Positive Geometry Of Scalar Theories

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DECLARATION

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In this thesis based on [47, 48] we extended the amplituhedron program to a large class of scalar field theories. We have shown that there are a class of convex polytopes (accordiohedra) which can be embedded in kinematic space. We provided a prescription to get the tree-level planar amplitude as a weighted sum of canonical functions associated with all the accordiohedra of a given level n for ϕ^p theories. We introduced the notion of primitive accordiohedra to simplify our computations and provided a formula for the number of primitives at arbitrary level n. We provided the results of the implementation of our prescription to compute weights.

The thesis contains the following results:

- **Planar amplitudes from accordiohedra :** We provide an embedding of the accordiohedron $\mathcal{AC}^P_{p,n}$ into kinematic space and show that a weighted sum of the canonical functions of all the primitive accordiohedra of a given dimension n does indeed produce the right planar $p + n(p-2)$ amplitude for ϕ^p interactions .

- **Formula for counting primitives:** We prove a formula to count the number of primitive accordiohedra of a given dimension n and classify them for $n \leq 3$ for any ϕ^p interactions.

- **Computation of weights:** We provide a prescription to find the weights for accordiohedra of any dimension n and demonstrate our prescription to compute weights for $n \leq 3$ and all $p \leq 12$.

- **Factorisation:** We prove that the accordiohedra $\mathcal{AC}^{P}_{p,n}$ factorise geometrically i.e. on any facet $X_{ij} = 0$, the accordiohedron $\mathcal{AC}^{P}_{p,n}$ factorises into product of lower dimensional accordiohedra

$$\mathcal{AC}^{P}_{p,n}\Big|_{X_{ij}\,=\,0} \equiv \mathcal{AC}^{P_1}_{p,m} \times \mathcal{AC}^{P_2}_{p,n+2-m}$$

  where P_1 and P_2 are such that $P_1 \cup P_2 \cup (ij) = P$.
  P_1 is the p-angulation of the polygon $\{i,\ i+1, \ldots,\ j\}$ and P_2 is the p-angulation of $\{j, j+1, \ldots, n, \ldots, i\}$.

Re-formulating scattering amplitudes as differential forms on positive geometries (succinctly called the Amplituhedron program) has had profound impact on how we understand Quantum field theories and how properties like unitarity and locality are a natural consequence of the positive geometries. In theories like $\mathcal{N} = 4$ Super Yang Mills theory, the Amplituhedron program offers conceptual as well as striking technical advancements in the understanding of planar S-matrix. In the non super-symmetric world, these ideas were extended to bi-adjoint scalar theory in [43] where it was shown that the corresponding amplituhedron is an associahedron in Kinematic space and the canonical form on this associahedron was proportional to the scattering amplitude.

These ideas were extended from cubic to quartic interactions in [47] where the underlying positive geometry was Stokes polytope. However unlike Associahedron, which is unique (in a given dimension), there are several Stokes polytopes in any given dimension and it was shown that one had to sum over canonical forms of all such polytopes to obtain scattering amplitude of ϕ^4 theory. Not all Stokes polytopes contributed equally but one had to assign different weights to each Stokes polytope. In [47] it was argued that these weights were not assigned to a given Stokes polytope but to an equivalence class of such polytopes which were related to each other by cyclic permutations and that in each such class, one could choose a representative that we called primitive. Whence the computation of scattering amplitude reduced to the problem of finding all the primitives and assigning weights to them.

In this thesis, continuing along the lines of [43] we extended the Amplituhedron program to (tree-level) planar amplitudes for massless scalar field theories with ϕ^p interactions. We have shown that the positive geometry underlying scattering amplitudes in this theory is a class of polytopes called accordiohedron. Accordiahedron is a family of polytopes whose members include associahedron and Stokes Polytope.

Just as in the case of quartic interactions there exists no single accordiohedron of a given dimension n and a weighted sum of canonical forms of all the accordiohedra of a given dimension n does indeed produce the full planar amplitude. This re-affirms and generalises the result we had obtained in in the case of quartic interactions.

We gave an enumeration of the number of primitives at arbitrary dimension n and a complete classification of primitive diagrams for $n \leq 3$. We then gave a prescription to compute the weights and provided the results for the weights obtained by using our prescription for all $p > 4$ in $n = 1, 2$ dimensions and $p \leq 12$ for $n = 3$.

Accordioheron is a very general polytope and one may wonder if they can be used to extend the Amplituhedron program to Scalar theories with mixed-vertices (e.g. theories with cubic as well as quartic interactions). It turns out that this is indeed the case [104]. Our work thus shows that the positive geometry underlying planar amplitudes in any scalar field theory is an Accordioheron.

There are several outstanding questions that arise out of our analysis. In [45] it was shown that the 1-loop integrand of ϕ^3 theory also corresponds to canonical form on a polytope which is well known in mathematical literature called Halohedron [105]. Whether this idea can be extended to 1-loop integrand of ϕ^p theories remains to be seen.

One of the most striking results obtained in [43] was the derivation of CHY formula for bi-adjoint scalar ϕ^3 interactions from the canonical form on kinematic space associahedron. Although CHY integrands exist for ϕ^p interactions for $p > 3$ they do not admit any such geometric interactions . Our hope is that understanding of kinematic space Accordiohedron is the first step in "geometrizing" the CHY formula for ϕ^p theories.

There is also an obvious question of how to go beyond planar amplitudes and is there a polytope realisation for full tree-level scattering amplitude of ϕ^p theory. In the massless ϕ^3 case, certain progress in this direction was already reported in [99,106,107]. It was shown that a wider class of amplitudes than simply planar ones could be computed with the corresponding polytopes being generalisation of associahedra known as Cayley polytopes, which is a member of a "complimentary" family of polytopes known as graph associahedra which also had deep connections with geometry of scattering amplitudes [99,100]. In contrast to accordiohedron, graph associahedra can not always be obtained by considering dissections of polygons. Graph associahedra is a set of polytopes which includes, associahedron, permutahedron , halohedron etc. Many of these members, e.g. Permutahedron and Halohedron are associated to amplitudes in bi-adjoint scalar theory with non-planar and 1-loop amplitudes respectively. It is intriguing that one class of polytopes helps one to move beyond tree-level and planarity in bi-adjoint ϕ^3 theories and the other class helps one move beyond cubic vertices. It will be interesting to see if by generalising accordiohedra (to more general polytopes associated with p-angulations) we can go beyond planar diagrams in ϕ^p theory.

Synopsis

Scattering amplitudes are at the heart of high energy physics. They lie at the intersection between quantum field theory and collider experiments. The usual method of computing scattering amplitudes using a Lagrangian and evaluating Feynman diagrams give us a nice physical picture and makes locality manifest [1]. It has been known for quite sometime now that the method of Feynman diagrams is not very efficient in computing amplitudes, as the number of diagrams grows rapidly with external particles though due to various seemingly miraculous cancellations the final answer obtained by summing over all the diagrams can be remarkably simple [2]. Feynman diagrams are also not very useful for revealing hidden symmetries/structures like the dual conformal invariance of planar $\mathcal{N} = 4$ SYM or the BCJ Double copy relations which relate Yang Mills and gravity amplitudes [3–14].

Over the years various on-shell methods have been developed to compute scattering amplitudes for various theories without using the Lagrangian which have been collectively called the *amplitudes program* [15–19]. The amplitudes program consists of various different methods like Unitarity cuts [20–28], BCFW recursion relations [29–34], CHY [35–38] etc it is not yet understood how these different methods are manifestly related though they all compute the same scattering amplitude. One of the major goals of the amplitudes program is to unify these seemingly different methods into a single framework.

One such possible candidate framework was proposed in [39–42] for planar $\mathcal{N} = 4$ SYM where scattering amplitudes were re-formulated in a space-time independent way as differential forms on a positive geometry living in an auxiliary grassmanian space the *Amplituhedron*. In this remarkable new picture scattering amplitudes are to be thought of more fundamentally as differential forms rather than functions, unitarity and locality of the theory emerged from the geometric properties of the amplituhedron rather than being inputs to the theory. It also made manifest the dual conformal invariance of $\mathcal{N} = 4$ SYM.

In the non-supersymmetric world this picture was also shown to be valid for tree level amplitudes for bi-adjoint ϕ^3 theory [43] where a precise connection was established between scattering forms and a polytope called the *associahedron* living in kinematic space. It was further shown that various properties like soft limits, recursion relations follow from the geometric properties of the asociahedron. Another beautiful result was established in [43] that gave a new understanding of the CHY formulae for tree-level scattering amplitudes. It was shown that the CHY integrand for ϕ^3 theory is a pushforward of the canonical scattering form on the associahedron. The program was further extended to 1-loop amplitudes in ϕ^3 theory [44, 45].

It is quite natural to ask for what class of theories does such a formulation exist. In particular since tree level CHY formulae exist for amplitudes in a wide class of quantum field theories including tree-level planar diagrams in scalar field theories with ϕ^p $(p > 3)$ interactions [46]. Thus it is a natural to see if the Amplituhedron program can be extended for all ϕ^p $(p > 3)$ theories.

In this thesis we answer this question in the affirmative by showing that there exists a precise connection between scattering forms and a polytope called the *Accordiohedron* living in kinematic space for all scalar ϕ^p interactions [47, 48].

The kinematic space $\mathcal{K}_n$ of n massless momenta is the $\frac{n(n-3)}{2}$ dimensional space spanned by the independent Mandelstram variables $s_{ij} = (p_i + p_j)^2$. A more natural basis for particles with a fixed ordering is given by the planar kinematic variables $X_{ij} = (p_i + p_{i+1} + ... + p_{j-1})^2$.

We can translate from s_{ij} to the X_{ij} basis using the following relation

$$s_{ij} = X_{i+1\ j} + X_{i\ j+1} - X_{i\ j} - X_{i+1\ j+1}. \tag{1}$$

The planar kinematic variables have a nice interpretation as the diagonal of polygon with sides $p_i, ..., p_{j-1}$. *There exists a one to one correspondence between planar tree level Feynman diagrams in ϕ^p theory with $p + n(p-2)$ external legs and p-angulations of a $p + n(p-2)$-gon.* We can define a planar scattering form Ω_n for each Feynman diagram with $p + n(p-2)$ legs as the n-form with simple poles when each of the n propagators $X_{i_1 j_1}, ..., X_{i_n j_n}$ goes on shell with residue ± 1. The full planar scattering form is obtained by summing over all Feynman diagrams and is not unique.

The accordiohedron $\mathcal{AC}^P_{p,n}$ is a combinatorial polytope associated with p-angulations of polygons [49, 50]. It is obtained by starting with any complete p-angulation P of a $p + n(p-2)$-gon and performing recursively a series of Q-flips till no new p-angulations are generated. There is an embedding of the accordiohedron in kinematic space $\mathcal{K}_n$. The accordiohedra $\mathcal{AC}^P_{p,n}$ of a given dimension n is not unique for $n \geq 2$ and depends on the reference p-angulation P (unless $p = 3$). When restricted to the subset of Feynman diagrams that are vertices of one of the accordiohedra $\mathcal{AC}^P_{p,n}$ then we can define a unique planar scattering form Ω^P_n by demanding that any two diagrams related to each other by replacing a diagonal with its Q-flip have opposite residues. This unique planar scattering form turns out to be the canonical form $\omega^P_{n,p}$ associated with the $\mathcal{AC}^P_{p,n}$. The canonical function $m^P_{n,p}$ can be obtained from the canonical form $\omega^P_{n,p}$ once we factor out the top form.

The weighted sum of the canonical functions $m^P_{n,p}$ of all the accordiohedra $\mathcal{AC}^P_{p,n}$ of a given dimension n corresponding to all possible p-angulations P with appropriate weights α_P when pulled back onto the accordionedra embedded in kinematic space gives the right scattering amplitude $\mathcal{M}_n$

$$\mathcal{M}_n = \sum_P \alpha_P \, m^{(P)}_{p,n}.$$

We can simplify this procedure by introducing the notion of *primitive* p-angulations which are the subset of rotationally inequivalent p-angulations from which all other p-angulations can be obtained via rotations .

The thesis contains the following results:

- **Planar amplitudes from accordiohedra :** We provide an embedding of the accordiohedron $\mathcal{AC}^P_{p,n}$ into kinematic space and show that a weighted sum of the canonical functions of all the primitive accordiohedra of a given dimension n does indeed produce the right planar $p + n(p - 2)$ amplitude for ϕ^p interactions .
- **Formula for counting primitives:** We prove a formula to count the number of primitive accordiohedra of a given dimension n and classify them for $n \leq 3$ for any ϕ^p interactions.
- **Computation of weights:** We provide a prescription to find the weights for accordiohedra of any dimension n and demonstrate our prescription to compute weights for $n \leq 3$ and all $p \leq 12$.
- **Factorisation:** We prove that the accordiohedra $\mathcal{AC}^P_{p,n}$ factorise geometrically i.e. on any facet $X_{ij} = 0$, the accordiohedron $\mathcal{AC}^P_{p,n}$ factorises into product of lower dimensional accordiohedra

  $$\mathcal{AC}^P_{p,n}\Big|_{X_{ij}\,=\,0} \equiv \mathcal{AC}^{P_1}_{p,m} \times \mathcal{AC}^{P_2}_{p,n+2-m}$$

  where P_1 and P_2 are such that $P_1 \cup P_2 \cup (ij) = P$.
  P_1 is the p-angulation of the polygon $\{i,\ i+1, \ldots,\ j\}$ and P_2 is the p-angulation of $\{j, j+1, \ldots, n, \ldots, i\}$.

We shall now provide a few technical definitions that will useful for elaborating our results.

Positive geometry

A positive geometry $\mathcal{A}$ is a closed geometry [51] with boundaries of all co-dimensions with a *unique* differential form $\Omega(\mathcal{A})$ called its canonical form that satisfies:

1. It has simple poles on the boundary $\mathcal{A}$ and only on the boundary of $\mathcal{A}$.

2. At every boundary $\mathcal{B}$, the residue of the canonical form is the canonical form of the boundary

$$Res_B\, \Omega(\mathcal{A}) = \Omega(B).$$

3. If $\mathcal{A}$ is a point then $\Omega(A) = \pm 1$ depending on the orientation.

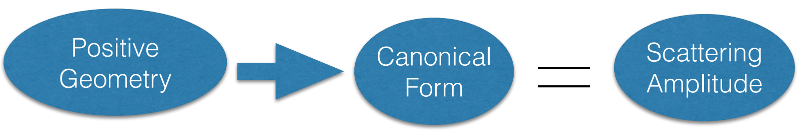
4. For any pair of positive geometries $\mathcal{A}$ and $\mathcal{B}$

$$\Omega(\mathcal{A} \times \mathcal{B}) = \Omega(\mathcal{A}) \wedge \Omega(\mathcal{B}).$$

Polytopes and grassmanians are examples of positive geometries.

The Amplituhedron program can be summarised as follows:

For a given theory there is some putative positive geometry living in kinematic space and when the canonical form is pulled back onto the geometry it gives the scattering amplitude.



Accordiohedron

Let A be a convex polygon. Let us consider the division of A into identical p-gons which we call p-angulation of A. We can represent A as a set of points on the unit circle oriented clockwise where the arcs represent edges of A and chords represent diagonals of A. The simplest example is the case where we divide $(2p-2)$-gon A into two p-gons. There are $(p-1)$ possible p-angulations which correspond to having the diagonals $\{(1,p),(2,p+1),\cdots,(p-1,2p-2)\}$.

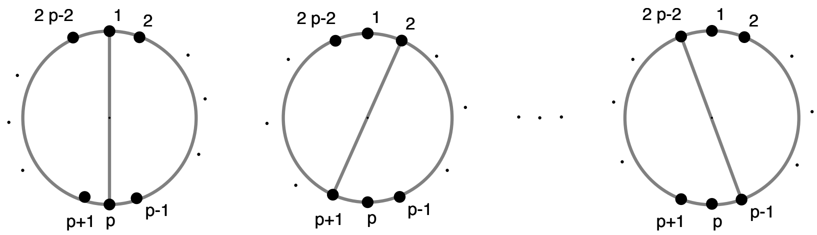


Figure 1: The (p-1) different p-angulations of A

We define a notion of *Q-flip* for each diagonal (i, j) as:

$$\begin{aligned} (i,j) &\rightarrow (k,l) \\ \text{with } (k,l) &= (\,Mod(i+p-2,2p-2), Mod(j+p-2,2p-2)). \end{aligned} \tag{2}$$

The diagonal (i, j) is said to be Q-compatbile with the diagonal (k,l). Q-compatability is not an equivalence relation. We can use Q-flips to define accordion lattices $\mathcal{AL}^P_{p,n}$ of dimension n associated with a reference p -angulation P as follows:

We can start with any p-angulation P of a convex polygon with n diagonals,

• *In the first step, for each of the n diagonals, we go to the unique* $(2p-2)$*-gon which contains it and replace it with its Q-compatible diagonal.*

• *In the second step, for each of the n p-angulations at the end of step one we choose one of the original* $(n-1)$ *diagonals and replace it with its Q-compatible diagonal as in step* *one.*

• *We repeat this till none of the original n diagonals remain in step n.*

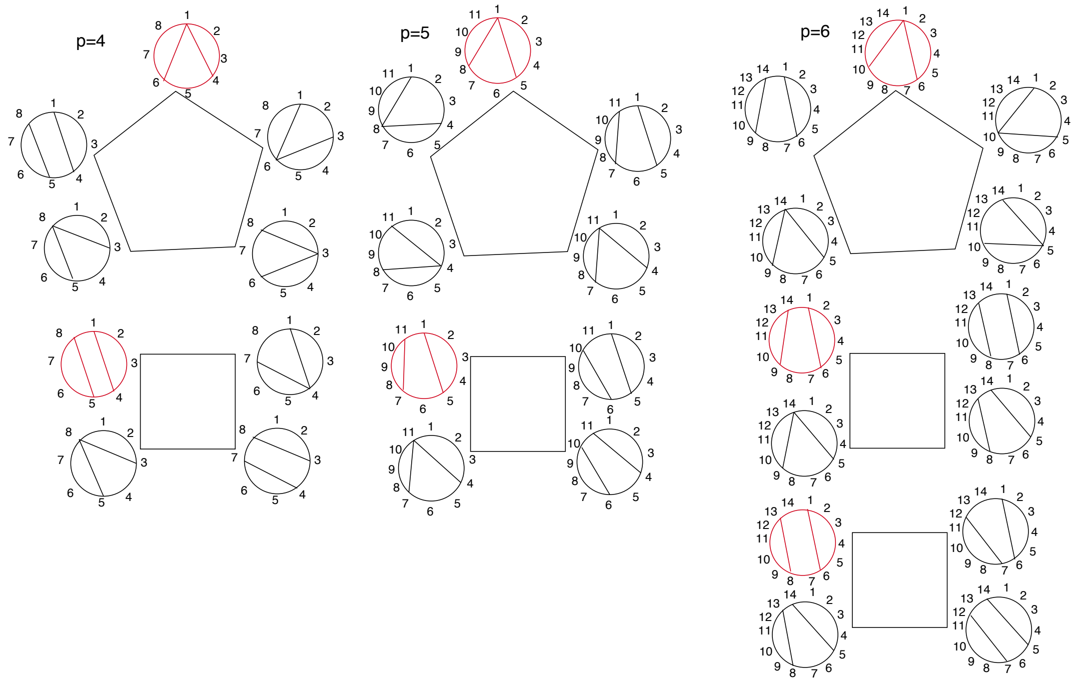


Figure 2: accordiohedra for the n=2 case. The red circles indicate the reference p-angulations.

This generates a graph which is the 1-skeleton of a convex polytope called the *Accordiohedron* [49, 50], which we shall denote by $\mathcal{AC}^{P}_{p,n}$. The correspondence between the faces of the accordiohedron and p-angulations is as follows

Vertices $\leftrightarrow$ **Complete p-angulations**

Edges $\leftrightarrow$ **Q-Flips between them**

k-Faces $\leftrightarrow$ **k-partial p-angulations**.

By a k-partial p-angulation we mean a dissection of the polygon that contains exactly k p-gons. A complete p-angulation contains maximal number of p-gons.

In the case of cubic interactions ($p = 3$), equation (3.1) reduces to $(i, j) \to (Mod(i + 1, 4), Mod(j + 1, 4))$ which is the usual *mutation* rule and the resulting accordiohedron

$\mathcal{AC}^{P}_{3,n}$ is the associahedron [43]

We shall now elaborate our results.

Planar scattering form for ϕ^p interactions

We would like to define a planar scattering form for ϕ^p interactions. We can associate to each planar graph g with propagators $X^{-1}_{i_1 j_1}, X^{-1}_{i_2 j_2}, \cdots, X^{-1}_{i_n j_n}$ a scattering form

$$\frac{\sigma(g)}{\prod_{k=1}^{n} X_{i_k j_k}} dX_{i_1 j_1} \wedge dX_{i_2 j_2} \wedge \cdots \wedge dX_{i_n j_n},$$

where $\sigma(g) = \pm 1$.

Thus, when we sum over all planar graphs we have several possible scattering forms [2]. We choose a particular reference graph g (equivalently a p-angulation P) and look at only those subset of graphs which are related to this graph by a sequence of Q-flips namely all the vertices of the *accordiohedron*. If a graph g' is related to g by an odd (even) number of Q-flips we can associate $-$ $(+)$ sign to it. Thus, we can define a P dependent planar scattering form Ω^P_n

$$\Omega^P_n = \sum_{Q-flips} \frac{(-1)^{\sigma(Q-flip)}}{\prod_{k=1}^{n} X_{i_k j_k}} dX_{i_1 j_1} \wedge dX_{i_2 j_2} \wedge \cdots \wedge dX_{i_n j_n}.$$

Since the Q-compatible p-angulations corresponding to any reference P does not exhaust all the p-angulations, we need to define such a planar scattering form for each P.

[2] We do not have a notion of *projectivity* except in the case of $p = 3$ which helps us fix a unique scattering form [43]

Accordiohedron as positive geometry of ϕ^p

In the thesis we show that the accordiohedron $\mathcal{AC}^{(P)}_{p,n}$ is the positive geometry associated to ϕ^p interactions. We shall do this by first embedding the accordiohedron into kinematic space and then showing that the canonical form of the accordiohedron when pulled back gives the right planar scattering amplitude for ϕ^p interactions.

Locating the accordiohedron inside kinematic space

We locate the *accordiohedron* $\mathcal{AC}^{(P)}_{p,n}$ inside the positive region of kinematic space $X_{ij} \geq 0$ for all $1 \leq i < j < p + (p-2)n$ by imposing the following constraints

$$\begin{aligned} s_{ij} &= -c_{ij}\,; \quad for \;\; 1 \leq i < j \leq p-1+(p-2)n, \; |i-j| \geq 2 \\ X_{r_i s_i} &= d_{r_i s_i}\,; \; s.t. \; P \cup_{i=1}^{n} \{(r_i, s_i)\} \text{ is a complete triangulation,} \end{aligned} \tag{3}$$

where $c_{ij}, d_{r_i s_i}$ are positive constants.

Physically we choose the above set of constraints as they do not appear as propagators of any ϕ^p graph. The first constraint above is the famous associahedron embedding [43]. We have thus embedded the accordiohedron inside the associahedron. The positivity of X_{ij}'s, the above constraints along with the equation (1) are a set of inequalities satisfied by the X_{ij} which makes the convexity of the accodiohedron manifest.

The full planar amplitude can be obtained as a weighted sum of canonical functions $m^{P}_{n,p}$ of all accordiohedra $\mathcal{AC}^{P}_{p,n}$ of dimension n which is obtained by pulling the planar scattering form Ω^{P}_{n} back onto (3.2) to get $\omega^{P}_{p,n}$ and factoring out the top form

$$\mathcal{M}_n = \sum_{P} \alpha_P \, m^{(P)}_{p,n}.$$

Primitives and Weights

We simplify our computation by considering a subset of p-angulations $\{P_1, \cdots, P_I\}$ called primitive p-angulations for which :

(a) no two members of the set are related to each other by cyclic permutations and

(b) all the other p-angulations can be obtained by a (sequence of) cyclic permutations of one of the P's belonging to the set.

The primitives are a class of rotationally inequivalent diagrams. Since, a rotation does not change the relative configuration of diagonals it is clear that accordiohedra remain the same for all the diagrams that belong to a primitive class and that the weights depend only on primitives

$$\mathcal{M}_n = \sum_{\substack{rotations\\ \sigma}} \sum_{\substack{primitives\\ P}} \alpha_P \, m_{p,n}^{(\sigma.P)}. \tag{4}$$

In the thesis we shall provide a formula for the number of primitives at arbitrary level n and also provide complete classification of primitives unto $n = 3$:

For $n = 1$ there is only one primitive.

For $n = 2$ there are $\lfloor \frac{p-2}{2} \rfloor$ primitives which we label as $(p - 2 - i, i)$.

For $n = 3$ there are $\frac{p(2p-1)}{3}$ primitives which can be divided into two types which are denoted as $[i, j]$ and (k_1, k_2, k_3).

Counting Primitives for ϕ^p case

The number of *primitives p-angulations* of an $p + (p - 2)n$-gon is the same as the number of orbits of the cyclic group $\mathbb{Z}_{p+(p-2)n}$ when it acts on the set of all p-angulations. In the thesis we provide a counting for the number of such orbits. The total number of such p-angulations is

$$p_n = \begin{cases} \frac{1}{(p-2)n+p} F_{n,p} + \frac{1}{2} F_{\frac{n+1}{2},p} + \frac{1}{p} \sum_{d|Gcd(n,p)} \phi(d) \tilde{F}_{n/d,p,p/d}, & \text{if n is odd} \\ \frac{1}{(p-2)n+p} F_{n,p} + \frac{1}{p} \sum_{d|Gcd(n,p)} \phi(d) \tilde{F}_{n/d,p,p/d}, & \text{if n is even,} \end{cases}$$

where, $\tilde{F}_{a,b,c} = \frac{c}{(b-2)a+c} \binom{(b-2)a+c}{a}$, $\tilde{F}_{a,b,1} = F_{a,b}$ and $\phi(d)$ is the Euler totient function.

Determination of the weights

We need to determine the weights as these form part of the data to determine the full amplitude. We shall provide a prescription to do this. As we had emphasised before the weights depend only on the primitive class to which a particular p-angulation belongs thus it is sufficient to determine the weights for the primitive p-angulations. We do this by demanding that

$$\sum_{i=1}^{l} n_p^i \alpha_p^i = 1 \; for\ each\ primitive\ 1 \leq i \leq l,$$

where n_p^i is number of times primitive i appears in the vertices of all accodiohedra.

We shall now state the results for the weights obtained by implementing our prescription corresponding to primitives for all $n \leq 3$:

- For any p with $n = 1$ there is only one primitive whose weight is

$$\alpha = \frac{1}{2}.$$

- For any p with $n = 2$ and the results are the following

For $p = 2k$

$$\alpha_{(p-2-i,i)} = \begin{cases} \frac{1}{6} \ , i\ even \\ \\ \frac{1}{3} \ , i\ odd. \end{cases}$$

and For $p = 2k + 1$

$$\alpha_{(p-2-i,i)} = \frac{k+1+i}{3p-4}$$

with $i = 0, ..., k-1$.

- The α's for $n = 3$ case with $p \leq 12$ are given below (for the sake of brevity we shall call α's corresponding to $[i, j], (k_1, k_2, k_3)$ as $[i, j], (k_1, k_2, k_3)$)

  If p is even then :

  $[i, i] = \frac{1}{24}, \frac{5}{24}, \frac{1}{24}, ...$; $[1, i] = \frac{3}{24}, \frac{1}{24}, \frac{3}{24}, ...$; $[2, i] = \frac{3}{24}, \frac{5}{24}, \frac{3}{24}, ...$; $[3, i] = \frac{3}{24}, \frac{1}{24}, \frac{3}{24}, ...$; ...

  $(k_1, k_2, k_3) = (k_2, k_1, k_3) = \frac{6}{24}, \frac{2}{24}, \frac{6}{24}, ...$; $(k_1, 0, k_2) = (0, k_1, k_2) = \frac{2}{24}$, $(0, 0, p-3) = \frac{2}{24}$

  If p **is odd then the results for the first few cases are** :

  p=5 : $[i, i] = \frac{1}{20}, \frac{3}{20}$ with $i = 1, 2$; $[1, 2] = \frac{2}{20}$; $(1, 1, 0) = \frac{2}{20}$; $(0, 0, 2) = \frac{2}{20}$.

  p=7 : $[i, i] = \frac{3}{64}, \frac{11}{64}, \frac{7}{64}$ with $i = 1, 2, 3$; $[1, j] = \frac{7}{64}, \frac{5}{64}$, with $j = 2, 3$; $[2, 3] = \frac{9}{64}$; $(1, 1, 2) = \frac{10}{64}$,

  $(0, 1, 3) = (1, 0, 3) = \frac{6}{64}$; $(2, 0, 2) = \frac{6}{64}$; $(0, 0, 4) = \frac{6}{64}$.

  p=9 : $[i, i] = \frac{2}{44}, \frac{8}{44}, \frac{4}{44}, \frac{6}{44}$, with $i = 1, 2, 3, 4$; $[1, j] = \frac{5}{44}, \frac{3}{44}, \frac{4}{44}$, with $j = 2, 3, 4$; $[2, j] = \frac{6}{44}, \frac{7}{44}$, with $j = 3, 4$; $[3, 4] = \frac{5}{44}$; $(2, 2, 2) = \frac{4}{44}$; $(1, 1, 4) = \frac{8}{44}$; $(1, 2, 3) = (2, 1, 3) = \frac{6}{44}$; $(3, 0, 3) = \frac{4}{44}$,

  $(1, 0, 5) = (0, 1, 5) = \frac{6}{64}$; $(2, 0, 4) = (0, 2, 4) = \frac{4}{44}$; $(0, 0, 6) = \frac{4}{44}$.

p=11: $[i,i] = \frac{10}{112}, \frac{21}{112}, \frac{9}{112}, \frac{17}{112}, \frac{13}{112}$ with $i = 1,2,3,4,5$; $[1,j] = \frac{13}{112}, \frac{7}{112}, \frac{11}{112}, \frac{9}{112}$, with $j = 2,3,4,5$;

$[2,j] = \frac{15}{112}, \frac{19}{112}, \frac{17}{112}$, with $j = 3,4,5$; $[3,j] = \frac{13}{112}, \frac{11}{112}$, with $j = 4,5$; $[4,5] = \frac{11}{112}$; $(1,1,6) = \frac{22}{112}$;

$(2,2,4) = \frac{10}{112}$; $(3,3,2) = \frac{14}{112}$; $(0,4,4) = \frac{10}{112}$; $(1,0,7) = (0,1,7) = \frac{10}{112}$; $(2,0,6) = (0,2,6) = \frac{10}{112}$;

$(3,0,5) = (0,3,5) = \frac{10}{112}$; $(1,2,5) = (2,1,5) = \frac{14}{112}$; $(1,3,4) = (3,1,4) = \frac{14}{112}$; $(0,0,8) = \frac{5}{112}$.

Factorisation

One of the remarkable consequences of relating tree level scattering amplitudes to positive geometries like associahedron, Stokes polytope is the fact that geometric factorisation implied physical factorisation of scattering amplitude. This in turn implied that tree-level unitarity and locality are emergent properties of the positive geometry [43]. In the thesis we show that this is indeed the case even for planar amplitudes in massless ϕ^p theory. We shall first argue that the geometric factorisation of accordiohedron holds and then show that this leads to the factorisation of the amplitude.

Conclusion

This synopsis contains a brief summary of our work [47, 48] which extends the amplituhedron program to a large class of scalar field theories. We have shown that there are a class of convex polytopes (accordiohedra) which can be embedded in kinematic space. We provided a prescription to get the tree-level planar amplitude as a weighted sum of canonical functions associated with all the accordiohedra of a given level n for ϕ^p theories. We introduced the notion of primitive accordiohedra to simplify our computations and provided a formula for the number of primitives at arbitrary level n. We provided the results of the implementation of our prescription to compute weights.

Plan of the thesis:

1. The first chapter will contain some mathematical preliminaries and a brief review of the amplituhedron program and the cubic case .

2. The second chapter will contain details of the quartic case namely the Stokes polytope, our prescription for computing the planar amplitude using primitives and weights,a formula for the number of primitives and proof of factorisation of Stokes polytopes.

3. The third chapter will contain details of the accordiohedron, our prescription for computing the planar amplitude using primitives and weights, a formula for the number of primitive accordiohedra and proof of factorisation of accordiohedra.

4. The fourth chapter will contain a classification of all primitives for $n \leq 3$ for arbitrary p and implementation of our prescription to compute the weights for $p \leq 12$ and $n \leq 3$.

5. The fifth chapter will contain a discussion of the results, open problems and general outlook.

Chapter 1

Introduction

Scattering amplitudes are arguably the most important observables in high energy physics. They describe the probability for a specific scattering process to happen. The probability can in turn be measured in a high energy experiment like the detectors at the LHC. To test a theory, we check whether the scattering amplitudes computed from it match the measured probabilities. This makes scattering amplitudes the link between theory and experiment. Therefore computing scattering amplitudes has a very practical purpose and computing them efficiently is important.

The usual method for computing amplitudes is using a Lagrangian and computing Feynman diagrams [1]. It is a very useful and as of now the only way we understand Quantum field theories (QFT's). Though this method provides a nice physical picture and makes locality manifest, it is not a very efficient approach for computing amplitudes. The Lagrangian formulation contains a lot of off-shell information which is not really needed to compute the on-shell amplitudes which we are interested in. The Lagrangian formulation is also plagued by redundancies due to field redefinitions and gauge invariances. By using field redefinitions we can map an action into an infinite set of different actions that describe identical physics. But this equivalence is not manifest at the level of Feynman diagrams and the Feynman diagrams corresponding to a field redefinition can be immensely complicated though the final result for the amplitude after tedious computations remains the same due to various seemingly miraculous cancellations. The Lagrangian for Yang-Mills (YM) has both cubic and quartic interaction vertices. It turns out that the cubic vertex has all the information needed to compute any n-point amplitude. The quartic vertex is only needed to ensure gauge invariance off-shell. Thus a poor choice of field basis or a gauge can make computations extremely tedious and complicated. A Lagrangian formulation can also obscure or conceal underlying structures in theories. For example the dual conformal invariance of planar $\mathcal{N} = 4$ SYM [3–8], colour kinematics duality [9, 10] or the KLT relations that relate the Yang-Mills and Gravity amplitudes [11–14] .

Over the years various on-shell methods have been developed that compute amplitudes without using the Lagrangian that have broadly been called the "Amplitudes program" [15–19]. The plan is to build an S-matrix from a series of postulates [52] :

1. **Poincare Invariance:** We try to describe scattering in flat Minkowski space whose isometries form the Poincare group.

2. **Existence of asymptotic one-particle states:** They describe the particles we scatter and are in one to one correspondence with irreducible representations of the Poincare group.

3. **Analyticity:** We shall require that the S-matrix is analytic in the external momenta, as we shall continue them to complex values and use their properties to constrain the S-matrix.

4. **Cluster decomposition:** It is a weak notion of locality, by which we mean that all the singularities of the S-matrix come from propagators.

The program attempts to build an S-matrix starting from these postulates. This approach is not new and was taken by the S-matrix program in the 60's, the modern amplitudes methods use new tools to address this line of thinking. We shall briefly review this and discuss its formal properties in the next few sections. For more details the reader can see [1, 52, 53].

We start with some notations and conventions. We work in $D+1$ dimensional Minkowski space-time with metric $\eta_{\mu\nu} = diag(+1, -1, \cdots, -1)$. The isometries are given by Poincare group $\mathbb{R}^{1,D} \ltimes SO(1, D)$ which is the semi-direct product of the group of translations and the Lorentz group. The generators of translations and Lorentz boosts are denoted by P^μ and $J^{\mu\nu}$ respectively which satisfy:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 && (1.1)\\
[P_\mu, J_{\rho\sigma}] &= -i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho) && (1.2)\\
[J_{\mu\nu}, J_{\rho\sigma}] &= -i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma}). && (1.3)
\end{aligned}
$$

We also assume that one-particle states exist, and that they form the basis of our Hilbert space. We denote them by $|\, p, a\,\rangle^{in}_{out}$ where p and a denote $(D+1)$-momenta and a set of quantum labels respectively.

We choose to normalise one particle states as $\langle p', a' \,|p,\ a\rangle = (2\pi)^D 2p^0 \delta^D(\vec{p} - \vec{p'})$ which is needed for Lorentz invariance. The one particle states satisfy :

(1) They are eigenstates of the translation generators

$$P^\mu|\, p, a\,\rangle = p^\mu|\, p, a\,\rangle. \qquad (1.4)$$

(2) Under Lorentz transformations they transform as

$$U(\Lambda)|\, p, a\,\rangle = \sum_{a'} D(\Lambda, p)_{aa'}|\,\Lambda p, a'\rangle, \;\; \Lambda \in \mathbb{R}^{D+1} \ltimes SO(1, D), \qquad (1.5)$$

where the $D(\Lambda, p)_{aa'}$ form a representation of the Poincare group. A basis for the full Hilbert space of physical states is called the Fock space and is built by combining single particle states $|p, a\,\rangle$ into multi-particle states $|\alpha\rangle = |p_{(1)}, a_{(1)}\,\rangle \otimes \cdots \otimes |p_{(n_\alpha)}, a_{(n_\alpha)}\,\rangle =$

$|\{p_{(i)}, a_{(i)}\}\ \rangle$.

