tag{2.2.27}$$

Here $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ are anti-symmetric 2×2 matrices

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{\dot{\alpha}\dot{\beta}}. \tag{2.2.28}$$

From the second term inside parentheses in Eq.(2.2.19) we get

$$\begin{aligned}
&\sum_{a=1}^{n-1}\frac{[sa]\langle na\rangle^2}{\langle sa\rangle\langle ns\rangle^2}\left(\frac{\langle ns\rangle}{\langle na\rangle}\tilde{\lambda}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot{\alpha}}}+\frac{\langle as\rangle}{\langle an\rangle}\tilde{\lambda}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_n^{\dot{\alpha}}}\right)\\
=\;&\sum_{a=1}^{n-1}\frac{[sa]\langle na\rangle}{\langle sa\rangle[ns]}\tilde{\lambda}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot{\alpha}}},
\end{aligned}\tag{2.2.29}$$

where we have used the conservation of momentum $\sum_{a=1}^{n-1}[sa]\langle an\rangle = 0$. It can be checked that Eq.(2.2.29) gives the sub-leading soft graviton factor

$$S^{(1)} = -i\sum_{a=1}^{n}\frac{\epsilon_{s\mu\nu}k_a^{\mu}q_{\rho}\hat{J}_a^{\rho\nu}}{q\cdot k_a} = \sum_{a=1}^{n-1}\frac{[sa]\langle na\rangle}{\langle sa\rangle[ns]}\tilde{\lambda}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot{\alpha}}}\tag{2.2.30}$$

in the particular choice of reference spinor of Eq.(2.2.21).

Gauge invariance Using the gauge transformation $\epsilon_{\mu\nu}(q) \to \epsilon_{\mu\nu} + q_{\mu}\Lambda_{\nu} + q_{\nu}\Lambda_{\mu}$ it is easy to verify the sub-leading soft graviton factor is invariant due to conservation of total angular momentum

$$\sum_{a=1}^{n}\hat{J}_a^{\mu\nu} = 0.\tag{2.2.31}$$

Third term Now we consider the third term in the parentheses which can be given by

$$\begin{aligned}
&\frac{1}{2}\sum_{a=1}^{n-1}\frac{[sa]\langle na\rangle^2}{\langle sa\rangle\langle ns\rangle^2}\left(\frac{\langle ns\rangle}{\langle na\rangle}\tilde{\lambda}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot{\alpha}}}+\frac{\langle as\rangle}{\langle an\rangle}\tilde{\lambda}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_n^{\dot{\alpha}}}\right)^2\\
=\;&\frac{1}{2}\sum_{a=1}^{n-1}\left[\frac{[sa]}{\langle sa\rangle}\tilde{\lambda}_s^{\dot{\alpha}}\tilde{\lambda}_s^{\dot{\beta}}\frac{\partial^2}{\partial\tilde{\lambda}_a^{\dot{\alpha}}\partial\tilde{\lambda}_a^{\dot{\beta}}}+2\frac{[sa]}{\langle ns\rangle}\frac{\partial^2}{\partial\tilde{\lambda}_a^{\dot{\alpha}}\partial\tilde{\lambda}_n^{\dot{\beta}}}+\frac{[sa]\langle sa\rangle}{\langle ns\rangle^2}\frac{\partial^2}{\partial\tilde{\lambda}_n^{\dot{\alpha}}\partial\tilde{\lambda}_n^{\dot{\beta}}}\right]\\
=\;&\frac{1}{2}\sum_{a=1}^{n-1}\frac{[sa]}{\langle sa\rangle}\tilde{\lambda}_s^{\dot{\alpha}}\tilde{\lambda}_s^{\dot{\beta}}\frac{\partial^2}{\partial\tilde{\lambda}_a^{\dot{\alpha}}\partial\tilde{\lambda}_a^{\dot{\beta}}},
\end{aligned}\tag{2.2.32}$$

where the second term vanishes due to conservation of angular momentum and the third term vanishes because of linear momentum conservation. Then the last expression gives the sub-sub-leading soft graviton factor

$$S^{(2)} = -\frac{1}{2}\sum_{a=1}^{n}\frac{\epsilon_{s\mu\nu}q_\rho q_\sigma \hat{J}_a^{\rho\mu}\hat{J}_a^{\sigma\nu}}{q\cdot k_a} = \frac{1}{2}\sum_{a=1}^{n-1}\frac{[sa]}{\langle sa\rangle}\tilde{\lambda}_s^{\dot{\alpha}}\tilde{\lambda}_s^{\dot{\beta}}\frac{\partial^2}{\partial\tilde{\lambda}_a^{\dot{\alpha}}\partial\tilde{\lambda}_a^{\dot{\beta}}} \tag{2.2.33}$$

in the particular choice of reference spinor of Eq.(2.2.21).

From Eq.(2.2.33) it is immediately obvious that if $\epsilon_{s\mu\nu}$ is replaced by $q_\mu\Lambda_\nu + q_\nu\Lambda_\mu$ the expression vanishes because indices are symmetric in q but anti-symmetric in J_a. Hence gauge invariance of the sub-sub-leading soft graviton factor does not require any conservation law.

It has been shown in [22] that second type of diagram does not contain any pole and remains finite as $\epsilon \to 0$. Therefore using Eq.(2.2.8) and Eq.(2.2.19) we obtain factorization of gravity amplitude in the holomorphic soft limit to sub-sub-leading order as

$$\mathcal{M}_{n+1} = \left(\frac{1}{\epsilon^3}S^{(0)} + \frac{1}{\epsilon^2}S^{(1)} + \frac{1}{\epsilon}S^{(2)}\right)\mathcal{M}_n + \mathcal{O}\left(\epsilon^0\right). \tag{2.2.34}$$

2.2.1 Comments on gauge invariance

Although the soft factors for gravitational amplitudes given in Eq.(2.2.23), Eq.(2.2.29) and Eq.(2.2.33) are written in a particular choice of gauge by fixing the reference spinors of the polarization tensor of the soft graviton, they can be expressed in a gauge invariant way in spinor helicity language in the following way

$$\begin{aligned}
S^{(0)} &= \sum_{a=1}^{n}\frac{[sa]\langle xa\rangle\langle ya\rangle}{\langle sa\rangle\langle xs\rangle\langle ys\rangle} \\
S^{(1)} &= \frac{1}{2}\sum_{a=1}^{n}\frac{[sa]}{\langle sa\rangle}\left(\frac{\langle xa\rangle}{\langle xs\rangle}+\frac{\langle ya\rangle}{\langle ys\rangle}\right)\tilde{\lambda}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot{\alpha}}} \\
S^{(2)} &= \frac{1}{2}\sum_{a=1}^{n}\frac{[sa]}{\langle sa\rangle}\tilde{\lambda}_s^{\dot{\alpha}}\tilde{\lambda}_s^{\dot{\beta}}\frac{\partial^2}{\partial\tilde{\lambda}_a^{\dot{\alpha}}\partial\tilde{\lambda}_a^{\dot{\beta}}}.
\end{aligned} \tag{2.2.35}$$

Here x and y are arbitrary reference spinors.

To check for explicit gauge invariance we can vary the terms in Eq.(2.2.35) with respect to x or y. Then we get

$$\begin{aligned}
\delta_x S^{(0)} &= \frac{1}{2}\sum_{a=1}^{n} \frac{[sa]\langle ya\rangle}{\langle sa\rangle\langle ys\rangle}\left(\frac{\langle xs\rangle\langle \delta x\ a\rangle - \langle xa\rangle\langle \delta x\ s\rangle}{\langle xs\rangle^2}\right) \\
&= \frac{1}{2}\sum_{a=1}^{n} \frac{[sa]\langle ya\rangle\langle x\ \delta x\rangle}{\langle xs\rangle^2\langle ys\rangle} \\
&= 0. \qquad (2.2.36)
\end{aligned}$$

To obtain the second equality we have used Schouten identity and the last equality follows from the conservation of momentum.

Similarly

$$\begin{aligned}
\delta_x S^{(1)} &= \frac{1}{2}\sum_{a=1}^{n} \frac{\langle x\ \delta x\rangle[sa]}{\langle xs\rangle^2}\tilde{\lambda}_s^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot{\alpha}}} \\
&= 0 \qquad (2.2.37)
\end{aligned}$$

follows from conservation of angular momentum.

The expression of $S^{(2)}$ is independent of any reference spinor and hence $S^{(2)}$ is gauge invariant term by term.

2.2.2 Remarks

Sen used covariantization method [8, 29] to derive soft graviton theorem to sub-leading order for one soft graviton and arbitrary number of finite energy external states of any masses and spins for any generic theory of quantum gravity for which perturbative analysis of S-matrix holds. In dimensions greater than or equal to five the graviton soft theorems are universal and do not receive any loop corrections. In lower dimensions there are issues of IR divergences [9]. Laddha and Sen [9] showed that sub-sub-leading soft graviton theorem is not universal. The amplitude with one soft graviton and other finite energy particles to sub-sub-leading order in soft momentum contains two parts: one universal part that depends only on the amplitude without the soft graviton and another non-universal part that depends on the details of the theory, particularly on the two and three point functions of the theory.

3 Cachazo He Yuan formalism

In a series of seminal papers [7, 61, 62, 66, 67] Cachazo, He and Yuan came up with a novel technique of calculating tree-level scattering amplitudes involving massless particles for a wide variety of theories in Quantum Field Theory. Inspired from String Theory, in this formalism scattering amplitudes can be expressed as integrals over moduli space of punctured complex Riemann spheres. In the conventional method of computing scattering amplitudes we generally have to take into account inconveniently huge number of Feynman diagrams, also Feynman rules are not unique because using field redefinitions one can write different Lagrangians for same theories. CHY formalism is lot simpler because we do not need Lagrangians and hence nor Feynman diagrams to calculate scattering amplitudes. Scattering amplitudes calculated in this method are valid for any arbitrary number of spacetime dimensions. This elegant formulation is likely to have serious ramifications for our understanding of Quantum Field Theory and their dependence on spacetime structures.

Many underlying relations between various theories, for example double-copy relations, Kawai-Lewellen-Tye orthogonality, color-kinematic dualities, etc which are difficult to see in the Lagrangian descriptions, become manifest in this formulation. Kawai, Lewellen and Tye [68] discovered that tree-level amplitudes of closed strings can be expressed as square of open string amplitudes. In the field theory limit these relations reduce to pure gravity amplitudes as "square" of Yang-Mills amplitudes [69, 70]. Cachazo, He and Yuan made this statement more precise by showing that the coefficient of the field-theory limit of KLT expansions are given by inverse of a matrix whose components are amplitudes of a particular type of cubic scalar theory called bi-adjoint scalar [66,71]. Schematically this can be expressed as Gravity $= \frac{\text{Yang-Mills}^2}{\text{bi-adjoint scalar}}$. These type of relations have also been found among different other theories.

Soft theorems and factorization properties of scattering amplitudes can be studied very easily using CHY formalism. Single soft theorems [7,23,24,27,28,61] as well as multiple soft theorems [14,63,72] have been studied for different theories. KLT orthogonality and double copy relations among soft factorizations of various theories have been explored in [42].

Although CHY formulation is most well developed for tree-level amplitudes for massless particles, loop-level extensions for scalars and gauge bosons have been done in [73,74].

We organize this chapter as follows: in Sec.(3.1) we give a brief review, mainly the essential construction of the Cachazo-He-Yuan formalism. In Sec.(3.2) we present some important examples of scattering amplitudes for scalar, gauge and gravity theories. Soft theorems and their derivations are given in Sec.(3.3). In Sec.(3.4) we describe the double copy relations between amplitudes using CHY formulation.

3.1 Basic ingredients

The essential feature of the formalism is to map the singularities of the scattering amplitude in the kinematic space of say, n massless particles to the singularity structure of an auxiliary space which is better understood. In this case Cachazo et al consider the moduli space of all n-punctured Riemann sphere, $\mathbb{CP}^1$. Let $\{k_1^\mu, k_2^\mu, \ldots k_n^\mu\}$ are the momenta of n massless particles in D dimension forming the kinematic space. The kinematic space configuration can be defined by

$$\mathcal{K} := \left\{\left(k_1^\mu, k_2^\mu, \ldots, k_n^\mu\right) | \sum_{a=1}^{n} k_a^\mu = 0, k_a^2 = 0, \forall a \in \{1, \ldots, n\}\right\} / \mathbb{SO}\,(1, D-1)\,. \tag{3.1.1}$$

We consider $\{\sigma_1, \sigma_2, \ldots \sigma_n\}$ to be holomorphic variables which parametrize the moduli space, $\mathfrak{M}_{0,n}$. The holomorphic variables specify the locations of punctures on the Riemann sphere. The mapping of the singularities is given by [75]

$$k_a^\mu = \frac{1}{2\pi i} \oint_{|z-\sigma_a|=\varepsilon} \mathrm{d}z \frac{f^\mu(z)}{\prod_{b=1}^{n}(z-\sigma_b)}, \qquad \forall a \in \{1, 2, \ldots n\} \tag{3.1.2}$$

where $f^\mu(z)$ is a D degree $n-2$ polynomials.

Clarification: Here $f^\mu(z)$ is a D dimensional vector with $\mu = 1, 2, \ldots D$ and each of $f^\mu(z)$ is a $n-2$ degree polynomial in z. Eq.(3.1.2) can be written more suggestively as

$$k_a^\mu = \frac{1}{2\pi i} \oint_{|z-\sigma_a|=\varepsilon} \mathrm{d}z \left[\sum_{b=1}^{n} \frac{k_b^\mu}{z-\sigma_b}\right] \tag{3.1.3}$$

Now the expression inside the square brackets can be expanded

$$\begin{aligned} & \frac{k_1^\mu}{z-\sigma_1} + \frac{k_2^\mu}{z-\sigma_2} + \ldots + \frac{k_n^\mu}{z-\sigma_n} \\ = \; & \frac{z^{n-1}\left(k_1^\mu + k_2^\mu + \ldots + k_n^\mu\right) + z^{n-2}(\ldots) + \ldots}{(z-\sigma_1)(z-\sigma_2)\ldots(z-\sigma_n)} \\ \equiv \; & \frac{f^\mu(z)}{(z-\sigma_1)(z-\sigma_2)\ldots(z-\sigma_n)} \end{aligned} \tag{3.1.4}$$

Due to momentum conservation coefficient of z^{n-1} vanishes. Hence $f^\mu(z)$ is a polynomial of degree $n-2$ in z.

$\mathbb{SL}\,(2, \mathbb{C})$ **invariance** Moduli space of all n-punctured Riemann spheres, $\mathfrak{M}_{0,n}$ is a $n-3$ dimensional complex space and is invariant under $\mathbb{SL}\,(2, \mathbb{C})$ transformations given by

$$\sigma_a \to \psi(\sigma_a) = \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \ \alpha\delta - \beta\delta = 1. \tag{3.1.5}$$

Because of this $\mathbb{SL}(2, \mathbb{C})$ redundancies we can fix any of the three punctures. Typically we will set $\sigma_1 = 0, \sigma_2 = 1$ and $\sigma_3 = \infty$.

3.1.1 Scattering equations

Imposing the conservation of momentum $\sum_{a=1}^{n} k_a = 0$ and the fact that particles are massless, $k_a^2 = 0, \ \forall a \in \{1, 2, \ldots, n\}$ one gets the following set of equations [7, 67]

$$\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \qquad \forall a \in \{1, 2, \ldots n\}. \tag{3.1.6}$$

These scattering equations can be checked to be invariant under $\mathbb{SL}(2, \mathbb{C})$ transformations of Eq.(3.1.2)

$$\begin{aligned} & \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\psi(\sigma_a) - \psi(\sigma_b)} \\ = & \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} (\gamma\sigma_a + \delta)(\gamma\sigma_b + \delta) \\ = & (\gamma\sigma_a + \delta)^2 \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} - (\gamma\sigma_a + \delta)\gamma \sum_{\substack{b=1 \\ b\neq a}}^{n} k_a \cdot k_b \\ = & \ 0. \end{aligned} \tag{3.1.7}$$

Out of the n equations in Eq.(3.1.6) not all are independent. There are three constraints satisfied by the following equations

$$\sum_{a=1}^{n} \sigma_a^m \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \qquad m = 0, 1, 2 \tag{3.1.8}$$

for any arbitrary values of σ.

For $m = 0$ it is easy to see that LHS of Eq.(3.1.8) is antisymmetric in a and b and hence vanishes. For $m = 1$ we get

$$\begin{aligned}\sum_{a=1}^{n}\sigma_a\sum_{\substack{b=1\\b\neq a}}^{n}\frac{k_a\cdot k_b}{\sigma_a-\sigma_b} &= \sum_{a=1}^{n}\sum_{\substack{b=1\\b\neq a}}^{n}k_a\cdot k_b+\sum_{a=1}^{n}\sum_{\substack{b=1\\b\neq a}}^{n}\sigma_b\frac{k_a\cdot k_b}{\sigma_a-\sigma_b}\\ \Rightarrow\frac{1}{2}\sum_{a=1}^{n}\sigma_a\sum_{\substack{b=1\\b\neq a}}^{n}\frac{k_a\cdot k_b}{\sigma_a-\sigma_b} &= \sum_{a=1}^{n}\sum_{\substack{b=1\\b\neq a}}^{n}k_a\cdot k_b\\ &= 0\end{aligned} \tag{3.1.9}$$

because of momentum conservation. For $m = 2$ we get

$$\begin{aligned}&\sum_{a=1}^{n}\sigma_a^2\sum_{\substack{b=1\\b\neq a}}^{n}\frac{k_a\cdot k_b}{\sigma_a-\sigma_b}\\ &= \sum_{a=1}^{n}\sigma_a\sum_{\substack{b=1\\b\neq a}}^{n}k_a\cdot k_b+\sum_{a=1}^{n}\sum_{\substack{b=1\\b\neq a}}^{n}\frac{\sigma_a\sigma_b k_a\cdot k_b}{\sigma_a-\sigma_b}\\ &= 0.\end{aligned} \tag{3.1.10}$$

Because of these constraints we have $n-3$ number of independent scattering equations. There are $(n-3)!$ set of solutions for $\{\sigma_1,\sigma_2,\ldots,\sigma_n\}$.

3.1.2 Scattering amplitude

Taking into account that scattering amplitude in quantum field theory is translation invariant, we can write the full scattering amplitude for n-particle scattering process in D spacetime dimension as

$$\mathcal{M}_n\left(\{k,\epsilon\}\right) = M_n\left(\{k,\epsilon\}\right)\delta^D\left(\sum_{a=1}^{n}k_a^\mu\right). \tag{3.1.11}$$

Conservation of total momentum is imposed through the D-dimensional delta function. Here M_n is called the momentum-stripped amplitude and depends on the kinematic data - momenta and polarizations. Cachazo-He-Yuan formalism provides an integral representation of this stripped amplitude M_n on the moduli space of n-punctured Riemann spheres.

For scattering of n particles M_n is given by

$$M_n = \int \frac{\mathrm{d}^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a.k_b}{\sigma_a - \sigma_b}\right) I_n(\{k,\epsilon,\sigma\}). \tag{3.1.12}$$

The measure of n σ variables are written as wedge products

$$\mathrm{d}^n\sigma = \mathrm{d}\sigma_1 \wedge \mathrm{d}\sigma_2 \wedge \ldots \wedge \mathrm{d}\sigma_n. \tag{3.1.13}$$

Because of the $\mathbb{SL}(2,\mathbb{C})$ redundancies we have to mod out vol$\mathbb{SL}(2,\mathbb{C})$ which is given by $\frac{\mathrm{d}\sigma_a \mathrm{d}\sigma_b \mathrm{d}\sigma_c}{(\sigma_a-\sigma_b)(\sigma_b-\sigma_c)(\sigma_c-\sigma_a)}$ for any a, b, c from the integration measure.

The delta functions are holomorphic delta functions whose arguments are the scattering equations. Due to the presence of these delta functions integrals are localized on the solutions of the scattering equations. Again because of the $\mathbb{SL}(2,\mathbb{C})$ redundancies we need $n-3$ delta functions to localize the integrals. Therefore the primed product is defined as

$$\prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a.k_b}{\sigma_a - \sigma_b}\right) := (\sigma_i - \sigma_j)(\sigma_j - \sigma_k)(\sigma_k - \sigma_i) \prod_{a\neq i,j,k} \delta\left(\sum_{b\neq a} \frac{k_a.k_b}{\sigma_a - \sigma_b}\right) \tag{3.1.14}$$

for any i, j, k.

The integrand I_n contains information about particular theories to be considered. Cachazo, He and Yuan have constructed a wide class of integrands for large varieties of theories including scalars, photons, gluons and gravitons. Before giving examples of I_n for some theories we can find some elementary properties of I_n in general from the $\mathbb{SL}(2,\mathbb{C})$ invariance of M_n in Eq.(3.1.12).

We will use convenient notation to express scattering amplitude as $M_n = \int d\mu_n I_n$. From the transformations given in Eq.(3.1.5) we find

$$\begin{aligned} d\sigma_a &\rightarrow \frac{d\sigma_a}{(\gamma\sigma_a+\delta)^2} \\ \delta\left(\sum_{b\neq a}\frac{k_a.k_b}{\sigma_a-\sigma_b}\right) &\rightarrow \frac{1}{(\gamma\sigma_a+\delta)^2}\delta\left(\sum_{b\neq a}\frac{k_a.k_b}{\sigma_a-\sigma_b}\right). \end{aligned} \tag{3.1.15}$$

Therefore under $\mathbb{SL}(2,\mathbb{C})$ transformations measure changes as

$$d\mu_n \rightarrow \prod_{a=1}^{n}(\gamma\sigma_a+\delta)^{-4}\, d\mu_n. \tag{3.1.16}$$

This implies that integrand must transforms as

$$I_n \rightarrow \prod_{a=1}^{n}(\gamma\sigma_a+\delta)^{4}\, I_n \tag{3.1.17}$$

to keep the scattering amplitude M_n invariant under $\mathbb{SL}(2,\mathbb{C})$ transformations.

In principle M_n can be expressed as sum over the $(n-3)!$ solutions of scattering equations as

$$M_n = \sum_{I=1}^{(n-3)!}\frac{i_n(\{k,\epsilon,\sigma\})}{J_n(\{k,\epsilon\})}\Big|_{i\text{–th solution}} \tag{3.1.18}$$

where J_n is the Jacobian factor. To calculate the Jacobian we define the following matrix

$$(\Phi_n)_{ab} = \begin{cases} \frac{k_a\cdot k_b}{(\sigma_a-\sigma_b)^2}, & a\neq b \\ -\sum\limits_{\substack{c=1\\c\neq a}}^{n}(\Phi_n)_{ac}\,, & a=b. \end{cases} \tag{3.1.19}$$

Then the Jacobian can be given by

$$J_n(\{k,\epsilon\}) = \det{}'\Phi_n = \frac{\det\,[\Phi_n]^{\hat{a}'\hat{b}'\hat{c}'}_{\hat{a}''\hat{b}''\hat{c}''}}{\sigma_{a'b'}\sigma_{b'c'}\sigma_{c'a'}\sigma_{a''b''}\sigma_{b''c''}\sigma_{c''a''}}, \tag{3.1.20}$$

where we have used the notation $\sigma_{ab} = \sigma_a - \sigma_b$ and $[\Phi_n]^{\hat{a}'\hat{b}'\hat{c}'}_{\hat{a}''\hat{b}''\hat{c}''}$ denotes the minor of Φ_n obtained by removing the rows labeled by $\{a', b', c'\}$ and the columns labeled by $\{a'', b'', c''\}$.

3.2 Examples

We will give some simpler examples of scattering amplitudes which can be expressed by Cachazo-He-Yuan formalism. There are several *building blocks* from which one can construct the integrand I_n. Various combinations of these building blocks make up different I_n which then correspond to specific theories of interest [61, 62].

3.2.1 Scalar

Let us consider the following quantity [61]

$$C_n(\alpha) = \frac{1}{\sigma_{\alpha(1)\alpha(2)}\sigma_{\alpha(2)\alpha(3)}\ldots\sigma_{\alpha(n)\alpha(1)}}. \tag{3.2.1}$$

Here α denotes the ordering of the punctures. $C_n(\alpha)$ is referred to as Parke-Taylor factor with α ordering. It can be checked that under $\mathbb{SL}(2,\mathbb{C})$ transformations

$$C_n(\alpha) \to \prod_{a=1}^{n} (\gamma\sigma_a + \delta)^2 C_n(\alpha). \tag{3.2.2}$$

Therefore if we consider product of two such Parke-Taylor factors then it will have the desired transformation weight as I_n. Indeed this integrand corresponds to cubic scalar interaction theory given by

$$I_n^{\phi^3}(\alpha|\beta) = C_n(\alpha)\, C_n(\beta). \tag{3.2.3}$$

These amplitudes have two copies of orderings and summing over all the orderings produce the full amplitude. We can refer to

$$m_n(\alpha|\beta) = \int \mathrm{d}\mu_\mu I_n^{\phi^3}(\alpha|\beta) \tag{3.2.4}$$

as partial amplitudes.

More generalized cubic scalar interactions can be considered called *bi-adjoint cubic scalar* theory whose Lagrangian is given by

$$\mathcal{L}^{\phi^3} = -\frac{1}{2}\partial_\mu \phi_{aa'} \partial^\mu \phi^{aa'} - \frac{\lambda}{3!} f_{abc} \tilde{f}_{a'b'c'} \phi^{aa'} \phi^{bb'} \phi^{cc'} \tag{3.2.5}$$

where the scalar fields live in the adjoint representation of two unitary groups, $U(N) \times U\left(\tilde{N}\right)$. f_{abc} and $\tilde{f}_{a'b'c'}$ are structure constants of $U(N)$ and $U\left(\tilde{N}\right)$ respectively. Tree level scattering amplitude for this theory can be given by

$$M_n = \sum_{\alpha \in S_n/Z_n} \sum_{\beta \in S_n/Z_n} \mathrm{Tr}\left(\tilde{T}^{b_{\alpha_1}} \tilde{T}^{b_{\alpha_2}} \ldots \tilde{T}^{b_{\alpha_n}}\right) \mathrm{Tr}\left(T^{a_{\beta_1}} T^{a_{\beta_2}} \ldots T^{a_{\beta_n}}\right) m_n(\alpha|\beta). \tag{3.2.6}$$

Here cyclic permutations given by Z_n have been excluded. T^a and $\tilde{T}^a$ are the generators of $U(N)$ and $U\left(\tilde{N}\right)$ respectively.

Furthermore an alternative color basis can be used to replace the traces by [76]

$$c_\alpha = \sum_{c_1, c_2, \ldots, c_{n-3}} f_{a_1 a_{\alpha_2} c_1} \ldots f_{c_{n-3} a_{\alpha(n-1)} a_n}, \tag{3.2.7}$$

where $\alpha \in S_{n-2}$. In this basis one can fix any two elements in the color ordering and permute the rest of them. Then using the Jacobian factor of Eq.(3.1.20) the scattering amplitude can be expressed as

$$M_n = \sum_{\{\sigma\} \in \mathrm{sol}} \frac{1}{\det' \Phi_n} \sum_{\alpha, \beta \in S_{n-2}} \frac{c_\alpha \tilde{c}_\beta}{\left(\sigma_{\alpha_1 \alpha_2} \ldots \sigma_{\alpha_n \alpha_1}\right)\left(\sigma_{\beta_1 \beta_2} \ldots \sigma_{\beta_n \beta_1}\right)}. \tag{3.2.8}$$

3.2.2 Gluon

To describe scattering amplitudes for particles with polarizations new building block has to be introduced. We have to consider the reduced pfaffian of the following $2n \times 2n$

anti-symmetric matrix

$$\Psi_n = \left(\begin{array}{c|c} A & -C^T \\ \hline C^T & B \end{array}\right) \tag{3.2.9}$$

where each of A, B and C is $n \times n$ matrix and the components are:

$$A_{ab} = \begin{cases} \frac{k_a.k_b}{\sigma_a - \sigma_b}, & a \neq b \\ 0, & a = b \end{cases} \qquad B_{ab} = \begin{cases} \frac{\epsilon_a.\epsilon_b}{\sigma_a - \sigma_b}, & a \neq b \\ 0, & a = b \end{cases} \qquad C_{ab} = \begin{cases} \frac{\epsilon_a.k_b}{\sigma_a - \sigma_b}, & a \neq b \\ -\sum\limits_{c \neq a} \frac{\epsilon_a.k_c}{\sigma_a - \sigma_c}, & a = b. \end{cases} \tag{3.2.10}$$

Pfaffian of an anti-symmetric matrix is defined as

$$\text{Pf}(E_n) := \sum_{\alpha \in \text{P.M.}} \text{sgn}(\alpha)(E_n)_{\alpha(1),\alpha(2)}(E_n)_{\alpha(3),\alpha(4)} \ldots (E_n)_{\alpha(n-1),\alpha(n)}, \tag{3.2.11}$$

where P.M. denotes perfect matchings for all possible decompositions into pairs. Here

$$\text{sgn}\,(\alpha) = \begin{cases} +1, & \alpha \in \text{even permutations} \\ -1, & \alpha \in \text{odd permutations.} \end{cases} \tag{3.2.12}$$

For anti-symmetric matrix Pfaffian is square root of its determinant

$$\text{Pf}\,(E_n) = \sqrt{\text{det} E_n}. \tag{3.2.13}$$

The $n \times n$ matrix A_n has a nontrivial kernel of dimension two spanned by the vectors

$$(\sigma_1^m, \sigma_2^m, \ldots, \sigma_n^m)^{\text{T}}, \qquad m = 0, 1. \tag{3.2.14}$$

For $m = 0$ the kernel satisfies scattering equations. For $m = 1$ it can be verified

$$\sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{k_a \cdot k_b \sigma_b}{\sigma_a - \sigma_b} = \sigma_a \sum_{\substack{b=1 \\ b \neq a}} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} - \sum_{\substack{b=1 \\ b \neq a}} k_a \cdot k_b = 0. \tag{3.2.15}$$

Hence Ψ_n also has the nontrivial kernel spanned by $\left(\sigma_a^m, \sigma_2^m, \ldots, \sigma_n^m, 0, 0, \ldots 0\right)$ for $m = 0, 1$. Therefore determinant and hence pfaffian of Ψ_n vanishes on the support of the scattering equations. So we can define an invariant quantity called *reduced Pfaffian* as

$$\mathrm{Pf}' \Psi_n := -\frac{(-1)^{a+b}}{\sigma_a - \sigma_b} \mathrm{Pf}\left[\Psi_n\right]_{\hat{a},\hat{b}}, \tag{3.2.16}$$

where $[\Psi_n]_{\hat{a},\hat{b}}$ is the minor of Ψ_n obtained after removing a-th row and b-th column with the condition that $1 \leq a, b \leq n$.

It can be checked that the matrix elements transform under $\mathbb{SL}(2, \mathbb{C})$ as
$(\Psi_n)_{ab} \to (\gamma\sigma_a + \delta)(\gamma\sigma_b + \delta)(\Psi_n)_{ab}$ except the diagonal components of C which transform as $C_{aa} \to (\gamma\sigma_a + \delta)^2 C_{aa}$. Therefore it can be checked, under $\mathbb{SL}(2, \mathbb{C})$ transformations

$$\mathrm{Pf}' \Psi_n \to \prod_{a=1}^{n} (\gamma\sigma_a + \delta)^2 \, \mathrm{Pf}' \Psi_n. \tag{3.2.17}$$

The integrand for color ordered n-point amplitude, $A_n(\alpha_1, \alpha_2, \alpha_n)$ for Yang-Mills theory is given by

$$I_n^{YM}(\alpha) = \frac{1}{\sigma_{\alpha(1)\alpha(2)}\sigma_{\alpha(2)\alpha(3)} \ldots \sigma_{\alpha(n)\alpha(1)}} \mathrm{Pf}' \Psi_n = C_n(\alpha) \, \mathrm{Pf}' \Psi_n. \tag{3.2.18}$$

Therefore

$$A_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = \int \mathrm{d}\mu_n C_n(\alpha) \, \mathrm{Pf}' \Psi_n \tag{3.2.19}$$

and the full tree level n gluon amplitude for pure Yang-Mills theory

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} \mathrm{tr}\left(F_{\mu\nu a} F^{\mu\nu a}\right) \tag{3.2.20}$$

is given by sum of the color ordered amplitudes as

$$\mathcal{A}_n(1, 2, \ldots, n) = g^{n-2} \sum_{S_n/Z_n} \mathrm{tr}\left(T^{\sigma_1} T^{\sigma_2} \ldots T^{\sigma_n}\right) A_n(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{3.2.21}$$

where g is the coupling constant.

3.2.3 Graviton

Gravitons are spin-2 particles whose polarizations can be given as direct product of two spin-1 polarization vectors. In general we can write

$$\epsilon_{\mu\nu} = \epsilon_\mu \tilde{\epsilon}_\nu. \tag{3.2.22}$$

In CHY formalism tree level S-matrix for gravity is proposed to be

$$M_n = \int \mathrm{d}\mu_n \, \mathrm{Pf}' \Psi_n \left(\{k, \epsilon, \sigma\}\right) \mathrm{Pf}' \tilde{\Psi}_n \left(\{k, \tilde{\epsilon}.\sigma\}\right) \tag{3.2.23}$$

In the simplest case we can take $\epsilon_{\mu\nu} = \epsilon_\mu \epsilon_\nu$ with the constraints

$$\epsilon_{\mu\nu}(k) k^\mu = 0, \qquad \epsilon_\mu{}^\mu = 0. \tag{3.2.24}$$

Then the integrand corresponding to tree level n-point graviton amplitude for pure gravity

$$\mathcal{L}_{\mathrm{GR}} = -\frac{1}{2}\sqrt{-g}R \tag{3.2.25}$$

is given by

$$I_n^{\mathrm{GR}} = \left(\mathrm{Pf}' \Psi_n \left(\{k, \epsilon, \sigma\}\right)\right)^2 = \det{}' \Psi, \tag{3.2.26}$$

where $\det{}' \Psi_n = 4\left(\sigma_i - \sigma_j\right)^{-2} \det\left[\Psi_n\right]_{ij}^{ij}$.

3.3 Soft limits

A remarkable aspect of CHY formalism is that soft limits of various scattering amplitudes can be studied very efficiently [7, 23, 24, 27, 28, 42, 61]. Many new soft theorems have been found out using this new technique of finding amplitudes. Also soft limits help us to verify the consistencies of several proposed formulas of scattering amplitudes for various theories in this formalism which otherwise are difficult to prove directly. In this section we will study behavior of scalar, gluon and graviton amplitudes under single soft limit, *i.e.* when one of the scattered particles in the scattering process is taken to be of infinitesimal energy compared to other particles.

Let us assume in n-point amplitude n-th particle is taken to be soft, its momentum scales as τp in the limit $\tau \to 0$. In this limit the scattering equations (3.1.6) can be expressed as

$$
\begin{aligned}
f_a &= \sum_{\substack{b=1 \\ b \neq a}}^{n-1} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} + \tau \frac{k_a \cdot p}{\sigma_a - \sigma_n}, \qquad a \in \{1, 2, \ldots, n-1\} \\
f_n &= \tau \sum_{b=1}^{n-1} \frac{p \cdot k_b}{\sigma_n - \sigma_b}.
\end{aligned} \tag{3.3.1}
$$

As τ goes to zero there $n-4$ independent scattering equations for $n-1$ variables σ_a and out of which three can be fixed due to $\mathbb{SL}(2, \mathbb{C})$ redundancies. Therefore there are $(n-4)!$ number of solutions for the set $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$. We have not yet specified σ_n. As τ tends to zero but as long as it does not exactly vanish the last equation of (3.3.1) becomes

$$
\sum_{b=1}^{n-1} p \cdot k_b \prod_{\substack{c=1 \\ c \neq b}}^{n-1} (\sigma_n - \sigma_c) = 0. \tag{3.3.2}
$$

So this is an equation in σ_n of order $n-2$. But due to momentum conservation the coefficient of leading order in σ_n is zero

$$
\sigma_n^{n-2} \sum_{b=1}^{n-1} p \cdot k_b = 0. \tag{3.3.3}
$$

Therefore Eq.(3.3.2) is actually of order $n-3$ in σ_n. Hence for every solution out of $(n-4)!$ solutions of $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ there are $n-3$ number of solution to σ_n. This counting also implies that total number of solutions is $(n-3)(n-4)! = (n-3)!$.

