Surface Operators, Seiberg-Dual Quivers and Contours

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis[1]

Journal

1. "Surface operators in $N = 2$ SQCD and Seiberg Duality"

   Sujay K. Ashok, ***Sourav Ballav****, Marialuisa Frau, Renjan Rajan John*;

   Eur. Phys. J. C (2019) 79: 372

2. "Modular properties of surface operators in $N = 2$ SQCD"

   Sourav Ballav*, Renjan Rajan John*;

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Journal

1. "Surface operators, dual quivers and contours"

   S.K. Ashok, ***S. Ballav*** *, M. Billo , E. Dell'Aquila , M. Frau, V. Gupta ,R.R. John, A. Lerda*;

   Eur. Phys. J. C (2019) 79: 278

Sourav Ballav

[1] As it is standard in the High Energy Physics Theory (hep-th) community the names of the authors on any paper appear in their alphabetical order.

Contents

List of Figures

Chapter 6

Conclusion

In the first part of the thesis, based on [2] we study the relation between two distinct realization of surface operators in $\mathcal{N} = 2$ SQCD as monodromy defects and flavor defects. We show that contours specified by a particular Jeffrey-Kirwan residue prescription in the localization analysis map to particular realizations of the surface operator as flavour defects. The localization integrand in the asymptotically conformal case differs from that of the pure theory only in the structure of the numerator and hence the set of poles picked by a given JK vector remains the same as in the theory without flavours. As a result, on the quiver side, the ranks of gauge nodes in the quiver remain the same. The ranks of flavour nodes are uniquely fixed by how the flavour symmetry is broken by the defect and by requiring conformality at each 2d gauge node. For each contour choice, we propose how to construct a 2d/4d quiver theory whose twisted superpotential, when evaluated on the solutions of the twisted chiral ring equations, matches the localization result after a suitable map of parameters. Some work in this direction appeared recently in [34], but our analysis of Seiberg duality has significant differences.

As in the case of the pure 4d gauge theory, distinct contour choices are equivalent and give rise to Seiberg-dual 2d/4d gauge theories. However, there is a new feature in the asymptotically conformal SQCD case: due to a non-vanishing residue at infinity, distinct contours are inequivalent. While the prepotential obtained from the instanton partition function is independent of the contour of integration, the twisted superpotential turns out to be different for distinct contour choices.

Our main focus has been understanding Seiberg duality for surface defects in SQCD which is consistent with such inequivalent contours . The ranks and connectivity of the quivers one gets by Seiberg duality are exactly those that correspond to the different JK prescriptions, but the effective twisted superpotentials on the 4d Coulomb branch for different JK prescriptions are not trivially related. The resolution to this is known for the case of 2d gauge theories in which the flavour group is not gauged [10]: the Lagrangian of the dual theory is modified by non-perturbative corrections. Our main result in this work is a proposal for a generalized Seiberg duality rule with further non-perturbative terms that applies to the case of surface operators realized as flavour defects. We derive this from the localization integrand by a careful analysis of the residue at infinity. With the modified duality rules that now also involve the 4d gauge coupling, the twisted superpotentials evaluated on the solutions of the chiral ring equations match for all dual pairs of theories.

In the second part of thesis, we considered surface defects in SU(2) theory with four fundamental flavours and studied modular properties the twisted chiral superpotential [6]. We matched the results for the superpotential obtained from localization methods and from the Seiberg-Witten data. The coefficients in the mass expansion satisfy a modular anomaly equation that allows one to solve for them in an iterative manner in terms of (quasi-) modular and elliptic functions. A key input here is the explicit localization results that are crucial to fix the purely modular and elliptic contributions. While such an equation was known for the $\mathcal{N} = 2^\star$ theory, the main difference now is that the variables in terms of which the resummation is done are not the bare couplings but the renormalized ones. This required us to write down the map that relates the bare and the renormalized variables. The map is verified using the Seiberg-Witten analysis in Appendix E.

In [14] it was shown that for the SU(2) $N_f = 4$ theory the instanton partition function in the presence of the defect is reproduced by a 4-point spherical conformal block in Liouville CFT with the insertion of a degenerate primary. This was studied in great detail in [15] and we have checked up to $n = 6$ that our resummed results for $\widetilde{w}'_n$ match the results one would obtain following the CFT analysis in [15].

Synopsis

Motivation and Introduction

Defects or nonlocal operators in QFT are disturbances supported on a submanifold in spacetime. They are good theoretical tools for studying non-perturbative effects of gauge theories and dualities. Defects are classified by the dimension of their support. Line operators (e.g. Wilson lines, 't Hooft loops) which are one dimensional defects, were first introduced to study the phase structure of gauge theories. Surface operators are 2-dimensional generalisations of 't Hooft and Wilson lines in gauge theories. Surface operators were first introduced in $\mathcal{N} = 4$ super Yang-Mills theories in [1] as solutions to Hitchin equations with isolated singularities on a two-dimensional submanifold of the four dimensional space-time. They serve as order parameters of the gauge theories. Once inserted in the path integral their correlation function give us valuable information about non-perturbative aspects of the gauge theory. Surface operators can also distinguish some topological phases of the gauge theory which line operators can not detect.

In this thesis, we study surface operators in $\mathcal{N} = 2$ supersymmetric QCD theories with gauge group SU(N) and 2N fundamental flavours in four dimensions [2]. The matter content of the theory ensures that it is conformal in the limit that the flavour masses are zero and is referred to as asymptotically conformal SQCD. The low energy physics of the gauge theory on the Coulomb branch in the presence of the defect is described by two holomorphic functions, the prepotential and the twisted chiral superpotential. While the prepotential describes the effective four dimensional theory in the absence of a defect, the twisted chiral superpotential describes the effective theory on the defect. So, our focus will be on computing the twisted superpotential and analysing it. We study surface operators following two different approaches, namely as monodromy defects [1,3]and flavor defects [4,5]. Our goal is to clarify the relationship between these different approaches of surface operators. In addition, we want to clarify Seiberg duality in the context of surface operators in $\mathcal{N} = 2$ SQCD.

We also study modular properties of simplest possible surface operator in four dimensional $\mathcal{N} = 2$ SQCD with gauge group SU(2) and four fundamental flavours [6]. It is well known that this theory enjoys S-duality. Using the constraints imposed by the S-duality, we show that the instanton contribution to the twisted chiral superpotential can be resummed into elliptic functions and (quasi-) modular forms of the duality group.

Surface operators as monodromy defects

In this chapter, we discuss surface operators as monodromy defects in $\mathcal{N} = 2$ SU(N) gauge theory with $2N$ fundamental flavours in four dimension. In this approach, the defect is defined by introducing a singularity structure for the gauge field near the location of the defect. If $r\,e^{i\theta}$ is the coordinate of the plane transverse to the defect D, then, as $r \to 0$, the gauge field has the following behaviour [7]:

$$A \sim \text{diag}\Big(\underbrace{\alpha_1,\ldots,\alpha_1}_{n_1},\underbrace{\alpha_2,\ldots,\alpha_2}_{n_2},\ldots\underbrace{\alpha_M,\ldots,\alpha_M}_{n_M}\Big)d\theta\,,$$

where the M integers n_I satisfy

$$\sum_{I=1}^{M} n_I = N\,.$$

In the path integral, one then integrates over all gauge field configurations with this prescribed singular boundary condition. In the presence of a surface operator the gauge group $SU(N)$ is broken to the Levi subgroup at the location of the defect:

$$\mathbb{L} = S\left[U(n_1) \times U(n_2) \times \ldots \times U(n_M)\right] .$$

In the path integral, one can also add a 2d topological term :

$$\exp\Big(2\pi\mathrm{i} \sum_{I=1}^{M} \eta_I \int_D \mathrm{Tr}\, F_{\mathrm{U}(n_I)}\Big)$$

where η_I's are constant parameters.

A surface operator is thus specified by the discrete lables $[n_1, n_2, \ldots, n_M]$ and the continous parameters : $(\alpha_1, \ldots, \alpha_M)$ and $(\eta_1, \ldots, \eta_M)$.

In the asymptotically conformal SQCD theory, surface operators also breaks the SU($2N$) flavour symmetry to the following subgroup at the location of the defect:

$$\mathbb{F} = S\left[U(n_1 + n_2) \times U(n_2 + n_3) \times \ldots \times U(n_M + n_1)\right] . \tag{1}$$

Due to supersymmetry, the prepotential $\mathcal{F}$ and the twisted chiral superpotential $\mathcal{W}$ receive three different contributions from classical, 1-loop, and instanton terms :

$$\mathcal{F} = \mathcal{F}^{\text{class}} + \mathcal{F}^{\text{1loop}} + \mathcal{F}^{\text{inst}}$$
$$\mathcal{W} = \mathcal{W}^{\text{class}} + \mathcal{W}^{\text{1loop}} + \mathcal{W}^{\text{inst}}$$

To obtain the instanton contributions to $\mathcal{F}$ and $\mathcal{W}$, we first compute the instanton partition function $\mathcal{Z}^{\text{inst}}[\vec{n}]$ in a Omega-deformed background using equivariant localization technique [8] in the presence of the surface defect. In the vanishing limit of Ω-deformation parameters ϵ_1 and ϵ_2 :

$$\lim_{\epsilon_i \to 0} \log(1 + \mathcal{Z}^{\text{inst}}[\vec{n}]) = -\frac{\mathcal{F}^{\text{inst}}}{\epsilon_1 \hat{\epsilon}_2} + \frac{\mathcal{W}^{\text{inst}}}{\epsilon_1} \tag{2}$$

where $\hat{\epsilon}_2 = \frac{\epsilon_2}{M}$.

The instanton partition function in the presence of such a surface operator, is given by a multi-dimensional contour integral [2]:

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_I\}} Z_{\{d_I\}}[\vec{n}] \quad \text{with} \quad Z_{\{d_I\}}[\vec{n}] = \prod_{I=1}^{M} \Big[\frac{(-q_I)^{d_I}}{d_I!} \int \prod_{\sigma=1}^{d_I} \frac{d\chi_{I,\sigma}}{2\pi \mathrm{i}} \Big] \, z_{\{d_I\}} \tag{3}$$

where

$$\begin{aligned} z_{\{d_I\}} = & \prod_{I=1}^{M} \prod_{\sigma,\tau=1}^{d_I} \frac{(\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma,\tau})}{(\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1)} \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{\rho=1}^{d_{I+1}} \frac{\big(\chi_{I,\sigma} - \chi_{I+1,\rho} + \epsilon_1 + \hat{\epsilon}_2\big)}{\big(\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\epsilon}_2\big)} \\ & \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \frac{\prod_{i \in \mathcal{F}_I} (\chi_{I,\sigma} - m_i)}{\prod_{s \in N_I} \big(a_s - \chi_{I,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big) \prod_{t \in N_{I+1}} \big(\chi_{I,\sigma} - a_t + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)} \, . \end{aligned} \tag{4}$$

Equation (4) is our proposal for the integrand for SQCD. In this case, the denominator and its singularity structure is same as that of pure gauge theory [9], the fundamental flavours only add factors in the numerator of the instanton partition function. To calculate the partition function one has to specify the contour of integration for the χ_I variables. The contour is specified by integrating $\chi_{I,\sigma}$ in the upper or lower half-plane and by choosing a definite order in the successive integrations. Equivalently, the contour of integration can be selected by specifying a Jeffrey-Kirwan reference vector. Localization methods therefore allows one to calculate the instanton partition function order by order in the instanton expansion. The new feature is that there are M instanton counting parameters q_I in this case. The product of all of these turn out to be related to the usual 4d instanton counting parameter.

$$q_{4d} = \prod_{I=1}^{M} q_I \, . \tag{5}$$

Due to this fractionation, the partition function in the presence of the surface operator is referred to in the mathematical literature as the ramified instanton partition function. The M positive integers d_I count the numbers of ramified instantons in the various sectors of the partition function.

Surface operators as flavor defects

In this chapter, we discuss surface operators as flavor defects. In this approach, surface operators are described as coupled 2d/4d systems realized as quiver gauge theories . To describe a surface operator of Levi type in pure SU(N) theory one considers a σ-model with target space $\mathcal{M} = \frac{SU(N)}{\mathbb{L}}$. It can be realized as the low-energy limit of a GLSM , whose gauge and matter content can be summarized in a quiver diagram like in Fig. 3.1.

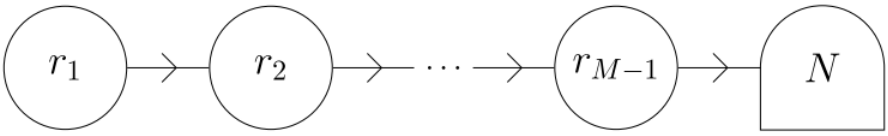


Figure 1: The quiver which describes the generic surface operator in pure SU(N) gauge theory.

Here, $r_I = n_1 + n_2 + ... + n_I$. This is a U($r_1$)× U($r_2$) $\times \ldots$ gauge theory in 2d with bi-fundamental matter and SU(N) flavour group. The Coulomb vevs of the 4d theory act as twisted masses for the chiral matter fields. So these can be integrated out, leading to an effective low energy theory in the infrared for the vector multiplet in 2d. The effective action of such a theory can be encoded in the effective twisted chiral superpotential and is given by [9]:

$$\mathcal{W} = 2\pi \mathrm{i} \sum_{I=1}^{M-1} \sum_{s=1}^{r_I} \tau_I \sigma_s^{(I)} - \sum_{I=1}^{M-2} \sum_{s=1}^{r_I} \sum_{t=1}^{r_{I+1}} \varpi(\sigma_s^{(I)} - \sigma_t^{(I+1)}) - \sum_{s=1}^{r_{M-1}} \Big\langle \mathrm{Tr}\, \varpi(\sigma_s^{(M-1)} - \Phi) \Big\rangle \tag{6}$$

where

$$\varpi(x) = x \left(\log \frac{x}{\mu} - 1 \right) ,$$

μ is the UV cut-off scale, and τ_I is the complexified FI parameter of the I^{th} node at the scale μ. The angular brackets in the last term of (3.1) correspond to a chiral correlator in the 4d SU(N) theory which implies that the coupling between the 2d and 4d theory is via the resolvent of the SU(N) gauge theory [5].

We then obtain the massive vacua by extremizing $\mathcal{W}(\sigma_s^{(I)})$ i.e. they are solutions of the twisted chiral ring equations:

$$\exp\left(\frac{\partial \mathcal{W}}{\partial \sigma_s^{(I)}}\right) = 1 \tag{7}$$

The evaluation of the twisted chiral superpotential on the $\sigma_{s,\star}$ that solves the chiral ring equations turns out to be identical to the $\mathcal{W}$ evaluated from the localization calculation.

In SQCD, the matter multiplets also provide flavours to the 2d gauge nodes. The additional constraint we impose is that for every quiver the complexified FI parameters of the 2d gauge nodes should not run, so that the 2d gauge theories are conformal. Combined with the breaking of the flavour symmetry to $\mathbb{F}$ as in (1) for a generic surface operator, this uniquely fixes how the broken 4d flavour group acts on the 2d gauge nodes. For a generic surface operator $[n_1, n_2, \ldots n_M]$ we show a particular 2d/4d realization below :

Now we follow the same prescription as the pure gauge theory for SCQD. We write down the effective action and extremize it to find the twisted chiral ring equations. We then solve them to find the massive vacuum. The evaluation of the twisted chiral superpotential in the massive vacuum once again reproduces the twisted superpotential $\mathcal{W}_{\text{inst}}$ calculated using localization.

Contour vs quiver and Seiberg duality

In this chapter, we discuss the precise relationship between two different formulations of surface operators discussed earlier. In an earlier work [9] on surface operators in the

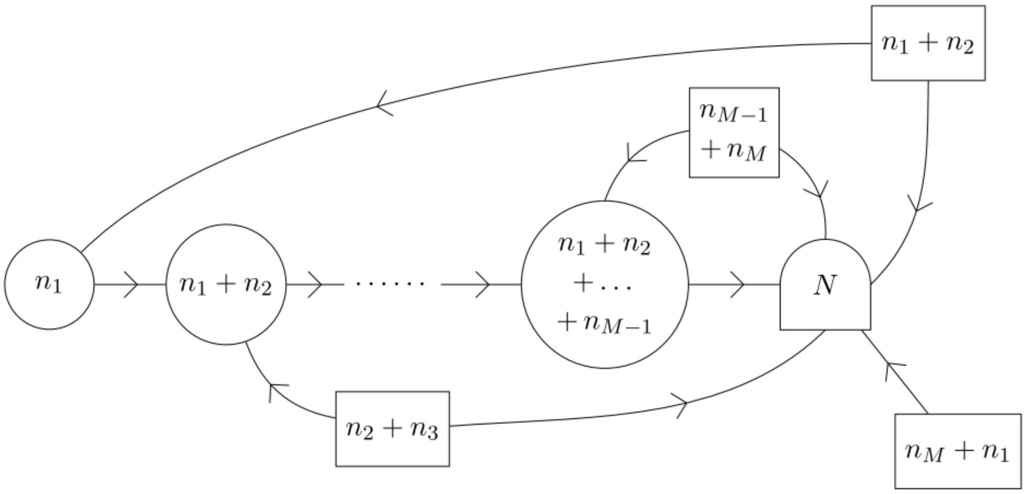


Figure 2: One realization of the $[n_1, n_2, \dots n_M]$ surface operator as a 2d/4d quiver in which the 2d gauge nodes are oriented. The 4d flavour groups provide matter for the 2d nodes such that the β-function of each FI parameter is zero.

pure 4d theory, it was shown that there is a one-to-one correspondence between massive vacua of the 2d/4d theory and surface operators as monodromy defects. For a 2d/4d quiver, the evaluation of the twisted chiral superpotential in a particular massive vacuum reproduces the twisted superpotential $\mathcal{W}_{\text{inst}}$ calculated using localization. In particular, a precise correspondence between different integration contour prescriptions in the ramified instanton partition function for a monodromy defect and a particular 2d/4d quiver was established.

In extending this correspondence to the asymptotically conformal SQCD case, we observe that the fundamental flavours only add factors in the numerator of the instanton partition function, leaving the denominator and its singularity structure unchanged. It therefore follows that the quiver we may associate to a given integration contour has the same 2d gauge content of the one in the corresponding case without flavour. Given a 2d/4d quiver, the evaluation of the twisted chiral superpotential in the chosen vacuum once again reproduces the twisted superpotential $\mathcal{W}_{\text{inst}}$ calculated using localization and with the particular JK prescription associated to the 2d/4d quiver.

In this context, there is an interesting issue that arises. There can be many equivalent 2d quivers that can be coupled to the $SU(N)$ flavour group. By equivalence it is meant that the infrared behaviour of the 2d quivers (in this case, massive vacua) can be mapped one-to-one, onto each other. Such quivers are related by 2d Seiberg duality. In this Chapter, we also study this issue of Seiberg duality in the 2d/4d quiver realization of the defect in SQCD and propose a relation between the twisted superpotentials of dual quivers. Seiberg duality has been studied in the pure 4d theory in [9]. It was shown that there can be different 2d/4d quivers which realize the same flavour defect and are related by 2d Seiberg duality [10]. Seiberg duality is an infrared equivalence such that for dual quivers the low energy effective superpotentials, evaluated in particular vacua, match. On the localization side, we propose that each such Seiberg dual realization of the surface operator is associated to a contour prescription and residue theorems guarantee the equality of the low energy effective superpotentials. In the case of the pure 4d gauge theory, distinct contour choices are equivalent and give rise to Seiberg-dual 2d/4d gauge theories.

However, in the asymptotically conformal SQCD case due to a non-vanishing residue at infinity, distinct contours are inequivalent. While the prepotential obtained from the instanton partition function is independent of the contour of integration, the twisted superpotential turns out to be different for distinct contour choices. Our main result is a proposal for a generalized Seiberg duality rule that the Lagrangian of the dual theory is modified by non-perturbative corrections that applies to the case of surface operators realized as flavour defects. The appearance of non-perturbative terms in the superpotential of the dual theory is in part already known in 2d conformal gauge theories [10]. Our result extends it to the case when the flavor node is gauged in the 2d/4d quiver realization of surface operator. When the flavor node is gauged, the twisted superpotential of the dual quiver is modified by the following non-perturbative correction [2] :

$$\delta W = \left[\log\left(1 - (-1)^{N_f}\, x\right) + \log\left(1 - (-1)^{N_f} \frac{q_0}{x}\right) \right] (\mathrm{Tr}\, \widetilde{m} - \mathrm{Tr}\, m)\,, \tag{8}$$

where x is the exponentiated FI parameter of the 2d gauge node that is dualized, and $\mathrm{Tr}\, m$

and $\mathrm{Tr}\,\widetilde{m}$ denote respectively the sum of twisted masses for all N_f fundamental and N_f anti-fundamental flavours attached to that node.