The normalisation of these states follows

$\langle\alpha'|\alpha\rangle = \delta_{n_\alpha n'_\alpha}(2\pi)^{Dn_\alpha}\sum_\sigma(-1)^{S_\sigma}\prod_{i=1}^{n_\alpha} 2p^0_{(i)}\delta^{(D)}(\vec{p_{(i)}} - \vec{p'}_{(\sigma(i))})\delta_{a_{(i)}a'_{\sigma(i)}} \equiv \langle\alpha|\alpha'\rangle$,

where $p_\alpha = \sum_{i=1}^{n_\alpha} p_{(i)}$.

The identity can be represented in the basis of multi particle states as

$\mathbb{I} = |0\rangle\langle 0| + \sum_{n_\alpha=1}^{\infty}\frac{1}{(2\pi)^{Dn_\alpha}}\int\prod_{i=1}^{n_\alpha}\frac{d^D\vec{p_{(i)}}}{2p^0_{(i)}}\sum_{a_{(i)}}|\{p_{(i)}, a_{(i)}\}\ \rangle\langle\{p_{(i)}, a_{(i)}\}\ | \equiv \int_\alpha d\alpha|\alpha\rangle\langle\alpha|$.

It is obvious how these states should transform under a Poincare transformation characterised by a translation z and a Lorentz transformation Λ

$U(\Lambda, z)|\{p_{(i)}, a_{(i)}\}\ \rangle = e^{-iz_\mu p^\mu_\alpha}\sum_{a'_{(1)},\cdots,a'_{(n_\alpha)}} D(\Lambda, p_{(1)})_{a_{(1)}a'_{(1)}}\cdots D(\Lambda, p_{(n_\alpha)})_{a_{(n_\alpha)}a'_{(n_\alpha)}}|\{\Lambda p_{(i)}, a'_{(i)}\}\ \rangle$.

When the representation $D(\Lambda, p)_{aa'}$ is irreducible we talk about an elementary particle. Thus, the unitary irreducible representations of $\mathbb{R}^{D+1}\ltimes SO(1, D)$ are in correspondence with elementary particles. We shall briefly review the irreducible representations of $\mathbb{R}^{D+1}\ltimes SO(1, D)$ which were classified by Wigner [1, 54].

1.1 Irreducible representations of Poincare group

We want to determine all possible matrices $D(\Lambda, k)_{aa'}$. Since we know that the translations $\Lambda : x^\mu \to x^\mu + z^\mu$ form an Abelian subgroup of the full Poincare group and they act on one-particle states as a phase

$$U(\Lambda)|\ p, a\ \rangle = e^{-iz_\mu P^\mu}|\ p, a\ \rangle. \tag{1.6}$$

We just need to find the action of Lorentz transformations. We can do this by fixing a reference momentum k and finding its orbit under the action of the Lorentz group. We can write any other vector p in this orbit as $p = \Lambda(p, k)k$, where $\Lambda(p, k)$ is a Lorentz transformation that takes k to p. For instance we can choose

$$k = \begin{cases} (m, 0, \cdots, 0), & \text{for massive particles} \\ (E, E, \cdots, 0), & \text{for massless particles.} \end{cases}$$

We can then define

$$| \Lambda(p,k)p, a \rangle = U(\Lambda(p,k))| p, a \rangle. \tag{1.7}$$

and use (1.5) to find $D(\Lambda, p)_{aa'}$. But this transformation $\Lambda(p,k)$ is not unique as we can include any other Lorentz transformation Λ_0 in the stability group of p i.e. the set of transformations which leave the momentum p fixed $\Lambda_0 p = p$. This set of transformations is called the little group LG_p.

$$LG_p \cong \begin{cases} ISO(D-1), & \text{for massive particles} \\ SO(D), & \text{for massless particles.} \end{cases}$$

The idea now is that any unitary irreducible representation of the little group induces a unitary representation of the Lorentz group. Given a unitary representation D of the little group,

$$D(l)| k, a \rangle = \sum_{a'} D(l)_{aa'} | k, a' \rangle, \tag{1.8}$$

we choose a specific $\Lambda_0(p,k)$ for each p in the orbit of k. Under an arbitrary Lorentz transformation the oneparticle states $| p, a \rangle$ transform as

$$\begin{aligned} U(\Lambda(p',p))| p, a \rangle &= U(\Lambda(p',p))U(\Lambda_0(p,k))| k, a \rangle \qquad (1.9) \\ &= U(\Lambda_0(p',k)) \left(U(\Lambda_0(p',k))^{-1} U(\Lambda(p',p)) U(\Lambda_0(p,k)) \right) | k, a \rangle. \end{aligned}$$

Since, the bracketed term in the above equation is a transformation that takes k to p to p' and back to k it is an element l of the little group.

$$U(\Lambda(p',p))|\,p,a\,\rangle \;=\; U(\Lambda_0(p',k))\sum_{a'} D(l)_{aa'}|\,k,a'\,\rangle \tag{1.10}$$

$$= \sum_{a'} D(l)_{aa'}|\,p',a'\,\rangle. \tag{1.11}$$

Thus, the unitary irreducible representations of the Lorentz group are in correspondence with the unitary irreducible representations of the little group. In the massive case, the representations of the $SO(D)$ are well known and a is called the spin index. In the massless case the group $ISO(D-1) = \mathbb{R}^{D-1} \ltimes SO(D-1)$ is a non-compact group and we shall only consider the $SO(D-1)$ part and a is called the helicity index.

Little group in four dimensions

The Lorentz group in 4d has six generators $J^{\mu\nu}$. The little group LG_p is generated by the independent components of the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu J_{\rho\sigma}$ given by $P\cdot W = 0$. From the Poincare algebra it follows that

$$[W^\mu, W^\nu] = -i\epsilon^{\mu\nu\rho\sigma}W_\rho P_\sigma. \tag{1.12}$$

For massive particles we choose $p = (m,0,0,0)$ and thus $W = m(0, J^{23}, J^{31}, J^{12})$ which form an $SO(3)$.

For massless particles we choose $p = (E,E,0,0)$ and thus $W^\mu = -p^\mu R + \epsilon_1^\mu T_1 + \epsilon_2^\mu T_2$ with $\epsilon_{1,2}.p = 0$ which form an $ISO(2)$ with T_1, T_2 acting as translations.

If we assume that the non-compact part acts trivially[1] then the little group is Abelian $SO(2) = U(1)$ and all irreducible representations are one-dimensional charecterized by a half integer h called helicity [1]. It is an integer for bosons and half-integer for fermions.

$$D(\Lambda, p)_{a,a'} = e^{ih\theta(\Lambda,p)}\delta_{a,a'}.$$

[1] If the non compact part is allowed to act non-trivially then we can have continuous spin particles [55,56].

1.2 The S-matrix

We have two bases to describe the Hilbert space of physical states namely *in* states $|\,\rangle^{in}$ and *out* states $|\,\rangle^{out}$. The matrix which takes in-states to out-states is called the S-matrix

$$S_{\beta\alpha} = {}^{out}\langle\beta|\alpha\rangle^{in}, \qquad (1.13)$$

$$S = \int_\alpha d\alpha|\alpha\rangle^{in\ out}\langle\alpha|. \qquad (1.14)$$

The inverse is given by the Hermitian conjugate $S^\dagger$:

$$S^\dagger_{\beta\alpha} = {}^{in}\langle\beta|\alpha\rangle^{out}, \qquad (1.15)$$

$$S^\dagger = \int_\alpha d\alpha|\alpha\rangle^{out\ in}\langle\alpha|. \qquad (1.16)$$

The S-matrix is unitary by construction

$$SS^\dagger = \int_\alpha d\alpha \int_\beta d\beta|\alpha\,\rangle^{in}{}_{out}\langle\alpha|\beta\rangle_{out}{}^{in}\langle\beta| = \int_\alpha d\alpha|\alpha\rangle^{in}{}_{in}\langle\alpha| = \mathbb{I} = S^\dagger S.$$

The matrix elements $S_{\alpha\beta}$ of the S-matrix are the transition amplitudes for the in states $|\alpha\rangle = |\{p_{(i)}, a_{(i)}\}\,\rangle$ with $i = 1, \cdots, n_\alpha$ to evolve into the out states $|\beta\rangle = |\{p'_{(i')}, a'_{(i')}\}\,\rangle$ with $i' = 1, \cdots, n_\beta$.

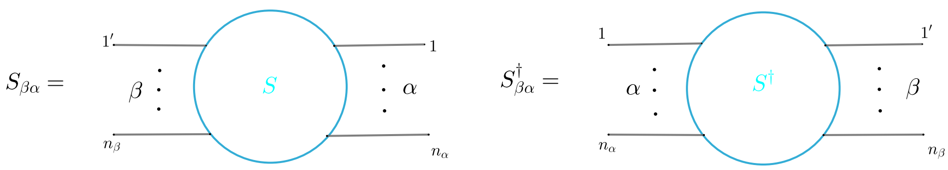


Figure 1.1: The S-matrix elements $S_{\beta\alpha}$ and $S^\dagger_{\beta\alpha}$.

If we were given a Hamiltonian $H = H_0 + V$ we could write a formal expression for the operator $S = U(-\infty, \infty)$, where $U(\tau, \tau_0) = e^{iH_0\tau}e^{-iH(\tau-\tau_0)}e^{-iH_0\tau_0}$. Then we can use time dependent perturbation theory to derive a Feynman diagrammatic expansion of the S-matrix.

However, as emphasised earlier we want to follow a different approach and determine the S-matrix from its analytic properties and symmetries without using a Lagrangian.

1.3 Symmetries and the S-matrix

A unitary transformation that acts the same way on the in and out states imposes constraints on the S-matrix: ${}^{out}\langle\beta|\alpha\rangle^{in} = {}^{out}\langle U\beta|U\alpha\rangle^{in}$. Let us look at a few examples:

- For translations, $U = e^{-iz_\mu P^\mu}$ then

$$S_{\beta\alpha} = e^{iz_\mu(p^\mu_{(\beta)} - p^\mu_{(\alpha)})} S_{\beta\alpha} \tag{1.17}$$

  which implies that $S_{\beta\alpha} \propto \delta^{(D+1)}(p_{(\beta)} - p_{(\alpha)})$.

- Any $U(1)$ symmetry $U = e^{-i\theta Q}$ would work like translations. If we assume the state $|p_{(i)}, a_{(i)}\rangle$ has a charge $q_{(i)}$ then we get charge conservation condition

$$S_{\beta\alpha} \propto \delta\left(\sum_{j=1}^{n_\beta} q_{(j)} - \sum_{i=1}^{n_\alpha} q_{(i)}\right). \tag{1.18}$$

- For Lorentz transformations we get the condition

$$\left(\prod_{j=1}^{n_\beta} D(\Lambda, p_{(j)})\right)^* \left(\prod_{i=1}^{n_\beta} D(\Lambda, p_{(i)})\right) S_{\beta\alpha} = S_{\beta\alpha}. \tag{1.19}$$

1.4 Analyticity properties of the S-matrix

We shall now describe the analyticity properties of the S-matrix and constraints arising from them. Although we will only consider massless scalars in this thesis for which there are various subtleties in the definition of the S-matrix due to infrared issues and understanding the loop level analytic structure of amplitudes. We shall not be concerned with these issues since we shall only consider tree level amplitudes in this thesis for which neither of these issues are relevant.

- **Connectedness and Cluster decomposition**

  When the interaction is trivial, we have $S_{\beta\alpha} = \delta(\alpha - \beta)$. When it is not $S_{\beta\alpha}$ still contains a $\delta(\alpha - \beta)$ term as there is a non-zero probability that the particles will not interact. We are more interested in the non-trivial part of the S-matrix so we remove this term from the S-matrix and define the *connected* part S^C which we do recursively starting with:

$$S_{\beta\alpha} = \delta(\beta - \alpha) \text{ when } \beta \text{ and } \alpha \text{ are } 1-\text{particle states} \tag{1.20}$$

$$S_{\beta\alpha} = S^C_{\beta\alpha} + \sum_P (-1)^{S_P} S^C_{\beta_1\alpha_1} \cdots S^C_{\beta_r\alpha_r} \tag{1.21}$$

where, P stands for partition of the initial α and final β sets of particles into $\alpha_1, \cdots, \alpha_r$ and $\beta_1, \cdots, \beta_r$ respectively and $S_P = 1$ if rearranging $\alpha \to \alpha_1 \cdots \alpha_r$, $\beta \to \beta_1 \cdots \beta_r$ involves rearranging an odd number of fermions and zero otherwise.

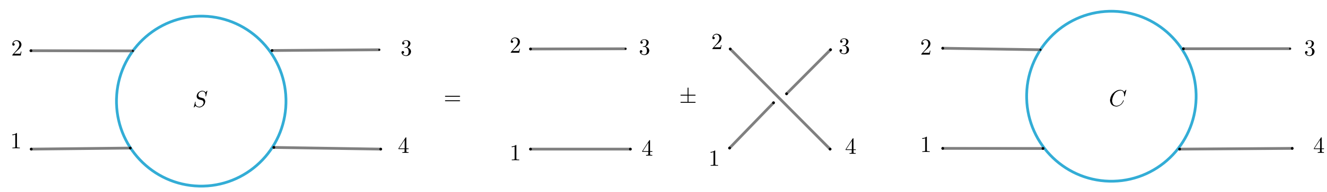


Figure 1.2: The connectedness for two-two scattering.

The *cluster decomposition principle* is the statement that $S^C_{x,\beta\alpha} = 0$ when at least one of the particles is far from the others. It is a weak form of locality and is formulated in position space by a Fourier-transform,

$$S^C_{x,\beta\alpha} = \frac{1}{(2\pi)^{n_\alpha n_\beta}} \int \prod_{i,j=1}^{n_\alpha,\, n_\beta} d^D p_{\vec{(i)}} d^D p_{\vec{(j)}} e^{-i\sum_i^{n_\alpha} p_{\vec{(i)}}\cdot x_{\vec{(i)}} + i\sum_j^{n_\beta} p_{\vec{(j)}}\cdot x_{\vec{(j)}}} S^C_{\beta\alpha}. \tag{1.22}$$

We see that the above statement imposes constraints on smoothness of $S^C_{\beta\alpha}$.

In particular, it implies that $S^C_{\beta\alpha}$ should not contain any other δ-functions than $\delta^{(D+1)}(p_{(\beta)} - p_{(\alpha)})$.

Thus we can define the *scattering amplitude* $M_{\beta\alpha}$ for the process $\alpha \to \beta$ by

$S^C_{\beta\alpha} = i(2\pi)^{(D)}\delta^{(D+1)}(p_{(\beta)} - p_{(\alpha)})M_{\beta\alpha}$.

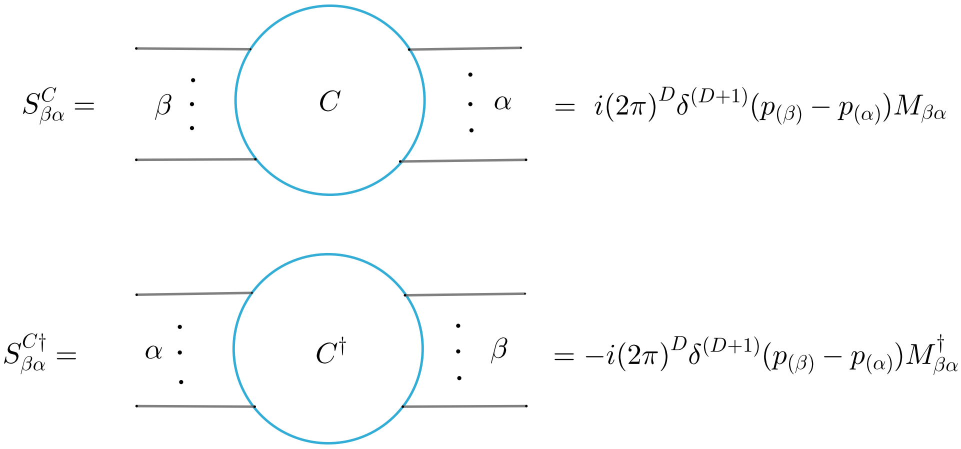


Figure 1.3: The Scattering amplitudes $M_{\beta\alpha}$ and $M^\dagger_{\beta\alpha}$.

As with any other QFT observable, we shall think of amplitudes perturbatively. If the theory has a dimensionless coupling constant g, then the scattering amplitude for n external legs M_n admits a perturbative expansion:

$M_n(g) = g^{n_{tree}} M_n^{tree} + g^{n_{1-loop}} M_n^{1-loop} + \cdots$

The exponents n_{tree}, n_{1-loop} etc depend on the theory. For theories with a cubic vertex $n_{L-loop} = g^{n-2+2L}$ and tree level is understood as $L = 0$ loops. So loop order is defined by the power of coupling constant that accompanies the amplitude. Since each loop order has a different singularity structure there is no need to think of the loop as a real loop in a Feynman diagram.

- **Unitarity and Factorisation**

Unitarity determines the factorisation properties of the amplitude. When several particles combine kinematically to produce another particle, the amplitude is singular. The singularity is a simple pole captured by the propagator of the internal particle. Unitarity also determines the residue to be the product of lower amplitudes with fewer external legs.

Consider a scattering process $\alpha = |\{p_{(i)}, a_{(i)}\}\rangle \to \beta = |\{p_{(j)}, a_{(j)}\}\rangle$.

If the physical momenta are such that $\left(\sum_{A\subset\alpha} p_{(i)} - \sum_{B\subset\beta} p_{(j)}\right)^2 = m^2$ where m is the mass of one of the physical 1-particle states $|p, a\rangle$ produced by the interaction of A and B then the following is possible

$A \to B + P$, $P + \bar{A} \to \bar{B}$, where $\bar{A} = \alpha \setminus A$, $\bar{B} = \beta \setminus B$.

Unitarity of the S-matrix determines that the amplitude has a simple pole when particle P goes on-shell and also determines its residue :

$$M_{\beta\alpha} = \sum_a M(A \to B + P_a)\frac{1}{p^2 - m^2 + i\epsilon} M(P_a + \bar{A} \to \bar{B}) + \begin{pmatrix}\text{regular terms} \\ in\ p^2 = m^2\end{pmatrix}. \tag{1.23}$$

- **Crossing Symmetry**

  The existence of anti-particles can be inferred from the S-matrix perspective as well.

  Let, us consider the scattering process $\alpha \to \beta$. We consider the division of α into A, $\bar{A}$ and β into B, $\bar{B}$ with a intermediate particle P with momentum $k = p_{(A)} - p_{(B)}$ and mass m as shown in the figure below.

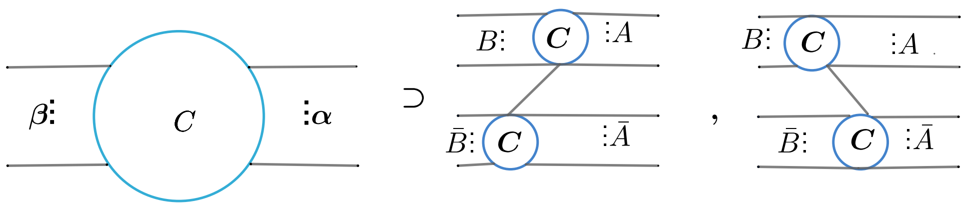


Figure 1.4: A graphical representation of the two complimentary processes that could contribute to the factorisation of the amplitude.

In the *physical region* of kinematic space: $\{\ p_{(i)}\ real\ , p^0_{(i)} > 0, p^2_{(i)} = m^2_{(i)}\ \}$ we interpret the pole in (1.23) as the process $A \to P + B, \bar{A} + P \to \bar{B}$ if $k^0 > 0$ as in this case we have:

$$\sqrt{-p^2_{(A)}} \geq \sqrt{-p^2_{(B)}} + m, \quad \sqrt{-p^2_{(\bar{B})}} \geq \sqrt{-p^2_{(\bar{A})}} + m.$$

If $k^0 < 0$ then we interpret this as the process $A + \bar{P} \to B$, $\bar{A} \to \bar{B} + \bar{P}$:

$$\sqrt{-p^2_{(A)}} \leq \sqrt{-p^2_{(B)}} + m, \quad \sqrt{-p^2_{(\bar{B})}} \leq \sqrt{-p^2_{(\bar{A})}} + m.$$

Where, $\bar{P}$ has mass m and satisfies $\bar{k}^2 = m^2$ with $\bar{k} = -k$ and with quantum numbers opposite to those of P. Thus, $\bar{P}$ is the antiparticle of P. Even though both the processes described above can never simultaneously occur and one of them is necessarily outside the physical region, we deal with this by analytically continuing the amplitude outside the physical region.

The factorisation of the amplitude is uniquely determined by (1.23) and since we have two possible interpretations we must have:

$$M(A \to P + B)M(\bar{A} + P \to \bar{B}) = M(A + \bar{P} \to B)M(\bar{A} \to \bar{P} + \bar{B}).$$

Since, the above equation is valid for any process we must have

$$M(\bar{A} + P \to \bar{B}) = \zeta M(\bar{A} \to \bar{P} + \bar{B}),\ M(A \to P + B) = \zeta^{-1} M(A + \bar{P} \to B).$$

We can now use the Hermitian analyticity property which states that the matrix elements of S and $S^\dagger$ are conjugate to each other and using this property we see that $|\zeta| = 1$. We can choose $\zeta = 1$ by appropriately choosing the phase of the one particle states $|p, a\rangle$. We then get

$$M(A \to P + B) = M(A + \bar{P} \to B). \tag{1.24}$$

Thus, using crossing symmetry we can consider processes where all particles are in going and denote the process by $\alpha \to \emptyset$ or just α for simplicity. This puts all particles on an equal footing and we follow this convention for the rest of this thesis.

The factorisation (1.23) of scattering amplitudes can now be re-expressed as

$$M_n(\alpha) = \sum_a M_{n-k+1}(A + P_a)\frac{1}{p^2_{(A)} - m^2}M_{k+1}(\bar{A} + \bar{P}_{\bar{a}}). \tag{1.25}$$

where $\bar{P}_{\bar{a}}$ is the antiparticle of P_a and $\bar{A} = \alpha \setminus A$. This is the form of unitarity we shall be using later in this thesis.

- **Soft Factorisation**

Consider a scattering process involving massless particles. If we parametrize the momentum of a massless particle P as τp^μ and let the momentum of P go to zero by taking $\tau \to 0$ then we call this the *soft limit*. Since, $(p_{(j)} + p)^2 = p^2_{(j)} + 2p_{(j)}.p \to m^2_{(j)}$ for any j then factorisation (1.25) implies that the amplitude must have a pole in the soft limit

$$M_{n+1}(P, P_1, \cdots, P_n) = \left(\sum_{j=1}^{n} \frac{1}{p_{(j)}.p} S_{(j)}(p, a, p_{(j)})\right) M_n(P_1, \cdots, P_n) + O(p^0). \tag{1.26}$$

The $S_{(j)}(p, a, p_{(j)})$ are called the *soft factors*. The soft factors $S_{(j)}$ can be determined by their Lorentz tensor properties. This was originally done by Weinberg for massless scalars, photons and gravitons [57, 58]. See [59–62] for more recent developments.

Scattering amplitudes have also nice factorisation properties under multi-particle and collinear limits [23, 63–65] which are not relevant for this thesis.

1.5 Amplitudes without Lagrangians

Having formally defined the S-matrix we would now like to see if we can reconstruct scattering amplitudes from its analytic structure. For our purposes we shall restrict to only tree level amplitudes for which the only singularities are simple poles coming from the factorisation channels and soft limits. The tree level amplitude is therefore a meromorphic function, in fact it is a rational function. There are several modern amplitude methods which build the amplitude without using a Lagrangian [15–19]. We shall describe three modern amplitude methods which will be most relevant for our purposes.

BCFW on-shell recursion relations

In this approach we would like to see if we can construct the amplitude which is a rational function from its singularities and residues. It is known that for functions of a single complex variable this is possible if the function vanishes at infinity. If the function does not vanish at infinity then we also need the values of the function at a few points to determine it. These points can be chosen to be some of the zeroes of the function. But the n-point amplitude is a rational function of $n(D+1)$ variables. The trick of mapping this problem in multi-variable complex analysis to a problem in a single complex variable was invented by Britto, Cachazo, Feng and Witten (BCFW) [30, 31].

We pick two particles P_i and P_j and deform their momenta $p_{(i)} \to p_{(i)} - zq$ and $p_{(j)} \to p_{(j)} + zq$ where $z \in \mathbb{C}$ and q is a momentum to be determined below. We leave the momenta of the other particles $p_{(k)}(z) = p_{(k)}\ for\ all\ k \neq i, j$ untouched. This is called the *BCFW* deformation or shift.

To interpret the process as scattering process after the deformation we need the following two conditions:

1. Momentum conservation: $\sum_i p_{(i)}(z) = 0$, which is clearly still satisfied.

2. On-shellness of deformed momenta: $p_{(i)}(z)^2 = -m^2_{(i)}$, $p_{(j)}(z)^2 = -m^2_{(j)}$ which is equivalent to $z^2q^2 - 2zq.p_{(i)} = 0$ and $z^2q^2 + 2zq.p_{(i)} = 0$ and can only be satisfied if :

$$q^2 = 0, \qquad q.p_{(i)} = q.p_{(j)} = 0. \tag{1.27}$$

We need to find a q satisfying the above conditions. We can then consider the on-shell amplitude which is a rational function of z, $M_n^{(i,j)}(z) \equiv M_n(\{p_{(k)}(z), a_{(k)}\})$ with simple poles. Since the amplitude is a rational function there must be an integer ν such that $M_n^{(i,j)}(z) \sim z^\nu$ when $z \to \infty$.

The exponent ν depends on the choice of particles (i, j) as well as the choice of q for the deformation. When a deformation generates an amplitude that vanishes at infinity we call it a *good* BCFW shift, and if $\nu \geq 0$ we call it a *bad* shift. When $\nu < 0$ no zeroes are needed and we get the celebrated BCFW recursion relation [34]. If $\nu \geq 0$ then the amplitude has a non-zero "boundary" contribution and we need the zeroes to get the generalised BCFW recursion relations [66].

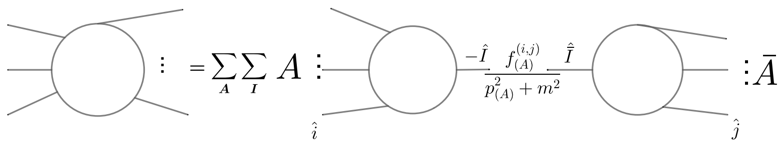


Figure 1.5: A graphical representation of the generalised BCFW recursion relations.

The result is as follows :

$$M_n = \sum_A \sum_I M_L^{(i,j)}(z_A) \frac{f_A^{(i,j)}}{p^2_{(A)} - m^2} M_R^{(i,j)}(z_A), \tag{1.28}$$

$$f_A^{(i,j)} = \begin{cases} 1 & , if \quad \nu < 0 \\ \prod_{l=1}^{\nu+1} \left(1 - \frac{p^2_{(A)} - m^2}{p^2_{(A)}(z_0^{(l)}) - m^2}\right) & , if \quad \nu \geq 0 \end{cases} \tag{1.29}$$

where, $\{z_0^{(l)}\}_{l=1}^{\nu+1}$ is a set of $\nu + 1$ zeroes of the deformed amplitude $M_n^{(i,j)}(z)$.

Starting with the smallest building block which is the smallest known non-zero amplitude in the theory as the initial value we can then determine all other higher point amplitudes. This has been realised for a few theories and is computationally more superior than Feynman diagrams [32,33]. For n-point MHV amplitudes in YM whose BCFW representation remarkably contains only one term for any n (as opposed to $o(e^n)$ Feynman diagrams) and gives a straightforward proof of the celebrated *Parke-Taylor* formula [2].

The major obstacles to this approach are the lack of a generic procedure to find the exponent ν and the zeroes $z_0^{(l)}$ for an arbitrary theory.

CHY representatation

In this formalism the singularities of scattering amplitudes involving massless particles in kinematic space are mapped onto an auxiliary space namely the *moduli space* of n-punctured Riemann sphere $\mathcal{M}_{0,n}$. Let $\{k_1^\mu, k_2^\mu, \cdots k_n^\mu\}$ of n massless particles in $D+1$ dimensions forming the kinematic space defined as:

$\mathcal{K} = \{(k_1^\mu, k_2^\mu, \cdots k_n^\mu) \mid \sum_{a=1}^n k_a^\mu = 0, k_a^2 = 0, \forall a \in \{1, 2, \cdots, n\}\}/\mathbb{SO}(1, D)$

The kinematic space $\mathcal{K}$ is a $\frac{n(n-3)}{2}$ dimensional and spanned by the Mandelstram variables $s_{ij} = k_i \cdot k_j$ for $1 \leq i < j < n$.

We consider $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ to be holomorphic coordinates the moduli space $\mathcal{M}_{0,n}$, which also specify the locations of punctures on the Riemann sphere. The moduli space of n-punctured Riemann sphere $\mathcal{M}_{0,n}$ is an $(n-3)$-dimensional complex space and is invariant under $\mathbb{SL}(2, \mathbb{C})$ transformations given by:

$$\sigma_i \to \psi(\sigma_i) = \frac{a\sigma_i + b}{c\sigma_i + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1 \tag{1.30}$$

The real part $\mathcal{M}_{0,n}(\mathbb{R})$ is the open string moduli space consisting of all distinct points σ_i

on the extended real line (with infinity) modulo $SL(2,\mathbb{R})$. This allows us to fix any three of the punctures to a fixed value using which one usually sets $\sigma_1 = 0,\ \sigma_{n-1} = 1,\ \sigma_n = \infty$, which we will use later in this thesis. In this section the choice of punctures is not relevant.

The mapping of singularities in kinematic space to the moduli space of Riemann sphere is given by [67]:

$$k_i^\mu = \frac{1}{2\pi i}\oint_{|z-\sigma_i|=\epsilon} dz \left[\sum_{j=1}^{n} \frac{k_j^\mu}{(z-\sigma_j)}\right] = \frac{1}{2\pi i}\oint_{|z-\sigma_i|=\epsilon} dz \left[\frac{f^\mu(z)}{\prod_{b=1}^{n}(z-\sigma_a)}\right]$$

The function $f^\mu(z)$ is a polynomial of degree $(n-2)$ as the coefficient of the leading term vanishes by momentum conservation.

Since, the momenta k_i^μ are null we must have $f(\sigma_i)^2 = 0$ for all i. But, as $f(z)^2$ has degree $(2n-4)$ and we need to know $(n-3)$ additional conditions as we only know n of its roots. The extra conditions imposed are that $f^\mu(z)$ remain null for all z i.e. $f(z)^2 = 0$ which in turn imply that $f(z)\cdot f'(z) = 0$. When we evaluate this condition on the n puncture locations σ_i we get the *scattering equations* [35, 68]:

$$E_i = \sum_{j,\ j\neq i} \frac{s_{ij}}{(\sigma_i - \sigma_j)} = 0, \qquad i = 1,\cdots,n \tag{1.31}$$

It can be easily checked that only $(n-3)$ of the scattering equations are independent as $\sum_i \sigma_i^m E_i = 0$ for $m = 0,1,2$. By *Bezout's* lemma [69] the scattering equations have $(n-3)!$ solutions.

The Cachazo-He-Yuan formalism provides an integral representation of the tree amplitude M_n of massless particles over the moduli space of n-punctured Riemann sphere as follows:

$$M_n = \int \frac{d^n\sigma}{vol\mathbb{SL}(2,\mathbb{C})} {\prod_i}' \delta\left(\sum_{j,j\neq i} \frac{s_{ij}}{(\sigma_i - \sigma_j)}\right) I_n(\{k,\epsilon,\sigma\}) = \int d\mu_n I_n \tag{1.32}$$

where, $\frac{d^n\sigma}{vol\mathbb{SL}(2,\mathbb{C})} = (\sigma_i - \sigma_j)(\sigma_j - \sigma_k)(\sigma_k - \sigma_i)d\sigma_1 \wedge \cdots d\hat{\sigma}_i \wedge \cdots d\hat{\sigma}_j \wedge \cdots d\hat{\sigma}_k \cdots \wedge d\sigma_n$ is the measure on $\mathcal{M}_{0,n}$ the hats indicate the corresponding coordinates have been removed for any three i, j, k.

Due to the presence of delta functions imposing the scattering equations as arguments the integral localises onto the solutions of the scattering equations and due $\mathbb{SL}(2,\mathbb{C})$ redundancies we need only $(n-3)$ delta functions to localise the integrals. The primed product is defined as:

$$\prod_i{}' \delta\left(\sum_{j,j\neq i} \frac{k_i.k_j}{(\sigma_i - \sigma_j)}\right) = (\sigma_k - \sigma_l)(\sigma_l - \sigma_m)(\sigma_m - \sigma_k) \prod_{i\neq k,l,m} \delta\left(\sum_{j,j\neq i} \frac{k_i.k_j}{(\sigma_i - \sigma_j)}\right)$$

for any k, l, m.

The integrand I_n contains the information about the particular theory being considered. Cachazo, He and Yuan have constructed the integrands for a wide class of massless theories including bi-adjoint scalars, YM and gravity [36, 38]. We shall briefly review only the scalar example as we shall need it in later sections.

Bi-adjoint cubic scalar theory : A field theory involving scalar fields $\phi = \phi^{aa'} T_a T'_{a'}$ that transform in the adjoint representation of two unitary groups $U(N) \times U(\tilde{N})$ where T_a and $T'_{a'}$ are generators of the two factors respectively with Lagrangian given by [36]:

$$\mathcal{L}^{\phi^3} = \frac{1}{2}\partial_\mu \phi_{aa'} \partial^\mu \phi^{aa'} - \frac{\lambda}{3!} f_{abc} \tilde{f}_{a'b'c'} \phi^{aa'} \phi^{bb'} \phi^{cc'} \tag{1.33}$$

where f_{abc} and $\tilde{f}_{a'b'c'}$ are the structure constants of $U(N)$ and $U(\tilde{N})$ respectively. Tree level scattering amplitudes for this theory can be decomposed in a double colour expansion as:

$$M_n = \sum_{\alpha\in S_n/Z_n} \sum_{\beta\in S_n/Z_n} Tr(T^{a_{\alpha(1)}} T^{a_{\alpha(2)}} \cdots T^{a_{\alpha(n)}}) Tr(\tilde{T}^{a_{\beta(1)}} \tilde{T}^{a_{\beta(2)}} \cdots \tilde{T}^{a_{\beta(n)}})\ m_n(\alpha|\beta) \tag{1.34}$$

The $m_n(\alpha\,|\,\beta)$'s are called double colour amplitudes. The double colour amplitude $m_n(\alpha|\beta)$ is given by the sum of all cubic diagrams that are compatible with both α and β orderings with an overall sign coming from (1.34).

If $\mathcal{T}(\alpha)$ and $\mathcal{T}(\beta)$ denote α and β coloured trivalent graphs (i.e. collection of graphs with α and β orderings) respectively then

$$m_n(\alpha|\beta) = (-1)^{n-3+n_{flip}(\alpha|\beta)} \sum_{g\in\mathcal{T}(\alpha)\cap\mathcal{T}(\beta)} \prod_{e\in E(g)} \frac{1}{s_e} \tag{1.35}$$

where $s_e = P_e^2$ for momentum P_e flowing along the edge e in the set of edges $E(g)$ for the Feynman diagram g and $n_{flip}(\alpha|\beta)$ is the number of ordering flips. (see [36] for details)

For $\alpha = \beta$ the double color amplitude is the sum over all cubic graphs and when $\alpha \neq \beta$ it is sum over a smaller subset of cubic graphs that are consistent with both α and β orderings. In particular for $\mathcal{T}(\alpha) \cap \mathcal{T}(\beta) = \emptyset$, $m_n(\alpha|\beta) = 0$. The CHY integrand $I_n^{\phi^3}$ for this theory is

$$I_n^{\phi^3}(\alpha|\beta) = C_n(\alpha)C_n(\beta), \qquad m_n(\alpha|\beta) = \int d\mu_n I_n^{\phi^3} \tag{1.36}$$

The $C_n(\alpha)$'s are called the *Parke-Taylor* factors and are defined as:

$$C_n(\alpha) = \frac{1}{(\sigma_{\alpha(1)} - \sigma_{\alpha(2)})(\sigma_{\alpha(2)} - \sigma_{\alpha(3)})\cdots(\sigma_{\alpha(n)} - \sigma_{\alpha(1)})} \tag{1.37}$$

For YM and gravity the corresponding integrands are given by replacing one of the Parke-Taylor factors $C_n(\alpha)$ by the reduced Pfaffian $Pf'\Psi_n(\{k,\epsilon,\sigma\})$:

$$I_n^{YM}(\alpha) = C_n(\alpha)Pf'\Psi_n(\{k,\epsilon,\sigma\}), \qquad I_n^{GR}(\alpha) = (Pf'\Psi_n(\{k,\epsilon,\sigma\}))^2 \tag{1.38}$$

where Ψ_n is the $2n \times 2n$ antisymmetric matrix $\Psi_n = \left(\begin{array}{c|c} A & -C^T \\ \hline C^T & B \end{array}\right)$ consisting of $n \times n$ matrices A, B and C defined as:

$$A_{ab} = \begin{cases} \frac{k_a.k_b}{\sigma_a-\sigma_b}, & a \neq b \\ 0, & a = b \end{cases} \qquad B_{ab} = \begin{cases} \frac{\epsilon_a.\epsilon_b}{\sigma_a-\sigma_b}, & a \neq b \\ 0, & a = b \end{cases} \qquad C_{ab} = \begin{cases} \frac{\epsilon_a.k_b}{\sigma_a-\sigma_b}, & a \neq b \\ -\sum_{c\neq a} \frac{\epsilon_a.k_c}{\sigma_a-\sigma_c}, & a = b \end{cases}$$

The Pfaffian $Pf(\Psi)$ is the square root of the determinant $Pf(\Psi) = \sqrt{Det(\Psi_n)}$ and the reduced Pfaffian $Pf'(\Psi_n)$ is given by :

$$Pf'\Psi_n = -\frac{(-1)^{a+b}}{\sigma_a - \sigma_b} Pf[\Psi_n]_{\hat{a},\hat{b}} \tag{1.39}$$

with $[\Psi_n]_{\hat{a},\hat{b}}$ being the minor of Ψ_n obtained after removing the a-th row and b-th column such that $1 \leq a, b \leq n$.

