BLACK HOLE MICROSTATES AND SOLUTION GENERATING TECHNIQUES

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution/University.

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List of Publications arising from the thesis

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Deepali Mishra

Deepali Mishra

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Deepali Mishra

Deepali Mishra

Contents

No tables and Figures in this thesis.

Chapter 1

Introduction

This Chapter is a basic introduction to black holes as solutions to Einstein general relativity and various problems associated with black holes. Also, we will briefly summarise aspects of string theory and its applications as a tool to solve various black hole problems. In particular, it describes the role of AdS/CFT.

1.1 Black holes in classical general relativity

Black holes are the endpoints of gravitational collapse of massive astronomical objects with masses more than around 5-10 solar masses. For such massive objects, the gravitational force dominates over all other fundamental forces and instead of attaining an equilibrium state, they keep collapsing until they end up with something called 'event horizon'. This is the hypothetical surface around black holes from which even light can't escape i.e. the escape velocity of an object $\left(v_e = \sqrt{\frac{2GM}{r}}\right)$ exceeds the speed of light. Starting from the LIGO (Laser Interferometer Gravitational-Wave Observatory) observation in 2015 [1], the existence of black holes in the observational universe are constantly being detected through the gravitational waves coming from black hole merging events. Recently on April 2019, the 'event horizon telescope (EHT)' was able to picture the image of the horizon of a supermassive black hole having mass nearly equal to 7 billion times the solar mass [2] [3].

From the theoretical point of view black holes are solutions to Einstein's equations in general relativity,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{1.1.1}$$

where G_N is the Newton's gravitational constant. The left-hand side of the equation is the gravity part whereas the right-hand side is the matter part. Λ is the cosmological constant. In the following discussions we are going to set $G_N = 1 = \hbar = c = k_B$.

The Einstein equations are highly non-linear and in general, cannot be solved exactly. However they have been solved in many special cases. The simplest spherically symmetric, non-rotating solution to the vacuum Einstein equations (i.e. for $T_{\mu\nu} = 0$, $\Lambda = 0$) that carries no charge is expressed in terms of Schwarzschild coordinates as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{1.1.2}$$

Here, the coordinates t and r are the Schwarzschild time and radial directions respectively. M is the mass of the corresponding gravitational object. The metric outside any spherically symmetric solution to the Einstein equations can be effectively described in terms of the Schwarzschild coordinates with $r \geq 2M$. However, for most of such solutions the Schwarzschild coordinates appear to be invalid at the radius $r = 2M$ as it lies inside the matter content[1] of the spherical object e.g. stars, planets etc. For some special solutions such as black holes where all the matter content is compressed into a single point, the Schwarzschild coordinates are valid at $r = 2M$ as well. From now on, we will consider the metric (1.1.2) as only a black hole solution and exclude other spherical solutions. We can see that the Schwarzschild black hole (1.1.2) has the g_{tt} component vanishing and g_{rr} component diverging at $r = 2M$. So $r = 2M$ is a radius of singularity. However, it is only a property of the Schwarzschild coordinate system and nothing special happens at $r = 2M$. There are other coordinate systems like Eddington-Finkelstein or Kruskal coordinates [4] with which we can describe the region inside a black hole as well. The hypothetical surface at $r = 2M$ around a Schwarzschild black hole behaves as a one-way membrane through which information can pass only into the interior and nothing can come out classically. This surface is known as the event horizon. We will discuss more on it shortly.

[1]Hence cannot be described by the vacuum solution anymore.

The metric (1.1.2) has another singularity at $r = 0$ which is a proper singularity[2] of the geometry and cannot be removed by any choice of coordinates. Under generic conditions, a collapsing star gives rise to black hole. According to the 'Cosmic censorship conjecture (CCC)' as proposed by Penrose [5], the singularities produced in the process of formation of black holes are hidden by the presence of event horizon. No observer can detect the presence of the singularity as no information can come out of the event horizon. In the absence of event horizon the singularity would be a naked singularity which is avoided according to CCC. We will see this event horizon plays a very important role in black hole physics.

Till now, we discused black hole as a classical geometry. Quantum mechanically black holes radiate and have characteristic thermal temperature. By semi-classical computations[3] Hawking [6] showed that black holes just like any thermal object radiate and are characterized by the characteristic Hawking temperature

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}. \tag{1.1.3}$$

where κ is the 'surface gravity' at the horizon as observed by an observer at infinity. Surface gravity can be intuitively understood as the measure of maximum force that has to be exerted by an observer at infinity to keep an object at the horizon in place. In addition to the original computation of Hawking involving quantum fields in classical blackhole background, there also have been computations of the Hawking temperature by other people using Euclideanisation techniques e.g. see [7]. In quantum field theory, for the Euclidean formulation of the path integral quantisation there is an identification between periodicity (P) of the Wick rotated time $\tau \sim \tau + P$ with the inverse temperature $\beta = \frac{1}{T}$ of a statistical mechanical system (in the units of $\hbar = k_B = 1$). Keeping this in mind, by Wick rotating the Schwarzschild time $\tau = it$, we can write the metric (1.1.2) as,

$$ds_{\mathrm{E}}^2 = \rho^2 d\omega^2 + \left(\frac{r}{2M}\right)^4 d\rho^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.1.4}$$

[2]The square of the Riemann tensor $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges at $r \to 0$.

[3]By 'Semi-classical computation' we mean quantum field theory effects are considered in a curved geometry.

where $\rho = 4M\sqrt{1-\frac{2M}{r}}$ and $\omega = \frac{\tau}{4M}$. In these new coordinates, the horizon corresponds to $\rho = 0$, which can be thought of as the origin of the (ρ, ω)-plane with ω being the angular coordinate and ρ being the radial direction. Just like the plane polar coordinate system, the Euclidean manifold can be covered completely with (ρ, ω)-coordinates, by choosing ω to be periodic with period 2π and including the additional point $\rho = 0$. This implies that the Euclidean time τ is periodic with period $P = 8\pi M$. So for the Schwarzschild solution, the thermal temperature associated with the event-horizon turns out to be

$$T_{\mathrm{H}} = \frac{1}{8\pi M}. \tag{1.1.5}$$

Schwarzschild black hole is the simplest black hole solution defined only in terms of its mass. In addition there are charged black holes that carry electric charge Q along with the mass M. They are solutions to Einstein gravity coupled to U(1) gauge field with the following action

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right). \tag{1.1.6}$$

The corresponding equations of motion are

$$\begin{aligned} G_{\mu\nu} &= 2\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}F_{\lambda\sigma}F^{\lambda\sigma}\right), && (1.1.7)\\ \nabla_\mu F^{\mu\nu} &= 0, && (1.1.8)\end{aligned}$$

which are non-linear equations implying the fact that the solution is non-perturbative i.e. they can't be obtained from Einstein equations linearised around flat spacetime. The spherically symmetric Reissner-Nordström (RN) solution to these equations is given by,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.1.9}$$

where $A = \frac{Q}{r}dt$ and $F = dA$ is the Maxwell electromagnetic field coupled to the black hole. Thus, Q is the associated electric charge. The g_{tt} component vanishes at $r = r_\pm$ where

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{1.1.10}$$

So there are two horizons of RN-black hole, outer horizon at $r = r_+$ and the inner horizon at $r = r_-$. The value of mass and charge for which $Q^2 > M^2$ causes naked singularity. Hence it is not allowed according to Cosmic Censorship conjecture. This restricts the values of M and Q to satisfy $M \geq Q$. The black holes that saturate the condition $Q = M$ are called extremal RN black holes, they are stable (soliton-like), are at zero thermal temperature and they don't radiate. For $M > Q$ (non-extremal) they have finite positive temperature and can Hawking radiate. For extremal black holes the inner horizon and the outer horizon coincides and are at $r = r_+ = r_- = M$.

Astrophysically stars and planets also have rotational degrees of freedom so must be true for blackholes as they are formed out of collapsing stars. Thus there are rotating black hole solutions of Einstein equations, which carries angular momentum in addition to mass and charge denoted by J. They can be either uncharged (Kerr metric) [8] or charged (Kerr-Newmann metric) [9] solutions. According to black hole uniqueness theorems these are the only stationary, asymptotically flat black hole solutions to the Einstein's field equations. While the non-rotating black holes are also static solutions, the rotating ones are non-static. The rotating stationary solutions are axisymmetric. For more details on black hole solutions, one can refer to the basic text books on general relativity [4, 7, 10].

Black hole dynamics and it's correspondence to the laws of classical thermodynamics was developed in 1970's. Especially, the analogous of the four laws of thermodynmics for classical black holes were developed by Bardeen, Carter and Hawking in 1973 [11]. The Bekenstein-Hawking entropy [12] of a black hole is directly related to the area of its event horizon and is given by

$$S_{\rm BH} = \frac{A}{4} \tag{1.1.11}$$

where A is the area of the horizon. This is also in line with the black hole 'area law' [13], according to which 'the area of the event horizon of a classical black hole never decreases'. This can be compared with the second law of thermodynamics, the total entropy associated with an isolated system never decreases.

From the statistical mechanical point of view, this amount of entropy requires $e^{S_{\rm BH}}$ numbers of microscopic states of a particular black hole. There are several no-hair theorems [4] according to which for classical black holes associated with usual matter content in $D = 4$ the solutions are uniquely defined in terms of their mass M, electromagnetic charge Q and angular momentum J. Thus construction of $e^{S_{\rm BH}}$ number of microstates to account for the Bekenstein-Hawking entropy (1.1.11) is not possible in the classical treatment of a black hole. Quantum effects has to be taken into account [14–16].[4].

Due to the Hawking radiation, a black hole eventually evaporates, leaving only the thermal radiation behind. Thus starting from a pure state the black hole can evolve into a mixed state which is a violation of unitarity, a basic property of physical states in quantum mechanics. The Hawking radiation from a black hole contains only the information about the mass and charge of the black hole. Any information about the initial data in the formation of black hole is lost completely after its evaporation. This is known as the 'information loss paradox'. To address this issues we need a quantum theory of gravity.

String theory is the theory which could be a correct theory of quantum gravity within which the issues like black hole microstates and information loss have been attempted and solved to some extent. We will not discuss much about the resolution to the information loss paradox in this thesis. Interested readers may refer to [17].

1.2 String Theory

In this section, we are going to discuss some general aspects of string theory. In particular, we discuss fundamental strings and branes giving special attention to D-branes. We show how the equivalence between two different descriptions of the D-branes motivates towards the formulation of AdS/CFT duality [18] [19], which is an important tool for the construction of black hole microstates. The basic and excellent books on string theory are [20, 21].

The fundamental objects of string theory are higher dimensional objects like one dimensional strings and branes of two or more dimensions. To be precise, strings are the fundamental objects in a perturbative string theory where as p-branes are non-perturbative objects[5]. A

[4]However another problem came into picture which is known as the "Universality" problem. It was not clear why the semi-classical computation (It is QFT on curved spacetime where gravity is not quantized) by Bekenstein-Hawking yields the same entropy as obtained from quantum gravity (gravity is also quantized).

[5]They have masses inversely proportional to the string coupling.

p-brane has p-spatial dimensions e.g. a two dimensional object is called a 2-brane or a membrane. There are also 3-branes, 4-branes and so on. There are also special kind of p-branes called as D-branes or Dp-branes on which end-points of fundamental strings can end. We will discuss more on D-branes in section 1.2.1. The string excitations give particles. There are both massive and massless excitations of the fundamental string. Among the massless excitations of the closed-string one is graviton, the quanta of gravitational field. String theory lives in higher dimensions and lower dimensional theories can be constructed by compactifying extra dimensions on small spaces[6](more on compactification is dicussed in section 1.3.1). The fundamental strings can be open strings having two free ends[7]. There are also fundamental closed strings having periodic/anti-periodic boundary conditions along their lengths depending on the type of the theory considered.

Initially, string theory was constructed considering only the bosonic excitations of a string. This is known as the "bosonic string theory". For consistency it requires 26 spacetime dimensions. Bosonic string theory may contain oriented or non oriented strings. The bosonic closed strings have periodic boundary conditions along their length. For all types of bosonic string theories, the ground state of the theory is tachyonic (-ve mass square state) which implies instability of the vacuum in these theories. To resolve this issue and also to add fermions to the theory which are fundamental objects of standard model, supersymmetry is incorporated into string theory. This lead to supersymmetric string theory, also known as "superstring theory".

The superstring theory requires 10-spacetime dimensions. The supersymmetric closed strings can have periodic boundary conditions (Ramond-sector) or anti-periodic boundary conditions (Neveu-Schwartz sector) along their length.

There are five consistent supersymmetric string theories namely,

1. Type-I

   It has $\mathcal{N} = 1$ supersymmetry in ten-dimensions. It includes both open strings and closed strings. Here the fundamental strings are unoriented and they can be open strings. Where as, in type-II superstring theories fundamental strings are always closed and oriented.

[6]Smaller radius than the Planck length l_P..

[7]They have to satisfy some boundary conditions which requires them to lie on some hypersurface.

2. Type-II A

   The bosonic field content of the low-energy theory is given by metric g_{MN}, dilaton ϕ, NS-NS two form field B_{MN} attached to the fundamental string along with the Ramond-Ramond one-form field $C^{(1)}$ and three-form field $C^{(3)}$. These Ramond-Ramond fields are associated with D-branes, the $C^{(1)}$-field is associated with a D0-brane where as the $C^{(3)}$-field is associated with a D2-brane. There are also electromagnetic duals of these R-R fields (see B.1).

3. Type-II B

   Here the bosonic field content of the theory in the low-energy limit is given by metric g_{MN}, dilaton ϕ, NS-NS two form field B_{MN} attached to the fundamental string along with the Ramond-Ramond zero-form field $C^{(0)}$, two-form field $C^{(2)}$, four form field $C^{(4)}$ and 6-form field $C^{(6)}$ along with their electromagnetic duals. The $C^{(0)}$ field is associated to a so called D-instanton that carries charge under axion field χ. Other RR-fields i.e. $C^{(2)}$, $C^{(4)}$ and $C^{(6)}$ are associated to D1-branes, D3-branes and D5-branes respectively.

4. Heterotic SO(32)

   They are another kind of supersymmetric string theories with $\mathcal{N} = 1$ supersymmetry in ten-dimensions as in case of type-I superstring theory. In this case there is an associated Yang-Mills gauge symmetry with the gauge group $SO(32)$.

5. Heterotic $E_8 \times E_8$.

   Here the Yang-Mills gauge symmetry is associated to the $E_8 \times E_8$ Lie group.

Out of these type-I and heterotic string theories have $\mathcal{N} = 1$ supersymmetry where as type-IIA and type-IIB string theories have $\mathcal{N} = 2$ supersymmetry in ten-dimensions. In case of type-IIA superstring theory left-moving and right-moving spinors have opposite chiralities where as in type-IIB they have the same chiralities. All these theories are not completely independent of each other rather related by various dualities such as S-duality, T-duality etc. A brief description of these dualities is given in the appendix B.

Based on dualities all the string theories mentioned above have been conjectured to be different limits of a special theory known as M-theory [22]. We don't know the full definition of M-theory yet (there are various proposals like matrix theory, membrane theory etc.). However, we do know that in the low-energy limit it reduces to 11-dimensional supergravity. The BPS solutions in this 11-dimensional supergravity are given by M2-brane and M5-brane which are of two and five spatial extensions respectively. Upon compactifying one extra dimension the 11-dimensional supergravity reduces to type-IIA supergravity. Various objects in type-IIA supergravity can be related to that of M-theory. For example, D0-branes are related to the Kaluza-Klein (KK) excitations along the compact 11-th direction where as D2-branes of type-IIA can be mapped to M2-branes wrapped along the compact direction. Similarly, D4-branes can be obtained from M5-branes by wrapping one of its spatial dimension along the 11-th direction. D6-branes being electromagnetic dual of D0-branes can be identified as KK-monopole in the 11-dimensional theory. We will discuss on KK-monopoles in Chapter 5. The $E_8 \times E_8$ heterotic string theory can also be related to M-theory in eleven-dimensions.

The low-energy effective action for string theories is given by supergravity. In this thesis we will be discussing only the supergravity limit where we have only massless fields. In particular we will be working with the bosonic sector of type-IIA and type-IIB supergravity only.

The type-II supergravity action that contains only metric $g_{\mu\nu}$, dilaton ϕ, one NS-NS 2-form field $B_{\mu\nu}$ and a set of R-R $p+1$-form fields $C^{(p+1)}$ is given in the string frame by [20] [21],

$$S_{II} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g_s} \Big\{ R(g_s) + 4(d\phi)^2 - \frac{1}{12}(dB)^2 - \frac{1}{2}\sum_p \frac{1}{(p+2)!}(F^{(p+2)})^2 \Big\} \tag{1.2.12}$$

The last expression also contains the Chern-Simons terms. Here p is even with values $p = 0, 2, 4$ for type-IIA and p is odd with values $p = 1, 3, 5$ for type-IIB. This is the low energy limit of string theory action containing only massless sector of the full theory. This limit also corresponds to only closed string excitations in the ten-dimensional bulk spacetime.

As mentioned earlier, we can also have open strings in the theory. As compared to closed strings which have their both ends joined to each other, open strings have their end points free in the vacuum and any excitation of the open strings would violate the conservation of momentum. This implies open string end points should lie on some higher dimensional surface. The end-point of an open string lying on a dynamical hyper-surface satisfies Dirichlet boundary condition in the directions perpendicular to the surface. Hence these higher dimensional surfaces are known as D-branes (D stands for Dirichlet) or Dp-branes, p denoting the spatial dimension of the D-brane. D-branes also appear in bosonic string theory however, only in superstring theories some of the D-branes carry charges and are stable. In another way D-branes are solitonic (massive) solutions of non-linear supergravity equations, with masses inversely proportional to coupling, which makes them non-perturbative in nature.

Along with the fundamental branes, the D-branes behave as fundamental objects in non-perturbative string theory.

In the following sections we are going to give a brief introduction to D-branes and fundamental strings as they are the essential tools in the construction of black hole microstates. We will also discuss how D-branes are related to the correspondence between supergravity on AdS-space and conformal field theories (CFT) on the boundary of AdS.

1.2.1 D-branes and AdS/CFT duality

D-brane solutions can be seen as a higher-dimensional generalization of the Reissner - Nordstrom black hole. The extremal RN-black hole metric can be written as

$$ds^2 = -\left(1 - \frac{Q}{r}\right)^2 dt^2 + \left(1 - \frac{Q}{r}\right)^{-2} dr^2 + r^2 d\Omega^2. \tag{1.2.13}$$

Performing a change of coordinates $r - Q = R$, the event horizon shiftes to $R = 0$ and we can write this as

$$ds^2 = -\left(1 - \frac{Q}{R}\right)^{-2} dt^2 + \left(1 - \frac{Q}{R}\right)^2 (dR^2 + R^2 d\Omega^2). \tag{1.2.14}$$

So if we consider a point sorce, then the function $\left(1 - \frac{Q}{R}\right)$ is a harmonic function in the transverse space. The first term corresponds to the world-volume of a point particle and the last term corresponds to metric in the transverse space. The above metric is similar to that of a 0-brane in string theory.

A Dp-brane is a p-dimensional generalisation of the extremal 0-dimensional RN black hole solution where $p \leq 10$. Similar to the black hole extremality condition, the D-branes saturates the Bogomol'nyi–Prasad–Sommerfield (BPS) bound and breaks half of the 32 components of

$\mathcal{N} = 2$ supersymmetry in ten-dimensional type-II supergravity. Compared to the dynamics of a point particle that is described on a 1-dimensional worldline, the dynamics of a string is described on the two-dimensional worldsheet. Likewise the world-volume of a Dp-brane is $(p+1)$-dimensional. The corresponding metric can be written as

$$ds^2 = H_p(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H_p(r)^{1/2}dz^i dz^i, \tag{1.2.15}$$

where in the the first term $\mu, \nu = 0, 1, \ldots, p$ and it corresponds to $p+1$-dimensional brane worldvolume. The $dz^i dz^i$ part with $i = p+1, \ldots, 9$ corresponds to the transverse space metric. The functions $H_p(r)$ are harmonic functions in the transverse radial coordinate $r^2 = \sum_{i=p+1}^{9} z_i^2$ and for asymptotically flat spacetime takes the form

$$H_p(r) = 1 + \left(\frac{L_p}{r}\right)^{7-p}. \tag{1.2.16}$$

Just like point particles couple to one-form gauge potential a Dp-brane is electrically coupled to a $(p+1)$-form Ramond-Ramond (R-R) gauge potential $C^{(p+1)}$ [23]. Thus the electric field strength tensor associated to a Dp-brane is a $(p+2)$-form $F^{(p+2)}$. The factor L_p appearing in the expression (1.2.16) is related to the corresponding charge on the Dp-brane. For N number of Dp-branes it can be computed from the R-R flux in the transverse space to be

$$L_p^{7-p} = (4\pi)^{(5-p)/2}\Gamma\left(\frac{7-p}{2}\right) g_s N \alpha'^{(7-p)/2}, \tag{1.2.17}$$

where g_s is the string coupling, the Regge slope parameter α' is the square of the string length l_s or can be written in terms of the string tension (T) as $\alpha' = \frac{1}{2\pi \mathrm{T}}$.

In case of magnetic coupling a Dp-brane couples to a $(7-p)$-form gauge potential or $(8-p)$-form magnetic field strength tensor [B.1]. This is the gravitational picture of a Dp-brane. However, there is another description given in terms of open strings. We will see that the AdS/CFT duality naturally emerge from these two completely different yet equivalent descriptions of D-branes.

Two different descriptions of D-branes

One approach is the (a) Closed string or supergravity description. Here the D-branes are solitonic solutions to the low energy string theory action. Supergravity is a good approximation in the strong coupling regime $g_s >> 1$. This is the geometrical description of D-branes, described as the curvature of spacetime. The geometry has a near horizon Anti-deSitter (AdS) spacetime (this is a solution to Einstein gravity with negative cosmological constant i.e. $\Lambda < 0$). The second picture is (b) brane-picture or open string description. Here D-branes are considered as higher dimensional objects of the string theory on which open strings end. In this case there are both open-string and closed-string excitations which are small perturbations around the D-branes i.e. $g_s << 1$. Open string excitations are the excitations of the brane worldvolume where as closed string excitations are in the ten-dimensional bulk spacetime. In the low energy limit these two different kind of excitations decouple from each other. In the low-energy limit, when we RG-flow to the IR fixed point, the field theory on the D-branes is a conformal field theory (CFT). In this limit there are no massive excitations in the theory.

According to the AdS/CFT duality these two descriptions of D-branes are equivalent which is a very strong and non-trivial statement. To have a more clear understanding of the dual picture of D-branes we consider the simplest example of D3-branes. A good reference on gauge/gravity duality is [24].

1.2.1.1 D3-branes and AdS_5/CFT_4

D3-branes are BPS solitonic solutions of type-IIB supergravity with 3-spatial dimensions and their trajectories in spacetime is given by a 3+1-dimensional worldvolume. Using harmonic superposition rule we can write the ten-dimensional metric to be of the following form,

$$ds_{10}^2 = H^{-1/2}(-dt^2 + d\vec{X}^2) + H^{1/2}(dr^2 + r^2 d\Omega_5^2), \tag{1.2.18}$$

where the $(t, \vec{X})$ part of the metric corresponds to the worldvolume of the D3-branes with the isometry group $SO(3,1)$. The remaining part of the metric (1.2.18) corresponds to six transverse spatial directions with isometry $SO(6)$. $X^i, i = 1, 2, 3$ corresponds to the directions along which the D3-brane extends, $d\Omega_5$ is the five dimensional transverse sphere with radius r. The harmonic function H is given by

$$H = 1 + \frac{L^4}{r^4}, \tag{1.2.19}$$

where L is related to the $U(1)$ charge and tension of the D3-branes. For N-coincident D3-branes the value of L is

$$L = (4\pi g_s N \alpha'^2)^{1/4}. \tag{1.2.20}$$

$r = 0$ corresponds to the source or the location of the D3-brane. The associated Ramond-Ramond field living on the 3+1 dimensional worldvolume of the D3-brane is given by a 4-form potential $C^{(4)}$ which gives a five form field strength $F^{(5)} = dC^{(4)}$. The electromagnetic dual [see B.1] of a five-form field strength in ten-dimensions is again a five form. Thus the D3-branes are self-dual. The solution (1.2.18) is asymptotically flat which can be seen by taking the limit $r \to \infty$ i.e. $H \sim 1$.

As we go to small r i.e. $r << L$, $H \sim \frac{L^4}{r^4}$ the metric looks like,

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + d\vec{X}^2) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2. \tag{1.2.21}$$

As we can see in the above metric as $r \to 0$ the worldvolume term goes to zero. The radial term $\frac{dr^2}{r^2}$ goes as $\log r$ which goes to infinity as $r \to 0$. So the D3-brane is infinitely far away. This is the near-horizon limit of the solution. The metric (1.2.21) can be reduced into two parts as follows

$$ds^2 = ds^2_{\text{AdS}} + L^2 d\Omega_5^2, \tag{1.2.22}$$

where with $u = L^2/r$

$$ds^2_{\text{AdS}} = \frac{r^2}{L^2}(-dt^2 + d\vec{X}^2) + \frac{L^2}{r^2}dr^2 = \frac{L^2}{u^2}(-dt^2 + d\vec{X}^2 + du^2), \tag{1.2.23}$$

is the metric for five dimensional AdS-space i.e. AdS_5 with radius L. So the near horizon limit gives $AdS_5 \times S^5$, S^5 having the constant radius L. Supergravity limit is a good approximation when the AdS radius L is much larger than the string length l_s i.e. small curvature of the AdS-spacetime. We can see from the expression for AdS-radius (1.2.20) that

$$\frac{L}{l_s} = (4\pi g_s N)^{1/4}. \tag{1.2.24}$$

So $L >> l_s$ corresponds to $g_s N >> 1$. Thus supergravity limit is also the strong coupling limit. In the supergravity description of D3-branes there are only closed string modes propagating in the curved spacetime (asymptotically flat) with the $F^{(5)}$ flux through S^5.[8] There are no D-branes or open-string excitations in this picture.

Now coming to the open-string picture, considering N-parallel D3-branes in ten dimensional space-time, the brane worldvolume is along t, X^1, X^2, X^3. Open-strings are small perturbations of the D-branes and this picture is reliable for weak string coupling $g_s << 1$. For N-coincident branes, the effective coupling is $g_s N$ and weak coupling limit corresponds to $g_s N << 1$. At low energies, excitations of the D-branes are given by supersymmetric gauge theories on the brane worldvolume. For open string excitations in the transverse directions, we get scalar fields on the brane worldvolume. For D3-branes there are six scalar fields, ϕ^i $i = 1, ..., 6$ corresponding to six transverse directions. There is a $U(1)$ gauge group for the open string excitations parallel to a D-brane. For N-coincident branes the gauge group is $U(N)$. On the 3+1-dimensional worldvolume of D3-branes we have $SU(N)$[9] super-Yang-Mills gauge theory with $\mathcal{N} = 4$ supersymmetry corresponding to 16 unbroken supercharges. In addition there are closed string excitations in the bulk spacetime, which gets decoupled from the open string excitations at low energies.

AdS/CFT duality [18] implies that the supergravity theory on $AdS_5 \times S^5$ is equivalent to the $\mathcal{N} = 4$, $SU(N)$ super-Yang-Mills theory on the 3+1-dimensional brane world-volume. It's strong statement, and it has only been proved completely in the large N-limit, which corresponds to the planar limit. The couplings on both the sides of the theory are related to each

[8]Associated to the R-R field $C^{(4)}$.

[9]If we remove one $U(1)$ degrees of freedom corresponding to the collective mode of N-coincident branes.

other as,

$$g_{\rm YM}^2 = 2\pi g_s, \qquad\qquad 2\lambda = \left(\frac{L}{l_s}\right)^4, \tag{1.2.25}$$

where $\lambda = g_{\rm YM}^2 N$ is the effective coupling in large N-limit, known as the 't Hooft coupling.

Most of the string theory black holes occurring in the construction of microstates have a near horizon AdS_3-geometry e.g. D1-D5 system has a AdS_3-near horizon geometry. So the matching between the logarithm of the number of microstates (field theory side) and the Bekenstein-Hawking entropy (gravity side) is a manifestation of the duality between supersymmetric gauge theory on AdS_3 and the CFT on its two-dimensional boundary. We will discuss about AdS_3/CFT_2 in (1.4.4) which is relevant to our future discussions.

We also encounter fundamental strings (also known as F1-strings) in the construction of black hole microstates. So we should get a brief idea about these fundamental objects in string theory.

1.2.2 Fundamental strings (F1)

These are the fundamental objects in perturbative string theory i.e. weakly coupled string theory. Using the harmonic superposition rule the fundamental strings or F1-strings can be written in terms of the following metric[10]

$$ds^2_{string} = H_1^{-1}[-dt^2 + dy^2] + \sum_{i=1}^{8} dx_i dx_i, \tag{1.2.26}$$

where the fundamental string is wrapped n_1 times around the compact circle S^1 along y. The periodicity of y is given by $y \sim y + 2\pi R$. x_i are the non-compact transverse directions. In terms of null coordinates $u = t + y$, $v = t - y$, the metric (1.2.26) looks like

$$ds^2_{string} = -H_1^{-1} du dv + \sum_{i=1}^{8} dx_i dx_i. \tag{1.2.27}$$

[10]The subscript 'string' refers to string frame.

H_1 is the harmonic function in the transverse directions given by,

$$H_1 = 1 + \frac{Q_1}{r^6}, \quad \text{where} \quad Q_1 = \frac{g\alpha'^3}{V} n_1. \tag{1.2.28}$$

Here $r = \sqrt{x_i x^i}$ is the transverse radial direction. The metric is associated with the Kalb-Ramond B-field and the dilaton field given by

$$B_{uv} = \frac{1}{2}(e^{2\Phi} - 1), \qquad e^{2\Phi} = H_1^{-1}. \tag{1.2.29}$$

We will use this metric (1.2.26) while constructing F1-P solution in section 1.4.2.1. These F1-P solutions are in turn useful for the black hole microstate construction in string theory.

In the following section we are going to give a brief review on the construction of black holes in the realm of string theory. Some of the good reviews are [25] [26].

1.3 Black holes in string theory

Construction of black hole microstates in string theory helped in solving black hole entropy problem as well as information loss paradox. Since superstring theory lives in ten-spacetime dimensions, and classical black holes are realized in 4-spacetime dimensions, the extra dimensions are preferably compactified in these constructions. These compactifications give rise to various additional fields in the lower-dimensional theory.

1.3.1 Compactification

String theories live in higher dimensional spacetimes e.g. superstring theory lives in ten spacetime dimensions, where as physical black holes exist in 3+1- dimensional spacetime. So a natural intuition would be to compactify extra directions into small spaces such as they cannot be detected in physical experiments. Compactification of higher dimensional theory to lower dimensions yields a set of scalar fields in the lower dimensional theory along with the lower dimensional metric and other associated vector fields. This is because of the breaking of the higher dimensional symmetry group into lower ones. These scalar fields form the moduli space of the lower dimensional theory. The original ideas on compactifications were proposed by T. Kaluza and O. Klein in 1920 [27]. To see how this works let's consider the ten-dimensional metric $g_{\mu\nu}$ which is given by a 10×10 symmetric matrix, with μ, ν taking the values $\mu, \nu = 0, 1, \ldots, 9$. Suppose we compactify the ninth-direction into a small compact circle, then we get the following fields in the remaining nine-dimensional theory,

$$\text{(metric) } g_{ij},\ i, j = 0, 1, \ldots, 8, \quad \text{(scalar/dilaton) } \phi = g_{99}, \text{ (vector field) } A_i = g_{i9}. \tag{1.3.30}$$

In addition to the metric the associated form fields also reduce to give additional scalar fields in the lower dimensional theory.

For the case of type-IIB string theory compactified on T^4 the moduli space is 25 - dimensional. It is parametrized by ten components of the 4×4 part of the full metric, six components of the 4×4 part of the antisymmetric B-field, the dilaton field ϕ, one torus direction component of the 4-form Ramond-Ramond field $C^{(4)}$, six components of the 4×4 part of the antisymmetric 2-form R-R field $C^{(2)}$ and the 1-component 0-form $C^{(0)}$. In the near-horizon limit this moduli space reduce to 20-dimensional manifold. In the near horizon limit the values of moduli fields depends only on certain charges irrespective of their asymptotic values, this is known as the 'attractor mechanism'.

1.3.2 Counting of states

The counting of black hole microstates using system of intersecting strings and branes was developed independently by Sen and Strominger, Vafa. The counting of microstates was done for fundamental strings carrying vibrations (F1-P) by A. Sen in 1995 [28]. Here, the ten-dimenisional string theory is compactified on $S^1 \times K3$ to five dimensions (This is dual to heterotic string theory compactified on five torus T^5). Strominger and Vafa did the microstates counting for D1-D5-P states [29]. In these analysis the microstates are counted in the weak string coupling limit, logarithm of which is matched to the Bekenstein-Hawking entropy of the corresponding black hole that exists in the strong coupling regime. Supersymmetry ensures that the number of microstates doesn't change with the change of coupling[11]. These results in addition to being useful for solving black hole entropy problem also give strong indications towards a duality between the supergravity theory on the AdS space which describes the near horizon geometry of the branes and the boundary CFT.

Then another question arises, how the different microstates appear in the gravitational picture. In 2001 attempt was made to give the gravitational picture of these microstates [30–32] by Mathur and Lunin and is still a developing field [33–36]. Some nice reviews on fuzzball proposal and it's recent developments are [37–40].

In the following section there will be a brief review on some important aspects of fuzzball proposal and some examples of some fuzzball solutions.

1.4 Fuzzball Program

According to 'no-hair theorems' after formation of the black hole in 3+1-spacetime dimensions, it is uniquely defined in terms of the conserved charges(mass M, charge Q, angular momentum J). However string theory is a higher-dimensional theory with higher degrees of freedom and can effectively describe black hole microstates. The idea is that black holes are solutions to low energy effective string theory action and they carry only few low energy fields such as metric, Maxwell fields etc. However fuzzballs are solutions to the whole string theory out of which a few low energy solutions correspond to black hole microstates. These low energy solutions doesn't carry only the fields associated with the black hole but other low energy fields of the whole theory as well. This makes them different from each other microscopically even if they describe the same macroscopic black hole. This solves the black hole entropy problem by providing a correct statistical description.

In the fuzzball program, individual microstates are considered as smooth, horizonless solutions of supergravity known as "fuzzballs". Here the effects of gravity is considered not just upto Planck scale but upto the horizon length scale. The blak hole geometry is considered to be the superposition of e^S number of smooth, horizonless microstate geometries. Horizons occur after "coarse graining" over all the microstates of the system. These microstates are also known

[11]To be precise, the object that is computed is an 'index' which remains invariant under the change of moduli

as 'hair' modes on the black hole. Asymptotically they look like black hole geometries however the geometries differ at the horizon scale. The gravitational description of all the microstates for 2-charge D1-D5 solutions has been obtained by Lunin, Mathur, see [30,41]. It was possible because the D1-D5 solution is related to the F1-P solution by a set of dualities. For the 3-charge D1-D5-P solutions as much microstates as possible are being constructed. There has been also construction of non-supersymmetric fuzzball solutions [36].

The following section covers a brief introduction to the D1-D5 solutions and their various properties.

1.4.1 The D1-D5 System

For these two-charge solutions all the fuzzball geometries can be constructed out of F1-P solutions by the use of dualities. The total entropy of the system is in terms of the D1 and D5 charges, Q_1 and Q_5 respectively and is given by $\sim \sqrt{Q_1 Q_5}$. However the entropy of these solutions doesn't correspond to that of a macroscopic black hole. An addition of a third charge i.e. momentum charge (P) gives solutions with the correct value of the Bekenstein-Hawking entropy.

In the closed string picture, the D1-D5 solutions can be thought of as solutions of type-IIB supergravity compactified on $T^4 \times S^1$. We denote the S^1 direction as y and the torus directions are given by z_α. The system consists of n_1 numbers of D1-branes, wrapped along S^1 and n_5 numbers of D5-branes, wrapped along $T^4 \times S^1$. x_is with $i = 1 \ldots 4$ are the transverse non-compact directions. The naive geometry of the D1-D5 system can be written using harmonic superposition rule as[12]

$$ds^2 = \frac{1}{\sqrt{H_1 H_5}}(-dt^2 + dy^2) + \sqrt{H_1 H_5}\sum_{i=1}^{4} dx_i dx_i + \sqrt{\frac{H_1}{H_5}}\sum_{\alpha=1}^{4} dz_\alpha dz_\alpha \tag{1.4.31}$$

where H_1 and H_5 are Harmonic functions in the transverse space given by,

$$H_1 = 1 + \frac{Q_1}{r^2}, \qquad H_5 = 1 + \frac{Q_5}{r^2}. \tag{1.4.32}$$

[12]This is the string frame metric.

with $r^2 = \sum_i x_i^2$. The integer number of D1, D5, and P branes n_1, n_5, respectively are related to the parameters appearing in the metric as follows,

$$Q_1 = \frac{g\alpha'^3}{V} n_1, \quad Q_5 = g\alpha' n_5 \tag{1.4.33}$$

where g is string coupling, volume of the torus T^4 is $(2\pi)^4 V$ and in terms of string length l_s, $\alpha' = l_s^2$ is the parameter that defines the tension on a fundamental string which is $T = 1/2\pi\alpha'$. The metric (1.4.31) is associated with a 3-form field strength $F^{(3)}$ and the associated dilaton field Φ is given by,

$$e^{2\Phi} = \frac{H_1}{H_5} \tag{1.4.34}$$

The naive D1-D5 metric has a zero sized horizon at $r = 0$ thus it gives vanishing Bekenstein-Hawking entropy. According to fuzzball proposal this metric is a superposition of the actual D1-D5 microstate geometries of a black hole. These microstate geometries are smooth and horizonless and can be obtained by a set of duality maps (S and T dualities) from the momentum carrying fundamental string (F1-P) solution [see (1.4.35)].

1.4.2 Obtaining the D1-D5 metric

By applying a set of S, T dualities (see Appendix B.2 and B.3) we can go from F1-P system In Type-IIB string theory to D1-D5 system in type-IIB string theory. By this duality transformations the bound state of a fundamental string wrapped n_5 times around the y-circle and carrying n_1 units of momentum, maps to the bound state of a D1-brane wrapped n_1 times around the y-circle and a D5-brane wrapped n_5 times around $T^4 \times S^1$. The set of S and T dualities are given by[13],

$$\left| \begin{matrix} P(y) \\ F1(y) \end{matrix} \right| \xrightarrow{S} \left| \begin{matrix} P(y) \\ D1(y) \end{matrix} \right| \xrightarrow{Tz_\alpha} \left| \begin{matrix} P(y) \\ D5(yz_\alpha) \end{matrix} \right| \xrightarrow{S} \left| \begin{matrix} P(y) \\ NS5(yz_\alpha) \end{matrix} \right| \xrightarrow{Tyz_1} \left| \begin{matrix} F1(y) \\ NS5(yz_\alpha) \end{matrix} \right| \xrightarrow{S} \left| \begin{matrix} D1(y) \\ D5(yz_\alpha) \end{matrix} \right| \tag{1.4.35}$$

[13]The expression inside the bracket '()' indicates the direction along which the respective brane or string is extended.

Also by performing duality maps on F1-P solution of heterotic string theory one can obtain D1-D5 solution of type-IIB on Kähler manifold $K3$ [38]. Now we are going to discuss the geometry of F1-P solution from which the D1-D5 microstate geometries are obtained.

1.4.2.1 F1-P solution

We have already discussed fundamental string solution in section 1.2.2. Now, we are going to add n_p units of left-moving momentum to the F1-solution along the y-direction. To do so, by performing a boost along y the metric (1.2.26) takes the following form,

$$ds^2_{string} = H_1^{-1}[-dt^2 + dy^2 + K(dt - dy)^2] + \sum_{i=1}^{8} dx_i dx_i, \tag{1.4.36}$$

where the function K is given by[14],

$$K = \frac{Q_p}{r^6}. \tag{1.4.37}$$

Or in terms of the null-coordinates

$$ds^2_{string} = H_1^{-1}[-dudv + Kdv^2] + \sum_{i=1}^{8} dx_i dx_i. \tag{1.4.38}$$

This metric (1.4.38) obtained by performing boost on the fundamental string solution has a horizon at $r = 0$ with vanishing area. The F1-P microstate geometries are constructed by Dabholkar and harvey [42]. These Dabholkar-Harvey solutions are singular corresponding to the location of the string source, $r = 0$ but they do not have any horizon. Using higher derivative corrections they can develop horizons. The F1-P solutions are constructed using Garfinkle-Vachaspati transform [43] on a fundamental string background (1.2.26). We can write (1.2.26) in terms of the null coordinates as

$$ds^2 = -e^{2\Phi} dudv + \sum_{i=1}^{8} dx_i dx_i. \tag{1.4.39}$$

[14] Q_p is related to n_p

The components of the associated two-form NS-NS field and dilaton field are given by,

$$B_{uv} = \frac{1}{2}(e^{2\Phi} - 1), \qquad e^{-2\Phi} = 1 + \frac{Q}{r^6}. \tag{1.4.40}$$

The metric has translation symmetries along the null directions with the Killing vectors $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$. Addition of travelling wave deformation to such a geometry can be done by using the Garfinkle-Vachaspati transformation [43]. First we shall discuss the basic formalism of the solution generating technique. More details can be found in section 2.2.

The Garfinkle-Vachaspati transformation (GV) is given by the following deformation of the metric,

$$g'_{\mu\nu} = g_{\mu\nu} + e^S \Psi k_\mu k_\nu, \tag{1.4.41}$$

where $g_{\mu\nu}$ is the background metric and k_μ is the background null Killing vector which is also hypersurface orthogonal satisfying the following equations

$$\nabla_{[\mu} k_{\nu]} = k_{[\mu} \nabla_{\nu]} S. \tag{1.4.42}$$

S is the scalar function appearing in (1.4.41). In other words, a vector k^μ is called a hypersurface orthogonal vector when the corresponding co-vector k_μ is proportional to $\nabla_\mu S$ for some scalar function S. The scalar function Ψ in the GV transform (1.4.41) satisfies massless scalar equation

$$\Box \Psi = 0, \tag{1.4.43}$$

with respect to the undeformed metric $g_{\mu\nu}$. With the above mentioned features, the GV transform generate a new solution $g'_{\mu\nu}$ to the Einstein equations of motion. We will discuss more on GV in section 2.2.