Modular properties of Surface operator

In this chapter, we study half-BPS surface operators in $\mathcal{N} = 2$ supersymmetric QCD in four dimensions with gauge group SU(2) and four fundamental flavours. It is well known from [11] that the SU(2) theory with $N_f = 4$ enjoys an S-duality symmetry. A lot of progress has been made in resumming the instanton contribution to the prepotential of a large class of theories into (quasi-) modular forms of their respective S-duality groups [12]. This was then extended to the case of the twisted chiral superpotential of $\mathcal{N} = 2^\star$ SU(N) theory in the presence of a surface defect in [13]. Here we do the same for SQCD. We use the constraints imposed by S-duality to show that $\mathcal{W}$ satisfies a modular anomaly equation and that the instanton expansion of $\mathcal{W}$ at each order in a mass expansion can be written in terms of elliptic functions and (quasi-) modular forms.

We compute the twisted chiral superpotential using equivariant localization as well as the Seiberg-Witten data. The instanton contributions to $\mathcal{F}$ and $\mathcal{W}$ are obtained from the instanton partition function $\mathcal{Z}^{\text{inst}}[1,1]$ as before using localization with choice of contour in the upper-half complex plane. In order to confirm the results for the twisted superpotential obtained via localization, we compute the same from the Seiberg-Witten (SW) curve of the gauge theory.

In [14], it was proposed that the twisted superpotential can be computed from the Seiberg-Witten curve and is given by the integral of the SW differential λ along an open path on the SW curve :

$$\mathcal{W}(x_0) = \int^{x_0} \lambda\,, \tag{9}$$

where x_0 is the continuous parameter that labels the surface operator, and is given by the location of the defect on the Riemann surface. The SW differential is obtained from the

Gaitto form of the curve as

$$\lambda = x\,dt = \sqrt{\phi_2(t)}dt\,. \tag{10}$$

where the Gaiotto form of the curve is

$$x^2 = \phi_2(t) \tag{11}$$

The twisted superpotential is obtained by performing the integral with the Gaiotto form of the SW curve for SU(2) theory with $N_f = 4$ flavours. We check that the instanton contributions to $\mathcal{W}$ obtained via localization matches the results from the SW data , provided one uses a suitable map between the instanton counting parameters (q_1, q_2) and the gauge theory parameters (q_0, x_0).

To resum the instanton contributions to $\mathcal{W}$ one expands the twisted superpotential in a mass expansion. It is convenient to work with the log x derivative of $\widetilde{\mathcal{W}}$ whose expansion is :

$$x\frac{\partial\widetilde{\mathcal{W}}}{\partial x} \equiv \widetilde{\mathcal{W}}' = \sum_{n=0}^{\infty} \widetilde{w}'_n \tag{12}$$

where $\widetilde{w}'_n \sim a^{1-n}$.

Using the S-duality symmetry of SQCD we show that the $\widetilde{w}'_n$ in (5.25) obey a modular anomaly equation :

$$\frac{\partial\widetilde{w}'_n}{\partial E_2} + \frac{1}{12}\sum_{l=0}^{n-1}\left(\frac{\partial\widetilde{w}'_\ell}{\partial a}\right)\left(\frac{\partial\widetilde{f}_{n-\ell}}{\partial a}\right) = 0 \tag{13}$$

To solve the modular anomaly equation one needs to fix the coefficients of the modular pieces that occur as integration constants. This we do by appealing to the explicit localization results. The final resummed results for low values of n are given as follows:

$$\widetilde{w}_2' = -\frac{1}{6a}\sum_{A=0}^{3} M_A^2 \left(E_2 + 12\widehat{\wp}(z+\omega_A)\right)$$

$$\begin{aligned}\widetilde{w}_4' &= -\frac{1}{72a^3}\Big(\sum_{A=0}^{3} M_A^4 \left(2E_2^2 - E_4 + 24E_2\widehat{\wp}(z+\omega_A) + 144\widehat{\wp}(z+\omega_A)^2\right)\\ &+ 2\sum_{A<B} M_A^2 M_B^2\Big(2E_2^2 - E_4 + 12E_2\widehat{\wp}(z+\omega_A) + 12E_2\widehat{\wp}(z+\omega_B)\\ &+ 144\widehat{\wp}(z+\omega_A)\widehat{\wp}(z+\omega_B)\Big) - 12T_1\theta_4^4(E_2 - 2\theta_2^4 - \theta_4^4) + 12T_2\theta_2^4(E_2 + \theta_2^4 + 2\theta_4^4)\Big)\end{aligned} \quad (14)$$

The resummed results match what one would obtain from the description of surface operators as the insertion of a degenerate operator in a spherical conformal block in Liouville CFT [15].

Conclusion

In this thesis, we study half-BPS surface operators in $\mathcal{N} = 2$ supersymmetric asymptotically conformal gauge theories in four dimensions with SU(N) gauge group and 2N fundamental flavours using localization methods and coupled 2d/4d quiver gauge theories. We show that contours in the localization analysis map to particular realizations of the surface operator as flavour defects. We study Seiberg duality of 2d/4d quivers. Dual quivers are mapped to contour deformations of the localization integral which involves a residue at infinity. The Lagrangian of the dual theory gets shifted by non-perturbative terms, which is referred to as modified Seiberg duality rule.The new rules, that depend on the 4d gauge coupling, lead to a match between the low energy effective twisted chiral superpotentials for any pair of dual 2d/4d quivers. We also study modular properties of half-BPS surface operators in $\mathcal{N} = 2$ SQCD in four dimensions with gauge group SU(2) and four fundamental flavours. We compute the twisted chiral superpotential that describes the effective theory on the surface operator using equivariant localization as well as the Seiberg-Witten data. We then use the constraints imposed by S-duality to resum the instanton contributions to the twisted superpotential into elliptic functions and (quasi-) modular forms.

Plan of the thesis

In this thesis, we study surface operators in $\mathcal{N} = 2$ SQCD theories with gauge group SU(N) and 2N fundamental flavours in four dimensions.

- Chapter 1 will provide general introduction and motivation for studying surface operators.
- Chapter 2 will discuss surface operators as monodromy defects in $\mathcal{N} = 2$ SQCD theories.
- Chapter 3 will discuss surface operators as Flavor defects in $\mathcal{N} = 2$ SQCD theories.
- Chapter 4 will discuss the relation between the above two approaches and Seiberg duality.
- Chapter 5 will study modular properties of surface operators in $\mathcal{N} = 2$ SQCD with SU(2) gauge group and four fundamental flavours.
- Chapter 6 will conclude with a discussion of the results as well as open problems.

Chapter 1

Motivation and Introduction

Defects or nonlocal operators in QFT are disturbances supported on a submanifold in spacetime. They are good theoretical tools for studying non-perturbative effects of gauge theories and dualities. Defects are classified by the dimension of their support. Line operators (e.g. Wilson lines, 't Hooft loops) which are one dimensional defects, were first introduced to study the phase structure of gauge theories. Surface operators are 2-dimensional generalisations of 't Hooft and Wilson lines in gauge theories. Surface operators were first introduced in $\mathcal{N} = 4$ super Yang-Mills theories in [1] as solutions to Hitchin equations with isolated singularities on a two-dimensional submanifold of the four dimensional space-time. They serve as order parameters of the gauge theories. Once inserted in the path integral their correlation function give us valuable information about non-perturbative aspects of the gauge theory. Surface operators can also distinguish some topological phases of the gauge theory which line operators can not detect.

In this thesis, we have two parts. In the first part, we study surface operators in $\mathcal{N} = 2$ SQCD theories with gauge group SU(N) and $2N$ fundamental flavours in four dimensions. The condition on the number of fundamental flavours ensures that in the limit they are massless the theory is super-conformal at the quantum level. We will refer to these as asymptotically conformal gauge theories. Our interest is in the low-energy effective action of such theories on the Coulomb branch, in the presence of a surface defect. This effective action is encoded in two holomorphic functions: the prepotential, which describes the four dimensional (4d) dynamics without the defect, and the twisted chiral superpotential, which describes the dynamics of the two dimensional (2d) theory on the defect.

In our study of surface operators we follow two approaches. In the first approach, we consider the ramified instanton partition function Z_{inst}, which is obtained by a suitable orbifold of the instanton moduli space of the 4d SQCD theory without the defect [8] (see also [13] for details). One way to realize the instanton moduli space is by considering the open string excitations of D(-1)/D3/D7-brane systems in an orbifold of type IIB string theory. In this realization, the ramified instanton moduli are open strings with at least one end-point on the D(-1)-branes and, using localization techniques, the partition function Z_{inst} can be written as a contour integral over those moduli which represent the position of the D(-1)-branes in the directions transverse to both the D3 and the D7-branes. For a particular contour whose residues have an interpretation as Young tableaux, this is interpreted as the partition function of a monodromy defect in the gauge theory [1, 3]. Such surface defects are labelled by Levi subgroups of SU(N) which are classified by partitions of N. For asymptotically conformal SQCD, it turns out that the flavour group SU($2N$) is also broken at the location of the defect into M factors, whose ranks are determined by the same partition of N. Both the prepotential and the twisted chiral superpotential on the Coulomb branch can then be extracted from Z_{inst} in the limit of vanishing Ω-deformation parameters [7, 14].

In the second approach, we describe surface defects as flavour defects, which are coupled 2d/4d systems realized as quiver gauge theories [4, 5]. Here the 2d sector is a (2,2) theory described by a gauged linear sigma model in the ultraviolet, in which the vacuum expectation values of the adjoint scalar of the 4d theory act as twisted masses [16, 17]. The 2d theory has a discrete set of massive vacua determined by solutions of twisted chiral ring equations that extremize the twisted superpotential. Our goal is to understand the precise relationship between these two distinct realization of surface operators, namely as monodromy defects and flavor defects.

Work along this direction has been pursued in the pure 4d theory in [9, 13, 18–20]. One of the main results in [9] is that there can be different 2d/4d quivers which realize the same flavour defect and are related by 2d Seiberg duality [10]. Seiberg duality is an infrared equivalence such that for dual quivers the low energy effective superpotentials, evaluated in particular vacua, match. These statements are reflected on the localization side in an elegant way: each Seiberg dual realization of the surface operator is associated to a contour prescription and residue theorems guarantee the equality of the low energy effective superpotentials. The contours are specified by the Jeffrey-Kirwan (JK) prescription [21] and each reference JK vector associated to a given 2d/4d quiver can be written unambiguously in terms of its Fayet-Iliopoulos (FI) parameters. So, in the case of the pure 4d gauge theory, distinct contour choices are equivalent and give rise to Seiberg-dual 2d/4d gauge theories. However, in the asymptotically conformal SQCD case: due to a non-vanishing residue at infinity, distinct contours are inequivalent. While the prepotential obtained from the instanton partition function is independent of the contour of integration, the twisted superpotential turns out to be different for distinct contour choices. Our main focus is to understand how Seiberg duality can be consistent with such inequivalent contours in the context of surface defects in asymptotically conformal SQCD.

In the second part of the thesis, we study modular properties of simplest possible surface operator in four dimensional $\mathcal{N} = 2$ SQCD with gauge group SU(2) and four fundamental flavours [6]. It is well known that this theory enjoys S-duality. In [12, 22–25], S-duality was used to constrain the prepotential of $\mathcal{N} = 2^{\star}$ theories with classical and exceptional gauge groups. The instanton expansion of the prepotential was resummed to a mass expansion such that the expansion coefficients were expressed as linear combinations of (quasi-) modular forms of the duality group. This was then done for asymptotically conformal SQCD with fundamental matter in [26, 27]. This program was later extended to the case of a gauge theory with a surface defect in $\mathcal{N} = 2^\star$ SU(N) theory to constrain the twisted superpotential [13]. We want to extend this one step further by using S-duality to constrain the twisted superpotential of the SU(2) theory with $N_f = 4$ fundamental flavours and resum the instanton contributions to the twisted superpotential.

The rest of the thesis is organized as follows. In Section 2, we introduce surface operators as monodromy defects and write the localization integrand from which the instanton partition function is obtained after specifying a contour of integration. In Section 3, we introduce surface operators as flavor defects. In section 4, we relate the different contours of integration to distinct 2d/4d quivers by studying the $[p, N - p]$ defect and we propose a generalized Seiberg duality move and show in the case of the simplest quiver in what manner the Lagrangian for the dual quiver is corrected from the perturbatively exact 1-loop result by non-perturbative terms. We analyze the 3-node quiver in detail and test successfully the rules laid out. In section 5, we study modular properties of simplest possible surface operator in four dimensional $\mathcal{N} = 2$ SQCD with gauge group SU(2) and four fundamental flavours [6]. Using the constraints imposed by the S-duality, we show that the instanton contribution to the twisted chiral superpotential can be resummed into elliptic functions and (quasi-) modular forms of the duality group. We provide a derivation of our proposal using localization methods in Appendix A , collect some details of the computations in the Appendices B and C, give some technical details on elliptic functions and modular forms in appendix D and verify the map between the resummed and the bare variables in appendix E.

Chapter 2

Surface operators as monodromy defects

In this chapter, first we briefly review surface operators as monodromy defects in pure $\mathcal{N} = 2$ theory with SU(N) gauge group in 4d [9, 20]. This will help us to introduce our notation. Next we move on to discuss our main interest : surface operators as monodromy defects in $\mathcal{N} = 2$ SQCD.

2.1 In pure $\mathcal{N} = 2$ theory

In this approach, a surface operator is defined by introducing a singularity structure for the gauge field on a two dimensional plane in four dimensional Euclidean spacetime. If $r\,e^{i\theta}$ is the coordinate of the plane transverse to the defect, then, as $r \to 0$, the gauge field has the following behaviour:

$$A = A_\mu dx^\mu \simeq \text{diag}\Big(\underbrace{\alpha_1,\ldots,\alpha_1}_{n_1},\underbrace{\alpha_2,\ldots,\alpha_2}_{n_2},\ldots\underbrace{\alpha_M,\ldots,\alpha_M}_{n_M}\Big)d\theta \tag{2.1}$$

, here the α_I's are real parameters, where $I = 1, \ldots, M$. The M integers n_I are such that

$$\sum_J n_J = N \,. \tag{2.2}$$

In the path integral, one integrates over all gauge field configurations with this prescribed singular boundary condition. In the presence of a surface operator, one can also add a topological phase factor to the path integral

$$S_{\text{top}}[\vec{n}] = \exp\Big(2\pi\mathrm{i}\sum_{I=1}^{M}\eta_I \int_D \mathrm{Tr}\, F_{\mathrm{U}(n_I)}\Big) \tag{2.3}$$

where η_I's are constant parameters.

As a monodromy defect, a surface operator in a 4d SU(N) theory is specified by a partition of N, denoted by $\vec{n} = [n_1, n_2, \ldots n_M]$, which corresponds to the breaking of the gauge group to a Levi subgroup

$$\mathbb{L} = \mathrm{S}\left[\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\ldots\mathrm{U}(n_M)\right] \tag{2.4}$$

at the location of the defect [1,3] and 2M real parameters (α_I, η_I). This also gives a natural partitioning of the classical Coulomb v.e.v.'s of the adjoint scalar Φ of the $\mathcal{N} = 2$ SU(N) theory as follows:

$$\langle\Phi\rangle = \{a_1, \ldots, a_{r_1}|\ldots\big|a_{r_{I-1}+1}, \ldots a_{r_I}\big|\ldots|a_{r_{M-1}+1}, \ldots, a_N\} \,. \tag{2.5}$$

Here we have defined the integers r_I according to

$$r_I = \sum_{J=1}^{I} n_J \,, \tag{2.6}$$

so that the I^{th} partition in (2.5) is of length n_I. Introducing the following set of numbers with cardinality n_I:

$$\mathcal{N}_I \equiv \{r_{I-1}+1, r_{I-1}+2, \ldots, r_I\}\,, \tag{2.7}$$

we define the $n_I \times n_I$ block-diagonal matrices $\mathcal{A}_I$ according to

$$\mathcal{A}_I \equiv \text{diag}\,(a_{s\in\mathcal{N}_I}) = \begin{pmatrix} a_{r_{I-1}+1} & 0 & 0 & \ldots \\ 0 & \ddots & 0 & \ldots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \ldots & a_{r_I} \end{pmatrix}. \tag{2.8}$$

With these conventions, the splitting in (2.5) can be written as

$$\langle \Phi \rangle = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \oplus \mathcal{A}_M\,. \tag{2.9}$$

We are interested in the low-energy effective action of such theories on the Coulomb branch, in the presence of a surface operator. The low energy effective action that governs the combined gauge theory/surface operator system is completely specified by two holomorphic functions, the prepotential $\mathcal{F}$ and the twisted chiral superpotential $\mathcal{W}$. The non-perturbative contributions to these functions are obtained by first computing the instanton partition function in a Omega-deformed background and then taking the vanishing limit of the Ω-deformation parameters:

$$\lim_{\epsilon_i\to 0} \log\left(1 + Z_{\text{inst}}[\vec{n}]\right) = -\frac{\mathcal{F}_{\text{inst}}}{\epsilon_1\epsilon_2} + \frac{\mathcal{W}_{\text{inst}}}{\epsilon_1}\,. \tag{2.10}$$

2.2 Monodromy defects in SQCD

The 4d $\mathcal{N} = 2$ gauge theory of interest is the asymptotically conformal SQCD, which is an SU(N) gauge theory with $2N$ fundamental flavours. We are interested in half-BPS surface operators in this gauge theory, whose classification is the same as that for the pure gauge theory studied in [9]. For every partition of N, given by

$$\sum_{I=1}^{M} n_I = N\,, \tag{2.11}$$

one obtains a surface operator, labelled by the Levi subgroup:

$$\mathbb{L} = S\left[U(n_1)\times U(n_2)\times\ldots\times U(n_M)\right]\,. \tag{2.12}$$

A new feature of surface operators in the asymptotically conformal SQCD theory is that it also breaks the SU(2N) flavour symmetry to the following subgroup:

$$\mathbb{F} = S\left[U(n_1+n_2)\times U(n_2+n_3)\times\ldots\times U(n_M+n_1)\right]\,. \tag{2.13}$$

To denote the blocks into which the flavour group is broken, it is useful to define

$$\mathcal{F}_I = \{r_{I-1}+r_I-r_1+1,\ldots,r_I+r_{I+1}-r_1\}\,, \tag{2.14}$$

which is a set of cardinality $n_I + n_{I+1}$. The breaking of flavour symmetry in the presence of the surface operator is represented in Fig. 2.1.

The instanton partition function in the presence of such a surface operator, which is also referred to as the ramified instanton partition function, can be derived from the moduli action of a D(-1)/D3/D7-brane system in an orbifold background that represents the surface defect. Given the breaking of the gauge and flavour symmetry groups, the analysis is very similar to what was carried out in [13] and therefore here we merely present the answer:

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_I\}} Z_{\{d_I\}}[\vec{n}] \quad \text{with} \quad Z_{\{d_I\}}[\vec{n}] = \prod_{I=1}^{M}\Big[\frac{(-q_I)^{d_I}}{d_I!}\int\prod_{\sigma=1}^{d_I}\frac{d\chi_{I,\sigma}}{2\pi\mathrm{i}}\Big]\,z_{\{d_I\}} \tag{2.15}$$

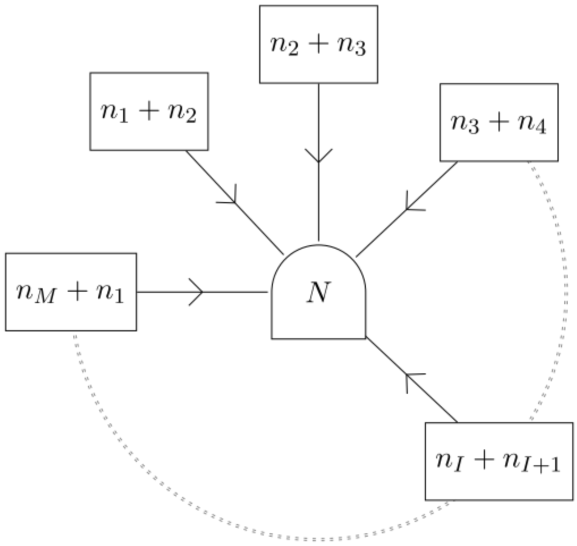


Figure 2.1: The asymptotically conformal 4d node with broken flavour symmetry. In the realization of surface operators as flavour defects, the 4d gauge node as well as the 4d flavour nodes act as matter for the gauge nodes of the 2d quiver.

where

$$z_{\{d_I\}} = \prod_{I=1}^{M}\prod_{\sigma,\tau=1}^{d_I}\frac{(\chi_{I,\sigma}-\chi_{I,\tau}+\delta_{\sigma,\tau})}{(\chi_{I,\sigma}-\chi_{I,\tau}+\epsilon_1)}\times\prod_{I=1}^{M}\prod_{\sigma=1}^{d_I}\prod_{\rho=1}^{d_{I+1}}\frac{\big(\chi_{I,\sigma}-\chi_{I+1,\rho}+\epsilon_1+\hat{\epsilon}_2\big)}{\big(\chi_{I,\sigma}-\chi_{I+1,\rho}+\hat{\epsilon}_2\big)}$$
$$\times\prod_{I=1}^{M}\prod_{\sigma=1}^{d_I}\frac{\prod_{i\in\mathcal{F}_I}(\chi_{I,\sigma}-m_i)}{\prod_{s\in\mathcal{N}_I}\big(a_s-\chi_{I,\sigma}+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\big)\prod_{t\in\mathcal{N}_{I+1}}\big(\chi_{I,\sigma}-a_t+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\big)}\,. \tag{2.16}$$

The M positive integers d_I count the numbers of ramified instantons in the various sectors and ϵ_1 and $\hat{\epsilon}_2 = \frac{\epsilon_2}{M}$ parametrize the Ω background introduced to localize the integration over the instanton moduli space [28, 29]. If one neglects the contribution of the flavours, namely the numerator factors in the second line of (2.16), the integrand is identical to that of the pure 4d theory. Note that the flavour factors are such that the breaking of the flavour symmetry is respected.