Notice that from (1.36), (1.38) CHY representation makes the *double copy* relations manifest – often stated as *pure gravity amplitudes are "square" of Yang-Mills amplitudes* and schematically shown as $Gravity = \frac{Yang-Mills^2}{bi-adjoint\ scalar}$ which are difficult to see in the Lagrangian formulation [70, 71].

Amplituhedron framework

In this framework the amplitude is thought of more fundamentally as a differential form as opposed to a function living in kinematic space. The differential form is naturally associated to a geometric object living in kinematic space which is called the *Amplituhedron* of the theory. A remarkable feature of this framework is that unitarity and locality emerge as natural consequences of the geometric properties of Amplituhedron as opposed to being inputs as we had outlined earlier for the S-matrix program.

The amplituhedron framework [39–41, 72] was originally formulated for tree and all loop N^k-MHV amplitudes in $\mathcal{N} = 4$ SYM and subsequently extended to tree level planar and 1-loop amplitudes in bi-adjoint ϕ^3 theory [43–45].

In this thesis we extend this framework to an infinite class of scalar theories namely tree level planar amplitudes in ϕ^p theories for all $p \geq 4$. In the rest of this chapter we shall first develop a few mathematical preliminaries that are needed to explain the *amplituhedron* framework and then we shall review the ϕ^3 case.

1.6 Positive geometries and Canonical forms

We begin by introducing the notion of *positive geometry* which is a geometry with boundaries of all co-dimensions [51, 73].

1.6.1 Positive Geometries

Let $\mathbb{P}^N$ denote the N dimensional complex projective space, X be *complex projective algebraic variety*, which is a solution set of a finite number of homogenous polynomial equations with real coefficients in $\mathbb{P}^N$. We denote by $X(\mathbb{R})$ the solution set of the same set of equations in the real projective space $\mathbb{P}^N(\mathbb{R})$.

A *semialgebraic set* in $\mathbb{P}^N(\mathbb{R})$ is a finite union of subsets of solutions of homogeneous real polynomial equations $\{x \in \mathbb{P}^N(\mathbb{R}) \mid p(x) = 0\}$ and homogeneous real polynomial inequalities $\{x \in \mathbb{P}^N(\mathbb{R}) \mid q(x) > 0\}$ (Since the inequality does not make sense in $\mathbb{P}^N(\mathbb{R})$ we first solve $q(x) > 0$ in $\mathbb{R}^{N+1} \setminus \{0\}$ and then project down to $\mathbb{P}^N(\mathbb{R})$).

Residue operator

Let ω be a meromorphic form on X, C is an irreducible subvariety of X and z is a holomophic coordinate whose zero set $z = 0$ locally parametrizes C and u are the rest of the collective holomorphic coordinates. Around the simple pole at C we can then expand ω as

$$\omega(u, z) = \omega'(u) \wedge \frac{dz}{z} + \cdots , \tag{1.40}$$

where $\omega^{'}(u)$ is a non-zero meromorphic form defined locally on the boundary component. Around such a simple pole we define the *residue operator* **Res** locally as :

$$Res_C \omega := \omega'. \tag{1.41}$$

If there is no such simple pole then we define the residue to be zero.

We define a D-dimensional *positive geometry* to be a pair $(X, X_{\geq 0})$ of irreducible complex projective variety X of complex dimension D and a nonempty oriented closed semialgebraic set $X_{\geq 0}$ of real dimension D together with the following:

- For $D = 0$: X is a single point and we must have $X_{\geq 0} = X$. We define the 0-form $\Omega(X, X_{\geq 0})$ on X to be ± 1 depending on the orientation of $X_{\geq 0}$.
- For $D > 0$: we have
  1. Every boundary component $(C, C_{\geq 0})$ of $(X, X_{\geq 0})$ is a positive geometry of dimension $D - 1$.
  2. There exists a unique nonzero rational D-form $\Omega(X, X_{\geq 0})$ on X constrained by the residue realtion $Res_C \Omega(X, X_{\geq 0}) = \Omega(C, C_{\geq 0})$ along every boundary component C and nowhere else.

X is called the embedding space and D is the dimension of the positive geometry. The form $\Omega(X, X_{\geq 0})$ is the *canonical form* of the positive geometry $(X, X_{\geq 0})$. The *codimension d boundary components* of a positive geometry $(X, X_{\geq 0})$ are the positive geometries obtained by recursively taking the boundary components d times.

Examples

- If $(X, X_{\geq 0})$ is a zero dimensional positive geometry, then both X and $X_{\geq 0}$ are points and we have $\Omega(X, X_{\geq 0}) = \pm 1$.
- If $(X, X_{\geq 0})$ is a one dimensional positive geometry then X is isomorphic to $\mathbb{P}^1$ and $X_{\geq 0}$ is isomorphic to a closed subset of $\mathbb{P}^1(\mathbb{R}) \cong S^1$ which is a union of closed intervals. A generic closed interval $[a, b] \subset \mathbb{P}^1(\mathbb{R})$ is the set of points $\{(1, x) \mid x \in [a, b]\} \subset \mathbb{P}^1(\mathbb{R})$, where $a < b$. The canonical form in this case is given by:

$$\Omega([a, b]) = \frac{dx}{x - a} - \frac{dx}{x - b} = \frac{b - a}{(b - x)(x - a)} dx \tag{1.42}$$

which is the unique 1-form whose leading residues are $Res_{x=a}\Omega([a,b]) = 1$ and $Res_{x=b}\Omega([a,b]) = -1$.

- Some examples of two dimensional positive geometries in $X = \mathbb{P}^2(\mathbb{R})$ are shown in the figure below :

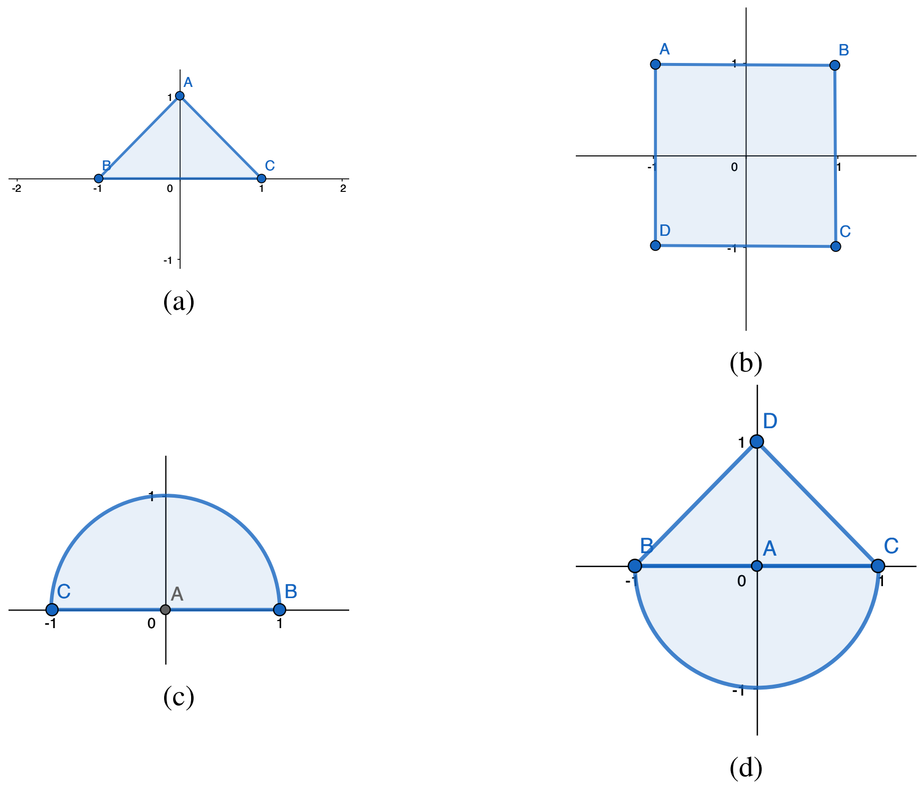


Figure 1.6: A few two dimensional positive geometries.

(a) A triangle $\mathcal{T} = (X, X_{\geq 0})$ with $X = \mathbb{P}^2(\mathbb{R})$ and $X_{\geq 0} = \{(1, x, y) \in \mathbb{P}^2(\mathbb{R}) \mid q_1 = y \geq 0,\ q_2 = 1 - x - y \geq 0,\ q_3 = 1 - y + x \geq 0\}$.

(b) A square $\mathcal{S} = (X, X_{\geq 0})$ with $X = \mathbb{P}^2(\mathbb{R})$ and $X_{\geq 0} = \{(1, x, y) \in \mathbb{P}^2(\mathbb{R}) \mid q_1 = y \geq -1,\ q_2 = x \geq -1,\ q_3 = -y \geq -1,\ q_4 = -x \geq -1\}$.

(c) A half disk $\mathcal{D} = (X, X_{\geq 0})$ with $X = \mathbb{P}^2(\mathbb{R})$ and $X_{\geq 0} = \{(1, x, y) \in \mathbb{P}^2(\mathbb{R}) \mid q_1 = y \geq 0,\ q_2 = 1 - x^2 - y^2 \geq 0\}$.

(d) A pizza slice $\mathcal{P} = (X, X_{\geq 0})$ with $X = \mathbb{P}^2(\mathbb{R})$ and $X_{\geq 0} = \{(1, x, y) \in \mathbb{P}^2(\mathbb{R}) \mid q_1 = -y \geq 0,\ q_2 = 1 - y - x \geq 0,\ q_3 = 1 - y + x \geq 0, q_4 = 1 - x^2 - y^2 \geq 0\}$.

- In higher dimensions *polytopes* [74, 75] which are generalisations of polygons to general dimensions are examples of positive geometries with non-curved boundaries. They are class of positive geometries that are relevant for this thesis and we shall discuss more about them in this section.

Let $Z_1, Z_2, \cdots, Z_n \in \mathbb{R}^{m+1}$ and denote by Z the $n \times (m+1)$ matrix whose rows are given by Z_i. We define a *convex projective polytope* $\mathcal{A} = \mathcal{A}(Z) \subset \mathbb{P}^m(\mathbb{R})$ as the convex hull

$$\mathcal{A} = Conv(Z) = \left\{ \sum_{i=1}^{n} C_i Z_i \in \mathbb{P}^m(\mathbb{R}) \mid C_i \geq 0 \ \forall i \right\} \tag{1.43}$$

We call $Z_1, Z_2, \cdots, Z_n$ the *vertices* of $\mathcal{A}$. More generally we define a *Face* F of $\mathcal{A}$ to be the intersection $F = \mathcal{A} \cap H$ with a liner hyperplane $H \subset \mathbb{R}^{m+1}$ such that $\mathcal{A}$ lies completely on one side of H. If H is defined as $v \cdot \alpha = 0$ for some $\alpha \in \mathbb{R}^{m+1}$ then we must have $Z_i \cdot \alpha \geq 0$ or $Z_i \cdot \alpha \leq 0$.

If $dim(F) = k$ then we call F a k-face of $\mathcal{A}$. The 0-faces, 1-faces and $(n-1)$-faces of $\mathcal{A}$ are called vertices, edges and facets respectively. It is clear from this definition that faces of a convex polytope which are co-dimension $(n-k)$ boundaries of $\mathcal{A}$ are also convex polytopes thus showing that convex polytopes are indeed positive geometries.

A polytope $\mathcal{A} \in \mathbb{P}^m$ is called *simple* if each facet is adjacent to exactly m vertices say $Z_{i_1}, \cdots, Z_{i_m}$ i.e. the facets are $\{W | \ W \cdot Z_i = 0 \ \text{ for } i = 1, ..., m\}$ and in such a case the facets have a very simple representation in terms of the vertices namely:

$$W_I = \epsilon_{I I_1 I_2 \ldots I_m} Z_{i_1}^{I_1} \cdots Z_{i_m}^{I_m} \tag{1.44}$$

We can also define the corresponding dual polytope $\mathcal{A}_Y^{\star}$ at a point Y living in the dual projective space of $\mathbb{P}^m(\mathbb{R})$ (also $\mathbb{P}^m(\mathbb{R})$) to be the convex hull of the facets W_i:

$$\mathcal{A}_Y^* = Conv(W_1, W_2, ..., W_n) = \Big\{ \sum_{i=1}^{n} C_i W_i \in \mathbb{P}^m(\mathbb{R}) \mid C_i \geq 0, i = 1, \cdots, n \Big\} \quad (1.45)$$

In the above definition we choose the signs of W_i such that $W_i \cdot Y > 0$, the relative signs of W_i is important in the sum above (1.45). The signs indicate the position of Y with respect to the facets which we indicate with the subscript Y for the dual.

When $Y \in Int(\mathcal{A})$ then $W_i \cdot Y > 0$ for all i and we can forget the Y dependence and simply call this the *dual* $\mathcal{A}^\star$ of $\mathcal{A}$ defined as:

$$\mathcal{A}^\star = \Big\{ W \in \mathbb{P}^m(\mathbb{R}) |\ W \cdot Y \geq 0 \text{ for all } Y \in \mathcal{A} \Big\} \quad (1.46)$$

To visualise a polytope it is better to look at the Euclidean version defined as follows:

We let $Z = (1, Z')$ where $Z' \in \mathbb{R}^m$ and the Euclidean polytope $\mathcal{A}$ is the convex combination:

$$\mathcal{A} = Conv(Z) = \Big\{ \sum_{i=1}^{n} C_i Z'_i \in \mathbb{R}^m(\mathbb{R}) |\ C_i \geq 0\ \forall i \text{ and } \sum_{i=1}^{n} C_i = 1 \Big\} \quad (1.47)$$

- *Simplexes* $\mathit{\Delta}$ are the simplest polytopes with vertices being the Euclidean basis vectors $Z_i^I = \delta_{iI}$. An n-simplex is convex polytope with $n + 1$ vertices. The first few simplexes are a point, a line, a triangle and a tetrahedron. An equivalent definition of the simplex in terms of its facets is:

$$\mathit{\Delta} = \{ Y \in \mathbb{P}^m(\mathbb{R}) |\ Y \cdot W_i \geq 0\ for\ i = 1, ..., m + 1 \} \quad (1.48)$$

Every convex polytope can be triangulated by simplexes (i.e. divided into simplexes with disjoint interiors). This is a very useful fact as we can translate every result about simplexes to the corresponding result for convex polytopes straightforwardly

[75].

- *Grassmann polytopes:* Let $M_{k,n}(\mathbb{R})$ be the set of $k\times n$ matrices of rank k (*with* $k \leq n$). The Grassmannian manifold $G(k,n)(\mathbb{R})$ is defined as $G(k,n)(\mathbb{R}) = M_{k,n}(\mathbb{R})/GL(k,\mathbb{R})$ i.e. for any $A_1, A_2 \in M_{k,n}(\mathbb{R})$ are equivalent if there is a $\Lambda \in GL(k,\mathbb{R})$ such that $A_1 = \Lambda A_2$.

  The positive Grassmannian $G_{>0}(k,n)(\mathbb{R})$ is the set of points in $G(k,n)(\mathbb{R})$ all of whose $k\times k$ minors called *Plucker coordinates* are positive. We call the closure of this set $G_{\geq 0}(k,n)(\mathbb{R})$. The collection $(G(k,n)(\mathbb{R}), G_{\geq 0}(k,n)(\mathbb{R}))$ is a positive geometry.

  Let $Z_1, Z_2, \cdots, Z_n \in \mathbb{R}^{k+m}$ be a collection of vertices. The linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ induces a map $Z : G(k,n) \to G(k,k+m)$ whose image we define as the *tree level Grassmann polytope* $Z(G_{\geq 0}(k,n)) = \{C\cdot Z \mid C \in G_{\geq 0}(k,n)\}$ [76–78].

  We can also define an *L-loop Grassmanian* $G(k,n;\underline{k})$ where $\underline{k} = (k_1, \cdots, k_L)$ to be a set of points which are a collection of linear subspaces $V_S \subset \mathbb{C}^n$ indexed by $S = \{s_1, \cdots, s_l\} \subset \{1,2,\cdots,L\}$ satisfying $k_S = k_{s_1} + \cdots + k_{s_L} \leq n-k$ along with $dimV_S = k + k_S$ and $V_S \subset V_{S'}\ for\ S \subset S'$ [76–78]. The collection $(G(k,n;\underline{k}), G_{\geq 0}(k,n;\underline{k}))$ is conjectured to be a positive geometry.

  We can analogously define the non-negative part $G_{\geq 0}(k,n;\underline{k})$ of $G(k,n;\underline{k})$ and use it to define the *loop level Grassmann polytope*. As before we have a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ which induces a map $Z : G(k,n;\underline{k}) \to G(k,k+m;\underline{k})$ whose image $Z(G_{\geq 0}(k,n;\underline{k}))$ we define as the *loop level Grassmann polytope*.

  The image of the L-loop Grassmanian polytope $G(k,n;l^L)$ where l^L denotes L-tuple $(l,l,\cdots,l)$ under the linear map defines the *L-loop Amplituhedron*.

$$\mathcal{A}(k,n,m;l^L) = Z(G_{\geq 0}(k,n;l^L)) \tag{1.49}$$

  In particular the 0-loop Amplituhedron is called the *tree amplituhedron* and simply denoted as $\mathcal{A}(k,n,m) = Z(G_{\geq 0}(k,n))$. As we shall see later (1.49) with $m = 4$ and $l = 2$ will play an important role for computing planar scattering amplitudes in $\mathcal{N} = 4$ SYM.

We shall shortly describe how to obtain the canonical forms corresponding to the above examples, but first let us describe a few more properties satisfied by the canonical forms which will prove pivotal in determining the canonical forms.

We could try to construct new positive geometries from old ones by taking disjoint unions, direct products and morphisms etc. The union of positive geometries however is not necessarily a positive a geometry, the classic example being union of two half disks giving a disk which doesn't have any 0-dimensional boundaries and hence is a not positive geometry. But if we allow some of the $X_{\geq 0}$ to be empty as well then we get what is called a *pseudo-positive geometry* whose disjoint union continues to be a pseudo-positive geometry.

The canonical form satisfies the following properties:

1. **Triangulation:** If $(X, X_{i,\geq 0})$ are pseudo-positive geometries whose interiors are disjoint $X_{i,>0} \cap X_{j,>0} = \emptyset \ \forall i \neq j$ then the union is a pseudo-positive geometry $(X, X_{\geq 0} = \cup_i X_{i,\geq 0})$ and the collection $\{X_{i,\geq 0}\}$ is called a *triangulation* of $X_{\geq 0}$

$$\Omega(X, \cup_i X_{i,\geq 0}) = \sum_i \Omega(X, X_{i,\geq 0}). \tag{1.50}$$

   The boundaries of $X_{i,\geq 0}$ which are also boundaries of $X_{\geq 0}$ are called physical boundaries and the other boundaries are called spurious boundaries.

   The poles corresponding to these boundaries are called physical and spurious poles respectively. The spurious poles cancel when we sum over all triangulations and the final result contains only physical poles.

2. **Direct Product:** If $(X, X_{\geq 0})$ and $(Y, Y_{\geq 0})$ are positive geometries then the cartesian

product $(Z, Z_{\geq 0}) = (X \times Y, X_{\geq 0} \times Y_{\geq 0})$ is a positive geometry and

$$\Omega(Z, Z_{\geq 0}) = \Omega(X, X_{\geq 0}) \wedge \Omega(Y, Y_{\geq 0}). \tag{1.51}$$

3. **Morphisms:** Given a morphism $\Phi : (X, X_{\geq 0}) \to (Y, Y_{\geq 0})$ of positive geometries

$$\Phi_\star(\Omega(X, X_{\geq 0})) = \Omega(Y, Y_{\geq 0}) \tag{1.52}$$

We could construct the canonical form $\Omega(X, X_{\geq 0})$ of a positive geometry by applying the axiomatic definition or by using the above properties in the following ways:

- **Direct construction from poles and zeros**

  We propose an ansatz for the canonical form as a rational function and impose residue constraints to determine the rational function.

  Suppose, $(X, X_{\geq 0})$ is a positive geometry of dimension m for which there is a morphism $\Phi : (\mathbb{P}^m, \mathcal{A}) \to (X, X_{\geq 0})$ for some positive geometry $\mathcal{A}$ in projective space defined by the homogenous inequalities $q_i(Y) \geq 0$ for $Y \in \mathbb{P}^m(\mathbb{R})$. Then we can make an ansatz for the canonical form

$$\Omega(\mathcal{A}) = \frac{q(Y)\langle Y d^m Y\rangle}{\prod_i q_i(Y)} = \underline{\Omega}(\mathcal{A})\langle Y d^m Y\rangle \tag{1.53}$$

  for some homogenous polynomial $q(Y)$ which has degree $deg\ q = \sum_i q_i - m - 1$ so that the form is invariant under local GL(1) action $Y \to \alpha(Y)Y$. The angular brackets denote the determinant $\langle Y d^m Y\rangle = \epsilon_{II_1 I_2 \cdots i_m} Y_0^I dY_1^{I_1} \cdots dY_m^{I_m}$ and we call the quantity $\underline{\Omega}(\mathcal{A})$ the *canonical rational function*.

  As we shall see later the canonical rational function plays a crucial role in the amplituhedron framework as they turn out to be scattering amplitudes. We can now impose residue constraints and determine $q(Y)$, this is called the method of *undetermined numerator*.

(a) Consider the triangle $\mathcal{T}$ shown in figure (1.6a), we begin by making an ansatz

$$\Omega(\mathcal{T}) = \frac{N dxdy}{y(1-x-y)(1+x-y)}. \tag{1.54}$$

Applying the residue constraints we get,

$$\begin{aligned}
Res_{y=0,x=1}\Omega(\mathcal{T}) &= \tfrac{N}{2} = 1 \\
Res_{y=0,x=-1}\Omega(\mathcal{T}) &= \tfrac{N}{2} = 1 \\
Res_{y=1,x=0}\Omega(\mathcal{T}) &= \tfrac{N}{2} = 1.
\end{aligned}$$

Thus $N = 2$ and the canonical form is

$$\Omega(\mathcal{T}) = \frac{2dxdy}{y(1-x-y)(1+x-y)}.$$

(b) Consider the square $\mathcal{S}$ shown in figure (1.6b) , we begin by making an ansatz

$$\Omega(\mathcal{S}) = \frac{(Ax+By+C)dxdy}{(1-x^2)(1-y^2)}$$

Applying the residue constraints we get,

$$\begin{aligned}
Res_{y=1,x=1}\Omega(\mathcal{S}) &= \tfrac{A+C+B}{2} = 1 \\
Res_{y=1,x=-1}\Omega(\mathcal{S}) &= \tfrac{B-A+C}{2} = 1 \\
Res_{y=-1,x=1}\Omega(\mathcal{S}) &= \tfrac{A+C-B}{2} = 1 \\
Res_{y=-1,x=-1}\Omega(\mathcal{S}) &= \tfrac{B+C-A}{2} = 1.
\end{aligned}$$

Solving we get $A = 0$, $B = 0$ and $C = 4$ which gives

$$\Omega(\mathcal{S}) = \frac{4dxdy}{(1-x^2)(1-y^2)}.$$

(c) For the half-disk $\mathcal{D}$ (1.6c)

$$\Omega(\mathcal{D}) = \frac{2dxdy}{y(1-x^2-y^2)}.$$

(d) For the pizza slice $\mathcal{P}$ (1.6c)

$$\Omega(\mathcal{P}) = \frac{(4-4y)dxdy}{(1-x-y)(1+x-y)(1-x^2-y^2)}.$$

In the example (d) we see that the canonical form vanished at $y = 1$. We call the class of positive geometries $(X, X_{\geq 0})$ whose canonical form does not vanish anywhere as a *generalised simplex* or *simplex-like*. The canonical form $\Omega(X, X_{\geq 0})$ for simplex like positive geometries can be readily written down as it is determined directly from its poles without any condition on its residues up to an overall constant.

To see this consider Ω_1 and Ω_2 be two rational forms on X with the same simple poles and no zeroes. Since the ratio Ω_1/Ω_2 is a holomorphic function on X which is projective and irreducible this ratio must be a constant $\Omega_1 = c\Omega_2$.

This simplifies the determination of canonical forms of generalised simplexes significantly. For example let us consider the simplest generalised simplex which is a projective simplex $(\mathbb{P}^m(\mathbb{R}), \Delta)$ defined in terms of its facets W_i by (1.48). Then since it has simple poles corresponding to $Y.W_i = 0$ we can write down the canonical form as:

$$\Omega(\Delta) = \frac{\langle W_1 W_2 ... W_{m+1}\rangle \, \langle Y d^m Y\rangle}{m! \, (Y \cdot W_1) \, (Y \cdot W_2) ... (Y \cdot W_m)}. \tag{1.55}$$

We have used the fact the form is projective and should be invariant under the scaling $W_i \to \alpha(W) W_i$ to determine the overall constant directly without using any residue constraints. We can rewrite this in terms of vertices Z_i using (1.44) as :

$$\Omega(\Delta) = \frac{\langle Z_1 Z_2 ... Z_{m+1}\rangle^m \langle Y d^m Y\rangle}{m! \langle Y Z_1 ... Z_m\rangle \langle Y Z_2 ... Z_{m+1}\rangle ... \langle Y Z_1 ... Z_{m-1}\rangle}. \tag{1.56}$$

- **Triangulations:** We triangulate the positive geometry and find the canonical form by summing over the canonical forms of the individual pieces using (1.50). Let us give a couple of examples.

(1) Consider a triangulation of a line segment $[a, b]$ by a sequence of connected segments:

$$[a, b] = \cup_{i=1}^{n} [c_{i-1}, c_i], \tag{1.57}$$

where $a = c_0 < c_1 < \cdots < c_n = b$.

We can see that

$$\Omega([a, b]) = \frac{(b-a)dx}{(b-x)(x-a)} = \sum_{i=1}^{n} \frac{(c_i - c_{i-1})dx}{(c_i - x)(x - c_{i-1})} = \sum_{i=1}^{n} \Omega([c_i, c_{i-1}]). \tag{1.58}$$

Here $x = a, b$ are physical poles and $x = c_i$ for all $1 \leq i \leq (n-1)$ are spurious poles.

(2) For the pizza slice $\mathcal{P}$ we could have also found the canonical form by triangulating it into a triangle $\mathcal{T}$: $X_{1,\geq 0} = \{(1, x, y) \in \mathbb{P}^2(\mathbb{R}) \mid q_1 = y \geq 0,\ q_2 = 1 - x - y \geq 0,\ q_3 = 1 - y + x \geq 0\}$ and a half disk $\mathcal{D}'$: $X_{2,\geq 0} = \{(1, x, y) \in \mathbb{P}^2(\mathbb{R}) \mid q_1 = -y \geq 0,\ q_2 = 1 - x^2 - y^2 \geq 0\}$

$$\begin{aligned} \Omega(\mathcal{P}) &= \Omega(\mathcal{T}) + \Omega(\mathcal{D}') \\ &= \frac{2dxdy}{y(1-x-y)(1+x-y)} - \frac{2dxdy}{y(1-x^2-y^2)} \\ &= \frac{(4-4y)dxdy}{(1-x-y)(1+x-y)(1-x^2-y^2)}. \end{aligned}$$

The spurious pole is at $y = 0$. The poles at $y \pm x = 1$ and $x^2 + y^2 = 1$ correspond to physical poles.

We can obtain the canonical form of any convex polytope using (1.50) and (1.56) as it can be triangulated by simplexes.

- **Direct products:** Whenever a positive geometry is a product of two lower dimensional positive geometries we can find the canonical form using (1.51).

  The square $(\mathbb{P}^2(\mathbb{R}), \mathcal{S})$ is the direct product of line segments $(\mathbb{P}^1(\mathbb{R}), X_{1,\geq 0})$ and $(\mathbb{P}^1(\mathbb{R}), X_{2,\geq 0})$ where $X_{1,\geq 0} = \{(1, x)|q_1 = x \geq 1 \text{ and } q_2 = -x \geq -1\}$ and $X_{2,\geq 0} = \{(1, y)|q_1 = y \geq 1 \text{ and } q_2 = -y \geq -1\}$.

  The canonical form of $\mathcal{S}$ can thus be obtained as

$$\begin{aligned} \Omega(\mathcal{S}) &= \Omega(X_{1,\geq 0}) \wedge \Omega(X_{2,\geq 0}) \\ &= \frac{-2dx}{(1-x^2)} \wedge \frac{-2dy}{(1-y^2)} \\ &= \frac{4dxdy}{(1-x^2)(1-y^2)}. \end{aligned}$$

- **Push-forwards:** We find morphisms from simpler positive geometries to more complicated positive geometries and find the canonical form using (1.52).

  We can find the canonical form for the half disk $\mathcal{D}$ by using the mapping $\Phi : T \to D$ of the triangle $\mathcal{T}$ onto it defined as $\Phi(1, x, y) = (1, x, \sqrt{2y - y^2})$.

  We can see that Φ takes boundary and interior of $\mathcal{T}$ to boundary and interior of $\mathcal{D}$ respectively. By setting $\Phi(1, x, y) = (1, x', y')$ and solving we get $x = x'$ and $y_\pm = 1 \pm \sqrt{1 - y'^2}$.

  We find the canonical form of $\mathcal{D}$ as the pushforward :

$$\begin{aligned} \Phi_\star(\Omega(\mathcal{D})) &= \sum_{\alpha=\pm} \frac{2dx_\alpha dy_\alpha}{y_\alpha(1 - x_\alpha - y_\alpha)(1 + x_\alpha - y_\alpha)} \\ &= \frac{2y'}{\sqrt{1-y'^2}} \left(\frac{-1}{(1+\sqrt{1-y'^2})(1-x'^2-y'^2)} + \frac{1}{(1-\sqrt{1-y'^2})(1-x'^2-y'^2)} \right) dx'dy' \\ &= \frac{2}{y'(1-x'^2-y'^2)}. \end{aligned}$$

- **Integral representations:** For convex polytopes there is simple connection between the canonical rational function and volume of the dual polytope. We can use this connection to find the canonical form by using various integral representations of the volume.

For a convex polytope $(\mathbb{P}^m(\mathbb{R}), \mathcal{A})$ and a point Y in its interior, the canonical rational function $\underline{\Omega}(\mathcal{A})$ at Y is given by the volume of the dual polytope $\mathcal{A}^*_Y$, i.e.,

$$\text{Vol}(\mathcal{A}^*_Y) = \underline{\Omega}(\mathcal{A})(Y) = \frac{1}{m!}\int_{W\in\mathcal{A}^*_Y} \frac{\langle W d^m W\rangle}{(Y.W)^{m+1}}. \tag{1.59}$$

In the integral expression above we have used the fact that since it is an integral on projective space, the integrand must be invariant under local scaling transformation $W \to \alpha(W)W$. Since we are integrating over all points inside $\mathcal{A}^*_Y$ it is clear that this is indeed the volume of the $\mathcal{A}^*_Y$, thus establishing the equivalence of the first and third statement. We shall argue that the second and third statements are equivalent for simplexes as the extension to any convex polytope is straightforward.

Let $Y \in Int(\Delta)$ for some simplex. The dual simplex Δ^*_Y has vertices $W_1, \cdots, W_{m+1}$ with $Y.W_i > 0$ for all $1 \leq i \leq m+1$. For any W we can write $W = \alpha_1 W_1 + \cdots + \alpha_{m+1}W_{m+1}$ for suitable $\alpha_i > 0$. Since, the integral (1.59) is scale invariant we can always set one of the α's to unity (say $\alpha_1 = 1$) and we can then replace the integral over W's by an integral over all α's.

$$\begin{aligned} Vol(\Delta^*_Y) &= \frac{1}{m!}\int_{W\in\mathcal{A}^*_Y} \frac{d^m\alpha\langle W_1\cdots W_{m+1}\rangle}{((Y.W_1)+\alpha_2(Y.W_2)+\cdots+\alpha_{m+1}(Y.W_{m+1}))^{m+1}} \\ &= \frac{1}{m!}\frac{\langle W_1\cdots W_{m+1}\rangle}{(Y.W_1)\cdots(Y.W_{m+1})}. \end{aligned}$$

In going from the second to the third line we have used the Feynman parametrisation to evaluate the integral over the α's. Since, every convex polytope can be triangulated by simplexes the above result (1.59) extends straightforwardly to all convex polytopes $\mathcal{A}$.

In this thesis we shall restrict to the simplest class of positive geometries (with straight boundaries) namely convex polytopes. For later purposes we would like to emphasise the difference between the convex polytopes we have defined in this section and a more general class of polytopes called *abstract polytopes* [75, 79].

1.7 Abstract polytopes

We would like to call the polytopes we have discussed till now as *geometric polytopes*. There is a more general class of polytopes which are called *abstract polytopes* that capture only combinatorial properties such as connections and incidences between various structural elements of the geometric polytope but not any geometric properties such as lengths and angles [79]. A geometric polytope is a realisation of the abstract polytope in Euclidean or projective space. The same abstract polytope can have several inequivalent realisations as Euclidean polytopes as can be seen for the case of the triangles in the figure below [79]:

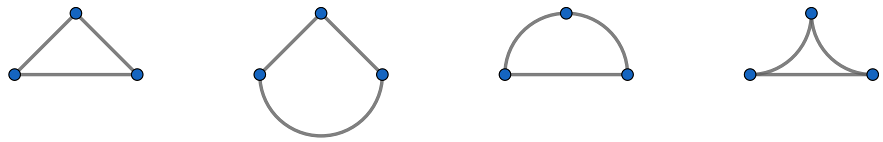


Figure 1.7: A few geometric triangles corresponding to an abstract triangle

In an abstract polytope each structural element like vertex, edge, facet etc is associated with corresponding member of the set. The term face refers to any such element of the set. The faces are ranked according to their real dimension: vertices have rank 0, edges have rank 1, facets have rank (n-1) etc. Faces of different rank can be ordered by the relation $F < G$ if F is a subface of G.

The faces of the polytope thus form a *lattice* with partial ordering determined by containment of faces. Since both the polytope itself and the empty set are faces, every pair of faces has a unique supremum and infimum. The whole polytope is the unique maximal element of the lattice and the empty set (the -1 dimensional face) is the unique minimal element of the lattice.

An abstract polytope $(P, <)$ is a partially ordered set, whose elements we call faces that satisfy the following:

1. It has a least face and a greatest face which are the null face $\emptyset$ and P respectively.

2. All maximal chains of totally ordered faces $(\emptyset, F', \cdots, F)$ called a *Flag* have an equal number of faces.

3. For any pair of faces F and G of P with $F \leq G$ there is a sequence of proper faces $H_1, \cdots, H_k$ such that $F = H_1 < H_2 < . \cdots < H_k = G$.

4. If ranks of two faces $b < a$ differ by 2, then there are exactly 2 faces that lie strictly between a and b.

A polytope of rank n is called an n-polytope. An abstract polytope is completely specified by its face lattice, and any two polytopes having the same face lattices are isomorphic to each other.

A simple way to visualize a polytope is using the concept of k-skeleton. A k-skeleton of an n-polytope is the collection of all faces of dimension up to k. For example the 0-skeleton is a discrete collection of vertices, 1-skeleton is the set of vertices and edges of the polytope which are graphs, and so on. To specify a generic polytope completely we would need to specify its n-skeleton. But for simple polytopes the 1-skeleton completely determines its face lattice [80, 81].

This is a crucial fact which we shall use throughout this thesis as all the polytopes we consider are simple polytopes. As we shall see later we define polytopes throughout this thesis by using their 1-skeleton which is in turn defined using a *flip graph*.

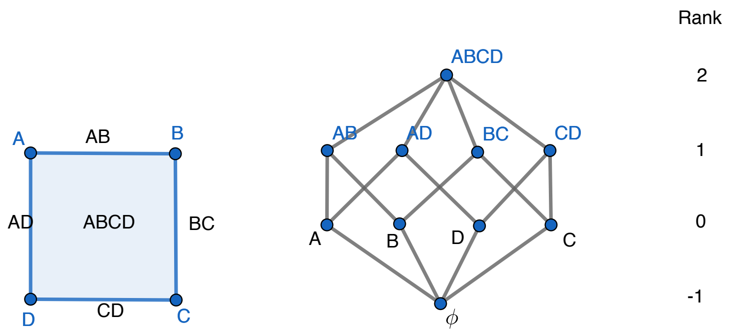


Figure 1.8: Hasse diagram of a quadrilateral

An abstract polytope can also be visualised using a *Hasse diagram*. The Hasse diagram is drawn by placing all faces of the same rank at the same vertical level and drawing edges to indicate containment of faces as shown for the the quadrilateral in figure (1.8).

The Hasse diagram defines a unique poset and therefore fully captures the structure of the polytope. Isomorphic polytopes give rise to isomorphic Hasse diagrams and vice versa.