Now to apply this method to the fundamental string solution (1.4.39) we consider the null, Killing vector $\frac{\partial}{\partial u}$. Lowering the index we can find the non-vanishing component of the corresponding co-vector to be

$$k_v = g_{vu} k^u == -\frac{1}{2} e^{2\Phi}. \tag{1.4.44}$$

Now using the Garfinkle-Vachaspati transform (1.4.41) we can write the deformed metric as,

$$g_{\mu\nu} \to g_{\mu\nu} + e^{-2\Phi} T'(v, x_i) k_\mu k_\nu, \tag{1.4.45}$$

where $S = -2\Phi$. Substituting for k_μ (1.4.44) this can be written as

$$g_{vv} \to g_{vv} + e^{-2\Phi} T'(v, x_i) k_v^2. \tag{1.4.46}$$

Thus the transformed metric looks like

$$ds^2 = -e^{2\Phi} dudv + T'(v, x_i) \frac{1}{4} e^{2\Phi} dv^2 + \sum_{i=1}^{8} dx_i dx_i, \tag{1.4.47}$$

Or we can write

$$ds^2 = -e^{2\Phi}(dudv - T(v, x_i) dv^2) + \sum_{i=1}^{8} dx_i dx_i, \tag{1.4.48}$$

where T satisfies wave equation with respect to the background i.e $\partial^2 T = 0$ where partial derivative is with respect to the 8-transverse directions. The additional dv^2 term implies we have a left-moving traveling wave on the fundamental string background. By considering the killing vector $k^v = 1$ we can add right-moving vibrations in a similar way. Since $\partial^2 T(v, \vec{x}) = 0$, we can expand T in terms of spherical harmonics in the 8-dimensional transverse space. Keeping only relevant terms that correspond to string sources the function can be written as,

$$T(v, \vec{x}) = \vec{f}(v).\vec{x} + \frac{p(v)}{r^6}. \tag{1.4.49}$$

The second term in the above expression is associated with gravitational wave and hence not attached to the string source. For oscillating string solution, we neglect this term and consider only the first term. Again as we can see, the first term is linear in x that means it doesn't go to zero as $x \to \infty$ and hence doesn't correspond to asymptotically flat geometry. However we can make the final solution (1.4.48) asymptotically flat by performing a set of diffeomorphisms which are a bit complicated and are given by

$$
\begin{aligned}
v &= v', && (1.4.50)\\
u &= u' - 2\dot{\vec{F}}.\vec{x}' + 2\dot{\vec{F}}.\vec{F} - \int^{v'} \dot{\vec{F}}^2 dv, && (1.4.51)\\
\vec{x} &= \vec{x}' - \vec{F}, && (1.4.52)
\end{aligned}
$$

where $f(v) = -2\ddot{F}(v)$. We will see that this $\vec{F}(v)$ defines the wave profile of the vibrating string. After performing the diffeomorphism the deformed solution takes the following form in the new coordinates $(u', v', \vec{x}')$

$$
ds^2 = -e^{2\Phi}du'dv' - (e^{2\Phi} - 1)\dot{F}^2 dv'^2 + 2(e^{2\phi} - 1)\dot{\vec{F}}.d\vec{x}'dv' + d\vec{x}'.d\vec{x}'. \quad (1.4.53)
$$

For convenience we replace $(u', v', \vec{x}')$ with $(u, v, \vec{x})$ and write the metric as

$$
ds^2 = -e^{2\Phi}dudv - (e^{2\Phi} - 1)\dot{F}^2 dv^2 + 2(e^{2\phi} - 1)\dot{\vec{F}}.d\vec{x}dv + d\vec{x}.d\vec{x}. \quad (1.4.54)
$$

The associated field components are given by

$$
B_{uv} = \frac{1}{2}(e^{2\Phi} - 1), \qquad B_{vi} = \dot{F}_i(e^{2\phi} - 1), \qquad e^{-2\Phi} = 1 + \frac{Q}{\mid \vec{x} - \vec{F} \mid^6}. \quad (1.4.55)
$$

As we can wee $\vec{x} = \vec{F}$ gives the location of the string and $\vec{F}(v)$ i.e $F_1(v), F_2(v), F_3(v), F_4(v)$ describes the transverse oscillation profile of the string. The dilaton field $e^{2\Phi} = 0$ on the $\vec{x} = \vec{F}$ surface. We can make the following identifications of the functions H^{-1}, K and the components of the one-form A

$$
A_i = \frac{Q\dot{F}_i}{\mid \vec{x} - \vec{F} \mid^6}, \qquad K = \frac{Q\dot{F}^2}{\mid \vec{x} - \vec{F} \mid^6}, \qquad H^{-1} = 1 + \frac{Q}{\mid \vec{x} - \vec{F} \mid^6}, \quad (1.4.56)
$$

with which the metric takes the following form

$$
ds^2 = H(-dudv + Kdv^2 + 2A_i dx_i dv) + \sum_{i=1}^{8} dx_i dx_i, \quad (1.4.57)
$$

with the associated field components given by

$$B_{uv} = -G_{uv} = \frac{H}{2}, \qquad B_{vi} = -G_{vi} = -HA_i, \qquad e^{2\Phi} = H^{-1}. \tag{1.4.58}$$

This form (1.4.57) of the F1-P metric is known as the chiral null model. The functions K and H are harmonic functions and satisfies the linear wave equation on the 8-dimensional transverse space. Thus in general we can take a superposition of the functions for different wave profiles $\vec{F}(v)$ of different strings. For m such strings having m different vibration profiles, by taking the superposition over different harmonics we can write the harmonic functions as

$$A_i = \sum_m \frac{Q\dot{F}_{mi}}{|\,\vec{x} - \vec{F}_m\,|^6}, \quad K = \sum_m \frac{Q\dot{F}_m^{\,2}}{|\,\vec{x} - \vec{F}_m\,|^6}, \quad H^{-1} = 1 + \sum_m \frac{Q}{|\,\vec{x} - \vec{F}_m\,|^6}. \tag{1.4.59}$$

To map this solution to D1-D5 frame usually four out of the 8-transverse directions are compactified on a four torus T^4. In this case we can write the metric (1.4.57) as

$$ds^2 = H(-dudv + Kdv^2 + 2A_i dx_i dv) + \sum_{i=1}^{4} dx_i dx_i + \sum_{\alpha=1}^{4} dz_\alpha dz_\alpha, \tag{1.4.60}$$

where z_α with $\alpha = 1, \ldots, 4$ are the compact torus directions with periodicities $z_\alpha \sim z_\alpha + 2\pi R_\alpha$. Thus we can smear the harmonic functions uniformly over the 4-torus T^4. Considering vibrations only in the non-compact directions they can be written as,

$$A_i = \sum_m \frac{Q\dot{F}_{mi}}{|\,\vec{x} - \vec{F}_m\,|^2}, \quad K = \sum_m \frac{Q\dot{F}_m^{\,2}}{|\,\vec{x} - \vec{F}_m\,|^2}, \quad H^{-1} = 1 + \sum_m \frac{Q}{|\,\vec{x} - \vec{F}_m\,|^2}. \tag{1.4.61}$$

The one-form A_i, where $i = 1, \ldots, 4$ can be thought of as a $U(1)$ gauge potential with the corresponding field strength tensor given by $F_{ij} = \partial_i A_j - \partial_j A_i$. Where the functions H^{-1}, K, F_{ij} satisfy the following equations, $\partial^2 H^{-1} = 0$, $\partial^2 K = 0$, $\partial^2 F_{ij} = 0$. For a multiwound F1-string wrapped along y with a large winding number n_5, the strands can be considered as closely packed. In that case the summation over different strands can be replaced with an integration along the length of the string which gives the following form of the harmonic functions

$$H^{-1} = 1 + \frac{Q}{L}\int_0^L \frac{dv}{\mid \vec{x} - \vec{F}(v) \mid^2}, \quad K = \frac{Q}{L}\int_0^L \frac{dv\dot{F}^2}{\mid \vec{x} - \vec{F}(v) \mid^2}, \quad A_i = -\frac{Q}{L}\int_0^L \frac{dv\dot{F}_i}{\mid \vec{x} - \vec{F} \mid^2}, \tag{1.4.62}$$

where $L = 2\pi n_5 R$ is the total length of the F1-string wound n_5 times along the y circle with radius R. The above configuration gives the level-matched F1-P solution [31] in type-IIB supergravity.

Now performing a series of S and T duality transformations (1.4.35) this solution can be mapped to the D1-D5 frame where it takes the following form

$$ds^2 = \sqrt{\frac{H}{1+K}}[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2] + \sqrt{\frac{1+K}{H}}dx_i dx_i + \sqrt{H(1+K)}dz_\alpha dz_\alpha. \tag{1.4.63}$$

with the associated fields,

$$\begin{aligned} e^{2\Phi} &= H(1+K), \qquad & C^{(2)}_{ti} &= \frac{B_i}{1+K}, \qquad & C^{(2)}_{ty} &= -\frac{K}{1+K} \\ C^{(2)}_{iy} &= -\frac{A_i}{1+K}, \qquad & C^{(2)}_{ij} &= C_{ij} + \frac{A_i B_j - A_j B_i}{1+K} \end{aligned} \tag{1.4.64}$$

where

$$dB = - *_4 dA, \qquad dC = - *_4 dH^{-1}. \tag{1.4.65}$$

Here $*_4$ is the duality operation in the 4d-transverse space. To get maximally rotating D1-D5 system we can choose the following vibration profile,

$$F_1 = a\cos\omega v, \qquad F_2 = a\sin\omega v, \qquad F_3 = 0, \qquad F_4 = 0. \tag{1.4.66}$$

where, $Q_1 = Q_5 = Q$, $a = \frac{Q}{R}$, $\omega = \frac{1}{n_5 R}$. After duality transformation this R goes to R' (in the D1-D5 frame). Then length of the string encircling S^1 in the D1-D5 frame is $2\pi R n_5 = \frac{2\pi n_5}{R'}$. We can write the metric (1.4.63) in polar coordinates (r, θ, ϕ, ψ) by using the following identification of the 4-dimensional Cartesian coordinates

$$x_1 = \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi}, \qquad x_2 = \tilde{r}\sin\tilde{\theta}\sin\tilde{\phi} \tag{1.4.67}$$

$$x_3 = \tilde{r}\cos\tilde{\theta}\cos\tilde{\psi}, \qquad x_4 = \tilde{r}\cos\tilde{\theta}\sin\tilde{\psi}. \tag{1.4.68}$$

For further simplification we need to apply the following coordinate transformation of $(\tilde{r},\tilde{\theta})$ to (r,θ)

$$\tilde{r} = \sqrt{r^2 + a^2\sin^2\theta}, \qquad \cos\tilde{\theta} = \frac{r\cos\theta}{\sqrt{r^2 + a^2\sin^2\theta}}, \tag{1.4.69}$$

upon which the functions and components of the 1-form take the following form

$$H^{-1} = 1 + \frac{Q}{r^2 + a^2\cos^2\theta} = 1 + \frac{Q}{f}, \qquad K = \frac{a^2}{n_5^2 R'^2}\frac{Q}{r^2 + a^2\cos^2\theta}. \tag{1.4.70}$$

For the gauge fields A_i we get

$$A_1 = -\frac{Q}{L}\int_0^L \frac{dv\dot{F}_1}{\mid \vec{x} - \vec{F_m} \mid^2} = \frac{Qa^2}{n_5 R'}\sin\tilde{\phi}\frac{\sin\theta}{(r^2 + a^2\cos^2\theta)}\frac{1}{\sqrt{r^2 + a^2}} \tag{1.4.71}$$

$$A_2 = -\frac{Q}{L}\int_0^L \frac{dv\dot{F}_2}{\mid \vec{x} - \vec{F_m} \mid^2} = -\frac{Qa^2}{n_5 R'}\cos\tilde{\phi}\frac{\sin\theta}{(r^2 + a^2\cos^2\theta)}\frac{1}{\sqrt{r^2 + a^2}} \tag{1.4.72}$$

and $A_3 = A_4 = 0$. Thus we get the maximally rotating D1-D5 solution from the F1-P solution.

1.4.3 Geometry of the D1-D5 solution

Here we consider n_1 numbers of D1 branes wrapped around S^1 with radius R_y, n_5 numbers of D5-branes wrapped around $T^4(Z^\alpha) \times S^1$ with volume of the torus given by $(2\pi)^4 V$. The brane numbers n_1 and n_5 are related to the Q_1, Q_5 charges appearing in the metric as follows

$$Q_1 = \frac{g\,\alpha'^3}{V}n_1, \qquad Q_5 = g\,\alpha'\,n_5. \tag{1.4.73}$$

To see the geometrical properties of the D1-D5 solution let's consider the general form of maximally-rotating D1-D5 metric is given by,

$$\begin{aligned} ds^2 &= -\frac{1}{h}\,(dt^2 - dy^2) + h\,f\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) \\ &\quad + h\left(r^2 + \frac{Q_1 Q_5\,a^2\,\cos^2\theta}{h^2\,f^2}\right)\cos^2\theta\,d\psi^2 + h\left(r^2 - \frac{Q_1 Q_5\,a^2\,\sin^2\theta}{h^2\,f^2}\right)\sin^2\theta\,d\phi^2 \\ &\quad - \frac{2\sqrt{Q_1 Q_5}}{h\,f}\,a\,\sin^2\theta\,d\phi\,dt - \frac{2\sqrt{Q_1 Q_5}}{h\,f}\,a\cos^2\theta\,d\psi\,dy + \sqrt{\frac{H_1}{H_5}}\,(dz^\alpha dz^\alpha)\,, \end{aligned} \tag{1.4.74}$$

where

$$\begin{aligned} a &= \frac{\sqrt{Q_1\,Q_5}}{R_y}\,, \qquad f = r^2 + a^2\,\cos^2\theta\,, \qquad e^{2\Phi} = \frac{H_1}{H_5}, \\ H_1 &= 1 + \frac{Q_1}{f}\,, \qquad H_5 = 1 + \frac{Q_5}{f}\,, \qquad h = \sqrt{H_1\,H_5}\,. \end{aligned} \tag{1.4.75}$$

This geometry can be obtained by dualities from F1-P system [see section 1.4.2.1]. The R-R two-form field associated with this configuration is given by,

$$\begin{aligned} C^{(2)} &= -\frac{\sqrt{Q_1 Q_5}\,\cos^2\theta}{H_1\,f}\,a\,dt\wedge d\psi - \frac{\sqrt{Q_1 Q_5}\,\sin^2\theta}{H_1\,f}\,a\,dy\wedge d\phi \\ &\quad - \frac{Q_1}{H_1\,f}\,dt\wedge dy - \frac{Q_5\,\cos^2\theta}{H_1\,f}\,(r^2 + a^2 + Q_1)\,d\psi\wedge d\phi\,, \end{aligned} \tag{1.4.76}$$

For the extremal case where $Q_1 = Q_5 = Q$ the dilaton vanishes with $e^{2\Phi} = 1$. The metric (1.4.74) is axi-symmetric along the angular directionsψ and ϕ.

These geometries represent only a subclass of D1-D5 ground states. We can see that the metric is asymptotically flat which can be seen by taking $r \to \infty$ limit. As proposed by Lunin, Mathur the metric is smooth and horizonless and ends with a smooth cap near $r \sim a$. The cap geometries differ from each other for different microstates. The region near $r \to 0$ limit where the different microstate geometries differ from each other corresponds to the possible horizon, after summing over all the microstates. The near-horizon geometry of the D1-D5 solution (1.4.74) is locally $AdS_3 \times S^3 \times T^4$ which is dual to the Ramond sector ground state $|0\rangle_{\rm R}$ in the CFT picture. I will discuss more on D1-D5 CFT in section 1.4.4. The asymptotically flat spacetime is connected to the inner cap region of the geometry by a AdS throat given by $a \ll r \ll \sqrt{Q}$ and a neck region near $r \sim \sqrt{Q}$. To summarize the D1-D5 geometry can be divided into two parts, inner part consisting of the "throat+ cap" and the outer part consisting of the "throat+ neck+ asymptotically flat region".

In the following sections we are going to discuss the D1-D5 geometry [31] in more detail. Here we will consider the extremal D1-D5 solution. The analysis is same for non-extremal solutions as well.

1.4.3.1 Outer region

In this region we have asymptotically flat spacetime with the neck. This can be obtained by taking $r \gg a$ limit.

1.4.3.2 Inner region

The inner region of the geometry consists of the AdS throat and the cap region of the geometry. This is given by the $r \ll \sqrt{Q}$ limit. To take this limit we want the asymptotically flat region to decouple from the inner region. The decoupling limit or the long AdS-throat limit is given by $\epsilon = \frac{a^2}{Q} \ll 1$ or after substituting a it is $\frac{Q}{R_y^2} \ll 1$. So the long AdS limit corresponds to taking large S^1 radius keeping the D1, D5 charges fixed. Thus the inner region metric is obtained from (1.4.74) by taking both $r \ll \sqrt{Q}$ as well as $\epsilon \ll 1$ upon which we get

$$\begin{aligned} ds^2 &= -\frac{r^2 + a^2\cos^2\theta}{Q}(dt^2 - dy^2) + Q(d\theta^2 + \frac{dr^2}{r^2+a^2}) - 2a(\cos^2\theta dy d\psi + \sin^2\theta dt d\phi) \\ &\quad + Q(\cos^2\theta d\psi^2 + \sin^2\theta d\phi^2) + dz_\alpha dz_\alpha. \end{aligned} \tag{1.4.77}$$

This geometry is locally $AdS_3 \times S^3$ which has a dual CFT picture which will be discussed in section 1.4.4. The AdS-geometry is more realised in a frame obtained by the following coordinate transformation

$$\psi_{NS} = \psi - \frac{y}{R}, \qquad \phi_{NS} = \phi - \frac{t}{R}, \tag{1.4.78}$$

after which the metric looks like

$$ds^2 = -\frac{r^2+a^2}{Q}dt^2 + \frac{r^2}{Q}dy^2 + Q\frac{dr^2}{r^2+a^2} + Q(d\theta^2 + cos^2\theta d\psi_{NS}^2 + sin^2\theta d\phi_{NS}^2) + ds_{T^4}^2. \quad (1.4.79)$$

As we can see the metric is $AdS_3 \times S^3 \times T^4$. The AdS part is defined in terms of the coordinates (t, y, r) and the 3-sphere coordinates are given by $(\theta, \psi_{NS}, \phi_{NS})$ both having the common radius $\sqrt{Q}$. The coordinate transformation (1.4.78) is kind of a rotation in S^3. In the CFT side this coordinate transformation is known as spectral flow transformation which is discussed in section 1.4.4. Under odd units of spectral flow transformation the Ramond sector CFT gets transformed to the NS-sector CFT.

Different regions of the D1-D5 geometry

We can summarize different regions of the D1-D5 geometry given by the metric (1.4.74). These terms defining different parts of the D1-D5 geometry will be frequently used in our thesis. The decoupling parameter is defined as

$$\epsilon \equiv \frac{\sqrt{Q}}{R_y} = \frac{a}{\sqrt{Q}}. \quad (1.4.80)$$

Here, R_y is the radius of the y-circle and $\sqrt{Q}$ is the AdS radius. This parameter when becomes very small $\epsilon \ll 1$, the geometry develops a long AdS throat which means there can be several units of AdS radius along the radial length of the throat i.e. $a \ll r \ll \sqrt{Q}$. In the decoupling limit the outer asymptotic flat spacetime corresponding to $r \sim \sqrt{Q}$ can be separated from the inner "cap $(r \sim a)$ + throat" part of the geometry. Thus for $\epsilon \ll 1$, the inner region corresponds to the metric (1.4.74) in the following limit

$$\frac{r}{\sqrt{Q}} << 1, \quad (1.4.81)$$

Similarly, the outer region is defined as

$$\frac{a}{r} << 1 \quad (1.4.82)$$

The 'throat' region is that part of the geometry where the inner and outer regions overlap. In the overlapping region

$$\epsilon = \frac{a}{\sqrt{Q}} << \left\{ \frac{a}{r}, \frac{r}{\sqrt{Q}} \right\} << 1 \tag{1.4.83}$$

The geometry of the cap region depends on the particular CFT state.

1.4.4 D1-D5 CFT

The D1-D5 solution has a near horizon $AdS_3 \times S^3 \times T^4$ geometry [section 1.4.3.2] which has a 1+1-dimensional dual CFT. The AdS_3/CFT_2 duality here plays a very important role for the identification of black hole microstates with corresponding brane geometries. There is a good review of D1-D5 CFT in [26]. The 9+1 dimensional spacetime is compactified as

$$\mathcal{M}^{1,9} \to \mathcal{M}^{1,4} \times S^1 \times T^4. \tag{1.4.84}$$

After the compactification the initial isometry group $SO(1,9)$ of the 9+1 dimensional spacetime is broken and now the isometry group is given by the $SO(4)_E \simeq SU(2)_L \times SU(2)_R$ group corresponding to the spatial directions of $\mathcal{M}^{1,4}$ and $SO(4)_I$ group corresponding to T^4-directions. We note that $SO(4)_I$ is broken due to toroidal compactification of the corresponding directions. Yet they are useful in organizing different states of the CFT. The $SO(4)_E$ corresponds to the R-symmetry group in the dual CFT.

From the brane-picture i.e the open string picture, there can be 3-types of open string excitations. The 1-1 strings which are the strings that start on D1-branes and end on D1-branes, 5-5 strings that start on D5-branes and end on D5-branes. Also there can be 1-5 (5-1) strings that start on D1-branes (D5-branes) and end on D5-branes (D1-branes). For 1-1 strings we have a $U(n_1)$ gauge theory in the 1+1 dimensional worldvolume. For 5-5 strings, we have a $U(n_5)$ gauge theory on the 5+1-dimensional worldvolume. Open strings with polarizations perpendicular to the brane directions have Dirichlet boundary conditions and they define transverse excitations of the corresponding D-brane. They give scalars on the worldvolume. In the Higgs phase the stack of D-branes are considered as coincident without any separation between them and being BPS saturated they break half of the 32 supercharges of type-IIB. The near horizon limit of the supergravity theory in the dual CFT corresponds to the IR limit in which closed string excitations of the bulk gets decoupled from the open string excitations of the D-branes and we get 1+1 dimensional $\mathcal{N} = (4,4)$ superconformal CFT. The $\mathcal{N} = (4,4)$ superconformal algebra is generated by the stress energy tensor $T(z)$, the $SU(2)$ supersymmetry generators $G^a(z)$ and the $SU(2)_L$ R-symmetry generators $J^i(z)$ along with their anti-holomorphic counterparts $(\bar{T}(\bar{z}), \bar{G}(\bar{z}), \bar{J}(\bar{z}))$ for the right-moving sector. For the right-moving sector the R-symmetry group is $SU(2)_R$.

It has been shown that for the near horizon AdS_3 geometry of the D1-D5 solution the dual CFT is a deformation of $\mathcal{N} = (4,4)$ superconformal sigma model with target space $(T^4)^N/S^N$ which corresponds to $N = n_1 n_5$ symmetrized copies of T^4 [44–47]. The base space of the sigma model is given by a cylinder (t, y), y being the direction along S^1. Each copy of the CFT has a central charge $c = 6$ corresponding to 4 bosonic and 4 fermionic degrees of freedom, resulting a total $c = 6n_1n_5$. There is a point in the 20-dimensional near-horizon moduli space (see section 1.3.1) where the dual CFT is just the $\mathcal{N} = (4,4)$ sigma model CFT with the orbifold target space $(T^4)^N/S^N$. This is known as the 'orbifold point' and most of the solutions studied are at this point. The orbifold point is kind of free field theory limit of the full moduli space. There are 20-marginal deformations in the CFT to deform away from this orbifold point. The symmetric permutation group S^N gives rise to twisting operations among different copies of the $c = 6$ CFT. Use of twist operators for the computation of correlation functions are discussed in [48, 49].

Depending on the periodicity of the fermions along y, we can have the CFT in Neveu-Schwarz (NS) sector (anti-periodic) or in the Ramond-sector (periodic). In the AdS/CFT dictionary, locally-AdS_3 geometry corresponds to Ramond-sector ground state $|0\rangle_{\mathrm{R}}$ of the dual CFT. The periodicity of the boundary fermions is a result of periodic fermions in the asymptotically flat bulk spacetime. However, in case of global AdS, the corresponding CFT state is in the NS-sector with anti-periodic fermions. The NS-sector states are related to R-sector sector states by spectral flow. Under α units of spectral flow the dimension and R-symmetry charge transforms as

$$h' = h - \alpha j + \alpha^2 \frac{c}{24}, \qquad\qquad j' = j - \alpha \frac{c}{12}. \tag{1.4.85}$$

Under odd units of spectral flow we can go from NS-sector to R-sector.

These twist operators give rise to NS-sector chiral primaries. The NS-sector vacuum $|0\rangle_{\rm NS}$ is the simplest state which has no twisting between the copies of the effective string

$$|0\rangle_{\rm NS} \; : \; h = \bar{h} = 0, \; j = m = 0, \; \bar{j} = \bar{m} = 0 \tag{1.4.86}$$

The dual geometry is global AdS. Here h is the conformal dimension and (j, m) are the $SU(2)_L$ quantum numbers which in the supergravity picture correspond to the rotations under $SO(4)_E$ in the non-compact directions. Next comes the chiral primaries which are defined by quantum numbers $(h, j, \bar{h}, \bar{j})$ satisfying

$$h = j, \; \bar{h} = \bar{j} \tag{1.4.87}$$

They can be obtained by operating 'twist operator' on the NS-sector vacuum $|0\rangle_{\rm NS}$.

The twist operator $\left[\sigma_k^{s,\bar{s}}\right]^{N/k}$ corresponds to N/k-component strings each wound k-times e.g for $k = 2$, we have twisting of each pair of component strings which results in $N/2$ component strings each wound 2-times.

For $su(2)$-charges $s = \bar{s} = \frac{k-1}{2}$, The chiral primary $\left[\sigma_k^{--}\right]^{N/k} |0\rangle_{\rm NS}$ is defined by,

$$h = j = \frac{N}{k}\frac{k-1}{2}, \qquad\qquad \bar{h} = \bar{j} = \frac{N}{k}\frac{k-1}{2} \tag{1.4.88}$$

NS-sector chiral primaries upon odd units $\alpha = (2m + 1)$ of spectral flow gives Ramond sector states.

By performing different units of spectral flow on the left sector and right sector we get,

$$\left[\sigma_k^{--}\right]^{N/k} |0\rangle_{\rm NS} \xrightarrow[\text{spectral flow}]{\alpha_L = 2m+1, \; \alpha_R = 1} |\Psi^{--}(k, m)\rangle_{\rm R} \tag{1.4.89}$$

with $n_p = h - \bar{h} = Nm\left(m + \frac{1}{k}\right)$

All the CFT microstates may not have classical geometrical analog, whenever we have a geometrical interpretation they corresponds to smooth, horizonless geometries with zero entropy. The CFT states corresponding to some supersymmetric fuzzball microstate geometries are already known [33–35]. For non-supersymmetric fuzzball solution of JMaRT [36] the CFT dual has been identified in [50].

1.4.5 3-charge solutions

Even though the 2-charge solutions describe black hole microstates effectively, but they do not give a classical sized horizon for which a third momentum charge is added. The microscopic counting of 3-charge D1-D5-P solutions has already been done by Strominger and Vafa [29] and it exactly matches with the Bekenstein-Hawking entropy in the weak coupling limit. The construction of 3-charge black hole microstate geometries has been carried out by Lunin and collaborators [41, 51]. In [41] the gravity solution is constructed for 1/4-BPS D1-D5 solutions carrying angular momentum.

However, unlike the 2-charge case where all the microstate geometries are already constructed [30, 41] the geometrical picture of the three charge microstates are still being constructed. All the two-charge solutions admit regular geometry [41]. Under dualities the D1, D5 charges and momentum charge P gets interchanged between themselves and they are difficult to study. To construct these three charge geometries we need to develop new solution generating techniques. The generalised Garfinkle-Vachaspati transform developed recently [52] [53] plays significant role in constructing new solutions from existing supersymmetric smooth D1-D5-P solutions. The naive geometry of the D1-D5-P geometry has horizon and singularity but the actual geometry was found to be smooth and horizonless [54].

There has been construction of three-charge type-IIB supergravity solutions in [54, 55] which can be written in terms of the following six-dimensional metric

$$\begin{aligned}
ds^2 &= -\frac{1}{h}(dt^2-dy^2)+\frac{Q_p}{hf}(dt-dy)^2+hf\left(\frac{dr^2}{r^2+(\gamma_1+\gamma_2)^2\eta}+d\theta^2\right)\\
&+h\Big(r^2+\gamma_1(\gamma_1+\gamma_2)\eta-\frac{Q^2(\gamma_1^2-\gamma_2^2)\eta\cos^2\theta}{h^2f^2}\Big)\cos^2\theta d\psi^2\\
&+h\Big(r^2+\gamma_2(\gamma_1+\gamma_2)\eta+\frac{Q^2(\gamma_1^2-\gamma_2^2)\eta\sin^2\theta}{h^2f^2}\Big)\sin^2\theta d\phi^2\\
&+\frac{Q_p(\gamma_1+\gamma_2)^2\eta^2}{hf}\left(\cos^2\theta d\psi+\sin^2\theta d\phi\right)^2\\
&-\frac{2Q}{hf}\left(\gamma_1\cos^2\theta d\psi+\gamma_2\sin^2\theta d\phi\right)(dt-dy)\\
&-\frac{2Q(\gamma_1+\gamma_2)\eta}{hf}\left(\cos^2\theta d\psi+\sin^2\theta d\phi\right)dy, \qquad (1.4.90)
\end{aligned}$$

The associated two-form field is of the following form

$$\begin{aligned}
C &= -\frac{Qc_\theta^2}{Q+f}(\gamma_2 dt+\gamma_1 dy)\wedge d\psi-\frac{Qs_\theta^2}{Q+f}(\gamma_1 dt+\gamma_2 dy)\wedge d\phi\\
&+\frac{(\gamma_1+\gamma_2)\eta Q_p}{Q+f}(dt+dy)\wedge(c_\theta^2 d\psi+s_\theta^2 d\phi)-\frac{Q}{Q+f}dt\wedge dy\\
&-\frac{Qc_\theta^2}{Q+f}(r^2+\gamma_2(\gamma_1+\gamma_2)\eta+Q)d\psi\wedge d\phi. \qquad (1.4.91)
\end{aligned}$$

These class of D1-D5-P solutions correspond to those Ramond-sector states of the CFT that can be obtained by $2m+1$ units of spectral flow of the NS-sector chiral primaries. The 3-charge states thus obtained carry an additional spectral flow parameter m that appears in the metric via the following terms

$$\gamma_1=-am, \qquad \gamma_2=a\left(m+\frac{1}{k}\right), \qquad (1.4.92)$$

where m takes integer values. More general family of 3-charge solutions are known with fractionated spectral flow parameter [36,56,57] which we won't discuss in this thesis. However the analysis in our thesis can be extended to those cases as well. Here k is the winding parameter that takes values $k=1,2,\ldots,n_1n_5$. It corresponds to the winding of component strings in the dual CFT picture (see section [1.4.4]). For $k=1$ we have singly wound D1-D5-P solution. For general purpose we take the range $m\geq 0, k>0\in\mathbb{Z}$.

The D1-D5-P solutions carry Q_1 and Q_5 charges which are related to the respective winding parameters by (1.4.73). In addition there is a third parameter Q_p which is related to the units of momentum n_p by

$$Q_p = \frac{g^2 \alpha'^4}{V R_y^2} n_p, \tag{1.4.93}$$

where n_1, n_5, n_p are integers. Q_p is related to the spectral flow parameter in the following way

$$Q_p = -\gamma_1 \gamma_2. \tag{1.4.94}$$

So for $m = 0$, we get $Q_p = 0$ and the three charge solution reduces to two-charge D1-D5 solution. Other parameters appearing in the metric are

$$f = r^2 + (\gamma_1 + \gamma_2)\,\eta\left(\gamma_1 \sin^2\theta + \gamma_2 \cos^2\theta\right), \quad h = 1 + \frac{Q}{f}, \quad \eta = \frac{Q}{Q + 2Q_p}. \tag{1.4.95}$$

The associated dilaton field Φ is given by

$$e^{2\Phi} = \frac{H_1}{H_5}, \tag{1.4.96}$$

which vanishes for $(Q_1 = Q_5 = Q)$ and for the moduli at infinity chosen appropriately.

1.5 Outline of the Thesis

This thesis is organized as follows. The Chapters 2-4 are based on our work in the two publications [52, 53]. In Chapter 2 I am going to discuss about the Garfinkle-Vachaspati transform and it's generalizations as a solution generating technique in minimal and non-minimal supergravity theories. Then in Chapter 3 the GGV will be applied to a class of smooth, supersymmetric D1-D5-P solutions, for minimal supergravity in section 3.2 and involving dilaton field in section 3.3. For non-minimal supergravity we will consider the GGV transform of the F1-P state in section 3.3.2. Chapter 4 is based on writing GGV in different M-theory frames. In our thesis we are mostly considering five-dimensional black hole solutions.

In Chapter 5 we will discuss deformation of KK-monopole geometry. This is an incomplete work during my PhD which imply some future directions to our project. In section 5.2.1 we have attempted to find the general form of the wave deformation on KK-monopole geometry by solving the scalar wave equation. Also in section 5.3 and section 5.4 we have written the KK-monopole solution and KK-P solution in the GMR form (see Appendix D) and applied GGV on such them.

Chapter 2

Generalised Garfinkle-Vachaspati Transform (GGV)

2.1 Solution Generating Techniques

In general, the supergravity equations are highly non-linear and solving them is a very difficult task. Solution generating techniques are useful in the sense that from an existing solution of the supergravity equations, new solutions can be obtained solving simpler equations for the original background.

From another point of view, construction of black hole microstates requires the development of such solution generating techniques. Especially for the construction of three-charge microstate geometries [see 1.4.5] we need the development and application of such techniques. There also has been developments involving group theoretic formalisms for generating new solutions [58]. Before going to the solution generating techniques developed in this thesis first we will discuss briefly the general prescription for generation of new solutions.

The key idea of the solution generating techniques [10] is that, for a manifold that admits two metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$, the respective covariant derivatives are related as follows

$$\nabla'_{\mu}\xi_{\nu} = \nabla_{\mu}\xi_{\nu} - \Omega^{\rho}_{\mu\nu}\xi_{\rho}, \tag{2.1.1}$$

where $\Omega^{\rho}_{\mu\nu} = \Gamma'^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}$ is symmetric in $\mu \leftrightarrow \nu$. It plays the role of a symmetric connection.

We know that the Riemann tensor can be written in terms of the commutator of the covariant derivatives as,

$$R_{\mu\nu\rho}{}^{\sigma}\xi_{\sigma} = [\nabla_{\mu}, \nabla_{\nu}]\, \xi_{\rho}. \tag{2.1.2}$$

Thus for the metric $g'_{\mu\nu}$ we can write

$$R'_{\mu\nu\rho}{}^{\sigma}\xi_{\sigma} = \left[\nabla'_{\mu}, \nabla'_{\nu}\right] \xi_{\rho}. \tag{2.1.3}$$

Substituting the expressions (2.1.1) and (2.1.2) in the above expression we can write the Riemann tensor corresponding to both the metrics to be related as

$$R'_{\mu\nu\rho}{}^{\sigma} = R_{\mu\nu\rho}{}^{\sigma} - \nabla_{\mu}\Omega^{\sigma}_{\nu\rho} + \nabla_{\nu}\Omega^{\sigma}_{\mu\rho} + \Omega^{\lambda}_{\rho\mu}\Omega^{\sigma}_{\nu\lambda} - \Omega^{\lambda}_{\rho\nu}\Omega^{\sigma}_{\mu\lambda}. \tag{2.1.4}$$

From the above expression contracting ν and σ we get the transformation of the Ricci tensor to be of the following form

$$R'_{\mu\rho} = R_{\mu\rho} - \nabla_{\mu}\Omega^{\nu}_{\nu\rho} + \nabla_{\nu}\Omega^{\nu}{}_{\mu\rho} + \Omega^{\lambda}{}_{\rho\mu}\Omega^{\nu}{}_{\nu\lambda} - \Omega^{\lambda}{}_{\rho\nu}\Omega^{\nu}{}_{\mu\lambda}. \tag{2.1.5}$$

Then the primed Ricci tensor $R'_{\mu\nu}$ along with the metric $g'_{\mu\nu}$ solves the corresponding Einstein equations. For example, using the above transformation rules (2.1.1)-(2.1.5), it can be checked that the rotating Kerr black hole solution is a solution to the vacuum Einstein equation in addition to the Minkowski spacetime $\eta_{\mu\nu}$. The Kerr black hole metric can be written in terms of flat metric $\eta_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + k_{\mu}k_{\nu}. \tag{2.1.6}$$

where k^{μ} is a null vector with respect to the flat spacetime $\eta_{\mu\nu}$. i.e. $\eta^{\mu\nu}k_{\mu}k_{\nu} = 0$. The expression (2.1.6) is known as the Kerr-Schild form of the metric.

One such widely used solution generating techniques is the Garfinkle-Vachaspati transform which was developed by D. Garfinkle and T. Vachaspati [43] (GV). Before going to the generalisation of GV transform, we are going briefly review the important aspects of the original Garfinkle-Vachaspati transform. In section [2.3] we will introduce the generalisation proposed in [52] where we have considered the GGV of metric and associated 2-form field. In section [2.4] we will discuss the GGV of supergravity solutions when there is non-vanishing dilaton coupling [53].

2.2 The Garfinkle-Vachaspati Transform

In the original paper [43] the Garfinkle-Vachaspati transformation (GV) was proposed for Yang-Mills-Higgs system coupled to gravity in four spacetime dimensions. There, GV was applied to "non-gravitating string" solutions given in terms of the flat metric η_{ab} with associated scalar field ϕ and gauge field A_a. The flat metric transforms under GV as follows

$$\tilde{\eta}_{ab} = \eta_{ab} + F k_a k_b, \tag{2.2.7}$$

where F is a scalar function and the vector k^a is a null covariantly constant vector with respect to the background metric η_{ab}. Also, k^a is orthogonal to $\nabla_a \phi$ and A_a. The solution after the transformation is given by $(\tilde{\eta}_{ab}, \phi, A_a)$. It preserves the null Killing symmetry of the background hence defines a travelling wave deformation on the background. For the case of gravitating string the flat metric η_{ab} is replaced with g_{ab}. For such backgrounds the null Killing vector k^a, is hypersurface orthogonal but no more covariantly constant. Although these requirements look very much non-trivial, still a wide range of gravity solutions are available which satisfy all these properties e.g. plane waves, fundamental string solution, pp-waves etc.

Formulation of GV transform to generate new solutions of low-energy string theory was developed in 1992 [59]. The background bosonic fields of the low-energy string theory are given in terms of the ten-dimensional metric g_{ab} with associated scalar field ψ and 2-form potential B_{ab}. Later, the technique was extended [60] to arbitrary spacetime dimensions D and generalised to include a set of p-form potentials. To see how the method works we consider D-dimensional gravity given in terms of the metric $g_{\mu\nu}$ coupled to an arbitrary set of scalar fields ϕ_a and a set of p-form potentials $A_{(p)}$[1]. The theory is defined in terms of the following

[1]The discussion in this section closely follows as done in [60].

generic action,

$$I = \frac{1}{16\pi G_D}\int d^Dx\sqrt{-g}\left(R(g) - \frac{1}{2}\sum_a h_a(\phi)(\nabla\phi_a)^2 - \frac{1}{2}\sum_p f_p(\phi)F^2_{(p+1)}\right), \qquad (2.2.8)$$

where G_D is the D-dimensional Newton's constant and $F_{(p+1)} = dA_{(p)}$ are the $p+1$-form field strength tensors. . For the case of $D = 10$ the above action (2.2.8) defines the generic massless sector of the full string theory action. Here the matter fields ϕ_a and $F_{(p+1)}$ have non-trivial couplings with the dilaton ϕ given in terms of the functions $h_a(\phi)$ and $f_p(\phi)$ respectively. These functions take different values for different string theories and different types of compactifications. The Einstein equations of motion for the action (2.2.8) are given by,

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = (8\pi G_D)T^\mu_\nu. \qquad (2.2.9)$$

where T^μ_ν is the mixed energy-momentum tensor which is of the following form,

$$\begin{aligned} T^\mu_\nu \ &= \ \sum_a h_a(\phi)\left(g^{\mu\rho}\partial_\rho\phi_a\partial_\nu\phi_a - \frac{1}{2}\delta^\mu_\nu(g^{\rho\lambda}\partial_\rho\phi_a\partial_\lambda\phi_a)\right) \\ &+ \ \sum_p f_p(\phi)\left((p+1)[F_{(p+1)}]^{\mu\rho_1\cdots\rho_p}[F_{(p+1)}]_{\nu\rho_1\ldots\rho_p} - \frac{1}{2}\delta^\mu_\nu[F_{(p+1)}]^{\rho_1\cdots\rho_{p+1}}[F_{(p+1)}]_{\rho_1\ldots\rho_{p+1}}\right) \end{aligned} \qquad (2.2.10)$$

In addition we can have equations of motion for the scalar fields and the $(p+1)$-form fields,

$$\begin{aligned} 0 \quad &= \partial_\mu(\sqrt{g}h_a(\phi)g^{\mu\nu}\partial_\nu\phi_a) - \frac{1}{2}\sqrt{g}\sum_b \frac{\partial h_p(\phi)}{\partial\phi_a}\nabla^\mu\phi_b\nabla_\mu\phi_a \\ &-\frac{1}{2}\sqrt{g}\sum_p \frac{\partial f_p(\phi)}{\partial\phi_a}[F_{(p+1)}]^{\rho_1\ldots\rho_{p+1}}[F_{(p+1)}]_{\rho_1\ldots\rho_{p+1}}. \qquad (2.2.11) \\ 0 \quad &= \partial_\mu(\sqrt{g}f_p(\phi)[F_{(p+1)}]^{\mu\rho_1\cdots\rho_p}). \qquad (2.2.12) \end{aligned}$$

Now to study GV, let's consider a solution $(g, \phi_a, A_{(p)})$ that admits a null, hypersurface orthogonal Killing vector k^μ. As discussed earlier in section 1.4.2.1 the GV transform is given by the following deformation of the metric (1.4.41)[2]

$$g'_{\mu\nu} = g_{\mu\nu} + e^S \Psi k_\mu k_\nu, \tag{2.2.13}$$

where k^μ is the background null Killing hypersurface orthogonal vector satisfying the conditions

$$k^\mu k_\mu = 0, \qquad \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0. \tag{2.2.14}$$

The hypersurface orthogonality condition is already written in (1.4.42). Ψ is the scalar function satisfying massless scalar equation (1.4.43). Furthermore, it can be checked that k has a vanishing Lie-derivative on S i.e. $\mathcal{L}_k S = k^\mu \partial_\mu S = 0$. To ensure that the Killing symmetries of the background solution along the null direction k^μ is preserved after the GV transform, the scalar function Ψ has to satisfy the following compatibility condition

$$k^\mu \partial_\mu \Psi = 0. \tag{2.2.15}$$

Also the null Killing vector k^μ has to satisfy the following conditions

$$\begin{aligned} \mathcal{L}_k \phi_a &= k^\mu \partial_\mu \phi_a = 0, & (2.2.16)\\ \mathcal{L}_k F_{(p+1)} &= (d i_k + i_k d) F_{(p+1)} = d i_k F_{(p+1)} = 0. & (2.2.17) \end{aligned}$$

This is needed for the wave description of the deformed solution. To be precise, these compatibility conditions ensure that after the GV transform the new solution has all the null Killing symmetries of the background preserved and it has additional momentum corresponding to a travelling wave on the particular background solution. The matter fields doesn't change under GV. So the new metric $g'_{\mu\nu}$ along with the background fields give a new solution that describes a travelling wave on the undeformed background.