To calculate the partition function one has to specify the contour of integration for the χ_I variables. The contour is specified by integrating $\chi_{I,\sigma}$ in the upper or lower half-plane and by choosing a definite order in the successive integrations. Equivalently, the contour of integration can be selected by specifying a Jeffrey-Kirwan reference vector. Localization methods therefore allows one to calculate the instanton partition function order by order in the instanton expansion. Unlike the case of surface operators in the pure theory, in asymptotically conformal SQCD, the instanton counting parameters q_I are dimensionless. The product of all of these turn out to be related to the usual 4d instanton counting parameter.

$$q_{4d} = \prod_{I=1}^{M} q_I \ . \tag{2.17}$$

Due to this fractionation, the partition function in the presence of the surface operator is referred to in the mathematical literature as the ramified instanton partition function. The M positive integers d_I count the numbers of ramified instantons in the various sectors of the partition function.

Chapter 3

Surface operators as Flavor defects

In this Chapter, following [9, 20] we briefly review surface operators as flavor defects in pure $\mathcal{N} = 2$ theory with SU(N) gauge group in 4d. Then we discuss how one can extend this formulation for surface operators as flavor defects in $\mathcal{N} = 2$ SQCD.

3.1 In pure $\mathcal{N} = 2$ theory

In this approach a surface operator is described by a non-linear sigma model. For a surface operator with a Levi subgroup $\mathbb{L}$ in a 4d theory with a gauge group G, the relevant sigma model is defined on the target space $G/\mathbb{L}$ [1, 3]. Such a space is, in general, a flag variety which can be realized as the low-energy limit of a GLSM [16, 17], whose gauge and matter content can be summarized in the quiver diagram of Fig. 3.1.

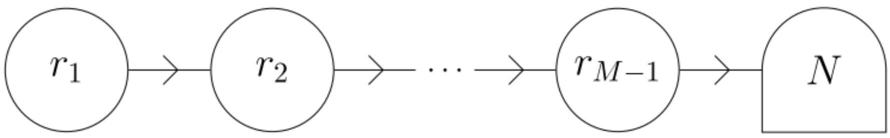


Figure 3.1: The quiver which describes the generic surface operator in pure SU(N) gauge theory.

Each circular node represents a 2d gauge group U(r_I) where the ranks r_I are as in (2.6), whereas the last node on the right hand side represents the 4d gauge group SU(N) which acts as a flavour symmetry group for the $(M-1)^{\text{th}}$ 2d node. The arrows correspond to matter multiplets which are rendered massive by non-zero v.e.v's of the twisted scalars $\sigma^{(I)}$ of the I^{th} node and of the 4d adjoint scalar Φ. The orientation of the arrows specifies whether the matter is in the fundamental (out-going) or in the anti-fundamental (in-going) representation.

The defect is 1/2-BPS and preserves (2, 2) supersymmetry in two dimensions. The effective action for the twisted chiral multiplets is obtained by integrating out the massive matter multiplets and, thanks to supersymmetry, can be encoded in the effective twisted chiral superpotential. For the quiver of Fig. 3.1, this is given by:

$$\mathcal{W} = 2\pi\mathrm{i}\sum_{I=1}^{M-1}\sum_{s=1}^{r_I}\tau_I\,\sigma_s^{(I)} - \sum_{I=1}^{M-2}\sum_{s=1}^{r_I}\sum_{t=1}^{r_{I+1}}\varpi(\sigma_s^{(I)}-\sigma_t^{(I+1)}) - \sum_{s=1}^{r_{M-1}}\Big\langle\mathrm{Tr}\,\varpi(\sigma_s^{(M-1)}-\Phi)\Big\rangle \tag{3.1}$$

where

$$\varpi(x) = x\left(\log\frac{x}{\mu}-1\right), \tag{3.2}$$

μ is the UV cut-off scale, and τ_I is the complexified FI parameter of the I^{th} node at the scale μ, namely

$$\tau_I = \frac{\theta_I}{2\pi} + \mathrm{i}\,\zeta_I \tag{3.3}$$

with θ_I and ζ_I being, respectively, the θ-parameter and the real FI parameter of the I^{th} gauge node. Finally, the angular brackets in the last term of (3.1) correspond to a chiral correlator in the 4d SU(N) theory. This correlator implies that the coupling between the 2d and 4d theory is via the resolvent of the SU(N) gauge theory [5], which in turn depends on the 4d dynamically generated scale $\Lambda_{4\text{d}}$.

Once the 4d Coulomb v.e.v.'s are given, the 2d Coulomb branch is completely lifted except for a finite number of discrete vacua. These are found by extremizing the twisted chiral superpotential $\mathcal{W}$, *i.e.* they are solutions of the twisted chiral ring equations [30, 31]

$$\exp\left(\frac{\partial \mathcal{W}}{\partial \sigma_s^{(I)}}\right) = 1 \,. \tag{3.4}$$

Once the solutions to the twisted chiral ring equations are obtained (order by order in the strong coupling scales of the 2d/4d theories), we evaluate the effective twisted chiral superpotential $\mathcal{W}$ on this particular solution, and verify that the non-perturbative contributions exactly coincide with the $\mathcal{W}_{\text{inst}}$ calculated using localization. In essence, this match provides a one-to-one map between 1/2-BPS defects in the 4d gauge theory and massive vacua in the coupled 2d/4d gauge theory.

3.2 Flavor defects in SQCD

In extending this correspondence to the asymptotically conformal SQCD case, we notice that the main difference with the pure case is that the matter multiplets now provide flavours to the 2d gauge nodes as well. The additional constraint we have to impose is that for every quiver the complexified FI parameters of the 2d gauge nodes should not run, so that the 2d gauge theories are conformal. Since the ranks of the 2d gauge nodes are fixed, the number of (anti-) fundamental flavours at each node is fixed by the necessity to cancel the contribution of the neighbouring gauge nodes (that also act as flavours) to the running of the FI coupling. Combined with the breaking of the flavour symmetry to $\mathbb{F}$ as in (5.2) for a generic surface operator, this uniquely fixes how the broken 4d flavour group acts on the 2d gauge nodes.

We illustrate the points above for a generic surface operator $[n_1, n_2, \ldots n_M]$ by giving a particular 2d/4d realization, shown in Figure 3.2 .

Given the resulting 2d/4d quiver, the evaluation of the twisted chiral superpotential in the chosen vacuum once again reproduces the twisted superpotential $\mathcal{W}_{\text{inst}}$ calculated using

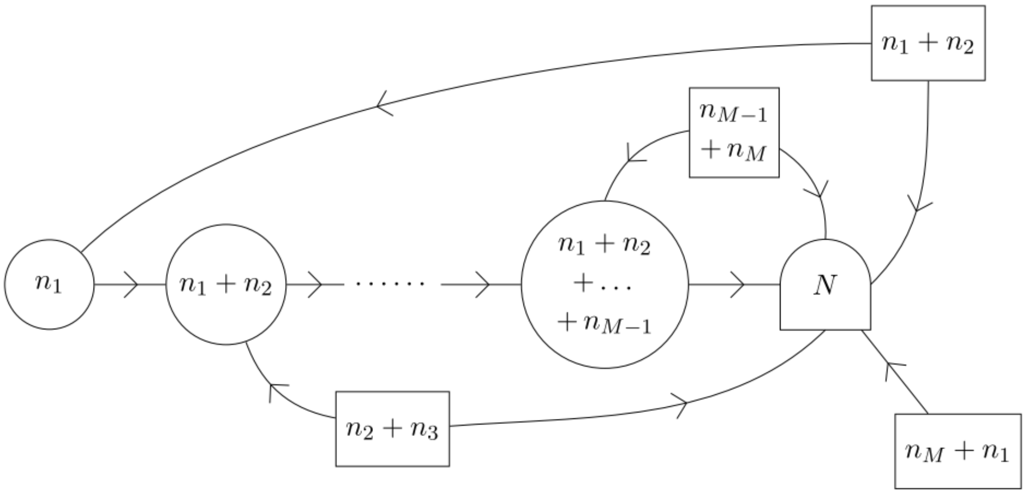


Figure 3.2: One realization of the $[n_1, n_2, \dots n_M]$ surface operator as a 2d/4d quiver in which the 2d gauge nodes are oriented. The 4d flavour groups provide matter for the 2d nodes such that the β-function of each FI parameter is zero.

localization and with the particular JK prescription associated to the 2d/4d quiver.

Now we follow the same prescription as the pure gauge theory for SCQD. We write down the effective action and extremize it to find the twisted chiral ring equations. We then solve them to find the massive vacuum. The evaluation of the twisted chiral superpotential in the massive vacuum once again reproduces the twisted superpotential $\mathcal{W}_{\text{inst}}$ calculated using localization.

Chapter 4

Contours vs quivers and Seiberg duality

In this chapter, we discuss the exact relationship between two different approaches of surface operators, name as Monodromy defects and Flavor defects in $\mathcal{N} = 2$ SQCD. We further discuss Seiberg duality in the context of SQCD.

4.1 Contours vs quivers

In order to extract the twisted chiral superpotential from the localization analysis one needs to provide a residue prescription to calculate the instanton partition function. This prescription is most succinctly specified via a JK reference vector [21] that uniquely specifies the set of poles chosen by the contour [1].

In our previous work [9] on surface operators in the pure 4d theory, it was shown that different JK prescriptions map to distinct 2d/4d quiver gauge theories. In that case the quivers are equivalent, and related to each other by Seiberg duality. For such quivers the ranks of the 2d gauge groups directly correlate with a choice of a massive vacuum and the evaluation of the twisted chiral superpotential in that particular massive vacuum reproduces the twisted superpotential $\mathcal{W}_{\text{inst}}$ calculated using localization. Despite the

[1] For applications to gauge theories see, for instance, [18, 32, 33].

equivalence between quivers, we could obtain an unambiguous map between contours and quivers, thanks to the match between the individual residues on the localization side and the individual terms in the solution of the chiral ring equations. In particular, the number of residues that contribute to the contour integrals are related to the ranks of the 2d nodes of the quivers, while the coefficients of the JK vector correspond to the FI parameters of the 2d gauge groups.

In extending this correspondence to the asymptotically conformal SQCD case, we find that given the resulting 2d/4d quiver, the evaluation of the twisted chiral superpotential in the chosen vacuum once again reproduces the twisted superpotential $\mathcal{W}_{\text{inst}}$ calculated using localization and with the particular JK prescription associated to the 2d/4d quiver.

We illustrate the points above for a generic surface operator $[n_1, n_2, \ldots n_M]$ by giving a particular 2d/4d realization, shown in Figure 3.2.

One property we would like to emphasize is that the map between a given JK contour and its corresponding quiver is unambiguous: the superpotential calculated from a particular localization prescription and that obtained from the twisted chiral ring equations of the 2d/4d quiver match. We will show this explicitly in the following sections in various examples.

It is important to stress that in our earlier discussion, nowhere did we use Seiberg duality rules for the asymptotically conformal SQCD theory or mention the equivalence between the different 2d/4d coupled theories. For surface operators in the pure 4d theory, the 2d/4d quivers related to distinct JK prescriptions were part of a duality chain in which each step is a particular 2d Seiberg duality move (for example, see Fig. 7 in [9]). In that case it was true that the twisted superpotential calculated using different contour choices were identical. However in asymptotically conformal SQCD there is an interesting twist to the story. By explicit calculation, one can check that, while the prepotential calculated using the above prescription is independent of the choice of surface operator and of the contour of integration, the twisted chiral superpotential calculated using different JK prescriptions, in fact, do *not* agree. We illustrate this point in the simple setting of the $[p, N-p]$ defect.

The $[p, N-p]$ defect

Let us consider the surface defect $[p, N-p]$, and focus for simplicity on the 1-instanton contribution to the partition function $Z_{\text{1-inst}}$.

In order to write it in compact form, we introduce the polynomials:

$$P_I(z) = \prod_{u\in\mathcal{N}_I}(z-a_u)\,, \qquad B_I(z) = \prod_{i\in\mathcal{F}_I}(z-m_i)\,. \tag{4.1}$$

In terms of these, $Z_{\text{1-inst}}$ for the 2-node defect takes the form

$$Z_{\text{1-inst}} = -\sum_{I=1}^{2}\frac{q_I}{\epsilon_1}\int\frac{d\chi_I}{2\pi\mathrm{i}}\,\frac{(-1)^{n_I}\,B_I(\chi_I)}{P_I\Big(\chi_I-\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\Big)\,P_{I+1}\Big(\chi_I+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\Big)}\,, \tag{4.2}$$

while, using (2.10), the 1-instanton twisted superpotential is given by

$$\mathcal{W}_{\text{1-inst}} = \lim_{\epsilon_i\to 0}\epsilon_1 Z_{\text{1-inst}}\,. \tag{4.3}$$

To compute the integrals in (4.2) we can use distinct JK prescriptions that simply correspond to integrating each χ_I along a closed contour in the upper (+) or lower (−) half-planes. According to the analysis of [9, 20], out of the four inequivalent possibilities, only two JK prescriptions are relevant and we denote them by (+−) and (−+), respectively.

With the (+−) prescription, the 1-instanton contribution to the twisted superpotential is

$$\mathcal{W}^{+-}_{\text{1-inst}} = (-1)^{p+1}q_1\sum_{u\in\mathcal{N}_1}\frac{B_1(a_u)}{P_1'(a_u)P_2(a_u)} + (-1)^{N-p}q_2\sum_{u\in\mathcal{N}_1}\frac{B_2(a_u)}{P_1'(a_u)P_2(a_u)}\,, \tag{4.4}$$

while with the $(-+)$ prescription we get

$$\mathcal{W}^{-+}_{\text{1-inst}} = (-1)^p q_1 \sum_{u\in\mathcal{N}_2} \frac{B_1(a_u)}{P_1(a_u)P_2'(a_u)} + (-1)^{N-p+1} q_2 \sum_{u\in\mathcal{N}_2} \frac{B_2(a_u)}{P_1(a_u)P_2'(a_u)}\,. \tag{4.5}$$

where the $'$ symbol denotes derivative. We can easily verify that the two superpotentials are different and that their difference is

$$\mathcal{W}^{-+}_{\text{1-inst}} - \mathcal{W}^{+-}_{\text{1-inst}} = (-1)^p q_1 \sum_{u\in\mathcal{N}_1\cup\mathcal{N}_2} \frac{B_1(a_u)}{P'(a_u)} + (-1)^{N-p+1} q_2 \sum_{u\in\mathcal{N}_1\cup\mathcal{N}_2} \frac{B_2(a_u)}{P'(a_u)}\,, \tag{4.6}$$

where $P(z)$ is the classical gauge polynomial given by

$$P(z) = \prod_{u=1}^{N} (z - a_u)\,. \tag{4.7}$$

It is simple to realize that, since the difference of the contours in the upper and lower half-planes is a contour around infinity, the difference (4.6) is due to non-vanishing residues at infinity in the integrand of $Z_{\text{1-inst}}$, a property which is characteristic of asymptotically conformal theories. So we can write:

$$\begin{aligned}\mathcal{W}^{-+}_{\text{1-inst}} - \mathcal{W}^{+-}_{\text{1-inst}} &= \int_{C_\infty} dz \left[(-1)^p q_1 \frac{B_1(z)}{P(z)} + (-1)^{N-p+1} q_2 \frac{B_2(z)}{P(z)}\right] \\ &= \left[(-1)^{p+1} q_1 + (-1)^{N-p+1} q_2\right] \sum_{i\in\mathcal{F}_1} m_i\end{aligned} \tag{4.8}$$

where C_∞ is a closed curve encircling infinity clockwise, and the second line follows from using the explicit expressions for the gauge and flavour polynomials.

In Appendix A, by lifting the model to five dimensions with one compact direction, we show that the existence of non-vanishing residues at infinity holds at every instanton number. This explains why twisted superpotentials evaluated with different contour prescriptions are generically different. For the 2-node defect $[p, N-p]$, we are able to resum the instanton expansion and obtain

$$\mathcal{W}_{\text{inst}}^{-+} - \mathcal{W}_{\text{inst}}^{+-} = -\Big[\log(1+(-1)^p q_1) + \log(1+(-1)^{N-p} q_2)\Big] \sum_{i\in\mathcal{F}_1} m_i\,. \tag{4.9}$$

As discussed earlier, we expect that different JK prescriptions map to distinct quivers related by 2d Seiberg duality, with equivalent superpotentials. This result therefore suggests that in SQCD with surface defects, the definition of what is the dual quiver necessarily involves non-perturbative modifications due to ramified instantons. We will discuss this issue in greater detail in the following section.

4.2 Seiberg duality

We now study Seiberg duality in the 2d/4d quiver realization of the defect and propose a relation between the twisted superpotentials of dual quivers. For the purely 2d case, this has been discussed in detail in [10]. We begin with a 2d U(N) gauge theory with N_f fundamental flavours and N_f anti-fundamental flavours shown in Figure 4.1. We now

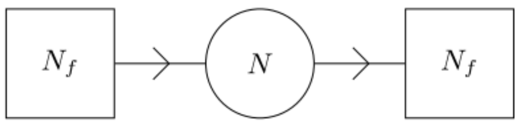


Figure 4.1: The gauge group is represented by a circle, and the flavour groups are represented by squares. The quiver diagram has a single 2d gauge node of rank N with N_f fundamental and N_f anti-fundamental flavours attached to it.

perform a Seiberg duality operation on the 2d gauge node, and obtain the quiver diagram shown in Figure 4.2.

Under the duality, the roles of the fundamental and anti-fundamental flavours are exchanged as denoted by the reversal of the arrows. There is also the addition of a mesonic field, as described by the line connecting the two flavour groups.

When these duality rules are applied to quiver theories, one has to take into account that

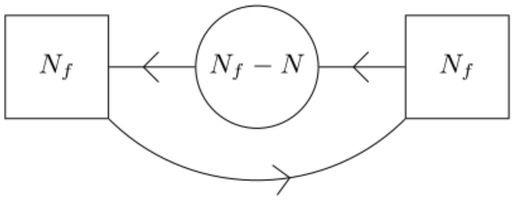


Figure 4.2: The quiver diagram obtained after a 2d Seiberg duality on the gauge node in Fig. 4.1.

for each 2d gauge node flavours can be provided by other 2d gauge nodes of the quiver; in such cases the extra mesonic field should be treated as just another chiral multiplet in the dual quiver. We will see several examples in later sections. So far, we have only shown how the quiver itself is modified by the action of duality, we still have to show how the duality acts on the parameters and Lagrangian of the quiver theories.

We again focus on the simplest 2-node case and solve the twisted chiral ring equations of the two purported dual quivers. Imposing duality will then allow us to find the rules.

4.3 The 2-node case

We consider the $[p, N-p]$ defect and describe its effective action. We remark that the results of this subsection have some partial overlap with those of the recent paper [34], but our analysis of Seiberg duality has significant differences.

The quiver Q_0: We first consider the realization of the defect as the quiver in Figure 4.3. After the massive chiral multiplets are integrated out, the twisted chiral superpotential takes the following form:

$$\mathcal{W}_{Q_0} = \log x \sum_{s\in\mathcal{N}_1} \sigma_s - \sum_{s\in\mathcal{N}_1}\sum_{i\in\mathcal{F}_1} \varpi(m_i - \sigma_s) - \sum_{s\in\mathcal{N}_1} \Big\langle \mathrm{Tr}\, \varpi(\sigma_s - \Phi) \Big\rangle\,, \tag{4.10}$$

where x is the exponentiated FI parameter of the 2d theory, σ_s are the scalars in the twisted chiral suprerfield that encodes the 2d vector multiplet and the m_i are the masses

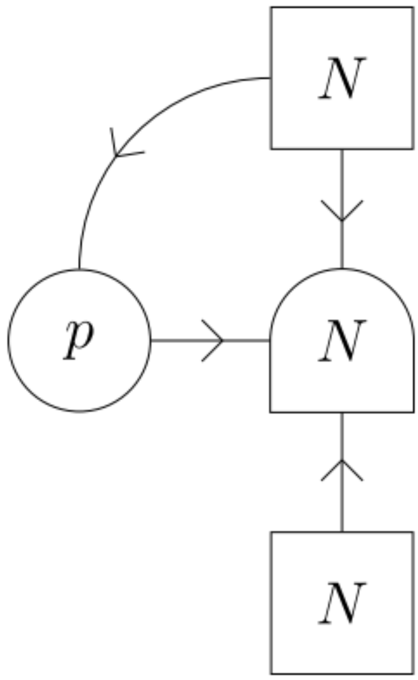


Figure 4.3: A 2d/4d quiver realization of the $[p, N-p]$ defect in SU(N) theory with $2N$ flavours

of the 4d flavours that also act as twisted masses for the 2d chiral multiplets. We have also introduced the function $\varpi(x) = x(\log\frac{x}{\mu} - 1)$, which is the result of integrating out a chiral multiplet of twisted mass x. In the last term of (4.10), the angular brackets denote a chiral correlator in the 4d SU(N) gauge theory, and Φ is the adjoint scalar in the vector multiplet. The twisted chiral ring equations are [30, 31]:

$$\exp\left(\frac{\partial \mathcal{W}_{Q_0}}{\partial \sigma_s}\right) = 1, \quad s \in \mathcal{N}_1\,, \tag{4.11}$$

and they explicitly read

$$\exp\left\langle \mathrm{Tr}\,\log(\sigma_s - \Phi)\right\rangle = (-1)^N\, x\, B_1(\sigma_s) \quad \text{for } s \in \mathcal{N}_1 \tag{4.12}$$

where $B_1(\sigma_s)$ is the polynomial defined in (4.1). For the asymptotically conformal case under consideration, the resolvent of the 4d gauge theory which determines the chiral correlator, has a non-trivial dependence on q_0, the instanton weight of the 4d SU(N) theory.