1.8 The Amplituhedron

Having discussed the concept of positive geometries we can now summarise the Amplituhedron framework as follows:

For a given theory there are some putative class of positive geometries of any dimension n living in kinematic space and when the canonical form is pulled back onto these geometries it gives the scattering amplitude of n particles.

$$\text{Positive Geometry} \rightarrow \text{Canonical Form} \rightarrow \text{Scattering amplitude}$$

This remarkable program began with planar $\mathcal{N} = 4$ SYM [39–41, 72] where a complete geometric formulation of planar $\mathcal{N} = 4$ SYM amplitudes was given as:

$$\text{the } n\text{-point tree level } N^k \text{ MHV amplitude} = \Omega(\mathcal{A}(n, k, 4))$$

$$\text{the integrand of } n\text{-point } L\text{-loop } N^k \text{ MHV amplitude} = \Omega(\mathcal{A}(n, k, 4; 2^L))$$

The required super amplitude can be obtained from canonical form which is purely "bosonic" quantity using a straightforward prescription that involves integrating out auxiliary Grassmann variables. The amplituhedron lives in the *momentum twistor space*.

Furthermore, universal properties of the amplitude such as locality and unitarity *emerge* readily from the geometry. This is due to the fact that the singularities of the amplitude are encoded in the boundaries of the geometry and the remarkable property that each boundary of the amplituhedron is the product of lower dimensional amplituhedra which immediately implies the factorisation of the scattering amplitudes.

This geometric formulation of planar $\mathcal{N} = 4$ SYM also makes the hidden dual super conformal symmetry manifest [82]. This picture also gives infinitely many BCFW representations of the scattering amplitude which correspond to different triangulations of the amplituhedron and thus provides a more intuitive understanding of the many different possible BCFW expansions in terms of the cancellation of spurious poles [76, 77, 83, 84]. It has been shown recently that this formulation can also be used to compute planar $\mathcal{N} = 4$ amplitudes efficiently [42, 85].

In [43] it was shown that remarkably such a picture exists for a non-supersymmetric theory too by providing an explicit connection between tree-level amplitudes in bi-adjoint ϕ^3 theory and a polytope called the associahedron. As in the case of amplituhedron unitarity and locality emerged from geometric properties of the associahedron. Furthermore various properties such as colour kinematic duality and soft limits were directly deduced from the geometry of the associahedron. It was also argued that the CHY integrand for bi-adjoint ϕ^3 theory was just a push forward of the canonical form of the associahedron thus providing a simple "proof" of the CHY formula for bi-adjoint ϕ^3 theory. The program was further extended recently to 1-loop amplitudes in bi-adjoint ϕ^3 theory [44, 45].

It is therefore quite natural to wonder for what class of theories does such a geometric formulation exist. In particular since tree level CHY formulae exist for amplitudes in a wide class of quantum field theories including tree-level planar amplitudes in scalar field theories with ϕ^p $(p > 3)$ interactions [46], it is a essential to ask if the amplituhedron program can be extended for all ϕ^p $(p > 3)$ theories.

In this thesis we answer this question in the affirmative by showing that there exists a precise connection between scattering forms and a polytope called the *Accordiohedron* living in kinematic space for all scalar ϕ^p interactions [47,48]. We shall first briefly review some aspects of [43] which we shall need to understand the extension to ϕ^p interactions.

1.9 Planar scattering form and associahedron

In this section, we summarise the key results of [43]. We review the construction of planar scattering form and kinematic associahedron for tree-level amplitudes $m_n(\alpha \,|\, \beta)$ in bi-adjoint ϕ^3 theory. For simplicity we shall consider only the canonical ordering $\alpha = \beta = (1, 2, \cdots, n)$, for which $m_n(\alpha \,|\, \beta)$ is the sum over all planar cubic Feynman diagrams[2] as we had seen in (1.5). The generalisation to other orderings is straightforward. For further details, we refer the reader to [43].

1.9.1 Kinematic space

Kinematic space ($\mathcal{K}_n$) of n-massless momenta p_i where $i = 1, 2, \ldots, n$ is spanned by $\binom{n}{2}$ Mandelstam variables,

$$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j. \tag{1.60}$$

We shall only consider spacetime dimensions $d \geq (n-1)$, for which all s_{ij}'s are not linearly independent and satisfy

$$\sum_{j=1;j\neq i}^{n} s_{ij} = 0, \quad i = 1, 2, \ldots n. \tag{1.61}$$

[2]By planar diagrams we mean diagrams with no crossing.

If $d < (n-1)$ there are additional conditions that s_{ij}'s need to satisfy and thus the number of independent variables is lower. Thus the dimensionality of the kinematic space $\mathcal{K}_n$ of n massless particles reduces to

$$dim(\mathcal{K}_n) = \binom{n}{2} - n = \frac{n(n-3)}{2} \tag{1.62}$$

For any set of particle labels $I \subset \{1, 2, \ldots n\}$ one can define Mandelstam variables as follows,

$$s_I = \Big(\sum_{i \in I} p_i\Big)^2 = \sum_{i,j \in I;\ i<j} s_{ij}. \tag{1.63}$$

For cyclically ordered particles it is useful to define planar kinematic variables,

$$X_{i,j} = s_{\{i,i+1,\ldots j-1\}}; \quad 1 \le i < j \le n. \tag{1.64}$$

From the definition it is easy to see that $X_{i,i+1} = 0$ and $X_{1,n} = 0$. The variables $X_{i,j}$ can be visualized as the diagonal between i^{th} and j^{th} vertices of the corresponding n-gon (see figure (1.9)). In other words $X_{i,j}$ are dual to $\frac{n(n-3)}{2}$ diagonals of n-gon made up of edges with momenta $p_1, p_2, \ldots p_n$.

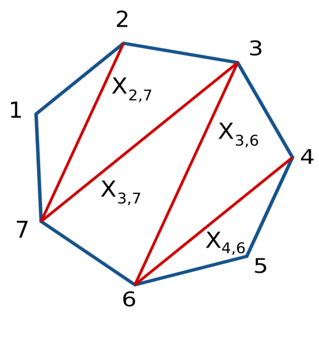


Figure 1.9: Planar variables.

These variables are related to Mandelstam variables via the following relation

$$s_{ij} = X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1}. \tag{1.65}$$

There exists an one-to-one correspondence between cuts of cubic graphs and complete triangulations of a n-gon. Each side of the n-gon corresponds to an external particle in the Feynman diagram and each diagonal *i.e* $X_{i,j}$ cuts the internal propagator of a Feynman diagram once (see figure (1.10)).

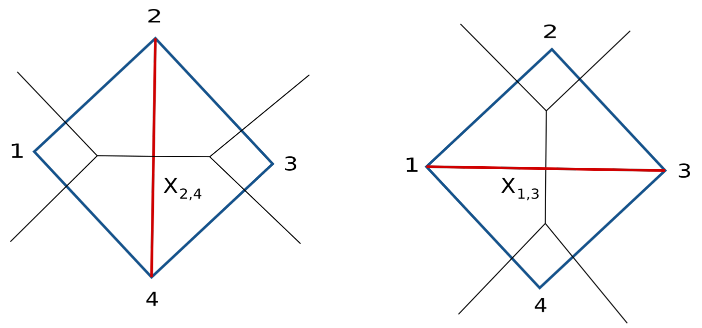


Figure 1.10: A planar variable cuts an internal propagator of the Feynman diagram once.

A partial triangulation of regular n-gon is a set of non-crossing diagonals which do not divide the n-gon into $(n-2)$ triangles. Here is an example of partial triangulation for a 5-gon.

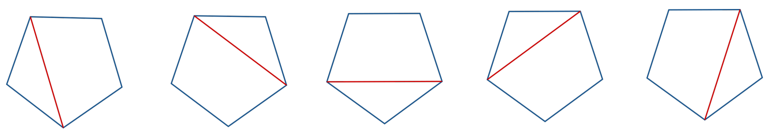


Figure 1.11: Partial triangulations of a pentagon.

We define a notion of *flip* diagonal for any given diagonal in a complete triangulation as the replacement of the diagonal by the conjugate diagonal inside the unique quadrilateral that contains it. For example in the figure (1.10) above the flip of diagonal $(2,4)$ is $(1,3)$ and vice versa. We can use this rule to define the associahedron $\mathcal{A}_n$ of dimension $(n-3)$ as follows:

We can start with any complete triangulation P of a convex polygon with $(n-3)$ diagonals,

• *In the first step for each of the $(n-3)$ diagonals, we go to the unique quadrilateral which contains it and replace it with the conjugate diagonal.*

• *In the second step for each of the* $(n-3)$ *triangulations at the end of step one we choose one of the original* $(n-4)$ *diagonals and replace it with its flipped diagonal as in step one.*

• *We repeat this till none of the original* $(n-3)$ *diagonals remain in step* $(n-3)$*.*

This generates a *flip graph* which is the 1-skeleton of a convex polytope called the *Associahedron* [86–88], which we shall also call $\mathcal{A}_n$. Since the associahedron is a simple polytope we can reconstruct the face lattice from its 1-skeleton. The *associahedron* of dimension $(n-3)$ is a polytope whose co-dimension d boundaries are in one-to-one correspondence with the partial triangulation by d diagonals (see figure (1.12)).

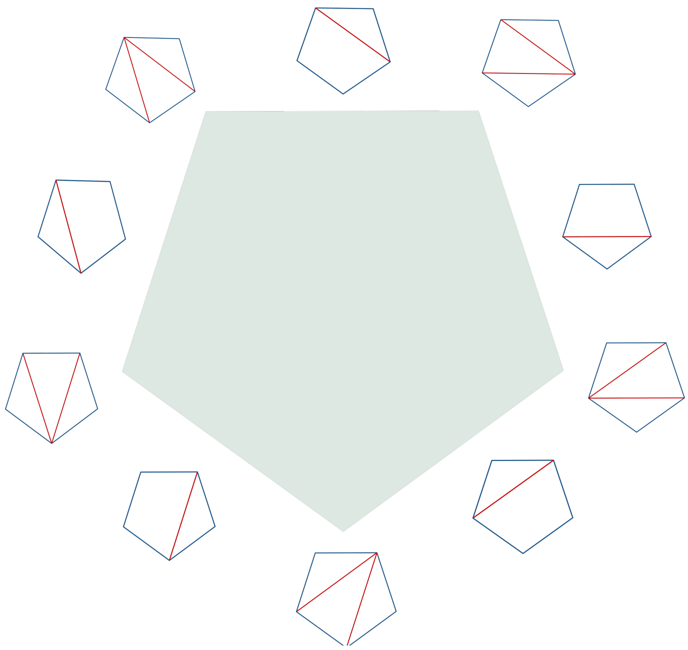


Figure 1.12: Two dimensional associahedron $\mathcal{A}_5$: 5 partial triangulations are represented by 5 diagonals. 5 complete triangulations are represented by 5 vertices.

We can sumarize the correspondence between faces of the associahedron and triangulations as

$$\begin{aligned} \textbf{Vertices} &\leftrightarrow \textbf{complete triangulations} \\ \textbf{Edges} &\leftrightarrow \textbf{Flips between them} \\ \textbf{k-Faces} &\leftrightarrow \textbf{k-partial triangulations}. \end{aligned}$$

The total number of ways to triangulate a convex n-gon by non-intersecting diagonals is the $(n-2)$-th Catalan number [89], $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$ (see Appendix (B) for a simpler proof). The dimension of the associahedron corresponding to a n-gon is $(n-3)$.

1.9.2 Planar scattering form

We now introduce the planar scattering form, a differential form on the space of kinematic variables $X_{i,j}$ that encodes information about on-shell tree-level scattering amplitudes of the scalar ϕ^3 theory. Let g denote a (tree) cubic graph with propagators X_{i_a,j_a} for $a = 1,\ldots,(n-3)$. The ordering is important here. For each ordering of these propagators, one assigns a value $\text{sign}(g) \in \{\pm 1\}$ to the graph with the property that flipping two propagators flips the sign. The form must have logarithmic singularities at $X_{i_a,j_a} = 0$. Therefore one assigns to the graph a $d\log$ form and thus defines the *planar scattering form* of rank $(n-3)$

$$\Omega_n^{(n-3)} := \sum_{\text{planar } g} \text{sign}(g) \bigwedge_{a=1}^{n-3} d\log X_{i_a,j_a}, \tag{1.66}$$

where the sum is over each planar cubic graph g. It's important to note that there are two sign choices[3] for each graph. Due to this fact there are potentially many different scattering forms. But one can fix the scattering form uniquely[4] if one demands projectivity of the differential form *i.e.* if one requires the form should be invariant under *local* $GL(1)$ transformations $X_{i,j} \to \Lambda(X)X_{i,j}$, for any index pair (i,j). We use this *projectivity* property to define a useful operation called *mutation*. Two planar graphs g and g' are related by a *mutation* if we can obtain one from the other just by exchanging four-point sub-graph channel (see figure (1.13)). In the figure (1.13), $X_{i,j}$ and $X_{i',j'}$ are the mutated propagators of the graphs g and g', respectively. Let's denote the rest of the (common) propagators as X_{i_b,j_b} with $b = 1,2,\ldots n-4$. Under a local GL(1) transformation, the $\Lambda(x)$ dependence of

[3]For 'clockwise' or 'anticlockwise' ordering of propagators $g = +1$ or -1, respectively.

[4]Actually the requirement of projectivity fixes the scattering form up to an overall sign which one ignores.

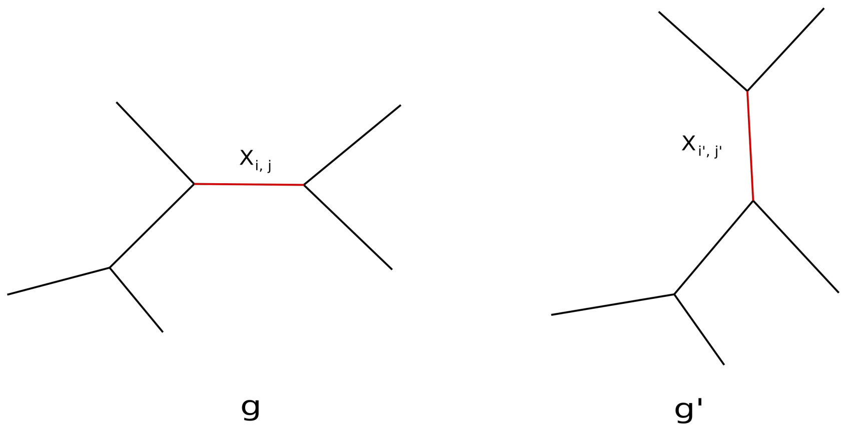


Figure 1.13: Two 5-point graphs related by mutation : $X_{i,j} \to X_{i',j'}$.

the scattering form becomes,

$$(\text{sign}(g) + \text{sign}(g'))\, d\log \Lambda \wedge \bigwedge_{a=1}^{n-4} d\log X_{i_a,j_a} + \dots . \tag{1.67}$$

But since we demand projectivity the form shouldn't have any $\Lambda(x)$ dependent piece and therefore,

$$\text{sign}(g') = -\,\text{sign}(g). \tag{1.68}$$

Note that projectivity ensures that the form should be ratios of Mandelstam variables. Here are few examples of $(n-3)$-forms in kinematic space of n particle scattering.

$$\Omega^{(1)}_{n=4} = d\log\left(\frac{s}{t}\right) = d\log\left(\frac{X_{1,3}}{X_{2,4}}\right), \tag{1.69}$$

$$\Omega^{(2)}_{n=5} = \quad d\log\frac{X_{1,3}}{X_{2,4}} \wedge d\log\frac{X_{1,3}}{X_{1,4}} + d\log\frac{X_{1,3}}{X_{2,5}} \wedge d\log\frac{X_{3,5}}{X_{2,4}} \tag{1.70}$$

and so on.

1.9.3 The kinematic associahedron

In the previous section we described how one gets an associahedron $\mathcal{A}_n$ in the kinematic space $\mathcal{K}_n$, nonetheless it is not evident how it should be embedded in $\mathcal{K}_n$, as $\mathcal{K}_n$ and $\mathcal{A}_n$ are of different dimensionality

$$dim(\mathcal{K}_n) = \frac{n(n-3)}{2} \tag{1.71}$$

$$dim(\mathcal{A}_n) = (n-3). \tag{1.72}$$

Ergo, one must to impose constraints to embed $\mathcal{A}_n$ inside $\mathcal{K}_n$. A natural choice is to demand all planar kinematic variables to be positive,

$$X_{i,j} \geq 0\,; \quad 1 \leq i < j \leq n. \tag{1.73}$$

These are $\frac{n(n-3)}{2}$ inequalities and thus cutout a big simplex Δ_n inside $\mathcal{K}_n$ which is still $\frac{n(n-3)}{2}$ dimensional. Therefore, one needs $\frac{n(n-3)}{2} - (n-3) = \frac{(n-2)(n-3)}{2}$ more constraints to embed the $\mathcal{A}_n$ inside $\mathcal{K}_n$. To that end, one imposes the following constraints [43, 90],

$$s_{ij} = -c_{ij}\,; \quad for\;\; 1 \leq i < j \leq n-1,\; |i-j| \geq 2, \tag{1.74}$$

where c_{ij} are positive constants.

These constraints give a space H_n of dimensions $(n-3)$ which is precisely the dimension of $\mathcal{A}_n$. The kinematic associahedron $\mathcal{A}_n$ now can be embedded in $\mathcal{K}_n$ as the intersection of the simplex Δ_n and the subspace H_n as follows

$$\mathcal{A}_n := H_n \cap \Delta_n. \tag{1.75}$$

It was conjectured in [43] that the convex polytope carved out by these conditions is a realization of the abstarct associahedron we had defined in the previous section. The conjecture was proved recently [90].

Once one has embedded the associahedron in $\mathcal{K}_n$, all one needs to do is to obtain its canonical form $\Omega(\mathcal{A}_n)$. Since associahedron is a simple polytope, one can directly write down its canonical form as follows [51]

$$\Omega(\mathcal{A}_n) = \sum_{\text{vertex } Z} \text{sign}(Z) \bigwedge_{a=1}^{n-3} d\log X_{i_a,j_a}, \tag{1.76}$$

where for each vertex Z, $X_{i_a,j_a} = 0$ denotes its adjacent facets[5] for $a = 1, \ldots, n-3$.

It was argued in [43] that the above differential form (1.76) is identical to the pullback of scattering form (1.66) in $\mathcal{K}_n$ to the subspace $\mathcal{A}_n$. We can justify this statement by identifying: $g \leftrightarrow Z$ and $\text{sign}(g) \leftrightarrow \text{sign}(Z)$ which follows from:

- There is a one-to-one correspondence between vertices Z and planar cubic graphs g. Also g and its corresponding vertex Z has same propagators X_{i_a,j_a}.
- Let Z and Z' be two vertices related by mutation. Note that mutation can also be framed in the language of triangulation. Two triangulations are related by a mutation if one can be obtained from the other by exchanging exactly one diagonal (see figure (1.14)).

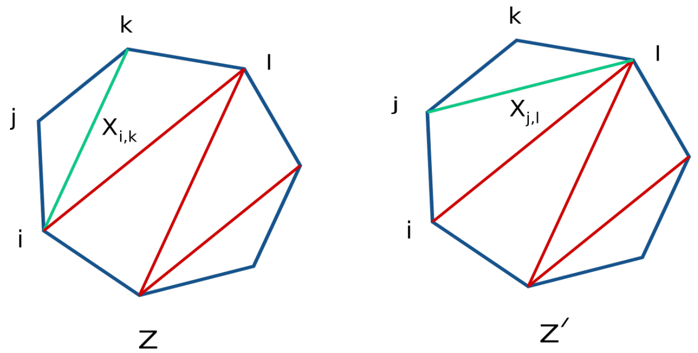


Figure 1.14: Two triangulations related by mutation : $X_{i,k} \to X_{j,l}$.

[5] One should be careful about the orientations of the facets. Depending on the ordering of the facets, we assign a sign(Z) $\in \{\pm 1\}$.

Thus for Z and Z' vertices we have

$$\bigwedge_{a=1}^{n-3} dX_{i_a,j_a} = -\bigwedge_{a=1}^{n-3} dX_{i'_a,j'_a} \tag{1.77}$$

which leads to sign-flip rule identical to g *i.e.* $\text{sign}(Z) = -\,\text{sign}(Z')$.

Therefore one can construct the following quantity (an $(n-3)$-form) which is independent of g on pullback.

$$d^{n-3}X := \text{sign}(g) \bigwedge_{a=1}^{n-3} dX_{i_a,j_a} \tag{1.78}$$

Substituting this in (1.76) one gets,

$$\Omega(\mathcal{A}_n) = \underbrace{\left(\sum_{\text{planar } g} \frac{1}{\prod_{a=1}^{n-3} X_{i_a,j_a}} \right)}_{m_n} d^{n-3}X, \tag{1.79}$$

where m_n is the expected tree level planar n-point amplitude for scalar cubic theory.

1.10 Factorisation and Soft limits

In this section we show two important properties of bi-adjoint amplitude follow readily from the geometric properties of the associahedron viz

1. The amplitude factorises on physical poles.
2. The amplitude vanishes in a soft limit.

Factorisation

We want to show that the bi-adjoint amplitude factorises as (1.25) directly from the geometry of the associahedron. We do this by first showing that the associahedron factorises combinatorially i.e. each facet is combinatorially identical to a product of lower dimensional associahedra.

On a facet $X_{ij} = 0$ we have

$$\mathcal{A}_n|_{X_{i,j}=0} \cong \mathcal{A}_L \times \mathcal{A}_R, \tag{1.80}$$

where $\mathcal{A}_L = \mathcal{A}(i, i+1, \cdots, j-1, \bar{I})$ and $\mathcal{A}_R = \mathcal{A}(1, \cdots, i-1, I, j, \cdots, j-1, n)$ for an intermediate particle I. Then by (1.51) we get

$$Res_{X_{i,j}=0}\Omega(\mathcal{A}_n) = \Omega(\mathcal{A}_L) \wedge \Omega(\mathcal{A}_R), \tag{1.81}$$

which implies the factorisation of the amplitude (1.25).

To prove (1.80) we begin by constructing a "left associahedron" $\mathcal{A}_L$ and a "right associahedron" $\mathcal{A}_R$ living in independent kinematic spaces. The left and right associahedra have a kinematic basis consisting of left variables $L_{a,b}$ and right variables $R_{a,b}$ respectively which correspond to some triangulation of the left and right sub-polygon obtained by omitting the diagonal (i, j)

$$\begin{aligned} \mathcal{A}_L \quad &: \quad L_{a,b}\ for\ i \leq a < b < j \\ \mathcal{A}_R \quad &: \quad R_{a,b}\ for\ 1 \leq a < b < n \text{ except } i \leq a < b < j \end{aligned}$$

We also assume that the two associahedra come with non-adjacent positive constants l_{ab} with $i \leq a < b < j$ and r_{ab} with $1 \leq a < b < n$ except $i \leq a < b < j$. Since the triangulations of the left and right sub polygons combine to form a partial triangulation of the n-gon with the diagonal (i, j) removed. These variables provide a basis for the subspace $\mathcal{H}_n|_{X_{ij}=0}$. By letting the non-adjacent constants match $c_{ab} = l_{ab}$ for all l_{ab} and $c_{ab} = r_{ab}$ for all $a, b \neq I$ we can write an obvious map $\mathcal{A}_L \times \mathcal{A}_R \to \mathcal{H}_n|_{X_{ij}=0}$:

$$\begin{aligned} X_{a,b} \quad &= \quad L_{a,b} \text{ for all left variables } L_{a,b} \\ X_{a,b} \quad &= \quad R_{a,b} \text{ for all right variables } R_{a,b} \end{aligned}$$

As such a map can first be defined for the left and right basis variables. But, since any left or right variables can be written as a linear combination of the basis variables and non-adjacent constants, the map can be extended to all left or right variables.

We now argue that the image of the embedding lies in the facet $\mathcal{A}_n|_{X_{i,j}=0}$ by showing that all planar variables apart from $X_{i,j} = 0$ are positive. It is sufficient to show this for diagonals (k,l) that cross (i,j) with $1 \leq i < k < j < l \leq n$ since all the others are positive by construction. By considering the following identity with $X_{i,j} = 0$

$$X_{k,l} + X_{i,j} = X_{k,j} + X_{i,l} + \sum_{\substack{i \leq a < k \\ j \leq b < l}} c_{ab}. \tag{1.82}$$

We see that $X_{k,l}$ is positive since the right hand is positive term by term as (k, j) and (i,l) are diagonals of the left and right sub-polygons the corresponding planar variables and the non-adjacent constants c_{ab}'s are all positive. Thus, proving that the map is one-one and thereby guaranteeing (1.80).

Soft limit

Let us consider the soft limit where we send momentum $p_i \to 0$ i.e. $c_{ij} \to 0$ for $j \neq i-1,\ i+1$ for the associahedron $\mathcal{A}_n$ which lives in the subspace $\mathcal{H}_n$ defined by non-adjacent structure constants c_{ij}, it follows from kinematic constraints that

$$X_{i,i+2} + X_{i-1,i+1} = s_{i\ i+1} + s_{i-1\ i} = -\sum_{j \neq i-1,\ i+1} s_{ij} = \sum_{j \neq i-1,\ i+1} c_{ij} \to 0$$

Since $X_{i,i+2},\ X_{i-1,i+1} \geq 0$ inside the associahedron this limit implies that $X_{i,i+2} = X_{i-1,i+1} = 0$. Thus the soft limit "squashes" the polytope to a lower dimensional one whose canonical form vanishes identically on $\mathcal{H}_n$ implying that the amplitude m_n is identically zero.

The vanishing of the amplitude m_n in the soft limit is a non-trivial fact that is not manifest from Feynman diagrams.

1.11 Worldsheet associahedron and CHY

We have seen that scattering amplitudes can be obtained from geometry of the associahedron. This however is not the first instance where the associahedron appears in physics. It has been known for a long time that the open string moduli space when suitably compactified has an associahedron associated to it. The canonical form associated with *worldsheet associahedron* turns out to be the famous *worldsheet Parke-Taylor form*. Recall that the Parke-Taylor form was associated to the CHY integrand for bi-adjoint ϕ^3 theory.

It was conjectured in [43] that the worldsheet associahedron and the kinematic associahedron are diffeomrphic to each other with the diffeomorphism being provided by the scattering equations. The conjecture was then verified numerically for a substantial amount of data. The canonical form of the kinematic associahedron can then be obtained as push forward of the worldsheet associahedron there by giving beautiful meaning to the scattering equations and a elegant geometric derivation of the CHY formula for bi-adjoint ϕ^3 theory.

The *positive moduli space* $\mathcal{M}^+_{0,n} = \{0 < \sigma_2 < \cdots < \sigma_{n-2} < 1\}$ is a positive subspace of the open string moduli space $\mathcal{M}_{0,n}(\mathbb{R})$. This is just one of the $\frac{(n-1)!}{2}$ distinct regions given by ordering σ_i variables [91]. The *positive moduli space* $\mathcal{M}^+_{0,n}$ corresponds to the standard ordering $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ where the $SL(2,\mathbb{R})$ redundancy has been used to set $\sigma_1 = 0, \sigma_{n-1} = 1$ and $\sigma_n = \infty$. $\mathcal{M}^+_{0,n}$ does not contain boundaries of all co-dimensions and hence is not a positive geometry. But, it can be made into one by compactifying it. This is done by introducing the variables u_{ij} for $1 \leq i < j-1 < n$ which are constrained to lie in the region $0 \leq u_{ij} \leq 1$, subject to the *non-crossing identity* [92, 93]

$$u_{i,j} = 1 - \prod_{(k,l)\in(i,j)^c} u_{k,l}. \tag{1.83}$$

The $u_{i,j}$'s are analogous to the planar kinematic variables $X_{i,j}$ and can also be visualised as the diagonal (i, j) of a convex n-gon with cyclically ordering. The above constraints imply that the u-space is $(n-3)$-dimensional. We can construct a map from the the positive moduli space $\mathcal{M}^{+}_{0,n}$ to the interior of the u-space

$$u_{i,j} = \frac{(\sigma_i - \sigma_{j-1})(\sigma_{i-1}\sigma_j)}{(\sigma_i - \sigma_j)(\sigma_{i-1}\sigma_{j-1})} = \frac{(i\ j-1)(i-1\ j)}{(i\ j)(i-1\ j-1)}. \tag{1.84}$$

We can take the closure in the u-space thereby compactifying the positive moduli space $\bar{\mathcal{M}}^{+}_{0,n}$ [94,95]. This is called the *u-space compactification* and it produces the same boundary structure as that of the associahedron. This was proved by noticing that on every boundary $u_{i,j} = 0$ for some i, j factors geometrically into the product of lower dimensional worldsheets

$$\partial_{(i,j)}\bar{\mathcal{M}}^{+}_{0,n} \cong \bar{\mathcal{M}}^{+}_{0,n_L} \times \bar{\mathcal{M}}^{+}_{0,n_R}, \tag{1.85}$$

where $\bar{\mathcal{M}}^{+}_{0,n_L} = \bar{\mathcal{M}}^{+}_{0,n}(i, \cdots, j-1, I)$ and $\bar{\mathcal{M}}^{+}_{0,n_R} = \bar{\mathcal{M}}^{+}_{0,n_R}(1, \cdots, i-1, I, j, \cdots, n)$.

The canonical form of the worldsheet associahedron can be found by a systematic blow-up procedure wherein the boundaries of the simplex are blown up to get an associahedron and it turns out be the Prake-Taylor form

$$\omega^{WS}_n = \frac{1}{vol SL(2)} \prod_{a=1}^{n} \frac{d\sigma_a}{\sigma_a - \sigma_{a+1}}. \tag{1.86}$$

The scattering equations E_i relate points in moduli space $\mathcal{M}_{0,n}$ to points in kinematic space $\mathcal{K}_n$, the key observation made in [43] is that the scattering equations also act as diffeomorphism between the two associahedra $\bar{\mathcal{M}}^{+}_{0,n}, \mathcal{A}_n$.

This was argued by first rewriting the planar variables $X_{a,b}$ interms of the u-coordinates and non-adjacent Mandelstram variables which have been set to constants $s_{ij} = -c_{ij}$:

$$X_{a,b} = \sum_{\substack{1\leq i<a \\ a<j<b}} \frac{(a\,j)(i\,n)}{(i\,j)(a\,n)} c_{ij} + \sum_{\substack{a\leq i<b \\ b\leq j<n}} \frac{(j\,n)(i\,b-1)}{(i\,j)(b-1\,n)} c_{ij} + \sum_{\substack{1\leq i<a \\ b\leq j<n}} \frac{(i\,n)(j\,n)(a\,b-1)}{(i\,j)(a\,n)(b-1\,n)} c_{ij}$$

which provides a map from $(\mathbb{P}^n(\mathbb{R}), \mathcal{A}_n) \to (\mathbb{P}^n(\mathbb{R}), \bar{\mathcal{M}}^+_{0,n})$ space that takes boundaries $u_{i,j} = 0$ of $\bar{\mathcal{M}}^+_{0,n}$ to boundaries of $X_{i,j} = 0$ of $\mathcal{A}_n$. It was further checked numerically for a substantial amount of data, that for every point in the interior of the kinematic associahedron exactly one of the $(n-3)!$ solutions of the scattering equations lies on the interior of the worldsheet associahedron. It was conjectured based on this strong evidence that scattering equations form a diffeomorphism.

Furthermore on using (1.52) we can write

$$\sum_{sol.\sigma} \omega_n^{WS} = m_n d^{n-3} X. \tag{1.87}$$

For a more general ordering α, β this generalises to

$$\sum_{sol.\sigma} \omega_n^{WS}[\alpha] = m_n[\alpha|\beta] d^{n-3} X. \tag{1.88}$$

Thus providing a simple and elegant "proof" of the CHY formula for bi-adjoint ϕ^3 amplitude.

Having reviewed the nessecary mathematical prelimaries and relevant details of the amplituhedron program for $\mathcal{N} = 4$ SYM and bi-adjoint ϕ^3 theory we are now ready to descibe the "amplituhedron" of ϕ^p interactions for all $p \geq 4$. We shall first consider the case of ϕ^4 interactions and subsequently generalize to all p.

The rest of the thesis is organised as follows. In chapter 2, we discuss the positive geometry of quartic interactions, namely, the Stokes polytope, our prescription for computing the planar amplitude using primitives and weights, a formula for the number of primitives and proof of factorisation of Stokes polytopes. In chapter 3, we discuss the positive geometry of ϕ^p interactions the accordiohedron, our prescription for computing the planar amplitude using primitives and weights, a formula for the number of primitive accordiohedra and proof of factorisation of accordiohedra. The chapter 4 contains a classification of all primitives for $n \leq 3$ for arbitrary p and implementation of our prescription to compute the weights for $p \leq 12$ and $n \leq 3$. We provide a discussion of our results and open questions in chapter 5. The appendices contain a collection of relevant mathematical results used in the thesis.

Chapter 2

Positive geometry for ϕ^4 interactions

As reviewed in the previous chapter, the relationship between (planar) Feynman graphs in ϕ^3 theory and positive geometry (namely associahedron) encapsulates a few intriguing features:

(1) There is a one to one correspondence between Feynman graphs with complete triangulations of a polygon.

(2) Dimension of the kinematic associahedron is the same as number of propagators in an n-particle scattering.

(3) Each co-dimension k-face of the associahedron is in one to one correspondence with a $(n-3-k)$-partial triangulation of the n sided polygon.

At first sight, it is tempting to consider a generalisation of these inter-relationships between polygons and planar (tree-level) amplitudes in ϕ^4 theory.

One immediately notices the following. Precisely as in the case of ϕ^3 theory and the triangulations of polygon, there is a one-to-one correspondence between planar tree-level diagrams of ϕ^4 theory and *complete quadrangulations*[1] of a polygon (see figure (2.1)).

[1]By complete Quadrangulation we just means decomposing a polygon into maximum number of quadrilaterals. We will refer to any subset of the diagonals which do not constitute a complete quadrangulation as partial quadrangulation.

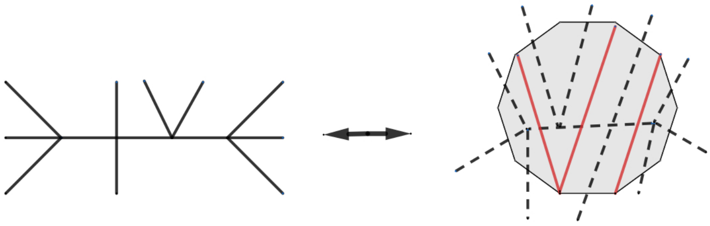


Figure 2.1: A one-to-one correspondence between Feynman graphs of ϕ^4 theory and quadrangulations of an even polygon.

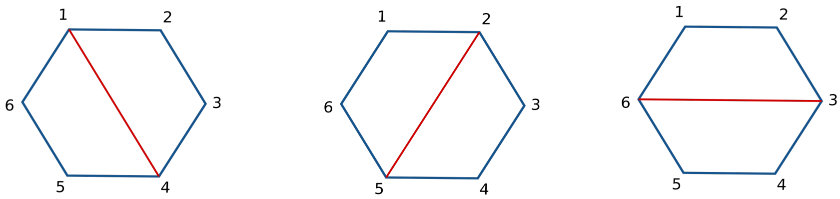


Figure 2.2: The 3 different planar channels for 6-point scattering.

A few facts about the quadrangulations are well known [96]. The total number of quadrangulations of an $n = (2N + 4)$-gon is given by the *Fuss-Catalan* number,

$$F_N = \tfrac{1}{2N+3}\, {}^{3N+3}C_{N+1}.$$

We can thus ask the following question. Is there a polytope $\mathcal{S}_n$ whose vertices are in $1 - 1$ correspondence with all quadrangulations of a polygon and whose dimension is same as the number of propagators in a single channel as in the associahedron case. Since, each quartic graph with $n = 2N + 4$ external legs has precisely N propagators,

$$dim(\mathcal{S}_n) = N.$$

We can now ask if there is a polytope whose dimension is N and number of vertices are same as F_N. Here we immediately run into an obstacle due to the fact that for the six-point scattering (*i.e.* $N = 1$) we should get a one dimensional polytope, which can only be a line segment with two boundaries but since there are in fact *three* planar scattering channels (see figure (2.2)) for the six-point diagram we cannot find such a polytope with boundaries which correspond to all three propagators going onshell. So, the only way to define a polytope is to exclude one of the channels using some systematic rule. This idea was precisely encapsulated in [97] in a different context and used to construct the Stokes polytope.

2.0.1 Stokes polytope

In order to introduce Stokes polytope, we first need to define a notion of *Q-compatibility* which selects, among the set of all (complete) quadrangulations of a polygon, a subset which will be in one-to-one correspondence with vertices of Stokes polytope.

Consider, a pair of quadrangulations Q and Q' of a regular $(2N+4)$-gon which we call *red* and *blue* respectively with diagonals directed from odd to even vertices (see figure (2.3)). We rotate Q' anti-clockwise and then superimpose it over Q so that the vertices now get interlaced. We then say Q' is Q-compatible with Q if and only if at each crossing of diagonals the pair (red,blue) in that order are oriented clockwise.

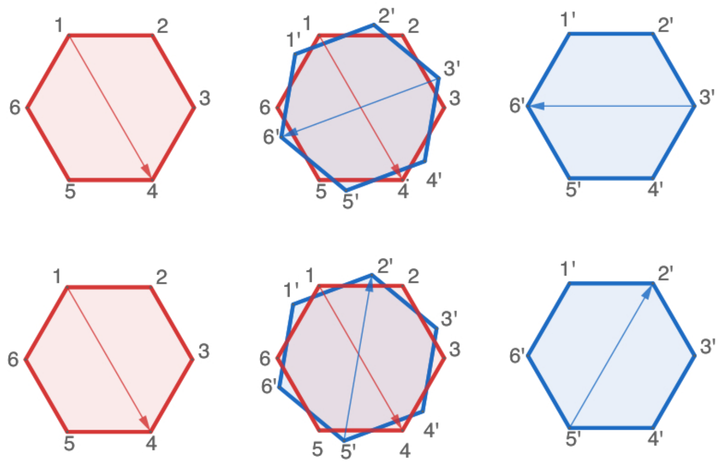


Figure 2.3: The above figure shows 36 is Q-compatible with 14 but 25 is not.