The delta functions for the hard scattering equations can be expanded as

$$\delta(f_a) = \delta\left(f_a^{n-1}\right) + \tau \frac{k_a \cdot p}{\sigma_a - \sigma_n} \delta'\left(f_a^{n-1}\right) + O\left(\tau^2\right), \tag{3.3.4}$$

where we denote $f_a^{n-1} = \sum_{\substack{b=1 \\ b\neq a}}^{n-1} \frac{k_a \cdot k_b}{\sigma_a}$. Then the product of delta functions can be expanded as

$$\begin{aligned}
\prod_{a=1}^{n-1}{}' \delta(f_a) &= \prod_{a=1}^{n-1}{}' \delta\left(f_a^{n-1}\right) + \tau \sum_{a=1}^{n-1}{}' \left[\prod_{\substack{b=1 \\ b\neq a}}^{n-1}{}' \delta\left(f_b^{n-1}\right)\right] \frac{k_a . p}{\sigma_a - \sigma_{n-1}} \delta'\left(f_a^{n-1}\right) + O(\tau^2) \\
&\equiv \delta^{(0)} + \tau \delta^{(1)} + O(\tau^2).
\end{aligned} \tag{3.3.5}$$

Now we can write Eq.(3.1.12) as

$$\begin{aligned}
M_n &= \int \frac{\mathrm{d}^{n-1}\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \int \mathrm{d}\sigma_n \prod_{a=1}^{n-1}{}' \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n-1} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} + \tau \frac{k_a \cdot p}{\sigma_a - \sigma_n}\right) \delta\left(\tau \sum_{b=1}^{n-1} \frac{p \cdot k_b}{\sigma_n - \sigma_b}\right) I_n(\{k, \epsilon, \sigma\}) \\
&= \frac{1}{\tau} \int \frac{\mathrm{d}^{n-1}\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \oint_{\{A_i\}} \frac{\mathrm{d}\sigma_n}{\sum_{b=1}^{n-1} \frac{p . k_b}{\sigma_n - \sigma_b}} \left(\delta^{(0)} + \tau \delta^{(1)} + O(\tau^2)\right) I_n.
\end{aligned} \tag{3.3.6}$$

Here delta function supported at the n-th scattering equation deforms the σ_n integral to a closed contour integral which encloses the zeros of $\sum_{b=1}^{n-1} \frac{p . k_b}{\sigma_n - \sigma_b}$ denoted by $\{A_i\}$.

In the soft limit if the integrand factorizes as a soft part and the integrand of $n-1$ hard particles then soft factorization of scattering amplitude follows. We will show soft factorization for some particular theories.

3.3.1 Scalar

We will consider soft limit of the bi-adjoint cubic scalar amplitude studied in Sec.(3.2.1). The n-point amplitude with one soft scalar can be expressed as

$$M_n = \frac{1}{\tau}\int \frac{\mathrm{d}^{n-1}\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})}\delta^{(0)} \oint_{\{A_i\}} \frac{\mathrm{d}\sigma_n}{\sum_{b=1}^{n-1}\frac{p.k_b}{\sigma_n-\sigma_b}} \sum_{\alpha,\beta\in S_{n-2}} c_\alpha \tilde{c}_\beta C_n(\alpha) C_n(\beta)\,. \tag{3.3.7}$$

Here we will calculate the soft factorization to leading order in τ only. For the α and β ordering we can fix $\alpha_1 = \beta_1 = 1$ and $\alpha_n = \beta_n = n$ and permute rest for S_{n-2}. Then we can write

$$C_n(\alpha) = \frac{\sigma_{\alpha_{n-1}} - \sigma_1}{(\sigma_{\alpha_{n-1}} - \sigma_n)(\sigma_n - \sigma_{\alpha_1})} C_{n-1}(\alpha)\,. \tag{3.3.8}$$

Let us denote $\alpha_{n-1} := i$ and similarly $\beta_{n-1} := j$. $(n-2)!$ permutations of the integrand in Eq.(3.3.7) can be expressed as

$$\begin{aligned}\sum_{\alpha,\beta\in S_{n-2}} c_\alpha\tilde{c}_\beta C_n(\alpha) C_n(\beta) &= \sum_{i,j=2}^{n-1}\frac{(\sigma_i-\sigma_1)\left(\sigma_j-\sigma_1\right)}{(\sigma_i-\sigma_n)\left(\sigma_j-\sigma_n\right)(\sigma_n-\sigma_1)^2}\sum_{\alpha^i,\beta^j\in S_{n-3}} c_{\alpha^i}\tilde{c}_{\beta^j}C_n\left(\alpha^i\right)C_n\left(\beta^j\right)\\ &= \sum_{i,j=2}^{n-1}\frac{(\sigma_i-\sigma_1)\left(\sigma_j-\sigma_1\right)}{(\sigma_i-\sigma_n)\left(\sigma_j-\sigma_n\right)(\sigma_n-\sigma_1)^2}\sum_{\alpha^i,\beta^j\in S_{n-3}} c_{\alpha^i}\tilde{c}_{\beta^j}I^{\phi^3}_{n-1}\left(\alpha^i,i;\beta^j,j\right).\end{aligned} \tag{3.3.9}$$

Now we can get the $(n-1)$-point amplitude

$$\int \frac{\mathrm{d}^{n-1}\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})}\delta^{(0)} I^{\phi^3}_{n-1}\left(\alpha^i,i;\beta^j,j\right) = M_{n-1}\left(\hat{\alpha}^i,\hat{\beta}^j\right). \tag{3.3.10}$$

Then from Eq.(3.3.7) we obtain

$$M_n = \frac{1}{\tau}\oint_{\{A_i\}} \frac{\mathrm{d}\sigma_n}{\sum_{b=1}^{n-1}\frac{p.k_b}{\sigma_n-\sigma_b}} \sum_{i,j=2}^{n-1}\frac{(\sigma_i-\sigma_1)\left(\sigma_j-\sigma_1\right)}{(\sigma_i-\sigma_n)\left(\sigma_j-\sigma_n\right)(\sigma_n-\sigma_1)^2}\sum_{\alpha^i,\beta^j\in S_{n-3}} c_{\alpha^i}\tilde{c}_{\beta^j}M_{n-1}\left(\hat{\alpha}^i,\hat{\beta}^j\right). \tag{3.3.11}$$

To evaluate the integral of σ_n we will deform the contour of integration to infinity and apply Cauchy's residue theorem. From Eq.(3.3.11) we can see that there are simple poles at $\sigma_n = \sigma_1$ and $\sigma_n = \sigma_i$ only when $i = j$. There are no poles at infinity. Therefore, to leading order in τ, we get

$$M_n = -\frac{1}{\tau}\sum_{c,d}\left(\frac{1}{p\cdot k_1}\sum_{i,j=2}^{n-1} f_{ca_ia_n}\tilde{f}_{db_jb_n}M_{n-1}\left(i^{ca_i}, j^{b_jd}\right) + \sum_{i=2}^{n-1}\frac{1}{p\cdot k_i} f_{ca_ia_n}\tilde{f}_{db_ib_n}M_{n-1}\left(i^{cd}\right)\right) \tag{3.3.12}$$

It can be shown that the first term in the parentheses is exactly equal i-th term in the second summation with $i = 1$ (this calculation is similar to that involved in Eq.(3.3.24)). Hence these terms combine to give

$$M_n = -\frac{1}{\tau}\sum_{c,d}\sum_{i=1}^{n-1}\frac{1}{p\cdot k_i} f_{ca_ia_n}\tilde{f}_{db_ib_n}M_{n-1}\left(i^{cd}\right). \tag{3.3.13}$$

This is the soft theorem for bi-adjoint cubic scalar theory.

3.3.2 Yang-Mills

Color ordered partial amplitudes for pure gluon scattering is given by Eq.(3.2.19). In terms of the partial amplitudes full amplitude for n-point gluon scattering can be given by

$$\mathcal{A}_n = g^{n-2}\sum_{\alpha\in\mathbb{S}_n/\mathbb{Z}_n}\mathrm{Tr}\left(T^{a_{\alpha(1)}}T^{a_{\alpha(2)}}\dots T^{a_{\alpha(n)}}\right)A_n\left[\alpha(1)\dots\alpha(n)\right]. \tag{3.3.14}$$

We will assume $(n+1)$-th gluon to be soft and its momentum will be denoted τk_{n+1}. We can write $(n+1)$-point gluon amplitude as

$$\mathcal{A}_{n+1} = g^{n-1}\sum_{i=1}^{n}\sum_{\alpha\in\mathbb{S}_{n-1}\setminus i}\mathrm{Tr}\left(T^{a_{n+1}}T^{a_i}T^{a_{\alpha(1)}}\dots T^{a_{\alpha(n-1)}}\right)A_{n+1}\left[n+1, i, \alpha(1)\dots\alpha(n-1)\right]. \tag{3.3.15}$$

In the sum over $n-1$ permutations i-th label has been excluded.

Let us first find out soft limit of A_{n+1}. Parke-Taylor factor can be written as

$$C_{n+1}(n+1,i,\alpha) = \frac{\sigma_{\alpha_{n-1}} - \sigma_i}{(\sigma_{n+1}-\sigma_i)(\sigma_{\alpha_{n-1}}-\sigma_{n+1})} C_n(i,\alpha)\,. \tag{3.3.16}$$

We can use the Pfaffian expansion given in (4.4.22) for any even dimensional matrix to write

$$\mathrm{Pf}'(\Psi_{n+1}) = \mathrm{Pf}'(\Psi_n) \sum_{b=1}^{n} \frac{\epsilon_{n+1}.k_b}{\sigma_{n+1}-\sigma_b} + O(\tau). \tag{3.3.17}$$

Therefore to leading order in τ we obtain

$$A_{n+1}[n+1,i,\alpha_1\ldots\alpha_{n-1}] = \frac{1}{\tau}\oint_{\{A_i\}} \frac{d\sigma_{n+1}}{\sum\limits_{b=1}^{n}\frac{k_{n+1}\cdot k_b}{\sigma_{n+1}-\sigma_b}} \left(\frac{\sigma_{\alpha_{n-1}}-\sigma_i}{(\sigma_{n+1}-\sigma_i)(\sigma_{\alpha_{n-1}}-\sigma_{n+1})} \sum_{b=1}^{n} \frac{\epsilon_{n+1}.k_b}{\sigma_{n+1}-\sigma_b} \right) A_n[i,\alpha_1\ldots\alpha_{n-1}]\,. \tag{3.3.18}$$

Deforming the contour of integration to infinity we find that there are poles at $\sigma_{n+1} = \sigma_i$ and $\sigma_{n+1} = \sigma_{\alpha_{n-1}}$ and there are no poles at infinity. Evaluating residues at these poles we get

$$A_{n+1}[n+1,i,\alpha_1\ldots\alpha_{n-1}] = -\frac{1}{\tau}\left(\frac{\epsilon_{n+1}\cdot k_i}{k_{n+1}\cdot k_i} - \frac{\epsilon_{n+1}\cdot k_{\alpha(n-1)}}{k_{n+1}\cdot k_{\alpha(n-1)}} \right) A_n[i,\alpha_1\ldots\alpha_{n-1}] \tag{3.3.19}$$

Full amplitude Using the color basis given in [76, 77] $(n+1)$-point gluon amplitude can be expressed as

$$\mathcal{A}_n = (ig)^{n-2} \sum_{\alpha\in S_{n-2}} \sum_{x_1\ldots x_{n-3}} f^{a_1 a_{\alpha_2} x_1} f^{x_1 a_{\alpha_3} x_2} \ldots f^{x_{n-3} a_{\alpha_{n-1}} a_n} A_n[1,\alpha_2\ldots\alpha_{n-1},a_n]\,. \tag{3.3.20}$$

In the single soft limit we can write

$$\mathcal{A}_{n+1} = (ig)^{n-1} \sum_{i=1}^{n-1} \sum_{\alpha\in S_{n-2}\backslash i} f^{a_{n+1} a_i x_1} f^{x_1 a_{\alpha_1} x_2} \ldots f^{x_{n-1} a_{\alpha_{n-2}} a_n} A_{n+1}[n+1,i,\alpha_1\ldots\alpha_{n-2},n]\,, \tag{3.3.21}$$

where $A_{n+1}[n+1,i,\alpha_1\ldots\alpha_{n-2},n]$ is given in Eq.(3.3.19). Then we obtain

$$\begin{aligned}-\tau\mathcal{A}_{n+1} &= ig\sum_b\sum_{i=1}^{n-1} f^{a_{n+1}a_ib}\frac{\epsilon_{n+1}\cdot k_i}{k_{n+1}\cdot k_i}\mathcal{A}_n\left(i^b\right)\\ &\quad -(ig)^{n-1}\frac{\epsilon_{n+1}\cdot k_n}{k_{n+1}\cdot k_n}\sum_{i=1}^{n-1}\sum_{\alpha\in S_{n-2}\backslash i} f^{a_{n+1}a_ix_1}f^{x_1a_{\alpha_1}x_2}\ldots f^{x_{n-1}a_{\alpha_{n-2}}a_n}A_n[i,\alpha_1\ldots\alpha_{n-2},n].\end{aligned} \tag{3.3.22}$$

Now we use the fact $if^{a_{n+1}a_ix_1} = \mathrm{Tr}\left([T^{a_{n+1}},T^{a_i}]T^{x_1}\right)$ and

$$\sum_{x_1}\mathrm{Tr}\left([T^{a_{n+1}},T^{a_i}]T^{x_1}\right)\mathrm{Tr}\left(T^{x_1}T^{a_{\alpha_1}}\ldots T^{a_n}\right) = \mathrm{Tr}\left(T^{a_{n+1}}T^{a_i}T^{a_{\alpha_1}}\ldots T^{a_n}\right)-\mathrm{Tr}\left(T^{a_{n+1}}T^{a_{\alpha_1}}\ldots T^{a_n}T^{a_i}\right). \tag{3.3.23}$$

Let us consider

$$\begin{aligned}& i\sum_{i=1}^{n}\sum_{\alpha\in S_{n-1}\backslash i}\sum_b f^{a_{n+1}a_ib}\mathrm{Tr}\left([T^{a_{n+1}},T^{a_i}]T^b\right)\mathrm{Tr}\left(T^bT^{a_{\alpha_1}}\ldots T^{a_{\alpha_{n-1}}}\right)A_n[i,\alpha_1\ldots\alpha_{n-2},\alpha_{n-1}]\\ &= \sum_{i=1}^{n}\sum_{\alpha\in S_{n-1}\backslash i}\left\{\mathrm{Tr}\left(T^{a_{n+1}}T^{a_i}T^{a_{\alpha_1}}\ldots T^{a_{\alpha_{n-1}}}\right)-\mathrm{Tr}\left(T^{a_{n+1}}T^{a_{\alpha_1}}\ldots T^{a_{\alpha_{n-1}}}T^{a_i}\right)\right\}A_n[i,\alpha_1\ldots\alpha_{n-2},\alpha_{n-1}]\\ &= 0\end{aligned} \tag{3.3.24}$$

Summing over S_{n-1} permutations along with $i=1\ldots n$ produce summation over S_n permutations. Both the terms in the braces give same expression and hence the sum vanishes. Therefore second term in Eq.(3.3.22) is equal to

$$ig\sum_b f^{a_{n+1}a_nb}\frac{\epsilon_{n+1}\cdot k_n}{k_{n+1}\cdot k_n}\mathcal{A}_n\left(n^b\right), \tag{3.3.25}$$

which together with the first term in Eq.(3.3.22) gives the factorization

$$\mathcal{A}_{n+1} = -\frac{1}{\tau}ig\sum_b\sum_{i=1}^{n} f^{a_{n+1}a_ib}\frac{\epsilon_{n+1}\cdot k_i}{k_{n+1}\cdot k_i}\mathcal{A}_n\left(i^b\right). \tag{3.3.26}$$

This is the leading order soft theorem for pure gluon amplitude. In Sec.(A.1) Feynman diagram analysis for soft factorization of Yang-Mills amplitude is given in Eq.(A.1.3) which matches with this result.

3.3.3 Gravity

For pure gravity amplitude for $(n+1)$ graviton scattering in CHY formalism is expressed by

$$M_{n+1} = \int \mathrm{d}\mu_{n+1} \det{}' \left(\Psi_{n+1}\right)^2 . \tag{3.3.27}$$

In the soft limit given by $k_{n+1} = \tau q$ with $\tau \to 0$ reduced Pfaffian can be expanded to leading order

$$\mathrm{Pf}'(\Psi_{n+1}) = \mathrm{Pf}'(\Psi_n) \sum_{b=1}^{n} \frac{\epsilon_{n+1}.k_b}{\sigma_{n+1} - \sigma_b} + O(\tau) \tag{3.3.28}$$

which implies

$$\det{}' \left(\Psi_{n+1}\right) = \left(\sum_{b=1}^{n} \frac{\epsilon_{n+1}.k_b}{\sigma_{n+1} - \sigma_b} \right)^2 \det{}' \left(\Psi_n\right) + O\left(\tau\right). \tag{3.3.29}$$

Then leading order soft factorization can be given by

$$M_{n+1} = \frac{1}{\tau} \oint_{\{A_i\}} \frac{\mathrm{d}\sigma_{n+1}}{\sum\limits_{a=1}^{n} \frac{q \cdot k_a}{\sigma_{n+1} - \sigma_a}} \left(\sum_{b=1}^{n} \frac{\epsilon_{n+1}.k_b}{\sigma_{n+1} - \sigma_b} \right)^2 \int \mathrm{d}\mu_n \det{}' \left(\Psi_n\right) + O\left(\tau^0\right). \tag{3.3.30}$$

We can deform the contour of the integration for σ_{n+1} away from the original contour to infinity, then we get poles at $\sigma_{n+1} = \sigma_b, \forall b \in \{1, 2, \ldots, n\}$. It can be checked that there are no poles at infinity. Evaluating the residues at the poles we get

$$M_{n+1} = -\frac{1}{\tau} \sum_{a=1}^{n} \frac{\left(\epsilon_{n+1} \cdot k_a\right)^2}{q \cdot k_a} M_n + O\left(\tau^0\right). \tag{3.3.31}$$

This is Weinberg's soft theorem for gravity amplitude.

Sub-leading soft graviton theorem using CHY formulation has been derived in [23, 24] and sub-sub-leading soft graviton theorem has been derived in [27, 28].

3.4 Double copy relations

In 1986 Kawai, Lewellen and Tye (KLT) [68] discovered that tree-level amplitudes of closed strings can be expressed as square of the open string amplitudes. Later these double copy relations have been shown to hold in Quantum Field Theories also such as pure gravity amplitudes can be constructed from two copies of pure Yang Mills theory [69, 70] at tree level. Cachazo, He and Yuan [7, 61, 62, 66] made these relations mathematically more precise by interpreting the coefficients of the field theory limit of KLT expansion as inverse of a matrix whose elements are scattering amplitudes of bi-adjoint scalar theory described in Sec.(3.2.1).

In terms of CHY representations relation between gravity and Yang-Mills amplitudes can be given by

$$\begin{aligned} M_n^{\mathrm{GR}} &= A_n^{\mathrm{YM}} \overset{\mathrm{KLT}}{\otimes} A_n^{\mathrm{YM}} \\ &= A_n^{\mathrm{YM}}(\beta)\, m^{-1}\left(\beta|\tilde{\beta}\right) A_n^{\mathrm{YM}}\left(\tilde{\beta}\right). \end{aligned} \tag{3.4.1}$$

Here $A_n^{\mathrm{YM}}(\beta)$ is a Yang-Mills partial amplitude with the color ordering β explained in Sec.(3.2.2) and $m\left(\beta|\tilde{\beta}\right)$ is the n-point bi-adjoint scalar partial amplitude explained in Sec.(3.2.1). $m^{-1}\left(\beta|\tilde{\beta}\right)$ denotes the inverse of the matrix whose rows and columns are labeled by bi-adjoint scalar amplitudes with the permutations β and $\tilde{\beta}$ respectively. Each of β and $\tilde{\beta}$ span over $(n-3)!$ permutations forming a BCJ basis [69] and hence the matrix $m\left(\beta|\tilde{\beta}\right)$ has dimension $(n-3)! \times (n-3)!$.

Let us consider an example of five point amplitudes to illustrate the KLT relations described above. We choose the orderings to be $\beta \in \{(12345), (12435)\}$ and $\tilde{\beta} \in \{(13254), (14253)\}$. It can be checked the non-zero elements of the KLT matrix are $m(12345|13254) = \frac{1}{s_{23}s_{45}}$

and $m(12435|14253) = \frac{1}{s_{24}s_{35}}$. Then we obtain the five-point gravity amplitude

$$\begin{aligned} M_5^{\mathrm{GR}} &= \left[A^{\mathrm{YM}}(12345) \quad A^{\mathrm{YM}}(12435)\right] \begin{bmatrix} \frac{1}{s_{23}s_{45}} & 0 \\ 0 & \frac{1}{s_{24}s_{35}} \end{bmatrix}^{-1} \begin{bmatrix} A^{\mathrm{YM}}(13254) \\ A^{\mathrm{YM}}(14253) \end{bmatrix} \\ &= s_{23}s_{45}A^{\mathrm{YM}}(12345)A^{\mathrm{YM}}(13254) + s_{24}s_{35}A^{\mathrm{YM}}(12435)A^{\mathrm{YM}}(14253). \end{aligned} \tag{3.4.2}$$

3.4.1 Soft limit

KLT orthogonality relation between gravity and Yang-Mills scattering amplitudes can be extended in the soft limits of the amplitudes also. Calculations become simpler if we choose the bases for n-point partial Yang-Mills amplitudes [42] as

$$\beta = (1, \omega, n{-}1, n), \qquad \tilde{\beta} = (1, \tilde{\omega}, n{-}1, n, b), \qquad \text{with} \quad \omega, \tilde{\omega} \in S_{n-4}, \quad a, b \in \{2, 3, \ldots, n{-}3\}. \tag{3.4.3}$$

S_{n-4} is the $(n-4)$ permutations which exclude a and b for ω and $\tilde{\omega}$ respectively. We will take the n-th particle to be soft and denotes its momentum by $\tau\hat{k}_n$. In this choice of basis KLT matrix becomes orthogonal to leading order:

$$m_n\left(1, \omega, n-1, a, n|1, \tilde{\omega}, n-1, n, b\right) = \frac{1}{\tau}\frac{\delta_{ab}}{\hat{s}_{an}} m_{n-1}\left(1, \omega, n-1, a|1, \tilde{\omega}, n-1, a\right) + O\left(\tau^0\right). \tag{3.4.4}$$

An n-point graviton amplitude can be given by

$$\begin{aligned} M_n^{\mathrm{GR}} &= \sum_{a,b=2}^{n-2} \sum_{\omega,\tilde{\omega}\in S_{n-4}} A_n^{\mathrm{YM}}\left(1, \omega, n-1, a, n\right) m_n^{-1} A_n^{\mathrm{YM}}\left(1, \tilde{\omega}, n-1, n, b\right) \\ &= \sum_{a,b=2}^{n-2} \sum_{\omega,\tilde{\omega}\in S_{n-4}} \frac{1}{\tau}\left(\frac{\epsilon_n\cdot k_a}{\hat{k}_n\cdot k_a} - \frac{\epsilon_n\cdot k_1}{\hat{k}_n\cdot k_1}\right) A_{n-1}^{\mathrm{YM}}\left(1, \omega, n-1, a\right) \\ &\qquad \times \tau\hat{s}_{an}\delta_{ab} m_{n-1}\left(1, \omega, n-1, a|1, \tilde{\omega}, n-1, a\right)^{-1} \\ &\qquad \times \frac{1}{\tau}\left(\frac{\epsilon_n\cdot k_{n-1}}{\hat{k}_n\cdot k_{n-1}} - \frac{\epsilon_n\cdot k_b}{\hat{k}_n\cdot k_b}\right) A_{n-1}^{\mathrm{YM}}\left(1, \tilde{\omega}, n-1, b\right) + O\left(\tau^0\right) \end{aligned} \tag{3.4.5}$$

Summing over ω and $\tilde{\omega}$ permutations give $(n-1)$-point graviton amplitude. Then we use momentum conservation to get

$$\begin{aligned} M_n^{\mathrm{GR}} &= \frac{1}{\tau}\sum_{a=2}^{n-2}\hat{s}_{an}\left(\frac{\epsilon_n\cdot k_a}{\hat{k}_n\cdot k_a}-\frac{\epsilon_n\cdot k_1}{\hat{k}_n\cdot k_1}\right)\left(\frac{\epsilon_n\cdot k_{n-1}}{\hat{k}_n\cdot k_{n-1}}-\frac{\epsilon_n\cdot k_b}{\hat{k}_n\cdot k_b}\right)M_{n-1}^{\mathrm{GR}}+\mathcal{O}\left(\tau^0\right) \\ &= -\frac{1}{\tau}\sum_{a=1}^{n-1}\frac{\epsilon_{\mu\nu}k_a^{\mu}k_a^{\nu}}{\hat{k}_n\cdot k_a}M_{n-1}^{\mathrm{GR}}+\mathcal{O}\left(\tau^0\right) \end{aligned} \tag{3.4.6}$$

which produces the soft graviton theorem to leading order.

KLT relations for various theories like NLSM, DBI and special Galilean theories in single soft limit have been studied in [42]. Double copy relations between gravity and Yang-Mills soft theorems at sub-leading order have been explored in [35].

4 Double soft graviton theorem

Recently double soft limits of scattering amplitudes have been explored for large variety of theories. In [72] double soft theorems for Yang Mills, supersymmetric gauge theories and open superstring theory have been studied. simultaneous and consecutive double soft limits of gluon and graviton amplitudes have been analyzed in [12]. Double soft theorems have also been studied in supergravity theories in [78, 79]. In [80] it has been shown that double soft theorems in nonlinear sigma model follow from a shift symmetry. Current-current algebra for double soft gluon amplitude at null infinity has been studied in [81,82] where it was shown to produce a level zero Kac-Moody current.

In [10, 11] we have studied double soft graviton theorem for simultaneous soft graviton scattering to sub-leading order using CHY formalism. Multiple soft graviton theorem for any generic theories of quantum gravity have been derived using Sen's covariantization method in [13] and using CHY formulation in [14]. Ward identities from BMS symmetries related to consecutive double soft graviton theorem have been derived in [83]. Classical limits of multiple soft graviton theorem have been studied in [84–86].

In this chapter we will use CHY formulation to study simultaneous double soft limits of some amplitudes in field theories. In Sec.(4.1) we review the analysis of taking double soft limits of amplitudes in CHY formalism. In [63] degenerate solutions of the scattering equations are considered in the double soft limit, here we analyze both non-degenerate solutions [11] and degenerate solutions of the scattering equations for taking double soft limit. In Sec.(4.2) we consider double soft theorem for Einstein-Maxwell theory. We derive double soft theorem to leading order for Yang-Mills theory in Sec.(4.3). In Sec.(4.4) we calculate double soft theorem for pure gravity to sub-leading order [10, 11].

4.1 Double soft limit from CHY formalism

In this section we will explain the method of taking soft limit to a scattering amplitude when the momenta of two of the scattered particles become infinitesimally small using the Cachazo-He-Yuan formalism. Our starting point is $n+2$ point amplitude which can be written as

$$M_{n+2} = \int \frac{d^n\sigma d\sigma_{n+1} d\sigma_{n+2}}{\text{vol}\mathbb{SL}(2,\mathbb{C})} \prod_{a=1}^{n}{}' \delta(f_a)\, \delta(f_{n+1})\, \delta(f_{n+2})\, I_{n+2}(\{k,\epsilon,\sigma\}). \tag{4.1.1}$$

Here we have separated out $(n+1)$th and $(n+2)$th integrals for reasons that will be clear shortly. In the soft limits where two of the particles' momenta tend to zero at the same rate (let us denote the soft momenta by $k_{n+1} = \tau p$ and $k_{n+2} = \tau q$ with $\tau \to 0$) the scattering equations f_α can be written as

$$f_\alpha = \begin{cases} \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a.k_b}{\sigma_a-\sigma_b} + \frac{\tau k_a.p}{\sigma_a-\sigma_{n+1}} + \frac{\tau k_a.q}{\sigma_a-\sigma_{n+2}}, & \alpha \in \{1,2,\cdots n\} \\ \sum_{b=1}^{n} \frac{\tau k_b.p}{\sigma_{n+1}-\sigma_b} + \frac{\tau^2 p.q}{\sigma_{n+1}-\sigma_{n+2}}, & \alpha = n+1 \\ \sum_{b=1}^{n} \frac{\tau k_b.p}{\sigma_{n+2}-\sigma_b} - \frac{\tau^2 p.q}{\sigma_{n+1}-\sigma_{n+2}}, & \alpha = n+2. \end{cases} \tag{4.1.2}$$

Thus effectively in the vanishing limit of τ there are n scattering equations for n hard particles. Solving these n equations one obtains solutions for $\sigma_1, \sigma_2, \cdots, \sigma_n$ and there are $(n-3)!$ such solution sets. The last two scattering equations do not provide any solution, rather they are used to transform the σ_{n+1} and σ_{n+2} integrals to contour integrals. As we will see there are subtleties in performing these contour integrals depending on the behavior of $|\sigma_{n+1} - \sigma_{n+2}|$. In the seminal paper [63] the authors classified the behavior in two categories: 1) *non-degenerate solutions* - when $|\sigma_{n+1} - \sigma_{n+2}| \sim \mathcal{O}(\tau^0)$ and 2) *degenerate solutions* - when $|\sigma_{n+1} - \sigma_{n+2}| \sim \mathcal{O}(\tau)$. It was shown that for theories like sGal, DBI, EMS, NLSM and YMS the leading order contribution come from the degenerate one. However in case of pure gravity (which is given by Einstein-Hilbert action) we find the opposite feature, non-degenerate contribution dominates over the degenerate one. We will elaborate on this issue in more details as we proceed.

4.1.1 Non-degenerate solutions

Here we consider the situation when $|\sigma_{n+1} - \sigma_{n+2}| \sim \mathcal{O}\left(\tau^0\right)$. This implies that the two soft punctures never overlap each other. The delta functions corresponding to the last two scattering equations, f_{n+1} and f_{n+2} now transform the integrations of σ_{n+1} and σ_{n+2} variables to independent contour integrals where σ_{n+1} and σ_{n+2} enclose around the solutions of the scattering equations f_{n+1} and f_{n+2} respectively.

In this case the scattering amplitude (4.1.1) takes the form

$$M_{n+2} \to \int \frac{\mathrm{d}^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \left[\sum_{i=1}^{n} \oint_{\{\mathcal{A}_i\}} \frac{\mathrm{d}\sigma_{n+1}}{f_{n+1}}\right] \left[\sum_{j=1}^{n} \oint_{\mathcal{B}_i} \frac{\mathrm{d}\sigma_{n+2}}{f_{n+2}}\right] \prod_{a=1}^{n}{}' \delta\left(f_a\right) I_{n+2}(\{k,\epsilon,\sigma\}). \tag{4.1.3}$$

Here $\{\mathcal{A}_i\}$ and $\{\mathcal{B}_i\}$ are the zeros of f_{n+1} and f_{n+2} respectively. From here onwards we will drop the summation signs and assume sum over the contour integrals is implied.

Both the measure and the integrand can be expanded in orders of τ parameter as follows.

$$\begin{aligned} \frac{1}{f_{n+1}}\frac{1}{f_{n+2}} &= \frac{1}{\tau^2}\frac{1}{\sum_a \frac{p\cdot k_a}{\sigma_{n+1}-\sigma_a}}\left[1-\tau\frac{\frac{p\cdot q}{\sigma_{n+1}-\sigma_{n+2}}}{\sum_{a'}\frac{p\cdot k_{a'}}{\sigma_{n+1}-\sigma_{a'}}}+\cdots\right]\frac{1}{\sum_b \frac{q\cdot k_b}{\sigma_{n+2}-\sigma_b}}\left[1+\tau\frac{\frac{p\cdot q}{\sigma_{n+1}-\sigma_{n+2}}}{\sum_{b'}\frac{q\cdot k_{b'}}{\sigma_{n+2}-\sigma_{b'}}}\cdots\right] \\ &\equiv \frac{1}{\tau^2}\frac{1}{C_1C_2} - \frac{1}{\tau}\frac{1}{C_1C_2}\left[\frac{1}{C_1}-\frac{1}{C_2}\right]\frac{p\cdot q}{\sigma_{n+1}-\sigma_{n+2}} + \mathcal{O}(\tau^0) \end{aligned} \tag{4.1.4}$$

where we define $C_1 := \sum\limits_{a=1}^{n} \frac{p\cdot k_a}{\sigma_{n+1}-\sigma_a}$ and $C_2 := \sum\limits_{b=1}^{n} \frac{q\cdot k_b}{\sigma_{n+2}-\sigma_b}$.

Product of the delta functions can be expanded as

$$\begin{aligned}\prod_{a=1}^{n}{}' \delta(f_a) &= \prod_{a=1}^{n}{}' \delta(f_a^n) + \tau \sum_{a=1}^{n}{}' \left[\prod_{\substack{b=1\\ b\neq a}}^{n}{}' \delta(f_b^n)\right]\left(\frac{k_a.p}{\sigma_a-\sigma_{n+1}} + \frac{k_a.q}{\sigma_a-\sigma_{n+2}}\right)\delta'(f_a^n) + \mathcal{O}(\tau^2) \\ &\equiv \delta^{(0)} + \tau\delta^{(1)} + \mathcal{O}(\tau^2) \end{aligned} \tag{4.1.5}$$

where $f_a^n = \sum\limits_{\substack{b=1\\ b\neq a}}^{n} \frac{k_a.k_b}{\sigma_a-\sigma_b}$. Prime denotes exclusion of any three delta functions due to $\mathbb{SL}(2,\mathbb{C})$ redundancy.

Similarly we can write the integrand as a Taylor series expansion

$$I_{n+2} = I_{n+2}^{(0)} + \tau I_{n+2}^{(1)} + \cdots \tag{4.1.6}$$

Therefore using Eq.(4.1.4), Eq.(4.1.5) and Eq.(4.1.6) expansion of M_{n+2} in Eq.(4.1.3) is given by

$$\begin{aligned} M_{n+2} \;\to\; & \frac{1}{\tau^2}\int \frac{d^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \oint_{\{\mathcal{A}_i\}} \frac{d\sigma_{n+1}}{C_1} \oint_{\{\mathcal{B}_i\}} \frac{d\sigma_{n+2}}{C_2}\, \delta^{(0)}\, I^{0}_{n+2}(\{k,\epsilon,\sigma\}) \\ & + \frac{1}{\tau}\int \frac{d^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \oint_{\{\mathcal{A}_i\}} \frac{d\sigma_{n+1}}{C_1} \oint_{\{\mathcal{B}_i\}} \frac{d\sigma_{n+2}}{C_2} \Big[\delta^{(1)} I^{(0)}_{n+2} + \delta^{(0)} I^{(1)}_{n+2} \\ & \qquad - \left(\frac{1}{C_1} - \frac{1}{C_2}\right) \frac{p\cdot q}{\sigma_{n+1}-\sigma_{n+2}}\, \delta^{(0)}\, I^{(0)}_{n+2}(\{k,\epsilon,\sigma\})\Big] \\ & + \mathcal{O}(\tau^0) \end{aligned} \tag{4.1.7}$$

In this approximation $\{\mathcal{A}_i\}$ and $\{\mathcal{B}_i\}$ are the zeros of C_1 and C_2 respectively.

4.1.2 Degenerate solutions

The case when $|\sigma_{n+1} - \sigma_{n+2}| \sim \mathcal{O}(\tau)$ has been studied in great detail in [63, 87]. A new pair of variables is defined

$$\sigma_{n+1} = \rho - \frac{\xi}{2}, \qquad \sigma_{n+2} = \rho + \frac{\xi}{2} \tag{4.1.8}$$

In terms of the new variables the scattering equations (4.1.2) now become

$$f_a = \begin{cases} \sum\limits_{\substack{b=1 \\ b\neq a}}^{n} \left(\frac{k_a.k_b}{\sigma_a - \sigma_b} + \frac{\tau k_a.p}{\sigma_a - \rho + \frac{\xi}{2}} + \frac{\tau k_a.q}{\sigma_a - \rho - \frac{\xi}{2}} \right), & a \neq n+1, n+2 \\ \sum\limits_{b=1}^{n} \left(\frac{\tau k_b.p}{\rho - \frac{\xi}{2} - \sigma_b} - \frac{\tau^2 p.q}{\xi} \right), & a = n+1 \\ \sum\limits_{b=1}^{n} \left(\frac{\tau k_b.p}{\rho + \frac{\xi}{2} - \sigma_b} + \frac{\tau^2 p.q}{\xi} \right), & a = n+2. \end{cases} \tag{4.1.9}$$

We can expand ξ perturbatively in τ as

$$\xi = \tau \xi_1 + \tau^2 \xi_2 + \mathcal{O}(\tau^3) \tag{4.1.10}$$

and using the last two scattering equations we get

$$\frac{1}{\xi_1} = \frac{1}{p.q} \sum_{b=1}^{n} \frac{k_b.p}{\rho - \sigma_b} = -\frac{1}{p.q} \sum_{b=1}^{n} \frac{k_b.q}{\rho - \sigma_b}. \tag{4.1.11}$$

With the change of variables in (4.1.8) the σ_{n+1} and σ_{n+2} integrals can be transformed as follows:

$$\begin{aligned} &\int \mathrm{d}\sigma_{n+1} \mathrm{d}\sigma_{n+2} \delta(f_{n+1}) \delta(f_{n+2}) \\ \rightarrow \quad &-2 \int \mathrm{d}\rho \mathrm{d}\xi \delta(f_{n+1} + f_{n+2}) \delta(f_{n+1} - f_{n+2}) \end{aligned}$$

$$
\begin{aligned}
&\to \quad -2\oint \frac{d\rho}{2\pi i} \sum_{\xi \text{ solutions}} \int d\xi \frac{1}{(f_{n+1}+f_{n+2})} \frac{1}{\frac{\partial}{\partial \xi}(f_{n+1}-f_{n+2})} \\
&\to \quad -2\oint \frac{d\rho}{2\pi i} \sum_{\xi \text{ solutions}} \int d\xi \frac{1}{\sum_{a=1}^{n} \tau\left(\frac{k_a.p}{\rho-\frac{\xi}{2}-\sigma_a}+\frac{k_a.q}{\rho+\frac{\xi}{2}-\sigma_a}\right)} \frac{2\,\delta(\xi-\xi_{\text{sol}})}{\sum_{b=1}^{n} \tau\left(\frac{k_b.p}{(\rho-\frac{\xi}{2}-\sigma_b)^2}+\frac{k_b.q}{(\rho+\frac{\xi}{2}-\sigma_b)^2}\right)+\frac{4\,\tau^2\,p.q}{\xi^2}}
\end{aligned}
\tag{4.1.12}
$$

where the first delta constraint is expressed as a contour integral for ρ wrapping around the solutions to the scattering equations and the second delta constraint localizes the ξ variable.