We refer the reader to Appendix B for details; here we merely present the result, namely

$$\left\langle \mathrm{Tr}\,\log\frac{z-\Phi}{\mu}\right\rangle = \log\left((1+q_0)\,\frac{\widehat{P}(z)+Y}{2\mu^N}\right). \tag{4.13}$$

Here, $\widehat{P}(z)$ is the quantum gauge polynomial corresponding to the classical one defined in (4.7):

$$\widehat{P}(z) = z^N + u_2 z^{N-2} + \ldots + (-1)^N u_N\,, \tag{4.14}$$

where u_k are the gauge invariant coordinates on moduli space, and the variable Y is given in terms of the Seiberg-Witten curve of the asymptotically conformal 4d gauge theory:

$$Y^2 = \widehat{P}(z)^2 - \frac{4q_0}{(1+q_0)^2}B(z)\,, \tag{4.15}$$

where $B(z)$ is the flavour polynomial. We refer the reader to Appendix B for details.

Exponentiating (4.13) and using (4.15), we can recast the twisted chiral ring equations (4.12) in the following form

$$(1+q_0)\,\widehat{P}(\sigma_s) \;=\; (-1)^N\left(x\,B_1(\sigma_s) \;+\; \frac{q_0}{x}\,B_2(\sigma_s)\right) \quad \text{for } s\in\mathcal{N}_1. \tag{4.16}$$

The classical vacuum about which we solve these equations is

$$\sigma_s = a_s + \delta\sigma_s \quad \text{for} \quad s\in\mathcal{N}_1\,, \tag{4.17}$$

and the solution in the 1-instanton approximation is

$$\delta\sigma_s = (-1)^N\frac{1}{P_1'(a_s)P_2(a_s)}\left[\,x\,B_1(a_s) \;+\; \frac{q_0}{x}\,B_2(a_s)\right] \quad \text{for } s\in\mathcal{N}_1\,. \tag{4.18}$$

Let us now evaluate the twisted superpotential (4.10) on this solution. This is a little tricky since one needs to expand the 4d chiral correlator $\left\langle \mathrm{Tr}\,\varpi(\sigma-\Phi)\right\rangle$ in powers of q_0. This is carried out in Appendix B; using those results, we find (neglecting the 1-loop contributions)

$$\mathcal{W}_{Q_0}(\sigma_\star) = \log x \sum_{s\in\mathcal{N}_1} a_s + (-1)^N x \sum_{s\in\mathcal{N}_1} \frac{B_1(a_s)}{P_1'(a_s)P_2(a_s)} + (-1)^{N+1}\frac{q_0}{x}\sum_{s\in\mathcal{N}_1}\frac{B_2(a_s)}{P_1'(a_s)P_2(a_s)}\,. \tag{4.19}$$

It can be easily checked that the 1-instanton terms match the localization result (4.4) with the $(+-)$ prescription, namely

$$\mathcal{W}_{Q_0}(\sigma_\star)\big|_{\text{1-inst}} = \mathcal{W}^{+-}_{\text{1-inst}}\,, \tag{4.20}$$

provided we make the following identifications:

$$q_1 = (-1)^{N+p+1}\,x\,, \quad q_2 = (-1)^{p+1}\,\frac{q_0}{x}\,. \tag{4.21}$$

We have checked that the match between the superpotential evaluated on the solution of twisted chiral ring equations and the localization results continues to hold up to 8 instantons for various low rank cases.

The quiver Q_1: Acting with the Seiberg duality rules on the quiver diagram of Fig. 4.3, one obtains the quiver diagram represented in Fig. 4.4.

After integrating out the massive chiral multiplets, the twisted chiral superpotential corresponding to this quiver diagram takes the following form:

$$\mathcal{W}_{Q_1} = \log y \sum_{s\in\mathcal{N}_2} \sigma_s - \sum_{s\in\mathcal{N}_2}\sum_{i\in\mathcal{F}_1} \varpi(\sigma_s - m_i) - \sum_{s\in\mathcal{N}_2} \Big\langle \mathrm{Tr}\,\varpi(\Phi - \sigma_s)\Big\rangle\,, \tag{4.22}$$

where we have denoted by y the exponentiated FI parameter of of the 2d gauge node. The

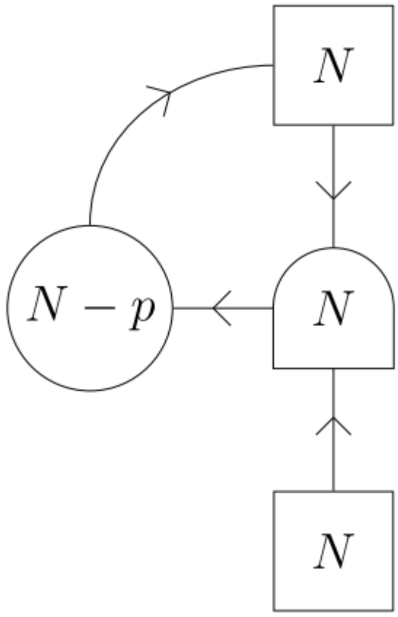


Figure 4.4: The 2d/4d quiver diagram obtained after the action of Seiberg duality on the 2d node in Fig. 4.3

chiral ring equations that follow from $\mathcal{W}_{Q_1}$ are:

$$(1+q_0)\ \widehat{P}(\sigma_s)\ =(-1)^N\left[\frac{1}{y}\,B_1(\sigma_s)\ +\ q_0\,y\ B_2(\sigma_s)\right]\quad \text{for } s\in\mathcal{N}_2\,. \tag{4.23}$$

We solve them in the vacuum given by

$$\sigma_s=a_s+\delta\sigma_s \quad \text{for} \quad s\in\mathcal{N}_2\,, \tag{4.24}$$

and in the 1-instanton approximation we obtain

$$\delta\sigma_s=(-1)^N\frac{1}{P_1(a_s)P_2'(a_s)}\left[\frac{1}{y}\,B_1(a_s)\ +\ q_0\,y\,B_2(a_s)\right]\quad \text{for } s\in\mathcal{N}_2\,. \tag{4.25}$$

Evaluating the twisted chiral superpotential on this solution, we find (neglecting the 1-

loop contributions as before)

$$\mathcal{W}_{Q_1}(\sigma_\star) = \log y \sum_{s\in\mathcal{N}_2} a_s + (-1)^{N+1}\frac{1}{y}\sum_{s\in\mathcal{N}_2}\frac{B_1(a_s)}{P_1(a_s)P_2'(a_s)} + (-1)^N q_0\, y \sum_{s\in\mathcal{N}_2}\frac{B_2(a_s)}{P_1(a_s)P_2'(a_s)}\,. \tag{4.26}$$

If we now impose that the classical contributions in $\mathcal{W}_{Q_0}$ and $\mathcal{W}_{Q_1}$ match, we find that the FI parameters of the pair of dual theories are related in the same way as in the pure theory, namely:

$$y = \frac{1}{x}\,. \tag{4.27}$$

Using this identification and the relations in (4.21), it can be checked that

$$\mathcal{W}_{Q_1}(\sigma_\star)\Big|_{\text{1-inst}} = \mathcal{W}^{-+}_{\text{1-inst}} \tag{4.28}$$

i.e. the 1-instanton contribution of $\mathcal{W}_{Q_1}$ matches the localization result (4.5) with the $(-+)$ prescription. We have checked in several examples that this match also occurs at higher instanton numbers.

4.4 The dual theory

In the previous section we have shown that, with an appropriate map of parameters, the twisted superpotentials of the quivers Q_0 and Q_1 match the results from localization obtained with two distinct JK prescriptions. Since these differ already at the 1-instanton level, as shown in (4.8), it is clear that the superpotentials $\mathcal{W}_{Q_0}$ and $\mathcal{W}_{Q_1}$ *do not match*. By studying a number of low-rank theories at the first few instanton orders, we find that the difference between the superpotentials of the two quivers can be written as

$$\mathcal{W}_{Q_1}(\sigma_\star) - \mathcal{W}_{Q_0}(\sigma_\star) = (-1)^{N+1}\Big(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\Big)\sum_{i\in\mathcal{F}_1} m_i$$

$$+ (-1)^{N+1}\Big(\frac{q_0}{x} + \frac{q_0^2}{2x^2} + \frac{q_0^3}{3x^3} + \ldots\Big)\sum_{i\in\mathcal{F}_1} m_i\,. \qquad (4.29)$$

After using the map (4.21) we observe that this is in complete agreement with the resummed result (4.9) obtained for the superpotentials calculated from the (+−) and (−+) prescriptions using localization. This not only supports our identification between contours and quivers, it also shows us the way to correctly identify dual pairs of quiver theories.

In fact, given the simple relation (4.29), it is natural to propose that the quiver theory that is actually dual to Q_0 and that we denote by $\widetilde{Q}_1$, is the one whose superpotential differs from that of Q_1 by non-perturbative corrections according to

$$\begin{aligned}\mathcal{W}_{\widetilde{Q}_1} = &-\log x \sum_{s\in\mathcal{N}_2}\sigma_s - \sum_{i\in\mathcal{F}_1}\sum_{s\in\mathcal{N}_2}\varpi(\sigma_s - m_i) - \sum_{s\in\mathcal{N}_2}\Big\langle \mathrm{Tr}\,\varpi(\Phi - \sigma_s)\Big\rangle \\ &+ \Big(\log\Big(1-(-1)^N x\Big) + \log\Big(1-(-1)^N\frac{q_0}{x}\Big)\Big)\sum_{i\in\mathcal{F}_1} m_i\,. \qquad (4.30)\end{aligned}$$

The first line of the right hand side of the equation above is identical to the superpotential of the quiver Q_1 given in (4.22), in which we have used the map (4.27). The second line in (4.30) encodes the non-perturbative corrections to the naive answer. Expanding the logarithm, we see that the power series coincides with the difference calculated in (4.29). Furthermore, this is precisely what we derive from first principles using contour deformation arguments in Appendix A.

The appearance of non-perturbative terms in the superpotential of the dual theory is in part already known in the context of conformal gauge theories in two dimensions. Indeed, as shown in [10], 2d Seiberg duality requires not only the inversion of the FI couplings, but also that the twisted superpotential of the dual quiver is modified by the following non-perturbative correction:

$$\delta W = \log\Big(1-(-1)^{N_f}\,x\Big)\left(\mathrm{Tr}\,\widetilde{m} - \mathrm{Tr}\,m\right). \qquad (4.31)$$

where x is the exponentiated FI parameter of the 2d gauge node that is dualized, and $\mathrm{Tr}\, m$ and $\mathrm{Tr}\, \widetilde{m}$ denote respectively the sum of twisted masses for all N_f fundamental and N_f anti-fundamental flavours attached to that node.

The purely 2d part of the non perturbative term in (4.30) is exactly δW in (4.31), written for the particular case we are considering. Note that in (4.30) it depends only on the anti-fundamental flavours attached to the dualized node in Fig. 4.3, because in Q_0 the contribution from fundamental flavours is solely due to the 4d node and this vanishes due to the tracelessness condition of $\mathrm{SU}(N)$. Our result shows that, whenever a 2d gauge node connected to a dynamical 4d gauge node is dualized, there is also an extra contribution that arises as a consequence of the non-trivial 4d dynamics. The modified Seiberg rule in (4.30) is thus a generalization of the one in (4.31) and represents the main result of this section.

4.5 Generalized Seiberg duality : basic rules

The definition of the dual quiver we have introduced might seem simply a change in nomenclature since the non-perturbative terms we have added are constant and do not affect the dynamics or twisted chiral ring equations. However, in a generic quiver with more nodes, the fundamental or anti-fundamental matter fields of a given 2d node are realized by other 2d gauge nodes; in this case the role of the twisted masses will be played by $\mathrm{Tr}\, \sigma$ of that gauge node and thus such terms do affect the dynamics. Indeed, they affect the form of the twisted chiral ring equations.

In summary the basic duality rules for the twisted chiral superpotentials of pairs of dual quivers are:

1. The ranks of the gauge and flavour nodes of the dual quiver are completely determined by the operation shown in Figures 4.1 and 4.2.

2. For such a duality move, the exponentiated FI couplings of the pair of dual quivers are related by inversion, as shown in (4.27).

3. If the dualized node is only connected to flavour or other 2d gauge nodes, the twisted chiral superpotential of the dual quiver is corrected by a non-perturbative piece given in (4.31). The twisted masses are replaced by the twisted scalars of the vector multiplet in case the flavour is realized by a 2d gauge node.

4. If the dualized node is connected to the dynamical 4d gauge node, the non-perturbative correction to the twisted superpotential takes the form:

$$\delta W = \left[\log\left(1 - (-1)^{N_f}\, x\right) + \log\left(1 - (-1)^{N_f}\frac{q_0}{x}\right)\right](\mathrm{Tr}\,\widetilde{m} - \mathrm{Tr}\,m)\,, \tag{4.32}$$

   where N_f is the number of (anti-) fundamental flavours attached to the dualized node. As before, when the flavour symmetry is realized by a 2d gauge node, the twisted masses are replaced by the twisted scalars in the 2d vector multiplet.

Given these duality rules and the resulting twisted superpotential of the dual quiver theory, we solve the twisted chiral ring equations order by order in the exponentiated FI couplings. Upon evaluating the superpotential on the solutions of the chiral ring equations, we find a perfect match with the evaluation of the superpotential on the corresponding massive vacuum of original quiver.

4.6 Seiberg duality for 3-node quivers

We now apply the duality rules derived in the previous section to quivers with two gauge nodes and one flavour node. We begin with the quiver denoted by Q_0 and perform the sequence of Seiberg dualities shown in Fig. 4.5.

The ranks and connectivity of the quivers are determined by the duality rules discussed

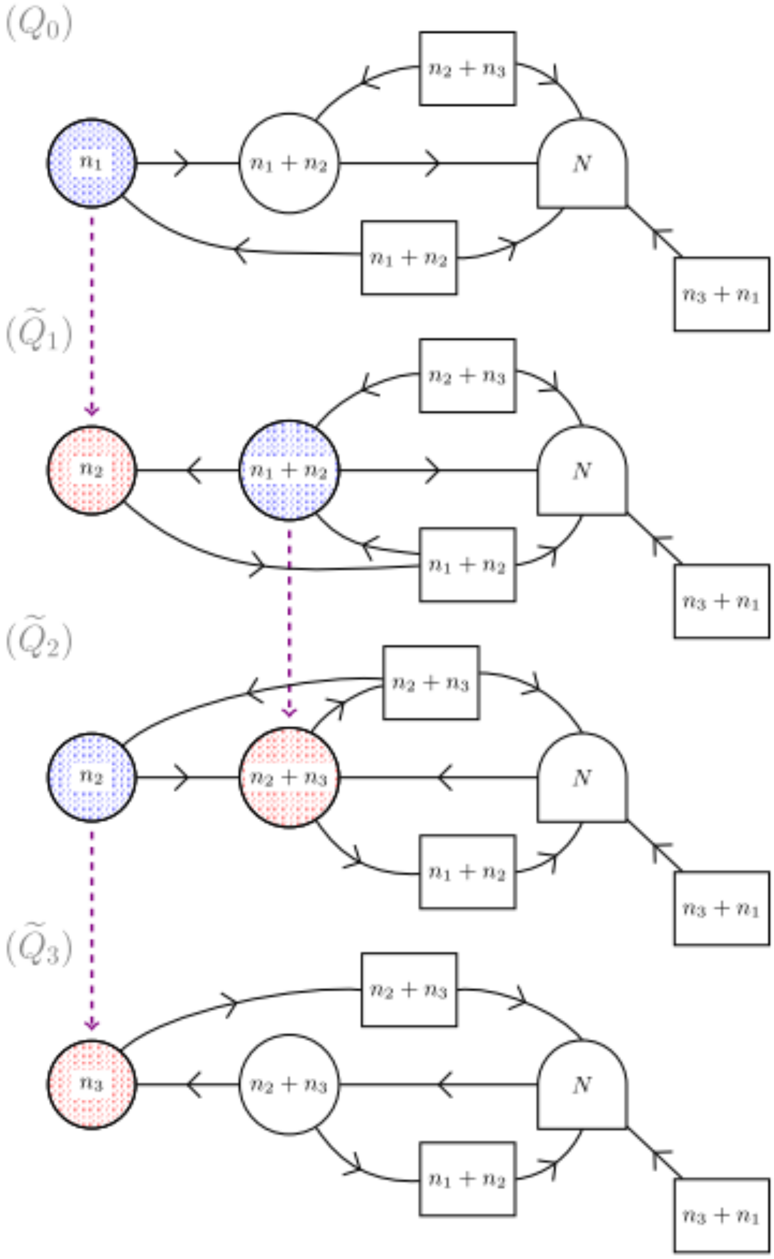


Figure 4.5: A sequence of dualities relating 2d/4d quivers corresponding to the $[n_1, n_2, n_3]$ defect in SU(N) asymptotically conformal SQCD, starting from the quiver Q_0.

in Section 4.2. These are sufficient to determine the classical and one-loop contributions to the twisted chiral superpotential. With these ingredients alone, starting from Q_0 we can obtain three quivers Q_ℓ (with $\ell = 1, 2, 3$), whose twisted chiral superpotential $\mathcal{W}_{Q_\ell}$ is computed, at 1-instanton level, in Appendix C. From our previous discussion of Seiberg duality we know however that each step of the duality chain induces additional non-perturbative corrections for the superpotential. We shall therefore use the notation

$\widetilde{Q}_\ell$, in order to indicate that while their twisted superpotentials share the classical and one-loop parts with those of Q_ℓ, they differ by non-perturbative terms.

The twisted chiral superpotential for the first quiver Q_0 is:

$$\begin{aligned}\mathcal{W}_{Q_0}(\{x\}) = \; &\log x_1 \operatorname{Tr}\sigma^{(1)} + \log x_2 \operatorname{Tr}\sigma^{(2)} - \sum_{s\in\mathcal{N}_1}\sum_{t\in\mathcal{N}_1\cup\mathcal{N}_2} \varpi(\sigma^{(1)}_s - \sigma^{(2)}_t) \\ &- \sum_{s\in\mathcal{N}_1}\sum_{i\in\mathcal{F}_1} \varpi(m_i - \sigma^{(1)}_s) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2}\sum_{i\in\mathcal{F}_2} \varpi(m_i - \sigma^{(2)}_s) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \Big\langle \operatorname{Tr}\varpi(\sigma^{(2)}_s - \Phi)\Big\rangle .\end{aligned} \tag{4.33}$$

We now perform a duality on the $\mathrm{U}(n_1)$ gauge node in Q_0 to obtain the quiver $\widetilde{Q}_1$ whose twisted superpotential is

$$\begin{aligned}\mathcal{W}_{\widetilde{Q}_1}(\{x\}) = &-\log x_1 \operatorname{Tr}\sigma^{(1)} + \log(x_1x_2) \operatorname{Tr}\sigma^{(2)} - \sum_{s\in\mathcal{N}_2}\sum_{i\in\mathcal{F}_1} \varpi(\sigma^{(1)}_s - m_i) \\ &- \sum_{s\in\mathcal{N}_2}\sum_{t\in\mathcal{N}_1\cup\mathcal{N}_2} \varpi(\sigma^{(2)}_t - \sigma^{(1)}_s) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2}\sum_{i\in\mathcal{F}_1\cup\mathcal{F}_2} \varpi(m_i - \sigma^{(2)}_s) \\ &- \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \Big\langle \operatorname{Tr}\varpi(\sigma^{(2)}_s - \Phi)\Big\rangle + \log\Big(1-(-1)^{n_1+n_2}\,x_1\Big)\Big(\sum_{i\in\mathcal{F}_1} m_i - \operatorname{Tr}\sigma^{(2)}\Big) .\end{aligned} \tag{4.34}$$

The last logarithmic term accounts for the non-perturbative corrections due to the standard duality rule (4.31) in which we have used $N_f = n_1 + n_2$ since it is the $\mathrm{U}(n_1 + n_2)$ gauge node that provides fundamental matter to the $\mathrm{U}(n_1)$ node that is dualized.