This solution generating method can be verified by computing the equations of motion explicitly. From the computation of the matter-field equations it was found that matter field

[2]The new metric is said to be in the generalized Kerr-Schild form (2.1.6).

do not transform under GV. In fact the form of the energy momentum tensor T^{μ}_{ν} after GV remains the same as long as a certain transversality condition is satisfied by the p-form fields. The transversality condition imposed on the $p+1$-form field strength tensor is an algebraic condition which is defined as follows,

$$i_k F_{(p+1)} = k \wedge \theta_{(p-1)} \ . \tag{2.2.18}$$

i_k is the interior product and $\theta_{(p-1)}$ is a $p-1$-form. Also note that $(i_k)^2 F_{(p+1)} = 0$ implies $i_k \theta_{(p-1)} = 0$.

In the next section, we will briefly outline the computational techniques to verify GV as a solution generating technique. Computing the mixed Ricci tensor R^{μ}_{ν} for the new metric $g'_{\mu\nu}$ we can check that it changes with an additional term proportional to $\nabla^2\Psi$. Computing the energy-momentum tensor with the $p+1$-form field satisfying the transversality condition (2.2.18) we found $\nabla^2\Psi = 0$ for the transformed fields $(g'_{\mu\nu}, \phi_a, A_{(p)})$ to satisfy the Einstein equations (2.2.9). Solving the wave equation for Ψ we can the generate new solution. The new solution represents a travelling wave on the initial background.

2.2.1 Verifying Einstein equations

To verify the technique we need to check the validity of the transformed Einstein's equations

$$R'^{\mu}_{\nu} - \frac{1}{2}\delta'^{\mu}_{\nu} R' = \frac{1}{2} T'^{\mu}_{\nu}. \tag{2.2.19}$$

Since the matter fields do not change in this transformation so the right hand side remains the same as long as the transversality condition (2.2.18) holds true. The only things remain to compute are the transformed Ricci tensor and Ricci scalar. Note that the background admits a null, hypersurface orthogonal Killing vector k^μ which implies, k^μ satisfies the following identities,

$$k^\mu k_\mu = 0, \qquad \nabla_{[\mu} k_{\nu]} = k_{[\mu}\nabla_{\nu]} S, \qquad \nabla_{(\mu} k_{\nu)} = 0. \tag{2.2.20}$$

The hypersurface orthogonality condition (1.4.42) is already discussed in section 1.4.35. Now, for computational purposes we note that the inverse transformed metric is of the following form

$$g'^{\mu\nu} = g^{\mu\nu} - e^S \Psi k^\mu k^\nu. \tag{2.2.21}$$

In addition S and Ψ satisfies the following compatibility conditions

$$k^\mu \partial_\mu S = 0, \qquad\qquad k^\mu \partial_\mu \Psi = 0. \tag{2.2.22}$$

These conditions ensures that k^μ is again a Killing symmetry for the transformed solution $g'_{\mu\nu}$. Which implies the deformed solution is a wave deformation on the background geometry.

For a transformation of metric $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, the covariant derivatives in the two frames are related by (see expression (2.1.1))

$$\Omega^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\nabla_\nu h_{\rho\sigma} + \nabla_\rho h_{\nu\sigma} - \nabla_\sigma h_{\nu\rho}). \tag{2.2.23}$$

Thus, for the transformation of the metric as in (2.2.13), we have $h_{\mu\nu} = e^S \Psi k_\mu k_\nu$ from which we can compute,

$$\Omega^\mu{}_{\nu\rho} = \frac{1}{2}\left[\nabla_\nu(e^S \Psi k^\mu k_\rho) + \nabla_\rho(e^S \Psi k^\mu k_\nu) - \nabla^\mu(e^S \Psi k_\nu k_\rho)\right]. \tag{2.2.24}$$

Simplifying the above expression we get,

$$\Omega^\mu{}_{\nu\rho} = \frac{1}{2}[k^\mu k_\nu e^S(\nabla_\rho \Psi) + k^\mu k_\rho e^S(\nabla_\nu \Psi) - k_\nu k_\rho e^S(\nabla^\mu \Psi) + e^S \Psi k_\nu k_\rho (\nabla^\mu S)]. \tag{2.2.25}$$

Now, to verify the Einstein equations we need to compute the transformed Ricci tensor given in (2.1.5). First note that in (2.2.25) by contracting μ and ν we have

$$\Omega^\mu{}_{\mu\rho} = 0. \tag{2.2.26}$$

Also, it can be readily checked that

$$\Omega^{\rho}{}_{\mu\lambda}\Omega^{\lambda}{}_{\rho\nu} = 0. \tag{2.2.27}$$

Thus, the non-vanishing terms in the transformed Ricci tensor (2.1.5) are

$$R'_{\lambda\nu} = R_{\lambda\nu} + \nabla_{\mu}\Omega^{\mu}{}_{\lambda\nu}. \tag{2.2.28}$$

To compute the only non-vanishing addition $\nabla_{\mu}\Omega^{\mu}{}_{\lambda\nu}$ to background Ricci tensor, we take the covariant derivative of (2.2.25) which upon simplification yields

$$\nabla_{\mu}\Omega^{\mu}{}_{\lambda\nu} = \frac{1}{2}[-k_{\lambda}k_{\nu}e^{S}(\nabla^2\Psi) + e^{S}\Psi k_{\lambda}k_{\nu}(\nabla^2 S)]. \tag{2.2.29}$$

Substituting the above expression in (2.2.28) the simplified expression for the transformed Ricci tensor is given by

$$R'_{\lambda\nu} = R_{\lambda\nu} - \frac{1}{2}k_{\lambda}k_{\nu}e^{S}(\nabla^2\Psi) + \frac{1}{2}e^{S}\Psi k_{\lambda}k_{\nu}(\nabla^2 S). \tag{2.2.30}$$

This expression contains both $\nabla^2\Psi$ and $\nabla^2 S$ terms. Computing the mixed Ricci tensor we can remove the $\nabla^2 S$ term. To do so we raise one of the indices

$$R'^{\mu}{}_{\nu} = g'^{\mu\lambda}R'_{\lambda\nu} = R^{\mu}{}_{\nu} - e^{S}\Psi k^{\mu}k^{\lambda}R_{\lambda\nu} - \frac{1}{2}k^{\mu}k_{\nu}e^{S}(\nabla^2\Psi) + \frac{1}{2}e^{S}\Psi k^{\mu}k_{\nu}(\nabla^2 S). \tag{2.2.31}$$

We need to use the following identity in order to compute the contraction of Ricci tensor with the Killing vector,

$$k^{\lambda}R_{\lambda\nu} = \nabla_{\mu}\nabla_{\nu}k^{\mu} = k_{\nu}\frac{\nabla^2 S}{2}. \tag{2.2.32}$$

It can be checked that the above term exactly cancels out the $\nabla^2 S$ term appearing in (2.2.31) and we get the final expression for the transformed mixed Ricci tensor to be,

$$R'^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}e^{S}k^{\mu}k_{\nu}\nabla^2\Psi. \tag{2.2.33}$$

Hence we get $R'^{\mu}{}_{\nu} = R^{\mu}{}_{\nu}$ by imposing

$$\nabla^2 \Psi = \Box \Psi = 0. \tag{2.2.34}$$

As we have mentioned before, the energy-momentum tensor do not transform under GV for matter fields satisfying the condition (2.2.18). Thus in order the transform (2.2.13) to generate a new solution the only condition is the wave equation (2.2.34) satisfied by Ψ with respect to the background solution. We also observe that Ricci scalar does not transform under GV i.e. $R' = R$.

To check the regularity of the deformed solution we can compute the determinant of the metric g which is related to $\Gamma^{\mu}{}_{\mu\nu}$ as follows

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g}. \tag{2.2.35}$$

As we have seen $\Omega^{\mu}_{\mu\nu} = 0$ which implies the determinant of the metric remains unchanged under this transform. Thus starting from a regular solution, for finite values of Ψ the transformed solution turns out to be regular.

The compatibility conditions satisfied by the matter fields to make sure that the final solution is a travelling wave deformation on the background are

$$\begin{aligned} \mathcal{L}_k \phi_a &= k^\mu \partial_\mu \phi_a = 0, && (2.2.36) \\ \mathcal{L}_k F_{(p+1)} &= (d\, i_k + i_k d) F_{(p+1)} = d\, i_k F_{(p+1)} = 0, && (2.2.37) \end{aligned}$$

where $\mathcal{L}_k$ is defined as $\mathcal{L}_k = d\, i_k + i_k d$ on forms, and i_v denotes the interior product. Also the $p+1$-form field strength satisfies the following Bianchi identities,

$$dF_{(p+1)} = 0 \tag{2.2.38}$$

Garfinkle-Vachaspati transformation had been applied successfully in varied contexts in literature [42, 61–64]. However they require the presence of a null, Killing, hypersurface orthogonal vector. The D1-D5 and D1-D5-P systems that are useful in studying black hole microstates do not admit any such a Killing vectors. Thus we need new solution generating techniques that can be applied to such solutions. We will see in the following sections how the generalised Garfinkle-Vachaspati (GGV) turns out to be such a solution generating technique.

2.3 Generalized Garfinkle-Vachaspati transform (no dilaton)

First we are going to verify the GGV as a solution generating technique for solutions given by metric $g_{\mu\nu}$ and a 2-form matter field $C_{\mu\nu}$[3]. For the simplest case we set other fields to zero [52]. In section 2.4 we will include dilaton as well.

Lunin, Mathur and Turton (LMT) [33] conjectured that the traveling wave deformation of a class of D1-D5-P solutions takes a form which is a variant of the GV transform. They considered the special class of smooth, supersymmetric D1-D5-P solutions constructed in [54, 55]. These solutions are of the form of minimal six-dimensional supergravity solutions of GMR [65] embedded in ten-dimensions. Supersymmetric solutions are easier to study because the BPS condition ensures that the number of microstates is invariant under the change of moduli fields one of which is the string coupling. Generally, computation of the number of microstates is done in the weak coupling regime and matched with the corresponding black hole entropy that occurs at strong coupling. The AdS/CFT correspondence plays the central role in all these techniques.

Also the GMR solutions are important for the black hole microstate construction program as the 2-charge and 3-charge solutions [51] admit this form. For the dilaton free case the six-dimensional solutions can be trivially lifted to ten-dimensions. By trivial lifting we mean (i) a four torus is added to the six-dimensional metric part and (ii) the field components remain the same with vanishing components along the additional torus.

As noticed by LMT, the wave deformation on this class of supersymmetric solutions is of

[3]It could be any solution with gravity coupled to matter field, not necessarily a supergravity solution.

the form of a generalisation of the Garfinkle-Vachaspati transform which can be written as

$$g'_{\mu\nu} = g_{\mu\nu} + 2\,\Psi\,k_{(\mu}l_{\nu)}, \tag{2.3.39}$$

$$C'_{\mu\nu} = C_{\mu\nu} - 2\,\Psi\,k_{[\mu}l_{\nu]}, \tag{2.3.40}$$

where k^μ is a null, Killing vector i.e. k satisfies the following conditions

$$k^\mu k_\mu = 0, \qquad \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0. \tag{2.3.41}$$

and l is unit-normalized covariantly constant (which makes it Killing) spacelike vector i.e. it satisfies

$$l^\mu l_\mu = 1, \qquad \nabla_\mu l_\nu = 0. \tag{2.3.42}$$

Unlike the Garfinkle-Vachaspati transform here k need not be hypersurface orthogonal. Thus GGV is applicable to more wide range of solutions compared to GV. Here l^μ can be chosen along any of the torus directions. The presence of this vector in the transformation rules is an additional requirement of GGV. As in the case of GV the scalar function Ψ satisfies the wave equation $\Box\Psi = 0$ with respect to the background metric. Also, as compared to the case of GV where only the metric transforms and the matter field remains the same, in GGV the associated 2-form field $C_{\mu\nu}$ also transforms. So far we have only studied the transformation of 2-form fields. For any general p-form field there will be different transformation rule. We have verified that the transform (2.3.39)–(2.3.40) generates a new solution of type-IIB supergravity by explicitly computing the Einstein equation (See appendix [C]). For the technique to be applied successfully the background matter fields need to satisfy the following transversality condition,

$$k^\mu F_{\mu\nu\rho} = -(\nabla_\nu k_\rho - \nabla_\rho k_\nu), \tag{2.3.43}$$

The above condition is a differential condition compared to the algebraic transversality condition of GV (2.2.18). To apply GGV, we do not require any supersymmetric analysis of the solutions involved. Just like the case of GV in case of GGV again the deformed solution describes a travelling wave on the corresponding background.

As done for GV in section 2.2.1 we are going to verify GGV as a solution generating technique. The following section gives a brief outline of the verification of the transformed Einstein equation and matter field equations. The details can be found in Appendix [C].

2.3.1 Verifying The Technique

The computations are a bit lengthy but straight forward. Setting the B-field and the dilaton to zero, and keeping only the 2-form R-R field i.e. $p = 1$. the equations of motion for the metric and the matter field are given by,

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{4} F_{\mu\lambda\sigma} F_{\nu}^{\lambda\sigma}, & (2.3.44)\\
0 &= \nabla_{\mu} F^{\mu\lambda\sigma}. & (2.3.45)
\end{aligned}
$$

where $F = dC$ and $F^2 = F_{\mu\lambda\sigma} F^{\mu\lambda\sigma} = 0$. These are the ten-dimensional Einstein frame equations. Now let $g_{\mu\nu}, C_{\mu\nu}$ be a solution to these equations. And let the solution admits a null, Killing vector k^{μ} and a spacelike, unit normalised, covariantly constant vector l^{ν} orthogonal to k^{μ}. We need to verify that the GGV transform as defined in (2.3.39)–(2.3.40) gives a new solution to the equations of motion (2.3.44)-(2.3.45). By explicit computations of the Einstein equations (2.3.44) it can be shown that the left hand side transform exactly the same way as the right hand side when Ψ satisfies,

$$
\Box \Psi = 0, \tag{2.3.46}
$$

with respect to the background metric $g_{\mu\nu}$ and is compatible with the Killing symmetries, i.e., $k^{\mu} \nabla_{\mu} \Psi = 0$ and $l^{\mu} \nabla_{\mu} \Psi = 0$. This ensures that after the deformation the Killing symmetries along k^{μ} and l^{μ} are preserved hence the deformation is a travelling wave deformation on the background.

By a lengthy but straight forward computation (see Appendix C) we note that the left hand side i.e. $R_{\mu\nu}$ simply transforms as,

$$
\begin{aligned}
R'_{\lambda\nu} &= R_{\lambda\nu} - l_\lambda[k^\mu(\nabla_\nu\nabla_\mu\Psi) + \Psi\Box k_\nu] - l_\nu[k^\mu(\nabla_\lambda\nabla_\mu\Psi) + \Psi\Box k_\lambda] \\
&\quad + \frac{1}{2}(\nabla_\rho\Psi)(\nabla^\rho\Psi)k_\lambda k_\nu - \Psi^2(\nabla_\mu k^\rho)(\nabla_\rho k^\mu)l_\lambda l_\nu. \qquad (2.3.47)
\end{aligned}
$$

Also, the right hand side of the Einstein equations (2.3.44) transforms exactly in the same way as long as the transversality condition (2.3.43) is satisfied which can also be written as,

$$
i_k(dC) = -dk. \qquad (2.3.48)
$$

By explicit computation of the matter field equation 2.3.45 for the 3-form field we can check that it transforms covariantly i.e.,

$$
\nabla_\mu F^{\mu\nu\rho} = 0 \implies \nabla'_\mu F'^{\mu\nu\rho} = 0. \qquad (2.3.49)
$$

For more than one covariantly constant spacelike vectors $l^\mu_{(a)}$ we can write the GGV in a more generalised form

$$
\begin{aligned}
g'_{\mu\nu} &= g_{\mu\nu} + \sum_a \Psi_{(a)}(k_\mu l^{(a)}_\nu + k_\nu l^{(a)}_\mu), \qquad (2.3.50) \\
C'_{\mu\nu} &= C_{\mu\nu} - \sum_a \Psi_{(a)}(k_\mu l^{(a)}_\nu - l^{(a)}_\mu k_\nu), \qquad (2.3.51)
\end{aligned}
$$

where $\Psi_{(a)}$ are scalars on the original background spacetime $g_{\mu\nu}$ satisfying ${}_{(a)} = 0$.

2.3.2 Comparison to Garfinkle-Vachaspati transform

In some ways our solution-generating approach is more restrictive compared to the Garfinkle-Vachaspati (GV) transformation. As shown in [60], certain algebraic transversality conditions must be met by the original matter fields for the GV technique to work. As long as those conditions are met, the matter fields do not change. Unlike the GV technique, in our technique the matter fields do transform. There is no uniform prescription for the transformation of all matter fields. We need to do a case by case analysis. For the two-form gauge field considered in this paper, the transformation is (2.3.40), provided the untransformed 3-form field strength satisfies the differential transversality condition (2.3.43). The differential transversality condition (2.3.43) is analogous to the transversality condition for the GV technique, though now it is a differential condition rather than an algebraic condition.

In the following subsection we show that the differential transversality condition (2.3.43) is satisfied for all supersymmetric solutions written in the GMR form. However, to the best of our understanding, conditions for having supersymmetric solutions are more extensive than just the above differential transversality condition. We speculate that our solution-generating technique finds applications in non-supersymmetric settings as well, provided the differential transversality condition (2.3.43) is satisfied, though we do not work out any non-supersymmetric example in this thesis.

The differential transversality condition is consistent with Einstein equations. To see this, let's contract equations (2.3.44) with the $k^\mu k^\nu$ as:

$$R_{\mu\nu}k^\mu k^\nu = \frac{1}{4}k^\mu F_{\mu\lambda\sigma}k^\nu F_\nu{}^{\lambda\sigma}, \tag{2.3.52}$$

From the fact that k^μ is a Killing vector, we have the identity

$$k^\lambda \Box k_\lambda = -R_{\lambda\rho}k^\lambda k^\rho. \tag{2.3.53}$$

From this, it follows that

$$\begin{aligned} R_{\lambda\rho}k^\lambda k^\rho &= -k^\lambda \Box k_\lambda & (2.3.54)\\ &= -\left(\nabla^\mu(k^\lambda\nabla_\mu k_\lambda) - (\nabla^\mu k^\lambda)(\nabla_\mu k_\lambda)\right) & (2.3.55)\\ &= \frac{1}{4}\left[(\nabla^\mu k^\lambda - \nabla^\lambda k^\mu)(\nabla_\mu k_\lambda - \nabla_\lambda k_\mu)\right], & (2.3.56) \end{aligned}$$

where we have used the fact that k^μ is null and Killing. Equating this with the right hand side of equation (2.3.52), we have

$$k^\mu F_{\mu\lambda\sigma}k^\nu F_\nu{}^{\lambda\sigma} = (\nabla^\lambda k^\sigma - \nabla^\sigma k^\lambda)(\nabla_\lambda k_\sigma - \nabla_\sigma k_\lambda), \tag{2.3.57}$$

which is the "square" of this differential transversality condition (2.3.43).

We have also established GGV as a solution generating technique for supergravity solutions involving dilaton [53] which we discuss in the following section. Inclusion of dilaton is a further generalisation of our solution generating technique. This is because, (i) most of the interesting and rich examples of D1-D5 geometries involve the dilaton, (ii) the presence of the dilaton allows one to convolute the GGV technique with S-duality. In the following section we discuss a generalisation of the generalised Garfinkle-Vachaspati transform with dilaton. As detailed in appendix C.2, it is quite non-trivial that the technique finds a generalisation with dilaton.

2.4 Generalised Garfinkle-Vachaspati transform with dilaton

The GGV transform can be established as an effective solution generating technique even beyond the minimal supergravity approximation. We consider the case of six-dimensional solutions having a non-trivial dilaton profile. For such cases there are two closely related set-ups for which this generalisation is developed in this section: (i) ten-dimensional type IIB Ramond-Ramond (RR) sector with dilaton Φ and the two-form RR field $C^{(2)}$, and (ii) ten-dimensional Neveu-Schwarz (NS-NS) sector with dilaton where we have the NS B-field or Kalb-Ramond fields $B_{\mu\nu}$. These two set-ups are related by S-duality (see appendix B.2). In section 2.4.1 we are going to discuss GGV in R-R sector and in section 2.4.2 NS-sector GGV is explored.

2.4.1 Transform for the type IIB R-R sector

The non-zero dilaton brings in several new elements. From the six-dimensional perspective, in general, we can no longer truncate to minimal supergravity. The simplest set-up in six dimensions that allows for the dilaton is minimal supergravity coupled to one self-dual tensor multiplet. The six-dimensional action is, see, e.g. [66],

$$S_6 = \frac{1}{16\pi G_6}\int d^6x\sqrt{-g}\left[R-(d\phi)^2-\frac{1}{12}e^{2\phi}F_{\mu\nu\rho}F^{\mu\nu\rho}\right]. \tag{2.4.58}$$

where $F^{(3)} = dC^{(2)}$. This is the 6d theory we will work with exclusively. For ten-dimensional fields we follow Polchinski's conventions [20,21]. The ten-dimensional IIB string frame action with R-R 2-form field is

$$S_{\rm RR} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left[e^{-2\Phi}[R + 4(d\Phi)^2] - \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} \right]. \tag{2.4.59}$$

where the 10-dimensional Newton's constant is given by $G_{10} = 8\pi^6 g_s^2 \alpha'^4$, g_s is the string coupling and $\alpha' = l_s^2$, l_s being the string length. Here we have set the two-form NS B-field to zero.

The embedding of interest of six-dimensional fields in ten-dimensions is

$$ds^2_{\rm (S)} = ds^2_6 + e^{\phi} ds^2_4, \tag{2.4.60}$$

where $ds^2_{\rm (S)}$ is the ten-dimensional string frame metric, $ds^2_4 = \sum_{i=1}^4 dz^i dz^i$ is the flat torus metric, ϕ is the six-dimensional dilaton. The ten-dimensional dilaton is same as the six-dimensional dilaton

$$\Phi = \phi, \tag{2.4.61}$$

and the ten-dimensional 2-form R-R field is also same as the six-dimensional 2-form field with zero components in the four torus directions.

The spacelike Killing vectors provided by the torus directions,

$$l_{(i)} = l^{\mu}_{(i)} \partial_\mu = \partial_{z^i}, \tag{2.4.62}$$

are normalised as $l^\mu l_\mu = e^{\phi}$. These vectors are not covariantly constant, unlike in the analysis of embedding of minimal supergravity as in section 2.3. Let k^μ be a null Killing vector of the six-dimensional metric ds^2_6, with the property that the dilaton is compatible with the null Killing symmetry[4]

$$k^\mu \partial_\mu \phi = 0. \tag{2.4.63}$$

[4]To avoid notational clutter we have not introduced separate indices that range over six-dimensional spacetime. Most of the equations we write are in ten-dimensions. It should be clear from the context when the indices run over six dimensions.

The generalised Garfinkle-Vachaspati transform takes the following form in the ten dimensional string frame

$$\begin{aligned} g_{\mu\nu} &\rightarrow g_{\mu\nu} + \Psi e^{-\phi}(k_\mu l_\nu + k_\nu l_\mu), && (2.4.64) \\ C &\rightarrow C - \Psi e^{-2\phi}(k_\mu l_\nu - k_\nu l_\mu). && (2.4.65) \end{aligned}$$

It is a valid solution generating technique provided

$$k^\mu F_{\mu\nu\rho} = -d(e^{-\phi}k)_{\nu\rho}, \tag{2.4.66}$$

is satisfied by the background solution. We refer to this condition as the transversality condition. The scalar Ψ should satisfy the following wave equation on the background spacetime,

$$\Box\Psi - 2(\partial_\mu\phi)g^{\mu\nu}(\partial_\nu\Psi) = 0. \tag{2.4.67}$$

This equation can equivalently be written as

$$\nabla_\mu(e^{-2\phi}g^{\mu\nu}\nabla_\nu\Psi) = 0. \tag{2.4.68}$$

In addition we also require that the scalar Ψ is compatible with the Killing symmetries, i.e.,

$$k^\mu\partial_\mu\Psi = 0, \qquad\qquad l^\mu\partial_\mu\Psi = 0. \tag{2.4.69}$$

To establish the above statements we present a detailed calculation in appendix C.2. There are two main steps involved in this computation. First, we do a conformal transformation such that the spacelike Killing vector l^μ becomes covariantly constant. Once this is achieved, we adapt technology from the previous section 2.3 to find the transformations of the left and right hand sides of the IIB equations of motion. We show that provided the transversality condition (2.4.66) and the wave equation (2.4.68) are satisfied, all ten-dimensional equations transform covariantly. Hence, we show the generalised Garfinkle-Vachaspati transform (2.4.64)–(2.4.65) is a valid solution generating technique with dilaton. We have also checked these computations independently using Cadabra [67, 68]. For the embedding (2.4.60), the ten-dimensional Einstein frame metric takes the form,

$$ds^2_{(\mathrm{E})} = e^{-\phi/2} ds^2_6 + e^{\phi/2} ds^2_4. \tag{2.4.70}$$

One can write the corresponding GGV in Einstein frame. Since for applications to the D1-D5 systems in Chapter 3 we only work with string frame metric, we relegate these details to appendix C.2.4.

2.4.2 Transform for the NS-NS sector

In our conventions the ten-dimensional NS-NS sector string frame action is,

$$S_{\mathrm{NS}} = \frac{1}{16\pi G_{10}} \int \sqrt{-g} e^{-2\Phi} \left[R + 4(d\Phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right], \tag{2.4.71}$$

where $H = dB$. The embedding of interest of six-dimensional theory (2.4.58) in the ten-dimensional NS-NS sector string frame is,

$$ds^2_{(\mathrm{S})} = e^{-\phi} ds^2_6 + ds^2_4, \tag{2.4.72}$$

with ten-dimensional dilaton

$$\Phi = -\phi. \tag{2.4.73}$$

The six-dimensional 2-form field is now the 2-form B-field with zero components in the four torus directions. In Einstein frame this embedding reads

$$ds^2_{(\mathrm{E})} = e^{-\Phi/2} ds^2_{(\mathrm{S})} = e^{\phi/2} ds^2_{(\mathrm{S})} = e^{-\phi/2} ds^2_6 + e^{\phi/2} ds^2_4. \tag{2.4.74}$$

Note that this metric is same as (2.4.70). In fact, the two embeddings are related by S-duality. S-duality relates the RR sector of IIB supergravity to the NS-NS sector. The S-duality transformation in Einstein frame is

$$g^{(\mathrm{E})}_{\mu\nu} \to g^{(\mathrm{E})}_{\mu\nu}, \qquad \Phi \to -\Phi, \qquad C_{\mu\nu} \to B_{\mu\nu}. \tag{2.4.75}$$

The equivalence of (2.4.74) and (2.4.70) is the reflection of the fact that the Einstein frame metric does not change under S-duality. We can adapt the GGV from the RR sector to the NS-NS sector, for details see appendix C.2.4. In string frame the generalised Garfinkle-Vachaspati transform takes the form,

$$g_{\mu\nu} \;\to\; g_{\mu\nu} + \Psi(k_\mu l_\nu + k_\nu l_\mu), \tag{2.4.76}$$

$$B_{\mu\nu} \;\to\; B_{\mu\nu} - \Psi(k_\mu l_\nu - k_\nu l_\mu). \tag{2.4.77}$$

The transversality condition reads,

$$k^\mu H_{\mu\nu\rho} = -(dk)_{\nu\rho}, \tag{2.4.78}$$

and the scalar wave equation for the field Ψ reads,

$$\nabla_\mu (e^{-2\phi} g^{\mu\nu} \nabla_\nu \Psi) = 0. \tag{2.4.79}$$

Thus we have established GGV as an effective solution generating technique both for solutions with vanishing dilaton and non-zero trivial.

In the following Chapter we give explicit examples of applications of this technique. We add travelling wave deformations on multi-wound round supertubes and on a class of D1-D5-P backgrounds, generalising examples considered in [33]. We pick these examples as their dual CFT interpretations are well understood. We also present CFT interpretation of the deformed solutions.

Chapter 3

GGV on supersymmetric solutions

In the previous Chapter we studied the basic formalism of GGV for both the case of solution involving dilaton and for vanishing dilaton. To summarize, the only conditions on a solution for the succesful application of GGV are (1) the presence of a null, Killing vector (2) at least one spacelike covariantly constant vector (3) fulfilment of the transversality condition.

In this Chapter we will study some such examples appearing in the context of black hole microstates. Especially we observed that the six-dimensional supersymmetric solutions of the Gutowski, Martelli and Reall (GMR) [65] when embedded in ten-dimensional supergravity, satisfy all the above mentioned criteria and are good candidates for the implementation of GGV. Also, in section 3.3 we will apply GGV to non-minimal six-dimensional supergravity solutions which can be written in the form of a so called generalised GMR. For these solutions the dilaton field is non-zero and GGV on them has been already studied in section 2.4. We will also apply the NS-NS sector GGV discussed in section 2.4.2 to F1-P solutions.

Before applying GGV the following section involves a brief introduction to the classification of general supersymmetric solutions in different spacetime dimensions.

3.1 Classification of Supersymmetric solutions

The advancement of general formalisms [65, 69–73] for the classification of supersymmetric solutions is of considerable importance for the construction of black hole microstates. These formalisms are based on the use of Killing spinor techniques that involves the construction of bosonic objects out of Killing spinors. A set of bosonic equations are the necessary and sufficient conditions for the solution to be supersymmetric. These classification schemes have been carried out in 4D [69], for minimal N=2 theory, in 5D [70–73]. In 6D the classification was carried out by Gutowski, Martelli and Reall (GMR) [65]. The GMR solution has been used extensively in this thesis.

The six-dimensional GMR solutions are written in terms of a 2D fibre over a 4D almost hyper-Kähler base. For such solutions the 6D supergravity equations are reducible to 4D base space equations which are easier to handle (for details see Appendix[D.1]). These six dimensional solutions always admit a null Killing vector. In addition, they can always be trivially lifted to type-IIB supergravity solutions by the addition of four torus T^4 directions. The torus directions provide covariantly constant spacelike vectors. This makes GMR solutions the perfect candidates for the application of GGV. It was also shown in [33] that the class of supersymmetric D1-D5-P solutions studied in [54, 55] are trivial lifting of six-dimensional solutions of the form of GMR. In the following section we are going to discuss the deformation of this class of solutions.

3.2 Minimal six dimensional supergravity

For the minimal supergravity the dilaton field is zero. The metric has an associated two-form matter field. Thus the bosonic field content of the theory is given by graviton $g_{\mu\nu}$, 2-form Ramond-Ramond field $C^{(2)}$ with self-dual field strength $F^{(3)} = dC^{(2)}$. The field equations satisfied these fields were already mentioned in the previous Chapter (2.3.44)(2.3.45).

We can trivially lift the six-dimensional solutions to ten dimensions, as follows [33, 65]

$$ds^2 = -H^{-1}(dv+\beta)\Big(du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\Big) + Hh_{mn}dx^m dx^n + dz_i dz_i. \qquad (3.2.1)$$

The six-dimensional metric and hence the ten-dimensional metric admits a null, Killing vector

$$k = \frac{\partial}{\partial u}, \qquad (3.2.2)$$

The ten-dimensional dilaton is also zero. Now, to perform GGV, we can pick any one of the torus directions to get a spacelike covariantly constant (Killing) vector. In general we can add deformations corresponding to all the four torus directions. For simplicity of the analysis we skip it for the time being and pick, say,

$$l = \frac{\partial}{\partial z_4}. \tag{3.2.3}$$

In addition, as discussed earlier, the background matter field needs to satisfy the transversality condition (2.3.43) for the application of the generalized GV sucsessfully. For GMR solutions this condition is automatically imposed by the Killing spinor equations [33] [65]. We can also explicitly verify this. To do so let's consider $k^\mu F_{\mu\nu\rho}$:

$$\begin{aligned} k^\mu F_{\mu\nu\rho} &= F_{u\nu\rho} && (3.2.4)\\ &= \partial_u C_{\nu\rho} + \partial_\rho C_{u\nu} + \partial_\nu C_{\rho u} && (3.2.5)\\ &= -(\partial_\nu C_{u\rho} - \partial_\rho C_{u\nu}). && (3.2.6)\end{aligned}$$

We see that the differential transversality condition is equivalent to showing $C_{u\nu} = k_\nu$, upto possible gauge transformations. For GMR solutions we see that indeed it is the case (see appendix D.1) since

$$C_{u\nu} dx^\nu = -\frac{1}{2H}(dv + \beta) = k_\nu dx^\nu. \tag{3.2.7}$$

3.2.1 Deformation of a class of D1-D5-P backgrounds

In this section we present explicit examples of our general construction. We consider two classes of examples: first the multi-wound D1-D5 round supertubes and secondly a class of D1-D5-P backgrounds. Throughout this section, $Q_1 = Q_5 = Q$ which corresponds to setting dilaton to zero. Some important identifications of the brane charges and the compact torus volume are as follows,

$$Q_1 = \frac{g\alpha^3}{V} n_1\,, \qquad Q_5 = g\alpha n_5\,, \qquad (2\pi)^4 V = \text{vol}(T^4). \tag{3.2.8}$$

Multi-wound D1-D5 round supertubes were constructed in [74,75]. This family is parametrised by an integer k via,

$$\gamma = \frac{1}{k}, \qquad k = 1, 2 \ldots, N, \qquad N = n_1 n_5. \tag{3.2.9}$$

The case $k = 1$ corresponds to singly wound D1-D5 supertube. This configuration is dual to the Ramond vaccum $|0\rangle_{\mathrm{R}}$ of the D1-D5 CFT. The $k \neq 1$ members of the family are obtained by acting with certain twist operator such that the resulting states have N/k component strings [30]. For $k \neq 1$ the geometries have conical singularities. The metric takes the form,

$$\begin{aligned} ds_0^2 &= -\frac{1}{h}(dt^2 - dy^2) + hf\left(\frac{dr^2}{r^2 + a^2\gamma^2} + d\theta^2\right) \\ &+ h\Big(r^2 + \frac{a^2\gamma^2\, Q^2\, \cos^2\theta}{h^2 f^2}\Big)\cos^2\theta d\psi^2 \\ &+ h\Big(r^2 + a^2\gamma^2 - \frac{a^2\gamma^2\, Q^2\, \sin^2\theta}{h^2 f^2}\Big)\sin^2\theta d\phi^2 \\ &- \frac{2a\gamma\, Q}{hf}(\cos^2\theta\, dy\, d\psi + \sin^2\theta\, dt\, d\phi) + dz_i dz_i, \end{aligned} \tag{3.2.10}$$

and the two-form field takes the form,

$$\begin{aligned} C^0_{ty} &= -\frac{Q}{Q+f}, & C^0_{t\psi} &= -\frac{Qa\gamma\cos^2\theta}{Q+f}, \\ C^0_{y\phi} &= -\frac{Qa\gamma\sin^2\theta}{Q+f}, & C^0_{\phi\psi} &= Q\cos^2\theta + \frac{Qa^2\gamma^2\sin^2\theta\cos^2\theta}{Q+f}, \end{aligned} \tag{3.2.11}$$

where

$$f = r^2 + a^2\gamma^2\cos^2\theta, \qquad h = 1 + \frac{Q}{f}. \tag{3.2.12}$$

The y coordinate is periodic with periodicity $2\pi R_y$, and the parameter a is related to the size R_y of the y-circle as,

$$a = \frac{Q}{R_y}. \tag{3.2.13}$$

In the large R_y limit, the above geometry has a long $AdS_3 \times S^3 \times T^4$ throat. The throat together with the cap region is described by the metric obtained by focusing on the region of the spacetime with $r \ll \sqrt{Q}$. In this limit the metric becomes locally $AdS_3 \times S^3$ with a $\mathbb{Z}_k$ orbifold at $r = 0,\ \theta = \frac{\pi}{2}$ which we can see as follows. The decoupled metric takes the following form,

$$\begin{aligned} ds_0^2 &= -\frac{f}{Q}(dt^2 - dy^2) + Q\left(\frac{dr^2}{r^2 + a^2\gamma^2} + d\theta^2\right) + Q\cos^2\theta d\psi^2 + Q\sin^2\theta d\phi^2 \\ &\quad -2a\gamma(\cos^2\theta\, dy\, d\psi + \sin^2\theta\, dt\, d\phi) + dz^i dz^i, \end{aligned} \tag{3.2.14}$$

where the periodicity of y, ψ, ϕ are given by,

$$y \to y + 2\pi R_y, \quad \psi \to \psi + 2\pi, \quad \phi \to \phi + 2\pi \tag{3.2.15}$$

We can diagonalize the above metric by performing the following coordinate transformations,

$$\tilde{t} = \frac{a\gamma}{Q}t, \quad \tilde{y} = \frac{a\gamma}{Q}y, \quad \tilde{r} = \frac{r}{a\gamma}, \quad \tilde{\theta} = \theta, \quad \tilde{\psi} = \psi - \tilde{y}, \quad \tilde{\phi} = \phi - \tilde{t}, \tag{3.2.16}$$

Then the diagonal form of the metric which is locally $AdS_3 \times S^3$ is given by,

$$ds_0^2 = -Q(\tilde{r}^2 + 1) + Q\tilde{r}^2 d\tilde{y}^2 + Q\left(\frac{d\tilde{r}^2}{\tilde{r}^2 + 1} + d\tilde{\theta}^2\right) + Qcos^2\tilde{\theta}d\tilde{\psi}^2 + Qsin^2\tilde{\theta}d\tilde{\phi}^2, \tag{3.2.17}$$

with the following identifications,

$$\left\{\tilde{y} \to \tilde{y} + 2\pi\gamma, \tilde{\psi} \to \tilde{\psi} - 2\pi\gamma\right\}, \quad \tilde{\psi} \to \tilde{\psi} + 2\pi, \quad \tilde{\phi} \to \tilde{\phi} + 2\pi \tag{3.2.18}$$

Now at $(r = 0,\ \theta = \frac{\pi}{2})$ which implies $f = 0$, the $\tilde{y}$ circle and $\tilde{\psi}$ circle shrinks to zero. This corresponds a conical defect. But these kind of singularities are allowed in string theory and we won't be concerned about it.

Linear deformation of the type obtained via our Garfinkle-Vachaspati transform on this solution were studied in [35]. We proceed by writing the linear perturbation from reference [35] in a suggestive form. We will then see that the deformation is valid non-linearly. To begin with, let us start by writing the background solution in GMR form (3.2.1):

$$ds_0^2 = -\frac{1}{h}\left[du + A\right]\left[dv + B\right] + h ds_{\text{base}}^2 + dz_i dz_i, \tag{3.2.19}$$

$$C_0 = \frac{1}{2h}[dv + B] \wedge [du + A] + Q\frac{(r^2 + a^2\gamma^2)}{f}\, c_\theta^2\, d\phi \wedge d\psi, \tag{3.2.20}$$

with

$$ds_{\text{base}}^2 = \frac{f}{r^2 + a^2\gamma^2} dr^2 + d\theta^2 + r^2 c_\theta^2 d\psi^2 + (r^2 + a^2\gamma^2) s_\theta^2 d\phi^2, \tag{3.2.21}$$

and one-forms

$$A = \frac{a\gamma Q}{f}\{s_\theta^2 d\phi - c_\theta^2 d\psi\}, \tag{3.2.22}$$

$$B = \frac{a\gamma Q}{f}\{s_\theta^2 d\phi + c_\theta^2 d\psi\}, \tag{3.2.23}$$

where $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$.

The linear perturbation in reference [35] was constructed with the gauge choice

$$h_{\mu z} + (C - C_0)_{\mu z} = 0, \tag{3.2.24}$$

where z is one of the four-torus coordinates. The explicit form of the solution with added linear perturbation is

$$ds^2 = ds_0^2 + 2\,\epsilon\, e^{-in\frac{v}{R_y}} \left(\frac{r^2}{r^2 + a^2\gamma^2}\right)^{\frac{nk}{2}} K\; dz, \tag{3.2.25}$$

$$C = C_0 + \epsilon\, e^{-in\frac{v}{R_y}} \left(\frac{r^2}{r^2 + a^2\gamma^2}\right)^{\frac{nk}{2}}\; dz\; \wedge K, \tag{3.2.26}$$

where

$$K = \frac{Q}{Q + f}\left[dv - a\gamma(c_\theta^2 d\psi + s_\theta^2 d\phi)\right] + \frac{ia\gamma Q}{r(r^2 + a^2\gamma^2)} dr. \tag{3.2.27}$$

We can simplify this form of the solution by adding a pure-gauge piece. We start by observing that K defined in (3.2.27) can also be written as

$$K = -\frac{f}{Q + f}\left[dv + B\right] + dv + \frac{ia\gamma Q}{r(r^2 + a^2\gamma^2)} dr. \tag{3.2.28}$$

Contribution to C, cf. (3.2.26), from the last two terms of K in the form of equation (3.2.28) can be identified as a complete differential

$$e^{-in\frac{v}{R_y}}\left(\frac{r^2}{r^2+a^2\gamma^2}\right)^{\frac{nk}{2}}\left[dv+\frac{ia\gamma Q}{r(r^2+a^2\gamma^2)}dr\right]\equiv d\Psi, \tag{3.2.29}$$

where

$$\Psi=\frac{iR_y}{n}e^{-in\frac{v}{R_y}}\left(\frac{r^2}{r^2+a^2\gamma^2}\right)^{\frac{nk}{2}}. \tag{3.2.30}$$

As a result we can gauge away these pieces. Specifically, consider the following diffeomorphism and the gauge transformation,

$$\xi_z = -\Psi, \tag{3.2.31}$$

$$\Lambda = \Psi dz. \tag{3.2.32}$$

Thus, the new metric

$$g^{\text{new}}_{\mu\nu} = g_{\mu\nu}+\epsilon\,\nabla_{(\mu}\xi_{\nu)}, \tag{3.2.33}$$

takes the following form

$$ds^2_{\text{new}} = g^{\text{new}}_{\mu\nu}dx^\mu dx^\nu \tag{3.2.34}$$

$$= ds_0^2+2\,\epsilon\,e^{-in\frac{v}{R_y}}\left(\frac{r^2}{r^2+a^2\gamma^2}\right)^{\frac{nk}{2}}\left\{-\frac{f}{Q+f}\left[dv+B\right]\right\}dz, \tag{3.2.35}$$

and the associated two-form field is now

$$C_{\text{new}} = C+\epsilon\,d\Lambda \tag{3.2.36}$$

$$= C_0+\epsilon\,e^{-in\frac{v}{R_y}}\left(\frac{r^2}{r^2+a^2\gamma^2}\right)^{\frac{nk}{2}}\left\{\frac{f}{Q+f}\left[dv+B\right]\right\}\wedge dz. \tag{3.2.37}$$

The configuration (3.2.35) and (3.2.37) is a generalised Garfinkle-Vachaspati transform of background (3.2.19)–(3.2.20). It is a non-linear solution of ten-dimensional IIB supergravity. Therefore, from now onwards we set $\epsilon=1$. Realising that $\frac{f}{Q+f}$ is simply $\frac{1}{h}$ we observe that the above solution is compatible with the form (3.2.19), provided we shift the one-form du as

$$
\begin{aligned}
du &\rightarrow du + \Psi\, dz, && (3.2.38)\\
\Psi &= 2\left(\frac{r^2}{r^2+a^2\gamma^2}\right)^{\frac{nk}{2}} e^{-in\frac{v}{R_y}}. && (3.2.39)
\end{aligned}
$$

The scalar field Ψ satisfies $\Box_0\Psi = 0$ with respect to the background metric ds_0^2. This deformation is therefore of the form of GGV (2.3.39)-(2.3.40). We can generalise the above deformation further. Instead of working with the specific solution (3.2.39), we can consider the most general u-independent solution of the wave equation $\Box_0\Psi = 0$ that remains finite everywhere. Such a solution can be written as a superposition

$$
\Psi = \sum_{n=-\infty}^{\infty} c_n \left(\frac{r^2}{r^2+a^2\gamma^2}\right)^{\frac{|n|k}{2}} e^{-in\frac{v}{R_y}}. \tag{3.2.40}
$$

The requirement that Ψ be real fixes $(c_n)^* = c_{-n}$.