For finite ρ contour the CHY expression for the scattering amplitude at tree level in the double soft limit as an expansion in the order of τ is given by [87]

$$
M_N = -\frac{1}{\tau}\oint_{\{C_i\}} \frac{d\rho}{2\pi i} \int d\mu_n \frac{\xi_1^2}{p.q \sum_{b=1}^{n} \frac{k_b.(p+q)}{\rho-\sigma_b}} \left(1-\frac{\tau\xi_1}{2}\frac{\sum_{b=1}^{n}\frac{k_b.(p+q)}{(\rho-\sigma_b)^2}}{\sum_{b=1}^{n}\frac{k_b.(p+q)}{\rho-\sigma_b}}+3\tau\frac{\xi_2}{\xi_1}+O(\tau^2)\right) I_N, \tag{4.1.13}
$$

where C_i are the zeros of $\sum_{b=1}^{n} \frac{k_b.(p+q)}{\rho-\sigma_b}$. Here we have use the notation $d\mu_n \equiv \frac{d^n\sigma}{\text{vol}\mathbb{SL}(2\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a.k_b}{\sigma_a-\sigma_b}\right)$. If the integrand can be written as a product like

$$
I_N(k,\sigma,\rho,\xi) = F(k,\sigma,\rho,\xi) I_n(k,\sigma) + (\text{sub-leading order}) \tag{4.1.14}
$$

then the previous expression at leading order simplifies to

$$
M_N = \left[-\frac{1}{\tau}\oint_{\{C_i\}} \frac{d\rho}{2\pi i} \frac{\xi_1^2}{p.q \sum_{b=1}^{n}\frac{k_b.(p+q)}{\rho-\sigma_b}} F(k,\sigma,\rho,\tau\xi_1)\right] M_n. \tag{4.1.15}
$$

The term in the square bracket gives the leading order double soft factor $S^*(0)$.

We will now compute examples of the double soft limit to scattering amplitudes for some specific theories before deriving the result for graviton scattering.

4.2 Double Soft Limit for Einstein Maxwell Theory

Now we will like to show how the double soft theorem follows from Feynman diagrams. As an example we consider Einstein Maxwell theory. In [87] the authors have investigated scattering amplitudes in Born Infeld and Einstein Maxwell theories in the double soft limit with two soft photons. The integrand for this class of theories is given by

$$I_N = (\text{Pf}X_N)^{-m}(\text{Pf}'A_N)^{2+m}\text{Pf}'\Psi_N \tag{4.2.1}$$

where $m = 0, -1$ denote BI and EM respectively. The result for EM theory with two soft photon emission is

$$S^{*(0)} = \frac{1}{\tau}\sum_{b=1}^{n}\frac{1}{k_b.(p+q)}\left[\frac{p.q\ \epsilon_{n+1}.\epsilon_{n+2} - \epsilon_{n+2}.p\ \epsilon_{n+1}.q}{4(p.q)^2}\{k_b.(p-q)\}^2 - \epsilon_{n+1}.p_b^{\perp}\ \epsilon_{n+2}.q_b^{\perp}\right] \tag{4.2.2}$$

which can be further simplified to[1]

$$S^{*(0)} = \frac{1}{\tau}\sum_{b=1}^{n}\frac{1}{k_b.(p+q)}\left[\frac{\epsilon_{n+1}.q\ \epsilon_{n+2}.k_b\ p.k_b + \epsilon_{n+1}.k_b\ \epsilon_{n+2}.p\ q.k_b - \epsilon_{n+1}.\epsilon_{n+2}\ p.k_b\ q.k_b}{p.q} - \epsilon_{n+1}.k_b\ \epsilon_{n+2}.k_b\right]. \tag{4.2.3}$$

In the following subsection we show that from Feynman diagrams we can reproduce the above expression modulo an overall constant factor.

4.2.1 Feynman Diagrams

The Einstein Maxwell action in four dimension is given by

[1] Here we have used $\sum_{b=1}^{n}\frac{\{k_b.(p-q)\}^2}{k_b.(p+q)} = -4\sum_{b=1}^{n}\frac{k_b.p\ k_b.q}{k_b.(p+q)} + O(\tau)$.

$$S_{EM} = \int \mathrm{d}^4x \left(-\frac{1}{4}\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{2}{\kappa^2}\sqrt{-g} R \right) \tag{4.2.4}$$

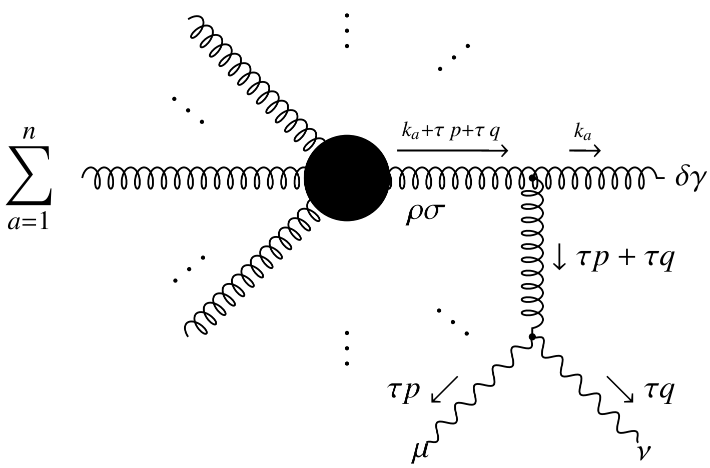
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and R is the Ricci scalar given by $R = g^{\mu\nu}\left(\Gamma^\lambda_{\mu\lambda,\nu} - \Gamma^\lambda_{\mu\nu,\lambda} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\sigma\lambda} \right)$.

In the linearized perturbative theory of gravity a small deviation $h_{\mu\nu}$ around flat Minkowski spacetime is considered such as

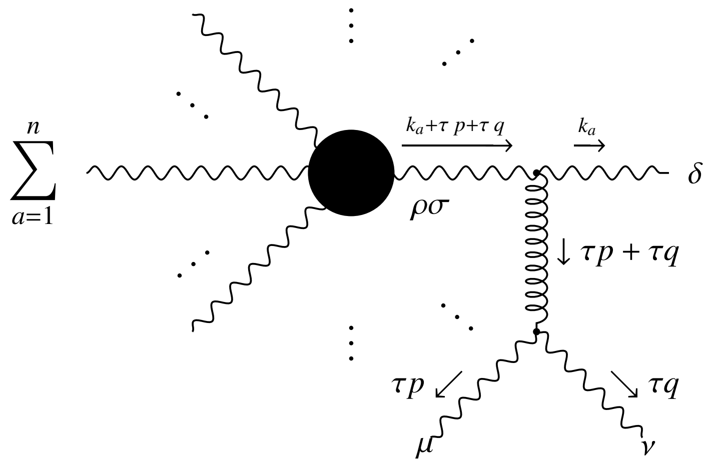
$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{4.2.5}$$

The Feynman rules for EM are given in sec(B.1.1). At leading order two soft photons can be emitted from either an external graviton leg through an internal graviton propagator or from an external photon leg through a graviton propagator. Both the processes involve two three-point AAg vertices. These give terms of $O\left(\frac{\kappa^2}{\tau}\right)$. There also exists a four-point $AAgg$ vertex through which two soft photons can come from an external graviton leg. This vertex comes from a term in Lagrangian of the form $\sim hh\partial A\partial A$ and thus lead to the order of $O(\tau)$ in the scattering amplitude. Therefore for our purpose of interest it suffices to compute the following Feynman diagrams:

- **photons emitted from an external graviton**



- **photons emitted from an external photon**



Both of these diagrams give exactly the same soft factor as in Eq.(4.2.3). The above analysis demonstrates the correspondence between the CHY integrand (4.2.1) with $m = -1$ and the Lagrangian description of EM theory (4.2.4) in the double soft limit.

4.3 Double soft limit in Yang Mills theory

Let us now find the double soft limit to Yang Mills amplitude. Scattering amplitude for Yang Mills theory can be expressed in terms of sum over colored ordered amplitudes, called partial amplitudes. In CHY representation the partial amplitude of a particular color ordering $(1, 2, \ldots N)$ can be given by

$$A_{YM}\,(1,2,\ldots,N) = \int \mathrm{d}\mu_N\; C_N\,(1,2,\ldots,N)\,\mathrm{Pf}'\,\Psi_N\,(\{\sigma_i, p_i, \epsilon_i\}) \tag{4.3.1}$$

where

$$C_N\,(1,2,\ldots,N) = \frac{1}{\sigma_{12}\sigma_{23}\ldots\sigma_{N1}}, \qquad \text{with} \quad \sigma_{ij} \equiv \sigma_i - \sigma_j \tag{4.3.2}$$

is the color ordered Parke-Taylor factor.

Because of the color ordering there are two ways of taking double soft limit to gluon amplitudes - 1) the soft gluons can be *adjacent* to each other in the color ordering, and 2) soft limits can be taken in *non-adjacent* external states. Soft limits in the non-adjacent legs are straightforward to calculate while soft limits in the adjacent legs are more non trivial. In this section we will deal with each of the two cases separately.

4.3.1 Double soft limit in adjacent legs

First let us consider the case when two soft gluons are adjacent in the color ordering. We will denote the momenta and polarizations of soft gluons by

$$\{\tau k_{n+1}, \epsilon_{n+1}\}; \qquad \{\tau k_{n+2}, \epsilon_{n+2}\} \tag{4.3.3}$$

where the parameter $\tau \to \infty$.

As we will see in detail, both the contributions of non-degenerate and degenerate solutions to double soft limit are at same order in τ in this case.

4.3.1.1 Non-degenerate solutions

Factorization of integrand at leading order can be given by

$$\begin{aligned} C_{n+2}\left(1,2,\ldots,n,n+1,n+2\right) &= \frac{\sigma_n - \sigma_1}{\left(\sigma_n - \sigma_{n+1}\right)\left(\sigma_{n+1} - \sigma_{n+2}\right)\left(\sigma_{n+2} - \sigma_1\right)} C_n\left(1,2,\ldots,n\right) \\ \mathrm{Pf}' \Psi_{n+2} &= -\left(\sum_{a=1}^{n} \frac{\epsilon_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)\left(\sum_{b=1}^{n} \frac{\epsilon_{n+2}\cdot p_b}{\sigma_{n+2}-\sigma_b}\right)\mathrm{Pf}' \Psi_n \end{aligned} \tag{4.3.4}$$

Therefore using Eq.(4.1.7) at leading order we obtain

$$\begin{aligned} \mathcal{B}_0 = -\frac{1}{\tau^2}\int \mathcal{D}\sigma\delta^{(0)} I_n \oint_{\{A_i\}} \mathrm{d}\sigma_{n+1}\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1} \oint_{\{B_i\}} \mathrm{d}\sigma_{n+2}\left(\sum_{b=1}^{n}\frac{k_{n+2}\cdot p_b}{\sigma_{n+2}-\sigma_b}\right)^{-1} \\ \left[\frac{\left(\sigma_n-\sigma_1\right)}{\left(\sigma_n-\sigma_{n+1}\right)\left(\sigma_{n+1}-\sigma_{n+2}\right)\left(\sigma_{n+2}-\sigma_1\right)}\left(\sum_{a=1}^{n}\frac{\epsilon_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)\left(\sum_{b=1}^{n}\frac{\epsilon_{n+2}\cdot p_b}{\sigma_{n+2}-\sigma_b}\right)\right] \end{aligned}$$

(4.3.5)

First we will deform the contour of σ_{n+2} away from the zeros contained in $\{B_i\}$ to infinity. In doing so we will encounter poles at $\sigma_{n+2} = \sigma_1, \sigma_{n+1}$. Contour of σ_{n+2} encloses these poles in clockwise direction and hence give negative residues. It can be checked that there is no pole at infinity. Then we get

$$\begin{aligned}
\mathcal{B}_0 &= \frac{1}{\tau^2}\int \mathcal{D}\sigma\delta^{(0)} I_n \oint_{\{A_i\}} \mathrm{d}\sigma_{n+1}\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1}\left(\sum_{a=1}^{n}\frac{\epsilon_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)\\
&\qquad \times\frac{(\sigma_n-\sigma_1)}{(\sigma_n-\sigma_{n+1})(\sigma_{n+1}-\sigma_1)}\frac{\epsilon_{n+2}\cdot p_1}{k_{n+2}\cdot p_1}\\
&\quad -\frac{1}{\tau^2}\int \mathcal{D}\sigma\delta^{(0)} I_n \oint_{\{A_i\}} \mathrm{d}\sigma_{n+1}\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1}\left(\sum_{b=1}^{n}\frac{k_{n+2}\cdot p_b}{\sigma_{n+1}-\sigma_b}\right)^{-1}\left(\sum_{c=1}^{n}\frac{\epsilon_{n+2}\cdot p_c}{\sigma_{n+1}-\sigma_c}\right)\\
&\qquad \times\left(\sum_{d=1}^{n}\frac{\epsilon_{n+1}\cdot p_d}{\sigma_{n+1}-\sigma_d}\right)\frac{(\sigma_n-\sigma_1)}{(\sigma_n-\sigma_{n+1})(\sigma_{n+1}-\sigma_1)}. \qquad (4.3.6)
\end{aligned}$$

To evaluate the first contour integration we deform the contour of σ_{n+1} away from the zeros of $\{A_i\}$ to infinity. Then the poles will be at $\sigma_{n+1} = \sigma_n, \sigma_1$ and it can be checked that there is no pole at infinity. For the second contour integration we can similarly deform the contour, but along with the poles at $\sigma_{n+1} = \sigma_n, \sigma_1$ there will be poles at the zeros of $\left(\sum_{b=1}^{n}\frac{k_{n+2}\cdot p_b}{\sigma_{n+1}-\sigma_b}\right)$.The last residues are hard to calculate and we will leave the expression as it is. Then we get

$$\begin{aligned}
\mathcal{B}_0 &= \frac{1}{\tau^2}\left[\frac{\epsilon_{n+1}\cdot p_n}{k_{n+1}\cdot p_n}-\frac{\epsilon_{n+1}\cdot p_1}{k_{n+1}\cdot p_1}\right]\frac{\epsilon_{n+2}\cdot p_1}{k_{n+2}\cdot p_1}M_n\\
&\quad -\frac{1}{\tau^2}\oint_{\{A_i\}} \mathrm{d}\sigma_{n+1}\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1}\left(\sum_{b=1}^{n}\frac{k_{n+2}\cdot p_b}{\sigma_{n+1}-\sigma_b}\right)^{-1}\left(\sum_{c=1}^{n}\frac{\epsilon_{n+2}\cdot p_c}{\sigma_{n+1}-\sigma_c}\right)\left(\sum_{d=1}^{n}\frac{\epsilon_{n+1}\cdot p_d}{\sigma_{n+1}-\sigma_d}\right)\\
&\qquad \frac{(\sigma_n-\sigma_1)}{(\sigma_n-\sigma_{n+1})(\sigma_{n+1}-\sigma_1)}M_n
\end{aligned}$$

(4.3.7)

This is the only term at $O\left(\tau^{-2}\right)$ coming from the non-degenerate solutions.

4.3.1.2 Degenerate solutions

Factorization of the building blocks in the integrand in this case can be given by

$$
\begin{aligned}
C_{n+2}(1,2,\ldots,n,n+1,n+2) &= \frac{-(\sigma_n-\sigma_1)}{\tau\xi_1(\sigma_n-\rho)(\rho-\sigma_1)}C_n(1,2,\ldots,n),\\
\mathrm{Pf}'\Psi_{n+2} &= \Bigg[-\left(\sum_{a=1}^{n}\frac{\epsilon_{n+1}\cdot p_a}{\rho-\sigma_a}\right)\left(\sum_{b=1}^{n}\frac{\epsilon_{n+2}\cdot p_b}{\rho-\sigma_b}\right)+\frac{1}{\xi_1}\Bigg\{\epsilon_{n+1}\cdot k_{n+2}\left(\sum_{a=1}^{n}\frac{\epsilon_{n+2}\cdot p_a}{\rho-\sigma_a}\right)\\
&\quad -\epsilon_{n+2}\cdot k_{n+1}\left(\sum_{b=1}^{n}\frac{\epsilon_{n+1}\cdot p_b}{\rho-\sigma_b}\right)\Bigg\}+\frac{\epsilon_{n+1}\cdot\epsilon_{n+2}\,k_{n+1}\cdot k_{n+2}}{\xi_1^2}\Bigg]\mathrm{Pf}'\Psi_n.
\end{aligned}
\tag{4.3.8}
$$

Calculation of leading soft factorization:

$$
\begin{aligned}
&-\frac{2}{\tau}\int\mathcal{D}\sigma\delta^{(0)}I_n\int\mathrm{d}\rho\,\delta\left(\sum_{b=1}^{n}\frac{(k_{n+1}+k_{n+2})\cdot p_b}{\rho-\sigma_b}\right)\frac{\xi_1^2}{2\,k_{n+1}\cdot k_{n+2}}\left[\frac{-(\sigma_n-\sigma_1)}{\tau\xi_1(\sigma_n-\rho)(\rho-\sigma_1)}\right]\\
&\Bigg[-\left(\sum_{a=1}^{n}\frac{\epsilon_{n+1}\cdot p_a}{\rho-\sigma_a}\right)\left(\sum_{b=1}^{n}\frac{\epsilon_{n+2}\cdot p_b}{\rho-\sigma_b}\right)+\frac{1}{\xi_1}\Bigg\{\epsilon_{n+1}\cdot k_{n+2}\left(\sum_{a=1}^{n}\frac{\epsilon_{n+2}\cdot p_a}{\rho-\sigma_a}\right)\\
&\quad -\epsilon_{n+2}\cdot k_{n+1}\left(\sum_{b=1}^{n}\frac{\epsilon_{n+1}\cdot p_b}{\rho-\sigma_b}\right)\Bigg\}+\frac{\epsilon_{n+1}\cdot\epsilon_{n+2}\,k_{n+1}\cdot k_{n+2}}{\xi_1^2}\Bigg]
\end{aligned}
\tag{4.3.9}
$$

where

$$
\xi_1=k_{n+1}\cdot k_{n+2}\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\rho-\sigma_a}\right)^{-1}=-k_{n+1}\cdot k_{n+2}\left(\sum_{a=1}^{n}\frac{k_{n+2}\cdot p_a}{\rho-\sigma_a}\right)^{-1}=2\,k_{n+1}\cdot k_{n+2}\left(\sum_{a=1}^{n}\frac{(k_{n+1}-k_{n+2})\cdot p_a}{\rho-\sigma_a}\right)^{-1}.
\tag{4.3.10}
$$

Using the delta function integration over ρ is converted to a contour integration

$$
\int\mathrm{d}\rho\,\delta\left(\sum_{b=1}^{n}\frac{(k_{n+1}+k_{n+2})\cdot p_b}{\rho-\sigma_b}\right)\rightarrow\oint_{\{C_i\}}\mathrm{d}\rho\left(\sum_{b=1}^{n}\frac{(k_{n+1}+k_{n+2})\cdot p_b}{\rho-\sigma_b}\right)^{-1}
\tag{4.3.11}
$$

where $\{C_i\}$ contain the zeros of the argument of the delta function.

Let us expand the expression in the second square bracket and evaluate each term separately.

First term

$$-\frac{1}{\tau^2}\oint\limits_{\{C_i\}} d\rho \left(\sum_{b=1}^{n}\frac{(k_{n+1}+k_{n+2})\cdot p_b}{\rho-\sigma_b}\right)^{-1}\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\rho-\sigma_a}\right)^{-1}\frac{(\sigma_n-\sigma_1)}{(\sigma_n-\rho)(\rho-\sigma_1)}\left(\sum_{a=1}^{n}\frac{\epsilon_{n+1}\cdot p_a}{\rho-\sigma_a}\right)\left(\sum_{b=1}^{n}\frac{\epsilon_{n+2}\cdot p_b}{\rho-\sigma_b}\right)M_n \tag{4.3.12}$$

Deforming the contour of ρ away from $\{C_i\}$ to infinity we encounter poles as $\rho=\sigma_n,\sigma_1$ and at the zeros of $\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\rho-\sigma_a}\right)$ which are same as $\{A_i\}$. Therefore residues on these poles are

$$\frac{1}{\tau^2}\oint\limits_{\{A_i\}} d\rho \left(\sum_{b=1}^{n}\frac{k_{n+2}\cdot p_b}{\rho-\sigma_b}\right)^{-1}\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\rho-\sigma_a}\right)^{-1}\frac{(\sigma_n-\sigma_1)}{(\sigma_n-\rho)(\rho-\sigma_1)}\left(\sum_{a=1}^{n}\frac{\epsilon_{n+1}\cdot p_a}{\rho-\sigma_a}\right)\left(\sum_{b=1}^{n}\frac{\epsilon_{n+2}\cdot p_b}{\rho-\sigma_b}\right)M_n$$
$$+\frac{1}{\tau^2}\left[\frac{\epsilon_{n+1}\cdot p_1\ \epsilon_{n+2}\cdot p_1}{(k_{n+1}+k_{n+2})\cdot p_1\ k_{n+1}\cdot p_1}-\frac{\epsilon_{n+1}\cdot p_n\ \epsilon_{n+2}\cdot p_n}{(k_{n+1}+k_{n+2})\cdot p_n\ k_{n+1}\cdot p_n}\right]M_n \tag{4.3.13}$$

Second term

$$\frac{1}{\tau^2}\frac{1}{k_{n+1}\cdot k_{n+2}}\oint\limits_{\{C_i\}} d\rho \left(\sum_{b=1}^{n}\frac{(k_{n+1}+k_{n+2})\cdot p_b}{\rho-\sigma_b}\right)^{-1}\frac{(\sigma_n-\sigma_1)}{(\sigma_n-\rho)(\rho-\sigma_1)}\Bigg\{\epsilon_{n+1}\cdot k_{n+2}\left(\sum_{a=1}^{n}\frac{\epsilon_{n+2}\cdot p_a}{\rho-\sigma_a}\right)$$
$$-\epsilon_{n+2}\cdot k_{n+1}\left(\sum_{b=1}^{n}\frac{\epsilon_{n+1}\cdot p_b}{\rho-\sigma_b}\right)\Bigg\}M_n \tag{4.3.14}$$

Deforming the contour and evaluating residues at the poles $\rho=\sigma_n,\sigma_1$ we get

$$\frac{1}{\tau^2}\frac{1}{k_{n+1}\cdot k_{n+2}}\left[\frac{\epsilon_{n+1}\cdot k_{n+2}\ \epsilon_{n+2}\cdot p_n-\epsilon_{n+2}\cdot k_{n+1}\ \epsilon_{n+1}\cdot p_n}{(k_{n+1}+k_{n+2})\cdot p_n}-\frac{\epsilon_{n+1}\cdot k_{n+2}\ \epsilon_{n+2}\cdot p_1-\epsilon_{n+2}\cdot k_{n+1}\ \epsilon_{n+1}\cdot p_1}{(k_{n+1}+k_{n+2})\cdot p_1}\right]M_n \tag{4.3.15}$$

Third term

$$\frac{1}{2\,\tau^2}\oint\limits_{\{C_i\}} d\rho \left(\sum_{b=1}^{n}\frac{(k_{n+1}+k_{n+2})\cdot p_b}{\rho-\sigma_b}\right)^{-1}\frac{(\sigma_n-\sigma_1)}{(\sigma_n-\rho)(\rho-\sigma_1)}\left(\sum_{a=1}^{n}\frac{(k_{n+1}-k_{n+2})\cdot p_a}{\rho-\sigma_a}\right)M_n \tag{4.3.16}$$

Deforming the contour and evaluating residues at the poles $\rho = \sigma_n, \sigma_1$ we get

$$\frac{1}{2\,\tau^2}\frac{\epsilon_{n+1}\cdot\epsilon_{n+2}}{k_{n+1}\cdot k_{n+2}}\left[\frac{(k_{n+1}-k_{n+2})\cdot p_n}{(k_{n+1}+k_{n+2})\cdot p_n}-\frac{(k_{n+1}-k_{n+2})\cdot p_1}{(k_{n+1}+k_{n+2})\cdot p_1}\right]M_n \tag{4.3.17}$$

Pole at infinity If we use the same ξ_1 of Eq.(4.3.10) and take the contour of ρ to infinity it can be checked that there cannot be any pole at infinity. This can be seen from the following argument: in the limit $\rho \to \infty$, we can make a change of variable

$$\rho = \frac{1}{\zeta}. \tag{4.3.18}$$

Then

$$\frac{1}{\rho^2}\mathrm{d}\rho = -\mathrm{d}\zeta. \tag{4.3.19}$$

It can be seen that taking $\rho \to \infty$ the expressions (4.3.12), (4.3.14) and (4.3.16) behave as $\frac{\mathrm{d}\rho}{\rho^2} \sim \mathrm{d}\zeta$, hence there is no pole at $\zeta = 0$ or $\rho = \infty$.

4.3.1.3 Result

Adding all the contributions coming from equations (4.3.7), (4.3.13), (4.3.15) and (4.3.17) we obtain the double Soft YM amplitude at leading order for adjacent soft legs can be expressed as

$$\begin{aligned}
&A_{YM}(p_1,\epsilon_1;p_2,\epsilon_2;\ldots;p_n,\epsilon_n;\tau k_{n+1},\epsilon_{n+1};\tau k_{n+2},\epsilon_{n+2})\\
=\;&\frac{1}{\tau^2}\Bigg[\left(\frac{\epsilon_{n+1}\cdot p_n}{k_{n+1}\cdot p_n}-\frac{\epsilon_{n+1}\cdot p_1}{k_{n+1}\cdot p_1}\right)\frac{\epsilon_{n+2}\cdot p_1}{k_{n+2}\cdot p_1}\\
&+\frac{\epsilon_{n+1}\cdot p_1\;\epsilon_{n+2}\cdot p_1}{(k_{n+1}+k_{n+2})\cdot p_1\;k_{n+1}\cdot p_1}-\frac{\epsilon_{n+1}\cdot p_n\;\epsilon_{n+2}\cdot p_n}{(k_{n+1}+k_{n+2})\cdot p_n\;k_{n+1}\cdot p_n}\\
&+\frac{1}{k_{n+1}\cdot k_{n+2}}\Bigg\{\frac{\epsilon_{n+1}\cdot k_{n+2}\;\epsilon_{n+2}\cdot p_n-\epsilon_{n+2}\cdot k_{n+1}\;\epsilon_{n+1}\cdot p_n}{(k_{n+1}+k_{n+2})\cdot p_n}\\
&-\frac{\epsilon_{n+1}\cdot k_{n+2}\;\epsilon_{n+2}\cdot p_1-\epsilon_{n+2}\cdot k_{n+1}\;\epsilon_{n+1}\cdot p_1}{(k_{n+1}+k_{n+2})\cdot p_1}\Bigg\}\\
&+\frac{1}{2}\frac{\epsilon_{n+1}\cdot\epsilon_{n+2}}{k_{n+1}\cdot k_{n+2}}\Bigg\{\frac{(k_{n+1}-k_{n+2})\cdot p_n}{(k_{n+1}+k_{n+2})\cdot p_n}-\frac{(k_{n+1}-k_{n+2})\cdot p_1}{(k_{n+1}+k_{n+2})\cdot p_1}\Bigg\}\Bigg]\;A_{YM}(p_1,\epsilon_1;p_2,\epsilon_2;\ldots,p_n,\epsilon_n)\,.
\end{aligned}$$

(4.3.20)

4.3.1.4 Comments

In the analysis of double soft limit to YM amplitude [72] , gauge choices have been made which in our notation translate to

$$\epsilon_{n+1} \cdot p_n = 0, \qquad \epsilon_{n+2} \cdot p_1 = 0. \tag{4.3.21}$$

With this choice first line in Eq.(4.3.7), which is a part of the contribution from non-degenerate solutions, vanishes. Also some contributions from degenerate solutions become zero in this gauge. But the term

$$\frac{1}{\tau^2} \oint_{\{A_i\}} \mathrm{d}\rho \left(\sum_{b=1}^{n} \frac{k_{n+2} \cdot p_b}{\rho - \sigma_b} \right)^{-1} \left(\sum_{a=1}^{n} \frac{k_{n+1} \cdot p_a}{\rho - \sigma_a} \right)^{-1} \frac{(\sigma_n - \sigma_1)}{(\sigma_n - \rho)(\rho - \sigma_1)} \left(\sum_{a=1}^{n} \frac{\epsilon_{n+1} \cdot p_a}{\rho - \sigma_a} \right) \left(\sum_{b=1}^{n} \frac{\epsilon_{n+2} \cdot p_b}{\rho - \sigma_b} \right) M_n \tag{4.3.22}$$

can not vanish in this gauge. This term is missed in the result presented in this paper.

This term actually plays a crucial role in maintaining gauge invariance of the contributions from non-degenerate and degenerate solutions separately. To illustrate this point, let us consider the contribution from non-degenerate solutions, given in Eq.(4.3.7). Substituting $\epsilon_{n+1} \to k_{n+1}$ we see the first line vanishes. The second line becomes

$$-\frac{1}{\tau^2} \oint_{\{A_i\}} \mathrm{d}\sigma_{n+1} \left(\sum_{b=1}^{n} \frac{k_{n+2} \cdot p_b}{\sigma_{n+1} - \sigma_b} \right)^{-1} \left(\sum_{c=1}^{n} \frac{\epsilon_{n+2} \cdot p_c}{\sigma_{n+1} - \sigma_c} \right) \frac{(\sigma_n - \sigma_1)}{(\sigma_n - \sigma_{n+1})(\sigma_{n+1} - \sigma_1)} M_n. \tag{4.3.23}$$

This contour integration vanishes because the poles which were contained in $\{A_i\}$ as zeros of $\left(\sum_{a=1}^{n} \frac{k_{n+1} \cdot p_a}{\sigma_{n+1} - \sigma_a} \right)$ have canceled with the numerator. Again substituting $\epsilon_{n+2} \to k_{n+2}$ we get

$$\frac{1}{\tau^2} \left[\frac{\epsilon_{n+1} \cdot p_n}{k_{n+1} \cdot p_n} - \frac{\epsilon_{n+1} \cdot p_1}{k_{n+1} \cdot p_1} \right] M_n$$

$$-\frac{1}{\tau^2}\oint_{\{A_i\}} d\sigma_{n+1}\left(\sum_{a=1}^{n}\frac{k_{n+1}\cdot p_a}{\sigma_{n+1}-\sigma_a}\right)^{-1}\left(\sum_{d=1}^{n}\frac{\epsilon_{n+1}\cdot p_d}{\sigma_{n+1}-\sigma_d}\right)\frac{(\sigma_n-\sigma_1)}{(\sigma_n-\sigma_{n+1})(\sigma_{n+1}-\sigma_1)}M_n. \tag{4.3.24}$$

Deforming the contour of integration away from $\{A_i\}$ to infinity, poles come at $\sigma_{n+1} = \sigma_n, \sigma_1$. Evaluating the residues gives same expression as in the first line but with negative sign. Hence the term vanishes. Similar check for gauge invariance can be shown for contribution from degenerate solutions also.

This gauge invariance at each stage (individually for the contributions from non-degenerate and degenerate solutions) is expected as well as important. At the level of integral expressions it can be checked that CHY representations of the amplitudes are gauge invariant on both non-degenerate and degenerate solutions of ξ separately. So even after evaluating the integrals gauge invariance should be preserved at every step.

4.3.2 Double soft limit in non-adjacent legs

In case of the soft gluons are in the non-adjacent legs of the color ordered partial amplitude the analysis becomes much easier. At leading order of the factorization we have to consider the contributions of non-degenerate solutions only.

We will denote the soft particles by i and j, where $i, j \notin \{1, 2, \ldots, n\}$.

Factorization of the building blocks in the double soft limit have been given in Sec.(B.2)

Then the leading order soft factorization for non-adjacent soft legs can be expressed as

$$\begin{aligned}\lim_{\tau\to 0} A_{YM}\left(p_1,\epsilon_1;\ldots\tau k_i,\epsilon_i;\ldots;\tau k_j,\epsilon_j;\ldots;p_n,\epsilon_n\right) &= \frac{1}{\tau^2}\left(\frac{\epsilon_i\cdot p_{i+1}}{k_i\cdot p_{i+1}}-\frac{\epsilon_i\cdot p_{i-1}}{k_i\cdot p_{i-1}}\right)\left(\frac{\epsilon_j\cdot p_{j-1}}{k_j\cdot p_{j-1}}-\frac{\epsilon_j\cdot p_{j+1}}{k_j\cdot p_{j+1}}\right)\\ &\quad\times A_{YM}\left(p_1,\epsilon_1;\ldots;p_n,\epsilon_n\right). \end{aligned} \tag{4.3.25}$$

Here we find that the double soft factor is the product of two single soft factor.

4.4 Double soft graviton theorem

The integrand for pure gravity theory is given in terms of reduced Pfaffian of an antisymmetric matrix in the following way

$$I_n = \left(\mathrm{Pf}'\Psi_n(\{k,\epsilon,\sigma\})\right)^2 . \tag{4.4.1}$$

In this section we will derive soft limit to the pure graviton scattering amplitude when two external gravitons are taken to be infinitesimally small energy. We will compute the double soft factor of this amplitude to sub-leading order in the energy of the soft particles.

In the calculation of double soft factorization of Yang Mills amplitude given in Sec.(4.3), particularly when adjacent legs are taken to be soft, we have seen how there are cancellation of terms coming from non-degenerate and degenerate solutions. We will see that similar type of analysis happen for double soft factorization of graviton from CHY prescription.

First we will compute the contribution from non-degenerate solutions followed by degenerate ones. Let us label the soft momenta as $k_{n+1} = \tau p$ and $k_{n+2} = \tau q$.