In order to see the effect of the duality more clearly, one can write the above superpotential using the variables that are natural for the quiver Q_1, by collecting the $\operatorname{Tr}\sigma^{(2)}$ terms together. Comparing with the superpotential $\mathcal{W}_{Q_1}$ given in (C.6), we have

$$\mathcal{W}_{\widetilde{Q}_1}(\{x\}) = \mathcal{W}_{Q_1}(\{y\}) + \log\Big(1-(-1)^{n_1+n_2}\,x_1\Big)\sum_{i\in\mathcal{F}_1} m_i\,, \tag{4.35}$$

where the FI parameters (y_1, y_2) appearing in $\mathcal{W}_{Q_1}$ are

$$y_1 = \frac{1}{x_1}\,, \quad y_2 = \frac{x_1 x_2}{1-(-1)^{n_1+n_2}\,x_1}\,. \tag{4.36}$$

Here we see how the Seiberg duality acts on the FI parameters when more than one gauge node is present [2].

The twisted chiral ring equations obtained from $\mathcal{W}_{Q_0}$ and $\mathcal{W}_{\widetilde{Q}_1}$ can be solved as usual by expanding about a particular classical vacuum that corresponds to the surface operator and performing an order-by-order expansion in the exponentiated FI couplings x_I. Upon evaluating the respective superpotentials on the resulting solutions, we find a perfect match up to purely q_0-dependent terms. We have checked this up to 8 (ramified) instantons for several low rank cases and this agreement is a confirmation of the proposal for 2d Seiberg duality at the level of the low energy effective action.

It is important to mention here that the twisted chiral ring equations one would write for $\widetilde{Q}_1$ are different from those that one would write for the quiver Q_1 on account of the non-perturbative corrections to the FI parameters of the dual theory. It is only with these corrections that the equality with the low-energy superpotential $\mathcal{W}_{Q_0}$ holds.

Along the same lines, we now consider the second and third dualities moves in the duality chain in Figure 4.5. In the former, the dualized 2d node is connected to the 4d gauge node, and thus the modified duality rules (4.32) have to be used. This duality step leads to the quiver $\widetilde{Q}_2$ and, collecting terms as before, we find

$$\begin{aligned}\mathcal{W}_{\widetilde{Q}_2}(\{x\}) = \mathcal{W}_{Q_2}(\{z\}) \;&+\; \Big[\log\Big(1-(-1)^{n_1+n_3}\,y_2\Big) + \log\Big(1-(-1)^{n_1+n_3}\,\frac{q_0}{y_2}\Big)\Big]\sum_{i\in\mathcal{F}_1\cup\mathcal{F}_2} m_i\\ &+\; \log\Big(1-(-1)^{n_1+n_2}\,x_1\Big)\sum_{i\in\mathcal{F}_1} m_i\,. \end{aligned}\tag{4.37}$$

The superpotential $\mathcal{W}_{Q_2}$ is defined in (C.11) in Appendix C and is determined purely by

[2] As shown in [10], it is possible to define cluster variables in terms of which the Seiberg duality action on the FI parameters can be recast as a cluster algebra.

the connectivity of the quiver Q_2 whose FI parameters we denote (z_1, z_2) are expressed in terms of those of the original quiver Q_0 via their dependence on y_I according to

$$z_1 = -\frac{y_1 y_2}{(1-y_2)(1-\frac{q_0}{y_2})}\,, \quad z_2 = \frac{1}{y_2}\,. \tag{4.38}$$

In (4.38) we see the appearance of q_0 since the dualized node is directly connected to the dynamical 4d node.

Finally, we perform the third duality move and obtain the quiver denoted by $\widetilde{Q}_3$ in Figure 4.5; its twisted superpotential is:

$$\begin{aligned} \mathcal{W}_{\widetilde{Q}_3}(\{x\}) = \mathcal{W}_{Q_3}(\{w\}) &+ \log\left(1-(-1)^{n_2+n_3}\, z_1\right)\sum_{i\in\mathcal{F}_2} m_i \\ &+ \left[\log\left(1-(-1)^{n_1+n_3}\, y_2\right) + \log\left(1-(-1)^{n_1+n_3}\, \frac{q_0}{y_2}\right)\right] \sum_{i\in\mathcal{F}_1\cup\mathcal{F}_2} m_i \\ &+ \log\left(1-(-1)^{n_1+n_2}\, x_1\right)\sum_{i\in\mathcal{F}_1} m_i\,. \end{aligned} \tag{4.39}$$

The superpotential $\mathcal{W}_{Q_3}$ is defined in (C.16) in Appendix C and its FI parameters (w_1, w_2) are

$$w_1 = \frac{1}{z_1} \qquad w_2 = \frac{z_1 z_2}{1-(-1)^{n_2+n_3}\, z_1}\,. \tag{4.40}$$

By successively composing the relations (4.38) and (4.36), one can express these FI couplings in terms of those of the original quiver Q_0.

Once the twisted chiral superpotentials of the dual quivers are obtained, we can solve the corresponding chiral ring relations as usual and evaluate the superpotentials on these solutions. Our calculations confirm the equality of these quantities and show that, up to purely q_0-dependent terms, the three quivers $\widetilde{Q}_{1,2,3}$ derived from Q_0, lead to the same low energy effective action on the Coulomb branch.

4.7 Rows of dual quivers

So far, we have worked out a single duality chain starting with the quiver Q_0 and shown that the twisted superpotentials evaluated on the solutions to the twisted chiral ring equations match for all these four 3-node quivers:

$$Q_0 \longleftrightarrow \widetilde{Q}_1 \longleftrightarrow \widetilde{Q}_2 \longleftrightarrow \widetilde{Q}_3 \, . \tag{4.41}$$

The arrows are double headed since dualities can be performed in either direction. The new result has been the second duality move, which is a generalized duality and involves a change in the superpotential as shown in (4.32).

These results can be easily generalized to the generic case in which the gauge nodes form a linear quiver. Our earlier work has shown that, for such an M-node case, there are 2^{M-1} possible Seiberg-dual quivers [9]. Each such quiver is labelled by a vector $(s_1, s_2, \ldots, s_{M-1})$ whose entries take values 0 or 1. For instance, for the 3-node cases studied in this work, we find the following set of quivers that map to distinct JK vectors on the localization side, that are completely determined by the permutation $\vec{s}$ (see [9] for details):

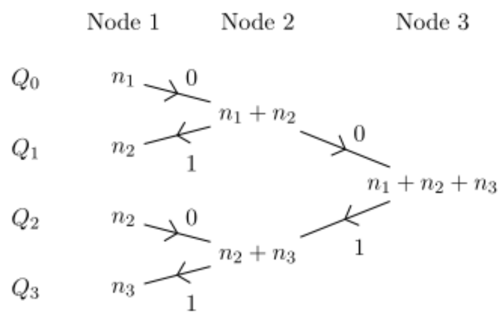


Figure 4.6: The linear 3-node quivers that are Seiberg-dual to the oriented quiver Q_0. Only the gauge nodes are shown, the flavour nodes can be assigned unambiguously such that each 2d gauge node is conformal. The s_I that label the quiver are drawn on the arrows linking the gauge nodes.

For the asymptotically conformal gauge theories, as we have seen, for each duality move, one has to add non-perturbative corrections in order to obtain the correct twisted superpotential of the dual quiver. So, given a dual quiver specified by a permutation $\vec{s}$, there are two steps to be carried out: first, one needs to find out the sequence of Seiberg-duality moves needed to connect the quiver Q_0 to any one of the quivers in the list. Secondly, one has to add appropriate non-perturbative corrections after each duality move.

The way Seiberg duality moves are encoded in terms of the permutation basis can be described by realizing that there are only $M-1$ basic duality moves, that correspond to dualizing one of the $M-1$ 2d gauge nodes. Given that the $M-1$ arrows of the quivers are also denoted by the same vector $\vec{s}$, and knowing the action of duality, which exchanges fundamental with anti-fundamental matter on the dualized node, it then follows that the basis of duality moves can be represented by the following actions on the vector $\vec{s}$:

$$\begin{aligned}
&\mathcal{D}_1 : (* * * \ldots 0) \to (* * * \ldots 1) \\
&\mathcal{D}_2 : (* * * \ldots 10) \to (* * * \ldots 01) \\
&\mathcal{D}_3 : (* * \ldots 10*) \to (* * \ldots 01*) \quad \text{and so on}\,.
\end{aligned} \tag{4.42}$$

In this way, it is easy to find out how any quiver labelled by $\vec{s}$ can be connected to Q_0 by a sequence of duality moves. Once this is done, one can add the appropriate non-perturbative corrections to the twisted superpotential after each duality using the rules explained in Section 4.2 and obtain a row of dual theories, just as before:

$$Q_0 \xleftrightarrow{\mathcal{D}_1} \widetilde{Q}_1 \xleftrightarrow{\mathcal{D}_2} \widetilde{Q}_2 \longleftrightarrow \cdots\,. \tag{4.43}$$

This solves the problem of finding dual quivers related to Q_0 for the generic linear quiver.

We conclude with the following observation: given the localization integrand, one could choose any JK prescription to evaluate the partition function. On the 2d/4d quiver side, this corresponds to choosing a particular quiver Q_k; one could then perform a set of

Seiberg dualities:

$$\widehat{Q}_0 \overset{\mathcal{D}_1}{\longleftrightarrow} \widehat{Q}_1 \overset{\mathcal{D}_2}{\longleftrightarrow} \widehat{Q}_2 \cdots \longleftrightarrow Q_k \longleftrightarrow \widehat{Q}_{k+1} \longleftrightarrow \cdots \tag{4.44}$$

In this duality chain, the quiver Q_k has a Lagrangian that one would write down purely from the quiver itself. All the others $\widehat{Q}_\ell$ are related to it by Seiberg-dualtiy and their superpotentials would differ from those one would write for the quiver Q_ℓ by non-perturbative pieces determined by the sequence of dualities involved. The low energy superpotentials for each quiver in the chain are identical to that obtained for Q_k (up to purely q_0-dependent terms). One can therefore write down 2^{M-1} such duality chains starting with any of the quivers corresponding to a given JK prescription. The results match along the rows of dual quiver: these are interpreted as the result of deforming the integration contour from one set of poles to another, keeping into account the residues at infinity.

Chapter 5

Modular properties of surface operators

In this chapter, we consider a simplest surface operator in $\mathcal{N} = 2$ supersymmetric SQCD with gauge group SU(2) and $N_f = 4$ fundamental flavours in four dimensions and study it's modular properties [6].

5.1 Localization analysis

For the SU(2) theory, there is one monodromy defect that breaks the gauge group on the defect to the Levi subgroup :

$$\mathbb{L} = U(1) \times U(1) \tag{5.1}$$

The defect also breaks the flavour symmetry to [2] :

$$\mathbb{F} = S\,[U(2) \times U(2)] \tag{5.2}$$

The prepotential $\mathcal{F}$ and the twisted chiral superpotential $\mathcal{W}$ receive contributions from classical, 1-loop, and instanton terms :

$$\begin{aligned}\mathcal{F} &= \mathcal{F}^{\text{class}} + \mathcal{F}^{\text{1loop}} + \mathcal{F}^{\text{inst}} \\ \mathcal{W} &= \mathcal{W}^{\text{class}} + \mathcal{W}^{\text{1loop}} + \mathcal{W}^{\text{inst}}\end{aligned} \tag{5.3}$$

The instanton contributions to $\mathcal{F}$ and $\mathcal{W}$ are obtained from the instanton partition function $\mathcal{Z}^{\text{inst}}$ as (2.10). In the presence of a co-dimension 2 surface defect, $\mathcal{Z}^{\text{inst}}$ is obtained by the orbifold procedure detailed in [7, 8, 13, 20, 35–38]. For the SU(2) theory with $N_f = 4$, $\mathcal{Z}^{\text{inst}}$ is given by equations 8-9 of [2] with $M = 2$ and $\vec{n} = [1, 1]$:

$$\mathcal{Z}^{\text{inst}}[1,1] = \sum_{\{d_1,d_2\}} \frac{(-q_1)^{d_1}}{d_1!}\frac{(-q_2)^{d_2}}{d_2!} \int \prod_{\sigma=1}^{d_1} \frac{d\chi_{1,\sigma}}{2\pi i} \int \prod_{\sigma=1}^{d_2} \frac{d\chi_{2,\sigma}}{2\pi i} z_{\{d_1,d_2\}}\,, \tag{5.4}$$

where q_1 and q_2 are the instanton counting parameters, d_1 and d_2 the number of ramified instantons, $\hat{\epsilon}_2 \equiv \frac{\epsilon_2}{2}$, $m_1, \ldots, m_4$ the masses of fundamental flavours, and

$$\begin{aligned} z_{\{d_1,d_2\}} = &\prod_{\sigma,\tau=1}^{d_1} \frac{(\chi_{1,\sigma} - \chi_{1,\tau} + \delta_{\sigma,\tau})}{(\chi_{1,\sigma} - \chi_{1,\tau} + \epsilon_1)} \prod_{\sigma,\tau=1}^{d_2} \frac{(\chi_{2,\sigma} - \chi_{2,\tau} + \delta_{\sigma,\tau})}{(\chi_{2,\sigma} - \chi_{2,\tau} + \epsilon_1)} \\ &\prod_{\sigma=1}^{d_1}\prod_{\rho=1}^{d_2} \frac{\left(\chi_{1,\sigma} - \chi_{2,\rho} + \epsilon_1 + \hat{\epsilon}_2\right)}{\left(\chi_{1,\sigma} - \chi_{2,\rho} + \hat{\epsilon}_2\right)} \prod_{\sigma=1}^{d_2}\prod_{\rho=1}^{d_1} \frac{\left(\chi_{2,\sigma} - \chi_{1,\rho} + \epsilon_1 + \hat{\epsilon}_2\right)}{\left(\chi_{2,\sigma} - \chi_{1,\rho} + \hat{\epsilon}_2\right)} \\ &\prod_{\sigma=1}^{d_1} \frac{(\chi_{1,\sigma} - m_1)(\chi_{1,\sigma} - m_2)}{\left(a_1 - \chi_{1,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\right)\left(\chi_{1,\sigma} - a_2 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\right)} \\ &\prod_{\sigma=1}^{d_2} \frac{(\chi_{2,\sigma} - m_3)(\chi_{2,\sigma} - m_4)}{\left(a_2 - \chi_{2,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\right)\left(\chi_{2,\sigma} - a_1 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\right)}\,. \end{aligned} \tag{5.5}$$

Here a_1 and a_2 are the Coulomb vev's and upon imposing the SU(2) constraint we have $a_1 = -a_2 = a$.

Our choice of contour is such that the integral picks the poles in the upper half $\chi_{1,2}$ plane

[18, 20]. We package the instanton contributions to $\mathcal{F}$ and $\mathcal{W}$ as [1]

$$\mathcal{F}^{\text{inst}} = \sum_{n=0}^{\infty} f_n^{\text{inst}}, \qquad \mathcal{W}^{\text{inst}} = \sum_{n=0}^{\infty} w_n^{\text{inst}} \tag{5.6}$$

where $f_n^{\text{inst}} \sim a^{2-n}$ and $w_n^{\text{inst}} \sim a^{1-n}$. From (5.4) and (2.10) one obtains :

$$\begin{aligned} f_{2k+1}^{\text{inst}} &= 0, \quad \forall \quad k \in \mathbb{Z}_{\geq 0} \\ w_{2k+1}^{\text{inst}} &= 0, \quad \forall \quad k \in \mathbb{Z}^{+} . \end{aligned} \tag{5.7}$$

The first few non-zero f_n^{inst} up to 4 ramified instantons are :

$$\begin{aligned} f_0^{\text{inst}} &= a^2 \left[\frac{q_1 q_2}{2} + \frac{13(q_1 q_2)^2}{64} \right] \\ f_2^{\text{inst}} &= \frac{q_1 q_2}{2} \sum_{i<j} m_i m_j + \frac{(q_1 q_2)^2}{64} \left(\sum_i m_i^2 + 16 \sum_{i<j} m_i m_j \right) \\ f_4^{\text{inst}} &= \frac{1}{2a^2} \left[q_1 q_2 m_1 m_2 m_3 m_4 + \frac{(q_1 q_2)^2}{32} \left(16 m_1 m_2 m_3 m_4 + \sum_{i<j} m_i^2 m_j^2 \right) \right] . \end{aligned} \tag{5.8}$$

We now give the first few non-zero w_n^{inst} up to 4 ramified instantons :

$$\begin{aligned} w_0^{\text{inst}} &= a \left[\frac{q_1}{2} + \frac{3q_1^2}{16} + \frac{5q_1^3}{48} + \frac{35q_1^4}{512} - (q_1 \to q_2) + \frac{q_1 q_2}{16} \left(q_1 + \frac{q_1^2}{2} - (q_1 \to q_2) \right) \right] \\ w_1^{\text{inst}} &= -\frac{m_1 + m_2}{2} \left(q_1 + \frac{q_1^2}{2} + \frac{q_1^3}{3} + \frac{q_1^4}{4} \right) - (m_{1,2} \to m_{3,4}, q_1 \to q_2) \\ w_2^{\text{inst}} &= \frac{1}{a} \left[\frac{\left(m_1^2 + m_2^2\right)}{16} \left(q_1^2 + q_1^3 + \frac{15}{16} q_1^4 - q_1^2 q_2 + \frac{q_1^2 q_2^2}{8} - \frac{q_1^3 q_2}{2} \right) - \left(m_{1,2} \to m_{3,4}, q_1 \leftrightarrow q_2 \right) \right. \\ &\quad + \frac{m_1 m_2}{2} \left(q_1 + \frac{q_1^2}{2} + \frac{3q_1^3}{8} + \frac{5q_1^4}{16} - \frac{q_1 q_2}{2} - \frac{q_1 q_2^2}{8} - \frac{q_1 q_2^3}{16} - \frac{3}{16} q_1^2 q_2^2 - \frac{q_1^3 q_2}{16} \right) \\ &\quad \left. - \left(m_{1,2} \to m_{3,4}, q_1 \leftrightarrow q_2 \right) \right] \\ w_4^{\text{inst}} &= -\frac{1}{16a^3} \left[\frac{1}{32} \left(m_1^4 + m_2^4 \right) q_1^4 - \left(m_3^4 + m_4^4 \right) q_2^4 + \frac{m_1 m_2 m_3 m_4}{2} \left(q_1^3 q_2 - q_1 q_2^3 \right) \right. \end{aligned}$$

[1] When we package the entire prepotential or the twisted chiral superpotential as in (5.6) we use no superscript.

$$+\left(m_1^3m_2+m_1m_2^3\right)\left(\frac{q_1^3}{3}-\frac{q_1^3q_2}{2}+\frac{q_1^4}{2}\right)-(m_{1,2}\to m_{3,4},q_1\leftrightarrow q_2)$$
$$+m_1^2m_2^2\left(q_1^2+q_1^3+\frac{9}{8}q_1^4-q_1^2q_2+\frac{q_1^2q_2^2}{4}-\frac{q_1^3q_2}{2}\right)-(m_{1,2}\to m_{3,4},q_1\leftrightarrow q_2)$$
$$+\left(m_1^2+m_2^2\right)m_3m_4\left(q_1^2q_2-\frac{q_1^2q_2^2}{2}+\frac{q_1^3q_2}{2}\right)-(m_{1,2}\leftrightarrow m_{3,4},q_1\leftrightarrow q_2)\Bigg] \quad (5.9)$$

where we have used $(\to,\leftrightarrow)$ to denote terms that are obtained by performing the switch indicated by the arrows on the immediately preceding terms.

5.2 Superpotential from Seiberg-Witten data

In this section we follow the proposal in [14] according to which the twisted chiral superpotential can be computed from the SW data. This helps us obtain the map that relates the gauge theory parameters to the instanton counting parameters and thereby verify the results from localization obtained in the previous section. According to the proposal in [14] the twisted superpotential is given by the integral of the SW differential λ along an open path on the SW curve :

$$\mathcal{W}(x_0)=\int^{x_0}\lambda\,, \quad (5.10)$$

where x_0 is the continuous parameter that labels the surface operator, and is given by the location of the defect on the Riemann surface.