We must emphasise that Q-compatability is *not* an equivalence relation and is very much dependent on the reference quadrangulation Q, as can be easily checked that 14 is compatible with 36, 25 with 14 and 36 with 25 [2].

[2] A simple way to remember this rule is that every diagonal is Q-compatible with every alternate diagonal when we move clockwise(14 with 36 , 25 with 41 and 36 with 52).

We can now define a *flip* as the replacement of a diagonal of any hexagon inside the quadrangulation of the polygon with its Q-compatible diagonal, this corresponds to changing to a compatible channel for any 6-point diagram inside our $(2N + 4)$-point diagram. This is the analogue of mutation for quartic case (see eqn. (1.77)).

We can now use the notion of flip to define the Stokes polytope $\mathcal{S}_n^Q$, by starting with a particular complete quadrangulation Q with diagonals $(i_1 j_1, ..., i_N j_N)$, and by performing flips on each diagonal $i_k j_k$ by going to the unique hexagon that contains $i_k j_k$ and replacing it with its Q-compatible diagonal iteratively.

• *In the first step for each of the N diagonals, we go to the unique hexagon which contains it and replace it with the Q-flipped diagonal.*

• *In the second step for each of the* $(N - 1)$ *quadrangulations at the end of step one we choose one of the original* $(N - 1)$ *diagonals and replace it with its Q-flipped diagonal as in step one.*

• *We repeat this till none of the original N diagonals remain in step N.*

We illustrate this for the $N = 2$ (8-point scattering) below.

We start with the $Q = \{14, 58\}$ and flip either 14 to 38 in $\{1, 2, 3, 4, 5, 8\}$ or 58 to 47 in $\{1, 4, 5, 6, 7, 8\}$ and to get $Q_1 = \{36, 58\}$ or $Q_2 = \{14, 47\}$ respectively, then a further flip of either 14 to 38 in $\{1, 2, 3, 4, 7, 8\}$ or 58 to 47 in $\{3, 4, 5, 6, 7, 8\}$ both give $Q_4 = \{16, 47\}$. Further flips do not give us any new quadrangulations. Thus the corresponding Stokes Polytope in this case has 4 vertices. This is shown in the left half of $n = 8$ in figure (2.4).

If we start with $Q = \{14, 16\}$ and flip either 14 to 36 in $\{1, 2, 3, 4, 5, 6\}$ or 16 to 58 in $\{1, 4, 5, 6, 7, 8\}$ to get $Q_1 = \{36, 16\}$ or $Q_2 = \{14, 58\}$ respectively, then further flips of 16 to 38 in $\{1, 2, 3, 6, 7, 8\}$ and 14 to 38 in $\{1, 2, 3, 4, 5, 8\}$ give $Q_4 = \{36, 38\}$ and $Q_5 = \{36, 58\}$. Further flips do not give us any new quadrangulations. Thus the corresponding Stokes polytope in this case has 5 vertices. This is shown in the right half of $n = 8$ in figure (2.4).

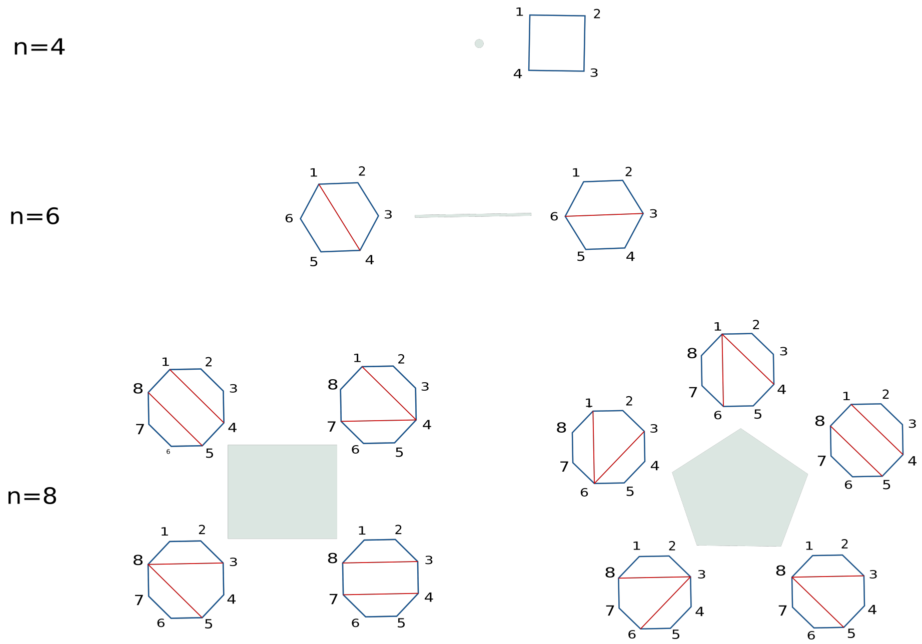


Figure 2.4: The first few Stokes polytopes. Note that for $n = 8$ there are two kinds of polytopes. This is one of the key features of the quartic case.

It can be checked that if we start with any of the $F_2 = 12$ *quadrangulations then the Stokes polytope we get is either a square or a pentagon. This is easily seen if we notice that the other* 10 *quadrangulations can be obtained from* $\{14, 16\}$ *and* $\{14, 58\}$ *by cyclic permutations and thus just amount to relabeling of the vertices.*

We can proceed along these lines to obtain Stokes polytopes for any $n = 2N + 4$, and there will be several Stokes polytopes depending on the reference quadrangulation Q we start with. Some of them do turn out be associahedra and we will say more about this in appendix (B). We can thus summarize the Stokes polytope in analogy with associahedron as follows:

Vertices $\leftrightarrow$ Q-compatible quadrangulations

Edges $\leftrightarrow$ Flips between them

k-Faces $\leftrightarrow$ k-partial quadrangulations

As we see, there are two key differences in the relationship of the Stokes polytope with quadrangulations from that of the associahedron and triangulations. First being, definition of Stokes polytope depends on the reference quadrangulation Q, and for each Q one has a Stokes polytope $\mathcal{S}_n^Q$. Secondly vertices of $\mathcal{S}_n^Q$ are not in 1-1 correspondence with all the quadrangulations of the polygon but only with a specific sub-set of them, namely Q-compatible quadrangulations. As all (planar) diagrams of a ϕ^4 theory are in 1-1 correspondence with set of *all* quadrangulations of a polygon, it is clear that a single $\mathcal{S}_n^Q$ can not be the amplituhedron for planar ϕ^4 theory.

However a rather enticing feature of definition of $\mathcal{S}_n^Q$ is a notion of the flip, which is analogous to mutation in the case of triangulations. As it was the mutation which was responsible for defining a unique scattering form in $\mathcal{K}_n$ in the ϕ^3 case, there is a possibility that the flip may do the same in this case. In the next section we propose just such a definition of planar scattering form for ϕ^4 theory in kinematic space, which however will depend on the reference quadrangulation Q.

2.1 Planar scattering form for ϕ^4 interactions

We consider tree level scattering amplitudes in a massless scalar field theory with quartic interactions. Given a specific ordering of external particles, we consider contribution of only planar diagrams which are consistent with this ordering. We refer to such amplitudes as *planar amplitudes* of massless ϕ^4 theory. These amplitudes can be thought of as analogs of the partial amplitudes $\mathcal{M}_n(\alpha|\alpha)$ in the context of bi-adjoint scalar ϕ^3 theory[3] which was considered in [43].

We would like to extend the idea of defining planar scattering form to planar amplitudes in massless ϕ^4 theory. However a quick look at the simplest example of six point amplitude shows us that such a form can not be projective. In general, for an n particle amplitude

[3]It is conceivable that the amplitudes we analyse can be considered as basic building blocks of amplitudes of a bi-adjoint scalar field theory with quartic interaction of the type $\mathrm{Tr}\left[[\phi,\phi]^2\right]$ where $[\phi,\phi]$ is the bi-adoijnt Lie bracket given by $f^{ijk}\tilde{f}^{i'j'k'}\phi^{ii'}\phi^{jj'}$. However as bi-adjoint scalar theory with quartic interaction has not been considered in literature so far, we will not refrain from exploring this point of view further.

in quartic theory, the number of planar diagrams can be even or odd and there is no sense in which projectivity can be employed to fix a unique scattering form. In the absence of projectivity, it is a priori not clear how do we define a planar scattering form for planar amplitudes in ϕ^4 theory. The hint in our case (that we alluded to in the previous section) comes from one of the key observations made in [43]. Namely, defining a scattering form projectively is equivalent to choosing the relative signs among various terms via mutation, which is in turn equivalent to flipping one of the diagonals in the triangulation of the n-gon.

For ϕ^4 interaction, even though mutation or projectivity do not appear to be relevant concepts, as we saw above, there is an analog. Given a reference quadrangulation Q, there is a set Q-compatible quadrangulations for which a notion of flip is well defined. Whence given a Q and its corresponding set of Q-compatible quadrangulations, we can define a planar scattering form on the kinematic space $\mathcal{K}_n$ as follows:

Let Q be a quadrangulation of an n-gon which is associated to an planar Feynman diagram and let $\{X_{i_1 j_1}, \ldots, X_{i_N j_N}\}$ denote Q-compatible quadrangulations with diagonals $\{i_1 j_1, \cdots, i_N j_N\}$, which are vertices of $\mathcal{S}_n^Q$. Then we define the (Q-dependent) planar scattering form as,

$$\Omega_n^Q \;=\; \sum_{\text{flips}} (-1)^{\sigma(\text{flip})} d\ln X_{i_1 j_1} \wedge \ldots d\ln X_{i_N j_N}, \tag{2.1}$$

where $\sigma(\text{flip}) \;=\; \pm 1$ depending on whether the quadrangulation $\{X_{i_1 j_1}, \ldots, X_{i_N j_N}\}$ can be obtained from Q by even or odd number of flips.

As the set of Q-compatible quadrangulations (for a given Q) does not exhaust all quadrangulations or equivalently, all the planar Feynman diagrams, the set of terms which appear in the planar scattering form in eqn. (2.1) does not correspond to all the diagrams of the theory. As an example consider $n = 6$ case and let $Q = 14$. Then the set of Q compatible quadrangulations are $\{(14,\ +),\ (36,\ -)\}$. We have attached a sign to each of the quadrangulation which measures the number of flips needed to reach it starting from reference

$Q = 14$. Whence the form Ω_6^Q on the kinematic space is given by,

$$\Omega_6^{Q=(14)} = (d\ln X_{14} - d\ln X_{36}). \tag{2.2}$$

It is clear that this form does not capture singularity associated to X_{25} channel for the 6 particle amplitude. Hence it may appear that eventually we may not recover full planar scattering amplitude from such a form. However there are two more Qs we need to consider. For $Q = 36$ the Q-compatible set is $\{(36, +), (25, -)\}$ and for $Q = 25$ the Q-compatible set is $\{(25, +), (14, -)\}$. The corresponding forms on Kinematic space are given by

$$\begin{aligned} \Omega_6^{Q=(36)} &= (d\ln X_{36} - d\ln X_{25}) \\ \Omega_6^{Q=(25)} &= (d\ln X_{25} - d\ln X_{14}). \end{aligned} \tag{2.3}$$

Hence we see that unlike the planar scattering form in the case of ϕ^3 interaction which is uniquely determined by requirement of projectivity, we have F_N planar scattering forms, one for each quadrangulation.

It can be easily checked that for all Q, Ω_n^Q in eqn. (2.1) factorises correctly when any one of the channels goes on-shell. For $i < j$,

$$\Omega_n^Q\Big|_{X_{ij} \to 0} = \Omega_{|j-i+1|}^{Q_1}(i, i+1, \ldots, j) \wedge \frac{dX_{ij}}{X_{ij}} \wedge \Omega_{n+2-|j-i+1|}^{Q_2}(j, \ldots, n, 1, \ldots, i), \tag{2.4}$$

where Q_1, Q_2 are quadrangulations associated to the polygons $\{(i, i{+}1, \ldots, j), (j, \ldots, n, 1, \ldots, i)\}$ respectively.

A happy fact about Ω_n^Q will emerge in the next section, paralleling the construction of [43] we will see how these forms naturally descends to the canonical form on a $\mathcal{S}_n^Q$. As Stokes polytope is a positive geometry, it has a canonical form associated to it which has (logarithmic) singularities on all the facets, such that the residue of restriction of this form on any of the facet equals the canonical form on the facet.

For Simple polytopes one can write down an explicit formula for the canonical form by embedding the polytope in projective space. Such an explicit formula for canonical form on $\mathcal{S}_n^Q$ does not seem to be available in the literature. The planar scattering form defined above however gives us precisely such a form on $\mathcal{S}_n^Q$. That is, we will take a cue from ideas of [43] and start with a definition of planar scattering form for ϕ^4 theory and show that it descends to a form on $\mathcal{S}_n^Q$ which satisfies all the properties required of the canonical form.

2.2 Locating the Stokes polytope in kinematic space

In this section we define kinematic Stokes polytopes $\{\mathcal{S}_6^Q \mid Q \in (14, 25, 36)\}$ for 6 particle amplitude and show how the planar scattering form Ω_n^Q defined above descends to the canonical form on $\mathcal{S}_6^Q$. We begin by fixing a reference quadrangulation Q in terms of kinematic data (*i.e.* a set of $X'_{ij}s$) and get a Stokes polytope $\mathcal{S}_n^Q$ in $\mathcal{K}_n$ which sits inside the positive region of kinematic space $\Delta_n = \{X_{ij} \geq 0, \forall\, i, j\}$. In fact, our definition of this kinematic Stokes polytope will be such that it is located inside the kinematic associahedron $\mathcal{A}_n$, thus ensuring that it lies in the positive region Δ_n.

For $Q_1 = (14)$ the Q_1 compatible set is given by $\{(14, +), (36, -)\}$. The corresponding Stokes polytope is one dimensional with two vertices. We locate this Stokes polytope inside the kinematic space via the following constraints

$$\begin{aligned} s_{ij} &= -c_{ij} \quad \forall\, 1 < i < j < n-1 = 5,\ |i-j| \geq 2 \\ X_{13} &= d_{13},\ X_{15} = d_{15}, \text{with } d_{13},\ d_{15} > 0. \end{aligned} \tag{2.5}$$

The first line of constraints are precisely the ones which define the three dimensional kinematic associahedron $\mathcal{A}_6$ inside $\mathcal{K}_6$. We have motivated the remaining two constraints as follows. We can adjoin, to the diagonal (14) any one out of the following pairs.

$\mathcal{I} = \{(13, 15), (24, 15), (13, 46), (24, 46)\}$ to form a complete triangulation of the hexagon. We pick *any one* of these pairs to impose further constraints on the kinematic data. From the perspective of Feynman diagrams, these constraints are rather natural as planar variables from this set can never occur in Feynman diagrams of ϕ^4 theory.

Using the above constraints, it can be easily checked that the planar kinematic variables satisfy,

$$\begin{aligned} X_{36} &= -X_{14} + c_{14} + c_{24} + c_{15} + c_{25} \geq 0 \\ X_{25} &= d_{15} + c_{14} - d_{13} + c_{13} \geq 0. \end{aligned} \tag{2.6}$$

We thus see that we have a (one dimensional) Stokes polytope $\mathcal{S}_6^{Q=(14)}$ whose vertices are given by $X_{14} = 0$ and $X_{36} = 0$ (which is when $X_{14} = c_{14} + c_{24} + c_{15} + c_{25}$) which correspond to the two Q-compatible quadrangulations. It can be readily verified that the kinematic Stokes polytope is insensitive to which of the pairs of diagonals in $\mathcal{I}$ above we choose to constrain [4]. We can now pull back the form given in eqn. (2.2) on $\mathcal{S}_6$

$$\omega_6^{Q_1} = \left(\frac{1}{X_{14}} + \frac{1}{X_{36}}\right) dX_{14} =: m_6(\mathcal{S}_6^{Q_1})\, dX_{14}, \tag{2.7}$$

where $m_6(Q_1)$ is the canonical rational function associated to the Stokes polytope $\mathcal{S}_6^{Q_1}$.

As a one dimensional Stokes polytope is also an associahedron (see appendix (B)), and as the form in eqn.(2.7) is the canonical form on associahedron, we have a canonical form on $\mathcal{S}_6^{Q=(14)}$.

The canonical rational function $m_6(Q_1)$ [5] is

[4] This is true only for Stokes polytopes which are hypercubes (B), however we claim that you can always find atleast one choice of diagonals which will carve out the Stokes polytope in kinematic space.

[5] For the sake of pedagogy, we are not differentiating between reference quadrangulation Q that we fix which is in rotated (blue) polygon and quadrangulations which generate stokes polytope which are quadrangulations of the red polygon [98].

$$m_6(Q_1) \;=\; \left(\frac{1}{X_{14}} \;+\; \frac{1}{X_{36}}\right). \tag{2.8}$$

We can now repeat the analysis with $Q_2 = (25)$ and $Q_3 = (36)$ analogously and it can be shown that the corresponding canonical forms on the Stokes polytopes are,

$$\begin{aligned} \omega_6^{Q_2} &= \left(\tfrac{1}{X_{25}} \;+\; \tfrac{1}{X_{14}}\right) dX_{25} \\ \omega_6^{Q_3} &= \left(\tfrac{1}{X_{36}} \;+\; \tfrac{1}{X_{25}}\right) dX_{36}. \end{aligned} \tag{2.9}$$

We now define a function $\widetilde{\mathcal{M}}_n$ on the kinematic space which is a weighted sum of the m_6 over all $\mathcal{S}_n^Q$. In the $n = 6$ case this function is defined as,

$$\widetilde{\mathcal{M}}_6 \;:=\; \alpha_{Q_1}\left(\tfrac{1}{X_{14}} \;+\; \tfrac{1}{X_{36}}\right) \;+\; \alpha_{Q_2}\left(\tfrac{1}{X_{25}} \;+\; \tfrac{1}{X_{14}}\right) \;+\; \alpha_{Q_3}\left(\tfrac{1}{X_{36}} \;+\; \tfrac{1}{X_{25}}\right). \tag{2.10}$$

Here α_{Q_i} are positive constants. It is immediately evident that if and only if $\alpha_{Q_1} = \alpha_{Q_2} = \alpha_{Q_3} = \frac{1}{2}$, $\widetilde{\mathcal{M}}_6 = \mathcal{M}_6$.

2.2.1 Eight particle scattering

Let us now consider the $n = 8$ case.

Our analysis will proceed along the same lines as in the previous section. Namely we first define planar scattering form on $\mathcal{K}_8^Q$ for all the quadrangulations. We will then show how all the kinematic Stokes polytopes $\mathcal{S}_8^Q$ sit inside the 5 dimensional associahedron $\mathcal{A}_8$ and then show how a weighted sum of canonical rational functions over all the polytopes leads to the planar scattering amplitude.

This computation can be made much easier by realising that all the quadrangulations of an octagon (and in general any polygon) can be obtained from cyclic permutations of a subset of quadrangulations. We call this set, set of *primitive quadrangulations*. More precisely,

Given a n sided polygon with labelled vertices, we call a set of quadrangulations $\{Q_1, \ldots, Q_I\}$ primitive if,

(a) no two members of the set are related to each other by cylic permutations and

(b) all the other quadrangulations can be obtained by a (sequence of) cyclic permutations of one of the Qs belonging to the set.

We note that, choice of which quadrangulations are called primitive is not unique but the cardinality of the set of primitive quadrangulations is uniquely fixed by n. In the $n = 6$ case, there is only one primitive Q and can be chosen to be $Q = (14)$.

As shown in section (2.0.1), there are two primitive Q's in this case. With out loss of generality we can take them to be $\{Q_1 = (14, 58),\ Q_2 = (14, 16)\}$.

As we have shown in figure (2.4),

Q_1 compatible quadrangulations are given by $S_1 = \{(14, 58; +),\ (14, 47; -),\ (83, 58; -),\ (83, 47; +)\}$,

Q_2 compatible quadrangulations are $S_2 = \{(14, 16; +),\ (14, 58; -),\ (36, 16; -),\ (36, 83; +),\ (58, 83; -)\}$. The signs associated to each quandrangulation is obtained by measuring the number of relative flips from the reference Q.[6]

Using eqn. (2.1), for each of the two sets $S_1,\ S_2$ we can define two distinct planar 2-forms on $\mathcal{K}_8$ as,

$$
\begin{aligned}
\Omega_8^{Q_1} &= (d\ln X_{14} \wedge d\ln X_{58} + d\ln X_{38} \wedge d\ln X_{47} - d\ln X_{14} \wedge d\ln X_{47} - d\ln X_{38} \wedge d\ln X_{58}) \\
\Omega_8^{Q_2} &= (d\ln X_{14} \wedge d\ln X_{16} - d\ln X_{14} \wedge d\ln X_{58} - d\ln X_{36} \wedge d\ln X_{16} + d\ln X_{36} \wedge d\ln X_{83} - d\ln X_{58} \wedge d\ln X_{83})\,.
\end{aligned}
$$

One can write down scattering forms for all other quadrangulations exactly analogously. The Stokes polytopes associated to S_1, S_2 are two dimensional positive geometries with

[6]It is important to maintain the order of the diagonals when a flip is taken as these denote the ordering of the wedge product ($(14, 58) \rightarrow d\ln X_{14} \wedge d\ln X_{58}$ etc.) and since this also contributes to the overall sign of the term when the Scattering form is written down.

four and five vertices respectively.

We now locate the two Stokes polytopes S_{Q_1} and S_{Q_2} inside the Kinematic space (in fact, inside the five dimensional associahedron $\mathcal{A}_8$) precisely in analogy with $n = 6$ case. Let T_1 and T_2 be *any* two sets of diagonals which are such that $T_1 \cup \{14, 58\}$ and $T_2 \cup \{14, 16\}$ are complete triangulations of the octagon (with labelled vertices). We choose T_1 and T_2 to be $\{13, 48, 57\}$ and $\{13, 46, 86\}$ respectively.

The constraints defining S_{Q_1} and S_{Q_2} inside the kinematic space are respectively given by

$$\begin{aligned} s_{ij} &= -c_{ij} \;\forall\; 1 \leq i < j \leq 7 \text{ with } |i-j| \geq 2 \\ X_{13} &= d_{13},\; X_{48} = d_{48}\,, X_{57} = d_{57}. \end{aligned} \tag{2.11}$$

$$\begin{aligned} s_{ij} &= -c_{ij} \;\forall\; 1 \leq i < j \leq 7 \text{ with } |i-j| \geq 2 \\ X_{13} &= d_{13},\; X_{46} = d_{46}\,, X_{68} = d_{68}. \end{aligned} \tag{2.12}$$

These constraints locate both the Stokes polytopes inside the five dimensional associahedron $\mathcal{A}_8$ and hence ensure that all the X_{ij}'s are positive in the interior of the Stokes polytopes.

Using these constraints it is simple algebraic exercise to show that on $\mathcal{S}_8^{Q_1}$, $\mathcal{S}_8^{Q_2}$ one has the following top forms obtained from Ω_{Q_i} on $\mathcal{K}_8$.

$$\begin{aligned} \omega_8^{Q_1} &= \left(\frac{1}{X_{14}X_{58}} + \frac{1}{X_{38}X_{47}} + \frac{1}{X_{14}X_{47}} + \frac{1}{X_{38}X_{58}}\right) dX_{14} \wedge dX_{58} \\ \omega_8^{Q_2} &= \left(\frac{1}{X_{14}X_{16}} + \frac{1}{X_{14}X_{58}} + \frac{1}{X_{36}X_{16}} + \frac{1}{X_{36}X_{83}} + \frac{1}{X_{58}X_{83}}\right) dX_{14} \wedge dX_{16}. \end{aligned} \tag{2.13}$$

The corresponding canonical functions m_8 are given by

$$\begin{aligned} m_8(Q_1) &= \left(\frac{1}{X_{14}X_{58}} + \frac{1}{X_{38}X_{47}} + \frac{1}{X_{14}X_{47}} + \frac{1}{X_{38}X_{58}}\right) \\ m_8(Q_2) &= \left(\frac{1}{X_{14}X_{16}} + \frac{1}{X_{14}X_{58}} + \frac{1}{X_{36}X_{16}} + \frac{1}{X_{36}X_{83}} + \frac{1}{X_{58}X_{83}}\right). \end{aligned} \tag{2.14}$$

As all the other quadrangulations can be obtained by cyclic permutations of (labels of) Q_1 and Q_2, we can easily write down the functions f associated to all the Stokes polytopes and substitute them in $\widetilde{\mathcal{M}}_8$

$$\widetilde{\mathcal{M}}_8 \;=\; \sum_{\sigma_1} \alpha_{\sigma_1 \cdot Q_1}\, m_8(\sigma_1 \cdot Q_1) \;+\; \sum_{\sigma_2} \alpha_{\sigma_2 \cdot Q_2}\, m_8(\sigma_2 \cdot Q_2), \tag{2.15}$$

where σ_1, σ_2 range over all the cyclic permutations which map Q_1 and Q_2 to distinct quadrangulations respectively.

Upon substituting the residues in eqn. (2.15), it can be easily checked that there is a unique choice of α s , namely $\alpha_{\sigma_1 \cdot Q_1} \;=\; \frac{2}{6}\; \forall\, \sigma_1$ and $\alpha_{\sigma_2 \cdot Q_2} \;=\; \frac{1}{6}\; \forall \sigma_2$, for which $\widetilde{\mathcal{M}}_8 \;=\; \mathcal{M}_8$ (see appendix (2.4)).

2.3 Computing $\mathcal{M}_n$ from the canonical forms

As we saw in the previous section, in both the $n = 6$ and $n = 8$ cases the scattering amplitude can be obtained from a weighted sum of rational functions (associated to canonical forms) over all the Stokes polytopes. A curious fact about the weights α was that the α s for which $\widetilde{\mathcal{M}}_n$ equals $\mathcal{M}_n$ were parametrized only by the primitive quadrangulations. In other words, in both the cases considered above,

$$\alpha_Q \;=\; \alpha_{\sigma \cdot Q}\; \forall\, \sigma. \tag{2.16}$$

We also formalize this observation into a constraint on the weights as

$$\alpha_Q \;=\; \alpha_{Q'} \text{ if } Q' \;=\; \sigma \cdot\; Q \text{ for a cyclic permutation } \sigma \tag{2.17}$$

That is if two quadrangulations are related by a cylic permutation of vertices of the polygon, then the corresponding α s should be equal.

The underlying motivation for the constraint in (2.17) is the following. Consider two quadrangulations Q and Q' which are cyclically related. From the perspective of kinematic Stokes polytope this means that the difference between $\mathcal{S}_{Q'}$ and $\mathcal{S}_Q$ is simply in how they are embedded in the kinematic space. Our constraints are based on our intuition (based on $n = 6,\ 8$ cases) that α_Q only depend on the intrinsic (combinatorial) property of $\mathcal{S}^Q$ and not on how it is embedded in $\mathcal{K}_n$. This dependence of α's on certain equivalence class of quadrangulations can be encapsulated by the notion of primitive quadrangulations.

We now propose a formula for evaluating the function $\widetilde{\mathcal{M}}_n$ for arbitrary n.

$$\widetilde{\mathcal{M}}_n \;=\; \sum_{Q \text{ primitive}} \sum_{\sigma} \alpha_Q\, m_n(\sigma \cdot Q). \tag{2.18}$$

The proposal (for computing the planar scattering amplitude $\mathcal{M}_n$) can thus be summarised as follows

For any n we first compute $m_n(\sigma \cdot Q)$ and substitute in eqn.(2.18). We conjecture that there is a unique choice of α's *which should be computed purely from combinatorics of Q s* such that for these α' s, $\widetilde{\mathcal{M}}_n \;=\; \mathcal{M}_n$. That is, there is a unique choice of α_Q $\forall$ primitive Q such that contribution of all the poles to $\widetilde{\mathcal{M}}_n$ with residue unity.

We should emphasize that to compute the scattering amplitude $\mathcal{M}_n$ from residues of the Stokes polytopes, we need an independent formula for α_Q which is consistent with eqn. (2.17), and such that all the kinematic channels give equal contribution of order unity. Computing α's at any given level N requires a complete list primitives and vertices of the Stokes polytopes of dimension N corresponding to them. Since, the number of Stokes polytopes proliferates very quickly with increasing N this seems to be computationally intractable (B). However, later in this thesis we shall derive a formula for the number of primitives p_N at any given level N and propose a iterative method to classify all the primitives. We do not have a formula for α's so far.

We shall shortly describe the computation of α's for $n = 10$ case and verify that our proposal leads to the correct scattering amplitude. But, first we shall describe a more convinient form of (2.18) that will help us in this regard.

$$\widetilde{\mathcal{M}}_n = \sum_Q \alpha_Q \, m_n(Q), \tag{2.19}$$

where one sums over all the Stokes polytopes (parametrized by Q), with the proviso that α_Q are same for any two quadrangulations which are related by cyclic permutation.

2.4 Some details : For $n = 8, 10$

Some details of the $n = 8$ case

We provide the details of the computation of the α factors for $n = 8$ case here. The functions m_8 corresponding to all $F_2 = 12$ quadrangulations are given below. There are 4 Stokes polytopes with 4 vertices and 8 Stokes polytopes with 5 vertices.

$$\begin{aligned}
m_8(Q_1) &= \left(\tfrac{1}{X_{14}X_{58}} + \tfrac{1}{X_{38}X_{47}} + \tfrac{1}{X_{14}X_{47}} + \tfrac{1}{X_{38}X_{58}}\right) \\
m_8(Q_2) &= \left(\tfrac{1}{X_{25}X_{16}} + \tfrac{1}{X_{25}X_{58}} + \tfrac{1}{X_{14}X_{58}} + \tfrac{1}{X_{14}X_{16}}\right) \\
m_8(Q_3) &= \left(\tfrac{1}{X_{36}X_{27}} + \tfrac{1}{X_{36}X_{16}} + \tfrac{1}{X_{25}X_{16}} + \tfrac{1}{X_{25}X_{27}}\right) \\
m_8(Q_4) &= \left(\tfrac{1}{X_{47}X_{38}} + \tfrac{1}{X_{47}X_{27}} + \tfrac{1}{X_{36}X_{27}} + \tfrac{1}{X_{36}X_{38}}\right)
\end{aligned}$$

$$
\begin{aligned}
m_8(Q'_1) &= \left(\frac{1}{X_{14}X_{16}} + \frac{1}{X_{14}X_{58}} + \frac{1}{X_{36}X_{16}} + \frac{1}{X_{36}X_{83}} + \frac{1}{X_{58}X_{38}} \right) \\
m_8(Q'_2) &= \left(\frac{1}{X_{25}X_{27}} + \frac{1}{X_{25}X_{16}} + \frac{1}{X_{14}X_{16}} + \frac{1}{X_{47}X_{14}} + \frac{1}{X_{47}X_{27}} \right) \\
m_8(Q'_3) &= \left(\frac{1}{X_{36}X_{38}} + \frac{1}{X_{36}X_{27}} + \frac{1}{X_{25}X_{27}} + \frac{1}{X_{58}X_{25}} + \frac{1}{X_{58}X_{38}} \right) \\
m_8(Q'_4) &= \left(\frac{1}{X_{47}X_{14}} + \frac{1}{X_{47}X_{38}} + \frac{1}{X_{36}X_{38}} + \frac{1}{X_{16}X_{36}} + \frac{1}{X_{16}X_{14}} \right) \\
m_8(Q'_5) &= \left(\frac{1}{X_{58}X_{25}} + \frac{1}{X_{14}X_{58}} + \frac{1}{X_{14}X_{47}} + \frac{1}{X_{27}X_{47}} + \frac{1}{X_{25}X_{27}} \right) \\
m_8(Q'_6) &= \left(\frac{1}{X_{16}X_{36}} + \frac{1}{X_{16}X_{25}} + \frac{1}{X_{25}X_{58}} + \frac{1}{X_{38}X_{58}} + \frac{1}{X_{36}X_{38}} \right) \\
m_8(Q'_7) &= \left(\frac{1}{X_{27}X_{47}} + \frac{1}{X_{27}X_{36}} + \frac{1}{X_{16}X_{36}} + \frac{1}{X_{14}X_{16}} + \frac{1}{X_{14}X_{47}} \right) \\
m_8(Q'_8) &= \left(\frac{1}{X_{38}X_{58}} + \frac{1}{X_{38}X_{47}} + \frac{1}{X_{27}X_{47}} + \frac{1}{X_{25}X_{27}} + \frac{1}{X_{25}X_{58}} \right)
\end{aligned}
$$

Every term in the above sum has either $X_{ii+3}X_{jj+3}$ or $X_{ii+3}X_{ii+5}$ in its denominator. We can see that each $X_{ii+3}X_{jj+3}$ term appears twice in the first list and twice in the second list. Similarly, each $X_{ii+3}X_{ii+5}$ term appears only once in the first list and four times in the second list. Thus, we have

$$
\begin{aligned}
2\alpha_{\sigma.Q} + 2\alpha_{\sigma'.Q'} &= 1 \\
\alpha_{\sigma.Q} + 4\alpha_{\sigma'.Q'} &= 1
\end{aligned}
$$

which gives $\alpha_{\sigma\cdot Q} = \frac{2}{6}\ \forall\, \sigma$ and $\alpha_{\sigma'\cdot Q'} = \frac{1}{6}\ \forall \sigma'$.

Scattering form and Stokes polytopes for the $n = 10$ case

We would like to provide the details of how to obtain the Scattering amplitude $\mathcal{M}_{10}$ by summing over the kinematic Stokes polytopes here. There are a total of $F_3 = 55$ quadrangulations the sum over all of them can equivalently be replaced with a sum over just the 7 primitive Stokes polytopes corresponding to the quartic graphs shown below (2.5) with appropriate coefficients. The reference quadrangulations for these primitves are $Q_1 = (14, 510, 69)$, $Q_2 = (14, 16, 18)$, $Q_3 = (14, 16, 69)$, $Q_4 = (14, 49, 69)$, $Q_5 = (14, 47, 710)$, $Q_6 = (14, 510, 710)$, $Q_7 = (14, 16, 710)$.

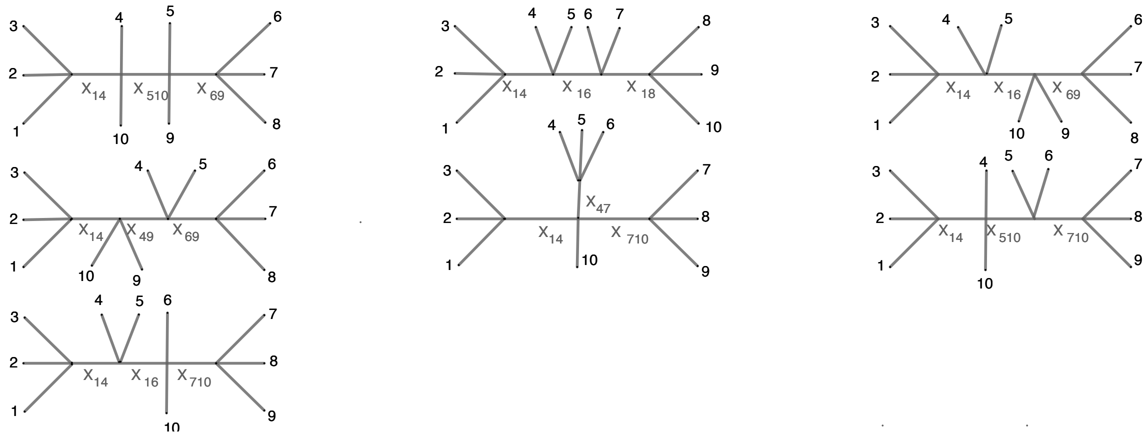


Figure 2.5: The primitive quartic graphs (in clockwise order) with corresponding Stokes polytopes being Cube, Associahedron(2-4), Lucas and Mixed Classes(6 and 7)

We first provide the details of these Stokes polytopes and demonstrate how to get the planar scattering form, which when pulled back gives the scattering amplitude.

We always impose the associahedron condtions

$$s_{ij} = -c_{ij} \text{ for } 1 \leq i < j \leq 2n+1, \quad |i-j| \geq 2 \tag{2.20}$$

and together with this we need to impose 4 additional conditions which carve out the Stokes polytope inside the associahedron. As explained in section (2.2) we consider the reference quadrangulation Q corresponding to each Stokes polytope and find a set of 4 other diagonals T that complete the triangulation of Q. We then set the X_{ij}'s corresponding to this set to positive constants d_{ij}'s, since these X_{ij}'s can never correspond to propagators of any quartic graph. This particular choice of additional contraints provides a particular embedding of the Stokes polytope into the associahedron. We illustrate this for all the four cases below.