4.4.1 Non-degenerate solutions

The building block Ψ_{n+2} in the gravity integrand can be expressed as

$$\Psi_{n+2}=\left(\begin{array}{ccc|ccc}
(A_n)_{ab} & \frac{\tau\, k_a.p}{\sigma_a-\sigma_{n+1}} & \frac{\tau\, k_a.q}{\sigma_a-\sigma_{n+2}} & (-C_n^T)_{ab} & \frac{-\epsilon_{n+1}.k_a}{\sigma_{n+1}-\sigma_a} & \frac{-\epsilon_{n+2}.k_a}{\sigma_{n+2}-\sigma_a} \\
\frac{\tau\, p.k_b}{\sigma_{n+1}-\sigma_b} & 0 & \frac{\tau^2\, p.q}{\sigma_{n+1}-\sigma_{n+2}} & \frac{-\tau\,\epsilon_b.p}{\sigma_b-\sigma_{n+1}} & -C_{n+1,n+1} & \frac{-\tau\,\epsilon_{n+2}.p}{\sigma_{n+2}-\sigma_{n+1}} \\
\frac{\tau\, q.k_b}{\sigma_{n+2}-\sigma_b} & \frac{\tau^2\, p.q}{\sigma_{n+2}-\sigma_{n+1}} & 0 & \frac{-\tau\,\epsilon_b.q}{\sigma_b-\sigma_{n+2}} & \frac{\tau\,\epsilon_{n+1}.q}{\sigma_{n+1}-\sigma_{n+2}} & -C_{n+2,n+2} \\
\hline
(C_n)_{ab} & \frac{\tau\,\epsilon_a.p}{\sigma_a-\sigma_{n+1}} & \frac{\tau\,\epsilon_a.q}{\sigma_a-\sigma_{n+2}} & (B_n)_{ab} & \frac{\epsilon_a.\epsilon_{n+1}}{\sigma_a-\sigma_{n+1}} & \frac{\epsilon_a.\epsilon_{n+2}}{\sigma_a-\sigma_{n+2}} \\
\frac{\epsilon_{n+1}.k_b}{\sigma_{n+1}-\sigma_b} & C_{n+1,n+1} & \frac{\tau\,\epsilon_{n+1}.q}{\sigma_{n+1}-\sigma_{n+2}} & \frac{\epsilon_{n+1}.\epsilon_b}{\sigma_{n+1}-\sigma_b} & 0 & \frac{\epsilon_{n+1}.\epsilon_{n+2}}{\sigma_{n+1}-\sigma_{n+2}} \\
\frac{\epsilon_{n+2}.k_b}{\sigma_{n+2}-\sigma_b} & \frac{\tau\,\epsilon_{n+2}.p}{\sigma_{n+2}-\sigma_{n+1}} & C_{n+2,n+2} & \frac{\epsilon_{n+2}.\epsilon_b}{\sigma_{n+2}-\sigma_b} & \frac{\epsilon_{n+2}.\epsilon_{n+1}}{\sigma_{n+2}-\sigma_{n+1}} & 0
\end{array}\right). \tag{4.4.2}$$

At leading order the gravity integrand, Pf$'$ $(\Psi_{n+2})^2$ becomes

$$I^{(0)}_{n+2}=\left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^2\left(\sum_b \frac{\epsilon_{n+2}\cdot k_b}{\sigma_{n+2}-\sigma_b}\right)^2 I_n. \tag{4.4.3}$$

Therefore the leading order double soft factor is

$$\begin{aligned}
S^{(0)}(p,q) &= \frac{1}{\tau^2}\oint\limits_{|\sigma_{n+1}-\sigma_i|\to 0} \mathrm{d}\sigma_{n+1}\frac{\left(\sum\limits_a \frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^2}{\sum\limits_{a'}\frac{p\cdot k_{a'}}{\sigma_{n+1}-\sigma_{a'}}}\oint\limits_{|\sigma_{n+2}-\sigma_j|\to 0} \mathrm{d}\sigma_{n+2}\frac{\left(\sum\limits_b \frac{\epsilon_{n+2}\cdot k_b}{\sigma_{n+2}-\sigma_b}\right)^2}{\sum\limits_{b'}\frac{q\cdot k_{b'}}{\sigma_{n+2}-\sigma_{b'}}} \\
&= \left(\sum_{a=1}^n \frac{\epsilon_{n+1,\,\mu\nu}k_a^\mu k_a^\nu}{k_a\cdot p}\right)\times\left(\sum_{b=1}^n \frac{\epsilon_{n+2,\,\mu\nu}k_b^\mu k_b^\nu}{k_a\cdot q}\right) \\
&= S^{(0)}(p)S^{(0)}(q)
\end{aligned} \tag{4.4.4}$$

which is product of two leading order single soft factors [5] as expected. Here σ_i are the solutions of scattering equations corresponding to the n number of finite energy external particles.

To compute Eq.(4.4.4) we have to deform the contour of integrations for the variables σ_{n+1} and σ_{n+2} independently away from their original contours (which enclose the zeros of A_i and B_i respectively) to infinity. In doing this there will be residues from the poles at infinity. Therefore in addition to the answer given in Eq.(4.4.4) the contour integration will give the following term

$$\begin{aligned} & -\frac{1}{\tau^2}\left[S^{(0)}(p)\frac{\left(\sum\limits_{a=1}^{n}\epsilon_{n+2}\cdot k_a\right)^2}{\sum\limits_{a=1}^{n} q\cdot k_a}+S^{(0)}(q)\frac{\left(\sum\limits_{a=1}^{n}\epsilon_{n+1}\cdot k_a\right)^2}{\sum\limits_{a=1}^{n} p\cdot k_a}\right] \\ = & \ \frac{1}{\tau\, p\cdot q}\left[S^{(0)}(p)\,(\epsilon_{n+2}\cdot p)^2+S^{(0)}(q)\,(\epsilon_{n+1}\cdot q)^2\right], \end{aligned} \tag{4.4.5}$$

where the last equality follows from conservation of total momentum of the scattered particles.

Let us now consider the terms in Eq.(4.1.7) which are combinations $\delta^{(1)}I^{(0)}_{n+2}$ and $\delta^{(0)}I^{(1)}_{n+2}$. We can compute these terms in the following way.

Sub-leading soft factor for gravity is given by [22]

$$S^{(1)}(p)=\sum_{a=1}^{n}\frac{\epsilon_{n+1,\mu\nu}k_a^{\mu}p_{\rho}\hat{J}_a^{\rho,\nu}}{p\cdot k_a},\qquad S^{(1)}(q)=\sum_{a=1}^{n}\frac{\epsilon_{n+2,\mu\nu}k_a^{\mu}q_{\rho}\hat{J}_a^{\rho,\nu}}{q\cdot k_a} \tag{4.4.6}$$

where $\hat{J}$ is a first order differential operator which acts on both momenta and polarizations.

In the subsequent steps we will closely follow the analysis of [24]. Acting $S^{(1)}(p)$ on M_n we get

$$\begin{aligned} S^{(1)}(p)M_n = & \int\frac{d^n\sigma}{\text{vol}\mathbb{SL}(2,\mathbb{C})}\sum_l{}'\left[\prod_{\substack{a\\ a\neq l}}{}'\delta\left(f_a^n\right)\right]\delta'\left(f_l^n\right)\sum_{\substack{b=1\\ b\neq l}}^{n}\frac{1}{\sigma_l-\sigma_b}\Big[2\epsilon_{n+1}\cdot k_b\;\epsilon_{n+1}\cdot k_l \\ & \qquad -\frac{(\epsilon_{n+1}\cdot k_b)^2\,p\cdot k_l}{p\cdot k_b}-\frac{(\epsilon_{n+1}\cdot k_l)^2\,p\cdot k_b}{p\cdot k_l}\Big]I_n \\ & +\int\frac{d^n\sigma}{\text{vol}\mathbb{SL}(2,\mathbb{C})}\prod_a{}'\,\delta\left(f_a^n\right)S^{(1)}(p)I_n. \end{aligned} \tag{4.4.7}$$

Let us now focus on the $\delta^{(1)} I^{(0)}_{n+2}$ term of Eq.(4.1.7) and compare with the first term of Eq.(4.4.7).

$$\begin{aligned}
&\int \frac{d^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \oint_{|\sigma_{n+1}-\sigma_l|\to 0} \frac{d\sigma_{n+1}}{\sum_{a=1}^{n}\frac{\tau k_a.p}{\sigma_{n+1}-\sigma_a}} \oint_{|\sigma_{n+2}-\sigma_m|\to 0} \frac{d\sigma_{n+2}}{\sum_{b=1}^{n}\frac{\tau k_b.q}{\sigma_{n+2}-\sigma_b}} \delta^{(1)} I^{(0)}_{n+2} \\
= &\int \frac{d^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})} \oint_{|\sigma_{n+1}-\sigma_l|\to 0} \frac{d\sigma_{n+1}}{\sum_{i=1}^{n}\frac{\tau k_i.p}{\sigma_{n+1}-\sigma_i}} \oint_{|\sigma_{n+2}-\sigma_m|\to 0} \frac{d\sigma_{n+2}}{\sum_{j=1}^{n}\frac{\tau k_j.q}{\sigma_{n+2}-\sigma_j}} \left(\sum_{a'} \frac{\epsilon_{n+1}\cdot k_{a'}}{\sigma_{n+1}-\sigma_{a'}}\right)^2 \\
&\times \left(\sum_{b'} \frac{\epsilon_{n+2}\cdot k_{b'}}{\sigma_{n+2}-\sigma_{b'}}\right)^2 \tau \sum_a{}' \left[\prod_{\substack{b\\ b\neq a}}{}' \delta\left(f^n_b\right)\right]\left(\frac{k_a.p}{\sigma_a-\sigma_{n+1}} + \frac{k_a.q}{\sigma_a-\sigma_{n+2}}\right)\delta'\left(f^n_a\right) I_n
\end{aligned} \tag{4.4.8}$$

Now using

$$\begin{aligned}
&\oint_{|\sigma_{n+1}-\sigma_l|\to 0} d\sigma_{n+1} \frac{k_a.p}{\sigma_a-\sigma_{n+1}} \frac{\left(\sum_{a'}\frac{\epsilon_{n+1}\cdot k_{a'}}{\sigma_{n+1}-\sigma_{a'}}\right)^2}{\sum_{i=1}^{n}\frac{\tau k_i.p}{\sigma_{n+1}-\sigma_i}} \\
= &\frac{1}{\tau}\sum_{\substack{b=1\\ b\neq a}}^{n}\left[-\frac{k_a\cdot p}{\sigma_a-\sigma_b}\frac{(\epsilon_{n+1}\cdot k_b)^2}{k_b\cdot p} + 2\frac{\epsilon_{n+1}\cdot k_a\ \epsilon_{n+1}\cdot k_b}{\sigma_a-\sigma_b} - \frac{(\epsilon_{n+1}\cdot k_a)^2\, k_b\cdot p}{(\sigma_a-\sigma_b)\, k_a\cdot p}\right]
\end{aligned} \tag{4.4.9}$$

and comparing with Eq.(4.4.7) it is evident that Eq.(4.4.8) becomes

$$S^{(0)}(q)\int \frac{d^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})}\left(S^{(1)}(p)\left[\prod_a{}' \delta\left(f^n_a\right)\right]\right) I_n + (p\leftrightarrow q). \tag{4.4.10}$$

In the rest of the analysis to make our calculations easier we will choose the following gauge fixing conditions

$$\begin{aligned}
\epsilon_{n+1}\cdot q &= 0 \\
\epsilon_{n+2}\cdot p &= 0 \\
\epsilon_a\cdot q &= 0, \qquad \forall a\in\{1,2,\cdots,n\}.
\end{aligned} \tag{4.4.11}$$

Now our task is to find $I^{(1)}_{n+2}$ and calculate the remaining term in Eq.(4.1.7). Taking derivative of the determinant and using Eq.(4.4.11) we get

$$
\begin{aligned}
\frac{\partial I_{n+2}}{\partial \tau}|_{\tau=0} \;=\; & \sum_{a=1}^{n}\Bigg[(-1)^{n+a+1}\frac{k_a\cdot p}{\sigma_a-\sigma_{n+1}}\tilde{\Psi}^{a}_{n+1}+(-1)^{n+a}\frac{k_a\cdot q}{\sigma_a-\sigma_{n+2}}\tilde{\Psi}^{a}_{n+2}+(-1)^{n}\frac{\epsilon_a\cdot p}{\sigma_a-\sigma_{n+1}}\tilde{\Psi}^{a}_{n+a} \\
& +(-1)^{n+1}\frac{\epsilon_a\cdot p}{\sigma_a-\sigma_{n+1}}\tilde{\Psi}^{n+2+a}_{a}+(-1)^{a+1}\frac{\epsilon_a\cdot p}{\sigma_a-\sigma_{n+1}}\tilde{\Psi}^{n+2+a}_{n+1}\Bigg] \\
& +(-1)^{n}\left[C_{n+1,n+1}\tilde{\Psi}^{2n+3}_{n+1}+C_{n+2,n+2}\tilde{\Psi}^{2n+4}_{n+2}\right] \qquad (4.4.12)
\end{aligned}
$$

where $\tilde{\Psi}^{a}_{b}$ denotes determinant of the reduced matrix with ath row and bth column removed. After expanding the reduced determinants the above equation can be written as

$$
\begin{aligned}
& I^{(1)}_{n+2} = \\
& (C_{n+2,n+2})^2\,C_{n+1,n+1}\sum_{a=1}^{n}\sum_{b=1}^{n}\Bigg[\frac{k_a\cdot p}{\sigma_a-\sigma_{n+1}}\left((-1)^{a+b+1}\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{n+1}-\sigma_b}\Psi^{a}_{b}+(-1)^{n+a+b+1}\frac{\epsilon_{n+1}\cdot \epsilon_b}{\sigma_{n+1}-\sigma_b}\Psi^{a}_{n+b}\right) \\
& \qquad +\frac{p\cdot k_b}{\sigma_{n+1}-\sigma_b}\left((-1)^{a+b+1}\frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\Psi^{a}_{b}+(-1)^{n+a+b}\frac{\epsilon_a\cdot \epsilon_{n+1}}{\sigma_a-\sigma_{n+1}}\Psi^{n+a}_{b}\right) \\
& \qquad +\frac{\epsilon_b\cdot p}{\sigma_b-\sigma_{n+1}}\left((-1)^{n+a+b}\frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\Psi^{a}_{n+b}+(-1)^{a+b+1}\frac{\epsilon_a\cdot \epsilon_{n+1}}{\sigma_a-\sigma_{n+1}}\Psi^{n+a}_{n+b}\right) \\
& \qquad +\frac{\epsilon_a\cdot p}{\sigma_a-\sigma_{n+1}}\left((-1)^{n+a+b+1}\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{n+1}-\sigma_b}\Psi^{n+a}_{b}+(-1)^{a+b+1}\frac{\epsilon_{n+1}\cdot \epsilon_b}{\sigma_{n+1}-\sigma_b}\Psi^{n+a}_{n+b}\right)\Bigg] \\
& +(-1)^{n}\,(C_{n+1,n+1})^2\,(C_{n+2,n+2})^2\sum_{a=1}^{n}\frac{\epsilon_a\cdot p}{\sigma_a-\sigma_{n+1}}\left(\Psi^{a}_{n+a}-\Psi^{n+a}_{a}\right) \\
& +(C_{n+1,n+1})^2\,C_{n+2,n+2}\sum_{a=1}^{n}\sum_{b=1}^{n}\Bigg[\frac{k_a\cdot q}{\sigma_a-\sigma_{n+2}}\left((-1)^{a+b+1}\frac{\epsilon_{n+2}\cdot k_b}{\sigma_{n+2}-\sigma_b}\Psi^{a}_{b}+(-1)^{n+a+b+1}\frac{\epsilon_{n+2}\cdot \epsilon_b}{\sigma_{n+2}-\sigma_b}\Psi^{a}_{n+b}\right) \\
& \qquad +\frac{q\cdot k_b}{\sigma_{n+2}-\sigma_b}\left((-1)^{a+b+1}\frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+2}-\sigma_a}\Psi^{a}_{b}+(-1)^{n+a+b}\frac{\epsilon_a\cdot \epsilon_{n+2}}{\sigma_a-\sigma_{n+2}}\Psi^{n+a}_{b}\right)\Bigg]. \qquad (4.4.13)
\end{aligned}
$$

Substituting $I^{(1)}_{n+2}$ into the relevant term in Eq.(4.1.7) and doing the contour integrals over σ_{n+1} and σ_{n+2} we get

$$
S^{(0)}(q)\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{2}{\sigma_a-\sigma_b}\quad\times\quad\Bigg[\left(\frac{\epsilon_{n+1}\cdot k_a}{p\cdot k_a}-\frac{\epsilon_{n+1}\cdot k_b}{p\cdot k_b}\right)(p\cdot k_a)\,(\epsilon_{n+1}\cdot k_b)\,(-1)^{a+b}\,\Psi^{a}_{b}
$$

$$
\begin{aligned}
& - \left(\frac{\epsilon_{n+1}\cdot k_a}{p\cdot k_a} - \frac{\epsilon_{n+1}\cdot k_b}{p\cdot k_b}\right)(\epsilon_{n+1}\cdot\epsilon_a)\,(\epsilon_b\cdot p)\,(-1)^{a+b}\Psi^{n+a}_{n+b} \\
& + \left(\frac{\epsilon_{n+1}\cdot k_a}{p\cdot k_a} - \frac{\epsilon_{n+1}\cdot k_b}{p\cdot k_b}\right)\{(p\cdot k_a)\,(\epsilon_{n+1}\cdot\epsilon_b) - (\epsilon_b\cdot p)\,(\epsilon_{n+1}\cdot k_a)\} \\
& \qquad \times(-1)^{n+a+b}\Psi^{a}_{n+b} \\
& + \Bigg\{\left(\frac{\epsilon_{n+1}\cdot k_a}{p\cdot k_a} - \frac{\epsilon_{n+1}\cdot k_b}{p\cdot k_b}\right)(\epsilon_a\cdot p)\,(\epsilon_{n+1}\cdot k_b) \\
& \quad + \left(\frac{\epsilon_{n+1}\cdot k_b}{p\cdot k_a} - \frac{(p\cdot k_b)\,(\epsilon_{n+1}\cdot k_a)}{(p\cdot k_a)^2}\right)(p\cdot k_a)\,(\epsilon_{n+1}\cdot\epsilon_a)\Bigg\}(-1)^n\Psi^a_{n+a}\Bigg] \\
+ S^{(0)}(p)\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n}\frac{2}{\sigma_a-\sigma_b} & \times \Bigg[\left(\frac{\epsilon_{n+2}\cdot k_a}{q\cdot k_a} - \frac{\epsilon_{n+2}\cdot k_b}{q\cdot k_b}\right)(q\cdot k_a)\,(\epsilon_{n+2}\cdot k_b)\,(-1)^{a+b}\Psi^a_b \\
& + \left(\frac{\epsilon_{n+2}\cdot k_a}{q\cdot k_a} - \frac{\epsilon_{n+2}\cdot k_b}{q\cdot k_b}\right)(q\cdot k_a)\,(\epsilon_{n+2}\cdot\epsilon_b)\,(-1)^{n+a+b}\Psi^a_{n+b} \\
& + \left(\frac{\epsilon_{n+2}\cdot k_b}{q\cdot k_a} - \frac{(q\cdot k_b)\,(\epsilon_{n+2}\cdot k_a)}{(q\cdot k_a)^2}\right)(q\cdot k_a)\,(\epsilon_{n+2}\cdot\epsilon_a)\,(-1)^n\Psi^a_{n+a}\Bigg].
\end{aligned}
\tag{4.4.14}
$$

Details of the above calculations are provided in Sec.(B.3.1). Finally it can be shown that Eq.(4.4.14) is equal to

$$
S^{(0)}(q)\int\frac{\mathrm{d}^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})}\prod_a{}'\,\delta\left(f^n_a\right)\left[S^{(1)}(p)I_n\right] + (p\leftrightarrow q) \tag{4.4.15}
$$

Adding together Eq.(4.4.10) and Eq.(4.4.15) we obtain

$$
S^{(0)}(p)S^{(1)}(q) + S^{(0)}(q)S^{(1)}(p). \tag{4.4.16}
$$

Now we will consider the first term in the last line of Eq.(4.1.7) which is of the form

$$
\begin{aligned}
-\frac{1}{\tau}\int\frac{\mathrm{d}^n\sigma}{\mathrm{vol}\mathbb{SL}(2,\mathbb{C})}\delta^{(0)}I^{(0)}_n\oint_{\{A_i\}}\mathrm{d}\sigma_{n+1}\oint_{\{B_i\}}\mathrm{d}\sigma_{n+2}\left(\sum_a\frac{p\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^{-2}\left(\sum_a\frac{q\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{-1} \\
\frac{p\cdot q}{\sigma_{n+1}-\sigma_{n+2}}\left(\sum_a\frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^2\left(\sum_a\frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^2.
\end{aligned}
\tag{4.4.17}
$$

We can first do the contour integral for σ_{n+1} by deforming the contour away from the zeros contained in $\{A_i\}$ to infinity. Then there are poles at $\sigma_{n+1} = \sigma_{n+2}, \ \infty$. Therefore from Eq.(4.4.17) we obtain

$$\begin{aligned}
&\frac{1}{\tau}\int \frac{d^n\sigma}{\text{vol}\mathbb{SL}(2,\mathbb{C})}\delta^{(0)}I_n^{(0)}\oint_{\{B_i\}} d\sigma_{n+2}\Bigg[\left(\sum_a \frac{p\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{-2}\left(\sum_a \frac{q\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{-1}\\
&\times p\cdot q\left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^2\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^2-\left(\sum_a \frac{q\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{-1}\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^2\\
&\times p\cdot q\left(\sum_a p\cdot k_a\right)^{-2}\left(\sum_a \epsilon_{n+1}\cdot k_a\right)^2\Bigg]\\
=\ &\frac{1}{\tau}\int \frac{d^n\sigma}{\text{vol}\mathbb{SL}(2,\mathbb{C})}\delta^{(0)}I_n^{(0)}\oint_{\{B_i\}} d\sigma_{n+2}\left(\sum_a \frac{p\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{-2}\left(\sum_a \frac{q\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{-1}\\
&\times p\cdot q\left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^2\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^2\\
&+\frac{1}{\tau}S^{(0)}(q)\frac{(\epsilon_{n+1}\cdot q)^2}{p\cdot q}M_n,
\end{aligned}\tag{4.4.18}$$

where in the last line we have calculated residues at $\sigma_{n+2} = \sigma_a$ and used total momentum conservation.

Similarly from the last term in Eq.(4.1.7) we get

$$\begin{aligned}
&\frac{1}{\tau}\int \frac{d^n\sigma}{\text{vol}\mathbb{SL}(2,\mathbb{C})}\delta^{(0)}I_n^{(0)}\oint_{\{A_i\}} d\sigma_{n+1}\left(\sum_a \frac{q\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^{-2}\left(\sum_a \frac{p\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^{-1}\\
&\times p\cdot q\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^2\left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^2\\
&+\frac{1}{\tau}S^{(0)}(p)\frac{(\epsilon_{n+2}\cdot p)^2}{p\cdot q}M_n
\end{aligned}\tag{4.4.19}$$

It is to be noted that the last terms of Eq.(4.4.18) and Eq.(4.4.19) and also the the term in Eq.(4.4.5) vanish because of the gauge conditions chosen in Eq.(4.4.11). Therefore adding together the results obtained in Eq.(4.4.4), Eq.(4.4.16), Eq.(4.4.18) and Eq.(4.4.19) total contribution coming from the non-degenerate solutions in the particular gauge conditions used can be given by

$$
\begin{aligned}
&\left[\frac{1}{\tau^2}S^{(0)}(p)S^{(0)}(q)+\frac{1}{\tau}\left\{S^{(0)}(p)S^{(1)}(q)+S^{(0)}(q)S^{(1)}(p)\right\}\right]M_n\\
&+\frac{1}{\tau}\int \mathrm{d}\mu_n I_n \oint_{\{B_i\}} \mathrm{d}\sigma_{n+2}\left(\sum_a \frac{p\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{-2}\left(\sum_a \frac{q\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{-1} p\cdot q\\
&\qquad\qquad\qquad\qquad \times\left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{2}\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+2}-\sigma_a}\right)^{2}\\
&+\frac{1}{\tau}\int \mathrm{d}\mu_n I_n \oint_{\{A_i\}} \mathrm{d}\sigma_{n+1}\left(\sum_a \frac{q\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^{-2}\left(\sum_a \frac{p\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^{-1} p\cdot q\\
&\qquad\qquad\qquad\qquad \left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^{2}\left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\right)^{2}. \qquad (4.4.20)
\end{aligned}
$$

4.4.2 Degenerate solutions

In this subsection we will derive the contribution coming from the degenerate solutions. At leading order in τ the structure of matrix Ψ_{n+2} is [87]

$$
\Psi_{n+2}\approx\left(\begin{array}{ccc|ccc}
(A_n)_{ab} & \frac{\tau\, k_a.p}{\sigma_a-\rho} & \frac{\tau\, k_a.q}{\sigma_a-\rho} & (-C_n^T)_{ab} & \frac{-\epsilon_{n+1}.k_b}{\rho-\sigma_b} & \frac{-\epsilon_{n+2}.k_b}{\rho-\sigma_b}\\
\frac{\tau\, p.k_b}{\rho-\sigma_b} & 0 & \frac{-\tau\, p.q}{\xi_1} & \frac{-\tau\, \epsilon_a.p}{\sigma_a-\rho} & -C_{n+1,n+1} & \frac{-\epsilon_{n+2}.p}{\xi_1}\\
\frac{\tau\, q.k_b}{\rho-\sigma_b} & \frac{\tau\, p.q}{\xi_1} & 0 & \frac{-\tau\, \epsilon_a.q}{\sigma_a-\rho} & \frac{\epsilon_{n+1}.q}{\xi_1} & -C_{n+2,n+2}\\
\hline
(C_n)_{ab} & \frac{\tau\, \epsilon_a.p}{\sigma_a-\rho} & \frac{\tau\, \epsilon_a.q}{\sigma_a-\rho} & (B_n)_{ab} & \frac{\epsilon_a.\epsilon_{n+1}}{\sigma_a-\rho} & \frac{\epsilon_a.\epsilon_{n+2}}{\sigma_a-\rho}\\
\frac{\epsilon_{n+1}.k_b}{\rho-\sigma_b} & C_{n+1,n+1} & \frac{-\epsilon_{n+1}.q}{\xi_1} & \frac{\epsilon_{n+1}.\epsilon_b}{\rho-\sigma_b} & 0 & \frac{\epsilon_{n+1}.\epsilon_{n+2}}{-\tau\, \xi_1}\\
\frac{\epsilon_{n+2}.k_b}{\rho-\sigma_b} & \frac{\epsilon_{n+2}.p}{\xi_1} & C_{n+2,n+2} & \frac{\epsilon_{n+2}.\epsilon_b}{\rho-\sigma_b} & \frac{\epsilon_{n+1}.\epsilon_{n+2}}{\tau\, \xi_1} & 0
\end{array}\right) \qquad (4.4.21)
$$

Pfaffian of any $2km \times 2m$ matrix can be expanded as follows

$$\mathrm{Pf}(E) = \sum_{q=1}^{2m} (-1)^q e_{pq} \mathrm{Pf}(E^{pq}_{pq}), \tag{4.4.22}$$

where e_{pq} is the element of the matrix E at the pth row and qth column.

First we make an expansion of the Pfaffian of Ψ_{n+2} along the $(n+2)$th row to leading order in τ

$$\mathrm{Pf}'\Psi_{n+2} = \frac{\tau\, p.q}{\xi_1}\mathrm{Pf}'(\Psi_{n+2})^{n+2,n+1}_{n+2,n+1} - \frac{\epsilon_{n+1}.q}{\xi_1}\mathrm{Pf}'(\Psi_{n+2})^{n+2,n+3}_{n+2,n+3} - C_{n+2,n+2}\mathrm{Pf}'(\Psi_{n+2})^{n+2,2n+4}_{n+2,2n+4}. \tag{4.4.23}$$

Again each of the reduced Pfaffians can be further expanded as

$$\begin{aligned}
\mathrm{Pf}'(\Psi_{n+2})^{n+2,n+1}_{n+2,n+1} &= -\frac{\epsilon_{n+1}.\epsilon_{n+2}}{\tau\,\xi_1}\mathrm{Pf}'\Psi_n + O(1) \\
\mathrm{Pf}'(\Psi_{n+2})^{n+2,2n+3}_{n+2,2n+3} &= -\frac{\epsilon_{n+2}.p}{\xi_1}\mathrm{Pf}'\Psi_n + O(\tau) \\
\mathrm{Pf}'(\Psi_{n+2})^{n+2,2n+4}_{n+2,2n+4} &= -C_{n+1,n+1}\mathrm{Pf}'\Psi_n + O(\tau).
\end{aligned} \tag{4.4.24}$$

The two diagonal terms of the matrix C_{n+2} can be approximated as

$$\begin{aligned}
C_{n+1,n+1} &= -\sum_{i=1}^{n} \frac{\epsilon_{n+1}.p_i^{\perp}}{\rho - \sigma_b} \\
C_{n+2,n+2} &= -\sum_{i=1}^{n} \frac{\epsilon_{n+2}.q_i^{\perp}}{\rho - \sigma_b}
\end{aligned} \tag{4.4.25}$$

where $p_i^{\perp} = k_i - \frac{p.k_i}{p.q}q$ and $q_i^{\perp} = k_i - \frac{q.k_i}{p.q}p$.

In this case the integrand takes the form

$$I_{n+2} = \left[\frac{\epsilon_{n+1}.q\ \epsilon_{n+2}.p - \epsilon_{n+1}.\epsilon_{n+2}\ p.q}{\xi_1^2} + \sum_{i,j=1}^{n} \frac{\epsilon_{n+1}.p_i^{\perp}\ \epsilon_{n+2}.q_j^{\perp}}{(\rho - \sigma_i)(\rho - \sigma_j)}\right]^2 (\mathrm{Pf}'\Psi_n)^2 + O(\tau). \tag{4.4.26}$$

Using Eq.(4.1.15) and Eq.(4.4.26) we get

$$M_{n+2} = -\frac{1}{\tau}\int d\mu_n I_n \oint_{\{C_i\}} d\rho \frac{\xi_1^2}{p.q\sum_{a=1}^{n}\frac{k_a.(p+q)}{\rho-\sigma_a}}\left[\frac{\epsilon_{n+1}.q\,\epsilon_{n+2}.p-\epsilon_{n+1}.\epsilon_{n+2}\,p.q}{\xi_1^2}+\sum_{i,j=1}^{n}\frac{\epsilon_{n+1}.p_i^{\perp}\,\epsilon_{n+2}.q_j^{\perp}}{(\rho-\sigma_i)(\rho-\sigma_j)}\right]^2 \tag{4.4.27}$$

To perform the integration over ρ we have to deform the contour away from the original contour of integration to infinity. Then we will get simple poles at $\rho = \sigma_a, \forall\ a \in \{1, \dots n\}$ along with other poles. Residues at the simple poles at $\rho = \sigma_a$ give the following result

$$\begin{aligned}
&\frac{1}{\tau}\sum_{a=1}^{n}\frac{1}{k_a.(p+q)}\Bigg[-\frac{(\epsilon_{n+1}.q\,\epsilon_{n+2}.p-\epsilon_{n+1}.\epsilon_{n+2}\,p.q)^2\,k_a.p\,k_a.q}{(p.q)^3}\\
&\qquad+2\frac{(\epsilon_{n+1}.q\,\epsilon_{n+2}.p-\epsilon_{n+1}.\epsilon_{n+2}\,p.q)\,\epsilon_{n+1}.p_a^{\perp}\,\epsilon_{n+2}.q_a^{\perp}}{p.q}\\
&\qquad-\frac{(\epsilon_{n+1}.p_a^{\perp}\,\epsilon_{n+2}.q_a^{\perp})^2\,p.q}{k_a.p\,k_a.q}\Bigg]M_n\\
=\ &\frac{1}{\tau}\sum_{a=1}^{n}\Bigg[\frac{1}{k_a.(p+q)\,p.q}\Big\{-(\epsilon_{n+1}.\epsilon_{n+2})^2\,k_a.p\,k_a.q\\
&\qquad+2\,\epsilon_{n+1}.\epsilon_{n+2}\,(\epsilon_{n+1}.q\,\epsilon_{n+2}.k_a\,k_a.p+\epsilon_{n+1}.k_a\,\epsilon_{n+2}.p\,k_a.q)\\
&\qquad-2\,\epsilon_{n+1}.q\,\epsilon_{n+2}.p\,\epsilon_{n+1}.k_a\,\epsilon_{n+2}.k_a+(\epsilon_{n+1}.q)^2\,(\epsilon_{n+2}.k_a)^2+(\epsilon_{n+1}.k_a)^2\,(\epsilon_{n+2}.p)^2\Big\}\\
&\qquad+\frac{1}{k_a.(p+q)}\Big\{-2\,\epsilon_{n+1}.\epsilon_{n+2}\,\epsilon_{n+1}.k_a\,\epsilon_{n+2}.k_a\\
&\qquad+2\,\epsilon_{n+1}.k_a\,\epsilon_{n+2}.k_a\left(\frac{\epsilon_{n+1}.q\,\epsilon_{n+2}.k_a}{k_a.q}+\frac{\epsilon_{n+1}.k_a\,\epsilon_{n+2}.p}{k_a.p}\right)\\
&\qquad-\frac{(\epsilon_{n+1}.k_a)^2\,(\epsilon_{n+2}.k_a)^2\,p.q}{k_a.p\,k_a.q}\Big\}\Bigg]M_n.
\end{aligned} \tag{4.4.28}$$

Here we have used the solutions of ξ_1 of Eq.(4.1.11) and used their product to substitute for ξ_1^2.

In Sec.(B.3.2) that there is also a pole at infinity but the contribution coming from that pole is at sub-sub-leading order in τ and hence is not relevant for our analysis.

There are other terms coming from the degenerate solutions which can be given by

$$
\begin{aligned}
&- \frac{1}{\tau}\int \mathrm{d}\mu_n I_n \oint_{\{A_i\}} \mathrm{d}\rho \left(\sum_a \frac{p\cdot k_a}{\rho-\sigma_a}\right)^{-1}\left(\sum_a \frac{q\cdot k_a}{\rho-\sigma_a}\right)^{-2} p\cdot q \left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\rho-\sigma_a}\right)^{2}\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\rho-\sigma_a}\right)^{2}\\
&+ \frac{2}{\tau}\int \mathrm{d}\mu_n I_n \oint_{\{A_i\}} \mathrm{d}\rho \left(\sum_a \frac{p\cdot k_a}{\rho-\sigma_a}\right)^{-1}\left(\sum_a \frac{q\cdot k_a}{\rho-\sigma_a}\right)^{-1} \epsilon_{n+2}\cdot p \left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\rho-\sigma_a}\right)^{2}\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\rho-\sigma_a}\right)\\
&- \frac{1}{\tau}\int \mathrm{d}\mu_n I_n \oint_{\{B_i\}} \mathrm{d}\rho \left(\sum_a \frac{p\cdot k_a}{\rho-\sigma_a}\right)^{-2}\left(\sum_a \frac{q\cdot k_a}{\rho-\sigma_a}\right)^{-1} p\cdot q \left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\rho-\sigma_a}\right)^{2}\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\rho-\sigma_a}\right)^{2}\\
&+ \frac{2}{\tau}\int \mathrm{d}\mu_n I_n \oint_{\{B_i\}} \mathrm{d}\rho \left(\sum_a \frac{p\cdot k_a}{\rho-\sigma_a}\right)^{-1}\left(\sum_a \frac{q\cdot k_a}{\rho-\sigma_a}\right)^{-1} \epsilon_{n+1}\cdot q \left(\sum_a \frac{\epsilon_{n+1}\cdot k_a}{\rho-\sigma_a}\right)\left(\sum_a \frac{\epsilon_{n+2}\cdot k_a}{\rho-\sigma_a}\right)^{2}.
\end{aligned}
\tag{4.4.29}
$$

The first and third line of the expression (4.4.29) cancel with the extra terms (given in the last two lines of (4.4.20)) in the contribution from non-degenerate solutions. In Sec.(4.4.1) we have calculated the contribution from the non-degenerate solutions using a specific gauge choice. On further analysis it can be seen that in general without fixing any gauge condition there are more terms in the contribution from the non-degenerate solutions which precisely cancel all the terms of the above expression (4.4.29) [14].

4.4.3 Total contribution

Adding together the total contributions from non-degenerate and degenerate solutions we obtain the final expression for the double soft limit limit of graviton scattering amplitude to sub-leading order in the energy of the soft particles which can be given by

$$
\begin{aligned}
M_{n+2} \;=\; & \Bigg[\frac{1}{\tau^2} S^{(0)}(p) S^{(0)}(q) \\
& + \frac{1}{\tau}\Big\{S^{(0)}(p)S^{(1)}(q) + S^{(0)}(q)S^{(1)}(p)\Big\}\Bigg] M_n \\
& + \frac{1}{\tau}\sum_{a=1}^{n}\Bigg[\frac{1}{k_a.(p+q)\; p.q}\Big\{-(\epsilon_{n+1}.\epsilon_{n+2})^2\, k_a.p\; k_a.q
\end{aligned}
$$

$$+ 2\, \epsilon_{n+1}.\epsilon_{n+2}\, (\epsilon_{n+1}.q\, \epsilon_{n+2}.k_a\, k_a.p + \epsilon_{n+1}.k_a\, \epsilon_{n+2}.p\, k_a.q)$$

$$- 2\, \epsilon_{n+1}.q\, \epsilon_{n+2}.p\, \epsilon_{n+1}.k_a\, \epsilon_{n+2}.k_a + (\epsilon_{n+1}.q)^2\, (\epsilon_{n+2}.k_a)^2 + (\epsilon_{n+1}.k_a)^2\, (\epsilon_{n+2}.p)^2 \Big\}$$

$$+ \frac{1}{k_a.(p+q)} \Big\{ -2\, \epsilon_{n+1}.\epsilon_{n+2}\, \epsilon_{n+1}.k_a\, \epsilon_{n+2}.k_a$$

$$+ 2\, \epsilon_{n+1}.k_a\, \epsilon_{n+2}.k_a \left(\frac{\epsilon_{n+1}.q\, \epsilon_{n+2}.k_a}{k_a.q} + \frac{\epsilon_{n+1}.k_a\, \epsilon_{n+2}.p}{k_a.p} \right)$$

$$- \frac{(\epsilon_{n+1}.k_a)^2\, (\epsilon_{n+2}.k_a)^2\, p.q}{k_a.p\, k_a.q} \Big\} \Bigg] M_n \tag{4.4.30}$$

4.4.4 Feynman diagrams

Here we will present some analysis for double soft limit of gravity amplitude from Feynman diagrams and compare with the result obtained in the previous subsection.