Let us now recall a few salient features of the SW solution of the SU(2) theory with $N_f=4$ flavours. We will work with the Gaiotto form of the curve as λ is easily extracted from there. The Gaiotto form of the curve is [39]

$$x^2=\phi_2(t)\,, \quad (5.11)$$

where $\phi_2(t)dt^2$ is a quadratic differential. The SW differential is readily given by [39]

$$\lambda = x\,dt = \sqrt{\phi_2(t)}dt\,. \tag{5.12}$$

Let us first analyse the case when the masses of the flavours are set to zero. In this limit, the Gaiotto curve is such that $\phi_2(t)$ takes the form [40]

$$\phi_2(t) = \frac{q_0(q_0-1)}{t(t-q_0)(t-1)}\frac{\partial f_0}{\partial q_0}\,. \tag{5.13}$$

where $q_0 = e^{\pi i \tau_0}$, such that $\tau_0 = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$ is the bare complexified gauge coupling and f_0 is the prepotential in the massless limit. After adding the classical and the 1 loop terms to the instanton contribution obtained via the SW analysis (see [40] for example) which matches the localization results obtained in the previous section we have :

$$\begin{aligned} f_0 &= a^2\log q_0 - a^2\log 16 + f_0^{\text{inst}} \\ &= a^2\left(\log q_0 - \log 16 + \frac{q_0}{2} + \frac{13q_0^2}{64}\right). \end{aligned} \tag{5.14}$$

We substitute for the SW differential from (5.12), (5.13) and (5.14), and perform the integral in (5.10) to obtain :

$$w_0 = a\log x_0 + a\left[\frac{x_0}{2} + \frac{3x_0^2}{16} + \frac{5x_0^3}{48} + \frac{35x_0^4}{512} - \left(x_0 \to \frac{q_0}{x_0}\right) + \frac{q_0}{16}\left(x_0 + \frac{x_0^2}{2} - \left(x_0 \to \frac{q_0}{x_0}\right)\right)\right]. \tag{5.15}$$

By comparing w_0^{inst} from (5.9) and w_0 obtained from the curve (5.15), we obtain the following map between the instanton counting parameters (q_1, q_2) and the gauge theory parameters (q_0, x_0) :

$$q_1 = x_0, \quad q_2 = \frac{q_0}{x_0}\,. \tag{5.16}$$

Note that in f_n^{inst} in (5.8) q_1 and q_2 always appear as the combination $q_1 q_2$ and powers thereof, thus ensuring that the prepotential depends only on q_0 and is independent of x_0.

We will now consider the case when all the masses are turned on. The Gaiotto form of the SW curve is still $x^2 = \phi_2(t)$, where [40]:

$$\phi_2(t) = \frac{q_0(q_0-1)}{t(t-1)(t-q_0)}\frac{\partial \mathcal{F}}{\partial q_0} + \frac{q_0(m_1+m_2)(m_3+m_4)}{2t(t-1)(t-q_0)} - \frac{(q_0-1)(m_3^2+m_4^2)}{2t(t-1)(t-q_0)} \\ - \frac{m_3^2+m_4^2+2m_1m_2}{2t(t-1)} + \frac{(m_3-m_4)^2}{4t^2} + \frac{(m_3+m_4)^2}{4(t-q_0)^2} + \frac{(m_1+m_2)^2}{4(t-1)^2} \,. \tag{5.17}$$

The twisted superpotential when the masses are turned on is obtained exactly as in the massless case by performing the integral in (5.10). One can easily check that the instanton contributions to $\mathcal{W}$ obtained via localization in the previous section matches the results from the SW data after the masses are turned on, provided one uses the map (5.16) between parameters. We have checked that the match holds up to w_8 to 8 ramified instantons.

Now that we have matched $\mathcal{W}$ obtained via localization and from the SW data we will shift gears and turn our attention to utilizing the S-duality symmetry of the theory to resum the instanton contributions.

5.3 Resumming the twisted chiral superpotential

A lot of progress has been made in resumming the instanton contribution to the prepotential of a large class of theories into (quasi-) modular forms of their respective S-duality groups [12, 22–27]. This was then extended to the case of the twisted chiral superpotential of $\mathcal{N} = 2^\star$ SU(N) theory in the presence of a surface defect in [13]. There it was shown that $\mathcal{W}$ satisfies a modular anomaly equation, and that the instanton expansion of $\mathcal{W}$ at each order in a mass expansion can be resummed into elliptic functions and (quasi-) modular forms. Since the SU(2) theory with $N_f = 4$ also has an S-duality symmetry we will now attempt to do the same in this theory.

5.3.1 Resummation variables

Unlike in the $\mathcal{N} = 2^\star$ theory, the gauge coupling and the continuous parameter that describes the surface defect are renormalized in asymptotically conformal SQCD theories. In the theory of interest to us this is already clear from the expressions for $\mathcal{F}$ and $\mathcal{W}$ in the massless limit in (5.14) and (5.15) respectively. We would like to resum the terms on the RHS of these equations to simple expressions in terms of the renormalized counterparts q and x of q_0 and x_0 respectively. The q_0 vs q relation has appeared in several references and is given by [12, 15, 27, 41] :

$$q_0 = \frac{e_3 - e_2}{e_1 - e_2}(q) = \frac{\theta_2^4(q)}{\theta_3^4(q)} \tag{5.18}$$

where $e_i \equiv \wp(\omega_i)$ denote the Weierstraß $\wp$ function evaluated at the half periods and θ_i are the Jacobi θ functions. We refer the reader to Appendix D for details on Jacobi theta functions and the Weierstraß $\wp$ function. The first few terms that appear in the expansion of (5.18) are :

$$q_0 = 16q(1 - 8q + 44q^2 - 192q^3 + \ldots) \tag{5.19}$$

One can now check that $\widetilde{f_0}$ which is the prepotential in the massless limit (5.14) when expressed in terms of q takes the expected form :

$$\widetilde{f_0} = a^2 \log q\,. \tag{5.20}$$

Note that here and henceforth we use the tilde symbol to denote quantities expressed in terms of the renormalized variables (q, x).

For the parameter x_0, following the analysis in [15] we have the following map to the resummed variable x :

$$x_0 = \frac{\wp(z + w_1|\tau) - e_2}{e_1 - e_2} \tag{5.21}$$

where τ and z are such that

$$q = \exp(\pi i\tau), \quad x = \exp(2\pi iz)\,. \tag{5.22}$$

A similar map was also used in the recent paper [42]. We verify this map in Appendix E using the SW analysis in the massless limit. Note that the q_0 vs q map in (5.18) is a special case of (5.21) for $z = w_2$. The first few terms that appear in the expansion of (5.21) are :

$$x_0 = 4(x - 2x^2 + 3x^3 - 4x^4) + 8q(1 - 4x + 8x^2 - 12x^3) + 4q^2\left(\frac{1}{x} - 12\right) + \ldots \tag{5.23}$$

With the above expansions for q_0 and x_0 one can check that up to purely q_0 dependent terms, $\widetilde{w}_0(q, x)$ which is the twisted superpotential in the massless limit (5.15) takes the expected form :

$$\widetilde{w}_0 = a\log x\,. \tag{5.24}$$

In the next section where we resum the instanton contributions to $\mathcal{W}$ we will find it more convenient to work with the $\log x$ derivative of $\widetilde{\mathcal{W}}$ whose expansion is :

$$x\frac{\partial\widetilde{\mathcal{W}}}{\partial x} \equiv \widetilde{\mathcal{W}}' = \sum_{n=0}^{\infty}\widetilde{w}'_n \tag{5.25}$$

where $\widetilde{w}'_n \sim a^{1-n}$. Clearly from (5.24) we have :

$$\widetilde{w}'_0 = a\,. \tag{5.26}$$

The next few non-zero $\widetilde{w}'_n$ obtained by substituting the expansions for q_0 and x_0 from (5.19) and (5.23) in (5.9) are :

$$\widetilde{w}'_1 = -2(m_1 + m_2)\left(x + x^3 - \frac{q^2}{x}\right) - 2(m_3 + m_4)\left(q\,x - \frac{q}{x}\right)$$

$$\widetilde{w}'_2 = \frac{1}{a}\Big[2(m_1^2+m_2^2)(x^2+2x^4) + \frac{2q^2}{x^2}(m_3^2+m_4^2) + 2m_1m_2\left(x+3x^3+\frac{q^2}{x}\right) + 2m_3m_4\left(\frac{q}{x}+q\,x\right)\Big]$$

$$\widetilde{w}'_4 = -\frac{1}{a^3}\Big[2(m_1^4+m_2^4)x^4 + 4m_1m_2(m_1^2+m_2^2)x^3 + 2m_1^2m_2^2(x^2+8x^4) + 2m_3^2m_4^2\frac{q^2}{x^2} + 4m_3m_4(m_1^2+m_2^2)q\,x + 4m_1m_2(m_3^2+m_4^2)\frac{q^2}{x} + 16m_1m_2m_3m_4\,q\,x^2\Big] \tag{5.27}$$

The above expressions will be useful in the next sub-section when we resum $\widetilde{w}'_n$ to linear combinations of elliptic functions and (quasi-) modular forms.

5.3.2 Modular Anomaly Equation for the twisted superpotential

It is well known from [11] that the SU(2) theory with $N_f = 4$ enjoys an S-duality symmetry under which the renormalized gauge coupling τ transforms as

$$\tau \to -\frac{1}{\tau}\,. \tag{5.28}$$

It was shown in [1] (see also [13]) that under this duality the variable z that is related to the continuous parameter x that labels the defect as in (5.22) transforms as

$$z \to -\frac{z}{\tau}\,. \tag{5.29}$$

The action of S-duality on the Coulomb vev a is such that

$$S(a) := a_D = \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial a} = \tau\left(a + \frac{\delta}{12}\frac{\partial f}{\partial a}\right) \tag{5.30}$$

where $\delta = \frac{6}{\pi i \tau}$ and $f = \mathcal{F}^{1\text{ loop}} + \mathcal{F}^{\text{inst}}$. The anomalous terms on the RHS arise solely from the dependence of the prepotential on the second Eisenstein series E_2 [12]. From the form of $\widetilde{w}'_0$ in (5.26), we see that it transforms exactly as in (5.30).

Motivated by the transformation of $\widetilde{\mathcal{W}'}^{\text{class}}$ we propose that, as in [13], $\widetilde{\mathcal{W}'}$ transforms under S-duality with weight one. The $\widetilde{w}'_n$ in (5.25) then obey a modular anomaly equation, the derivation of which proceeds exactly as in the case of the $\mathcal{N} = 2^\star$ theory in [13]. The anomaly equation is :

$$\frac{\partial \widetilde{w}'_n}{\partial E_2} + \frac{1}{12}\sum_{l=0}^{n-1}\left(\frac{\partial \widetilde{w}'_\ell}{\partial a}\right)\left(\frac{\partial \widetilde{f}_{n-\ell}}{\partial a}\right) = 0 \tag{5.31}$$

Since $\widetilde{w}_1$ and $\widetilde{w}'_1$ are independent of a, they do not contribute to the IR dynamics and we start our analysis at $n = 2$. For $n = 2$ the equation takes the form :

$$\frac{\partial \widetilde{w}'_2}{\partial E_2} + \frac{1}{12}\left(\frac{\partial \widetilde{w}'_0}{\partial a}\right)\left(\frac{\partial \widetilde{f}_2}{\partial a}\right) = 0 \tag{5.32}$$

The prepotential for this theory was resummed in [12] and in particular :

$$\widetilde{f}_2 = 2R\log\left(\frac{a}{\Lambda}\right). \tag{5.33}$$

where

$$R = \frac{1}{2}\sum_{f=1}^{4} m_f^2 . \tag{5.34}$$

We substitute for $\widetilde{w}'_0$ from (5.26) and $\widetilde{f}_2$ from (5.33) and solve (5.32) to obtain,

$$\widetilde{w}'_2 = -\frac{E_2 R}{6a} + \frac{1}{a}\,(\text{modular term}) \tag{5.35}$$

Since the modular terms that one must add to (5.35) must have weight two, one arrives at the following ansatz for $\widetilde{w}'_2$:

$$\widetilde{w}'_2 = -\frac{E_2 R}{6a} + \frac{1}{a}\sum_{A=0}^{3} c_A\,\wp(z+\omega_A) \tag{5.36}$$

The coefficients c_A are fixed by comparing the expansion of the RHS of the above equation with the first few terms in the localization result for the same expressed in terms of (q, x)

in (5.27). This leads to :

$$\widetilde{w}_2' = -\frac{1}{6a}\sum_{A=0}^{3} M_A^2\left(E_2 + 12\widehat{\wp}(z+\omega_A)\right) \tag{5.37}$$

where M_A are the following mass combinations :

$$M_0 = -\frac{(m_1+m_2)}{2},\;\; M_1 = \frac{(m_1-m_2)}{2},\;\; M_2 = \frac{(m_3+m_4)}{2},\;\; M_3 = \frac{(m_3-m_4)}{2}\,, \tag{5.38}$$

which appear as residues of the quadratic differential in the SW data. From the resummed result for w_2' in (5.37), one can see that under the combined action of S-duality on the gauge coupling and the triality transformation on the masses of the fundamental flavours, the a independent part transforms as a quasi-modular form of weight two.

We performed a similar analysis of (5.31) at the next two levels. This required the following resummed expressions for the prepotential at $n = 4, 6$ [12] :

$$\begin{aligned}
\widetilde{f}_4 &= -\frac{R^2 E_2}{6} + T_1\theta_4^4 - T_2\theta_2^4 \\
\widetilde{f}_6 &= -\frac{R^3\left(5E_2^2 + E_4\right)}{180a^4} - \frac{NE_4}{5a^4} \\
&\quad + \frac{RT_1\theta_4^4\left(2E_2 + 2\theta_2^4 + \theta_4^4\right)}{6a^4} - \frac{RT_2\theta_2^4\left(2E_2 - 2\theta_4^4 - \theta_2^4\right)}{6a^4}
\end{aligned} \tag{5.39}$$

where

$$\begin{aligned}
T_1 &= \frac{1}{12}\sum_{f<f'=1}^{4} m_f^2 m_{f'}^2 - \frac{1}{24}\sum_{f=1}^{4} m_f^4 \\
T_2 &= -\frac{1}{24}\sum_{f<f'=1}^{4} m_f^2 m_{f'}^2 + \frac{1}{48}\sum_{f=1}^{4} m_f^4 - \frac{1}{2}m_1m_2m_3m_4 \\
N &= \frac{3}{16}\sum_{f<f'<f''=1}^{4} m_f^2 m_{f'}^2 m_{f''}^2 - \frac{1}{96}\sum_{f\neq f'=1}^{4} m_f^2 m_{f'}^4 + \frac{1}{96}\sum_{f=1}^{4} m_f^6\,.
\end{aligned} \tag{5.40}$$

Here R, T_i, and N are the first few mass invariants that transform under the triality action as :

$$R \to R, \quad T_1 \leftrightarrow T_2, N \to N\,. \tag{5.41}$$

Solving the modular anomaly equation (5.31) at $n = 4, 6$ we obtained the following resummed results for $\widetilde{w}'_4$ and $\widetilde{w}'_6$:

$$\begin{aligned}
\widetilde{w}'_4 = -\frac{1}{72a^3}\Big(&\sum_{A=0}^{3} M_A^4\left(2E_2^2 - E_4 + 24E_2\widehat{\wp}(z+\omega_A) + 144\widehat{\wp}(z+\omega_A)^2\right)\\
&+ 2\sum_{A<B} M_A^2 M_B^2\Big(2E_2^2 - E_4 + 12E_2\widehat{\wp}(z+\omega_A) + 12E_2\widehat{\wp}(z+w_B)\\
&+ 144\widehat{\wp}(z+\omega_A)\widehat{\wp}(z+\omega_B)\Big) - 12T_1\theta_4^4(E_2 - 2\theta_2^4 - \theta_4^4) + 12T_2\theta_2^4(E_2 + \theta_2^4 + 2\theta_4^4)\Big)\\
\widetilde{w}'_6 = -\frac{1}{432a^5}&\left(\sum_{A=0}^{3} M_A^2\left(E_2 + 12\widehat{\wp}(z+\omega_A)\right)\right)\Big(\sum_{B=0}^{3} M_B^4\Big(2E_2^2 - E_4 + 24E_2\widehat{\wp}(z+\omega_B)\\
&+ 144\widehat{\wp}(z+\omega_B)^2\Big) + 2\sum_{B<C} M_B^2 M_C^2\Big(2E_2^2 - E_4 + 12E_2\widehat{\wp}(z+\omega_B) + 12E_2\widehat{\wp}(z+w_C)\\
&+ 144\widehat{\wp}(z+\omega_B)\widehat{\wp}(z+\omega_C)\Big) - 12T_1\theta_4^4(E_2 - 2\theta_2^4 - \theta_4^4) + 12T_2\theta_2^4(E_2 + \theta_2^4 + 2\theta_4^4)\Big)\\
&- \frac{R^3}{720a^5}(5E_2^3 - E_2E_4 - 2E_6) - \frac{N}{15a^5}(E_2E_4 - E_6) + \frac{R}{12a^5}(T_1\theta_4^4 - T_2\theta_2^4)(E_2^2 - E_4)
\end{aligned} \tag{5.42}$$

Note that as in the case of $\widetilde{w}'_2$, under the combined action of S-duality and triality, $\widetilde{w}'_4$ and $\widetilde{w}'_6$ transform as expected. The above resummed results have been matched with explicit results from localization expressed in terms of the renormalized variables (q, x) up to 8 ramified instantons.

Appendix A

Contour deformations

In this section, we derive the 2d/4d Seiberg duality rule in SQCD by lifting the theory to three dimensions and studying the partition function of the surface operator of type $[n_1, n_2]$ with support $\mathbb{R}^2 \times S^1_\beta$ in $\mathbb{R}^4 \times S^1_\beta$. As we shall see, the extra circle direction allows us to relate the partition functions for the $(+-)$ and $(-+)$ contours up to all orders in the instanton expansion. We follow the basic ideas in [43] though we will keep the 4d instanton weight $q_0 \neq 0$. In the end we will take the four dimensional limit $\beta \to 0$ and set the Ω-deformation parameters ϵ_i to zero in order to read off how the 2d twisted superpotentials obtained using the two prescriptions are related.

The partition function for the $(+-)$ contour in the 2-node case is given by

$$Z^{+-} = \sum_{d_1,d_2} \frac{(-q_1)^{d_1}}{d_1!} \frac{(-q_2)^{d_2}}{d_2!} \int_+ \prod_{\sigma=1}^{d_1} \frac{d\chi_{1,\sigma}}{2\pi \mathrm{i}} \int_- \prod_{\rho=1}^{d_2} \frac{d\chi_{2,\rho}}{2\pi \mathrm{i}} z_{\{d_I\}}\,, \tag{A.1}$$

where the integrand takes the following form:

$$\begin{aligned} z_{\{d_I\}} = &\prod_{I=1}^{2} \prod_{\sigma,\tau=1}^{d_I} \frac{\sinh\frac{\beta}{2}(\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma,\tau})}{\sinh\frac{\beta}{2}(\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1)} \\ &\times \prod_{\sigma=1}^{d_1} \prod_{\rho=1}^{d_2} \frac{\sinh\frac{\beta}{2}\left(\chi_{1,\sigma} - \chi_{2,\rho} + \epsilon_1 + \hat{\epsilon}_2\right)}{\sinh\frac{\beta}{2}\left(\chi_{1,\sigma} - \chi_{2,\rho} + \hat{\epsilon}_2\right)} \frac{\sinh\frac{\beta}{2}\left(\chi_{2,\rho} - \chi_{1,\sigma} + \epsilon_1 + \hat{\epsilon}_2\right)}{\sinh\frac{\beta}{2}\left(\chi_{2,\rho} - \chi_{1,\sigma} + \hat{\epsilon}_2\right)} \end{aligned} \tag{A.2}$$

$$\times \prod_{I=1}^{2} \prod_{\sigma=1}^{d_I} \frac{\prod_{i\in\mathcal{F}_I} \sinh\frac{\beta}{2}(\chi_{I,\sigma}-m_i)}{\prod_{s\in\mathcal{N}_I} \sinh\frac{\beta}{2}\left(a_s-\chi_{I,\sigma}+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\right)\prod_{t\in\mathcal{N}_{I+1}} \sinh\frac{\beta}{2}\left(\chi_{I,\sigma}-a_t+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\right)} \,.$$

This is obtained from the integrand in (2.16) by lifting rational functions to trigonometric functions. Since we are eventually interested only in the strict 4d limit, we have not turned on either 3d or 5d Chern-Simons levels. Given this starting point, our goal is to deform the contour to obtain Z_{-+}, knowing that as emphasized in [43], due to a non-trivial residue at infinity, one should obtain a wall-crossing type pre-factor.