1. **Cube type :** The corresponding Polytope is a cube with 8 vertices as shown in the figure (2.6).The set of Q_1 compatible quadrangulations are given by:

$$S_1 = \{(14, 510, 69, +),\ (310, 510, 69, -),\ (14, 49, 69, -),\ (14, 510, 58, -),$$
$$(14, 49, 58, +), (310, 510, 58, +), (310, 49, 69, +),\ (310, 49, 58, -)\}.$$

One set of diagonals which triangulate Q_1 are $T_1 = \{13, 410, 59, 68\}$ which we set

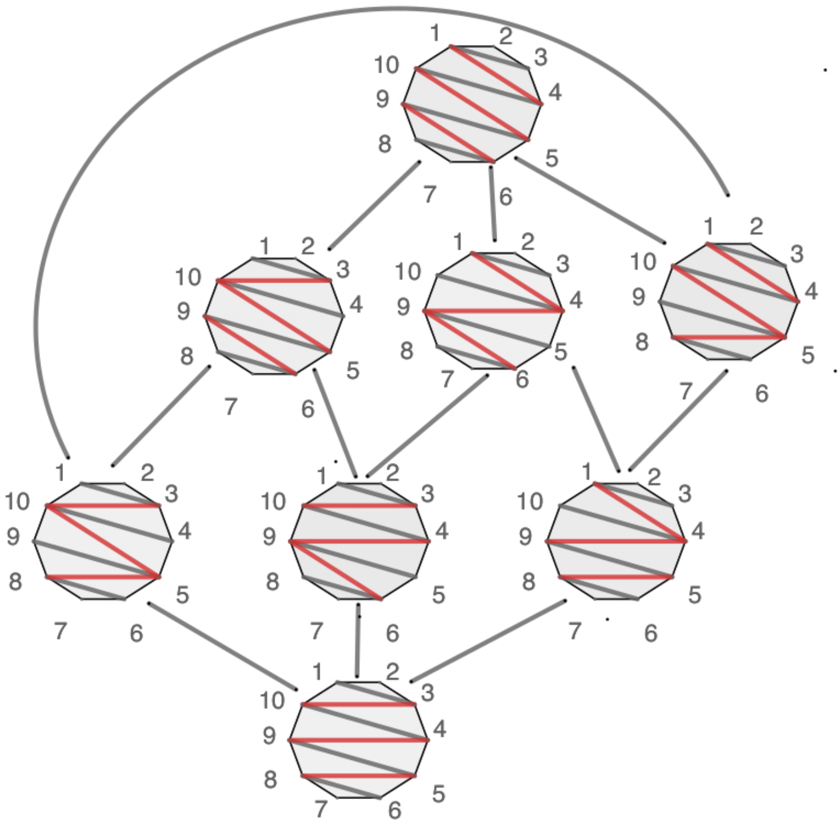


Figure 2.6: The Polytope is a cube as can been seen above each quadrangulation is a vertex and the lines joining them represent edges, each closed loop represents a face. The set of common diagonals which complete the triangulation are shown in grey.

to positive constants to get an embedding

$$X_{13} = d_{13},\quad X_{410} = d_{410},\quad X_{59} = d_{59},\quad X_{68} = d_{68}. \tag{2.21}$$

The planar scattering form for this case is given by:

$$\Omega_{10}^{Q_1} = (d\ln X_{14}\wedge d\ln X_{510}\wedge d\ln X_{69} - d\ln X_{310}\wedge d\ln X_{510}\wedge d\ln X_{69} - d\ln X_{14}\wedge d\ln X_{49}\wedge d\ln X_{69}$$
$$-d\ln X_{310}\wedge d\ln X_{510}\wedge d\ln X_{58} + d\ln X_{14}\wedge d\ln X_{49}\wedge d\ln X_{58} + d\ln X_{310}\wedge d\ln X_{510}\wedge d\ln X_{58}$$
$$+d\ln X_{310}\wedge d\ln X_{49}\wedge d\ln X_{69} - d\ln X_{310}\wedge d\ln X_{49}\wedge d\ln X_{58}).$$

When pulled back onto the space of constraints ((2.20), (2.21)) gives the canonical form for the cube :

$$\omega_{10}^{Q_1} = \left(\frac{1}{X_{14}X_{510}X_{69}} + \frac{1}{X_{310}X_{510}X_{69}} + \frac{1}{X_{14}X_{49}X_{69}} + \frac{1}{X_{14}X_{510}X_{58}} + \frac{1}{X_{14}X_{49}X_{58}} + \frac{1}{X_{310}X_{510}X_{58}} + \frac{1}{X_{310}X_{49}X_{69}} + \frac{1}{X_{310}X_{49}X_{58}}\right) dX_{14}\wedge dX_{510}\wedge dX_{69}.$$

2. **Snake type :** The corresponding polytope is an associahedron $\mathcal{A}_6$ with 14 vertices (see figure (2.7)). As Explained above there are three quadrangulations that correspond to this case namely $Q_2 = (14, 16, 18)$, $Q_3 = (14, 16, 69)$, $Q_4 = (14, 49, 69)$. We show how to get the planar scattering form and canonical form for Q_2 below:

The set of Q_2 compatible quadrangulations are given by:

$$S_2 = \{(14,16,18,+),(36,16,18,-),(14,58,18,-),(14,16,710,-),(36,16,710,+),(36,38,18,+)$$
$$,(14,58,510,+),(38,58,18,+),(14,510,710,+),(36,310,710,-),(36,38,310,-),(310,58,510,-)$$
$$,(38,58,310,-),(310,510,710,-)\}.$$

One set of diagonals which triangulates the reference quadrangulation Q_2 is $T_2 = \{13, 46, 68, 810\}$ which we set to positive constants to get an embedding:

$$X_{13} = d_{13}\ ,\ X_{46} = d_{46}\ ,\ X_{68} = d_{68}\ ,\ X_{810} = d_{810}. \tag{2.22}$$

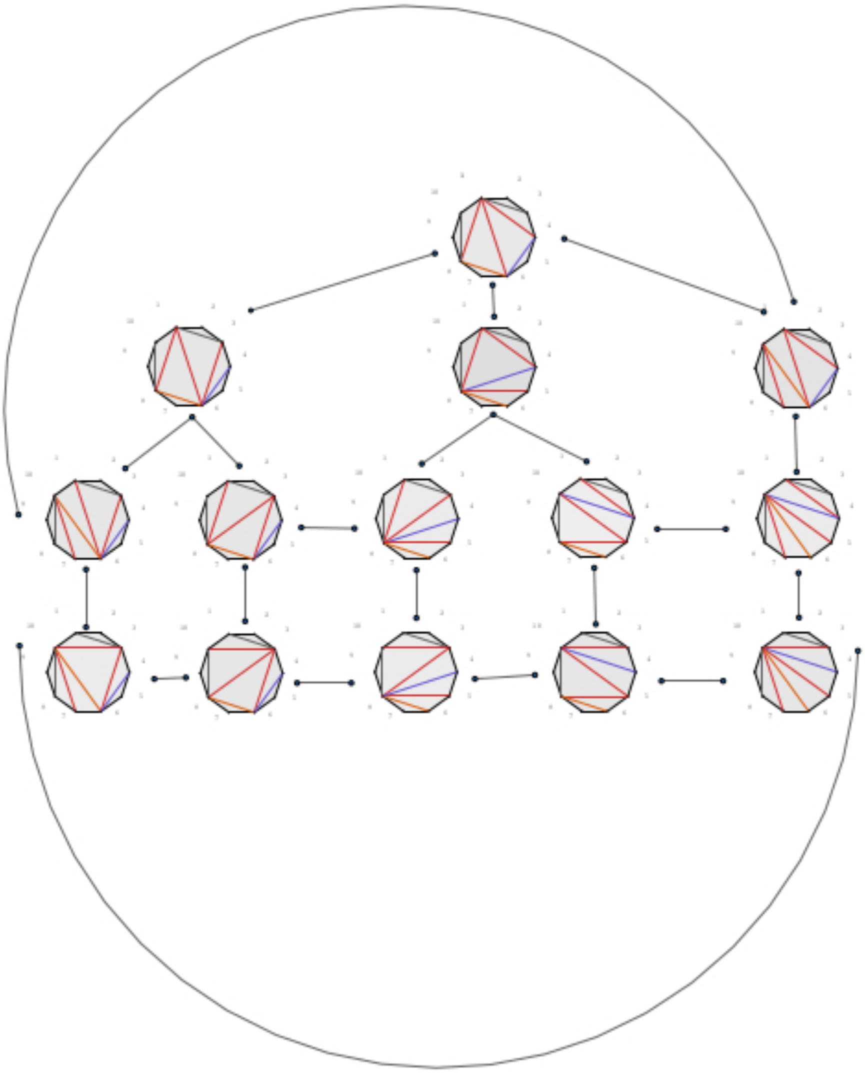


Figure 2.7: In the Snake case the corresponding Stokes polytope is an associahedron $\mathcal{A}_6$.

The planar scattering form for this case is given by,

$$\begin{aligned}\Omega_{10}^{Q_2} &= d\ln X_{14}\wedge d\ln X_{16}\wedge d\ln X_{18} - d\ln X_{36}\wedge d\ln X_{16}\wedge d\ln X_{18}\\ &-d\ln X_{14}\wedge d\ln X_{58}\wedge d\ln X_{18} - d\ln X_{14}\wedge d\ln X_{16}\wedge d\ln X_{710} + d\ln X_{36}\wedge d\ln X_{16}\wedge d\ln X_{710}\\ &+d\ln X_{36}\wedge d\ln X_{38}\wedge d\ln X_{18} + d\ln X_{14}\wedge d\ln X_{58}\wedge d\ln X_{510} + d\ln X_{38}\wedge d\ln X_{58}\wedge d\ln X_{18}\\ &+d\ln X_{14}\wedge d\ln X_{510}\wedge d\ln X_{710} - d\ln X_{36}\wedge d\ln X_{310}\wedge d\ln X_{710} - d\ln X_{36}\wedge d\ln X_{38}\wedge d\ln X_{310}\\ &-d\ln X_{310}\wedge d\ln X_{58}\wedge d\ln X_{510} - d\ln X_{38}\wedge d\ln X_{58}\wedge d\ln X_{310} - d\ln X_{310}\wedge d\ln X_{510}\wedge d\ln X_{710}.\end{aligned}$$

When pulled back onto the space of constraints eqn. (2.20) and eqn. (2.22) we get the canonical form:

$$\omega_{10}^{Q_2} = \Bigg(\frac{1}{X_{14}X_{16}X_{18}} + \frac{1}{X_{36}X_{16}X_{18}} + \frac{1}{X_{14}X_{58}X_{18}} + \frac{1}{X_{14}X_{16}X_{710}} + \frac{1}{X_{36}X_{16}X_{710}}$$
$$+\frac{1}{X_{36}X_{38}X_{18}} + \frac{1}{X_{14}X_{58}X_{510}} + \frac{1}{X_{38}X_{58}X_{18}} + \frac{1}{X_{14}X_{510}X_{710}} + \frac{1}{X_{36}X_{310}X_{710}}$$
$$+\frac{1}{X_{36}X_{38}X_{310}} + \frac{1}{X_{310}X_{58}X_{510}} + \frac{1}{X_{38}X_{58}X_{310}} + \frac{1}{X_{310}X_{510}X_{710}}\Bigg)dX_{14} \wedge dX_{16} \wedge dX_{18}.$$

Similarly,

$$\omega_{10}^{Q_3} = \Bigg(\frac{1}{X_{14}X_{49}X_{69}} + \frac{1}{X_{310}X_{49}X_{69}} + \frac{1}{X_{14}X_{16}X_{69}} + \frac{1}{X_{14}X_{49}X_{58}} + \frac{1}{X_{36}X_{310}X_{69}}$$
$$+\frac{1}{X_{310}X_{49}X_{58}} + \frac{1}{X_{16}X_{36}X_{69}} + \frac{1}{X_{14}X_{16}X_{18}} + \frac{1}{X_{14}X_{18}X_{58}} + \frac{1}{X_{36}X_{38}X_{310}}$$
$$+\frac{1}{X_{38}X_{310}X_{58}} + \frac{1}{X_{16}X_{18}X_{36}} + \frac{1}{X_{18}X_{38}X_{58}} + \frac{1}{X_{18}X_{36}X_{38}}\Bigg)dX_{14} \wedge dX_{16} \wedge dX_{18}.$$

$$\omega_{10}^{Q_4} = \Bigg(\frac{1}{X_{14}X_{16}X_{69}} + \frac{1}{X_{16}X_{36}X_{69}} + \frac{1}{X_{14}X_{510}X_{69}} + \frac{1}{X_{14}X_{16}X_{18}} + \frac{1}{X_{16}X_{18}X_{36}}$$
$$+\frac{1}{X_{36}X_{310}X_{69}} + \frac{1}{X_{310}X_{510}X_{69}} + \frac{1}{X_{14}X_{58}X_{510}} + \frac{1}{X_{14}X_{18}X_{58}} + \frac{1}{X_{18}X_{36}X_{38}}$$
$$+\frac{1}{X_{36}X_{38}X_{310}} + \frac{1}{X_{310}X_{58}X_{510}} + \frac{1}{X_{18}X_{38}X_{58}} + \frac{1}{X_{38}X_{310}X_{58}}\Bigg)dX_{14} \wedge dX_{16} \wedge dX_{18}.$$

3. **Lucas type :** In this the corresponding Stokes Polytope has Lucas number $L_3 = 12$ vertices (see figure (2.8)). The set of Q_5 compatible quadrangulations are given by:

$$S_5 = \{(14, 47, 710, +), (310, 47, 710, -), (14, 16, 710, -), (14, 47, 49, -), (310, 49, 47, +), (36, 310, 710, +)$$
$$, (36, 16, 710, +), (14, 16, 69, +), (14, 49, 69, +), (310, 49, 69, -), (310, 36, 69, -), (36, 16, 69, -)\}.$$

One set of diagonals which triangulates the reference quadrangulation Q_5 is $T_3 = \{13, 46, 79, 410\}$ which we set to positive constants to get an embedding:

$$X_{13} = d_{13}\ ,\ X_{46} = d_{46}\ ,\ X_{79} = d_{79}\ ,\ X_{410} = d_{410}. \tag{2.23}$$

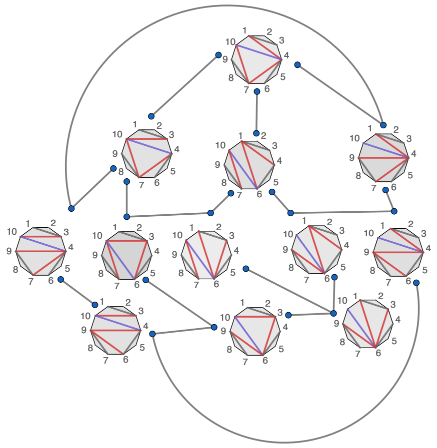


Figure 2.8: In the Lucas case the corresponding polytope has 12 vertices, 18 edges and 8 faces.

The planar scattering form for this case is given by,

$$\begin{aligned}\Omega_{10}^{Q_5} = d\ln X_{14}\wedge d\ln X_{47}\wedge d\ln X_{710} - d\ln X_{310}\wedge d\ln X_{47}\wedge d\ln X_{710}\\ -d\ln X_{14}\wedge d\ln X_{16}\wedge d\ln X_{710} - d\ln X_{14}\wedge d\ln X_{47}\wedge d\ln X_{49} + d\ln X_{310}\wedge d\ln X_{49}\wedge d\ln X_{47}\\ +d\ln X_{36}\wedge d\ln X_{310}\wedge d\ln X_{710} + d\ln X_{36}\wedge d\ln X_{16}\wedge d\ln X_{710} + d\ln X_{14}\wedge d\ln X_{16}\wedge d\ln X_{69}\\ +d\ln X_{14}\wedge d\ln X_{49}\wedge d\ln X_{69} - d\ln X_{310}\wedge d\ln X_{49}\wedge d\ln X_{69} - d\ln X_{310}\wedge d\ln X_{36}\wedge d\ln X_{69}\\ -d\ln X_{36}\wedge d\ln X_{16}\wedge d\ln X_{69}.\end{aligned}$$

When pulled back onto the space of constraints eqn. (2.20) and eqn. (2.23) we get the canonical form:

$$\begin{aligned}\omega_{10}^{Q_5} = \Bigg(&\frac{1}{X_{14}X_{47}X_{710}} + \frac{1}{X_{310}X_{47}X_{710}} + \frac{1}{X_{14}X_{16}X_{710}} + \frac{1}{X_{14}X_{47}X_{49}} + \frac{1}{X_{310}X_{49}X_{47}}\\ &+\frac{1}{X_{36}X_{310}X_{710}} + \frac{1}{X_{36}X_{16}X_{710}} + \frac{1}{X_{14}X_{16}X_{69}} + \frac{1}{X_{14}X_{49}X_{69}} + \frac{1}{X_{310}X_{49}X_{69}}\\ &+\frac{1}{X_{310}X_{36}X_{69}} + \frac{1}{X_{36}X_{16}X_{69}}\Bigg)dX_{14}\wedge dX_{47}\wedge dX_{710}.\end{aligned}$$

4. **Mixed type :** In this case the stokes polytope is just product of lower dimensional stokes polytopes $S_1 \times S_2$ hence has 10 vertices (see figure (2.9)). As Explained above there are two quadrangulations that correspond to this case namely $Q_6 = (14, 510, 710)$, $Q_7 = (14, 16, 710)$. We show how to get the planar scattering form and canonical form for Q_6 below:

   The set of Q_6 compatible quadrangulations are given by:

$$S_6 = \{(14, 510, 710, +), (310, 510, 710, -), (14, 47, 710, -), (14, 510, 69, -), (310, 47, 710, +),$$
$$(310, 510, 69, +), (14, 47, 49, +), (14, 49, 69, +), (310, 49, 69, -), (310, 47, 49, -)\}$$

   . One set of diagonals which triangulates the reference quadrangulation Q_6 is $T_6 = \{13, 410, 79, 57\}$ which we set to positive constants to get an embedding:

$$X_{13} = d_{13}\ ,\ X_{410} = d_{410}\ ,\ X_{79} = d_{79}\ ,\ X_{57} = d_{57}. \tag{2.24}$$

   The planar scattering form for this case is,

$$\begin{aligned}\Omega_{10}^{Q_6} = (&d\ln X_{14} \wedge d\ln X_{510} \wedge d\ln X_{710} - d\ln X_{310} \wedge d\ln X_{510} \wedge d\ln X_{710} \\ &- d\ln X_{14} \wedge d\ln X_{47} \wedge d\ln X_{710} - d\ln X_{14} \wedge d\ln X_{510} \wedge d\ln X_{69} + d\ln X_{310} \wedge d\ln X_{47} \wedge d\ln X_{710} \\ &+ d\ln X_{310} \wedge d\ln X_{510} \wedge d\ln X_{69} + d\ln X_{14} \wedge d\ln X_{47} \wedge d\ln X_{49} + d\ln X_{14} \wedge d\ln X_{49} \wedge d\ln X_{69} \\ &- d\ln X_{310} \wedge d\ln X_{49} \wedge d\ln X_{69} - d\ln X_{310} \wedge d\ln X_{47} \wedge d\ln X_{49}).\end{aligned}$$

   When pulled back onto the space of constraints (2.20,2.24) we get the canonical form:

$$\begin{aligned}\omega_{10}^{Q_6} = \Bigg(&\frac{1}{X_{14}X_{510}X_{710}} + \frac{1}{X_{310}X_{510}X_{710}} + \frac{1}{X_{14}X_{47}X_{710}} + \frac{1}{X_{14}X_{510}X_{69}} + \frac{1}{X_{310}X_{47}X_{710}} \\ &+ \frac{1}{X_{310}X_{510}X_{69}} + \frac{1}{X_{14}X_{47}X_{49}} + \frac{1}{X_{14}X_{49}X_{69}} + \frac{1}{X_{310}X_{49}X_{69}} + \frac{1}{X_{310}X_{47}X_{49}}\Bigg) \\ &dX_{14} \wedge dX_{510} \wedge dX_{710}\end{aligned}$$

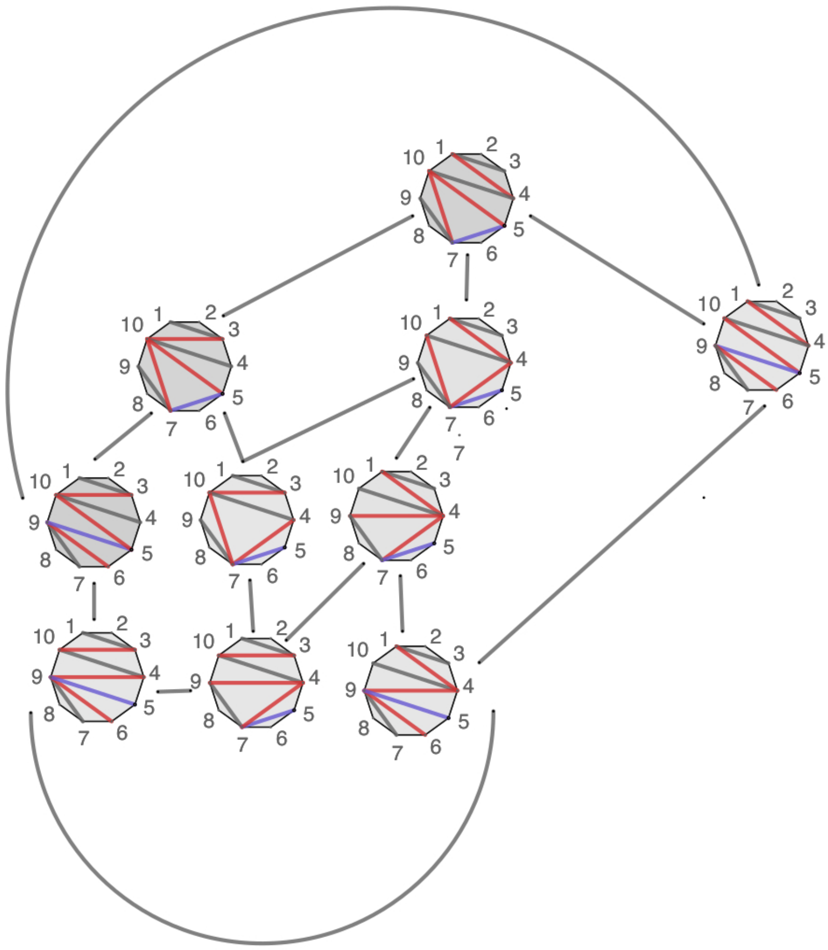


Figure 2.9: In the mixed case the corresponding polytope has 10 vertices, 15 edges and 7 faces.

Similarly,

$$\omega_{10}^{Q_7} = \Bigg(\frac{1}{X_{14}X_{16}X_{710}} + \frac{1}{X_{16}X_{36}X_{710}} + \frac{1}{X_{14}X_{510}X_{710}} + \frac{1}{X_{14}X_{16}X_{69}} + \frac{1}{X_{36}X_{310}X_{710}}$$

$$+\frac{1}{X_{16}X_{36}X_{69}} + \frac{1}{X_{310}X_{510}X_{710}} + \frac{1}{X_{14}X_{510}X_{69}} + \frac{1}{X_{36}X_{310}X_{69}} + \frac{1}{X_{310}X_{510}X_{69}}\Bigg)$$

$$dX_{14} \wedge dX_{16} \wedge dX_{710}.$$

Upon substituting the corresponding m_{10} in eqn.(8), it can be checked that for $\alpha_{Q_1} = \frac{5}{24}$, $\alpha_{Q_2} = \alpha_{Q_3} = \alpha_{Q_4} = \frac{1}{24}$, $\alpha_{Q_5} = \frac{2}{24}$ and $\alpha_{Q_6} = \alpha_{Q_7} = \frac{3}{24}$ the sum over all the residues give $\mathcal{M}_{10}$.

2.5 Factorisation

One of the remarkable consequences of relating tree level scattering amplitudes to positive geometries like associahedron is the fact that geometric factorisation of the associahedron implied physical factorisation of scattering amplitude (1.10). In this section we will try to argue that this is indeed the case even for planar amplitudes in massless ϕ^4 theory. Namely that, there is a combinatorial factorisation of Stokes polytope and that exactly as in the case of associahedron, it implies amplitude factorisation.

Our first assertion is the following. Given any diagonal (ij), consider *all* Q which contains ij and the consider all the corresponding kinematic Stokes polytopes $\mathcal{S}_n^Q$. We contend that for each of these Stokes polytopes, the corresponding facet $X_{ij} = 0$ is a product of lower dimensional Stokes polytopes

$$\mathcal{S}_n^Q\Big|_{X_{ij} = 0} \equiv \mathcal{S}_m^{Q_1} \times \mathcal{S}_{n+2-m}^{Q_2}, \tag{2.25}$$

where Q_1 and Q_2 are such that $Q_1 \cup Q_2 \cup (ij) = Q$. Q_1 is the quadrangulation of the polygon $\{i,\ i+1,\ldots,\ j\}$ and Q_2 is the quadrangulation of $\{j, j+1,\ldots,n,\ldots,i\}$. Now we know that, on $\mathcal{S}_n^Q$ any planar scattering variable X_{kl} is a linear combination of X_{ij} and remaining X's which constitute Q. Hence in order to prove this assertion we need to show that any X_{kl} with $i \leq k < l \leq j$ can be written as a linear combination of X_{ij} and elements of Q_1 and similarly any variable in the complimentary set can be written in terms of X_{ij} and elements of Q_2.

However this is immediate since we know from the factorisation property of associahedron proven in (1.10) that any $X_{kl} = X_{ij} + \sum_{i<m<n<j} X_{mn}$. some of these $X_{mn} \in Q_1$ and the others are constrained via $X_{mn} = d_{mn}$. This proves our assertion. Thus $X_{ij} = 0$ facet factorises into two lower dimensional Stokes polytopes.

Our second assertion is that the geometric factorisation implies amplitude factorisation of quartic theory. This assertion is based on the following two facts

(1) As Stokes polytope is a positive geometry , we know that it's canonical form satisfies the following properties satisfed by canonical form on any positive geometry $\mathcal{A}$

$$\text{Res}_H\, \omega_{\mathcal{A}} \;=\; \omega_{\mathcal{B}}, \tag{2.26}$$

where we think of $\omega_{\mathcal{A}}$ as defined on the embedding space and H is any subspace in the embedding space which contains the face $\mathcal{B}$. It is also known that if $\mathcal{B} \;=\; \mathcal{B}_1 \times \mathcal{B}_2$ then

$$\omega(\mathcal{B}) \;=\; \omega(\mathcal{B}_1) \,\wedge\, \omega(\mathcal{B}_2). \tag{2.27}$$

Thus we immediately see that

$$\text{Res}_{X_{ij}=0}\;\; \omega(\mathcal{S}_n^Q) \;=\; \omega_m^{Q_1} \,\wedge\, \omega_{n+2-m}^{Q_2} \;\forall\, Q, \tag{2.28}$$

where $m \;=\; j - i + 1$.

We thus see that residue over each Stokes polytope which contains a boundary $X_{ij} \;\rightarrow\; 0$ factorises into residues over lower dimensional Stokes polytopes. This factorisation property naturally implies factorisation of amplitudes as follows. Consider the n-gon with a diagonal (ij) (with i, j such that this diagonal can be part of a quadrangulation). This diagonal subdivides the n-gon into a two polygons with vertices $\{i, \ldots, j\}$ and $\{j, \ldots, n, 1, \ldots i\}$ respectively. By considering all the kinematic Stokes polytopes associated to these polygons, we can evaluate $\widetilde{M}_{|j-i+1|}$, $\widetilde{M}_{n+2-(|j-i+1|)}$ which correspond to left and right sub-amplitudes respectively. This immediately implies that

$$\widetilde{\mathcal{M}}_n\bigg|_{X_{ij}=0} \;=\; \widetilde{\mathcal{M}}_L\,\frac{1}{X_{ij}}\,\widetilde{\mathcal{M}}_R. \tag{2.29}$$

This proves physical factorisation. We also note that, eqns. (5.1) and (3.12) imply following constraints on α s

$$\sum_{Q \text{ containing}(ij)} \alpha_Q = \sum_{Q_L, Q_R} \alpha_{Q_L} \alpha_{Q_R}, \tag{2.30}$$

where Q_L and Q_R range over all the quadrangulations of the two polygons to the left and right of diagonal (ij) respectively.

It can be verified that in the case of $n = 6, 8,$ and 10 particles α_Q's do indeed satisfy these constraints.

2.6 Relationship with planar scattering form for cubic coupling

Planar tree-level diagrams of massless ϕ^4 theory can be obtained from diagrams of a theory with cubic interactions $\psi\phi^2$ which contains two scalar fields ϕ and ψ, where ϕ is massless and ψ is massive. Consider an (ordered) n-point amplitude in this theory $\mathcal{M}^{\phi^2\psi}(p_1, \ldots, p_n)$ in which all the external particles are ϕ-particles. The super-script on the amplitudes indicates the coupling we are considering. It is easy to see that in all the Feynman graphs associated to such an amplitude, the ϕ-propagators precisely correspond to the ϕ-propagators in the corresponding diagrams in ϕ^4 theory. Remaining propagators are propagators associated to ψ field and hence upon integrating out this massive field, one recovers planar amplitudes in massless ϕ^4 theory.

Whence one may wonder if the canonical form we obtained on Stokes polytopes, $\mathcal{S}_n^Q$ could be obtained from the planar scattering form associated to the theory with $\psi\phi^2$ interaction. [7] We show below that this is not the case.

[7] We are indebted to Nemani Suryanarayana and Suresh Govindarajan for raising this question. We also note that this issue was already raised in [43].

We can postulate a planar scattering form in the kinematic space associated to $\psi\phi^2$ coupling, in which all the log singularities associated to ψ fields are absent[8]. On restricting this form to $\mathcal{S}_n^Q$, we can observe that the corresponding form is not the canonical form on $\mathcal{S}_n^Q$.

Let us illustrate this idea in the simplest of examples, namely $n = 6$ case. We thus consider planar scattering form on $\mathcal{K}_6$ which is obtained by summing over 12 planar graphs[9].

This form is given by

$$\begin{aligned}
&\Omega_{n=6}^{\psi\phi^2} = \\
&dX_{24} \wedge d\ln X_{14} \wedge [dX_{15} - dX_{46}] + dX_{26} \wedge d\ln X_{36} \wedge [dX_{46} - dX_{35}] \\
&- dX_{13} \wedge d\ln X_{36} \wedge [dX_{46} - dX_{35}] - dX_{26} \wedge d\ln X_{25} \wedge [dX_{24} - dX_{35}] \\
&+ dX_{15} \wedge d\ln X_{25} \wedge [dX_{24} - dX_{35}] - dX_{13} \wedge d\ln X_{14} \wedge [dX_{15} - dX_{46}],
\end{aligned} \tag{2.31}$$

where singularities associated to ψ propagators are absent.

On restricting this form to $\mathcal{S}_6^{Q=(14)}$ using eqn. (2.5), we get

$$\tilde{\Omega}_{N=6}\Big|_{\mathcal{S}_6^{Q=(14)}} = 2\left[\frac{1}{X_{14}} + \frac{1}{X_{25}} + \frac{1}{X_{36}}\right] dX_{13} \wedge dX_{14} \wedge dX_{15}. \tag{2.32}$$

We thus see that projection of $\Omega_{n=6}^{\psi\phi^2}$ onto $\mathcal{S}_6^{Q=(14)}$ is not the same as its canonical form. This is because the form in eqn.(2.32) has an additional singularity at $X_{25} \rightarrow 0$. Thus from the perspective of positive geometry there does not seem to be a direct relationship between quartic interactions and cubic interactions with two scalar fields. Of course in hindsight, this is not too surprising as integrating out the ψ field reproduces all (planar)

[8]This is how we implement "integrating out the ψ-field" in language of scattering forms.

[9]In the case of ϕ^3 coupling, one has to sum over 14 graphs, however two of these do not arise if we instead consider $\psi\phi^2$ coupling. Whence the corresponding form on $\mathcal{K}_6$ is not projective! In the context of triangulation, what this means is that we consider only those triangulations which has *at least* one partial triangulation which can be part of a quadrangulation.

diagrams in ϕ^4 theory and this is precisely reflected in the presence of $\frac{1}{X_{25}}$ in eqn. (2.32) above. However as the $X_{25} \rightarrow 0$ singularity is not on one of the vertices of the Stokes polytope, this form is not the canonical form on the Stokes polytope.

In summary we have shown that given any quadrangulation Q of an n-sided polygon, one can define a unique planar scattering form on the kinematic space $\mathcal{K}_n$. We then showed how this form naturally descends to the canonical form on the Stokes polytope $\mathcal{S}_n^Q$ such that the corresponding canonical rational function m_n gives a partial contribution to planar scattering amplitude in ϕ^4 theory. Thus an individual Stokes polytope is not quite the same as an amplituhedron which as a single geometric object contained information about complete scattering amplitude. However the families of all Stokes polytope does contain complete information about $\mathcal{M}_n$. We proposed a formula for obtaining $\mathcal{M}_n$ as a weighted sum over $m_n(Q)$ of all the primitve Stokes polytopes and have shown it to be valid for 6, 8 and 10 particle amplitudes. We finally showed that the Stokes polytope factorises geometrically and just as in the associahedron case this immedeately implies factorisation of the amplitude.

We would now like to address the general ϕ^p for $p > 4$ case in the next chapter for which all these features continue to persist. Infact we shall see that the class of polytopes which we shall consider reduce to associahedra and Stokes polytopes for $p = 3, 4$ respectively and thus allow us to treat ϕ^p $\forall p \geq 3$ in a unified manner.

Chapter 3

Positive geometry of ϕ^p interactions

We shall begin by defining an object called the *accordiohedron* [49, 50] associated with general dissections of polygons. We shall then show how this reduces to the associahedron and Stokes polytopes for cubic and quartic interactions respectively. We shall then argue that the *accordiohedron* is thus the natural candidate for the positive geometry of all ϕ^p interactions.

3.1 Accordion lattices and Accordiohedra

Let A be a convex polygon. Let us consider the division of A into p-gons which we call p-angulation of A. We can represent A as a set of points on the unit circle oriented clockwise where the arcs represent edges of A and chords represent diagonals of A. The simplest example is the case where we divide $(2p-2)$-gon A into two p-gons (see figure (3.1)). We define a notion of *Q-compatible diagonal* as [1]:

$$(i, j) \rightarrow (\mathrm{Mod}(i + p - 2, 2p - 2), \mathrm{Mod}(j + p - 2, 2p - 2)). \tag{3.1}$$

[1] In [49, 50] there is different definition of compatability, but these two definitions can be shown to be equivalent to each other and we shall use the definition (3.1) as its most suited for our purposes. We thank Alok Laddha for explaining this fact to us.

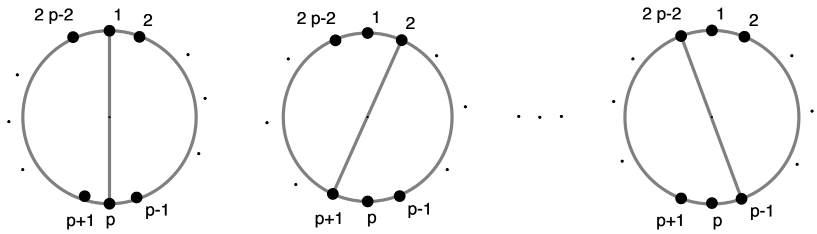


Figure 3.1: The (p-1) different p-angulations of A

We can use this rule to define accordion lattices $\mathcal{AL}^P_{p,n}$ of dimension n associated with a reference p-angulation P [2] as follows:

We can start with any p-angulation P of a convex polygon with n diagonals,

- *In the first step for each of the n diagonals, we go to the unique* $(2p-2)$*-gon which contains it and replace it with its Q-compatible diagonal.*

- *In the second step for each of the n p-angulations at the end of step one we choose one of the original* $(n-1)$ *diagonals and replace it with its Q-compatible diagonal as in step one.*

- *We repeat this till none of the original n diagonals remain in step n.*

This generates a flip graph which is the 1-skeleton of a convex polytope called the *Accordiohedron* [49, 50], which we shall also call $\mathcal{AC}^P_{p,n}$.

The correspondence between k-faces of the accordiohedron and p-angulations is

Vertices	$\leftrightarrow$	**Q-compatible p-angulations**
Edges	$\leftrightarrow$	**Flips between them**
k-Faces	$\leftrightarrow$	**k-partial p-angulations**.

[2] We consider only the case where we divide the polygon into p-gons in this thesis, but accordion lattices are defined for arbitrary dissections.

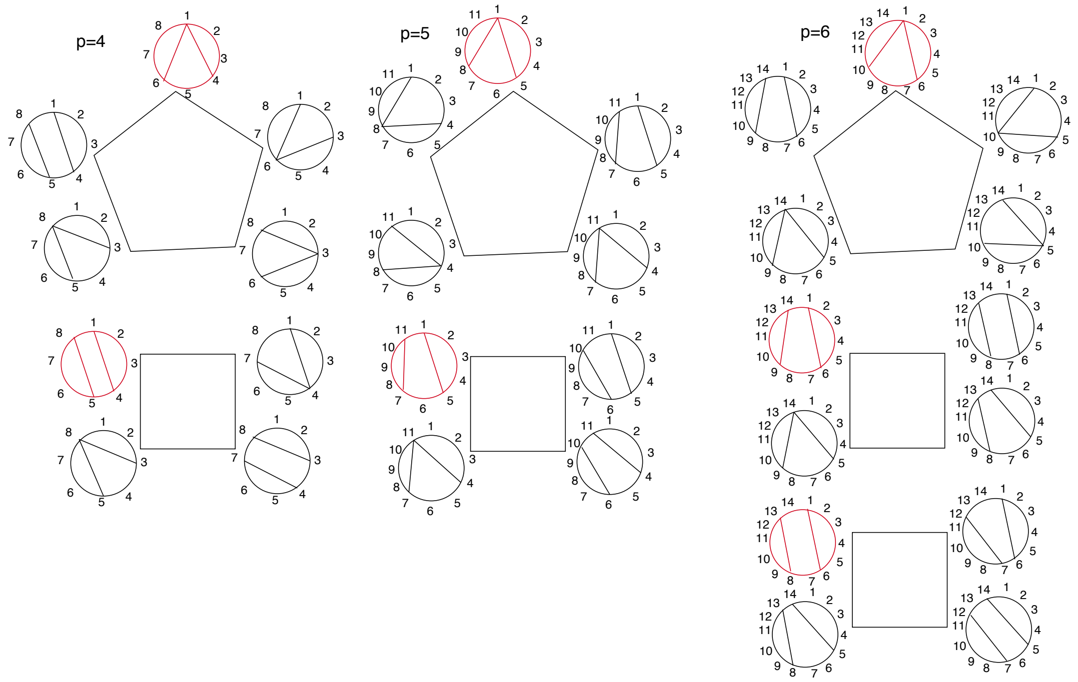


Figure 3.2: accordiohedra for the n=2 case. The red circles indicate the reference p-angulations.