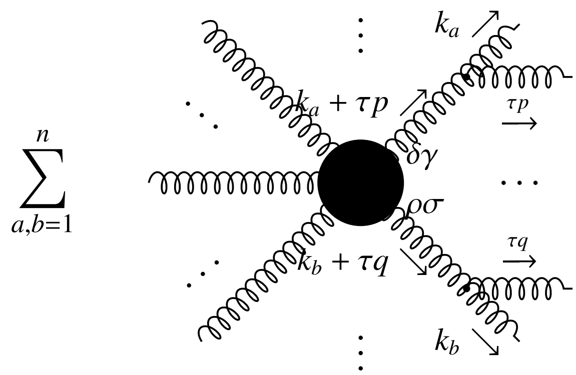
The action for Einstein Hilbert gravity in four dimension is

$$S_{EH} = \frac{2}{\kappa^2} \int \mathrm{d}^4 x\, \sqrt{-g} R. \tag{4.4.31}$$

The Feynman rules for linearized gravity (4.2.5) are given in [88] where the conventions of [89] are used. Every three-point vertex is of $O(\kappa)$ and four-point vertex is of $O(\kappa^2)$. When looking for scattering amplitude in the double soft limit it is sufficient for our purpose to consider only three and four point vertices because they are the ones to contribute at the leading order. Given n external legs there can not be any higher than four-point vertex in each leg from where two soft gravitons can be emitted because in that case number of hard particles will exceed n. In the perturbative linearized gravity in the double soft limit there are two parameters, coupling constant, κ and energy scale of soft gravitons, τ and the dominating term, as we will see below, is of $O\left(\frac{\kappa^2}{\tau}\right)$.

Following are the relevant Feynman diagrams (we will use the convention that momenta at all external legs are outgoing):

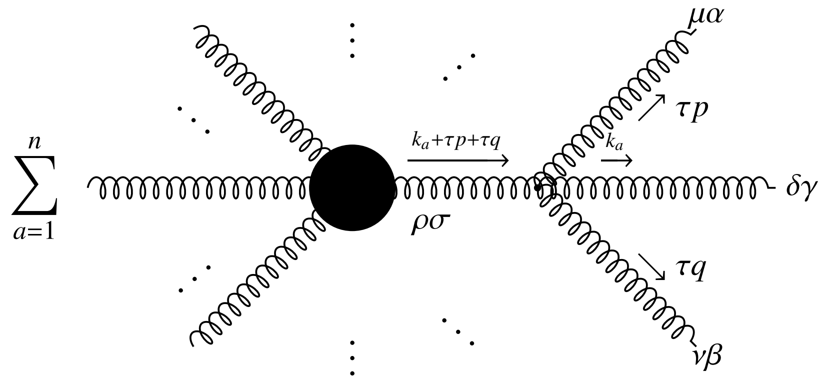
- The leading order term in the double soft factorization which is the product of two single soft factors comes from this diagram



$$= \frac{\kappa^2}{\tau^2} \sum_{\substack{a,b=1 \\ a\neq b}}^{n} \frac{(\epsilon_{n+1}.k_a)^2(\epsilon_{n+2}.k_b)^2}{k_a.p\, k_b.q} M_n + O\left(\frac{1}{\tau}\right). \tag{4.4.32}$$

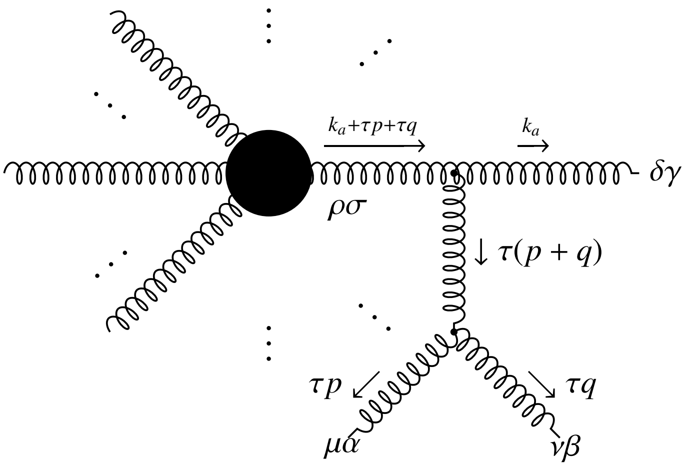
Therefore we find that leading order term coming from this diagram gives the factor $\frac{1}{\tau^2}S^{(0)}(p)S^{(0)}(q)$. Taylor series expansion of this digram also produce the subleading terms of the form $\frac{1}{\tau}\left(S^{(0)}(p)S^{(1)}(q) + S^{(0)}(q)S^{(1)}(p)\right) M_n$.

- **4 point vertex**



$$\approx -4\frac{\kappa^2}{\tau}\sum_{a=1}^{n}\frac{\epsilon_{n+1}.\epsilon_{n+2}\ \epsilon_{n+1}.k_a\ \epsilon_{n+2}.k_a}{k_a.(p+q)}(M_n)_{\rho\sigma}\epsilon_a^{\rho\sigma}+\kappa^2 O(1) \tag{4.4.33}$$

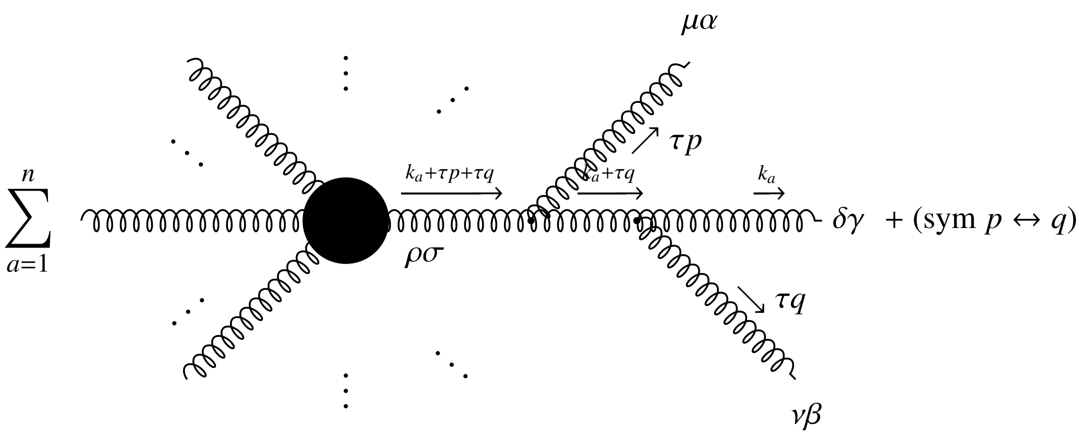
- **3 point vertex (I)**

$\sum_{a=1}^{n}$ 

$$\begin{aligned}\approx\ & \frac{\kappa^2}{\tau}\sum_{a=1}^{n}\frac{1}{p.q\ k_a.(p+q)}\Big[-(\epsilon_{n+1}.\epsilon_{n+2})^2 k_a.p\ k_a.q+(\epsilon_{n+1}.q)^2(\epsilon_{n+2}.k_a)^2+(\epsilon_{n+1}.k_a)^2(\epsilon_{n+2}.p)^2\\ &-2\epsilon_{n+1}.\epsilon_{n+2}\Big\{\epsilon_{n+1}.k_a\ \epsilon_{n+2}.p\ k_a.p+\epsilon_{n+1}.q\ \epsilon_{n+2}.k_a\ k_a.q\Big\}\\ &-2\ \epsilon_{n+1}.q\ \epsilon_{n+2}.p\ \epsilon_{n+1}.k_a\ \epsilon_{n+2}.k_a\Big](M_n)_{\rho\sigma}\epsilon_a^{\rho\sigma}\\ &+2\frac{\kappa^2}{\tau}\sum_{a=1}^{n}\frac{\epsilon_{n+1}.\epsilon_{n+2}\ \epsilon_{n+1}.k_a\ \epsilon_{n+2}.k_a}{k_a.(p+q)}(M_n)_{\rho\sigma}\epsilon_a^{\rho\sigma}+\kappa^2 O(1)\end{aligned} \tag{4.4.34}$$

- **3 point vertex (II)**

We consider the subleading order term from this diagram because the leading term has already been accounted in (4.4.32). Therefore we obtain



$$\approx \frac{\kappa^2}{\tau} \sum_{a=1}^{n} \frac{1}{k_a.(p+q)} \left[\frac{2\, \epsilon_{n+1}.q\, \epsilon_{n+1}.k_a (\epsilon_{n+2}.k_a)^2}{k_a.q} + \frac{2\, (\epsilon_{n+1}.k_a)^2\, \epsilon_{n+2}.p\, \epsilon_{n+2}.k_a}{k_a.p} - \frac{(\epsilon_{n+1}.k_a)^2 (\epsilon_{n+2}.k_a)^2 p.q}{k_a.p\, k_a.q} \right] (M_n)_{\rho\sigma} \epsilon_a^{\rho\sigma} + \kappa^2 O(1) \quad (4.4.35)$$

The last term comes from the expansion of the propagator in the denominator

$$\frac{1}{(k_a + \tau p + \tau q)^2} \approx \frac{1}{\tau\, k_a.(p+q)} \left[1 - \frac{\tau\, p.q}{k_a.(p+q)} + O(\tau^2) \right]. \quad (4.4.36)$$

Adding together the expressions (4.4.32), (4.4.33), (4.4.34) and (4.4.35) we recover the CHY expression for the double soft factor (4.4.30).

4.4.5 Gauge invariance

Gauge invariance of the double soft graviton factor can be checked by making suitable transformations of the polarization tensors of the soft gravitons. We can make following shifts

$$\epsilon_{n+1,\mu\nu} \to \xi_{1,\mu} p_\nu + \xi_{1,\nu} p_\mu \quad \text{or} \quad \epsilon_{n+2,\mu\nu} \to \xi_{2,\mu} q_\nu + \xi_{2,\nu} q_\mu, \quad (4.4.37)$$

with $\xi_1 \cdot p = 0$ and $\xi_2 \cdot q = 0$. Because of the diffeomorphism invariance under the above transformations the result in Eq.(4.4.30) should not change.

To check for the gauge invariance we have to consider the full amplitude including the momentum conserving delta function. We can multiply both sides of Eq.(4.4.30) by $\delta^{(D)}(k_1 + k_2 + \ldots k_n + \tau p + \tau q)$ and then expand the delta function around any of the hard momenta k_a to obtain scattering amplitude $\mathcal{M}_n$ on the right hand side of Eq.(4.4.30). However it is easier to start with the full n-point amplitude $\mathcal{M}_n$ instead on M_n on the RHS of Eq.(4.4.30) and show gauge invariance.

Let us consider the transformation of $\epsilon_{1,\mu\nu}$. Then $S^{(0)}(p)$ will become $2\sum_{a=1}^{n} \xi_1 \cdot k_a$. Due to the momentum conserving delta function $\delta^{(D)}(k_1 + k_2 + \ldots k_n)$, first term in Eq.(4.4.30) vanishes. Also second term in the second line vanishes by angular momentum conservation: $\sum_{a=1}^{n} J_a^{\rho\nu}\mathcal{M}_n = 0$. There is subtlety involved with the first term in the second line of Eq.(4.4.30). $S^{(2)}(q)$ contains angular momentum operator J_a which acts on the full momentum $\mathcal{M}_n$ which includes the delta function. Therefore $S^{(0)}(p)$ can not gauge invariant by itself as the momentum conserving delta function has to pass through the angular momentum operators contained in $S^{(1)}(q)$. After substituting $\epsilon_{1,\mu\nu} \to \xi_\mu p_\nu + \xi_\nu p_\mu$, first term in the second line of Eq.(4.4.30) becomes

$$2\sum_{a=1}^{n}(\xi_1 \cdot k_a)\sum_{b=1}^{n}\frac{\epsilon_{n+2,\mu\nu}k_b^\mu q_\rho}{k_b \cdot q}\left(k_b^\nu\frac{\partial}{\partial k_{b\rho}} - k_b^\rho\frac{\partial}{\partial k_{b\nu}}\right)\{M_n\delta^{(D)}(k_1 + k_2 + \ldots + k_n)\}. \tag{4.4.38}$$

We consider only the orbital angular momentum part here because it contains the derivative operators, spin angular momentum part does not have any derivative operator acting on the delta function and is thus trivial to handle. Second term in the parentheses in expression (4.4.38) vanishes because of

$$\sum_{a=1}^{n}\frac{\epsilon_{n+2,\mu\nu}k_a^\mu q_\rho}{k_a \cdot q}k_a^\rho\frac{\partial}{\partial k_{a,\nu}}\delta^{(D)}(k_1 + k_2 + \ldots + k_n) = 0 \tag{4.4.39}$$

which is a consequence of tracelessness of polarization tensor, ${\epsilon_{n+2,\mu}}^{\mu} = 0$. Using the result $\frac{\delta}{\delta k_{a,\mu}} k_b^\nu = \delta_{ab}\delta_\mu^\nu$ we then obtain from the expression (4.4.38)

$$-2\sum_{a=1}^{n} \xi_1 \cdot q \frac{\epsilon_{n+2,\mu\nu} k_a^\mu k_a^\nu}{k_a \cdot q} M_n \delta^{(D)}(k_1 + k_2 + \ldots + k_n)\,. \tag{4.4.40}$$

It can be checked that under the transformation $\epsilon_{n+1,\mu\nu} \to \xi_{1,\mu} p_\nu + \xi_{1,\nu} p_\mu$ the contact term in Eq.(4.4.30) changes as exactly with the opposite sign of the expression given in (4.4.40). Hence the full amplitude in Eq.(4.4.30) is invariant under $\epsilon_{n+1,\mu\nu} \to \xi_{1,\mu} p_\nu + \xi_{1,\nu} p_\mu$. In the same way one can show invariance of Eq.(4.4.30) under the transformation $\epsilon_{2,\mu\nu} \to \xi_{2,\mu} q_\nu + \xi_{2,\nu} q_\mu$. Thus we have shown the gauge invariance of gravity amplitude in the double soft limit.

4.5 Discussion

In [63] double soft limits of scattering amplitudes involving soft scalars have been studied. It was found that leading order contributions for scalar double soft theorems come from degenerate solutions of the scattering equations. From our analysis of double soft limits of Yang-Mills and gravity amplitudes we can see that non-degenerate solutions also contribute to leading order. Moreover our calculations show that to obtain physically meaningful results (refer to Eq.(4.3.20) and Eq.(4.4.30)) there have to be cancellations among some terms coming from non-degenerate and degenerate solutions separately.

As we see from Eq.(4.4.30) double soft factor of gravity amplitude has three terms at the sub-leading order - two of the terms can be expressed as products of leading and sub-leading soft factors, $S^{(0)}$ and $S^{(1)}$ respectively and a third term, called contact term, whose expression is given in Eq.(4.4.28). Presence of the contact term is due to the fact that leading and sub-leading soft factors do not commute. Also the term $\Big\{S^{(0)}(p)S^{(1)}(q) + S^{(0)}(q)S^{(1)}(p)\Big\}M_n$ in Eq.(4.4.30) is not gauge invariant, the contact term is needed to maintain gauge invariance of the amplitude to sub-leading order in the double soft limit.

5 Supertranslation symmetries and soft graviton theorem

In early sixties Bondi, Metzner, Sachs and Van der Burg [1, 2] discovered that the asymptotic symmetry group of asymptotically flat spacetimes in four dimension is not just the Poincaré group but an infinite dimensional group, popularly known as BMS group which consists of angle dependent translations along the null directions, called supertranslations acting semi-directly on the Lorentz group. Supertranslations form an infinite dimensional normal subgroup of BMS group. Recently Strominger and his collaborators [3, 56] have established that Weinberg's soft graviton theorem [4, 5] can be expressed as Ward identities derived from supertranslation invariance of gravitational S-matrix. Since then there have been progressively growing interests in the studies of asymptotic symmetries for asymptotically flat spacetimes mainly in four dimension in the context of soft graviton theorems. Asymptotic symmetries related to sub-leading and sub-sub-leading soft graviton theorems [22–28] have been explored in [15, 58–60]. Asymptotic symmetries for massive scalars coupled to gravity have been considered in [57]. Connection between soft graviton theorem, Ward identity and gravitational memory effect [90] can be beautifully depicted in the form of a "infra-red triangle" in four dimension.

Since soft graviton theorems [8–11, 13, 14, 23, 24, 27–29] are valid in any spacetime dimensions it is an obvious question to ask whether the relations between soft graviton theorems and asymptotic symmetries exist for asymptotically flat spacetimes beyond four dimension. To answer this question one should in principle first study the asymptotic symmetry group for asymptotically flat spacetime in higher dimensions [91] in full generality, however such analysis is highly nontrivial. Due to our lack of understanding of full symmetry group in higher dimensions (analogous to BMS group in four dimension), we will address a much smaller subset of the question: can we relate Weinberg's soft graviton theorem to supertranslation symmetries in arbitrary dimensions?

There have been some studies in this context. In [16] Weinberg's soft graviton theorem has been recast as supertranslation Ward identity in higher even dimensions. Gravitational memory effects in higher dimensions have been studied in [19, 92, 93]. Asymptotic charges from scalar soft theorems in higher even dimensions have been constructed in [94]. In odd spacetime dimensions the analysis is subtle; the main problem is the non-existence of conformal null infinity for spacetime with radiation in odd dimensions [64, 95–97]. Due to the fractional power fall-off of metric perturbations in radial coordinates one can not show smoothness of Einstein's equations at null infinity. Our analysis will be in linearized perturbative gravity and hence we will not have to worry about such non-smoothness issues of conformal null infinity.

In this chapter our goal is to understand the subtleties associated with supertranslation symmetries and soft graviton theorem for asymptotically flat spacetimes in higher dimensions. Here we follow the approach of [60] to study large diffeomorphisms related to supertranslation symmetries in higher spacetime dimensions. Using the covariant phase space methods [17, 98, 99] (a concise review of covariant phase space formalism can be found in [100]) we calculate the conserved charges for large gauge transformations at asymptotic null infinity. In Bondi coordinates we consider the null infinity in the limit radial coordinate is taken to infinity while keeping retarded time coordinate and angular coordinates held fixed. Topologically in D dimension null infinity is described as $\mathbb{R}\times\mathbb{S}^{D-2}$. The conserved charges have two parts: the gravitational radiation part - which contains radiative data for free metric perturbations and the matter part - which contains radiative data for matter fields. In four dimensions radiative modes of graviton have fall-off behavior at null infinity such that shift due to supertranslation vector fields occur at same order in the radial coordinate. In higher dimensions radiative modes and shift of metric perturbations due to supertranslations occur at different orders in the radial coordinate. This is a main reason of inconsistencies in the description of supertranslation symmetries in higher dimensions. Most of the analyses in the literature [18, 19, 64, 65, 90, 93, 95, 101] have ruled out the existence of supertranslation symmetries and BMS in higher dimensional asymptotically flat spacetimes based on the fall-off behavior of metric components at null infinity. Here we try to explore the possibility of recovering supertranslation Ward identities for soft graviton theorem by considering different boundary conditions on metric perturbations. Fall-off behavior of radiative modes of the metric perturbations can be obtained from saddle-point analysis at null infinity. These fall-off behavior are too restrictive and one needs to relax the boundary conditions in order to have supertranslations in higher dimensions. However we find that relaxing these fall-off conditions leads to the appearance of divergent terms in the asymptotic conserved charge at null infinity. At this stage it is not clear to us how to show that divergent terms in the charge vanish and the finite part of the asymptotic charge gives rise to Weinberg's soft graviton theorem. Another possibility is to impose the restrictive boundary conditions and this leads to vanishing of the gravitational part of the charge. This result is in agreement with the analysis of [18]. The only possible gauge transformations at null infinity in this case are that of spacetime translations which give rise to finite matter part of the charge implying global energy-momentum conservation.

This chapter is organized as follows: in Sec.(5.1) we present basic details of perturbative gravity coupled to massless scalar field for flat spacetime in Bondi coordinates. We use de Dender gauge choice to obtain the linearized gravity equations. In Sec.(5.2) we derive the conserved charges at asymptotic null infinity from covariant phase space methods. In Sec.(5.3) we review the analysis of Ward identities from supertranslation symmetries related to Weinberg's soft graviton theorem in four dimension. In Sec.(5.4) we present some basic calculations involving supertranslation symmetries in higher dimensional spacetimes. We show the subtleties with restricted boundary conditions following from saddle point analysis and try to find appropriate fall-off behavior of the metric perturbations compatible with supertranslations.

5.1 Linearized gravity

Let us consider metric perturbations given by $h_{\mu\nu}$ around flat spacetime metric $\eta_{\mu\nu}$ such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{5.1.1}$$

At this stage we will consider the flat metric of the form $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, \ldots, 1)$. But as we will see later flat spacetime can be given in different coordinate systems. For any arbitrary metric representation of Minkowski spacetime we have to replace partial derivatives with covariant derivatives.

Christoffel symbols to linear order in the metric fluctuations can be expressed as

$$\Gamma^{\alpha}_{\mu\nu}\,[h] = \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right). \tag{5.1.2}$$

Ricci tensors are given by

$$R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}. \tag{5.1.3}$$

The last two terms are quadratic in h and so can be neglected at linearized order. Therefore Eq.(5.1.3) takes the form

$$R_{\mu\nu}\,[h] = \frac{1}{2}\partial^\alpha\left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu}\right) - \frac{1}{2}\partial^\alpha\partial_\alpha h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h. \tag{5.1.4}$$

Here h is the trace of $h_{\mu\nu}$.

We now define trace-reversed metric perturbations as

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{5.1.5}$$

which implies in D dimensions

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}\bar{h}. \tag{5.1.6}$$

We can choose *harmonic gauge* which is an off-shell gauge condition and is given by

$$\partial^{\mu}\bar{h}_{\mu\nu} = 0. \tag{5.1.7}$$

There are D number of such gauge conditions.

In this gauge choice Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5.1.8}$$

at linearized order can be expressed as

$$\partial^{\alpha}\partial_{\alpha}\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{5.1.9}$$

5.1.1 Residual gauge symmetries

Under diffeomorphism induced by a vector field $\vec{\xi}$ metric perturbations transform as follows

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} \tag{5.1.10}$$

which implies

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \partial_{\alpha}\xi^{\alpha}\eta_{\mu\nu}. \tag{5.1.11}$$

Since Einstein's equation is diffeomorphism invariant, harmonic gauge condition of Eq.(5.1.7) then imposes the following constraints

$$\partial^\alpha \partial_\alpha \xi^\mu = 0. \tag{5.1.12}$$

These provide us with additional D gauge fixing choices for the metric perturbations $h_{\mu\nu}$ which we call residual gauge conditions.

It is to be noted that total number of components of a rank 2 symmetric tensor in D dimensions is $\frac{D(D+1)}{2}$ and total number of gauge fixing conditions are $2D$. Hence there are $\frac{D(D-3)}{2}$ number of independent components which are correspond to the polarizations of the metric perturbations.

5.1.2 Bondi coordinates

For the rest of our analysis we will work in a particular coordinate system, called *Bondi coordinates*. The studies of asymptotically flat spacetimes become much simpler in these coordinates. Here $u = t - r$ is the retarded time, r is the radial coordinate and γ_{AB} are the metric components on the $(D-2)$ dimensional sphere.

In terms of the Bondi coordinates Minkowski metric written can be expressed as

$$ds^2 = -du^2 - 2dudr + r^2 \gamma_{AB} dz^A dz^B. \tag{5.1.13}$$

It is easy to compute the inverse metric components are

$$\eta^{rr} = 1, \qquad \eta^{ur} = \eta^{ru} = -1, \qquad \eta^{AB} = \frac{1}{r^2}\gamma^{AB} \tag{5.1.14}$$

and the non-zero Christoffel symbols are

$$\Gamma^A_{rB} = \frac{1}{r}\delta^A_B, \qquad \Gamma^r_{AB} = -r\gamma_{AB}, \qquad \Gamma^u_{AB} = r\gamma_{AB} \qquad \text{and} \quad \Gamma^A_{BC}. \tag{5.1.15}$$

Trace of metric perturbations in these coordinates is

$$h = h_{rr} - 2h_{ur} + \frac{1}{r^2}\gamma^{AB}h_{AB}. \tag{5.1.16}$$

A similar analysis can be done using the advanced time coordinate $v = t + r$. In this case the flat metric is given by

$$ds^2 = -dv^2 + 2dvdr + r^2\gamma_{AB}dz^A dz^B. \tag{5.1.17}$$

For any asymptotically flat spacetimes one can take the limit $r \to \infty$, $t \to \infty$ such that u is constant to reach the future asymptotic *null infinity* denoted by $\mathscr{I}^+$. The null infinity is topologically $\mathbb{R} \times \mathbb{S}^{D-2}$ and is parametrized by (u, z^A) coordinates. Similarly by taking the limit $t \to -\infty$, $r \to \infty$ with constant v one will reach past null infinity, $\mathscr{I}^-$. Geodesics of any massless particles will begin at $\mathscr{I}^-$ and end at $\mathscr{I}^+$. We can expand the fields in inverse powers of the radial coordinate with suitable boundary conditions of the fields specified at null infinity.

5.1.3 Equations of motion

We can write the harmonic gauge condition of Eq.(5.1.7) in a covariant way as $\nabla^\mu \bar{h}_{\mu\nu} = 0$. Therefore we obtain the following set of equations [92]:

$$\begin{aligned}
\nabla^\mu \bar{h}_{\mu u} &= -\partial_u \bar{h}_{ur} - \partial_r \bar{h}_{uu} + \partial_r \bar{h}_{ur} - \frac{(D-2)}{r}\left(\bar{h}_{uu} - \bar{h}_{ur}\right) + \frac{1}{r^2}D^A \bar{h}_{uA}, \\
\nabla^\mu \bar{h}_{\mu r} &= -\partial_u \bar{h}_{rr} - \partial_r \bar{h}_{ur} + \partial_r \bar{h}_{rr} - \frac{(D-2)}{r}\left(\bar{h}_{ur} - \bar{h}_{rr}\right) + \frac{1}{r^2}D^A \bar{h}_{rA} - \frac{1}{r^3}\gamma^{AB}\bar{h}_{AB}, \\
\nabla^\mu \bar{h}_{\mu A} &= -\partial_u \bar{h}_{rA} - \partial_r \bar{h}_{uA} + \partial_r \bar{h}_{rA} - \frac{(D-2)}{r}\left(\bar{h}_{uA} - \bar{h}_{rA}\right) + \frac{1}{r^2}D^B \bar{h}_{BA}.
\end{aligned} \tag{5.1.18}$$

Here D^A denotes the covariant derivative with respect to the sphere metric γ^{AB}. Harmonic gauge condition imposes the following constraint on the residual diffeomorphisms $\vec{\xi}$ given by $\Box\xi^\mu = 0$. Thus we get

$$\begin{aligned}
r\Box\xi^u &= \partial_r^2(r\xi^u) - 2\partial_u\partial_r(r\xi^u) + (D-4)(\partial_r - \partial_u)\xi^u + 2D_A\xi^A + \frac{1}{r}\left[\Delta\xi^u + (D-2)\xi^r\right] = 0,\\
r\Box\xi^r &= \partial_r^2(r\xi^r) - 2\partial_u\partial_r(r\xi^r) + (D-4)(\partial_r - \partial_u)\xi^r - 2D_A\xi^A + \frac{1}{r}\left[\Delta - (D-2)\right]\xi^r = 0,\\
r^2\Box\xi^A &= \partial_r^2(r^2\xi^A) - 2\partial_u\partial_r(r^2\xi^A) + (D-4)r(\partial_r - \partial_u)\xi^A + \frac{2}{r}D^A\xi^r + \left[\Delta + (D-5)\right]\xi^A = 0.
\end{aligned} \tag{5.1.19}$$

Here Δ is the Laplacian on the $(D-2)$ sphere.

In any coordinate systems the linearized gravity equation can be expressed

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{5.1.20}$$

In the presence of gravity, metric perturbations consist of two parts - $h^{(C)}_{\mu\nu}$ which corresponds to the gravitational wave and satisfies homogeneous equation

$$\Box\bar{h}^{(C)}_{\mu\nu} = 0 \tag{5.1.21}$$

and $h^{(\phi)}_{\mu\nu}$ which is determined by the matter content. Thus we can write $h_{\mu\nu} = h^{(C)}_{\mu\nu} + h^{(\phi)}_{\mu\nu}$. The matter part typically has faster fall-off conditions than the gravitational wave part. In later part of our analysis we will consider massless scalar field minimally coupled to gravity and calculate stress energy tensor from it.

The linearized gravity equation in vacuum satisfied by the homogeneous part of metric perturbations can be written as (we omit the superscript C for convenience)

$$\begin{aligned}
\Box\bar{h}_{uu} &= \left(\partial_r^2 - 2\partial_r\partial_u - \frac{(D-2)}{r}(\partial_u - \partial_r) + \frac{1}{r^2}\Delta\right)\bar{h}_{uu},\\
\Box\bar{h}_{ur} &= \left(\partial_r^2 - 2\partial_r\partial_u + \frac{1}{r^2}\Delta\right)\bar{h}_{ur} + \frac{(D-2)}{r^2}\left(\bar{h}_{uu} - \bar{h}_{ur}\right) + \frac{(D-2)}{r}(\partial_u - \partial_r)\bar{h}_{ur} - \frac{2}{r^3}D^A\bar{h}_{uA},\\
\Box\bar{h}_{rr} &= \left(\partial_r^2 - 2\partial_r\partial_u + \frac{1}{r^2}\Delta\right)\bar{h}_{rr} - \frac{4}{r^3}D^A\bar{h}_{Ar} - \frac{(D-2)}{r}\left(\partial_u\bar{h}_{rr} - \partial_r\bar{h}_{rr}\right) + \frac{2(D-2)}{r^2}\left(\bar{h}_{ur} - \bar{h}_{rr}\right)
\end{aligned}$$

$$
\begin{aligned}
& \qquad +\frac{2}{r^4}\gamma^{CB}\bar{h}_{CB}, \\
\Box\bar{h}_{uA} &= \left(\partial_r^2 - 2\partial_r\partial_u + \frac{1}{r^2}\Delta\right)\bar{h}_{uA} - \frac{(D-4)}{r}\left(\partial_u - \partial_r\right)\bar{h}_{uA} - \frac{2}{r}D_A\left(\bar{h}_{uu} - \bar{h}_{ur}\right) + \frac{(3-D)}{r^2}\bar{h}_{uA}, \\
\Box\bar{h}_{rA} &= \left(\partial_r^2 - 2\partial_r\partial_u + \frac{1}{r^2}\Delta\right)\bar{h}_{rA} - \frac{(D-4)}{r}\left(\partial_u - \partial_r\right)\bar{h}_{rA} - \frac{2}{r^3}D^C\bar{h}_{CA} - \frac{2}{r}D_A\left(\bar{h}_{ru} - \bar{h}_{rr}\right) \\
& \qquad +\frac{D}{r^2}\left(\bar{h}_{Au} - \bar{h}_{Ar}\right) + \frac{1}{r^2}\bar{h}_{rA} - \frac{(D-2)}{r^2}\bar{h}_{rA}, \\
\Box\bar{h}_{AB} &= \left(\partial_r^2 - 2\partial_r\partial_u + \frac{1}{r^2}\Delta\right)\bar{h}_{AB} - \frac{2}{r}D_A\left(\bar{h}_{uB} - \bar{h}_{rB}\right) - \frac{2}{r}D_B\left(\bar{h}_{uA} - \bar{h}_{rA}\right) - \frac{2(D-4)}{r^2}\bar{h}_{AB} + \\
& \qquad \frac{(D-6)}{r}\partial_r\bar{h}_{AB} - \frac{(D-6)}{r}\partial_u\bar{h}_{AB} + 2\gamma_{AB}\left(\bar{h}_{uu} - 2\bar{h}_{ur} + \bar{h}_{rr}\right). \quad (5.1.22)
\end{aligned}
$$

From the above equations it can be concluded that not all the components of the metric perturbations can be solved independently. We can perturbatively expand the metric fluctuations in terms of radial coordinate off the null infinity. Then some components will specify the free data and other components can be determined in terms of these free data. We will see some examples later.

5.1.4 Massless scalar field

For the matter contribution we will consider massless scalar field minimally coupled to gravity whose action is given by

$$S_{\text{matter}} = -\frac{1}{2}\int \mathrm{d}^D x\,\sqrt{-g}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi. \quad (5.1.23)$$

The scalar field will satisfy Klein Gordon equation given by $\Box\Phi = 0$ which can be expressed as

$$\left[\partial_r^2 - 2\partial_u\partial_r + \frac{(D-2)}{r}\left(\partial_r - \partial_u\right) + \frac{1}{r^2}\Delta\right]\Phi = 0. \quad (5.1.24)$$

From this action (5.1.23) we can calculate the stress energy tensor to be

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

$$= \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi. \tag{5.1.25}$$

Components of $T_{\mu\nu}$ for the scalar field Φ are then given by

$$\begin{aligned}
T_{uu} &= (\partial_u \Phi)^2 + \frac{1}{2}\left[-2\partial_u \Phi \partial_r \Phi + (\partial_r \Phi)^2 + \frac{1}{r^2} q^{AB} \partial_A \Phi \partial_B \Phi\right], \\
T_{ur} &= \frac{1}{2}\left[(\partial_r \Phi)^2 + \frac{1}{r^2} q^{AB} \partial_A \Phi \partial_B \Phi\right], \\
T_{uA} &= \partial_u \Phi \partial_A \Phi, \\
T_{rr} &= (\partial_r \Phi)^2, \\
T_{rA} &= \partial_r \Phi \partial_A \Phi, \\
T_{AB} &= \partial_A \Phi \partial_B \Phi - \frac{1}{2} r^2 q_{AB}\left[-2\partial_u \Phi \partial_r \Phi + (\partial_r \Phi)^2 + \frac{1}{r^2} q^{AB} \partial_A \Phi \partial_B \Phi\right].
\end{aligned} \tag{5.1.26}$$

It can be checked that in D dimension free data has a fall off behavior of $r^{-\frac{(D-2)}{2}}$ at asymptotic null infinity. Therefore we can make an expansion of the scalar field as

$$\Phi = \frac{\phi}{r^{\frac{(D-2)}{2}}} + \ldots \tag{5.1.27}$$

where ϕ specifies the free data. Therefore the fall off conditions of the components of stress energy tenor can be given by

$$\begin{aligned}
&T_{uu} \sim r^{-2m}, \qquad T_{ur} \sim r^{-(2m+2)}, \qquad T_{rr} \sim r^{-(2m+2)}, \\
&T_{uA} \sim r^{-2m}, \qquad T_{rA} \sim r^{-(2m+1)}, \qquad T_{AB} \sim r^{-(2m-1)}.
\end{aligned} \tag{5.1.28}$$

Here we have denoted the dimension of spacetime as $D = 2 + 2m$ for convenience. This convention will be helpful for later sections where we will mainly focus on even spacetime dimensions.

5.2 Asymptotic conserved charges

Given a vector field ξ^μ we can derive a conserved charge Q_ξ at asymptotic null infinity using the covariant phase space formalism. We will consider massless scalar field minimally coupled to gravity in $D = 2 + 2m$ dimension. The radiative phase space at null infinity is of the form $\Gamma = \Gamma_{\text{grav}} + \Gamma_\Phi$. The covariant charge will consist of two parts - 1) *soft charge* which is due to the gravitational radiation and corresponds to the metric fluctuations that satisfy homogeneous linearized gravity equation, and 2) *hard charge* which comes from the matter contribution, massless scalar in this case.

5.2.1 Gravitational charge

We will follow the approach of [60] to derive the gravitational contribution to the asymptotic conserved charge. In [60] soft charge for large gauge transformations corresponding to supertranslation and superroation vector fields in four dimensional have been derived. Here we will consider only supertranslation vector fields and generalize the analysis to arbitrary dimensions.

Einstein-Hilbert action in D dimension is given by

$$S_{\text{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^D x\,\sqrt{-g}R, \tag{5.2.1}$$

where R is the Ricci scalar.

Variation of this action (5.2.1) is given by

$$\delta S_{\text{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^D x\,\sqrt{-g}\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right]. \tag{5.2.2}$$

The first term of Eq.(5.2.2) inside the parentheses yields Einstein's field equation. Variation of Ricci tensor is given by

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}. \tag{5.2.3}$$

Therefore the remaining term in Eq.(5.2.2) produces a boundary term which can be expressed as

$$\begin{aligned}
[\delta S_{EH}]_{\text{b'dy}} &= \frac{1}{16\pi G}\int \mathrm{d}^D x \sqrt{-g} g^{\mu\nu}\left(\nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}\right) \\
&= \frac{1}{16\pi G}\int \mathrm{d}^D x \sqrt{-g} \nabla_\lambda \left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu}\right) \\
&= \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^{D-1} x \sqrt{|\hat{g}|} n_\mu \left(\hat{g}^{\nu\lambda}\delta\Gamma^\mu_{\nu\lambda} - \hat{g}^{\mu\lambda}\delta\Gamma^\nu_{\nu\lambda}\right),
\end{aligned} \tag{5.2.4}$$

where $\hat{g}_{\mu\nu}$ is the induced metric on the Σ hypersurface and n_μ is normal to the hypersurface.