After exploring GGV for the two-charge multiwound D1-D5 solutions we can discuss more general three-charge solutions. We have already given a brief introduction to three-charge microstate geometries (see section 1.4.5). There we discussed the general class of multiwound supersymmetric D1-D5-P solutions constructed in [54, 55]. These family of solutions can be written in the GMR form (3.2.1) by identifying the quantities H, $\mathcal{F}$, β, ω as [76],

$$
\begin{aligned}
H &= h, && (3.2.41)\\
\mathcal{F} &= -\frac{2Q_p}{f}, && (3.2.42)\\
\beta &= \frac{Q}{f}(\gamma_1+\gamma_2)\,\eta\,(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi), && (3.2.43)\\
\omega &= \frac{Q}{f}\Big[\Big(2\gamma_1 - (\gamma_1+\gamma_2)\,\eta\Big(1-2\frac{Q_p}{f}\Big)\Big)\cos^2\theta\, d\psi \\
&\qquad + \Big(2\gamma_2 - (\gamma_1+\gamma_2)\,\eta\Big(1-2\frac{Q_p}{f}\Big)\Big)\sin^2\theta\, d\phi\Big], && (3.2.44)
\end{aligned}
$$

and the base metric h_{mn} given as,

$$
\begin{aligned}
ds^2_{\text{base}} &= h_{mn}dx^m dx^n = f\left(\frac{dr^2}{r^2+(\gamma_1+\gamma_2)^2\,\eta}+d\theta^2\right) \\
&+\frac{1}{f}\Big[\left[r^4+r^2\,(\gamma_1+\gamma_2)\,\eta\,(2\gamma_1-(\gamma_1-\gamma_2)\cos^2\theta)+(\gamma_1+\gamma_2)^2\,\gamma_1^2\,\eta^2\,\sin^2\theta\right]\cos^2\theta\,d\psi^2 \\
&+\left[r^4+r^2\,(\gamma_1+\gamma_2)\,\eta\,(2\gamma_2+(\gamma_1-\gamma_2)\sin^2\theta)+(\gamma_1+\gamma_2)^2\,\gamma_2^2\,\eta^2\,\cos^2\theta\right]\sin^2\theta\,d\phi^2 \\
&-2\gamma_1\gamma_2\,(\gamma_1+\gamma_2)^2\,\eta^2\,\sin^2\theta\cos^2\theta\,d\psi d\phi\Big]. \qquad (3.2.45)
\end{aligned}
$$

On this rather complicated configuration one can add a general deformation as,

$$
du \;\to\; du+\Psi_i\,dz_i, \qquad (3.2.46)
$$

$$
\Psi_i \;=\; \sum_{n=-\infty}^{\infty} c^i_n\left(\frac{r^2}{r^2\left(1+\frac{2a^2}{Q}m\left(m+\frac{1}{k}\right)\right)+\frac{a^2}{k^2}}\right)^{\frac{|n|k}{2}} e^{-in\frac{v}{R_y}}. \qquad (3.2.47)
$$

As we know here the index i refers to the four-torus directions. One can easily check that $\Box\Psi_i = 0$ with respect to the background metric (1.4.90). Note that when $m = 0$, scalar (3.2.47) reduces to deformation scalar (3.2.40); when $k = 1$ it reduces to the deformation considered in section 5 of [33]. The deformed two-form field is,

$$
\begin{aligned}
C &= -\frac{1}{2h}[du+\Psi_i\,dz_i]\wedge dv+\frac{(\gamma_1+\gamma_2)}{hf}\left(\eta Q_p-\frac{Q}{2}\right)[du+\Psi_i\,dz_i]\wedge(c_\theta^2 d\psi+s_\theta^2 d\phi) \\
&-\frac{Q}{2hf}(\gamma_2-\gamma_1)dv\wedge(c_\theta^2 d\psi-s_\theta^2 d\phi) \\
&-\frac{Q}{hf}c_\theta^2(r^2+\gamma_2(\gamma_1+\gamma_2)\eta+Q)d\psi\wedge d\phi. \qquad (3.2.48)
\end{aligned}
$$

The deformed solution has flat asymptotics, however it is not manifest in the above coordinates. In the next section we find a set of coordinates that makes the asymptotic flatness of the solution manifest and read off the charges of the solution. In the following section we identify the CFT states dual to the deformed spacetimes.

3.2.2 Global properties and smoothness of deformed spacetimes

In this section we present a discussion on asymptotics, ADM charges, smoothness and some other global properties and of the deformed spacetime. The following discussion is a generalisation of the corresponding discussion in [33] of D1-D5-P geometries with $k = 1$ to D1-D5-P orbifolds parametrised by integer $k \neq 1$. We write out calculations where our analysis offers a simplification, or a different perspective, or fixes typos/errors over the corresponding discussion in that reference.

3.2.2.1 Asymptotics

To find the map between the deformed spacetime and the CFT states, we need to evaluate charges of the deformed spacetime. We first evaluate the charges in the asymptotically flat setting, and in the next section in the $AdS_3 \times S^3 \times T^4$ setting. We assume that $c_0^i = 0$ in (3.2.47). A constant term in Ψ can be removed by shifting the u-coordinate. However, since y and z_i are periodic coordinates, such a shift does have an effect on the global properties of the solution. For simplicity we do not analyse the constant terms in Ψ_i here, and assume they are set to zero. At infinity metric of the deformed spacetime takes the form

$$ds^2 = -\left[du + f_i(v)dz_i\right]dv + dr^2 + r^2 d\Omega_3^2 + dz_i dz_i, \tag{3.2.49}$$

where

$$f_i(v) = \lim_{r\to\infty} \Psi_i(r,v) = \sum_{n\neq 0} c_n^i \left(1 + \frac{2a^2}{Q} m \left(m + \frac{1}{k}\right)\right)^{-\frac{|n|k}{2}} e^{-in\frac{v}{R_y}}. \tag{3.2.50}$$

The diffeomorphism that puts the metric (3.2.49) in a standard asymptotically flat form and has the property that the new time-coordinate is single valued is:

$$\begin{aligned} z_i' &= z_i - \frac{1}{2}\int_0^v f_i(\tilde{v})d\tilde{v}, & (3.2.51)\\ u' &= \lambda\left[u + \frac{1}{4}\int_0^v f_i(\tilde{v})f_i(\tilde{v})d\tilde{v}\right], & (3.2.52)\\ v' &= \frac{v}{\lambda}, & (3.2.53)\end{aligned}$$

with the value of λ is fixed by the requirement that the new time coordinate $t' = \frac{1}{2}(u' + v')$ is a single valued function under $y \sim y + 2\pi R_y$. This is achieved as follows:

$$\begin{aligned} t'(y = 2\pi R_y) - t'(y = 0) &= \lambda \left[\pi R_y + \frac{1}{8} \int_t^{t - 2\pi R_y} f_i(\tilde{v}) f_i(\tilde{v}) d\tilde{v} \right] - \frac{\pi R_y}{\lambda} && (3.2.54) \\ &= \pi R_y \left[\lambda - \frac{1}{\lambda} \right] + \frac{\lambda}{8} \int_0^{-2\pi R_y} f_i(\tilde{v}) f_i(\tilde{v}) d\tilde{v} && (3.2.55) \\ &= \pi R_y \left[\lambda - \frac{1}{\lambda} \right] - \frac{\lambda}{8} \int_0^{2\pi R_y} f_i(\tilde{v}) f_i(\tilde{v}) d\tilde{v}, && (3.2.56) \end{aligned}$$

where in going from the first step to the second we have used the fact that since $f_i(\tilde{v})$ are periodic functions in $\tilde{v} \sim \tilde{v} - 2\pi R_y$, the limit of integration $(t, t - 2\pi R_y)$ can be changed to $(0, -2\pi R_y)$. In going from the second step to the third step, we have once again used the periodic property of the functions $f_i(\tilde{v})$ and converted the limit of integration to $(0, 2\pi R_y)$. This fixes the value of λ to be:

$$\lambda^{-2} = \left[1 - \frac{1}{8\pi R_y} \int_0^{2\pi R_y} f_i(\tilde{v}) f_i(\tilde{v}) d\tilde{v} \right]. \qquad (3.2.57)$$

This expression differs from the one written in equation (4.12) of [33]; also the value of the function $f_i(v)$ in (3.2.50) is different from equation (6.2) of [33] when $k = 1$.

In new coordinates, the asymptotic metric (3.2.49) is

$$ds^2 = -(dt')^2 + (dy')^2 + dr^2 + r^2 d\Omega_3^2 + dz_i' dz_i'. \qquad (3.2.58)$$

The z_i' coordinates have the same periodicity as the z_i coordinates. The periodicity of the y' coordinate is

$$\begin{aligned} y'(y = 2\pi R_y) - y'(y = 0) &= \lambda \left[\pi R_y + \frac{1}{8} \int_t^{t - 2\pi R_y} f_i(\tilde{v}) f_i(\tilde{v}) d\tilde{v} \right] + \frac{\pi R_y}{\lambda} && (3.2.59) \\ &= \pi R_y \left[\lambda + \frac{1}{\lambda} \right] + \frac{\lambda}{8} \int_0^{-2\pi R_y} f_i(\tilde{v}) f_i(\tilde{v}) d\tilde{v} && (3.2.60) \\ &= \pi R_y \left[\lambda + \frac{1}{\lambda} \right] - \frac{\lambda}{8} \int_0^{2\pi R_y} f_i(\tilde{v}) f_i(\tilde{v}) d\tilde{v} && (3.2.61) \\ &= \frac{2\pi R_y}{\lambda}. && (3.2.62) \end{aligned}$$

This implies that the deformed solution has asymptotic radius $y' \sim y' + 2\pi R$, with

$$R = \frac{R_y}{\lambda}. \tag{3.2.63}$$

The picture is as follows: deformations of a given state are constructed by introducing functions Ψ_i, while keeping n_1, n_5, m, k and asymptotic radius R fixed. In order to work with radius R (as opposed to R_y) we introduce

$$h_i(v') = f_i(v) = f_i(\lambda v'). \tag{3.2.64}$$

and we also note that

$$\lambda^{-2} = 1 - \frac{1}{8\pi R}\int_0^{2\pi R} h_i(\tilde{v}')h_i(\tilde{v}')d\tilde{v}'. \tag{3.2.65}$$

3.2.2.2 ADM Charges

Now that we know the coordinate transformations that bring the metric in the standard flat form asymptotically, we can work out the charges. We extend the diffeomorphism (3.2.51)–(3.2.53) to finite radial coordinates as:

$$z_i' = z_i - \frac{1}{2}\int_0^v \Psi_i(\tilde{v})d\tilde{v}, \tag{3.2.66}$$

$$u' = \lambda\left[u + \frac{1}{4}\int_0^v \Psi_i(\tilde{v})\Psi_i(\tilde{v})d\tilde{v}\right], \tag{3.2.67}$$

$$v' = \frac{v}{\lambda}. \tag{3.2.68}$$

This choice simplifies the extraction of charges. At large values of r we find[1],

$$g_{t't'} = -1 + \frac{1}{r^2}\left(Q + \lambda^2 Q_p + \frac{1}{4}\lambda^2 Q h_i h_i\right) + \dots \quad (3.2.69)$$

$$g_{t'y'} = -\frac{\lambda^2}{r^2}\left(Q_p + \frac{1}{4}Q h_i h_i\right) + \dots \quad (3.2.70)$$

$$g_{y'y'} = 1 + \frac{1}{r^2}\left(-Q + \lambda^2 Q_p + \frac{1}{4}\lambda^2 Q h_i h_i\right) + \dots \quad (3.2.71)$$

$$g_{t'z_i} = \frac{\lambda Q}{2r^2} h_i + \dots \quad (3.2.72)$$

$$g_{t'\phi} = -\frac{\lambda Q}{r^2} s_\theta^2 \left(\gamma_2 - \frac{\gamma_1 + \gamma_2}{2}\eta\left(1 - \frac{1}{4}h_i h_i - \frac{1}{\lambda^2}\right)\right) + \dots \quad (3.2.73)$$

$$g_{t'\psi} = -\frac{\lambda Q}{r^2} c_\theta^2 \left(\gamma_1 - \frac{\gamma_1 + \gamma_2}{2}\eta\left(1 - \frac{1}{4}h_i h_i - \frac{1}{\lambda^2}\right)\right) + \dots . \quad (3.2.74)$$

From these components we can extract the charges. The ADM momenta of the solution are given by

$$P_i = -\frac{\pi}{4G_N}\int_0^{2\pi R} dy\, r^2\, \delta g_{t'z_i} = 0, \quad (3.2.75)$$

$$P_{y'} = -\frac{\pi}{4G_N}\int_0^{2\pi R} dy\, r^2\, \delta g_{t'y'} = \frac{\pi\lambda^2}{4G_N}\left(2\pi R\, Q_p + \frac{1}{4}Q\int_0^{2\pi R} h_i h_i dy'\right), \quad (3.2.76)$$

where we have used the fact that $c_0^i = 0$ and where $G_N = \frac{\pi^2\alpha'^4 g^2}{2V}$ is the six-dimensional Newton's constant. The ADM mass is [77]

$$M = \frac{\pi}{8G_N}\int_0^{2\pi R} dy\, r^2\, (3\delta g_{t't'} - \delta g_{y'y'}) \quad (3.2.77)$$

$$= \frac{\pi}{4G_N}(2Q)(2\pi R) + \frac{\pi\lambda^2}{4G}\left(2\pi R\, Q_p + \frac{1}{4}Q\int_0^{2\pi R} h_i h_i dy'\right) \quad (3.2.78)$$

$$= \frac{\pi}{4G_N}(2Q)(2\pi R) + P_{y'}. \quad (3.2.79)$$

Not surprisingly, the BPS bound is saturated; addition of momentum shifts the mass by $P_{y'}$. Using (3.2.8) can rewrite the ADM momentum $P_{y'}$ as

$$P_{y'} = \frac{n_1 n_5}{R}\left[m\left(m + \frac{1}{k}\right) + \frac{Q}{4a^2}\frac{1}{2\pi R}\int_0^{2\pi R} dy' h_i h_i\right]. \quad (3.2.80)$$

[1]In the following equations, we only write components of the metric that are relevant for the computation of the gravitational charges. The are other components with $\frac{1}{r^2}$ terms.

To extract angular momenta, we use

$$
\begin{aligned}
J_\phi &= -\frac{\pi}{8G_N}\int_0^{2\pi R} dy'\, r^2 \frac{\delta g_{t'\phi}}{\sin^2\theta}, && (3.2.81)\\
J_\psi &= -\frac{\pi}{8G_N}\int_0^{2\pi R} dy'\, r^2 \frac{\delta g_{t'\psi}}{\cos^2\theta}. && (3.2.82)
\end{aligned}
$$

A simple calculation then gives,

$$
\begin{aligned}
J_\phi &= \frac{\pi\lambda Q}{8G_N}\int_0^{2\pi R} dy' \left(\gamma_2 - \frac{\gamma_1+\gamma_2}{2}\eta\left(1-\frac{1}{4}h_i h_i - \frac{1}{\lambda^2}\right)\right) && (3.2.83)\\
&= \frac{\pi\lambda Q}{8G_N}\gamma_2(2\pi R) \;=\; \frac{n_1 n_5}{2}\left(m+\frac{1}{k}\right), && (3.2.84)
\end{aligned}
$$

where we have used expression for λ^{-2} (3.2.65) in going from the first to the second step. Similarly, we have

$$
J_\psi \;=\; \frac{\pi\lambda Q}{8G_N}\gamma_1(2\pi R) \;=\; -\frac{n_1 n_5}{2}m. \qquad (3.2.85)
$$

To summarise, the deformed state saturates the BPS bound and has charges

$$
\begin{aligned}
&P_{y'} = \frac{n_1 n_5}{R}\left[m\left(m+\frac{1}{k}\right) + \frac{Q}{4a^2}\frac{1}{2\pi R}\int_0^{2\pi R} dy' h_i h_i\right], && J_\phi = \frac{n_1 n_5}{2}\left(m+\frac{1}{k}\right), && (3.2.86)\\
&P_i = 0, && J_\psi = -\frac{n_1 n_5}{2}m. && (3.2.87)
\end{aligned}
$$

3.2.2.3 Smoothness of the deformed solution

Remarkably, the determinant of metric of the deformed solution gets no contribution from the scalars Ψ_i:

$$
\det g = -\frac{1}{4}\cos^2\theta\sin^2\theta h^2 f^2. \qquad (3.2.88)
$$

Therefore, as long as Ψ_i remain finite, the potential singularities can only occur at places where the background geometry can become singular. The vicinity of these potentially dangerous points is analysed in [55] for the undeformed solution. The analysis of that reference applies almost verbatim to our case together with the fact that the scalars (3.2.47) remain finite everywhere.

This is perfectly in agreement with the conjecture of reference [33] which states that any regular solution of the D1-D5 system can be deformed into a regular solution via the GGV transform provided, (i) Ψ_i satisfies $\Box\Psi_i = 0$, (ii) Ψ_i remains finite everywhere, (iii) Ψ_i approaches a regular function $f_i(v)$ as $r \to \infty$ on the four-dimensional base space. Clearly all these conditions are met for the specific class of D1-D5-P solutions studied in this thesis.

3.2.3 Decoupling Limit and Identifying CFT states

To map the deformed geometries into states in the dual CFT, we need to evaluate charges in the AdS region rather than the asymptotically flat region. Such a computation is possible only when the deformed geometry has a large AdS region; and a decoupling limit can be taken. The geometry develops a large AdS region when we take

$$\epsilon \equiv \frac{a^2}{Q} \ll 1. \tag{3.2.89}$$

To take the decoupling limit we must take $\epsilon \to 0$ while keeping the AdS radius $\sqrt{Q}$ fixed. The relation (3.2.13) implies that the size of the y-circle R_y should go to infinity. We introduce

$$\bar{u} = \frac{u}{R_y}, \qquad \bar{v} = \frac{v}{R_y}, \qquad \bar{r} = \frac{r}{a}, \tag{3.2.90}$$

and take the limit $R_y \to \infty$.

Without the deformation (i.e., with $\Psi_i = 0$) the decoupling limit gives

$$\begin{aligned} ds^2 \;&=\; Q\left[-\bar{r}^2 d\bar{u}d\bar{v} - \frac{1}{4}(d\bar{u}+d\bar{v})^2 + \frac{d\bar{r}^2}{\bar{r}^2+k^{-2}}\right] \\ &\quad + Q\left[d\theta^2 + c_\theta^2\left(d\psi - \frac{1}{2k}(d\bar{u}-d\bar{v}) + m d\bar{v}\right)^2 + s_\theta^2\left(d\phi - \frac{1}{2k}(d\bar{u}+d\bar{v}) - m d\bar{v}\right)^2\right] \\ &\quad + dz_i dz_i\,. \end{aligned} \tag{3.2.91}$$

To understand the decoupling limit while the scalars Ψ_i are turned on, we start by noting that in order to maintain ADM momentum (3.2.86) finite at $R_y \to \infty$, in addition to the scaling (3.2.90) of the coordinates we must scale the scalars Ψ_i as well. The appropriate scaling is given by

$$\Psi_i = \frac{a}{\sqrt{Q}}\,\bar{\Psi}_i = \frac{\sqrt{Q}}{R_y}\,\bar{\Psi}_i. \tag{3.2.92}$$

With this, terms of the form

$$[du + \Psi_i dz_i], \tag{3.2.93}$$

behave as

$$du + \Psi_i dz_i = R_y\, d\bar{u} + \frac{\sqrt{Q}}{R_y}\,\bar{\Psi}_i\, dz_i, \tag{3.2.94}$$

which in the decoupling limit $R_y \to \infty$ simply becomes

$$R_y\, d\bar{u}. \tag{3.2.95}$$

Thus it seems that in the decoupling limit all Ψ_i terms scale out, and we once again we get the decoupled metric (3.2.91). However, there is one subtlety. As we saw in the previous section to get a manifestly asymptotically flat deformed metric we should use the z'_i, t', y' coordinates instead of z_i, t, y so that we can connect the decoupled region to the asymptotically flat region. We will see that through this change of coordinates the scalars reappear. In order to implement these coordinate transformations, we first observe that in the decoupling limit λ from equation (3.2.57) simplifies to unity,

$$\begin{aligned}\lambda^{-2} &= \lim_{R_y\to\infty}\left[1 - \frac{1}{4}\frac{Q}{R_y^2}\left(\frac{1}{2\pi R_y}\int_0^{2\pi R_y} \bar{f}_i(\tilde{v})\bar{f}_i(\tilde{v})d\tilde{v}\right)\right] \\ &= 1.\end{aligned} \tag{3.2.96}$$

Since λ scales to unity, the transformations (3.2.66)–(3.2.68) simplify to

$$z'_i = z_i - \frac{1}{2}\sqrt{Q}\int_0^{\bar{v}} \bar{\Psi}_i\, d\tilde{\bar{v}}, \qquad u' = u, \qquad v' = v. \tag{3.2.97}$$

As a result, in primed coordinates the decoupled metric is

$$\begin{aligned} ds^2 &= Q\left[-\bar{r}^2 d\bar{u}d\bar{v} - \frac{1}{4}(d\bar{u}+d\bar{v})^2 + \frac{d\bar{r}^2}{\bar{r}^2+k^{-2}}\right] \\ &\quad + Q\left[d\theta^2 + c_\theta^2\left(d\psi - \frac{1}{2k}(d\bar{u}-d\bar{v}) + md\bar{v}\right)^2 + s_\theta^2\left(d\phi - \frac{1}{2k}(d\bar{u}+d\bar{v}) - md\bar{v}\right)^2\right] \\ &\quad + \left(dz_i' + \frac{1}{2}\sqrt{Q}\bar{\Psi}_i d\bar{v}\right)^2 . \end{aligned} \tag{3.2.98}$$

We can now read off the charges. We find

$$P_{y'} = \frac{n_1 n_5}{R}\left[m\left(m+\frac{1}{k}\right) + \frac{1}{8\pi}\int_0^{2\pi} d\bar{y}\bar{f}_i\bar{f}_i\right], \qquad J_\phi = \frac{n_1 n_5}{2}\left(m+\frac{1}{k}\right), \tag{3.2.99}$$

$$P_i = 0, \qquad J_\psi = -\frac{n_1 n_5}{2}m. \tag{3.2.100}$$

These charges agree with (3.2.86)–(3.2.87) in the $R_y \to \infty$ limit.

3.2.3.1 Deformed states in the D1-D5 CFT

The expression for the momentum $P_{y'}$, cf. (3.2.99), can be compared with momentum of the CFT state,

$$|\Psi\rangle = N \exp\left[\sum_{n>0}\mu_n^i J_{-n}^i\right]|\psi\rangle, \tag{3.2.101}$$

where $|\psi\rangle$ is the undeformed state and J_{-n}^i are the modes of the four U(1) currents of the D1-D5 CFT. Assuming that the state $|\psi\rangle$ is unit normalised, $\langle\psi|\psi\rangle = 1$, we can fix the normalisation constant N using the commutation relations,

$$[J_m^i, J_n^j] = m\frac{n_1 n_5}{2}\delta^{ij}\delta_{m+n}. \tag{3.2.102}$$

Define $A^\dagger = \sum_{n>0}\mu_n^i J_{-n}^i$. Using the fact that the commutator

$$[A, A^\dagger] = \frac{n_1 n_5}{2}\sum_{n>0} n(\mu_n^i)^*\mu_n^i \tag{3.2.103}$$

is a c-number, a small calculation shows that the normalisation constant N is given by

$$1 = \langle\Psi|\Psi\rangle = N^2\langle\psi|e^A e^{A^\dagger}|\psi\rangle = N^2 e^{[A,A^\dagger]}\langle\psi|e^{A^\dagger}e^A|\psi\rangle = N^2 e^{[A,A^\dagger]}, \tag{3.2.104}$$

where we have used $e^A|\psi\rangle = |\psi\rangle$ (which follows from $J^i_n|\psi\rangle = 0$ for positive n). This gives

$$N = \exp\left[-\frac{n_1 n_5}{4}\sum_{n>0} n(\mu^i_n)^*\mu^i_n\right]. \tag{3.2.105}$$

To find the momentum, we compute the expectation value of L_0 and $\bar{L}_0$. Since right moving sector is untouched, we simply have

$$\langle\Psi|\bar{L}_0|\Psi\rangle = \langle\psi|\bar{L}_0|\psi\rangle. \tag{3.2.106}$$

For the left sector, we need to do a computation. A simple way to organise this computation is as follows. Using the commutation relations,

$$[L_m, J^i_n] = -nJ^i_{m+n}, \tag{3.2.107}$$

in particular, $[L_0, J^i_{-n}] = nJ^i_{-n}$, we get

$$[L_0, A^\dagger] = \sum_{n>0}\mu^i_n[L_0, J^i_{-n}] = \sum_{n>0} n\mu^i_n J^i_{-n} =: B^\dagger. \tag{3.2.108}$$

To calculate $\langle\Psi|L_0|\Psi\rangle$ we observe

$$\langle\Psi|L_0|\Psi\rangle = N^2\langle\psi|e^A L_0 e^{A^\dagger}|\psi\rangle = N^2\langle\psi|e^A e^{A^\dagger}e^{-A^\dagger}L_0 e^{A^\dagger}|\psi\rangle. \tag{3.2.109}$$

Now we can use Baker–Campbell–Hausdorff formula to write $e^{-A^\dagger}L_0 e^{A^\dagger} = L_0 + B^\dagger$. We also use $e^A e^{A^\dagger} = e^{A^\dagger}e^A e^{[A,A^\dagger]}$ and the fact that $N^2 e^{[A,A^\dagger]} = 1$ as shown earlier. We get

$$\langle\Psi|L_0|\Psi\rangle = N^2\langle\psi|e^A e^{A^\dagger}(L_0 + B^\dagger)|\psi\rangle = \langle\psi|e^{A^\dagger}e^A(L_0 + B^\dagger)|\psi\rangle. \tag{3.2.110}$$

Now we use $[L_0, A]|\psi\rangle = B|\psi\rangle = 0$, as B contains only J_n^i with positive n, we get

$$\begin{aligned} \langle\Psi|L_0|\Psi\rangle &= \langle\psi|L_0|\psi\rangle + \langle\psi|[A, B^\dagger]|\psi\rangle && (3.2.111) \\ &= \langle\psi|L_0|\psi\rangle + \sum_{n>0} \frac{n^2 n_1 n_5}{2} (\mu_n^i)^* \mu_n^i, && (3.2.112) \end{aligned}$$

We conclude that,

$$\langle\Psi|L_0 - \bar{L}_0|\Psi\rangle = RP_{y'} = \langle\psi|L_0 - \bar{L}_0|\psi\rangle + \sum_{n>0} \frac{n^2 n_1 n_5}{2} (\mu_n^i)^* \mu_n^i. \qquad (3.2.113)$$

Upon doing the Fourier expansion of (3.2.99) in the decoupling limit, we get

$$RP_{y'} = \langle\psi|L_0 - \bar{L}_0|\psi\rangle + \sum_{n>0} \frac{n_1 n_5}{2} \frac{Q}{a^2} \left((c_n^i)^* c_n^i\right). \qquad (3.2.114)$$

Therefore, the map between the quantities c_n^i and μ_n^i is

$$\mu_n^i = \frac{1}{n} \sqrt{\frac{Q}{a^2}} c_n^i. \qquad (3.2.115)$$

Let us remark that in the computations of this subsection the only property of the undeformed state $|\psi\rangle$ we have used is that it is annihilated by A and B operators. The above analysis is therefore applicable to a large class of states. Although matching of the charges is no proof that the identified states are dual to the gravity deformation considered above; it is a strong indicator.

Having explored the application of GGV to the class of supersymmetric D1-D5-P solutions with vanishing dilaton, the most relevant extension of is the inclusion of the dilaton. In the following sections we are going to apply GGV on the D1-D5-P solutions with non-zero dilaton. Most of the analysis are quite similar to those done in this section. We will also discuss its applications to F1-P solutions.

3.3 GGV on Supergravity Solutions with Dilaton

In section 2.4 we postulate our generalised Garfinkle-Vachaspati transform (GGV) with dilaton. Two set-ups related by S-duality, namely, ten-dimensional type IIB Ramond-Ramond (R-R) sector with dilaton and the ten-dimensional Neveu-Schwarz (NS-NS) sector with dilaton, are addressed in sections 2.4.1 and 2.4.2 respectively.

Now, we are going to use the techniques to the class of D1-D5-P solutions discussed in the previous section without setting dilaton to zero. In section 3.3.1 we work out travelling-wave deformation involving the torus directions on a class of supersymmetric D1-D5-P orbifold geometries. The deformed solutions are given in terms of solutions of a (non-minimally coupled) scalar field on the background geometry. When the background contains a large AdS region, the deformed states are identified in the D1-D5 CFT as an action of a U(1) current on the undeformed state. In section 3.3.2 application of the GGV technique to the F1-P system is discussed.

3.3.1 Deformation of a class of D1-D5-P backgrounds

In this section we explore applications of the GGV transform to the class of multiwound supersymmetric D1-D5-P geometries with non-zero dilaton profile. As described in the vanishing dilaton case, the system consists of type IIB string theory compactified on S$^1\times$ T^4. Similar to the last section, we consider traveling wave deformation along the torus directions on the class of D1-D5-P backgrounds constructed in [54, 55]. In case of non-zero dilaton $\Phi \neq 0$, for the D1-D5-P solutions we have $Q_1 \neq Q_5$. One main difference from the last section due to presence of dilaton is that now the string frame and Einstein frame has to be treated diffrently. We will mostly stick to the string frame. A standard form [55] for the string frame metric, RR

2-form and dilaton for this background is,

$$\begin{aligned}
ds^2 &= -\frac{1}{h}\,(dt^2 - dy^2) + \frac{Q_p}{h\,f}\,(dt-dy)^2 + h\,f\left(\frac{dr^2}{r^2 + (\gamma_1+\gamma_2)^2\,\eta} + d\theta^2\right) \\
&\quad + h\left(r^2 + \gamma_1\,(\gamma_1+\gamma_2)\,\eta - \frac{Q_1 Q_5\,(\gamma_1^2-\gamma_2^2)\,\eta\,\cos^2\theta}{h^2\,f^2}\right)\cos^2\theta\,d\psi^2 \\
&\quad + h\left(r^2 + \gamma_2\,(\gamma_1+\gamma_2)\,\eta + \frac{Q_1 Q_5\,(\gamma_1^2-\gamma_2^2)\,\eta\,\sin^2\theta}{h^2\,f^2}\right)\sin^2\theta\,d\phi^2 \\
&\quad + \frac{Q_p\,(\gamma_1+\gamma_2)^2\,\eta^2}{h\,f}\,(\cos^2\theta\,d\psi + \sin^2\theta\,d\phi)^2 \\
&\quad - \frac{2\,\sqrt{Q_1 Q_5}}{h\,f}\,(\gamma_1\,\cos^2\theta\,d\psi + \gamma_2\,\sin^2\theta\,d\phi)\,(dt-dy) \\
&\quad - \frac{2\,\sqrt{Q_1 Q_5}\,(\gamma_1+\gamma_2)\,\eta}{h\,f}\,(\cos^2\theta\,d\psi + \sin^2\theta\,d\phi)\,dy + \sqrt{\frac{H_1}{H_5}}\,(dz^i dz^i)
\end{aligned} \tag{3.3.116}$$

The associated two-form field is given by,

$$\begin{aligned}
C^{(2)} &= -\frac{\sqrt{Q_1 Q_5}\,\cos^2\theta}{H_1\,f}\,(\gamma_2\,dt + \gamma_1\,dy)\wedge d\psi - \frac{\sqrt{Q_1 Q_5}\,\sin^2\theta}{H_1\,f}\,(\gamma_1\,dt + \gamma_2\,dy)\wedge d\phi \\
&\quad + \frac{(\gamma_1+\gamma_2)\,\eta\,Q_p}{\sqrt{Q_1 Q_5}H_1\,f}\,(Q_1\,dt + Q_5\,dy)\wedge(\cos^2\theta\,d\psi + \sin^2\theta\,d\phi) \\
&\quad - \frac{Q_1}{H_1\,f}\,dt\wedge dy - \frac{Q_5\,\cos^2\theta}{H_1\,f}\,(r^2 + \gamma_2\,(\gamma_1+\gamma_2)\,\eta + Q_1)\,d\psi\wedge d\phi\,,
\end{aligned} \tag{3.3.117}$$

$$e^{2\Phi} = \frac{H_1}{H_5}\,, \tag{3.3.118}$$

where

$$\gamma_1 = -am\,, \quad \gamma_2 = a\left(m + \frac{1}{k}\right) \tag{3.3.119}$$

and

$$\begin{aligned}
&a = \frac{\sqrt{Q_1\,Q_5}}{R_y}\,, \quad Q_p = -\,\gamma_1\,\gamma_2 \quad \eta = \frac{Q_1\,Q_5}{Q_1\,Q_5 + Q_1\,Q_p + Q_5\,Q_p}\,, \\
&f = r^2 + a^2\,(\gamma_1+\gamma_2)\,\eta\,(\gamma_1\,\sin^2\theta + \gamma_2\,\cos^2\theta)\,, \\
&H_1 = 1 + \frac{Q_1}{f}\,, \quad H_5 = 1 + \frac{Q_5}{f}\,, \quad h = \sqrt{H_1\,H_5}\,.
\end{aligned} \tag{3.3.120}$$

This configuration carries D1, D5, and P charges. The integer number of D1, D5, and P branes n_1, n_5, n_p, respectively are related to the parameters appearing in the metric as follows

$$Q_1 = \frac{g\,\alpha'^3}{V}\,n_1, \qquad Q_5 = g\,\alpha'\,n_5, \qquad Q_p = \frac{g^2\,\alpha'^4}{V\,R_y^2}\,n_p. \tag{3.3.121}$$

The metric (3.3.116) and associated two-form field (3.3.117) can be written in a generalised GMR form as a 2D fiber over a 4D almost hyper-Kähler base space. Note that we call it a generalised GMR form, as the nomenclature GMR form typically refers to supersymmetric solutions of minimal 6D supergravity [65]. For non-minimal 6D supergravity supersymmetric solutions have been recently studied in [78, 79]. As a 2D fibre over a 4D base space the string frame metric takes the form

$$ds^2 = -h^{-1}(dv+\beta)\Big(du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\Big) + h h_{mn}dx^m dx^n + \sqrt{\frac{H_1}{H_5}}(dz^i dz^i), \tag{3.3.122}$$

where $u = t + y$ and $v = t - y$, and

$$\mathcal{F} = -\frac{2Q_p}{f}, \tag{3.3.123}$$

$$\beta = \frac{\sqrt{Q_1 Q_5}}{f}\,(\gamma_1+\gamma_2)\,\eta\,(\cos^2\theta\,d\psi + \sin^2\theta\,d\phi), \tag{3.3.124}$$

$$\begin{aligned}\omega = \frac{\sqrt{Q_1 Q_5}}{f}\,\Big[&\Big(2\gamma_1 - (\gamma_1+\gamma_2)\,\eta\,\Big(1-2\frac{Q_p}{f}\Big)\Big)\,\cos^2\theta\,d\psi \\ &+\Big(2\gamma_2 - (\gamma_1+\gamma_2)\,\eta\,\Big(1-2\frac{Q_p}{f}\Big)\Big)\,\sin^2\theta\,d\phi\Big],\end{aligned} \tag{3.3.125}$$

and the base metric h_{mn} given as,

$$\begin{aligned} ds^2_{\text{base}} = {}& h_{mn}dx^m dx^n = f\left(\frac{dr^2}{r^2+(\gamma_1+\gamma_2)^2\,\eta} + d\theta^2\right) \\ &+\frac{1}{f}\Big[[r^4 + r^2\,(\gamma_1+\gamma_2)\,\eta\,(2\gamma_1 - (\gamma_1-\gamma_2)\cos^2\theta) + (\gamma_1+\gamma_2)^2\,\gamma_1^2\,\eta^2\,\sin^2\theta]\cos^2\theta\,d\psi^2 \\ &+[r^4 + r^2\,(\gamma_1+\gamma_2)\,\eta\,(2\gamma_2 + (\gamma_1-\gamma_2)\sin^2\theta) + (\gamma_1+\gamma_2)^2\,\gamma_2^2\,\eta^2\,\cos^2\theta]\sin^2\theta\,d\phi^2 \\ &-2\gamma_1\gamma_2\,(\gamma_1+\gamma_2)^2\,\eta^2\,\sin^2\theta\cos^2\theta\,d\psi d\phi\Big]. \end{aligned} \tag{3.3.126}$$

The above configuration has

$$k = \frac{\partial}{\partial u}, \tag{3.3.127}$$

as the appropriate null Killing vector and

$$l^{(i)} = \frac{\partial}{\partial z^i}, \tag{3.3.128}$$

as the appropriate spacelike Killing vector for the application of the generalised Garfinkle-Vachaspati transform. The background configuration also satisfies the transversality condition, cf. (2.4.66),

$$k^\mu F_{\mu\nu\rho} = -(d(e^{-\Phi}k))_{\nu\rho}. \tag{3.3.129}$$

A general solution to the scalar equation, cf. (2.4.68),

$$\partial_\mu \left[e^{-2\Phi} \sqrt{-g} g^{\mu\nu} \partial_\nu \Psi \right] = 0, \tag{3.3.130}$$

can be obtained using the ansatz

$$\Psi = \sum_{n=-\infty}^{\infty} f_n(r) \exp\left[-in \frac{v}{R_y} \right]. \tag{3.3.131}$$

Upon substituting this ansatz we get ordinary differential equations for the functions $f_n(r)$, which can be readily solved. We find

$$\Psi_i(r, v) = \sum_{n=-\infty}^{\infty} c_n^i \left(\frac{r^2}{r^2 \left(1 + a^2 \frac{(Q_1+Q_5)}{Q_1 Q_5} m \left(m + \frac{1}{k} \right) \right) + \frac{a^2}{k^2}} \right)^{\frac{|n|k}{2}} e^{-in\frac{v}{R_y}}. \tag{3.3.132}$$

where the index i refers to the four-torus coordinates z^i. One can see that upon setting $Q_1 = Q_5 = Q$ the above expression reduces to (3.2.47), which further validates our technique.

Here, we are considering all the four torus directions where the generalised Garfinkle-

Vachaspati transform is of the form

$$
\begin{aligned}
g_{\mu\nu} &\rightarrow g_{\mu\nu} + \sum_{i=1}^{4} \Psi_i e^{-\Phi}(k_\mu l^{(i)}_\nu + k_\nu l^{(i)}_\mu), && (3.3.133)\\
C_{\mu\nu} &\rightarrow C_{\mu\nu} - \sum_{i=1}^{4} \Psi_i e^{-2\Phi}(k_\mu l^{(i)}_\nu - k_\nu l^{(i)}_\mu), && (3.3.134)
\end{aligned}
$$

and it simply corresponds to replacing

$$
du \rightarrow du + \Psi_i dz^i, \tag{3.3.135}
$$

in the background metric (3.3.122). For the form field, the recipe is the same, but there are some details. The two form field written above can also be written as

$$
\begin{aligned}
C^{(2)} &= -\frac{1}{2}\left(H_1^{-1}\right) du \wedge dv \\
&+ (\gamma_1+\gamma_2)\frac{(\eta\, Q_p\,(Q_1+Q_5) - Q_1\, Q_5)}{2\sqrt{Q_1\, Q_5}\, H_1\, f}\, du \wedge (\cos^2\theta\, d\psi + \sin^2\theta\, d\phi) \\
&+ (\gamma_1+\gamma_2)\,\frac{\eta\, Q_p\,(Q_1 - Q_5)}{2\sqrt{Q_1\, Q_5}\, H_1\, f} dv \wedge (\cos^2\theta\, d\psi + \sin^2\theta\, d\phi) \\
&+ (\gamma_1-\gamma_2)\,\frac{\sqrt{Q_1\, Q_5}}{2\, H_1\, f}\, dv \wedge (\cos^2\theta\, d\psi - \sin^2\theta\, d\phi) \\
&- \frac{Q_5\, \cos^2\theta}{H_1\, f}\,(r^2 + \gamma_2\,(\gamma_1+\gamma_2)\,\eta + Q_1)\, d\psi \wedge d\phi\,,
\end{aligned} \tag{3.3.136}
$$

where we have removed a constant term proportional to $du \wedge dv$ by a gauge transformation. In this form the deformation of $C^{(2)}$ also simply corresponds to replacing

$$
du \rightarrow du + \Psi_i dz^i. \tag{3.3.137}
$$

When $Q_1 = Q_5$ the deformation reduces to to the one considered in section 3.2. In the following section we will discuss about the asymptotics and smoothness of the deformed solution and compute the ADM charges as we have done for the vanishing dilaton case.