In the case of cubic interactions ($p = 3$), (3.1) reduces to $(i, j) \rightarrow (\mathrm{Mod}(i + 1, 4), \mathrm{Mod}(j + 1, 4))$ which is the *mutation* rule and the resulting accordiohedron $\mathcal{AC}^{P}_{3,n}$ is the associahedron [43].

In the case of quartic interactions ($p = 4$), (3.1) reduces to $(i, j) \rightarrow (\mathrm{Mod}(i+2, 6), \mathrm{Mod}(j+2, 6))$ which was the Q−compatibility rule defined in [97] and the accordiohedron $\mathcal{AC}^{P}_{4,n}$ was shown to be the Stokes polytope [47].

Thus the Accordiohedra $\mathcal{AC}^{P}_{p,n}$ are a general class of polytopes which contain both associahedron and Stokes polytopes as special cases when the p-angulations corresponds to triangulations and quadrangulations respectively. The accordiohedra $\mathcal{AC}^{P}_{p,n}$ with $p > 4$ also retains many of the features of the Stokes polytopes we had discussed earlier in **??** including the fact that the accordiohedron $\mathcal{AC}^{P}_{p,n}$ of a given dimension n is not unique and depends on the reference p-angulation P. This is due to the fact that (3.1) is not an equivalence relation as $(1, p) \rightarrow (p - 1, 2p - 2)$, but $(p - 1, 2p - 2) \rightarrow (p - 2, 2p - 3) \neq (1, p)$ except when $p = 3$.

The case $p = 3$ is special in this sense as for $\mathcal{AC}^P_{3,n}$ is independent of P, as every diagonal is Q-compatible with every other diagonal and thus we could start with any triangulation P and we would generate all possible triangulations.

The accordiohedron obtained by starting with a particular p-angulation is also completely determined by the relative configuration of diagonals. [3]

The $n = 1$ accordiohedron $\mathcal{AC}^{(i,p+i)}_{p,1}$ are lines with vertices $(i, p + i)$ and $(\text{Mod}(i + p - 2, 2p - 2), \text{Mod}(i + 2p - 2, 2p - 2))$ for $i = 1, ..., p - 1$.

The $n = 2$ case the accordiohedron can be either pentagons or squares depending on whether the two diagonals meet or don't meet respectively (see fig(3.2)) just as in the case of Stokes polytopes. In other words $\mathcal{AC}^{(P)}_{p,2} \cong \mathcal{AC}^{(Q)}_{4,2}$ for all p provided both P and Q have the same configuration of diagonals, we shall prove this in a later section (4.1) by establishing the precise maps between vertices of the Stokes polytope and that of the accordiohedron.

The $n = 3$ case the accordiohedra continue to be one of the four $n = 3$ Stokes polytopes with different multiplicities i.e. $\mathcal{AC}^{(P)}_{p,3} \cong \mathcal{AC}^{(Q)}_{4,3}$ for all p provided P and Q have the same configuration of diagonals. We elaborate on this in section (4.2). At higher n new polytopes which are not one of the Stokes polytopes will be eventually generated.

3.2 Positive geometry for ϕ^p interactions

We would like to show that the accordiohedron $\mathcal{AC}^{(P)}_{p,n}$ is the positive geometry associated to ϕ^p interactions. We shall do this by first embedding the accordiohedron into kinematic space and then showing that the canonical form of the accordiohedron when pulled back gives the right planar scattering amplitude for ϕ^p interactions. We start by noting the following facts:

[3] From the perspective of Feynman graphs this is equivalent to saying that there is an accordiohedron for each topological class of graphs.

- The only tree level amplitudes consistent with ϕ^p interactions have $p + (p - 2)n$ external legs for $n + 1$ vertices.

- Analogously to the cubic and quartic cases there is a 1-1 correspondence between planar tree level Feynman graphs and dissections of $p + (p - 2)n$- gon into p-gons.

- We also require the accordiohedron $\mathcal{AC}_{p,n}^{(P)}$ to have dimension n, which is the number of propagators. [4]

3.3 Planar scattering form for ϕ^p interactions

We would like to define a planar scattering form for ϕ^p interactions. We can associate to each planar graph g with propagators $X_{i_1 j_1}, X_{i_2 j_2} \cdots X_{i_n j_n}$ a scattering form

$$\frac{\sigma(g)}{\prod_{k=1}^{n} X_{i_k j_k}} dX_{i_1 j_1} \wedge dX_{i_2 j_2} \wedge \cdots \wedge dX_{i_n j_n},$$

where $\sigma(g) = \pm 1$

Thus, when we sum over all planar graphs g we have several possible scattering forms. Since we do not have a notion of *projectivity* except in the case of $p = 3$ which helps us fix a unique scattering form [43]. We can choose a particular reference graph g (equivalently a p-angulation P) and look at only those subset of graphs which are related to this graph by a sequence of Q-flips namely all the vertices of the *accordiohedron*. If a graph g' is related to g by an odd (even) number of Q-flips we can associate $-(+)$ sign to it. Thus, we can define a p-angulation P dependent planar scattering form Ω_n^P

$$\Omega_n^P = \sum_{flips} \frac{(-1)^{\sigma(flip)}}{\prod_{k=1}^{n} X_{i_k j_k}} dX_{i_1 j_1} \wedge dX_{i_2 j_2} \wedge \cdots \wedge dX_{i_n j_n}.$$

Since, the Q-compatible p-angulations corresponding to any reference p-angulation P

[4] This is because we require the top-form on the positive geometry, once embedded in kinematic space to produce the right scattering amplitude.

does not exhaust all the p-angulations, we need to define such a planar scattering form for each P.

In the $n = 1$ case the set of Q-compatible p-angulations are $\{(1\ p;+),(p-1\ 2p-2;-)\},\{(2\ p+1;+),(p\ 2p-1;-)\},\cdots,\{(p-1\ 2p-2;+),(p-2\ 2p-3;-)\}$[5] the planar scattering forms for which are

$$\Omega^{(ij)}_{2p-2} = d\ln X_{i\ j} - d\ln X_{i+p-2\ j+p-2},$$

where $i, j = 1, \cdots, (p-1)$ Mod $(2p-2)$ with $|i-j| = (p-1)$

We now turn to embedding the accordiohedron in kinematic space and showing that when the planar scattering form is pulled back onto the accordiohedron it gives the canonical form of the accordiohedron.

3.4 Locating the accordiohedron inside kinematic space

We now define the kinematic accordiohedron $\mathcal{AC}^{(P)}_{p,n}$. We locate the *accordiohedron* inside the positive region of kinematic space $X_{ij} > 0$ for all $1 \leq i < j \leq p + (p-2)n$ by imposing the following constraints:

$$\begin{aligned} s_{ij} &= -c_{ij}\,; \quad for\ \ 1 \leq i < j \leq p-1+(p-2)n,\ |i-j| \geq 2 \\ X_{r_i s_i} &= d_{r_i s_i}\,;\ s.t\ P \cup_{i=1}^{n} \{(r_i, s_i)\}\ is\ a\ complete\ triangulation, \end{aligned} \tag{3.2}$$

where $c_{ij}, d_{r_i s_i}$ are positive constants. [6]

Physically we choose the above set of constraints as they do not appear as propagators of

[5]Here the signs denote $(-1)^{\sigma(flip)}$, when we have multiple diagonals we need to carefully maintain the order of diagonals when we flip as it contibutes to the sign.

[6]In the case when $p = 4$ that is when the accordiohedron is a Stokes polytope, there is a canonical choice for the additional constraints if the stokes polytope is itself not an associahedron [47]. However for $p > 4$ we do not have any canonical choice of these constraints. As we show in this section, there is at least one choice which consistently embeds the Accordiohedron in the Kinematic space.

any ϕ^p graph. The first constraint above is the famous associahedron embedding [43]. We have thus embedded the accordiohedron inside the associahedron. The positivity of X_{ij}'s, the above constraints along with the equation (1) are a set of inequalities satisfied by the X_{ij} which makes the convexity of the accodiohedron manifest.

We first consider the $n = 1$ case with reference p-angulations to be $P = \{(1, p)\}$ for $p = 5, 6$.

For $p = 5, 6$ we can choose $\cup_i\{r_i\ s_i\}$ to be $\{24, 25, 17, 57\}$ and $\{35, 36, 26, 17, 19, 79\}$ respectively. The above constraints then translate to:

p=5: $X_{48} = \sum_{i=1}^{3} c_{i5} + c_{i6} + c_{i7} - X_{15}$ which a line with boundaries at $X_{15}, X_{48} = 0$ provided the following are satisfied [7]

$$\begin{gathered}
\sum_{i=1}^{3} c_{i7} \leq d_{17} \leq \sum_{i=1}^{5} c_{i7} \\
\sum_{i=5}^{7}(c_{2i} + c_{3i}) \leq d_{25} \leq \sum_{i=3}^{7} c_{1i} + \sum_{i=5}^{7}(c_{2i} + c_{3i}) \\
0 \leq d_{24} \leq d_{25} + c_{24} \\
0 \leq d_{57} \leq c_{46}
\end{gathered}$$

p=6: $X_{510} = \sum_{i=1}^{4} c_{i6} + c_{i7} + c_{i8} + c_{i9} - X_{16}$ which a line with boundaries at $X_{16}, X_{510} = 0$ provided the following are satisfied

$$\begin{aligned}
\sum_{i=1}^{4} c_{i9} &\leq d_{19} \leq \sum_{i=1}^{7} c_{i9} \\
\sum_{i=1}^{4}\sum_{j=7}^{9} c_{ij} &\leq d_{17} \leq \sum_{i=1}^{4}\sum_{j=7}^{9} c_{ij} + d_{19} \\
\sum_{i=2}^{4}\sum_{j=6}^{9} c_{ij} &\leq d_{26} \leq \sum_{i=2}^{4}\sum_{j=6}^{9} c_{ij} + \sum_{i=3}^{9} c_{1i} \\
\sum_{i=3}^{4}\sum_{j=6}^{9} c_{ij} &\leq d_{36} \leq c_{24} + c_{25} + d_{26} \\
0 &\leq d_{79} \leq \sum_{i=1}^{6} c_{i8} + d_{19} \\
0 &\leq d_{35} \leq c_{35} + d_{36}.
\end{aligned}$$

[7] since we are slicing the associahedron using some hyperplanes $X_{r_i s_i} = d_{r_i s_i}$ to get the accordiohedron these constraints tell us how the slicing should be made. For higher n we shall not state these constraints for brevity but we shall assume that they are satisfied.

The above equations define lines with Q-compatible vertices $\{15, 48\}$ and $\{16, 510\}$ for $p = 5$ and $p = 6$ respectively. We can trivially repeat this exercise for any other reference p-angulation $P = \{i, i + p\}$, the results of which can be obtained by taking $k \to k + i - 1$ in the above equations. We can now pull back the scattering form onto the *accordiohedron* $\mathcal{AC}^P_{p,n}$ as

$$\omega^P_{p,n} = \left(\frac{1}{X_{i\ i+p-1}} + \frac{1}{X_{i+p-2\ i+2p-3}}\right) d\ln X_{ii+p-1} := m^P_{p,n}(\mathcal{AC}_n) d\ln X_{ii+p-1},$$

with $i = 1, ..., p - 1$.

As before to get the full amplitude $\mathcal{M}_n$ we consider a weighted sum $\tilde{\mathcal{M}}_n$ of $m^P_{p,n}$ over all P

$$\tilde{\mathcal{M}}_n = \sum_{i=1}^{p-1} \alpha_i \left(\frac{1}{X_{i\ i+p-1}} + \frac{1}{X_{i+p-2\ i+2p-3}}\right). \tag{3.3}$$

It is clear that $\mathcal{M}_n = \tilde{\mathcal{M}}_n$ if and only if $\alpha_i = \frac{1}{2}$ for all $i = 0, ..., p - 1$.

Thus, we can simplify our computation by considering a subset of p-angulations $\{P_1, \ldots, P_I\}$ *called primitive p-angulations for which :*

(a) no two members of the set are related to each other by cyclic permutations and

(b) all the other p-angulations can be obtained by a (sequence of) cyclic permutations of one of the Ps belonging to the set.

The primitives are the class of rotationally inequivalent diagrams. Since, a rotation does not change the relative configuration of diagonals it is clear that accodiohedra remain the same for all the diagrams that belong to a primitive class and that the weights depend only on primitives. We shall say more about primitives in section (3.5).

$$\mathcal{M}_n = \sum_{\substack{\text{rotations} \\ \sigma}} \sum_{\substack{\text{primitives} \\ P}} \alpha_P\, m^{(\sigma.P)}_{p,n}. \tag{3.4}$$

For now let us look at a couple of examples to see how finding primitive accordiohedra and their weights help us in getting the scattering amplitude.

In the $n = 1$ case above there was only one primitive $P = \{(1, p)\}$.

We consider the $n = 2$ case for $p = 5, 6$ for which we now have the set of primitives as (see figure(3.2)) $\{(15, 610), (15, 18)\}$ and $\{(16, 914), (16, 110), (16, 813)\}$ respectively. The set of Q-compatible p-angulations for these are

$$\begin{aligned}
\mathbf{p = 5}: \quad S_5^1 &= \{(15, 711; +), (411, 711; -), (15, 610; -), (411, 610; +)\}, \\
S_5^2 &= \{(15, 18; +), (18, 48; -), (15, 711; -), (411, 711; +), (411, 48; +)\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{p = 6}: \quad S_6^1 &= \{(16, 914; +), (514, 914; -), (16, 813; -), (514, 813; +)\}, \\
S_6^2 &= \{(16, 813; +), (514, 813; -), (16, 712; -)(514, 712; +)\}, \\
S_6^3 &= \{(16, 110; +), (16, 914; -), (110, 510; -), (510, 514; +), (514, 914; +)\}.
\end{aligned}$$

The embedding constraints (3.2) can be solved to obtain:

p=5: For P=(15,711), with $\cup_i\{r_i\ s_i\}$ to be $\{13, 35, 17, 57, 810, 710\}$

$$\begin{aligned}
X_{411} &= \sum_{i=1}^{3}\sum_{j=5}^{10} c_{ij} - X_{15} \\
X_{610} &= \sum_{i=1}^{5}\sum_{j=7}^{10} c_{ij} + c_{610} - d_{17} + d_{710} - X_{711}.
\end{aligned}$$

The positivity of X_{610}, X_{411} carves out a square region in the X_{15}, X_{711} space. The accordiohedron in this case is also a square as we had emphasised in section (3.1).

For P=(15,18), with $\cup_i\{r_i\ s_i\}$ to be $\{13, 35, 16, 68, 810, 110\}$

$$\begin{aligned} X_{48} &= \sum_{i=1}^{3}\sum_{j=5}^{7} c_{ij} - X_{15} + X_{18} \\ X_{411} &= \sum_{i=1}^{3}\sum_{j=5}^{10} c_{ij} - X_{15} \\ X_{711} &= \sum_{i=1}^{6}\sum_{j=8}^{10} c_{ij} - X_{18}. \end{aligned}$$

Similarly, for the $p = 6$ case we get ,

p=6: For P=(16,914), with $\cup_i\{r_i\ s_i\}$ to be $\{13, 35, 15, 911, 912, 913, 814, 17, 714\}$

$$\begin{aligned} X_{514} &= \sum_{i=1}^{4}\sum_{j=6}^{13} c_{ij} - X_{16} \\ X_{610} &= c_{813} + d_{814} + d_{913} - X_{914}. \end{aligned}$$

For P=(16,813), with $\cup_i\{r_i\ s_i\}$ to be $\{13, 35, 15, 713, 912, 911, 812, 17, 714\}$

$$\begin{aligned} X_{514} &= \sum_{i=1}^{4}\sum_{j=6}^{13} c_{ij} - X_{16} \\ X_{712} &= c_{712} + d_{713} + d_{812} - X_{813}. \end{aligned}$$

For P=(16,110), with $\cup_i\{r_i\ s_i\}$ to be $\{13, 35, 15, 17, 18, 19, 113, 1113, 111\}$

$$\begin{aligned} X_{510} &= \sum_{i=1}^{4}\sum_{j=6}^{9} c_{ij} - X_{16} + X_{110} \\ X_{514} &= \sum_{i=1}^{4}\sum_{j=6}^{13} c_{ij} - X_{16} \\ X_{914} &= \sum_{i=1}^{8}\sum_{j=10}^{13} c_{ij} - X_{110}. \end{aligned}$$

When pulled back onto constraints the corresponding $m^P_{p,n}$'s are

p = 5:

$$m^1_{5,2} = \left(\frac{1}{X_{15}X_{610}} + \frac{1}{X_{411}\,X_{610}} + \frac{1}{X_{15}\,X_{711}} + \frac{1}{X_{411}\,X_{711}}\right) dX_{15} \wedge X_{610}$$
$$m^2_{5,2} = \left(\frac{1}{X_{15}X_{18}} + \frac{1}{X_{18}\,X_{48}} + \frac{1}{X_{15}\,X_{711}} + \frac{1}{X_{411}\,X_{711}} + \frac{1}{X_{48}\,X_{411}}\right) dX_{15} \wedge X_{18}.$$

Plugging the above forms into equation (3.4) with weights $\alpha^{P_1}_{5,2} = \frac{3}{11}, \alpha^{P_2}_{5,2} = \frac{2}{11}$ (see section (4.3) for details).

p = 6:

$$m^1_{6,2} = \left(\frac{1}{X_{16}X_{914}} + \frac{1}{X_{514}\,X_{914}} + \frac{1}{X_{16}\,X_{813}} + \frac{1}{X_{514}\,X_{813}}\right) dX_{16} \wedge X_{914}$$
$$m^2_{6,2} = \left(\frac{1}{X_{16}X_{813}} + \frac{1}{X_{514}\,X_{813}} + \frac{1}{X_{16}\,X_{712}} + \frac{1}{X_{514}\,X_{712}}\right) dX_{16} \wedge X_{813}$$
$$m^3_{6,2} = \left(\frac{1}{X_{16}X_{110}} + \frac{1}{X_{16}\,X_{914}} + \frac{1}{X_{110}\,X_{510}} + \frac{1}{X_{510}\,X_{514}} + \frac{1}{X_{514}\,X_{914}}\right) dX_{16} \wedge X_{110}$$

Plugging the above forms into equation (3.4) with weights $\alpha^{P_1}_{6,2} = \frac{1}{3}, \alpha^{P_2}_{6,2} = \frac{1}{6}, \alpha^{P_3}_{6,2} = \frac{1}{3}$ (see section (4.3) for details).

3.5 Analysing the combinatorics of Accordiohedra

A complete computation of the amplitude from the geometry of the polytope requires determination of all the primitives of a given dimension n and computation of the corresponding weights. We shall address the problem in this section. We emphasise that this a purely combinatorial problem and hence does not depend on the construction of kinematic space accordiohedron. In sections (3.5.1) ,(3.5.2) we shall first derive formulae to count the number of primitive accordiohedra of a given dimension n. Then in sections (4.1) ,(4.2) we provide a complete classification of primitive accordiohedra for $n \leq 3$ and compute the corresponding weights for any ϕ^p interactions. Let us first consider the quartic case.

3.5.1 Counting primitives for the quartic case

In this section we shall address the quartic case first and provide a formula for the number of primitive Stokes polytopes p_n of a given dimension n. The main result of this section is :

$$p_n = \begin{cases} \frac{1}{2n+4}F_{n+1} + \frac{1}{2}F_{\frac{n+1}{2}}, \ n = 2k+1 \\ \frac{1}{2n+4}F_{n+1} + \frac{1}{4}\tilde{F}_{\frac{n}{2}}, \ n = 2k \text{ with k odd} \\ \frac{1}{2n+4}F_{n+1} + \frac{1}{2}F_{\frac{n}{4}}, \ n = 4k \end{cases}$$

Where $F_n = \frac{1}{2n+1}\binom{3n}{n}$ and $\tilde{F}_n = \frac{2}{2n+2}\binom{3n+1}{n}$

We shall now prove this result.

We shall consider a $(4+2n)$-gon as equally spaced points a_i on the circle i.e. $a_i = \exp\frac{2\pi i}{4+2n}$ with $i = 0, ..., 2n+3$. The edges of the polygon correspond to arcs $a_i a_{i+1}$ on the circle and the diagonals of a quadrangulation correspond to chords.

There is a natural action of the dihedral group $\mathbb{D}_{2n+4}$ on any given quadrangulation which is generated by rotation and reflections about a given diagonal. We are interested in counting *primitive quadrangulations*, no two of which are related to each other by a cyclic permutation, which corresponds to a rotation on the circle. Thus, it is sufficient to consider only the cyclic group $\mathbb{Z}_{2n+4}$ for our purposes. The problem of counting *primitive quadrangulations* is thus equivalent to finding the number orbits of the set of all quadrangulations of a $(4+2n)$-gon under the action of the cyclic group $\mathbb{Z}_{2n+4}$.

We shall do this by using the celebrated *Burnside's lemma* [101], which is the standard way to count the number of orbits G/G_X for the action of any finite group G on a set X. It states that the number of orbits is equal to the average number of points that remain invariant when acted on by elements of G.

$$|G/G_X| = \frac{1}{|G|} \sum_{g \,\epsilon\, G} |\{e \,\epsilon\, X| \, g.e = \, e\}|. \tag{3.5}$$

Thus, to count the number of *primitive quadrangulations* we just need to find the subset of quadrangulations that are invariant under some rotation. This problem has been addressed by [102] using the method of generating functions, but we shall take a simpler approach here following [103].

We can consider the division of $(2n + 4)$-gon into $n + 1$ quadrilaterals. We first note that the centre of the circle is left invariant by the action of the cyclic group $\mathbb{Z}_{2n+4}$. The centre of the circle can lie on :

(1) A diameter. This can only happen when n is odd since, the relative angle between the end points a_i, a_j of this diameter has to π.

(2) The midpoint of an invariant cell i.e. on the point of intersection of the diagonals of a centre square which remains invariant.

In case (1) the diameter forms an axis of symmetry and has to be left invariant by rotations and it is clear that the only possible rotation which does this is by π and the quadrangulation Q consists of a left and a right part where the left part is a rotation of the right one.(see figure(3.3)).

In case (2) the diagonals can either be rotated to themselves or into each other. This can only be accomplished by rotations of $\pi, \pm\frac{\pi}{2}$ and the corresponding quadrangulations are shown in the figure (3.3).

The number of quadrangualtions of $(2n + 4)$-gon into $n + 1$ quadrangles is given by the Fuss-Catalan number $F_n = \frac{1}{2n+1}\binom{3n}{n}$ [89] (which we also derive in appendix B). The number of quadrangulations of type (1) is $nF_{\frac{n+1}{2}}$, as we can choose a diameter in $n + 2$ ways and for each choice of the diameter there $F_{\frac{n+1}{2}}$ sub-quadrangulations A.

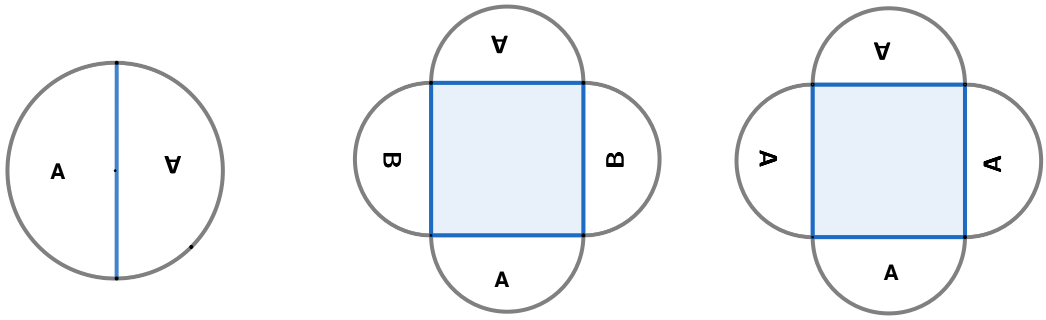


Figure 3.3: All the quadrangulations invariant under some rotation

The number of quadrantulations of type (2) depends on whether n is divisible by 2 or 4 and is given by $\frac{(n+2)}{2}\tilde{F}_{\frac{n}{2}}$ and $(n+2)F_{\frac{n}{4}}$ respectively. In the case where $n = 2k$ we can divide k into k_1, k_2 which we call A and B in the third figure of (3.3) and the number of such quadrangulations would correspond to F_{k_1} and F_{k-k_1} respectively. The total number of such quadrangulations would then be $\sum_{k_1=0}^{k} F_{k_1} F_{k-k_1}$and since there are $\frac{n+2}{2}$ ways we can relabel the invariant square. Using the following combinatorial identity (see appendix C).

$$\tilde{F}_n = \sum_{k_1=0}^{n} F_{k_1} F_{n-k_1} = \frac{2}{2n+2}\binom{3n+1}{n}.$$

We have $\frac{(n+2)}{2}\tilde{F}_{\frac{n}{2}}$ invariant quadrangulations under a rotation by π.

When $n = 4k$ we have $F_{\frac{n}{4}}$ subquadrangulations A as shown in figure (3.3). There are also $(n + 2)$ ways to relabel the invariant cell and thus there are a total of $(n + 2)F_{\frac{n}{4}}$ quadrangulations that are invariant under a rotation by $\pm\frac{\pi}{2}$. Thus, after also including the identity rotation which leaves all the elements invariant we get the total number of *primitive quadrangulations* p_n is given by:

$$p_n = \begin{cases} \frac{1}{2n+4}F_{n+1} + \frac{1}{2}F_{\frac{n+1}{2}}, \ n = 2k+1 \\ \frac{1}{2n+4}F_{n+1} + \frac{1}{4}\tilde{F}_{\frac{n}{2}}, \ n = 2k \text{ with k odd} \\ \frac{1}{2n+4}F_{n+1} + \frac{1}{2}F_{\frac{n}{4}}, \ n = 4k. \end{cases}$$

We can easily check the above formula for $n = 1, 2, 3$ cases by using $F_n = 1, 3, 12, 55$ and $\tilde{F}_n = 1, 2, 7, 30$ for $n = 1, 2, 3, 4$. The set of invariant quadrangulations is shown in the figure (3.4) below.

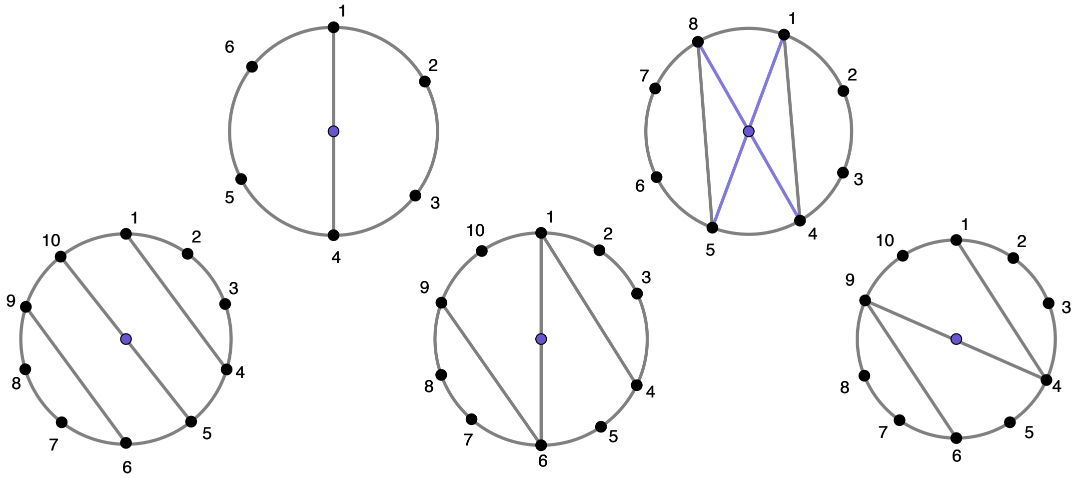


Figure 3.4: invariant quadrangulations for n=1,2,3

n=1: We have 3 quadrangulations $\{14, 25, 36\}$ which remain invariant under rotation by π.

$$p_1 = \frac{1}{6}(3 + 3) = 1$$

n=2: There are 4 quadrangulations $\{(1+i\ 4+i\ , 5+i\ 8+i))\}$ with $i = 0, .., 3$ which remain invariant under rotation by $\pm\frac{\pi}{2}$.

$$p_2 = \frac{1}{8}(12 + 4) = 2$$

n=3: There are 15 quadrangulations $\{(1+i\ 4+i, 5+i\ 10+i, 6+i\ 9+i), (1+i\ 4+i, 1+i\ 6+i, 6+i\ 9+i), (1+i\ 4+i, 4+i\ 9+i, 6+i\ 9+i)\}$ with $i = 0, ..., 4$

$$p_3 = \frac{1}{10}(55 + 5 + 5 + 5) = 7$$

3.5.2 Counting primitives for ϕ^p case

We shall now extend our analysis for the quartic case to any general p and provide a formula for the number of primitive accordiohedra of dimension n. The number of *primitives p-angulations* of an $(p-2)n+p$-gon is the same as the number of orbits of the cyclic group $\mathbb{Z}_{(p-2)n+p}$ when it acts on the set of all p-angulations. There number of such orbits can be straightforwardly computed from *Burnside's lemma* just as we had done in (3.5.1). We proceed analogously to the quartic case (3.5.1) by noting that the centre of the circle is invariant under any rotation and can lie:

(1) On a diameter, this happens only when n is odd and leaves the p-angulation invariant under a rotation by π (see figure (3.5)).

(2) Inside an invariant cell, in this case we have p-angulations for every $d \mid Gcd(p, n)$ which is invariant under rotation by $\frac{2\pi}{d}$ (see figure (3.5)).

The total number of p-angulations of an $(p-2)n + p$-gon into $(n + 1)$ p-gons is given by the Fuss Catalan number $F_{p,n} = \frac{1}{(p-2)n+p}\binom{(p-1)n}{n}$ [89] (see appendix B for a proof of this).

In case (1) there are $\frac{(p-2)n+p}{2}$ choices for the diameter and $F_{p,(n+1)/2}$ choices for A. Thus, there are a total of $\frac{(p-2)n+p}{2}F_{p,(n+1)/2}$ invariant p-angulations under a rotation by π.

In case (2) there is an invariant cell and the remaining n cells can be divided into $i = \frac{p}{d}$ parts for every $d \mid Gcd(p, n)$ in $F_{k_1,p}$, $F_{k_2,p}$,...$F_{k_i,p}$ ways s.t. $k_1 + k_2 + ... + k_i = \frac{n}{d}$ which we call A, B, C etc. For each such d there are $\phi(d)$ p-angulations which remain invariant under a $\frac{2\pi}{d}$ rotation, where $\phi(d)$ is the Euler totient function which counts positive integers up to d which are relatively prime to it.

For if $d = p_1^{a1}p_2^{a_2}...p_r^{a_r}$ is the prime factorisation of d then $\phi(d)$ is given by:

$$\phi(d) = d\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)...\left(1 - \frac{1}{p_r}\right)$$

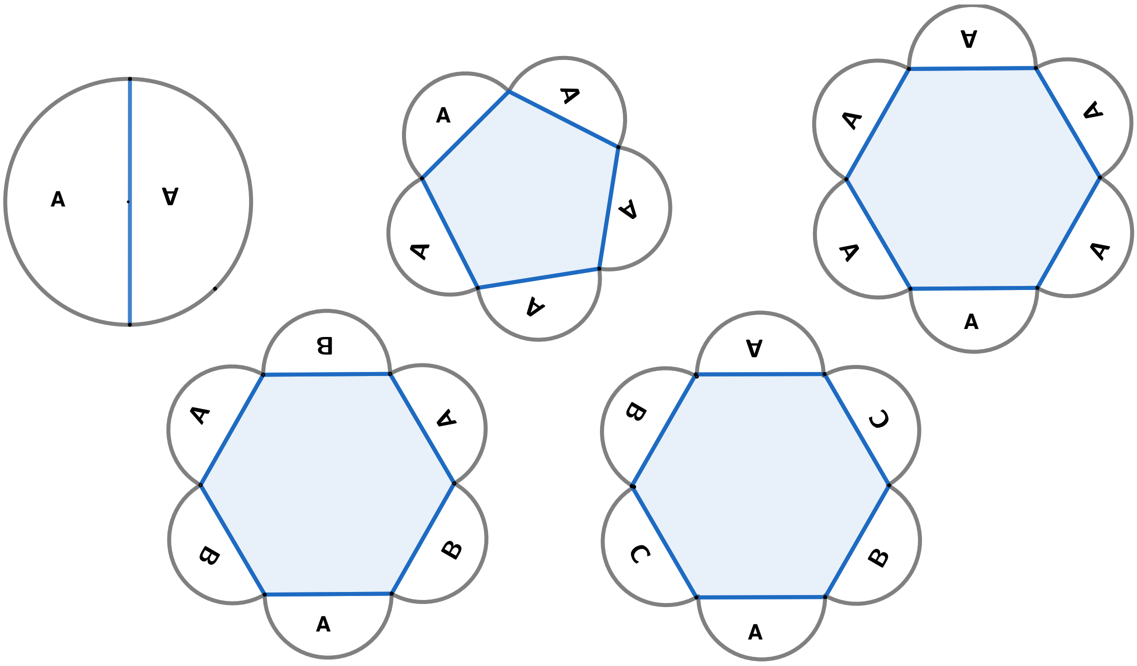


Figure 3.5: The dissections invariant under some rotation for $p = 5$ it is clear that $d = 1, 5$ and are as shown in the first two diagrams starting clockwise from the left corner . For $p = 6$ with $d = 1, 2, 3, 6$ and the invariant dissections are the ones shown in first and last three diagrams of figure(3.5)

Thus, the total number of such p-angulations once we also include identity rotation is

$$p_n = \begin{cases} \frac{1}{(p-2)n+p} F_{n,p} + \frac{1}{2} F_{\frac{n+1}{2},p} + \frac{1}{p} \sum_{d|Gcd(n,p)} \phi(d) \sum_{k_1+\ldots+k_i=n/d} F_{k_1,p} F_{k_2,p} \ldots F_{k_i,p}, & \text{if n is odd} \\ \frac{1}{(p-2)n+p} F_{n,p} + \frac{1}{p} \sum_{d|Gcd(n,p)} \phi(d) \sum_{k_1+\ldots+k_i=n/d} F_{k_1,p} F_{k_2,p} \ldots F_{k_i,p}, & \text{if n is even} \end{cases} \tag{3.6}$$

$$p_n = \begin{cases} \frac{1}{(p-2)n+p} F_{n,p} + \frac{1}{2} F_{\frac{n+1}{2},p} + \frac{1}{p} \sum_{d|Gcd(n,p)} \phi(d) \tilde{F}_{n/d,p,p/d}, & \text{if n is odd} \\ \frac{1}{(p-2)n+p} F_{n,p} + \frac{1}{p} \sum_{d|Gcd(n,p)} \phi(d) \tilde{F}_{n/d,p,p/d}, & \text{if n is even,} \end{cases}$$

where, we have used the combinatorial identity

$$\tilde{F}_{n/d,p,p/d} = \sum_{k_1+\ldots+k_i=n/d} F_{k_1,p} F_{k_2,p} \ldots F_{k_i,p} = \frac{p/d}{(p-2)n+p} \binom{(p-1)n + \frac{p}{d}}{n} \tag{3.7}$$

with $\tilde{F}_{n,p,1} = F_{n,p}$ which we shall prove in appendix C.

3.6 Factorisation

In this section we will try to argue that the accordiohedra factorise geometrically and this directly implies factorisation for planar amplitudes in massless ϕ^p theory. We shall first argue that the geometric factorisation of accordiohedron holds and then show how this leads to the factorisation of the amplitude.

In **??** it was shown how the factorisation of Stokes polytope leads to a recursions relation on α's. We shall see that even for more general Accordiohedra α's are required to satisfy analogous recursion relations.

Our first assertion is the following. Given any diagonal (ij), consider *all* P which contains ij and the consider all the corresponding kinematic accordiohedron $\mathcal{AC}^P_{p,n}$. We contend that for each accordiohedron , the corresponding facet $X_{ij} = 0$ is a product of lower dimensional accordiohedra

$$\mathcal{AC}^P_{p,n}\Big|_{X_{ij}\,=\,0} \equiv \mathcal{AC}^{P_1}_{p,m} \times \mathcal{AC}^{P_2}_{p,n+2-m}, \tag{3.8}$$

where P_1 and P_2 are such that $P_1 \cup P_2 \cup (ij) = P$.

P_1 is the p-angulation of the polygon $\{i,\ i+1, \dots,\ j\}$ and P_2 is the p-angulation of $\{j, j+1, \dots, n, \dots, i\}$. Now we know that, on $\mathcal{AC}^P_{p,n}$ any planar scattering variable X_{kl} is a linear combination of X_{ij} and remaining X's which constitute P. Hence in order to prove this assertion we need to show that any X_{kl} with $i \leq k < l \leq j$ can be written as a linear combination of X_{ij} and elements of P_1 and similarly any variable in the complimentary set can be written in terms of X_{ij} and elements of P_2.