From the boundary term we can define a *symplectic potential density* as follows

$$\theta^\mu(\delta) := \frac{1}{16\pi G}\sqrt{|\hat{g}|}\left(\hat{g}^{\nu\lambda}\delta\Gamma^\mu_{\nu\lambda} - \hat{g}^{\mu\lambda}\delta\Gamma^\nu_{\nu\lambda}\right). \tag{5.2.5}$$

We can then define a covariant phase space charge for the vector field ξ^μ on the Σ hypersurface as

$$\delta Q_\xi = \int_\Sigma \mathrm{d}S_\mu \left[\delta\theta^\mu\left(\delta_\xi\right) - \delta_\xi \theta^\mu(\delta)\right]. \tag{5.2.6}$$

Here δ is any arbitrary variation while δ_ξ is the variation induced by the vector field ξ^μ.

We will choose Σ to be a constant time hypersurface and then take the limit $t \to \infty$ while keeping u to be constant. In this way we can define the phase space structure at future null infinity $\mathscr{I}^+$.

Keeping terms to linear order in metric perturbations, variation of Christoffel symbols can be expressed as

$$\delta\Gamma^\mu_{\nu\lambda} = \frac{1}{2}\eta^{\mu\sigma}\left(\nabla_\nu \delta h_{\sigma\lambda} + \nabla_\lambda \delta h_{\sigma\nu} - \nabla_\sigma \delta h_{\nu\lambda}\right). \tag{5.2.7}$$

Using Eq.(5.2.7) we can show

$$\begin{aligned}
\delta\theta^{\mu}\left(\delta_{\xi}\right) &= \frac{1}{16\pi G}\left[-\delta\bar{h}^{\nu\lambda}\delta_{\xi}\Gamma^{\mu}_{\nu\lambda} + \delta\bar{h}^{\mu\lambda}\partial_{\lambda}\left(\nabla_{\alpha}\xi^{\alpha}\right)\right], \\
\delta_{\xi}\theta^{\mu}\left(\delta\right) &= -\frac{1}{16\pi G}\left[\frac{1}{2m}\delta_{\xi}\bar{h}^{\mu\lambda}\partial_{\lambda}\delta\bar{h} + \delta_{\xi}\bar{h}^{\nu\lambda}\delta\Gamma^{\mu}_{\nu\lambda}\right].
\end{aligned} \tag{5.2.8}$$

To obtain these equations we have used $\delta\Gamma^{\nu}_{\nu\lambda} = \frac{1}{2}\partial_{\lambda}\delta h$ and $\bar{h} = -mh$.

Now if we substitute Eq.(5.2.8) in Eq.(5.2.6) and neglect terms beyond linear order in variations then we obtain an expression which is a total variation in δ. We can eventually derive the covariant charge at null infinity which is given by

$$Q^{grav}_{\xi} = \lim_{t\to\infty}\frac{1}{16\pi G}\int_{\mathscr{I}^{+}} du d\Omega_{2m}\, r^{2m}\left[\Gamma^{t}_{\nu\lambda}\delta_{\xi}\bar{h}^{\nu\lambda} - \bar{h}^{\nu\lambda}\delta_{\xi}\Gamma^{t}_{\nu\lambda} + \frac{1}{2m}\delta_{\xi}\bar{h}^{t\lambda}\partial_{\lambda}\bar{h} + \bar{h}^{t\lambda}\partial_{\lambda}\left(\nabla_{\alpha}\xi^{\alpha}\right)\right]. \tag{5.2.9}$$

With appropriate fall off conditions imposed on the metric perturbations there will be a finite contribution to Q^{grav}_{ξ} which will then give the desired gravitational charge at asymptotic null infinity.

Large gauge transformation The diffeomorphisms which induce non-trivial transformations at null infinity are called large gauge transformations. The covariant phase space charges at null infinity vanishes for residual diffeomorphisms implying the fact these gauge transformations are pure and physically irrelevant. On the contrary large gauge transformations, if exist, will contribute to non trivial conserved charges at asymptotic null infinity. Such transformations are physical because they relate one space time to another which are physically distinct.

5.2.2 Matter charge

We will calculate the conserved Noether's charge for the matter content. Let us denote the matter action by S_{matter}, then the stress energy tensor is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{5.2.10}$$

Under diffeomorphism metric transforms as

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{5.2.11}$$

Therefore variation of the action gives a boundary term of the form

$$\begin{aligned}
\delta S_{\text{matter}} &= -\frac{1}{2}\int \mathrm{d}^D x\,\sqrt{-g}\delta_\xi g^{\mu\nu} T_{\mu\nu} \\
&= -\frac{1}{2}\int \mathrm{d}^D x\,\sqrt{-g}\,(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu)\,T_{\mu\nu} \\
&= -\int \mathrm{d}^D x\,\sqrt{-g}\nabla^\mu\left(\xi^\nu T_{\mu\nu}\right) \\
&= -\int_\Sigma \mathrm{d}^{D-1} x n_\mu\,\sqrt{|\hat{g}|} T^\mu{}_\nu \xi^\nu.
\end{aligned} \tag{5.2.12}$$

In the third step we have used conservation of stress energy tensor, $\nabla_\mu T^{\mu\nu} = 0$. Choosing a constant time hypersurface and taking the limit $t \to \infty$ we can derive conserved charge at asymptotic null infinity as

$$Q_\xi^{\text{matter}} = -\lim_{t\to\infty}\int_{\mathscr{I}^+} \mathrm{d}u\mathrm{d}\Omega_{2m} r^{2m} T^t{}_\nu \xi^\nu. \tag{5.2.13}$$

It can be checked easily

$$T^t{}_\mu = T^u{}_\mu + T^r{}_\mu = -T_{u\mu}. \tag{5.2.14}$$

Given the form of the vector field ξ^μ and using the expressions of stress energy tensors for massless scalar field in Eq.(5.1.26) we can then calculate matter contribution to the asymptotic charge.

5.3 Supertranslation symmetries in four dimension

In addition to the space time translations there are also infinite number of supertranslation symmetries present in four dimensional asymptotically flat spacetimes. These supertranslation vector fields are arbitrary functions of $\mathbb{S}^2$ coordinates and produce angle dependent translations at null infinity.

A two dimensional sphere is parametrized by angular coordinates (θ, ϕ). Using stereographic projections one can map any point on $\mathbb{S}^2$ to a corresponding point on complex plane parametrized by $(z, \bar{z})$ coordinates. The mapping can be given by

$$z = \tan\frac{\theta}{2} e^{i\phi} \qquad \text{and} \qquad \bar{z} = \tan\frac{\theta}{2} e^{-i\phi}. \tag{5.3.1}$$

The mapping between the Euclidean coordinates (x^1, x^2, x^3) and $(z, \bar{z})$ can be expressed as

$$x^1 + ix^2 = \frac{2rz}{1 + z\bar{z}}, \qquad x^3 = \frac{r(1 - z\bar{z})}{1 + z\bar{z}}, \tag{5.3.2}$$

such that $\vec{x} \cdot \vec{x} = r^2$.

Having set this parametrization we will use $(u, z, \bar{z})$ coordinates for $\mathscr{I}^+$.

5.3.1 Gravitational radiation

We can use the residual gauge conditions to fix some components of the metric perturbations. Since our goal is to derive the covariant phase space charge which is gauge independent so residual gauge fixing does not affect our result and eventually simplifies the computations. In particular we will choose *radiation gauge* for the free metric perturbations $h^{(C)}_{\mu\nu}$ (we will drop the superscript C for convenience). Radiation gauge choice can be expressed by the following conditions

$$h_{\mu u} = 0,\ \mu \in \{r, z, \bar{z}\} \qquad \text{and} \qquad \eta^{\mu\nu} h_{\mu\nu} = 0. \tag{5.3.3}$$

It can be checked that from the equation[1] $\Box h_{uu} = 0$ it also follows that $h_{uu} = 0$.

We can consistently solve the linearized gravity equations (5.1.22) in vacuum by imposing the following fall off conditions

$$\begin{aligned} h_{AB} &= r h^{(-1)}_{AB} + h^{(0)}_{AB} + \dots \qquad A, B \in (z, \bar{z}) \\ h_{Ar} &= \frac{1}{r} h^{(1)}_{Ar} + \frac{1}{r^2} h^{(2)}_{Ar} + \dots \\ h_{rr} &= \frac{1}{r^3} h^{(3)}_{rr} + \frac{1}{r^4} h^{(4)}_{rr} + \dots \end{aligned} \tag{5.3.4}$$

Then we obtain

$$\begin{aligned} \partial_u h^{(0)}_{AB} &= \left(-\frac{1}{2}\Delta + 1\right) h^{(-1)}_{AB}, \\ \partial_u h^{(1)}_{AB} &= -\frac{1}{4}\Delta h^{(0)}_{AB} - \frac{1}{2}\left(D_A h^{(1)}_{Br} + D_B h^{(1)}_{Ar}\right), \\ \partial_u h^{(1)}_{Ar} &= D^B h^{(-1)}_{AB}, \\ \partial_u h^{(2)}_{Ar} &= \frac{1}{4}(3 - \Delta) h^{(1)}_{Ar} + \frac{1}{2} D^C h^{(0),}_{AC} \\ \partial_u h^{(3)}_{rr} &= D^A h^{(1)}_{Ar}. \end{aligned} \tag{5.3.5}$$

From the above equations it can be inferred that free data can be specified by the components $h^{(-1)}_{AB}$ and all other components of metric fluctuations can be determined in terms of $h^{(-1)}_{AB}$. Therefore $h^{(-1)}_{AB}$ correspond to the graviton modes at $\mathscr{I}^+$ which we will denote by C_{zz} and $C_{\bar{z}\bar{z}}$.

[1] $\Box h_{uu} = \left[\partial_r^2 - 2\partial_u\partial_r - \frac{2}{r}(\partial_u - \partial_r) + \frac{1}{r^2}\Delta\right] h_{uu} = 0$. So h_{uu} can not be expressed in terms of radiative data $h^{(-1)}_{AB}$.

Saddle point analysis We can also determine the radiative data from the gravitational radiative phase space defined at null infinity. Let us consider the mode expansions of graviton in terms of creation and annihilation operators given by

$$h_{\mu\nu}(x) = \sum_{\alpha=\pm} \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\omega_q} \left[\varepsilon^{\alpha\,*}_{\mu\nu}(\vec{q})\, a^{\rm out}_{\alpha}(\vec{q})\, e^{iq\cdot x} + \varepsilon^{\alpha}_{\mu\nu}(\vec{q})\, a^{{\rm out}\,\dagger}_{\alpha}(\vec{q})\, e^{-iq\cdot x}\right], \tag{5.3.6}$$

where α denotes the helicity of the graviton and ω_q is the energy. Since we are interested in the phase space at null infinity we have to do a stationary phase approximation of this mode expansion as $r \to \infty$. In the retarded time coordinate $q \cdot x = -\omega_q u - \omega_q r\,(1 - \cos\theta)$, therefore saddle points are at $\theta = 0$ and π. But at $\theta = \pi$ the integrand is oscillating and therefore vanishes at large r. Dominant contribution to the saddle point approximation comes from $\theta = 0$ and is given by

$$\lim_{r\to\infty} h_{\mu\nu}(x) = -\frac{i}{8\pi^2 r} \sum_{\alpha=\pm} \int_0^\infty d\omega_q \left[\varepsilon^{\alpha\,*}_{\mu\nu}(\hat{q})\, a^{\rm out}_{\alpha}\left(\omega_q \hat{q}\right) e^{-i\omega u} - \varepsilon^{\alpha}_{\mu\nu}(\hat{q})\, a^{{\rm out}\,\dagger}_{\alpha}\left(\omega_q \hat{q}\right) e^{i\omega u}\right]. \tag{5.3.7}$$

Therefore it is immediately obvious that radiative data falls off as r^{-1} in four dimension and should be identified with $h^{(-1)}_{AB}$.

Now taking the projections of $h_{\mu\nu}$ on the $\mathbb{S}^2$ at null infinity we can write

$$C_{zz}(u, z, \bar{z}) = \kappa \lim_{r\to\infty} \frac{1}{r} \partial_z x^\mu \partial_z x^\nu h^{\rm out}_{\mu\nu}(x). \tag{5.3.8}$$

Here κ is the gravitational coupling constant and is given by $\kappa = \sqrt{32\pi G}$.

We can parametrize the null vectors in terms of the $\mathbb{S}^2$ coordinates as

$$\begin{aligned} x^\mu(z,\bar{z}) &= r\left(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i\,(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right), \\ q^\mu(\omega,\bar{\omega}) &= \omega_q\left(1, \frac{\omega+\bar{\omega}}{1+\omega\bar{\omega}}, \frac{-i\,(\omega-\bar{\omega})}{1+\omega\bar{\omega}}, \frac{1-\omega\bar{\omega}}{1+\omega\bar{\omega}}\right), \\ \varepsilon^{+\mu}(\vec{q}) &= \frac{1}{\sqrt{2}}\left(\bar{\omega}, 1, -i, -\bar{\omega}\right), \qquad \varepsilon^{+}(\vec{q})^{*} = \varepsilon^{-}(\vec{q}). \end{aligned} \tag{5.3.9}$$

Here the graviton polarizations are given by $\varepsilon^{\alpha}_{\mu\nu} = \varepsilon^{\alpha}_{\mu}\varepsilon^{\alpha}_{\nu}$. Then we get $\varepsilon^{\alpha}_{z} = \partial_z x^{\mu}\varepsilon^{\alpha}_{\mu}$. Therefore

$$\varepsilon^{+}_{z}(\vec{q}) = \frac{\sqrt{2}r\bar{z}(\bar{\omega}-\bar{z})}{(1+z\bar{z})^2}, \qquad \varepsilon^{-}_{z}(\vec{q}) = \frac{\sqrt{2}r(1+\omega\bar{z})}{(1+z\bar{z})^2}. \tag{5.3.10}$$

In the large r limit $\omega \to z$, hence ε^{+}_{z} vanishes. Therefore we can express C_{zz} as

$$C_{zz}(u,z,\bar{z}) = -\frac{i\kappa}{4\pi^2(1+z\bar{z})^2}\int_0^{\infty} \mathrm{d}\omega_q \left[a^{\text{out}}_{+}\left(\omega_q\hat{x}\right)e^{-i\omega_q u} - a^{\text{out}\,\dagger}_{-}\left(\omega_q\hat{x}\right)e^{i\omega_q u}\right]. \tag{5.3.11}$$

Similarly we can find out the mode expansions of $C_{\bar{z}\bar{z}}$ at $\mathscr{I}^+$. At past null infinity $\mathscr{I}^-$ the graviton modes will be denoted by D_{zz} and $D_{\bar{z}\bar{z}}$.

Next we will study how these radiative data transform under large diffeomorphisms, particularly under supertranslations. It is to be noted that the global Poincare transformations which include space time translations and Lorentz transformations do not shift the radiative data.

5.3.2 Supertranslations

Here we will derive the solution for ξ^{μ} which induces large diffeomorphisms at $\mathscr{I}^+$. Like the components of metric perturbations we can expand ξ^{μ} in orders of radial components as follows

$$\begin{aligned}
\xi^{u} &= \xi^{u\,(0)} + \frac{1}{r}\xi^{u\,(1)} + \ldots, \\
\xi^{r} &= \xi^{r\,(0)} + \frac{1}{r}\xi^{r\,(1)} + \ldots, \\
\xi^{A} &= \frac{1}{r}\xi^{A\,(1)} + \frac{1}{r^2}\xi^{A\,(2)} + \ldots, \qquad A \in (z,\bar{z}).
\end{aligned} \tag{5.3.12}$$

With this ansatz we will solve for Eq.(5.1.19) under the gauge fixing conditions chosen in Eq.(5.3.3).

Since the trace of the metric perturbations, h is set to zero therefore we have

$$\nabla_\mu \xi^\mu = 0 \qquad \Rightarrow \qquad \partial_u \xi^u + \partial_r \xi^r + \frac{2}{r}\xi^r + D_A \xi^A = 0. \tag{5.3.13}$$

This implies $\partial_u \xi^{u\,(0)} = 0$. Therefore we see $\xi^{u\,(0)}$ is an arbitrary function of $\mathbb{S}^2$ coordinates and can be expressed as

$$\xi^{u\,(0)} = f(z, \bar{z}). \tag{5.3.14}$$

Now from the fall off condition $h_{Ar} \sim O\left(r^{-1}\right)$ we can determine that

$$\mathcal{L}_\xi g_{Ar} = r^2 \gamma_{AC} \partial_r \xi^C - D_A \xi^u \qquad \Rightarrow \qquad \xi^{A\,(1)} = -D^A f(z, \bar{z}). \tag{5.3.15}$$

Again from the equation $\Box \xi^u = 0$ we can obtain

$$\xi^{r\,(0)} = \frac{1}{2} \Delta f(z, \bar{z}). \tag{5.3.16}$$

Therefore the vector field producing large gauge transformations can be given by

$$\vec{\xi} = f(z, \bar{z})\, \partial_u + \frac{1}{2} \Delta f(z, \bar{z})\, \partial_r - \frac{1}{r} D^A f(z, \bar{z})\, \partial_A + \ldots \tag{5.3.17}$$

The ellipses denote the subleading components which can be determined by the residual gauge conditions, but will not be required for our analysis.

It can be checked that the vector field obtained in Eq.(5.3.17) acts at $\mathscr{I}^+$ and transforms the null coordinates as follows

$$u \to u + f(z, \bar{z}), \qquad z \to z, \qquad \bar{z} \to \bar{z}. \tag{5.3.18}$$

So these are angle dependent translations along retarded time coordinates keeping the angular coordinates fixed and are called supertranslations.

Global Poincare translations of the form $x^\mu \to x^\mu + l^\mu$ can be realized at $\mathscr{I}^+$ by the following functional form of $f(z,\bar{z})$

$$f(z,\bar{z}) = \frac{\left(l^0 - l^3\right) - \left(l^1 - il^2\right) z - \left(l^1 + il^2\right) \bar{z} + \left(l^0 + l^3\right) z\bar{z}}{1 + z\bar{z}}. \tag{5.3.19}$$

Now let us find out how the free data C_{zz} and $C_{\bar{z}\bar{z}}$ shift under supertranslations given in Eq.(5.3.17). From the relation

$$\mathcal{L}_\xi g_{AB} = r^2 \left(\gamma_{AC} D_B + \gamma_{BC} D_A\right) \xi^C + 2r\gamma_{AB}\xi^r, \qquad A, B \in (z, \bar{z}) \tag{5.3.20}$$

it is easy to see that

$$\begin{aligned} \delta_f C_{zz} &= -2D_z^2 f(z,\bar{z}), \\ \delta_f C_{\bar{z}\bar{z}} &= -2D_{\bar{z}}^2 f(z,\bar{z}). \end{aligned} \tag{5.3.21}$$

Therefore under supertranslations vacuum of one asymptotically flat spacetime which is characterized by the radiative data is mapped to the vacuum of another asymptotically flat spacetime. This is the reason of enhanced infinite dimensional asymptotic symmetry group for asymptotically flat spacetimes in four dimensions. It can be easily checked that for Poincare translations there is no shift in the radiative data of the gravitational radiation.

5.3.3 Weinberg's soft graviton theorem from supertranslations

We will now study how Weinberg's soft graviton theorem is related to supertranslation symmetries in four dimension. Here we review the analysis of [3, 56]. In Sec.(5.2.1) we have derived the covariant phase space charge at asymptotic null infinity in $D = 2 + 2m$ dimensions. Here we will use the expression of the asymptotic charge given in Eq.(5.2.9) to calculate the charges corresponding to supertranslation symmetries in four dimension. This charge acts on the Fock states of gravitational radiative phase space at null infinity and creates a soft graviton. However it is to be remembered that there are infinite number of asymptotic charges giving rise to infinite number of soft gravitons at null infinity.

Soft charge We will use the radiation gauge condition given in Eq.(5.3.3). Then last two terms in Eq.(5.2.9) vanish and we get

$$Q_\xi^{grav\,+} = \lim_{t\to\infty} \frac{1}{16\pi G} \int_{\mathscr{I}^+} du d\Omega_2 \; r^2 \left[\Gamma^t_{\nu\lambda} \delta_\xi h^{\nu\lambda} - h^{\nu\lambda} \delta_\xi \Gamma^t_{\nu\lambda} \right]. \tag{5.3.22}$$

Let us now calculate each of these terms separately.

- First term:

$$\begin{aligned} \Gamma^t_{\mu\nu} &= \Gamma^u_{\mu\nu} + \Gamma^r_{\mu\nu} \\ &= \frac{1}{2} \left(\eta^{u\sigma} + \eta^{r\sigma}\right) \left(\nabla_\mu h_{\sigma\nu} + \nabla_\nu h_{\sigma\mu} - \nabla_\sigma h_{\mu\nu} \right) \\ &= \frac{1}{2} \nabla_u h_{\mu\nu} \\ &= \frac{1}{2} \partial_u h_{\mu\nu}. \end{aligned} \tag{5.3.23}$$

So the non zero components are

$$\begin{aligned} \Gamma^t_{rr} &= \frac{1}{2} \partial_u h_{rr} \sim \mathcal{O}\left(r^{-3}\right), \\ \Gamma^t_{rA} &= \frac{1}{2} \partial_u h_{rA} \sim \mathcal{O}\left(r^{-1}\right), \\ \Gamma^t_{AB} &= \frac{1}{2} \partial_u h_{AB} \sim \mathcal{O}\left(r\right). \end{aligned} \tag{5.3.24}$$

Again we have

$$\begin{aligned} \delta_\xi h^{rr} &= 2 \left(\partial_r - \partial_u\right) \xi^r, \\ \delta_\xi h^{Ar} &= \left(\partial_r - \partial_u\right) \xi^A + \frac{1}{r^2} D^A \xi^r, \\ \delta_\xi h^{AB} &= \frac{1}{r^2} \left(D^A \xi^B + D^B \xi^A \right) + \frac{2}{r^3} \gamma^{AB} \xi^r. \end{aligned} \tag{5.3.25}$$

Therefore combining Eq.(5.3.24) and Eq.(5.3.25) we can obtain

$$r^2\Gamma^t_{\nu\lambda}\delta_\xi h^{\nu\lambda} = -\partial_u h^{(-1)}_{AB} D^A D^B f + O\left(r^{-1}\right) \tag{5.3.26}$$

- Second term:

$$\begin{aligned} h^{rr} &= h^{uu} = -h^{ru} = h_{rr} \sim O\left(r^{-3}\right), \\ h^{rA} &= -h^{uA} = \frac{1}{r^2}\gamma^{AB}h_{rB} \sim O\left(r^{-1}\right), \\ h^{AB} &= \frac{1}{r^4}\gamma^{AM}\gamma^{BN}h_{MN} \sim O\left(r^{-3}\right). \end{aligned} \tag{5.3.27}$$

Variation of Christoffel symbols are given by

$$\begin{aligned} \delta_\xi \Gamma^t_{ur} &= \partial_r\partial_u\left(\xi^u + \xi^r\right), \\ \delta_\xi \Gamma^t_{uu} &= \partial_u^2\left(\xi^u + \xi^r\right), \\ \delta_\xi \Gamma^t_{uA} &= D_A\partial_u\left(\xi^u + \xi^r\right), \\ \delta_\xi \Gamma^t_{rr} &= \partial_r^2\left(\xi^u + \xi^r\right), \\ \delta_\xi \Gamma^t_{Ar} &= r\partial_r\left(\frac{1}{r}D_A\left(\xi^u + \xi^r\right)\right), \\ \delta_\xi \Gamma^t_{AB} &= D_A D_B\left(\xi^u + \xi^r\right) + r\gamma_{AB}\left(\partial_r - \partial_u\right)\left(\xi^u + \xi^r\right). \end{aligned} \tag{5.3.28}$$

Combining Eq.(5.3.27) and Eq.(5.3.28) we find there is no finite contribution to the term $r^2 h^{\nu\lambda}\delta_\xi\Gamma^t_{\nu\lambda}$ in the limit $r \to \infty$.

Finally we find the soft part of the asymptotic conserved charge at $\mathscr{I}^+$ to be given by

$$Q^{grav\,+}_\xi = -\frac{1}{16\pi G}\int_{\mathscr{I}^+} du d^2z\,\sqrt{\gamma}\left[D^z D^z f(z,\bar z)\,\partial_u C_{zz} + D^{\bar z} D^{\bar z} f(z,\bar z)\,\partial_u C_{\bar z\bar z}\right]. \tag{5.3.29}$$

Here γ is the determinant of the metric on $\mathbb{S}^2$ whose components are given by

$$\gamma_{z\bar z} = \frac{2}{(1+z\bar z)^2}, \qquad \gamma^{z\bar z} = \frac{(1+z\bar z)^2}{2}. \tag{5.3.30}$$

Similarly we can derive the asymptotic conserved gravitational charge at past null infinity $\mathscr{I}^-$.

Soft graviton theorem We now consider S-matrix with the soft charge inserted in it. $Q_\xi^{\text{grav}\,+}$ acts on the asymptotic scattering out-states at $\mathscr{I}^+$ and $Q_\xi^{\text{grav}\,-}$ acts on the in-states at $\mathscr{I}^-$. We will now calculate the quantity $\langle\text{out}|\left[Q_\xi^{\text{grav}}, S\right]|\text{in}\rangle$.

Let us recall that at $\mathscr{I}^+$ graviton mode expansion can be given by

$$C_{zz}(u, z, \bar{z}) = -\frac{i\kappa}{4\pi^2(1+z\bar{z})^2}\int_0^\infty \mathrm{d}\omega_q \left[a_+^{\text{out}}\left(\omega_q\hat{x}\right)e^{-i\omega_q u} - a_-^{\text{out}\,\dagger}\left(\omega_q\hat{x}\right)e^{i\omega_q u}\right]. \tag{5.3.31}$$

which implies

$$\int_{-\infty}^\infty \mathrm{d}u\,\partial_u C_{zz} = -\frac{\kappa}{2\pi(1+z\bar{z})^2}\lim_{\omega\to 0}\int_0^\infty \mathrm{d}\omega_q\left[\omega_q a_+^{\text{out}}\left(\omega_q\hat{x}\right)\delta\left(\omega_q-\omega\right) + \omega_q a_-^{\text{out}\,\dagger}\left(\omega_q\hat{x}\right)\delta\left(\omega_q+\omega\right)\right]. \tag{5.3.32}$$

We now define *News tensor* as

$$N_{zz}^\omega(z, \bar{z}) := \int_{-\infty}^\infty \mathrm{d}u e^{i\omega u}\partial_u C_{zz}. \tag{5.3.33}$$

In the limit $\omega \to 0$ the News tensor can be defined in a hermitian way as

$$N_{zz}^0(z, \bar{z}) := \lim_{\omega\to 0}\frac{1}{2}\left(N_{zz}^\omega + N_{zz}^{-\omega}\right), \qquad \text{with} \quad \omega > 0. \tag{5.3.34}$$

So the zero mode of the News tensor can be expressed as

$$N_{zz}^0(z, \bar{z}) = -\frac{\kappa}{4\pi(1+z\bar{z})^2}\lim_{\omega\to 0}\omega\left[a_+^{\text{out}}(\omega\hat{x}) + a_-^{\text{out}\,\dagger}(\omega\hat{x})\right]. \tag{5.3.35}$$

Annihilation operator a acting on the radiative phase space annihilates the vacuum, $a|0\rangle =$

0. Therefore we get the soft mode insertion from

$$\begin{aligned}
\lim_{\omega\to 0}\omega\langle z_1^{\text{out}}\dots|a_+^{\text{out}}S|z_1^{\text{in}}\dots\rangle &= \lim_{\omega\to 0}\frac{\kappa\omega}{2}\left[\sum_{k\in\text{out}}\frac{p_k^\mu p_k^\nu\varepsilon^+_{\mu\nu}(\vec{q})}{p_k\cdot q}-\sum_{k'\in\text{in}}\frac{p_{k'}^\mu p_{k'}^\nu\varepsilon^+_{\mu\nu}(\vec{q})}{p_{k'}\cdot q}\right]\langle z_1^{\text{out}}\dots|S|z_1^{\text{in}}\dots\rangle\\
&= -\frac{\kappa}{2}\left[\sum_{k\in\text{out}}E_k\frac{(1+z\bar{z})(\bar{z}-\bar{z}_k)}{(1+z_k\bar{z}_k)(z-z_k)}-\sum_{k'\in\text{in}}E_{k'}\frac{(1+z\bar{z})(\bar{z}-\bar{z}_{k'})}{(1+z_{k'}\bar{z}_{k'})(z-z_{k'})}\right]\\
&\qquad\langle z_1^{\text{out}}\dots|S|z_1^{\text{in}}\dots\rangle.
\end{aligned}\tag{5.3.36}$$

To get the last equality we have used the parametrization of finite-energy momentum vector as

$$p_k(z_k,\bar{z}_k)=E_k\left(1,\frac{z_k+\bar{z}_k}{1+z_k\bar{z}_k},\frac{-i(z_k-\bar{z}_k)}{1+z_k\bar{z}_k},\frac{1-z_k\bar{z}_k}{1+z_k\bar{z}_k}\right).\tag{5.3.37}$$

Eq.(5.3.36) is the statement of Weinberg's soft graviton theorem for a single soft outgoing graviton with positive helicity. Negative helicity soft graviton theorem will follow from considering $C_{\bar{z}\bar{z}}$. Because of *Christodolou-Klainerman constraints* [102] S-matrix with a soft graviton of positive helicity insertion is same as that with a soft graviton of negative helicity insertion. As we will see later this condition ensures that same Ward identities hold for both positive and negative helicities of soft graviton.

Therefore we get

$$\begin{aligned}
&\langle z_1^{\text{out}}\dots|Q_\xi^{\text{grav}\,+}S|z_1^{\text{in}}\dots\rangle\\
&= -\frac{1}{16\pi G}\langle z_1^{\text{out}}\dots|\int_{\mathscr{I}^+}du\,d^2z\,\sqrt{\gamma}\left[D^zD^zf(z,\bar{z})\,\partial_uC_{zz}+D^{\bar{z}}D^{\bar{z}}f(z,\bar{z})\,\partial_uC_{\bar{z}\bar{z}}\right]S|z_1^{\text{in}}\dots\rangle\\
&= -\frac{1}{8\pi G}\langle z_1^{\text{out}}\dots|\left(\int_{\mathbb{S}^2}d^2z\,\sqrt{\gamma}D^zD^zf(z,\bar{z})\,N^0_{zz}\right)S|z_1^{\text{in}}\dots\rangle\\
&= \frac{1}{8\pi G}\langle z_1^{\text{out}}\dots|\left(\int_{\mathbb{S}^2}d^2z\,\sqrt{\gamma}D^zD^zf(z,\bar{z})\,\frac{\kappa}{4\pi(1+z\bar{z})^2}\lim_{\omega\to 0}\omega\left[a_+^{\text{out}}(\omega\hat{x})+a_-^{\text{out}\,\dagger}(\omega\hat{x})\right]\right)S|z_1^{\text{in}}\dots\rangle\\
&= -\frac{\kappa^2}{64\pi G}\int d^2z\,f(z,\bar{z})\,D^2_{\bar{z}}\left[\sum_{k\in\text{out}}E_k\frac{(1+z\bar{z})(\bar{z}-\bar{z}_k)}{(1+z_k\bar{z}_k)(z-z_k)}-\sum_{k'\in\text{in}}E_{k'}\frac{(1+z\bar{z})(\bar{z}-\bar{z}_{k'})}{(1+z_{k'}\bar{z}_{k'})(z-z_{k'})}\right]
\end{aligned}$$

$$\times\langle z_1^{\text{out}}\ldots|S|z_1^{\text{in}}\ldots\rangle$$

$$= -\frac{1}{2}\left[\sum_{k=\in\text{out}} E_k f(z_k,\bar{z}_k) - \sum_{k'\in\text{in}} E_{k'} f(z_{k'},\bar{z}_{k'})\right]\langle z_1^{\text{out}}\ldots|S|z_1^{\text{in}}\ldots\rangle, \tag{5.3.38}$$

where we have used the identity [57] (see Sec.(C.1) for details)

$$D_{\bar{z}}^2\left[\frac{(1+z\bar{z})(\bar{z}-\bar{z}_k)}{(1+z_k\bar{z}_k)(z-z_k)}\right] = 2\pi\delta^{(2)}(z-z_k)\,. \tag{5.3.39}$$

A similar construction for the asymptotic charge $Q_\xi^{\text{grav}^-}$ acting on the ingoing scattering states, using metric component D_{zz} in the advanced coordinate ($v = t + r$) can be done on past null infinity $\mathscr{I}^-$. Taking the time ordered product of the soft charge with S-matrix we can show that gravitational part of the asymptotic charge satisfies the following identity

$$\langle\text{out}|Q_\xi^{\text{grav}\,+}S - S\,Q_\xi^{\text{grav}\,-}|\text{in}\rangle = -\left[\sum_{k\in\text{out}} E_k f(z_k,\bar{z}_k) - \sum_{k'\in\text{out}} E_{k'} f(z_{k'},\bar{z}_{k'})\right]\langle\text{out}|S|\text{in}\rangle. \tag{5.3.40}$$

Therefore we see that the soft charge acts of the asymptotic scattering states at null infinity to insert a soft graviton in the S-matrix. Soft factor appearing due to insertion of the soft graviton is related to the supertranslation symmetries.

5.3.4 Supertranslation Ward identity

We will consider the matter part of the asymptotic charge at null infinity and calculate the its effect on the Fock states at null infinity. As stated earlier we will consider massless scalar field minimally coupled to gravity. For massless scalar the action is

$$S_{\text{matter}} = -\frac{1}{2}\int d^4x\,\sqrt{-g}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi \tag{5.3.41}$$

Then the symplectic potential for matter field can be expressed as

$$\theta^\mu_{\text{matter}}(\delta) = -\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu\Phi\delta\Phi \tag{5.3.42}$$

where $\hat{g}$ is the induced metric on the three dimensional hypersurface of four dimensional space. We will choose constant time slice, Σ_t and then take $r \to \infty$ keeping u constant to reach null infinity.

We consider the scalar field Φ to be a function of u and r. Φ has a fall off behavior of the form $\frac{\phi(u)}{r}$. The free data at $\mathscr{I}^+$ can be calculated from saddle point approximation as

$$\phi(u, z, \bar{z}) = -\frac{i}{8\pi^2} \int_0^\infty \mathrm{d}E_p \left[a^{\text{out}}\left(E_p \hat{x}\right) e^{-iE_p u} - a^{\text{out}\dagger}\left(E_p \hat{x}\right) e^{iE_p u}\right] \tag{5.3.43}$$

Then symplectic structure of the matter phase space at null infinity is given by

$$\Omega_{\text{matter}}(\delta, \delta') = \lim_{r\to\infty} \int r^2 \mathrm{d}u \mathrm{d}^2 z \sqrt{\gamma} \left[\delta\Phi \delta' \dot{\Phi} - \delta' \Phi \delta \dot{\Phi}\right] \tag{5.3.44}$$

where dot means derivative with respect to u. Then ϕ and $\dot{\phi}$ satisfy the following classical Poisson bracket

$$\{\phi(u, z, \bar{z}), \dot{\phi}(u', z', \bar{z}')\}_{\text{P.B}} = \frac{1}{\sqrt{\gamma}} \delta(u - u') \delta^{(2)}(z - z') \tag{5.3.45}$$

Creation and annihilation operators corresponding to the scalar field can be expressed in terms of these conjugate variables as

$$\begin{aligned} a_E(z, \bar{z}) &= 4\pi i \int_\infty^\infty \mathrm{d}u \, \phi(u, z, \bar{z}) e^{iEu}, \\ a_E^\dagger(z, \bar{z}) &= -\frac{4\pi}{E} \int_\infty^\infty \mathrm{d}u \, \dot{\phi}(u, z, \bar{z}) e^{-iEu}, \end{aligned} \tag{5.3.46}$$

and they satisfy the classical Poisson bracket relation compatible with Eq.(5.3.45)

$$\{a_E(z, \bar{z}), a_{E'}^\dagger(z', \bar{z}')\}_{\text{P.B}} = -i \frac{4(2\pi)^3}{\sqrt{\gamma} E} \delta(E - E') \delta^{(2)}(z - z'). \tag{5.3.47}$$

The normal ordered number operator, constructed from these creation and annihilation operators, acting on the Fock states is given by

$$\int_{-\infty}^{\infty} \mathrm{d}u\dot{\phi}^2\left(u,z,\bar{z}\right) = \frac{2\pi}{\left(8\pi^2\right)^2}\int_{0}^{\infty}\mathrm{d}E\ E^2 a_E^{\dagger}\left(z,\bar{z}\right) a_E\left(z,\bar{z}\right). \tag{5.3.48}$$

On $\mathscr{I}^+$ hard charge acts on the asymptotic outgoing scattering states. From Eq.(5.2.13) we can derive the expression of matter charge at null infinity corresponding to the supertranslation vector field. The matter part of the charge at $\mathscr{I}^+$ is then given by

$$Q_\xi^{\text{matter}\,+} = \int_{\mathscr{I}^+} \mathrm{d}u\mathrm{d}\Omega_2\ f\left(z,\bar{z}\right)\dot{\phi}^2, \tag{5.3.49}$$

where dot represents derivative with respect to u.