3.3.1.1 Global properties and smoothness

Although it is not manifest in the above coordinates, the deformed solution has flat asymptotics [33, 52]. Much of the following discussion in this section parallels the corresponding discussions in those references, so we shall be brief. At infinity the metric of the deformed solution looks like,

$$ds^2 = -[du + f_i(v)dz^i]dv + dr^2 + r^2 d\Omega_3^2 + dz^i dz^i, \tag{3.3.138}$$

where

$$f_i(v) = \lim_{r\to\infty} \Psi_i(r,v) = \sum_{n\neq 0} c_n^i \left(1 + \frac{a^2(Q_1+Q_5)}{Q_1 Q_5} m\left(m + \frac{1}{k}\right)\right)^{-\frac{|n|k}{2}} e^{-in\frac{v}{R_y}}. \tag{3.3.139}$$

As done in subsection 3.2.2, for simplicity, from now onwards we assume $c_0^i = 0$. The coordinate transformation that puts the deformed spacetime in an asymptotically flat form and simplifies the extraction of charges is

$$\begin{aligned} z'_i &= z_i - \frac{1}{2}\int_0^v \Psi_i(r,\tilde{v})d\tilde{v}, & (3.3.140)\\ u' &= \lambda\left[u + \frac{1}{4}\int_0^v \Psi_i(r,\tilde{v})\Psi_i(r,\tilde{v})d\tilde{v}\right], & (3.3.141)\\ v' &= \frac{v}{\lambda}. & (3.3.142)\end{aligned}$$

In the $r \to \infty$ limit, this transformation simplifies to,

$$\begin{aligned} z'_i &= z_i - \frac{1}{2}\int_0^v f_i(\tilde{v})d\tilde{v}, & (3.3.143)\\ u' &= \lambda\left[u + \frac{1}{4}\int_0^v f_i(\tilde{v})f_i(\tilde{v})d\tilde{v}\right], & (3.3.144)\\ v' &= \frac{v}{\lambda}, & (3.3.145)\end{aligned}$$

where

$$\lambda^{-2} = 1 - \frac{1}{8\pi R_y}\int_0^{2\pi R_y} f_i(\tilde{v})f_i(\tilde{v})d\tilde{v}. \tag{3.3.146}$$

The value of λ is fixed by the requirement that the new time coordinate $t' = \frac{1}{2}(u' + v')$ is a single valued function under $y \sim y + 2\pi R_y$ at infinity. In new coordinates, the asymptotic metric (3.3.138) is manifestly flat,

$$ds^2 = -(dt')^2 + (dy')^2 + dr^2 + r^2 d\Omega_3^2 + dz'^i dz'^i. \tag{3.3.147}$$

The z'^i coordinates have the same periodicity as the z^i coordinates. The periodicity of the $y' = \frac{1}{2}(u' + v')$ coordinate is $y' \sim y' + 2\pi R$, with $R = \lambda^{-1} R_y$. In the rest of the section we exclusively work with R as opposed to R_y. Following the analysis of the dilaton free case in 3.2.2 and also as considered in [33], we introduce

$$h_i(v') := f_i(v) = f_i(\lambda v'). \tag{3.3.148}$$

In terms of the parameter R we have,

$$\lambda^{-2} = 1 - \frac{1}{8\pi R} \int_0^{2\pi R} h_i(\tilde{v}') h_i(\tilde{v}') d\tilde{v}'. \tag{3.3.149}$$

Now we would like to extract the ADM quantities. We find that it is most convenient to do this computation in six-dimensions as the ten-dimensional string frame metric is directly related to the six-dimensional Einstein frame metric. At large values of r we find that the relevant terms of the ten-dimensional string frame metric admits an expansion of the form,

$$g_{t't'} = -1 + \frac{1}{r^2}\left(\frac{Q_1 + Q_5}{2} + \lambda^2 Q_p + \frac{1}{4}\lambda^2 Q_1 h_i h_i\right) + \dots \tag{3.3.150}$$

$$g_{t'y'} = -\frac{\lambda^2}{r^2}\left(Q_p + \frac{1}{4} Q_1 h_i h_i\right) + \dots \tag{3.3.151}$$

$$g_{y'y'} = 1 + \frac{1}{r^2}\left(-\frac{Q_1 + Q_5}{2} + \lambda^2 Q_p + \frac{1}{4}\lambda^2 Q_1 h_i h_i\right) + \dots \tag{3.3.152}$$

$$g_{t'\phi} = -\frac{\lambda\sqrt{Q_1 Q_5}}{r^2} s_\theta^2 \left(\gamma_2 - \frac{\gamma_1 + \gamma_2}{2}\eta\left(1 - \frac{1}{4} h_i h_i - \frac{1}{\lambda^2}\right)\right) + \dots \tag{3.3.153}$$

$$g_{t'\psi} = -\frac{\lambda\sqrt{Q_1 Q_5}}{r^2} c_\theta^2 \left(\gamma_1 - \frac{\gamma_1 + \gamma_2}{2}\eta\left(1 - \frac{1}{4} h_i h_i - \frac{1}{\lambda^2}\right)\right) + \dots \ . \tag{3.3.154}$$

From these components we can extract (six-dimensional) ADM quantities. The ADM momenta in the y'-direction is

$$P_{y'} = -\frac{\pi}{4G}\int_0^{2\pi R} dy\, r^2\, \delta g_{t'y'} = \frac{\pi\lambda^2}{4G}\left(2\pi R\, Q_p + \frac{1}{4}Q_1\int_0^{2\pi R} dy' h_i h_i\right) \quad (3.3.155)$$

$$= \frac{n_1 n_5}{R}\left[m\left(m+\frac{1}{k}\right) + \frac{Q_1}{4a^2}\frac{1}{2\pi R}\int_0^{2\pi R} dy' h_i h_i\right], \quad (3.3.156)$$

where we have used the 6-dimensional Newton's constant to be $G = \frac{\pi^2\alpha'^4 g^2}{2V}$ together with (3.3.121). The ADM mass is [52]

$$M = \frac{\pi}{8G}\int_0^{2\pi R} dy\, r^2\, (3\delta g_{t't'} - \delta g_{y'y'}) \quad (3.3.157)$$

$$= \frac{\pi}{4G}(Q_1 + Q_5)(2\pi R) + P_{y'}. \quad (3.3.158)$$

We note that the BPS bound is saturated; addition of momentum shifts the mass by $P_{y'}$. To extract angular momenta, we use

$$J_\phi = -\frac{\pi}{8G}\int_0^{2\pi R} dy'\, r^2 \frac{\delta g_{t'\phi}}{\sin^2\theta} = \frac{n_1 n_5}{2}\left(m + \frac{1}{k}\right), \quad (3.3.159)$$

$$J_\psi = -\frac{\pi}{8G}\int_0^{2\pi R} dy'\, r^2 \frac{\delta g_{t'\psi}}{\cos^2\theta} = -\frac{n_1 n_5}{2} m. \quad (3.3.160)$$

To analyse the smoothness of the spacetime, we start by looking at the determinant of the deformed metric. The determinant of the deformed metric remains the same as the undeformed metric. Furthermore, since the scalar (3.3.132) is finite everywhere, potential singularities can only occur at places where the background solution becomes singular. In the background solution, there are no such points [54, 55]. Hence the solution remains smooth even after the deformation.

3.3.1.2 Decoupling limit

Just like the case of zero dilaton, again the *undeformed* geometry develops a large AdS region when,

$$\epsilon \equiv \frac{\sqrt{Q_1 Q_5}}{R_y^2} = \frac{a^2}{\sqrt{Q_1 Q_5}} \ll 1. \quad (3.3.161)$$

To obtain the decoupled metric we introduce,

$$\bar{u} = \frac{u}{R_y}, \qquad \bar{v} = \frac{v}{R_y}, \qquad \bar{r} = \frac{r}{a}, \tag{3.3.162}$$

and take the limit $R_y \to \infty$ keeping Q_1 and Q_5 fixed. Due to rescaling of coordinates (3.3.162) we get a metric that describes the inner $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ region of the geometry,

$$\begin{aligned} ds^2 &= \sqrt{Q_1 Q_5}\left[-\bar{r}^2 d\bar{u}d\bar{v} - \frac{1}{4}(d\bar{u}+d\bar{v})^2 + \frac{d\bar{r}^2}{\bar{r}^2+k^{-2}}\right] \\ &\quad + \sqrt{Q_1 Q_5}\Big[d\theta^2 + c_\theta^2\left(d\psi - \frac{1}{2k}(d\bar{u}-d\bar{v}) + m d\bar{v}\right)^2 \\ &\quad + s_\theta^2\left(d\phi - \frac{1}{2k}(d\bar{u}+d\bar{v}) - m d\bar{v}\right)^2\Big] + \sqrt{\frac{Q_1}{Q_5}}\, dz^i dz^i\,. \end{aligned} \tag{3.3.163}$$

To obtain the decoupled metric with the deformation turned on we proceed in exactly the same way as for dilaton free case in section 3.2.3 [33,52]. In order to maintain ADM momentum (3.3.156) finite as R_y becomes large, we must scale the scalars appropriately with R_y. In the present set-up, scalars should scale as

$$\Psi_i =: \frac{a}{\sqrt{Q_1}}\,\bar{\Psi}_i = \frac{\sqrt{Q_5}}{R_y}\,\bar{\Psi}_i. \tag{3.3.164}$$

With this rescaling, terms in the metric of the form $du + \Psi_i dz^i$ behave as

$$du + \Psi_i dz^i = R_y\, d\bar{u} + \frac{\sqrt{Q_5}}{R_y}\,\bar{\Psi}_i\, dz^i. \tag{3.3.165}$$

In the limit $R_y \to \infty$ such terms simply become $R_y\, d\bar{u}$, i.e., scalars Ψ_i all scale out. Once again we get the decoupled metric (3.3.163).

However again recalling from the analysis of the dilaton free case, that the deformed metric is not manifestly asymptotically flat in coordinates z_i, t, y; it is manifestly asymptotically flat in z_i', t', y'. The decoupled metric in the z_i', t', y' coordinates is naturally glued to the asymptotically flat region. Therefore, we should write the decoupled metric in these coordinates. This change of coordinates reintroduces scalars. In the $R_y \to \infty$ limit transformations (3.3.140)–

(3.3.142) simplify to

$$z_i' = z_i - \frac{1}{2}\sqrt{Q_5}\int_0^{\bar{v}} \bar{\Psi}_i \, d\bar{\tilde{v}}, \qquad \bar{u}' = \bar{u}, \qquad \bar{v}' = \bar{v}. \tag{3.3.166}$$

The decoupled metric takes the form,

$$\begin{aligned} ds^2 &= \sqrt{Q_1 Q_5}\left[-\bar{r}^2 d\bar{u}d\bar{v} - \frac{1}{4}(d\bar{u}+d\bar{v})^2 + \frac{d\bar{r}^2}{\bar{r}^2+k^{-2}}\right] \\ &\quad + \sqrt{Q_1 Q_5}\left[d\theta^2 + c_\theta^2\left(d\psi - \frac{1}{2k}(d\bar{u}-d\bar{v}) + md\bar{v}\right)^2 + s_\theta^2\left(d\phi - \frac{1}{2k}(d\bar{u}+d\bar{v}) - md\bar{v}\right)^2\right] \\ &\quad + \sqrt{\frac{Q_1}{Q_5}}\left(dz_i' + \frac{1}{2}\sqrt{Q_5}\,\bar{\Psi}_i d\bar{v}\right)^2 . \end{aligned} \tag{3.3.167}$$

We can now read off the charges, say, by comparing the above metric to a standard form of asymptotic form of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$. We find,

$$\begin{aligned} P_{y'} &= \frac{n_1 n_5}{R}\left[m\left(m+\frac{1}{k}\right) + \frac{1}{8\pi}\int_0^{2\pi} d\bar{y}\bar{f}_i\bar{f}_i\right], && (3.3.168) \\ J_\phi &= \frac{n_1 n_5}{2}\left(m+\frac{1}{k}\right), && (3.3.169) \\ J_\psi &= -\frac{n_1 n_5}{2}m. && (3.3.170) \end{aligned}$$

These charges agree with expressions (3.3.156), (3.3.159), (3.3.160) in the $R_y \to \infty$ limit.

3.3.1.3 Deformed states in the D1-D5 CFT

Let $|\psi\rangle$ be the normalised state in the D1-D5 CFT that describes the dual to the undeformed gravity configuration with ADM momentum

$$\frac{n_1 n_5}{R}\left[m\left(m+\frac{1}{k}\right)\right]. \tag{3.3.171}$$

Then the expression for the momentum $P_{y'}$, cf. (3.3.168), can be compared with the momentum of the normalised deformed CFT state as done for minimal-supergravity solution [33, 52],

$$|\Psi\rangle = \exp\left[-\frac{n_1 n_5}{4}\sum_{n>0} n(\mu_n^i)^*\mu_n^i\right]\exp\left[\sum_{n>0}\mu_n^i J_{-n}^i\right]|\psi\rangle, \tag{3.3.172}$$

where J^i_{-n} are the modes of the four U(1) currents of the D1-D5 CFT, and the parameters μ^i_n are determined below. The momentum of the deformed state turns out to be,

$$RP_{y'} := \langle\Psi|L_0 - \bar{L}_0|\Psi\rangle = \langle\psi|L_0 - \bar{L}_0|\psi\rangle + \sum_{n>0} \frac{n^2 n_1 n_5}{2}(\mu^i_n)^* \mu^i_n \tag{3.3.173}$$

$$= n_1 n_5 \left[m \left(m + \frac{1}{k} \right) \right] + \sum_{n>0} \frac{n^2 n_1 n_5}{2}(\mu^i_n)^* \mu^i_n \tag{3.3.174}$$

Upon doing the Fourier expansion of (3.3.168) in the decoupling limit, we get (assuming Ψ_i's are real scalars),

$$RP_{y'} = n_1 n_5 \left[m \left(m + \frac{1}{k} \right) \right] + \sum_{n>0} \frac{n_1 n_5}{2} \frac{Q_1}{a^2} \left((c^i_n)^* c^i_n \right). \tag{3.3.175}$$

From the matching between the gravity and the CFT answers we arrive at the relation between the quantities c^i_n and μ^i_n,

$$\mu^i_n = \frac{1}{n} \frac{\sqrt{Q_1}}{a} c^i_n. \tag{3.3.176}$$

With this identification we have singled out a state in the D1-D5 CFT that has the same charges as the deformed gravity solutions.

3.3.2 Application to the F1-P system

In this section, we briefly discuss application of the GGV transform to the F1-P system. Since the most general vibrating fundamental string solution with momentum modes added on top is already well known [42, 80],[2] we do not expect that the GGV technique would allow us to discover something novel. Nonetheless, the F1-P system is well suited for an application of the GGV transform in the NS sector.[3]

We start with the chiral null model for the NS sector of type II supergravity in Einstein frame [81, 82]. We take ∂_u as the null Killing vector. Metric and the supporting matter fields

[2] Also discussed in section 1.4.2.1.

[3] Application to the NS1-NS5 bound states is also a possibility, but this set-up is related to the D1-D5 set-up by S-duality.

take the form,

$$ds^2 = H^{3/4}(-dudv + Kdv^2 + 2A_i dx^i dv) + H^{-1/4}(dx^i dx^i + dz^j dz^j), \tag{3.3.177}$$
$$B_{uv} = \frac{H}{2}\,, \qquad B_{vi} = -HA_i\,, \qquad e^{2\phi} = H, \tag{3.3.178}$$

where x^i with $i = 1, \ldots, 4$ are non-compact cartesian coordinates on $\mathbb{R}^4$ and z^j with $j = 1, \ldots, 4$ are the T^4 coordinates. The supergravity equations of motion are satisfied provided H^{-1} and K are harmonic functions on the transverse space $\mathbb{R}^4 \times \mathrm{T}^4$ parameterised by (x^i, z^j). Upon smearing these functions on T^4, they become harmonic functions on $\mathbb{R}^4$. The coefficients and the sources for these harmonic functions can depend on v. The functions A_i can be thought of as a gauge field. The chiral null model equations of motion require that it satisfies source-free Maxwell equation, see e.g., appendix C of [30]. In general the gauge field can also have components in the T^4 directions; in the above metric we have written only for the $\mathbb{R}^4$ directions. Let $u = t + y, v = t - y$ with the coordinate y be periodic with length $L_y = 2\pi R_y$. Level matched F1-P configurations with smeared harmonic functions over the four-torus and the y direction are described by [30],

$$H^{-1} = 1 + \frac{Q}{L_y}\int_0^{L_y} \frac{dv}{|x - F(v)|^2}, \quad A_i = -\frac{Q}{L_y}\int_0^{L_y} \frac{dv\dot{F}_i(v)}{|x - F(v)|^2}, \quad K = \frac{Q}{L_y}\int_0^{L_y} \frac{dv\dot{F}_i(v)\dot{F}_i(v)}{|x - F(v)|^2}, \tag{3.3.179}$$

where $|x - F(v)|^2 = \sum_{i=1}^4 (x_i - F_i(v))^2$.

We wish to apply the generalised Garfinkle-Vachaspati transform to such a smeared chiral null model solution, with $k^\mu = (\partial_u)^\mu$ as the null Killing vector and $l^\mu_{(i)} = (\partial_{z^i})^\mu$ as the spacelike Killing vector. Under the GGV transform, Einstein frame metric and the 2-form field transform as (see appendix C.2.4),

$$g_{\mu\nu} \;\rightarrow\; g_{\mu\nu} + \Psi_{(j)} e^{\phi/2}(k_\mu l^{(j)}_\nu + k_\nu l^{(j)}_\mu)\,, \tag{3.3.180}$$
$$B_{\mu\nu} \;\rightarrow\; B_{\mu\nu} - \Psi_{(j)} e^{\phi}(k_\mu l^{(j)}_\nu - k_\nu l^{(j)}_\mu)\,, \tag{3.3.181}$$

where functions $\Psi_{(j)}$ satisfy $\Box\Psi_{(j)} = 0$ for $j = 1, \ldots, 4$. Applying this transformation to configuration (3.3.177)–(3.3.178) we get the transformed metric and 2-form field as

$$ds^2 = H^{3/4}(-dudv + Kdv^2 + 2A_i dx^i dv - \Psi_{(j)} dz^j dv) + H^{-1/4}(dx^i dx^i + dz^j dz^j) \tag{3.3.182}$$

$$B_{uv} = \frac{H}{2}\,, \qquad B_{vi} = -HA_i\,, \tag{3.3.183}$$

$$B_{vz_j} = H\,\Psi_{(j)}\,, \qquad e^{2\phi} = H\,. \tag{3.3.184}$$

The harmonic functions $\Psi_{(j)}$ in equations (3.3.182)–(3.3.184) can be interpreted as additional components of the gauge field A_i with components in the torus directions, see, e.g., [38, 41]. The final solution is also a chiral null model solution. In general, it does not satisfy the level matching condition [80]. One can always perform an ordinary GV transform to add appropriate momentum to get a solution that satisfies the level matching condition.

Alternatively, taking

$$\Psi_{(j)}(v, x_i) = -2H^{-1}p_j(v), \tag{3.3.185}$$

we can arrive at a slightly different interpretation as follows. The transformed metric takes the form,

$$ds^2 = H^{3/4}(-dudv + Kdv^2 + 2A_i dx^i dv) + H^{-1/4}(dx^i dx^i) + H^{-1/4}(dz^j dz^j + 2p_j dz^j dv). \tag{3.3.186}$$

Introducing new coordinates $z'^j = z^j + \int_0^v p^j(v')dv'$, we can write the above metric as,

$$ds^2 = H^{3/4}(-dudv + Kdv^2 + 2A_i dx^i dv - H^{-1}p^2(v)dv^2) + H^{-1/4}(dx^i dx^i + dz'^j dz'^j). \tag{3.3.187}$$

The two-form B-field and the dilaton remain unchanged under this coordinate transformation,

$$B_{uv} = \frac{H}{2}\,,\ B_{vi} = -HA_i\,,\ B_{vz'^j} = -2p_j(v)\,,\ e^{2\phi} = H. \tag{3.3.188}$$

The component $B_{vz'^j} = -2p_j(v)$ can be removed by a gauge transformation $B'_{ab} = B_{ab} + \partial_a\Lambda_b - \partial_b\Lambda_a$ with gauge function $\Lambda_{z'^j} = 2\int_0^v p_j(v')dv'$ (no other component changes under this gauge transformation). We finally have

$$ds^2 = H^{3/4}(-dudv + Kdv^2 + 2A_i dx^i dv - H^{-1}p^2(v)dv^2) + H^{-1/4}(dx^i dx^i + dz'^j dz'^j), \tag{3.3.189}$$

$$B_{uv} = \frac{H}{2}\,, \qquad B_{vi} = -HA_i\,, \qquad e^{2\phi} = H. \tag{3.3.190}$$

This metric can be readily interpreted as pp-wave added to the above F1-P chiral null model with matter fields remaining unchanged.[4] The final metric is simply the ordinary GV transform (2.2.13) on the F1-P chiral null model (3.3.177)-(3.3.179) with harmonic function $\Psi = H^{-1}$. In the terminology of [42], this choice of the harmonic function Ψ in the GV transform corresponds to adding "momentum waves without oscillations." Due to the constant term in the harmonic function H^{-1}, metric (3.3.189) is not manifestly asymptotically flat. It can be made asymptotically flat by shifting u appropriately. In general, the final metric does not satisfy the level matching condition.

[4]This is similar to the interpretation given in section 7.1 of reference [33], though for a different set-up where matter fields do change.

Chapter 4

Dualities and the generalized Garfinkle-Vachaspati transform

In this Chapter we are going to explore further applications of the generalized Garfinkle-Vachaspati transform and related solution generating techniques. By the application of a set of dualities, we write deformed Bena-Warner solutions in various M2-M5-P duality frames. The supersymmetric eleven-dimensional solutions of Bena-Warner (BW) are discussed in appendix D.2. These intersecting M2-M2-M2 solutions are mapped to D1-D5-P solutions. The set of dualities is worked out in appendix D.2. First, we are going to perform GGV on the BW form of D1-D5-P solutions which can in turn be mapped to eleven-dimensional M2-M5-P system. We will see that in the M-theory frame the deformation also has the form of GGV.

4.1 GGV on Bena-Warner form of the D1-D5-P solution

The string frame D1-D5-P metric obtained from the BW solution can be written in the following form, cf. (D.2.34),

$$ds_{10}^2 = -\frac{1}{Z_3 Z_1}(dt+\kappa)^2 + Z_1 h_{mn} dx^m dx^n + \frac{Z_3}{Z_1}(dz_5 + A_\mu^{(3)} dx^\mu)^2 + (dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2), \tag{4.1.1}$$

where

$$A_\mu^{(3)} dx^\mu = -\frac{dt+\kappa}{Z_3} + \omega_3. \tag{4.1.2}$$

The RR two-form field associated with this solution is given by, cf. (D.2.41),

$$C = -\left(\frac{dt+\kappa}{Z_1} - \omega_1\right) \wedge (dz_5 + \omega_3) + \sigma. \tag{4.1.3}$$

where the two-form σ satisfies equation (D.2.42).

For the application of the generalized Garfinkle-Vachaspati transform first we note that the null Killing vector and the spacelike Killing vector for the D1-D5-P metric (4.1.1) are as follows,

$$k = \frac{\partial}{\partial t}, \qquad\qquad l = \frac{\partial}{\partial z_4}, \tag{4.1.4}$$

$$k_\mu dx^\mu = -Z_1^{-1}(dz_5 + \omega_3), \qquad\qquad l_\mu dx^\mu = dz_4, \tag{4.1.5}$$

from which it can be readily checked that the GV-transformed metric takes the following form,

$$(ds'_{10})^2 = ds_{10}^2 - 2Z_1^{-1}\Phi(dz_5 + \omega_3)dz_4, \tag{4.1.6}$$

along with the transformed C-field of the form,

$$C' = C + \frac{\Phi}{Z_1}(dz_5 + \omega_3) \wedge dz_4. \tag{4.1.7}$$

Now, by performing dualities we will write the transformed metric in various other M-theory frames.

4.2 T-duality along z_1-direction and M-theory lift

The first duality frame we discuss is obtained from the D1-D5-P system by T-duality along z_1-direction followed by an M-theory lift along z_6:

$$\begin{aligned} \mathrm{D1}_{z_5} - \mathrm{D5}_{z_1 z_2 z_3 z_4 z_5} - \mathrm{P}_{z_5} \quad &\xrightarrow{\mathrm{T}_{z_1}} \quad \mathrm{D2}_{z_1 z_5} - \mathrm{D4}_{z_2 z_3 z_4 z_5} - \mathrm{P}_{z_5} \\ &\xrightarrow{\text{M-theory lift}} \quad \mathrm{M2}_{z_1 z_5} - \mathrm{M5}_{z_2 z_3 z_4 z_5 z_6} - \mathrm{P}_{z_5}. \end{aligned} \tag{4.2.8}$$

Performing these dualities, the final metric is obtained as

$$ds_{11}^2 = ds_{10}^2 - 2Z_1^{-1}\Phi(dz_5 + \omega_3)dz_4 + dz_6^2,$$

together with the 3-form field

$$\mathcal{A}^{(3)} = \left(C + \frac{\Phi}{Z_1}(dz_5 + \omega_3) \wedge dz_4\right) \wedge dz_1. \tag{4.2.9}$$

In this duality frame, the transformation is intrinsically of the form of the generalised Garfinkle-Vachaspati transform. It is normal to speculate that a solution generating technique similar to generalised Garfinkle-Vachaspati transform exist in (an appropriate truncation of) M-theory.

4.3 T-dualities along z_1, z_2, z_3 and M theory lift

The next duality frame we learn about is obtained by T-dualities along z_1, z_2, z_3-directions followed by an M-theory lift along z_6:

$$\begin{aligned} \mathrm{D1}_{z_5} - \mathrm{D5}_{z_1z_2z_3z_4z_5} - \mathrm{P}_{z_5} \quad &\xrightarrow{\mathrm{T}_{z_1z_2z_3}} \quad \mathrm{D4}_{z_1z_2z_3z_5} - \mathrm{D2}_{z_4z_5} - \mathrm{P}_{z_5} \\ &\xrightarrow{\text{M-theory lift}} \quad \mathrm{M5}_{z_1z_2z_3z_5z_6} - \mathrm{M2}_{z_4z_5} - \mathrm{P}_{z_5}. \end{aligned} \tag{4.3.10}$$

Performing these dualities, the eleven-dimensional M2-M5-P metric is,

$$ds_{11}^2 = ds_{10}^2 - \frac{2\Phi}{Z_1}(dz_5 + \omega_3)dz_4 + dz_6^2,$$

together with the six-form field $\mathcal{A}^{(6)}$ in eleven-dimensions, which is thought of as the electromagnetic dual of $\mathcal{A}^{(3)}$ (B.1):

$$\mathcal{A}^{(6)} = \left(C + \frac{\Phi}{Z_1}(dz_5 + \omega_3) \wedge dz_4\right) \wedge dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_6. \tag{4.3.11}$$

Even in this duality frame, the transformation is fundamentally of the form of the generalised Garfinkle-Vachaspati transform which indicates the existence of a solution generating technique akin to generalised Garfinkle-Vachaspati transform in this set-up.

4.4 T-duality along z_4-direction and M-theory lift

Next, we discuss the duality frame obtained by T-duality along z_4-directions followed by an M-theory lift along z_6. This case is little bit different from the previous ones in the sense that z_4 is the same spacelike direction used for the generalised Garfinkle-Vachaspati transform, cf. (4.1.4). Here the duality sequence is:

$$\begin{aligned} \mathrm{D1}_{z_5} - \mathrm{D5}_{z_1z_2z_3z_4z_5} - \mathrm{P}_{z_5} \quad &\xrightarrow{\mathrm{T}_{z_4}} \quad \mathrm{D2}_{z_4z_5} - \mathrm{D4}_{z_1z_2z_3z_5} - \mathrm{P}_{z_5} \\ &\xrightarrow{\text{M-theory lift}} \quad \mathrm{M2}_{z_4z_5} - \mathrm{M5}_{z_1z_2z_3z_5z_6} - \mathrm{P}_{z_5}. \end{aligned} \tag{4.4.12}$$

We can see that the final M2-M5-P solution obtained has the same construction as in (4.3.10) even if the way of obtaining it is different. To be precise, in the process of duality transformations, after the T-duality along z_4 the IIA ten-dimensional metric for D2-D4-P system in the string frame is,

$$ds^2_{10} = -2Z_1^{-1}(dt+k)(dz_5+\omega_3) + \frac{Z_3}{Z_1}\left(1-\frac{\Phi^2}{Z_1Z_3}\right)(dz_5+\omega_3)^2 + Z_1 h_{mn}dx^m dx^n + ds^2_{\mathrm{T}^4}, \tag{4.4.13}$$

with the associated form-fields are,

$$C^{(3)} = C\wedge dz_4, \qquad C^{(1)} = \frac{\Phi}{Z_1}(dz_5+\omega_3), \qquad B^{(2)} = \frac{\Phi}{Z_1}(dz_5+\omega_3)\wedge dz_4. \tag{4.4.14}$$

The dilaton remains the same, i.e., $e^{2\phi} = 1$. After lifting to the M-theory frame the GV-transformed solution takes the following form,

$$ds^2_{11} = ds^2_{10} + \frac{2\Phi}{Z_1}(dz_5+\omega_3)dz_6 + dz_6^2, \tag{4.4.15}$$

$$\mathcal{A}^{(3)} = C^{(3)} + \frac{\Phi}{Z_1}(dz_5+\omega_3)\wedge dz_4\wedge dz_6. \tag{4.4.16}$$

In this duality frame too, the transformation is essentially of the generalised Garfinkle-Vachaspati form.

Similarly, one can consider another duality chain to another M2-M5-P frame as follows

$$\mathrm{D1}_{z_5}-\mathrm{D5}_{z_1 z_2 z_3 z_4 z_5}-\mathrm{P}_{z_5} \xrightarrow{\mathrm{T}_{z_1 z_2 z_4}} \mathrm{D4}_{z_1 z_2 z_4 z_5}-\mathrm{D2}_{z_3 z_5}-\mathrm{P}_{z_5} \xrightarrow{\text{M-theory lift}} \mathrm{M5}_{z_1 z_2 z_4 z_5 z_6}-\mathrm{M2}_{z_3 z_5}-\mathrm{P}_{z_5}.$$

Even in this duality frame the transformation is essentially of the Garfinkle-Vachaspati form. It is tempting to speculate that some solution generating techniques akin to generalised Garfinkle-Vachaspati transform exist for these set-ups as well.

Chapter 5

Adding momentum to KK-monopole solution

The pure gravitational theory in five-dimensions admits solitonic solutions. The 'magnetic monopole' is one such solitonic solution to the field equations which is also termed as 'Kaluza-Klein (KK) monopole'. These solutions can also be embedded in ten-dimensional string theory by adding compact directions. In string theory, these KK-monopoles are fundamental objects and they are related to D-branes by duality(this is already discussed in section 1.2). The five-dimensional KK-monopole solutions are first obtained in [83] by Gross and Perry, also independently by R. Sorkin in [84] by adding one extra time direction to the four-dimensional Euclidean Taub-NUT spacetime.

There has been construction of KK-P solutions where momentum has been added to the KK-monopole solution using GV transform [85]. In the construction of black hole microstates the KK-P solutions play significant role as they are dual to the 2-charge F1-P solutions. The set of duality transformations is also discussed in [85]. To apply GV transform (2.2.13) on the KK-monopole background, the scalar wave equation $\Box\Psi = 0$ was solved for special cases which we will discuss in section 5.2. The procedure was applied to both single monopole and multiple KK-monopole cases.

In this Chapter we are going to do the following:

1. We try to obtain more general solution to the wave equation including the vibration along

the fibre direction as well.

2. We will also see in section 5.3 that the KK-monopole metric is of the GMR form (3.2.1) and GGV can be readily applied to it.

3. We also wrote KK-P metric in the GMR-form in section 5.4 and performed GGV on it.

The work presented in this Chapter are unpublished yet.

In the following section we are going to give a brief introduction to KK-monopole solution and discuss some of its characteristic properties. The next section involves computation of general solution to the wave equation.

5.1 KK-monopole

The five-dimensional KK-monopole solution can be obtained from four the four-dimensional Euclidean Taub-NUT solution, by adding one time direction t as follows

$$ds^2 = -dt^2 + ds^2_{\text{TN}}, \tag{5.1.1}$$

where the four-dimensional Taub-NUT spacetime is defined as

$$ds^2_{\text{TN}} = V^{-1}[ds + \chi]^2 + V[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{5.1.2}$$

Here, V and χ are given by

$$V = 1 + \frac{Q_K}{r}, \qquad\qquad \vec{\nabla} \times \chi = -\vec{\nabla} V. \tag{5.1.3}$$

The second equation in (5.1.3) gives the magnetic field associated with the KK-monopole solution with potential given by V and Q_K corresponds to the monopole charge. The curl and grad operations are with respect to the transverse directions. By solving the χ-equation we get

$$\chi_\phi = Q_K \cos\theta. \tag{5.1.4}$$

For the KK-monopole solution 5.1.1, $r = 0$ is the point of singularity which corresponds to the monopole source. The periodicity of s-circle ensures the regularity of the solution as $r = 0$. The KK-monopole charge Q_K is related to the number of KK-monopoles N_K as

$$Q_K = \frac{1}{2} N_K R_K, \tag{5.1.5}$$

where R_K is the asymptotic radius of the s-circle at $r \to \infty$. For $N_K = 1$, we get the solution corresponding to a single KK-monopole with $Q_K = \frac{R_K}{2}$ and the radius of the s-circle goes smoothly to zero at $r \to 0$. For $N_k \neq 1$, the solution has Z_{N_K} orbifold singularity in the $r \to 0$ limit.

This five-dimensional solution can be embedded in ten-dimensional string theory by adding five extra compact directions: one circle $S^1(y)$ and a four torus $T^4(z_i)$. The ten-dimensional metric is then of the following form

$$ds^2 = -dt^2 + dy^2 + V^{-1}[ds + \chi]^2 + V[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + \sum_{i=6}^{9} dz^i dz_i, \tag{5.1.6}$$

where y-direction is compactified on a circle of radius R_5 i.e. $y \sim y + 2\pi R_5$. The compact torus T^4 directions are given by z_i, $i = 6, 7, 8, 9$. Introducing the null coordinates, $u = t + y$ and $v = t - y$ we can write the metric as

$$ds^2 = -du dv + V^{-1}[ds + \chi_\phi d\phi]^2 + V[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + \sum_{i=6}^{9} dz^i dz_i. \tag{5.1.7}$$

The determinant of the metric is $\sqrt{-g} = V r^2 \sin\theta$.

As mentioned earlier, $r = 0$ is a singularity corresponding to the source. However, it is a coordinate singularity and can be removed by proper choice of coordinates upon which we get standard flat metric in the $r \to 0$ limit. This is discussed in the following section.

Removing singularity at $r = 0$

To get the proper coordinate system in which $r = 0$ is not a singularity any more, first we write the KK-monopole metric (5.1.7) in the limit $r \to 0$. In this limit the function V can be approximated as

$$V \approx \frac{Q_K}{r}. \tag{5.1.8}$$

Thus, the KK-monopole metric takes the following form[1]

$$ds^2 = -dudv + ds^2_{\mathrm{T}_4} + \frac{r}{Q_K}[ds + \chi_\phi d\phi]^2 + \frac{Q_K}{r}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{5.1.9}$$

Next, we perform the following coordinate transformations

$$dr^{*2} = \frac{dr^2}{r} \Rightarrow r^* = 2\sqrt{r}, \qquad \tilde{\theta} = \theta/2, \tag{5.1.10}$$

upon which the metric in terms of the $(r*, \tilde{\theta})$-coordinates can be written as

$$ds^2 = -dudv + ds^2_{\mathrm{T}_4} + \frac{r^{*2}}{4Q_K}\left[ds + Q_K\cos(2\tilde{\theta})d\phi\right]^2 + Q_K\left[dr^{*2} + r^{*2}\left(d\tilde{\theta}^2 + \frac{\sin^2(2\tilde{\theta})}{4}d\phi^2\right)\right], \tag{5.1.11}$$

where we have substituted $\chi_\phi = Q_K\cos\theta$. On further simplification the expression (5.1.11) reduces to

$$ds^2 = -dudv + ds^2_{\mathrm{T}_4} + \frac{r^{*2}}{4Q_K}[\cos^2\tilde{\theta}(ds + Q_K d\phi)^2 + \sin^2\tilde{\theta}(ds - Q_K d\phi)^2] + Q_K(dr^{*2} + r^{*2}d\tilde{\theta}^2). \tag{5.1.12}$$

Now, defining a new set of new coordinates as follows

$$s = Q_K(\tilde{\phi} + \tilde{\psi}), \qquad \phi = \tilde{\phi} - \tilde{\psi}, \qquad \tilde{r} = \sqrt{Q_K}r^*, \tag{5.1.13}$$

we can make the following substitutions

$$d\tilde{\phi}^2 = \frac{1}{4Q_K^2}(ds + Q_K d\phi)^2, \quad d\tilde{\psi}^2 = \frac{1}{4Q_K^2}(ds - Q_K d\phi)^2, \quad d\tilde{r}^2 = Q_K dr^{*2}. \tag{5.1.14}$$

[1]We are writting the torus part $\sum_{i=6}^{9} dz^i dz_i$ as $ds^2_{\mathrm{T}_4}$.

The final metric in terms of the $(\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{\psi})$ coordinates is then a standard flat metric

$$ds^2 = -dudv + \sum_{i=6}^{9} dz^i dz_i + d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \cos^2\tilde{\theta} d\tilde{\phi}^2 + \sin^2\tilde{\theta} d\tilde{\psi}^2). \tag{5.1.15}$$

Here there is no s-coordinate in the metric any more. Instead there is the new angular coordinate $\tilde{\psi}$. We can also define Cartesian coordinates in four-dimensions as

$$x_1 = \tilde{r}\cos\tilde{\theta}\cos\tilde{\phi}, \quad x_2 = \tilde{r}\cos\tilde{\theta}\sin\tilde{\phi}, \quad x_3 = \tilde{r}\sin\tilde{\theta}\cos\tilde{\psi}, \quad x_4 = \tilde{r}\sin\tilde{\theta}\sin\tilde{\psi}. \tag{5.1.16}$$

In terms of the Cartesian coordinates we can write the metric (5.1.15) as

$$ds^2 = -dudv + \sum_{i=6}^{9} dz^i dz_i + \sum_{i=1}^{4} dx_i^2. \tag{5.1.17}$$

Now, on this KK-monopole geometry (5.1.7) we want to add travelling wave deformation. For which there has been application of GV transform in [85] and the momentum was added in one of the isometry directions. For the application of GV method (2.2.13) the scalar wave equation $\Box\Psi = 0$ is solved for Ψ where the $\Box$ is with respect to the transverse directions. Also the scalar function satisfy the compatibility condition (2.2.15).

In the following section we are going to briefly outline the KK-P solution constructed in [85] and discuss about finding more general solution to the scalar wave equation $\Box\Psi = 0$.

5.2 KK-P solution

By performing GV transform (2.2.13) on the KK-monopole metric (5.1.7) one can obtain KK-P metric of the following form

$$ds^2 = -(dudv + \Psi dv^2) + V^{-1}[ds + \chi_j dx^j]^2 + V[\sum_{j=1}^{3} dx_j^2] + \sum_{i=6}^{9} dz^i dz_i, \tag{5.2.18}$$

where we have written the transverse directions (r, θ, ϕ) in terms of Cartesian coordinates

$$x_1 = r \sin\theta \sin\phi, \qquad x_2 = r \sin\theta \cos\phi, \qquad x_3 = r\cos\theta. \tag{5.2.19}$$

Here, Ψ solves the background wave equation[2]. In [85] the solution is obtained without considering s-dependence. It is denoted as $T(v, \vec{x})$, where T satisfies three-dimensional Laplace equation. In terms of spherical harmonics Y_{lm} the general form of the solution is as follows [85]

$$T(v, \vec{x}) = \sum_{l\geq 0} \sum_{m=-l}^{l} [a_l(v) r^l + b_l(v) r^{-l+1}] Y_{lm}. \tag{5.2.20}$$

For a regular and asymptotically flat KK-P solution $T(v, \vec{x})$ has the simple expression $T(v, \vec{x}) = f(v).\vec{x}$.

To construct more general solution we try to solve for Ψ having dependence on the fibre direction s as well.

5.2.1 General solution to the wave equation

Here we will be brief. More details can be found in appendix E. The wave equation can be written as,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = 0, \tag{5.2.21}$$

where μ, ν are the indices denoting components with respect to the background spacetime. Recall the background spacetime which is of the form

$$ds^2 = -du dv + V^{-1}[ds + \chi_\phi d\phi]^2 + V[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + \sum_{i=6}^{9} dz^i dz_i. \tag{5.2.22}$$

The metric has a null Killing vector along u. From the compatibility condition $k^\mu \partial_\mu \Psi = 0$ we can see that Ψ is independent of u. We also assume that the solution is independent of the torus directions. Thus, the wave equation (5.2.21) is now a five-dimensional equation which solution Ψ is a function of (s, v, r, θ, ϕ)-coordinates. To express the equation (5.2.21) in terms of the

[2] No dependence on the torus directions.

determinant and components of the inverse metric we note the following

$$\sqrt{-g} = Vr^2 \sin\theta, \qquad g^{uv} = 2, \qquad g^{rr} = \frac{1}{V}, \qquad g^{\theta\theta} = \frac{1}{Vr^2}, \tag{5.2.23}$$

$$g^{\phi\phi} = \frac{1}{Vr^2 \sin^2\theta}, \qquad g^{s\phi} = -\frac{Q_K \cos\theta}{Vr^2 \sin^2\theta} = g^{\phi s}, \qquad g^{ss} = V + \frac{Q_K^2 \cot^2\theta}{Vr^2}. \tag{5.2.24}$$

Taking into account the periodicity of the v-coordinate we can write $\Psi(s, v, \vec{x})$ to have the following form,

$$\Psi(s, v, \vec{x}) = T(s, r, \theta, \phi) e^{i\omega v}, \tag{5.2.25}$$

where w takes integer values. Thus, $T(s, r, \theta, \phi)$ solves four-dimensional Laplace equation.

Substituting the determinant and components of the inverse metric from equations (5.2.23) and (5.2.24) we can write the four-dimensional wave equation for T as

$$\begin{aligned} \frac{1}{r^2}\partial_r(r^2\partial_r T) \;&+\; \frac{1}{r^2}\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta T) + \frac{1}{\sin^2\theta}(Q_K^2\partial_s^2 T + \partial_\phi^2 T - 2Q_K cos\theta\partial_\phi\partial_s T)\right) \\ &+\; \frac{1}{r^2}(V^2r^2 - Q_K^2)\partial_s^2 T = 0, \end{aligned} \tag{5.2.26}$$

where we have taken out a factor of $r^2 \sin\theta$ from each of the terms. This equation is similar to the hydrogen atom solution comparing with which we can write the angular momentum operator L^2 as

$$-L^2 T = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta T) + \frac{1}{\sin^2\theta}(Q_K^2\partial_s^2 T + \partial_\phi^2 T - 2Q_K \cos\theta\partial_\phi\partial_s T). \tag{5.2.27}$$

Solution to the angular part of the equation is discussed in appendix .