However this is immediate since we know from the factorisation property of associahedron proven in [43] that any $X_{kl} = X_{ij} + \sum_{i<m<n<j} X_{mn}$. some of these $X_{mn} \in Q_1$ and the others are constrained via $X_{mn} = d_{mn}$. This proves our assertion. Thus $X_{ij} = 0$ facet

factorises into two lower dimensional accordiohedra.

Our second assertion is that the geometric factorisation implies amplitude factorisation of ϕ^p theory. This assertion is based on the following two facts.
(1) As the accordiohedra is a positive geometry , we know that it's canonical form satisfies the following properties satisfed by canonical form on any positive geometry $\mathcal{A}$

$$\mathrm{Res}_H \omega_{\mathcal{A}} \; = \; \omega_{\mathcal{B}}, \tag{3.9}$$

where we think of $\omega_{\mathcal{A}}$ as defined on the embedding space and H is any subspace in the embedding space which contains the face $\mathcal{B}$. It is also known that if $\mathcal{B} \; = \; \mathcal{B}_1 \times \mathcal{B}_2$ then

$$\omega(\mathcal{B}) \; = \; \omega(\mathcal{B}_1) \; \wedge \; \omega(\mathcal{B}_2). \tag{3.10}$$

Thus we immediately see that

$$\mathrm{Res}_{X_{ij} \; = \; 0} \;\; \omega(\mathcal{A}C^{P}_{p,n}) \; = \; \omega^{P_1}_{m} \; \wedge \; \omega^{P_2}_{n+2-m} \, \forall \, P. \tag{3.11}$$

where $m \; = \; j - i + 1$.

We thus see that residue over each accordiohedron which contains a boundary $X_{ij} \; \rightarrow \; 0$ factorises into residues over lower dimensional accordiohedra. This factorisation property naturally implies factorisation of amplitudes as follows. Consider the n-gon with a diagonal (ij) (with i, j such that this diagonal can be part of a p-angulation). This diagonal subdivides the n-gon into a two polygons with vertices $\{i, \ldots, j\}$ and $\{j, \ldots, n, 1, \ldots i\}$ respectively. By considering all the kinematic accordiohedra associated to these polygons, we can evaluate $\widetilde{M}_{|j-i+1|}, \; \widetilde{M}_{n+2-(|j-i+1|)}$ which correspond to left and right sub-amplitudes respectively. This immediately implies that

$$\widetilde{\mathcal{M}}_n|_{X_{ij}\ =\ 0} \;=\; \widetilde{\mathcal{M}}_L \frac{1}{X_{ij}} \widetilde{\mathcal{M}}_R \tag{3.12}$$

This proves physical factorisation. We also note that, eqns. (5.1) and (3.12) imply following constraints on α's.

$$\sum_{P\ \text{containing}(ij)} \alpha_P \;=\; \sum_{P_L,P_R} \alpha_{P_L}\alpha_{P_R} \tag{3.13}$$

The left hand side of the above equation involves sum over all accordiohedra $\mathcal{AC}^P_{p,n}$ for which $(ij)\epsilon\ P$ and the right hand side involves sum over P_L and P_R which range over all the p-angulations of the two polygons to the left and right of the diagonal (ij) respectively.

It can be verified that in all the examples up to $p = 12$ and $n = 3$ the α_P's do indeed satisfy these constraints.

For $n = 1$ there is only one diagonal which we can to be $(1, p)$. The accordiohedra is always a line as we had emphasised in section (3.3) and $(1, p)$ appears in the vertex of exactly two of these lattices namely $\{1\ p,\ p-1\ 2p-2\}$ and $\{p+1\ 2p+1,\ 1\ p\}$. There is only way to divide P into P_L and P_R and both these are trivial have $\alpha = 1$. Thus, the above equation (3.13) gives

$$\begin{aligned} 2\alpha_P &= 1 \\ \alpha_P &= \frac{1}{2}\ for\ all\ P. \end{aligned}$$

We could expect that the set of equations (3.13) would help in determining all the weights. But, as we shall now show this is not the case as (3.13) provides *too few* equations . In other words the set of equations (3.13) provide a set of necessary but not sufficient conditions.

Let, us consider the case $n = 2$ for $p = 5$ and the diagonal (15), in this case the 11-gon gets divided into a 5-gon and 8-gon and we have weights $\alpha_{P_L} = \frac{1}{2}$ and $\alpha_{P_R} = 1$. There are exactly 4 squares and 5 pentagons which contain the diagonal (15) in their vertices. Thus, we have

$$4\alpha_5^1 + 5\alpha_5^2 = 2. \tag{3.14}$$

We can check that for any other choice of diagonal (ij) we get the same equation. It is clear that this is not sufficient to solve for α_5^1, α_5^2. The solution we had obtained using our prescription in section () namely $\alpha_5^1 = \frac{3}{11}, \alpha_5^2 = \frac{2}{11}$ does indeed satisfy (3.14).

Chapter 4

Primitives and Weights

To determine the weights we would need to actually identify the primitive accordiohedra , then we need to find all the vertices of the accordiohedra starting with these primitives as the reference p-angulations. There is no general classification for primitives of an arbitrary dimension n to our knowledge since they grow as p^n. We provide a complete classification for $n \leq 3$ and give the compute the corresponding weights.

4.1 Primitives and Weights for $n = 2$ case

We would like to provide the details of the primitives and weights for ϕ^p interactions for the $n = 2$ (2d case). In this case there are 3 vertices with $3p - 4$ legs. We could try to recursively construct these graphs from the $n = 1$ graphs. So without loss of generality we consider only Feynman graphs in which two of the vertices lie in a line as shown in the figure(4.1). The 3rd vertex can then be made to lie on the central line connecting the first two vertices and it can be either above or below this line. These graphs can be denoted as (k_1, k_2) such that $k_1 + k_2 = p - 2$, where the 3rd vertex has k_1 legs above and k_2 legs below the central line.

Since, we are only interested in primitives the graphs for which 3rd vertices are (k_1, k_2)

and (k_2, k_1) correspond to the same primitive graph. Thus, without loss of generality we can choose the diagrams in which the 3rd vertex has more legs above the central vertex than below it to be primitive graphs namely $(p-2, 0)$, $(p-3, 1)$,..., $(\left\lceil \frac{p-2}{2} \right\rceil, \left\lfloor \frac{p-2}{2} \right\rfloor)$. We shall call them [1],[2],...,[k], where $k = \left\lfloor \frac{p}{2} \right\rfloor$.

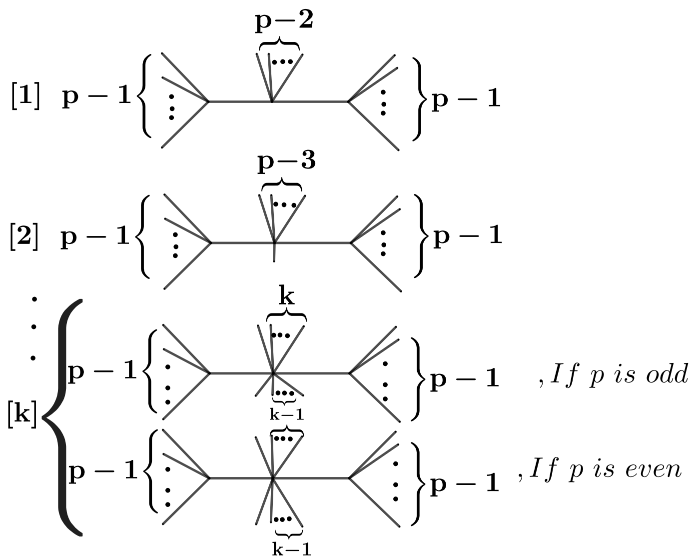


Figure 4.1: The primitive graphs for n=2 case.

The primitives are all shown in the above figure(4.1). We shall now show that these are the only primitives. It is clear that all the graphs above are inequivalent under cyclic permutations. As explained earlier the total number of such graphs is given by the Fuss-Catalan number

$$\frac{1}{3(p-2)+1}\binom{3(p-1)}{3} = \frac{(p-1)(3p-4)}{2}$$

We shall show that if we perform the channel sum starting with these primitives we generate all the graphs. A cyclic permutation corresponds to a clockwise rotation of the labels and every graph returns to itself after a rotation of period $3p-4$ but a graph for which the 3rd vertex is symmetric about the central line returns to itself after only half a rotation i.e has a period $\frac{3p-4}{2}$. The only such graph is the last graph in the case when $p = 2k$. Thus, the total number of graphs generated by performing sum over all the channels is :

$$\begin{cases} k(3p-4) & ,\text{if } p = 2k+1 \\ (k-1)(3p-4) + \frac{3p-4}{2} & ,\text{if } p = 2k \end{cases}$$

using

$$\begin{cases} k = \frac{p-1}{2} & ,\text{if } p = 2k+1 \\ k = \frac{p}{2} & ,\text{if } p = 2k \end{cases}$$

we get the total number of graphs to be $\frac{(p-1)(3p-4)}{2}$ which agrees with our results from (3.6). As explained in the previous subsection the accordiohedra generated by starting with a particular graph (or p-angulation) depends only on the relative configuration of the diagonals and in this case since there are 2 diagonals the only possibilities are :

1. The diagonals meet as in [1], in this case the accordiohedron is an associahedron A_n.(see (3.2))

2. The diagonals do not meet as in [2],... ,[k], in all these other cases the accordiohedron turns out to be a square.

We can provide a mapping between the vertices of the Stokes polytope $AC^P_{4,2}$ and $AC^P_{p,2}$ for the $n = 2$ case as follows: 1.) When the two diagonals meet then $P = \{i\ p+i, i\ i+2p-2\}$ with $i = 1, ..., 3p-4$ and we could map the vertices of the Stokes polytope which is a pentagon in case with the pentagon corresponding to the accordiohedra once we notice that the all $i = 1, ..., 8$ which is part of some diagonal ij do not appear in a vertex of the Stokes polytope. For example in the case of $P = \{(1\ 4, 1\ 6\}$ only $i = 1, 3, 4, 5, 6, 8$ appear and similarly in the case of the accordiohedron exactly 6 $i = 1,\ p-1,\ p,\ 2p-3,\ 2p-4,\ 3p-4$ out of a possible $3p-4$ appear thus we could trivially define a map between the two as follows:

$$
\begin{aligned}
i &\rightarrow i + f(i) \\
with\ f(i) &= a\begin{cases} 1 \quad , & \text{if } i = 1 \\ (p-4), & \text{if } i = 3,4, \\ 2(p-4), & \text{if } i = 5,6 \\ 3(p-4), & \text{if } i = 8 \end{cases}
\end{aligned}
$$

2.) When the two diagonals do not meet then we notice that there are several possible choices of the diagonals for $p > 5$, but we notice that the maximum number of i's which can appear for any choice of P is 8, since each diagonal appears twice thus there are only 4 possible ij's. We can thus identify these i's with $i = 1, ..., 8$ and define a mapping.

For example we provide such a mapping between the Stokes polytope corresponding to $Q = \{(14, 58)\}$ and the accordiohedra corresponding to $p = 5, 6$ (see (3.2))below:

p=5: With $P = \{(15, 610)\}$

$$
f(i) = \begin{cases} 1 \quad , & \text{if } i = 1 \\ (p-4), & \text{if } i = 2,3,4,5 \\ 2(p-4), & \text{if } i = 6 \\ 3(p-4), & \text{if } i = 7,8 \end{cases}
$$

p=6: With $P = \{(16, 914)\}$

$$
f(i) = \begin{cases} 1 \quad , & \text{if } i = 1 \\ (p-3), & \text{if } i = 2,3,4,5,6 \\ 2(p-3), & \text{if } \end{cases}
$$

It is straightforward to find such a mapping for general $p > 4$ and a general choice of P. [1]

4.2 Primitives for $n = 3$ case

We would now try to find all primitive graphs for the $n = 3$ case. In this case there are 4 vertices with $4p - 6$ legs. As before we can try to recursively construct primitive graphs from $n = 2$ case.

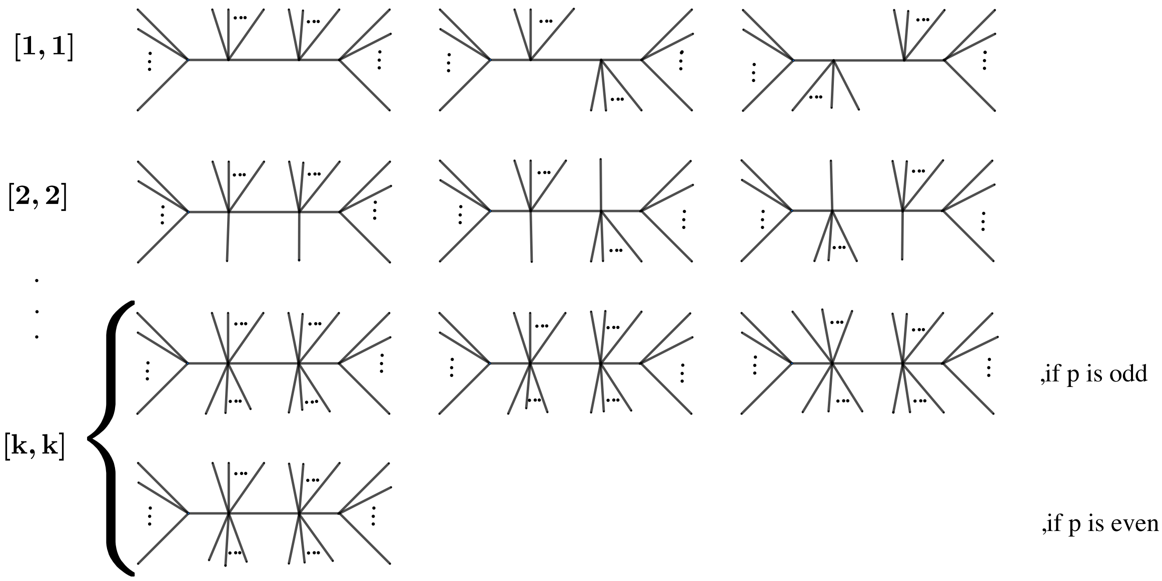


Figure 4.2: The primitive graphs of the type $[i, i]$.

There are two possible ways we add the 4th vertex which could be any of the $k = \left\lfloor \frac{p}{2} \right\rfloor$ types in the $n = 2$ case :

1. We can add another central vertex either above or below the central line to a $n = 2$ graph. We call these graphs $[i, i]$ and $[i, j]$. (see figures (4.2), (4.4)).

[1]It is a well known fact that there is a unique 2d convex polytope with a given number of vertices, so the above mapping is not really needed here but in the case of 3 and higher dimensional polytopes ,there could be several polytopes with same f-vector, thus we would need such a mapping to be sure that the polytopes are isomorphic.

2. We can add the vertex to any one of the external legs of the 3rd vertex. We denote these graphs by (k_1, k_2, k_3), where $k_1 + k_2 + k_3 = p - 3$

There are 3 primitives of the type $[i, i]$ for each $i = 1, ..., k$ since the graphs where both vertices are down are just cyclic permutations of the graph with both vertices up. In case where p is even you also have a vertex with equal number of legs above and below the central line, thus there is only one such primitive corresponding to this case. The graphs with one central vertex up and the other down (the 2nd and 3rd graphs for each $[i, i]$) have half periods i.e. under cyclic permutations they go back to themselves after $2p - 3$ operations. The same is also true for the symmetric vertex when p is even. All other graphs have full period of $4p - 6$.

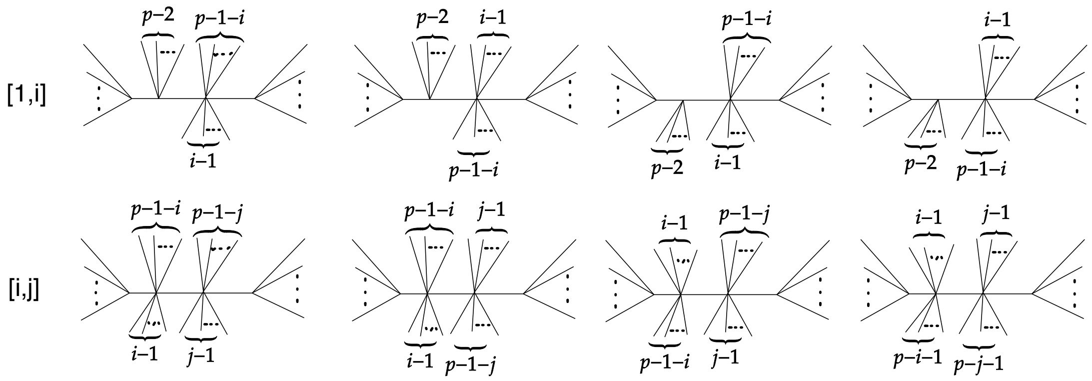


Figure 4.3: The primitive graphs of the type $[i, j]$.

There are 4 primitives for each $[i, j]$ since now the graphs with both vertices down are inequivalent to the ones with both vertices up under cyclic permutations. When p is even we have only 2 primitives of the type $[i, k]$ since $[k]$ is symmetric. All these graphs all have a period of $4p - 6$.

These possibilities are summarised in the table below:

We could also consider graphs of the type (k_1, k_2, k_3) such that $k_1 + k_2 + k_3 = p - 3$. Since there are $\binom{n-1}{r-1}$ non zero solutions of $x_1 + ... + x_r = n$, in this case we have the following possibilities:

Type of primitive	number of primitives	period of the primitive
$[i, i]$ with $i = 1, ..., k-1$	3	1 with $4p-6$ 2 with $\frac{4p-6}{2}$
$[k, k]$	1 , if p is even 3 , if p is odd	$\frac{4p-6}{2}$ 1 with $4p-6$ 2 with $\frac{4p-6}{2}$
$[i, j]$ with $i, j = 1, ..., k-1$ $i \neq j$	4	$4p-6$
$[i, k]$ with $i = 1, ..., k-1$	2 , if p even 4 , if p odd	$4p-6$ $4p-6$

Table 4.1: Primitive graphs of type 1.

- We have a graph of the type (0,0,p-3) with period $4p-6$.

- We can have a two graphs $(k_1, 0, k_3)$ and $(0, k_1, k_3)$ for each $k_1 + k_3 = p-3$ with $k_1 \neq k_3$ (which are inequivalent since they are reflections of each other) and when p is odd we also have one graph $(k_1, 0, k_3)$ with $k_1 = k_3$ with period $4p-6$. Thus, there are

$$\begin{cases} \binom{p-4}{1} \quad = (p-4), \; if \; p \; is \; odd \\ \binom{p-4}{1} - 1 = (p-5), \; if \; p \; is \; even \end{cases}$$

  such diagrams.

- We have one graph for each (k_1, k_2, k_3) with $k_1 = k_2 \neq k_3$ with period $4p-6$. In this case we have

$$\begin{cases} \lfloor \frac{p-3}{2} \rfloor, \qquad if \; p \neq 3k \\ \lfloor \frac{p-3}{2} \rfloor - 1, \; if \; p = 3k \end{cases}$$

  such diagrams.

- When $p = 3k$ we have exactly one graph with $k_1 = k_2 = k_3$ which has a period $\frac{4p-6}{3}$.
- We have two graphs (k_1, k_2, k_3) and (k_2, k_1, k_3) for each $k_1 \neq k_2 \neq k_3$. In this case we have

$$\begin{cases} \frac{\frac{(p-4)(p-5)}{2} - 3\lfloor \frac{p-3}{2} \rfloor}{3}, & if\ p \neq 3k \\ \frac{\frac{(p-4)(p-5)}{6} - 3\left(\lfloor \frac{p-3}{2} \rfloor - 1\right) - 1}{3}, & if\ p = 3k \end{cases}$$

These possibilities are summarised in the table below:

Type of primitive	number of primitives	period of the primitive
$(0, 0, p-3)$	1	$4p-6$
$(k_1, 0, k_3)$ $k_1 = k_3 = \frac{p-3}{2}$	1 , if p is odd 0 , if p is even	$4p-6$
$(k_1, 0, k_3)$ $k_1 \neq k_3$	$p-4$, if p is even $p-5$, if p is odd	$4p-6$
(k_1, k_2, k_3) $k_1 = k_2$	$\frac{p-4}{2}$, if p is even $p \neq 3k$ $\frac{p-5}{2}$, if p is odd $p \neq 3k$ $\lfloor \frac{p-3}{2} \rfloor - 1$, if $p = 3k$	$4p-6$ $4p-6$ $4p-6$
(k_1, k_2, k_3) $k_1 = k_2 = k_3$	1 , if $p = 3k$ 0 , otherwise	$\frac{4p-6}{3}$
(k_1, k_2, k_3) $k_1 \neq k_2 \neq k_3$	$\frac{(p-4)(p-5)}{6} - \frac{p-4}{2}$, if p is even $p \neq 3k$ $\frac{(p-4)(p-5)}{6} - \frac{p-5}{2}$, if p is odd $p \neq 3k$ $\frac{(p-4)(p-5)}{6} - \lfloor \frac{p-3}{2} \rfloor + \frac{2}{3}$, if $p = 3k$	$4p-6$ $4p-6$ $4p-6$

Table 4.2: Primitive graphs of type 2.

We can now find the total number of p-angulations by summing over all channels by multiplying columns two and three of the tables above and adding them up.

$$\text{number of p-angulations} = \sum_{\sigma} (\text{ primitive } \sigma) \times (\text{period of } \sigma)$$

The result of this exercise turns out to be $\frac{(p-1)(2p-3)(4p-5)}{3}$ which matches with the expected

Fuss-Catalan number which agrees with equation (3.6)

$$\frac{1}{4(p-2)+1}\binom{4(p-1)}{4} = \frac{(p-1)(2p-3)(4p-5)}{3}$$

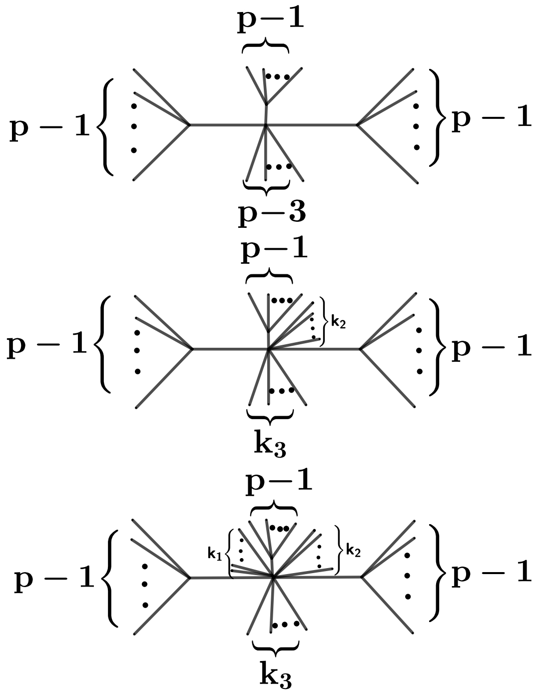


Figure 4.4: The primitive graphs of the type (k_1, k_2, k_3).

Since, there are 3 diagonals now the relative configuration of diagonals can be of one of the following types:

1. None of the diagonals meet - in this case the corresponding accordiohedron is a cube. There are $1 + \left\lfloor \frac{2(p-3)^2}{3} \right\rfloor$ graphs of this type namely $[i, i]$, $[i, j]$ with $i, j = 2, ..., k$ and (k_1, k_2, k_3) with $k_1, k_2, k_3 \neq 0$.

2. Two of the diagonals meet - in this case the corresponding accordiohedron is of the mixed type. There are $3p - 10$ graphs of this type namely $[1, j]$ with $j = 2, ..., k$ and $(k_1, 0, k_3)$ with $k_1, k_3 \neq 0$.

3. All three diagonals meet at a vertex or form zig-zag configuration in this case the corresponding accordiohedron is an associahedron. There are 3 graphs of this type namely $[1, 1]$.

4. All three diagonals meet and form an inverted U configuration in this case the corresponding accordiohedron is of the Lucas type. There is exactly one graph of this type which is $(0, 0, p-3)$.

Thus the total number of primitives is :

$$3p - 5 + \left\lfloor \frac{2(p-3)^2}{3} \right\rfloor = \left\lceil \frac{(p-1)(2p-1)}{3} \right\rceil$$

which agrees with we our general formula (3.6).

The accordiohedra for $n = 3$ we get continue to be one of the four kinds of Stokes polytopes. We could define a function from vertices of the Stokes polytopes to that of the accordiohedra as we had done in the $n = 2$ case to establish that this is indeed the case. We can thus continue to use the same names Lucas, Mixed etc for the $n = 3$ Stokes polytopes for accordiohedra as well. We expect that at sufficiently higher n, accordiohedra will be generated which do not correspond to any Stokes polytope.

4.3 Determination of the weights

In this section we shall provide a simple method to determine the weights for the general case and demonstrate the method in a few examples. We recall that we had the reduced amplitude $\tilde{\mathcal{M}}_n$ which is a weighted sum of canonical forms of all the primitive accordiohedra of a given dimension n. We would like to determine the weights such that this gives the full amplitude i.e. $\tilde{\mathcal{M}}_n = \mathcal{M}_n$.

The full amplitude $\mathcal{M}_n$ is given by :

$$\mathcal{M}_n = \sum_{all\ i_k j_k} \prod_{k=1}^{n} \frac{1}{X_{i_k j_k}}$$

where, the sum is over all $(i_1 j_1, ..., i_n j_n)$ that form a complete p-angulation.

Thus, to get the full amplitude from the partial amplitude we need to impose the constraint that each $\prod_{k=1}^{n} \frac{1}{X_{i_k j_k}}$ appears exactly once.

But, as we had emphasised before the accordiohedron depends only on the relative configuration of diagonals of the reference p-angulation which does not change under rotations and thus it is sufficient to impose these constraints for the primitive p-angulations.

$$\sum_{i=1}^{l} n_p^i \alpha_p^i = 1 \ for\ each\ primitive\ 1 \leq i \leq l$$

where, n_p^i is number of times primitive i appears in the vertices of all accodiohedra,

α_p^i are the corresponding weights

Since, we have managed to classify all the primitives unto $n = 3$ we should be able to implement this straightforward procedure to get all the weights and we shall now discuss our results.

We shall first see what these conditions are for $n = 2$ in the $p = 5, 6$ cases.

p = 5 : In this case there are two primitives as we had explained in the section (4.1) and we get:

$$\begin{aligned} 3\alpha_5^1 + \alpha_5^2 &= 1 \\ \alpha_5^1 + 4\alpha_5^2 &= 1 \end{aligned}$$

which can be solved to give:

$$\alpha_5^1 = \frac{3}{11}, \alpha_5^2 = \frac{2}{11}$$

p = 6 : In this case there are 3 primitives and we get:

$$\begin{aligned} 2\alpha_6^1 + \alpha_6^2 + 2\alpha_6^3 &= 1 \\ \alpha_6^1 + 4\alpha_6^2 &= 1 \\ \alpha_6^1 + 2\alpha_6^3 &= 1 \end{aligned}$$

which can be solved to give:

$$\alpha_6^1 = \frac{1}{3}\ \alpha_6^2 = \frac{1}{6}\ , \alpha_6^3 = \frac{1}{3}$$

We can similarly do this for any p with $n = 2$ and the results are the following :

For $p = 2k$

$$\alpha_{(p-2-i,i)} = \begin{cases} \frac{1}{6}\ , i\ even \\ \frac{1}{3}\ , i\ odd \end{cases}$$

and For $p = 2k + 1$

$$\alpha_{(p-2-i,i)} = \frac{k+1+i}{3p-4}$$

with $i = 0, ..., k - 1$.

The α's for $n = 3$ case with $p \leq 12$ are given below (for the sake of brevity we shall call α's corresponding to $[i, j], (k_1, k_2, k_3)$ as $[i, j], (k_1, k_2, k_3)$):

If p is even then :

$[i, i] = \frac{1}{24}, \frac{5}{24}, \frac{1}{24}, ...$; $[1, i] = \frac{3}{24}, \frac{1}{24}, \frac{3}{24}, ...$; $[2, i] = \frac{3}{24}, \frac{5}{24}, \frac{3}{24}, ...$; $[3, i] = \frac{3}{24}, \frac{1}{24}, \frac{3}{24}, ...$; ...

$(k_1, k_2, k_3) = (k_2, k_1, k_3) = \frac{6}{24}, \frac{2}{24}, \frac{6}{24}, ...$; $(k_1, 0, k_2) = (0, k_1, k_2) = \frac{2}{24}$, $(0, 0, p - 3) = \frac{2}{24}$

If p **is odd then the results for the first few cases are** :

p=5 : $[i,i] = \frac{1}{20}, \frac{3}{20}$ with $i = 1, 2$; $[1,2] = \frac{2}{20}$; $(1,1,0) = \frac{2}{20}$; $(0,0,2) = \frac{2}{20}$.

p=7 : $[i,i] = \frac{3}{64}, \frac{11}{64}, \frac{7}{64}$ with $i = 1, 2, 3$; $[1, j] = \frac{7}{64}, \frac{5}{64}$, with $j = 2, 3$; $[2,3] = \frac{9}{64}$; $(1,1,2) = \frac{10}{64}$,

$(0,1,3) = (1,0,3) = \frac{6}{64}$; $(2,0,2) = \frac{6}{64}$; $(0,0,4) = \frac{6}{64}$.

p=9 : $[i,i] = \frac{2}{44}, \frac{8}{44}, \frac{4}{44}, \frac{6}{44}$, with $i = 1, 2, 3, 4$; $[1, j] = \frac{5}{44}, \frac{3}{44}, \frac{4}{44}$, with $j = 2, 3, 4$; $[2, j] = \frac{6}{44}, \frac{7}{44}$, with $j = 3, 4$; $[3,4] = \frac{5}{44}$; $(2,2,2) = \frac{4}{44}$; $(1,1,4) = \frac{8}{44}$; $(1,2,3) = (2,1,3) = \frac{6}{44}$; $(3,0,3) = \frac{4}{44}$,

$(1,0,5) = (0,1,5) = \frac{6}{64}$; $(2,0,4) = (0,2,4) = \frac{4}{44}$; $(0,0,6) = \frac{4}{44}$.

p=11: $[i,i] = \frac{10}{112}, \frac{21}{112}, \frac{9}{112}, \frac{17}{112}, \frac{13}{112}$ with $i = 1, 2, 3, 4, 5$; $[1, j] = \frac{13}{112}, \frac{7}{112}, \frac{11}{112}, \frac{9}{112}$, with $j = 2, 3, 4, 5$;

$[2, j] = \frac{15}{112}, \frac{19}{112}, \frac{17}{112}$, with $j = 3, 4, 5$; $[3, j] = \frac{13}{112}, \frac{11}{112}$, with $j = 4, 5$; $[4,5] = \frac{11}{112}$; $(1,1,6) = \frac{22}{112}$;

$(2,2,4) = \frac{10}{112}$; $(3,3,2) = \frac{14}{112}$; $(0,4,4) = \frac{10}{112}$; $(1,0,7) = (0,1,7) = \frac{10}{112}$; $(2,0,6) = (0,2,6) = \frac{10}{112}$;

$(3,0,5) = (0,3,5) = \frac{10}{112}$; $(1,2,5) = (2,1,5) = \frac{14}{112}$; $(1,3,4) = (3,1,4) = \frac{14}{112}$; $(0,0,8) = \frac{5}{112}$.

List of Publications arising from the thesis[1]

Journal

1. "Stokes Polytopes : The positive geometry for ϕ^4 interactions"
   Pinaki Banerjee, Alok Laddha and ***Prashanth Raman***;
   JHEP **1908**(2019)067

2. "The positive geometry for ϕ^p interactions"
   Prashanth Raman;
   JHEP **1910**(2019)271

List of other Publications, Not included in the thesis

Journal

1. "Holographic Conformal Partial Waves as Gravitational Open Wilson Networks"
   Atanu Bhatta, ***Prashanth Raman*** *and Nemani V. Suryanarayana*;
   JHEP **1606** (2016) 119

2. "Scalar Blocks as Gravitational Wilson Networks"
   Atanu Bhatta, ***Prashanth Raman*** *and Nemani V. Suryanarayana*;
   JHEP **1812** (2018) 125

[1] As it is standard in the High Energy Physics Theory (hep-th) community the names of the authors on any paper appear in their alphabetical order.

Conferences and workshops attended

1. “Spring School on Superstring Theory and Related Topics” at “International Centre for Theoretical Physics”, Trieste from 28th March- 05th April, 2019.

2. “String Days IV: Soft Holography” a discussion meeting held at “Indian Institute of Science Education and Research(IISER)”, Pune from 02-04th March, 2019.

3. “Indian String Meeting 2018” at “Indian Institute of Science Education and Research(IISER)”, Trivandrum from 16-21 December 2018.

4. “Chennai Strings Meeting 2018” at “Institute of Mathematical Sciences”, Chennai 5-6 October, 2018.

5. “AdS/CFT at 20 and Beyond” a discussion meeting held at “International Centre for Theoretical Sciences”, Bengaluru from 21st May -2nd June, 2018.

6. “The 12th Kavli Asian Winter School on Strings, Particles and Cosmology” at “International Centre for Theoretical Sciences”, Bengaluru from 08-18th January, 2018.

7. “National String Meeting 2017” at “National Institute of Science Education and Research(NISER)”, Bhuvaneshwar from 05-10 December 2017.

8. “Student Talks on Trending Topics in Theory” at “Chennai Mathematical Institute” from 08-19 May,2017.

9. “School and Workshop on Modular Forms and Black Holes” at “National Institute of Science Education and Research”, Bhuvaneshwar from January 05 - 14, 2017.

10. “Indian Strings Meeting 2016” at “Indian Institute of Science Education and Research(IISER)”, Pune from 15-21 December, 2016.

Seminars presented

1. Invited talk in Oct, 2018 at the conference, **Chennai Strings Meeting**, Institute of Mathematical Sciences, Chennai, India.
   Talk title: “Scattering Forms and Stokes Polytopes."

2. Invited talk in December, 2018 at the conference, **Indian Strings Meeting**, Indian Institute of Science Education and Research, Trivandrum, India.
   Talk title: “Holographic Conformal Partial Waves as Gravitational Open Wilson Networks."

3. Invited seminar in March, 2019 at the conference, **Stringy Days IV: Soft Holography**, Indian Institute of Science Education and Research, Pune, India.
   Talk title: “Scattering Forms and Stokes Polytopes."

4. Invited seminar in April, 2019 at the University of Torino, Turin, Italy.
   Talk title: “Scattering Forms and Stokes Polytopes."

<u>Thesis Highlight</u>

Name of the Student: Prashanth Raman

Name of the CI/OCC:IMSc, Chennai **EnrolmentNo.: PHYS10201404003**

Thesis Title: Positive Geometry Of Scalar Theories

Discipline: Physical Sciences **Sub-Area of Discipline:** String Theory

Date of viva voce: 20/03/2020

The Amplituhedron is a remarkable space-time independent framework that has been developed by Arkani-Hamed and collaborators over the last decade. In this framework, each theory is associated to a putative family of geometric objects living kinematic space. This geometric object has a unique differential form associated to it called the canonical form which has logarithmic singularities on the boundary of the geometry. The scattering amplitude is obtained from the canonical form by pulling it back onto the geometry. In this formulation unitarity and locality emerge from properties of the geometry rather than being physical inputs to the theory. Locality emerges from the fact that the only physical poles correspond to boundaries of the geometry and Unitarity follows because each boundary of the geometry is a product of lower dimensional geometries of the same kind which directly implies the physical factorization of the corresponding amplitude. This framework was established for all loop N^kM H V amplitudes in N=4 SYM **and tree level and 1-loop amplitudes in bi-adjoint ϕ^3 theory.**

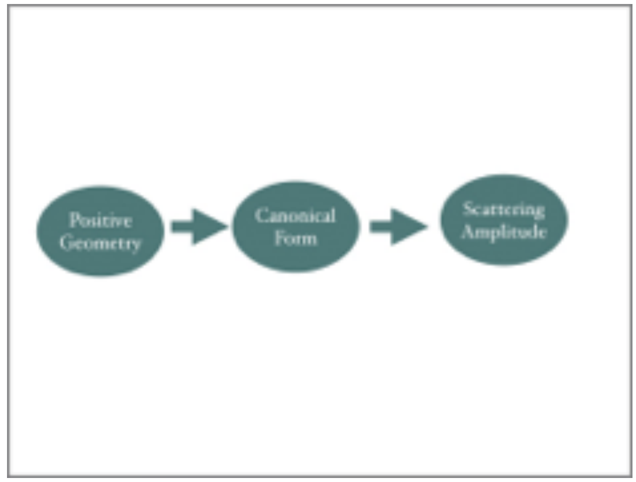


Figure 1.Schematic summarising the Amplituhedron Programme

In this thesis the formulation was extended to tree amplitudes in planar $\phi^p (p \geq 4)$ theories by establishing a precise connection between scattering forms and a polytope called `Stokes polytopes' for** p=4 **( more generally `Accordiohedron' for $p \geq 4$) living in kinematic space. It was shown in the thesis that unlike the case of ϕ^3 interactions there is no single simple polytope which can be the amplituhedron of these theories, and whose canonical form yields the planar ϕ^p amplitude.

However, there are several simple polytopes associated with the different topological classes of ordered p+(p-2)n**-point Feynman diagrams for each dimension** n **which can be embedded in kinematic space. A weighted sum of the canonical forms of all these accordiohedra does indeed give the right planar amplitude. The weights are unique positive rational numbers and can be determined by a simple prescription provided in the thesis.**