Let us review how this works for the case in which $q_2 = 0$. The second line in (A.2) is not present in such a case and only terms with $I = 1$ survive. The integral receives contributions from multiple residues at the singularities of the integrand. When we reverse the contour, among the possible singularities to consider there are both the ones at finite points and the ones at infinity. In the particular case of our integrand, there are residues at both asymptotes $\pm\infty$. Out of the d_1 integration variables, let us assume that p_1 of them are evaluated at their poles in the asymptotes. We will eventually sum over all values of p_1 from 0 up to d_1. The integrand breaks up naturally into three sets of terms: the first involves just the p_1 variables that approach infinity; a second, which involves only the complementary set and a last piece, which involves both; after taking the limit in which the $\chi_{1,\sigma}$ are taken in this last piece, we find the following result for the residue:

$$\begin{aligned}\mathrm{Res}_{(+)}z_{d_1} = \sum_{p=0}^{d_1} \mathrm{Res}_{\cap_\sigma \chi_{1,\sigma}\in\mathrm{Asymp}\pm}\left[\prod_{\sigma,\tau=1}^{p_1} \frac{\sinh\frac{\beta}{2}\left(\chi_{1,\sigma}-\chi_{1,\tau}+\delta_{\sigma,\tau}\right)}{\sinh\frac{\beta}{2}\left(\chi_{1,\sigma}-\chi_{1,\tau}+\epsilon_1\right)}\right] \\ \times(-1)^{p_1 n_1} e^{-\frac{\beta}{2}p_1\left(\sum_{j\in\mathcal{F}_1} m_j-\sum_{u\in\mathcal{N}_1\cup\mathcal{N}_2} a_u\right)}\,\mathrm{Res}_{(-)}z_{d_1-p_1}\,. \qquad \text{(A.3)}\end{aligned}$$

The sum over a_u gives zero due to the tracelessness condition. The way to deal with the residue coming from the asymptotic region is identical to what is calculated in [43] and we refer the reader to that reference for the details. The final result for the case when there are equal numbers of fundamental and anti-fundamental 2d flavours is given as follows (see equation (4.44) of [43]):

$$Z^+ = Z^- \times \mathrm{PE}\left[\frac{2(-1)^{n_1} q_1 \left(e^{-\frac{\beta}{2}\sum_{i\in\mathcal{F}_1} m_i} - e^{+\frac{\beta}{2}\sum_{i\in\mathcal{F}_1} m_i}\right)}{(1 - e^{2\beta\epsilon_1})}\right] \tag{A.4}$$

Here we have written the result in terms of the plethystic exponential. For a function $f(t)$ given by a series expansion:

$$f(q_1) = \sum_{n=0}^{\infty} f_n q_1^n \implies \mathrm{PE}\left[f(q_1)\right] = \frac{1}{\prod_{n=1}^{\infty}(1 - q_1^n)^{f_n}} . \tag{A.5}$$

For our case, the function whose plethystic exponential is taken is a linear one and we consider a series expansion in $(-1)^{n_1+1} q_1$. In order to understand what this means for the superpotential that governs 4d effective action, we take the $\beta \to 0$ limit and find

$$Z^+ = Z^- \times \mathrm{PE}\left[(-1)^{n_1} q_1 \frac{1}{\epsilon_1} \sum_{i\in\mathcal{F}_1} m_i\right] \tag{A.6}$$

The $\epsilon_1 \to 0$ limit then allows one to extract the low energy twisted chiral superpotential from the partition function via[1]

$$\lim_{\epsilon_1 \to 0} Z^{\pm} = e^{\frac{\mathcal{W}^{\pm}}{\epsilon_1}} . \tag{A.7}$$

Putting all this together, we find that

$$\mathcal{W}^+ - \mathcal{W}^- = \log(1 + (-1)^{n_1} q_1) \sum_{i\in\mathcal{F}_1} m_i . \tag{A.8}$$

Using the relation (4.21) between the vortex counting parameter q_1 and the exponentiated FI parameter of the quiver Q_0, we find that (A.8) is the same result proposed in [10] for the twisted chiral superpotentials of quivers related by Seiberg duality using the S^2 partition function. Here we have shown that this can be derived from a simple contour deformation argument.

[1] Since $q_0 = 0$, the prepotential is zero.

We now turn to generalize this to the case in which $q_0 \neq 0$, starting from the integrand in (A.2). We begin with the $(+-)$ contour and deform it to the $(-+)$ contour; in the deformation process, one picks up contributions from the asymptotes $\chi_{I,\sigma} \to \pm\infty$. Let us consider the term in which, out of the (d_1, d_2) integration variables, we let (p_1, p_2) of them to approach infinity. As before, the integrand breaks up into three sets of terms: the first involves only those $\chi_{I,\sigma}$ that take asymptotic values; another set that involves the complementary $\chi_{I,\sigma}$ that take finite values and lastly, those that take values in both sets. After taking the asymptotic limit in this last piece and summing over all the possible values for p_1 and p_2, we find the following result for the residue:

$$\begin{aligned}
\mathrm{Res}_{(+-)} z_{d_1,d_2} = \sum_{p_1=0}^{d_1} \sum_{p_2=0}^{d_2} \mathrm{Res}_{\cap_\sigma \chi_{1,\sigma} \in \mathrm{Asymp}\pm} \Bigg[\prod_{I=1}^{2} \prod_{\sigma,\tau=1}^{p_1} \frac{\sinh \frac{\beta}{2} (\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma,\tau})}{\sinh \frac{\beta}{2} (\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1)} \\
\times \prod_{\sigma=1}^{p_1} \prod_{\rho=1}^{p_2} \frac{\sinh \frac{\beta}{2} \left(\chi_{1,\sigma} - \chi_{2,\rho} + \epsilon_1 + \hat{\epsilon}_2\right)}{\sinh \frac{\beta}{2} \left(\chi_{1,\sigma} - \chi_{2,\rho} + \hat{\epsilon}_2\right)} \frac{\sinh \frac{\beta}{2} \left(\chi_{2,\rho} - \chi_{1,\sigma} + \epsilon_1 + \hat{\epsilon}_2\right)}{\sinh \frac{\beta}{2} \left(\chi_{2,\rho} - \chi_{1,\sigma} + \hat{\epsilon}_2\right)} \Bigg] \\
\times (-1)^{p_1 n_1} e^{-\frac{\beta}{2} p_1 \sum_{i\in\mathcal{F}_1} m_i} (-1)^{p_2 n_2} e^{-\frac{\beta}{2} p_2 \sum_{i\in\mathcal{F}_2} m_i} \, \mathrm{Res}_{(-+)} z_{d_1-p_1, d_2-p_2} \, .
\end{aligned} \tag{A.9}$$

In the asymptotic residue, there is now a mixed term between the $\chi_{1,\sigma}$ and $\chi_{2,\rho}$; however, the key observation is that we are only interested in how the twisted chiral superpotential changes across the contour deformation and not the whole partition function, which is obtained by setting $\epsilon_2 \to 0$. In this limit, the mixed term is an even function of $\chi_{1,\sigma} - \chi_{2,\rho}$ and does not lead to any new pole that might contribute to the twisted chiral superpotential. As a result, the residue calculation factorizes into a contribution from the $\chi_{1,\sigma}$ integrals and that from the $\chi_{2,\rho}$ integrals; the calculation for each set is identical to that done for the purely 2d case and we obtain in the 4d limit,

$$Z^{+-} = Z^{-+} \times \mathrm{PE}\left[(-1)^{n_1} q_1 \frac{1}{\epsilon_1} \sum_{i\in\mathcal{F}_1} m_i \right] \times \mathrm{PE}\left[(-1)^{n_2} q_2 \frac{1}{\epsilon_1} \sum_{i\in\mathcal{F}_2} m_i \right] \tag{A.10}$$

By using the formula for the plethystic exponential, the tracelessness of the flavour group

SU(2N), and the form of the instanton partition function in the limit $\epsilon_1 \to 0$ we finally obtain

$$\mathcal{W}^{+-} - \mathcal{W}^{-+} = \big(\log(1 + (-1)^{n_1} q_1) + \log(1 + (-1)^{n_2} q_2)\big) \sum_{i \in \mathcal{F}_1} m_i \, . \tag{A.11}$$

Using the map between the q_I and the exponentiated FI parameters and the 4d couplings we derived in (4.21), and by identifying the $(+-)$ and $(-+)$ contours with the corresponding quivers, we derive the following rule for how the twisted superpotential transforms under the action of Seiberg duality:

$$\delta W = \left[\log(1 - (-1)^{N_f} x) + \log\left(1 - (-1)^{N_f} \frac{q_0}{x}\right)\right] \sum_{i \in \mathcal{F}_1} m_i \, , \tag{A.12}$$

where N_f denotes the number of (anti-) fundamental flavours attached to the 2d gauge node.

Appendix B

4d corrections to the 2d Lagrangian

In this section we show how to evaluate the 4d instanton corrections to the 2d twisted chiral superpotential due to the presence of the chiral correlator $\langle \mathrm{Tr}\, \varpi(\sigma - \Phi) \rangle$. We write this function as follows:

$$\langle \, \mathrm{Tr} \; \varpi(z - \Phi) \, \rangle = \int^{z} dz' \left\langle \mathrm{Tr} \;\; \log \frac{(z' - \Phi)}{\mu} \right\rangle . \tag{B.1}$$

We observe that the 4d observable on the R.H.S. is itself the integral of the generating function of the chiral correlators in the 4d gauge theory, and is referred to as the resolvent of the 4d theory. So we begin with a brief review of known results regarding the resolvent of the $\mathcal{N} = 2$ supersymmetric SQCD gauge theory (we follow the discussion in [44]). We then show how the quantum gauge polynomial can be written in terms of the chiral correlators of the gauge theory and finally we show how the 2d Lagrangian is affected by the coupling to the four dimnensional theory.

B.1 Resolvents and chiral correlators in 4d asymptotically conformal SQCD

The Seiberg-Witten curve of the asymptotically conformal SU(N) gauge theory with $N_f = 2N$ fundamental flavours is given by

$$Y^2 = \widehat{P}(z)^2 - g^2 B(z)\,, \tag{B.2}$$

where the characteristic gauge polynomial is given by

$$\widehat{P}(z) = z^N + u_2 z^{N-2} + \ldots + (-1)^N u_N\,, \tag{B.3}$$

and the flavour polynomial is given by

$$B(z) = \prod_{i=1}^{2N} (z - m_i)\,. \tag{B.4}$$

The constant g^2 is related to the Nekrasov counting parameter q_0 by

$$g^2 = \frac{4q_0}{(1+q_0)^2}\,. \tag{B.5}$$

The Seiberg-Witten differential is given by

$$\lambda_{SW} = z\,\frac{d\Psi(z)}{dz}dz\,, \tag{B.6}$$

where the function $\Psi(z)$ is

$$\Psi(z) = \log\left(\frac{\widehat{P}(z) + Y}{\mu^N}\right). \tag{B.7}$$

The chiral correlators of the gauge theory $\left\langle \mathrm{Tr}\, \Phi^{\ell} \right\rangle$ can be obtained by expanding (for large z) the resolvent:

$$\left\langle \mathrm{Tr}\, \frac{1}{z-\Phi} \right\rangle = \frac{d\Psi(z)}{dz} \,. \tag{B.8}$$

Integrating with respect to z, we find that the integral of the resolvent has a simple form in terms of the function $\Psi(z)$:

$$\left\langle \mathrm{Tr}\, \log \frac{z-\Phi}{\mu} \right\rangle = \log \left((1+q_0) \frac{\widehat{P}+Y}{2\mu^N} \right) . \tag{B.9}$$

The constant log piece added on the R.H.S ensures that the large-z expansion of both sides match.

B.2 Chiral correlators vs. quantum gauge polynomial

Given the gauge polynomial in (B.3) and using equation (B.9), it is possible to write the coefficients that appear in the gauge polynomial in terms of the chiral correlators of the quantum gauge theory, which can be calculated from first principles using localization methods [44–47]. Unlike the case of pure gauge theory, in the asymptotically conformal case, this relation is subtle due to the presence of the dimensionless coupling q_0 that appears non-trivially in the resolvent. To extract this relation, it is convenient to use an equivalent expression for the resolvent [44]:

$$\left\langle \mathrm{Tr}\, \log \frac{z-\Phi}{\mu} \right\rangle = \frac{1}{2} \log \frac{\widehat{P}(z)+Y}{\widehat{P}(z)-Y} + \frac{1}{2} \log \frac{B(z)}{\mu^{2N}} + \frac{1}{2} \log q_0 \,. \tag{B.10}$$

Expanding the R.H.S of (B.10) for large z and equating the coefficients of z^{-k} on both sides of the equation allows us to express the u_k purely in terms of the $\langle \mathrm{Tr}\, \Phi^k \rangle$. In order to write down compact expressions, we express the flavour polynomial also in terms of the symmetric polynomials of the masses S_k:

$$B(z) = z^{2N} + \sum_{j=2}^{2N} (-1)^j S_j z^{2N-j} \,. \tag{B.11}$$

For the lowest orders, following this procedure, we find:

$$\begin{aligned}
u_2 &= -\frac{1}{2}\left(\frac{1-q_0}{1+q_0}\right)\langle \mathrm{Tr}\,\Phi^2\rangle + \frac{q_0}{1+q_0} S_2 \\
u_3 &= +\frac{1}{3}\left(\frac{1-q_0}{1+q_0}\right)\langle \mathrm{Tr}\,\Phi^3\rangle + \frac{q_0}{1+q_0} S_3 \\
u_4 &= -\frac{1}{4}\left(\frac{1-q_0}{1+q_0}\right)\langle \mathrm{Tr}\,\Phi^4\rangle + \frac{1}{2}\langle \mathrm{Tr}\,\Phi^2\rangle\left(\frac{1}{4}\langle \mathrm{Tr}\,\Phi^2\rangle + \frac{q_0 S_2}{1+q_0}\right) + \frac{q_0}{1+q_0} S_4 \\
u_5 &= +\frac{1}{5}\left(\frac{1-q_0}{1+q_0}\right)\langle \mathrm{Tr}\,\Phi^5\rangle - \frac{1}{3}\langle \mathrm{Tr}\,\Phi^3\rangle\left(\frac{1}{2}\langle \mathrm{Tr}\,\Phi^2\rangle + \frac{q_0 S_2}{1+q_0}\right) + \frac{q_0 S_3}{2(1+q_0)}\langle \mathrm{Tr}\,\Phi^2\rangle + \frac{q_0}{1+q_0} S_5 \\
u_6 &= -\frac{1}{6}\left(\frac{1-q_0}{1+q_0}\right)\langle \mathrm{Tr}\,\Phi^6\rangle + \frac{1}{4}\langle \mathrm{Tr}\,\Phi^4\rangle\left(\frac{1}{2}\langle \mathrm{Tr}\,\Phi^2\rangle + \frac{q_0 S_2}{1+q_0}\right) + \frac{1}{3}\langle \mathrm{Tr}\,\Phi^3\rangle\left(\frac{1}{6}\langle \mathrm{Tr}\,\Phi^3\rangle - \frac{q_0 S_3}{1+q_0}\right) \\
&\quad - \frac{1}{2}\left(\frac{1-q_0}{1+q_0}\right)\langle \mathrm{Tr}\,\Phi^2\rangle\left(\frac{1}{24}\left(\langle \mathrm{Tr}\,\Phi^2\rangle\right)^2 - \frac{q_0 S_2}{4(1-q_0)}\langle \mathrm{Tr}\,\Phi^2\rangle - \frac{q_0}{(1-q_0)} S_4\right) + \frac{q_0}{1+q_0} S_6 \,.
\end{aligned} \tag{B.12}$$

B.3 Weak coupling expansions and 4d corrections to the 2d superpotential

In this section we expand the resolvent in (B.10) as an expansion in small q_0. As we have seen, the coefficients u_k in the gauge polynomial $P(z)$ have a q_0-expansion; let us formally expand the gauge polynomial as follows:

$$\widehat{P}(z) = P(z) + \sum_{n=1}^{\infty} p_n(z) q_0^n \,, \tag{B.13}$$

where $P(z)$ is the classical gauge polynomial defined in (4.7), and the $p_n(z)$'s can be calculated using (B.12). Then we find that the resolvent has the following instanton expansion (we suppress the z-dependence of the polynomials in order to have compact expressions):

$$\left\langle \mathrm{Tr}\,\log\frac{z-\Phi}{\mu}\right\rangle = \log\frac{P}{\mu} + q_0\left(1+\frac{p_1}{P}-\frac{B}{P^2}\right) + q_0^2\left(-\frac{3B^2}{2P^4}+\frac{2B\,(P+p_1)}{P^3}-\frac{P^2-2p_2P+p_1^2}{2P^2}\right)+\dots\,. \tag{B.14}$$

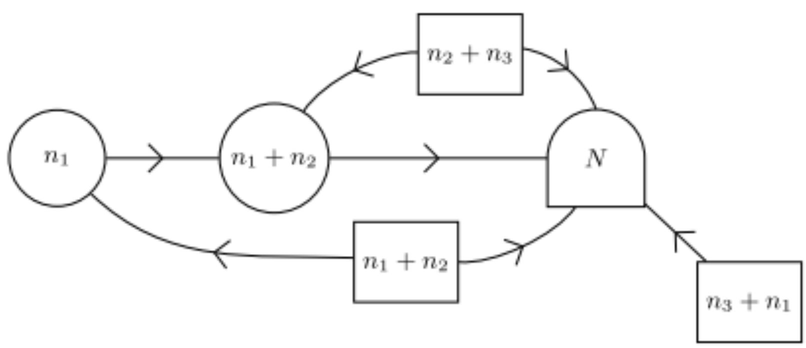
Substituting this into (B.1) and performing the integral, we obtain the one and two instanton corrections to the twisted superpotential of the 2d quiver due to the 4d theory.

Appendix C

Chiral ring equations and superpotentials at the 1-instanton level

In this section we study the four quivers shown in Figure 4.5 in turn, write down the twisted chiral ring equations and calculate the 1-instanton result for the low energy superpotential, which is the evaluation of the twisted chiral superpotential in a particular vacuum. Given these results one can check explicitly that the low energy superpotential for the distinct quivers are different already at the 1-instanton level.

Quiver Q_0



The twisted superpotential is

$$
\begin{aligned}
\mathcal{W}_{Q_0}(x) = \log x_1 \sum_{s\in\mathcal{N}_1} \sigma_s^{(1)} + \log x_2 \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \sigma_s^{(2)} \\
- \sum_{t\in\mathcal{N}_1\cup\mathcal{N}_2} \sum_{s\in\mathcal{N}_1} \varpi(\sigma_s^{(1)} - \sigma_t^{(2)}) - \sum_{i\in\mathcal{F}_1} \sum_{s\in\mathcal{N}_1} \varpi(m_i - \sigma_s^{(1)}) \\
- \sum_{i\in\mathcal{F}_2} \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \varpi(m_i - \sigma_s^{(2)}) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \langle \mathrm{Tr}\, \varpi(\sigma_s^{(2)} - \Phi) \rangle \, . \qquad \text{(C.1)}
\end{aligned}
$$

The chiral ring equations are:

$$
\begin{aligned}
G_2(\sigma_s^{(1)}) &= (-1)^{n_1+n_2}\; x_1 B_1(\sigma_s^{(1)}) \quad \text{for} \quad s \in \mathcal{N}_1 \, , \\
(1+q_0)\widehat{P}(\sigma_s^{(2)}) &= (-1)^N \left(x_2\, G_1(\sigma_s^{(2)}) B_2(\sigma_s^{(2)}) + \frac{q_0}{x_2} \frac{B_1(\sigma_s^{(2)}) B_3(\sigma_s^{(2)})}{G_1(\sigma_s^{(2)})} \right) \text{for } s \in \mathcal{N}_1 \cup \mathcal{N}_2 \, ,
\end{aligned} \qquad \text{(C.2)}
$$

where $G_1(z)$ and $G_2(z)$ are the 2d gauge polynomials for the quiver. We solve the equations about the following vacuum:

$$
\begin{aligned}
\sigma_s^{(1)} &= a_s \quad \text{for} \quad s \in \mathcal{N}_1 \, , \\
\sigma_s^{(2)} &= a_s \quad \text{for} \quad s \in \mathcal{N}_1 \cup \mathcal{N}_2 \, .
\end{aligned} \qquad \text{(C.3)}
$$

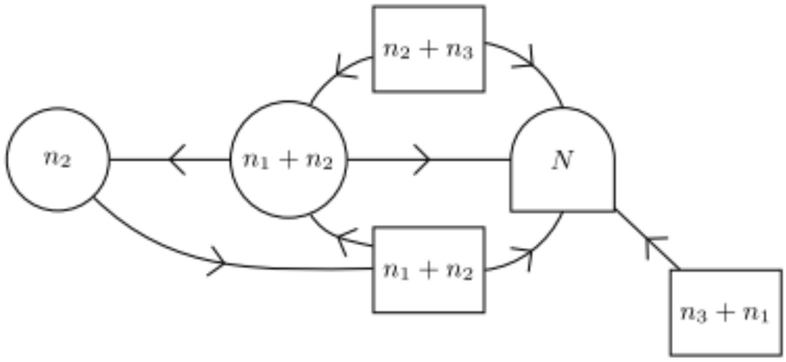
Then at 1-instanton the twisted superpotential evaluated on the solution is

$$
\begin{aligned}
\mathcal{W}\Big|_{\sigma_\star} = (-1)^{n_1+n_2} x_1 \sum_{s\in\mathcal{N}_1} \frac{B_1(a_s)}{P_1'(a_s) P_2(a_s)} + (-1)^{N+1} x_2 \sum_{s\in\mathcal{N}_2} \frac{B_2(a_s)}{P_2'(a_s) P_3(a_s)} \\
+ (-1)^{n_3} \frac{q_0}{x_1 x_2} \sum_{s\in\mathcal{N}_1} \frac{B_3(a_s)}{P_3(a_s) P_1'(a_s)} \, . \qquad \text{(C.4)}
\end{aligned}
$$

This matches the 1-instanton results from localization at 1-instanton using the contour $(++-)$ if we use the map:

$$
q_1 = (-1)^{n_2+1} x_1, \quad q_2 = (-1)^{n_1+n_3+1} x_2, \quad q_3 = \frac{q_0}{x_1 x_2} \, . \qquad \text{(C.5)}
$$