We can redefine the derivative operator ∂_s as $Q_K\partial_s = \partial_\xi$. Then the wave equation (5.2.26) after substituting (5.2.27) takes the following form

$$\frac{1}{r^2}\partial_r(r^2\partial_r T) - \frac{L^2}{r^2}T + \frac{1}{Q_K^2 r^2}(V^2r^2 - Q_K^2)\partial_\xi^2 T = 0. \tag{5.2.28}$$

In another way it can also be written as

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} - \frac{L^2}{r^2}T + \frac{1}{Q_K^2 r^2}(V^2 r^2 - Q_K^2)\partial_\xi^2 T = 0. \tag{5.2.29}$$

We can also define the third-component of the angular momentum operator, L_3 as $L_3 = -i\partial_\phi$. Similarly, the linear momentum operator along ξ can be defined as $P_\xi = -i\partial_\xi$. It can be checked that L^2, L_3 and P_ξ forms a complete set of commuting operators, thus can be simultaneously diagonalized. We denote the simultaneous eigenvector to be $Y^q_{lm}(s, \theta, \phi)$. The eigenvalue equations for L^2, L_3 and P_s can be written as

$$L^2 Y^q_{lm} = l(l+1)Y^q_{lm}, \qquad L_3 Y^q_{lm} = mY^q_{lm}, \qquad P_s Y^q_{lm} = qY^q_{lm}. \tag{5.2.30}$$

Here, l is the angular momentum quantum number taking values $l \geq 0$. m is the L_3 quantum number which takes values from $-l$ to $+l$. In addition, we have the linear momentum quantum number q. Now, ϕ and ξ are Killing directions corresponding to rotational and translational symmetries respectively. Thus, we can write the corresponding eigenfunctions as

$$\Psi_\phi = N_m e^{im\phi}, \qquad \Psi_s = N_q e^{iq\xi}. \tag{5.2.31}$$

where N_m and N_q are the normalization constants. Now, since ϕ is periodic within the range $0 \leq \phi < 2\pi$, from $\Psi_\phi(\phi + 2\pi) = \Psi_\phi(\phi)$ it can be checked that m takes integer values. Similarly, s is periodic within the range $0 \leq s < 2\pi R_K$. That means ξ has the periodicity $0 \leq \xi < 2\pi R_K/Q_K$ which by using the expression (5.1.5) for monopole charge can also be written as $0 \leq \xi < 4\pi/N_K$. Thus, from $\Psi_s(\xi + \frac{4\pi}{N_K}) = \Psi_s(\xi)$ it can be concluded that q takes values

$$|q| = N_K \frac{n}{2}, \tag{5.2.32}$$

where n takes integer values. Thus, for single monopole where $N_K = 1$, q takes the values $q = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots$. For any general N_K we have $q = 0, \pm N_K/2, \pm N_K, \pm 3N_K/2, \ldots$.

The solution $T(s, r, \theta, \phi)$ to the four-dimensional Laplace equation (5.2.29) can be reduced into angular and radial equations which can be treated separately. To do so we write the function

$T(s, r, \theta, \phi)$ as

$$T = \sum_{lmq} f_q^{lm}(r) Y_{lm}^q, \tag{5.2.33}$$

where f is the radial part and Y_{lm}^q are the four-dimensional spherical harmonics which is the solution to the angular equation (5.2.27). Substituting (5.2.33) into the wave equation (5.2.29) and using the eigenvalue equations (5.2.30) we can write the radial equation as follows

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{l(l+1)}{r^2} f + \frac{1}{Q_K^2 r^2}\left(V^2 r^2 - Q_K^2\right)\left(-\frac{q^2}{Q_K^2}\right) f = 0. \tag{5.2.34}$$

The solutions to the angular equation and radial equation are discussed in appendix E.

The solution $Y_{lm}^q(s, \theta, \phi)$ to the angular equation (5.2.27) can be written in terms of $\Psi_\theta(\theta)$ and Ψ_ϕ, Ψ_s given in (5.2.31). Then the θ-dependent part of the solution is given in terms of the hypergeometric functions ${}_2F_1(a, b; c; x)$ by

$$\begin{aligned}\Psi(x) \;\; = \;\; & (-1)^{\frac{1}{2}(-m+q+1)}\left(\frac{1-x}{x}\right)^{\frac{1}{2}(m+q)}\Big[C_1(-1)^q x^q {}_2F_1(q-l, l+q+1; -m+q+1; x) \\ & +C_2(-1)^m x^m {}_2F_1(m-l, l+m+1; m-q+1; x)\Big].\end{aligned} \tag{5.2.35}$$

where $x = \sin^2\frac{\theta}{2}$.

Similarly, solution to the radial equation (5.2.34), worked out in appendix E is given in terms of the confluent hypergeometric function of second kind $U(a, b, x)$ and associated Laguerre polynomial $L_\nu^\lambda(x)$ by

$$f = \left(\frac{-2q}{Q}\right)^{l+1} r^l \Big[C_{1q}^{lm} e^{\frac{qr}{Q}} U(1+l-q, 2l+2, -\frac{2qr}{Q}) + e^{-\frac{qr}{Q}} C_{2q}^{lm} L_{-q-l-1}^{2l+1}\left(\frac{2qr}{Q}\right)\Big]. \tag{5.2.36}$$

Thus, we have constructed more general solution to the scalar wave equation having the radial part (5.2.36) and angular part given by (5.2.35). The asymptotic limits and finiteness of the solutions at $r = 0$ are discussed in appendix E.

Next, we will try to write the KK-P solution discussed in section 5.2 in the GMR form. As we have discussed earlier in section 2.4.64, for any solutions of the GMR form, GGV is a valid solution generating technique. Before going to KK-P solution we will see that the KK-monopole solution without momentum is already in the GMR form and we can perform GGV on it.

5.3 KK-monopole in GMR form

In this section we are going write the KK-monopole solution (5.1.7) in the GMR form (discussed in appendix (D)). To do so let's rewrite the KK-monopole solution embedded in ten-dimensional string theory given as

$$ds^2 = -dudv + V^{-1}[ds + \chi_\phi d\phi]^2 + V[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] + \sum_{i=6}^{9} dz^i dz_i. \quad (5.3.37)$$

The general GMR form of the ten-dimensional metric is (3.2.1)

$$ds^2 = -H^{-1}(dv+\beta)\Big(du + \omega + \frac{\mathcal{F}}{2}(dv+\beta)\Big) + Hh_{mn}dx^m dx^n + dz_i dz_i. \quad (5.3.38)$$

Comparing the above with the KK-monopole metric we can make the following identifications

$$H = 1, \qquad \beta = 0, \qquad \omega = 0, \qquad \mathcal{F} = 0. \quad (5.3.39)$$

with the base metric given by the four-dimensional Euclidean Taub-NUT

$$ds_4^2 = V^{-1}[ds + \chi_\phi d\phi]^2 + V[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (5.3.40)$$

Then the GMR equations (D.1.4)-(D.1.7) which have the following general form

$$\begin{aligned} \star d \star d\mathcal{F} - \frac{1}{2}(\mathcal{G}^+)^2 &= 0, & (5.3.41)\\ d \star dH + \frac{d\beta \wedge \mathcal{G}^+}{2} &= 0, & (5.3.42)\\ d\beta - \star d\beta &= 0, & (5.3.43)\\ d\mathcal{G}^+ &= 0, & (5.3.44) \end{aligned}$$

where $\mathcal{G}^+$ is given by[3]

$$\mathcal{G}^+ = \frac{1}{2H}\left(d\omega + \star_4\, d\omega + \mathcal{F} d\beta\right), \tag{5.3.45}$$

become trivial ones. Thus, the KK-monopole solution is trivially of the GMR form. We can add GGV on this metric (5.3.37).

5.3.1 GGV on KK-monopole

Consider the null KIlling vector

$$k = \frac{\partial}{\partial u}. \tag{5.3.46}$$

The torus directions provide the spacelike covariantly constant vectors $l^{(i)}$. The GGV is then given as (2.3.39)-(2.3.40)

$$g'_{\mu\nu} = g_{\mu\nu} + 2\,\Psi\, k_{(\mu} l_{\nu)}, \tag{5.3.47}$$

$$C'_{\mu\nu} = C_{\mu\nu} - 2\,\Psi\, k_{[\mu} l_{\nu]}, \tag{5.3.48}$$

where Ψ satisfies the background wave equation $\Box\Psi = 0$ and the compatibility condition $k^\mu \partial_\mu \Psi = 0$.

However, note that KK-monopole solution is purely gravitational solution so for the transversality condition (2.3.43) to be satisfied in these cases we have

$$dk = 0. \tag{5.3.49}$$

i.e. the null Killing vector k^μ should be covariantly constant. Since $\nabla_\mu k_\nu$ can readily be checked as zero for the KK-monopole solution, thus the transversality condition holds true.

One should also note that by performing GGV, since the matter field also transforms this gives rise to additional matter field content to the GGV transformed KK-monopole solution which was initially purely gravitational.

A similar analysis can be done on the KK-P solution constructed in [85] which we are going to discuss in the next section. First, we will write the KK-P solution in GMR form and then

[3]Here 'dot' represents derivative with respect to v.

apply GGV on it.

5.4 KK-P metric in GMR form

The KK-P metric is given by

$$ds^2 = -(dudv + T(v, \vec{x}dv^2) + V^{-1}[ds + \chi_j dx_j]^2 + V[\sum_{j=1}^{3} dx_j^2] + \sum_{i=6}^{9} dz^i dz_i, \qquad (5.4.50)$$

where $T(v, \vec{x})$ satisfies the Laplace equation with respect to the three-dimensional transverse space. To write it in the GMR form we need to make the following identifications

$$H = 1, \qquad \omega = 0, \qquad \beta = 0, \qquad \frac{\mathcal{F}}{2} = T(v, \vec{x}). \qquad (5.4.51)$$

The base metric is again the four-dimensional Taub-NUT metric.

We also need to check the equations of motion. From all the GMR-equations (D.1.4)-(D.1.7), one can readily see that the only non-trivial equation here is

$$\star d \star d\mathcal{F} = 0, \qquad (5.4.52)$$

which holds true since $\nabla^2 T(v, \vec{x}) = 0$ for the GV-transform on KK-monopole metric. Thus, KK-P solution is also of the GMR form with vanishing matter fields. We can perform GGV on it.

5.4.1 GGV on KK-P solution

Again considering the null KIlling vector

$$k = \frac{\partial}{\partial u}, \qquad (5.4.53)$$

and the spacelike covariantly constant vectors along the torus $l^{(i)}$, GGV-transformed KK-P solution can be written as

$$ds^2 = -(dudv + T(v, \vec{x})dv^2 + \sum_{i=6}^{9} \Psi_{(i)} dudz_i) + V^{-1}[ds + \chi_j dx_j]^2 + V[\sum_{j=1}^{3} dx_j^2] + \sum_{i=6}^{9} dz^i dz_i, \tag{5.4.54}$$

where each $\Psi_{(i)}$ satisfies the background wave equation $\Box\Psi_{(i)} = 0$ and the compatibility condition $k^\mu \partial_\mu \Psi_{(i)} = 0$.

One can readily check that the transversality condition (5.3.49) is also satisfied for KK-P solution.

Thus, we constructed more general solution to the scalar wave equation in the KK-monopole geometry. The solutions are in terms of Hypergeometric and Laguerre functions. We also wrote KK-monopole and KK-P solutions in the GMR form, on which GGV is applied successfully. One just needs to solve the scalar wave equation $\Box\Psi = 0$ to deform the solutions suitably. The analysis of the regularity and supersymmetries of the deformed solution is not done in this thesis.

Chapter 6

Conclusions and Future directions

In this thesis, we mainly discussed the development the generalized Garfinkle-Vachaspati transform (or GGV) as a solution generating technique based on our two published papers [52, 53]. We studied both the dilaton free and non-zero dilaton cases. For the first case [52] we established GGV as a solution generating technique for a six-dimensional theory with dilaton that is embedded in ten-dimensions via addition of four torus. The technique is verified in Chapter 2 by direct computations of the equations of motion for the transformed system. For non-zero dilaton unlike the case of dilaton-free case there are two S-duality related set-ups namely, (i) R-R sector and (ii) NS-NS sector[1]. We established GGV in both the set-ups where we used S-duality explicitly. In Chapter 3 we discussed the applications of GGV to a class of supersymmetric D1-D5-P orbifold solutions. We studied this class of D1-D5-P solution with both zero dilaton case and solution with non-trivial dilaton profile as well. The dilaton free case is a special case of the more general non-zero dilaton case and one can obtain all the results of dilaton-free case by setting $\Phi = 0$ in all the results of [53]. We also briefly discussed the application of the GGV technique to the F1-P system. We also obtained the GGV transform in certain M-theory frames by dualising them from D1-D5-P string theory frames in Chapter 4. The final solutions obtained are various M2-M5-P systems. It is natural to expect that some variant of the generalised Garfinkle-Vachaspati transform also exist for these M-theory set-ups.

On a different approach, we discussed KK-monopole solution, KK-P solution and applications of GGV on it in Chapter 5. This work is not yet published. Previously, the KK-P

[1] "NS" stands for Neveu-Schwartz

solution [85] obtained via the application of GV on KK-monopole background doesn't involve the dependence of the scalar wave function on the fibre direction. We worked out more general solution to the wave equation including the fibre direction as well. We also related the KK-monopole solution and KK-P solution to the GMR form of supersymmetric solutions. We performed GGV on these GMR form of KK and KK-P solutions.

Our work provides us with some potential future directions which we are going to briefly discuss below. So far we have only considered those particular D1-D5-P microstates obtained from NS-chiral primaries via odd units of spectral flow. More general class of supersymmetric three-charge solutions are constructed by fractionated spectral flow parameters [36, 56, 57] for which the GGV techniques can be extended. We also expect that the GGV solution generating technique may admit a further generalisation to non-supersymmetric settings as well. Few of such non-supersymmetric geometries have been constructed in [36, 57]. Studying GGV on these solutions is difficult compared to supersymmetric solutions we have studied so far[2]. Another more natural direction to explore is the supersymmetry properties of the deformed solutions. In [33] it was shown that under the GGV transform, supersymmetric solutions of minimal six-dimensional supergravity are deformed into supersymmetric solutions of ten-dimensional IIB supergravity. It is natural to conjecture that the GGV deformations of supersymmetric solutions of non-mininal six-dimensional supergravity are also supersymmetric in ten dimensions.

For GV on KK, we need to match with brane side with a supertube kind of analysis. Similar discussions has been done before for D1-D5-KK microstate solutions of Bena and Kraus [87]. These solutions are smooth and contribute to microscopic degeneracy. We need to find if more general solutions with momentum along fibre direction still stays regular. In this thesis, we have not discussed the solution to the scalar wave equation required to perform GGV on the KK-monopole and KK-P backgrounds which is one of the potential future directions in this project. We have not studied the regularity and supersymmetric properties of the constructed GGV solutions. This needs to be done. We can study microscopic interpretation of GGV on KK and KK-P. We can try to find out if these modes contribute to microscopic index or if they are

[2]A different, but related, type of deformation on [36] was studied in [86]. This particular non-supersymmetric solution is the simplest case. It is tempting to speculate, given the analysis [34, 86], that a variant of the above analysis finds application to non-supersymmetric settings.

like hair modes which need to be excluded from counting. Using our techniques, deformations of further examples can be considered, including microstates of the D1-D5-KK system [87]. These Bena and Kraus D1-D5-KK solutions are known along with their CFT interpretation. We would also like to perform GGV on this and interpret it both on gravity and CFT side.

In a related line of investigation, reference [78] studied a class of BPS black string solutions with traveling waves. The horizon of these solutions turns out to be singular. It will be interesting to understand if our technique allows one to add non-singular travelling wave deformations on black string solutions lifted to ten-dimensions. Such hair will also be of interest with regard to the 4D-5D connection in IIB compactification on $T^4 \times S^1$, cf. [88, 89].

It will be useful to explore GGV technique in other duality frames, in particular, say for solutions of five-dimensional STU supergravity embedded in M-theory. The structure of embedding, see, e.g., [90], is very similar as in the present work, though because of the presence of different matter fields details are likely to be different. If successfully implemented, the technique will allow us to add hair modes associated to various U(1) currents of the MSW CFT on the MSW microstates [91].

Our generalized Garfinkle-Vachaspati transformation is an example of the extended Kerr-Schild metrics considered in [92] and [93]. Due to the assumption that the null and spacelike vectors are Killing, our analysis is more restrictive and hence our final results are much simpler. In addition, we have non-trivial matter present compared to the general extended Kerr-Schild forms considered in those references. It will be interesting to see if we can further relax our conditions on null and spacelike vectors and relate our analysis to theirs.

Since the number of Killing symmetries do not change under our generalized Garfinkle-Vachas-pati deformation, it is natural to ask whether the deformation has a simple group theory interpretation from the hidden symmetry point of view of type IIB theory. Hidden symmetries under null reduction of gravity theories have not been fully explored. Some general results are known [94]. It can be useful to explore the null reduction further and find the interpretation of (generalised) Garfinkle-Vachaspati transform from the hidden symmetry point of view. We hope to return to some of the above problems in our future work.

THESIS SUMMARY

In this thesis, we mainly discussed the development of a new solution generating technique termed as the generalized Garfinkle-Vachaspati transform (or GGV). These are based on our two published papers [52, 53]. We studied both the dilaton free [52] and non-zero dilaton [53] cases. For the first case [52] we established GGV as a solution generating technique for a six-dimensional theory with dilaton that is embedded in ten-dimensions via addition of four torus. Field content of the six-dimensional theory is given in terms of metric $g_{\mu\nu}$ and associated Ramond-Ramond (R-R) two form field $C_{\mu\nu}$. Both the metric components and the components of the R-R field gets transformed in well defined ways under GGV. The technique is verified by direct computations of the equations of motion for the transformed system. In [53] we did similar analysis for solutions with non-zero dilaton. Unlike the dilaton-free case here there are two S-duality related set-ups namely, (i) R-R sector and (ii) NS-NS sector[1]. We established GGV in both the set-ups where we used S-duality explicitly. Here the R-R sector verification method is a bit different from the earlier zero dilaton case.

The technique has been successfully applied to a class of supersymmetric D1-D5-P orbifold solutions. These class of solutions belong to the form of general supersymmetric solution of minimal six dimensional supergravity as proposed by Gutowski, Martelli and Reall (GMR) [65]. We studied this class of D1-D5-P solution with non-trivial dilaton profile as well. These class of supersymmetric solutions are obtained from NS-sector chiral primaries of the corresponding CFT by odd units of spectral flow (discussed in section 1.4.4) [54, 55]. In the type IIB Ramond-Ramond embedding, the technique allows us to add travelling-wave deformations involving the torus directions on this class of D1-D5-P geometries. The deformed solutions are given in terms of a scalar field on the background geometry. In the large AdS-limit, similar to the case of the background geometry, the deformed solution also gets its inner part of the geometry decoupled from the asymptotic flat spacetime and we can identify the deformed states in the D1-D5 CFT as an action of a U(1) current on the undeformed state. The dilaton free case [33, 52] is a special case of the more general non-zero dilaton case [53] and one can obtain all the results of dilaton-free case by setting $\Phi = 0$ in the later [53]. We also

[1] "NS" stands for Neveu-Schwartz

briefly discussed the application of the GGV technique to the F1-P system.

Also, we tried to explore further applications of the generalized Garfinkle-Vachaspati transform in other dual frames, especially in eleven-dimensional M-theory frames. We obtained the GGV transform in certain M-theory frames by dualising them from D1-D5-P string theory frames. The final solutions obtained are various M2-M5-P systems.

On a different approach, we discussed KK-monopole solution, KK-P solution and applications of GGV on it. This work is not yet published. The KK-P solution is obtained in [85] via the application of Garfinkle-Vachaspati transform [43]. There, the scalar wave equation corresponding to Garfinkle-Vachaspati transform was solved for special cases where the scalar field was considered to be independent of the fibre direction. We worked out more general solution to the wave equation including the fibre direction as well. We also related the KK-monopole solution and KK-P solution to the GMR form of supersymmetric solutions. Being pure gravitational solutions, the matter field contents associated to the GMR form of KK and KK-P solutions are set to zero. We performed GGV on these GMR form of KK and KK-P solutions.

Appendices

Appendix A

Differential forms

A p-form denoted as $A^{(p)}$ can be written in terms of its components as

$$A^{(p)} = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}. \tag{A.0.1}$$

The components $A_{\mu_1 \ldots \mu_p}$ of the p-form $A^{(p)}$ are anti-symmetric in all their indices.

Exterior derivative (denoted as d) on a p-form is a $(p+1)$-form $F^{(p+1)} = dA^{(p)}$ with components

$$F^{(p+1)}_{\mu_1 \cdots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A^{(p)}_{\mu_2 \ldots \mu_{p+1}]}. \tag{A.0.2}$$

Exterior product or wedge product of a p-form $A^{(p)}$ with a q-form $B^{(q)}$ is a $p+q$-form denoted as $A \wedge B$ and it has components

$$(A \wedge B)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\mu_{p+1} \ldots \mu_{p+q}]}. \tag{A.0.3}$$

Interior product of a vector v^μ with a p-form $A^{(p)}$ is denoted as $i_v A$ which is a $(p-1)$-form and its components are given by

$$(i_v A)_{\mu_1 \ldots \mu_{p-1}} = \frac{1}{(p-1)!} v^\mu A_{\mu \mu_1 \ldots \mu_{p-1}}. \tag{A.0.4}$$

Lie derivative of a scalar function f with respect to vector field v is denoted as $\mathcal{L}_v f$ and defined in terms of exterior derivative as

$$\mathcal{L}_v f = i_v df = v^\mu \partial_\mu f. \tag{A.0.5}$$

Lie derivative of a p-form $A^{(p)}$ with respect to the a vector field v is given by

$$\mathcal{L}_v A^{(p)} = i_v dA^{(p)} + d(i_v A^{(p)}). \tag{A.0.6}$$

This is known as Cartan's magic formula.

Hodge star of a n-form $F^{(n)}$ in a D-dimensional manifold is a $(D-n)$-form $\tilde{F}^{(D-n)}$ given by

$$\tilde{F} = \star F = \frac{1}{(D-n)!}(\star F)_{\mu_{n+1}\cdots\mu_D} dx^{\mu_{n+1}} \wedge \ldots \wedge dx^{\mu_n}, \tag{A.0.7}$$

where the components of $\star F$ are given by,

$$(\star F)_{\mu_{n+1}\cdots\mu_D} = \frac{\sqrt{\det(g_{\mu\nu})}}{n!}\epsilon_{\mu_1\cdots\mu_n} F^{\mu_1\cdots\mu_n}. \tag{A.0.8}$$

Here ϵ is the Levi-Civita symbol with the signature $\epsilon_{1\ldots n} = 1$.

Appendix B

Dualities in string theory

There are various dualities in physics that relates two completely different theories. We are going to give a brief outline of some important dualities appearing in string theory.

B.1 Electromagnetic duality

As we know the Dp-branes occuring in string theory are associated with $U(1)$ charges that can be both electrical and magnetic in nature. If we consider a point particle then it's worldline is electrically coupled to a 1-form gauge potential A_μ which appears in the action as

$$Q \int A_\mu, \tag{B.1.1}$$

where Q is the Maxwell charge associated with the point particle. Similarly, for a p-brane the worldvolume is coupled to a $p+1$-form gauge potential $A_{(p+1)}$ under $p+2$-form electric field strength tensor given by,

$$F_{(p+2)} = dA_{(p+1)} \tag{B.1.2}$$

The magnetic dual of the field strength tensor is given by the Hodge star operation as,

$$^\star F_{(p+2)} = \tilde{F}_{D-(p+2)} \tag{B.1.3}$$

where D is the spacetime dimension. For string theory $D = 10$ which implies the magnetic field strength associated to a p-brane is given by the $(8-p)$-form tensor $\tilde{F}_{(8-p)}$. The gauge potential is a $(7-p)$-form. In other words we can say, a $p+1$-form gauge potential is electrically coupled to a p-brane where as it's magnetically coupled to a $(6-p)$-brane. In other words we can say that, a p-brane is electromagnetic dual of a $(6-p)$-brane. In that sense D0-brane is dual to D6-brane, D1-brane is dual to D5-brane, D2-brane is dual to D4-brane and D3-brane is self-dual.

B.2 S-duality

This duality is a symmetry of type-IIB string theory, since under S-duality type-IIB string theory maps into itself. Also, under S-duality type-I superstring theory can be mapped to $SO(32)$ heterotic string theory. Under this duality two type-IIB string theories are equivalent if their fields are related in the following way

$$g^{(\mathrm{E})}_{\mu\nu} \to g^{(\mathrm{E})}_{\mu\nu}, \qquad \Phi \leftrightarrow -\Phi, \qquad B_{\mu\nu} \leftrightarrow C_{\mu\nu}, \tag{B.2.4}$$

where $g^{(\mathrm{E})}_{\mu\nu}$ is Einstein frame metric, Φ is the dilaton and $B_{\mu\nu}, C_{\mu\nu}$ are the associated two form fields. This is a kind of strong/weak duality as the asymptotic value Φ_0 of string coupling is related to dilaton by the following relation,

$$g_s = e^{\Phi_0} \tag{B.2.5}$$

Thus the coupling g_s in one theory goes to $1/g_s$ in the dual one. Also as the two-form NS-NS field $B_{\mu\nu}$ and the two-form Ramond-Ramond fields $C_{\mu\nu}$ get interchanged under S-duality, it can be interpreted as the S-duality interchanges fundamental strings and branes with D-branes. For example under S-duality a fundamental string $F1$ along z transforms to $D1$-brane along z,

$$F1(z) \leftrightarrow D1(z), \tag{B.2.6}$$

and a $NS5$ along $z_1 \leftrightarrow z_5$ brane gets transformed to a $D5$ along the same directions.

$$NS5(z_1z_2z_3z_4z_5) \leftrightarrow D5(z_1z_2z_3z_4z_5). \tag{B.2.7}$$

B.3 T-duality

This is a duality between type-IIA superstring theory and type-IIB superstring theory. Under T-duality along a compact direction of radius R, closed strings in one theory mapped to the same closed strings in a theory where the compact direction now have a radius of α'/R. Similarly if we consider a compact volume V then under T-duality it maps to α'^4/V .

The changing of the background fields under T-duality is governed by the Buscher rules. These rules are for low energy effective action of string theory. According to these rules, at lowest order the change of background fields for a T-duality along z-direction is given by,

$$
\begin{aligned}
G'_{zz} &= \frac{1}{G_{zz}}, && \text{(B.3.8)}\\
G'_{\mu z} &= \frac{B_{\mu z}}{G_{zz}}, && \text{(B.3.9)}\\
G'_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu z}G_{\nu z} - B_{\mu z}B_{\nu z}}{G_{zz}}, && \text{(B.3.10)}\\
B'_{\mu z} &= \frac{G_{\mu z}}{G_{zz}}, && \text{(B.3.11)}\\
B'_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu z}G_{\nu z} - G_{\mu z}B_{\nu z}}{G_{zz}}, && \text{(B.3.12)}\\
e^{2\phi'} &= \frac{e^{2\phi}}{G_{zz}}, && \text{(B.3.13)}\\
C'^{(n)}_{\mu\ldots\nu\alpha z} &= C^{(n-1)}_{\mu\ldots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|z}G_{|\alpha]z}}{G_{zz}}, && \text{(B.3.14)}\\
C'^{(n)}_{\mu\ldots\nu\alpha\beta} &= C^{(n+1)}_{\mu\ldots\nu\alpha\beta z} + nC^{(n-1)}_{[\mu\ldots\nu\alpha}B_{\beta]z} + n(n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|z}B_{|\alpha|z}G_{|\beta]z}}{G_{zz}}. && \text{(B.3.15)}
\end{aligned}
$$

Appendix C

Verifying GGV by explicit computation of equations of motion

In this Chapter the details of the computations of equations of motion are presented. In section C.1 the minimal supergravity equations are considered for which dilaton is set to zero. Here the computations are a bit simpler. The more general supergravity equations with non-zero dilaton are considered in C.2.

C.1 Dilaton is zero

To establish the generalised Garfinkle-Vachaspati transform as a valid solution generating technique we computed the equations of motions via a brute force calculation. By doing so we found that under GGV, the left and the right hand side of the Einstein equations transform in the exactly the same way. This proves new solutions can be generated from existing solutions by deforming with GGV. In our convention, Einstein equations are

$$R_{\mu\nu} = \frac{1}{4} F_{\mu\lambda\sigma} F_{\nu}{}^{\lambda\sigma}, \tag{C.1.1}$$

and matter field equations are

$$\nabla_{\mu} F^{\mu\nu\rho} = 0. \tag{C.1.2}$$

We need to show that the above equations transform covariantly under GGV. The calculations are very straight-forward yet tedious. We can organize the computations as follows: (1) The left hand side of the Einstein equations are analysed in section C.1.1, (2) Then we analysed the right hand side of the Einstein equations in section C.1.2, (3) and finally in section C.1.3 matter equations are analysed.

As discussed earlier in the main text, the generalised Garfinkle-Vachaspati transform of the metric is given by,

$$g'_{\mu\nu} = g_{\mu\nu} + \Psi(k_\mu l_\nu + k_\nu l_\mu). \tag{C.1.3}$$

Here Ψ is a massless scalar that satisfies the wave equation with respect to the original background spacetime $g_{\mu\nu}$,

$$\Box\Psi = 0. \tag{C.1.4}$$

The transformation (C.1.3) involves a null Killing vector k^μ with satisfying the following conditions

$$k^\mu k_\mu = 0, \qquad \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0, \tag{C.1.5}$$

and another spacelike, unit normalised covariantly constant vector l^μ that satisfies the following:

$$l^\mu l_\mu = 1, \qquad k^\mu l_\mu = 0, \qquad \nabla_\mu l_\nu = 0. \tag{C.1.6}$$

Also note that l^μ is orthogonal to k^μ. The compatibility of Ψ with the Killing symmetries can be written as,

$$k^\mu \nabla_\mu \Psi = 0, \qquad l^\mu \nabla_\mu \Psi = 0, \tag{C.1.7}$$

these compatibility conditions preserve the Killing symmetries of the background.

C.1.1 Left hand side of Einstein equations

The goal of this subsection is to obtain the expression for the transformation of the left hand side of the Einstein equations (C.1.1). For this we need to compute the transformation of the Ricci tensor which is given by

$$R'_{\lambda\nu} = R_{\lambda\nu} - \nabla_\lambda \Omega^\mu{}_{\mu\nu} + \nabla_\mu \Omega^\mu{}_{\lambda\nu} + \Omega^\mu{}_{\mu\rho}\Omega^\rho{}_{\lambda\nu} - \Omega^\rho{}_{\mu\lambda}\Omega^\mu{}_{\rho\nu}, \tag{C.1.8}$$

where the change $\Omega^\mu_{\lambda\nu}$ in metric compatible connection :

$$\Gamma'^\mu_{\lambda\nu} = \Gamma^\mu_{\lambda\nu} + \Omega^\mu_{\lambda\nu}, \tag{C.1.9}$$

is given by

$$\Omega^\mu_{\lambda\nu} = \frac{1}{2} g'^{\mu\sigma} \left(\nabla_\lambda g'_{\nu\sigma} + \nabla_\nu g'_{\sigma\lambda} - \nabla_\sigma g'_{\nu\lambda} \right). \tag{C.1.10}$$

The objective is to compute various pieces in equation (C.1.8) and then obtain the complete expression for the transformed Ricci tensor.

By looking at the transformed metric (C.1.3) we can observe that the inverse transformed metric is simply

$$g'^{\mu\nu} = g^{\mu\nu} + \Psi^2 k^\mu k^\nu - \Psi S^{\mu\nu}. \tag{C.1.11}$$

Next, we introduce the notation,

$$S_{\mu\nu} = k_\mu l_\nu + k_\nu l_\mu, \tag{C.1.12}$$

$$h_{\mu\nu} = \Psi S_{\mu\nu}, \tag{C.1.13}$$

$$n_{\mu\nu} = \nabla_\mu k_\nu - \nabla_\nu k_\mu. \tag{C.1.14}$$

The change in the metric compatible connection, $\Omega^\mu_{\lambda\nu}$, can be organised in two terms,

$$\Omega^\mu_{\lambda\nu} = \Xi^\mu_{\lambda\nu} + \frac{1}{2}(\Psi^2 k^\mu k^\alpha - \Psi S^{\mu\alpha})(\nabla_\lambda h_{\nu\alpha} + \nabla_\nu h_{\alpha\lambda} - \nabla_\alpha h_{\lambda\nu}), \tag{C.1.15}$$

where the first term $\Xi^\mu_{\lambda\nu}$ is the combination that appears in the original Garfinkle-Vachaspati transform:

$$\Xi^{\mu}_{\lambda\nu} = \frac{1}{2} g^{\mu\alpha}(\nabla_\lambda h_{\nu\alpha} + \nabla_\nu h_{\alpha\lambda} - \nabla_\alpha h_{\lambda\nu}). \tag{C.1.16}$$

For the sake of convenience we make the following identification,

$$K^{\mu}_{\nu\lambda} := \nabla_\nu S^{\mu}_{\lambda} + \nabla_\lambda S^{\mu}_{\nu} - \nabla^{\mu} S_{\lambda\nu}, \tag{C.1.17}$$

using which it follows that

$$\Xi^{\mu}_{\lambda\nu} = \frac{1}{2}\left((\nabla_\nu \Psi) S^{\mu}_{\lambda} + (\nabla_\lambda \Psi) S^{\mu}_{\nu} - (\nabla^{\mu}\Psi) S_{\nu\lambda} + \Psi K^{\mu}_{\nu\lambda}\right), \tag{C.1.18}$$

and therefore,

$$\Omega^{\mu}_{\lambda\nu} = \Xi^{\mu}_{\lambda\nu} - \frac{1}{2}\Psi k^{\mu}(k_\nu \nabla_\lambda \Psi + k_\lambda \nabla_\nu \Psi). \tag{C.1.19}$$

The trace of $\Omega^{\mu}_{\lambda\nu}$is easily seen to be zero

$$\Omega^{\mu}_{\mu\lambda} = 0. \tag{C.1.20}$$

Which implies the following simplified form of the transformation of the Ricci tensor (C.1.8)

$$R'_{\lambda\nu} = R_{\lambda\nu} + \nabla_\mu \Omega^{\mu}{}_{\lambda\nu} - \Omega^{\rho}{}_{\mu\lambda}\Omega^{\mu}{}_{\rho\nu}. \tag{C.1.21}$$

We compute the above by computing the individual terms $\nabla_\mu \Omega^{\mu}{}_{\lambda\nu}$ and $\Omega^{\rho}{}_{\mu\lambda}\Omega^{\mu}{}_{\rho\nu}$. We can first show that

$$2\nabla_\mu \Xi^{\mu}_{\lambda\nu} = (\nabla_\mu \nabla_\nu \Psi) S^{\mu}_{\lambda} + (\nabla_\mu \nabla_\lambda \Psi) S^{\mu}_{\nu} - (\nabla^{\mu}\Psi)(\nabla_\mu S_{\nu\lambda}) + (\nabla_\mu \Psi) K^{\mu}_{\nu\lambda} + \Psi(\nabla_\mu K^{\mu}_{\nu\lambda}), \tag{C.1.22}$$

where we have used $\nabla_\mu S^{\mu}_{\nu} = 0$ and the massless scalar field equation (C.1.4) for Ψ. With this it's easy to see that the first three terms of (C.1.22) combine to zero,

$$(\nabla_\mu \nabla_\nu \Psi) S^{\mu}_{\lambda} + (\nabla_\mu \nabla_\lambda \Psi) S^{\mu}_{\nu} - (\nabla^{\mu}\Psi)(\nabla_\mu S_{\nu\lambda}) = 0. \tag{C.1.23}$$

For further simplification of (C.1.22) some of the identities required are analysed below

$$K^{\mu}_{\nu\lambda} = (\nabla_\nu k^\mu - \nabla^\mu k_\nu) l_\lambda + (\nabla_\lambda k^\mu - \nabla^\mu k_\lambda) l_\nu \tag{C.1.24}$$

$$= n_\nu{}^\mu l_\lambda + n_\lambda{}^\mu l_\nu. \tag{C.1.25}$$

It then follows that the fourth term of (C.1.22) simplifies to

$$(\nabla_\mu \Psi) K^{\mu}_{\nu\lambda} = -2k^\mu [(\nabla_\nu \nabla_\mu \Psi) l_\lambda + (\nabla_\lambda \nabla_\mu \Psi) l_\nu], \tag{C.1.26}$$

where we have used

$$(\nabla_\mu \Psi) n_\nu{}^\mu = -2k^\mu (\nabla_\nu \nabla_\mu \Psi). \tag{C.1.27}$$

Inserting (C.1.25) in $(\nabla_\mu K^{\mu}_{\nu\lambda})$, the last term of (C.1.22) simplifies to

$$\nabla_\mu K^{\mu}_{\nu\lambda} = -2(\Box k_\nu) l_\lambda - 2(\Box k_\lambda) l_\nu, \tag{C.1.28}$$

where we have also used

$$\nabla_\mu n_\nu{}^\mu = -2\Box k_\nu. \tag{C.1.29}$$

When the dust settles, we get a simplified expression for equation (C.1.22):

$$\nabla_\mu \Xi^{\mu}_{\lambda\nu} = -l_\lambda [k^\mu (\nabla_\nu \nabla_\mu \Psi) + \Psi \Box k_\nu] - l_\nu [k^\mu (\nabla_\lambda \nabla_\mu \Psi) + \Psi \Box k_\lambda]. \tag{C.1.30}$$

From (C.1.19) it then follows that

$$2\nabla_\mu \Omega^{\mu}_{\lambda\nu} = 2\nabla_\mu \Xi^{\mu}_{\lambda\nu} - \Psi k^\mu [k_\nu (\nabla_\mu \nabla_\lambda \Psi) + k_\lambda (\nabla_\mu \nabla_\nu \Psi)]. \tag{C.1.31}$$

Similarly we can compute straightforwardly the other piece $\Omega^{\rho}_{\mu\lambda}\Omega^{\mu}_{\rho\nu}$ that is required in

(C.1.21). We start by observing that

$$4\Omega^{\rho}_{\mu\lambda}\Omega^{\mu}_{\rho\nu} = [2\Xi^{\rho}_{\mu\lambda} - \Psi k^{\rho}(k_{\mu}\nabla_{\lambda}\Psi + k_{\lambda}\nabla_{\mu}\Psi)][2\Xi^{\mu}_{\rho\nu} - \Psi k^{\mu}(k_{\rho}\nabla_{\nu}\Psi + k_{\nu}\nabla_{\rho}\Psi)] \tag{C.1.32}$$

$$= 4\Xi^{\rho}_{\mu\lambda}\Xi^{\mu}_{\rho\nu}. \tag{C.1.33}$$

The combination $\Xi^{\rho}_{\mu\lambda}\Xi^{\mu}_{\rho\nu}$ is,

$$\begin{aligned} 4\Xi^{\rho}_{\mu\lambda}\Xi^{\mu}_{\rho\nu} &= [(\nabla_{\mu}\Psi)S^{\rho}_{\lambda} + (\nabla_{\lambda}\Psi)S^{\rho}_{\mu} - (\nabla^{\rho}\Psi)S_{\mu\lambda} + \Psi K^{\rho}_{\mu\lambda}] \\ &\quad \times [(\nabla_{\rho}\Psi)S^{\mu}_{\nu} + (\nabla_{\nu}\Psi)S^{\mu}_{\rho} - (\nabla^{\mu}\Psi)S_{\rho\nu} + \Psi K^{\mu}_{\rho\nu}. \end{aligned} \tag{C.1.34}$$

For further simplification, we use the following non-trivial identities, which can be easily established:

$$S^{\rho}_{\mu}K^{\mu}_{\rho\nu} = 0, \qquad S_{\mu\lambda}K^{\mu}_{\rho\nu} = 0, \tag{C.1.35}$$

$$S^{\mu}_{\nu}K^{\rho}_{\mu\lambda} = k_{\nu}n_{\lambda}{}^{\rho}, \qquad K^{\rho}_{\mu\lambda}K^{\mu}_{\rho\nu} = 4(\nabla_{\mu}k^{\rho})(\nabla_{\rho}k^{\mu})l_{\lambda}l_{\nu}. \tag{C.1.36}$$

After all these simplifications, we get

$$\Omega^{\rho}_{\mu\lambda}\Omega^{\mu}_{\rho\nu} = -\frac{1}{2}(\nabla_{\rho}\Psi)(\nabla^{\rho}\Psi)k_{\lambda}k_{\nu} - \frac{1}{2}\Psi k^{\mu}[k_{\lambda}(\nabla_{\mu}\nabla_{\nu}\Psi) + k_{\nu}(\nabla_{\mu}\nabla_{\lambda}\Psi)] + \Psi^{2}(\nabla_{\mu}k^{\rho})(\nabla_{\rho}k^{\mu})l_{\lambda}l_{\nu}. \tag{C.1.37}$$

Therefore, a final simplified expression for the transformed Ricci tensor is

$$\begin{aligned} R'_{\lambda\nu} &= R_{\lambda\nu} - l_{\lambda}[k^{\mu}(\nabla_{\nu}\nabla_{\mu}\Psi) + \Psi\Box k_{\nu}] - l_{\nu}[k^{\mu}(\nabla_{\lambda}\nabla_{\mu}\Psi) + \Psi\Box k_{\lambda}] \\ &\quad + \frac{1}{2}(\nabla_{\rho}\Psi)(\nabla^{\rho}\Psi)k_{\lambda}k_{\nu} - \Psi^{2}(\nabla_{\mu}k^{\rho})(\nabla_{\rho}k^{\mu})l_{\lambda}l_{\nu}. \end{aligned} \tag{C.1.38}$$

In the next subsection we verify that the right hand side of the Einstein equations (C.1.1) also transform in exactly the same way.