Therefore we get

$$\begin{aligned} &\langle z_1^{\text{out}}\ldots|\int_{\mathscr{I}^+}\mathrm{d}u\mathrm{d}^2z\,\sqrt{\gamma}f\left(u,z,\bar{z}\right)\dot{\phi}^2\left(u,z,\bar{z}\right)S\left|z_1^{\text{in}}\ldots\right\rangle \\ =\ &\frac{2\pi}{\left(8\pi^2\right)^2}\int_{\mathbb{S}^2}\mathrm{d}^2z\,\sqrt{\gamma}f\left(z,\bar{z}\right)\int_0^{\infty}\mathrm{d}E\ E^2\prod_{k\in\text{in}}\langle 0|a_{E_k}\left(z_k,\bar{z}_k\right)a_E^{\dagger}\left(z,\bar{z}\right)a_E\left(z,\bar{z}\right)S\left|z_1^{\text{in}}\ldots\right\rangle \\ =\ &\sum_{k\in\text{out}}E_k f\left(z_k,\bar{z}_k\right)\langle z_1^{\text{out}}\ldots|S\left|z_1^{\text{in}}\ldots\right\rangle, \end{aligned} \tag{5.3.50}$$

where we have used the commutation relation

$$\left[a_{E_k}\left(z_k,\bar{z}_k\right), a_E^{\dagger}\left(z,\bar{z}\right)\right] = \frac{4\left(2\pi\right)^3}{\sqrt{\gamma}E}\delta\left(E-E_k\right)\delta^{(2)}\left(z-z_k\right). \tag{5.3.51}$$

Similarly we can find the hard charge $Q_\xi^{\text{matter}\,-}$ acting on the ingoing scattering states at $\mathscr{I}^-$. Then matter part of the asymptotic charge satisfies the following relation with S-matrix

$$\langle z_1^{\text{out}}\ldots|Q_\xi^{\text{matter}\,+}S - S\,Q_\xi^{\text{matter}\,-}|z_1^{\text{in}}\ldots\rangle = \left[\sum_{k=\in\text{out}}E_k f\left(z_k,\bar{z}_k\right) - \sum_{k'\in\text{in}}E_{k'}f\left(z_{k'},\bar{z}_{k'}\right)\right]\langle z_1^{\text{out}}\ldots|S\left|z_1^{\text{in}}\ldots\right\rangle. \tag{5.3.52}$$

Therefore we find the same factorization of the S-matrix without soft graviton from the matter part of the asymptotic charge.

The total covariant phase space charge at null infinity for the radiative phase space of $\Gamma_{\text{grav}} \times \Gamma_{\Phi}$ can be given by $Q_\xi = Q_\xi^{\text{grav}} + Q_\xi^{\text{matter}}$. From Eq.(5.3.40) and Eq.(5.3.52) we obtain the following Ward identity at null infinity

$$\langle \text{out}| Q_\xi^+ S - S Q_\xi^- |\text{in}\rangle = 0. \tag{5.3.53}$$

This is the supertranslation Ward identity relating S-matrix with and without insertion of a soft graviton. There are infinite number of asymptotic charges related to the supertranslation symmetries at null infinity of asymptotically flat spacetimes and hence there are infinite such Ward identities associated to such symmetries. This establishes the fact that gravitational S-matrix is invariant under supertranslation symmetries (more generally under BMS transformations). The gravitational part of the charge generates supertranslation on the radiative phase space Γ_{grav} and inserts a soft graviton with polarization $D_z^2 f(z, \bar{z})$. This soft graviton can be interpreted as the Goldstone mode which arises when one vacuum is shifted to another vacuum under supertranslation. It should be noted that the four spacetime translations do not generate any nontrivial gauge transformations on the radiative phase space at null infinity and eventually leads to vanishing of the gravitational part of asymptotic conserved charge, however there will be finite matter charge which corresponds to the conservation of energy and momenta. Therefore we see supertranslation symmetries are related to the infrared divergence properties of gravitational scattering.

5.4 Comments on higher dimensions

In dimensions greater than four the existence of infinite dimensional supertranslation symmetries for asymptotically flat spacetimes is debatable. In a number of contemporary literatures [64, 65, 93, 95, 97, 101] it has been claimed that supertranslation symmetries exist only in four dimensional asymptotically flat spacetimes and that the asymptotic symmetry group in higher dimensions is the Poincare group. In this section we will try to understand the subtleties that are involved with the infinite dimensional asymptotic symmetry group in higher dimensions.

Motivation We have seen in Sec.(5.3.3) that infrared properties of gravitational scattering are related to the infinite dimensional asymptotic symmetries in four dimensional asymptotically Minkowskian spacetimes. On the other hand we have also learnt that soft graviton theorems exist in all spacetime dimensions ($D \geq 4$). So obviously it is a natural question to ask if the infrared properties in higher dimensions also have any relations to spacetime symmetries. This is the main motivation behind our exploration presented in the remaining part of this chapter.

5.4.1 Radiative data at null infinity

Let us find out the fall-off behavior of the free radiative modes of graviton at asymptotic null infinity. In $D = 2 + 2m$ dimension mode expansion of graviton can be given by

$$\begin{aligned} h_{\mu\nu}(x) &= \sum_{\alpha} \int \frac{\mathrm{d}^{1+2m}\vec{q}}{(2\pi)^{1+2m}} \frac{1}{2\omega_q} \left[\varepsilon^{\alpha\,*}_{\mu\nu}(\vec{q})\, a_\alpha(\vec{q})\, e^{iq\cdot x} + \varepsilon^{\alpha}_{\mu\nu}(\vec{q})\, a^\dagger_\alpha(\vec{q})\, e^{-iq\cdot x} \right] \\ &= \sum_{\alpha} \int_0^\infty \frac{\omega_q^{2m} \mathrm{d}\omega_q}{(2\pi)^{1+2m} 2\omega_q} \int_{\mathbb{S}^{2m}} \mathrm{d}\Omega_{2m} \left[\varepsilon^{*\,\alpha}_{\mu\nu}(\vec{q}) a_\alpha(\vec{q}) e^{-i\omega_q u - i\omega_q r(1-\hat{q}\cdot\hat{r})} \right. \\ &\qquad \left. + \varepsilon^{\alpha}_{\mu\nu}(\vec{q})\, a^\dagger_\alpha(\vec{q})\, e^{i\omega_q u + i\omega_q r(1-\hat{q}\cdot\hat{r})} \right]. \end{aligned} \tag{5.4.1}$$

Here the index α runs over the $\frac{D(D-3)}{2}$ number of polarization degrees of freedom. In the limit $r \to \infty$ we need to do the integration over $\mathbb{S}^{2m}$ using saddle point approximation. Around the saddle point $\hat{q} = \hat{r}$ one can expand $\hat{q} \cdot \hat{r}$ in terms of the angular coordinates as [103]

$$\hat{q}\cdot\hat{r} = 1 - \frac{1}{2}\sum_{i=1}^{2m}\theta_i^2, \tag{5.4.2}$$

where θ_i parametrizes the $2m$ angular coordinates of $\mathbb{S}^{2m}$. There is another saddle at $\hat{q} = -\hat{r}$, but in the limit $r \to \infty$ this gives oscillatory value and hence is neglected. To evaluate the first saddle we can use the integration

$$\int \mathrm{d}\theta e^{-\frac{i\omega r}{2}\theta^2} = \left(\frac{2\pi}{i\omega r}\right)^{\frac{1}{2}} \tag{5.4.3}$$

to obtain

$$\lim_{r\to\infty} h_{\mu\nu}(x) = \frac{1}{r^m}\sum_\alpha \int_0^\infty \frac{(-i\omega)^m}{(2\pi)^{1+2m}}\frac{\mathrm{d}\omega_q}{2\omega_q}\left[\varepsilon^{*\,\alpha}_{\mu\nu}(\omega_q\hat{r})a_\alpha(\omega_q\hat{r})e^{-i\omega_q u} + \varepsilon^{\alpha}_{\mu\nu}\left(\omega_q\hat{r}\right)a^\dagger_\alpha\left(\omega_q\hat{r}\right)e^{i\omega_q u}\right]. \tag{5.4.4}$$

Using the embedding coordinates we can find the projection of $h_{\mu\nu}(x)$ on $\mathbb{S}^{2m}$ at null infinity will give the components of free radiative data

$$\begin{aligned} h^{(m-2)}_{AB}(z) &= \lim_{r\to\infty} r^2\partial_A\hat{x}^\mu\partial_B\hat{x}^\nu h_{\mu\nu}(x) \\ &= \frac{1}{r^{m-2}}\partial_A\hat{x}^\mu\partial_B\hat{x}^\nu\sum_\alpha\int_0^\infty \frac{(-i\omega)^m}{(2\pi)^{1+2m}}\frac{\mathrm{d}\omega_q}{2\omega_q}\left[\varepsilon^{*\,\alpha}_{\mu\nu}(\omega_q\hat{r})a_\alpha(\omega_q\hat{r})e^{-i\omega_q u}\right. \\ &\qquad \left. +\varepsilon^{\alpha}_{\mu\nu}\left(\omega_q\hat{r}\right)a^\dagger_\alpha\left(\omega_q\hat{r}\right)e^{i\omega_q u}\right]. \end{aligned} \tag{5.4.5}$$

Here z^A represents the embedding coordinates on $\mathbb{S}^{2m}$.

Therefore we find that in $D = 2 + 2m$ dimensions the radiative modes are determined by the $h^{(m-2)}_{AB}$ components which have fall-off condition of $r^{-(m-2)}$ at null infinity.

5.4.2 Boundary conditions from saddle

We can adopt the following fall off conditions for the metric perturbations in the large radial coordinate limit

$$h_{uu} \sim O\left(r^{-2m+1}\right), \qquad h_{ur} \sim O\left(r^{-2m}\right), \qquad h_{uA} \sim O\left(r^{-2m+1}\right),$$
$$h_{rr} \sim O\left(r^{-m-2}\right), \qquad h_{rA} \sim O\left(r^{-m}\right), \qquad h_{AB} \sim O\left(r^{-m+2}\right). \tag{5.4.6}$$

These fall-off conditions are compatible with the saddle point analysis at large r. Using these boundary conditions we can consistently solve for the components of the metric perturbations from the linearized gravity equations (5.1.22) in terms of the free radiative data $h_{AB}^{(m-2)}$. To see this let us expand the components in terms of radial coordinate as follows

$$h_{uu} = \sum_{n=2m-1}^{\infty} \frac{h_{uu}^{(n)}}{r^n}, \qquad h_{ur} = \sum_{n=2m}^{\infty} \frac{h_{ur}^{(n)}}{r^n}, \qquad h_{uA} = \sum_{n=m+2}^{\infty} \frac{h_{uA}^{(n)}}{r^n},$$
$$h_{rr} = \sum_{n=m+2}^{\infty} \frac{h_{rr}^{(n)}}{r^n}, \qquad h_{rA} = \sum_{n=m}^{\infty} \frac{h_{rA}^{(n)}}{r^n}, \qquad h_{AB} = \sum_{n=m-2}^{\infty} \frac{h_{AB}^{(n)}}{r^n}. \tag{5.4.7}$$

From linearized Einstein's equations and using the de Donder gauge condition given by $\nabla^\mu \bar{h}_{\mu\nu} = 0$ we can obtain the following equations

$$h_{rr}^{(n)} + \gamma^{AB} h_{AB}^{(n-2)} = 0, \tag{5.4.8}$$

$$\left(\Delta - (n-2m)^2 - n\right) h_{rr}^{(n)} + 2\,(m-n)\,\gamma^{AB} h_{AB}^{(n-2)} - 2\,(m-n+1)\,D^A h_{rA}^{(n-1)} = 0, \tag{5.4.9}$$

$$\left(\Delta - (2m-n-1)\,(2m-n) + 2\,(n-1)\right) D^A h_{rA}^{(n-1)} + 2\,(n-m)\,D^A D^B h_{AB}^{(n-2)} + 2\Delta h_{rr}^{(n)} = 0, \tag{5.4.10}$$

where $m \leq n \leq 2m-1$.

From the above set of equations it is evident the radiative modes $h_{AB}^{(m-2)}$ are the free radiative data and all other metric components can be specified in terms of $h_{AB}^{(m-2)}$.

Residual gauge It can be checked that if we choose radiation gauge condition given in Eq.(5.3.3) to fix residual gauge degrees of freedom the above set of equations are still satisfied. The traceless condition of metric perturbations, *i.e.* $\eta^{\mu\nu}h_{\mu\nu} = 0$ together with $h_{u\mu} = 0$ immediately implies Eq.(5.4.8). The other two equations (5.4.9) and (5.4.10) also follow from this gauge choice.

5.4.3 Large gauge transformations

In this subsection we will explore the possibility of having large diffeomorphisms for the asymptotically flat spacetimes in higher dimensions under the boundary conditions chosen in Eq.(5.4.6). In particular we are interested in finding out if supertranslation symmetries exist in higher dimensions then what will be the corresponding transformations of the metric perturbations at null infinity.

We can generalize the expression of the vector field given in Eq.(5.3.17) for higher dimensions ($D = 2 + 2m$) as

$$\vec{\xi} = f(z)\,\partial_u + \frac{1}{2m}\Delta f(z)\,\partial_r - \frac{1}{r}D^A f(z)\,\partial_A + \ldots \tag{5.4.11}$$

Here z are the $2m$ angular coordinates on $\mathbb{S}^{2m}$. It can be checked that this vector field satisfies the residual gauge condition, $\Box\xi^\mu = 0$.

We can calculate shift in h_{AB} components under $\vec{\xi}$ (variations of metric perturbations under any arbitrary vector fields are given in Sec.(C.2)) and we find

$$\delta_\xi h_{AB}^{(-1)} = \frac{1}{m}\gamma_{AB}\Delta f(z) - (D_A D_B + D_B D_A)\,f(z)\,. \tag{5.4.12}$$

Therefore we notice that the components $h_{AB}^{(-1)}$ which are not radiative data in higher dimensions are shifted under the diffeomorphisms induced by the vector field in Eq.(5.4.11). In fact in higher dimensions ($m > 1$) the boundary conditions of Eq.(5.4.6) imply that components of h_{AB} at $\mathcal{O}(r)$ should vanish, hence right hand side of Eq.(5.4.12) should be zero. This imposes a constraint on the functional form of f, which now can not be any arbitrary function of $2m$-sphere coordinates. *The only allowed solutions of the vector field $\vec{\xi}$ are those of D number of Poincare translations.*

So we find that supertranslation symmetries are not compatible because of the boundary conditions specified in Eq.(5.4.6) for higher dimensions. In four dimension the order of the radial coordinate at which radiative modes occur and supertranslation vector field acts on the radiative phase space at null infinity are exactly same. This is the reason for the existence of enhanced asymptotic symmetry group in four dimensional asymptotically flat spacetimes. Clearly in higher dimensions existence of supertranslation symmetries depend on the boundary conditions specified - usual fall-off conditions that follow from saddle point analysis rule out supertranslation symmetries but there may be other boundary conditions that can allow supertranslation symmetries to exist. We explore such possibilities in the next sub-section.

5.4.4 Relaxed boundary conditions

For supertranslations to exist in higher dimensions we have to relax the boundary conditions specified in Eq.(5.4.6) and choose less restrictive fall off conditions of metric perturbations. Here we show how to obtain the appropriate boundary conditions to meet the purpose.

From the analysis in the previous sub-section it is understood that we can not set the components $h_{AB}^{(-1)}$ to zero. Therefore our first try will be to keep the $\mathcal{O}(r)$ components of h_{AB} non-zero in higher dimensions while keeping the fall-off conditions of the other metric perturbations same as in Eq.(5.4.6). We present the calculations of linearized gravity equations in six dimension in Sec.(C.3). In six dimensions we find $h^{(-1)}_{AB}$ components are independent of u and $h^{(0)}_{AB}$ contain the free propagating modes of graviton as expected from the saddle point analysis. But there are problems coming from the equations satisfied by $h^{(-1)}_{AB}$ which are

$$h^{(-1)}_{AB} = \frac{1}{4}\Delta h^{(-1)}_{AB}, \qquad D^B h^{(-1)}_{AB} = 0. \tag{5.4.13}$$

These equation imply that $h^{(-1)}_{AB}$ can have only particular solutions of sphere coordinates and hence the functional form of f in Eq.(5.4.11) can not be arbitrary. Therefore supertranslations can again be ruled out in this case[2].

Hence we find that in any dimesnions keeping only $h^{(-1)}_{AB}$ non vanishing without relaxing the fall off conditions of other metric perturbations is not helpful because we get the following constraint on $f(z)$

$$D_A\left(\frac{1}{2m}\Delta f(z) + f(z)\right) = 0 \tag{5.4.14}$$

which follows from the divergence-free condition, $D^A h^{(-1)}_{AB} = 0$. Therefore we need to look for the boundary conditions that will remove the constraint given in Eq.(5.4.14) on $f(z)$.

Supertranslations in higher dimensions

The above constraint can be avoided by modifying the fall off behavior of h_{Ar} as $h_{Ar} \sim \mathcal{O}\left(r^{-1}\right)$ in higher dimensions. This removes the divergence-free condition of $D^A h^{(-1)}_{AB} = 0$ and we get

$$D^B h^{(-1)}_{AB} - \partial_u h^{(1)}_{Ar} = 0. \tag{5.4.15}$$

[2]One may think of modifying the equations (5.4.13) by allowing $\mathcal{O}\left(r^0\right)$ term $h^{(0)}_{uA}$ but because of covariant gauge condition, $\nabla^\mu \bar{h}_{\mu A} = 0$ we get back equations (5.4.13).

In this way we can continue to find the suitable fall off conditions of the metric perturbations, mostly by trial and error method, without spoiling the supertranslation symmetries in higher dimensions. In general we can adopt the fall-off conditions as

$$h_{Ar} \sim O\left(r^{-1}\right), \qquad h_{rr} \sim O\left(r^{-1}\right), \qquad h_{AB} \sim O(r)\,. \tag{5.4.16}$$

Some details of this calculation have been presented in Sec.(C.4.1). *It is evident that to have supertranslation symmetries in higher dimensions boundary conditions should be such that along with the propagating modes of graviton (which are u dependent) many non-propagating metric fluctuations also need to be present.*

Covariant phase space charge Let us calculate the covariant phase space charge from the vector field described in Eq.(5.4.11) at the asymptotic null infinity. We can still choose radiation gauge given in Eq.(5.3.3) to fix the residual gauge degrees of freedom, but there is a subtlety involved here. We have to keep $h^{(0)}_{uA}$ non-zero because otherwise we will get

$$\Box\xi^A = 0 \quad \Rightarrow \quad (1-m)\left[D^A\left(\frac{1}{2m}\Delta f + f\right) + \partial_u \xi^{A\,(2)}\right] = 0, \tag{5.4.17}$$

$$\mathcal{L}_\xi g_{uA} = 0 \quad \text{at} \quad O(1) \quad \Rightarrow \quad D_A\left(\frac{1}{2m}\Delta f + f\right) - \gamma_{AC}\partial_u \xi^{C\,(2)} = 0. \tag{5.4.18}$$

Eq.(5.4.17) and Eq.(5.4.18) combined together give back Eq.(5.4.14), $D_A\left(\frac{1}{2m}\Delta f(z) + f(z)\right) = 0$ for $m \neq 1$. It can be checked that the components $h^{(0)}_{uA}$ can not be expressed in terms of the radiative modes $h^{(m-2)}_{AB}$ and hence are also non-propagating like $h^{(-1)}_{AB}$.

Details of calculation of the gravitational part of the covariant phase space charge at null infinity for higher dimensions have been given in Sec.(C.4.2). Because of the presence of non-propagating modes of metric perturbations calculation of gravitational contribution to the asymptotic charge at null infinity is more complicated in higher dimensions than that in four dimension. One major problem is many divergent terms appear in the computation of the charge. It is expected that divergent terms vanish to give finite asymptotic charge and the finite charge should then give rise to soft graviton theorem. In Sec.(C.4.3) we have made an attempt to compute the asymptotic soft charge in six dimension. At this stage we are unable to derive the expected expression of the charge corresponding to the soft graviton theorem and leave it for further investigation.

Hard charge Calculation of matter contribution to the asymptotic covariant charge is straightforward. For massless scalar coupled to gravity described in Sec.(5.2.2) the hard charge at $\mathscr{I}^+$ can be given by

$$Q_\xi^{\text{matter}\,+} = \int_{\mathscr{I}^+} \mathrm{d}u \mathrm{d}\Omega_{2m} f(z)\, \dot{\phi}^2. \tag{5.4.19}$$

Similar expression can be obtained at past null infinity, $\mathscr{I}^-$ also.

Note If we choose the stringent boundary conditions (5.4.6) following from saddle point analysis then we obtain the trivial result that gravitational part of the asymptotic charge vanishes [18]. This is expected because the allowed translations at null infinity are only rigid Poincare translations which satisfy the following equations

$$D^B\left[\frac{1}{m}\gamma_{AB}\Delta f(z) - (D_A D_B + D_B D_A) f(z)\right] = D_A\left[\frac{1}{2m}\Delta f(z) + f(z)\right] = 0. \tag{5.4.20}$$

Under the rigid spacetime translations radiative data remain unaffected in higher dimensions, hence there is no physical gauge transformations happening in the radiative phase space at null infinity.

However we will still get hard part of the asymptotic charge coming from the matter contribution. These conserved charges satisfy the trivial Ward identities at asymptotic null infinity

$$\langle \text{out}| Q_\xi^{\text{matter}\,+} S - S\, Q_\xi^{\text{matter}\,-} |\text{in}\rangle = 0 \tag{5.4.21}$$

which imply the conservation of total energy and momenta.

5.4.5 Discussion

In this section we have explored the possibilities of having supertranslation symmetries for asymptotically flat spacetimes in dimensions greater than four. From the analysis in Sec.(5.4.2) it is clear that with the boundary conditions adopted from the saddle point approximation near null infinity supertranslation symmetries can be shown to be inconsistent in higher dimensions. To allow for the existence of supertranslations we have to specify alternative boundary conditions. In higher dimensions free radiative components of metric fluctuations are given by the spherical components, $h_{AB}^{(m-2)}$ which fall off as $r^{-(m-2)}$ as $r \to \infty$ but supertranslations act on $h_{AB}^{(-1)}$ which are linear in r. Therefore the boundary conditions be such that $h_{AB}^{(-1)}$ should be present although these components do not depend on the retarded time u and hence are not dynamical. Then we see in Sec.(5.4.4) that for consistencies we also need to relax fall-off behavior of other components of metric perturbations. Relaxed boundary conditions lead to appearance of divergent terms in the asymptotic conserved charges at null infinity. It is expected that divergent part of the charges vanishes and the finite contribution in the limit $r \to \infty$ matches with the soft charges [16].

In our analysis we have considered Laurent series expansions of metric perturbations in radial coordinates. However the fall-off conditions may also involve transcendental functions. We leave it for future study to show if it is possible to obtain the correct expression of the soft charges using any transcendental functional form of metric perturbations.

In principle our analysis is valid for odd dimensions also. Although the are subtleties with the conformal null infinity in odd dimensions [64, 95–97] but in perturbative gravity we can consider asymptotic null infinity as the limit $r \to \infty$. In odd dimensions metric perturbations will have half-integer fall-off in radial coordinate. In the large r limit there is a $(D-2)$-dimensional sphere integral which contains $r^{(D-2)}$, because of this it is hard to obtain any finite expression for soft charge as $r \to \infty$. If we consider only spacetime translations then there are finite asymptotic hard charges which correspond to conservation of energy and momenta.

6 Conclusion

In this thesis we have studied some aspects of soft graviton theorems in arbitrary dimensions. Modern techniques of calculating scattering amplitudes in Quantum Field Theory have played significant roles in the development and better understanding of soft theorems in general. We have focused on one such remarkable formalism pioneered by Cachazo, He and Yuan to compute scattering amplitudes for several massless theories containing scalars, gluons and gravitons using mathematics of moduli spaces and complex analysis. Cachazo-He-Yuan (CHY) formalism provides a sophisticated tool for studying scattering amplitudes without going through the cumbersome computations of Feynman diagrams. On the other hand several important aspects of scattering amplitudes have become manifest from this formalism. Dualities and double-copy relations among theories like gravity and Yang-Mills can be made more precise, Kawai-Lewellen-Tye (KLT) orthogonality relations which were originally discovered in the context of open and closed string amplitudes can be shown to hold in the field theory limit in CHY representations. In the field theory limit elements of KLT matrix is related to S-matrix elements of a particular type of scalar theory with cubic interactions known as bi-adjoint ϕ^3 where the scalar fields, $\phi^{aa'}$ live in the adjoint representations of two unitary groups $U(N) \times U(\tilde{N})$. Scattering equations and building blocks are the essential ingredients of CHY formulation. Punctures on the Riemann sphere correspond to external massless states, using the scattering equations one can solve for the location of the punctures in terms of kinematic variables of the external states. Building blocks are expressed in terms of the punctures and kinematic variables of the external particles. These building blocks are different for different theories.

Another remarkable outcome of CHY formalism is that soft factorization properties of scattering amplitudes can be studied with great convenience. In the soft limits integrations on the moduli space become contour integrals in the complex plane. In chapter 4 we have studied double soft limits of scattering amplitudes for Yang-Mills and pure gravity. Punctures on the complex spheres are denoted by σ_a, for n external particles carrying finite energies $a \in \{1, 2, \ldots, n\}$ and for soft particles $a \in \{n+1, n+2\}$. Here we consider simultaneous soft limits where both the soft momenta k_{n+1} and k_{n+2} scale as τ and we take the limit $\tau \to 0$. Then we have to consider two situations:

- *non-degenerate case* - when $|\sigma_{n+1} - \sigma_{n+2}| \sim O\left(\tau^0\right)$, then contour integrals for σ_{n+1} and σ_{n+2} can be calculated independently of each other,
- *degenerate case* - when $|\sigma_{n+1} - \sigma_{n+2}| \sim O(\tau)$, then the two punctures coalesce. There is now effectively one contour integration.

In case of double soft scalar theorems contributions from degenerate solutions of scattering equations dominate over that from non-degenerate solutions. But for Yang-Mills and gravity we find contributions from non-degenerate solutions appear at leading order. In fact many terms coming from non-degenerate and degenerate contributions mutually cancel each other.

We obtain the double soft factorization of gravity amplitude to sub-leading order as

$$\lim_{\tau\to 0} M_{n+2}\left(\{k_a\}, \tau p, \tau q\right) = \left[\frac{1}{\tau^2} S^{(0)}(p)\, S^{(0)}(q) + \frac{1}{\tau}\left\{S^{(0)}(p)\, S^{(1)}(q) + S^{(0)}(q)\, S^{(1)}(p) + S_{\text{contact}}\right\}\right] M_n\left(\{k_a\}\right) \quad . \tag{6.0.1}$$

$S^{(0)}$ and $S^{(1)}$ are the leading and sub-leading single soft factors respectively. At sub-leading order of double soft factor there is a contact term whose expression is given in

Eq.(4.4.28). This third term appears because $S^{(0)}$ and $S^{(1)}$ factors do not commute, S_{contact} is required to maintain gauge invariance of the sub-leading double soft factor. We have also given interpretations of the terms in the double soft factor for gravity from Feynman diagrammatic. It will be interesting to derive Ward identities of gravitational S-matrix for the double soft theorem in the simultaneous soft limit from asymptotic symmetries at null infinity of asymptotically flat spacetimes.

In the next part of the thesis we have explored some issues related to asymptotic symmetries, particularly supertranslations, for asymptotically flat spacetimes in dimensions greater than four. In four dimension due to the existence of supertranslations there is an enhancement of asymptotic symmetry group acting at null infinity. Under supertranslations $h^{(-1)}_{AB}$ components of metric perturbations are shifted as

$$\delta_f h^{(-1)}_{AB} = \frac{2}{D-2}\gamma_{AB}\Delta f - (D_A D_B + D_B D_A)\, f, \tag{6.0.2}$$

where D is the dimension of spacetime, γ_{AB} is the metric on $(D-2)$-dimensional sphere, Δ and D^A are laplacian and covariant derivative compatible with γ_{AB} and f is any arbitrary function of sphere coordinates corresponding to supertranslations. In four dimension $h^{(-1)}_{AB}$ are the free unconstrained data representing graviton modes. One can construct conserved charges at asymptotic null infinity from the supertranslation symmetries using covariant phase space formalism. These charges act on the incoming and outgoing scattering states at $\mathscr{I}^-$ and $\mathscr{I}^+$ respectively and insert soft graviton mode in the S-matrix elements. In this way Weinberg's soft graviton theorem and supertranslation symmetries are related to each other in four dimension.

In chapter 5 we have tried to obtain asymptotic charges from supertranslation symmetries in higher dimensions. In dimensions greater than four $h^{(-1)}_{AB}$ are not the radiative components, they do not depend on retarded time u; free radiative data are given by $h^{(m-2)}_{AB}$ which have asymptotic fall-off behavior of $r^{-(m-2)}$ with $m > 1$. It is evident that if we adopt boundary conditions compatible with fall-off behavior following from saddle point analysis at large r then supertranslation symmetries can be ruled out. To allow for supertranslations in higher dimensions $h_{AB}^{(-1)}$ components have to unconstrained. However we find boundary conditions of other components of metric perturbations also have to be relaxed, otherwise we get a constraint on $h_{AB}^{(-1)}$ given by

$$D^A h_{AB}^{(-1)} = 0, \tag{6.0.3}$$

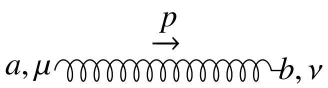
which will again rule out existence of supertranslations.

With the restricted boundary conditions following from saddle point analysis, Poincare symmetries are the only allowed asymptotic symmetries. Then it can be checked $\delta_f h_{AB}^{(-1)} = 0$ follow from rigid translations. Since there is non-trivial diffeomorphisms therefore gravitational charges vanish at null infinity. If we relax the boundary conditions consistently then we recover supertranslation symmetries but then there are many non-propagating components of metric perturbations which are u independent. The problem with relaxing the fall-off conditions is that while calculating asymptotic charges there appear divergent terms. It is expected that divergent part of the charges will vanish and finite contribution will produce the correct expression of the gravitational charges at null infinity which will match with leading soft graviton theorem in higher dimensions. It will be an interesting future study to obtain such finite gravitational asymptotic charges from supertranslation symmetries.

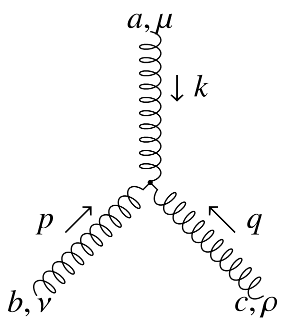
A Appendix of chapter 2

A.1 Soft gluon theorem from Feynman diagram

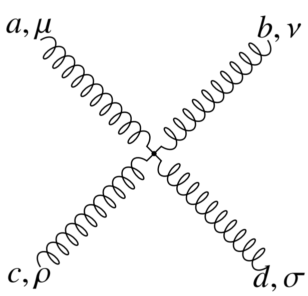
We will need following Feynman rules



$$= \quad -i\delta^{ab}\frac{\eta_{\mu\nu}}{p^2-i\epsilon}$$



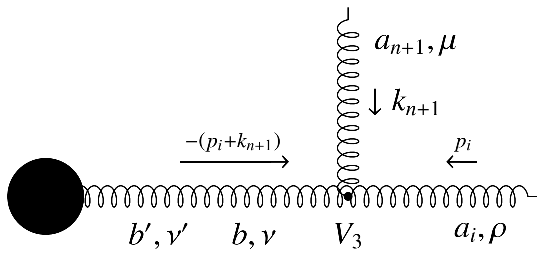
$$= \quad gf^{abc}\left[\eta^{\mu\nu}(k-p)^{\rho}+\eta^{\nu\rho}(p-q)^{\mu}\right.$$
$$\left.+\eta^{\rho\mu}(q-k)^{\nu}\right]$$



$$= \quad -ig^2\left[f^{abe}f^{cde}\left(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}\right)\right.$$
$$+f^{ace}f^{bde}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}\right)$$
$$\left.+f^{ade}f^{bce}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}\right)\right]$$

In the single soft limit the relevant Feynman diagram is

A.1.1 Feynman diagram



$\lim k_{n+1} \to 0$ **:**

$$\begin{aligned} V_3^{\mu\nu\rho} &= g f^{a_{n+1} b a_i} \left[\eta^{\mu\nu} (2k_{n+1} + p_i)^\rho + \eta^{\nu\rho} (-2p_i - k_{n+1})^\mu + \eta^{\rho\mu} (p_i - k_{n+1})^\nu\right] \\ &= g f^{a_{n+1} b a_i} \left[\eta^{\mu\nu} p_i^\rho - 2\eta^{\nu\rho} p_i^\mu + \eta^{\rho\mu} p_i^\nu\right] + O(k_{n+1}) . \end{aligned} \tag{A.1.1}$$

The above diagram can be evaluated as

$$\begin{aligned} & \mathcal{A}_n \left(i^{b'}\right)_\nu \frac{(-i)\, \delta^{b'b} \eta_{\nu'\nu}}{2k_{n+1} \cdot p_i} g f^{a_{n+1} b a_i} \left[\eta^{\mu\nu} p_i^\rho - 2\eta^{\nu\rho} p_i^\mu + \eta^{\rho\mu} p_i^\nu\right] \varepsilon_{n+1,\mu} \varepsilon_{i,\rho} \\ = \; & i g f^{b a_i a_{n+1}} \frac{\varepsilon_{n+1} \cdot p_i}{k_{n+1} \cdot p_i} \varepsilon_i^\nu \mathcal{A}_n \left(i^b\right)_\nu \\ = \; & i g f^{b a_i a_{n+1}} \frac{\varepsilon_{n+1} \cdot p_i}{k_{n+1} \cdot p_i} \mathcal{A}_n \left(i^b\right) . \end{aligned} \tag{A.1.2}$$

The last term in the square bracket vanishes due to Ward identity. Therefore soft factorization is given by

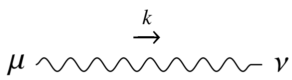
$$\mathcal{A}_{n+1} = \sum_b \sum_{i=1}^{n} i g f^{b a_i a_{n+1}} \frac{\varepsilon_{n+1} \cdot p_i}{k_{n+1} \cdot p_i} \mathcal{A}_n \left(i^b\right) . \tag{A.1.3}$$

B Appendix of chapter 3

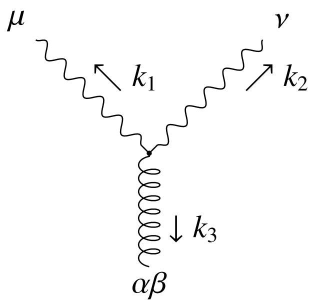
B.1 Feynman rules

B.1.1 Feynman Rules for EM

The Feynman rules for Yang-Mills theory coupled to gravity have been derived in [104]. In the same way Feynman rules for Einstein Maxwell theory coupled to gravity can be derived.



$$-\frac{i\,\eta_{\mu\nu}}{k^2+i\epsilon}$$



$$-i\,\kappa\Big[(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta})k_1.k_2+\eta_{\mu\nu}(k_{1\alpha}k_{2\beta}+k_{1\beta}k_{2\alpha})$$
$$-k_{1\nu}k_{2\mu}\eta_{\alpha\beta}-k_{1\nu}(\eta_{\mu\alpha}k_{2\beta}+\eta_{\mu\beta}k_{2\alpha})-k_{2\mu}(\eta_{\nu\alpha}k_{1\beta}+\eta_{\nu\beta}k_{1\alpha})\Big]$$

B.1.2 Feynman Rules for Gravity

The Feynman rules for three and four point vertices are given in [88]. Using Eq.(4.2.5) we can write

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa\, h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_\alpha^{\ \nu} + O(\kappa^3) \tag{B.1.1}$$

and

$$\sqrt{-g} = 1 + \frac{\kappa}{2} h + \frac{\kappa^2}{8}\left(h^2 - 2\, h^{\alpha\beta} h_{\alpha\beta}\right) + O(\kappa^3) \tag{B.1.2}$$

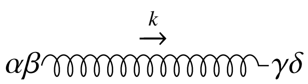
where $h = h^\alpha_{\ \alpha}$. The Ricci scalar can be expanded as

$$\begin{aligned} R \;=\; & \kappa\left(\Box h - \partial_\mu \partial_\nu h^{\mu\nu}\right) \\ & + \kappa^2 \left(\frac{1}{4}\partial_\mu h \partial^\mu h - \partial_\mu h^{\mu\nu}\partial_\nu h + \partial_\mu h^{\mu\nu}\partial^\rho h_{\nu\rho} - \frac{3}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \right. \\ & \left. + \frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + 2\, h^{\mu\nu}\partial_\mu \partial^\rho h_{\rho\nu} - h^{\mu\nu}\Box h_{\mu\nu}\right) + O(\kappa^3). \end{aligned} \tag{B.1.3}$$

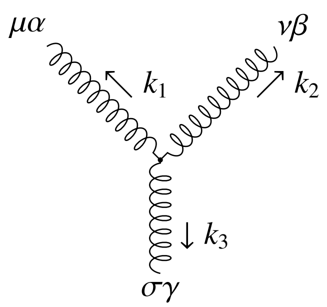
We work in harmonic (de Donder) gauge, where

$$h^\alpha_{\ \mu,\alpha} - \frac{1}{2} h_{,\mu} = 0. \tag{B.1.4}$$

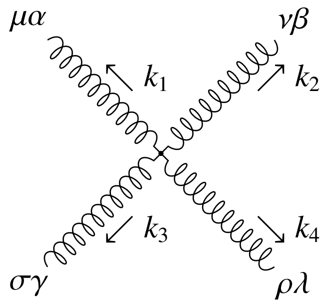
Then the resulting graviton propagator, three and four point vertices are:



$$= -\frac{i}{2}\frac{\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}}{k^2 - i\epsilon}$$



$$= \text{sym}\Big[-\frac{1}{2}P_3(k_1.k_2\ \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \frac{1}{2}P_3(k_1.k_2\ \eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})$$
$$+ P_6(k_1.k_2\ \eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})$$
$$+ P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})$$
$$+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1.k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big]$$



$$= \text{sym}\Big[-\frac{1}{4}P_6(k_1.k_2\ \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda}) - \frac{1}{4}P_{12}(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}\eta_{\rho\lambda}) - \frac{1}{2}P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})$$
$$+ \frac{1}{4}P_6(k_1.k_2\ \eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda}) + \frac{1}{2}P_6(k_1.k_2\ \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda}) + \frac{1}{2}P_{12}(k_{1\nu}k_{2\beta}\eta_{\mu\alpha}\eta_{\sigma\rho}\eta_{\gamma\lambda})$$
$$+ P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda}) - \frac{1}{2}P_6(k_1.k_2\ \eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda}) + \frac{1}{2}P_{24}(k_1.k_2\ \eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}\eta_{\rho\lambda})$$
$$+ \frac{1}{2}P_{24}(k_{1\nu}k_{1\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\rho\lambda}) + \frac{1}{2}P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\rho\lambda}) + P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\mu}\eta_{\alpha\gamma}\eta_{\rho\lambda})$$
$$- P_{12}(k_1.k_2\ \eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu}\eta_{\rho\lambda}) + P_{12}(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}\eta_{\rho\lambda}) + P_{12}(k_{1\nu}k_{1\sigma}\eta_{\beta\gamma}\eta_{\mu\alpha}\eta_{\rho\lambda})$$
$$- P_{24}(k_1.k_2\ \eta_{\mu\alpha}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\nu}) - 2P_{12}(k_{1\nu}k_{1\beta}\eta_{\alpha\sigma}\eta_{\gamma\rho}\eta_{\lambda\mu}) - 2P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\alpha\rho}\eta_{\lambda\nu}\eta_{\beta\mu})$$
$$- 2P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\rho}\eta_{\lambda\mu}\eta_{\alpha\gamma}) - 2P_{12}(k_{1\sigma}k_{2\rho}\eta_{\gamma\nu}\eta_{\beta\mu}\eta_{\alpha\lambda}) + 2P_6(k_1.k_2\ \eta_{\alpha\sigma}\eta_{\gamma\nu}\eta_{\beta\rho}\eta_{\lambda\mu})$$
$$- 2P_{12}(k_{1\nu}k_{1\sigma}\eta_{\mu\alpha}\eta_{\beta\rho}\eta_{\lambda\gamma}) - P_{12}(k_1.k_2\ \eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\nu\rho}\eta_{\beta\lambda}) - 2P_{12}(k_{1\nu}k_{1\sigma}\eta_{\beta\gamma}\eta_{\mu\rho}\eta_{\alpha\lambda})$$
$$- P_{12}(k_{1\sigma}k_{2\rho}\eta_{\gamma\lambda}\eta_{\mu\nu}\eta_{\alpha\beta}) - 2P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\mu}\eta_{\alpha\rho}\eta_{\lambda\gamma}) - 2P_{12}(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\alpha})$$
$$+ 4P_6(k_1.k_2\ \eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\mu})\Big]$$

where "sym" stands for symmetrization between (μ, α); (ν, β); (σ, γ); (ρ, λ) and the symbol P_m denotes m number of distinct permutations between the indices (k_1, μ, α); (k_2, ν, β); (k_3, σ, γ); (k_4, ρ, λ).