Quiver Q_1



The twisted superpotential is

$$\begin{aligned}\mathcal{W}_{Q_1}(y) = &\log y_1 \sum_{s\in\mathcal{N}_2} \sigma_s^{(1)} + \log y_2 \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \sigma_s^{(2)} \\ &- \sum_{t\in\mathcal{N}_1\cup\mathcal{N}_2} \sum_{s\in\mathcal{N}_2} \varpi(\sigma_t^{(2)} - \sigma_s^{(1)}) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \langle \mathrm{Tr}\, \varpi(\sigma_s^{(2)} - \Phi)\rangle \\ &- \sum_{i\in\mathcal{F}_1} \sum_{s\in\mathcal{N}_2} \varpi(\sigma_s^{(1)} - m_i) - \sum_{i\in\mathcal{F}_1\cup\mathcal{F}_2} \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \varpi(m_i - \sigma_s^{(2)})\,. \end{aligned} \tag{C.6}$$

The chiral ring equations are:

$$\begin{aligned} G_2(\sigma_s^{(1)}) &= \frac{(-1)^{n_1+n_2}}{y_1} B_1(\sigma_s^{(1)}) \quad \text{for} \quad s \in \mathcal{N}_2\,, \\ (1+q_0)\widehat{P}(\sigma_s^{(2)}) &= (-1)^{n_1+n_3}\Big(y_2 \frac{B_1(\sigma_s^{(2)})B_2(\sigma_s^{(2)})}{G_1(\sigma_s^{(2)})} + \frac{q_0}{y_2} B_3(\sigma_s^{(2)}) G_1(\sigma_s^{(2)})\Big) \\ &\qquad\qquad \text{for } s \in \mathcal{N}_1 \cup \mathcal{N}_2\,. \end{aligned} \tag{C.7}$$

We solve the equations about the following vacuum:

$$\begin{aligned} \sigma_s^{(1)} &= a_s \quad \text{for} \quad s \in \mathcal{N}_2\,, \\ \sigma_s^{(2)} &= a_s \quad \text{for} \quad s \in \mathcal{N}_1 \cup \mathcal{N}_2\,. \end{aligned} \tag{C.8}$$

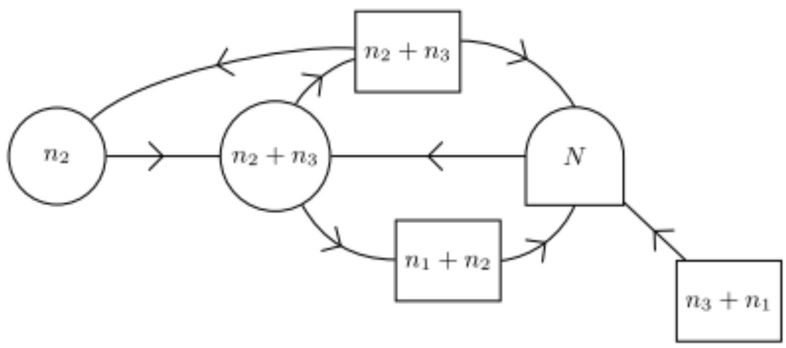
Then at 1-instanton the twisted superpotential evaluated on the solution is:

$$\mathcal{W}\Big|_{\sigma_\star} = \frac{(-1)^{n_1+n_2+1}}{y_1} \sum_{s\in\mathcal{N}_2} \frac{B_1(a_s)}{P_1(a_s)P_2'(a_s)} + (-1)^{n_2+n_3+1} y_1\, y_2 \sum_{s\in\mathcal{N}_2} \frac{B_2(a_s)}{P_2'(a_s)P_3(a_s)}$$
$$+ (-1)^{n_1+n_3+1} \frac{q_0}{y_2} \sum_{s\in\mathcal{N}_1} \frac{B_3(a_s)}{P_3(a_s)P_1'(a_s)} . \tag{C.9}$$

This matches the 1-instanton results from localization using the contour $(- + -)$ and the map:

$$q_1 = \frac{(-1)^{n_2+1}}{y_1}, \quad q_2 = (-1)^{n_3} y_1\, y_2, \quad q_3 = (-1)^{n_1+1} \frac{q_0}{y_2} . \tag{C.10}$$

Quiver Q_2



The twisted superpotential is

$$\mathcal{W}_{Q_2}(z) = \log z_1 \sum_{s\in\mathcal{N}_2} \sigma_s^{(1)} + \log z_2 \sum_{s\in\mathcal{N}_2\cup\mathcal{N}_3} \sigma_s^{(2)}$$
$$- \sum_{t\in\mathcal{N}_2\cup\mathcal{N}_3} \sum_{s\in\mathcal{N}_2} \varpi(\sigma_s^{(1)} - \sigma_t^{(2)}) - \sum_{s\in\mathcal{N}_2\cup\mathcal{N}_3} \langle \operatorname{Tr} \varpi(\Phi - \sigma_s^{(2)}) \rangle$$
$$- \sum_{i\in\mathcal{F}_2} \sum_{s\in\mathcal{N}_2} \varpi(m_i - \sigma_s^{(1)}) - \sum_{i\in\mathcal{F}_1\cup\mathcal{F}_2} \sum_{s\in\mathcal{N}_2\cup\mathcal{N}_3} \varpi(\sigma_s^{(2)} - m_i) . \tag{C.11}$$

The chiral ring equations are:

$$G_2(\sigma_s^{(1)}) = (-1)^{n_2+n_3} z_1 B_2(\sigma_s^{(1)}) \quad \text{for} \quad s \in \mathcal{N}_2 ,$$

$$(1+q_0)\widehat{P}(\sigma_s^{(2)}) = (-1)^{n_1+n_3}\left(\frac{B_1(\sigma_s^{(2)})B_2(\sigma_s^{(2)})}{z_2 G_1(\sigma_s^{(2)})} + q_0\, z_2\, B_3(\sigma_s^{(2)})G_1(\sigma_s^{(2)})\right) \text{ for } s \in \mathcal{N}_2 \cup \mathcal{N}_3\,. \tag{C.12}$$

We solve the equations about the following vacuum:

$$\begin{aligned} \sigma_s^{(1)} &= a_s \quad \text{for} \quad s \in \mathcal{N}_2\,, \\ \sigma_s^{(2)} &= a_s \quad \text{for} \quad s \in \mathcal{N}_2 \cup \mathcal{N}_3\,. \end{aligned} \tag{C.13}$$

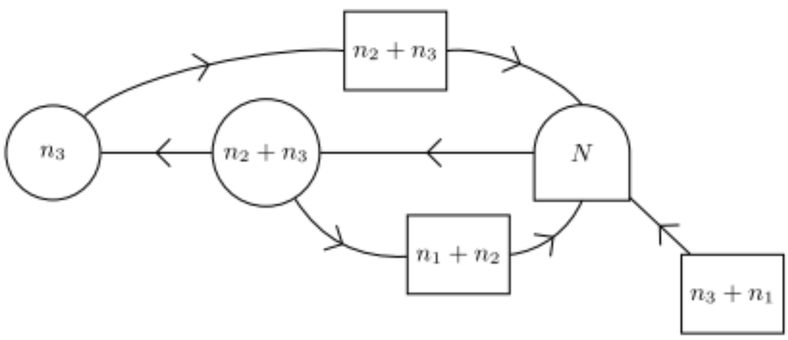
Then at 1-instanton the twisted superpotential evaluated on the solution is:

$$\begin{aligned} \mathcal{W}\Big|_{\sigma_\star} = &\frac{(-1)^{n_1+n_2}}{z_1 z_2}\sum_{s\in\mathcal{N}_2}\frac{B_1(a_s)}{P_1(a_s)P_2'(a_s)} + (-1)^{n_2+n_3} z_1 \sum_{s\in\mathcal{N}_2}\frac{B_2(a_s)}{P_2'(a_s)P_3(a_s)} \\ &+ (-1)^{n_3+1} q_0\, z_2 \sum_{s\in\mathcal{N}_3}\frac{B_3(a_s)}{P_3'(a_s)P_1(a_s)}\,. \end{aligned} \tag{C.14}$$

This matches the 1-instanton results from localization following the contour $(-++)$ and the map

$$q_1 = \frac{(-1)^{n_2}}{z_1\, z_2}, \quad q_2 = (-1)^{n_3+1} z_1, \quad q_3 = (-1)^{n_1+1} q_0\, z_2\,. \tag{C.15}$$

Quiver Q_3



The twisted superpotential is

$$\begin{aligned}\mathcal{W}_{Q_3}(w) = \log w_1 \sum_{s\in\mathcal{N}_3} \sigma_s^{(1)} + \log w_2 \sum_{s\in\mathcal{N}_2\cup\mathcal{N}_3} \sigma_s^{(2)} \\ - \sum_{t\in\mathcal{N}_2\cup\mathcal{N}_3} \sum_{s\in\mathcal{N}_3} \varpi(\sigma_t^{(2)} - \sigma_s^{(1)}) - \sum_{s\in\mathcal{N}_2\cup\mathcal{N}_3} \langle \mathrm{Tr}\, \varpi(\Phi - \sigma_s^{(2)})\rangle \\ - \sum_{i\in\mathcal{F}_2} \sum_{s\in\mathcal{N}_3} \varpi(\sigma_s^{(1)} - m_i) - \sum_{i\in\mathcal{F}_1} \sum_{s\in\mathcal{N}_2\cup\mathcal{N}_3} \varpi(\sigma_s^{(2)} - m_i)\,. \end{aligned} \tag{C.16}$$

The chiral ring equations are:

$$\begin{aligned} G_2(\sigma_s^{(1)}) &= (-1)^{n_2+n_3} \frac{B_2(\sigma_s^{(1)})}{w_1} \quad \text{for} \quad s \in \mathcal{N}_3\,, \\ (1+q_0)\widehat{P}(\sigma_s^{(2)}) &= (-1)^N \left(\frac{G_1(\sigma_s^{(2)}) B_1(\sigma_s^{(2)})}{w_2} + q_0\, w_2 \frac{B_2(\sigma_s^{(2)})\, B_3(\sigma_s^{(2)})}{G_1(\sigma_s^{(2)})} \right) \text{for } s \in \mathcal{N}_2 \cup \mathcal{N}_3\,. \end{aligned} \tag{C.17}$$

We solve the equations about the following vacuum:

$$\begin{aligned} \sigma_s^{(1)} &= a_s \quad \text{for} \quad s \in \mathcal{N}_3\,, \\ \sigma_s^{(2)} &= a_s \quad \text{for} \quad s \in \mathcal{N}_2 \cup \mathcal{N}_3\,. \end{aligned} \tag{C.18}$$

Then at 1-instanton the twisted superpotential evaluated on the solution is:

$$\begin{aligned} \mathcal{W}\Big|_{\sigma_\star} = \frac{(-1)^{N+1}}{w_2} \sum_{s\in\mathcal{N}_2} \frac{B_1(a_s)}{P_1(a_s)P_2'(a_s)} + \frac{(-1)^{n_2+n_3+1}}{w_1} \sum_{s\in\mathcal{N}_3} \frac{B_2(a_s)}{P_2(a_s)P_3'(a_s)} \\ + (-1)^{n_1+1} q_0\, w_1\, w_2 \sum_{s\in\mathcal{N}_3} \frac{B_3(a_s)}{P_3'(a_s)P_1(a_s)}\,. \end{aligned} \tag{C.19}$$

This matches the 1-instanton results from localization following the contour $(--+)$ and the map

$$q_1 = \frac{(-1)^{n_2+n_3+1}}{w_2}, \quad q_2 = \frac{(-1)^{n_3+1}}{w_1}, \quad q_3 = (-1)^{n_1+n_3} q_0\, w_1\, w_2\,. \tag{C.20}$$

Appendix D

Useful formulas for modular forms and elliptic functions

The Jacobi θ-functions are

$$\begin{aligned}
\theta_1(z|\tau) &= \sum_{n=-\infty}^{\infty} q^{\left(n-\frac{1}{2}\right)^2}(-x)^{n-\frac{1}{2}} \\
\theta_2(z|\tau) &= \sum_{n=-\infty}^{\infty} q^{\left(n-\frac{1}{2}\right)^2} x^{n-\frac{1}{2}} \\
\theta_3(z|\tau) &= \sum_{n=-\infty}^{\infty} q^{n^2} x^n \\
\theta_4(z|\tau) &= \sum_{n=-\infty}^{\infty} q^{n^2}(-x)^n
\end{aligned} \tag{D.1}$$

where $x = e^{2\pi i z}$ and $q = e^{\pi i \tau}$. At $z = 0$, θ_2, θ_3 and θ_4 give the following expansions

$$\begin{aligned}
\theta_2(0|\tau) \equiv \theta_2(q) &= 2q^{1/4}(1 + q^2 + q^6 + \ldots) \\
\theta_3(0|\tau) \equiv \theta_3(q) &= 1 + 2q + 2q^4 + 2q^9 + \ldots \\
\theta_4(0|\tau) \equiv \theta_4(q) &= 1 - 2q + 2q^4 - 2q^9 + \ldots
\end{aligned} \tag{D.2}$$

Under $\tau \to \tau' = -\frac{1}{\tau}$ these transform as follows :

$$\theta_2^4 \to -\tau^2\theta_4^4, \quad \theta_3^4 = -\tau^2\theta_3^4, \quad \theta_4^4 = -\tau^2\theta_2^4 \tag{D.3}$$

The expansions to the first few orders of the first three Eisenstein series are given by

$$\begin{aligned} E_2 &= 1 - 24q^2 - 72q^4 + \dots \\ E_4 &= 1 + 240q^2 + 2160q^4 + \dots \\ E_6 &= 1 - 504q^2 - 16632q^4 + \dots \end{aligned} \tag{D.4}$$

While $E_4(\tau)$ and $E_6(\tau)$ transform as modular forms with weight 4 and 6 respectively, $E_2(\tau)$ is quasi-modular of degree 2. Under $\tau \to \tau' = -\frac{1}{\tau}$ we have the following transformations :

$$\begin{aligned} E_2(\tau') &= \tau^2 E_2(\tau) + \frac{6}{i\pi}\tau \\ E_4(\tau') &= \tau^4 E_4(\tau) \\ E_6(\tau') &= \tau^6 E_6(\tau) \end{aligned} \tag{D.5}$$

The Weierstraß $\wp$-function is defined as

$$\wp(z|\tau) = -\frac{\partial^2}{\partial z^2}\log\theta_1(z|\tau) - \frac{\pi^2}{3}E_2(\tau)\,. \tag{D.6}$$

In many of our formulas the following rescaled $\wp$-function appears:

$$\widehat{\wp}(z|\tau) := \frac{\wp(z,\tau)}{4\pi^2} = x\frac{\partial}{\partial x}\Big(x\frac{\partial}{\partial x}\log\theta_1(z|\tau)\Big) - \frac{1}{12}E_2(\tau)\,. \tag{D.7}$$

Under S duality, this transforms as

$$\widehat{\wp}(z|\tau) \to \tau^2\,\widehat{\wp}(z|\tau)\,. \tag{D.8}$$

A few terms that appear in the expansion of $\widehat{\wp}(z|\tau)$ are as follows :

$$\widehat{\wp}(z|\tau) = -\frac{1}{12} - (x + 2x^2 + 3x^3 + 4x^4) + q^2\left(2 - \frac{1}{x}\right) + \dots \tag{D.9}$$

There are also the $\wp$ functions with arguments shifted by half-periods $z \to z + \omega_i$, where

$$\omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{\tau}{2}, \quad \omega_3 = \frac{\tau + 1}{2} \tag{D.10}$$

On x these correspond to the following transformations respectively,

$$x \to -x, \quad x \to qx, \quad x \to -qx \tag{D.11}$$

The expansions for $\widehat{\wp}(z + \omega_i|q)$ are easily obtained by performing (D.11) in (D.9). The expression for $\widehat{\wp}$ function evaluated at the half-periods ω_i (D.10) are denoted as $\widehat{e_i}$ and they satisfy the following relations :

$$\begin{aligned} \widehat{e}_1 - \widehat{e}_2 &= \frac{\theta_3^4}{4} \\ \widehat{e}_3 - \widehat{e}_2 &= \frac{\theta_2^4}{4} \\ \widehat{e}_1 - \widehat{e}_3 &= \frac{\theta_4^4}{4} \end{aligned} \tag{D.12}$$

Appendix E

Verifying the resummation map

Let us now verify that (5.21) is indeed the correct map that relates the bare and the renormalized variables. We start by expressing the twisted superpotential in the massless limit as the integral of the SW differential as described in Section 5.2. We substitute (5.12) and (5.13) in (5.10) to get :

$$w_0 = \int^{x_0} \sqrt{q_0(q_0-1)\frac{\partial f_0}{\partial q_0}} \frac{dt}{\sqrt{t(t-q_0)(t-1)}} \tag{E.1}$$

We notice that when expressed in terms of q using (5.18) or its expansion in (5.19) we have the following:

$$\begin{aligned} q_0 \frac{\partial f_0}{\partial q_0} &= \frac{a^2}{\theta_4^4}\,, \\ q_0 - 1 &= \frac{e_3 - e_1}{e_1 - e_2}(q) = -\frac{\theta_4^4(q)}{\theta_3^4(q)}\,. \end{aligned} \tag{E.2}$$

We substitute this in (E.1) to get the following expression for w_0 :

$$w_0 = \frac{ia}{\theta_3^2} \int^{x_0} \frac{dt}{\sqrt{t(t-q_0)(t-1)}} \tag{E.3}$$

Let us now look at w_0 expressed in terms of the renormalized variables, i.e. $\widetilde{w}_0$ in (5.24) and perform some simple manipulations :

$$\widetilde{w}_0 = 2\pi i a \int^z dz = 2\pi i a \int^{x_0} \frac{dx_0}{\frac{dx_0}{dz}} = 2\pi i a\, \pi^2 \theta_3^4 \int^{x_0} \frac{dx_0}{\wp'} \,. \tag{E.4}$$

In the final equality we have used the map (5.21). The Weierstraß $\wp$-function satisfies the differential equation :

$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \tag{E.5}$$

which can be expressed as

$$\wp' = 2\pi^3 \theta_3^6 \sqrt{x_0(x_0 - q_0)(x_0 - 1)} \tag{E.6}$$

using (5.21) and (D.12). We substitute the above in (E.4) and obtain (E.3) which was arrived at using only the well established q_0 vs q map in (5.18). This confirms the x_0 vs x map in (5.21).

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Thesis Highlight

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Thesis Title: Surface Operators, Seiberg-Dual Quivers and Contours

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Surface operators in gauge theories are non-local operators supported on co-dimension two sub-manifolds of spacetime. There are different types of surface operator which can be studied from various approaches. In one approach, surface operator is studied as monodromy defects, where one specifies how the gauge fields are affected by the presence of the surface operator by imposing suitable boundary conditions in the path-integral. In this framework the non-perturbative effects are described in terms of ramified instantons whose partition function can be computed using equivariant localization methods. In another approach, surface operators are described as flavor defects, which are coupled 2d/4d quiver gauge theories. Quiver realizations of surface defects can be compared with the localization approach by considering the low energy twisted chiral superpotential.

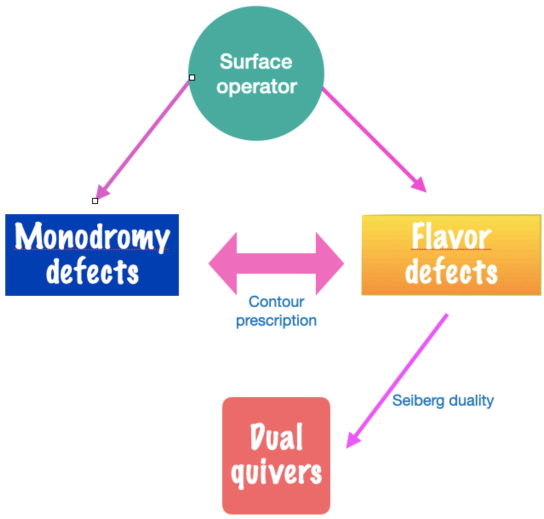

Figure: Schematic summary of surface operator

In this thesis, we study half-BPS surface operators in N=2 supersymmetric asymptotically conformal gauge theories in four dimensions with SU(N) gauge group and 2N fundamental flavours using localization methods and coupled 2d/4d quiver gauge theories. We show that contours in the localization analysis map to particular realizations of the surface operator as flavour defects. We study Seiberg duality of 2d/4d quivers. Dual quivers are mapped to contour deformations of the localization integral which involves a residue at infinity. The Lagrangian of the dual theory gets shifted by non-perturbative terms, which is referred to as modified Seiberg duality rule. The new rules, that depend on the 4d gauge coupling, lead to a match between the low energy effective twisted chiral superpotentials for any pair of dual 2d/4d quivers. We also study modular properties of half-BPS surface operators in N=2 SQCD in four dimensions with gauge group SU(2) and four fundamental flavours. We compute the twisted chiral superpotential that describes the effective theory on the surface operator using equivariant localization as well as the Seiberg-Witten curve. We then use the constraints imposed by S-duality to resum the instanton contributions to the twisted superpotential into elliptic functions and (quasi-) modular forms.