C.1.2 Right hand side of Einstein equations

For the computation of the right hand side first recall that under generalised Garfinkle-Vachaspati transform (GGV) the two-form field transforms as

$$C \to C' = C - \Psi\, k_\mu dx^\mu \wedge l_\nu dx^\nu. \tag{C.1.39}$$

Here we need to use the transversality conditions (2.3.43) in order to match the transformed right hand side of the Einstein equations (C.1.1) with the transformed left hand side (C.1.38). So we write them again:

$$k^\mu F_\mu{}^{\nu\rho} = -n^{\nu\rho}, \tag{C.1.40}$$

$$l^\mu F_\mu{}^{\nu\rho} = 0. \tag{C.1.41}$$

As mentioned in the main text, a large class of minimal six-dimensional supergravity solutions uplifted to ten-dimensional type-IIB supergravity, satisfy these conditions. Introducing the notation

$$m_{\mu\nu} = k_\mu l_\nu - k_\nu l_\mu, \tag{C.1.42}$$

we have

$$C'_{\mu\nu} = C_{\mu\nu} - \Psi(k_\mu l_\nu - k_\nu l_\mu) \tag{C.1.43}$$

$$= C_{\mu\nu} - \Psi m_{\mu\nu}. \tag{C.1.44}$$

It then simply follows that

$$F'_{\mu\nu\rho} = \partial_\mu C_{\nu\rho} + \partial_\rho C_{\mu\nu} + \partial_\nu C_{\rho\mu} - \partial_\mu(\Psi m_{\nu\rho}) - \partial_\rho(\Psi m_{\mu\nu}) - \partial_\nu(\Psi m_{\rho\mu}) \tag{C.1.45}$$

$$= \partial_\mu C_{\nu\rho} + \partial_\rho C_{\nu\mu} + \partial_\mu C_{\rho\nu} - Q_{\mu\nu\rho} - \Psi P_{\mu\nu\rho} \tag{C.1.46}$$

$$= F_{\mu\nu\rho} - Q_{\mu\nu\rho} - \Psi P_{\mu\nu\rho}, \tag{C.1.47}$$

where

$$Q_{\mu\nu\rho} = (\partial_\mu \Psi) m_{\nu\rho} + (\partial_\rho \Psi) m_{\mu\nu} + (\partial_\nu \Psi) m_{\rho\mu}, \tag{C.1.48}$$

$$P_{\mu\nu\rho} = \partial_\mu m_{\nu\rho} + \partial_\rho m_{\mu\nu} + \partial_\nu m_{\rho\mu}. \tag{C.1.49}$$

Inserting (C.1.42) in (C.1.49) we get,

$$\begin{aligned} P_{\mu\nu\rho} &= \partial_\mu (k_\nu l_\rho - k_\rho l_\nu) + \partial_\rho (k_\mu l_\nu - k_\nu l_\mu) + \partial_\nu (k_\rho l_\mu - k_\mu l_\rho) &\text{(C.1.50)}\\ &= (\partial_\mu k_\nu - \partial_\nu k_\mu) l_\rho + (\partial_\rho k_\mu - \partial_\mu k_\rho) l_\nu + (\partial_\nu k_\rho - \partial_\rho k_\nu) l_\mu &\text{(C.1.51)}\\ &= n_{\mu\nu} l_\rho + n_{\rho\mu} l_\nu + n_{\nu\rho} l_\mu. &\text{(C.1.52)} \end{aligned}$$

For the computation of the right hand side we need to obtain the expression for the transformed three-form field with two of the indices raised i.e.$F_\nu{}^{\rho\sigma}$. To do this we raise the indices on the three-form field $F_{\mu\nu\lambda}$ each at a time. Raising the first index we get

$$\begin{aligned} F'^{\sigma}{}_{\nu\rho} &= g'^{\mu\sigma} F'_{\mu\nu\rho} &\text{(C.1.53)}\\ &= (g^{\mu\sigma} + \Psi^2 k^\mu k^\sigma - \Psi S^{\mu\sigma})(F_{\mu\nu\rho} - Q_{\mu\nu\rho} - \Psi P_{\mu\nu\rho}), &\text{(C.1.54)} \end{aligned}$$

which can be simplified to

$$F'^{\sigma}{}_{\nu\rho} = F^{\sigma}{}_{\nu\rho} - Q^{\sigma}{}_{\nu\rho} - \Psi P^{\sigma}{}_{\nu\rho} + \Psi l^\sigma n_{\nu\rho} + \Psi k^\sigma [(\partial_\nu \Psi) k_\rho - (\partial_\rho \Psi) k_\nu]. \tag{C.1.55}$$

by the use of the following identities

$$k^\mu Q_{\mu\nu\rho} = 0, \qquad k^\mu P_{\mu\nu\rho} = 0, \tag{C.1.56}$$

$$S^{\mu\sigma} F_{\mu\nu\rho} = -l^\sigma n_{\nu\rho}, \qquad S^{\mu\sigma} P_{\mu\nu\rho} = k^\sigma n_{\nu\rho}, \tag{C.1.57}$$

$$S^{\mu\sigma} Q_{\mu\nu\rho} = k^\sigma [k_\rho (\partial_\nu \Psi) - k_\nu (\partial_\rho \Psi)]. \tag{C.1.58}$$

Similarly raising the second index we get,

$$\begin{aligned}
F'^{\sigma\eta}{}_{\rho} &= g'^{\eta\nu}F'^{\sigma}{}_{\nu\rho} \\
&= F^{\sigma\eta}{}_{\rho} - Q^{\sigma\eta}{}_{\rho} - \Psi P^{\sigma\eta}{}_{\rho} + \Psi l^{\sigma}n^{\eta}{}_{\rho} + \Psi k^{\sigma}[(\partial^{\eta}\Psi)k_{\rho} - (\partial_{\rho}\Psi)k^{\eta}] \\
&\quad -\Psi l^{\eta}(n^{\sigma}{}_{\rho}) - \Psi k^{\eta}[(\partial^{\sigma}\Psi) - k^{\sigma}(\partial_{\rho}\Psi)]. \qquad \text{(C.1.59)}
\end{aligned}$$

With the above expressions we can readily compute the right hand side of the Einsteins equations. However, it turns out by raising all the three indices of the three-form field we get a much simpler expression for F' which appears to be easier to work with in various computations. Hence, before turning to the Einstein equations first we write the expression for F' with all three indices raised. It can be straightforwardly computed that

$$\begin{aligned}
F'^{\sigma\eta\alpha} &= g'^{\rho\alpha}F'^{\sigma\eta}{}_{\rho} \qquad \text{(C.1.60)} \\
&= F^{\sigma\eta\alpha} - Q^{\sigma\eta\alpha} - \Psi P^{\sigma\eta\alpha} + \Psi l^{\sigma}n^{\eta\alpha} + \Psi k^{\sigma}[(\partial^{\eta}\Psi)k^{\alpha} - (\partial^{\alpha}\Psi)k^{\eta}] \\
&\quad -\Psi l^{\eta}(n^{\sigma\alpha}) - \Psi k^{\eta}[k^{\alpha}(\partial^{\sigma}\Psi) - k^{\sigma}(\partial^{\alpha}\Psi)]) + \Psi^2 k^{\alpha}k^{\rho}F^{\sigma\eta}{}_{\rho} - \Psi S^{\alpha\rho}F^{\sigma\eta}{}_{\rho} \\
&\quad +\Psi S^{\alpha\rho}Q^{\sigma\eta}{}_{\rho} + \Psi^2 S^{\alpha\rho}P^{\sigma\eta}{}_{\rho} \qquad \text{(C.1.61)} \\
&= F^{\sigma\eta\alpha} - Q^{\sigma\eta\alpha} - \Psi(n^{\sigma\eta}l^{\alpha} + n^{\alpha\sigma}l^{\eta} + n^{\eta\alpha}l^{\sigma}) + \Psi l^{\sigma}n^{\eta\alpha} \\
&\quad +\Psi k^{\sigma}[(\partial^{\eta}\Psi)k^{\alpha} - (\partial^{\alpha}\Psi)k^{\eta}] - \Psi l^{\eta}(n^{\sigma\alpha}) - \Psi k^{\eta}[k^{\alpha}(\partial^{\sigma}\Psi) - k^{\sigma}(\partial^{\alpha}\Psi)]) \\
&\quad +\Psi l^{\alpha}(n^{\sigma\eta}) + \Psi k^{\alpha}[k^{\eta}(\partial^{\sigma}\Psi) - k^{\sigma}(\partial^{\eta}\Psi)] \\
&= F^{\sigma\eta\alpha} - Q^{\sigma\eta\alpha} + \Psi k^{\sigma}[(\partial^{\eta}\Psi)k^{\alpha} - (\partial^{\alpha}\Psi)k^{\eta}] - \Psi k^{\eta}[k^{\alpha}(\partial^{\sigma}\Psi) - k^{\sigma}(\partial^{\alpha}\Psi)]) \\
&\quad +\Psi k^{\alpha}[k^{\eta}(\partial^{\sigma}\Psi) - k^{\sigma}(\partial^{\eta}\Psi)] \qquad \text{(C.1.62)} \\
&= F^{\sigma\eta\alpha} - Q^{\sigma\eta\alpha}, \qquad \text{(C.1.63)}
\end{aligned}$$

which is a remarkably simple equation. With this now we can compute the transformed right hand side of (C.1.1). By the use of the following identities

$$\begin{aligned}
-F_{\lambda\alpha\beta}Q^{\delta\alpha\beta} - Q_{\lambda\alpha\beta}F^{\delta\alpha\beta} &= -4[l^{\delta}(\nabla_{\lambda}\nabla_{\beta}\Psi) + l_{\lambda}(\nabla^{\delta}\nabla_{\beta}\Psi)]k^{\beta}, \qquad \text{(C.1.64)} \\
Q_{\lambda\alpha\beta}Q^{\delta\alpha\beta} &= 2(\partial_{\beta}\Psi)(\partial^{\beta}\Psi)k_{\lambda}k^{\delta}, \qquad \text{(C.1.65)}
\end{aligned}$$

$$P_{\lambda\alpha\beta}Q^{\delta\alpha\beta} = 4k^\delta k^\alpha(\nabla_\lambda\nabla_\alpha\Psi), \qquad P_{\lambda\alpha\beta}F^{\delta\alpha\beta} = 4l_\lambda\Box k^\delta, \tag{C.1.66}$$

we get the following simplified expression for the right hand side of the Einstein's equations

$$\begin{aligned}\frac{1}{4}F'_{\lambda\alpha\beta}F'^{\delta\alpha\beta} &= \frac{1}{4}F_{\lambda\alpha\beta}F^{\delta\alpha\beta} - [l^\delta(\nabla_\lambda\nabla_\beta\Psi) + l_\lambda(\nabla^\delta\nabla_\beta\Psi)]k^\beta \\ &+\frac{1}{2}(\nabla_\beta\Psi)(\nabla^\beta\Psi)k_\lambda k^\delta + \Psi k^\delta k^\alpha(\nabla_\lambda\nabla_\alpha\Psi) - \Psi l_\lambda\Box k^\delta. \qquad \text{(C.1.67)}\end{aligned}$$

From this expression it can be easily seen that $F'_{\lambda\alpha\beta}F'^{\lambda\alpha\beta} = F_{\lambda\alpha\beta}F^{\lambda\alpha\beta} = 0$. Moreover,

$$\begin{aligned}\frac{1}{4}g'_{\nu\delta}F'_{\lambda\alpha\beta}F'^{\delta\alpha\beta} &= \frac{1}{4}(g_{\nu\delta} + \Psi S_{\nu\delta})F'_{\lambda\alpha\beta}F'^{\delta\alpha\beta} \qquad \text{(C.1.68)}\\ &= \frac{1}{4}F_{\lambda\alpha\beta}F_\nu{}^{\alpha\beta} - [l_\nu(\nabla_\lambda\nabla_\mu\Psi) + l_\lambda(\nabla_\nu\nabla_\mu\Psi)]k^\mu + \frac{1}{2}(\nabla_\rho\Psi)(\nabla^\rho\Psi)k_\lambda k_\nu \\ &-\Psi l_\lambda\Box k_\nu - \Psi l_\nu\Box k_\lambda + \Psi^2 l_\lambda l_\nu(\nabla^\alpha k_\delta)(\nabla_\alpha k^\delta), \qquad \text{(C.1.69)}\end{aligned}$$

where we have used the identities

$$F_{\lambda\alpha\beta}n^{\alpha\beta} = 4\Box k_\lambda, \tag{C.1.70}$$

$$S_{\nu\delta}\Box k^\delta = l_\nu k_\delta\Box k^\delta. \tag{C.1.71}$$

We see that the right hand side matches with the left hand side. Next we compute the matter field equations.

C.1.3 Matter field equations

The matter field equations are

$$\nabla_\mu F^{\mu\nu\rho} = 0. \tag{C.1.72}$$

Under the deformation of the metric (C.1.3) and the form field (C.1.39) the left hand side of the above equation transforms as

$$\nabla'_\mu F'^{\mu\nu\rho} = \nabla_\mu F'^{\mu\nu\rho} + \Omega^\mu_{\mu\sigma} F'^{\sigma\nu\rho} + \Omega^\nu_{\mu\sigma} F'^{\mu\sigma\rho} + \Omega^\rho_{\mu\sigma} F'^{\mu\nu\sigma} \tag{C.1.73}$$

$$= \nabla_\mu F'^{\mu\nu\rho} \tag{C.1.74}$$

$$= \nabla_\mu F^{\mu\nu\rho} - \nabla_\mu Q^{\mu\nu\rho}. \tag{C.1.75}$$

Using the background matter field equation (C.1.72) the first term in the above expression is readily zero. For the second term in (C.1.75), we have via (C.1.48)

$$Q^{\mu\nu\rho} = g^{\mu\sigma} g^{\nu\eta} g^{\rho\alpha} Q_{\sigma\eta\alpha} \tag{C.1.76}$$

$$= (\nabla^\mu \Psi) m^{\nu\rho} + (\nabla^\nu \Psi) m^{\rho\mu} + (\nabla^\rho \Psi) m^{\mu\nu}. \tag{C.1.77}$$

Applying the covariant ∇_μ on this expression we find,

$$\begin{aligned}\nabla_\mu Q^{\mu\nu\rho} &= (\Box\Psi) m^{\nu\rho} + (\nabla^\mu \Psi)[l^\rho(\nabla_\mu k^\nu) - l^\nu(\nabla_\mu k^\rho)] + (\nabla_\mu \nabla^\nu \Psi)(k^\rho l^\mu - k^\mu l^\rho) \\ &+ (\nabla_\mu \nabla^\rho \Psi)(k^\mu l^\nu - k^\nu l^\mu).\end{aligned} \tag{C.1.78}$$

Using

$$\Box\Psi = 0, \tag{C.1.79}$$

$$l^\mu(\nabla_\mu \nabla^\nu \Psi) = 0, \tag{C.1.80}$$

$$k^\mu(\nabla_\mu \nabla^\nu \Psi) = (\nabla^\mu \Psi)(\nabla_\mu k^\nu), \tag{C.1.81}$$

we get

$$\nabla'_\mu F'^{\mu\nu\rho} = \nabla_\mu Q^{\mu\nu\rho} = 0. \tag{C.1.82}$$

This proves the covariance of the matter field equations (C.1.72). Thus under GGV solutions of type-IIB supergravity are mapped to new solutions of type-IIB supergravity.

C.2 Non-zero dilaton

In this section we establish that generalised Garfinkle-Vachaspati (GGV) transform is a valid solution generating technique for more general supergravity solutions involving non-zero dilaton. Here the computation is a bit tricky yet one can relate them to the dilaton free case in a special frame so-called conformal frame which I shall introduce below.

The main text has GGV presented in the string frame which is useful in the sense that the examples we studied in Chapter 3 are easier to work with in the string frame. However, for the verification of the technique it is most convenient to work in the so-called conformal frame. After establishing the technique in the conformal frame, it can be readily written in any frame we like. In section C.2.4 we provide a brief summary of GGV in string, Einstein, and what we call conformal frame.

As in case of vanishing dilaton studied in section C.1, we explicitly compute and show the matching between the transformed left hand side and the transformed right hand side of the IIB equations.

The system of interest consists of a six-dimensional metric with associated antisymmetric 2-form field $C_{\mu\nu}$ and dilaton ϕ which is lifted to ten-dimensional IIB theory compactified on T^4. The minimal six-dimensional supergravity coupled to one self-dual tensor multiplet has the following supergravity action [66]

$$S_6 = \frac{1}{16\pi G_6}\int d^6x\sqrt{-g}\left[R-(d\phi)^2-\frac{1}{12}e^{2\phi}F_{\mu\nu\rho}F^{\mu\nu\rho}\right], \tag{C.2.83}$$

For ten-dimensional fields we follow Polchinski's conventions in which we can write the ten-dimensional IIB string frame action as

$$S_{\mathrm{RR}} = \frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{-g}\left[e^{-2\Phi}[R+4(d\Phi)^2]-\frac{1}{12}F_{\mu\nu\rho}F^{\mu\nu\rho}\right]. \tag{C.2.84}$$

where F is the self-dual three-form field strength corresponding to the RR 2-form potential $C^{(2)}$. The six-dimensional fields are embedded in ten-dimensions as

$$ds^2_{(\mathrm{S})} = ds^2_6 + e^{\phi}ds^2_4, \tag{C.2.85}$$

where $ds^2_{(\mathrm{S})}$ is the ten-dimensional metric and $ds_4^2 = \sum_{i=1}^4 dz^i dz^i$ is the flat torus metric. The subscript (S) denotes that the metric is in string frame. The six-dimensional dilaton, ϕ is defined as the scalar function that relates the six-dimensional string frame action with the Einstein frame action. For the particular embedding of interest, the ten-dimensional dilaton is same as the six-dimensional dilaton

$$\Phi = \phi, \tag{C.2.86}$$

and the ten-dimensional 2-form R-R field is also the same as the six-dimensional 2-form field with the torus components set to zero. In the Einstein-frame the ten-dimensional metric takes the form,

$$ds^2_{(\mathrm{E})} = e^{-\phi/2} ds_6^2 + e^{\phi/2} ds_4^2. \tag{C.2.87}$$

In a standard IIB conventions [20, 21], the Einstein frame bosonic field equations are

$$R_{\mu\nu} = \frac{1}{2}\nabla_\mu \Phi \nabla_\nu \Phi + \frac{1}{4} e^{\Phi} \left(F_{\mu\rho\sigma} F_\nu{}^{\rho\sigma} - \frac{1}{12} g_{\mu\nu} F_{\rho\sigma\kappa} F^{\rho\sigma\kappa} \right), \tag{C.2.88}$$

$$\nabla_\mu \left(e^{\Phi} F^{\mu\rho\sigma} \right) = 0, \tag{C.2.89}$$

$$\Box \Phi = \frac{1}{12} e^{\Phi} F_{\rho\sigma\kappa} F^{\rho\sigma\kappa}. \tag{C.2.90}$$

We need to check the covariance of the above equations under the GGV transform (which we are going to define). Then, by computing the transformed left hand side and the right hand side of the Einstein equations (C.2.88) we will verify that GGV is an effective solution generating technique including dilaton as well.

C.2.1 Left hand side of Einstein equations

For the implementation of Generalised Garfinkle-Vachaspati transform we need a null Killing vector and a spacelike Killing vector. The analysis of the equations of motion in case of non-zero dilaton can be closely related to that in section C.1 by performing the computations in the so-called conformal frame. This is because here for non-zero dilaton we do not have any covariantly constant vectors along the torus directions neither in the Einstein frame nor in the string frame. By performing a conformal transformation we can get covariantly constant spacelike vectors $l^{\mu}_{(i)}$ along torus direction which is discussed below.

The conformal transformation that provides us with spacelike covariantly constant vectors is defined by multiplication of the Einstein frame metric with a factor of $e^{-\phi/2}$. The new metric thus obtained is called as the "conformal frame" metric. It takes the form,

$$ds^2_{(\mathrm{C})} = e^{-\phi/2} ds^2_{(\mathrm{E})} = e^{-\phi} ds^2_6 + ds^2_4 = \tilde{g}_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{C.2.91}$$

For k^{μ} being a null Killing vector of the six-dimensional metric ds^2_6, it is also a null Killing vector for the ten-dimensional string frame metric $ds^2_{(\mathrm{S})}$ cf. (C.2.85). The compatibility property of dilaton with the Killing symmetry is given by

$$k^{\mu} \partial_{\mu} \phi = 0. \tag{C.2.92}$$

It can be readily seen that the conformal frame metric $ds^2_{(\mathrm{C})}$ also admits the same Killing vector. The contravariant (upper index) form of the Killing vector k^{μ} always has the same components irrespective of the metric under consideration. However, it's not the same for the covariant form of the Killing vector k_{μ} and we need to use proper notations for the Killing vectors in different frames of the metric to avoid any confusion. We use the notation $\tilde{k}_{\mu}$ for the conformal frame Killing 1-form. For the contravariant form of the vector we use $\tilde{k}^{\mu}$, but note that $\tilde{k}^{\mu} \equiv k^{\mu}$. This is a convenient notation. It follows that

$$0 = (\mathcal{L}_{\tilde{k}} \tilde{g})_{\mu\nu} = \tilde{\nabla}_{\mu} \tilde{k}_{\nu} + \tilde{\nabla}_{\nu} \tilde{k}_{\mu}, \tag{C.2.93}$$

where $\tilde{\nabla}_{\mu}$ is the conformal frame metric compatible covariant derivative. In this transformed frame the torus directions provide covariantly constant unit normalised spacelike (Killing) vectors orthogonal to $\tilde{k}^{\mu}$:

$$l_{(i)} = \tilde{l}^{\mu}_{(i)} \partial_{\mu} = \partial_{z^i}. \tag{C.2.94}$$

Furthermore, we have that

$$\tilde{l}^{\mu} \partial_{\mu} \phi = 0. \tag{C.2.95}$$

Now in the conformal frame we are equipped with a null Killing vector $\tilde{k}^{\mu}$ and covariantly constant spacelike vectors $\tilde{l}^{\mu}_{(i)}$. Using the same methods as in C.1 we can compute the left and right hand sides of the transformed equations. By doing so, we obtained (i) the right set of transformation rules for the metric and the associated matter fields, (ii) equations to be satisfied by the background spacetime for the technique to work, and (iii) the correct scalar field equation for Ψ. Once we establish the technique in conformal frame, we can go to string or Einstein frame metric by appropriate conformal transformation.

Before going into the computations of the equations of motion let us write the expressions for the transformation of the covariant derivative and that of the Ricci tensor under a conformal transformation of the metric. We are going to need these rules for our computations. For the conformal transformation of the n-dimensional metric $\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, the covariant derivative transforms as

$$\widetilde{\nabla}_{\mu}\omega_{\nu} = \nabla_{\mu}\omega_{\nu} - C^{\rho}_{\mu\nu}\omega_{\rho}, \tag{C.2.96}$$

where

$$C^{\rho}_{\mu\nu} = \delta^{\rho}_{\mu}\nabla_{\nu}(\ln\Omega) + \delta^{\rho}_{\nu}\nabla_{\mu}(\ln\Omega) - g_{\mu\nu}g^{\rho\sigma}\nabla_{\sigma}(\ln\Omega), \tag{C.2.97}$$

and the Ricci tensor transforms as,

$$\begin{aligned}\widetilde{R}_{\mu\nu} \;=\;& R_{\mu\nu} - (n-2)\nabla_{\mu}\nabla_{\nu}(\ln\Omega) - g_{\mu\nu}g^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}(\ln\Omega) + (n-2)(\nabla_{\mu}\ln\Omega)(\nabla_{\nu}\ln\Omega)\\ &-(n-2)g_{\mu\nu}g^{\rho\sigma}(\nabla_{\rho}\ln\Omega)(\nabla_{\sigma}\ln\Omega).\end{aligned} \tag{C.2.98}$$

In our case $\Omega = e^{-\phi/4}$ and the conformal frame metric is related to the Einstein frame metric by

$$\tilde{g}^{(\mathrm{C})}_{\mu\nu} = e^{-\phi/2}g^{(\mathrm{E})}_{\mu\nu}. \tag{C.2.99}$$

Using the expressions in C.2.97 and C.2.98, the Einstein equations take the following form in the conformal frame

$$\tilde{R}_{\mu\nu} = 2\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\phi + \frac{1}{4}\tilde{g}^{\rho\alpha}\tilde{g}^{\sigma\beta}F_{\mu\rho\sigma}F_{\nu\alpha\beta}. \tag{C.2.100}$$

So the generalised Garfinkle-Vachaspati (GGV) in the conformal frame should be of a form such that the above equations C.2.100 remain covariant under the transform. Keeping this in mind, we postulate the GGV in the conformal frame to be,

$$\tilde{g}_{\mu\nu} \quad \to \quad \tilde{g}'_{\mu\nu} = \tilde{g}_{\mu\nu} + \Psi(\tilde{k}_\mu \tilde{l}_\nu + \tilde{k}_\nu \tilde{l}_\mu), \tag{C.2.101}$$

$$C_{\mu\nu} \quad \to \quad C'_{\mu\nu} = C_{\mu\nu} - \Psi\,(\tilde{k}_\mu \tilde{l}_\nu - \tilde{k}_\nu \tilde{l}_\mu). \tag{C.2.102}$$

We will see from the analysis below that the above transformation generates a new solution when the background spacetime configuration satisfies a "transversality" condition:

$$\tilde{k}^\mu \tilde{F}_{\mu\nu\rho} = -(d\tilde{k})_{\nu\rho}, \tag{C.2.103}$$

and the scalar Ψ satisfies,

$$\tilde{\Box}\Psi + 2(\partial_\mu \phi)\tilde{g}^{\mu\nu}(\partial_\nu \Psi) = 0. \tag{C.2.104}$$

We can see that by setting $\phi = 0$, the scalar field equation (C.2.104) reduces to the minimally coupled massless scalar equation (C.1.4) for the dilaton free case. And also the transversality condition (C.2.103) has exactly the same form as for the dilaton free case (C.1.40) which is a remarkable feature of this conformal frame. The consistency of the transversality condition with the Einstein equations (C.2.100) can be checked straightforwardly as in the dilaton free case. Even for the present case it turns out to be the "square root" of a doubly contracted Einstein equations. To see this we first contract the left hand side of the Einstein equations (C.2.100) twice with the null Killing vector $\tilde{k}^\mu$, which on simplification,

$$\tilde{R}_{\mu\nu}\tilde{k}^\mu \tilde{k}^\nu = -\tilde{k}^\lambda \tilde{\Box}\tilde{k}_\lambda = \frac{1}{4}(\tilde{\nabla}^\rho \tilde{k}^\sigma - \tilde{\nabla}^\sigma \tilde{k}^\rho)(\tilde{\nabla}_\rho \tilde{k}_\sigma - \tilde{\nabla}_\sigma \tilde{k}_\rho). \tag{C.2.105}$$

Contracting the right hand side similarly, we first note that,

$$\tilde{k}^\mu \tilde{k}^\nu \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi = 0. \tag{C.2.106}$$

Therefore, the contracted Einstein equations simply reduce to

$$(\tilde{\nabla}^\rho \tilde{k}^\sigma - \tilde{\nabla}^\sigma \tilde{k}^\rho)(\tilde{\nabla}_\rho \tilde{k}_\sigma - \tilde{\nabla}_\sigma \tilde{k}_\rho) = (\tilde{k}^\mu \tilde{F}_{\mu\rho\sigma})(\tilde{k}^\nu \tilde{F}_\nu{}^{\rho\sigma}), \tag{C.2.107}$$

which is just the square of the transversality condition (C.2.103).

Let the conformal frame Einstein equations (C.2.100) after perfoming GGV takes the following covariant form,

$$\tilde{R'}_{\mu\nu} = 2\tilde{\nabla}'_\mu \tilde{\nabla}'_\nu \phi + \frac{1}{4} \tilde{g'}^{\rho\alpha} \tilde{g'}^{\sigma\beta} F'_{\mu\rho\sigma} F'_{\nu\alpha\beta}. \tag{C.2.108}$$

The computation of the left hand side of the GGV trasnformed Einstein equation (C.2.108) involves the computation of the transformed Ricci tensor which is straightforward and we can simply follow the steps from C.1. The transformed Ricci tensor in the conformal frame turns out to be[1],

$$\begin{aligned} \tilde{R}'_{\lambda\nu} &= \tilde{R}_{\lambda\nu} - \tilde{l}_\lambda[\tilde{k}^\mu(\tilde{\nabla}_\nu \tilde{\nabla}_\mu \Psi) + \Psi \tilde{\Box} \tilde{k}_\nu] - \tilde{l}_\nu[\tilde{k}^\mu(\tilde{\nabla}_\lambda \tilde{\nabla}_\mu \Psi) + \Psi \tilde{\Box} \tilde{k}_\lambda] \\ &\quad + \frac{1}{2}(\tilde{\nabla}_\rho \Psi)(\tilde{\nabla}^\rho \Psi)\tilde{k}_\lambda \tilde{k}_\nu - \Psi^2(\tilde{\nabla}_\mu \tilde{k}^\rho)(\tilde{\nabla}_\rho \tilde{k}^\mu)\tilde{l}_\lambda \tilde{l}_\nu - \frac{1}{2}\tilde{\Box}\Psi \tilde{S}_{\lambda\nu}, \end{aligned} \tag{C.2.109}$$

where we have introduced the notation

$$\tilde{S}_{\mu\nu} = \tilde{k}_\mu \tilde{l}_\nu + \tilde{k}_\nu \tilde{l}_\mu. \tag{C.2.110}$$

By contracting two of the indices in C.2.109 it can be readily checked that $\tilde{R}' = \tilde{R}$ i.e Ricci scalar remains invariant under the GGV transform.

C.2.2 Right hand side of Einstein equations

The GGV transform (C.2.101)–(C.2.102) does not alter dilaton in the theory. The two-form field transforms as (C.2.102). Note that under conformal transformation the form fields in their covariant form do not change. Only the metric, hence the associated covariant derivative, metric compatible connection, and Ricci tensor changes. Thus we do not need to use different notations for the two-form fields in different frames. However, since the raising and lowering of indices completely depends on the metric so to avoid confusion, we put tildes on the two-form

[1] We have confirmed this transformation using Cadabra [67, 68] too.

field in the conformal frame. The GGV on the conformal frame two-form is then,

$$\tilde{C} \to \tilde{C}' = \tilde{C} - \Psi\, \tilde{k}_\mu dx^\mu \wedge \tilde{l}_\nu dx^\nu. \tag{C.2.111}$$

To compute the transformation of the right hand side of the Einstein equation (C.2.100), we simply follow the steps from C.1. We introduce

$$\tilde{m}_{\mu\nu} = \tilde{k}_\mu \tilde{l}_\nu - \tilde{k}_\nu \tilde{l}_\mu, \tag{C.2.112}$$

and write the transformed R-R two-form field (C.2.111) as

$$\tilde{C}'_{\mu\nu} = \tilde{C}_{\mu\nu} - \Psi \tilde{m}_{\mu\nu}. \tag{C.2.113}$$

We then find that

$$\tilde{F}'_{\mu\nu\rho} = \tilde{F}_{\mu\nu\rho} - \tilde{Q}_{\mu\nu\rho} - \Psi \tilde{P}_{\mu\nu\rho}, \tag{C.2.114}$$

where

$$\tilde{Q}_{\mu\nu\rho} = (\partial_\mu \Psi)\tilde{m}_{\nu\rho} + (\partial_\rho \Psi)\tilde{m}_{\mu\nu} + (\partial_\nu \Psi)\tilde{m}_{\rho\mu}, \tag{C.2.115a}$$

$$\tilde{P}_{\mu\nu\rho} = \partial_\mu \tilde{m}_{\nu\rho} + \partial_\rho \tilde{m}_{\mu\nu} + \partial_\nu \tilde{m}_{\rho\mu}. \tag{C.2.115b}$$

As done before in C.1, we need to raise the indices on $\tilde{F}_{\mu\nu\rho}$ and contract them appropriately with another $\tilde{F}_{\mu\nu\rho}$. We find

$$\tilde{F}'^{\sigma\eta\alpha} = \tilde{F}^{\sigma\eta\alpha} - \tilde{Q}^{\sigma\eta\alpha}, \tag{C.2.116}$$

and

$$\begin{aligned} \frac{1}{4}\tilde{F}'_{\lambda\alpha\beta}\tilde{F}'^{\delta\alpha\beta} &= \frac{1}{4}\tilde{F}_{\lambda\alpha\beta}\tilde{F}^{\delta\alpha\beta} - [\tilde{l}^\delta(\tilde{\nabla}_\lambda \tilde{\nabla}_\beta \Psi) + \tilde{l}_\lambda(\tilde{\nabla}^\delta \tilde{\nabla}_\beta \Psi)]\tilde{k}^\beta \\ &\quad + \frac{1}{2}(\tilde{\nabla}_\beta \Psi)(\tilde{\nabla}^\beta \Psi)\tilde{k}_\lambda \tilde{k}^\delta + \Psi \tilde{k}^\delta \tilde{k}^\alpha(\tilde{\nabla}_\lambda \tilde{\nabla}_\alpha \Psi) - \Psi \tilde{l}_\lambda \Box \tilde{k}^\delta. \end{aligned} \tag{C.2.117}$$

From this last expression it can be easily seen that

$$\tilde{F}'_{\alpha\beta\gamma}\tilde{F}'^{\alpha\beta\gamma} = \tilde{F}_{\alpha\beta\gamma}\tilde{F}^{\alpha\beta\gamma}. \tag{C.2.118}$$

Similarly, the transformation of the dilaton term in (C.2.100) turns out to be:

$$2\tilde{\nabla}'_\mu\tilde{\nabla}'_\nu\phi = 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi + \tilde{g}^{\rho\sigma}(\tilde{\nabla}_\sigma\Psi)\tilde{S}_{\mu\nu}(\partial_\rho\phi) - 2\Psi[(\tilde{\nabla}_\mu\tilde{k}^\rho)\tilde{l}_\nu + (\tilde{\nabla}_\nu\tilde{k}^\rho)\tilde{l}_\mu](\tilde{\nabla}_\rho\phi). \tag{C.2.119}$$

Comparing with the left hand side (C.2.109), it is confirmed that both sides transform exactly in the same way provided,

$$-\frac{1}{2}\tilde{S}_{\mu\nu}\tilde{\Box}\Psi = \tilde{g}^{\rho\sigma}(\tilde{\nabla}_\sigma\Psi)\tilde{S}_{\mu\nu}(\partial_\rho\phi) - 2\Psi[(\tilde{\nabla}_\mu\tilde{k}^\rho)\tilde{l}_\nu + (\tilde{\nabla}_\nu\tilde{k}^\rho)\tilde{l}_\mu](\tilde{\nabla}_\rho\phi), \tag{C.2.120}$$

which looks like a non-trivial tensor equation for the scalar Ψ. Fortunately, we can simplify the above expression by making use of the background Einstein equations. The term $2(\tilde{\nabla}_\mu\tilde{k}^\rho)(\tilde{\nabla}_\rho\phi)$ is simplified to

$$2(\tilde{\nabla}_\mu\tilde{k}^\rho)(\tilde{\nabla}_\rho\phi) = -2\tilde{k}^\rho(\tilde{\nabla}_\mu\tilde{\nabla}_\rho\phi). \tag{C.2.121}$$

The right hand side of the above equation can be written as,

$$-2\tilde{k}^\rho(\tilde{\nabla}_\mu\tilde{\nabla}_\rho\phi) = -2\tilde{k}^\rho\left(\tilde{R}_{\mu\rho} - \frac{1}{4}\tilde{F}_\mu{}^{\alpha\beta}\tilde{F}_{\rho\alpha\beta}\right) = -2\tilde{k}^\rho\tilde{R}_{\mu\rho} + \frac{1}{2}\tilde{F}_\mu{}^{\alpha\beta}(-\tilde{n}_{\alpha\beta}) \tag{C.2.122}$$

where

$$\tilde{n}_{\mu\nu} = \tilde{\nabla}_\mu\tilde{k}_\nu - \tilde{\nabla}_\nu\tilde{k}_\mu \tag{C.2.123}$$

and we have used background Einstein equation (C.2.100). Using the identities

$$\tilde{k}^\rho\tilde{R}_{\mu\rho} = -\tilde{\Box}\tilde{k}_\mu, \qquad \tilde{n}_{\alpha\beta}\tilde{F}_\mu{}^{\alpha\beta} = 4\tilde{\Box}\tilde{k}_\mu, \tag{C.2.124}$$

we get,

$$-2\tilde{k}^\rho(\tilde{\nabla}_\mu\tilde{\nabla}_\rho\phi) = 0. \tag{C.2.125}$$

Similarly,

$$2(\tilde{\nabla}_\nu \tilde{k}^\rho)(\tilde{\nabla}_\rho \phi) = 0. \tag{C.2.126}$$

As a result of these simplifications, the complicated tensor equation (C.2.120) for Ψ becomes a much simpler scalar equation,

$$\tilde{\Box}\Psi + 2(\partial_\mu \phi)\tilde{g}^{\mu\nu}(\partial_\nu \Psi) = 0, \tag{C.2.127}$$

equivalently,

$$\tilde{\nabla}_\mu (e^{2\phi}\tilde{g}^{\mu\nu}(\partial_\nu \Psi)) = 0. \tag{C.2.128}$$

Next we compute the matter field equation.

C.2.3 Matter field equations

The 2-form field equation in Einstein frame is (C.2.89). Note that under conformal transformation the metric compatible covariant derivative transforms as

$$\tilde{\nabla}_\mu \omega^{\mu\nu\rho} = \nabla_\mu \omega^{\mu\nu\rho} + C^\mu_{\mu\gamma}\omega^{\gamma\nu\rho} + C^\nu_{\mu\alpha}\omega^{\mu\alpha\rho} + C^\rho_{\mu\beta}\omega^{\mu\nu\beta}, \tag{C.2.129}$$

with $C^\nu_{\mu\alpha}$ given in equation (C.2.97). For the transformation from Einstein frame to the conformal frame $\ln \Omega = -\frac{\phi}{4}$. With these equations it is straightforward to find that the 2-form field equation in the conformal frame is,

$$\tilde{\nabla}_\mu (e^{2\phi}\tilde{F}^{\mu\nu\rho}) = 0. \tag{C.2.130}$$

We need to check the covariance of the matter field equation (C.2.130) in order to confirm the validity of our generalised GV transform. The dilaton field does not transform under the generalised GV transform. The transformation of the 3-form field strength as computed in the previous section (C.2.116) where all the three indices are contravariant (upper) is,

$$\tilde{F}'^{\mu\nu\rho} = \tilde{F}^{\mu\nu\rho} - \tilde{Q}^{\mu\nu\rho}. \tag{C.2.131}$$

Therefore, under the GGV transformation the left hand side of the matter field equation (C.2.130) transforms as

$$\tilde{\nabla}'_{\mu}(e^{2\phi}\tilde{F}'^{\mu\nu\rho}) = \tilde{\nabla}_{\mu}(e^{2\phi}(\tilde{F}^{\mu\nu\rho} - \tilde{Q}^{\mu\nu\rho})), \tag{C.2.132}$$

where

$$\tilde{Q}^{\mu\nu\rho} = (\tilde{\nabla}^{\mu}\Psi)\tilde{m}^{\nu\rho} + (\tilde{\nabla}^{\nu}\Psi)\tilde{m}^{\rho\mu} + (\tilde{\nabla}^{\rho}\Psi)\tilde{m}^{\mu\nu}, \tag{C.2.133}$$

and recall that $\tilde{m}^{\mu\nu} = \tilde{k}^{\mu}\tilde{l}^{\nu} - \tilde{k}^{\nu}\tilde{l}^{\mu} = m^{\mu\nu}$. The above expression (C.2.132) is obtained exactly in the same way as for the case of vanishing dilaton C.1. Expanding the above expression and using (C.2.127) one can show that

$$\tilde{\nabla}'_{\mu}(e^{2\phi}\tilde{F}'^{\mu\nu\rho}) = \tilde{\nabla}_{\mu}(e^{2\phi}\tilde{F}^{\mu\nu\rho}) = 0. \tag{C.2.134}$$

Obtaining the rules of generalised Garfinkle-Vachaspati transform in the conformal frame now we can easily convert them to other frames. In the following section we shall summarize the GGV in different frames.

C.2.4 Summary in different R-R and NS-NS frames

GGV is a valid solution generating technique which is made clear by explicitly verifying the equations of motion in the conformal frame. For the application of GGV to solutions of interest it is easier to work in string frame or Einstein frame for which we need to perform the inverse conformal transformation on the expressions obtained in the conformal frame.

Also the Ramond-sector equations can be transformed to NS-sector by the use of S-duality [B.2].

First let us summarize the conformal frame set of equations:

Conformal frame: The generalised Garfinkle-Vachaspati transform is

$$\tilde{g}_{\mu\nu} \quad\rightarrow\quad \tilde{g}'_{\mu\nu} = \tilde{g}_{\mu\nu} + \Psi(\tilde{k}_{\mu}\tilde{l}_{\nu} + \tilde{k}_{\nu}\tilde{l}_{\mu}), \tag{C.2.135}$$

$$C_{\mu\nu} \quad\rightarrow\quad C'_{\mu\nu} = C_{\mu\nu} - \Psi\,(\tilde{k}_{\mu}\tilde{l}_{\nu} - \tilde{k}_{\nu}\tilde{l}_{\mu}). \tag{C.2.136}$$

with scalar Ψ satisfying

$$\tilde{\nabla}_\mu(e^{2\phi}\tilde{g}^{\mu\nu}(\partial_\nu\Psi)) = 0, \tag{C.2.137}$$

and the 2-form C-field satisfying the transversality condition

$$\tilde{k}^\mu \tilde{F}_{\mu\nu\rho} = -(d\tilde{k})_{\nu\rho}. \tag{C.2.138}$$

In this frame the vector $\tilde{l}^\mu$ is covariantly constant and unit normalised.

By performing a conformal transformation by the factor $g^{(\mathrm{E})}_{\mu\nu} = e^{\phi/2}\tilde{g}_{\mu\nu}$ GGV can be rewritten in the Einstein frame which we discuss below:

Einstein frame: In this frame the spacelike vector l^μ is not covariantly constant rather satisfies

$$\nabla^{(\mathrm{E})}_\mu l^{(\mathrm{E})}_\nu = \frac{1}{4}[l^{(\mathrm{E})}_\nu(\partial_\mu\phi) - l^{(\mathrm{E})}_\mu(\partial_\nu\phi)]. \tag{C.2.139}$$

It of course satisfies the Killing equation,

$$\nabla^{(\mathrm{E})}_\mu l^{(\mathrm{E})}_\nu + \nabla^{(\mathrm{E})}_\nu l^{(\mathrm{E})}_\mu = 0, \tag{C.2.140}$$

and is normalised as $l^\mu_{(\mathrm{E})} l^{(\mathrm{E})}_\mu = e^{\phi/2}$.

After the conformal transformation from the conformal frame to Einstein frame Ψ satisfies the following equation:

$$\Box^{(\mathrm{E})}\Psi = 0. \tag{C.2.141}$$

The transversality condition (C.2.103) becomes

$$k^\mu_{(\mathrm{E})} F_{\mu\nu\rho} = -d(e^{-\phi/2}k^{(\mathrm{E})})_{\nu\rho}. \tag{C.2.142}$$

The GGV transform takes the form

$$g^{(\mathrm{E})}_{\mu\nu} \rightarrow g^{(\mathrm{E})}_{\mu\nu} + \Psi e^{-\phi/2}(k^{(\mathrm{E})}_\mu l^{(\mathrm{E})}_\nu + k^{(\mathrm{E})}_\nu l^{(\mathrm{E})}_\mu), \tag{C.2.143}$$

$$C \rightarrow C - \Psi e^{-\phi}(k^{(\mathrm{E})}_\mu l^{(\mathrm{E})}_\nu - k^{(\mathrm{E})}_\nu l^{(\mathrm{E})}_\mu). \tag{C.2.144}$$

String frame: Similarly the string frame GGV can be obtained by performing the conformal transformation by the factor $g^{(\mathrm{S})}_{\mu\nu} = e^{\phi/2} g^{(\mathrm{E})}_{\mu\nu}$. This is the frame which we have used exclusively in Chapter 3 of the main text. So we present some more details in this case. The spacelike vector $l^{\mu}_{(\mathrm{S})}$ is again not covariantly constant. It satisfies

$$\nabla^{(\mathrm{S})}_{\mu} l^{(\mathrm{S})}_{\nu} = \frac{1}{2}[l^{(\mathrm{S})}_{\nu}(\partial_\mu \phi) - l^{(\mathrm{S})}_{\mu}(\partial_\nu \phi)], \tag{C.2.145}$$

$$\nabla^{(\mathrm{S})}_{\mu} l^{(\mathrm{S})}_{\nu} + \nabla^{(\mathrm{S})}_{\nu} l^{(\mathrm{S})}_{\mu} = 0, \tag{C.2.146}$$

It is now normalised as $l^{\mu}_{(\mathrm{S})} l^{(\mathrm{S})}_{\mu} = e^{\phi}$.