B.2 Yang Mills

$$
\Psi_{n+2}\,(1,2,\ldots,i,\ldots,j\ldots,n) =
$$

$$
\left[\begin{array}{ccccccc|ccccccc}
0 & \cdots & \frac{\tau\, p_1\cdot k_i}{\sigma_1-\sigma_i} & \cdots & \frac{\tau\, p_1\cdot k_j}{\sigma_1-\sigma_j} & \cdots & \frac{p_1\cdot p_n}{\sigma_1-\sigma_n} & -C_{11} & \cdots & -\frac{\epsilon_i\cdot p_1}{\sigma_i-\sigma_1} & \cdots & -\frac{\epsilon_j\cdot p_1}{\sigma_j-\sigma_1} & \cdots & -\frac{\epsilon_n\cdot p_1}{\sigma_n-\sigma_1} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\tau\, k_i\cdot p_1}{\sigma_i-\sigma_1} & \cdots & 0 & \cdots & \frac{\tau^2\, k_i\cdot k_j}{\sigma_i-\sigma_j} & \cdots & \frac{\tau\, k_i\cdot p_n}{\sigma_i-\sigma_n} & -\frac{\tau\, \epsilon_1\cdot k_i}{\sigma_1-\sigma_i} & \cdots & -C_{ii} & \cdots & -\frac{\tau\, \epsilon_j\cdot k_i}{\sigma_j-\sigma_i} & \cdots & -\frac{\tau\, \epsilon_n\cdot k_i}{\sigma_n-\sigma_i} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\tau\, k_j\cdot p_1}{\sigma_j-\sigma_1} & \cdots & \frac{\tau^2\, k_j\cdot k_i}{\sigma_j-\sigma_i} & \cdots & 0 & \cdots & \frac{\tau\, k_j\cdot p_n}{\sigma_j-\sigma_n} & -\frac{\tau\, \epsilon_1\cdot k_j}{\sigma_1-\sigma_j} & \cdots & -\frac{\tau\, \epsilon_i\cdot k_j}{\sigma_i-\sigma_j} & \cdots & -C_{jj} & \cdots & -\frac{\tau\, \epsilon_n\cdot k_j}{\sigma_n-\sigma_j} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{p_n\cdot p_1}{\sigma_n-\sigma_1} & \cdots & \frac{\tau\, p_n\cdot k_i}{\sigma_n-\sigma_i} & \cdots & \frac{\tau\, p_n\cdot k_j}{\sigma_n-\sigma_j} & \cdots & 0 & -\frac{\epsilon_1\cdot p_n}{\sigma_1-\sigma_n} & \cdots & -\frac{\epsilon_i\cdot p_n}{\sigma_i-\sigma_n} & \cdots & -\frac{\epsilon_j\cdot p_n}{\sigma_j-\sigma_n} & \cdots & -C_{nn} \\
\hline
C_{11} & \cdots & \frac{\tau\, \epsilon_1\cdot k_i}{\sigma_1-\sigma_i} & \cdots & \frac{\tau\, \epsilon_1\cdot k_j}{\sigma_1-\sigma_j} & \cdots & \frac{\epsilon_1\cdot p_n}{\sigma_1-\sigma_n} & 0 & \cdots & \frac{\epsilon_1\cdot \epsilon_i}{\sigma_1-\sigma_i} & \cdots & \frac{\epsilon_1\cdot \epsilon_j}{\sigma_1-\sigma_j} & \cdots & \frac{\epsilon_1\cdot \epsilon_n}{\sigma_1-\sigma_n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\epsilon_i\cdot p_1}{\sigma_i-\sigma_1} & \cdots & C_{ii} & \cdots & \frac{\tau\, \epsilon_i\cdot k_j}{\sigma_i-\sigma_j} & \cdots & \frac{\epsilon_i\cdot p_n}{\sigma_i-\sigma_n} & \frac{\epsilon_i\cdot \epsilon_1}{\sigma_i-\sigma_1} & \cdots & 0 & \cdots & \frac{\epsilon_i\cdot \epsilon_j}{\sigma_i-\sigma_j} & \cdots & \frac{\epsilon_i\cdot \epsilon_n}{\sigma_i-\sigma_n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\epsilon_j\cdot p_1}{\sigma_j-\sigma_1} & \cdots & \frac{\tau\, \epsilon_j\cdot k_i}{\sigma_j-\sigma_i} & \cdots & C_{jj} & \cdots & \frac{\epsilon_j\cdot p_n}{\sigma_j-\sigma_n} & \frac{\epsilon_j\cdot \epsilon_1}{\sigma_j-\sigma_1} & \cdots & \frac{\epsilon_j\cdot \epsilon_i}{\sigma_j-\sigma_i} & \cdots & 0 & \cdots & \frac{\epsilon_j\cdot \epsilon_n}{\sigma_j-\sigma_n} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\epsilon_n\cdot p_1}{\sigma_n-\sigma_1} & \cdots & \frac{\tau\, \epsilon_n\cdot k_i}{\sigma_n-\sigma_i} & \cdots & \frac{\tau\, \epsilon_n\cdot k_j}{\sigma_n-\sigma_j} & \cdots & C_{nn} & \frac{\epsilon_n\cdot \epsilon_1}{\sigma_n-\sigma_1} & \cdots & \frac{\epsilon_n\cdot \epsilon_i}{\sigma_n-\sigma_i} & \cdots & \frac{\epsilon_n\cdot \epsilon_j}{\sigma_n-\sigma_j} & \cdots & 0
\end{array}\right] \tag{B.2.1}
$$

In the double soft limit Pfaffian can be expressed as

$$
\text{Pf'}\,\Psi_{n+2} \quad = \quad -C_{ii}C_{jj}\text{Pf'}\,\Psi_n
$$

$$= -\left(\sum_{a=1}^{n}\frac{\epsilon_i\cdot k_a}{\sigma_i-\sigma_a}\right)\left(\sum_{b=1}^{n}\frac{\epsilon_j\cdot k_b}{\sigma_j-\sigma_b}\right)\text{Pf'}\Psi_n. \tag{B.2.2}$$

The Park-Taylor factor can be factorized as

$$C_{n+2}=\frac{(\sigma_{i-1}-\sigma_{i+1})\left(\sigma_{j-1}-\sigma_{j+1}\right)}{(\sigma_{i-1}-\sigma_i)(\sigma_i-\sigma_{i+1})\left(\sigma_{j-1}-\sigma_j\right)\left(\sigma_j-\sigma_{j+1}\right)}C_n. \tag{B.2.3}$$

B.3 Details of double soft factorization for gravity

B.3.1 Non-degenerate solutions

The reduced determinants in Eq.(4.4.12) are given by

$$\begin{aligned}
\tilde{\Psi}^a_{n+1} &= (-1)^n\left(C_{n+2,n+2}\right)^2C_{n+1,n+1}\sum_{b=1}^{n}\left[(-1)^b\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{n+1}-\sigma_b}\Psi^a_b+(-1)^{n+b}\frac{\epsilon_{n+1}\cdot\epsilon_b}{\sigma_{n+1}-\sigma_b}\Psi^a_{n+b}\right]\\
\tilde{\Psi}^a_{n+2} &= (-1)^{n+1}\left(C_{n+1,n+1}\right)^2C_{n+2,n+2}\sum_{b=1}^{n}\left[(-1)^b\frac{\epsilon_{n+2}\cdot k_b}{\sigma_{n+2}-\sigma_b}\Psi^a_b+(-1)^{n+b}\frac{\epsilon_{n+2}\cdot\epsilon_b}{\sigma_{n+2}-\sigma_b}\Psi^a_{n+b}\right]\\
\tilde{\Psi}^{2n+3}_{n+1} &= \left(C_{n+2,n+2}\right)^2\sum_{a=1}^{n}\sum_{b=1}^{n}\left[\frac{p\cdot k_b}{\sigma_{n+1}-\sigma_b}\left((-1)^{n+a+b+1}\frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\Psi^a_b+(-1)^{a+b}\frac{\epsilon_a\cdot\epsilon_{n+1}}{\sigma_a-\sigma_{n+1}}\Psi^{n+a}_b\right)\right.\\
&\qquad\left.+\frac{\epsilon_b\cdot p}{\sigma_b-\sigma_{n+1}}\left((-1)^{a+b}\frac{\epsilon_{n+1}\cdot k_a}{\sigma_{n+1}-\sigma_a}\Psi^a_{n+b}+(-1)^{n+a+b+1}\frac{\epsilon_a\cdot\epsilon_{n+1}}{\sigma_a-\sigma_{n+1}}\Psi^{n+a}_{n+b}\right)\right]\\
\tilde{\Psi}^{2n+4}_{n+2} &= \left(C_{n+1,n+1}\right)^2\sum_{a=1}^{n}\sum_{b=1}^{n}\frac{q\cdot k_b}{\sigma_{n+2}-\sigma_b}\left[(-1)^{n+a+b+1}\frac{\epsilon_{n+2}\cdot k_a}{\sigma_{n+2}-\sigma_a}\Psi^a_b+(-1)^{a+b}\frac{\epsilon_a\cdot\epsilon_{n+2}}{\sigma_a-\sigma_{n+2}}\Psi^{n+a}_b\right]\\
\tilde{\Psi}^{n+2+a}_{n+1} &= (-1)^nC_{n+1,n+1}\left(C_{n+2,n+2}\right)^2\sum_{b=1}^{n}\left[(-1)^b\frac{\epsilon_{n+1}\cdot k_b}{\sigma_{n+1}-\sigma_b}\Psi^{n+a}_b+(-1)^{n+b}\frac{\epsilon_{n+1}\cdot\epsilon_b}{\sigma_{n+1}-\sigma_b}\Psi^{n+a}_{n+b}\right]\\
\tilde{\Psi}^a_{n+a} &= \left(C_{n+1,n+1}\right)^2\left(C_{n+2,n+2}\right)^2\Psi^a_{n+a}\\
\tilde{\Psi}^{n+2+a}_a &= \left(C_{n+1,n+1}\right)^2\left(C_{n+2,n+2}\right)^2\Psi^{n+a}_a
\end{aligned} \tag{B.3.1}$$

Plugging back these expressions we obtain Eq.(4.4.13).Now using the relation $\Psi^a_b = -\Psi^b_a$ which holds because of the antisymmetric property and the following result

$$I_{ab} = \oint d\sigma_{n+1} \frac{\sum_{c=1}^{n} \frac{\epsilon_{n+1}\cdot k_c}{\sigma_{n+1}-\sigma_c}}{\sum_{d=1}^{n} \frac{p\cdot k_d}{\sigma_{n+1}-\sigma_d}} \frac{1}{(\sigma_{n+1}-\sigma_a)(\sigma_{n+1}-\sigma_b)}$$
$$= \begin{cases} \frac{\epsilon_{n+1}\cdot k_a}{p\cdot k_a}\frac{1}{\sigma_a-\sigma_b} + \frac{\epsilon_{n+1}\cdot k_b}{p\cdot k_b}\frac{1}{\sigma_b-\sigma_a}, & a \neq b \\ \sum_{\substack{d=1\\ d\neq a}}^{n} \frac{1}{\sigma_a-\sigma_d}\left[\frac{\epsilon_{n+1}\cdot k_d}{p\cdot k_a} - \frac{(p\cdot k_d)(\epsilon_{n+1}\cdot k_a)}{(p\cdot k_a)^2}\right], & a = b \end{cases} \tag{B.3.2}$$

we get the first part in Eq.(4.4.14) multiplying $S^{(0)}(q)$ factor. Similarly one can obtain the other part.

Now we consider the action of $S^{(1)}$ factor on the determinant I_n.

$$\begin{aligned} S^{(1)}(p)I_n = 2\sum_{\substack{a=1\\ a\neq b}}^{n}\sum_{b=1}^{n}\frac{1}{\sigma_a-\sigma_b}\Big(& S^{(1)}_b(k_b\cdot k_a)(-1)^{a+b}\Psi^a_b + S^{(1)}_b(\epsilon_b\cdot\epsilon_a)(-1)^{a+b}\Psi^{n+a}_{n+b} \\ &+ \left[S^{(1)}_b(\epsilon_b\cdot k_a) + S^{(1)}_a(k_a\cdot\epsilon_b)\right](-1)^{n+a+b}\Psi^a_{n+b} \\ &+ \left[S^{(1)}_b(k_b\cdot\epsilon_a) + S^{(1)}_a(\epsilon_a\cdot k_b)\right](-1)^{n}\Psi^a_{n+a}\Big) \end{aligned} \tag{B.3.3}$$

The action of $S^{(1)}_b$ on the momentum part and polarization part are given by

$$\begin{aligned} S^{(1)}_b(p)k^\beta_b &= \frac{\epsilon_{n+1,\,\alpha\nu}k^\alpha_b p_\mu}{p\cdot k_b}k^{[\mu}_b\frac{\partial k^\beta_b}{\partial k_{b,\nu]}} \\ S^{(1)}_b(p)\epsilon^\beta_b &= \frac{\epsilon_{n+1,\,\alpha\nu}k^\alpha_b p_\mu}{p\cdot k_b}\left(\eta^{\nu\beta}\delta^\mu_\sigma - \eta^{\mu\beta}\delta^\nu_\sigma\right)\epsilon^\sigma_b \end{aligned} \tag{B.3.4}$$

The gauge fixing conditions reduce Eq.(B.3.3) to the first term of Eq.(4.4.14). The other term can be calculated in analogous way.

B.3.2 Soft Factor from Pole at Infinity

Deforming the contour around the pole at infinity, the leading order expression can be derived to be

$$(M_N)_\infty = \oint \frac{d\rho}{2\pi i} \int d\mu_n \frac{-2\rho^{-3}}{3\tau^4 (p.q)^2} \left(I_N|_{\xi=2i\rho} + I_N|_{\xi=-2i\rho} \right). \tag{B.3.5}$$

In case of gravity there exists a simple pole at $\rho = \infty$. The equation $f_{n+1} - f_{n+2} = 0$ leads to $\xi = \pm 2i\rho + O(1)$. Then the Pfaffian of Ψ_{n+2} can be expanded as

$$\text{Pf}'(\Psi_{n+2}) = \frac{\tau^2 \, p.q \; \epsilon_{n+1}.\epsilon_{n+2}}{4\rho^2} \text{Pf}'(\Psi_n) + O\left(\frac{1}{\rho^4}\right). \tag{B.3.6}$$

Then from Eq.(B.3.5) we get

$$(M_{n+2})_\infty = -\frac{1}{3} (\epsilon_{n+1}.\epsilon_{n+2})^2 M_n \tag{B.3.7}$$

which is clearly sub-leading in order τ as compared to Eq.(4.4.28).

C Appendix of chapter 4

C.1 Derivation of delta function in Eq.(5.3.39)

u integration over the charge gives a tensor structure of the form

$$s_{zz}(z,\bar{z};z_k,\bar{z}_k) = \frac{1}{(1+z_k\bar{z}_k)}\frac{(\bar{z}-\bar{z}_k)}{(1+z\bar{z})(z-z_k)}. \tag{C.1.1}$$

Covariant derivatives act on this tensor as

$$D^z D^z s_{zz} = \gamma^{z\bar{z}}\gamma^{z\bar{z}} D^2_{\bar{z}} s_{zz} = \gamma^{z\bar{z}}\partial_{\bar{z}}\left(\gamma^{z\bar{z}}\partial_{\bar{z}} s_{zz}\right). \tag{C.1.2}$$

The only non-zero Christoffel is $\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \gamma^{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}$. Now using $\partial_{\bar{z}}\frac{1}{(z-z_k)} = 2\pi\delta^{(2)}(z-z_k)$ we obtain

$$\partial_{\bar{z}}\left(\gamma^{z\bar{z}}\partial_{\bar{z}} s_{zz}\right) = \frac{1}{2}\left(\frac{1+z\bar{z}}{1+z_k\bar{z}_k}\right)2\pi\delta^{(2)}(z-z_k). \tag{C.1.3}$$

C.2 Metric variations

For any general vector fields, $\vec{\xi} = \xi^u\partial_u + \xi^r\partial_r + \xi^A\partial_A$ variations of metric perturbations under $\vec{\xi}$ can be given by the following equations:

$$\mathcal{L}_\xi g_{AB} = r^2(\gamma_{AC}D_B + \gamma_{BC}D_A)\xi^C + 2r\gamma_{AB}\xi^r$$

$$
\begin{aligned}
\mathcal{L}_\xi g_{Ar} &= r^2\gamma_{AC}\partial_r\xi^C - D_A\xi^u \\
\mathcal{L}_\xi g_{rr} &= -2\partial_r\xi^u \\
\mathcal{L}_\xi g_{uu} &= -2\partial_u\left(\xi^u+\xi^r\right) \\
\mathcal{L}_\xi g_{ur} &= -\partial_u\left(\xi^u+\xi^r\right) - \partial_r\xi^r \\
\mathcal{L}_\xi g_{uA} &= r^2\gamma_{AC}\partial_u\xi^C - D_A\left(\xi^u+\xi^r\right).
\end{aligned} \tag{C.2.1}
$$

Here A denotes the spherical coordinates.

C.3 Calculations of linearized gravity equations

In six dimension we can expand the metric perturbations as

$$
\begin{aligned}
h_{rr} &= \frac{1}{r^4}h^{(4)}_{rr} + \ldots \\
h_{Ar} &= \frac{1}{r^2}h^{(2)}_{Ar} + \frac{1}{r^3}h^{(3)}_{Ar} + \ldots \\
h_{AB} &= rh^{(-1)}_{AB} + h^{(0)}_{AB} + \frac{1}{r}h^{(1)}_{AB} + \ldots
\end{aligned} \tag{C.3.1}
$$

Then from the linearized gravity equations (5.1.22) we get the following relations

- $\Box h_{AB} = 0$

$$
\begin{aligned}
\mathcal{O}(1) &: \partial_u h^{(-1)}_{AB} = 0 \\
\mathcal{O}(r^{-1}) &: h^{(-1)}_{AB} = \frac{1}{4}\triangle h^{(-1)}_{AB} \\
\mathcal{O}(r^{-2}) &: \partial_u h^{(1)}_{AB} = (-\frac{1}{2}\triangle + 2)h^{(0)}_{AB} \\
\mathcal{O}(r^{-3}) &: \partial_u h^{(2)}_{AB} = -\frac{1}{4}\triangle h^{(1)}_{AB} + \frac{1}{2}h^{(1)}_{AB} - \frac{1}{2}\left(D_A h^{(2)}_{Br} + D_B h^{(2)}_{Ar}\right)
\end{aligned} \tag{C.3.2}
$$

- $\Box h_{Ar} = 0$

$$\begin{aligned}
\mathcal{O}(r^{-2}) \ &: \ D^B h^{(-1)}_{AB} = 0 \\
\mathcal{O}(r^{-3}) \ &: \ \partial_u h^{(2)}_{Ar} = D^B h^{(0)}_{AB} \\
\mathcal{O}(r^{-4}) \ &: \ \partial_u h^{(3)}_{Ar} = \frac{1}{4}(7-\triangle)h^{(2)}_{Ar} + \frac{1}{2}D^B h^{(1)}_{AB}
\end{aligned} \tag{C.3.3}$$

- $\Box h_{rr} = 0$

$$\mathcal{O}(r^{-5}) \ : \ \partial_u h^{(4)}_{rr} = D^A h^{(2)}_{Ar} - \frac{1}{2}q^{AB}h^{(1)}_{AB} \tag{C.3.4}$$

From the above equations it is evident that $h^{(-1)}_{AB}$ is independent of u and no other components of metric perturbations depend on $h^{(-1)}_{AB}$.

C.4 Perturbative gravity in higher dimensions

We assume the fall off conditions are

$$\begin{aligned}
h_{uu} &= \sum_{n=1}^{\infty} r^{-n} h^{(n)}_{uu}, \qquad h_{ur} = \sum_{n=1}^{\infty} r^{-n} h^{(n)}_{ur}, \qquad h_{uA} = \sum_{n=1}^{\infty} r^{-n} h^{(n)}_{uA}, \\
h_{rr} &= \sum_{n=1}^{\infty} r^{-n} h^{(n)}_{rr}, \qquad h_{rA} = \sum_{n=1}^{\infty} r^{-n} h^{(n)}_{rA}, \qquad h_{AB} = \sum_{n=-1}^{\infty} r^{-n} h^{(n)}_{AB}.
\end{aligned} \tag{C.4.1}$$

C.4.1 Leading order computation

Matter part of stress energy tensor has faster fall-off in r. Therefore from linearized gravity equations (5.1.22) we obtain

$$\begin{aligned}
(1-m)\,\partial_u \bar{h}^{(1)}_{uu} &= 0 \\
(1-m)\,\partial_u \bar{h}^{(1)}_{ur} &= 0
\end{aligned}$$

$$
\begin{aligned}
2\,(2-m)\,\partial_u\bar{h}^{(2)}_{rr} + (2-6m+\Delta)\,\bar{h}^{(1)}_{rr} + 4m\bar{h}^{(1)}_{ur} + 2\gamma^{CB}\bar{h}^{(-1)}_{CB} &= 0\\
(2-m)\,\partial_u\bar{h}^{(1)}_{uA} - \partial_A\left(\bar{h}^{(1)}_{uu} - \bar{h}^{(1)}_{rr}\right) &= 0\\
(2-m)\,\partial_u\bar{h}^{(1)}_{rA} - \partial_A\left(\bar{h}^{(1)}_{ur} - \bar{h}^{(1)}_{rr}\right) - D^C\bar{h}^{(-1)}_{CA} &= 0\\
(1-m)\,\partial_u\bar{h}^{(-1)}_{AB} &= 0. \qquad \text{(C.4.2)}
\end{aligned}
$$

Harmonic gauge condition yields

$$
\begin{aligned}
\partial_u\bar{h}^{(1)}_{ur} &= 0\\
\partial_u\bar{h}^{(1)}_{rr} &= 0\\
\partial_u\bar{h}^{(1)}_{rA} - D^B\bar{h}^{(-1)}_{BA} &= 0. \qquad \text{(C.4.3)}
\end{aligned}
$$

Since $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = \bar{h}_{rr} - 2\bar{h}_{ur} + r^{-2}\gamma^{AB}\bar{h}_{AB}$, so combining above equations we can show for $m \neq 1$

$$
\partial_u\bar{h}^{(1)} = 0 \qquad \Rightarrow \qquad \partial_u h^{(1)} = 0. \qquad \text{(C.4.4)}
$$

This in turn implies $h^{(1)}_{uu}, h^{(1)}_{ur}, h^{(1)}_{rr}$ and $h^{(-1)}_{AB}$ are independent of u.

C.4.2 Charge in higher dimensions ($m \geqslant 2$)

We will use the radiation gauge fixing condition, however in higher dimensions $h^{(0)}_{uA}$ can not be set to zero which otherwise rules out the existence of supertranslation symmetry. Equation of motion implies $\partial_u h^{(0)}_{uA} = 0$ for $m \neq 1$.

Expression of asymptotic conserved charge in $D = 2 + 2m$ dimension is

$$
Q_\xi = \lim_{t\to\infty} \frac{1}{16\pi G} \int_{\mathscr{I}^+} du d^{2m}z\; r^{2m}\left[\Gamma^t_{\nu\lambda}\delta_\xi h^{\nu\lambda} - h^{\nu\lambda}\delta_\xi\Gamma^t_{\nu\lambda}\right].
$$

Non-zero components of $\Gamma^t_{ab}[h]$ are

$$\begin{aligned}
\Gamma^t_{rr} &= \frac{1}{2}\partial_u h_{rr} \\
\Gamma^t_{Ar} &= \frac{1}{2}\partial_u h_{Ar} + \frac{1}{r}h^{(0)}_{uA} \\
\Gamma^t_{AB} &= \frac{1}{2}\partial_u h_{AB} - \frac{1}{2}\left(D_A h^{(0)}_{uB} + D_B h^{(0)}_{uA}\right).
\end{aligned} \tag{C.4.5}$$

and

$$\begin{aligned}
\delta_\xi h^{rr} &= 2\left(\partial_r - \partial_u\right)\xi^r = -\sum_{n=1}^{\infty}\frac{2n}{r^{n+1}}\xi^{r\,(n)} - \sum_{n=1}^{\infty}\frac{2}{r^n}\partial_u\xi^{r\,(n)}, \\
\delta_\xi h^{Ar} &= \left(\partial_r - \partial_u\right)\xi^A + \frac{1}{r^2}D^A\xi^r = \frac{2}{r^2}D^A\left(\frac{1}{2m}\Delta f + f\right) - \sum_{n=2}^{\infty}\frac{1}{r^{n+1}}\left[n\xi^{A\,(n)} + \partial_u\xi^{A\,(n+1)}\right] \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{r^n}D^A\xi^{r\,(n)}, \\
\delta_\xi h^{AB} &= \frac{1}{r^2}\left(D^A\xi^B + D^B\xi^A\right) + \frac{2}{r^3}\gamma^{AB}\xi^r = -\frac{1}{r^3}\left[\left(D^A D^B + D^B D^A\right)f - \frac{1}{m}\gamma^{AB}\Delta f\right] \\
&\quad + \sum_{n=2}^{\infty}\frac{1}{r^{n+2}}\left[D^A\xi^{B\,(n)} + D^B\xi^{A\,(n)}\right] + \sum_{n=1}^{\infty}\frac{2}{r^{n+3}}\gamma^{AB}\xi^{r\,(n)}.
\end{aligned} \tag{C.4.6}$$

Variation of Christoffel symbols in the second term of the charge can be calculated to be

$$\begin{aligned}
\delta_\xi\Gamma^t_{ur} &= \partial_r\partial_u\left(\xi^u + \xi^r\right) = \sum_{n=1}^{\infty}\frac{1}{r^n}\partial_r\partial_u\left(\xi^{u\,(n)} + \xi^{r\,(n)}\right), \\
\delta_\xi\Gamma^t_{uu} &= \partial_u^2\left(\xi^u + \xi^r\right) = \sum_{n=1}^{\infty}\frac{1}{r^n}\partial_u^2\left(\xi^{u\,(n)} + \xi^{r\,(n)}\right), \\
\delta_\xi\Gamma^t_{uA} &= D_A\partial_u\left(\xi^u + \xi^r\right) = \sum_{n=1}^{\infty}\frac{1}{r^n}D_A\partial_u\left(\xi^{u\,(n)} + \xi^{r\,(n)}\right), \\
\delta_\xi\Gamma^t_{rr} &= \partial_r^2\left(\xi^u + \xi^r\right) = \sum_{n=1}^{\infty}\frac{n\,(n+1)}{r^{n+2}}\left[\xi^{u\,(n)} + \xi^{r\,(n)}\right], \\
\delta_\xi\Gamma^t_{Ar} &= r\partial_r\left(\frac{1}{r}D_A\left(\xi^u + \xi^r\right)\right) = -\frac{1}{r}D_A\left(f + \frac{1}{2m}\Delta f\right) - (n+1)\sum_{n=1}^{\infty}\frac{1}{r^{n+1}}D_A\left(\xi^{u\,(n)} + \xi^{r\,(n)}\right), \\
\delta_\xi\Gamma^t_{AB} &= D_A D_B\left(\xi^u + \xi^r\right) + r\gamma_{AB}\left(\partial_r - \partial_u\right)\left(\xi^u + \xi^r\right) = D_A D_B\left(f + \frac{1}{2m}\Delta f\right) \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{r^n}\left(D_A D_B - n\gamma_{AB}\right)\left(\xi^{u\,(n)} + \xi^{r\,(n)}\right) - \sum_{n=1}^{\infty}\frac{1}{r^{n-1}}\partial_u\left(\xi^{u\,(n)} + \xi^{r\,(n)}\right).
\end{aligned}$$

(C.4.7)

Again we know

$$\begin{aligned} h^{rr} &= h^{uu} = -h^{ru} = h_{rr}, \\ h^{rA} &= -h^{uA} = \frac{1}{r^2}\gamma^{AB}h_{rB}, \\ h^{AB} &= \frac{1}{r^4}\gamma^{AM}\gamma^{BN}h_{MN}. \end{aligned} \tag{C.4.8}$$

C.4.3 Charge in 6 dimension

The finite contribution is

$$\begin{aligned} Q_{\xi,\,\text{finite}} &= \frac{1}{16\pi G}\int_{\mathscr{I}^+} dud^4z\Big[-\partial_u h^{(2)}_{rr}\left(\xi^{r\,(1)} + \partial_u\xi^{r\,(2)}\right) - \partial_u h^{(3)}_{rr}\partial_u\xi^{r\,(1)} \\ &+\partial_u h^{(2)}_{Ar}D^A\left(\frac{1}{4}\Delta f + f\right) - \partial_u h^{(1)}_{Ar}\left(\xi^{A\,(2)} + \frac{1}{2}\partial_u\xi^{A\,(3)}\right) + \frac{1}{2}\partial_u h^{(1)}_{Ar}D^A\xi^{r\,(3)} + \frac{1}{2}\partial_u h^{(2)}_{Ar}D^A\xi^{r\,(2)} \\ &+\frac{1}{2}\partial_u h^{(3)}_{Ar}D^A\xi^{r\,(1)} - h^{(0)}_{uA}\left(2\xi^{A\,(2)} + \partial_u\xi^{A\,(3)} - D^A\xi^{r\,(3)}\right) \\ &-\frac{1}{2}\partial_u h^{(1)}_{AB}\left(D^AD^Bf + D^BD^Af - \frac{1}{2}\gamma^{AB}\Delta f\right) + \frac{1}{2}\partial_u h^{(0)}_{AB}\left(D^A\xi^{B\,(2)} + D^B\xi^{A\,(2)}\right) \\ &-\frac{1}{2}\left(D_A h^{(0)}_{uB} + D_B h^{(0)}_{uA}\right)\left(D^A\xi^{B\,(2)} + D^B\xi^{A\,(2)} - 2\gamma^{AB}\xi^{r\,(1)}\right) \\ &+h^{(1)}_{rr}\partial_r\partial_u\left(\xi^{u\,(3)} + \xi^{r\,(3)}\right) + h^{(2)}_{rr}\partial_r\partial_u\left(\xi^{u\,(2)} + \xi^{r\,(2)}\right) + h^{(3)}_{rr}\partial_r\partial_u\left(\xi^{u\,(1)} + \xi^{r\,(1)}\right) \\ &-h^{(1)}_{rr}\partial_u^2\left(\xi^{u\,(3)} + \xi^{r\,(3)}\right) - h^{(2)}_{rr}\partial_u^2\left(\xi^{u\,(2)} + \xi^{r\,(2)}\right) - h^{(3)}_{rr}\partial_u^2\left(\xi^{u\,(1)} + \xi^{r\,(1)}\right) \\ &+\gamma^{AB}h^{(1)}_{rB}\Big\{D_A\partial_u\left(\xi^{u\,(1)} + \xi^{r\,(1)}\right) + D_A\left(\frac{1}{4}\Delta f + f\right)\Big\} - 2h^{(1)}_{rr}\left(\xi^{u\,(1)} + \xi^{r\,(1)}\right) \\ &-\gamma^{Am}\gamma^{BN}h^{(0)}_{MN}\Big\{D_A\left(\frac{1}{4}\Delta f + f\right) - \partial_u\left(\xi^{u\,(1)} + \xi^{r\,(1)}\right)\Big\}\Big] \end{aligned} \tag{C.4.9}$$

There are other divergent terms also.

If we restrict the fall-off conditions on the metric perturbations such that $h_{rr} \sim O\left(r^{-4}\right)$ *and* $h_{Ar} \sim O\left(r^{-2}\right)$ *we find the terms containing subleading vector fields vanish and this also forces supetranslation symmetries to be killed. The other finite terms which contain*

f become zero because of the condition

$$D^A \left(\frac{1}{4} \varDelta f + f \right) = 0. \tag{C.4.10}$$

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