The scalar equation satisfied by Ψ in the string frame looks like,

$$\Box^{(\mathrm{S})}\Psi - 2(\partial_\mu \phi) g^{\mu\nu}_{(\mathrm{S})}(\partial_\nu \Psi) = 0, \tag{C.2.147}$$

equivalently

$$\nabla^{(\mathrm{S})}_{\mu}(e^{-2\phi} g^{\mu\nu}_{(\mathrm{S})} \partial_\nu \Psi) = 0. \tag{C.2.148}$$

The transversality condition (C.2.103) becomes

$$k^{\mu}_{(\mathrm{S})} F_{\mu\nu\rho} = -d(e^{-\phi} k^{(\mathrm{S})})_{\nu\rho}. \tag{C.2.149}$$

The GGV transform takes the form

$$g^{(\mathrm{S})}_{\mu\nu} \;\rightarrow\; g^{(\mathrm{S})}_{\mu\nu} + \Psi e^{-\phi}(k^{(\mathrm{S})}_{\mu} l^{(\mathrm{S})}_{\nu} + k^{(\mathrm{S})}_{\nu} l^{(\mathrm{S})}_{\mu}), \tag{C.2.150}$$

$$C \;\rightarrow\; C - \Psi e^{-2\phi}(k^{(\mathrm{S})}_{\mu} l^{(\mathrm{S})}_{\nu} - k^{(\mathrm{S})}_{\nu} l^{(\mathrm{S})}_{\mu}). \tag{C.2.151}$$

For convenience we write the string frame equations of motion by omitting the superscript (S). The IIB string frame action with RR 2-form $C^{(2)}$ with $F^{(3)} = dC^{(2)}$ is

$$S = \frac{1}{16\pi G}\int d^{10}x\sqrt{-g}\left[e^{-2\Phi}[R + 4(d\Phi)^2] - \frac{1}{12}F_{\mu\nu\rho}F^{\mu\nu\rho}\right], \tag{C.2.152}$$

and the resulting equations of motion are

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi = \frac{1}{4}e^{2\Phi}\left(F_{\mu\rho\sigma}F_\nu{}^{\rho\sigma} - \frac{1}{6}F_{\rho\sigma\kappa}F^{\rho\sigma\kappa}g_{\mu\nu}\right), \tag{C.2.153}$$

$$\nabla_\mu F^{\mu\nu\rho} = 0, \tag{C.2.154}$$

$$R + 4\nabla^2\Phi - 4(\nabla\Phi)^2 = 0. \tag{C.2.155}$$

In case of non-zero dilaton we have two different sectors namely, the Ramond-Ramond sector and the NS-NS sector which are related by duality. For the application of GGV to F1-P system 3.3.2 we need the set of rules in the NS-NS sector as well. It can be obtained by performing a S-duality which is discussed in B.2.

NS-NS sector string frame: Embedding of interest of the six-dimensional theory (C.2.83) in the ten-dimensional NS-NS sector string frame is as follows

$$ds^2_{(\mathrm{S})} = e^{-\phi}ds_6^2 + ds_4^2, \tag{C.2.156}$$

with ten-dimensional dilaton,

$$\Phi = -\phi. \tag{C.2.157}$$

The six-dimensional 2-form field is now the 2-form B-field with zero components in the four torus directions.

In this embedding the torus Killing vectors are unit normalised and are covariantly constant. In this set-up the GGV transform takes the form,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \Psi(k_\mu l_\nu + k_\nu l_\mu) \tag{C.2.158}$$

$$B_{\mu\nu} \rightarrow B_{\mu\nu} - \Psi(k_\mu l_\nu - k_\nu l_\mu) \tag{C.2.159}$$

The transversality condition reads,

$$k^\mu H_{\mu\nu\rho} = -(dk)_{\nu\rho}, \tag{C.2.160}$$

and the scalar wave equation for the field Ψ becomes

$$\nabla_\mu(e^{-2\phi}g^{\mu\nu}\nabla_\nu\Psi) = 0. \tag{C.2.161}$$

NS-NS sector Einstein frame: In Einstein frame, embedding (C.2.156) reads,

$$ds^2_{(\mathrm{E})} = e^{-\Phi/2}ds^2_{(\mathrm{S})} = e^{\phi/2}ds^2_{(\mathrm{S})} = e^{-\phi/2}ds^2_6 + e^{\phi/2}ds^2_4. \tag{C.2.162}$$

We note that this metric is same as (C.2.87). The two embeddings are related by S-duality:

$$g^{(\mathrm{E})}_{\mu\nu} \to g^{(\mathrm{E})}_{\mu\nu}, \qquad \Phi \to -\Phi, \qquad C_{\mu\nu} \to B_{\mu\nu}. \tag{C.2.163}$$

In this set-up the GGV transform takes the form,

$$g^{(\mathrm{E})}_{\mu\nu} \;\to\; g^{(\mathrm{E})}_{\mu\nu} + \Psi e^{\phi/2}(k^{(\mathrm{E})}_\mu l^{(\mathrm{E})}_\nu + k^{(\mathrm{E})}_\nu l^{(\mathrm{E})}_\mu), \tag{C.2.164}$$

$$B_{\mu\nu} \;\to\; B_{\mu\nu} - \Psi e^{\phi}(k^{(\mathrm{E})}_\mu l^{(\mathrm{E})}_\nu - k^{(\mathrm{E})}_\nu l^{(\mathrm{E})}_\mu). \tag{C.2.165}$$

The transversality condition reads, $k^\mu_{(\mathrm{E})}H_{\mu\nu\rho} = -(d(e^{\phi/2}k^{(\mathrm{E})}))_{\nu\rho}$, and the scalar wave equation for the field Ψ becomes $\Box^{(\mathrm{E})}\Psi = 0$.

Appendix D

BW and GMR formalisms

In this appendix, we are going to give a brief review of the Gutowski-Martelli-Reall (GMR) and the Bena-Warner (BW) formalisms, then obtained the relation between the two notations. Similar computations were also done in [95–97].

D.1 Gutowski-Martelli-Reall formalism

In the GMR formalism [65], we work with minimal six-dimensional supergravity. The notations we are going to use are the same as followed in the appendix A of reference [33]. The bosonic field content of this theory consists of metric $g_{\mu\nu}$ and a self-dual three-form $G_{\mu\nu\rho}$. GMR showed that any 6D supergravity solution preserving some amount of supersymmetries can be written in the following general form

$$ds^2 = -H^{-1}(dv+\beta)\left(du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right)+Hh_{mn}dx^m dx^n, \tag{D.1.1}$$

where h_{mn} is a metric on a four-dimensional almost hyper-Kähler base manifold. These manifolds are Calabi-Yau manifolds and characterized in terms of three complex quaternionic structures (for more details see [22]). β and ω are one-forms on this base space and independent of the u-direction. Similarly $\mathcal{F}$ and H are functions on the base space. In general h_{mn}, β, ω $\mathcal{F}$

and H can be v-dependent. In that case the only null Killing vector is given by

$$k = \frac{\partial}{\partial u}, \tag{D.1.2}$$

However, to compare with the Bena-Warner formalism [72], we restrict ourselves only to v-*independent* solutions. Then the six-dimensional field strength G takes the form

$$\begin{aligned} F \ = \ 2G \ &= \ \star dH - H^{-1}(dv+\beta)\wedge\left(\frac{d\omega - \star d\omega}{2}\right) \\ &\quad + H^{-1}\left(du + \omega + \frac{\mathcal{F}}{2}(dv+\beta)\right)\wedge\left(d\beta + H^{-1}(dv+\beta)\wedge dH\right). \end{aligned} \tag{D.1.3}$$

In the GMR formalism using Killing spinor techniques the 6D equations of motion reduce to 4D equations which are easier to solve. By analysing the Killing spinor equations, the equations of motion can be written as

$$\star d\star d\mathcal{F} - \frac{1}{2}(\mathcal{G}^+)^2 \ = \ 0, \tag{D.1.4}$$

$$d\star dH + \frac{d\beta\wedge\mathcal{G}^+}{2} \ = \ 0, \tag{D.1.5}$$

$$d\beta - \star d\beta \ = \ 0, \tag{D.1.6}$$

$$d\mathcal{G}^+ \ = \ 0. \tag{D.1.7}$$

Here the Hodge star is with respect to 4-dimensional base metric $h_{\mu\nu}$ and self-dual two-form $\mathcal{G}^+$ is defined as

$$\mathcal{G}^+ = \frac{1}{2H}\left(d\omega + \star d\omega + \mathcal{F}d\beta\right). \tag{D.1.8}$$

We also note that $\star d\star d\mathcal{F} = -\nabla^2\mathcal{F}$ and $(\mathcal{G}^+)^2 = (\mathcal{G}^+)^{mn}(\mathcal{G}^+)_{mn}$.

D.2 Bena-Warner formalism

As shown in [72] by Bena and Warner, solutions that preserve same supersymmetries as those of three charge black hole and black ring admit a general form in which one forms are defined on a four dimensional hyper-Kähler base space. This simplest formalism developed by Bena and Warner have the most symmetric form in the eleven-dimensional M-theory frame. There we have intersecting branes on the six-torus with coordinates $(z_1, \ldots, z_6)$, denoted as M2(12)–M2(34)–M2(56). Further details on brane intersection can be found in the review [98]. The eleven-dimensional metric takes the following symmetrical form,

$$ds^2_{11} = ds^2_5 + ds^2_{\mathrm{T}^6}, \tag{D.2.9}$$

where $ds^2_{\mathrm{T}^6}$ is metric on the six-torus,

$$ds^2_{\mathrm{T}^6} = (Z_2 Z_3 Z_1^{-2})^{\frac{1}{3}}(dz_1^2 + dz_2^2) + (Z_1 Z_3 Z_2^{-2})^{\frac{1}{3}}(dz_3^2 + dz_4^2) + (Z_1 Z_2 Z_3^{-2})^{\frac{1}{3}}(dz_5^2 + dz_6^2), \tag{D.2.10}$$

and ds^2_5 is the metric on five-dimensional transverse spacetime,

$$ds^2_5 = -(Z_1 Z_2 Z_3)^{-\frac{2}{3}}(dt + \kappa)^2 + (Z_1 Z_2 Z_3)^{\frac{1}{3}} h_{mn} dx^m dx^n, \tag{D.2.11}$$

where h_{mn} is the metric on a 4-dimensional hyper-Kähler base space. The three-form potential $\mathcal{A}$ associated to the metric (D.2.9) is written in terms of three one-form potentials $A^{(I)}$ which depend on the non-compact five-dimensional spacetime with metric ds^2_5. The potential is of the following symmetric form,

$$\mathcal{A} = A^{(1)} \wedge dz_1 \wedge dz_2 + A^{(2)} \wedge dz_3 \wedge dz_4 + A^{(3)} \wedge dz_5 \wedge dz_6, \tag{D.2.12}$$

The one-forms $A^{(I)}$ on the non-compact space in turn take the form,

$$A^{(I)} = -\frac{(dt + \kappa)}{Z_I} + \omega_I, \tag{D.2.13}$$

where κ and ω_I are one-forms on the four-dimensional base space while Z_I are functions on the base space.

The functions Z_I and the one-forms κ and ω_I are satisfy the BW equations [72]:

$$\begin{aligned} d\omega_I &= \star d\omega_I, && \text{(D.2.14)} \\ d\kappa + \star d\kappa &= Z_I d\omega_I, && \text{(D.2.15)} \\ \nabla^2 Z_I &= \frac{1}{2}|\epsilon_{IJK}| \star (d\omega_J \wedge d\omega_K), && \text{(D.2.16)} \end{aligned}$$

where the Hodge star is with respect to the four-dimensional base metric h_{mn}.

In order to compare with the GMR formalism which is in six-spacetime dimensions, we first need to perform a dimensional reduction on the M-theory form of the Bena-Warner solution for its reduction to ten-dimensions. After the dimensional reduction the solution reduces to the intersecting D-brane solution D2(12)–D2(34)–F1(5) of type-IIA theory. Upon performing a set of dualities it can be mapped to type IIB D1-D5-P solution.

Performing a dimensional reduction on (D.2.9) along the z_6-direction the resulting IIA metric in the string frame is,

$$\begin{aligned} ds_{10}^2 = & - \frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+\kappa)^2 + \sqrt{Z_1 Z_2} h_{mn} dx^m dx^n \\ & + \sqrt{\frac{Z_2}{Z_1}}(dz_1^2 + dz_2^2) + \sqrt{\frac{Z_1}{Z_2}}(dz_3^2 + dz_4^2) + \frac{\sqrt{Z_1 Z_2}}{Z_3} dz_5^2, \qquad \text{(D.2.17)} \end{aligned}$$

with IIA dilaton,

$$e^{2\phi} = \frac{\sqrt{Z_1 Z_2}}{Z_3}, \qquad \text{(D.2.18)}$$

and with three-form RR field,

$$\begin{aligned} C_{\mu z_1 z_2} &= A_\mu^{(1)}, && \text{(D.2.19)} \\ C_{\mu z_3 z_4} &= A_\mu^{(2)}, && \text{(D.2.20)} \end{aligned}$$

and two-form NS-NS B-field,

$$B_{\mu z_5} = A_\mu^{(3)}. \qquad \text{(D.2.21)}$$

This is the D2(12)–D2(34)–F1(5) solution of type IIA theory. To go to type-IIB theory we need to perform a set of T-duality transformations (B.3) on it. We perform T-dualities along z_3, z_4 and z_5 directions upon which we get D5(12345)–D1(5)–P(5) solution. We recall the T-duality rules (B.3) for a duality along z-direction:

$$G'_{zz} = \frac{1}{G_{zz}}, \tag{D.2.22}$$

$$G'_{\mu z} = \frac{B_{\mu z}}{G_{zz}}, \tag{D.2.23}$$

$$G'_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu z}G_{\nu z} - B_{\mu z}B_{\nu z}}{G_{zz}}, \tag{D.2.24}$$

$$B'_{\mu z} = \frac{G_{\mu z}}{G_{zz}}, \tag{D.2.25}$$

$$B'_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu z}G_{\nu z} - G_{\mu z}B_{\nu z}}{G_{zz}}, \tag{D.2.26}$$

$$e^{2\phi'} = \frac{e^{2\phi}}{G_{zz}}, \tag{D.2.27}$$

$$C'^{(n)}_{\mu\ldots\nu\alpha z} = C^{(n-1)}_{\mu\ldots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|z}G_{|\alpha]z}}{G_{zz}}, \tag{D.2.28}$$

$$C'^{(n)}_{\mu\ldots\nu\alpha\beta} = C^{(n+1)}_{\mu\ldots\nu\alpha\beta z} + nC^{(n-1)}_{[\mu\ldots\nu\alpha}B_{\beta]z} + n(n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|z}B_{|\alpha|z}G_{|\beta]z}}{G_{zz}}. \tag{D.2.29}$$

Now, we first perform T-dualities along z_3, z_4 directions after which we get the following fields:

$$\begin{aligned} ds^2_{10} = & - \frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+\kappa)^2 + \sqrt{Z_1 Z_2}h_{mn}dx^m dx^n \\ & + \sqrt{\frac{Z_2}{Z_1}}(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) + \frac{\sqrt{Z_1 Z_2}}{Z_3}dz_5^2, \end{aligned} \tag{D.2.30}$$

$$e^{2\phi} = \frac{Z_2^{3/2}}{Z_3\sqrt{Z_1}}, \tag{D.2.31}$$

$$C^{(5)}_{\mu z_1 z_2 z_3 z_4} = A^{(1)}_\mu, \qquad C^{(1)}_\mu = -A^{(2)}_\mu, \qquad B_{\mu z_5} = A^{(3)}_\mu. \tag{D.2.32}$$

Next, with another T-duality along z_5-direction, we get the required D1-D5-P configuration.

For the final configuration the IIB dilaton reads:

$$e^{2\phi} = \frac{Z_2}{Z_1}, \tag{D.2.33}$$

and the metric takes the form,

$$\begin{aligned} ds^2_{10} &= -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+\kappa)^2 + \sqrt{Z_1 Z_2}\, h_{mn} dx^m dx^n \\ &\quad + \frac{Z_3}{\sqrt{Z_1 Z_2}}(dz_5 + A^{(3)}_\mu dx^\mu)^2 + \sqrt{\frac{Z_2}{Z_1}}(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2), \end{aligned} \tag{D.2.34}$$

together with the associated RR-field components,

$$\begin{aligned} C^{(6)} &= A^{(1)}_\mu dx^\mu \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 + A^{(1)}_\mu A^{(3)}_\nu dx^\mu \wedge dx^\nu \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \\ C^{(2)} &= -A^{(2)}_\mu dx^\mu \wedge dx^5 - A^{(2)}_\mu A^{(3)}_\nu dx^\mu \wedge dx^\nu. \end{aligned} \tag{D.2.35}$$

The above solution can be considered in terms of a six-dimensional part and the remaining four-torus part. The six-form potential $C^{(6)}$ is just the electromagnetic dual of a two-form potential $C^{(2)}$ (refer to B.1). However, we do not need to follow the tedious steps to convert it to a two-form as by comparing the metric (D.2.34) with the GMR form (D.1.1), a complete dictionary between the variables in both the set-ups can be obtained. Also by using this dictionary the GMR form of the field strength (D.1.3) can be written in terms of the BW variables. The same results can be expected from electromagnetic duality.

The GMR formalism corresponds to minimal supergravity solution where the dilaton field is set to zero. In the present set-up (D.2.34) which we obtained from the Bena-Warner form of the solution, the dilaton can be set to zero by taking $Z_1 = Z_2$. Inserting $A^{(3)}_\mu dx^\mu$ from (D.2.13) in metric (D.2.34) we get,

$$ds^2_{10} = -2Z_1^{-1}(dt+\kappa)(dz_5+\omega_3) + Z_3 Z_1^{-1}(dz_5+\omega_3)^2 + Z_1 h_{mn} dx^m dx^n + ds^2_{\mathrm{T}_4}, \tag{D.2.36}$$

where

$$ds^2_{\mathrm{T}_4} = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2, \tag{D.2.37}$$

is the metric on the four-torus. Now with the following identifications

$$\begin{aligned} z_5 &= v, & Z_1 &= H, \\ Z_3 &= 1 - \frac{\mathcal{F}}{2}, & \omega_3 &= \beta, \\ \kappa &= \frac{\beta+\omega}{2}, & t &= \frac{u+v}{2}. \end{aligned} \tag{D.2.38}$$

we can match the Bena-Warner solution (D.2.36) with the GMR form (D.1.1). Also according to the dictionary (D.2.38), the GMR field strength (D.1.3) takes the following form :

$$\begin{aligned} G &= \frac{1}{2}\star dZ_1 - \frac{1}{4Z_1}(dz_5+\omega_3)\wedge[d\kappa - \star d\kappa] + \frac{1}{2Z_1}[(dt+\kappa) - \frac{Z_3}{2}(dz_5+\omega_3)]\wedge d\omega_3 \\ &\quad -\frac{1}{2Z_1^2}(dz_5+\omega_3)\wedge(dt+\kappa)\wedge dZ_1, \end{aligned} \tag{D.2.39}$$

which using the BW equations of motion simplifies to

$$2G = \star dZ_1 + d\left[(dz_5+\omega_3)]\wedge\left(\frac{dt+\kappa}{Z_1} - \omega_1\right)\right] + \omega_1\wedge d\omega_3. \tag{D.2.40}$$

The RR field strength in ten dimensions is normalised as $F = 2G$, with the associated 2-form field

$$C = -\left[\left(\frac{dt+\kappa}{Z_1} - \omega_1\right)\wedge(dz_5+\omega_3)\right] + \sigma, \tag{D.2.41}$$

where an explicit expression for σ cannot be obtained in general. It satisfies,

$$d\sigma = \star dZ_1 + \omega_1\wedge d\omega_3. \tag{D.2.42}$$

One can easily check that the three form $\star dZ_1 + \omega_1 \wedge d\omega_3$ appearing on the right hand side of equation (D.2.42) is exact due to BW equations of motion for Z_1.

D.3 Relation between GMR and BW

Now that we have a simple dictionary (D.2.38) we can easily relate BW and GMR equations of motion. On the GMR side, we look at v-independent solutions while on the BW side we consider solutions with $Z_1 = Z_2$ and $\omega_1 = \omega_2$.

We consider BW equations and using the dictionary transform them into GMR equations. Consider BW equation (D.2.15),

$$d\kappa + \star d\kappa = 2Z_1 d\omega_1 + Z_3 d\omega_3. \tag{D.3.43}$$

Rewriting this equation using dictionary (D.2.38), we have

$$\begin{aligned} 2d\omega_1 &= \frac{1}{Z_1}\left(d\kappa + \star d\kappa - Z_3 d\omega_3\right) && \text{(D.3.44)} \\ &= \frac{1}{2H}\left(d\omega + \star d\omega + 2(1-Z_3)d\beta\right) = \frac{1}{H}\left(d\omega + \star d\omega + \mathcal{F}d\beta\right) = \mathcal{G}^+, && \text{(D.3.45)} \end{aligned}$$

where we have used the fact that $d\beta = d\omega_3$ is self dual, cf. (D.2.14). It then immediately follows that $d\mathcal{G}^+ = 0$, which is one of the GMR equations, cf. (D.1.7). Similarly, from the BW scalar equations (D.2.16) for Z_1 we have,

$$\nabla^2 Z_1 = \nabla^2 H = -\star d \star dH = \star(d\omega_3 \wedge d\omega_2) = \star\left(\frac{d\beta \wedge \mathcal{G}^+}{2}\right), \tag{D.3.46}$$

which implies (D.1.5). Similarly,

$$\nabla^2 Z_3 = -\frac{1}{2}\nabla^2 \mathcal{F} = \star(d\omega_1 \wedge d\omega_2) = \star\left(\frac{\mathcal{G}^+ \wedge \mathcal{G}^+}{4}\right), \tag{D.3.47}$$

which implies (D.1.4).

Appendix E

General solution for wave equation in KK background

In this appendix we are going to discuss solution to the general wave equation 5.2.29 which we rewrite here

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} - \frac{L^2}{r^2}T + \frac{1}{Q_K^2 r^2}(V^2 r^2 - Q_K^2)\partial_\xi^2 T = 0. \tag{E.0.1}$$

where the angular momentum operator L^2 satisfies the following equation

$$-l(l+1)Y_{lm}^q = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y_{lm}^q) + \frac{1}{\sin^2\theta}(Q_K^2\partial_s^2 Y_{lm}^q + \partial_\phi^2 Y_{lm}^q - 2Q_K\cos\theta\partial_\phi\partial_s Y_{lm}^q). \tag{E.0.2}$$

Similarly we rewrite the radial equation (5.2.34)

$$\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{l(l+1)}{r^2}f + \frac{1}{Q_K^2 r^2}\left(V^2 r^2 - Q_K^2\right)\left(-\frac{q^2}{Q_K^2}\right)f = 0. \tag{E.0.3}$$

E.1 Angular Equation

Coming to the angular part which is given by the equation (E.0.2), writing the s, θ and ϕ-eigen vectors as Ψ_s, Ψ_θ and Ψ_ϕ respectively and using the eigenvalue equations (5.2.30) it reduces to just the θ-equation

$$-l(l+1)\Psi_\theta = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Psi_\theta) + \frac{1}{\sin^2\theta}(-q^2 - m^2 + 2qm\cos\theta)\Psi_\theta. \tag{E.1.4}$$

In the limit $q = 0$, the above angular equation reduces to

$$-l(l+1)\Psi_\theta = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Psi_\theta) - \frac{m^2}{\sin^2\theta}\Psi_\theta, \tag{E.1.5}$$

solution to which is given in terms of associated Legendre polynomial of first kind P_l^m and associated Legendre polynomial of second kind Q_l^m as

$$\Psi_\theta = c_1 P_l^m(\cos\theta) + c_2 Q_l^m(\cos\theta). \tag{E.1.6}$$

The second function $Q_l^m(\cos\theta)$ diverges at $\theta = 0$ so we set $c_2 = 0$. Thus the complete angular solution for $q = 0$ is given by

$$Y_l^m(\theta,\phi) = \Psi_\theta\Psi_\phi = Ce^{im\phi}P_l^m(\cos\theta), \tag{E.1.7}$$

C being the normalization constant.

Now, coming to the $q \neq 0$ case, making the change of variables $x = \sin^2\frac{\theta}{2}$ in equation (E.1.4) it reduces to

$$\Big[x(1-x)\frac{d^2}{dx^2} + (1-2x)\frac{d}{dx} - \frac{q^2+m^2-2qm(1-2x)}{4x(1-x)} + l(l+1)\Big]\Psi(x) = 0. \tag{E.1.8}$$

where now Ψ_θ is a function of x and denoted as $\Psi(x)$.

Taking the ansatz

$$\Psi(x) = x^p(1-x)^k F, \tag{E.1.9}$$

where F is a function of x, the angular equation (E.1.8) reduces

$$\begin{aligned} x(1-x)\frac{d^2F}{dx^2} &+ [2p(1-x) - 2kx + 1 - 2x]\frac{dF}{dx} + \Big[p(p-1)\frac{1-x}{x} + k(k-1)\frac{x}{1-x} \\ &-2pk + \frac{1-2x}{x}p - \frac{1-2x}{1-x}k - \frac{q^2+m^2-2qm(1-2x)}{4x(1-x)} + l(l+1)\Big]F = 0. \end{aligned} \tag{E.1.10}$$

This equation though looks very complicated yet can be solved using Mathematica. Solving this we get the solution in terms of hypergeometric functions ${}_2F_1$ and is of the form

$$\begin{aligned} F(x) &= (-1)^{\frac{(-m+q+1)}{2}} x^{-p}(1-x)^{-k}\left(\frac{1-x}{x}\right)^{\frac{(m+q)}{2}}\Big[C_1(-1)^q x^q {}_2F_1(q-l, l+q+1; \\ &\quad -m+q+1; x) + C_2(-1)^m x^m {}_2F_1(m-l, l+m+1; m-q+1; x)\Big], \end{aligned} \quad \text{(E.1.11)}$$

where C_1, C_2 are constant coefficients. ${}_2F_1$ is the hypergeometric function. From this we can write $\Psi(x)$ as

$$\begin{aligned} \Psi(x) &= (-1)^{\frac{1}{2}(-m+q+1)}\left(\frac{1-x}{x}\right)^{\frac{1}{2}(m+q)}\Big[C_1(-1)^q x^q {}_2F_1(q-l, l+q+1; -m+q+1; x) \\ &\quad + C_2(-1)^m x^m {}_2F_1(m-l, l+m+1; m-q+1; x)\Big]. \end{aligned} \quad \text{(E.1.12)}$$

Here, $x = \sin^2\left(\frac{\theta}{2}\right)$ which takes values $0 \leq x \leq 1$. The hypergeometric function ${}_2F_1(a, b; c; x)$ with $|x| < 1$ converges for $c \neq 0, -1, -2, \ldots$ which implies for the first term in (E.1.12) we have $q > m - 1$. For $l = 1$, we have $m = -1, 0, +1$. Thus we have a convergent series for the first term if $q > 0$. Similarly, for the second term we have $q < m + 1$ which implies $q < 0$. If we choose only $q < 0$ case then we can set $C_1 = 0$.

For $|x| = 1$, the hypergeometric function ${}_2F_1(a, b; c; x)$ converges for $c > a + b$ which implies for the solution to converge we need

$$q < -m. \quad \text{(E.1.13)}$$

Thus for the solution to be convergent we have the following values of q

$$q = -3/2, -2, \ldots. \quad \text{(E.1.14)}$$

The complete angular solution can be written as

$$Y_{lm}^q(s, \theta, \phi) = \Psi_s \Psi_\theta \Psi_\phi = N e^{im\phi} e^{iq\xi}\, \Psi\left(\sin^2\frac{\theta}{2}\right), \quad \text{(E.1.15)}$$

N is the normalization constant. This gives the angular solution to the wave equation (5.2.29). Next, we come to solving the radial part (5.2.34).

E.2 Radial Equation (r-equation)

The radial equation is given by (5.2.34)

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{l(l+1)}{r^2} f + \frac{q^2}{r^2}\left(1 - \frac{V^2 r^2}{Q^2}\right) f = 0. \tag{E.2.16}$$

In the limit $q = 0$ the above equation reduces to

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{l(l+1)}{r^2} f = 0, \tag{E.2.17}$$

solution to which is the same as in []given by

$$f(r) = \sum_{l \geq 0} [A r^l + B r^{-l-1}]. \tag{E.2.18}$$

Thus the complete s-independent solution to the Laplace equation is given in terms of angular part (E.1.7) and radial part (E.2.18).

We are trying to construct solutions which depends on the fibre direction s as well. We have already the form of the spherical harmonics Before considering the general radial let's consider the simplest case when $V = \frac{Q}{r}$. Then, the radial equation takes the following form,

$$r^2 \frac{d^2 f}{dr^2} + 3r\frac{df}{dr} - l(l+1) f = 0. \tag{E.2.19}$$

If we choose $f = r^\lambda$ the equation becomes

$$[\lambda(\lambda - 1) + 3\lambda - l(l+1)] f = 0 \tag{E.2.20}$$

For lowest harmonics choosing $l = 0$, solving $\lambda(\lambda - 1) + 3\lambda - l(l+1) = 0$ we get the radial function f to have the following form

$$f = C_1 + C_2 r^{-2}, \tag{E.2.21}$$

for some constant coefficients C_1 and C_2. However, for the general case when we have asymptotically flat spacetime

$$V = 1 + \frac{Q}{r}, \tag{E.2.22}$$

the radial equation (5.2.34) looks like

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{l(l+1)}{r^2} f - \frac{q^2}{Q^2} f - 2\frac{q^2}{rQ} f = 0. \tag{E.2.23}$$

This equation looks like radial Schrödinger equation in the $\hbar^2 = 2m$ limit. The associated energy and charge are given by

$$E = -\frac{q^2}{Q^2}, \qquad e^2 = -\frac{2q^2}{Q}. \tag{E.2.24}$$

Substituting the above expressions and taking the ansatz $f = \frac{S}{r}$, equation (E.2.23) reduces to

$$\frac{d^2 S}{dr^2} - \frac{l(l+1)}{r^2} S + \frac{e^2}{r} S + ES = 0. \tag{E.2.25}$$

Solving the above equation we get

$$S(r) = c_1 M_{-\sqrt{q^2}, l+\frac{1}{2}} \left(\frac{2\sqrt{q^2} r}{Q} \right) + c_2 W_{-\sqrt{q^2}, l+\frac{1}{2}} \left(\frac{2\sqrt{q^2} r}{Q} \right), \tag{E.2.26}$$

where M and W are the Whittaker functions [99] [100]. These functions can in turn be written in terms of confluent hypergeometric functions by the following relations

$$M_{k,n}(x) = e^{-x/2} x^{n+1/2} {}_1F_1(n - k + \frac{1}{2}; 2n+1; x), \tag{E.2.27}$$

$$W_{k,n}(x) = e^{-x/2} x^{n+1/2} U(n - k + \frac{1}{2}, 2n+1, x), \tag{E.2.28}$$

where ${}_1F_1$ is the confluent hypergeometric function of first kind and U is the confluent hypergeometric function of second kind. So from the expression (E.2.26) we can write the solution to the radial equation as,

$$f = \frac{S}{r} = e^{-\frac{\sqrt{q^2}r}{Q}} \left(\frac{2\sqrt{q^2}}{Q}\right)^{l+1} r^l \Bigg[C^{lm}_{1q}\,{}_1F_1(1+l+\sqrt{q^2}; 2l+2; \frac{2\sqrt{q^2}r}{Q}) + C^{lm}_{2q} U(1+l+\sqrt{q^2}, 2l+2, \frac{2\sqrt{q^2}r}{Q}) \Bigg]. \tag{E.2.29}$$

Here, q^2 can have both the roots $+q$ and $-q$. Note that, the confluent hypergeometric function ${}_1F_1$ and U become polynomials when $1 + l + \sqrt{q^2} \leq 0$. Here, we have $l = 1$ which means we need $\sqrt{q^2} \leq -2$. which is only possible if we choose

$$\sqrt{q^2} = q = -2, -5/2, -3, \dots \tag{E.2.30}$$

Or in other words we take only the negative square root part of $\sqrt{q^2} = -\tilde{q}$ where, $\tilde{q} = 2, 5/2, 3, \dots$. Thus we can write the radial solution as

$$f = e^{\frac{\tilde{q}r}{Q}} \left(\frac{-2\tilde{q}}{Q}\right)^{l+1} r^l \Bigg[C^{lm}_{1\tilde{q}}\,{}_1F_1(1+l-\tilde{q}; 2l+2; -\frac{2\tilde{q}r}{Q}) + C^{lm}_{2\tilde{q}} U(1+l-\tilde{q}, 2l+2, -\frac{2\tilde{q}r}{Q}) \Bigg]. \tag{E.2.31}$$

To avoid notational clutter henceforth we simply write q instead of $\tilde{q}$. However, we need to remember that unlike the q appearing in equation (E.1.14) of the angular solution part here q takes the values

$$q = 2, 5/2, 3, \dots. \tag{E.2.32}$$

Again for the choice of parameters, the confluent hypergeometric functions of the first kind ${}_1F_1$ can be related to the associated Laguerre functions $L^k_n(x)$ by the following relation

$${}_1F_1(-n; k+1; x) = \frac{n!k!}{(n+k)!} L^k_n(x), \tag{E.2.33}$$

with which we can write

$$
{}_1F_1\left(1+l-q;2l+2;-\frac{2qr}{Q}\right)=\frac{(2l+1)!(q-l-1)!}{(l+q)!}\quad L^{2l+1}_{q-l-1}\left(-\frac{2qr}{Q}\right). \tag{E.2.34}
$$

Associated Laguerre function $L^{\lambda}_{\nu}(x)$ becomes a polynomial if ν is a non-negative integer which implies in our case

$$
q-l-1\geq 0. \tag{E.2.35}
$$

with $l=1$ this is consistent with (E.2.32). Thus, f has the following form

$$
f=e^{\frac{qr}{Q}}\left(\frac{-2q}{Q}\right)^{l+1}r^l\Big[C^{lm}_{1q}U(1+l-q,2l+2,-\frac{2qr}{Q})+C^{lm}_{2q}e^{-\frac{2qr}{Q}}L^{2l+1}_{-q-l-1}\left(\frac{2qr}{Q}\right)\Big] \tag{E.2.36}
$$

where we have used

$$
L^{\lambda}_{-\nu-\lambda-1}(z)=(-1)^{\lambda}e^{z}L^{\lambda}_{\nu}(-z) \tag{E.2.37}
$$

Thus simplifying we can write

$$
f=\left(\frac{-2q}{Q}\right)^{l+1}r^l\Big[C^{lm}_{1q}e^{\frac{qr}{Q}}U(1+l-q,2l+2,-\frac{2qr}{Q})+e^{-\frac{qr}{Q}}C^{lm}_{2q}L^{2l+1}_{-q-l-1}\left(\frac{2qr}{Q}\right)\Big]. \tag{E.2.38}
$$

E.2.1 Asymptotic Limits

We need our solution to admit finite values at infinity. We need to consider the asymptotic limits of the special functions. First we will discuss the asymptotic properties of these functions in general.

Expanding the associated Laguerre $L^k_n(x)$ at $x\to\infty$ we have the following general expression

$$
\begin{aligned}
L_n^k(x) \quad \to \quad & e^x x^{-k-n}\Bigg(- \frac{\Gamma(k+n+1)\sin(n\pi)}{\pi x} - \frac{(n+1)(k+n+1)\Gamma(k+n+1)\sin(n\pi)}{\pi x^2} \\
& - \frac{(n+1)(n+2)(k+n+1)(k+n+2)\Gamma(k+n+1)\sin(n\pi)}{(2\pi)x^3} \\
& - \frac{(n+1)(n+2)(n+3)(k+n+1)(k+n+2)(k+n+3)\Gamma(k+n+1)\sin(n\pi)}{(6\pi)x^4} \\
& + O\left(\frac{1}{x^5}\right) + ...\Bigg) \\
& + x^n\Bigg(\frac{(-1)^n}{\Gamma(n+1)} + \frac{(-1)^{n+1}n(k+n)}{\Gamma(n+1)x} + \frac{(-1)^n(n-1)n(k+n-1)(k+n)}{2\Gamma(n+1)x^2} \\
& - \frac{(-1)^n(n-2)(n-1)n(k+n-2)(k+n-1)(k+n)}{6\Gamma(n+1)x^3} + O\left(\frac{1}{x^4}\right) + ...\Bigg).
\end{aligned}
\tag{E.2.39}
$$

As $x \to \infty$ neglecting higher order terms in $\frac{1}{x}$ for both the bracketed expressions, we are left with

$$
L_n^k(x) \to \left(-e^x x^{-k-n-1}\frac{\Gamma(k+n+1)\sin(n\pi)}{\pi} + \frac{(-x)^n}{\Gamma(n+1)}\right). \tag{E.2.40}
$$

Similarly, the general asymptotic expansion of the confluent hypergeometric function $U(n, k, x)$ at $x \to \infty$ is given by

$$
U(n, k, x) \to x^{-n}\left(1 + \frac{-n + kn - n^2}{x} + O\left(\frac{1}{x^2}\right) + ...\right). \tag{E.2.41}
$$

So asymptotically $U(n, k, x)$ goes as x^{-n}. Thus, asymptotically the solution (E.2.38) takes the form;

$$
\begin{aligned}
f \quad \sim \quad r^l\Bigg[& C_{1q}^{lm} e^{\frac{qr}{Q}}\left(-\frac{2qr}{Q}\right)^{q-l-1} + C_{2q}^{lm} e^{-\frac{qr}{Q}}\Bigg(\frac{\left(-\frac{2qr}{Q}\right)^{-q-l-1}}{\Gamma(-q-l)} \\
& -e^{\frac{2qr}{Q}}\left(\frac{-2qr}{Q}\right)^{q-l-1}\frac{\sin[(-q-l-1)\pi]\Gamma(-q+l+1)}{\pi}\Bigg)\Bigg]
\end{aligned}
\tag{E.2.42}
$$

By putting $l = 1$ and using the fact that $\Gamma(n)$ diverges for $n \leq 0$ we can check that the first term in the C_2 part vanishes where as the last term in the C_2 part diverges for all allowed values of q i.e. $q = 2, 5/2, 3, \ldots$ and no finite solution is possible at infinity. If we take $C^{lm}_{2q} = 0$ then the asymptotic radial solution becomes

$$f \sim rC^{lm}_{1q} e^{\frac{qr}{Q}} \left(-\frac{2qr}{Q}\right)^{q-2} \sim r^{q-1} e^{\frac{qr}{Q}}, \tag{E.2.43}$$

which again diverges at $r \to \infty$ for all possible values of q.

E.2.1.1 $r \to 0$ limit

In the $z \to 0$ limit, the Hypergeometric $U(a, b; z)$ function expands as;

$$\begin{aligned} U(a,b,z) \;\; &\sim \;\; z^{-b}\left(\frac{\Gamma(-1+b)z}{\Gamma(a)} + \frac{(-1-a+b)\Gamma(-1+b)z^2}{(-2+b)\Gamma(a)} + \ldots\right) \\ &\qquad + \left(\frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \frac{a\Gamma(1-b)z}{b\Gamma(1+a-b)} + \ldots\right) \end{aligned} \tag{E.2.44}$$

In our case $a = 1 + l - q$, $b = 2l + 2$ and $z = -\frac{2qr}{Q}$.

So putting $l = 1$ at $r \to 0$ we have for the C_1-term in (E.2.38),

$$\begin{aligned} re^{\frac{qr}{Q}} U(2-q, 4, -\frac{2qr}{Q}) \;\; &\sim \;\; re^{\frac{qr}{Q}} \left(-\frac{2qr}{Q}\right)^{-4} \left(\frac{\Gamma(3)\left(-\frac{2qr}{Q}\right)}{\Gamma(2-q)} + \frac{(q+1)\Gamma(3)}{2\Gamma(2-q)}\left(-\frac{2qr}{Q}\right)^2 + O(r^3) + \ldots\right) \\ &\quad + re^{\frac{qr}{Q}} \left[\left(\frac{\Gamma(3)}{\Gamma(-1-q)} + \frac{(2-q)\Gamma(3)}{4\Gamma(-1-q)}\left(-\frac{2qr}{Q}\right) + O(r^2) + \ldots\right)\right] \end{aligned} \tag{E.2.45}$$

Now since for $q = 2, 5/2, 3, \ldots$, $\Gamma(2-q)$ and $\Gamma(-1-q)$ diverges so we get all terms vanishing in the $r \to 0$ expansion of C_1-term.

Now coming to the C_2-term which for $l = 1$ is of the form

$$r^l e^{-\frac{qr}{Q}} L^{2l+1}_{-q-l-1}\left(\frac{2qr}{Q}\right) = re^{-\frac{qr}{Q}} L^3_{-q-2}\left(\frac{2qr}{Q}\right) = -re^{\frac{qr}{Q}} L^3_{q-2}\left(\frac{-2qr}{Q}\right) \tag{E.2.46}$$

where in the last step we have used (E.2.37).

For $q = 2$ the series expansion of $L^3_0\left(-\frac{4r}{Q}\right)$ at $r = 0$ gives $\frac{1}{\Gamma(1)} = 1$. So we can see for $q = 2$ that at $r \to 0$ the radial solution (E.2.38) converges. Similarly, we can check for other values of q.

Thesis Highlight

Name of the Student: Deepali Mishra

Name of the CI/OCC: NISER **Enrolment No.: PHYS11201404001**

Thesis Title: Black hole Microstates and Solution Generating Techniques

Discipline: Physics **Sub-Area of Discipline:** Theoretical High Energy Physics

Date of viva-voce : 17/11/2020

We have developed a solution generating technique, named as Generalised Garfinkle-Vachaspati transform (GGV) which effectively generates new supergravity solutions from the existing ones. The necessary conditions for the implementation of this technique are that the background solution admits a null Killing vector k^{μ} and one or more covariantly constant spacelike vectors $l^{\mu}{}_{(i)}$. Then by suitable transformations of the metric $g_{\mu\nu}$ and the associated matter fields the GGV can generate a new solution as long as the matter field satisfies some additional constraint. We have successfully verified our technique by direct computation of the equations of motion for the case of supergravity solutions that are given in terms of metric $g_{\mu\nu}$, associated two-form matter field $C_{\mu\nu}$ and dilaton field φ . Then GGV on the string frame fields as given by,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \Psi\, e^{-\varphi}\, (k_{\mu} l_{\nu} + k_{\nu} l_{\mu}),$$

$$C_{\mu\nu} \rightarrow C_{\mu\nu} - \Psi\, e^{-2\varphi}\, (k_{\mu} l_{\nu} - k_{\nu} l_{\mu}),$$

is a valid solution generating technique as long as the associated 3-form field strength $F = dC$ satisfies the following transversality condition

$$k^{\mu} F_{\mu\nu\rho} = -d(e^{-\varphi} k)_{\nu\rho}.$$

Here, Ψ is a scalar field that satisfies the following wave equation with respect to the background metric i.e.

$$\nabla_{\mu}(e^{-2\varphi} g_{\mu\nu} \nabla_{\nu} \Psi) = 0,$$

and it is compatible with the Killing symmetries of the background solution for which it needs to satisfy $k^{\mu}\partial_{\mu}\Psi = 0, l^{\mu}\partial_{\mu}\Psi$= 0.

We also studied the applications of GGV on a class of D1-D5-P solutions which are considered to have contribution to black hole microstates and hence black hole entropies.